

Applications of chiral perturbation theory to lattice QCD

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ABSTRACT

These notes contain the written version of lectures given at the 2009 Les Houches Summer School “Modern perspectives in lattice QCD: Quantum field theory and high performance computing.” The goal is to provide a pedagogical introduction to the subject, and not a comprehensive review. Topics covered include a general introduction, the inclusion of scaling violations in chiral perturbation theory, partial quenching and mixed actions, chiral perturbation theory with heavy kaons, and the effects of finite volume, both in the p - and ϵ -regimes.

I. WHY CHIRAL PERTURBATION THEORY FOR LATTICE QCD?

It is often claimed that lattice QCD provides a tool for computing hadronic quantities numerically, with fully controlled systematic errors, from first principles. This assertion is of course based on the fact that lattice QCD provides a nonperturbative definition of QCD (in fact, the only one to date). So, why not choose the parameters (in particular, the quark masses) at, or very near, their physical values, and compute all quantities of interest?

There are at least two major obstacles to this. The first obstacle is that lattice QCD is formulated in euclidean space, rather than in Minkowski space. This is a necessary restriction if one wants to use Monte Carlo methods in order to evaluate expectation values of operators from which one extracts physical quantities. In euclidean space correlation functions do not give direct access to physical scattering amplitudes; they first need to be continued to Minkowski space. If one is interested in physics at some scale for which an effective field theory (EFT) is available, one can use this EFT in order to match to euclidean lattice correlation functions in the regime of validity of the EFT. In the case of the chiral EFT for QCD, the form of the correlation functions is predicted in terms of a finite number of coupling constants to a given finite order in a momentum expansion. Once the value of these is known from lattice QCD computations, the EFT can then be used to continue to Minkowski space. Chiral perturbation theory (ChPT) provides this EFT framework for the low-energy physics of the (pseudo-) Nambu–Goldstone bosons of QCD.

A second obstacle is that in practice lattice QCD computations are carried out at values of the up and down quark mass larger than those observed in nature,¹ because of limits on the size of the physical volumes that can be handled with presently available computers, and because of the rapid increase in algorithmic cost with decreasing quark masses. If the quark masses are nevertheless still small enough for ChPT to be applicable, it can be used to extrapolate to the physical values for these light quark masses, again using ChPT to connect an “unphysical” lattice computation (namely, at values of the light quark masses larger than their real-world values) to physical quantities (those observed in nature).

It should be said that lattice QCD is moving toward the physical point, *i.e.*, that numerical computations are being done with the light (up and down) quark masses at or very near the physical values. If lattice quark masses are very near the physical values, simple smooth extrapolations are in principle enough to obtain hadronic quantities of interest at the physical point, and the use of ChPT may become less important in this respect. I will return to this point below.

Both of the obstacles described above are examples of how ChPT (and, in general, EFTs) can be used to connect “unphysical” computations with physical quantities. This is particularly helpful in the case of lattice QCD, where often computations are easier, or even only possible, in some unphysical regime. The use of ChPT is by no means limited to extrapolations in quark masses, or continuation from euclidean space. Other examples of the use of ChPT to bridge the gap between the lattice and the real world are, in increasing order of boldness:

1. Nonzero lattice spacing. While in state-of-the-art lattice computations the lattice spacing is small (in units of Λ_{QCD}), scaling violations are still significant. It is possible

¹ The strange quark mass can be taken at its physical value, although it is theoretically interesting to see what happens if one varies the strange quark mass as well.

to extend ChPT to parametrize scaling violations afflicting the physics of Nambu–Goldstone bosons. In particular, depending on what lattice fermions one uses, chiral symmetry (and rotational invariance!) are broken on the lattice, and such breakings can be parametrized in ChPT through new coupling constants that vanish in the continuum limit. This will be the topic of the second lecture, Sec. III. Another example in this category is the use of a finite volume in numerical computations. The dominant effects of this can also be studied in ChPT, as we will see in Sec. VI.

2. Partial quenching and the use of mixed actions. As we will see, on the lattice one is free to choose the valence quark masses to be different from sea quark masses. This is of course completely unphysical (it violates unitarity!), but it turns out to be very practical – it is (often much) cheaper to vary only valence quark masses than to vary the sea (or “dynamical”) quark mass, which affects the ensemble of gauge configurations. Moreover, as we will see, varying valence and sea quark masses independently from each other gives us extra handles on the theory, making it easier to determine the couplings of ChPT than it would be without this possibility. This trick is known as “partial quenching.”² One may even go further: one can use different discretizations for sea quarks (typically a computationally less expensive method, such as staggered or Wilson quarks) and for valence quarks (a discretization using lattice fermions with very good chiral and flavor symmetries — operator mixing depends primarily on the symmetries of those operators, which are constructed out of the valence quarks). This generalization of partially-quenched QCD (PQQCD) is now often referred to as “mixed-action” QCD. We will discuss these methods in the third lecture, Sec. IV.
3. Other unphysical constructions used in lattice QCD; a prominent example is the use of “fourth-rooted” staggered fermions. Again, EFT techniques provide much insight into the nature of this method, and the comparison of EFT calculations with numerical results can be used to test the validity of such methods.

Even if one is only interested in physical quantities which can directly be computed on the lattice, such as (stable) hadron masses at the physical values of the quark mass for a very small lattice spacing, ChPT may still be used to test the results. By varying the quark mass away from the physical value, one might discover that the numerical results do not match the predictions of continuum ChPT, for instance because the lattice spacing is still too large, the volume too small, or because of some other systematic effect. In other words, ChPT provides a useful tool for the validation of results obtained with lattice QCD.

Extending this latter argument, it can also be very interesting to use lattice QCD as a laboratory to explore what strong interaction physics would look like at parameter values different from those in the real world. Again, interpreting the results of numerical computations in lattice QCD through the help of EFT techniques can be very useful in this respect. An example of this is the question about whether it is possible that the up-quark mass m_u could be zero.³ This question has now been answered with the help of lattice QCD

² “Quenched QCD” corresponds to the case in which the sea quark masses are taken to infinity, making the fermion determinant constant. This is of course unphysical “beyond repair”: all effects of quark loops are irrevocably gone.

³ Which would solve the strong CP problem.

computations at *nonzero* m_u (the answer is that $m_u = 0$ is now excluded with a very high confidence level). This is only possible with assistance of ChPT, which makes it possible to extrapolate to what physics would have looked like at $m_u = 0$, from computations done at nonzero values of m_u .

Finally, with limited computational resources, there is the important question as to how to use them most judiciously. Does one use all resources to do a few, very expensive, computations at physical quark masses, in very large volume? While this sounds very attractive, restrictions on available computational resources may force one to choose only larger lattice spacings, and one thus pays with the increase of one systematic error for the removal of other ones. Or does one try to strike a balance between the various systematic errors which afflict lattice computations, by using larger-than-physical quark masses, and less expensive fermions with symmetry properties that are not as nice? What choices to make depends very much on the physics one is interested in, and they are to some extent subjective. However, ChPT and its extension to unphysical situations helps us with making such choices, and it therefore is, and will likely remain, a very important tool for lattice QCD practitioners.

The aim of these lectures is not to give a review of all the work that has been done in ChPT and its applications to lattice QCD. Rather, the aim is to provide a basic introduction to ChPT, with emphasis on its applications to lattice QCD, geared at readers with a general knowledge of quantum field theory and the basics of lattice QCD. The list of references included is heavily biased toward the papers from which I learned the subject. It will not be possible to refer to all papers that have been written on this and closely related topics and applications; there are probably at least a few hundred *more* than the references I did include. The references I did include are primarily those in which new concepts or methods are introduced and explained, and I strongly recommend them for delving deeper into the many aspects and applications of ChPT in its relation to QCD on the lattice.

It is not even possible to cover all the relevant and important applications that have been considered in the literature. Notable examples of topics left out are applications to baryons (including EFTs for two-nucleon systems), and to hadrons containing heavy quarks, although some aspects of these topics are shared by “heavy-kaon” ChPT, which we consider in Sec. V. The topics I have chosen to include are a general introduction to ChPT in Sec. II, the incorporation of scaling violations in ChPT in Sec. III, partially quenched and mixed-action ChPT in Sec. IV, two-flavor ChPT with a heavy kaon in Sec. V, and ChPT in finite volume in Sec. VI.

Good reviews of continuum chiral perturbation theory can for instance be found in Ref. [1], and, at the level of a textbook, Refs. [2, 3]. For a more general introduction to EFT methods, including ChPT, see Ref. [4]. For applications of EFT methods to lattice QCD, see Ref. [5]; for a more detailed review of ChPT in the context of lattice QCD, see Ref. [6]. Another lattice-oriented introduction, with applications of ChPT to lattice calculations of weak matrix elements, is Ref. [7]. For a recent review of applications of ChPT to hadron phenomenology, see Ref. [8].

II. CONTINUUM CHIRAL PERTURBATION THEORY

We begin with a review of ChPT in the continuum, in infinite volume. This is a vast subject in itself, with many important applications to phenomenology. I will not review any of those here; the idea is to get an understanding of the basics, which we will need for

applications to lattice QCD.

A. Chiral symmetry

Let us consider QCD with $N_f = 3$ flavors, the up, down and strange quarks. (We will also have reason to consider the $N_f = 2$ theory, with only up and down quarks, in situations where the strange quark can be considered heavy.) Of course, heavier quarks exist, but we will be interested in physics at energy scales well below the charm mass, so we may consider the theory in which the charm, bottom and top quarks have been decoupled by integrating them out. In deference to the lattice, we will work in euclidean space throughout these lectures, continuing back to Minkowski space when necessary. The fermion part of the QCD lagrangian is

$$\mathcal{L} = \bar{q}_L \not{D} q_L + \bar{q}_R \not{D} q_R + \bar{q}_L M q_R + \bar{q}_R M^\dagger q_L , \quad (2.1)$$

in which q_i , $i = u, d, s$ is the three-flavor quark field. The subscripts L and R denote left- and right-handed projectors:

$$\begin{aligned} q_L &= \frac{1}{2}(1 - \gamma_5) q , & q_R &= \frac{1}{2}(1 + \gamma_5) q , \\ \bar{q}_L &= \bar{q} \frac{1}{2}(1 + \gamma_5) , & \bar{q}_R &= \bar{q} \frac{1}{2}(1 - \gamma_5) . \end{aligned} \quad (2.2)$$

Note that we may define $\bar{q}_{L,R}$ as we like, since the Grassmann fields q and \bar{q} are independent in euclidean space; we have chosen them to be consistent with $\bar{q} \rightarrow q^\dagger \gamma^0$ in the operator formalism. M is the quark mass matrix, equal to

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} . \quad (2.3)$$

For massless QCD, with $M = 0$, the lagrangian is invariant under the symmetry group $U(3)_L \times U(3)_R$:

$$\begin{aligned} q_L &\rightarrow U_L q_L , & q_R &\rightarrow U_R q_R , \\ \bar{q}_L &\rightarrow \bar{q}_L U_L^\dagger , & \bar{q}_R &\rightarrow \bar{q}_R U_R^\dagger , \end{aligned} \quad (2.4)$$

with $U_{L,R} \in U(3)_{L,R}$. We observe that the full QCD lagrangian is invariant if we also let M transform as

$$M \rightarrow U_L M U_R^\dagger . \quad (2.5)$$

Of course, the quark masses do not transform, but this ‘‘spurion’’ trick, of letting some parameters transform under the symmetry group, will be useful later, for instance in Sec. II E. Note that in euclidean space, \mathcal{L} (with Eq. (2.5)) is invariant under a larger group, $GL(3, \mathbb{C})_L \times GL(3, \mathbb{C})_R$. (If we consider this larger group, U_L^\dagger and U_R^\dagger in Eqs. (2.4) and (2.5) have to be replaced by U_L^{-1} and U_R^{-1} , respectively.) But if we identify \bar{q} with $q^\dagger \gamma^0$, this reduces this larger group to $U(3)_L \times U(3)_R$, from which it can be shown that the larger symmetry group in euclidean space has no additional physical consequences.

It is well known, of course, that ‘‘axial $U(1)$,’’ *i.e.*, transformations with $U_L = U_R^\dagger = \exp(i\theta)\mathbf{1}$, are not preserved when QCD is quantized, even when $M = 0$. The actual symmetry group is therefore $SU(3)_L \times SU(3)_R \times U(1)$, where the $U(1)$ factor is quark number.

On mesons, the topic of these lectures, quark-number $U(1)$ is trivially realized, and we may drop it from consideration.

In the real world, we find that there exists an octet of pseudoscalar mesons with masses smaller than any other hadron masses. Moreover, one observes that relations between their masses and interactions are reasonably well described by assuming an approximate $SU(3)$ flavor symmetry.⁴ Under this $SU(3)$, the pseudoscalar mesons form an octet

$$\phi = \begin{pmatrix} \frac{\pi_0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi_0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \sim \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}, \quad (2.6)$$

where we also indicated the (valence) quark content.⁵ (In the theory with only two light flavors, ϕ reduces to the two-by-two upper left-hand block, with the η omitted.) Under $SU(3)$ this octet transforms as

$$\phi \rightarrow U\phi U^\dagger, \quad (2.7)$$

and it is clear that this $SU(3)$ can be identified with the diagonal subgroup $SU(3)_V$ of $SU(3)_L \times SU(3)_R$, for which $U_L = U_R = U$ in Eq. (2.4).

If the full symmetry group $SU(3)_L \times SU(3)_R$ were realized “manifestly” in nature, we would observe larger hadronic multiplets, and in particular, we would observe “parity partners,” *i.e.*, pairs of hadrons with the same mass but opposite parity, since parity takes left-handed quarks into right-handed ones and *vice versa*.⁶ Such parity partners are in general not observed, and specifically, there are no scalar mesons with masses near those of the pseudoscalar multiplet of Eq. (2.6). Instead, it is universally believed (with very strong evidence from both the real world and lattice computations) that, in a (hypothetical!) world with massless quarks, the chiral group is spontaneously broken to the diagonal subgroup, $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$. This requires eight Nambu–Goldstone bosons (NGBs), which are identified with the pions, eta and kaons of Eq. (2.6). Since the broken generators distinguish between left- and right-handed quarks, these NGBs have to be pseudoscalars. In the real world, quarks are not massless, but if the quark masses $m_{u,d,s}$ are small compared to Λ_{QCD} , they can be treated as a perturbation (as we will see below), giving the NGBs a small mass. Indeed, the pions have a very small mass of about 140 MeV, while the kaons and eta have a mass of about 500 MeV, which is at least smallish compared to other hadron masses. For a proof that $SU(3)_V$ does not undergo spontaneous symmetry breaking in continuum QCD, see Ref. [9].

Our interest in these lectures will be in the physics of these NGBs, at energies below those at which any other hadrons can be produced. We therefore expect that it should be possible to find an EFT for the physics of NGBs. What that means is that it should be possible to write down a local lagrangian in terms of the field ϕ that, in some systematic approximation, reproduces correlation functions involving only NGBs, with restrictions imposed by the principles of quantum field theory. The chiral lagrangian is precisely this EFT.

⁴ We will return to the issue that the strange quark mass is much larger than the up and down quark masses, see Sec. V.

⁵ We ignore isospin breaking, which causes the fields π^0 and η of Eq. (2.6) to mix, in most of these lectures.

⁶ In mathematical terms, parity is an automorphism of the group $SU(3)_L \times SU(3)_R$.

B. The chiral lagrangian

Let us therefore start with a closer look at spontaneous symmetry breaking of chiral symmetry, for the case that $M = 0$. An order parameter for this breaking is the renormalized condensate

$$\langle \bar{q}_{Ri} q_{Lj} \rangle = \langle \bar{q}_{Li} q_{Rj} \rangle \propto \Lambda_{QCD}^3 \delta_{ij} , \quad (2.8)$$

where the power of Λ_{QCD} estimates the magnitude of the condensate. As is always the case with spontaneous symmetry breaking, there is a manifold of equivalent vacua, and indeed, one may rotate this condensate as

$$\Omega_{ij} = \langle q_{Li} \bar{q}_{Rj} \rangle \rightarrow U_L \Omega U_R^\dagger , \quad (2.9)$$

where I indicated how Ω transforms under the chiral group (here I sum over spin and color indices, so that Ω is a color and spin singlet). These vacua are all equivalent, and they are rotated into each other by elements of the coset $SU(3)_L \times SU(3)_R / SU(3)_V$, which happens to be isomorphic to the group $SU(3)$. The standard choice of Eq. (2.8) leaves the diagonal subgroup $SU(3)_V$ invariant, but this does not mean that with another choice of condensate there would be no $SU(3)$ invariance: the unbroken $SU(3)$ would simply be differently embedded in $SU(3)_L \times SU(3)_R$.⁷

The color-singlet operators $\text{tr}(\Gamma q_{Li} \bar{q}_{Rj})$ and their parity conjugates, when acting on the vacuum, create mesons with flavor quantum numbers corresponding to the flavor indices i and j , and spin-parity corresponding to the matrix Γ , which is some product of Dirac gamma matrices. Since we are interested here in scalar and pseudoscalar mesons, we choose $\Gamma = 1$. Introducing a complex scalar field H_{ij} with the same quantum numbers, we can write down an effective lagrangian of the form

$$\mathcal{L}_{eff} = \text{tr}(\partial_\mu H^\dagger \partial_\mu H) + \mathcal{L}_{int}(H, H^\dagger) + \mathcal{L}_{other}(H, H^\dagger, \text{other hadrons}) . \quad (2.10)$$

This lagrangian should obey the same symmetries as the QCD lagrangian, and in particular, it should be invariant under $SU(3)_L \times SU(3)_R$, with $H \rightarrow U_L H U_R^\dagger$. We can now decompose $H = R \Sigma$ with R hermitian and positive, and Σ unitary. Symmetry breaking with the pattern $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$ implies that R picks up an expectation value, which again we can take to be proportional to the unit matrix, as in Eq. (2.8). Σ parametrizes the vacuum manifold, and thus describes the NGBs predicted by chiral symmetry breaking. If all other hadrons are massive, and we are interested only in the physics of NGBs below the typical hadronic scale, we can integrate out all non-Goldstone fields, leaving us with a low-energy effective lagrangian in terms of only Σ . This lagrangian is local at scales below the hadronic scale. Because of the anomalous axial $U(1)$, the meson associated with this symmetry is also heavy, and we may take Σ in $SU(3)$ rather than $U(3)$, and parametrize it as

$$\Sigma = \exp(2i\phi/f) \rightarrow U_L \Sigma U_R^\dagger , \quad (2.11)$$

with ϕ the field of Eq. (2.6). The field Σ inherits its transformation under the chiral group from the chiral transformation of H , as indicated above. We inserted a dimensionful parameter $f \propto \Lambda_{QCD}$ so that the field ϕ has the canonical mass dimension of a scalar field.

⁷ If we choose some other value Ω for the condensate, the subgroup leaving this invariant is the subgroup for which $U_L = \Omega U_R \Omega^{-1}$, which is isomorphic to $SU(3)_V$. Note that for a different value of Ω the definition of parity would also need to be modified.

Observe that Σ precisely parametrizes the vacuum manifold as in Eq. (2.9) for the choice $\Omega_{ij} = -\langle \bar{q}q \rangle \delta_{ij} / (2N_f)$. The unbroken group $SU(3)_V$ is linearly realized: Eq. (2.11) implies indeed that ϕ transforms as in Eq. (2.7) under $SU(3)_V$. This is not true for the full chiral group: the field ϕ transforms nonlinearly under all symmetries outside the subgroup $SU(3)_V$. For more discussion, see Sec. V.

Our lagrangian thus simplifies to

$$\mathcal{L}_{eff} = \frac{1}{8} f^2 \text{tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma) + \mathcal{L}_{int}(\Sigma, \Sigma^\dagger) \quad (2.12)$$

where I have normalized the first term such that, upon expanding Σ in terms of ϕ , the meson fields of Eq. (2.6) have properly normalized kinetic terms. While this is progress, we clearly need more input to turn this into any practical use.

The EFT does not have to be renormalizable. In fact, one expects the presence of a cutoff of order 1 GeV, because we integrated out all other hadrons, with masses of order 1 GeV and up. But we do expect that pion⁸ interactions obey all fundamental properties of quantum field theory: unitarity and causality, crossing symmetry, clustering and Lorentz invariance [10], of course all to the precision with which the EFT reproduces the physics predicted by QCD. We will assume⁹ that all these properties are satisfied if we take \mathcal{L}_{eff} to be the most general local, Lorentz (or euclidean) invariant function of the field Σ and its derivatives [10].¹⁰ Moreover, since the EFT should obey all symmetries of the underlying theory, it should be invariant under $SU(3)_L \times SU(3)_R$.

Our task is therefore to construct local invariants out of the field Σ . Because of the transformation rule for Σ , *cf.* Eq. (2.11), such invariants can only be formed by alternating Σ with Σ^\dagger , and taking a trace of such a product. One may also multiply such traces together to form new invariants. However, all such invariants collapse to a constant, because $\Sigma \Sigma^\dagger = \mathbf{1}$! The only way out is to allow for derivatives on the fields, as in the first term of Eq. (2.12). Because of Lorentz invariance, we need at least two such derivatives, and we can organize terms in our lagrangian into groups of terms with the same number of derivatives. The unique term with just two derivatives is the one shown as the first term in Eq. (2.12).

This construction leads to several fundamental consequences. First, since we need at least two derivatives in all terms, pion interactions vanish when their momenta vanish, and thus they become weak for small momenta (and masses, as we will see below). This, in turn, implies that we may organize our EFT in terms of a derivative expansion: pion correlation functions with small momenta on the external legs can be expanded in terms of a small parameter p/Λ_χ , where p is a typical pion momentum, and Λ_χ is the typical hadronic scale, of order 1 GeV.¹¹

These observations give us a very powerful method for constructing the chiral lagrangian. If our aim is to work only to leading order in the derivative expansion, the chiral lagrangian is simply given by the first term in Eq. (2.12):

$$\mathcal{L}^{(2)} = \frac{1}{8} f^2 \text{tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma) . \quad (2.13)$$

⁸ I will often use “pion” to refer to all (pseudo-) Nambu–Goldstone bosons, kaons and eta included.

⁹ I am not aware of any general proof from the underlying theory, QCD.

¹⁰ A more constructive argument, in which clustering plays a central role, was given in Ref. [11].

¹¹ This assumes that there are no other “accidentally” massless (or very light) hadrons other than the pions.

Note that this is an interacting theory, because Σ is nonlinear in the pion fields ϕ . It therefore describes all pion physics to leading order in pion momenta. For example, writing ϕ as $\phi = \phi_a T_a$, with T_a the generators of $SU(3)$ obeying

$$\text{tr}(T_a T_b) = \delta_{ab} \ , \quad [T_a, T_b] = \sqrt{2} i f_{abc} T_c \ , \quad (2.14)$$

one finds the leading-order prediction for the pion scattering amplitude from a tree-level calculation using Eq. (2.13):

$$\mathcal{A}(ab \rightarrow cd) = \frac{1}{3f^2} \{ (t-u) f_{abe} f_{cde} + (s-u) f_{ace} f_{bde} + (s-t) f_{ade} f_{bce} \} \ , \quad (2.15)$$

in which s , t and u are the Mandelstam variables. This is our first prediction from ChPT.

Beyond leading order, we need to worry about two different (but, as we will see, intricately related) issues. We need to construct the most general term of order p^4 , *i.e.*, with four derivatives, and we also need to worry about unitarity (up to a given order). A nice way of building terms with higher derivatives starts from the observation that we can define an object with one derivative that transforms only under $SU(3)_L$, or, analogously, an object which transforms only under $SU(3)_R$:

$$\begin{aligned} L_\mu &= \Sigma \partial_\mu \Sigma^\dagger = -\partial_\mu \Sigma \Sigma^\dagger \ , \\ R_\mu &= \Sigma^\dagger \partial_\mu \Sigma = -\partial_\mu \Sigma^\dagger \Sigma = -\Sigma^\dagger L_\mu \Sigma \ . \end{aligned} \quad (2.16)$$

The most general lagrangian with four derivatives is [12]

$$\mathcal{L}^{(4)} = -L_1 (\text{tr}(L_\mu L_\mu))^2 - L_2 \text{tr}(L_\mu L_\nu) \text{tr}(L_\mu L_\nu) - L_3 \text{tr}(L_\mu L_\mu L_\nu L_\nu) \ , \quad (2.17)$$

in which $L_{1,2,3}$ are new (dimensionless) coupling constants. For general N_f , there is an additional term of the form $\text{tr}(L_\mu L_\nu L_\mu L_\nu)$, but for $N_f = 3$ this can be written in terms of the three terms appearing in Eq. (2.17) [12]. Even with only three light quarks, this is not true in the partially-quenched case, see Sec. IV.

We see that the new terms in $\mathcal{L}^{(4)}$ will contribute to pion scattering at tree level: each of these terms starts off with a $(\partial\phi/f)^4$ term when we expand Σ in terms of ϕ . Thus, indeed, such contributions will be of order $(p/f)^4$, *i.e.*, of order $(p/f)^2$ relative to Eq. (2.15) – apparently f plays the role of the typical hadronic scale ~ 1 GeV, and $(p/f)^2$ is the expansion parameter of ChPT – see below.

In order to preserve unitarity, we should also calculate loop contributions, insofar as they contribute to order p^4 . They arise when we consider the one-loop scattering diagrams with two four-pion vertices from $\mathcal{L}^{(2)}$. This leads to contributions of the form $(s/4\pi f^2)^2 \log(-s/\Lambda^2)$ (and more such terms also involving t and u , as dictated by crossing symmetry), where Λ is the cutoff needed in order to define the theory, and the factor $1/(4\pi)^2$ comes from the loop integral. The appearance of the logarithm is interesting for two reasons: it shows that, as mandated by unitarity, there is a two-particle cut in the pion scattering amplitude. Second, it is clear that a cutoff is needed in order to define theory.

Physical quantities, such as the scattering amplitude, cannot depend on this cutoff, which has only been introduced in order to define the effective theory. But indeed, since also polynomial terms from $\mathcal{L}^{(4)}$ appear at this order (order p^4), the cutoff dependence can be removed by taking $L_{1,2,3}$ to be dependent on Λ , such that the scattering amplitudes (and other physical quantities) are not. We see that in order to renormalize the theory defined

by $\mathcal{L}^{(2)}$, we need to introduce $\mathcal{L}^{(4)}$. The new constants $L_{1,2,3}$ represent the effect of the underlying physics that has been integrated out in the EFT. This is the beginning of a general pattern, as we will argue in Sec. II C below.

Before we discuss higher orders, let us consider order p^4 in some more detail. First, note that the one-loop contribution is in fact of order $p^2/(4\pi f)^2$ relative to Eq. (2.15). Any EFT involves an expansion in a ratio of scales, and in ChPT the higher scale in this ratio appears in the form $4\pi f$. The size of f can be estimated from pion scattering (using Eq. (2.15)); for another method, see Sec. II D. At the level of ChPT, the “low-energy” constants (LECs) f and L_i are free parameters, which can only be determined either by comparison with experiment, or by matching ChPT to a lattice QCD computation. We can say something about the “generically expected” values of the L_i . Since the L_i absorb the scale dependence, we expect that their magnitude changes by an amount of order $1/(4\pi)^2 \log(\Lambda/\Lambda')$ when we change the scale $\Lambda \rightarrow \Lambda'$. Since the cutoff Λ appears because we integrated out all heavier hadrons, a physically sensible choice for Λ is the typical hadronic scale of 1 GeV. With this interpretation, varying the cutoff within an order of magnitude is reasonable. This shifts the L_i by an amount of order $1/(4\pi)^2$, and gives us a sense of what one expects the values of these LECs to be.

C. Power counting

Now, let us consider the derivative expansion more systematically, and show that indeed the chiral theory is a proper EFT. What this means is that there exists a systematic power counting, and that to any order in the expansion we need only a finite number of coupling constants to define the theory.

Consider an amputated connected diagram with V_d vertices with d derivatives. Collectively denoting the external momenta by p , this diagram is of order $\sum_d dV_d - 2I + 4L$ in p , where I is the number of internal lines, and L is the number of loops. This follows from simply counting powers of p . Using, as usual, that the number of loops $L = I - \sum_d V_d + 1$, we can rewrite this as $\sum_d (d-2)V_d + 2L + 2$. Our diagram is thus of order

$$f^2 p^2 \left(\frac{p^2}{f^2}\right)^N \left(\frac{1}{f}\right)^E, \quad (2.18)$$

with

$$N = \sum_d \frac{1}{2}(d-2)V_d + L, \quad (2.19)$$

and E the number of external legs. The powers of f in this result follow from dimensional analysis. Ignoring the cutoff for now, this is the only other scale in the problem, if we express all other LECs as products of dimensionless constants times the appropriate power of f .

We see that all contributions to a certain amplitude of a fixed order in external momenta correspond to a fixed value of N . For instance, for $N = 0$, only tree-level diagrams, calculated from $\mathcal{L}^{(2)}$ contribute, because $N = 0$ requires that $L = 0$ and $V_d = 0$ for $d > 2$. For $N = 1$, one-loop diagrams coming from $\mathcal{L}^{(2)}$ combine with tree-level diagrams coming from $\mathcal{L}^{(4)}$, consistent with our discussion in the previous section. In general, if we calculate to some fixed order in N , we need the chiral lagrangian only up to $\mathcal{L}^{(2N+2)}$. In other words, while our EFT is not renormalizable, it is nevertheless predictive if we work to a fixed order in the

derivative expansion, since to that order only a finite number of LECs occur in the chiral lagrangian [10].¹²

Each loop provides a factor $1/(4\pi)^2$. If we assume (*cf.* Sec. IIB) that this number also sets the natural size of all dimensionless LECs, that turns the momentum expansion into an expansion in powers of $p^2/(4\pi f)^2$. Then, so far we have ignored the fact that loops lead to divergences. That means that there is another scale, the cutoff, that can appear in our result. Putting everything together, we therefore amend our power-counting result: our diagram takes the schematic form (restoring a delta function for momentum conservation)

$$(2\pi)^4 \delta\left(\sum_{i=1}^E p_i\right) f^2 p^2 \left(\frac{p^2}{(4\pi f)^2}\right)^N \left(\frac{1}{f}\right)^E F(p^2/\Lambda^2), \quad (2.20)$$

with F a dimensionless function. Contributions to F can be of three types. First, any positive powers of p^2/Λ^2 can be ignored, as they correspond to higher-order contributions. Then, depending on the regulator, negative powers may occur (*i.e.*, positive powers of the cutoff); those correspond to power divergences. Since all divergences have to be local [13], such divergences can be absorbed into lower-order LECs. Finally, there can be logarithmic divergences, which, as we have seen, can be renormalized by LECs in $\mathcal{L}^{(2N+2)}$ if they occur at N loops. In dimensional regularization, power divergences do not occur, making this regulator the most practical one for calculations in ChPT.¹³

The conclusion of this section is that a well-defined power-counting scheme exists, turning our EFT, *i.e.*, ChPT, into a systematic and practically useful tool. This observation will play an important role when we start using ChPT in applications to lattice QCD.

D. Conserved currents

It follows from Noether's theorem that there are sixteen conserved currents, associated with the generators of the group $SU(3)_L \times SU(3)_R$. At the level of QCD, these currents are fixed by the lagrangian and the symmetry group, and the same should thus be true at the level of the chiral lagrangian. In both cases, these currents can be calculated following Noether's procedure. Equivalently, one may couple the QCD lagrangian to external gauge fields, ℓ_μ and r_μ (taken to be hermitian), which transform under local chiral transformations as

$$\begin{aligned} \ell_\mu &\rightarrow U_L \ell_\mu U_L^\dagger - i\partial_\mu U_L U_L^\dagger, \\ r_\mu &\rightarrow U_R r_\mu U_R^\dagger - i\partial_\mu U_R U_R^\dagger, \end{aligned} \quad (2.21)$$

with the covariant derivatives in Eq. (2.1) turning into $D_\mu \rightarrow D_\mu - i\ell_\mu$ when acting on left-handed quarks, and $D_\mu \rightarrow D_\mu - ir_\mu$ when acting on right-handed quarks. This substitution makes the QCD lagrangian (we still are considering the case that $M = 0$!) invariant under local chiral transformations, and thus the chiral lagrangian should also be made invariant under *local* chiral transformations (in addition to Lorentz (or euclidean) invariance and parity). This can be accomplished by replacing

$$\partial_\mu \Sigma \rightarrow D_\mu \Sigma = \partial_\mu \Sigma - i\ell_\mu \Sigma + i\Sigma r_\mu \rightarrow U_L D_\mu \Sigma U_R^\dagger \quad (2.22)$$

¹² The number of LECs grows rapidly with N , rendering our EFT practically useless beyond $N = 2$.

