ULB-TH 95/02
UMH-MG 95/01
LPTHENS
January 1995

A Primer for Black Hole Quantum Physics

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And so long as we have this itch
for explanations, must we not
always carry around with us this
cumbersome but precious bag of
clues called History?
— G. Swift, *Waterland* (1983)

A Primer for Black Hole Quantum Physics

Abstract: The mechanisms which give rise to Hawking radiation are revealed by analyzing in detail pair production in the presence of horizons. In preparation for the black hole problem, three preparatory problems are dwelt with at length: pair production in an external electric field, thermalization of a uniformly accelerated detector and accelerated mirrors. In the light of these examples, the black hole evaporation problem is then presented.

The leitmotif is the singular behavior of modes on the horizon which gives rise to a steady rate of production. Special emphasis is put on how each produced particle contributes to the mean albeit arising from a particular vacuum fluctuation. It is the mean which drives the semiclassical back reaction. This aspect is analyzed in more detail than heretofore and in particular its drawbacks are emphasized. It is the semiclassical theory which gives rise to Hawking's famous equation for the loss of mass of the black hole due to evaporation $dM/dt \simeq -1/M^2$. Black hole thermodynamics is derived from the evaporation process whereupon the reservoir character of the black hole is manifest. The relation to the thermodynamics of the eternal black hole through the Hartle–Hawking vacuum and the Killing identity are displayed.

It is through the analysis of the fluctuations of the field configurations which give rise to a particular Hawking photon that the dubious character of the semiclassical theory is manifest. The present frontier of research revolves around this problem and is principally concerned with the fact that one calls upon energy scales that are greater than Planckian and the possibility of a non unitary evolution as well. These last subjects are presented in qualitative fashion only, so that this review stops at the threshold of quantum gravity.

Contents

Acknowledgments

Out of little acorns big oaks grow. This review grew out of a short series of lectures by R. Brout at the University of Crete, Iraklion in May 1993. The topic was “Vacuum Instability in the Presence of Horizons”. It has grown into a rather complete introduction to quantum black hole physics as we now see it, a half baked but intensely interesting subject.

R. Brout would like to express his gratitude to Professor Floratos and the other members of the physics faculty at Crete, for their heartwarming hospitality.

We are beholden to many colleagues for the generous offerings of their wisdom. They are too numerous to enumerate, but we wish to cite in particular: R. Balbinot, J. Beckenstein, A. Casher, P. Grove, J. Katz, A. Ottewill, S. Popescu, T. Piran, G. Venturi and most of all F. Englert for his interest, advice and fruitfull discussions.

Introduction

This review is conceived as a pedagogical essay on black hole quantum physics.

Black hole physics has a rich and varied history. Indeed a complete coverage of the subject would run parallel to a good part of the development of modern theoretical physics, embracing as it does general relativity, quantum field theory and thermodynamics. The present review, a primer that is designed to bring the reader to the threshold of current research, covers the middle ground. It is a review of that aspect of the theory that dates from Hawking's seminal work on black hole evaporation up to but not including quantum gravity. This latter chapter has been the subject of intense investigation over the past few years and is still at a very speculative stage, certainly not yet in a state to receive consecration in a review article. What has emerged from investigation of present quantum field theory, in the presence of a black hole is the awareness that the problem cannot be confronted without at least some aspects that emerge from the (as yet unknown) quantum theory of gravity. It is our intention here to develop the theory to the point where the reader will have a clear idea of the nature of the unsolved problems given the present state of our ignorance. An optimist would say that the solution to these problems leads to the discovery of the quantum theory of gravity. But even a minimalist will admit that a good deal is to be learnt from black hole physics at its present stage of development. It is perhaps not too much to hope that a careful presentation in terms of present concepts can illuminate the way to the unknown. Such is our endeavor.

There exist several situations in quantum field theory which give rise to phenomena which are similar to black hole evaporation. Our own experience has been that it is very helpful to analyze those examples in detail. They furnish a body of information so as to constitute a useful theoretical laboratory for the study of the more complex issues which arise in the black hole problem. To this end the initial chapters of this review are devoted to these laboratory exercises : pair production in a static electric field, accelerated systems which become spontaneously excited in Minkowski-space and accelerated mirrors. Black hole evaporation is then presented in the light of these examples.

The common leitmotif that runs through all of this will gradually emerge upon reading. The two central elements are :

1) The existence of a horizon in each case, a separation of space-time into portions, which would have been in causal contact in other circumstances, that are no longer so. Straight trajectories become curved so as to approach or recede from the horizon exponentially slowly as seen by an external observer. The result of this law of approach is a modification of the spectral decomposition of the quantum matter fields in such manner as to give rise to a steady state of production.

2) The mechanism of production is the steady conversion of vacuum fluctuations into particles. A large part of the analysis is devoted to the description of how a particular fluctuation gives rise to a particular production event.

In this way, we reveal the mechanisms at work in black hole evaporation. These are understood in terms of usual quantum mechanics. For the black hole problem they turn out to be surprisingly complex in that they involve a subtle interplay between the conversion of vacuum fluctuations to physical particles and a restructuration of the vacuum itself. Each gives its contribution to the energy momentum tensor that provides the source of gravitational feed-back. Some of this richness is already present in the semi-classical approximation wherein the source of Einstein's equations is given by the mean energy-momentum tensor. But this approximation will certainly break down at later stages, obviously at the Planckian stage, but possibly before, even much before. What we have set out to do is display the structures at work in the hope that this approach will sharpen the focus on the types of problems which will have to be confronted at later stages. The hope is that, from thence, quantum gravity will emerge. But, as introduced in one BBC radio program, that was next week's news!

Considerable effort has been put into exhibiting the configurations of field energy which contribute to the production mechanism. This is the content of Section 1.4, 2.5.3, 2.6 and 3.5. Particularly relevant for black hole physics is Section 2.5.3. Of necessity, the analysis is not without subtlety (though rigorous and mathematically straightforward). The reader, who wants to get into the midsts of the black hole problem immediatly, may choose to skip Sections 2.5.3 and 2.6 in a first reading. For his benefit Section 3.5 contains a summary of all the necessary information related to the configurations of the fluctuations. The reader may continue on from this point (and of course then return if he wishes to the demonstrations and more detailed conceptual discussion of Sections 2.5.3 and 2.6.

One word on presentation. Generally speaking, the type of physics discussed is more conceptual than technical in character. As such the mathematical formalism is usually quite simple, unencumbered by extensive algebraic manipulation. Nevertheless, from time to time, the algebra does become heavy. When we have judged that the details of the mathematics interrupts smooth reading, we begin the topics under discussion with a resume of the results and relegate details to a passage enclosed within square brackets, or to an appendix.

Chapitre 1

Pair Production in a Static Electric Field

1.1 Qualitative Survey

It was shown early on by Heisenberg and collaborators [50] that the vacuum of quantum field theory in the presence of a static electric field is unstable against the creation of charged pairs. Subsequently using techniques of functional integration in the context of the action formalism, Schwinger [86] showed in an almost one line proof that the overlap function between vacuum at the time the field was turned on with vacuum at subsequent times decays exponentially fast.

After Hawking discovered black hole radiation [44] several authors [67], [87],[15],[73] pointed out the analogy with this kind of pair production. Though the analogy is not strict the formalisms which are used to encode these two phenomena have many points in common thereby making the electric pair production problem a fruitful exercise. To introduce the problem we first give a brief sketch of the physics in terms of a given mode (solution of the wave equation).

We shall work in one dimension with a charged scalar field. This contains enough physics to make the exercise applicable to the black hole problem. For definiteness we choose the gauge $A_t = Ex$, $A_x = 0$ (with the charge absorbed into the definition of E) whereupon the Klein-Gordon equation is

$$\Box\varphi = \left[+\partial_x^2 - (\partial_t - iEx)^2\right]\varphi = m^2\varphi \qquad . \tag{1.1}$$

Since the electric field is static, in this time independent gauge the operator

∂_t commutes with the d'Alembertian operator $\Box$, and modes can be taken of the form $\varphi_\omega(t,x) = \mathrm{e}^{-i\omega t} f_\omega(x)$ where $f_\omega(x)$ satisfies the equation

$$\left[\partial_x^2 + (\omega + Ex)^2\right] f_\omega(x) = m^2 f_\omega(x) \tag{1.2}$$

Upon dividing eq. (1.2) by the factor $2m$, one comes upon a non relativistic Schrödinger equation in the presence of an upside down oscillator potential centred at the point $x_c = -\omega/E$. The eigenvalue for the effective energy, is $-m/2$. Therefore the solution tunnels between two turning points ($x_c \pm m/E = x_c \pm a^{-1}$) where a is the classical acceleration due to the field.

Take the case $E > 0$. Then classically a particle is uniformly accelerated to the right, following the classical trajectory $(x - x_c)^2 - (t - t_c)^2 = a^{-2}$. So if it comes in asymptotically from the right along its past horizon $(x - x_c) = -(t - t_c)$ it will turn around at $x = x_c + a^{-1}$ and $t = t_c$. In quantum mechanics this is translated into the following description. A wave packet centred for example around $\omega = 0$ (i.e. $x_c = 0$) will come in from the right asymptotically at the speed of light, slow down and turn around at $x = a^{-1}$ and fly off to the right approaching once more the speed of light along its future horizon. The incident packet is localized around the classical orbit $x^2 - (t - t_c)^2 = a^{-2}$. Near the point $x = +a^{-1}$ there will be some amplitude to tunnel through to $x = -a^{-1}$ and then continue from that point on to accelerate to the left. Since $E > 0$, the tunneled particle must therefore have been "mesmerized" into an antiparticle in the tunneling region. This situation can only be met by second quantizing φ (confronting the Klein paradox [56]). The "mesmerization" is simply reflecting the fact that there is a probability amplitude to create a pair in the tunneling region. Then the above scattering description must be amended to: the initial particle is scattered as classically, but its final flux after the scattering is increased by a factor α^2 to accommodate for the creation of an antiparticle on the other side. The flux of this latter is denoted by $|\beta|^2$. Thus we see that $|\alpha|^2 - |\beta|^2 = 1$ is the statement of charge conservation.

In Fig. 1.1.

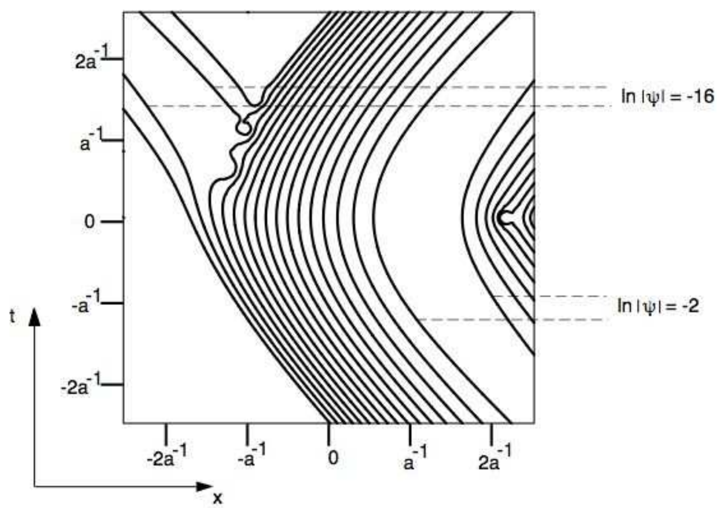

Fig. 1.1 *The function* $\ln|\psi(t,x)|$ *is plotted for an incoming wave packet solution of eq. (1.1). It is a minimal packet as described subsequently in the context of eq. (1.54). We use the logarithm because of the very small amplitude of the produced particle (*$e^{-\pi m/2a}$ *where we have taken* $m/a = 9$*). The particle (on the right) is centered on its classical trajectory* $x^2 - t^2 = a^{-2}$*. The wave function of the antiparticle behaves classically for* $t > 0$ *and* $x < -a^{-1}$ *and vanishes in the past.*

we have plotted the modulus of a wave packet. One sees both the classical path about which the packet is centered and the created pair. Note that the separation between the turning points $\pm a^{-1}$, the length of the region from which the pair emanates, coincides with the distance $|\Delta x|$ necessary to have the electrostatic energy compensate the rest mass ($E|\Delta x| = 2m$), so as to make possible on mass shell propagation from these points outward. It is the realization of this possibility, without energy cost, that causes the vacuum instability. Note also that since the amplitude of probability for a vacuum fluctuation to have its particle-antiparticle components separated by a space-like interval, $|\Delta x|$, is $\exp(-m|\Delta x|)$, it may be anticipated that the amplitude for production is $\beta \sim \exp\left[-C(m/a)\right]$ where C is some constant of $O(1)$ to be calculated by the formalism, and found to be equal to π. We have thus shown how a wave packet not only describes the classical orbit but gives an idea of the location of the produced pair as well as its probability amplitude.

1.2 Mode Analysis

We shall start with the analysis of eq. (1.2) considered as the Schrödinger equation of a fictitious non-relativistic problem. The solutions are well known [99]. We will nevertheless present them by a method [73] which is suitable for subsequent use in that it displays the singularity of the solution along the horizons. It is this singular behavior which in every case (pair production in an electric field, accelerating observer and mirror, black hole) encodes the production phenomena that is peculiar to quantum physics in the presence of horizons.

The function f_ω, solution of eq. (1.2) is the x representation of a ket $|\Xi\rangle$, i.e. $f_\omega(x) = \langle x|\Xi\rangle$. Go over to dimensionless variables centered at the origin by introducing

$$\xi = \sqrt{E}\,(x + \omega/E) \tag{1.3}$$

so as to write (1.2) in the suggestive form

$$\frac{\pi^2 - \xi^2}{2}\langle\xi|\Xi\rangle = -\varepsilon\langle\xi|\Xi\rangle \tag{1.4}$$

where

$$\begin{aligned} &\varepsilon = m^2/2E = m/2a \qquad\qquad (1.5)\\ &[\pi,\xi] = -i \quad (\pi = -i\partial/\partial\xi \text{ in } \xi\text{-representation}) \end{aligned}$$

Now introduce the new canonical variables [5], analogous to the standard annihilation and creation operators used to quantize the harmonic oscillator, defined by

$$\begin{aligned} \left.\begin{matrix} U \\ V \end{matrix}\right\} &= \frac{1}{\sqrt{2}}\,[\pi \mp \xi] \\ [U,V] &= -i \end{aligned} \tag{1.6}$$

so that (1.4) becomes

$$u\frac{\partial}{\partial u}\langle u|\Xi\rangle = \left(+i\varepsilon - \frac{1}{2}\right)\langle u|\Xi\rangle \tag{1.7}$$

We can go back to ξ-representation through

$$\langle\xi|\Xi\rangle = \int du\langle\xi|u\rangle\langle u|\Xi\rangle \tag{1.8}$$

where we have adopted the notation $U|u\rangle = u|u\rangle$, defining the u representation whereupon $V = i\partial_u$. To find the kernel $\langle\xi|u\rangle$ of this unitary transformation one must solve the conditions

$$\langle\xi\left|\frac{V-U}{\sqrt{2}}\right|u\rangle = \frac{1}{\sqrt{2}}\left(-i\frac{\partial}{\partial u} - u\right)\langle\xi|u\rangle = \xi\langle\xi|u\rangle$$
$$\langle\xi\left|\frac{(\pi-\xi)}{\sqrt{2}}\right|u\rangle = \frac{1}{\sqrt{2}}\left(-i\frac{\partial}{\partial \xi} - \xi\right)\langle\xi|u\rangle = u\langle\xi|u\rangle$$

to yield

$$\langle\xi|u\rangle = \frac{1}{\sqrt{2^{1/2}\pi}}\exp i[\xi^2/2 + \sqrt{2}\xi u + u^2/2] \tag{1.9}$$

where we have supplied the norm to make the transformation unitary. The general solution of (1.7) is

$$\lambda_\varepsilon(u) \equiv \langle u|\Xi\rangle = A\left[\theta(u)\frac{u^{i\varepsilon-1/2}}{\sqrt{2\pi}}\right] + B\left[\theta(-u)\frac{(-u)^{i\varepsilon-1/2}}{\sqrt{2\pi}}\right] \tag{1.10}$$

The constants A, B follow from initial conditions. Consider the classical orbits associated with eq. (1.4), described by the Hamiltonian problem $H = uv$. They are given by $u = u_0 e^{-\tau}, v = v_0 e^{\tau}$ with $u_0 v_0 = -\varepsilon$. At early times τ the orbits located near $u = -\infty$, $v = 0^+$ (i.e. incoming from $\xi = +\infty$) are described by $A = 0$ and the ones near $u = +\infty$, $v = 0^-$ by $B = 0$. These two sets of solutions (labeled by ε) are orthogonal, and complete. In the effective Schrödinger problem which we are discussing their norm (with respect to the measure du) is such that $\int_{-\infty}^{+\infty} du \lambda_\varepsilon^* \lambda_{\varepsilon'} = \delta(\varepsilon - \varepsilon')$. This fixes $|A|^2 + |B|^2 = 1$.

Inserting each of the above solutions and (1.9) into (1.8) gives an integral representation for the scattered mode $\langle\xi|\Xi\rangle$. Reference [99], identifies these as integral representations of parabolic cylinder functions, more succinctly, Whittaker functions $D_\nu(z)$ (with $\nu = -i\varepsilon - 1/2$) and provides their properties and connection formulae. For our purposes however it is instructive to continue to work as "quantum mechanics" (see ref. [73]) and show how to relate asymptotic amplitudes to on-shell quanta. The effort is worthwhile because of remarkable analogs with the modes that arise in black hole physics (the reason being that the singular character of the solution eq. (1.10) near the origin encodes the existence of a horizon).

Take the solution (1.10) with $B = 0$. We then must analyze

$$\langle\xi|\Xi\rangle = \int_0^\infty \frac{du}{2^{3/4}\pi} u^{i\varepsilon-1/2}\exp i[\xi^2/2 + \sqrt{2}\xi u + u^2/2] \tag{1.11}$$

$$\Big[\quad = \quad \frac{1}{2^{3/4}\pi} e^{-\varepsilon\pi/4} e^{i\pi/8} \Gamma(1/2+i\varepsilon) D_{-i\varepsilon-1/2}(\sqrt{2}e^{-i\pi/4}\xi)\Big]$$

in the limit $\xi \to \pm\infty$. The saddle points of the integral lie at

$$u^\star = \frac{-\sqrt{2}\xi \pm \sqrt{2\xi^2 - 4\varepsilon}}{2} \tag{1.12}$$

For $\xi \to -\infty$, the saddle at $u^\star = -\sqrt{2}\xi + \varepsilon/(\sqrt{2}\xi)$ may be used to evaluate the integral 1.12 by the saddle point method since it lies well within the limits of integration and one has therefore the reliable asymptotic estimate (W.K.B. approximation)

$$\langle \xi | \Xi \rangle_I = \frac{1}{\sqrt{2\pi|\xi|}} \exp -i \left[\xi^2/2 - \varepsilon \ln(\sqrt{2}|\xi|) - \pi/4 \right] \tag{1.13}$$

Since the solution that we are describing ($B = 0$ in eq. (1.10)) corresponds to an incoming particle coming from the left, hence with $u_{classical} \to \infty$ as $\xi \to -\infty$ for the incident beam, eq. (1.13) is to be identified with the incident wave function (I). For $\xi \to +\infty$, the saddle at $u^\star = -\sqrt{2}\xi + \varepsilon/(\sqrt{2}\xi)$ gives no contribution since it lies completely outside the limits of integration.

The other saddles at $u^\star = \varepsilon/(\sqrt{2}\xi)$ cannot be used to evaluate the integral by saddle point integration since, for $\xi \to \pm\infty$, they lie at the edge of the integration domain. However we expect that, since classically, $v \to +\infty$ for the transmitted solution and $v \to -\infty$ for the reflected one, one might fruitfully exploit the v-representation. This is indeed the case. The v-representation of $|\Xi\rangle$ is

$$\begin{aligned}
\langle v | \Xi \rangle &= \int_{-\infty}^{+\infty} \langle v | u \rangle \langle u | \Xi \rangle du \\
&= \int_0^{+\infty} \frac{e^{ivu}}{\sqrt{2\pi}} \frac{u^{i\varepsilon - 1/2}}{\sqrt{2\pi}} du \\
&= e^{i\pi/4} e^{-\varepsilon\pi/2} \frac{\Gamma(1/2+i\varepsilon)}{\sqrt{2\pi}} \qquad (1.14)
\end{aligned}$$

$$Bigl[\theta(v)\frac{v^{-i\varepsilon-1/2}}{\sqrt{2\pi}}\Big]$$

$$\begin{aligned}
&+ e^{-i\pi/4} e^{\varepsilon\pi/2} \frac{\Gamma(1/2+i\varepsilon)}{\sqrt{2\pi}} \Big[\theta(-v)\frac{(-v)^{-i\varepsilon-1/2}}{\sqrt{2\pi}}\Big] \\
\equiv \; & T \left[\theta(v)\frac{v^{-i\varepsilon-1/2}}{\sqrt{2\pi}}\right] + R \left[\theta(-v)\frac{(-v)^{-i\varepsilon-1/2}}{\sqrt{2\pi}}\right] \qquad (1.15)
\end{aligned}$$

This result follows from standard analysis, most easily by deformation of the contour from the real axis to the positive (resp. negative) imaginary axis for positive (resp. negative) values of v. It is this difference that gives the different weights R, T of the reflected to transmitted amplitudes, the ratio of which is

$$|\frac{R}{T}| = e^{\varepsilon\pi} \tag{1.16}$$

This exponential ratio is generic to all problems having horizons since the behavior of the wave functions at the horizon gives rise to the singular form $u^{i\epsilon}$, hence analytic continuation as exhibited in 1.15.

Since we have normed $|\Xi\rangle$, the unitarity condition $|R|^2 + |T|^2 = 1$ is fulfilled by eq. (1.15), as is easily checked thanks to the formula $|\Gamma(1/2 + i\varepsilon)|^2 = \pi/\cosh(\pi\varepsilon)$.

The identification of the waves multiplied by R and T as the reflected and transmitted waves, thereby giving a precise expression to the physics of the tunneling process, follows from the asymptotic behavior of $\langle\xi|\Xi\rangle$. Indeed

$$\begin{aligned}
\langle\xi|\Xi\rangle &= \int_{-\infty}^{+\infty} \langle\xi|v\rangle\langle v|\Xi\rangle dv \\
&= T\int_0^{+\infty} \frac{e^{i[-\xi^2/2+\sqrt{2}\xi v-v^2/2+\pi/4]}}{2^{3/4}\pi} v^{-i\varepsilon-1/2} dv \\
&\quad +R\int_{-\infty}^{0} \frac{e^{i[-\xi^2/2+\sqrt{2}\xi v-v^2/2+\pi/4]}}{2^{3/4}\pi} (-v)^{-i\varepsilon-1/2} dv \qquad (1.17)\\
\Big[&= \frac{1}{2^{3/4}\sqrt{2\pi}} \frac{e^{\varepsilon\pi/4}e^{-i\pi/8}}{\cosh\pi\varepsilon} \Big(ie^{-\varepsilon\pi} D_{i\varepsilon-1/2}(\sqrt{2}e^{-3i\pi/4}\xi) \\
&\quad +D_{i\varepsilon-1/2}(\sqrt{2}e^{i\pi/4}\xi)\Big)\Big] \quad . \qquad (1.18)
\end{aligned}$$

The phase of $\langle\xi|v\rangle$ ($= -\xi^2/2 + \sqrt{2}\xi v - v^2/2 + \pi/4$) is fixed by the relation $\langle\xi|v\rangle = \int \langle\xi|u\rangle\langle u|v\rangle\, du$. The saddle points are now located at

$$v^\star = \frac{\sqrt{2}\xi \pm \sqrt{2\xi^2 - \varepsilon}}{2} \tag{1.19}$$

The saddles that go to $\pm\infty$ as $\xi \to \pm\infty$ are responsible for the transmitted and reflected solution respectively. For these the integration can be estimated by a saddle point approximation to give

$$\langle\xi|\Xi\rangle_T \underset{\xi\to+\infty}{=} T\frac{e^{i[\xi^2/2-\varepsilon\ln\sqrt{2}\xi]}}{\sqrt{2\pi\xi}} \quad , \tag{1.20}$$

$$\langle\xi|\Xi\rangle_R \underset{\xi\to-\infty}{=} R\frac{e^{i[\xi^2/2-\varepsilon\ln|\sqrt{2}\xi|]}}{\sqrt{2\pi|\xi|}} \quad . \tag{1.21}$$

The other saddles ($v \to \pm 0$ as $\xi \to \pm\infty$) are concerned with the incident solution which has already been analyzed in u-representation. This is fortunate in that the width of these saddles extend outside the domains of integration in eq. (1.17). We call attention to the complementary rôles of the saddle points in u and v-representations in isolating the asymptotic incident wave on one hand and the transmitted and reflected parts of the solution on the other. As previously mentioned, this is due to the fact that the locus of saddle points is along classical trajectories.

1.3 Vacuum Instability

The above solution and notation describes the effective Schrödinger wave function which solves (1.2). We now analyze how it encodes vacuum instability and pair production. As we have announced the transmitted wave must have opposite charge. This can seen by following the movement of physical wave packets, solution of the Klein-Gordon equation (1.1). These are of the form

$$\int d\omega f(\omega-\omega_0)e^{-i\omega(t-t_0)}\chi_\varepsilon(\sqrt{E}(x+\omega/E)) \quad . \tag{1.22}$$

Note the novelty of the construction wherein the eigenvalue, ω, appears in the argument of the function $\chi_\varepsilon(\xi) = \langle\xi|\Xi\rangle$ since $\xi = \sqrt{E}\,(x+\omega/E)$. It is this circumstance that allows for the correct charge and current assignments of each asymptotic part of the wave (1.13), (1.21), (1.20) :

$$e^{-i\omega t}\begin{cases} 1\; e^{-iE(x+\omega/E)^2/2}|x+\omega/E|^{im^2/2E-1/2} & \equiv \mathcal{I}\;(x\to-\infty, t\to+\infty) \\ Re^{iE(x+\omega/E)^2/2}|x+\omega/E|^{-im^2/2E-1/2} & \equiv \mathcal{R}\;(x\to-\infty, t\to-\infty) \\ Te^{iE(x+\omega/E)^2/2}(x+\omega/E)^{-im^2/2E-1/2} & \equiv \mathcal{T}\;(x\to+\infty, t\to+\infty) \end{cases} \tag{1.23}$$

The motion of each branch is given by the group velocity obtained by setting the derivative with respect to ω of the phases equal to zero. Thus, $\mathcal{I}$ corresponds to motion to the left at late times, whereas $\mathcal{R}$ describes motion to the right at early times and $\mathcal{T}$ to the right, but at late times. The rôles of $\mathcal{I}$ and $\mathcal{R}$ have been swapped once we consider the dynamics with respect to the physical time appearing in the Klein-Gordon equation (1.1) instead of the fictuous dynamics associated to the solutions of the Schrödinger equation (1.7).

Charge assignments follow from the sign of the charge density

$$J^t = -J_t \equiv \varphi_\omega^* i \overleftrightarrow{\mathcal{D}}_t \varphi_\omega = \varphi_\omega^* (i \overleftrightarrow{\partial_t} + 2Ex)\varphi_\omega = 2(\omega + Ex)\chi_\varepsilon^* \chi_\varepsilon \qquad (1.24)$$

Thus $\mathcal{I}$ and $\mathcal{R}$ which live at large negative x have negative charge whereas $\mathcal{T}$ has positive charge. In summary, for $E > 0$ the solution of eq. (1.10) with $B = 0$ represents an incoming anti-particle of amplitude R, and outgoing reflected anti-particle of amplitude 1 and a transmitted particle of amplitude T. The unitarity relation $|R|^2 + |T|^2 = 1$ is then best recast into a form out of which one reads charge conservation rather than conservation of probability. Divide this relation by $|R|^2$ and recast it into the form

$$\begin{gathered} -|\alpha|^2 + |\beta|^2 = -1 \\ |\alpha|^2 = 1/|R|^2 \quad ; \quad |\beta|^2 = |T|^2/|R|^2 \end{gathered} \qquad (1.25)$$

The right hand side then corresponds to incident flux of unit negative charge coming from $x = -\infty$, $|\beta|^2$ is the outgoing transmitted flux of positive charge going to $+\infty$ and $(|\alpha|^2 - 1)$ is the increase of flux of the reflected wave necessary to implement charge conservation.

For each value of ω, in addition to the above mode, there is its parity conjugate, obtained from the transformation $(x + \omega/E) \to -(x + \omega/E)$ and complex conjugation. Clearly it represents the charge conjugate since one need only run through its mirror image about the axis $x_c (= -\omega/E)$. It corresponds to the solution of eq. (1.4) proportional to $\theta(v)$. The modulus of a wave packet built of this latter mode is given in Fig. 1.1.

From these considerations, it follows that these basis functions for the quantized field, whose quanta are single particles in the past (formed from wave packets), lose this property in the future due the production. More specifically let us envisage the situation wherein E is switched on during a time interval $0 \le t \le T$, over a length L. Then the modes fall into two classes:

- Class I: Those that scatter ($\beta \neq 0$) i.e. whose turning points lie within this space-time domain of area LT.

- Class II: Those which are only deflected, so as to follow the classical trajectory before they reach the would-be turning point (in the sense of wave packets). For these latter $\beta = 0$.

Indeed, we shall see subsequently that, for modes in the first class, if a pair is formed, it will emerge from a region of area $\mathcal{O}(a^{-1} \times a^{-1})$ whereas in the second class no pair is formed to within "edge corrections". By this, we mean for $L >> a^{-1}$ and $T >> a^{-1}$, no pair is formed except near the edges. Thus to obtain asymptotic results of $\mathcal{O}(L \times T)$ for the production of pairs, it is a legitimate idealization to divide the modes into these two classes. For the remainder of this chapter, all considerations are devoted to those in the non trivial ($\beta \neq 0$) class I. For these modes, the future fate of a single particle mode in the past is to become a many particle mode, hence not convenient to describe the result of a counter experiment in the future. Thus, in addition to the modes previously presented – denoted by "in-modes" – we shall be obliged to introduce "out-modes" as well. These latter will represent single-particle quanta in the future of their turning points.

It is in the framework of second quantized field theory that this discussion makes sense. Therefore we begin by quantizing the field operators in the in-modes. The complete set is given by

$$\varphi^{in}_{p,\omega}(t,x) = \frac{1}{(4E)^{1/4} R^*} e^{-i\omega t} \chi^*_\varepsilon(-x - \omega/E) \tag{1.26}$$

$$\varphi^{in*}_{a,\omega}(t,x) = \frac{1}{(4E)^{1/4} R} e^{-i\omega t} \chi_\varepsilon(x + \omega/E) \tag{1.27}$$

Note that for completeness, ω should span the range $[-\infty, +\infty]$ contrary to the case without E-field where the energy has to be taken positive only. The wave functions have been normed according to the Klein-Gordon scalar product

$$\begin{aligned} \int dx \varphi^{in}_{p,\omega}(t,x) i\overleftrightarrow{\mathcal{D}}_t \varphi^{in*}_{p,\omega}(t,x) &= +\delta(\omega - \omega') \\ \int dx \varphi^{in}_{a,\omega}(t,x) i\overleftrightarrow{\mathcal{D}}_t \varphi^{in*}_{a,\omega}(t,x) &= -\delta(\omega - \omega') \qquad . \end{aligned} \tag{1.28}$$

Here the label p designates particle and a anti-particle (described by the parity conjugate as previously discussed). The reason for the complex conjugation in the l.h.s of eq. (1.27) is that the function $\varphi^{in}_{a,\omega}$ is a solution of the charge conjugate field equation (here obtained by changing the sign of the charge, i.e. $E \to -E$ in the field equation). Then $\varphi^{in}_{p,\omega}$ and $\varphi^{in*}_{a,\omega}$ obey the same equation and the second quantized field is written in the in-basis,

$$\Phi(t,x) = \int d\omega (a^{in}_\omega \varphi^{in}_{p,\omega}(t,x) + b^{in\dagger}_\omega \varphi^{in*}_{a,\omega}(t,x)) \tag{1.29}$$

with the usual commutation relations

$$\left[a^{in}_\omega, a^{in\dagger}_{\omega'}\right] = \left[b^{in}_\omega, b^{in\dagger}_{\omega'}\right] = \delta(\omega - \omega') \tag{1.30}$$

The out-basis is obtained simply by observing that if one follows a single outgoing branch backwards in time it will trace out the same backwards history as that of one of the in-modes when it moves forward in time. These modes at fixed ω are obtained by sending t into $-t$ and complex conjugating so that the new functions remain solutions of the field equation (1.1).

$$\begin{aligned} \varphi^{out}_{p,\omega}(t,x) &= \varphi^{in*}_{p,\omega}(-t,x) \\ \varphi^{out*}_{a,\omega}(t,x) &= \varphi^{in}_{a,\omega}(-t,x) \end{aligned} \tag{1.31}$$

Furthermore, since for each ω, the two sets of in-modes eqs (1.26,1.27) are complete, these out-modes must be linear combinations of them. In fact they are given (after choices of phases) by

$$\varphi^{out}_{p,\omega} = \alpha\ \varphi^{in}_{p,\omega} - \beta^*\ \varphi^{in*}_{a,\omega} \tag{1.32}$$

$$\varphi^{out*}_{a,\omega} = \alpha^*\ \varphi^{in*}_{a,\omega} - \beta\ \varphi^{in}_{p,\omega} \tag{1.33}$$

where α and β are the coefficients introduced in (1.25, 1.25), now fixed precisely as:

$$\beta = T/R\ , \quad \alpha = e^{i\pi/4}/R \tag{1.34}$$

To understand this result it suffices to follow the histories of the φ^{in}'s as wave packets [16]. In eq. (1.32), for example, $\varphi^{in}_{p,\omega}$ will give rise to a reflected wave of amplitude α^* on the right and a transmitted wave of amplitude β^* on the left, whereas $\varphi^{in^*}_{a,\omega}$ gives a transmitted wave of amplitude β on the right and reflected wave of amplitude α on the left. The contributions on the left therefore cancel and those on the right have total amplitude $|\alpha|^2 - |\beta|^2$ $(= 1)$ as required for a particle out-mode. The assiduous reader will check that the right hand side of eqs (1.32,1.33) are indeed the complex conjugates of the χ_ε functions defined in the in-modes (all multiplied by $e^{i\omega t}$). The easiest route to obtain these relations is to use eqs (1.11,1.17) in the context of integral representations for $\langle\xi|\Xi\rangle$:

$$\chi_\varepsilon(\xi) = e^{\frac{i\pi}{4}}\left[T\chi_\varepsilon^*(-\xi) + R\chi_\varepsilon^*(\xi)\right] \tag{1.35}$$

and the correct identification of these wave functions as used in second quantization.

One develops Φ in the out basis by inverting eqs (1.32, 1.33),

$$\begin{aligned} \varphi^{in}_{p,\omega} &= \alpha^*\ \varphi^{out}_{p,\omega} + \beta^*\ \varphi^{out*}_{a,\omega} \\ \varphi^{in*}_{a,\omega} &= \alpha\ \varphi^{out*}_{a,\omega} + \beta\ \varphi^{out}_{p,\omega} \qquad , \end{aligned} \tag{1.36}$$

and substituting into eq.1.29 to give

$$\Phi = \int d\omega(a^{out}_\omega \varphi^{out}_{p,\omega} + b^{out\dagger}_\omega \varphi^{out^*}_{a,\omega}) \tag{1.37}$$

where

$$\begin{aligned} a^{out}_\omega &= \alpha^*\ a^{in}_\omega + \beta\ b^{in\ \dagger}_\omega \\ b^{out\ \dagger}_\omega &= \alpha\ b^{in\ \dagger}_\omega + \beta^* a^{in}_\omega \end{aligned} \tag{1.38}$$

These equations (1.38) define a Bogoljubov transformation [10] wherein eq. (1.25) ensures that a^{out} and b^{out} obey the correct commutation relation. Historically it came up when Bogoljubov [12] noted that the free particle Bose-Einstein ground state was unstable against creation of pairs of equal and opposite momentum once interparticle interactions were introduced. The interested reader will find a brief account of Bogoljubov's considerations in Appendix A.

From (1.38) it is clear that the in-vacuum $|0, \mathrm{in}\rangle$ (i.e. the state annihilated by in-annihilation operators) will contain out-particles, ie. we set up this Heisenberg state at early times and count the mean number of out particles that are realized from it at some later time.

$$\langle n_\omega\rangle = \langle 0, \mathrm{in}|a^{out\dagger}_\omega a^{out}_\omega|0, \mathrm{in}\rangle = |\beta|^2 \qquad . \tag{1.39}$$

One can express the in-vacuum state as the linear combination [55] of out states since the Bogoljubov transformation is unitary.

$$|0, \mathrm{in}\rangle \;=\; N^{-1/2}\exp\left[\left(\frac{\beta}{\alpha}\right)\sum_\omega a^{out\ \dagger}_\omega b^{out\ \dagger}_\omega\right]\ |0, \mathrm{out}\rangle \tag{1.40}$$

This result is obtained by setting $|0,\mathrm{in}\rangle = f(a^{out\ \dagger}, b^{out\ \dagger})|0,\mathrm{out}\rangle$ and imposing that $0 = a^{in}|0,\mathrm{in}\rangle = (\alpha a^{out} - \beta b^{out\ \dagger}) f|0,\mathrm{out}\rangle = (\alpha[a^{out}, f] - \beta b^{out\ \dagger} f)|0,\mathrm{out}\rangle$. This in turn implies that $\alpha(\partial f/\partial a^{out\ \dagger}) - \beta b^{out\ \dagger} = 0$, hence that f has the form written in 1.40. Note that the creation operators $a_\omega^{out\ \dagger}, b_\omega^{out\ \dagger}$ appear only as a product. Hence to each produced particle in mode p, ω, there corresponds one and only one antiparticle in mode a, ω. From what has been described these particles are born in pairs of opposite values of their conserved quantum numbers such as charge and energy . Thus the energy of the states containing pairs spontaneously created is equal to the energy of the in-vacuum. This is readily seen by expressing the hamiltonian of the field in terms of the a^{out}, b^{out} operators. One finds

$$H = \int_{-\infty}^{\infty} d\omega\omega(a_\omega^{out\dagger} a_\omega^{out} - b_\omega^{out\dagger} b_\omega^{out}) \tag{1.41}$$

To compute the normalization factor N in equation 1.40 we have

$$\begin{aligned} \langle 0,\mathrm{in}|0,\mathrm{in}\rangle &= 1 = N^{-1} \prod_\omega \sum_n \left|\frac{\beta}{\alpha}\right|^{2n} = N^{-1} \prod_\omega \frac{1}{1 - |\beta/\alpha|^2} = N^{-1} \prod_\omega |\alpha|^2 \\ N &= \prod_\omega |\alpha|^2 \end{aligned} \tag{1.42}$$

From this it follows that

$$|\langle 0,\mathrm{out}|0,\mathrm{in}\rangle|^2 = \frac{1}{N} = \prod_\omega \frac{1}{|\alpha|^2} = \exp - \sum_\omega \ln(1 + |\beta|^2) \tag{1.43}$$

thereby delivering the probability to find no pairs at future times.

To complete the calculation we must now give a meaning to the $\sum_\omega$ i.e. we must count the number of modes in the non trivial class (those which turn around in the interval $0 \le t \le T$ and $0 \le x \le L$).

The density of orthogonal modes of frequency ω in the interval $0 \le t \le T$ is $\frac{T}{2\pi} d\omega$. To calculate $\int d\omega$, we note that the scattering centers of non-trivial modes are at the points $x = -\omega/E$ where $0 \le x \le L$. Therefore the total number of non trivial modes is $(T/2\pi)(EL)$. The values of ω which contribute to $\int d\omega$ can also be obtained from classical mechanics. Indeed the classical equation of motion is $md^2t/d\tau^2 = E\ dx/d\tau$ which integrates to $mdt/d\tau = Ex + \omega$ in the region $0 \le x \le L$. The turning point is given by $dt/d\tau = 1$ (i.e. by $\omega = -E(x - a^{-1})$). Therefore the modes that have turning points in the region $0 \le x \le L$ have energies in the interval

$-EL \leq \omega \leq 0$, so that for $L >> a^{-1}$ the number of modes in class I is $T/2\pi \int_{-EL}^{0} d\omega = ETL/2\pi$ [1]. Of course the same result can be obtained in the gauge $A_x = -Et$, $A_t = 0$.

From this discussion a physical picture of production emerges wherein a certain class of vacuum fluctuations (class I) in the past contains the potentiality of making pairs. This comes about because these fluctuations fall into resonance with a state which contains pairs of out-quanta. The pair production is possible because of the degeneracy of states of zero energy (see eq. (1.41)). In each case that we review in this article the same physics is repeated. Only the details of the mechanism of conversion from vacuum to physically propagating states changes from problem to problem. Moreover the selection of those fluctuations that are predestined to become physical particles is such as to give rise to a steady rate of production, the common ground being the constancy of acceleration.

The Schwinger formula [86] now follows from (1.43)

$$|\langle 0, \mathrm{out}|0, \mathrm{in}\rangle|^2 = \exp - \frac{ELT}{2\pi} \ln \left(1 + e^{-m^2\pi/E}\right) \tag{1.44}$$

where we have used (1.16) $|\beta|^2 = |T/R|^2$. Actually Schwinger worked out the case of $3+1$ dimensions. This is obtained by replacing m^2 by the transverse mass squared ($= m^2 + k_\perp^2$) and integrating over $k_\perp (\int d^2k_\perp/2\pi)$ in the exponential so as to replace $ETL/2\pi$ by $E^2TV/4\pi^2$). For the interested reader Appendix B contains a modified version of Schwinger's derivation obtained by functional integration. The mode by mode analysis given here is obviously far more detailed in revealing physical mechanisms and hence of more pedagogical value for the black hole problem.

The physical interpretation of eq. (1.44) is illuminating when one recalls that the mean number of quanta produced in the mode ω is $|\beta|^2 = \langle n_\omega \rangle$ (c.f. eq. (1.39). The argument of the exponential in eq. (1.44) is thus $\sum_\omega \ln(1 + \langle n_\omega \rangle)$, as in a partition function.

It is interesting to remark that the population ratio of produced pairs of two charged fields one of mass m and the other of mass $m + \Delta m$ is given, in

[1] It is amusing that it is precisely the same consideration that affords a very simple proof of the chiral anomaly of Fermi fields in 2 dimensions in the presence of a magnetic field H. In its Minkowski version, replace H by E and chirality ($\gamma_5 = \gamma_1\gamma_2$) by velocity ($= \gamma_0\gamma_1$). Then the change in velocity of the vacuum due to E ($= \int dx \overline{\psi}\gamma_0\gamma_1\psi$) is $(1/2\pi)\int E dx dt$. See for example [59]

the case $\Delta m << m$ and $m^2/E << 1$, by

$$\frac{\langle n\rangle_{m+\Delta m}}{\langle n\rangle_m} = e^{-2\pi\Delta m/a} \tag{1.45}$$

i.e. the ratio of the mean number of created pairs with neighbouring masses is Bolzmannian with temperature $a/2\pi$. That the physics of accelerated systems is associated with this temperature is the subject of the next chapter. From 1.45 and Section 2, we can say that particles are born in equilibrium.

1.4 Pair Production as the Source of Back Reaction

Following [16] let us examine more closely the physical situation that arises when the E field is turned off after a finite time lapse ($E \neq 0$; in the interval $0 \leq t \leq T$), wherein the (Heisenberg) state is the vacuum $|0, \text{in}\rangle$. This state, at time T, the content of which is then measured by a counter system (i.e. a measuring device sensitive to out-quanta), is expressed in eq. (1.40). It appears as a linear superposition of different outcomes according to the number and nature (mode number) of produced pairs. For example eq. (1.44) gives the probability to find no pairs and eq. (1.39) gives the mean number of pairs in a mode. These pairs produce a non vanishing expectation value for the electric current at intermediate time t:

$$\begin{aligned}\langle J_\mu(t,x)\rangle &\equiv \langle 0,\text{in}|J_\mu(t,x)|0,\text{in}\rangle \\ &= \sum_m A_m^* \langle m,\text{out}|J_\mu(t,x)|0,\text{in}\rangle\end{aligned} \tag{1.46}$$

where the second equality results from the insertion of a complete set of states in the out-basis. The coefficients A_m $(= \langle m, \text{out}|\text{in}\rangle)$ are the probability amplitudes to find the system in the state $|m, \text{out}\rangle$ at time T. For a system with many pairs, the state $|m, \text{out}\rangle$, is very complicated due to interactions among these pairs. The whole development of the previous sections has been devoted to the production of non interacting pairs and we shall continue to work in this approximation valid for sufficiently small times T or large masses m. More precisely we require

$$\frac{ET}{2\pi} e^{-\pi m^2/E}[a^{-1} + T] << 1 \quad . \tag{1.47}$$

This inequality ensures negligible overlap among the pairs produced in time T, hence negligible interaction[2]. Equation (1.47) comes about by multiplying the density of pairs ($= (ET/2\pi)e^{-\pi m^2/E}$) by their mean separation ($= a^{-1} + T$) since they are produced at separation a^{-1} whereupon the members of the pair fly apart at the speed of light.

The opposite limit of large density, where mean field approximation is valid has been the subject of detailed investigation [25] where the back reaction has been solved. Unfortunately many body effects which give rise to plasma oscillations are an important element of this development. Hence it is not a relevant laboratory for the study of black hole radiation. Therefore we shall continue with the approximation of non interacting pairs, which turns out to be more relevant to the study of black hole evaporation. When eq. (1.47) is valid, the relative fluctuations of density of pairs are large, so the mean can no longer be used. It is no longer appropriate for calculating the effect of back reaction on the production. For this reason it becomes of interest to take apart eq. (1.46) term by term.