¹³ For a different chirally invariant regularization using a lattice cutoff, see Ref. [14].

everywhere in the chiral lagrangian, given to order p^4 by the sum of Eqs. (2.13) and (2.17). The second arrow in Eq. (2.22) indicates how the covariant derivative of Σ transforms; $U_{L,R}$ are now local transformations in $SU(3)_{L,R}$. In addition, more invariant terms can be constructed, if we use also the building blocks

$$\begin{aligned} L_{\mu\nu} &= \partial_\mu \ell_\nu - \partial_\nu \ell_\mu - i[\ell_\mu, \ell_\nu] , \\ R_{\mu\nu} &= \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu] . \end{aligned} \quad (2.23)$$

At order p^2 there are no new terms (because these field strengths have two Lorentz indices), but at order p^4 several new terms arise:

$$\begin{aligned} iL_9 \operatorname{tr} \left(L_{\mu\nu} D_\mu \Sigma (D_\nu \Sigma)^\dagger + R_{\mu\nu} (D_\mu \Sigma)^\dagger D_\nu \Sigma \right) - L_{10} \operatorname{tr} (L_{\mu\nu} \Sigma R_{\mu\nu} \Sigma^\dagger) \\ - H_1 \operatorname{tr} (L_{\mu\nu} L_{\mu\nu} + R_{\mu\nu} R_{\mu\nu}) . \end{aligned} \quad (2.24)$$

With the chiral lagrangian as constructed above, we have that

$$\log Z_{ChPT}(\ell_\mu, r_\mu) = \log Z_{QCD}(\ell_\mu, r_\mu) + \text{constant} , \quad (2.25)$$

up to the order at which we work in the chiral expansion. Since ℓ_μ and r_μ couple to the left- and right-handed Noether currents, (connected) correlation functions of these currents are generated by taking derivatives with respect to ℓ_μ and r_μ . In words, Eq. (2.25) states that these correlation functions, as calculated in ChPT, equal those of QCD, to some given order in the derivative expansion.

By taking one derivative of the chiral lagrangian with respect to ℓ_μ or r_μ , and setting these sources equal to zero, we find the conserved left- and right-handed currents. Showing only the lowest-order explicitly:

$$\begin{aligned} J_\mu^L &= \frac{i}{4} f^2 \partial_\mu \Sigma \Sigma^\dagger + \dots = -\frac{1}{2} f \partial_\mu \phi + \dots , \\ J_\mu^R &= -\frac{i}{4} f^2 \Sigma^\dagger \partial_\mu \Sigma + \dots = \frac{1}{2} f \partial_\mu \phi + \dots . \end{aligned} \quad (2.26)$$

Therefore, to lowest order in the derivative expansion, and writing

$$\begin{aligned} \phi &= \phi_a T_a , \\ J_\mu^{L,R} &= J_{a\mu}^{L,R} T_a , \end{aligned} \quad (2.27)$$

with T^a the $SU(3)$ generators normalized, as before, through $\operatorname{tr}(T_a T_b) = \delta_{ab}$, we get for the pion to vacuum matrix element of the axial current

$$\langle 0 | J_{a\mu}^R(x) - J_{a\mu}^L(x) | \phi_b(p) \rangle = -i p_\mu f \delta_{ab} e^{-ipx} , \quad (2.28)$$

and we conclude that f is equal to the pion decay constant, f_π , in the chiral limit.¹⁴ We also find another prediction: to leading order in the chiral expansion $f_\pi = f_K = f_\eta$.

We end this section with a number of remarks:

¹⁴ Our normalization is such that in the real world $f_\pi = 130.4$ MeV. Another common convention uses a value smaller by a factor $\sqrt{2}$.

1. The observation that we can obtain conserved currents in the effective theory just as well from the Noether procedure or from the “source method” that I described above is correct, but both methods do not in general lead to the same current. At order p^2 the currents obtained using either of these methods are the same, but at order p^4 they differ by a term proportional to L_9 . Clearly, if we do not introduce the sources ℓ_μ and r_μ at all, the L_9 term in Eq. (2.24) never appears, and therefore the Noether current has no term proportional to L_9 . But if we use the source method described above Eq. (2.26) and apply it to the terms in Eq. (2.24), we find additional terms

$$\begin{aligned}\Delta J_\mu^L &= iL_9 \partial_\nu [L_\mu, L_\nu] , \\ \Delta J_\mu^R &= iL_9 \partial_\nu [R_\mu, R_\nu] .\end{aligned}\tag{2.29}$$

However, these extra terms are automatically conserved, $\partial_\mu \Delta J_\mu^{L,R} = 0$ identically. Therefore, the most general form of the conserved current is that provided by the source method.

2. When we gauge the group $SU(3)_L \times SU(3)_R$, as we did above, it is in fact anomalous. In order to reproduce the “nonabelian” anomaly in the chiral theory, we need to add the gauged Wess–Zumino–Witten term [15]. Since this part of the chiral lagrangian will play no role in the rest of these lectures, we will not discuss this any further.
3. While (apart from the anomaly), the derivation given above looks very straightforward, in fact the freedom to perform field redefinitions and to add total-derivative terms is needed to complete the proof that the correct recipe is to just choose the chiral lagrangian to be locally invariant [11].
4. A new term proportional to H_1 , not containing the pion fields Σ , shows up. This term can only contribute contact terms to correlation functions and is thus not physical. For instance, if one considers the two-point function $\langle J_\mu^L(x) J_\nu^L(y) \rangle$ at one loop, this has a logarithmic divergence, which can be absorbed into H_1 . Such parameters in the chiral lagrangian are sometimes referred to as “high-energy constants.”
5. The coupling of the chiral lagrangian to the sources ℓ_μ and r_μ tells us how to couple pions to the photon and to (virtual) W and Z mesons. For instance, one obtains the photon coupling by taking

$$\ell_\mu = r_\mu = -\frac{1}{3} e a_\mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ,\tag{2.30}$$

with a_μ the photon field and e the charge of the electron.

Generally, electromagnetic effects in hadronic quantities are of approximately the same size as the isospin breaking coming from the fact that in nature $m_u \neq m_d$. This implies that ultimately electromagnetic effects will have to be taken into account in lattice QCD computations. For explorations in this direction, see Ref. [16].

E. Quark masses

It is time to remember that in the real world the masses of the up, down and strange quark do not vanish. Generalizing Eq. (2.5), the mass terms in Eq. (2.1) can be replaced by source terms, with hermitian scalar and pseudoscalar sources $s(x)$ and $p(x)$:

$$\bar{q}_L(s + ip)q_R + \bar{q}_R(s - ip)q_L , \quad (2.31)$$

with $SU(3)_L \times SU(3)_R$ transformation rules

$$s + ip \rightarrow U_L(s + ip)U_R^\dagger , \quad s - ip \rightarrow U_R(s - ip)U_L^\dagger . \quad (2.32)$$

We recover the quark mass terms by setting $s = M$ with M as in Eq. (2.3) and $p = 0$.

This gives us a new building block for constructing terms in the chiral lagrangian. To lowest order in s and p , using parity symmetry, there is a unique operator that can be added to $\mathcal{L}^{(2)}$, which now becomes

$$\begin{aligned} \mathcal{L}^{(2)} &= \frac{1}{8}f^2 \text{tr} \left((D_\mu \Sigma)^\dagger D_\mu \Sigma \right) - \frac{1}{8}f^2 \text{tr} \left(\chi^\dagger \Sigma + \Sigma^\dagger \chi \right) , \\ \chi &\equiv 2B_0(s + ip) , \end{aligned} \quad (2.33)$$

with B_0 a new LEC. Setting $\chi = \chi^\dagger = 2B_0M$, and expanding Σ to quadratic order in ϕ , we can read off the (tree-level) pion masses in terms of the quark masses:

$$\begin{aligned} m_{\pi^+}^2 &= B_0(m_u + m_d) , \\ m_{K^+}^2 &= B_0(m_u + m_s) , \\ m_{K^0}^2 &= B_0(m_d + m_s) , \\ m_{\pi^0}^2 &= B_0 \left(m_u + m_d + O \left(\frac{(m_u - m_d)^2}{m_s} \right) \right) , \\ m_\eta^2 &= \frac{1}{3}B_0 \left(m_u + m_d + 4m_s + O \left(\frac{(m_u - m_d)^2}{m_s} \right) \right) . \end{aligned} \quad (2.34)$$

This gives our third prediction from chiral symmetry: five meson masses are expressed in terms of three parameters. If we ignore isospin breaking (*i.e.*, set $m_\ell \equiv m_u = m_d$), we find that $m_{\pi^0} = m_{\pi^+}$ and $m_\eta^2 = (2(m_{K^+}^2 + m_{K^0}^2) - m_{\pi^+}^2)/3$, relations which agree with experiment at the few percent level. The latter relation is the well-known Gell-Mann–Okubo relation.

Since the masses on the left-hand side are physical quantities, also the expressions on the right-hand side should be physical; in particular, they should not depend on the renormalization scale of the underlying theory. Indeed, by taking a derivative with respect to s and then setting all sources equal to zero, we find that in the chiral limit

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle = -f^2 B_0 , \quad (2.35)$$

and we then use that $m_u \langle \bar{u}u \rangle$, *etc.*, are scale independent. The quark masses appear in the chiral lagrangian only in the scale-independent combinations

$$\chi_u \equiv 2B_0 m_u , \quad \chi_d \equiv 2B_0 m_d , \quad \chi_s \equiv 2B_0 m_s . \quad (2.36)$$

From this tree-level exercise, we see that for onshell momenta, one power of the quark mass should be counted as order p^2 , because for an onshell pion we have that $p^2 = -m_{\pi^+}^2$,

etc. With this power counting for the quark masses, the whole discussion of Sec. II C applies, with the proviso that p^2 in formulas like Eq. (2.20) can now also stand for any of the squared meson masses. With $f = 130$ MeV, we note that the expansion parameters of the chiral expansion are

$$\frac{m_\pi^2}{(4\pi f)^2} \approx 0.007, \quad \frac{m_K^2}{(4\pi f)^2} \approx 0.09, \quad (2.37)$$

which gives one hope that even for kaons the chiral expansion may be well behaved. We will return to this issue in Sec. V. As an aside, we also see why the heavy quark masses (charm *etc.*) cannot be accounted for in ChPT: The corresponding “expansion” parameters would not be small. In fact, the situation is “upside down”: one may instead consider expansions in Λ_{QCD}/m_{heavy} , leading to Heavy-Quark EFT.¹⁵

In order to find the meson masses, we expanded the nonlinear Σ around $\mathbf{1}$. Indeed, when all masses in Eq. (2.3) are positive, $\Sigma = \mathbf{1}$ is the minimum of the potential, which is given by the second term in Eq. (2.33). When we allow two of the masses to be negative, the situation is essentially the same. For instance, if $m_{u,d} < 0$ while $m_s > 0$, the vacuum is

$$\Sigma_{vac} = \text{diag}(-1, -1, 1), \quad (2.38)$$

which is equivalent to the trivial vacuum by an $SU(3)$ rotation. When, however, an odd number of quark masses are negative, there is no $SU(3)$ rotation relating the vacuum to the trivial vacuum, and the theory can be in a different phase, in which CP is broken because of the appearance of a θ -term with $\theta = \pi$ [17]. Since in nature all quark masses are positive (or equivalently, $\theta = 0$), we will only consider the case of positive quark masses throughout this talk.¹⁶ However, we will see in Sec. III C that a nontrivial vacuum structure is nevertheless possible for some discretizations of QCD at nonzero lattice spacing.

Of course, with the scalar source χ as a new building block, more terms can appear in $\mathcal{L}^{(4)}$ as well. Gauging Eq. (2.17) and including Eq. (2.24), the most general form becomes¹⁷

$$\begin{aligned} \mathcal{L}^{(4)} = & -L_1 \left(\text{tr} \left(D_\mu \Sigma (D_\mu \Sigma)^\dagger \right) \right)^2 - L_2 \text{tr} \left(D_\mu \Sigma (D_\nu \Sigma)^\dagger \right) \text{tr} \left(D_\mu \Sigma (D_\nu \Sigma)^\dagger \right) \\ & - L_3 \text{tr} \left(D_\mu \Sigma (D_\mu \Sigma)^\dagger D_\nu \Sigma (D_\nu \Sigma)^\dagger \right) \\ & + L_4 \text{tr} \left(D_\mu \Sigma (D_\mu \Sigma)^\dagger \right) \text{tr} (\chi^\dagger \Sigma + \Sigma^\dagger \chi) + L_5 \text{tr} \left(D_\mu \Sigma (D_\mu \Sigma)^\dagger (\chi^\dagger \Sigma + \Sigma^\dagger \chi) \right) \\ & - L_6 \left(\text{tr} (\chi^\dagger \Sigma + \Sigma^\dagger \chi) \right)^2 - L_7 \left(\text{tr} (\chi^\dagger \Sigma - \Sigma^\dagger \chi) \right)^2 - L_8 \text{tr} (\chi^\dagger \Sigma \chi^\dagger \Sigma + \Sigma^\dagger \chi \Sigma^\dagger \chi) \\ & + i L_9 \text{tr} \left(L_{\mu\nu} D_\mu \Sigma (D_\nu \Sigma)^\dagger + R_{\mu\nu} (D_\mu \Sigma)^\dagger D_\nu \Sigma \right) - L_{10} \text{tr} (L_{\mu\nu} \Sigma R_{\mu\nu} \Sigma^\dagger) \\ & - H_1 \text{tr} (L_{\mu\nu} L_{\mu\nu} + R_{\mu\nu} R_{\mu\nu}) - H_2 \text{tr} (\chi^\dagger \chi). \end{aligned} \quad (2.39)$$

The full lagrangian $\mathcal{L}^{(2)} + \mathcal{L}^{(4)}$ of Eqs. (2.33) and (2.39) is invariant under the local group $SU(3)_L \times SU(3)_R$. We note that five new LECs and one new “high-energy” constant have been added. The term proportional to H_2 contributes to the mass dependence of the chiral condensate, and is needed for instance in order to match lattice QCD, where the condensate includes a term proportional to m/a^2 (with m the relevant quark mass).

¹⁵ For an introduction to heavy quarks, see the lectures by Rainer Sommer at this school.

¹⁶ As mentioned already in the first section, the scenario with $m_u = 0$ is ruled out [18, 19].

¹⁷ Some terms in Eq. (2.39) have been removed by field redefinitions.

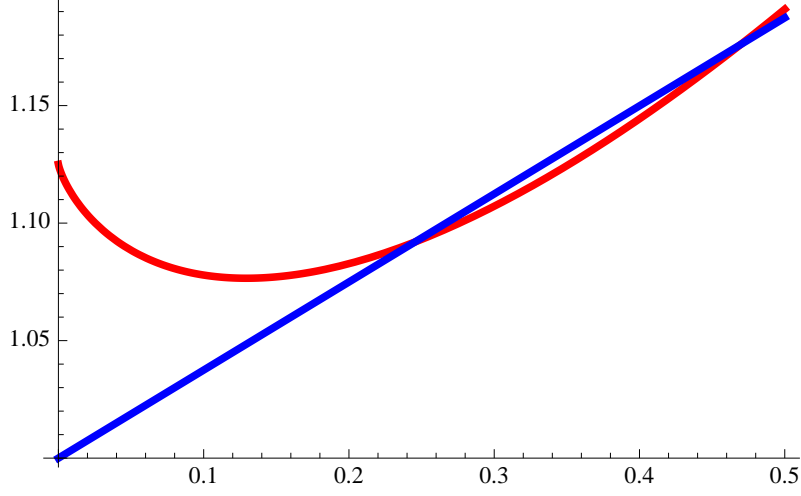


FIG. 1: The functions $1.125 + L(m_\pi^2)$ (red curve) and $1 + m_\pi^2/(4\pi f)^2$ (blue line), as a function of m_π^2 in GeV^2 . I took $\Lambda = m_\rho = 770$ MeV and $f = 130$ MeV.

As an application, we quote some results of the quark-mass dependence of various physical quantities to order p^4 [12], in the isospin limit $m_u = m_d$:

$$\begin{aligned}
f_\pi &= f \left\{ 1 + \frac{8L_5}{f^2} \hat{m}_\ell + \frac{8L_4}{f^2} (2\hat{m}_\ell + \hat{m}_s) - 2L(m_\pi^2) - L(m_K^2) \right\}, & (2.40) \\
\frac{f_K}{f_\pi} &= 1 + \frac{4L_5}{f^2} (\hat{m}_s - \hat{m}_\ell) + \frac{5}{4} L(m_\pi^2) - \frac{1}{2} L(m_K^2) - \frac{3}{4} L(m_\eta^2), \\
\frac{f_\eta}{f_\pi} &= \left(\frac{f_K}{f_\pi} \right)^{4/3} \left\{ 1 + \frac{1}{48\pi^2 f^2} \left(3m_\eta^2 \log \frac{m_\eta^2}{m_K^2} + m_\pi^2 \log \frac{m_\pi^2}{m_K^2} \right) \right\}, \\
\frac{m_\pi^2}{2\hat{m}_\ell} &= 1 + \frac{8(2L_8 - L_5)}{f^2} 2\hat{m}_\ell + \frac{16(2L_6 - L_4)}{f^2} (2\hat{m}_\ell + \hat{m}_s) + L(m_\pi^2) - \frac{1}{3} L(m_\eta^2), \\
\frac{m_K^2}{\hat{m}_\ell + \hat{m}_s} &= 1 + \frac{8(2L_8 - L_5)}{f^2} (\hat{m}_\ell + \hat{m}_s) + \frac{16(2L_6 - L_4)}{f^2} (2\hat{m}_\ell + \hat{m}_s) + \frac{2}{3} L(m_\eta^2),
\end{aligned}$$

in which

$$\begin{aligned}
\hat{m}_\ell &= 2B_0 m_u = 2B_0 m_d, & \hat{m}_s &= 2B_0 m_s, & (2.41) \\
L(m^2) &= \frac{m^2}{(4\pi f)^2} \log \left(\frac{m^2}{\Lambda^2} \right),
\end{aligned}$$

and where we can use the tree-level expressions (2.34) inside the logarithms. Because all meson interactions that follow from the chiral lagrangian are even in the meson fields,¹⁸ and

¹⁸ This is not true for the Wess–Zumino–Witten term, which however at this order does not contribute to the quantities of Eq. (2.40) to order p^4 .

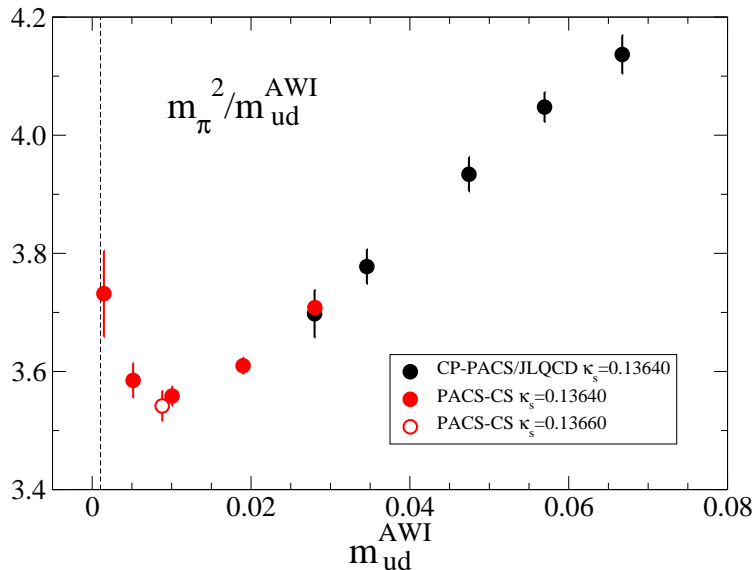


FIG. 2: Comparison of the PACS-CS (red) and the CP-PACS/JLQCD (black) results for $m_\pi^2/m_{\text{ud}}^{\text{AWI}}$ as a function of the light quark mass $m_{\text{ud}}^{\text{AWI}}$ (the so-called “axial Ward identity” definition of the quark mass). The vertical line denotes the physical point. From Ref. [22].

the calculation of both masses and decay constants involves two-point functions, the only diagrams that appear at one loop are tadpole diagrams.

Again, ChPT makes a prediction: if we use $f_K/f_\pi = 1.2$ as input, one finds that (to this order in ChPT) $f_\eta/f_\pi = 1.3$. With these values for f_K/f_π and f_η/f_π we see that ChPT to order p^4 works reasonably well for quantities involving the strange quark, with order- p^4 corrections of about 20–30%.¹⁹ For a discussion of meson masses and decay constants to order p^6 in ChPT, see Ref. [21]. For a recent review of the phenomenological status of ChPT, and many references, see Ref. [8].

An important observation is that all LECs, f , B_0 and the L_i ’s, are independent of the quark mass; they only depend on the number of flavors, N_f .²⁰ All quark-mass dependence in Eq. (2.40) is explicit. It follows from this that the values of the LECs can, in principle, be obtained from lattice QCD with unphysical values of the quark masses (as long as they are small enough for ChPT to be valid — an important restriction). Thus, if one finds a set of quantities through which all LECs can be determined from the lattice, one can then use these values to calculate other quantities that are less easily accessible on the lattice. A simple example of this is that one can extrapolate physical quantities to the physical values of the light quark masses, from lattice computations at larger values of m_ℓ in the isospin limit.²¹ Unfortunately, it is not possible to extract all the L_i from two-point functions, since the constants $L_{1,2,3}$ do not show up in masses and decay constants.

¹⁹ For a recent discussion of the experimental value of f_η , including mixing because of $SU(3)$ breaking, see Ref. [20].

²⁰ For the relation between $N_f = 3$ LECs and $N_f = 2$ LECs, see Ref. [12].

²¹ Since also EM interactions break isospin, they also will have to be taken into account if one reaches a precision at which isospin breaking becomes significant.

We conclude this section with an important lesson about the use of ChPT for fitting the quark mass dependence of a hadronic quantity on the lattice. As we see in Eq. (2.40), nonanalytic terms show up, here in the form of so-called “chiral logarithms.” Fig. 1 shows how important it can be to include such nonanalytic terms in chiral fits, by comparing the functions $1.125 + L(m_\pi^2)$ with $1 + m_\pi^2/(4\pi f)^2$. The logarithm has a dramatic effect: it looks nothing like the linear curve in the region around the physical pion mass, $m_\pi^2 = 0.02 \text{ GeV}^2$. If one would only have lattice data points in the region $m_\pi^2 \gtrsim 0.2 \text{ GeV}^2$ with error bars that would more or less overlap with both curves, and one would perform a linear fit,²² we see that this might lead to errors of order 10% in the value at the physical pion mass. In order to confirm the existence of the chiral logarithm in the data, clearly data points in the region of the curvature of the logarithm are needed. In the cartoon example of Fig. 1 this means that we need data points in the region down to $m_\pi \sim 200 \text{ MeV}$. In addition, data points with very good statistics help.

For a realistic example, see Fig. 2, which shows m_π^2/m_ℓ as a function of the light quark mass, m_ℓ , for fixed (physical) strange mass, from Ref. [22] (to which I refer for details). If one would only consider the large m_π points (the black points), one would not be able to reliably fit the chiral logarithm in Eq. (2.40), and thus also not the linear combination of LECs that accompany the logarithm. Only with the red points, which were obtained for small values of m_π , or equivalently m_ℓ , can one hope to perform sensible chiral fits. For a discussion of such fits, see Ref. [22]. Here we only note that both $O(a^2)$ effects and finite volume can in principle modify the chiral logarithms. These issues can be systematically studied with the help of ChPT, as we will see in Secs. III and VI.

III. CHPT AT NONZERO LATTICE SPACING

If we compute hadronic quantities on the lattice, the values we obtain will differ from their continuum values by scaling violations — terms of order a^n with some positive integer power n (possibly modified by logarithms), where a is the lattice spacing. For example, for the mass of some hadron evaluated on the lattice at some nonzero a we expect that

$$M_{lattice} = M_{continuum}(m_{quark}) + c_1 a \Lambda_{QCD}^2 + O(a^2, am_{quark}), \quad (3.1)$$

with c_1 at most logarithmically dependent on a .²³ In Eq. (3.1) I indicated schematically that the hadron mass also depends on the quark masses in the theory.

Evidently, at nonzero lattice spacing, there is a “new” infrared scale $a\Lambda_{QCD}^2$ that emerges from combining the physical infrared scale Λ_{QCD} with the ultraviolet scale a . Physical quantities thus depend not only on the quark masses, but also on this new scale, and one expects that a combined expansion in terms of m_{quark} and $a\Lambda_{QCD}^2$ should be possible when these parameters are both small enough. The goal of this section is to see how ChPT can be extended to provide this combined expansion.

Before we begin doing that, there are two important observations. First, ChPT is constructed by making use of the chiral symmetry of the underlying theory — it is nothing more

²² Both functions can be thought of as two-parameter fits. In the case of the logarithm, the two parameters are the constant and Λ , in the case of the linear function, they are the constant and the slope.

²³ For lattice regulators with an exact chiral symmetry, one expects scaling violations of order a^2 instead of order a . See for example Sec. III F.

or less than a very efficient way of implementing the Ward identities for chiral symmetry. This implies that if we use a chirally invariant regulator on the lattice, such as domain-wall fermions²⁴ or overlap fermions, there will be almost no change in the form of the chiral lagrangian. Almost no change, because for any lattice regulator continuous rotational invariance is broken, so one expects that new operators will appear in the chiral lagrangian that break continuous rotational invariance, but respect the lattice subgroup (usually the group of hypercubic rotations). Quite generally, in the construction of the chiral lagrangian for Nambu–Goldstone bosons, the only quantity transforming as a four-vector is the derivative, so such terms will have to have at least four derivatives. Combined with the fact that they need to carry a positive power of the lattice spacing, they are usually of quite high order in the chiral expansion, as we will see. The only other difference is that the continuum LECs themselves are afflicted by scaling violations, which can in principle be removed by an extrapolation to $a = 0$.

It is only for regulators that break some of the continuum chiral symmetries that the form of the chiral lagrangian will be different. The two most important examples are Wilson-like fermions (including also clover fermions and twisted-mass QCD), for which vectorlike flavor symmetries are preserved but chiral symmetries are broken, and staggered fermions, for which flavor symmetries are broken, and only one chiral symmetry is preserved. Here I will primarily discuss Wilson fermions, since they are conceptually the simplest. Section III F is devoted to a brief overview of ChPT for QCD with staggered quarks.

A second point is that we will be dealing with an expansion in two small parameters, m_{quark} and $a\Lambda_{QCD}^2$, and we thus have to define a power counting — we have to compare their relative size in order to know to what relative order in each parameter we have to expand. Typical quark masses in lattice computations range between 10 and 100 MeV, and typical lattice spacings vary between $a^{-1} \sim 2 - 3$ GeV. With $\Lambda_{QCD} \sim 300$ MeV, this means that m_{quark} and $a\Lambda_{QCD}^2$ are of the same order, and this is what I will assume to get the discussion started, until Sec. III C.

QCD with Wilson fermions has no chiral symmetry, and it thus seems that our guiding principle for constructing the chiral effective theory is completely lost: we should allow for all kinds of operators in the chiral lagrangian that are only invariant under $SU(3)_V$, and the only thing we know is that any such operator that is forbidden in the continuum theory is multiplied by some positive power of the lattice spacing. It thus seems that, while an EFT at low energies should exist for small enough lattice spacing, it will not be of much practical use. It turns out, however, that this is not true.

The solution to this apparent roadblock consists of first considering a “low-energy” continuum EFT of quarks and gluons, where “low energy” means that we want to consider correlation functions of quarks and gluons at momenta $\Lambda_{QCD} \ll p \ll 1/a$. Since at such momenta the degrees of freedom in this EFT are the same as in the lattice theory, no powers of Λ_{QCD} can appear in this EFT, and any operator of order a^n ($n \geq 0$) thus has to be multiplied by an operator of mass dimension $4 + n$, so that the full lagrangian always has dimension four. This means that, for small values of n the operators that can appear in this EFT are very constrained, and, as we will see, this allows us to also restrict the form of the operators for small values of n in the chiral theory. The idea of considering the quark-gluon continuum EFT for the lattice theory is due to Symanzik [23], and we will refer to this EFT as the Symanzik effective theory (SET). The insight that this can be used to develop ChPT

²⁴ At very small $m_{residual}$; see David Kaplan’s lectures at this school.

including scaling violations systematically is due to Sharpe and Singleton [24].

A. Symanzik effective theory

Let us begin with a brief review of the construction of the SET for lattice QCD with Wilson fermions.²⁵ The Symanzik expansion is an expansion in powers of a ,

$$\mathcal{L}_{SET} = \mathcal{L}_S^{(4)} + a\mathcal{L}_S^{(5)} + a^2\mathcal{L}_S^{(6)} + \dots \quad (3.2)$$

The first term, $\mathcal{L}_S^{(4)}$, is the continuum theory,

$$\mathcal{L}_S^{(4)} = \mathcal{L}_{gluons} + \bar{q}\not{D}q + \bar{q}Mq, \quad (3.3)$$

with M the quark mass matrix given in Eq. (2.3).²⁶ In fact, because of the lack of chiral symmetry of Wilson fermions, the first term in the Symanzik expansion is a term of dimension three of the form $(c/a)\bar{q}q$ with c a numerical constant. The quark masses in M thus have to be defined as the difference between the lattice quark masses and the power divergence c/a :

$$m_i = Z_S^{-1}(m_{0,i} - c)/a, \quad i = u, d, s, \quad (3.4)$$

where $m_{0,i}$ is the bare lattice mass for flavor i in lattice units, and Z_S is the multiplicative renormalization factor needed to relate lattice masses to the continuum regulator that we use to define the SET. It is conceptually easiest to think of the continuum regulator as dimensional regularization (in some minimal subtraction scheme). This brings us to an important point: the fact that, for momenta $\Lambda_{QCD} \ll p \ll 1/a$, an effective theory of the form (III A) can be used to represent the lattice theory was only argued to exist using perturbation theory.²⁷ It is an assumption, essentially based on locality, that the SET is also valid nonperturbatively: the claim is that discretization effects in correlation functions with all momenta $\Lambda_{QCD} \ll p \ll 1/a$ are given by insertions of local operators, to any finite order in an expansion in ap . We note that this is nothing else than the basic assumption underlying the construction of all EFTs, if we replace ap by the ratio of scales that provides the small parameter for the EFT.

Of course, in order to “think” about the SET outside perturbation theory, one would have to use a nonperturbative regulator — such as the lattice! One can imagine using a lattice regulator to define the SET with a lattice spacing a' much smaller than a , so that $p \ll 1/a \ll 1/a'$, and since $a' \ll a$ one does not have to include powers of a' in the expansion (3.2). This ignores the possibility of power divergences in the smaller lattice spacing of the form $1/a'$ or $1/a'^2$, which one might have to subtract in order to define the SET. Since I will only use the form of the SET, without ever doing any calculations with it, I will sidestep this issue, and assume that we can work with the expansion (3.2).²⁸

²⁵ For a more detailed discussion of Symanzik effective theories in the context of improvement, see the lectures of Peter Weisz at this school.

²⁶ For the application of these ideas to twisted-mass QCD, see the review of Ref. [6].

²⁷ And, in the original work by Symanzik, only for ϕ^4 theory [23] and the two-dimensional nonlinear sigma model [25], although there is little doubt that the ideas carry over to gauge theories as well.

²⁸ Since we are considering a renormalizable theory, there should only be a finite number of power divergences in a' . For more discussion of related issues in the context of staggered fermions, see Ref. [26].

Next, let us consider $\mathcal{L}_S^{(5)}$. This consists of all dimension five operators consistent with the exact symmetries of the underlying lattice theory. For Wilson fermions

$$\mathcal{L}_S^{(5)} = b_1 \bar{q} i \sigma_{\mu\nu} G_{\mu\nu} q + b_2 \bar{q} D_\mu D_\mu q + b_3 \bar{q} M \not{D} q + b_4 \bar{q} M^2 q + b_5 \text{tr}(M) \mathcal{L}_{gluons} , \quad (3.5)$$

in which the b_i are real dimensionless coefficients, independent of quark masses, that can be calculated in perturbation theory, or, nonperturbatively, on the lattice. $G_{\mu\nu}$ is the gluon field-strength tensor. Note that the b_i are not really constant: they depend on logarithms of the ratio of the scales used to define the lattice theory and the SET. If dimensional regularization is used for the SET, $b_i = b_i(g^2(\mu), \log a\mu)$. We will ignore this additional logarithmic dependence on a , as it should be much milder than the explicit power dependence on a in Eq. (3.2).

Since the b_3 and b_5 terms are of the form quark mass times terms which already occur in $\mathcal{L}_S^{(4)}$, they will translate into $O(p^6)$ terms in ChPT. The counting here works as follows. Terms in $\mathcal{L}^{(4)}$ translate into $O(p^2)$ terms in ChPT, *cf.* Eq. (2.13). The explicit factors of quark mass and a add two powers of p^2 , since we chose a power counting scheme in which $p^2 \sim m_{quark} \Lambda_{QCD} \sim a \Lambda_{QCD}^3$. Since here we will work to order p^4 , we may drop these terms from consideration. The same is true for the b_4 term, which is of order $a(m_{quark})^2$.

At the next order in the Symanzik expansion, the number of terms of dimension six, consistent with all exact lattice symmetries, proliferates, and I will not list them all [27]. There are purely gluonic operators, that do not break chiral symmetry, but they may break Lorentz (or, rather, euclidean) invariance, such as

$$\sum_{\mu, \kappa} \text{tr}(D_\mu G_{\mu\kappa} D_\mu G_{\mu\kappa}) , \quad (3.6)$$

where for clarity I showed index sums explicitly. While this term would thus lead to new terms in the chiral lagrangian, it is easy to see that these are higher order than p^4 . As we already noted before, in order to construct a chiral operator that breaks euclidean invariance, one needs at least four derivatives. With the additional two powers of a that multiply terms in $\mathcal{L}_S^{(6)}$, this makes such terms $O(p^8)$ in our power counting.²⁹ Note that this observation hinges on the fact that in pion ChPT the derivative is the only object with a Lorentz index. For instance, in baryon ChPT this is not true (there is also the four-velocity of the heavy baryon, which is order one in power counting), and the effects of the breaking of continuous rotational symmetry already shows up at order p^4 [28].