For the case of negligible interaction the normalized states $|m, \mathrm{out}\rangle$ are direct products over pair states :

$$|m, \mathrm{out}\rangle = \prod_\omega \frac{\left[a^{out\dagger}_{p,\omega} b^{out\dagger}_{a,\omega}\right]^{n_\omega}}{n_\omega!} |0, \mathrm{out}\rangle \qquad . \tag{1.48}$$

Since J_μ is quadratic in the field operator Φ, it contains creation and annihilation operators in a double sum over modes. It is easy to check that the diagonality of the Bogoljubov transformation (1.38) not mixing different values of ω, reduces the double sum to a single one. Thus the in-vacuum expectation value $\langle J_\mu \rangle$ in eq. (1.46) is expressible as a single sum over ω. It is given by

$$\begin{aligned} \langle J_\mu(t,x)\rangle &= \frac{\langle 0, \mathrm{out}|J_\mu(t,x)|0, \mathrm{in}\rangle}{\langle 0, \mathrm{out}|0, \mathrm{in}\rangle} \\ &\quad - \sum_\omega \left(|\frac{\beta}{\alpha}|^2 \right) \left(\frac{\alpha^*}{\beta^*} \varphi^{in*}_{p,\omega}(t,x) \overleftrightarrow{\mathcal{D}}_\mu \, \varphi^{in*}_{a,\omega}(t,x) \right) \qquad , \end{aligned} \tag{1.49}$$

which may be found by either hard work from eq. (1.40) or using the identity for the Feynman propagator:

$$G_F(t,x;t',x') = \frac{\langle 0, \mathrm{out}|\Phi(t,x)\Phi^\dagger(t',x')|0, \mathrm{in}\rangle}{\langle 0, \mathrm{out}|0, \mathrm{in}\rangle}$$

[2]Recall that the pairs are one-dimensional dipoles

$$= \quad \langle 0,\text{in}|\Phi(t,x)\Phi^\dagger(t',x')|0,\text{in}\rangle + \sum_\omega \frac{\beta}{\alpha}\varphi^{in*}_{a,\omega}(t,x)\varphi^{in*}_{p,\omega}(t',x') \tag{1.50}$$

easily obtained by expressing $\Phi^\dagger$ in the in-basis, and expanding the out-vacuum state $\langle 0,\text{out}|$ up to the term quadratic in the in-annihilation operators acting on the in-vacuum :

$$\langle 0,\text{out}| = N^{-1/2}\langle 0,\text{in}|\exp\left[\left(\frac{\beta}{\alpha}\right)\sum_\omega a^{in}_{\omega,p}b^{in}_{\omega,a}\right] \tag{1.51}$$

The first term on the r.h.s. of eq. (1.49) is the current contained in the state $|0,\text{in}\rangle$ which arises when no pairs are produced. Hence it should vanish. It does. There are two proofs. The first one is related to subtraction problems in general. This method will be presented in Section 3.3 (see ref. [16] for an explicit treatment in the case of a background electric field). The other is to prove that upon expanding J_μ into modes, each term is separately zero. We leave this as an exercise to the reader (see also ref. [16]). This latter proof is completely satisfactory since the summation over the alternating series is absolutely convergent due to the finite domain of space-time $L\times T$ in which the E field is non vanishing.

The second term on the r.h.s. of eq. (1.49) is the mean current, expressed as a weighted sum over the contribution of the current carried by pairs arising from the mode ω. The weight $|\beta/\alpha|^2$ is the probability that at least one pair be produced in this mode (since $1/|\alpha|^2$ is the probability that none be produced). The c-number in parenthesis which multiplies this weight will be called a weak-value. It is related to what one measures in an experiment in which the presence of a pair is detected, thereby making contact with the weak measurement theory of Aharonov et al. [3]. In our situation, the post-selection (i.e. the detection of a localized pair) is physically performed by registering the click in a counter. More generally, whether or not the pairs are treated as independent, one may write $\langle J_\mu\rangle$ as (see eq. (1.46)

$$\langle J_\mu(t,x)\rangle = \sum_m P_m\left[\frac{\langle m,\text{out}|J_\mu(t,x)|0,\text{in}\rangle}{A_m}\right] \quad . \tag{1.52}$$

where $P_m = |A_m|^2$ is the probability to find the system in the state $|m,\text{out}\rangle$. In other words the expectation value of J_μ is a weighted sum of non diagonal matrix elements (weak values) in exactly the same way as usual conditional probabilities. The physical relevance of these matrix elements as well as the concepts of post selection and weak value are discussed in more detail at the end of this section and in Appendix C.

For the nonce, we shall enquire into the properties of these matrix elements using well localized wave packets To this end a Gaussian envelop turns out to be most convenient for the following reason. First consider the asymptotic regions. For simplicity we take a mode which is centered on the space-time origin. Then, see eqs (1.24,1.26), we obtain :

$$\begin{aligned}
\lim_{t\to-\infty}\psi_p^{in} &\simeq \int d\omega e^{-\omega^2/2\sigma^2} e^{-i\omega t} e^{-iE(x+\omega/E)^2/2} \\
&\simeq e^{-iEx^2/2} e^{-(x+t)^2/2\Sigma_+^2} \\
\lim_{t\to+\infty}\psi_p^{in} &\simeq \int d\omega e^{-\omega^2/2\sigma^2} e^{-i\omega t}\left[\alpha e^{+iE(x+\omega/E)^2/2} + \beta^* e^{-iE(x+\omega/E)^2/2}\right] \\
&\simeq \alpha e^{+iEx^2/2} e^{-(x-t)^2/2\Sigma_-^2} + \beta^* e^{-iEx^2/2} e^{-(x+t)^2/2\Sigma_+^2} \qquad (1.53)
\end{aligned}$$

where

$$\Sigma_\pm^2 = \left[\frac{1}{\sigma^2} \pm \frac{i}{E}\right] \qquad (1.54)$$

and inessential phases factors have not been taken into account. The asymptotic widths are given by $\left[\mathrm{Re}(1/\Sigma_\pm^2)\right]^{-1/2}$. If $\mathrm{Im}\sigma \neq 0$ the two outgoing branches in (1.53) will have different widths. Charge symmetry then dictates the choice $\mathrm{Im}\sigma = 0$ whereupon once sees that the width ($= \left[(E^2+\sigma^4)/E^2\sigma^2\right]^{1/2}$) is minimized by the choice $\sigma^2 = E$. In what follows, wave packets of such width will be qualified as minimal. They are not only optimally localized, but they constitute a good approximation to a complete orthogonal set. This follows from the fact that their width is $\sim E^{-1/2}$ about the classical orbit so that the number of such (asymptotically) non overlapping packets is $\mathcal{O}(ELT)$ as is required for the correct count of non trivial modes. With a little tinkering on their size and shape they could be shaped into a rigorous complete orthogonal set [66], but for our purposes the minimal packet is sufficient to show how the back reaction field emerges from a single pair.

The subsequent analysis is facilitated by using the integral representation (1.11) which from (1.9) is a Gaussian transform of $\langle u|\chi\rangle$. Hence with ξ given by (1.3) the Gaussian integral over ω is trivial and one finds once more that the packet is nothing but another Whittaker function. With $\sigma^2 = E$, one obtains

$$\psi_a^{in}(t,x) = \int_{-\infty}^{+\infty} e^{-\omega^2/2E} e^{-i\omega t} \langle \xi | \Xi \rangle \, d\omega$$
$$= \frac{e^{i\pi/4}\Gamma(1/2+i\varepsilon)}{2\pi} e^{-(x^2+t^2+i2xt)E/4} D_{-i\varepsilon-1/2}[e^{i\pi/4}\sqrt{E/2}(t-ix)] \tag{1.55}$$

and ψ_p^{in} is given by the same superposition in terms of χ_ε^* see eq. (1.26). The algebraic details are given in ref. [16]. We have plotted in Fig. 1.1 the (logaritm) of the modulus of ψ_p^{in}.

The current (1.49) has been written as a sum over modes. this can be rewritten as a sum over packets. The contribution to this sum from ψ_p^{in} and ψ_a^{in} is

$$|\alpha|^2 J_\mu^\psi(t,x) = \frac{\alpha^*}{\beta^*} \psi_p^{in*}(t,x) i \overleftrightarrow{\mathcal{D}}_\mu \, \psi_a^{in*}(t,x) \tag{1.56}$$

(The factor $|\alpha|^2$ on the l.h.s. is introduced here for subsequent physical interpretation, see eq. (1.58)). We emphasize that this current is a product of two localized wave packets. It is a contribution to the non-vanishing second term of eq. (1.49) when the set of modes is recast into a set of packets.

In Fig. 1.2 we have plotted the real and the imaginary part of J_μ^ψ. Note in particular that this product vanishes in the remote past since we start from vacuum at early times. Indeed each factor (ψ_p^{in} and ψ_a^{in} respectively) of this product is a minimal packet centered on the space-time origin and they do not overlap before they reach their common turning point. Being in-modes, one comes in from the left only and the other from the right only. This is at is should be, the current produced by the pair should not exist before it is produced.

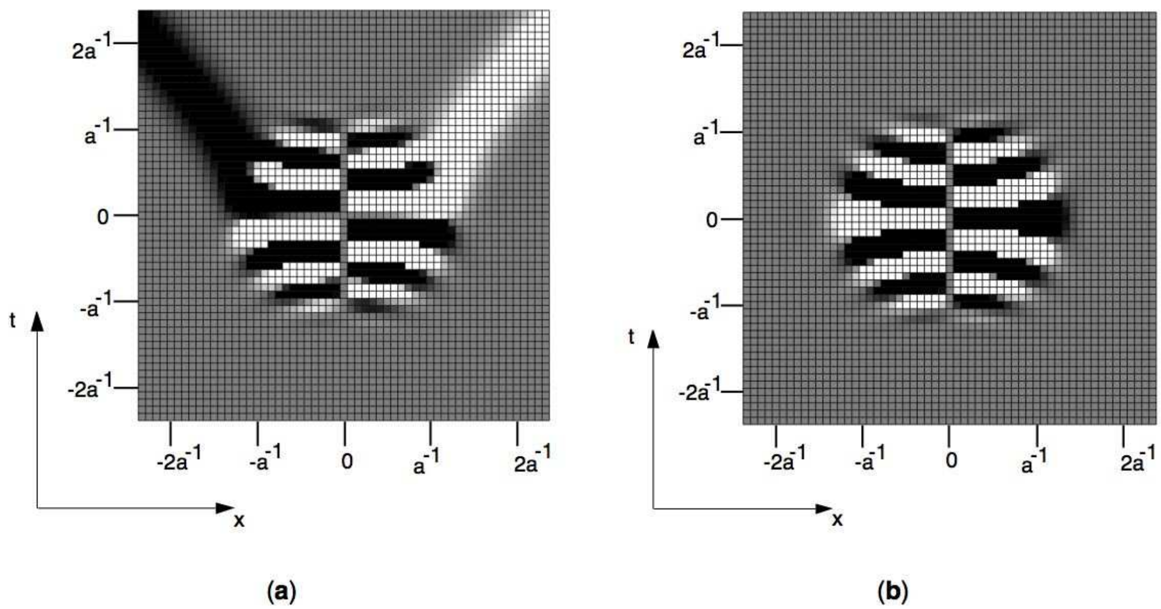

Fig. 1.2 *The real (fig.* **a***) and imaginary (fig.* **b***) parts of J_0^ψ are plotted for a minimal wave packet where we have taken $m/a = 9$. In the figure zero is grey, positive is white and negative is black. Within the production region $\Delta t \times \Delta x = 2a^{-1} \times 2a^{-1}$ the real and imaginary parts are comparable and oscillate. Outside the production region, the quanta are on mass shell and propagate classically: the imaginary part vanishes and the real part takes its classical value.*

Moreover, in the far future, from the inverse of eqs. (1.32, 1.33) taken together with the vanishing overlap of $\psi_p^{out}$and ψ_a^{out}, (where ψ_p^{out} is the same wave packet as ψ_p^{in} but made of out-modes) we find

$$\lim_{t\to+\infty} \quad |\alpha|^2 \quad \left[\frac{\psi_p^{in*}(x) i \overleftrightarrow{\mathcal{D}}_x \, \psi_a^{in*}(x)}{\alpha\beta^*}\right] =$$
$$|\alpha|^2 \quad \left[\psi_p^{out*}(x) i \overleftrightarrow{\mathcal{D}}_x \, \psi_p^{out}(x) + \psi_a^{out}(x) i \overleftrightarrow{\mathcal{D}}_x \, \psi_a^{out*}(x)\right] \quad (1.57)$$

This once more is at it should be. The future current is the sum of currents carried by the particle and anti-particle separately according to one's classical expectation. Remark on the importance of the denominator in eq. (1.49) (or more generally in eq. (1.52)), wherein $\langle m, out|$ is given by $\langle 1, \psi^{out}|$ the out state which contains only the pair described by ψ_p^{out} and ψ_a^{out}. Hence it is given by

$$\frac{\langle 1, \psi^{out}|J_\mu|0, \text{in}\rangle}{\langle 1, \psi^{out}|0, \text{in}\rangle} = \frac{1}{\alpha\beta^*}\psi_p^{in*}(t,x) i \overleftrightarrow{\mathcal{D}}_\mu \, \psi_a^{in*}(t,x) = J_\mu^\psi \tag{1.58}$$

It is the denominator $\alpha\beta^*$ that garantees that the current carried by the detected out pair is unity. The multiplicative factor $|\alpha|^2$ in eq. (1.57) is there to account for the fact that more than one pair may be produced in the packet. It is equal to the "induced emission" factor $\langle n+1\rangle$ where $\langle n\rangle \equiv |\beta|^2$ is the mean density number of pairs produced. It is not present in eq. (1.58) since in that case only one out pair characterizes the out-state.

Having understood that asymptopia goes according to rights, we note further features seen in Fig. 1.2

1) The region of production lies within a circle of radius $2/a$,

2) Within this region the contribution to J_μ is complex and it oscillates.

This is the quantum region. The oscillations occur because for x and $t \ll a^{-1}$ the modes oscillate with frequencies of $\mathcal{O}(m)$. This is seen most clearly in the gauge where $A_t = 0$, $A_x = -Et$. Then the packets are built out of $\chi_{-\varepsilon}(t + k/E)\exp(ikx)$ and these[3] oscillate near the origin with frequency $\sqrt{m^2+k^2}$. The amplitudes of these oscillations is very strong near the origin (like $\exp E(a^2 - t^2 - x^2)$) and then fade out as the particles settle down to get on to their mass shell. We are seeing how the transients work themselves out so as to arrive at the completion of a quantum event.

Note however that in the sum for $\langle J_\mu\rangle$ eq. (1.49), the imaginary part vanishes identically and furthermore, since E is constant in the box $\langle J_\mu(x)\rangle$ is homogeneous, so that the oscillations drop out as well. Thus one should question the physical relevance of these complex oscillations. We therefore conclude this section with a few remarks on this count, including a short discussion of the back reaction problem.

The simplest way to put them in evidence in a physical amplitude is by considering the change in the probability to find the ψ pair upon slightly

[3] Which are the analytic continuation from m^2 to $-m^2$ of the functions introduced in eq. (1.4).

modifying the electric field, i.e. by replacing E by $E + \delta E(x,t)$. Since the change in the action is given by $\delta S = \int dtdx J_\mu \delta A^\mu$, the change of the probability to find the ψ pair is given, in first order in $\delta E(x,t)$ by

$$\begin{aligned} P_{E+\delta E} &= |\langle 1, \psi^{out}| e^{i\delta S}|0, in\rangle|^2 \\ &= |\langle 1, \psi^{out}|(1 + i\delta S)|0, in\rangle|^2 \\ &= P_E\left(1 + 2\int dtdx \delta A^\mu(t,x) \mathrm{Im}\left[\frac{\langle 1, \psi^{out}|J_\mu|0, \mathrm{in}\rangle}{\langle 1, \psi^{out}|0, \mathrm{in}\rangle}\right]\right) \end{aligned} \quad (1.59)$$

(We recall that $P_E = |\langle 1, \psi^{out}|0, in\rangle|^2 = |\beta/\alpha|^2/N$ where N is given by eq. (1.42)). Hence it is the imaginary part of the weak current (given in eq. (1.58)) which furnishes the change of the probability when one compares two neighboring E-external fields. But given that each created pair carries an electric field these weak values will also govern the modification of multi-pair production due to current-current interaction in the usual perturbative approach. One way to proceed is to include the interaction term in the hamiltonian $\frac{1}{2}\int\int dxdx' J_t(x) v(x - x') J_t(x')$ where v is the Coulomb potential. It can be shown that once one pair is produced, the electric field which acts on fluctuations to give rise to subsequent pairs is of the form $E + \int dx' J_t^\psi(x')$ where $J_t^\psi(x')$ is the matrix element of the charge density due to the pair which has been created. The self interaction of a pair as it is produced is given by a more complicated loop correction. The statement of the back reaction problem is thus given in terms of a self consistent highly non trivial problem. One will then be led to treat the complete wave function of the system as a function of the configuration of both the matter and electric field. This will then give a generalization of the usual semi-classical theory [$\partial_\mu F^{\mu\nu} = \langle \mathrm{in}|J^\nu|\mathrm{in}\rangle$, wherein $F^{\mu\nu}$ on the l.h.s. is classical, but modified due to the mean of J^ν], to a full quantum Heisenberg equation among operators $\partial_\mu F^{\mu\nu} = J^\nu$.

The hard job then remains. How do these individual production acts affect subsequent production. This is the true feed-back problem and it has not been solved. Current research is now under way to find an approximate solution which is better than the mean field approximation (see for instance ref. [26]) wherein fluctuations and correlations among pairs are taken into account.

Another way to understand the status of the matrix elements eq. (1.58) and the relevance of the complex oscillations is by appealing to the formalism developed by Aharanov et al [3]. In this formalism, it is shown how those matrix elements give the result of a "weak measurement". This development contains two elements that we briefly expose. The interested reader will consult Appendix C as well.

- A weak detector, a quantum device in weak interaction with the system, where weak means that one can correctly work in first order in the interaction. The detector wave function is spread out so that first order perturbation theory is valid. Then one picks up not only the detector's displacement (in position) as in the von Neuwmann model of measurement [95], but one finds also that the imaginary part of the matrix element (here of J_μ) imparts momentum to the detector. For the case we discuss here the detector could be for example a test charge whose path is deflected by the external E field, modified by the back reaction field (induced by the weak value of the current) due to the selected pair, i.e. the modification of the E field is given by Gauss' law: $\Delta E(x,t) = \int^x dx' J_t^\psi(x',t)$, where $J_t^\psi(x',t)$ is the weak value of the current. Its statistical interpretation in the context of measurement theory is given below.

- A post selection, the specification of the final state of the system, here the state $|m, \mathrm{out}\rangle$. In the context of our wave packet development with no interaction among the pairs a possible out state is the one post-selected by the click of a localized counter. One selects, thus, that part of the wave function $|0,\mathrm{in}\rangle$ which causes the counter to click. Physically this post selection consists of registering all results of the weak detector (as one usually does to establish mean values, in the manner prescribed by the Copenhagen interpretation of quantum mechanics) and then to keep only the results when the particular localized click is registered (as one does in constructing conditional probabilities).

Chapitre 2

Accelerating Systems

2.1 The Accelerated Detector

A system in constant acceleration through empty Minkowski space, coupled to a field whose state is quantum vacuum in Minkowski space, will heat up and its internal degrees of freedom will become thermally distributed with a temperature given by

$$T = \beta^{-1} = a/2\pi \qquad . \tag{2.1}$$

More succinctly : an accelerated detector perceives the vacuum as a thermal bath. This remarkable observation is due to Unruh [91], who thereby gave substance to an equally remarkable observation by Fulling [35] who showed that quantization of a field in Rindler coordinates [81] is inequivalent to the usual Minkowski quantization.

The definition of Rindler coordinates ρ, τ in $1+1$ dimensions is

$$\begin{array}{c} t = \rho \ \mathrm{sinh} a\tau \quad , \quad x = \rho \ \mathrm{cosh} a\tau \\ \rho > 0 \quad , \quad -\infty < \tau < +\infty \end{array} \tag{2.2}$$

hence applicable to the quadrant $t > 0$, $x > 0$, hereafter referred to as the right quadrant (R). These coordinates are naturally associated to a uniformly accelerated system with acceleration a in the sense that its trajectory is $\rho = a^{-1} = const$; its proper time is τ. In addition the Minkowski metric expressed in coordinates ρ, τ

$$ds^2 = -dt^2 + dx^2 = -\rho^2 a^2 d\tau^2 + d\rho^2 \tag{2.3}$$

is static, translations in τ (boosts) being Lorentz transformations.

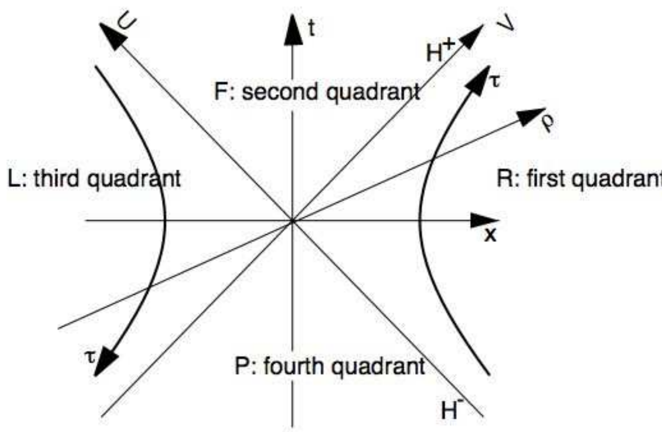

Fig. 2.1 *The coordinate system (ρ, τ), the trajectory of a uniformly accelerated detector, the four quadrants of Minkowski space and the horizons $H_{\pm}$ (U=0,V=0) which separate them.*

The remarkable result, now stated with more precision is that Minkowski vacuum is described in R (or alternatively in the left quadrant (L) wherein $t = -\rho\ \sinh a\tau$, $x = -\rho\ \cosh a\tau$, see Fig.2.1.) by a thermal bath in the quantization scheme based on the coordinates ρ, τ defined by eq. (2.2). The connection between the remarks of Unruh and Fulling is through the proper local dynamics of the accelerator when it is coupled to the radiation field so as to undergo transitions. Conventional emission and absorption is then given in terms of Rindler quanta

The easiest way to see this phenomenon is by looking at rates of absorption and emission of an accelerated two level detector. One may for example take a two level ion whose center of mass is described by the wave function in the W.K.B. approximation to the minimal wave packet of Section 1.4. The type of idealization envisioned is to let the mass of the ion tend to infinity at fixed acceleration, a, i.e. $M \to \infty$, $E \to \infty$, $E/M \to a$. The packet then describes a δ function along a classical orbit whose center is used to define the origin of space-time

$$x^\mu x_\mu = -t^2 + x^2 = a^{-2}$$
$$\begin{cases} t = a^{-1}\mathrm{sinh} a\tau \\ x = a^{-1}\mathrm{cosh} a\tau \end{cases} \tag{2.4}$$

(Acceleration is constant in the sense that $a_\mu a^\mu = a^2$ where $a^\mu = d^2x^\mu/d\tau^2 = a^2 x^\mu$ with τ the proper time on the path of the accelerator, i.e. $u_\mu u^\mu = (dx^\mu/d\tau)(dx_\mu/d\tau) = -1$.) The two levels are separated by the mass difference ΔM, taken to be finite.

We take the example of a massless scalar field (here chosen hermitian) in $1+1$ dimension, coupled to the two level accelerating detector [91],[28],[31]. In the interaction representation, the interacting hamiltonian of the system is

$$\begin{aligned} H &= g\left[e^{i\Delta M\tau}\sigma_+ + e^{-i\Delta M\tau}\sigma_-\right]\phi(x^\mu(\tau)) \\ &= g\int \frac{dk}{\sqrt{4\pi\omega}}\left[e^{i\Delta M\tau}\sigma_+ + e^{-i\Delta M\tau}\sigma_-\right]\left[e^{-ik_\mu x^\mu(\tau)}a_k + e^{ik_\mu x^\mu(\tau)}a_k^\dagger\right] \end{aligned} \tag{2.5}$$

where $\sigma_\pm$ are the operators which send the system from ground (excited) state to excited (ground) state and where $k_0 = \omega = |k|$,$k_1 = k$ and $a_k^\dagger$ (a_k) are the creation (destruction) operators of field quanta of momentum k.

Standard golden rule physics gives in lowest order perturbation theory the following formula for the rates expressed in terms of the proper time τ, of absorption (–) and emission (+)

$$R_\mp = g^2\int_{-\infty}^{+\infty} d\Delta\tau\ e^{\mp i\Delta M\Delta\tau} W(\Delta\tau - i\epsilon) \tag{2.6}$$

where W is the Wightman function between two points on the orbit separated by proper time $\Delta\tau$. For the reader unfamiliar with this formula we sketch a brief derivation

Take the case of absorption, then we begin at $\tau = 0$ with the atom in its ground state and the radiation field in vacuum. The amplitude to find it in the excited state is the matrix element of $-i\int_0^\tau H_{int}(\tau')d\tau'$. The probability is obtained by squaring the amplitudes and by integrating over all k and is therefore equal to

$$g^2\int_0^\tau d\tau_2\int_0^\tau d\tau_1\ e^{-i\Delta M(\tau_2-\tau_1)}\left[\langle 0_M|\varphi\left(x^\mu(\tau_2)\right)\varphi\left(x^\mu(\tau_1)\right)|0_M\rangle\right] \tag{2.7}$$

where $|0_M\rangle$ is Minkowski vacuum. For emission change the phase $e^{-i\Delta M(\tau_2-\tau_1)}$ to $e^{+i\Delta M(\tau_2-\tau_1)}$. The quantity in brackets is called the Wightman function $W(\tau_2,\tau_1)$ and the positivity of the Minkowski frequencies (ω) encoded in its decomposition into annihilation and creation operators impose an integration prescription in the complex plane given by $i\epsilon$ in eq. (2.6)

The Wightman function for the case of a massless field in $1+1$ dimension is $-(4\pi)^{-1}\ln\left[(\Delta t - i\epsilon)^2 - \Delta x^2\right)]$ where for constant acceleration eq. (2.4), we have

$$\begin{aligned}(\Delta t - i\epsilon)^2 - \Delta x^2 &= a^{-2}\left[(\mathrm{sinh} a\tau_2 - \mathrm{sinh} a\tau_1 - i\epsilon)^2 - (\mathrm{cosh} a\tau_2 - \mathrm{cosh} a\tau_1)^2\right] \\ &= 4a^{-2}\mathrm{sinh}^2 a(\Delta\tau - i\epsilon)/2 \qquad (2.8)\end{aligned}$$

Therefore for this case, as well as for the case where the path is inertial ($u^\mu =$ const), the integrand in eq. (2.7) is a function of $\tau_2 - \tau_1$ only[1]. Take τ in eq. (2.7) such that $\tau >> \mathrm{Max}(\Delta M^{-1}, a^{-1})$. [The usual condition to establish the golden rule is $\tau >> \Delta M^{-1}$ (for spontaneous emission) and $\tau >> \beta$ (for absorption of photons from a thermal bath). In the case of uniform acceleration this implies $\tau >> a^{-1}$] One may change variables to $(\tau_2+\tau_1)/2$ and $\tau_2 - \tau_1 = \Delta\tau$. Since the integral comes from the finite region around $\tau_2 - \tau_1 = O(\Delta M^{-1})$, the integral over the latter may have its limits extended to $\pm\infty$. (For $(\tau_2+\tau_1)/2 \leq \Delta M$ a small error is made which does not contribute to the rate.) Then integration over $(\tau_2+\tau_1)/2$ gives τ thereby yielding a rate formula per unit propertime. The rate is given by eq. (2.6). The reader unfamiliar with this formalism will check out the usual golden rule in terms of density of states for the inertial case. Also he will relate eq. (2.6) to the imaginary part of the self energy.

Substituting for W in eq. (2.6) we have

$$\begin{aligned}R_\mp &= -g^2\frac{1}{4\pi}\int_{-\infty}^{+\infty} d\tau\ e^{\mp i\Delta M\tau}\ln\left[\mathrm{sinh}^2\left(a\frac{\tau}{2} - i\epsilon\right)\right] \\ &= \pm ig^2\frac{a}{4\pi}\int_{-\infty}^{+\infty} d\tau\ \frac{e^{\mp i\Delta M\tau}}{\Delta M}\frac{\mathrm{cosh}\frac{a}{2}\tau}{\mathrm{sinh}\left(\frac{a}{2}\tau - i\epsilon\right)} \qquad (2.9)\end{aligned}$$

where we have integrated by parts accompanied by the usual prescription of setting infinite oscillating functions to zero. We remark that in $3+1$ dimensions the same type of formula obtains with W given by $a^2\mathrm{sinh}^{-2}(a\tau/2 - i\varepsilon)$.

[1] The origin of this miracle is that both kinds of trajectories are orbits of the Lorentz group. In the accelerated case τ translations are boosts.

From eq. (2.9), it is seen that for the minus sign (case of absorption), complexifying τ and extending the domain of integration to a closed contour in the lower half complex plane picks up poles at $a\tau = 2\pi n i$ ($n = -1, -2, \ldots$). For the plus sign (emission) the contributing poles lie in the upper half plane, at $a\tau = +i\epsilon$ and $a\tau = 2\pi n i$ ($n = 1, 2, \ldots$) whence the ratio

$$\frac{R_-}{R_+} = \frac{\sum_{n=1}^{\infty} e^{-\beta \Delta M n}}{\sum_{n=0}^{\infty} e^{-\beta \Delta M n}} = e^{-\beta \Delta M} \tag{2.10}$$

with $\beta = 2\pi/a$. Since R_- is proportional to $\langle n \rangle$ and R_+ to $\langle n \rangle + 1$ one obtains, following Einstein's famous argument, $\langle n \rangle = (e^{\beta \Delta M} - 1)^{-1}$ as in a thermal bath. Explicit evaluation yields

$$\begin{aligned} R_- &= \frac{g^2}{\Delta M}(e^{\beta \Delta M} - 1)^{-1} \qquad , \\ R_+ &= \frac{g^2}{\Delta M}(1 - e^{-\beta \Delta M})^{-1} \end{aligned} \tag{2.11}$$

Note that these rates coincide exactly with the rates given in an inertial thermal bath for the model considered. This occurs for massless fields in 2 and 4 dimensions only.

Equation (2.10), in fact, results from a general property of the propagator W appearing in eq. (2.6) related to the periodicity of the orbit eq. (2.4) when τ is imaginary, hence transforming the orbit to a circle. To see how this is related to thermal properties we first present the thermal Wightman function for a general system wherein H may include interaction of the ϕ field with itself

$$W_{\text{thermal}} = \text{tr}\left[e^{-\beta H} \phi(t_1) \phi(t_2)\right] \tag{2.12}$$

$$= \text{tr}\left[e^{-\beta H} \phi(\Delta t) \phi(0)\right] \tag{2.13}$$

$$= \text{tr}\left[e^{-H(\beta - i\Delta t)} \phi(0) e^{-iH\Delta t} \phi(0)\right] \tag{2.14}$$

where eqs.(2.13) and (2.14) follow from $\phi(t) = e^{iHt}\phi(0)e^{-iHt}$. Because of the positivity of the spectrum of H (i.e. $E_n - E_0 \geq 0$), $W_{\text{thermal}}(\Delta\tau)$ is defined by eq. (2.14) in the strip in the complex plane

$$-\beta + \epsilon < \text{Im}\Delta\tau < -\epsilon \tag{2.15}$$

In this strip one has the identity

$$W_{\text{thermal}}(t) = W_{\text{thermal}}(-i\beta - t) \tag{2.16}$$

obtained by using the cyclic invariance of the trace. Under general conditions W_{thermal} is analytic in the strip. [For a general discussion of thermal Green's functions see ref. [37]].

Analyticity and eq. (2.16) then imply eq. (2.10). This is derived by integrating $\oint dz\ e^{i\Delta M z} W_{\text{thermal}}(z) = 0$ over the contour surrounding the strip (2.15), and we assume that there is no contribution from the ends at Re$z = \pm\infty$. The integral along Im$z = -\epsilon$ is then $-R_+$ and along Im$z = -\beta + \epsilon$ is $e^{\beta\Delta M} R_-$ thereby recovering eq. (2.10).

The point of all this is to note that W for the accelerating system enjoys the property (2.16) since it is a function of $(\Delta x^\mu)^2 = -4a^{-2}\sinh^2 a\Delta\tau/2$ and this is true not only for a free field but in general. Thus the ratio of the accelerating detector rates of absorption and emission is given by eq. (2.10) simply in consequence of the imaginary periodicity of the orbit. We emphasize that it is not necessary that $W(\Delta\tau)$ be equal to an inertial thermal propagator eq. (2.12) and indeed for free field theory one may check that apart from $d = 2$ and $d = 4$ it is not since the density of Rindler modes of energy ΔM differs from the inertial one. However eq. (2.16) and therefore eq. (2.10) always obtains.

2.2 Quantization in Rindler Coordinates

We now show how it is possible to interpret these results in terms of annihilation and creation of Rindler quanta (Rindlerons) in situ. Rindler coordinates are defined by eq. (2.2) in the first quadrant and it is in this quadrant where we situate the accelerator. We make this point somewhat more explicit. The transformation eq. (2.2) is analogous to euclidean polar coordinates, with the rôle of the angle being played by $a\tau$ (hence Imτ has period $2\pi/a$). However whereas euclidean space is completely covered by polar coordinates, Minkowski space is only partially covered in one quadrant: $x > 0$, $x > |t|$ designated in Fig. 2.1 by R.

The d'Alembertian $-\partial_t^2 + \partial_x^2$ is $[-\rho^{-2}a^{-2}\partial_\tau^2 + \rho^{-1}\partial_\rho \rho \partial_\rho]$ so that for massless particles the modes in these coordinates are solutions of

$$\left[-\frac{1}{a^2}\frac{\partial^2}{(\partial\tau)^2} + \frac{\partial^2}{\left[\partial(\ln a\rho)\right]^2} \right] \phi = 0 \tag{2.17}$$

In R, a complete set of eigenmodes of $i\partial_\tau$, solutions of eq. (2.17), is

$$\left.\begin{array}{r}\varphi_{\lambda,R}(u)=\theta(-U)e^{-i\lambda u}/\sqrt{4\pi\lambda}=\theta(-U)(-aU)^{i\lambda/a}/\sqrt{4\pi\lambda}\\ \tilde{\varphi}_{\lambda,R}(v)=\theta(V)e^{-i\lambda v}/\sqrt{4\pi\lambda}=\theta(V)(aV)^{-i\lambda/a}/\sqrt{4\pi\lambda}\end{array}\right. \quad , \ \lambda>0 \tag{2.18}$$

taken together with their complex conjugates. Here

$$\left.\begin{array}{c}u\\ v\end{array}\right\} \;=\; \tau \mp a^{-1}\ln a\rho \tag{2.19}$$

$$\left.\begin{array}{c}U\\ V\end{array}\right\} \;=\; t\mp x=\left\{\begin{array}{l}-a^{-1}e^{-au}\\ a^{-1}e^{av}\end{array}\right. \tag{2.20}$$

The modes are normalized according to the Klein Gordon norm which for u modes can be written as

$$\int_{-\infty}^{+\infty} du\ \varphi^*_{\lambda',R}\ i\overleftrightarrow{\partial_u}\ \varphi_{\lambda,R}=\delta(\lambda-\lambda') \tag{2.21}$$

and similarily for v. We note that even though u, v are defined in R only by eq. (2.19), by trivial extension $\varphi_{\lambda,R}(u)$ is also defined in P since u is finite on the past horizon $V=0$. Similarly $\tilde{\varphi}_{\lambda,R}(v)$ is defined as well in F as well as R.

In L, the corresponding complete set of modes of positive Rindler frequency is

$$\left.\begin{array}{r}\varphi_{\lambda,L}(u_L)=\theta(U)e^{+i\lambda u_L}/\sqrt{4\pi\lambda}=\theta(U)(aU)^{-i\lambda/a}/\sqrt{4\pi\lambda}\\ \tilde{\varphi}_{\lambda,L}(v_L)=\theta(-V)e^{+i\lambda v_L}/\sqrt{4\pi\lambda}=\theta(-V)(-aV)^{+i\lambda/a}/\sqrt{4\pi\lambda}\end{array}\right. \quad , \ \lambda>0 \tag{2.22}$$

taken together with their complex conjugates where in L we have

$$\begin{array}{c}t=-\rho\mathrm{sinh}a\tau\\ x=-\rho\mathrm{cosh}a\tau\end{array} \tag{2.23}$$

and u_L, v_L in L are also given in terms of τ, ρ by eq. (2.19). Hence in L

$$\left.\begin{array}{c}U\\ V\end{array}\right\} \;=\; t\mp x=\left\{\begin{array}{l}a^{-1}e^{-au_L}\\ -a^{-1}e^{av_L}\end{array}\right. \quad . \tag{2.24}$$

Once more, by extension $\varphi_{\lambda,L}(u_L)$ is valid in L and F and $\tilde{\varphi}_{\lambda,L}(v_L)$ is valid in L and P. Note that in L, $dt/d\tau < 0$ whereas in R, $dt/d\tau > 0$. Therefore the modes (2.22) have Rindler frequency opposite to the usual one. The convenience of the convention (2.23) is that a given boost is represented by the same displacement in τ in both L and R. In addition the mapping of R into L by reflection through $x = 0,\ t = 0$ is very simply described by the analytic continuation $\tau \to \tau + i\pi/a$, $u \to u + i\pi/a = u_L$, $v \to v + i\pi/a = v_L$ which maps (2.2) into (2.23) and (2.20) into (2.24) respectively. If one performs this inversion twice one obtains that R is invariant under $\tau \to \tau + 2i\pi/a$ which is the essential ingredient used to obtain the thermal properties in the discussion following eq. (2.11).

These sets of modes are the Rindler versions of the Minkowski light-like modes

$$\begin{aligned} \xi_\omega(U) &= e^{-i\omega U}/\sqrt{4\pi\omega} \\ \tilde{\xi}_\omega(V) &= e^{-i\omega V}/\sqrt{4\pi\omega} \end{aligned} \tag{2.25}$$

and their conjugates.

The advantage of the use of these light-like variables is that the right movers (functions of u or U) and left movers (functions of v or V) do not mix under Bogoljubov transformations (e.g. $\varphi_{\lambda,R}(u)$ is a linear combination of $\xi_\omega(U)$ only). For the remainder of this section we shall work with the right movers only. The left follow suit.

We have two representations of the right moving part of the field

$$\begin{aligned} \phi(U) &= \int_0^\infty d\omega \, [a_\omega \xi_\omega(U) + \text{h.c.}] \\ &= \theta(-U) \int_0^\infty d\lambda \, [c_{\lambda,R}\varphi_{\lambda,R}(U) + \text{h.c.}] \\ &\quad + \theta(U) \int_0^\infty d\lambda \, [c_{\lambda,L}\varphi_{\lambda,L}(U) + \text{h.c.}] \end{aligned} \tag{2.26}$$

To find the Bogoljubov transformation in R we express $\xi_\omega(U)$ as linear combination of $\varphi_{\lambda,R}(U)$

$$\theta(-U)\xi_\omega(U) = \int_0^\infty d\lambda \ \left[\alpha^R_{\lambda\omega}\varphi_{\lambda,R}(U) + \beta^R_{\lambda\omega}\varphi^*_{\lambda,R}(U)\right] \tag{2.27}$$

$$\alpha^R_{\lambda\omega} = \int_{-\infty}^0 dU \ \varphi^*_{\lambda,R}(U) \ i\overleftrightarrow{\partial_U} \ \xi_\omega(U)$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} du\ \varphi^*_{\lambda,R}(u)\ i\overleftrightarrow{\partial_u}\ \xi_\omega(U(u)) \\
&= \frac{1}{2\pi}\frac{\sqrt{\lambda}}{\sqrt{\omega}}\int_{-\infty}^{0} dU(-aU)^{-i\lambda/a-1}e^{-i\omega U} \\
&= \frac{1}{2\pi a}\sqrt{\frac{\lambda}{\omega}}\Gamma(-i\lambda/a)\left(\frac{a}{\omega}\right)^{-i\lambda/a} e^{\pi\lambda/2a} \\
\text{and}\quad \beta^R_{\lambda\omega} &= \int_{-\infty}^{+\infty} du\ \varphi_{\lambda,R}(u)(-i\overleftrightarrow{\partial_u})\xi_\omega(U(u)) \\
&= \frac{1}{2\pi a}\sqrt{\frac{\lambda}{\omega}}\Gamma(i\lambda/a)\left(\frac{a}{\omega}\right)^{i\lambda/a} e^{-\pi\lambda/2a} \qquad (2.28)
\end{aligned}
$$

In terms of operators eq. (2.26) gives the Bogoljubov transformation

$$
c_{\lambda,R} = \int_0^\infty d\omega \left[\alpha^R_{\lambda\omega}a_\omega + \beta^{R*}_{\lambda\omega}a^\dagger_\omega\right] \qquad (2.29)
$$

From the explicit values (2.28), or more simply from their integral representations, or more formally from the canonical commutation relations of $c_{\lambda,R}$ and $c^\dagger_{\lambda,R}$ one checks out the completeness of ξ_ω

$$
\begin{aligned}
\int_0^\infty d\omega\ \left[\alpha^R_{\lambda\omega}\alpha^{R*}_{\lambda'\omega} - \beta^{R*}_{\lambda\omega}\beta^R_{\lambda'\omega}\right] &= \delta(\lambda-\lambda') \quad , \\
\int_0^\infty d\omega\ \left[\alpha^R_{\lambda\omega}\beta^{R*}_{\lambda'\omega} - \beta^{R*}_{\lambda\omega}\alpha^R_{\lambda'\omega}\right] &= 0 \quad . \qquad (2.30)
\end{aligned}
$$

Of course, the set $\varphi_{\lambda,R}$ is complete only in R. For complete completeness its complement $\varphi_{\lambda,L}$, as expressed in eq. (2.26), is required. Then the operators a_ω may be expressed as linear combinations of $c_{\lambda,R}$ and $c_{\lambda,L}$. So the totality of Bogoljubov transformations is eq. (2.29), its analog in L (wherein one has the simple rule $\alpha^L_{\lambda\omega} = \alpha^{R*}_{\lambda\omega}$ and $\beta^L_{\lambda\omega} = \beta^{R*}_{\lambda\omega}$), and the inverse given by

$$
a_\omega = \int_0^\infty d\lambda\ \left[\alpha^{R*}_{\lambda\omega}c_{\lambda,R} + \alpha^{L*}_{\lambda\omega}c_{\lambda,L} - \beta^R_{\lambda\omega}c^\dagger_{\lambda,R} - \beta^L_{\lambda\omega}c^\dagger_{\lambda,L}\right] \quad . \qquad (2.31)
$$

To calculate the Rindler content of Minkowski vacuum we use eq. (2.29) to obtain the mean number of Rindler particles in Minkowski vacuum

$$
\begin{aligned}
\langle 0_M|c^\dagger_{\lambda R}c_{\lambda' R}|0_M\rangle &= \int_0^\infty d\omega\beta^R_{\lambda\omega}\beta^{R*}_{\lambda'\omega} = \delta(\lambda-\lambda')n(\lambda) \\
n(\lambda) &= (e^{\beta\lambda}-1)^{-1} \qquad (2.32)
\end{aligned}
$$

where we have used 2.28 and $|\Gamma(ix)|^2 = \pi/x\sinh\pi x$ and $n(\lambda)$ is the Planck distribution. One finds also

$$\langle 0_M | c_{\lambda R} c_{\lambda' R} | 0_M \rangle = \int_0^\infty d\omega \alpha^R_{\lambda\omega} \beta^{R*}_{\lambda'\omega} = 0 \tag{2.33}$$

In this way one understands the thermal propagator W of eq. (2.6). Indeed since W has been calculated only in R, the accelerator perceives Minkowski vacuum in R (to which it is confined) as a thermal bath (of Rindlerons since in the proper frame of the accelerator it is they which are in resonance with the excitations of the accelerator).

2.3 Unruh Modes

There is an interesting and powerful technique of handling the Bogoljubov transformation 2.28 which is due to Unruh [91]. It couples together Rindler modes which lie in R and L respectively ($\varphi_{\lambda,R}$ and $\varphi_{\lambda,L}$) so as to yield a density matrix description for what happens in one quadrant when one traces over the states in the other. This technique makes contact with the physics of pair production in a constant electric field and finds important use in the black hole problem. It is implemented by inverting 2.27 (using 2.30) to give

$$\varphi_{\lambda,R}(U) = \int_0^\infty d\omega \left[\alpha^{R*}_{\omega\lambda} \xi_\omega(U) - \beta^{R*}_{\omega\lambda} \xi^*_\omega(U) \right] \tag{2.34}$$

Though the inversion has been carried out in R, eq. (2.34) can be extended into L (since the $\xi_\omega(U)$ are defined there as well). It will be checked out below that this is perfectly consistent since we will find automatically that in this continuation the right hand side of eq. (2.34) vanishes in L. We rewrite eq. (2.34) in the form

$$\varphi_{\lambda,R} = \alpha_\lambda \hat{\varphi}_\lambda(U) - \beta_\lambda \hat{\varphi}^*_{-\lambda}(U) \tag{2.35}$$

where

$$\begin{aligned} \hat{\varphi}_\lambda(U) &= \frac{1}{\alpha_\lambda} \int_0^\infty d\omega \; \alpha^{R*}_{\omega\lambda} \; \xi_\omega(U) \\ \hat{\varphi}^*_{-\lambda}(U) &= \frac{1}{\beta_\lambda} \int_0^\infty d\omega \; \beta^{R*}_{\omega\lambda} \; \xi^*_\omega(U) \end{aligned} \tag{2.36}$$

and

$$
\begin{aligned}
\alpha_\lambda &= \frac{1}{\sqrt{1-e^{-\beta\lambda}}} \\
\beta_\lambda &= \frac{1}{\sqrt{e^{\beta\lambda}-1}}
\end{aligned}
\tag{2.37}
$$

so that the modes $\hat{\varphi}_\lambda$ are normed in the usual way, i.e. $\int_{-\infty}^{+\infty} dU \hat{\varphi}^*_\lambda(U)\ i\overleftrightarrow{\partial_U}$ $\hat{\varphi}_{\lambda'}(U) = \delta(\lambda-\lambda')$. Substituting eq. (2.28) into eq. (2.36) gives

$$
\begin{aligned}
\hat{\varphi}_\lambda &= \frac{1}{\alpha_\lambda}\frac{1}{2\pi a}\sqrt{\frac{|\lambda|}{4\pi}} e^{\pi\lambda/2a}\Gamma(i\lambda/a)\int_0^\infty \frac{d\omega}{\omega}\left(\frac{\omega}{a}\right)^{-i\lambda/a} e^{-i\omega U} \\
&= \begin{cases} \alpha_\lambda\theta(-U)(-aU)^{i\lambda/a}/\sqrt{4\pi\lambda} + \beta_\lambda\theta(U)(aU)^{i\lambda/a}/\sqrt{4\pi\lambda} & \lambda > 0 \\ \beta_{|\lambda|}\theta(-U)(-aU)^{i\lambda/a}/\sqrt{4\pi\lambda} + \alpha_{|\lambda|}\theta(U)(aU)^{i\lambda/a}/\sqrt{4\pi\lambda} & \lambda < 0 \end{cases} \\
&= \begin{cases} \alpha_\lambda\varphi_{\lambda,R} + \beta_\lambda\varphi^*_{\lambda,L} & \lambda > 0 \\ \beta_{|\lambda|}\varphi^*_{|\lambda|,R} + \alpha_{|\lambda|}\varphi_{|\lambda|,L} & \lambda < 0 \end{cases}
\end{aligned}
\tag{2.38}
$$

We shall call the modes, $\hat{\varphi}_\lambda$, Unruh modes. As announced the linear combination (2.35) does vanish in L.

Unruh modes enjoy the following properties:

1) They are eigenfunctions of $iaU\partial_U$ with eigenvalue λ in both R and L.

2) $\hat{\varphi}_\lambda$ are manifestly positive Minkowski frequency modes for both signs of λ (c.f. eqs (2.36)). Together with their conjugates they form a complete orthogonal set just as plane waves. This is proven trivially by direct computation.

3) They are linear combinations of the Rindler modes φ^R_λ and φ^L_λ given by the Bogoljubov linear combination (2.38) (we remind the reader once more that the mode $U^{i\lambda/a}\theta(U), \lambda > 0$ is a negative frequency mode in L (eq. (2.22)).