Operators involving quarks are of two types: bilinear and quartic in the quark fields. In order to generate new terms in the chiral lagrangian, they have to break one of the symmetries of the continuum theory. There are bilinears which break euclidean invariance, which, as we argued above, we do not have to consider at order p^4 . Dimension-six operators that break chiral symmetry (and are not of the form $\text{tr}(M) \mathcal{L}_S^{(5)}$) are, in the basis of Ref. [29],

$$\begin{aligned} O_5^{(6)} &= (\bar{q}q)^2 , & O_{10}^{(6)} &= (\bar{q}t_r q)^2 , \\ O_6^{(6)} &= (\bar{q}\gamma_5 q)^2 , & O_{11}^{(6)} &= (\bar{q}t_r \gamma_5 q)^2 , \\ O_9^{(6)} &= (\bar{q}\sigma_{\mu\nu} q)^2 , & O_{14}^{(6)} &= (\bar{q}t_r \sigma_{\mu\nu} q)^2 , \end{aligned} \quad (3.7)$$

where t_r are the $SU(3)_{color}$ generators.

²⁹ Such terms would be $O(p^6)$ in a power counting in which $m_{quark} \sim a^2 \Lambda_{QCD}^3$.

B. Transition to the chiral theory

Now we are ready to make the transition to the chiral theory. First, consider terms of order a , corresponding to $\mathcal{L}_S^{(5)}$ in Eq. (3.2). Using a trick similar to the one we used translating quark-mass terms to ChPT, we introduce a spurion field A , and write the first term in Eq. (3.5) as

$$ab_1 \bar{q}i\sigma_{\mu\nu}G_{\mu\nu}q \rightarrow b_1 \left(\bar{q}_L i\sigma_{\mu\nu}G_{\mu\nu}Aq_R + \bar{q}_R i\sigma_{\mu\nu}G_{\mu\nu}A^\dagger q_L \right) , \quad (3.8)$$

which is invariant under $SU(3)_L \times SU(3)_R$ if we let A transform as (*cf.* Eqs. (2.5) and (2.32))

$$A \rightarrow U_L A U_R^\dagger . \quad (3.9)$$

Note that we recover the term on the left of Eq. (3.8) by setting $A = a\mathbf{1}$.

We see that the b_1 term, from the point of view of chiral symmetry, has exactly the same structure as the quark mass term. At lowest order in ChPT, we thus find a new term at leading order in the chiral lagrangian of the form

$$-\frac{1}{4}f^2W_0 \text{tr}(A^\dagger\Sigma + \Sigma^\dagger A) , \quad (3.10)$$

in which W_0 is a new LEC analogous to B_0 . The b_2 term in Eq. (3.5) has exactly the same chiral structure, and thus does not lead to any new operator in ChPT. This is an example of the fact that there is in general not a one-to-one correspondence between operators in the SET and in ChPT.

Once we set $A = a\mathbf{1}$, the new term can actually be completely absorbed into the mass term in Eq. (2.33), if we shift the quark mass matrix by

$$M \rightarrow M' \equiv M + \frac{W_0}{B_0}a\mathbf{1} , \quad (3.11)$$

or $m_i \rightarrow m'_i = m_i + \frac{W_0}{B_0}a$, $i = u, d, s$.

If we determine, nonperturbatively, the lattice quark masses by requiring the pion mass to vanish,³⁰ then this determines M' , rather than M — in other words, the shift of order a in Eq. (3.11) is automatically taken into account in the determination of the critical quark mass for Wilson fermions. Note that this does not remove all $O(a)$ effects from the theory when we extend it to include other hadrons,³¹ or operators other than the action, *cf.* Sec. III E.

Next, we consider terms of order $ap^2 \sim am_{quark} \sim a^2$, all of order p^4 in our power counting (appropriate powers of Λ_{QCD} are implicit). These come from two sources: higher order terms in the spurion A , as well as terms arising from $\mathcal{L}_S^{(6)}$. However, both of these lead to the same new $O(p^4)$ terms in the chiral lagrangian. The point is that the same spurion field can be used to also make the operators in Eq. (3.7) invariant under $SU(3)_L \times SU(3)_R$:

$$a^2(\bar{q}q)^2 \rightarrow (\bar{q}_L A q_R + \bar{q}_R A^\dagger q_L)^2 , \quad (3.12)$$

³⁰ Or the PCAC quark mass to go to zero.

³¹ For instance, the nucleon mass has $O(a)$ corrections which are not removed by the shift (3.11) [28, 30].

and similar for the other operators in Eq. (3.7). All these operators can thus be made invariant using the “composite” spurions

$$\begin{aligned}
A \otimes A &\rightarrow U_L A U_R^\dagger \otimes U_L A U_R^\dagger, \\
A \otimes A^\dagger &\rightarrow U_L A U_R^\dagger \otimes U_R A^\dagger U_L^\dagger, \\
A^\dagger \otimes A &\rightarrow U_R A^\dagger U_L^\dagger \otimes U_L A U_R^\dagger, \\
A^\dagger \otimes A^\dagger &\rightarrow U_R A^\dagger U_L^\dagger \otimes U_R A^\dagger U_L^\dagger.
\end{aligned} \tag{3.13}$$

Using these spurions, we find the following new terms in the chiral lagrangian

$$\begin{aligned}
\text{tr}(A\Sigma^\dagger) \text{tr}(A\Sigma^\dagger) + \text{h.c.} &\rightarrow a^2 \left(\text{tr}(\Sigma^\dagger) \right)^2 + \text{h.c.}, \\
\text{tr}(A\Sigma^\dagger A\Sigma^\dagger) + \text{h.c.} &\rightarrow a^2 \text{tr}(\Sigma^\dagger \Sigma^\dagger) + \text{h.c.}, \\
\text{tr}(A\Sigma^\dagger) \text{tr}(A^\dagger \Sigma) &\rightarrow a^2 \text{tr}(\Sigma^\dagger) \text{tr}(\Sigma).
\end{aligned} \tag{3.14}$$

Setting sources ℓ_μ and r_μ equal to zero,³² the complete set of additional terms in the chiral lagrangian at order p^4 , including new LECs multiplying each of the new operators is [29]

$$\begin{aligned}
\Delta\mathcal{L}^{(4)} &= \hat{a}W_4 \text{tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) \text{tr}(\Sigma + \Sigma^\dagger) + \hat{a}W_5 \text{tr} \left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger (\Sigma + \Sigma^\dagger) \right) \\
&\quad - \hat{a}W_6 \text{tr}(\hat{M}\Sigma^\dagger + \Sigma\hat{M}) \text{tr}(\Sigma + \Sigma^\dagger) - \hat{a}W_7 \text{tr}(\hat{M}\Sigma^\dagger - \Sigma\hat{M}) \text{tr}(\Sigma - \Sigma^\dagger) \\
&\quad - \hat{a}W_8 \text{tr}(\hat{M}\Sigma^\dagger \Sigma^\dagger + \Sigma\Sigma\hat{M}) \\
&\quad - \hat{a}^2 W'_6 \left(\text{tr}(\Sigma + \Sigma^\dagger) \right)^2 - \hat{a}^2 W'_7 \left(\text{tr}(\Sigma - \Sigma^\dagger) \right)^2 - \hat{a}^2 W'_8 \text{tr}(\Sigma\Sigma + \Sigma^\dagger \Sigma^\dagger),
\end{aligned} \tag{3.15}$$

in which

$$\hat{M} \equiv 2B_0 M, \quad \hat{a} \equiv 2W_0 a. \tag{3.16}$$

If one considers (nonperturbatively) $O(a)$ improved QCD with Wilson fermions, one can use the same chiral lagrangian. It simply means that $W_0 = 0$ in Eq. (3.10). Note that this does *not* mean that we should set all terms of order a^2 equal to zero, despite our convention to include a factor of W_0 in the definition of \hat{a} ! The reason is the following. If the underlying theory is $O(a)$ improved, all terms of order a in the SET (*cf.* Eq. (3.5)) vanish, and there is no need to introduce the spurion A at all at this order. However, this does not mean that there are no $O(a^2)$ terms in the SET, so we do need to introduce $O(a^2)$ spurions as in Eq. (3.12). It just so happens that we can “recycle” the same spurion A at order a^2 by taking tensor products as in Eq. (3.13). If underlying lattice theory is not improved, there are also contributions at order a^2 proportional to the $O(a)$ coefficients of Eq. (3.5), in addition to those proportional to appearing in the SET at order a^2 . At the level of $\Delta\mathcal{L}^{(4)}$ both types of contribution at order a^2 take the same form, given in Eq. (3.15).

As an application, let us consider the modifications to the $O(p^4)$ result for the pion mass, *cf.* Eq. (2.40) [29]. First, the overall factor \hat{m}_ℓ gets replaced by $\hat{m}_\ell + \hat{a} = 2B_0 m'_\ell$, with m'_ℓ the shifted light quark masses of Eq. (3.11), and likewise, the logarithms get replaced by

$$L(m_\pi^2) \rightarrow L(2B_0 m_\ell + 2W_0 a) = L(2B_0 m'_\ell), \tag{3.17}$$

etc. Both of these modifications are a consequence of the fact that the $O(a)$ term (3.10) can be absorbed into the shifted quark masses of Eq. (3.11). In this case, as we already saw

³² For the case including all sources and new “high-energy” constants, see Ref. [31].

above, the quark masses that one determines in a numerical computation are the shifted m' masses, so the $O(a)$ shift in Eq. (3.17) is not observable.

Second, the terms proportional to the L_i 's in Eq. (2.40) also change. Not surprisingly, instead of only terms proportional to $L_i m_{quark}^2$, one now also finds terms of the form $W_i a m_{quark}$ and $W_i' a^2$. At order a , the chiral lagrangian only “knows” about the shifted quark masses m'_i of Eq. (3.11), as the $O(a)$ operator (3.10) is absorbed into the quark masses before tuning them to be near the critical value. The same mechanism does not work at order a^2 . One may thus imagine working with quark masses so small that the shifted light quark mass m'_ℓ becomes of order $a^2 \Lambda_{QCD}^2$. This would imply that the power counting has to be modified; for instance, the $W'_{6,7,8}$ terms in Eq. (3.15) now become leading order. This is the so-called LCE (large cutoff effects) regime; the regime in which $m_\ell \sim m'_\ell \sim a \Lambda_{QCD}^2$ is often referred to as the GSM (generic small (quark) mass) regime.³³ So far, our discussion has been in the GSM regime. Before we discuss some of the ChPT results in the LCE regime, however, we consider the phase diagram of the theory, as it turns out that the competition between the $O(m'_\ell)$ terms and $O(a^2)$ terms in the chiral lagrangian can lead to various nontrivial phase structures, depending on the sign of an $O(a^2)$ LEC.

C. Phase diagram

So far, we have been expanding Σ around the trivial vacuum in order to calculate various physical quantities, such as in Eq. (2.40). Indeed, with $\chi = 2B_0 M$ with M as in Eq. (2.3) with all quark masses positive, $\Sigma = \mathbf{1}$ corresponds to the minimum of the classical potential energy, given by the second term in Eq. (2.33). For $a = 0$, higher order terms in the chiral expansion of the potential are smaller, and thus do not change this observation.

When we include scaling violations, there are two small parameters, m_{quark} and $a \Lambda_{QCD}^2$. In the previous section, we took these two parameters to be of the same order, and again the leading order nonderivative terms in Eqs. (2.33) and (3.10) constitute the leading-order classical potential. As long as the shifted quark mass of Eq. (3.11) stays positive and in the GSM regime, the vacuum remains trivial. However, if we now make the shifted quark mass smaller, so that at some point $m'_{quark} \sim a^2 \Lambda_{QCD}^3$, the terms with LECs $W'_{6,7,8}$ in Eq. (3.15) become comparable in size, and should be taken as part of the leading-order classical potential that determines the vacuum structure of the theory. This is the regime that we denoted as the LCE regime at the end of Sec. III B [24] (see also Ref. [32]).

The strange quark mass is much larger than the up or down quark masses, so it is interesting to consider the case that $m_\ell = m_u = m_d \ll m_s$, with $m_\ell \sim a^2 \Lambda_{QCD}^2$, while m_s stays in the GSM regime.³⁴ We then expect the vacuum value of Σ to take the form

$$\Sigma_{vacuum} = \begin{pmatrix} \Sigma_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.18)$$

in which Σ_2 denotes an $SU(2)$ valued matrix. The interesting phase structure is thus essentially that of a two-flavor theory, and in the rest of this subsection we will analyze the phase structure for $N_f = 2$. For $N_f = 2$, the classical potential, which consists of the second term

³³ The LCE regime is also commonly referred to as the Aoki regime.

³⁴ I will drop the primes on the shifted quark masses in the rest of this section.

in Eq. (2.33), Eq. (3.10), and the $W'_{6,7,8}$ terms in Eq. (3.15), becomes

$$V = -\frac{1}{4}f^2\hat{m}_\ell \operatorname{tr}(\Sigma_2 + \Sigma_2^\dagger) - \hat{a}^2(W'_6 + \frac{1}{2}W'_8) \left(\operatorname{tr}(\Sigma_2 + \Sigma_2^\dagger)\right)^2 + \text{constant} , \quad (3.19)$$

where we have used the $SU(2)$ relation

$$\operatorname{tr}(\Sigma_2\Sigma_2 + \Sigma_2^\dagger\Sigma_2^\dagger) = \frac{1}{2} \left(\operatorname{tr}(\Sigma_2 + \Sigma_2^\dagger)\right)^2 - 4 . \quad (3.20)$$

All other terms in the chiral potential are of higher order.

This potential exhibits an interesting phase structure. First, consider the case that $W' \equiv 32(W'_6 + \frac{1}{2}W'_8)(2W_0)^2/f^2 > 0$. In that case, the potential is minimized for

$$\Sigma_2 = \begin{cases} +1 , & \hat{m}_\ell > 0 \\ -1 , & \hat{m}_\ell < 0 \end{cases} \quad (W' > 0) . \quad (3.21)$$

The pion mass is given (to leading order) by

$$m_{\pi,LO}^2 = 2|\hat{m}_\ell| + 2a^2W' \quad (3.22)$$

(where LO remind us that this is the pion mass to leading order), and attains a nonzero minimum value at $\hat{m}_\ell = 0$. The phase transition is first order, with the vacuum expectation value Σ_2 exhibiting a discontinuous jump across $\hat{m}_\ell = 0$.

Equation (3.22) suggests what will happen for $W' < 0$.³⁵ The pion mass-squared turns negative when $|\hat{m}_\ell| < a^2|W'|$, which signals the spontaneous breakdown of a symmetry. Indeed, writing

$$\Sigma_2 = \sigma + i\vec{\tau} \cdot \vec{\pi} , \quad \sigma^2 + \vec{\pi}^2 = 1 \quad (3.23)$$

(with $\vec{\tau}$ the Pauli matrices), we find that the potential is minimized for

$$\sigma = \begin{cases} +1 , & \hat{m}_\ell \geq a^2|W'| \\ \frac{\hat{m}_\ell}{a^2|W'|} , & |\hat{m}_\ell| < a^2|W'| \\ -1 , & \hat{m}_\ell \leq -a^2|W'| \end{cases} \quad (W' < 0) . \quad (3.24)$$

A phase of width a^3 in lattice units opens up, in which isospin and parity are spontaneously broken, because of the formation of a pion condensate [24]. We can take the condensate in the $\pi_3 = \pi_0$ direction in isospin space, in which case the charged pions become massless, becoming the exact Goldstone bosons for the breakdown of isospin. Since the condensate now varies continuously across the phase transition, this is a second order transition. The possibility of such a phase was first noticed by Aoki [35], and provides a possible mechanism for pions to become exactly massless with Wilson fermions at nonzero lattice spacing by tuning the quark mass. At the phase transition the charged pions are still massless, and because of isospin symmetry at that point, the neutral pion also has to be massless at the transition. Note however, that if $W' > 0$, pions will not be massless at the phase transition for nonzero a .

There is a lot more than can be said about the phase structure. We will not do so in these lectures, but end this section with a few comments:

³⁵ For $W' = 0$ higher order terms in the potential would have to be considered [33].

1. An important observation is that the chiral lagrangian does not only provide us with a systematic expansion in the small infrared scales of the theory (the quark masses and $a\Lambda_{QCD}^2$), but it also provides nonperturbative information about the phase structure of the theory.
2. In two-flavor QCD with Wilson fermions, one may consider a more general quark-mass matrix

$$M = m_\ell \mathbf{1} + i\mu\tau_3 . \quad (3.25)$$

This leads to tmQCD, Wilson-fermion QCD with a twisted mass. In the continuum, one can perform a chiral rotation to make the mass matrix equal to $M = \sqrt{m^2 + \mu^2} \mathbf{1}$, but with Wilson fermions there is no chiral symmetry, and the theory for $\mu \neq 0$ differs from the one with $\mu = 0$. We note that for $\mu \neq 0$ this generalized mass matrix breaks isospin symmetry explicitly, and μ thus serves as a “magnetic field” pointing the condensate in the τ_3 direction in the limit $\mu \rightarrow 0$ (after the thermodynamic limit has been taken). It is straightforward to develop ChPT for this case; one simply substitutes this mass matrix into the (two-flavor) chiral lagrangian. For detailed discussions of tmQCD, see for instance the reviews in Refs. [6, 34].

D. The LCE regime

We have seen that $O(a)$ effects, while present for (unimproved) Wilson fermions, can be completely absorbed into a shifted quark mass in the pion chiral lagrangian. While we already considered $O(a^2)$ effects in the pion masses in the GSM regime toward the end of Sec. III B, it is interesting to consider effects from $O(a^2)$ scaling violations in the LCE regime, where $m'_\ell \sim a^2\Lambda_{QCD}^3$. We will always work with m'_ℓ , *cf.* Eq. (3.11), and in this section we will again restrict ourselves to $N_f = 2$, thinking of the strange quark as heavy. This makes sense, as we may expect that numerical computations with very small light quark masses, which may then indeed turn out to be of order $a^2\Lambda_{QCD}^3$, will become more prominent in the near future. The results we consider in this section have been obtained in Ref. [36].³⁶ They are valid in the phase with unbroken isospin and parity. In other words, we will choose m'_ℓ such that $m_{\pi,LO}^2$ of Eq. (3.22) is always nonnegative.

The first thing to note is that, since our power counting changes relative to the GSM regime, new terms need to be added to the chiral lagrangian if one wishes to work to order p^4 :

$$\begin{aligned} \text{LCE regime: } O(p^2) &: p^2, m'_\ell, a^2 , \\ O(p^3) &: ap^2, am'_\ell, a^3 , \\ O(p^4) &: p^4, p^2m'_\ell, (m'_\ell)^2, a^2p^2, a^2m'_\ell, a^4 . \end{aligned} \quad (3.26)$$

Since we will be interested in the nonanalytic terms at one loop, which follow from the lowest-order chiral lagrangian, I just refer to Ref. [36] for the explicit form for all new terms not present in Eq. (3.15), which does already include all $O(p^2)$ terms.

³⁶ For earlier work, see Ref. [37].

One finds for the pion mass to order p^4 [36]:

$$m_\pi^2 = \tilde{m}_{\pi,LO}^2 \left\{ 1 + \frac{1}{16\pi^2 f^2} \left((\tilde{m}_{\pi,LO}^2 - 10a^2 W') \log \left(\frac{\tilde{m}_{\pi,LO}^2}{\Lambda^2} \right) \right) \right. \quad (3.27)$$

$$\left. + \text{analytic terms proportional to } \tilde{m}_{\pi,LO}^2 \text{ and } a \right\}.$$

Here $\tilde{m}_{\pi,LO}^2$ is equal to $m_{\pi,LO}^2$ (which is given in Eq. (3.22)) up to $O(a^3)$ and $O(a^4)$ terms that have been absorbed into the definition of the quark mass. This corresponds to defining the critical quark mass as the value where the pion mass vanishes, but one should keep in mind that these scaling violations are not universal.

This result is interesting because it shows that the continuum value of the coefficient of the chiral logarithm, which is a prediction of ChPT, can be significantly modified by scaling violations in the LCE regime. It also shows that at nonzero lattice spacing the chiral limit is unphysical: for $\tilde{m}_{\pi,LO}^2 \rightarrow 0$ the chiral logarithm diverges, unless we take the lattice spacing to zero as well, at a rate not slower than the square root of the quark mass. Note however, that if $W' > 0$, there is a nonvanishing minimum value of the pion mass which is of order a^2 , *cf.* Eq. (3.22), so that this divergence cannot occur. In the case $W' < 0$ the divergence can happen, and the only way to make mathematical sense of the result (3.27) is to resum higher powers of $a^2 \log(\tilde{m}_{\pi,LO}^2)$.³⁷ Similar effects also show up in pion scattering lengths [36].³⁸ Because of the absence of chiral symmetry at nonzero a , pion interactions do not vanish for vanishing external momenta; the leading term is of order a^2 , and divergent chiral logarithms occur at order a^4 . Note that the results of Ref. [36] are obtained approaching the phase transition from the phase in which isospin is unbroken. If, for $W' < 0$, one calculates scattering lengths approaching the phase transition from the other side, one would expect them to vanish. The reason is that on that side isospin is broken, and two of the three pions are thus genuine Goldstone bosons at nonzero a , with vanishing scattering lengths. Since isospin is restored at the phase transition, all pion scattering lengths should thus vanish approaching the phase transition from the phase with broken isospin. This is not a contradiction: scattering lengths are discontinuous across the phase transition between a symmetric and a broken phase.

An important general lesson of this section is that Wilson ChPT allows us to choose quark masses so small that $m_\pi^2 \sim a^2 \Lambda_{QCD}^4$, *i.e.*, in a regime where physical and unphysical effects compete. In contrast, if we only had continuum ChPT as a tool, we would have to first extrapolate to the continuum before doing any chiral fits. However, the examples we have discussed make it clear that, in general, we would not want to take the chiral and continuum limits “in the wrong order”: if one first takes the physical infrared scale m_π to zero at fixed nonzero value of the unphysical infrared scale $a\Lambda_{QCD}^2$, ChPT tells us that infrared divergences may occur, as we have seen above.

We end this section by observing that the result for the GSM regime to $O(p^4)$ can be recovered from Eq. (3.27) by expanding in a^2 and dropping all terms of order a^3 , $a^2 m'_\ell$ and

³⁷ It is not clear to me that this would lead to a less divergent result after resummation for all physical quantities in which such enhanced logarithms may occur, as it turns out to be the case for the pion mass [37].

³⁸ This reference contains an interesting idea on how to determine the value of W' from the $I = 2$ pion scattering length.

a^4 .

E. Axial current

Before leaving Wilson fermions for staggered fermions, let us briefly look at currents in the presence of scaling violations; as an example I will consider the nonsinglet axial current, and we will work to order a , *i.e.*, we are back in the GSM regime. I will highlight some of the arguments presented in Ref. [38], which contains references to earlier work, and which also discusses the vector current.

The issue is that for $a \neq 0$, the axial current for QCD with Wilson fermions is not conserved. Since there is no Noether current at the lattice QCD level, there isn't one at the SET level, nor in ChPT. Since there is no conserved current, the lattice current that is usually employed is the local current³⁹

$$A_\mu^a(x) = \bar{q}(x)\gamma_\mu\gamma_5 T^a q(x) , \quad (3.28)$$

in which the T^a are $SU(3)$ or $SU(2)$ flavor generators. First, this current needs a finite renormalization, in order to match to the continuum-regularized (partially) conserved axial current [39],

$$A_{\mu,ren}^a = Z_A A_\mu^a , \quad (3.29)$$

with Z_A a renormalization constant determined nonperturbatively by enforcing Ward identities. Since Z_A is determined nonperturbatively, it includes scaling violations. Then, since A_μ^a is not a Noether current following from the lattice action, the effective current that represents the axial current in the SET also does not follow from the SET action; instead, it is just an “external” operator for which we have to find the corresponding expression in the SET. To order a , the “Symanzik” current that represents the lattice current A_μ^a in the SET is⁴⁰

$$\begin{aligned} A_{\mu,SET}^a &= \frac{1}{Z_A^0} (1 + \bar{b}_A am) \left(A_{\mu,cont}^a + a\bar{c}_A \partial_\mu P_{cont}^a \right) , \\ A_{\mu,cont}^a &= \bar{q}\gamma_\mu T^a q , \quad P_{cont}^a = \bar{q}\gamma_5 T^a q , \end{aligned} \quad (3.30)$$

in which \bar{b}_A and \bar{c}_A are Symanzik coefficients that depend on the underlying lattice action; “*cont*” indicates continuum operators. (In a fully $O(a)$ improved theory both \bar{b}_A and \bar{c}_A vanish.) By combining Eqs. (3.29) and (3.30), we see that the overall multiplicative renormalization factor is $(1 + \bar{b}_A am)Z_A/Z_A^0$. Z_A^0 is the all-orders perturbative matching factor needed to convert a lattice current into the properly normalized (partially conserved) continuum current; we have that $Z_A = Z_A^0 + O(a)$.⁴¹

At this stage, two steps need to be carried out to translate this current to ChPT. First, the operator $A_{\mu,cont}^a + a\bar{c}_A \partial_\mu P_{cont}^a$ needs to be mapped into ChPT. Then, the renormalization factor following from Eqs. (3.29) and (3.30) has to be determined in exactly the same way as the lattice current is matched to the renormalized continuum current in an actual lattice computation. This matching has to be done to order a as well, if one consistently wants to

³⁹ Or an improved version of the local current.

⁴⁰ See the lectures by Peter Weisz at this school.

⁴¹ Only the ratio Z_A/Z_A^0 appears when we combine Eqs. (3.29) and (3.30).

include all $O(a)$ corrections in the chiral theory. Since this involves the matching of scaling violations, one does not expect the outcome of this second step to be universal, and indeed, Ref. [38] finds that already at order a the matching depends on the precise correlation functions used for the matching, with a nonuniversal result for the ratio Z_A/Z_A^0 ; I refer to Ref. [38] for further discussion of this second step. Here, I will address the first step: the mapping to operators in the chiral theory, to order a .⁴²

In order to find the ChPT expression for the axial current, Ref. [38] first observes that, in the case that $\bar{c}_A = 0$, $\bar{b}_A = 0$ and $Z_A^0 = 1$, the Symanzik current is just the continuum axial current, which can be rotated into the vector continuum current by a chiral transformation. This fixes the $\bar{c}_A = 0$ part of the axial current in ChPT in terms of the vector current. It turns out that the vector current is given by the Noether current of the chiral theory, because the $O(a)$ terms in the vector current are of the form $a\partial_\nu(\bar{q}\sigma_{\mu\nu}T^a q)$, which is automatically conserved.⁴³ The Noether current following from $\mathcal{L}^{(2)} + \Delta\mathcal{L}^{(4)}$ (cf. Eqs. (2.33) and (3.15)) is

$$V_\mu = (J_\mu^R + J_\mu^L) \left(1 + \frac{8W_4}{f^2} \text{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \right) + \frac{4W_5}{f^2} \{ J_\mu^R + J_\mu^L, \hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A} \} , \quad (3.31)$$

with $J_\mu^{R,L}$ defined to lowest order in Eq. (2.26); we omitted terms coming from $\mathcal{L}^{(4)}$ of Eq. (2.17). Here $\hat{A} \equiv 2W_0 A$ with A the spurion of Eq. (3.9). We should set $\hat{A} = \hat{a}$ of course, but only *after* doing the chiral rotation to find the $\bar{c}_A = 0$ part of the axial current. We thus find

$$A_\mu = (J_\mu^R - J_\mu^L) \left(1 + \frac{8\hat{a}W_4}{f^2} \text{tr}(\Sigma + \Sigma^\dagger) \right) + \frac{4\hat{a}W_5}{f^2} \{ J_\mu^R - J_\mu^L, \Sigma + \Sigma^\dagger \} \quad (\bar{c}_A = 0) , \quad (3.32)$$

where now we have set $\hat{A} = \hat{a}$.

Finally, we have to worry about the \bar{c}_A term in Eq. (3.30). This is rather simple, because for this we need the lowest-order ChPT operator for the pseudoscalar density, which is $-i(\Sigma - \Sigma^\dagger)$. We need to introduce a new LEC W_A for this contribution, since, unlike the contribution in Eq. (3.32), this term does not relate in any way to the Symanzik effective action, and the same is thus true in ChPT. We thus find for the complete expression for the axial current to order a :

$$A_\mu = (J_\mu^R - J_\mu^L) \left(1 + \frac{8\hat{a}W_4}{f^2} \text{tr}(\Sigma + \Sigma^\dagger) \right) + \frac{4\hat{a}W_5}{f^2} \{ J_\mu^R - J_\mu^L, \Sigma + \Sigma^\dagger \} \quad (3.33)$$

$$-4i\hat{a}W_A\bar{c}_A\partial_\mu(\Sigma - \Sigma^\dagger) \quad (\bar{c}_A \neq 0) ,$$

where, following Ref. [38], I kept \bar{c}_A outside W_A .⁴⁴ In closing this section, I should emphasize that the derivation given here is the “quick and dirty” derivation — for a careful analysis of both vector and axial currents, I refer to Ref. [38].

F. Staggered fermions

Before developing ChPT for staggered QCD, I give a brief review of the definition and symmetry properties of lattice QCD with staggered fermions, to make these lectures more

⁴² This step was already carried out in Ref. [40] using the source method; see also Refs. [6, 38].

⁴³ There is a vector Noether current on the lattice [39], but often the local vector current is used instead.

⁴⁴ This result can now be multiplied again by the factor $(1 + \bar{b}_A a)/Z_A^0$.

self-contained.⁴⁵

1. Brief review

The staggered action is (for the time being I will set the lattice spacing $a = 1$, but I will restore it when we get to the SET and ChPT)

$$S = \sum_{x,\mu,i} \frac{1}{2} \eta_\mu(x) \bar{\chi}_i(x) \left[U_\mu(x) \chi_i(x + \mu) - U_\mu^\dagger(x - \mu) \chi_i(x - \mu) \right] + \sum_{x,i} m_i \bar{\chi}_i(x) \chi_i(x) ,$$

$$\eta_\mu(x) = (-1)^{x_1 + \dots + x_{\mu-1}} . \quad (3.34)$$

Here χ_i is a lattice fermion field with an explicit flavor index $i = 1, \dots, N_f$, an implicit color index, and *no* Dirac index; $U_\mu(x)$ are the gauge-field links, with values in $SU(3)_{color}$.

The action (3.34) has species doubling, and because of the hypercubic structure of the theory, this doubling is sixteen-fold. The sixteen components which emerge in the continuum limit can be accounted for by an index pair α, a , with each index running over the values $1, \dots, 4$. It turns out that the first index can be interpreted as a Dirac index, while the second index constitutes an additional flavor label; the continuum limit looks like [42–44]

$$S_{cont} = \int d^4x (\bar{q}_{i\alpha a} \gamma_{\mu,\alpha\beta} D_\mu q_{i\beta a} + m_i \bar{q}_{i\alpha a} q_{i\alpha a}) , \quad (3.35)$$

in which repeated indices are summed. In brief, this structure emerges as follows. Species doublers live near the corners

$$\pi_A \in \{(0, 0, 0, 0), (\pi, 0, 0, 0), \dots, (\pi, \pi, \pi, \pi)\} , \quad A = 1, \dots, 16 , \quad (3.36)$$

in the Brillouin zone. It thus makes sense to split the lattice momenta p into

$$p = \pi_A + k , \quad -\frac{\pi}{2} \leq k_\mu \leq \frac{\pi}{2} . \quad (3.37)$$

We thus identify sixteen different fermion fields in momentum space, $\chi(p) = \chi(\pi_A + k) \equiv \chi_A(k)$, which represent the sixteen doublers. Since the phases $\eta_\mu(x)$ insert momenta with values in the set (3.36), it follows that the operation

$$T_{\pm\mu} : \chi_i(x) \rightarrow \eta_\mu(x) \chi_i(x \pm \mu) \quad (3.38)$$

mixes the sixteen doublers. In momentum space, after a basis transformation, the operations T_μ can be represented by

$$T_{\pm\mu} : q_{i\alpha a}(k) \rightarrow \gamma_{\mu,\alpha\beta} e^{\pm i k_\mu} q_{i\beta a}(k) , \quad (3.39)$$

with the γ_μ a set of Dirac matrices. That these four matrices satisfy a Dirac algebra follows from the fact that the T_μ anticommute:

$$T_\mu T_\nu = -T_\nu T_\mu , \quad \mu \neq \nu , \quad (3.40)$$

⁴⁵ For more introduction, see the lectures by Pilar Hernández at this school, as well as the recent review by the MILC collaboration [41].

and that

$$T_{\pm\mu}^2 : \chi_i(x) \rightarrow \chi_i(x \pm 2\mu). \quad (3.41)$$

Using these ingredients, we can see how Eq. (3.35) follows from Eq. (3.34). Using Eqs. (3.38) and (3.39), the free kinetic term in Eq. (3.34) can be written as (for one flavor)

$$\sum_{x,\mu} \frac{1}{2} \bar{\chi}(x) (T_\mu - T_{-\mu}) \chi(x) = \int \frac{d^4k}{(2\pi)^4} \sum_\mu \bar{q}(k) \gamma_\mu i \sin(k_\mu) q(k). \quad (3.42)$$

In the continuum limit, $\sin(k_\mu) \rightarrow k_\mu$; note that k is restricted to the reduced Brillouin zone (*cf.* Eq. (3.37)), on which no species doublers reside. Finally, since Eq. (3.34) is gauge invariant, the continuum limit (3.35) has to be gauge invariant as well. It follows that, at least classically, Eq. (3.35) is the continuum limit of Eq. (3.34).

The observation that the continuum limit of staggered QCD is given by Eq. (3.35) has been proven to one loop in perturbation theory, and this result has been understood in terms of the exact lattice symmetries, making it likely that this proof can be extended to all orders [43]. There is also extensive numerical evidence [41]. We conclude that the lattice theory defined by Eq. (3.34) constitutes a regularization of QCD with $4N_f$ flavors. In modern parlance, “flavor” is now usually used to denote the explicit multiplicity associated with the flavor index i , while “taste” is used to denote the implicit multiplicity associated with species doubling. If indeed the continuum limit is given by Eq. (3.35), this implies that a full $U(4)_{taste}$ symmetry emerges in the continuum limit, for each staggered flavor. This taste symmetry is not present in the lattice theory — we will return to this shortly.

We note that the taste degeneracy is somewhat of an embarrassment, if one wishes to use the flavor index i as physical flavor, as is usually done in practice. While this is not a problem for valence quarks (one simply picks out, say, taste $a = 1$ in order to construct operators, for example), this means that there are too many sea quarks — too many quarks on the internal fermion loops. This problem is solved in practice by taking the (positive) fourth root of the staggered determinant for each physical flavor. While this makes sense intuitively, it is a serious modification of the theory, which entails a violation of locality and unitarity at nonzero lattice spacing [45]. Much work has recently gone into attempts to show that the procedure is nevertheless correct in the sense that it yields the desired continuum limit, in which EFT techniques in fact play an important role. There is no space here to discuss this very interesting (and important!) topic; I refer to the recent reviews of Refs. [41, 46]. Here I will focus on the construction of staggered ChPT for unrooted staggered fermions.

To construct the SET for staggered QCD it is important to use all exact symmetries of the lattice theory, in order to restrict the form of the operators that will appear in the Symanzik expansion, Eq. (3.2). The action (3.34) has an exact vectorlike $SU(N_f)$ symmetry, broken only if the quark masses m_i are nondegenerate. Furthermore, there are spacetime symmetries. In addition to hypercubic rotations and parity, the action is invariant under shift symmetry [43, 47]:

$$\begin{aligned} S_{\pm\mu} : \chi_i(x) &\rightarrow \zeta_\mu(x) \chi_i(x \pm \mu), & \zeta_\mu(x) &= (-1)^{x_{\mu+1} + \dots + x_4}, \\ S_{\pm\mu} : U_\nu(x) &\rightarrow U_\nu(x \pm \mu), \end{aligned} \quad (3.43)$$

because of the fact that

$$\zeta_\nu(x) \eta_\mu(x) \zeta_\nu(x + \mu) = \eta_\mu(x + \nu). \quad (3.44)$$

On the basis $q_{i\alpha a}$ shift symmetry is represented by

$$S_{\pm\mu} : q_{i\alpha a}(k) \rightarrow \xi_{\mu,ab} e^{\pm ik_\mu} q_{i\alpha b}(k) , \quad (3.45)$$

with ξ_μ a different set of Dirac matrices (like Eq. (3.40) we have that $S_\mu S_\nu = -S_\nu S_\mu$ for $\mu \neq \nu$) acting on the taste index, commuting with the γ_ν of Eq. (3.39), because of Eq. (3.44).

Finally, if $m_i = 0$ for a certain flavor, there is a chiral symmetry (“ $U(1)_\epsilon$ symmetry”) [48]

$$\begin{aligned} \chi_i(x) &\rightarrow e^{i\theta_i \epsilon(x)} \chi_i(x) , & \bar{\chi}_i(x) &\rightarrow e^{i\theta_i \epsilon(x)} \bar{\chi}_i(x) , \\ \epsilon(x) &= (-1)^{x_1+x_2+x_3+x_4} . \end{aligned} \quad (3.46)$$

Since this symmetry is broken by a nonzero m_i , it should be interpreted as an axial symmetry of the theory. In addition, since it is an exact symmetry of the lattice theory (for $m_i = 0$) it has to correspond to a nonsinglet axial symmetry in the continuum limit, because the continuum singlet axial symmetry is anomalous. If all quarks are massless, it is not difficult to show that this enlarges to an exact $U(N_f)_L \times U(N_f)_R$ symmetry.⁴⁶ The axial symmetries in this group are nonsinglet symmetries on the basis $q_{i\alpha a}$, because the phase $\epsilon(x)$ inserts momentum, and thus acts nontrivially on the index pair α, a : Since

$$\epsilon(x) \chi_i(x) = \prod_\mu T_{-\mu} S_\mu \chi_i(x) , \quad (3.47)$$

it follows that $U(1)_\epsilon$ symmetry is represented on the $q_{\alpha a}$ basis as

$$q_{i\alpha a} \rightarrow \left(e^{i\theta_i \gamma_5 \xi_5} \right)_{\alpha a, \beta b} q_{i\beta b} . \quad (3.48)$$

In staggered QCD there is extensive evidence that these axial symmetries are spontaneously broken, and for each such broken symmetry there is thus an exact Nambu–Goldstone boson. When m_i is turned on, this NG boson picks up a mass-squared proportional to m_i , as we will see below. Contrary to the case of Wilson fermions, we thus know the critical values of the quark masses m_i — the massless limit is at $m_i = 0$. Interpolating fields for these Nambu–Goldstone bosons are $\epsilon(x) \bar{\chi}_i(x) \chi_j(x)$ in position space, or $\bar{q}_i \gamma_5 \xi_5 q_j$ in momentum space.

As can be seen from Eq. (3.35), these lattice symmetries enlarge to $SU(4N_f)_L \times SU(4N_f)_R$ (and the usual euclidean spacetime symmetries) in the continuum limit. In the continuum limit, one thus expects $(4N_f)^2 - 1$ pions, associated with the spontaneous breakdown $SU(4N_f)_L \times SU(4N_f)_R \rightarrow SU(4N_f)_V$. At nonzero a , there are only N_f^2 exact Nambu–Goldstone bosons; the rest will pick up additional contributions to their mass-squared of order a^2 , as we will see in more detail below. These lattice-artifact mass splittings are usually referred to as “taste splittings,” since they occur because of the breakdown of $U(4)_{\text{taste}}$ on the lattice. Indeed, this taste breaking is seen on the lattice very clearly in the meson spectrum, which is shown in Fig. 3. Clearly, we would like to understand these results with help of a ChPT framework.

⁴⁶ $U(N_f)$ instead of $SU(N_f)$, because, as we just noted, the axial symmetries in this group are all nonsinglet!

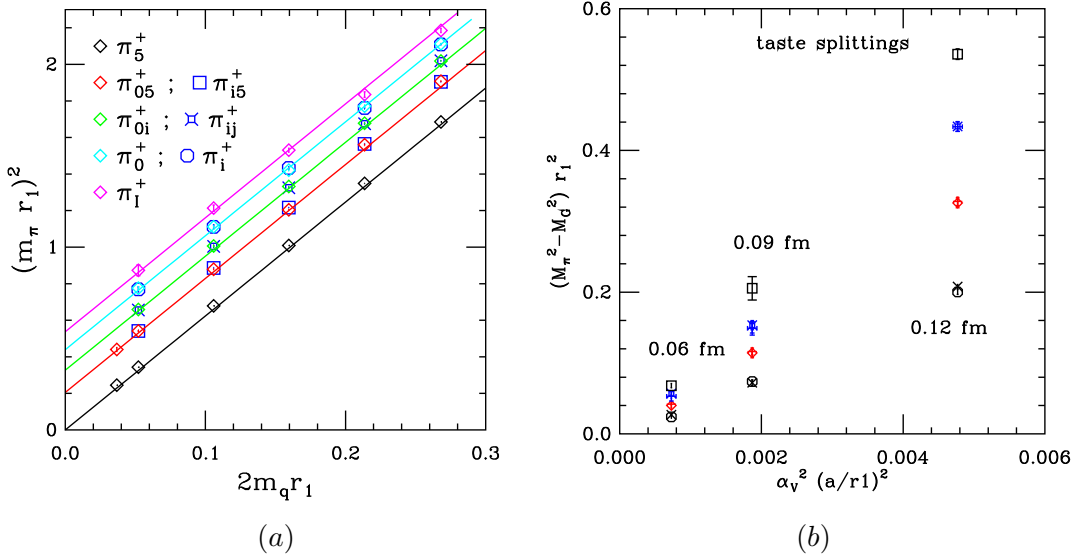


FIG. 3: (a) Taste splittings between pions made out of an up and (anti)down staggered quarks, in different representations of the staggered symmetry group, at fixed lattice spacing, $a = 0.125$ fm. The black points correspond to the exact Nambu–Goldstone boson, lines are chiral fits using staggered ChPT. (b) Taste splittings between the various nonexact Nambu–Goldstone bosons and the exact Nambu–Goldstone boson as a function of lattice spacing. These figures show the behavior predicted by Eq. (3.63). From Ref. [18], to which I refer for a detailed explanation.

2. Staggered ChPT

The SET is a continuum theory. This means that it is invariant under translations over an arbitrary distance; in particular, it is invariant under translations over a distance a in any direction.⁴⁷ It is also invariant under any of the exact lattice symmetries. We may thus combine a shift S_μ in the positive μ direction by a translation over a distance a in the negative μ direction. Combining these two, we see that the SET is invariant under [26, 49]

$$q_i(k) \rightarrow \xi_\mu q_i(k) . \quad (3.49)$$

At the level of the SET, the “translation” and “taste” parts of shift symmetry thus decouple, and we find that the SET thus has to be invariant under the 32-element group Γ_4 of discrete taste transformations generated by ξ_μ , which is a subgroup of $U(4)_{taste}$.⁴⁸ We can thus use this group in order to restrict operators at each order in the Symanzik expansion, along with the other exact lattice symmetries.

The first term in the Symanzik expansion, $\mathcal{L}_S^{(4)}$, is given by (the integrand of) Eq. (3.35). There are no dimension-five operators that can be constructed from the quark fields q and

⁴⁷ In contrast, staggered QCD is only invariant under ordinary translations over a distance $2a$ in any of the principal lattice directions.

⁴⁸ Note that this group does not become larger due to the presence of N_f flavors, because the gluon field also transforms nontrivially under shifts, *cf.* Eq. (3.43). Thus all quark flavors must be shifted together. But because U_μ shifts by a pure translation, the analogue of Eq. (3.49) for gluons is trivial: $U_\mu(p) \rightarrow U_\mu(p)$.

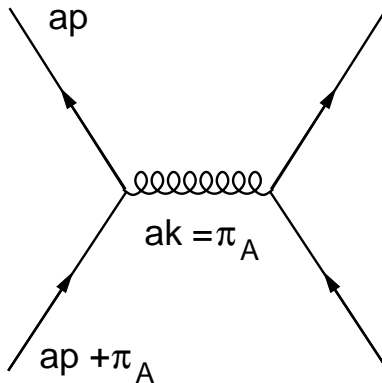


FIG. 4: Generation of taste-breaking four-fermion vertices through short-distance gluon exchange.

\bar{q} that respect the lattice symmetries [50]. Ignoring $U(1)_\epsilon$ symmetry, the possible dimension five operators are those of Eq. (3.5). The first two operators in this list are immediately excluded by Eq. (3.48). The three other terms involve powers of the quark mass, and in order to make use of $U(1)_\epsilon$ symmetry, we thus have to make the quark mass a spurion, as in Eqs. (2.1) and (2.5), but we now do it with L and R projectors on the quark fields defined with $\gamma_5 \xi_5$, instead of γ_5 . It is then not difficult to see that also the three last operators in Eq. (3.5) are excluded.

This implies that scaling violations for staggered QCD start at order a^2 . In fact, before we start the discussion of $\mathcal{L}^{(6)}$, we should rethink power counting. If we use GSM power counting, all terms of order a^2 would be $O(p^4)$. However, it turns out that scaling violations with staggered fermions, even though formally $O(a^2)$, are numerically rather large, even with improved staggered actions, at current values of the lattice spacing.⁴⁹ From the taste splittings in pseudoscalar mesons, it seems that the contribution of the quark masses and $O(a^2)$ scaling violations to their masses is roughly equal, and we will therefore use a power counting in which $p^2 \sim m_{quark} \Lambda_{QCD} \sim a^2 \Lambda_{QCD}^4$. This means that $O(a^2)$ effects are of leading order in ChPT, and we would like to construct at least the $O(a^2)$ part of the chiral lagrangian, because from the $O(p^2)$ lagrangian we can obtain the $O(p^2)$ and nonanalytic $O(p^4)$ parts of physical quantities in terms of quark masses and the lattice spacing. We thus set out to find the $O(a^2)$ part of $\mathcal{L}^{(2)}$, *cf.* Eq. (2.33). This new part contains no quark masses, and no derivatives, each of which would make such terms higher order in the chiral expansion.

The $O(a^2)$ part of the SET, $\mathcal{L}^{(6)}$, contains a large number of terms [49, 51], and I will only give some examples here. First, there are the purely gluonic terms which we already discussed in the context of Wilson ChPT, and thus do not have to revisit again. Similarly, fermion bilinears in $\mathcal{L}_S^{(6)}$ do not lead to any new terms at the desired order in ChPT [49]. The remaining terms are all four-fermion operators.

Taste-breaking, short distance four-fermion operators occur in the theory because of

⁴⁹ This state of affairs may change in the relatively near future, with more highly improved lattice actions, and even smaller lattice spacings.

diagrams as shown in Fig. 4. The gluon carries a momentum near one of the values in Eq. (3.36), with $n > 0$ components near π/a , thus changing the taste of the staggered quarks at both vertices. For such gluon momenta, we have that the gluon propagator

$$\frac{1}{\sum_{\mu} \frac{4}{a^2} \sin^2(\frac{1}{2}ap_{\mu})} \approx \frac{a^2}{4n} . \quad (3.50)$$

Effectively, this generates a four-fermion operator in the SET (in which gluons with momenta of order $1/a$ have been integrated out), for instance of the form⁵⁰

$$\mathcal{O}_1 = (\bar{q}_i \xi_{5\nu} q_i) (\bar{q}_j \xi_{\nu 5} q_j) , \quad (3.51)$$

in which $-\xi_{\nu 5} = \xi_{5\nu} = \xi_5 \xi_{\nu}$, and all repeated indices are summed. This operator is invariant under the group Γ_4 , but not under the full continuum taste group $U(4)_{taste}$ (it is, in fact, invariant under $SO(4)_{taste} \subset U(4)_{taste}$, accidentally). Since the gluon carries no flavor, the flavor index structure has to be as indicated. Note that the spin \otimes taste matrix appearing in each quark bilinear has to correspond to an odd number of applications of T_{μ} or S_{μ} , because gluons couple only to the kinetic term in Eq. (3.34), which is invariant under $U(1)_{\epsilon}$.

Note that the appearance of four-fermion operators due to the exchange of “heavy” gluons is analogous to the appearance of four-fermion terms with coefficients of order g_{weak}^2/M_W^2 (the Fermi EFT) in the Standard Model at low energy because of the exchange of the heavy W and Z bosons. Likewise, in the case of hand, the four-fermion operators in the SET are of order α_s , the strong “fine-structure” constant. Light gluons, with momenta $p \ll 1/a$ are still present in the SET, *i.e.*, they have not been integrated out.

The spin matrices γ_{μ} can also appear in these four-fermion operators. This can happen in two ways, here is an example of each:

$$\mathcal{O}_2 = \sum_{\mu} (\bar{q}_i \gamma_{\mu} \xi_5 q_i) (\bar{q}_j \gamma_{\mu} \xi_5 q_j) , \quad (3.52)$$

$$\mathcal{O}_3 = \sum_{\mu < \nu} (\bar{q}_i \gamma_{\nu} \xi_{\mu\nu} q_i) (\bar{q}_j \gamma_{\nu} \xi_{\nu\mu} q_j) , \quad (3.53)$$

in which $\xi_{\mu\nu} = \xi_{\mu} \xi_{\nu}$ ($\mu \neq \nu$), and where we now showed the sums over Lorentz indices explicitly. As before, neither of these operators is invariant under $U(4)_{taste}$, and both are invariant under Γ_4 (as well as all other lattice symmetries, in particular hypercubic rotations). The first operator is separately invariant under $SO(4)_{euclidean} \times SO(4)_{taste}$, the second is not. Operators of the second type can only be represented in ChPT with at least two derivatives (and one factor a^2) [52], which are of order p^4 .⁵¹ Only operators of the first type have $O(p^2)$ representations in ChPT. An interesting observation is thus that at lowest order staggered ChPT therefore has $SO(4)_{taste}$ symmetry, and not just the required Γ_4 symmetry [49].

⁵⁰ Four-fermion operators can be color unmixed or mixed. For instance, Eq. (3.51) can take the form $(\bar{q}_{ia} \xi_{5\nu} q_{ia}) (\bar{q}_{jb} \xi_{\nu 5} q_{jb})$ or $(\bar{q}_{ia} \xi_{5\nu} q_{ib}) (\bar{q}_{jb} \xi_{\nu 5} q_{ja})$, where a and b are color indices. We see that color indices can be contracted in two different ways. They will be omitted here, because it makes no difference for our analysis.

⁵¹ Now both derivatives and ξ_{μ} matrices can provide Lorentz indices, so the argument following Eq. (3.6) does not apply.

Finally, we need to make the transition from the SET to ChPT, for operators such as (3.51) and (3.52). For $N_f = 3$, our theory is now an $SU(12)$ theory, because of the three flavors times four tastes. The nonlinear field

$$\Sigma_{ia,jb} \sim q_{iaL} \bar{q}_{jbR} , \quad (3.54)$$

is thus an $SU(12)$ valued field, with an index structure as shown. In order to streamline our notation, we introduce matrices [51]

$$\xi_\mu^{(3)} = \begin{pmatrix} \xi_\mu & 0 & 0 \\ 0 & \xi_\mu & 0 \\ 0 & 0 & \xi_\mu \end{pmatrix} , \quad (3.55)$$

so that we can write for example Eq. (3.51) as $(\bar{q}\xi_{5\nu}^{(3)}q)(\bar{q}\xi_{\nu 5}^{(3)}q)$. We now rewrite

$$\begin{aligned} -(\bar{q}\xi_{5\nu}^{(3)}q)(\bar{q}\xi_{\nu 5}^{(3)}q) &= (\bar{q}_R \xi_{5\nu}^{(3)} q_L + \bar{q}_L \xi_{5\nu}^{(3)} q_R)^2 \\ &\rightarrow (\bar{q}_R X_R q_L + \bar{q}_L X_L q_R)^2 , \end{aligned} \quad (3.56)$$

where in the last line we introduced two spurions transforming as

$$X_L \rightarrow U_L X_L U_R^\dagger , \quad X_R \rightarrow U_R X_R U_L^\dagger , \quad (3.57)$$

while they are interchanged by parity. With the field Σ we can construct three types of terms in ChPT with these spurion fields that do not contain the quark mass matrix or any derivatives:

$$(\bar{q}\xi_{5\nu}^{(3)}q)(\bar{q}\xi_{\nu 5}^{(3)}q) \rightarrow \begin{cases} \text{tr}(X_R \Sigma) \text{tr}(X_L \Sigma^\dagger) , \\ \left(\text{tr}(X_R \Sigma)\right)^2 + \left(\text{tr}(X_L \Sigma^\dagger)\right)^2 , \\ \text{tr}(X_R \Sigma X_R \Sigma) + \text{tr}(X_L \Sigma^\dagger X_L \Sigma^\dagger) , \end{cases} \quad (3.58)$$

which, setting the spurions equal to the values that reproduce \mathcal{O}_1 , gives the three ChPT operators

$$\text{tr}(\xi_{5\nu}^{(3)}\Sigma) \text{tr}(\xi_{5\nu}^{(3)}\Sigma^\dagger) , \quad \left(\text{tr}(\xi_{5\nu}^{(3)}\Sigma)\right)^2 + \left(\text{tr}(\xi_{5\nu}^{(3)}\Sigma^\dagger)\right)^2 , \quad \text{tr}(\xi_{5\nu}^{(3)}\Sigma \xi_{5\nu}^{(3)}\Sigma) + \text{tr}(\xi_{5\nu}^{(3)}\Sigma^\dagger \xi_{5\nu}^{(3)}\Sigma^\dagger) , \quad (3.59)$$

in which sums over the index ν are understood.

Introducing spurions $Y_{L,R}$ for the taste matrices $\xi_5^{(3)}$ in \mathcal{O}_2 , which transform as $Y_{L,R} \rightarrow U_{L,R} Y_{L,R} U_{L,R}^\dagger$, and working through a similar exercise, one finds the (unique) ChPT operator representing \mathcal{O}_2 :

$$\sum_\mu (\bar{q}_i \gamma_\mu \xi_5^{(3)} q_i) (\bar{q}_j \gamma_\mu \xi_5^{(3)} q_j) \rightarrow \text{tr}(\xi_5^{(3)} \Sigma \xi_5^{(3)} \Sigma^\dagger) . \quad (3.60)$$

Referring to Ref. [51] for further details, we quote the $O(a^2)$ contribution to the chiral lagrangian, to be added to Eq. (2.33). Writing this part as $a^2 \mathcal{V} = a^2 (\mathcal{U} + \mathcal{U}')$, one finds that

$$\begin{aligned} -\mathcal{U} &= C_1 \text{tr}(\xi_5^{(3)} \Sigma \xi_5^{(3)} \Sigma^\dagger) + \frac{1}{2} C_3 \sum_\nu \left[\text{tr}(\xi_\nu^{(3)} \Sigma \xi_\nu^{(3)} \Sigma) + \text{h.c.} \right] \\ &\quad + \frac{1}{2} C_4 \sum_\nu \left[\text{tr}(\xi_{5\nu}^{(3)} \Sigma \xi_{\nu 5}^{(3)} \Sigma) + \text{h.c.} \right] + C_6 \sum_{\mu < \nu} \text{tr}(\xi_{\mu\nu}^{(3)} \Sigma \xi_{\nu\mu}^{(3)} \Sigma^\dagger) , \\ -\mathcal{U}' &= \frac{1}{4} C_{2V} \sum_\nu \left[\text{tr}(\xi_\nu^{(3)} \Sigma) \text{tr}(\xi_\nu^{(3)} \Sigma) + \text{h.c.} \right] + \frac{1}{4} C_{2A} \sum_\nu \left[\text{tr}(\xi_{5\nu}^{(3)} \Sigma) \text{tr}(\xi_{\nu 5}^{(3)} \Sigma) + \text{h.c.} \right] \\ &\quad + \frac{1}{2} C_{5V} \sum_\nu \text{tr}(\xi_\nu^{(3)} \Sigma) \text{tr}(\xi_\nu^{(3)} \Sigma^\dagger) + \frac{1}{2} C_{5A} \sum_\nu \text{tr}(\xi_{5\nu}^{(3)} \Sigma) \text{tr}(\xi_{\nu 5}^{(3)} \Sigma^\dagger) . \end{aligned} \quad (3.61)$$

Let us briefly consider some of the physics results that can be obtained from the $O(p^2)$ chiral lagrangian, which is the sum of Eqs. (2.33) (for a twelve-flavor continuum theory) and (3.61). First, consider meson masses at tree level. Here I will only quote results for flavored (*i.e.*, off-diagonal in Eq. (2.6)) masses. The flavor-neutral sector is more complicated, and I refer to Ref. [51] for details.

Each of the entries in Eq. (2.6) is now a four-by-four taste matrix, and we can thus expand each entry on a set of $U(4)_{taste}$ generators:

$$\begin{aligned} \phi_{ij} &= \frac{1}{2} \sum_{F=1}^{16} \Xi_F \phi_{ij}^F, \quad i, j = u, d, s, \\ \Xi_F &\in \{\xi_5, i\xi_{5\mu}, i\xi_{\mu\nu}, \xi_\mu, \mathbf{1}\}. \end{aligned} \quad (3.62)$$

By expanding Σ to quadratic order, we can then read off the meson masses from Eqs. (2.33) and (3.61). In the flavored sector, only \mathcal{U} contributes, and we find

$$m_{Fij}^2 = B_0(m_i + m_j) + a^2 \Delta(\Xi_F), \quad i, j = u, d, s, \quad i \neq j, \quad (3.63)$$

in which

$$\begin{aligned} \Delta(\xi_5) &= 0, \\ \Delta(i\xi_{5\mu}) &= \frac{16}{f^2} (C_1 + 3C_3 + C_4 + 3C_6), \\ \Delta(i\xi_{\mu\nu}) &= \frac{16}{f^2} (2C_3 + 2C_4 + 4C_6), \\ \Delta(\xi_\mu) &= \frac{16}{f^2} (C_1 + C_3 + 3C_4 + 3C_6), \\ \Delta(\xi_1) &= \frac{16}{f^2} (4C_3 + 4C_4). \end{aligned} \quad (3.64)$$

As we expect, there is one exact flavor multiplet of Nambu–Goldstone bosons, because of $U(1)_\epsilon$ symmetry, with a mass-squared proportional to the quark masses, as in the continuum. All other meson masses are shifted by terms of order a^2 , and Fig. 3 is in good agreement with the behavior predicted by Eq. (3.63). The numerical results show that C_4 must be dominant among $C_{1,3,4,6}$. This is consistent with the fact that, as previously mentioned, $U(4)_{taste}$ is broken at this order in ChPT to $SO(4)$, not all the way down to Γ_4 . Thus there are only five nondegenerate meson masses [49], whereas the irreducible representations of the staggered symmetry group would allow eight [53].

We end this section by quoting a one-loop result that can be obtained from our lowest-order staggered chiral lagrangian. But before we do this, we have to address the fact that the $SU(12)$ theory has too many sea quarks, which on the lattice side is fixed by taking the fourth root of each staggered fermion determinant. The question is how to carry over this procedure to staggered ChPT. The way this can be done is through the “replica trick.” One starts with *adding more* fermions to the theory by using n_r instead of one staggered fermions for each physical flavor. For each of the n_r up quarks, the quark mass is kept degenerate — all up quarks have mass m_u , and likewise for down and strange quarks. In the lattice QCD path integral, this results in raising the staggered determinant for each flavor to the n_r -th power. The idea is now to do all calculations keeping the n_r dependence, and then at the end set $n_r = 1/4$, which corresponds precisely to taking the fourth root of the determinants.

As we already mentioned, it is by no means obvious that the fourth-root procedure is field-theoretically legitimate. However, if it is, it is possible to show that the replica trick is the correct way of incorporating the effects of the fourth root in staggered ChPT (and EFTs for staggered QCD in general) [26, 54]. Assuming that this all works, we quote the $O(p^4)$ result for the two-flavor ξ_5 -pion decay constant (which is the real Nambu–Goldstone boson for the breakdown of $U(1)_\epsilon$ symmetry), with equal light quark masses, $m_u = m_d = m_\ell$:⁵²

$$\begin{aligned} \frac{f_\pi^{\xi_5}}{f} = 1 & - \frac{1}{8} \sum_B L(2B_0 m_\ell + a^2 \Delta(\Xi_B)) \\ & - 4 \left(L(2B_0 m + a^2 \delta'_V) + L(2B_0 m + a^2 \delta'_A) - 2L(2B_0 m) \right) \\ & + \frac{16B_0 m}{f^2} (2L_4 + L_5) + a^2 F , \end{aligned} \quad (3.65)$$

in which

$$\begin{aligned} \delta'_V &= \frac{16}{f^2} (C_{2V} - C_{5V}) , \\ \delta'_A &= \frac{16}{f^2} (C_{2A} - C_{5A}) , \end{aligned} \quad (3.66)$$

and F is a linear combination of $O(p^4)$ LECs.⁵³ Note that now the LECs are low-energy constants in the $N_f = 2$ theory. The LECs $C_{2V,A}$ and $C_{5V,A}$ only appear in flavor-neutral meson masses at tree level (as can be seen by expanding out \mathcal{U}'), and their appearance reflects the fact that flavor-neutral mesons appear in the one-loop corrections. Flavored mesons also appear in the loops; producing the logarithms in the sum over B .

We note that the masses as they appear in chiral logarithms get $O(a^2)$ corrections. This is of course natural, because all lattice mesons can in principle appear on loops, and most of these masses get $O(a^2)$ corrections, already at lowest order in ChPT. This change in the logarithms points at a reason as to why it is important to incorporate scaling violations in ChPT systematically: as long as $m_{quark} \sim a^2 \Lambda_{QCD}^3$, one cannot expand the logarithm in Eq. (3.65) in powers of a^2 . If we would have naively assumed that all scaling violations can be understood in terms of simple powers of a , we would have missed this nonanalytic dependence on the lattice spacing. The $O(a^2)$ term inside the logarithm reduces the curvature for small quark masses. Knowing the field-theoretical form of this nonanalytic behavior makes it thus feasible to explore smaller quark masses at a given lattice spacing than otherwise would be possible. In Fig. 5, I show a plot of the staggered ChPT fits of the meson masses from the MILC collaboration (which, of course, use the full nondegenerate 2 + 1-flavor and partially quenched formulas of Ref. [55]). The indices x and y denote the flavors of the valence quarks, which for most of the data points in this figure have masses different from the sea quarks, a generalization we will consider in the next section. In this figure, we see that indeed the curvature on the lattice is significantly different from that of the continuum curve.

Finally, we observe that the form of the tree-level masses, Eq. (3.63), is very similar to Eq. (3.22). That raises the question whether, since we are working in the LCE regime in

⁵² This result can be derived from Eq. (27) of Ref. [55], by making the strange quark heavy (a situation we will discuss more in Sec. 5). For nondegenerate results, see Ref. [55], for $O(p^4)$ meson masses, see Ref. [51].

⁵³ The complete $O(p^4)$ staggered chiral lagrangian has been worked out in Ref. [52].

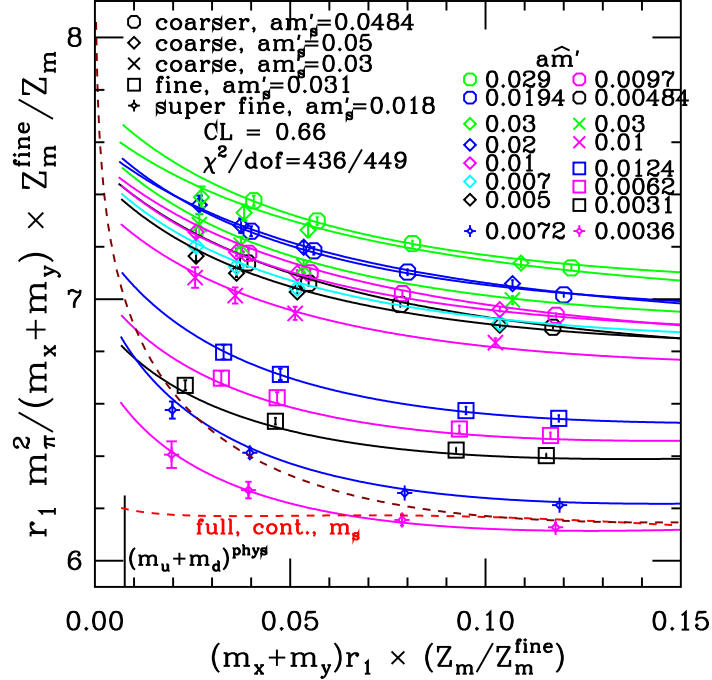


FIG. 5: $m_\pi^2/(m_x + m_y)$ as a function of $m_x + m_y$. The continuous curves are (partially quenched) chiral fits at fixed lattice spacing; the dashed line is the (partially quenched) continuum line (with sea quark masses corresponding to those of the “superfine” 0.018/0.0072 ensemble), which has a much stronger curvature. Note how the curvature is reduced by nonvanishing lattice spacing. The red dashed curve is the fully unquenched continuum curve. m'_s is the strange sea quark mass and \hat{m}' is the light seaquark mass. For more explanation of the meaning of all symbols, see Ref. [18]. Figure courtesy C. Bernard.

which $m_{quark} \sim a^2 \Lambda_{QCD}^3$, Aoki-like phases could occur for small enough quark mass. Using techniques similar to those used in QCD inequalities⁵⁴ it can be argued that all $\Delta(\Xi_B) \geq 0$, excluding this route to a possible Aoki-like phase. However, this is not true for the $O(a^2)$ corrections in the flavor-neutral sector, which are governed by the LECs $\delta'_{V,A}$. While no sign of an Aoki-like phase has been observed in numerical computations, this remains an interesting theoretical possibility [51, 57].