An independent and interesting derivation of eq. (2.38) is obtained by appeal to analyticity in the lower half U plane. This is because the Minkowski modes eq. (2.25) are analytically extendable in the lower half plane for positive frequencies only. This expresses the stability of the ground state (Minkowski vacuum) of the theory. Eigenfunctions of $i\partial_u$ considered as functions of U obey the differential equation

$$
-iU\partial_U\chi_\lambda = \frac{\lambda}{a}\chi_\lambda \tag{2.39}
$$

having solutions

$$\begin{aligned}\chi_\lambda &= A\left[\theta(-U)\frac{(-U)^{i\lambda/a}}{\sqrt{4\pi\lambda}}\right] + B\left[\theta(U)\frac{U^{i\lambda/a}}{\sqrt{4\pi\lambda}}\right] \\ &= A\varphi_{\lambda,R} + B\varphi^*_{\lambda,L} \end{aligned} \tag{2.40}$$

(Note the similarity to eq. (1.10). This is no accident. Any problem with an exponential approach to a horizon in the classical theory will be reflected in such a singular differential equation when expressed in terms of global coordinates.)

To determine A and B in eq. (2.40) we require that χ_λ be a positive frequency Minkowski mode i.e. an Unruh function. Therefore set U equal to $|U|e^{i\theta}$ with $-\pi \leq \theta \leq 0$ so as to continue the function $U^{i\lambda/a}$ analytically in the lower half complex plane to go from L to R. The boundaries at $\theta = -\pi, 0$ then give the ratio $B/A = e^{-\pi\lambda/a} = e^{-\beta\lambda/2}$ (compare with eq. (1.16). Normalization then gives $\chi_\lambda = \hat{\varphi}_\lambda$. This technique has been fruitfully used by Unruh [91] and Damour and Ruffini [27] in the black hole problem.

The Unruh modes can then be synthetically written as

$$\hat{\varphi}_\lambda = \frac{1}{\sqrt{4\pi\lambda(e^{\beta\lambda}-1)}}\left(\frac{U-i\epsilon}{a}\right)^{i\lambda/a} \tag{2.41}$$

with the $i\epsilon$ encoding how the function should be continued in the complex plane ($U - i\epsilon = U$ for $U > 0$ and $U - i\epsilon = |U|e^{-i\pi}$ for $U < 0$). This expression for $\hat{\varphi}_\lambda$ has the added luster that for small but finite ϵ the high frequency behavior of the modes in the vicinity of $U = 0$ has been regularized [71]. This will be seen to have important consequences when evaluating energy densities near and on the horizon, see Section 2.6.

The quantized field ϕ can be decomposed in Unruh modes everywhere according to

$$\phi = \int_{-\infty}^{+\infty} d\lambda \ \left[\hat{a}_\lambda\hat{\varphi}_\lambda + \text{h.c.}\right] \quad . \tag{2.42}$$

The bogoljuobov transformation among Rindler and Unruh annihilation and creation operators is

$$\begin{array}{l} \hat{a}_\lambda = \alpha_\lambda c_{\lambda,R} - \beta_\lambda c^\dagger_{\lambda,L} \\ \hat{a}^\dagger_{-\lambda} = \alpha_\lambda c^\dagger_{\lambda,L} - \beta_\lambda c_{\lambda,R} \end{array} \quad , \ \lambda > 0 \tag{2.43}$$

where we have used eq. (2.38).

In this way Minkowski vacuum can be viewed as a linear combination of pairs of L and R Rindlerons. Defining $|0_{Rindler}\rangle$ as the direct product $|0_R\rangle|0_L\rangle$ of Rindler vacuum in the left and right quadrant ($c_{\lambda,L}|0_L\rangle = 0$ and $c_{\lambda,R}|0_R\rangle = 0$), we have following eq. (1.40)

$$|0_M\rangle = Z^{-1/2}\exp\left[\sum_{\lambda=0}^{\infty}\frac{\beta_\lambda}{\alpha_\lambda}c^\dagger_{\lambda,R}c^\dagger_{\lambda,L}\right]|0_{Rindler}\rangle \tag{2.44}$$

where Z is given by (see eq. (1.42))

$$Z = \prod_\lambda|\alpha_\lambda|^2 = \exp\left[\frac{L}{2\pi}\int_0^\infty d\lambda\ \ln(1+\beta_\lambda^2)\right] = \exp\left[L\frac{\pi}{12}\frac{1}{\beta}\right] \tag{2.45}$$

which is the partition function of a massless gaz in 1+1 dimensions in a volume of size L. Thus for an operator O_R localized in R one then has

$$\begin{aligned}
\langle O_R\rangle &= Z^{-1}\langle 0_{Rindler}|\exp\left[\sum_{\lambda=0}^{\infty}\frac{\beta_\lambda}{\alpha_\lambda}c_{\lambda,R}c_{\lambda,L}\right]O_R\exp\left[\sum_{\lambda=0}^{\infty}\frac{\beta_\lambda}{\alpha_\lambda}c^\dagger_{\lambda,R}c^\dagger_{\lambda,L}\right]|0_{Rindler}\rangle\\
&= Z^{-1}\sum_{\{n_\lambda\}}\left(\frac{\beta_\lambda}{\alpha_\lambda}\right)^{2n_\lambda}\langle\{n_\lambda\}|O_R|\{n_\lambda\}\rangle\\
&= Z^{-1}\sum_{\{n_\lambda\}}e^{-\beta\sum_\lambda\lambda n_\lambda}\langle\{n_\lambda\}|O_R|\{n_\lambda\}\rangle\\
&= \frac{\mathrm{tr}e^{-\beta H_R}O_R}{\mathrm{tr}e^{-\beta H_R}}
\end{aligned}\tag{2.46}$$

where H_R is the Rindler hamiltonian generator of τ translations restricted to R. So single quadrant operators have their means given by a thermal density matrix. Note that the Rindler hamiltonian (the generator of boosts) is equal to $-iaU\partial_U$ and is given in terms of the operators $c_{\lambda,R}, c_{\lambda,L}$ by

$$\begin{aligned}
H_{Rindler} &= H_R - H_L\\
&= \theta(-U)\int_{-\infty}^{+\infty}du\ T_{uu} - \theta(+U)\int_{-\infty}^{+\infty}du_L\ T_{u_Lu_L}\\
&= \int_0^{+\infty}d\lambda\lambda(c^\dagger_{\lambda,R}c_{\lambda,R} - c^\dagger_{\lambda,L}c_{\lambda,L})
\end{aligned}\tag{2.47}$$

and possess therefore the same structure as the hamiltonian in the E-field given in eq. (1.41). Thus the pairs of Rindlerons in eq. (2.44) have zero

Rindler energy. This degeneracy allows for the creation of pairs of Rindler quanta.

Of course everything that has been derived for u modes applies equally for v modes where the rôle of P and F are interchanged hence the past and future horizons $H_\pm$ (see fig. 2.1). Also one can perform the analysis for fermions and include the effects of a mass and of higher dimensionality (see for instance [88], [85]).

It is more interesting (and relevant for blackholology as well) to inquire into the physical condition of the radiation due to its interaction with the accelerator. We shall therefore in the next section delve more intimately into the details of the transition amplitude.

2.4 Spontaneous Emission of Photons by an Accelerated Detector

We here dissect the rate formula eq. (2.11) by displaying the transitions of the atom as resonance phenomena with Doppler shifted Minkowski photons [74]. This will introduce in a natural way the conversion of a particular Minkowski vacuum fluctuation into an on mass shell quantum. Similar resonance phenomena constitute the dynamical origin of particle emission from a non inertial mirror (Section 2.5) and of black hole radiation.

We shall see that in this formulation the rate formula comes about by a somewhat different mechanism from that of usual golden rule analysis, i.e. in terms of density of states. It is rather a consequence of the steady process in which photons are brought into resonance through the ever–changing Doppler shift occasioned by the accelerator, much like the analysis of Section 1.3 (paragraph following equation (1.43)), wherein different ω values brought different vacuum fluctuations into resonance with real pairs giving rise to a time sequence of produced pairs.

We focus for definiteness on right movers. In lowest order in g, the amplitude for a transition in time τ is

$$\begin{aligned} A_\mp(\omega,\tau) &= -i\langle\pm|\langle 0_M|a_\omega \int_0^\tau d\tau' H_{int}(\tau')|0_M\rangle|\mp\rangle \\ &= \frac{-ig}{\sqrt{4\pi\omega}}\int_0^\tau d\tau' e^{\pm i\Delta M\tau'} e^{-i\omega a^{-1}e^{-a\tau'}} \end{aligned} \qquad (2.48)$$

where the plus (minus) subscript of A corresponds to spontaneous deexcitation (excitation) and the ket $|\pm\rangle$ means excited (ground) state. Begin with the former, spontaneous deexcitation. The integrand presents a saddle point at τ^* given by

$$\Delta M = \omega e^{-a\tau^*(\omega)} \quad . \tag{2.49}$$

If ω is such that $\tau^*(\omega)(= a^{-1}\ln \omega/\Delta M)$ lies well within the domain of integration in eq. (2.48) (i.e. the gaussian width $= (\Delta M a)^{-1/2}$ does not overlap the limits of integration), and furthermore if higher derivatives of the exponential are relatively small (i.e. $(\Delta M/a) >> 1$), then the saddle point estimate is valid and one finds

$$A_+(\omega,\tau) \simeq \frac{-ig}{\sqrt{4\pi\omega}}\sqrt{\frac{2\pi}{-i\Delta M a}}e^{-i\Delta M a^{-1}}\left(\frac{\Delta M}{\omega}\right)^{i\Delta M a^{-1}} \quad . \tag{2.50}$$

The crucial inequality for legitimization of eq. (2.50) is that ω should be bounded by

$$0 \leq a^{-1}\ln\frac{\omega}{\Delta M} \leq \tau \tag{2.51}$$

but we shall shortly loosen up on the condition $(\Delta M/a) >> 1$.

Once more there is, as for the electric case, a division into class I, the frequencies ω which resonate, i.e. satisfy eq. (2.51) and class II modes which do not resonate, i.e. lie outside the range (2.51) and for which $A_+(\omega,\tau)\simeq 0$. And once again as in Section 1.3 this must be qualified by "apart from edge effects". Clearly τ has to be sufficiently large for these asymptotic estimates to make sense, as in all "golden rule" type estimates.

For spontaneous excitation the saddle point condition has a minus sign on the r.h.s. of eq. (2.49) so that the saddle point lies at

$$\begin{aligned} \mathrm{Re}\tau^*(\omega) &= a^{-1}\ln\omega/\Delta M \\ \mathrm{Im}\tau^*(\omega) &= \pi/a \end{aligned} \tag{2.52}$$

whereupon one has

$$\begin{aligned} A_-(\omega,\tau) &= e^{-\beta\Delta M/2}\frac{-ig}{\sqrt{4\pi\omega}}\sqrt{\frac{2\pi}{i\Delta M a}}e^{i\Delta M a^{-1}}\left(\frac{\Delta M}{\omega}\right)^{i\Delta M a^{-1}} \\ &= -A_+^*(\omega,\tau)e^{-\beta\Delta M/2} \end{aligned} \tag{2.53}$$

with $\beta = 2\pi/a$. Equation (2.53) is valid only for class I modes; for class II, $A_-(\omega,\tau)\simeq 0$ as well. Squaring these formulae one recovers the thermal ratio of rates gven in eq. (2.10). Integrating ω over the bounds set by eq. (2.51) and dividing by τ, one recovers half of $R_\pm$ given by eq. (2.11) provided one limits oneself to the contribution from the leading poles in the integrand (2.9) [$\tau = -i\epsilon$ for emission and $2\pi i/a$ for absorption, consistent with the condition $\Delta M/a >> 1$]. The other half is due to the left movers. Note how the rate formula comes about, e.g.

$$\begin{aligned} P_-(\tau) &= \int_{\Delta M}^{\Delta M e^{a\tau}} d\omega |A_-(\omega,\tau)|^2 \\ &= \frac{g^2 e^{-\beta\Delta M}}{2\Delta M a} \int_{\Delta M}^{\Delta M e^{a\tau}} \frac{d\omega}{\omega} \\ &= \frac{g^2 e^{-\beta\Delta M}}{2\Delta M a} \int_0^\tau d\tau' \end{aligned} \tag{2.54}$$

The integral over ω is $\int d\ln\omega$ hence an integral over saddle times $\mathrm{Re}\tau^*(\omega)$. Neglecting edge effects this is equal to τ. The physical interpretation is clear. At each time $\tau^*(\omega)$, the Minkowski frequency ω enters into resonance because of the changing Doppler shift occasioned by the acceleration ($\omega_{\mathrm{Res}} = \Delta M e^{a\tau^*}$).

It is interesting to try to interpret the complex saddle encountered in eq. (2.52) (the case of absorption). In the complex τ plane the contour has to be deformed from $\mathrm{Im}\tau = 0$ so as to go through $\mathrm{Im}\tau = \pi/a$ (as in the construction of Unruh modes–paragraph after eq. (2.39). In both cases the voyage in the complex plane encodes the positivity of Minkowski frequencies). Reference to eq. (2.20) then indicates that there has been a voyage from U to $-U$ (or $x \to -x$, $t \to -t$), hence from a point in the quadrant R to its inversion in the third quadrant L where τ runs backwards (i.e. $d\tau/dt < 0$) hence where Rindler energy conservation is satisfied since the frequency ω appears as carrying negative Rindler energy (since it is given by $iaU\partial_U$ see eq. (2.47). In Sections 2.6 and 2.6.3 it will be shown by more detailed considerations that in fact the Minkowski photon which is emitted when the atom is excited arises from a vacuum fluctuation one part of which exists in the quadrants L and F, the other part lives in P and R. This latter has crossed the past horizon so as to be absorbed by the atom and the former continues out to infinity in quadrant F, i.e. it arises in L and radiates from there into F. It is this effective emission act in L which is encoded in the the rough Born approximation saddle point integral of saddle point eq. (2.52). The reader at this point can pick up some flavor for the true state of affairs by peeking ahead at Fig.2.5.

One can offer oneself the luxury of completing this type of analysis to get the full expressions for $R_\pm$ (i.e. by relaxing the condition $\Delta M/a >> 1$) by noting that for any value of $\Delta M/a$ the contribution to the integrand from a given ω is dominated by the region (gaussian width) around $\tau^*(\omega)$. So provided one maintains eq. (2.51) one can extend the τ' integration over the whole real axis. In this limit the transition amplitudes eqs (2.48) and (2.53) become

$$\begin{aligned} A_+ &= -ig\sqrt{\frac{\pi}{\Delta M}}\alpha^{R*}_{\Delta M\omega} \\ A_- &= -ig\sqrt{\frac{\pi}{\Delta M}}\beta^{R*}_{\Delta M\omega} \end{aligned} \tag{2.55}$$

thereby giving a dynamical content to the Bogoljubov coefficients (2.28). Squaring and integrating over ω within the bounds limited by eq. (2.51) leads to the exact rates given in eqs (2.53) multiplied by the interval τ. This is how the Golden rule comes about in this approach, successive resonances with the Doppler shifted frequencies of the proper atomic frequency (here $= \Delta M$).

For the skeptical reader we now present a more rigorous analysis of this argument as it plays an important role in the black hole problem as well. At the same time we shall clearly exhibit the difference between the class I and class II modes.

[The amplitude to emit a photon of frequency ω in the interval $(-\tau/2, \tau/2)$ is

$$A_+(\omega,\tau) = \frac{-ig}{\sqrt{4\pi\omega}}\left(\frac{\omega}{a}\right)^{-i\Delta M/a}\int_{\omega a^{-1}e^{-a\tau/2}}^{\omega a^{-1}e^{a\tau/2}} dx\ e^{-ix}x^{i\Delta M/a-1} \tag{2.56}$$

$$\begin{aligned} &= \frac{-ig}{\sqrt{4\pi\omega}}\left(\frac{\omega}{a}\right)^{-i\Delta M/a}\frac{e^{-\Delta M\pi/2a}}{a^2}\Big[\gamma(i\Delta M/a, i\omega a^{-1}e^{a\tau/2}) \\ &\qquad -\gamma(i\Delta M/a, i\omega a^{-1}e^{-a\tau/2})\Big] \end{aligned} \tag{2.57}$$

where the interval is taken symmetric around $\tau = 0$ for mathematical convenience; the physics is unmodified by this translation in τ and where γ is the incomplete gamma function [99] which for large and small values of its argument takes the form

$$\gamma(i\mu, ix) \underset{x\to\infty}{\simeq} \Gamma(i\mu) - ie^{-\pi\mu/2}x^{i\mu}e^{-ix}x^{-1} \tag{2.58}$$

$$\gamma(i\mu, ix) \underset{x\to 0}{\simeq} -ie^{-\pi\mu/2}x^{i\mu} \tag{2.59}$$

Before analyzing equation (2.57) we discuss the unphysical infra-red divergence which arises as $\omega \to 0$. This divergence exists in the inertial case as well. Indeed as $\omega \to 0$ one has $A_+ = 2ig \sin(\Delta M\tau/2)/\sqrt{4\pi\omega}$ independent of the acceleration. It is unphysical because in one dimension the coupling of the massless field to the atom becomes strong as $\omega \to 0$. Therefore the perturbation theory which has been used is inadequate. No doubt a resummation of all terms is possible to give the true infra red physics (like the Bloch Nordziek [11] theory in QED). But this is irrelevant to our purpose, since these infra red photons do not contribute to the rate.

One way to get rid of the problem is to subtract off the inertial amplitude. We shall do the computation in another manner so as to obtain an infra red finite answer. This is accomplished by adiabatically switching on and off the coupling to the field by a function $f(\tau)$. Then the amplitude for $\omega \to 0$ will decrease as a function of the period T of switch on and off. To see this we remark that

1) for $\omega \to 0$, A_+ is given by the Fourier transform of f $(A_+(\omega \simeq 0, \tau) \simeq \int d\tau e^{-i\Delta M\tau} f(\tau)/\sqrt{\omega})$.

2) If the interval τ during which f is constant is equal to $\tau = 2\pi k/\Delta M$ with k an integer then $A_+(\omega \simeq 0, \tau)$ is independent of k.

3) Therefore we can take $k = 0$.

4) If f can be differentiated n times, the Fourier transform of f decreases for large ΔM as ΔM^{-n}.

5) Since the only dimensional parameter is T, the Fourier transform of f must be of the form $T^{-n+1}\Delta M^{-n}$.

The correct procedure to calculate the rate of excitation consists in first taking the adiabatic limit $T \to \infty$ (with however the condition $T << \tau$) and only then performing the integral over ω. We shall show that with this order of operations the concept of resonant frequency described in the main text appears.

To perform thus the adiabatic limit it proves rather more convenient to introduce a time average of the amplitude

$$\bar{A}_+(\omega, \tau) = \int d\tau' \; g(\tau') A_+(\omega, \tau') \tag{2.60}$$

where $g(\tau')$ is a n times differentiable function, centered on $\tau' = \tau$, with width T, and normalized such that $\int d\tau \; g(\tau) = 1$. One verifies by permuting the integral over τ in eq. (2.60) and the integral over x in the definition of

A_+ (eq. (2.56) that the averaging of A_+ is equivalent to the introduction of a switch function $f(\tau)$ with the same regularity as g.

We now permute the series expansions eqs (2.58) and (2.59) with the averaging in eq. (2.60) to obtain that $\bar{A}_+$ is given by a similar expression to eq. (2.57) with γ replaced by $\bar{\gamma}$ where (see eq. (2.59))

$$\bar{\gamma}(i\Delta M/a, i\omega a^{-1}e^{\pm a\tau/2}) \underset{\frac{\omega e^{\pm a\tau/2}}{a}\to\infty}{\simeq} \Gamma(i\Delta M/a) + +O\left(\frac{e^{-\pi\Delta M/2a}}{\omega a^{-1}e^{\pm a\tau/2}}\right) \quad (2.61)$$

$$\bar{\gamma}(i\Delta M/a, i\omega a^{-1}e^{\pm a\tau/2}) \underset{\frac{\omega e^{\pm a\tau/2}}{a}\to 0}{\simeq} O\left(e^{-\pi\Delta M/2a}\frac{ae^{\pm i\Delta M\tau/2}}{T^{n-1}\Delta M^n}\right) \quad (2.62)$$

We are now in a position to rederive the results obtained in the main text. Three cases are to be considered according to the values of ω:

1) $\omega < ae^{-a\tau/2}$. Then both γ functions in eq. (2.57) are given by eq. (2.62), hence their difference yields
$\bar{A}_+(\omega,\tau) \simeq O[ae^{-\pi\Delta M/2a}\sin(\Delta M\tau/2)/(T^{n-1}\Delta M^n)]$.

2) $ae^{-a\tau/2} < \omega < ae^{a\tau/2}$. Then the first γ function in eq. (2.57) is given by eq. (2.61) and the second by eq. (2.62, hence their difference is $\Gamma(i\Delta M/a) + O[ae^{-\pi\Delta M/2a}/T^{n-1}\Delta M^n]$ and $\bar{A}_+(\omega,\tau)$ is given by eq. (2.55) as announced.

3) $\omega > a^{a\tau/2}$. Then both gamma functions are given by eq. (2.61) and A_+ vanishes once more.

In Fig.2.2 we have plotted a numerical calculation of $|\gamma(i\Delta M/a, i\omega a^{-1}e^{a\tau/2}) - \gamma(i\Delta M/a, i\omega a^{-1}e^{-a\tau/2})|^2$ as a function of τ and $\ln\omega$. The plateau of this function when $-a\tau/2 < \ln\omega/a < a\tau/2$ is clearly apparent. The oscillations on the plateau are due to the fact that the adiabatic limit has not been taken in this figure.

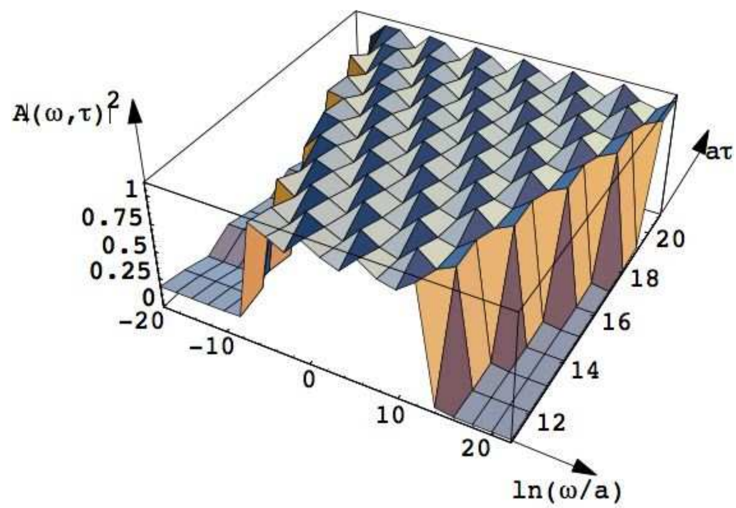

Fig. 2.2 *The square of the amplitude $A(\tau, \omega)$ to have made a transition in time τ by emitting a photon of Minkowski frequency ω. plotted as a funtion of τ and $\ln \omega$. The plateau that arrises when the resonance condition $-a\tau/2 < \ln \omega/\Delta M < a\tau/2$ is satisfied is clearly seen.*

In this way we have confirmed that eq. (2.48) arises from the saddle point region to yield the result (2.55). Hence that the successive resonances do build up to yield a rate.]

2.5 The Accelerating Mirror

2.5.1 General Description

A highly instructive chapter in the physics of accelerating systems is that of the accelerating mirror[36], [29], [10] in that the analogy to the production of Hawking radiation is remarkably close.