IV. CHOOSING VALENCE AND SEA QUARKS TO BE DIFFERENT

Consider the two-point function for a charged pion in euclidean QCD,

$$\begin{aligned} \langle \pi^+(x) \pi^-(y) \rangle &= \langle (\bar{u}(x) \gamma_5 d(x)) (\bar{d}(y) \gamma_5 u(y)) \rangle \\ &= \frac{1}{Z} \int \prod_\mu [dU_\mu] \prod_i [d\bar{q}_i] [dq_i] \exp(-S_{QCD}) \bar{u}(x) \gamma_5 d(x) \bar{d}(y) \gamma_5 u(y) \end{aligned} \quad (4.1)$$

⁵⁴ For a review, see Ref. [56].

$$= -\frac{1}{Z} \int \prod_{\mu} [dU_{\mu}] \text{Det}(\mathcal{D} + M) \text{tr} \left[\gamma_5 (\mathcal{D} + m_u)^{-1} (y, x) \gamma_5 (\mathcal{D} + m_d)^{-1} (x, y) \right] .$$

In an actual lattice computation, one usually does not use point sources and sinks; instead one sums with some weight over the timeslices at $t = x_4$ and $t' = y_4$, but this will not affect the observation we want to make.

In the third line of Eq. (4.1), the quarks enter in two clearly distinct ways: through the determinant, and through the quark propagators that follow from carrying out the Wick contractions. We refer to quarks of the first type as sea quarks, and those of the latter type as valence quarks. Of course, in nature these two types of quarks are the same, and they have to be in order to preserve unitarity of the theory. But nothing stops us on the lattice from taking the quark masses in the propagators unequal to those in the determinant, or, even more drastically, from choosing a completely different discretization for \mathcal{D} in the propagators and in the determinant! Let us first consider whether this seemingly sacrilegious idea might have any advantages, and if so, whether we can make any field-theoretical sense out of it.

First, consider only taking valence quark masses unequal to sea quark masses. This is a situation commonly referred to as partial quenching. This name has its origin in the fact that as we send the sea quark masses to infinity, we effectively remove the determinant from the path integral, thus reverting to the quenched approximation. Unlike the quenched approximation, however, as long as we keep the sea quark masses in the theory, we can always recover the physical theory by choosing the valence and sea quark masses equal (for each flavor) [58]. In other words, real or “full” QCD is a “special case” of partially quenched QCD (PQQCD).⁵⁵

This observation should then carry over to any EFT for PQQCD, and thus, in particular, to partially quenched ChPT (PQChPT). (Of course, we will have to reconsider the arguments for the existence of ChPT, a point to which we will return below.) What we will see is that, if both sea and valence quark masses are light enough for the chiral expansion to apply, being able to vary the valence and sea quark masses independently is very useful: It gives us an extra tool for matching lattice QCD computations to ChPT, and thus determine the numerical values of the LECs at some scale — which pin down the EFT quantitatively. The key observation here is that those LECs are the *same* in the full and partially quenched theories, because they are independent of the quark masses, and thus also of the distinction that we make between valence and sea quark masses in the partially quenched generalization of QCD [59]. In addition, a very practical consideration is that in most cases it is much less expensive in terms of computational cost to vary the valence quark masses (which only show up in the propagators coming from the external operators) than the sea quark masses, which are part of the effective gauge action used to generate an ensemble of gauge-field configurations. We will consider partial quenching in detail in Secs. IV B and IV C.

The motivation for choosing even the lattice Dirac operators different for valence and sea quarks is different. Of course, the basic assumption will be that in the continuum limit the difference in discretization disappears if the bare valence and sea quark masses are tuned to yield the same renormalized quark mass — an extension of the notion of

⁵⁵ If one chooses the number of light sea quarks to be equal to three! In that case, the name “partially quenched” is slightly unfortunate. However, plenty of lattice computations are still really partially quenched, in that they include only two sea quarks.

universality. But, the symmetry properties of a correlation function such as Eq. (4.1) are determined by the symmetry properties of the valence Dirac operator, and in many cases it is a great advantage to be able to use a discretization with symmetries (in particular chiral symmetry) as close as possible to those of the continuum. Often, because of the smaller amount of symmetry on the lattice, operator mixing is more severe on the lattice, making computations of, for instance, weak matrix elements much harder, if not impossible. More symmetry, in particular more chiral and flavor symmetry, means less mixing. However, lattice Dirac operators with good symmetry properties, such as domain-wall and overlap operators, are numerically very expensive. Since a large part of the computational effort goes into evaluating the determinant, it makes sense to choose a less expensive lattice version for the sea quark Dirac operator, such as Wilson or staggered operators. This “mixed-action” option will be discussed in Sec. IV D.

Having extolled the possible virtues, we should address the question whether these generalizations of QCD make any field-theoretical sense.

A. Path integral

Of course, operationally, the procedure of choosing different Dirac operators for the valence and sea quarks outlined in the previous section is well defined. But, does it make field-theoretical sense? For instance, can we hope to develop EFTs for these generalized, but — at this stage — suspect versions of QCD? Clearly, in order to address these questions, it would help if we can write down a path integral expression that reflects the distinction between valence and sea quarks. This can be done using a trick proposed by Morel [60]. Obviously, separate fermionic fields are needed for sea and valence quarks, but doing just this would lead to fermion determinants for both. The trick is to cancel the valence determinant by introducing yet another set of quark fields, with Dirac operator and mass matrix identical to those of the valence quarks, but with opposite (bosonic) statistics. We will refer to these fields as ghosts.⁵⁶ The fermionic part of the QCD lagrangian generalizes to⁵⁷

$$\mathcal{L} = \bar{q}_s(D_s + M_s)q_s + \bar{q}_v(D_v + M_v)q_v + \tilde{q}^\dagger(D_v + M_v)\tilde{q}, \quad (4.2)$$

in which \tilde{q} denotes the bosonic ghost-quark fields; D_s is the lattice Dirac operator for sea quarks, and D_v that for valence quarks. In general, we will label sea quantities with subscript s , valence quantities with subscript v , and put tildes on entities referring to ghosts. Note that the ghost and antighost fields, \tilde{q} and \tilde{q}^\dagger , are not independent. The path integral for this theory is

$$Z = \int \prod_\mu [dU_\mu] \prod_i [d\bar{q}_{s,i}] [dq_{s,i}] \prod_j [d\bar{q}_{v,j}] [dq_{v,j}] [d\tilde{q}_j^\dagger] [d\tilde{q}_j] \quad (4.3)$$

$$\exp \left(-S_{gauge} - \sum_x [\bar{q}_s(D_s + M_s)q_s + \bar{q}_v(D_v + M_v)q_v + \tilde{q}^\dagger(D_v + M_v)\tilde{q}] \right)$$

⁵⁶ These ghosts have nothing to do with Faddeev–Popov ghosts! They are called ghosts, because they share the similarly “incorrect” spin-statistics properties.

⁵⁷ I write the Dirac operators D_s and D_v without a slash, because not all lattice Dirac operators are of the form $\not{D} = \gamma_\mu D_\mu$.

$$= \int \prod_{\mu} [dU_{\mu}] \exp(-S_{gauge}) \text{Det} (D_s + M_s) ,$$

which follows because of the exact cancellation of the valence and ghost determinants. Intuitively, at the level of diagrams, valence and ghost loops cancel, because they always come in pairs with opposite signs between them. Clearly, it is possible to couple the path integral to sources for all quark fields, and we can thus generate correlation functions for operators made out of any of these fields. In numerical computations, of course the ghost fields are never introduced in the first place. But one still has access to correlation functions that include ghost quarks, because after doing the Wick contractions one simply finds that each ghost-antighost contraction is replaced by a valence propagator. For example, if one considers the pion two-point function of Eq. (4.1) for pions made entirely out of ghost quarks, one finds exactly the same expression as on the last line of Eq. (4.1), except without the minus sign, which would be absent because ghost fields commute, rather than anticommute.

In order for definition (4.3) to make sense, we have to make sure that the integral over ghost fields converges. This is the case if D_v is antihermitian, as it is in the continuum and for staggered fermions, for instance. In addition, we have to require that the mass matrix, M_v , has strictly positive eigenvalues. For Wilson or domain-wall fermions the Dirac operator is not antihermitian (or hermitian), and the real part of its eigenvalues can be of both signs. For a method of defining PQ or mixed-action QCD with valence Wilson fermions, I refer to Ref. [61].⁵⁸ For overlap fermions, the eigenvalues are also complex, but one can prove that if the overlap operator satisfies $D^{\dagger} = \gamma_5 D \gamma_5$, all eigenvalues have a nonnegative real part. Thus, the ghost integral converges again as long as M_v has only positive eigenvalues.

It is clear that, while a euclidean path-integral definition can be given, this theory is sick. The ghost quarks have the wrong statistics for a spin-1/2 field. Remember that if you try to quantize a spin-1/2 field with the wrong statistics, one of the problems that occurs is that the hamiltonian of the theory is unbounded from below [62]! This and other [63] aspects indicate that it may be problematic to continue the theory to a theory in Minkowski space with a positive hamiltonian. But even if we could continue to Minkowski space, there is a problem with unitarity if one considers correlation functions with valence quarks on the external lines, while *different* sea quarks run on the loops.⁵⁹

Fortunately, lattice QCD, as applied in modern numerical computations, does not want to be in Minkowski space. It is perfectly fine to match euclidean lattice computations to a euclidean EFT. Once the EFT has been completely defined by obtaining the values of all its LECs, we can take the continuum limit, set valence quark masses equal to sea quark masses, and continue to Minkowski space. The fact that PQQCD and mixed-action QCD can only be defined in euclidean space is not a problem.

However, for all this to work, we do need to develop ChPT for partially quenched and mixed-action theories, and it is not obvious that this can be done. The generalized theory does not satisfy some of the cherished properties of a healthy quantum field theory, in particular unitarity,⁶⁰ which are usually invoked to argue that a local, unitary EFT such as ChPT has to exist, and, is in fact the most general possibility [10]. We will say more about this in Sec. IV B for PQQCD, and in Sec. IV D for mixed-action QCD.

⁵⁸ A similar method as described in Ref. [61] exists for domain-wall fermions.

⁵⁹ It is clear that this sickness already occurs in weak-coupling perturbation theory.

⁶⁰ And causality, if these generalized theories only live in euclidean space. However, since PQQCD and mixed-action QCD are local theories, we expect that the cluster property is realized on correlation functions.

Before we delve into the specifics, let us discuss some generalities considering the symmetries of Eq. (4.2). If we have N_s sea quarks (we will always take $N_s = 3$ or 2 , as in previous sections) and N_v valence quarks (we can always pick as many valence quarks as we need, so N_v is arbitrary), it looks like the chiral symmetry group of Eq. (4.2) (for $M_s = 0$ and $M_v = 0$) is $[U(N_s)_L \times U(N_s)_R] \times [U(N_v|N_v)_L \times U(N_v|N_v)_R]$.⁶¹ Here $U(N|N)$ is a “graded” group, which we will discuss in more detail in the next section;⁶² for now, all we need is that the symmetry group of the ghost part of the lagrangian Eq. (4.2) alone would be the subgroup $U(N_v)_L \times U(N_v)_R$. It turns out that this is not quite correct [61, 64, 65], as I will now explain.

Since \tilde{q}^\dagger is not independent of \tilde{q} , it follows that the ghost part of Eq. (4.2) is invariant under

$$\tilde{q}_L \rightarrow V\tilde{q}_L, \quad \tilde{q}_R \rightarrow V^{\dagger-1}\tilde{q}_R, \quad (4.4)$$

with $V \in GL(N_v)$. Here

$$\tilde{q}_L = \frac{1}{2}(1 - \gamma_5)\tilde{q}, \quad \tilde{q}_R = \frac{1}{2}(1 + \gamma_5)\tilde{q}, \quad (4.5)$$

from which it follows that (unlike for quark fields \bar{q} and q , for which the chiral projections can be chosen independently, *cf.* Eq. (2.2))

$$\tilde{q}_L^\dagger = \tilde{q}^\dagger \frac{1}{2}(1 - \gamma_5), \quad \tilde{q}_R^\dagger = \tilde{q}^\dagger \frac{1}{2}(1 + \gamma_5). \quad (4.6)$$

Note that the definition of the projected q^\dagger 's is opposite to that of the projected \bar{q} 's. Equation (4.4) then follows from the fact that

$$\tilde{q}^\dagger D_v \tilde{q} = \tilde{q}_L^\dagger D_v \tilde{q}_R + \tilde{q}_R^\dagger D_v \tilde{q}_L. \quad (4.7)$$

Here I assumed that $\{D_v, \gamma_5\} = 0$, which holds in the continuum. The lattice equivalent of the decomposition (4.7) will depend on the type of lattice fermion employed, as already noted above.⁶³ But in all cases, the fact that \tilde{q}^\dagger is not independent of \tilde{q} determines the precise form of the chiral symmetry group, and in all cases it has to coincide with $GL(N_v)$ in the continuum limit.

The vector subgroup, defined as the group which leaves the ghost condensate $\tilde{q}_L^\dagger \tilde{q}_L + \tilde{q}_R^\dagger \tilde{q}_R$ invariant, is the unitary subgroup, $U(N_v) \subset GL(N_v)$. If we restrict ourselves momentarily to the ghost sector only, this implies that Σ has to be an element of the coset $GL(N_v)/U(N_v)$. If we want to write Σ in terms of a meson field $\tilde{\phi}$ analogous to ϕ in Eq. (2.6), but with its meson component fields “made out of ghost quarks and antiquarks,” we should write $\Sigma = \exp(2\tilde{\phi}/f)$, with $\tilde{\phi}$ a *hermitian* matrix. The number of mesonic degrees of freedom is the same as it would be for the coset $[U(N_v)_L \times U(N_v)_R]/U(N)_V$, but the expression for Σ in terms of the meson fields is different.⁶⁴

⁶¹ In the partially quenched case, for which $D_s = D_v$ in Eq. (4.2), the symmetry group is larger, as we will discuss in Sec. IV B.

⁶² See the text starting around Eq. (4.10).

⁶³ For instance, for staggered fermions, one can define \tilde{q}_R and \tilde{q}_L using the generator $\epsilon(x)$ of $U(1)_\epsilon$ transformations instead of γ_5 .

⁶⁴ I am skipping a number of further technical details about the precise symmetry group of Eq. (4.2). Also, when all is said and done, the anomaly will have to be taken into account, as usual. For the latter, see Sec. IV B.

For nonperturbative issues, it is crucial to work with the correct symmetry group. This can already be seen from Eq. (4.2): If we introduce spurion fields M and \overline{M} to write the ghost mass term as

$$\tilde{q}^\dagger M_v \tilde{q} \rightarrow \tilde{q}_L^\dagger M \tilde{q}_L + \tilde{q}_R^\dagger \overline{M} \tilde{q}_R, \quad (4.8)$$

transforming as

$$M \rightarrow V^{\dagger-1} M V^{-1}, \quad \overline{M} \rightarrow V \overline{M} V^\dagger, \quad (4.9)$$

we see that both M and \overline{M} remain strictly positive under $GL(N_v)$ transformations, consistent with the convergence of the ghost path integral. This would not be true if we replace $GL(N_v)$ by $U(N_v)_L \times U(N_v)_R$.

One may want to use ChPT to find the phase structure of a quenched or partially quenched theory, as we did for $N_f = 2$ Wilson fermions in Sec. III C [61]. In that case, one will have to be careful to start from the correct symmetries in constructing the chiral lagrangian. However, if one is in a phase in which the correct vacuum has been determined (and usually in applications, is given by $\Sigma = \mathbf{1}$), and one is only interested in calculating the chiral expansion of physical quantities, it can be shown that instead the symmetry group $U(N_v|N_v)_L \times U(N_v|N_v)_R$ can be used for constructing the chiral lagrangian [61, 66]. This is what we will do in the next section. The reason is that *in perturbation theory*, which is defined by expanding Σ in terms of $\tilde{\phi}$, we can perform a field redefinition $\tilde{\phi} \rightarrow i\tilde{\phi}$ that brings Σ into the “standard form,” $\Sigma = \exp(2i\tilde{\phi}/f)$, *i.e.*, the form that would have followed from starting with the ghost-sector symmetry group $U(N_v)_L \times U(N_v)_R$.

As an aside for the interested reader, there is another way to define PQQCD and mixed-action QCD, through the replica trick [67]. Instead of introducing ghost quarks, one only introduces separate valence quarks, n_i for each flavor. One then takes the limit $n_i \rightarrow 0$ at the end of a calculation.⁶⁵ Formally, this sends the valence determinant to a constant, thus removing it from the QCD partition function. For all positive integer values of the n_i , this defines a physically sensible theory, in which we just happen to discretize different quarks in different ways. The disadvantage is that not much is known about the properties of the limit $n_i \rightarrow 0$. In this case one bases the construction of the chiral lagrangian on the symmetry group $[U(N_s)_L \times U(N_s)_R] \times [U(N_v)_L \times U(N_v)_R]$ (with $N_v = \sum_i n_i$), and considers ChPT in the $n_i \rightarrow 0$ limit. In all cases where explicit calculations have been done, the results agree from those obtained with Morel’s trick, using ghost quarks.

B. Partially quenched ChPT

In this section, we will consider ChPT for PQQCD, the generalization of QCD in which we only choose the masses of valence and sea quarks to be independent, while using the same discretization of the Dirac operator. The first question is whether indeed EFTs such as ChPT can be constructed for PQQCD. We will assume that it can, as a euclidean EFT for the underlying euclidean theory. There exists no proof of this assertion. Some observations in support can be made, however.

⁶⁵ The idea is similar to the usage of the replica trick in defining the fourth root with staggered fermions, hence the same name. Note however, that the aim is different: here we take $n_i \rightarrow 0$, rather than continuing it to 1/4.

1. One expects the SET to exist, with almost the same level of rigor as in the physical case. This expectation is based on two observations: First, I expect that the perturbative construction of the SET goes through just the same for the partially quenched case. Then, the fact that the SET also is valid beyond perturbation theory amounts basically to the assumption that the coefficients in the Symanzik expansion have a well-defined meaning beyond perturbation theory,⁶⁶ and it does not look like a drastic step to expect that this assumption is also valid in the partially quenched case. While this observation does not directly address the validity of partially quenched ChPT (PQChPT), the SET is a “stepping stone,” and its validity is important for the construction of the appropriate chiral theory.
2. PQQCD contains full QCD as a subset: setting a valence quark mass equal to a sea quark mass makes that valence quark the same as a sea quark, and correlation functions with only sea quarks on the external lines are exactly those of full QCD. The collection of all correlation functions of PQQCD thus contains all correlation functions of full QCD as a subset. This allows us to make two observations: First, chiral symmetry breaking in the sea sector (*i.e.*, through a sea-quark condensate) takes place as usual. In PQQCD, one can use vectorlike symmetries to rotate sea quarks into valence quarks, thus producing also a valence condensate (and, by extension, a ghost condensate equal to the valence condensate).⁶⁷ Second, for full QCD the SET and ChPT exist, and it seems not unreasonable to assume that the match between PQQCD and PQChPT can be extended away from $m_v = m_s$, at least as long as the difference $m_v - m_s$ does not become too large, relative to m_s and m_v . Below, we will have more to say about what “too large” means.
3. The assumption that EFTs such as SET and ChPT exist for purely euclidean theories appears rather natural if one would imagine constructing the EFTs through the renormalization group. In this framework one would expect the EFT to be local below the scale above which one has integrated out all modes through some type of renormalization group blocking procedure. To be sure, there is no proof that the EFT obtained if one actually knew how to carry out the renormalization group blocking is local. But it is a “folklore” that seems equally applicable to PQQCD as to the standard case of full QCD.

Based on these observations, we will assume that, as before, the lagrangian of the EFT of interest should be taken to be the most general local function of the fields consistent with the symmetries of the theory. In fact, it is interesting to follow this route, and see where it will lead us. Hopefully, PQChPT will exhibit, in a more concrete form, what the diseases of the theory are, and, hopefully, these will be the only diseases! If this is true, this makes it possible to use PQChPT in the interpretation of lattice results (read: use the predictions of PQChPT to fit data). If this works to a high degree of precision, that constitutes a nontrivial and important test of PQChPT, and thus of our understanding of PQQCD. As we will see, the tests become nontrivial once numerical computations “can

⁶⁶ An assumption that is used in nonperturbative improvement.

⁶⁷ The argument that vector symmetries are not spontaneously broken [9] extends to continuum PQQCD. However, not all discretizations obey the conditions for this theorem, as the possible existence of the Aoki phase in QCD with Wilson fermions shows [24].

see” the nonanalytic terms predicted by PQChPT. Numerical evidence for such nonanalytic behavior exists, and I expect that it will fairly rapidly become more extensive, at least in the Nambu–Goldstone-boson sector.

PQQCD has a much larger symmetry than Eq. (4.2), because now sea, valence, and ghost quarks can all be rotated into each other. (Only the mass matrices cause a soft breaking of this symmetry.) Following our earlier discussion, we will now think of \tilde{q}^\dagger as independent of \tilde{q} , just as \bar{q} is independent of q . The full chiral symmetry group (ignoring the mass matrix) is then $SU(N_s + N_v|N_v)_L \times SU(N_s + N_v|N_v)_R$, where we now omit an anomalous axial $U(1)$, as well as the $U(1)$ for quark number, which is trivially represented on mesons. The nonlinear field $\Sigma = \exp(2i\Phi/f)$ now lives in the coset $[SU(N_s + N_v|N_v)_L \times SU(N_s + N_v|N_v)_R]/SU(N_s + N_v|N_v)_V$, and correspondingly, the field Φ describes $(N_s + 2N_v)^2 - 1$ mesons, counting precisely the number of pseudoscalar meson fields one can construct from the sea, valence, and ghost quarks and antiquarks when we leave out the singlet field that corresponds to the axial $U(1)$ (see below).

Before we continue, let us have a brief look at the graded group $SU(N_s + N_v|N_v)$ [68]. An element $U \in SU(N_s + N_v|N_v)$ can be written in block form as

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (4.10)$$

in which A is an $(N_s + N_v) \times (N_s + N_v)$ matrix of commuting numbers, D an $N_v \times N_v$ matrix of commuting numbers, while B and C are $(N_s + N_v) \times N_v$ and $N_v \times (N_s + N_v)$ matrices of anticommuting numbers, respectively. It is straightforward to check that if Σ has this structure, also Φ has to have this structure. The quark fields

$$Q = \begin{pmatrix} q_s \\ q_v \\ \tilde{q} \end{pmatrix}, \quad \bar{Q} = (\bar{q}_s \quad \bar{q}_v \quad \tilde{q}^\dagger) \quad (4.11)$$

transform in the fundamental and antifundamental representations of $SU(N_s + N_v|N_v)$,⁶⁸ and in terms of these “super” quark fields, the PQQCD lagrangian takes the simple form

$$\mathcal{L}_{PQQCD} = \mathcal{L}_{gauge} + \bar{Q}(D + \mathcal{M})Q, \quad (4.12)$$

in which \mathcal{M} is the mass matrix

$$\mathcal{M} = \begin{pmatrix} M_s & 0 & 0 \\ 0 & M_v & 0 \\ 0 & 0 & M_v \end{pmatrix}. \quad (4.13)$$

In the chiral theory, invariants are constructed using traces and determinants, and we need the generalization of these to graded groups. The “supertrace” of a matrix U as in Eq. (4.10) is defined as

$$\text{str}(U) = \text{tr}(A) - \text{tr}(D). \quad (4.14)$$

⁶⁸ Not all products of irreducible representations of graded groups are fully reducible. This plays a (minor) role for weak matrix elements in the partially-quenched theory with $N_s = 2$, see Ref. [69], and references therein.

The minus sign maintains the cyclic property, $\text{str}(U_1 U_2) = \text{str}(U_2 U_1)$. The “superdeterminant” is then defined through

$$\text{sdet}(U) = \exp(\text{str} \log(U)) = \det(A - BD^{-1}C) / \det(D) , \quad (4.15)$$

for U of the form Eq. (4.10); this definition implies that $\text{sdet}(U_1 U_2) = \text{sdet}(U_1) \text{sdet}(U_2)$. To understand this expression, first decompose U as

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix} . \quad (4.16)$$

Now, using the definition $\text{sdet}(U) = \exp(\text{str} \log(U))$, the result follows, with the superdeterminant of the first and last factors on the right-hand side being equal to one.

Hermitian conjugation is defined as usual, with the proviso that complex conjugation reorders: $(ab)^* = b^* a^*$,⁶⁹ from which it follows that $(U_1 U_2)^\dagger = U_2^\dagger U_1^\dagger$. The group $SU(N_s + N_v | N_v)$ is now defined as the group of all unitary graded $(N_s + 2N_v) \times (N_s + 2N_v)$ matrices, with grading as in Eq. (4.10), and superdeterminant equal to one. Clearly, if $\Sigma = \exp(2i\Phi/f)$ has $\text{sdet}(\Sigma) = 1$, this is equivalent with $\text{str}(\Phi) = 0$.

The matrix Φ can be written in block form

$$\Phi = \begin{pmatrix} \phi & \eta \\ \bar{\eta} & \tilde{\phi} \end{pmatrix} , \quad (4.17)$$

in which ϕ contains all meson fields made out of fermionic quarks, $\tilde{\phi}$ contains all fields made out of ghost quarks, η contains those made out of a fermionic quark and a ghost antiquark, whereas $\bar{\eta}$ contains meson fields made out of a ghost quark and a fermionic antiquark. Note that ϕ is not the same as Eq. (2.6), because it also contains meson fields made out of valence and sea quarks (“mixed” pions) and meson fields made out of valence quarks only (“valence” pions). The upper lefthand $N_s \times N_s$ block inside ϕ can be identified with Eq. (2.6), for $N_s = 3$.

In practical applications, it turns out to be useful to relax the restriction that $\text{str}(\Phi) = 0$, which makes $\Sigma \in U(N_s + N_v | N_v)$ instead of $SU(N_s + N_v | N_v)$. In conjunction, it is also useful to parametrize the diagonal of Φ not in terms of physical fields (*i.e.*, those found by diagonalizing the mass matrix), but in terms of “single-flavor” fields:

$$\text{diag}(\Phi) = (U, D, S, X, Y, \dots, \tilde{X}, \tilde{Y}, \dots) . \quad (4.18)$$

Here we have introduced a now commonly used notation for the valence quarks x, y, \dots , with $U \sim u\bar{u}$, $X \sim x\bar{x}$, *etc.*, instead of u_v, d_v, \dots , to save writing indices. One can have as many valence quarks as one needs — since they do not contribute to the dynamics, their number need not be fixed. This explains the dots in Eq. (4.18); of course, the number of ghost quarks needs to be equal to the number of valence quarks.⁷⁰ The “super”-singlet field (the “super- η ”),

$$\Phi_0 \equiv \text{str}(\Phi) = U + D + S + X + Y + \dots - (\tilde{X} + \tilde{Y} + \dots) , \quad (4.19)$$

⁶⁹ Of course, this only makes a difference when both a and b are Grassmann variables.

⁷⁰ Within these lectures, I’ll need only two valence quarks, x and y . But for instance for nonleptonic kaon decays, one needs three.

is not a Goldstone meson, because of the axial anomaly. This can easily be seen from the contribution of all quarks to the triangle diagram: in order for the ghost-quark contributions to *add* to the anomaly, one should include their contributions with an explicit minus sign, because they do not get a sign from the loop. This means that we should really remove the field Φ_0 from the chiral lagrangian (as we have been doing thus far). We will do this by giving it a large mass, that we will eventually send to infinity, thus decoupling the super- η' .⁷¹

It is time to get to the partially quenched chiral lagrangian! The lowest order form is

$$\mathcal{L}^{(2)} = \frac{1}{8}f^2 \text{str}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma) - \frac{1}{8}f^2 \text{str}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) + \frac{1}{6}m_0^2 (\text{str}(\Phi))^2, \quad (4.20)$$

in which χ is a spurion for the quark masses, that should be set equal to $2B_0\mathcal{M}$. We note that the field $\Phi_0 = \text{str}(\Phi)$ is not constrained by symmetry (except parity), and thus we should really multiply every term in Eq. (4.20) by an arbitrary (even) function of Φ_0 .⁷² However, we will be decoupling the super- η' , and that removes the dependence on all parameters contained in these potentials [66].

From this lagrangian, we can read off the lowest-order expression for all meson masses in the theory. This is straightforward for the off-diagonal fields in Φ . By expanding Eq. (4.20) to quadratic order, one find that the mass of a meson with flavors i and j is given by

$$m_{ij}^2 = B_0(m_i + m_j). \quad (4.21)$$

The only thing “different” is that the ghost-meson propagators get an extra minus sign because of the supertrace in Eq. (4.20),

$$\langle \tilde{\phi}_{ij}(p) \tilde{\phi}_{ji}(q) \rangle = \frac{-1}{p^2 + m_{ij}^2} \delta(p - q). \quad (4.22)$$

This is a first example of how PQChPT makes the “diseases” of PQQCD visible. For fermionic mesons, one has to remember that the ordering of η and $\bar{\eta}$ matters. The fermionic nature of these fields will provide minus signs when calculating loops, thus providing a mechanism through which the cancellation between valence quark and ghost quark loops is built into PQChPT.

The flavor-neutral sector is a little more complicated, because of the m_0^2 term (or, equivalently, because of mixing between the U , D , *etc.* fields). Let us calculate the propagator $\langle \Phi_{ii} \Phi_{jj} \rangle$. From the first two terms in Eq. (4.20), there is a contribution

$$G_{ij} \equiv \langle \Phi_{ii} \Phi_{jj} \rangle = \frac{\epsilon_i \delta_{ij}}{p^2 + m_{ii}^2}, \quad (4.23)$$

$$\epsilon_i = \begin{cases} +1, & i \text{ sea or valence} \\ -1, & i \text{ ghost} \end{cases},$$

from the first two terms in Eq. (4.20), with a minus sign as in Eq. (4.22) if i refers to a ghost. There are additional contributions coming from the m_0^2 term, that can be found by treating this term as a two-point vertex, and doing the geometric sums. In detail, we can rewrite

$$\frac{1}{6}m_0^2 (\text{str}(\Phi))^2 = \frac{1}{6}m_0^2 \Phi_{ii} K_{ij} \Phi_{jj}, \quad (4.24)$$

⁷¹ For a proof that this is a correct procedure, see Ref. [66].

⁷² And add more terms, see Refs. [12, 70].

with

$$K_{ij} = \epsilon_i \epsilon_j . \quad (4.25)$$

One finds, using that

$$(K GK)_{ij} = K_{ij} \sum_k G_{kk} , \quad (4.26)$$

for the contribution proportional to m_0^2 [58, 59]:

$$\begin{aligned} & -m_0^2/3 (GKG)_{ij} + \left(-m_0^2/3\right)^2 (GKGKG)_{ij} + \dots \\ &= \frac{-m_0^2/3}{(p^2 + m_{ii}^2)(p^2 + m_{jj}^2)} \left(\frac{1}{1 + \sum_{k=u,d,s} \frac{m_0^2/3}{p^2 + m_{kk}^2}} \right) \\ &= \frac{-m_0^2/3}{(p^2 + m_{ii}^2)(p^2 + m_{jj}^2)} \frac{(p^2 + m_U^2)(p^2 + m_D^2)(p^2 + m_S^2)}{(p^2 + m_{\pi_0}^2)(p^2 + m_\eta^2)(p^2 + m_{\eta'}^2)} \\ &\rightarrow \frac{-1/3}{(p^2 + m_{ii}^2)(p^2 + m_{jj}^2)} \frac{(p^2 + m_U^2)(p^2 + m_D^2)(p^2 + m_S^2)}{(p^2 + m_{\pi_0}^2)(p^2 + m_\eta^2)} . \end{aligned} \quad (4.27)$$

Let us step through this equation line by line. In the first expression on the right-hand side, the sum in the denominator really extends over all flavors, but the ghost terms cancel the valence terms, reducing the sum to run over the sea flavors u , d and s only. The second expression is just a rewriting of the first line. It is clear that the denominator of the second factor is a third-order polynomial in p^2 , and that it has to have poles at the physical masses (this factor only refers to the sea sector). These are of course the masses of the π^0 , η and η' . The η' mass-squared contains a term proportional to m_0^2 ,

$$m_{\eta'}^2 = \frac{N_s}{3} m_0^2 + \text{terms proportional to sea quark masses} , \quad (4.28)$$

whereas the π_0 and η become independent of m_0^2 when we take it to infinity. This observation explains the last line in Eq. (4.27). In Eq. (4.28) I made the dependence on the number of sea quarks, N_s , explicit — of course, in our explicit example above $N_s = 3$.

This result for the neutral propagators exhibits a new property of the partially quenched theory. The easiest way to see this is to note that, if we take $i = j$ to be a valence flavor, there is a double pole in Eq. (4.27). This is a clear sickness of the partially quenched theory, reflecting the fact that it does not correspond to a standard field theory in Minkowski space. We should expect, however, that if we take both i and j to correspond to sea flavors, that everything works as in the full theory: by design the sea sector does not know about valence and ghost quarks! Indeed, if we pick both i and j one of u, d, s , we see that at least one of these poles cancel against a factor in the numerator, leaving us with an expression that can be written as a sum over single poles. (This also works if there is a degeneracy in the sea sector.) It is a nice exercise to work out the propagators for the sea π^0 and η , and find that indeed they agree with the expressions one finds in the full theory. A very simple example is the case of degenerate sea masses, $m_U = m_D = m_S \equiv m_{sea}$, for which we obtain for the neutral sea propagator the expression

$$\langle \Phi_{sea} \Phi_{sea} \rangle = \left(1 - \frac{1}{3}\right) \frac{1}{p^2 + m_{sea}^2} , \quad (4.29)$$

where the “1” term comes from the first two terms in Eq. (4.20), and the “1/3” term comes from Eq. (4.27), and projects out the η' .

So, while the sea sector is healthy, as it should, the neutral valence sector is sick. So what? Maybe this just tells us that we should not consider flavor-neutral valence pions.⁷³ However, even if we make this choice, the sickness permeates to the flavored sector as well, as I will now argue.

We should be able to calculate chiral logarithms in meson masses, decay constants, *etc.*, from Eq. (4.20) as usual. The four-point vertices following from $\mathcal{L}^{(2)}$ have a flavor structure $\Phi_{ij}\Phi_{jk}\Phi_{kl}\Phi_{li}$, dressed up with derivatives or quark mass factors. Consider a one-loop contribution to $\langle\Phi_{ij}\Phi_{ji}\rangle$, with $i \neq j$ so as to stay away from the dangerous neutral sector. Because we only have four-point vertices, all one-loop diagrams are tadpoles, with Φ_{ij} and Φ_{ji} connected to the same vertex. One of the Wick contractions is the one in which the index structure on the vertex is $\Phi_{ij}\Phi_{jj}\Phi_{ji}\Phi_{ii}$, where Φ_{ii} and Φ_{jj} have to contract and form the loop! We see that the neutral propagator terms of Eq. (4.27) unavoidably show up, if we want to make use of partial quenching at all. For simplicity, let us work out the degenerate case $m_U = m_D = m_S = m_{sea}$, $m_{ii} = m_{jj} \equiv m_{val}$. Simplifying Eq. (4.27) accordingly, and integrating over p , as one would in the one-loop diagram, we obtain

$$\begin{aligned}
-\frac{1}{3} \int \frac{d^4p}{(2\pi)^4} \frac{p^2 + m_{sea}^2}{(p^2 + m_{val}^2)^2} &= \frac{1}{3} \frac{d}{dm_{val}^2} \int \frac{d^4p}{(2\pi)^4} \frac{p^2 + m_{sea}^2}{p^2 + m_{val}^2} \\
&\rightarrow \frac{1}{48\pi^2} (m_{sea}^2 - 2m_{val}^2) \log\left(\frac{m_{val}^2}{\Lambda^2}\right).
\end{aligned} \tag{4.30}$$

The arrow in the second line indicates that I only kept the chiral logarithm. This chiral logarithm is not of the structure we encounter in full QCD, where no infrared divergences can occur. Here there is an infrared divergence: Eq. (4.30) diverges when we take the valence mass to zero at fixed sea mass. Only when we take these masses in a fixed ratio can we define the limit. This means that the chiral expansion in the partially quenched theory will not converge if we take the sea and valence masses too different.⁷⁴ We should not only keep m_{val}^2 and m_{sea}^2 small in the sense of the chiral expansion, but we should also keep the unphysical infrared scale $m_{sea}^2 - m_{val}^2$ small compared to either of those, $|m_{sea}^2 - m_{val}^2| \lesssim \min(m_{sea}^2, m_{val}^2)$. For current numerical simulations, these conditions are well satisfied, so that the infrared divergences of the partially quenched theory do not constitute a problem in practice.

There are many examples of the unphysical infrared behavior of the partially quenched theory — the phenomenon is generic. Here I briefly mention two of them; for detailed explanations, see the original papers quoted below.

The first example occurs in scalar-isoscalar and scalar-isovector propagators [72]. In that case, there are one-loop contributions with $\pi - \pi$ or $\pi - \eta$ intermediate states. In the isopin limit, Eq. (4.27) contributes to the η internal line. With the additional pion internal line, this leads to an integral like Eq. (4.30), but with three powers of $p^2 + m_{val}^2$ in the denominator.⁷⁵ This leads to an infrared divergence (at fixed m_{sea}^2) that goes like $(m_{sea}^2 - m_{val}^2)/m_{val}^2$, worse than the logarithmic divergence of Eq. (4.30).

⁷³ They are numerically hard, because of disconnected diagrams, anyway.

⁷⁴ For a heuristic argument that such infrared divergences are a property of QCD, and not an artifact of ChPT, in the quenched case, see Ref. [71].

⁷⁵ Using the quark-flow picture of the next section, it is straightforward to see that all propagators in this contribution are valence propagators.

A second example where the double pole has a dramatic effect is in the contribution from one-pion exchange to the nucleon-nucleon potential. In full QCD one-pion exchange leads to a Yukawa potential, $\exp(-m_{val}r)/r$, whereas double-pole terms in the exchange lead, instead, to a potential of the form $(m_{sea}^2 - m_{val}^2)\exp(-m_{val}r)/m_{val}$, as can be seen by differentiating with respect to m_{val}^2 , like we did in Eq. (4.30) [73]. The double pole thus leads to an unphysical interaction that dominates the physical term at large distance.

In all cases, the unphysical infrared effects are, of course, proportional to $m_{sea}^2 - m_{val}^2$. In general, the message is that partial quenching can be a useful tool, but in order to make use of PQChPT, we do not only need m_{sea}^2 and m_{val}^2 to be small enough, but also their difference, $m_{sea}^2 - m_{val}^2$.

Having done all the preparatory work, let us consider the $O(p^4)$ result for the valence pion mass, in the case of degenerate sea quarks, and for $m_x = m_y$ [74]:⁷⁶

$$m_X^2 = 2B_0m_x \left\{ 1 + \frac{2B_0}{24\pi^2 f^2} \left((2m_x - m_{sea}) \log \left(\frac{2B_0m_x}{\Lambda^2} \right) + m_x - m_{sea} \right) + \frac{32B_0}{f^2} ((2L_8 - L_5)m_x + 3(2L_6 - L_4)m_{sea}) \right\}. \quad (4.31)$$

The chiral logarithms in this expression precisely come from the “double-pole” part of the neutral propagator, Eq. (4.27), *i.e.*, they are of the type Eq. (4.30).⁷⁷

This result, finally, allows us to return to the point as to why partial quenching is useful. As we have seen, full QCD is contained in PQQCD — the only difference is that we choose different masses for the valence and the sea quarks. Since the LECs are, by construction, independent of the quark masses, the LECs in PQChPT and full ChPT are identical. In other words, partially quenched lattice computations give us access to the real-world values of the LECs of ChPT [59]; the fact that we can vary the valence quark masses independently simply gives us another “knob to turn.” This can be very useful: it is clear that by varying the valence quark masses independently, one can extract $2L_8 - L_5$ from the pion mass, without varying the sea quark masses. This stands in contrast to the full theory: as can be seen from Eq. (2.40), in order to separate $2L_8 - L_5$ from $2L_6 - L_4$, one would have to vary the sea quark mass. It is important to note that the LECs *do* depend on the number of “dynamical” flavors, N_s — one still has to do the numerical lattice computation with the correct number of sea quarks.

While it is true that the *values* of the LECs in PQChPT and full ChPT are the same, there can be *more* LECs in the partially quenched case. This already shows up at order p^4 : the operator $\text{tr}(L_\mu L_\nu L_\mu L_\nu)$ is not independent from those in Eq. (2.17) for $SU(3)$ *cf.* Sec. II B, but $\text{str}(L_\mu L_\nu L_\mu L_\nu)$ is for $SU(3 + N_v | N_v)$ [76].⁷⁸ One thus obtains the most general $O(p^4)$ lagrangian for PQChPT by replacing traces with supertraces in Eq. (2.39), and by

⁷⁶ See also Ref. [75], which also discusses the case in which χ_{sea} is not very small compared to m_0^2 ; this can lead to a reduction of the coefficients of chiral logarithms.

⁷⁷ All contributions to the self energy with mesons containing sea quarks on the loop get absorbed by the wavefunction renormalization.

⁷⁸ For another example relevant for weak matrix elements, see Ref. [77]. For a discussion of the implications of graded group representation theory for “external operators” such as electroweak operators, see Ref. [69].

adding

$$L_{PQ} \left(\text{str}(L_\mu L_\nu L_\mu L_\nu) + 2 \text{str}(L_\mu L_\mu L_\nu L_\nu) + \frac{1}{2} (\text{str}(L_\mu L_\mu))^2 - \text{str}(L_\mu L_\nu) \text{str}(L_\mu L_\nu) \right) , \quad (4.32)$$

where the new operator is written in this way because this combination vanishes for $SU(3)$. This new operator does not contribute at tree level, because at tree level there is no distinction with full QCD, for which $SU(3)$ is the relevant symmetry group. It can contribute to one-loop diagrams, and does so in the case of pion scattering, for instance [76], where, of course, this only happens if valence masses are chosen different from sea masses.

Finally, we remark that quenched QCD, and thus quenched ChPT corresponds to the special case that $N_s = 0$ [70, 78]. While we are not pursuing quenched QCD in these lectures, it is worth noting that this special case has, in fact, some special properties. For example, one cannot take the limit $m_0^2 \rightarrow \infty$, as inspection of Eq. (4.27) reveals (the factor in parentheses on the first line gets replaced by one). Of course, quenched QCD does not have the correct number of light sea quarks, and is thus not only sick, but terminally ill. No physical result with controlled errors can ever be obtained from the quenched approximation.

C. Quark flow

In the previous section, we have seen that PQChPT can be systematically developed for PQQCD. Once we assume that this EFT exists just as in the full-QCD case, the construction is quite straightforward. But, one may like to have a more pictorial understanding how it all works, and this is provided by the quark-flow picture (which, in a sense, is formalized through the replica trick, but here I will not attempt to present it that way).

We have two types of quarks: valence and sea quarks. Valence quarks are represented by propagators, as in Eq. (4.1), and in a diagrammatic language, can be represented by valence quark lines. Each valence quark line has to have a beginning and an end at some external source or sink. Valence quark loops can be formed for instance if more than one valence line connects the same source and sink, as in the example of Eq. (4.1). The “valence quark part” of a contribution to a correlation function can thus be precisely defined, and the topology of valence lines represents the various possible Wick contractions which contribute to a particular correlation function.

In perturbation theory, the (logarithm of the) fermion determinant, which comes only from sea quarks, can be represented by an infinite sum over one-loop diagrams with any number of gluon lines attached to the sea-quark loop. One thus imagines the diagram coming from a certain Wick contraction in the valence sector to be dressed up with gluons and sea quark loops in all possible ways. In this way one can also represent sea quark contributions through loops, but clearly this picture is not rigorous outside perturbation theory. In this picture, however, the reason that only internal (*i.e.*, not connected to a source or sink) sea quark loops occur is because valence and ghost loops, which in principle also are present,⁷⁹ cancel each other. It is instructive to consider PQChPT in terms of quark loops.

Figure 6 shows quark-flow representations of the meson propagators at tree level, as well as the four-point meson vertex. The flavored propagators that follow from Eq. (4.20) are

⁷⁹ For instance, they would be present if one takes the ghost masses different from the valence masses.

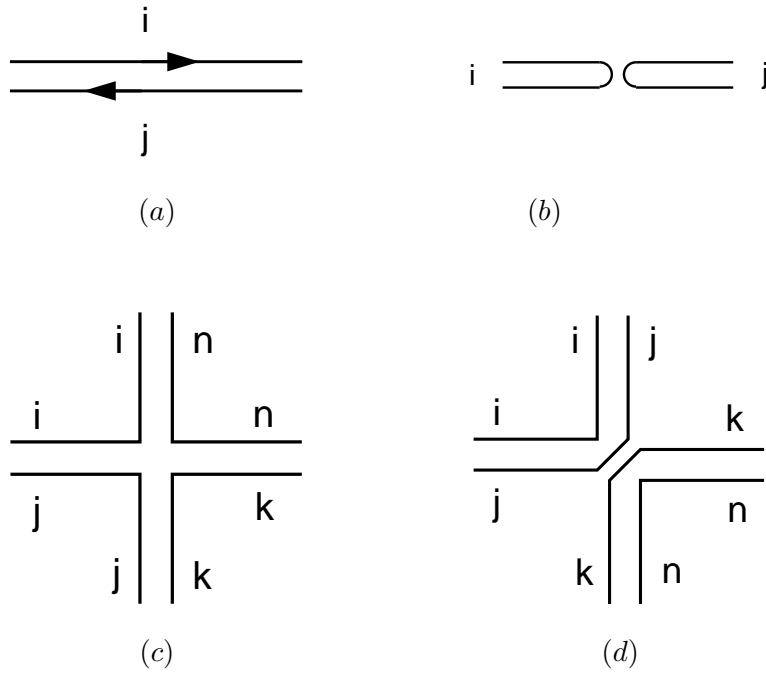


FIG. 6: Quark flow diagrams for ChPT Feynman rules. (a): flavored propagator, (b): m_0^2 part of the flavor-neutral propagator, *cf.* Eq. (4.27), (c): single-trace four-point vertex, (d): double-trace four-point vertex. Arrows on quark lines are shown in (a), and understood in the other figures. See text.

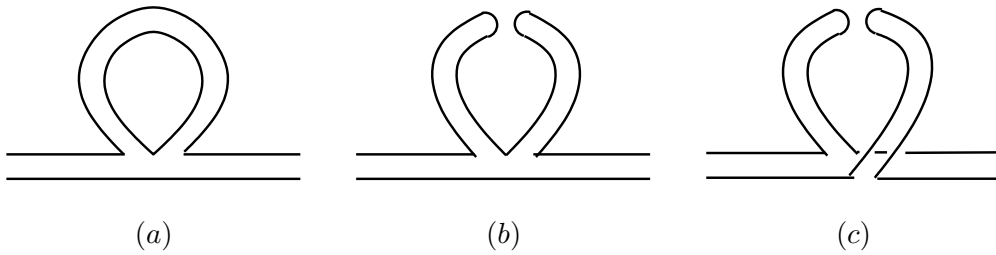


FIG. 7: Quark-flow diagrams for various one-loop contributions to the pion self energy.

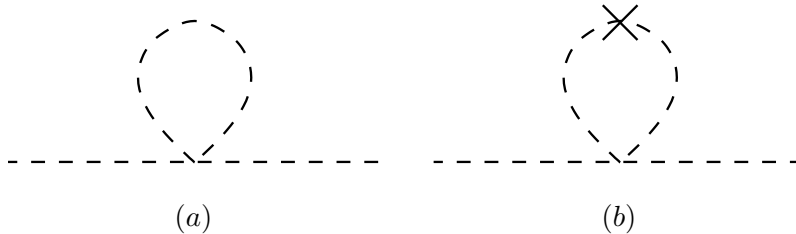


FIG. 8: ChPT diagrams for the pion one-loop self energy. A dashed line stands for one of the meson fields in Φ . The cross stand for an insertion of Eq. (4.27).

represented in “double-line” notation, with each line denoting a quark of a certain flavor, Fig. 6a. Note that we only draw valence lines — one is to imagine these diagrams arbitrarily dressed up with gluons and sea quark loops. In the flavor-neutral sector, we have seen that there are additional contributions, which follow from iterating the two-point vertices proportional to m_0^2 . Flavor-neutral mesons get a contribution as in Fig. 6a (just set $i = j$), but also get a contribution from Fig. 6b, where the “double hairpin” represents the m_0^2 two-point vertices. This is what we usually call a “disconnected” diagram in lattice QCD, because it represents a disconnected valence Wick contraction, but again, it is not really disconnected in terms of gluons. Note that the double hairpin stands for the full geometric sum (4.27) of all insertions of the m_0^2 vertices, where each additional insertion of a double-hairpin vertex corresponds to another sea-quark loop inserted between the two hairpins.

Figure 6c represents a vertex with flavor structure $\Phi_{ij}\Phi_{jk}\Phi_{kn}\Phi_{ni}$, which follows from the single-trace terms in $\mathcal{L}^{(2)}$, whereas Fig. 6d represents the flavor structure $\Phi_{ij}\Phi_{ji}\Phi_{kn}\Phi_{nk}$, which is a double-trace term. The latter do not occur in $\mathcal{L}^{(2)}$, but they do occur in $\mathcal{L}^{(4)}$. Various one-loop contributions to a pion propagator are shown in Fig. 7. Each internal quark loop has to be summed over only the sea quarks, because valence and ghost internal loops cancel each other. It so happens that only diagrams of type 7c contribute at order p^4 to meson masses; more types contribute to decay constants.

From these examples, it is clear that there is no one-to-one correspondence between PQChPT diagrams and quark-flow diagrams. The PQChPT diagram representing all diagrams of Fig. 7 is just the tadpole shown in Fig. 8a. One can refine this a little by representing the hairpin vertex by a cross, as in Fig. 8b. Of course, one can also use the quark-flow picture for correlation functions in full ChPT, and one can use those as a starting point to develop PQChPT intuitively, as was originally done for quenched ChPT [78]. It can be very helpful in seeing how ChPT should work for some “modified” version of QCD. For an example of applying this to staggered ChPT for staggered QCD with the fourth-root procedure, see Ref. [51]. However, it is important to have a path-integral representation for each modified version of QCD,⁸⁰ as this gives us the opportunity to use many of the standard tools of field theory. Examples of these tools are the precise definition and role of the symmetries of the

⁸⁰ This is the problem with staggered QCD with the fourth-root procedure, see Ref. [46].

modified theory, and the availability of field redefinitions [58, 70].

Double-hairpin diagrams as in Fig. 6b contribute only to flavor-neutral propagators (at tree level). The quarks on each of the hairpin lines are valence quarks, and the quark-antiquark pair that each of these hairpins represents gets bound into a valence meson — we thus expect each of these hairpins to represent a single meson pole with the mass of a valence pion. Any other quark loops in this diagram are not explicitly shown, and can only come from the fermion determinant. In other words, any such quark-loop contributions can only involve the sea quarks, and, if sea quark masses do not equal valence quark masses, nothing in the diagram can remove the two valence-pion poles. We conclude, at this heuristic level, that double pole terms have to occur in neutral pion propagators in PQQCD, and this is, of course, confirmed in PQChPT.⁸¹ Because double poles cannot occur in full QCD, we also conclude that the residue of these double poles has to be proportional to $m_{sea} - m_{val}$.

Another observation that is easily made in the quark flow picture is how PQQCD gives access to Wick contractions not available in full QCD. Suppose we do take all valence masses equal to the sea masses, so that the theory is not really partially quenched. The only effect of partial quenching is that we have more flavors available to put on the external lines than run around on the internal loops which come from the fermion determinant. We can use this, for example, to separate the disconnected part, Fig. 6b, from the connected part, Fig. 6a, of any flavor-neutral meson propagator. In other words, partial quenching separates the contribution of different Wick contractions to the same correlation function. At the level of quark flow diagrams, this is obvious. But since all that happens at the quark flow level is encoded in PQChPT, we can use this to obtain systematic chiral expansions for these separate Wick contractions. A simple example follows from Eqs. (4.23) and (4.27). In QCD with one flavor, these two contributions cannot be separated. But, one can add valence quarks to the theory (keeping the number of sea quarks at just one), and use this to separate the connected and disconnected parts (4.23) and (4.27) even in QCD with one flavor. This can for instance be used to define what it means to set the quark mass to zero, by requiring that the pion mass obtained from the connected part vanishes [79].

Another example of this is given in Ref. [66], in which it is shown how this can be used for a determination of L_7 . For other work using the “extra” quarks available in the partially quenched theory to separate connected and disconnected Wick contractions, see Ref. [80] for an application to electromagnetic properties of baryons, Ref. [81] for an application to quenched and partially quenched nonleptonic kaon decays, and Ref. [82] in the context of the hadronic contribution to $g - 2$.

D. ChPT for mixed actions

QCD with a mixed action is a generalization of PQQCD, because not only do we treat the valence and sea quark masses as independent, but also their lattice Dirac operators. An important consequence is that mixed-action QCD has fewer flavor symmetries than its partially quenched cousin [83]. Revisiting the general observations in support of the validity of PQQCD in Sec. IV B, we note that observations 1 and 3 still apply, but that observation 2 has to be modified. It is still true that full QCD is a subset of mixed-action QCD, but it is no longer true that one can rotate sea quarks into valence quarks.

⁸¹ For a much more detailed argument along these lines, see Section 6 of Ref. [66].

If we would consider a theory with two sets of sea quarks, with each set discretized with a different lattice Dirac operator, there would be little doubt that this constitutes a valid discretization of QCD, because of universality. One would expect that the framework of Sec. III applies to such a theory as well, with the necessary technical modifications. In mixed-action QCD, however, one of these sets is quenched, making it less obvious that a universal continuum limit exists. But if it exists, it will clearly have to be the corresponding partially quenched version of QCD. Hence, if one combines the assumption that PQChPT is the correct EFT for PQQCD with the notion of universality of different discretizations, it appears reasonable that we can combine the techniques of Secs. III and IV B to construct a chiral EFT for mixed-action QCD.

The relevant symmetry group clearly depends on what discretization we use for the valence and sea quarks. For definiteness, let us consider the case that we choose Wilson fermions for the sea quarks, and overlap fermions⁸² for the valence quarks.⁸³ The lattice symmetry group (ignoring the soft breaking by quark masses, and removing “overall” $U(1)$ factors as usual) is then $SU(N_s)_V \times [SU(N_v|N_v)_L \times SU(N_v|N_v)_R]$. With overlap fermions, the symmetry group in the valence sector is the same as in the continuum, but Wilson fermions have no chiral symmetry, and the symmetry group is thus the smaller group of vector-like transformations only.

We can now construct the chiral lagrangian (valid in the GSM regime to order p^4 , and in the LCE regime to order p^2) by writing down the SET for the mixed-action theory, introducing spurions, and making the transition to the chiral lagrangian as we did in Sec. III, and this has been done systematically in Ref. [29]. However, it is rather simple to understand the result, so, rather than repeat the whole analysis, I will first quote the result, and then comment on the various terms. The result is

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_{cont}^{PQQCD}(2B_0M_s \rightarrow 2B_0M'_s \equiv 2B_0M_s + \hat{a}\mathbf{1}) \\ & + \Delta\mathcal{L}^{(4)}(\hat{a} \rightarrow \hat{a}P_s) - \hat{a}^2 W_M \text{str}(P_s \Sigma P_s \Sigma^\dagger), \end{aligned} \quad (4.33)$$

with $\Delta\mathcal{L}^{(4)}$ given by Eq. (3.15), and the subscript *cont* stands for “continuum.” Here P_s is the projector on the sea sector, defined by $P_s q_s = q_s$ and $P_s q_v = 0$, $P_s \tilde{q} = 0$. While this equation contains a lot of information, it is easy to understand this result. First, since the valence quarks have exactly the same chiral symmetries as in the continuum, there are no explicit $O(a^2)$ terms in the valence sector of the chiral lagrangian. The only such terms come from the Wilson sea sector, and this explains the appearance of the projector P_s in Eq. (4.33), and the fact that only the sea quark masses get shifted by an $O(a)$ term, as in Eq. (3.11). Of course, there are scaling violations in the valence sector as well, but they can only occur as $O(a^2)$ corrections to the continuum LECs, f , B_0 and L_i ; but in the GSM regime these are of order p^6 .⁸⁴ Rotational symmetry is always broken on the lattice, but, as we have seen before, the effects of that only shows up at higher orders in the chiral lagrangian. Essentially, the form of the lagrangian (4.33) is thus that of the continuum lagrangian, plus the symmetry-breaking lattice part for the sea sector. However, one completely new term does appear in Eq. (4.33), and that is the term with the new LEC W_M . This term is allowed,

⁸² Or domain-wall fermions with a very small residual mass.

⁸³ For the case of overlap fermions on a staggered sea, see Ref. [84].

⁸⁴ To the best of my knowledge, the $O(p^4)$ chiral lagrangian with LCE power counting has not been constructed.

because there are no symmetries that connect the valence and sea sectors. The new term does not break (continuum) chiral symmetry restricted to the sea sector, so a similar term also appears if we choose different types of sea quarks, such as staggered or domain-wall quarks. Since this new term does not break chiral symmetry, a similar term also arises in the valence sector. But since we have that $P_v = \mathbf{1} - P_s$ and $\Sigma\Sigma^\dagger = \mathbf{1}$, we can write

$$\text{str}(P_v\Sigma P_v\Sigma^\dagger) = \text{str}(P_s\Sigma P_s\Sigma^\dagger) + \text{str}(\mathbf{1}) - 2\text{str}(P_s) , \quad (4.34)$$

so only one of these terms is independent. If we take the valence quarks to be the same as the sea quarks, the projector P_s is replaced by the unit matrix, and the new term is an irrelevant constant. But with a mixed action this is not the case, and the new term should be taken into account.

Let us consider the $O(p^2)$ expressions for meson masses in the LCE regime, beginning with the flavored ones. By expanding the lagrangian to quadratic order in Φ , we find, using now Latin indices for the valence flavors and Greek indices for sea flavors, in order to distinguish them more clearly:

$$\begin{aligned} m_{ij}^2 &= B_0(m_i + m_j) , \\ m_{\alpha\beta}^2 &= B_0(m'_\alpha + m'_\beta) + \frac{32\hat{a}^2}{f^2} (N_s W'_6 + W'_8) , \\ m_{i\alpha}^2 &= B_0(m_i + m'_\alpha) + \frac{4\hat{a}^2}{f^2} (4N_s W'_6 + 2W'_8 + W_M) . \end{aligned} \quad (4.35)$$

The factors N_s in these expressions come from factors $\text{str}(P_s)$ which occur working out the expansion in Φ . The new LEC W_M only shows up in the “mixed” pion mass, *i.e.*, the pion made of a valence and a sea quark, hence there is no $O(p^2)$ relation between the pure valence, pure sea, and mixed pion masses.⁸⁵ In any application of QCD with a mixed action, the mixed pion mass gives thus an important indication whether one should use GSM or LCE power counting.

In the flavor-neutral sector, all the complications of PQQCD of course are inherited by the mixed-action case, including the appearance of double-pole terms, but a new issues also arises. In the sea sector, to quadratic order, there is an additional contribution to the m_0^2 term in Eq. (4.20), coming from the W'_7 term in Eq. (3.15) shifting the value of m_0^2 *only in the sea sector* (because of the P_s projectors, *cf.* Eq. (4.33)) effectively to

$$m_{0,eff}^2 = m_0^2 + \frac{96\hat{a}^2}{f^2} W'_7 . \quad (4.36)$$

For a theory with degenerate valence and degenerate sea quarks, one finds for the flavor-neutral propagator in the valence sector [85]

$$\begin{aligned} \langle \phi_{ii} \phi_{jj} \rangle &= \frac{\delta_{ij}}{p^2 + m_{val}^2} - \frac{1}{3} \frac{m_0^2(p^2 + m_{sea}^2) + (N_s m_0^2/3)(m_{0,eff}^2 - m_0^2)}{(p^2 + m_{val}^2)^2(p^2 + m_{sea}^2 + N_s m_{0,eff}^2/3)} \\ &\rightarrow \left(\delta_{ij} - \frac{1}{N_s} \right) \frac{1}{p^2 + m_{val}^2} - \frac{(m_{sea}^2 - m_{val}^2)/N_s + 32\hat{a}^2 W'_7/f^2}{(p^2 + m_{val}^2)^2} , \end{aligned} \quad (4.37)$$

⁸⁵ In the GSM regime there is, because the $O(a^2)$ terms are of higher order in that regime. In the GSM regime we do thus have that $m_{mixed}^2 = (m_{sea}^2 + m_{val}^2)/2$ to lowest order.

where the second line is obtained by taking $m_0 \rightarrow \infty$ in the first line of this equation.

The importance of this result lies in the fact that scaling violations are not universal. Naively, one might think that the best way to conduct a mixed-action lattice computation is to tune the quark masses such that the valence pion mass is equal to the sea pion mass. However, comparison between Eqs. (4.35) and (4.37) shows that this is not the same as tuning quark masses such that the residue of the double pole vanishes. If one tunes quark masses to make sea and valence pion masses equal, there will still be a double pole in the theory, with a residue of order a^2 . Since the double pole can lead to infrared “enhancement” effects (such as the enhanced chiral logarithms and other such examples we encountered in Sec. IV B), such scaling violations could be larger than one would expect based on a simple estimate of the size of $(a\Lambda_{QCD})^2$. This phenomenon does not occur at this order in the case that one uses overlap valence quarks on a staggered sea; because both valence and staggered quarks have exact chiral symmetries, in that case the effect can occur at the earliest at order $a^2 m_{quark}^2$ [84, 85].

1. One-loop universality

As soon as the lattice spacing is large enough that we need to include it as a variable in chiral fits, the version of ChPT we need to use will depend in some detail on what kind of lattice action is being used. For instance, ChPT for Wilson fermions is quite different from ChPT for staggered fermions. This is in general also true when a mixed action is used. This is not a problem of principle, but for each new type of lattice action, the necessary ChPT calculations need to be repeated.

It would be very nice if some general prescription existed that allowed one to take a calculation carried out in continuum PQQCD, and modify it using some simple rules into the appropriate mixed-action ChPT expressions. While this is clearly not possible in general, it is possible under some limitations, which however cover many cases of practical interest.

As we noted before, the use of a mixed action is particularly useful when the valence fermions have exact chiral symmetry (broken only by the physical quark masses), while the sea sector has much less symmetry. If we allow only valence quarks on the external lines of any correlation function of interest, in order to access what we will refer to as “valence quantities,” and we work to leading order (*i.e.*, tree level) in ChPT, one clearly needs just that part of the $O(p^2)$ chiral lagrangian that refers only to the valence sector — one can simply set all sea-meson and mixed-meson fields equal to zero. This is nothing else than the observation that sea quarks (and thus sea and mixed mesons) can only appear on loops. If the lattice valence quarks have exact chiral symmetry, the $O(p^2)$ chiral lagrangian is the same as in the continuum, *i.e.*, it takes the form of Eq. (2.33).

What is remarkable is that a similar argument holds at order p^4 , *i.e.*, to one loop [86]. For one-loop contributions we need only the $O(p^2)$ lagrangian, and, in one-loop diagrams with *only valence quarks on the external legs*, only one sea quark loop can occur, as can easily be seen in the quark-flow picture.⁸⁶ Let us see how this works in the case that the sea quarks are Wilson-like, and that we are in the LCE regime. To order p^2 , the chiral

⁸⁶ For disconnected contributions, *cf.* the second term in Eq. (4.37), see below.

lagrangian is then

$$\begin{aligned}
\mathcal{L}_{LO} = & \frac{1}{8}f^2 \text{str} \left((D_\mu \Sigma)^\dagger D_\mu \Sigma \right) - \frac{1}{4}B_0 f^2 \text{str} \left(\mathcal{M} \Sigma + \Sigma^\dagger \mathcal{M} \right) \\
& - \hat{a}^2 W'_6 \left(\text{str}(P_s \Sigma + \Sigma^\dagger P_s) \right)^2 - \hat{a}^2 W'_7 \left(\text{str}(P_s \Sigma - \Sigma^\dagger P_s) \right)^2 \\
& - \hat{a}^2 W'_8 \text{str}(P_s \Sigma P_s \Sigma + \Sigma^\dagger P_s \Sigma^\dagger P_s) - \hat{a}^2 W_M \text{str}(P_s \Sigma P_s \Sigma^\dagger) ,
\end{aligned} \tag{4.38}$$

from Eqs. (2.33) and (3.15), where I have inserted the projectors P_s , and in which the $O(a)$ term has been absorbed into the sea-quark mass matrix, as in Eq. (4.33). As noted, we can have only one sea quark on the loop. This implies that if we expand out Σ in terms of Φ , we will only use those terms in the expansion of any of the $O(a^2)$ operators in which two of the indices on the Φ fields correspond to sea quarks (these two indices correspond to the “beginning” and the “end” of the sea quark loop), with the rest of the indices referring to valence quarks. Because of the P_s projectors in the $O(a^2)$ terms, such terms only arise when we set one Σ (or one Σ^\dagger) equal to one, in all possible ways in which this can be done. Applying this “1-loop valence” rule to Eq. (4.38), it simplifies to

$$\begin{aligned}
\mathcal{L}_{LO} \Big|_{1\text{-loop valence rule}} = & \frac{1}{8}f^2 \text{str} \left((D_\mu \Sigma)^\dagger D_\mu \Sigma \right) - \frac{1}{4}B_0 f^2 \text{str} \left(\mathcal{M} \Sigma + \Sigma^\dagger \mathcal{M} \right) \\
& - \hat{a}^2 (4N_s W'_6 + 2W'_8 + W_M) \text{str} \left(P_s \Sigma + \Sigma^\dagger P_s \right) .
\end{aligned} \tag{4.39}$$

In other words, all dependence (to one loop) on the LECs $W'_{6,8}$ and W_M can be absorbed completely into a shift of the sea-quark mass matrix of the form [87]

$$\text{1-loop valence rule: } B_0 M_s \rightarrow B_0 M_s + \frac{4\hat{a}^2}{f^2} (4N_s W'_6 + 2W'_8 + W_M) \mathbf{1} . \tag{4.40}$$

Note, however, that this simplified form *cannot* be used to calculate any quantity involving sea quarks! For instance, the leading-order meson mass for a meson made only out of sea quarks is given by the middle equation of Eq. (4.35), which does *not* correspond to shifting the sea quark masses as in Eq. (4.40); in particular, the LEC W_M does not occur. It also does not apply to the disconnected term in Eq. (4.37), in which again the sea meson masses are given by Eq. (4.35). However, since this is the only place where the sea meson mass shows up at one loop in valence quantities, the LEC W'_7 can be absorbed into the sea meson mass-squared, for the one-loop contributions to valence quantities.