We have shown that a uniformly accelerating system with internal degrees of freedom displays thermal properties due to the exponentially changing Doppler shift (eq. (2.49)) which relates the inertial frequencies to its local resonant frequency. Similarly a non inertial mirror scatters modes with the changing Doppler shift associated with its trajectory, hence giving rise to physical particles. If the mirror trajectory is such that the Doppler shift is identical to eq. (2.49) the particles emitted by the mirror will be thermally distributed as well. We write this Doppler shift as

$$\omega e^{-aU} = k \tag{2.63}$$

where the energy difference ΔM is now replaced by the frequency of the produced particle k, and ω is, as before, the frequency of the Minkowski vacuum fluctuation in resonance with the produced quantum which is reflected from the point U on the mirror. In the next paragraph the order of magnitude of k is taken to be of $O(a/2\pi)$ whereas ω varies strongly. We now determine the mirror trajectory which leads to this Doppler shift.

The general solution of $\Box\phi = \partial^2/\partial_U\partial_V\phi = 0$ is

$$\phi = F(V) + G(U) \tag{2.64}$$

The reflection condition is that ϕ vanish on the mirror hence the solutions take the form

$$\phi = F(V) - F(V_m(U)) \tag{2.65}$$

where the mirror trajectory is expressed as

$$V = V_m(U) \tag{2.66}$$

To determine the Doppler shift associated with this trajectory we take an incoming Minkowski mode: $e^{-i\omega V}/\sqrt{4\pi\omega}$. The reflected wave is then $-e^{-i\omega V_m(U)}/\sqrt{4\pi\omega}$. This mode should be decomposed into the inertial outgoing modes $e^{-ikU}/\sqrt{4\pi k}$ so as to determine its particle content. The scattering amplitude (the Bogoljubov coefficient $\alpha_{\omega k}$) is the overlap of the modes

$$\alpha_{k\omega} = \int dU \frac{e^{ikU}}{\sqrt{4\pi k}} i\overleftrightarrow{\partial}_U \frac{e^{-i\omega V_m(U)}}{\sqrt{4\pi\omega}} \tag{2.67}$$

The resonance condition given by the stationary phase of the integrand is

$$k = \omega \frac{dV_m}{dU} \tag{2.68}$$

Hence in order to recover eq. (2.63) we must take as trajectory for the mirror

$$V_m(U) = -\frac{1}{a} e^{-aU} \tag{2.69}$$

where we have set to zero an irrelevant integration constant. This trajectory tends exponentially fast towards the asymptote $V = 0$ which is the last reflected ray. It plays the role of a horizon in this problem (see Fig 2.3). It is by construction that this mirror trajectory has the same expression as an inertial trajectory expressed in Rindler coordinates u, v as it approaches the future horizon $u = \infty$ (see reference [10] for a mapping of the one problem into the other by a conformal transformation).

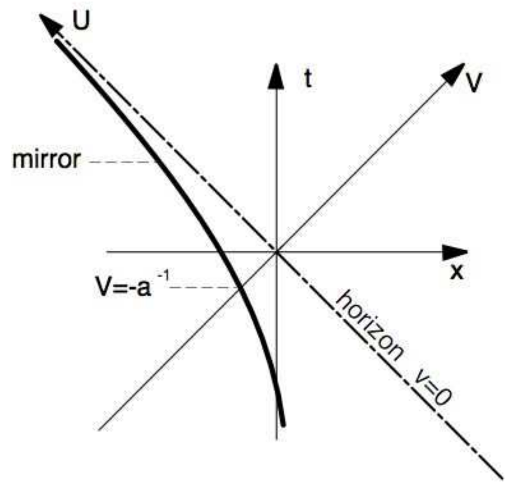

Fig. 2.3 *The mirror trajectory $V_m(U) = -a^{-1}e^{-aU}$ which gives rise to a steady thermal flux.*

The mirror therefore follows a very different trajectory from that of the uniformly accelerated detector.

In order to describe the particles emitted by the mirror it is necessary to work in the second quantized context. As in the electric field problem one introduces two bases. The initial one is given by

$$\varphi_\omega^{in} = \frac{1}{\sqrt{4\pi\omega}}(e^{-i\omega V} - e^{-i\omega a^{-1}e^{-aU}}) \qquad \omega > 0 \tag{2.70}$$

and corresponds on the past null infinity surface $\mathcal{I}^-$ $(U = -\infty)$ to the usual Minkowski basis. The in-vacuum is the state annihilated by the destruction operators a_ω^{in} associated with these modes:

$$\phi = \int_0^\infty d\omega \; (a_\omega^{in} \varphi_\omega^{in} + \text{h.c.}) \tag{2.71}$$

The final basis corresponds to the usual Minkowski basis on the future null infinity surface $\mathcal{I}^+$ ($V = +\infty$), hence given by

$$\varphi_k^{outL} = \frac{1}{\sqrt{4\pi k}} (|aV|^{ik/a} \theta(-V) - e^{-ikU}) \qquad k > 0 \tag{2.72}$$

The V part of φ^{outL} is identical to the left Rindler modes expressed in Minkowski coordinates (see eq. (2.22)). Similarly the U part of φ^{in} is the same function as a Minkowski mode when it is expressed in Rindler coordinates. Hence the mathematics as well as the physics of Sections 2.1 to 2.4 apply to obtain the number and type of particles emitted from the mirror. Indeed the alpha Bogoljubov coefficient (2.67) and the beta Bogoljubov coefficient

$$\beta_{k\omega} = \int_{-\infty}^{\infty} dU \frac{e^{-ikU}}{\sqrt{4\pi k}} i \overleftrightarrow{\partial}_U \frac{e^{-i\omega V_m(U)}}{\sqrt{4\pi\omega}} \tag{2.73}$$

are identical to the Bogoljubov coefficients obtained in eq. (2.28). In particular the ratio $|\beta_{\omega k}/\alpha_{\omega k}| = e^{-\pi k/a}$ obtains and implies a constant rate of particle production in a thermal spectrum of produced particles at temperature $T = a/2\pi$.

The mechanism wherein one finds a constant rate of emission is the same as in Section 2.4. It arises from the successive resonances of in-modes of varying energy ω with the emitted photons of energy k. To a photon in a wave packet of energy k emitted around the value $U = U_0$ corresponds one and only one V-mode. Its frequency is $\omega = ke^{aU_0}$ as in equation eq. (2.49). And the number of photons of energy k emitted during a certain ΔU lapse is given by the integral of $|\beta_{\omega k}|^2 = (1/\omega) n(k)$ (where $n(k) = (e^{k/T} - 1)^{-1}$) over the resonant frequencies ω. This integral yields $\Delta U n(k)$, i.e. a thermal flux times the time lapse, as in eq. (2.54).

We now calculate the energy momentum carried by these quanta. In the next subsection the mean energy will be obtained by two different techniques. In the third subsection we shall describe the fluctuations of the energy density: we shall obtain the energy momentum correlated to the observation of an outgoing particle around a particular value of U. We wish to emphasize at this point that we are now seriously trespassing into the domain of the black hole problem. The next two sections contain a great deal of the essential physics of black hole evaporation.

2.5.2 The Mean Energy Momentum Tensor

To start, from eq. (2.70) we have

$$\langle 0_{in}|T_{VV}|0_{in}\rangle|_{U=-\infty} = 0 \tag{2.74}$$

The surface $\mathcal{I}^-$ ($U = -\infty$) is where Minkowski vacuum is laid down. It is the Cauchy surface from which emanates the modes (2.70). This defines the Heisenberg state $|0_{in}\rangle$ which is annihilated by the operators a^{in}_ω. From eq. (2.70) the modes φ^{in}_ω are pure V-like on $\mathcal{I}^-$ (in the sense of broad wave packets) and are Minkowski modes. Thus eq. (2.74) is true in the usual sense of normal ordering. Furthermore since $\partial_U T_{VV} = 0$ for massless fields in Minkowski space it follows that eq. (2.74) remains true for all points (U, V) which lie to the right of the mirror's trajectory (2.66):

$$\langle 0_{in}|T_{VV}|0_{in}\rangle|_{V>V_m(U)} = 0. \tag{2.75}$$

More interesting is $\langle T_{UU}\rangle_{in}$. We calculate it in two different ways:
1) by mode analysis
2) more synthetically through use of Green's functions.
In the first method we have

$$\begin{aligned} \langle 0_{in}|T_{UU}|0_{in}\rangle \quad &- \quad \langle 0_{out}|T_{UU}|0_{out}\rangle = \\ \int_0^\infty d\omega\ \partial_U\varphi^{in}_\omega \partial_U\varphi^{in*}_\omega \quad &- \quad \int_0^\infty dk\ \partial_U\varphi^{outL}_k \partial_U\varphi^{outL*}_k \end{aligned} \tag{2.76}$$

wherein we have implemented the prescription of normal ordering by making a subtraction of the value of $\langle T_{UU}\rangle$ in Minkowski vacuum. Indeed by definition the U part of the out modes eq. (2.72) is Minkowski in character. We use the Bogoljubov coefficients (2.67) and (2.73) to express φ^{in}_ω in terms of φ^{outL}_k

$$\varphi^{in}_\omega = \int_0^\infty dk\ \alpha_{k\omega}\varphi^{out}_k + \beta_{k\omega}\varphi^{out*}_k \tag{2.77}$$

Thus obtaining

$$\begin{aligned} &\langle 0_{in}|T_{UU}|0_{in}\rangle - \langle 0_{out}|T_{UU}|0_{out}\rangle = \\ \int_0^\infty dk \int_0^\infty dk' \Big[&\left(\int_0^\infty d\omega\alpha_{k\omega}\alpha^*_{k'\omega}\right)\partial_U\varphi^{outL}_k \partial_U\varphi^{outL*}_{k'} \ + \\ &\left(\int_0^\infty d\omega\beta_{k\omega}\beta^*_{k'\omega}\right)\partial_U\varphi^{outL*}_k \partial_U\varphi^{outL}_{k'} \ + \end{aligned}$$

$$
\begin{aligned}
&\left(\int_0^\infty d\omega \alpha_{k\omega}\beta^*_{k'\omega}\right)\partial_U\varphi_k^{outL}\partial_U\varphi_{k'}^{outL} \;+\\
&\left(\int_0^\infty d\omega \beta_{k\omega}\alpha^*_{k'\omega}\right)\partial_U\varphi_k^{outL*}\partial_U\varphi_{k'}^{outL*}\Big]\\
&\qquad - \int_0^\infty dk\ \partial_U\varphi_k^{outL}\partial_U\varphi_{k'}^{outL*}
\end{aligned}
\tag{2.78}
$$

The unitary relations (2.30) simplify the result to the form

$$
\begin{aligned}
= \ &\int_0^\infty dk\int_0^\infty dk'\ 2\mathrm{Re}\left[\left(\int_0^\infty d\omega\beta_{k\omega}\beta^*_{k'\omega}\right)\partial_U\varphi_k^{outL*}\partial_U\varphi_{k'}^{outL}\right.\\
&\left.+\left(\int_0^\infty d\omega\alpha_{k\omega}\beta^*_{k'\omega}\right)\partial_U\varphi_k^{outL}\partial_U\varphi_{k'}^{outL}\right]
\end{aligned}
\tag{2.79}
$$

We now recall eqs. (2.32, 2.33)

$$
\begin{aligned}
\int_0^\infty d\omega\beta_{k\omega}\beta^*_{k'\omega} &= \frac{e^{-\pi(k+k')/(2a)}\Gamma(ik/a)\Gamma(-ik'/a)\sqrt{kk'}}{(2\pi a)^2}\int_0^\infty (d\omega/\omega)(a/\omega)^{i(k-k')/a}\\
&= \delta(k-k')n(k)
\end{aligned}
\tag{2.80}
$$

where

$$
n(k)=\frac{e^{-\pi k/a}|\Gamma(ik/a)|^2k}{(2\pi a)}=\frac{1}{(e^{2\pi k/a}-1)} \tag{2.81}
$$

One also verifies that

$$
\int_0^\infty d\omega\beta_{k\omega}\alpha^*_{k'\omega}=\frac{e^{-\pi(k-k')/(2a)}\Gamma(ik/a)\Gamma(ik'/a)\sqrt{kk'}}{(2\pi a)^2}\int_0^\infty (d\omega/\omega)(a/\omega)^{i(k+k')/a} \tag{2.82}
$$

is proportional to $\delta(k+k')$ hence does not contribute to eq. (2.79). The vanishing of this interference term is the expression in the particular case we are considering of the general theorem proven in eq. (2.46) that expectation values of operators restricted to one quadrant are given by their average in a thermal density matrix (see also eqs (2.32, 2.33)). This theorem is applicable here since the U part of each mode arises from the part of the mode to the left of the horizon ($V<0$).

The final answer is thus

$$
\begin{aligned}
\langle 0_{in}|T_{UU}|0_{in}\rangle-\langle 0_{out}|T_{UU}|0_{out}\rangle &= \int_0^\infty dk\ 2\ n(k)\ \partial_U\varphi_k^{outL}\partial_U\varphi_k^{outL*}\\
&= \frac{1}{2\pi}\int_0^\infty dk\ k\ n(k)=\frac{\pi}{12}\left(\frac{a}{2\pi}\right)^2
\end{aligned}
\tag{2.83}
$$

which is a thermal flux in one dimension with $T = a/2\pi$ as announced.

The other technique to calculate the flux relies only on the trajectory of the mirror eq. (2.69) [30]. We now pass it in review.

In usual global null coordinates the metric of Minkowski space reads as

$$ds^2 = -dU dV \tag{2.84}$$

and the modes used to quantize about the usual Minkowski vacuum are $e^{-i\omega U}$ and $e^{-i\omega V}$. The U part of the reflected wave in eq. (2.70) is in terms of modes $e^{-i\omega f(U)}$ with $f(U) = -a^{-1}e^{-aU}$. If we adopt $f(U)$ in place of U as coordinate we have

$$ds^2 = -\frac{dU}{df} df dV \tag{2.85}$$

The U part of the Green's function in usual Minkowski vacuum is $-\frac{1}{4\pi}\ln|U - U'|$ whereas in the modes $e^{-i\omega f(U)}$ it is $-\frac{1}{4\pi}\ln|f(U) - f(U')|$. Thus the difference in $\langle T_{UU}\rangle = \langle \partial_U\phi\partial_U\phi\rangle$ between the two is

$$\begin{aligned} \Delta\langle T_{UU}\rangle &= -\frac{1}{4\pi}\lim_{U\to U'} \partial_U\partial_{U'}\left[\ln|f(U) - f(U')| - \ln|U - U'|\right] \\ &= \frac{1}{12\pi}\left[f'^{1/2}\partial_U^2 f'^{-1/2}\right] \end{aligned} \tag{2.86}$$

where $f' = df/dU$ and where the second line is obtained by expanding f to third order in $U - U'$. In the present case $f(U) = V_m(U)$ which yields

$$\langle 0_{in}|T_{UU}|0_{in}\rangle - \langle 0_{out}|T_{UU}|0_{out}\rangle = \frac{a^2}{48\pi} = \frac{\pi}{12}T^2 \quad (T = a/2\pi) \tag{2.87}$$

as required. The second line of eq. (2.86) is a remarkably elegant formula which relates the average energy momentum to the coordinate transformation $f(U)$ only. It does not refer to the presence of the mirror. It suffices to refer to the non inertial coordinates used in the expression (2.85) of the metric. We shall see in Section (3.3) that there is a natural generalization to curved space which is of important use in the black hole problem.

Note that the complete in-in Green's function involves terms in V, V' and mixed terms U, V' due to the linear combinations which appear in eqs (2.70) and (2.72). The term in V, V' gives $\langle T_{VV}\rangle$ unchanged from Minkowski vacuum, eq. (2.75). The U, V term gives U, V energy-energy correlations but no contribution to $\langle T_{\mu\nu}\rangle$ itself since the trace ($4\ \langle T_{UV}\rangle = m^2\langle\phi^2\rangle$) vanishes identically for a massless field in Minkowski space (see Section (3.3) and ref. [10])).

2.5.3 The Fluctuations around the Mean

In preparation for black hole physics, we now ask a more detailed question, concerning fluctuations. We wish to display the field configuration in vacuum that is responsible for the emission of a particular photon whose wave function is localized around $U = U_0$. [Recall that the pure state Minkowski vacuum is a linear superposition of field configurations (it is the product of the ground states of the field oscillators: $|0_M\rangle = \Pi e^{-|\phi_n(x)|^2 \omega_n}$ in ϕ representation)]. In particular we shall describe the energy distributions which characterize this fluctuation (we consider this quantity since in the black hole problem the energy momentum is the source of the back reaction). . At the same time we will also answer the complementary question, to wit: what is the energy distribution when no photon is observed around U_0 (since we know the average). The calculation is straightforward but somewhat lengthy. We therefore sketch below the main points and then go on to the proofs.

The Heisenberg state $|0_{in}\rangle$ is a linear superposition of out states. It is a straightforward exercise to describe the contribution of each final state to the energy on $\mathcal{I}^+$ (in order to do this in a local manner it is necessary to use localized wave packets). In addition to the positive energy density on $\mathcal{I}^+$ associated to the production of a particle at $U = U_0$, there is correlated to it a "partner" which is a bump of field that propagates along the mirror (hence with $V > 0$). Before reflection there is also a bump in the region $V < 0$. This is the "ancestor" of the produced photon. The total energy carried by this pair of bumps vanishes as behooves a vacuum fluctuation. It does so in rather subtle fashion. It is positive definite for $V > 0$ whereas in the region $V < 0$, there is a positive energy bump, mirror image of the former as well as an oscillating broader piece which is negative. The sum of all these contributions vanishes on $\mathcal{I}^-$ and therefore for all U until the wave packet starts reflecting.

If two (or in general n) photons are produced around U_0 then the ancestors carry twice (n times) the energy if one photon is produced. If no photons are produced there is also a vacuum fluctuation which is proportional to minus the energy if one photon is produced. The coefficient is such that upon averaging the energy over the production of zero, one, two ... photons one recovers the mean, (zero for $\langle T_{VV}\rangle$, see eq. (2.75) and the thermal flux for T_{UU} (see eq. (2.83)).

To see all of this we first display the pair. This stands in strong analogy to Section 2.3. We then consider the non-diagonal matrix elements of $T_{\mu\nu}$

associated to the produced photon at U_0.

To display the pair we introduce the analog of the Unruh modes (Section 2.3) since they diagonalize the Bogoljubov coefficients. An out mode (eq. (2.72)) is proportional to $\theta(-V)$. By adding to it a piece proportional to $\theta(+V)$ one can obtain a purely positive frequency mode on $\mathcal{I}^-$. Writing only the V part of the modes we have

$$\hat{\varphi}_k = \frac{1}{\sqrt{4\pi k}} \Big(\alpha_k |aV|^{ik/a} \theta(-V) + \beta_k (aV)^{ik/a} \theta(+V) \Big) \quad k > 0 \tag{2.88}$$

The U part is given by the reflection condition (2.65). Analyticity in the lower half of the complex V plane (i.e. positive frequency ω in eq. (2.70)) fixes the ratio (see eqs (2.39) et seq.)

$$\frac{\beta_k}{\alpha_k} = e^{-\pi k/a} \tag{2.89}$$

and the normalization of $\hat{\varphi}_k$ with the Klein Gordon scalar product fixes $\alpha_k^2 - \beta_k^2 = 1$. To obtain a complete orthogonal set of positive frequency modes one must include the modes for $k < 0$

$$\hat{\varphi}_k = \frac{1}{\sqrt{4\pi |k|}} \Big(\beta_{|k|} |aV|^{ik/a} \theta(-V) + \alpha_{|k|} (aV)^{ik/a} \theta(+V) \Big) \quad k < 0 \tag{2.90}$$

From eqs (2.88) and (2.90) it is seen that the set of "Rindler" type modes given by eq. (2.72) and the mode given by

$$\varphi_k^{outR} = \theta(+V) \frac{1}{\sqrt{4\pi k}} (aV)^{-ik/a} \tag{2.91}$$

constitute a complete orthonormal set. It is to be noted that the modes (2.91) do not have a U part since they don't reflect (when the mirror follows eq. (2.69) forever).

The quantum number k in eqs (2.88) to (2.91) is not the usual Minkowski energy. Rather it is the eigenvalue of the boost operator $iaV\partial_V$ whereas the energy ω is the eigenvalue of $i\partial_V$. (We shall call it therefore "Rindler energy" see eq. (2.47).) However upon reflection the time dependent Doppler shift (2.63) gives k the meaning of Minkowski energy for out modes. It is then the eigenvalue of $i\partial_U$.

Equations (2.88,2.90) therefore are to be read as the Bogoljubov transformation (written in terms of eqs (2.72,2.91)):

$$\left.\begin{array}{rcl} \hat{\varphi}_k & = & \alpha_k \varphi_k^{outL} + \beta_k \varphi_k^{outR*} \\ \hat{\varphi}_{-k} & = & \beta_k \varphi_k^{outL*} + \alpha_k \varphi_k^{outR} \end{array}\right\} k > 0 \tag{2.92}$$

or in terms of operators:

$$\left.\begin{array}{rcl} \hat{a}_k & = & \alpha_k a_k^{outL} - \beta_k a_k^{outR\dagger} \\ \hat{a}_{-k} & = & -\beta_k a_k^{outL\dagger} + \alpha_k a_k^{outR} \end{array}\right\} k > 0 \tag{2.93}$$

The creation of particles of energy k by the moving mirror gives physical content to the modes (2.91) as the partners of the created particles since the Minkowski vacuum on $\mathcal{I}^-$ ($|0_{in}\rangle$) can be expressed (see eq. (2.44)) as

$$|0_{in}\rangle = \frac{1}{\sqrt{Z}} \prod_{k>0} \exp^{\frac{\beta_k}{\alpha_k} a_k^{outL\dagger} a_k^{outR\dagger}} |0_{out}\rangle \tag{2.94}$$

where $|0_{out}\rangle = |0_{outL}\rangle \otimes |0_{outR}\rangle$ with $a_k^{outL}|0_{outL}\rangle = 0$ and $a_k^{outR}|0_{outR}\rangle = 0$. In this way one sees that to each produced φ_k^{outL} particle corresponds a partner φ_k^{outR} living on the other side of the horizon ($V > 0$) with the opposite Rindler energy. Upon tracing over the states of different R-quanta in eq. (2.94) one obtains a thermal density matrix, with temperature $T = a/2\pi$, defined on the subspace of L-quanta states. This is exactly what was seen upon computing the flux on $\mathcal{I}^+$ in eqs ($2.76 \rightarrow 2.83$).

As stated, to each particle created on $\mathcal{I}^+$ there corresponds a bump of something on the other side ($V > 0$) which is correlated to it. To understand that this correlated bump is really present, consider a charged field and a measurement that reveals the production after reflection of a quantum of positive charge. Then the correlated bump necessarily has unit negative charge.