In summary, the one-loop expressions of continuum PQChPT for valence quantities, such as meson masses, decay constants and scattering amplitudes [88], carry over to the mixed case, if one takes the mixed meson mass to be given by the last line of Eq. (4.35),

$$m_{i\alpha}^2 = B_0(m_i + m'_\alpha) + \frac{4\hat{a}^2}{f^2} (4N_s W'_6 + 2W'_8 + W_M) , \tag{4.41}$$

and the sea meson mass by

$$m_{\alpha\beta}^2 = B_0(m'_\alpha + m'_\beta) + \frac{32\hat{a}^2}{f^2} (N_s W'_6 + W'_8 + N_s W'_7) , \tag{4.42}$$

by combining Eqs. (4.35) and (4.37). It is only through these masses that the LECs $W'_{6,7,8}$ and W_M occur.

In a calculation to order p^4 , there are also tree-level contributions originating from terms in the chiral lagrangian of order p^3 and p^4 , *cf.* Eq. (3.26). Single-trace terms containing a P_s thus do not contribute. In double-trace terms any Σ multiplied by a P_s should be set equal to one. That means that such terms do contribute, but again they can be absorbed into a redefinition of other LECs. For instance, the W_4 and W_6 terms in Eq. (3.15), reduce to

$$\begin{aligned} & \hat{a}W_4 \operatorname{str} \left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger \right) \operatorname{str} \left(P_s \Sigma + \Sigma^\dagger P_s \right) \\ & - 2\hat{a}B_0W_6 \operatorname{str} \left(\mathcal{M}\Sigma^\dagger + \Sigma\mathcal{M} \right) \operatorname{str} \left(P_s \Sigma + \Sigma^\dagger P_s \right) \\ \rightarrow & 2N_s \hat{a}W_4 \operatorname{str} \left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger \right) - 4N_s \hat{a}B_0W_6 \operatorname{str} \left(\mathcal{M}\Sigma^\dagger + \Sigma\mathcal{M} \right) , \end{aligned} \quad (4.43)$$

or

$$f^2 \rightarrow f^2 + \frac{16\hat{a}}{f^2} N_s W_4 , \quad B_0 \rightarrow B_0 \left(1 + \frac{16\hat{a}}{f^2} N_s W_6 \right) . \quad (4.44)$$

All double-trace terms in the chiral lagrangian need to be treated this way, leading to more redefinitions of the $O(p^2)$ LECs B_0 and f^2 . In this way the redefined LECs B_0 and f also pick up dependence on the sea quark masses, from the L_4 and L_6 terms in Eq. (2.39). Whether this is convenient or not depends on the application. Note that if one trades sea quark masses for sea-meson masses, one has to account for the leading-order $O(a^2)$ shifts of Eq. (4.35). For more discussion, including the example of $I = 2$ $\pi\pi$ scattering to order p^4 , see Ref. [86].

In summary, what we find is that, to order p^4 , the formulas of continuum PQChPT can be used for correlation functions with only valence quarks on the external legs, with a suitable redefinition of the mixed and sea meson masses and the $O(p^2)$ LECs. In other words, once these redefinitions have been made, the theory does not remember the fact that the sea quarks are Wilson like. That means that we uncovered something universal [86]: The same argument should also apply when the sea quarks are staggered! And indeed, it does: For example, it is straightforward to check from Eq. (3.61) that the shift in M_s corresponding to Eq. (4.40) is

$$\text{1-loop valence rule: } B_0 M_s \rightarrow B_0 M_s + \frac{4a^2}{f^2} (C_1 + 4C_3 + 4C_4 + 6C_6 + C_{mix}) , \quad (4.45)$$

where C_{mix} is the staggered equivalent of W_M [84].

V. $SU(2)$ VERSUS $SU(3)$: PHYSICS WITH HEAVY KAONS

So far, we have mostly discussed ChPT for QCD with three flavors to order p^4 . An important question is whether this is sufficiently high order for practical applications to lattice QCD, and the answer is, unfortunately, that in general it is not. It is a fact that the kaon mass is not so small in the real world, and even if ChPT applies to the strange quark mass as well, the expansion to order p^4 may not suffice. Indeed, $m_K^2/m_\pi^2 \approx 13$, suggesting that an expansion that works very well for m_π^2 may not work so well when m_K^2 is the expansion parameter.

One option for dealing with this problem is to go to higher order in the chiral expansion. In applications to phenomenology, there has been much work to extend ChPT calculations to order p^6 [89]. For applications to the lattice, $O(p^6)$ expressions, which involve two-loop

calculations (as can be seen from Eq. (2.18)), need to be extended to the partially quenched case, and ideally, also to include scaling-violation effects. Meson masses and decay constants have been calculated in partially quenched, three-flavor ChPT to order p^6 [90], and it will be interesting to see what happens if the two-loop chiral logarithms are included in fits to lattice data. Scaling violation effects have not been calculated in ChPT to the same order.⁸⁷ While this means that in principle lattice results would have to be first extrapolated to the continuum, before chiral fits are performed, it may be interesting to include “what we know,” *i.e.*, the continuum part at order p^6 , because the continuum $O(p^6)$ terms are expected to be important at larger quark masses, where scaling violations are less important. Up to date, something similar has been done by MILC, which included all the analytic terms to order p^8 in their mass and decay constant fits, but no logarithms beyond order p^4 (yet).⁸⁸ For an early attempt to fit $N_f = 2 + 1$ full-QCD overlap results to $O(p^6)$ ChPT including two-loop nonanalytic terms, see Ref. [91].

It has recently been suggested that kaon loops are not a reliable part of ChPT [92], because the relevant scale is $2m_K \sim 1$ GeV, rather than m_K itself. Diagrams can be reconstructed from their cuts and poles, and, for a meson of mass m , these cuts typically start at $4m^2$, and not m^2 . For instance, the contribution of a kaon pair to the imaginary part of the $\pi\pi$ scattering amplitude starts at $s = 4m_K^2$; for other examples (F_π and the pion electromagnetic form factor), see Ref. [92]. Of course, $2m_K \sim 1$ GeV is too large to trust a chiral expansion in that scale. However, this appears to be a quantity-dependent observation. It applies to purely pionic quantities, since (virtual) kaons always have to contribute in pairs to these quantities. But if we consider for instance πK scattering, the cut starts at $(m_\pi + m_K)^2 \approx m_K^2$, and the chiral expansion for this scattering amplitude as a function of m_K may have better convergence properties. A similar argument applies to F_K .

For lattice computations relevant for phenomenology, there is a different option. It is not difficult to do lattice computations with the strange quark mass adjusted to (close to) its physical value, so that only extrapolations in terms of the light quarks masses (m_u and m_d) are needed. In this case, we do not need three-flavor ChPT, but, rather, we can work with two-flavor ChPT, treating the strange quark as “heavy.” Note, however, that we do not integrate out the strange quark, which we cannot do, because m_s is not large compared to Λ_{QCD} . The actual lattice computations still have to be done with three flavors, up, down and strange.

In the pion sector it is straightforward to develop ChPT again, one simply starts from the group $SU(2)_L \times SU(2)_R$, instead of $SU(3)_L \times SU(3)_R$.⁸⁹ But the properties of kaons (its mass, decay constant, the kaon B parameter, *etc.*) will depend on m_u and m_d through interactions of kaons with pions, and since lattice computations will still typically use unphysical values for the up and down quark masses, we will still need to know how to extrapolate kaon properties in those light quark masses. We thus need to extend two-flavor ChPT systemat-

⁸⁷ In the GSM regime, $m_{quark} \sim a\Lambda_{QCD}^2$, this would be easier to carry out than in the LCE regime, $m_{quark} \sim a^2\Lambda_{QCD}^3$.

⁸⁸ This means that the fitted LECs at order p^6 from the lattice cannot be compared to any continuum results. While this “hybrid” fitting method might also affect lower-order LECs, they are expected to be much less sensitive, because their values are predominantly determined by lattice results at lower quark masses, where the $O(p^6)$ effects are less important [18].

⁸⁹ In this section we will use the word “pion” to refer to π^\pm and π^0 , because we will not think of the kaon as an approximate Nambu–Goldstone boson.

ically to include the interactions of kaons with pions. How to do this will be the topic of this section. Studies using this EFT in application to three-flavor lattice data were recently carried out in Refs. [93, 94]. While the case of kaons is particularly simple, the basic setup also applies to EFTs for the coupling of baryons and heavy-light mesons to pions.

While this alternative approach may turn out to be a useful approach, it is a more limited approach, because the kaons are not treated as Nambu–Goldstone bosons. For instance, while the dependence of the kaon mass on m_u and m_d is predicted by this EFT, it will not go to zero in the chiral limit. Instead, the chiral-limit value of the kaon mass is a new parameter in the EFT. Another example is that in this approach, one cannot ask certain questions, such as the value of the three-flavor condensate in the chiral limit. This particular question is interesting, because it has been argued that if this condensate were very small, that would lead to the need to redefine the power counting of ChPT, and thus rearrange the chiral lagrangian order by order [95]. To answer this question, one needs to work with the three-flavor theory, and vary the strange mass. In the setup we will investigate in this section, one only gets to ask about the two-flavor condensate, at the physical value of the strange mass.⁹⁰

A. Including a kaon in two-flavor ChPT

We need to develop two-flavor ChPT for the three pions, coupled to the isospin-1/2 kaon. In the special case of the kaon, it should be possible to deduce the results from ChPT with three flavors. For instance, one may take the result for f_K/f_π in Eq. (2.40), and expand it in the light quark mass for fixed strange quark mass. One then absorbs the strange quark mass dependence into the LECs, and this parameterizes the dependence of f_K/f_π on the light quark mass, while giving up information about the dependence on the strange quark mass. However, this method is not available for other hadrons such as baryons, and it is useful and instructive to develop a method that does not rely on three-flavor ChPT. After developing the method, we will return to the matching with three-flavor ChPT in some examples.

The systematic method for coupling pions to a non-Goldstone hadron was worked out a long time ago in Ref. [96], which shows how to couple any field with a given isospin to pions in the most general way. In fact, the method generalizes to other groups as well: The general case is that of some continuous group G that is spontaneously broken to some subgroup H . Since only H is realized in Wigner mode, all fields need to transform in representations of H , and not G , as we will see in more detail below. Here we will restrict ourselves to $G = SU(2)_L \times SU(2)_R$, and $H = SU(2)_V$, *i.e.*, isospin.

We will present the construction as explained in Ref. [97]. As we have seen in Sec. II B, the pion fields parametrize the coset $[SU(2)_L \times SU(2)_R]/SU(2)_V$ (G/H in the general case), because $SU(2)_V$ transformations leave the vacuum invariant. An element of the coset can be represented by picking an element (u_L, u_R) of the full symmetry group $SU(2)_L \times SU(2)_R$, and defining an equivalence class by making any two such elements equivalent when they differ by a transformation in $SU(2)_V$. Each equivalence class is an element of the coset. We can pick a “standard” representative of each equivalence class by imposing a condition on

⁹⁰ Recent results for both condensates in the three-flavor theory can be found in Ref. [19]. It is found that while the three-flavor condensate is smaller than the two-flavor condensate, it is not very small.

u_L and u_R ; we pick the condition that $u_L = u_R^\dagger \equiv u$. We can do this: by definition, if we multiply (u_L, u_R) on the right by an element $(h^\dagger, h^\dagger) \in SU(2)_V$, obtaining $(u_L h^\dagger, u_R h^\dagger)$, this group element is in the same equivalence class, and thus represents the same coset element. If we pick h such that it solves $u_R = h u_L^\dagger h$, this brings the coset element into standard form. If we now multiply the group element (u_L, u_R) by another group element, (v_L, v_R) :

$$(u_L, u_R) \rightarrow (v_L u_L, v_R u_R) , \quad (5.1)$$

the new group element in general corresponds to a different element of the coset. But again we can choose a $(h^\dagger, h^\dagger) \in SU(2)_V$ such that

$$(u_L, u_R) \rightarrow (v_L u_L h^\dagger, v_R u_R h^\dagger) \quad (5.2)$$

keeps the transformed element in standard form, where, of course, $h = h(u, v_{L,R})$ now depends both on u and (v_L, v_R) . If we choose $(v_L, v_R) \in SU(2)_V$, *i.e.*, $v_L = v_R = v$, we can choose the ‘‘compensator’’ field h equal to v , and it does not depend on u . It follows that, as expected, an $SU(2)_V$ transformation stays in the same equivalence class.

We now may define Σ in terms of u by⁹¹

$$\begin{aligned} \Sigma &= u_L u_R^\dagger = u^2 \rightarrow v_L \Sigma v_R^\dagger , \\ u &= \exp(i\phi/f) , \end{aligned} \quad (5.3)$$

where now

$$\phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix} , \quad (5.4)$$

because then Σ also parametrizes the coset, and has the correct transformation properties under the chiral group if we identify $v_{L,R} = U_{L,R}$, *cf.* Eq. (2.11).

This construction provides a straightforward way to include the kaon. Introducing a field K in the isospin-1/2 representation of $SU(2)_V$, we can extend the nonlinear representation of the chiral group to

$$\begin{aligned} u &\rightarrow v_L u h^\dagger(u, v_{L,R}) = h(u, v_{L,R}) u v_R^\dagger , \\ K &\rightarrow h(u, v_{L,R}) K . \end{aligned} \quad (5.5)$$

These two fields, their (covariant, see below) derivatives and the sources ℓ_μ, r_μ, χ and χ^\dagger now form the set of building blocks from which to construct the chiral lagrangian.

Note that we only have to specify the $SU(2)_V$ representation of the kaon field K , and not the representation of the full chiral group $SU(2)_L \times SU(2)_R$. There is a simple intuitive reason for that [4]: Suppose, for example, we specify that a kaon field K' transforms in the $(1/2, 0)$ representation of $SU(2)_L \times SU(2)_R$, $K' \rightarrow v_L K'$. Then, performing the field redefinition $K'' = \Sigma^\dagger K' = (1 + O(\pi)) K'$ changes the representation to $(0, 1/2)$, since $K'' \rightarrow v_R K''$. We see that the presence of pions (or, in general, the Goldstone bosons associated with G/H) give us the freedom to choose any representation of $SU(2)_L \times SU(2)_R$ we want! The field K

⁹¹ Whether with the symbol ‘‘ u ’’ I refer to the up quark or the nonlinear pion field should always be clear from the context.

used in Eq. (5.5) is obtained by defining $K = u^\dagger K'$.⁹² We note that the field K' transforms under parity to

$$K' \rightarrow K'^{(P)} = -\Sigma^\dagger K' , \quad (5.6)$$

because the parity transform of K' should transform as $K'^{(P)} \rightarrow v_R K'^{(P)}$ under the chiral group; the minus sign indicates that the kaon is a pseudoscalar. The field K transforms into $-K$ under parity, consistent with its transformation rule Eq. (5.5) under the chiral group ($h \in SU(2)_V$).

In order to construct the chiral lagrangian, we need to construct invariants out of u and K , which transform as in Eq. (5.5). Since h is local, we will need a gauge connection for $SU(2)_V$, constructed from u . We thus introduce

$$\begin{aligned} u_\mu^L &= u^\dagger \partial_\mu u , \\ u_\mu^R &= u \partial_\mu u^\dagger , \end{aligned} \quad (5.7)$$

which transform into each other under parity. Under Eq. (5.5), the parity odd and even combinations transform as

$$\begin{aligned} \Delta_\mu &\equiv \frac{1}{2} (u_\mu^R - u_\mu^L) \rightarrow \frac{1}{2} h (u_\mu^R - u_\mu^L) h^\dagger , \\ &\frac{1}{2} (u_\mu^R + u_\mu^L) \rightarrow \frac{1}{2} h (u_\mu^R + u_\mu^L) h^\dagger - \partial_\mu h h^\dagger . \end{aligned} \quad (5.8)$$

We see that Δ_μ transforms homogeneously, while the other combination transforms as an $SU(2)_V$ gauge connection. The latter can thus be used to define a covariant derivative for K :

$$D_\mu K = \partial_\mu K + \frac{1}{2} (u^\dagger \partial_\mu u + u \partial_\mu u^\dagger) K . \quad (5.9)$$

If we wish to include the sources ℓ_μ and r_μ , this can be done by replacing $\partial_\mu \rightarrow \partial_\mu - i\ell_\mu$ in u_μ^L and $\partial_\mu \rightarrow \partial_\mu - ir_\mu$ in u_μ^R .

The chiral lagrangian consists of three parts. First, there is the purely pionic part of the lagrangian, which is given by $\mathcal{L}^{(2)} + \mathcal{L}^{(4)}$ of Eqs. (2.33) and (2.39). For two flavors, a number of simplifications occur because of special properties of $SU(2)$. Then, there is the quadratic term in the kaon fields that defines the kaon propagator in ChPT,

$$\mathcal{L}_K^{(0)} = (D_\mu K)^\dagger D_\mu K + M^2 K^\dagger K . \quad (5.10)$$

Here M is a free parameter equal to the kaon mass in the chiral limit.

As already noted, even with $\ell_\mu = r_\mu = 0$ the covariant derivative (5.9) is required, because h in Eq. (5.5) is a local, field-dependent transformation. Interactions between pions and kaons are therefore necessarily present in our EFT. In this respect, it is instructive to see how this same conclusion would have been reached if we started from the field $K' = uK$ discussed above. It looks like we can write down a kaon kinetic term without any interactions in terms of this field: $\partial_\mu K'^\dagger \partial_\mu K'$ is invariant under the chiral group. However, this is not

⁹² It can be shown that any other nonlinear representation of the group $SU(2)_L \times SU(2)_R$ with the same field content can be brought into the form (5.5) by a field redefinition [96].

invariant under parity. Using the parity transform $K'^{(P)}$ introduced in Eq. (5.6), a lagrangian invariant under the chiral group and parity is

$$\frac{1}{2} \left(\partial_\mu K'^\dagger \partial_\mu K' + \partial_\mu (K'^\dagger \Sigma) \partial_\mu (\Sigma^\dagger K') \right) , \quad (5.11)$$

which, using $K' = uK$, is equal to the first term in Eq. (5.10), modulo a term proportional to $K^\dagger L_\mu L_\mu K = 2 \text{tr}(\Delta_\mu \Delta_\mu) K^\dagger K$ that can be absorbed into the A_1 term in Eq. (5.14) below.

Many more terms involving pion-kaon couplings can be constructed. For a more extensive discussion of building blocks and possible terms in the chiral lagrangian, I refer to Ref. [97].⁹³ Here I give only one more building block:

$$\chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u , \quad (5.12)$$

which is even under parity (the odd combination, with a minus sign instead of a plus sign, appears in the chiral lagrangian at order p^4).

With these definitions, noting that $\Delta_\mu = \frac{1}{2} u^\dagger L_\mu u$, with $L_\mu = \Sigma \partial_\mu \Sigma^\dagger$ (*cf.* Eq. (2.16)), Eq. (2.33) can be written as

$$\mathcal{L}^{(2)} = -\frac{1}{2} f^2 \text{tr}(\Delta_\mu \Delta_\mu) - \frac{1}{8} f^2 \text{tr}(\chi_+) , \quad (5.13)$$

and more terms involving the kaon field are given by

$$\begin{aligned} \mathcal{L}_K^{(2)} = & A_1 \text{tr}(\Delta_\mu \Delta_\mu) K^\dagger K + A_2 \text{tr}(\Delta_\mu \Delta_\nu) (D_\mu K)^\dagger D_\nu K \\ & - A_3 K^\dagger \chi_+ K - A_4 \text{tr}(\chi_+) K^\dagger K \\ & + A'_2 \text{tr}(\Delta_\nu \Delta_\nu) (D_\mu K)^\dagger D_\mu K + A'_3 (D_\mu K)^\dagger \chi_+ D_\mu K + A'_4 \text{tr}(\chi_+) (D_\mu K)^\dagger D_\mu K , \end{aligned} \quad (5.14)$$

where $A_{1,2,3,4}$ and $A'_{2,3,4}$ are seven new LECs.⁹⁴ Formally, we see that $\mathcal{L}_K^{(0)}$ and $\mathcal{L}_K^{(2)}$ are the first two terms in a small-derivative/light quark mass expansion, because only derivatives on the pion field u should be counted as small, and not those acting on the kaon field K . We note that some of the terms in $\mathcal{L}_K^{(2)}$ would be counted as $O(p^4)$ if the kaon is treated as a Nambu–Goldstone boson, *cf.* Sec. II.

B. Power counting

At this point, we need to revisit power counting. First, there are now three scales in the problem, m_π , m_K and $4\pi f$, *cf.* Sec. II C, and we are now *not* assuming that $m_K^2/(4\pi f)^2$ is small. Therefore, it is not obvious that a systematic chiral expansion in $m_\pi^2/(4\pi f)^2$ and/or m_π^2/m_K^2 can be set up. Second, our new lagrangian also contains interactions contributing for example to the scattering process $\pi^+ \pi^- \rightarrow K^+ K^-$. But, clearly, with the kaon mass not treated as a small parameter, the initial-state pions in this process also do not have small momenta in the sense of a chiral expansion. The prediction for this process from our new lagrangian can thus not be argued to be the leading order in a systematic expansion.

⁹³ Note that conventions in Ref. [97] differ from those used here.

⁹⁴ The primed A 's are not present in $\mathcal{L}_K^{(2)}$ of Ref. [97], and can probably be traded for higher-order terms by field redefinitions. However, this is not mandatory, and I choose to keep them in $\mathcal{L}_K^{(2)}$.

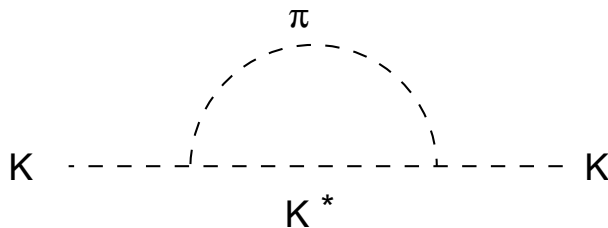


FIG. 9: One-loop contribution to the kaon self energy with a $K^*\pi$ intermediate state.

We conclude that “kaon ChPT” cannot be used for any process near or above the two-kaon threshold. However, it can be used to calculate correlation functions in which a kaon goes in and comes out, so that the large energy residing in the kaon mass does not get converted to pions. This means that one can use the new lagrangian to calculate the dependence of kaonic quantities, such as its mass and decay constant, on the light quark masses m_u and m_d . Note that in the leptonic decay of a kaon, its mass does get converted into the energy of the outgoing leptons, but that does not affect the calculation of its dependence on light quark masses.⁹⁵

While this claim turns out to be correct, we still need to deal with the first problem mentioned above, *i.e.*, the appearance of a new “large” scale m_K , in order to show its validity. From Eq. (5.10) it follows that the parameter M^2 only shows up in the kaon propagator. It can therefore only get promoted to the numerator of some contribution to a physical quantity by appearing in a loop. The simplest case is that of a closed kaon loop; for instance, the kaon tadpole diagram contribution to the pion self-energy that would follow from $\mathcal{L}_K^{(0)} + \mathcal{L}_K^{(2)}$. In dimensional regularization this would give a contribution of order $m_K^2 \log(m_K^2/\Lambda^2)$, which is not small. However, such contributions can be absorbed into the low-energy constants of the pion lagrangian, because they are independent of the pion mass.⁹⁶ We conclude from this example that we can thus *omit* closed kaon loops from the calculation of correlation functions with kaon ChPT.⁹⁷ With this rule that we omit closed kaon loops, we conclude that correlation functions with only pions on the external legs should be calculated with the pion chiral lagrangian only.

In effect, because the kaon is heavy compared to the pion, we have integrated it out, and including closed kaon loops as discussed above would amount to double-counting. This is of course the reason that we can use two-flavor ChPT for pion physics well below the kaon mass. The new element is that we can also consider processes with a kaon going in and coming out — as long as no kaons get annihilated or produced, the energy stored in the kaon mass does not play a dynamical role. An example is $\pi K \rightarrow \pi K$ scattering: As long as the energy of the incoming pion is small, the energy of the outgoing pion also has to be small, and two-flavor ChPT (in the presence of a “heavy” kaon) should apply. For a detailed analysis of this scattering process, see Ref. [97].

⁹⁵ Using kaon ChPT for semi-leptonic decays only works when the momentum of the outgoing pion is small.

⁹⁶ Similar to the way that the physics of ρ mesons and other heavy hadrons appears through the values of the LECs in the pion lagrangian.

⁹⁷ The same argument was used for baryon ChPT already in Ref. [98].

There are also loops with internal kaon lines where the kaon does not form a closed loop, for instance in the just-mentioned example of kaon-pion scattering. To one-loop order, they do not occur in loop corrections to pion and kaon masses and decay constants, because all one-loop diagrams for these quantities are tadpole diagrams, which follows from the fact that there are no three-meson vertices in our EFT. However, such vertices do occur in the kaon theory if we also include the K^* , described by a vector field K_μ also in the isospin-1/2 representation of $SU(2)_V$. In this extended version of kaon ChPT, a possible invariant is

$$-igfK_\mu^\dagger(u^\dagger\partial_\mu u)K = gK_\mu^\dagger K\partial_\mu\phi + \dots, \quad (5.15)$$

i.e., now the theory does contain a $K^*K\pi$ three-meson vertex.

Let us consider an example. The vertex (5.15) leads to a correction to the kaon self energy through the diagram of Fig. 9. The amputated part of this diagram stands for an integral of the form [93] (using dimensional regularization)

$$I = \int \frac{d^d q}{(2\pi)^d} \frac{q_\mu q_\nu (\delta_{\mu\nu} + (p-q)_\mu (p-q)_\nu / M_*^2)}{(q^2 + m_\pi^2)((p-q)^2 + M_*^2)}, \quad (5.16)$$

in which M_* is the mass of the K^* . We need this integral for an onshell kaon, so we need to set $p = Mv$, with v a four-vector with $v^2 = -1$. The denominator of the K^* propagator then becomes

$$q^2 - 2Mv \cdot q + M_*^2 - M^2. \quad (5.17)$$

Nonanalytic dependence on m_π comes from the region $q \sim m_\pi$. The behavior of the integral now depends on the various other scales. If M_* and M were the masses of spin-zero and spin-one heavy-light mesons (instead of the K^* and the kaon), the term linear in M would be the dominant term, because $M_*^2 - M^2 \sim \Lambda_{QCD}^2$ and $M_* \sim M \gg \Lambda_{QCD}$.⁹⁸ Instead, for the KK^* system both masses as well as their difference are of order Λ_{QCD} , and the difference $M_*^2 - M^2$ is the dominant term. This leads to a different infrared behavior for each of these cases. For the heavy-light case, the denominator (5.17) can be of order m_π (times M),⁹⁹ while for the kaonic case, the denominator does not become small. In the latter case, we can thus effectively replace this denominator by Λ_{QCD}^2 , and read off the infrared behavior from the rest of the integral, which gives rise to a contribution of order $m_\pi^4 \log(m_\pi^2/\Lambda^2)$ [93]. The power m_π^4 has to occur because of dimensional analysis, and the fact that the kaon propagator, even though it appears inside a loop integral, can be “power-counted” as order Λ_{QCD}^{-2} . I would expect this type of argument to extend to higher loop contributions, thus explaining how higher-loop contributions lead to nonanalytic terms at higher orders in m_π^2 . For an extensive discussion of all this in the context of baryon ChPT, I refer to Refs. [99, 101].

It is straightforward to show that there are no $m_\pi^2 \log(m_\pi^2/\Lambda^2)$ terms coming from $\mathcal{L}_K^{(0)} + \mathcal{L}_K^{(2)}$, and thus [97]

$$m_K^2 = M^2 - 2(A_3 + 2A_4)\hat{m}_\ell + O(\hat{m}_\ell^2 \log(\hat{m}_\ell)). \quad (5.18)$$

⁹⁸ Of course, $Mv \cdot q$ can be smaller than Λ_{QCD}^2 , but the q integral gets cut off in the infrared by m_π . The situation is more complicated if there are also other infrared scales present, such as momentum transfer in a form factor or scattering amplitude. For detailed discussions, see for instance Refs. [99, 100].

⁹⁹ This leads to the terms of order m_π^3 which appear in the heavy-light and baryon cases.

This is consistent with the result for m_K^2 shown in Eq. (2.40), which does not contain a term of the form $m_\pi^2 \log(m_\pi^2/\Lambda^2)$.¹⁰⁰ It should be noted that Ref. [97] obtains this result in the theory without the K^* . Indeed, since the K^* is significantly heavier than the kaon, we should be able to integrate it out. Doing this leads to an effective coupling of the form

$$g^2 \frac{f^2}{M_*^2} K^\dagger \Delta_\mu \Delta_\mu K = g^2 \frac{f^2}{2M_*^2} \text{tr}(\Delta_\mu \Delta_\mu) K^\dagger K \quad (5.19)$$

(where we used that $\Delta_\mu = -u^\dagger \partial_\mu u$ is an element of the Lie algebra for $SU(2)$), and we see that this is of the form of the first term in Eq. (5.14).

C. An application: decay constants

The pion decay constant follows simply from Eq. (2.40), by expanding out $L(m_K^2)$ in \hat{m}_ℓ :

$$f_\pi = f' \left(1 + \frac{4l_4}{f^2} B_0 m_\ell - 2L(2B_0 m_\ell) + O(\hat{m}_\ell^2) \right), \quad (5.20)$$

in which we can match the LECs f' and l_4 to their three-flavor counterparts [12],

$$\begin{aligned} f' &= f \left(1 - L(B_0 m_s) + \frac{16L_4}{f^2} B_0 m_s \right), \\ l_4 &= 4(L_5 + 2L_4) - \frac{1}{(4\pi)^2} \left(\log \left(\frac{B_0 m_s}{\Lambda^2} \right) + 1 \right). \end{aligned} \quad (5.21)$$

The matching formulas (5.21) only make sense if m_s is small enough for three-flavor ChPT to apply; otherwise f' and l_4 should be treated as two-flavor LECs that depend in an unknown way on the strange quark mass. The expressions (5.21) give the first few orders of the expansion of f' and l_4 in m_s . It is possible that the strange quark mass is so large that this expansion simply does not converge. But it is also possible that this expansion does (asymptotically) converge, but too slowly for the expressions (5.21) to be of practical use. In that case, one should either use kaon ChPT, or three-flavor ChPT to higher order than p^4 . Whether the expansion in the strange quark mass converges to some order in ChPT or not is quantity dependent.

In order to obtain the expansion of f_K in terms of m_ℓ , we can follow two routes. One is to simply use the three-flavor result for f_K in Eq. (2.40), and expand in m_ℓ , as we did above for f_π . However, it is instructive to see how one obtains this expansion in kaon ChPT independently of the three-flavor ChPT calculation [93]. In order to do this, we need to translate the currents

$$\begin{aligned} J_\mu^{L,R} &= \bar{q}_{L,R} \gamma_\mu S_{L,R}, \\ \bar{q} &= (\bar{u} \quad \bar{d}), \end{aligned} \quad (5.22)$$

into the appropriate operators in kaon ChPT. These currents now do not correspond to the Noether currents of (softly broken) symmetries, because we are not assuming that the

¹⁰⁰ Expanding $L(m_\eta^2)$ in m_π^2/m_K^2 gives rise to analytic terms in m_π^2 only.

strange quark is small. Therefore, their translations into kaon ChPT do not follow from the lagrangian $\mathcal{L}_K^{(0)} + \mathcal{L}_K^{(2)}$. We thus introduce spurions $H_{L,R}$ into these currents, with transformation rules such as to make these currents invariant under $SU(2)_L \times SU(2)_R$:

$$J_\mu^{L,R} = \bar{q}_{L,R} H_{L,R} \gamma_\mu s_{L,R}, \quad \text{with} \quad H_{L,R} \rightarrow v_{L,R} H_{L,R}, \quad (5.23)$$

in which $v_{L,R} \in SU(2)_{L,R}$. Making the transition to kaon ChPT, the first two terms in the expansion in terms of derivatives of the pion fields are

$$\begin{aligned} J_\mu^R &= C_1 (D_\mu K)^\dagger u H_R + i C_2 K^\dagger \Delta_\mu u H_R, \\ J_\mu^L &= -C_1 (D_\mu K)^\dagger u^\dagger H_L + i C_2 K^\dagger \Delta_\mu u^\dagger H_L, \end{aligned} \quad (5.24)$$

where $C_{1,2}$ are new LECs (not present in the lagrangian). The left- and right-handed currents are related by parity. In these ChPT expressions for the currents, one can now set $H_{L,R} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or $H_{L,R} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in order to pick out the charged or neutral kaon, corresponding to picking out the up or down quark in Eq. (5.23), respectively. Taking the matrix element of the axial current $J_\mu^R - J_\mu^L$ between a kaon state and the vacuum, one finds that

$$f_K = 2C_1 \left(1 + \frac{C}{f^2} \hat{m}_\ell - \frac{3}{4} L(2\hat{m}_\ell) + O(\hat{m}_\ell^2) \right), \quad (5.25)$$

where C is a linear combination of LECs in the kaon EFT lagrangian.¹⁰¹ Note that the LEC C_1 is unrelated to f' in Eq. (5.20), because there is no $SU(3)$ symmetry in kaon ChPT. If we do use $SU(3)$, we may again match the LECs in Eq. (5.25) to the three-flavor ChPT result, from which we find

$$\begin{aligned} C_1 &= \frac{1}{2} f \left(1 - 3L(\hat{m}_s) + \frac{8(2L_4 + L_5)}{f^2} \hat{m}_s \right), \\ C &= 8(4L_4 + L_5) - \frac{5}{2} \frac{1}{(4\pi)^2} \left(\log \left(\frac{\hat{m}_s}{\Lambda^2} \right) + 1 \right). \end{aligned} \quad (5.26)$$

As it should, the coefficient of $L(m_\pi^2)$ in Eq. (5.25) matches that following from Eq. (2.40).

We close this section with another simple example of the translation of a weak operator into kaon ChPT [93]. The $\Delta S = 2$ operator $(\bar{s}_L \gamma_\mu d_L)(\bar{s}_L \gamma_\mu d_L)$ can be written in $SU(2)_L$ -invariant form by again using the spurion H_L as in

$$(\bar{s}_L \gamma_\mu H_L^\dagger q_L)(\bar{s}_L \gamma_\mu H_L^\dagger q_L) \rightarrow B(H_L^\dagger u K)(H_L^\dagger u K), \quad (5.27)$$

where in the second step I gave the leading-order translation into kaon ChPT, introducing another weak LEC B . Finally, one sets $H_L^\dagger = (0 \ 1)$, because for this choice $H_L^\dagger q_L = d_L$. This technique for finding the ChPT representations of electro-weak operators is of course not special to kaon ChPT, and the same method can also be used in three-flavor ChPT [7].

It turns out that $SU(2)$ ChPT can also be applied to semi-leptonic and nonleptonic kaon decays, despite the fact that the final-state pions can have an energy up to half the kaon mass. This is not at all obvious, because in the $SU(2)$ framework the kaon mass is not a small parameter. For more on this topic, see Refs. [102, 103].

¹⁰¹ It turns out that C_2 does not contribute to this order.

VI. FINITE VOLUME

As a final topic, I would like to touch, rather briefly, on the fact that all lattice QCD computations are, necessarily, performed in a finite spatial volume, and with a finite maximal extent L_4 in the (euclidean) time direction.

Usually, the time dimension of the four-dimensional euclidean volume is chosen larger than the spatial directions; large enough so that only the lowest (or lowest few) intermediate states contribute to the correlation functions of interest at large times. In order to minimize, or eliminate, systematic effects from the finite extent in spatial directions, the conceptually easiest method would be to choose the spatial volume of a cubic box $V_3 = L^3$ (where L is the linear size of the spatial box, and we will assume periodic boundary conditions for all fields), sufficiently large that one may ignore the effects of L being finite. However, the cost of numerical computations grows rapidly with L , and one would thus like to choose the volume just large enough that finite-volume effects can be neglected, but not much larger. The question then arises whether we have any theoretical insight into what “just large enough” means.

In fact, as we will see, it is possible to develop systematic expansions which provide quantitative information on finite-volume effects, without the introduction of any new parameters in the chiral lagrangian. This allows us to quantitatively estimate the effects of working in a finite volume, and even to make use of finite-volume effects as a probe of hadronic quantities such as the pion decay constant and the chiral condensate.

The reason that again ChPT gives us the tool to develop these systematic expansions is that pions¹⁰² are the lightest hadrons. Thus, if we imagine reducing L from some very large value, the pions feel the effects of being in a finite volume first. Therefore, ChPT should give us access to the dominant finite-volume effects on many hadronic quantities, if we know how to apply it in this setting.

It already follows from this qualitative discussion that the “figure of merit” for what “large” or “small” volume means is the linear size of the box in terms of the wavelength of the pion, $z = m_\pi L$. Clearly, large volume means $z \gg 1$. But only a more quantitative analysis can tell us whether this means, say, $z \gtrsim 3$, or whether maybe $z \gtrsim 10$ is needed.

The key theorem in setting up finite-volume ChPT states that the lagrangian to be used is the infinite-volume lagrangian [104]. The argument goes as follows. First assume that the spatial volume is infinite, but that the euclidean time extent L_4 is finite, and that the pion fields (5.4) obey periodic boundary conditions in that direction. The pion effective theory with these boundary conditions describes the theory at a finite temperature $T = 1/L_4$. The important observation here is that to calculate pion correlation functions at finite temperature, one uses the *same* chiral lagrangian as at zero temperature (*i.e.*, infinite L_4). The temperature, and thus the dependence on L_4 enters only through the boundary conditions.¹⁰³ But, if this is the case, this then also has to be true for the dependence on the three spatial dimensions of the box, if we use periodic boundary conditions also in those directions [105]. In other words, the LECs in a finite-volume calculation should be taken the same as in infinite volume.

This result makes sense from the EFT point of view. If we remember that the LECs

¹⁰² In this section, I will use the word “pions” to refer only to π^\pm and π^0 , and assume that their masses are equal, *i.e.*, that isospin is conserved.

¹⁰³ For field theory at finite temperature, see the lectures by Owe Philipsen at this school.

originated from integrating out high-energy degrees of freedom, they can pick up exponentially suppressed finite-volume corrections of order $\exp(-EL)$, where E is the energy of the integrated mode, relative to their infinite-volume values. If we take for example $E = m_\rho$ and $L = 2$ fm, $\exp(-EL) = 4.5 \times 10^{-4}$, which is very small in comparison with the present precision of our knowledge of LECs. We conclude that we can safely ignore such effects, and thus take the LECs to be those of the infinite-volume theory.¹⁰⁴ This more intuitive argument also shows that the theorem should hold for more general boundary conditions, such as anti-periodic ones.

It follows that the way in which the volume dependence enters the Feynman rules is through the propagator, which now is periodic, since it solves, for particle with mass m_π ,

$$(-\square + m_\pi^2)G_L(x - y) = \bar{\delta}(x - y) = \prod_\mu \sum_{n_\mu} \delta(x_\mu - y_\mu + n_\mu L_\mu) , \quad (6.1)$$

where L_μ is the linear extent of the volume in the μ direction, and $\bar{\delta}$ is the periodic delta function, with period L_μ in the μ direction. The solution is, of course,

$$G_L(x - y) = \sum_{n_\mu} G_\infty(x - y + nL) , \quad (6.2)$$

with G_∞ the infinite-volume propagator, and in which nL is the four-vector with components $n_\mu L_\mu$. In words, finite volume effects occur because pions can “travel around the world,” multiple times in each direction, in a world that is a box with periodic boundary conditions. Because of the periodicity, momenta are quantized, $p_\mu = 2\pi n_\mu / L_\mu$, with n_μ integer, and the propagator can also be represented by

$$G_L(x - y) = \frac{1}{L_1 L_2 L_3 L_4} \sum_p \frac{e^{ip(x-y)}}{p^2 + m_\pi^2} , \quad (6.3)$$

where the sum is over the discrete momenta. Feynman rules for vertices are the same as in the infinite-volume theory. With these rules, one may extend the calculations of meson masses, decay constants, and other quantities to include finite-volume (and/or finite-temperature) corrections. We will return to this in Sec. VI A.

However, before we extend ChPT calculations to finite volume in this way, we need to address a serious physics issue: it is well known that in the chiral limit, the chiral condensate vanishes in a finite volume. In order to obtain a nonvanishing condensate, one should take the chiral limit after taking the infinite-volume limit, and not before; these two limits do not commute. But, if there is no chiral symmetry breaking, that brings into question the whole framework of ChPT!

Let us see what happens in the chiral theory at zero quark mass in finite volume. The condensate can be found from the logarithmic derivative of the partition function with respect to χ^\dagger :

$$\langle \bar{q}_R q_L \rangle = -\frac{2B_0}{Z} \text{tr} \left(\frac{\partial Z}{\partial \chi^\dagger} \right) \Big|_{\chi=0} = -\frac{1}{4} f^2 B_0 \langle \text{tr}(\Sigma) \rangle , \quad (6.4)$$

¹⁰⁴ A corollary is that the chiral theory at nonzero temperature only makes sense for small T , so that $\exp(-E/T)$ is small.

where in the second step we made the transition to the chiral theory. We can decompose

$$\Sigma(x) = U \exp(2i\phi(x)/f) , \quad (6.5)$$

with U a constant $SU(N_f)$ matrix, and in which ϕ does not contain any constant mode, *i.e.*, $\int d^4x \phi(x) = 0$.¹⁰⁵ The matrix U represents an element of the coset $[SU(N_f)_L \times SU(N_f)_R]/SU(N_f)$, and varying U over the group $SU(N_f)$ covers the vacuum manifold, *cf.* Sec. II B. Since U represents an $SU(N_f)_L$ symmetry transformation, it follows that for vanishing quark masses, the chiral lagrangian does not depend on U . This, in turn, implies that the condensate (6.4) vanishes, because

$$\int dU U = 0 , \quad (6.6)$$

with dU the Haar measure on the group $SU(N_f)$.

What we see is that ChPT reproduces the correct behavior in finite volume. The intuitive insight is that as long as the volume is large compared to the typical hadronic scale, $L \gg 1/\Lambda_{QCD}$, QCD is nonperturbative, and hadrons form. The lightest hadrons are the pions, and all that happens if we lower the value of $z = m_\pi L$ is that the pions get very distorted. However, if this is the correct picture, ChPT should still be the correct effective theory. Only when we make $\Lambda_{QCD}L$ too small QCD becomes perturbative, and ChPT is no longer the appropriate effective theory. As long as $\Lambda_{QCD}L \gg 1$, ChPT is the appropriate EFT for the low-energy physics of QCD, even if L is small in units of m_π .

When we turn on quark masses, the chiral condensate no longer vanishes. For a degenerate quark mass m and a zero mode $\Sigma_0 \equiv (1/V) \int d^4x \text{tr}(\Sigma)$, the size of the leading-order mass term in the chiral lagrangian is of order

$$N_f f^2 B_0 m V \Sigma_0 , \quad (6.7)$$

with $V = L^4$ the four-dimensional finite volume, if we take our volume to be a four-dimensional box with equal size in each (euclidean) direction. We may now consider different regimes. If we keep Eq. (6.7) fixed, we choose $m \sim 1/L^4$, or, equivalently, $m_\pi = \sqrt{2B_0 m} \sim 1/L^2$; we will refer to this regime as the “ ϵ -regime” [106].¹⁰⁶ This is a different regime from the large-volume regime we considered in Sec. II, in which we found that $m_\pi \sim p \sim 1/L$, and thus $m \sim 1/L^2$; this regime is usually referred to as the “ p -regime.” In the ϵ -regime, with $\epsilon \equiv 1/L$, we have that $p \sim \epsilon$, while $m_\pi \sim \epsilon^2 \ll p$, while in the p -regime they are of the same order.

The kinetic term of the chiral lagrangian, $\int d^4x \frac{1}{2} \text{tr}(\partial_\mu \phi \partial_\mu \phi)$, suppresses the nonzero modes ϕ , and limits the fields ϕ to be of order $1/L$. If we expand

$$\Sigma = U \exp(2i\phi/f) = U \left(1 + \frac{2i}{f} \phi - \frac{2}{f^2} \phi^2 + \dots \right) , \quad (6.8)$$

¹⁰⁵ Here I have arbitrarily placed U on the left of $\exp(2i\phi(x)/f)$. More generally, we can write $\Sigma(x) = U_L \exp(2i\phi(x)/f) U_R$, but if we set $U_L = U U_R^\dagger$, we have that $\Sigma(x) = U \exp(2iU_R^\dagger \phi(x) U_R / f)$, and $U_R^\dagger \phi U_R$ does not contain a constant mode if ϕ does not.

¹⁰⁶ In this small four-dimensional volume, this basically constitutes our definition of the pion mass, as we can clearly not define it as the parameter characterizing the large- t behavior of the euclidean pion two-point function.

it is easy to see that, with $m_u = m_d \equiv m$, the first term in

$$\int d^4x \frac{1}{4} f^2 m B_0 \operatorname{tr}(\Sigma + \Sigma^\dagger) = \frac{1}{4} f^2 B_0 m V \operatorname{tr}(U + U^\dagger) \quad (6.9)$$

$$- \int d^4x \frac{1}{2} m B_0 \left(\operatorname{tr}(U + U^\dagger) \phi^2 \right) + O(\phi^4)$$

is of order L^0 in the ϵ -regime, while the second term is of order $1/L^2$. To leading order, U is nonperturbative: The zero modes in Σ do not have a gaussian damping, and can thus not be taken into account through standard perturbation theory. When we increase the quark mass from $m \sim 1/L^4$ to $m \sim 1/L^2$, the first term in Eq. (6.9) grows to be of order L^2 , still dominating the second term, which is now of order L^0 . This forces U to approach $\mathbf{1}$, making also the zero mode perturbative. We thus make the transition from the ϵ -regime to the p -regime. This is also reflected by the behavior of the $p = 0$ mode contribution to the pion propagator (6.3) in finite volume, which, taking all $L_\mu = L$, is equal to $1/(L^4 m_\pi^2)$. For $m_\pi^2 = 2B_0 m \sim 1/L^4$, this is of order one, whereas all other momentum modes contribute terms of order $1/L^2$. When m_π is of order $1/L$, also the zero-momentum contribution is of order $1/L^2$, and it does not dominate the propagator.

There is thus a region in which the chiral expansion in the ϵ -regime coincides with that in the p -regime. Both representations have to agree for $m \sim 1/L^2$, making it possible to match correlation functions between the two regimes [106]. It is only in the p -regime that the Feynman rules in infinite volume, where only the propagator (6.2) knows about the finite-volume, apply.

Next, we will consider each of these two regimes separately. I will only consider the finite-volume physics of pions, because in the real world they are much lighter than kaons, and this is now also the case in state-of-the-art lattice QCD computations. Therefore, finite-volume effects for kaons and other hadrons will be dominated by their interactions with pions. To put it differently, the kaons are always in the p -regime ($m_K \gg 1/L$; for example, for $L = 2$ fm, $m_K L \approx 5$), and in the p -regime finite-volume effects due to kaons traveling around the world are much suppressed compared to those due to pions.¹⁰⁷

A. p -regime

In the p -regime, it is quite straightforward to adapt the infinite-volume expressions of ChPT to finite volume. Here, for definiteness, we will take a spatial volume $V_3 = L^3$, with periodic boundary conditions, while, for simplicity, we will assume that the (euclidean) time extent is large enough to be taken as infinity. We will also assume that only finite-volume effects due to pions are significant.

Finite-volume effects start occurring at one loop, because in the loops the pion propagator gets replaced by its finite-volume version, Eq. (6.2). For quantities like meson masses and decay constants, *cf.* Eq. (2.40), all one-loop integrals are tadpole integrals, for instance of the form

$$G_\infty(0) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m_\pi^2} \doteq \frac{m_\pi^2}{16\pi^2} \left(\frac{2}{d-4} + \log \left(\frac{m_\pi^2}{\Lambda^2} \right) + \text{finite} \right), \quad (6.10)$$

¹⁰⁷ For an investigation including both pions in the ϵ -regime and kaons in the p -regime, see Ref. [107].

where the second equation holds using dimensional regularization, and, of course, we have to stay away from the pole at $d = 4$. In finite volume, this gets replaced by

$$G_L(0) = G_\infty(0) + (G_L(0) - G_\infty(0)) . \quad (6.11)$$

Using that

$$G_\infty(x) = \frac{1}{4\pi^2} \frac{m_\pi}{\sqrt{x^2}} K_1(m_\pi \sqrt{x^2}) , \quad (6.12)$$

in which K_1 is the modified Bessel function of the second kind of order one, we find that

$$\begin{aligned} G_L(0) - G_\infty(0) &= \sum_{\vec{n}} G_\infty(x = (\vec{n}L, 0)) - G_\infty(0) \\ &= \sum_{\vec{n} \neq 0} G_\infty(x = (\vec{n}L, 0)) \\ &= \sum_{n=1}^{\infty} k(n) \frac{1}{4\pi^2} \frac{m_\pi}{\sqrt{n}L} K_1(\sqrt{n}m_\pi L) . \end{aligned} \quad (6.13)$$

Here $k(n)$ counts the number of vectors \vec{n} with integer-valued components of length \sqrt{n} . For instance, $k(1) = 6$, $k(2) = 12$, $k(3) = 8$, $k(4) = 6$, *etc.* Note that, as one would expect, the UV divergence of $G_L(0)$ is the same as in infinite volume, because finite-volume effects come from pions traveling a “long” distance. Of course, in the one-loop calculations leading to Eq. (2.40), we also encounter integrals such as Eq. (6.10) with an extra p^2 in the numerator. Here we can use that, for $x \neq 0$,

$$\int \frac{d^4p}{(2\pi)^4} \frac{p^2 e^{ipx}}{p^2 + m_\pi^2} = -m_\pi^2 \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2 + m_\pi^2} = -\frac{1}{4\pi^2} \frac{m_\pi^3}{\sqrt{x^2}} K_1(m_\pi \sqrt{x^2}) . \quad (6.14)$$

We thus find, for example, for the finite-volume pion and kaon masses and decay constants to one loop [104]

$$\begin{aligned} m_\pi(L) &= m_\pi \left(1 + \frac{m_\pi^2}{(4\pi f_\pi)^2} \sum_{n=1}^{\infty} \frac{4k(n)}{\sqrt{n}m_\pi L} K_1(\sqrt{n}m_\pi L) \right) \\ &= m_\pi \left(1 + \frac{m_\pi^2}{(4\pi f_\pi)^2} \frac{24}{(m_\pi L)^{3/2}} \sqrt{\frac{\pi}{2}} e^{-m_\pi L} + O(e^{-\sqrt{2}m_\pi L}) \right) , \\ f_\pi(L) &= f_\pi \left(1 - 2 \frac{m_\pi^2}{(4\pi f_\pi)^2} \sum_{n=1}^{\infty} \frac{4k(n)}{\sqrt{n}m_\pi L} K_1(\sqrt{n}m_\pi L) \right) \\ &= f_\pi \left(1 - \frac{m_\pi^2}{(4\pi f_\pi)^2} \frac{48}{(m_\pi L)^{3/2}} \sqrt{\frac{\pi}{2}} e^{-m_\pi L} + O(e^{-\sqrt{2}m_\pi L}) \right) , \\ f_K(L) &= f_K \left(1 - \frac{3}{4} \frac{m_\pi^2}{(4\pi f_\pi)^2} \sum_{n=1}^{\infty} \frac{4k(n)}{\sqrt{n}m_\pi L} K_1(\sqrt{n}m_\pi L) \right) \\ &= f_K \left(1 - \frac{m_\pi^2}{(4\pi f_\pi)^2} \frac{18}{(m_\pi L)^{3/2}} \sqrt{\frac{\pi}{2}} e^{-m_\pi L} + O(e^{-\sqrt{2}m_\pi L}) \right) , \end{aligned} \quad (6.15)$$

where m_π and f_π are the pion mass and decay constant in infinite volume. The kaon mass does not receive any finite-volume corrections due to pions, as we can infer from the fact that the infinite-volume one-loop contribution has no contribution from pion loops (but

only from an η' loop, *cf.* Eq. (2.40)). In the second of each of these equations, we used the asymptotic expansion $K_1(z) \sim \sqrt{\pi/(2z)} \exp(-z)$. The dominant corrections do not come from higher orders in the asymptotic expansion of K_1 , but from the term $n = 2$ in the sum in Eq. (6.15). For $m_\pi \approx 300$ MeV and $m_\pi L \approx 3$, the size of the corrections in Eq. (6.15) is of order one percent; making L smaller the corrections grow rapidly in size. Detailed studies of finite-volume effects for the pion and kaon masses and decay constants as well as the η mass have been made in Ref. [108] using a partial resummation of higher loops based on a different approach [109], and, for the pion mass to two loops in ChPT [110].¹⁰⁸

Many investigations of finite-volume effects in the p -regime have been carried out, using both ChPT and other techniques, and it is beyond the scope of these lectures to cover this topic in any more detail. However, there is one application in which ChPT can play a uniquely important role, and that is the effect of partial quenching on the study of two-particle correlation functions in a finite box.

In a finite volume, the energy spectrum of QCD is discrete, and that also holds for two-particle (and many-particle) states. In fact, the energy levels of two-particle states in finite volume give information on scattering phase shifts in infinite volume [111]. The intuitive idea is the following. The normalized wave function of a particle in a box of dimension L^3 is of order $L^{-3/2}$. If two such particles are put inside the box, and they interact, one thus expects a shift in the two-particle energy of order the product of the interaction strength time overlap of the two one-particle wave functions, which is of order $1/L^3$. Therefore, the dependence of the two-particle energy shifts on the volume gives us information about the interactions between the two particles. Unitarity plays an important role in the systematic analysis [111], making it not straightforward to generalize these ideas to PQQCD or QCD with mixed actions. However, ChPT gives us a handle on this problem, since, as we have seen, it can be extended (in euclidean space) to the partially quenched and mixed-action cases. The two-particle correlation functions, from which in the unitary case the two-particle energy levels are determined, can thus be calculated in PQChPT (or mixed-action ChPT). One finds that the double-hairpin contribution to flavor-neutral propagators gives rise to unphysical phenomena such as enhanced finite-volume effects that can be estimated using ChPT [58, 73, 85, 112]. More study of these intricacies would be interesting.

B. ϵ -regime

In this section, we will restrict ourselves to two flavors, under the assumption that the strange quark is not in the ϵ -regime. It can thus be taken into account for instance by the method described in Sec. V. Alternatively, one can start with $N_f = 3$ ChPT, but keep all fields describing mesons containing strange quarks in the p -regime.

Since in this section we will restrict ourselves to two flavors, $\Sigma \in SU(2)$, and instead of the parametrization (6.8) it is easier to use

$$\begin{aligned} \Sigma &= U \left(\sigma + i \frac{\sqrt{2}}{f} \vec{\tau} \cdot \vec{\pi} \right), \\ \sigma &= \sqrt{1 - \frac{2\vec{\pi} \cdot \vec{\pi}}{f^2}}, \end{aligned} \tag{6.16}$$

¹⁰⁸ For earlier work, see these references.

with $\vec{\tau}$ the Pauli matrices, and $\int d^4x \pi_i(x) = 0$. With $m_u = m_d = m$, $V = L^4$, and using Eq. (6.16), we expand the chiral lagrangian to second order in the pion field:

$$\begin{aligned} \int d^4x \mathcal{L} &= -\frac{1}{4} f^2 B_0 m V \operatorname{tr}(U + U^\dagger) \\ &+ \frac{1}{2} \int d^4x \left[\partial_\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} + \frac{1}{2} B_0 m \vec{\pi} \cdot \vec{\pi} \operatorname{tr}(U + U^\dagger) \right] + O(\pi^3). \end{aligned} \quad (6.17)$$

In the ϵ -regime, as we have seen already, the first term and the kinetic term are of order one. This makes the mass-dependent $O(\pi^2)$ term a term of order $1/L^2$, and the whole $O(p^4)$ lagrangian (2.39) of order $1/L^4$. At this point, we observe several differences between the p -regime and the ϵ -regime. First, in the p -regime, finite volume effects are exponentially suppressed, while in the ϵ -regime, they are suppressed by powers of $1/(fL)$. Then, we also see that the chiral expansion in the ϵ -regime rearranges itself: the LECs L_i only show up at next-to-next-to-leading order in the expansion. To leading order, we thus find for the partition function, as a function of the quark mass:

$$Z(m) = \int [d\Sigma] \exp \left(\frac{1}{4} f^2 B_0 m V \operatorname{tr}(U + U^\dagger) - \frac{1}{2} \int d^4x \partial_\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} \right). \quad (6.18)$$

Higher order terms in Eq. (6.17) can be expanded out from the exponent in the integrand, and thus be taken into account systematically. The notation $[d\Sigma]$ indicates the $SU(2)$ -invariant Haar measure for each $\Sigma(x)$.

In order to proceed, we need to split the measure $[d\Sigma]$ into an integration over U and an integration over $\vec{\pi}$. Following Ref. [113], we insert

$$\begin{aligned} 1 &= \int dH_0 dH_1 dH_2 dH_3 \delta \left(H - \frac{1}{V} \int d^4x \Sigma(x) \right) \\ &= \int h^3 dh dU \delta \left(hU - \frac{1}{V} \int d^4x \Sigma(x) \right), \\ H &= H_0 + i\vec{\tau} \cdot \vec{H}, \end{aligned} \quad (6.19)$$

where, with $h = |H| \equiv \sqrt{\det(H)}$, we can write $H = hU$ with $U \in SU(2)$, and with dU the Haar measure on $SU(2)$. Now we transform variables $\Sigma = U\tilde{\Sigma}$, so that

$$\tilde{\Sigma} = \sigma + i \frac{\sqrt{2}}{f} \vec{\tau} \cdot \vec{\pi}, \quad (6.20)$$

and use the fact that $d\Sigma = d\tilde{\Sigma}$ to obtain

$$\begin{aligned} Z(m) &= \int dU \int [d\tilde{\Sigma}] \prod_{i=1}^3 \delta \left(\frac{\sqrt{2}}{fV} \int d^4x \pi_i(x) \right) \\ &\times \exp \left(\frac{1}{4} f^2 B_0 m V \operatorname{tr}(U + U^\dagger) - \frac{1}{2} \int d^4x \partial_\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} + 3 \log \left(\frac{1}{V} \int d^4x \sigma(x) \right) \right) \\ &= \mathcal{N} \int dU \exp \left(\frac{1}{4} f^2 B_0 m V \operatorname{tr}(U + U^\dagger) \right), \end{aligned} \quad (6.21)$$

where \mathcal{N} is independent of the quark mass m . Carrying out the integral over U with

$$\begin{aligned} U &= \cos \theta \mathbf{1} + i \sin \theta \vec{n} \cdot \vec{\tau}, \quad |\vec{n}| = 1, \quad 0 \leq \theta \leq \pi, \\ dU &= \frac{1}{4\pi^2} d\Omega(\vec{n}) \sin^2(\theta) d\theta, \end{aligned} \quad (6.22)$$

we find

$$Z(m) = \mathcal{N} \frac{2}{f^2 B_0 m V} I_1(f^2 B_0 m V) , \quad (6.23)$$

where I_1 is the modified Bessel function of order one. From this, we find for the chiral condensate

$$\begin{aligned} \langle \bar{q}q \rangle &= -\frac{1}{ZV} \frac{dZ}{dm} = -f^2 B_0 \left(\frac{I_1'(f^2 B_0 m V)}{I_1(f^2 B_0 m V)} - \frac{1}{f^2 B_0 m V} \right) \\ &\stackrel{m \rightarrow 0}{=} -\frac{1}{4} f^4 B_0^2 m V . \end{aligned} \quad (6.24)$$

Consistent with our previous discussion, the condensate vanishes for $m \rightarrow 0$. For $m \neq 0$ fixed and $V \rightarrow \infty$, Eq. (6.24) reproduces the infinite-volume result $\langle \bar{q}q \rangle = -f^2 B_0$, *cf.* Eq. (2.35). Note that the dimensionless variable $f^2 B_0 m V$ can take any value; to be in the ϵ -regime, the requirements are only that $f^2 L^2 \gg 1$ and $B_0 m L^2 \ll 1$. This does not restrict the size of their product.

Equation (6.24) provides an example of a ChPT calculation in the ϵ -regime to lowest order. Starting from Eq. (6.17), it is in principle straightforward to go to higher orders in $1/(fL)$ [113, 114]. For recent work discussing the ChPT calculation of the chiral condensate, see Ref. [115].

For instance, it is rather easy to see that the $m \vec{\pi} \cdot \vec{\pi}$ term in Eq. (6.17) leads to a renormalization of the first term in that lagrangian by [113]

$$\rho \equiv 1 - \frac{1}{f^2 V} \int d^4x \langle \vec{\pi}(x) \cdot \vec{\pi}(x) \rangle = 1 - \frac{3}{f^2 V} \sum_{p \neq 0} \frac{1}{p^2} = 1 - \frac{3\beta_1}{f^2 L^2} , \quad (6.25)$$

where β_1 is a numerical constant [113], and we used that $p_\mu = 2\pi n_\mu/L$ with n_μ integer.¹⁰⁹

As we have noted before, the $O(p^4)$ LECs L_i only come into play at order $1/L^4$, *i.e.*, at next-to-next-to-leading order in an expansion in $1/(f^2 L^2)$. This is to be contrasted with the chiral expansion in the p -regime, where they appear already at next-to-leading order. This might be helpful for a precise determination of the leading-order LECs (f and B_0), because only these two parameters appear in fits using next-to-leading order expressions in the ϵ -regime. This is another example in which we can extract physical quantities (f and B_0) from unphysical computations — clearly, a small, euclidean, four-dimensional, finite volume with periodic boundary conditions is unphysical, but accessible to lattice QCD.

This concludes my brief overview of the application of ChPT to the study of volume dependence. For applications to the distribution of topological charge, see Ref. [116]. I have already mentioned the extension of the chiral expansion to the partially quenched case with some quark masses are in the ϵ -regime, with others in the p -regime (with the obvious application to QCD with $2+1$ flavors with only the strange quark mass in the p -regime), see Ref. [107]. For a recent extension including lattice-spacing artifacts for Wilson fermions, see Ref. [117]; for a recent higher-order study in a partially-quenched setting, see Ref. [118]. These references contain fairly complete pointers to earlier work in the ϵ -regime.

¹⁰⁹ The sum over p is of course infinite, but we are interested here only in the $1/L^2$ part, which is finite.

The volume-independent infinite part can be absorbed into a multiplicative renormalization of m . In dimensional regularization, the “infinite” part actually vanishes.

VII. CONCLUDING REMARKS

Clearly, ChPT is a very useful, and, in practice, indispensable tool for extracting hadronic physical quantities from lattice QCD, which, in almost all cases can only obtain correlation functions in some unphysical regime. We have seen that this includes nonzero lattice spacing and finite volume, necessarily, but also “much more” unphysical situations, such as independent choices for valence and sea quarks on the lattice. The chiral lagrangian, as well as other EFTs, such as baryon ChPT, heavy-light ChPT and two-nucleon EFT, are local theories, defined in terms of a number of LECs. Once lattice QCD determines the values of these LECs, even if this is by matching unphysical correlation functions, they can be used to evaluate any hadronic quantity accessible to the various EFTs. While we have restricted ourselves in these lectures to the physics of Nambu–Goldstone bosons, the conceptual points all carry over to EFTs that include the interactions of other hadrons with Nambu–Goldstone bosons.

There are many applications of ChPT, relevant for lattice QCD, that were left out of these lectures. As already mentioned during these lectures, one can extend ChPT to include the interactions of pions with heavier hadrons. While we briefly discussed this in the context of “heavy” kaons in Sec. V, the methods described there can be extended to baryons [99, 101] and hadrons containing heavy quarks [119, 120]. For a recent review of hadron interactions, including π - π and baryon-baryon scattering, see Ref. [121].

Some exercises

Most explicit equations in these lecture notes can be derived with relative ease. None of them require more than a one-loop calculation, and in all cases except Eq. (5.16) the corresponding diagrams are tadpole diagrams. Here are a few more exercises:

1. Derive Eq. (2.15), which is, in fact, valid for an arbitrary number of flavors. Simplify for $N_f = 2$.
2. Show that $\text{tr}(L_\mu L_\nu L_\mu L_\nu)$ can be written as a linear combination of the terms in Eq. (2.17), using the Cayley–Hamilton theorem, which, for a three-by-three matrix A says that

$$A^3 - \text{tr}(A) A^2 + \frac{1}{2} (\text{tr}(A)^2 - \text{tr}(A^2)) A - \det(A) \mathbf{1} = \mathbf{0} . \quad (7.1)$$

[Hint: one way to proceed is to consider $\text{tr}((L_\mu + L_\nu)^4)$.] Simplify Eq. (2.17) for $N_f = 2$.

3. Show that the scattering lengths for pion scattering in the $O(N)$ linear sigma model are discontinuous across the phase transition between the symmetric phase and the phase in which $O(N)$ breaks down to $O(N - 1)$ (*cf.* discussion after Eq. (3.27)).
4. Consider quenched ChPT (*i.e.*, no sea quarks at all), with one valence quark and ignore the anomaly. Show that using the naive (but wrong!) symmetry group $SU(1|1)_L \otimes SU(1|1)_R$ for constructing the chiral lagrangian leads to nonsense for the vacuum. See Ref. [61] for more discussion.

5. Verify cyclicity of the supertrace on 2×2 graded matrices.
6. Derive Eq. (4.35). Show that in the LCE regime all other terms in the chiral lagrangian contribute at next-to-leading order.
7. Investigate possible higher order terms, to order p^4 , in the LCE regime for Wilson ChPT (see Ref. [36]).
8. Show that a solution of the equation $u_R = hu_L^\dagger h$ exists, for given u_R and u_L . Here $u_{R,L}$ and h are unitary matrices with determinant one.
9. Show that Eq. (5.11) is equivalent to Eq. (5.10) plus a term that can be absorbed into the A_1 term of Eq. (5.14).
10. Calculate the integral in Eq. (5.16) and convince yourself of the correctness of the discussion following Eq. (5.17).
11. Extend the discussion of Sec. III C to $N_f = 3$ degenerate light quarks (in the LCE regime). I do not know the answer — as far as I know, you can publish the result!

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