To exhibit the correlations between the produced particle and its partner configuration we consider

a packet localized around the line U_0, having mean energy k_0. This packet issues from a vacuum fluctuation which is propagating in the V direction and which (from the reflection condition) is centered around $V = -a^{-1}e^{-aU_0}$. From the perfect symmetry ($V \rightarrow -V$) between φ_k^{outL} and φ_k^{outR}, the "partner fluctuation" is centered around $V = +a^{-1}e^{-aU_0}$. This particular configuration of the field gives one of the contributions to $\langle 0_{in}|T_{\mu\nu}|0_{in}\rangle$. Our object therefore is to decompose this in-vacuum expectation value into its component parts, these latter constituting a complete set of post-selected photons, i.e. out states. Thus we introduce a complete set of localized out-states (wave packets), considering the tensor products of all possible states of arbitrary numbers of right and left quanta, we write (as in eq. (1.46))

$$\langle 0_{in}|T_{\mu\nu}|0_{in}\rangle = \sum_{\substack{\{n_k^{outL}\}\\ \{n_{k'}^{outR}\}}} \langle 0_{in}|\{n_k^{outL}\}\{n_{k'}^{outR}\}\rangle\langle\{n_k^{outL}\}\{n_{k'}^{outR}\}|T_{\mu\nu}|0_{in}\rangle \tag{2.95}$$

Let us consider that piece of this expression wherein the post selected states are of the form

$$\left(\int_0^\infty dk\ c_k a_k^{outL\dagger}\right)|0_{outL}\rangle|\{n_{k'}^{outR}\}\rangle \tag{2.96}$$

i.e. where we have specified that the *outL* state factor contains one particle in the packet

$$\psi = \int_0^\infty dk\ c_k^* \varphi_k^{outL} \tag{2.97}$$

(with the normalization $\int dk|c_k|^2 = 1$) but the number or type of R-quanta in $|\{n_{k'}^{outR}\}\rangle$ not prescribed. This partial specification is appropriate in the present situation wherein only the *out* L quanta are realized on-shell [63]. However we shall soon prove that, due to the correlations in the pure state $|0_{in}\rangle$ the configuration in R is automatically specified as well.

It is convenient to define the projection operator

$$\Pi = \mathrm{I}_{outR} \otimes \int_0^\infty dk\ c_k a_k^{outL\dagger}|0_{outL}\rangle\langle 0_{outL}|\int_0^\infty dk\ c_k^* a_k^{outL} \tag{2.98}$$

which projects onto states of the form (2.96) since I_{outR} is the identity operator in the $V > 0$ region. One may then rewrite the decomposition (2.95) in such manner as to isolate the contribution from Π:

$$\langle 0_{in}|T_{\mu\nu}|0_{in}\rangle \;=\; \langle 0_{in}|\Pi T_{\mu\nu}|0_{in}\rangle + \langle 0_{in}|(I-\Pi)T_{\mu\nu}|0_{in}\rangle \tag{2.99}$$

We rewrite the first term on the right hand side of eq. (2.99) as

$$\langle 0_{in}|\Pi T_{\mu\nu}|0_{in}\rangle \;=\; \langle 0_{in}|\Pi|0_{in}\rangle\left[\frac{\langle 0_{in}|\Pi T_{\mu\nu}|0_{in}\rangle}{\langle 0_{in}|\Pi|0_{in}\rangle}\right] \tag{2.100}$$

so as to express it as the product of the probability of being in the state (2.96) times the weak value of $T_{\mu\nu}$ in that state (see eq. (1.52)). In the sequel it is this non diagonal matrix element (weak value)

$$\langle T_{\mu\nu}\rangle_w = \frac{\langle 0_{in}|\Pi T_{\mu\nu}|0_{in}\rangle}{\langle 0_{in}|\Pi|0_{in}\rangle} \tag{2.101}$$

which we shall analyze, since from the above analysis this is the value of $T_{\mu\nu}$ which corresponds to the final state $\Pi|0_{in}\rangle$ which we now construct explicitly.

We first check that the specification of the *out* L quantum uniquely fixes the configuration in R to be its partner. To this end we change basis from the waves φ_k^{outL} to a complete set of wave packets labeled by i. The matrix of this change of basis is the unitary matrix γ_{ik}. We shall take the wave packet $i = 0$ to be that specified in eq. (2.96), i.e. $\gamma_{0k} = c_k$. We now rewrite the argument of the exponential in eq. (2.94) so as to isolate the creation operators of the wave packets labeled by i:

$$\int dk\ \frac{\beta_k}{\alpha_k} a_k^{outR\dagger} a_k^{outL\dagger} = \int dk \int dk' \sum_i \frac{\beta_k}{\alpha_k} a_k^{outR\dagger} \gamma_{ik}^* \gamma_{ik'} a_{k'}^{outL\dagger} \tag{2.102}$$

Hence eq. (2.94) becomes

$$\begin{aligned} |0_{in}\rangle \;&=\; \frac{1}{\sqrt{Z}} \exp\left(\int dk' c_{k'} a_{k'}^{outL\dagger} \int dk\ c_k^* \frac{\beta_k}{\alpha_k} a_k^{outR\dagger} \right) \\ &\qquad \otimes \prod_{i\neq 0} \exp\left(\int dk' \gamma_{ik'} a_{k'}^{outL\dagger} \int dk\ \gamma_{ik}^* \frac{\beta_k}{\alpha_k} a_k^{outR\dagger} \right) |0_{out}\rangle\ . \end{aligned} \tag{2.103}$$

Note that we have arranged this construction so as to put into evidence the combination that creates the observed wave packet ($\int dk' c_{k'} a_{k'}^{outL\dagger}$). This construction shows clearly the asymmetry between the particle and its partner wave functions induced by the presence of β_k/α_k. This will be crucial in what follows.

Since by construction all the states created by the operators $\int dk' \gamma_{ik'} a_{k'}^{outL\dagger}$ ($i \neq 0$) are orthogonal to the states involving $\int dk' c_{k'} a_{k'}^{outL\dagger}$, the state $\Pi|0_{in}\rangle$ is easily found to be

$$\Pi|0_{in}\rangle = \frac{1}{\sqrt{Z}} \int dk' c_{k'} a_{k'}^{outL\dagger} |0_{outR}\rangle \int dk\ c_k^* \frac{\beta_k}{\alpha_k} a_k^{outR\dagger} |0_{outL}\rangle\ . \tag{2.104}$$

In this way we see that in the projection Π of eq. (2.98) onto $|0_{in}\rangle$ there is an implied specification of the partner. This EPR [79] effect results from the global structure of the Heisenberg state $|0_{in}\rangle$.

All is now ready for the evaluation of eq. (2.101). In order to simplify the notation we shall calculate $\phi(x)\phi(x')$ rather than $T_{\mu\nu}$. The latter is obtained by taking derivatives with respect to x, x' and then the coincidence limit. By expressing the *out* operators which appear on the right hand side of eq. (2.104) in terms of *in* operators and writing $\phi(x)\phi(x')$ in terms of the *in* basis a straightforward calculation yields

$$\begin{aligned}
\langle\phi(x)\phi(x')\rangle_w &= \frac{\langle 0_{in}|\Pi\phi(x)\phi(x')|0_{in}\rangle}{\langle 0_{in}|\Pi|0_{in}\rangle} \\
&= \frac{\langle 0_{out}|\phi(x)\phi(x')|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} \\
&\quad + \Big[\frac{\left(\int_0^\infty dk\ (c_k^*/\alpha_k)\hat{\varphi}_k^*(x)\right)\left(\int_0^\infty dk'\ c_{k'}(\beta_{k'}/\alpha_{k'}^2)\hat{\varphi}_{-k'}^*(x')\right)}{\int_0^\infty dk\ |c_k|^2(\beta_k/\alpha_k)^2} \\
&\qquad + (x) \leftrightarrow (x')\Big] \qquad . \qquad (2.105)
\end{aligned}$$

To derive eq. (2.105) we use eq. (2.93) to obtain the sequence of equalities:

$$\begin{aligned}
\langle 0_{out}|a_p^R a_q^L \phi\phi|0_{in}\rangle &= \langle 0_{out}|\frac{1}{\alpha_p}\hat{a}_{-p}\ \frac{1}{\alpha_q}(\hat{a}_q + \beta\hat{a}_q^{R\dagger})\phi\phi|0_{in}\rangle \\
&= \frac{\beta_q}{\alpha_q}\delta(p-q)\langle 0_{out}|\phi\phi|0_{in}\rangle + \langle 0_{out}|\frac{1}{\alpha_p\alpha_q}\hat{a}_{-p}\hat{a}_q\phi\phi|0_{in}\rangle
\end{aligned} \qquad (2.106)$$

Expanding ϕ in $\hat{\varphi}_k$ and making the packet construction indicated in eqs (2.103) and (2.104) yields the two terms in eq. (2.105) when the denominator $\langle 0_{in}|\Pi|0_{in}\rangle$ is taken into account. This denominator gives the probability to find the state eq. (2.104) on $\mathcal{I}^+$. It is given by

$$\begin{aligned}
\langle 0_{in}|\Pi|0_{in}\rangle &= |\langle 0_{out}|0_{in}\rangle|^2 \int_0^\infty dk\ |c_k|^2(\beta_k/\alpha_k)^2 \\
&= \frac{1}{Z}\int_0^\infty dk\ |c_k|^2(\beta_k/\alpha_k)^2.
\end{aligned} \qquad (2.107)$$

(as for the expression for P_E given after eq. (1.59)).

The first term in eq. (2.105) is background. It is evaluated by expressing $\phi(x)$ in terms of *out*-modes and $\phi(x')$ in terms of *in*-modes and one obtains

$$\begin{aligned}
\frac{\langle 0_{out}|\phi(x)\phi(x')|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} &= \langle 0_{in}|\phi(x)\phi(x')|0_{in}\rangle \\
&\quad - \int_0^\infty dk\frac{\beta_k}{\alpha_k}[\hat{\varphi}_k^*(x)\hat{\varphi}_{-k}^*(x') + (x \leftrightarrow x')]
\end{aligned} \qquad (2.108)$$

The first term gives the mean value of $T_{\mu\nu}$ calculated in the previous subsection.

The second term is equal to $-2\int_0^\infty \beta_k^2(|\varphi_k^L|^2+|\varphi_k^R|^2)dk$. Taking derivatives with respect to U and the coincidence limit, gives $-\int_0^\infty kn(k)dk$ which is the negative Rindler energy density in Rindler vacuum. Indeed, in Rindler vacuum (paragraph after eq. (2.43)), using Rindler coordinates (2.20), one has

$$\langle 0_{Rindler}|T_{uu}|0_{Rindler}\rangle = -(\pi/12)T^2 \tag{2.109}$$

This can be interpreted as the removal of the thermal distribution of Rindler quanta present in Minkowski vacuum. In the present case it corresponds to the removal of the energy of all the produced quanta.

Putting the two contributions together, we thus have

$$\frac{\langle 0_{out}|T_{UU}|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} = \frac{\langle 0_{out}|\partial_U\phi\partial_U\phi|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} = 0 \tag{2.110}$$

$$\frac{\langle 0_{out}|T_{VV}|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} = \frac{\langle 0_{out}|\partial_V\phi\partial_V\phi|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} = -\frac{\pi}{12}\frac{1}{(2\pi a)^2}\frac{1}{(aV)^2} \tag{2.111}$$

since $dU/dV = 1/aV$ see eq. (2.69). The absence of outgoing flux in the in-out matrix element (2.110) is natural since it corresponds to a state with no out-particle produced. This specification implies that before reflection the vacuum fluctuations leading to the production of out-particles be absent, hence to the negative Rindler energy (2.111). We note that we have neglected to treat properly the singularity at $V = 0$. This shall be analyzed subsequently.

We now consider the second term of eq. (2.105) (hereafter denoted $\langle T_{\mu\nu}\rangle_\psi$) which contains the contribution of the selected particle ψ eq. (2.97). Its contribution to $\langle T_{UU}\rangle_w$ is

$$\langle T_{UU}\rangle_\psi = 2\frac{\left(\int_0^\infty dk\ c_k(\beta_k^2/\alpha_k^2)\partial_U\varphi_k^{outL}\right)\left(\int_0^\infty dk'\ c_{k'}^*\partial_U\varphi_{k'}^{outL*}\right)}{\int_0^\infty dk\ |c_k|^2(\beta_k^2/\alpha_k^2)} \tag{2.112}$$

To calculate the energy of the particle we shall take a gaussian packet $c_k = e^{-ikU_0}e^{-\Delta^2(k-k_0)^2/2}$ where the phase factor e^{-ikU_0} locates the produced particle around $U = U_0$, its energy being approximatively k_0. Then one verifies by saddle point integration that $\langle T_{UU}\rangle_\psi$ is also located around $U = U_0$ and carries also the energy

$$\int_{-\infty}^{+\infty} dU \langle T_{UU}\rangle_\psi = \frac{(\int_0^\infty dk\ k|c_k|^2(\beta_k^2/\alpha_k^2))}{(\int_0^\infty dk\ |c_k|^2(\beta_k^2/\alpha_k^2))} \simeq k_0 \tag{2.113}$$

As in the electric field, once the selected particle is on mass shell the ψ part of the weak value behaves classically.

We now consider the contribution of ψ to $\langle T_{VV}\rangle_w$. Before reflection ($U < U_0$), there is a piece both for $V < 0$ (which upon reflection becomes $\langle T_{UU}\rangle_\psi$) and for $V > 0$ (the partner contribution):

$$\begin{aligned}\langle T_{VV}\rangle_\psi \quad &= \quad 2\theta(-V)\frac{\left(\int_0^\infty dk\ c_k(\beta_k^2/\alpha_k^2)\partial_V\varphi_k^{outL}\right)\left(\int_0^\infty dk'\ c_{k'}^*\partial_V\varphi_{k'}^{outL*}\right)}{\int_0^\infty dk\ |c_k|^2(\beta_k^2/\alpha_k^2)}\ + \\ &\quad 2\theta(+V)\frac{\left(\int_0^\infty dk\ c_k(\beta_k/\alpha_k)\partial_V\varphi_k^{outR}\right)\left(\int_0^\infty dk'\ c_{k'}^*(\beta_{k'}/\alpha_{k'})\partial_V\varphi_{k'}^{outR*}\right)}{\int_0^\infty dk\ |c_k|^2(\beta_k^2/\alpha_k^2)}\end{aligned} \tag{2.114}$$

After reflection, for $U > U_0$, only the $\theta(+V)$ piece remains since the $\theta(-V)$ is reflected and gives $\langle T_{UU}\rangle_\psi$. To grasp the energy content of the particle and partner let us first check that each "Rindler" piece (proportional to $\theta(+V)$ and $\theta(-V)$ respectively) carries the same "Rindler" energy (i.e. $iaV\partial_V$) approximatively equal to k_0. To this end we introduce $v = a^{-1}\ln|aV|$ and analyse the "Rindler" flux T_{vv}. One has: $\int_{-\infty}^{+\infty} dv\ \langle T_{vv}\rangle = \int_0^{\pm\infty} dV\ aV\langle T_{VV}\rangle$ with the $\pm$ corresponding to the $\theta(\pm V)$ pieces respectively. In both cases one finds

$$\int_{-\infty}^{+\infty} dv\ \langle T_{vv}\rangle = \left(\int_0^\infty dk\ k|c_k|^2(\beta_k^2/\alpha_k^2)\right) / \left(\int_0^\infty dk\ |c_k|^2(\beta_k^2/\alpha_k^2)\right) \simeq k_0 \tag{2.115}$$

For the $\theta(-V)$ piece this simply follows from eq. (2.113) since $v = U$ upon reflection. For the $\theta(+V)$ piece it follows from the orthogonality rules enjoyed by the φ^{outR} modes.

Thus $\langle T_{vv}\rangle_w$ is simply obtained by adding the "Rindler energy" density of the ψ-Rindler pair to the background, i.e. the first term of eq. (2.105) which we have explained gives the Rindler vacuum see eq. (2.111).

However $\langle T_{VV}\rangle_\psi$ is quite asymmetric and its analysis is more subtle. This asymmetry is already present in eq. (2.104) where the partner wave function is given explicitly. The source of the asymmetry in the formalism comes from the use of packets which is necessary to exhibit correlations in space-time.

Indeed the detection of a particle by a counter requires a description in terms of a broad packet for this outgoing particle. Once this is done the partner's packet becomes complicated: the convolution of the particle Fourrier components with the energy dependence of the ratio β/α, see eq. (2.104). One loses therefore the simplicity of the Bogoljubov transformation between Unruh and Rindler modes (eq. (2.92)). Hence the two terms on the right hand side of eq. (2.114) are not symmetric with respect to $V = 0$.

Continuing with the configuration before reflection we emphasize that

$$\int_{-\infty}^{+\infty} dV \ \langle T_{VV} \rangle_\psi = 0 \tag{2.116}$$

$$\int_{-\infty}^{+\infty} dV \ \frac{\langle 0_{out} | T_{VV} | 0_{in} \rangle}{\langle 0_{out} | 0_{in} \rangle} = 0 \tag{2.117}$$

for $U \leq U_0$. To see this, recall that $\int_{-\infty}^{+\infty} dV \ T_{VV} |0_{in}\rangle = 0$ on $\mathcal{I}^-$ in virtue of normal ordering (i.e. subtraction of the zero point energy of each mode, here the energy carried by each orthogonal packet). Thus $\int_{-\infty}^{+\infty} dV \ T_{VV} |0_{in}\rangle$ vanishes mode by mode on $\mathcal{I}^-$.

Equation (2.117) implies that the energy distribution in eq. (2.111) has a positive singular contribution at $V = 0$ which exactly compensates the negative energy density for $V \neq 0$ [71], [63]. This singularity will come up once again in Sections 2.6 and 3.5 where it will play a critical role in ensuring the consistency of the theory.

Let us now decompose $\int_{-\infty}^{+\infty} dV \ \langle T_{VV} \rangle_\psi$ into the contribution from the particle $\int_{-\infty}^{0} dV \ \langle T_{VV} \rangle_\psi$ and from the partner $\int_{0}^{+\infty} \langle T_{VV} \rangle_\psi$.

Starting with the latter one sees that the integral is positive since the integrand is (as is seen by inspection of the second term in the right hand side of eq. (2.114)). One estimates $\int_{0}^{+\infty} dV \ \langle T_{VV} \rangle_\psi \simeq k_0 e^{aU_0}$ (since we know that $\int_{0}^{+\infty} dV \ aV \langle T_{VV} \rangle_\psi \simeq k_0$ and it may be checked by stationary phase that the main contribution comes from $V_0 \simeq a^{-1} e^{-aU_0}$). This relation between "Rindler" and Minkowski energy is exactly that given by the Doppler shift equation (2.68).

These properties imply that $\int_{-\infty}^{0} dV \ \langle T_{VV} \rangle_\psi$ is negative and $\simeq -k_0 e^{aU_0}$. But we have seen that the "Rindler" energy $\int_{-\infty}^{0} dV \ |aV| \langle T_{VV} \rangle_\psi \simeq k_0$ is positive. How is that possible? The answer is that $\theta(-V) \langle T_{VV} \rangle_\psi$ is not positive definite and contains strong negative oscillations for small values of V. When evaluating $\int_{-\infty}^{0} dV \ (-aV) \langle T_{VV} \rangle_\psi$ (or upon reflection, $\int_{-\infty}^{+\infty} dU \langle T_{UU} \rangle_\psi$) these oscillations give negligible contributions. But in the Minkowski energy the oscillations near $V = 0$ play a dominant rôle (because of the weight factor $|V|^{-1}$ in going from Rindler to Minkowski energy) in such fashion as to make the Minkowski energy of particle plus partner vanish. We shall discuss similar oscillatory effects in quantitative detail in the next section.

In conclusion the post selection of an outgoing photon of frequency k_0 entails a distribution of T_{VV} which exists on both sides of the horizon $V = 0$. The partner piece, localized around the line $V_0 = +a^{-1}e^{-aU_0}$ propagates forever and has Minkowski energy equal to the Doppler shifted value given by $k_0 e^{aU_0}$. The particle piece propagates out to $U = U_0$ and is then reflected. Before reflection it carries energy equal and opposite to that of the partner. This energy has a positive piece equal to $k_0 e^{aU_0}$ centered around $V = -a^{-1}e^{-aU_0}$ and a broader oscillating piece which is net negative in such fashion that the sum of all contributions vanishes. After reflection this piece has gotten converted into T_{UU}. The energy carried in the reflected wave is almost all in the center positive piece (see eq. (2.112)). The oscillating negative piece is diffuse and negligible. For $U > U_0$ the partner continues alone carrying a net positive energy always equal to $k_0 e^{aU_0}$. However it is not a quantum in the usual sense. It does not manifest itself on the average since the field configurations, on the average, are still Minkowski (by causality). To pick up the effect of the partner requires an EPR correlation type experiment. What is important is that the fluctuations in play when an outgoing particle is produced have energies which blow up exponentially and which hugs the horizon at exponentially small distances. It is this circumstance which constitutes a major hiatus in the more realistic case of black hole collapse.

Had we post selected the absence of an outgoing photon we would come upon an "anti-partner" whose energy is negative being the weighted sum of the energies of all the negative energy corresponding to the absence of of $1, 2, \ldots, n, \ldots$ photons. As previously calculated (eq. (2.111) this is precisely the energy of the Rindler vacuum (i.e. the absence of the average thermal energy). The sum of all weak values is of course the net average as given in Section (2.5.2).

Perhaps a more physical way to exhibit the correlations between the emitted U photons and the V partners is to decelerate the mirror after a while. As seen in Fig 2.4,

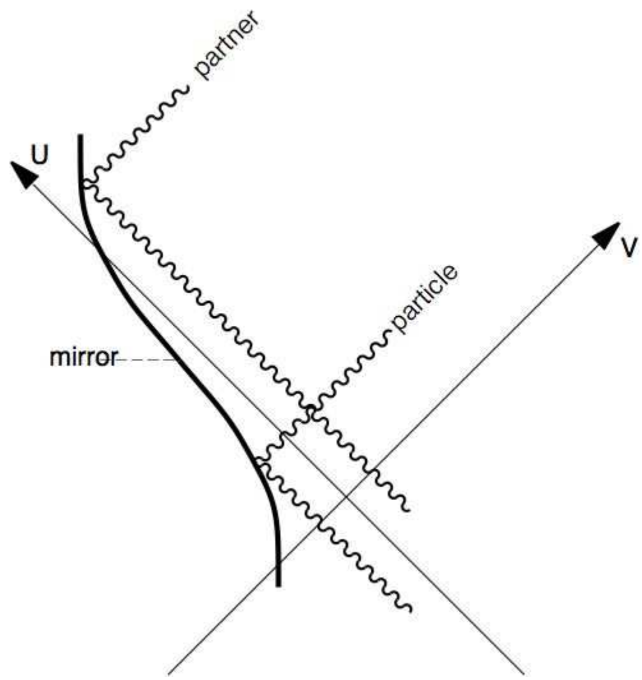

Fig. 2.4 *The trajectory of a mirror which decelerates after a while. The classical trajectories (stationnary phase) of a pair of particles are indicated by wavy lines. It is apparent that the deceleration of the mirror allows the partners to be realized on shell.*

φ_k^{outR} will then be reflected at late times, transformed into a real quantum and can therefore be observed, in particular in coincidence with φ_k^{outL}. One could choose a trajectory which starts from rest $V - U = const$ and come back to rest after having followed the trajectory eq. (2.69) for a while. The main point is that after the mirror becomes inertial again the system goes back to vacuum plus the radiation that goes out to infinity, always a pure state. This was pointed out by Carlitz and Willey in [21] who suggested possible applicability to the black hole problem (this was also mentioned in [74] and discussed in [100]).

2.6 The Energy Emitted by the Accelerated Detector

2.6.1 Introduction and Qualitative Description

Though not directly related to black hole physics we include this section in our review, not only because it is of interest in itself, given the rather stormy history of the problem, but also because one comes upon concepts that arise in the black hole problem as well.

We shall present in detail the energy emitted by the accelerated detector. The first subsection is qualitative and shows how the paradoxes which have been raised (see for instance and in historical order [94], [41], [80], [92], [64], [4], [63]) can be rationalized. In the second subsection the energy emitted is computed in perturbation theory and particular attention is paid to how it is correlated to the final state of the atom. In the third subsection the same decomposition of final states is used to display the vacuum fluctuations which induce the transitions giving rise to these final states.

It has been seen in Section 2.4 that in perturbation theory the leading order in H_{int} corresponds to photon *emission* in both cases, excitation and deexcitation, as it should be since one perturbs Minkowski vacuum. Nevertheless as Grove pointed out [41], when equilibrium is reached there is no net change of the state of the radiation in the quadrant, R, of the accelerator except for transient effects. We refer to this as Grove's theorem. His argument is the following. The accelerator feels the effect of a thermal bath. So first consider the inertial two level system in thermal equilibrium. The principal ingredients which guarantee the absence of net energy flux to or from the atom is the time independence of the Hamiltonian and the stationarity of the state of the atom (in the thermodynamic sense), so that each photon which is absorbed is re-emitted with the same energy, i.e. energy conservation results from time translational symmetry. Equilibrium is maintained through the implementation of the Einstein conditions.

This argument is immediately applicable to the accelerator since the fact that a = constant implies that his physics is translationally symmetric in his proper time (ie. invariance under boosts). Since Minkowski vacuum is also an eigenstate of the boost operator, the implication is that the total eigenvalue of $\partial/\partial\tau$ is conserved. The dynamical realization is the time averaged conservation of the energy of the totality of Rindler quanta which are absorbed and emitted, in strict analogy to thermal equilibrium. This implies no net energy flux.

The above considerations result in a dilemma since, as we said, both excitation and deexcitation of the atom leads to emission of a Minkowski photon (see for instance the amplitude eq. (2.48) wherein the first order interaction of the atom with the field always results in the creation of a photon of frequency ω). The resolution of the dilemma will be shown to lie in a global treatment of the radiation field which also takes into account the transients due to switching on and off the detector. These transients play a rôle similar to the oscillations encountered near $V = 0$ in equation (2.116) and (2.117) which ensure the global vanishing of the energy.

The above general considerations have been verified in an exact model, that of an accelerating harmonic oscillator coupled linearly to the radiation field [80], [92], [64]. It is noteworthy that in this work Heisenberg equations of motion have been integrated to give a long time steady state solution wherein initial conditions become irrelevant. This is what makes the analogy to thermal equilibrium possible. Rather than describe the exact oscillator system we shall continue with the two level atom in perturbation theory (since this is more relevant for the understanding of some corresponding problems which come up in the black hole problem).

A detailed picture of the steady state emerges from the following consideration. Focus on the ground state of the accelerator which excites by absorbing a Rindleron coming in from its left. Then the field configurations to its right is depleted of this Rindleron. Since this Rindleron carried positive energy, its removal can be described as the emission of negative energy to the right. In equilibrium there is also to be considered the process of deexcitation corresponding to the emission of positive energy to the right. The Einstein relation eq. (2.10) guarantees that the two cancel.

A nice way to express the physics is to appeal to exact eigenstates of the photon field, scattering states. Their energy (Minkowski energy for the inertial atom, Rindler energy for the accelerating one) is the same as that of the free states and their number is also the same since the scattering matrix is unitary. Therefore the average energy of the radiation field of any (mixed) state is unperturbed by the scatterer.

We now discuss qualitatively the transient behavior in both the Rindler and Minkowski representation of the radiation field by calculating the energy density emitted in terms of the mean energy momentum tensor, $\langle T_{\mu\nu} \rangle$. Take, for example, right movers. The relevant energy density is $\langle T_{UU} \rangle =$

$\langle(\partial\phi/\partial U)^2\rangle$.

By Grove's theorem, one should have $\langle T_{uu}\rangle$, the energy density measured by a co–accelerator, equal to zero in the absence of transients. (Once more we normalize the energy so that the expectation values of T_{UU} and T_{VV} vanish in Minkowski vacuum, therefore of T_{uu} and T_{vv} as well). Now consider the transients, with switch on (off) time at τ_i (τ_f) modeled by some function $f(\tau)$ which vanishes outside the interval (τ_i,τ_f) and is equal to 1 inside the interval except for a time $\Delta\tau$ during which f passes smoothly from 0 to 1. We choose $\tau_f-\tau_i >> \mathrm{Max}(\Delta M^{-1},a^{-1})$ which is the time necessary to establish the Golden rule. In addition we take $\tau_f-\tau_i >> \Delta\tau$. The compensation mechanism based, as it is, on translational invariance in time is then no longer operative. Insofar as T_{uu} is concerned this will introduce minor effects, but these become dramatic for T_{UU} owing to the exponential character of the Doppler shift measured by the inertial observer. Indeed the total Minkowski energy emitted is

$$E_M=\int_{-\infty}^{0}dU < T_{UU} >=\int_{-\infty}^{+\infty}du < T_{uu} > \frac{du}{dU} \tag{2.118}$$

where $du/dU=e^{au}$. The first integral in eq. (2.118) is limited to the domain $U<0$ because the accelerator lives in R thereby confining right movers (i.e. those which contribute to T_{UU} in the integrand) to the quadrants P and R. To pick up the total energy emitted, one must integrate along the surface $V=V_0$ with $V_0>a^{-1}e^{a\tau_f}$ so that all emitted right movers cross the surface (see Fig. (2.5)). From what has been said the integrand vanishes except at the endpoints u_i $(=\tau_i)$ and u_f $(=\tau_f)$ whereupon eq 2.118 integrates to a form

$$E_M=C_fe^{a\tau_f}-C_ie^{a\tau_i} \tag{2.119}$$

where C_i and C_f depend on the exact form of the switching function as it turns on and off the interaction.

Most surprisingly eq. (2.119) can be approximately written as an integral over the naive rates of absorption and of emission of Rindler photons taking into account that each transition, be it excitation or deexcitation, is accompanied by the emission of a Doppler shifted Minkowski photon, according to the resonance condition $\omega(\tau)=\Delta Me^{a\tau}$ (eq; (2.49)) i.e.

$$\begin{aligned}E_M &= \int_{\tau_i}^{\tau_f}d\tau\ f(\tau)(R_+p_-+R_-p_+)\Delta Me^{a\tau}\\ &= (R_+p_-+R_-p_+)\Delta M(e^{a\tau_f}-e^{a\tau_i})+(C'_fe^{a\tau_f}-C'_ie^{a\tau_i})\end{aligned} \tag{2.120}$$

Here $R_\pm$ are rates of excitation (deexcitation) and $p_\pm$ are probabilities of being in excited (ground) states (Einstein's condition eq. (2.10) is $R_+p_- = R_-p_+ =$constant). The constants C'_i and C'_f depend on the form of the function f near the endpoints. So the integral eq. (2.120) is of the same form as eq. (2.119). In the subsequent development we shall see how it is that the energy density $< T_{UU} >$ which appears in eq. (2.118) can be decomposed into a positive steady piece and an interference term. The positive piece has an integral of the form eq. (2.120) and should be interpreted in a similar way. The interference term carries no net energy, however it plays an essential role in ensuring that causality be respected. In the interval (τ_i, τ_f) far from the transients the interference term is negative and exactly compensates the positive piece so as to recover Grove's theorem in the region where it should hold, this being characterized in good approximation by translational symmetry. This detailed method proceeds event by event so as to provide a description of the radiation field in all four quadrants and its correlations to the detector.

[This state of affairs wherein transients have a global content which depends on the whole history also occurs in the problem of the classical electromagnetic field emitted by a uniformly accelerated charge in 3 dimensional Minkowski space. Here also one can argue, in a way strongly reminiscent of Grove's result, that no radiation should be emitted. To wit, the equivalence principle asserts that a uniformly accelerated charge is equivalent to a static charge in a uniform gravitational field. In a static situation no energy should be emitted (this is confirmed by everyday experience at the earth's surface) hence none should be emitted in the accelerated case. On the other hand the charge should radiate at a constant rate given by Larmor's formula

$$\frac{dE}{d\tau} = \frac{2}{3}\left(\frac{e^2}{4\pi}\right)\dot{u}^2 u^0 \tag{2.121}$$

where $\frac{dE}{d\tau}$ is the energy radiated per unit proper time of the particle, $\dot{u}^2$ is the square of the acceleration of the particle.

This dilemma was resolved by Boulware [14] who showed that both results are compatible. The crucial remark is that the equivalence principle is valid only "inside" the right quadrant (where the static Rindler coordinates eq. (2.2) are valid) and indeed inside R the field is the Coulomb field of the charge with no radiation part. However Maxwell's equations imply that along the past horizon $U = 0$ there is a delta like singularity in the field. This singularity can be interpreted as the infinitely blue shifted transient which occurred when the charge was set into acceleration. This singularity along the past horizon is unobservable by a coaccelerator, but in the future quadrant it propagates and gives rise to a flux of radiation at exactly the rate eq. (2.121). Hence once more transients in the accelerated frame acquire a global content for the Minkowski observer.]

2.6.2 The energy emitted at $O(g^2)$

We first consider the energy emitted during spontaneous excitation of the atom. The coupled state of atom and field is

$$\begin{aligned} |\psi_-\rangle &= \mathrm{T}e^{-i\int H_{\mathrm{int}}d\tau'}|0_M\rangle|-\rangle \\ &= |0_M\rangle|-\rangle - ig\int_{\tau_i}^{\tau_f} d\tau' e^{i\Delta M\tau'}\phi(\tau')|0_M\rangle|+\rangle \\ &\quad -g^2\int_{\tau_i}^{\tau_f} d\tau_2 \int_{\tau_i}^{\tau_2} d\tau_1 e^{-i\Delta M\tau_2}\phi(\tau_2)e^{i\Delta M\tau_1}\phi(\tau_1)|0_M\rangle|-\rangle + O(g^3) \end{aligned} \tag{2.122}$$

where T is the time ordering operator. $|+\rangle$ ($|-\rangle$) refer to excited (ground) state of the atom respectively. The mean energy momentum is (considering once more only right movers and therefore T_{UU})

$$\langle T_{UU}\rangle_- = \langle\psi_-|T_{UU}|\psi_-\rangle \tag{2.123}$$

The mean T_{UU} eq. (2.123) is dissected into its constituent parts by considering the final state of the atom [94], [41], [63] (i.e. by making a post selection similar to eq. (2.99) where the final state of the radiation field was specified). This is realized by inserting into eq. (2.123) the projectors Π_+ ($\Pi_- = 1-\Pi_+$) onto the excited (ground) state of the atom:

$$\langle T_{UU}\rangle_- = \langle\psi_-|\Pi_+T_{UU}|\psi_-\rangle + \langle\psi_-|\Pi_-T_{UU}|\psi_-\rangle \tag{2.124}$$

We have

$$\langle\psi_-|\Pi_+T_{UU}|\psi_-\rangle = g^2\langle 0_M|\int_{\tau_i}^{\tau_f} d\tau_1 e^{-i\Delta M\tau_1}\phi(\tau_1)T_{UU}\int_{\tau_i}^{\tau_f} d\tau_2 e^{i\Delta M\tau_2}\phi(\tau_2)|0_M\rangle \tag{2.125}$$

Equation (2.125) is the energy emitted due to excitation of the atom as it would be calculated in lowest order perturbation theory ($O(g)$ in the wave function, $O(g^2)$ in the energy). But, to $O(g^2)$, one requires the correction to the wave function as well, thereby leading to interference terms. These are in the Π_- term of eq. 2.124 corresponding to emission and reabsorption in the wave function. This term is

$$\langle\psi_-|\Pi_- T_{UU}|\psi_-\rangle = -2g^2 \mathrm{Re}\langle 0_M|\int_{\tau_i}^{\tau_f} d\tau_2 e^{-i\Delta M\tau_2}\phi(\tau_2)\int_{\tau_2}^{\tau_f} d\tau_1 e^{i\Delta M\tau_1}\phi(\tau_1) T_{UU}|0_M\rangle \tag{2.126}$$

It is convenient to reexpress eq. (2.126) as

$$\langle\psi_-|\Pi_- T_{UU}|\psi_-\rangle = -g^2 \mathrm{Re}[D(U)+\mathcal{F}(U)] \tag{2.127}$$

where

$$\begin{aligned} D(U) &= \langle 0_M|\int_{\tau_i}^{\tau_f} d\tau_2 e^{-i\Delta M\tau_2}\phi(\tau_2)\int_{\tau_i}^{\tau_f} d\tau_1 e^{i\Delta M\tau_1}\phi(\tau_1) T_{UU}|0_M\rangle \\ \mathcal{F}(U) &= \langle 0_M|\left[\int_{\tau_i}^{\tau_f} d\tau_2 \int_{\tau_i}^{\tau_f} d\tau_1 \epsilon(\tau_2-\tau_1) e^{-i\Delta M\tau_2}\phi(\tau_2) e^{i\Delta M\tau_1}\phi(\tau_1), T_{UU}\right]|0_M\rangle \end{aligned} \tag{2.128}$$

Complications from time ordering are no longer present in $D(U)$. The term $\mathcal{F}(U)$ plays no important role in the physics. Indeed a detailed analysis shows that it enjoys the following properties:

1) It is smaller than $D(U)$ by a factor $1/\Delta M(\tau_f - \tau_i)$ except in the transitory regime where it is comparable to D.

2) It vanishes in the non causal domain $U > 0$.

3) The integral $\int dU \mathcal{F}(U)$ vanishes (ie. it does not contribute to $E_M = \int dU T_{UU}$).

4) The integral $\int du (dU/du)^2 \mathcal{F}(U(u))$ vanishes also (ie. It does not contribute to the Rindler energy $E_R = \int du T_{uu}$).

All these properties are also valid when one introduces a switch on and off function $f(\tau)$ as in equation eq. (2.142) below. Hence from now on we shall drop the term $\mathcal{F}(U)$.

To execute the calculation, and so check out Grove's theorem at this order, we need to evaluate $\langle\psi_-|\Pi_+ T_{UU}|\psi_-\rangle$ and $D(U)$ in the limit $\tau_f - \tau_i \to \infty$. Begin with eq. (2.125) and expand $\phi(\tau_2)$ and $\phi(\tau_1)$ in Unruh modes (eq. (2.43)). These are the most convenient because their vacuum is Minkowski. Then $\phi(\tau_2)$ (and $\phi(\tau_1)$) contain the creation of an Unruh mode in R. Since $\Delta M > 0$, the part of $\phi(\tau_2)$ which contributes to the integral (i.e. resonates with the factor $e^{i\Delta M\tau_2}$) is that which corresponds to the annihilation of a

Rindler mode in R, hence carrying a factor β_λ in $\hat{\varphi}_\lambda$ (eq. (2.43)). The result is thus (compare with eq. (2.55))

$$\int d\tau e^{i\Delta M\tau}\phi(\tau)|0_M\rangle = \beta_{\Delta M}\sqrt{\pi/\Delta M}\,\hat{a}^\dagger_{\Delta M}|0_M\rangle. \tag{2.129}$$

This Unruh creation operator must be contracted with the corresponding Unruh annihilation operator appearing in T_{UU} to give

$$\begin{aligned}\langle\psi_-|\Pi_+T_{UU}|\psi_-\rangle &= \frac{\pi}{\Delta M}2g^2\beta^2_{\Delta_M}\partial_U\hat{\varphi}_{-\Delta M}(U)\partial_U\hat{\varphi}^*_{-\Delta M}(U)\\ &= \frac{g^2}{2}\frac{1}{(aU)^2}\left(\theta(-U)\beta^4_{\Delta_M}+\theta(U)\beta^2_{\Delta M}\alpha^2_{\Delta M}\right)\end{aligned} \tag{2.130}$$

The calculation of eq. (2.126) proceeds along similar lines and yields

$$\begin{aligned}\langle\psi_-|\Pi_-T_{UU}|\psi_-\rangle &= -\frac{\pi}{\Delta M}2g^2\beta^2_{\Delta_M}\mathrm{Re}\left[\partial_U\hat{\varphi}^*_{\Delta M}(U)\partial_U\hat{\varphi}^*_{-\Delta M}(U)\right]\\ &= -\frac{g^2}{2}\frac{1}{(aU)^2}\beta^2_{\Delta M}\alpha^2_{\Delta M}\end{aligned} \tag{2.131}$$

The important minus sign is the same as comes up in the verification of unitarity wherein to $O(g^2)$ the interference term cancels the direct term in

$$\begin{aligned}1 &= \langle\psi|\psi\rangle = \langle\psi_0+i\delta\psi|\psi_0-i\delta\psi\rangle\\ &= \langle\psi_0|\psi_0\rangle + \langle\delta\psi|\delta\psi\rangle + 2\mathrm{Im}\,\langle\psi_0|\delta\psi\rangle = \langle\psi_0|\psi_0\rangle\end{aligned} \tag{2.132}$$

This minus sign is in fact essential to satisfy causality which requires that all expectation values of T_{UU} vanish for $U>0$. The photons emitted cannot affect the region $U>0$. Indeed there is a rigorous theorem stating that $\langle\psi_-|T_{UU}|\psi_-\rangle = 0$ for $U>0$ which we now prove [94]:

$$\begin{aligned}\langle T_{UU}(U>0)\rangle_- &= \langle -|\langle 0_M|Te^{i\int d\tau H_{int}}T_{UU}(U>0)Te^{-i\int d\tau H_{int}}|0_M\rangle|-\rangle\\ &= \langle -|\langle 0_M|T_{UU}(U>0)Te^{i\int d\tau H_{int}}Te^{-i\int d\tau H_{int}}|0_M\rangle|-\rangle\\ &= \langle -|\langle 0_M|T_{UU}(U>0)|0_M\rangle|-\rangle\\ &= 0\end{aligned} \tag{2.133}$$

where we have used the commutativity of $\phi(\tau)$ appearing in H_{int} (i.e. on the accelerating trajectory) with $\phi(U)$ for $U>0$ appearing in T_{UU}. The same proof applies immediately if the initial state is $|\psi_+\rangle = |0_M\rangle|+\rangle$. [This proof may be also generalized to massive fields and interacting fields for $T_{UU}(t,x)$ with (t,x) in L since then it is space like separated from $\phi(\tau)$ but in general we will have $\langle T_{UU}(t,x)\rangle \neq 0$ for (t,x) in F.]

The sum of eqs. (2.130) and 2.131 gives a negative result to the average energy density

$$\langle T_{UU}\rangle_{-} = -\frac{g^2}{2}\beta^2_{\Delta M}\frac{\theta(-U)}{(aU)^2} \tag{2.134}$$

in accord with the expectation that absorption of a Rindleron diminishes the energy density. More precisely the total reduction of Rindler energy in the interval (τ_i, τ_f) at $O(g^2)$ due to absorption of right movers is

$$\begin{aligned}\int \langle T_{uu}\rangle_{-} du &= \int \langle T_{UU}\rangle_{-}(\frac{dU}{du})^2 du = -\frac{g^2}{2}\beta^2_{\Delta M}(\tau_f - \tau_i)\\ &= -\frac{R_+}{2}\Delta M(\tau_f - \tau_i)\end{aligned} \tag{2.135}$$

where we have used eq. (2.11). Dividing eq. (2.135) by the probability of excitation $P_+ = R_+(\tau_f - \tau_i)$ yields the energy emitted in U modes if the atom is found excited

$$\frac{1}{P_+}\int \langle T_{uu}\rangle_{-} du = -\Delta M/2 \tag{2.136}$$

Thus one obtains a steady absorption of energy exactly as in the usual golden rule. The factor 1/2 in eq. (2.135) arises because we have taken into account right movers only. We have set the integral $\int_{-\infty}^{+\infty} du = \tau_f - \tau_i$. This very reasonable result can be obtained rigorously by going to the limit $\tau_f - \tau_i \to \infty$ in a more controlled manner (see [63]).

The same procedure as that discussed in the paragraph preceding equation eq. (2.130) applies to the energy emitted if the initial state is $|\psi_+\rangle$. One finds

$$\langle \psi_+ | T_{UU} | \psi_+\rangle = \frac{g^2}{2}\alpha^2_{\Delta M}\frac{\theta(-U)}{(aU)^2} \tag{2.137}$$

whence

$$\int \langle \psi_+ | T_{uu} | \psi_+\rangle du = \frac{R_-}{2}\Delta M(\tau_f - \tau_i) \tag{2.138}$$

We can now check out Grove's theorem for the equilibrium situation. In the inside region $(\tau_i < \tau < \tau_f)$ one has at thermal equilibrium

$$p_+/p_- = R_+/R_- = e^{-\beta\Delta M} = \beta^2_{\Delta M}/\alpha^2_{\Delta M} \tag{2.139}$$

where p_- (p_+) is the probability to find the atom in the ground (excited) state. This guarantees no net flux:

$$p_+\langle T_{uu}\rangle_+ + p_-\langle T_{uu}\rangle_- = 0 \tag{2.140}$$

It is important to remark that the separate contributions to $\langle T_{UU}\rangle_-$ given by eq. (2.130) and eq. (2.131) are each non causal. Each has $\theta(+U)$ contributions which cancel in the sum. These non causal contributions occur because in the calculation of eq. (2.125) and eq. (2.126) we have introduced the projectors Π_+ and Π_- (the Π_+ contribution was already anticipated in the simple saddle point approximation (Section 2.4). They encode the correlations in Minkowski vacuum between L and R Rindlerons (recall eq. (2.44)). Thus in the description wherein an atom excites (deexcites) by means of Rindleron absorption (emission) its transitions correlate to the presence (absence) of the corresponding space like separated Rindler quantum in the other quadrant ($U > 0$). This is summarized in Fig. (2.5) which is presented at the end of this section.

Instrumental in the explicit realization of Grove's theorem is the negativity of $\langle T_{uu}\rangle_-$ and hence of $\langle T_{UU}\rangle_-$ in the region where translational symmetry is valid[2] (recall we have made the computation in the limit $\tau_f - \tau_i \to \infty$). But one has $\int_{-\infty}^{+\infty} dU \langle T_{UU}\rangle_- > 0$ since to order g^2 the contribution to it from eq. (2.131) must in fact vanish on the basis of a rigorous theorem

$$\int_{-\infty}^{+\infty} dU \;\langle\psi_-|\Pi_- T_{UU}|\psi_-\rangle = 0 \tag{2.141}$$

[2] After this manuscript was completed, further research on this subject was done. It has now been proven [72] that for a detector of finite mass M (but nevertheless with $M/a >> 1$) the interference term becomes negligible after a few transitions, thereby reinstating the validity of the naive Born approximation for the energy emitted. This occurs because, when recoil is taken into account, the detector shifts its orbit from $\rho = a^{-1}$ to an orbit characterized by a new horizon (ie. the center of the hyperbola which describes the detector's trajectory shifts). Thus one loses the translational invariance in τ (boost invariance) and Grove's theorem is no longer applicable. More formally, the term eq. (2.131) exists because the atom which has emitted a photon and then reabsorbed it interferes with the atom which has not made any transitions. When the atom recoils these two amplitudes no longer interfere destructively. Recoil induces decoherence. This decoherence occurs after a logarithmically short time $\tau \simeq a \ln(M/\Delta M)$. This very short time is a manifestation of the exponentially large frequencies which resonate with the accelerated atom at early times (see eq. (2.49)). The same exponential rise of frequencies in the Hawking radiation is a source of anguish when the gravitational back-reaction of these frequencies is considered. More on this in Section 3.7.

This is because $\int_{-\infty}^{+\infty} dU\ T_{UU}$ is the total energy operator carried by right movers; hence it annihilates the vacuum. (Recall we are normalizing the vacuum energy to zero). Furthermore the integral $\int_{-\infty}^{+\infty} dU\ \langle\psi_-|\Pi_+ T_{UU}|\psi_-\rangle$ is strictly positive (it is the expectation value of a positive definite operator: the hamiltonian, see eq. (2.125)).

Thus the sum of the two is positive, as stated. Therefore there is positive energy density which does not appear in eq. (2.134) and which can only come from transient behavior at the end points.

In order to study the details of the energy distribution one must introduce explicitly a switching function, for instance by taking the interaction Hamiltonian to be of the form

$$H_{int}(\tau) = f(\tau) e^{-i\Delta M\tau}\sigma_-\phi(\tau) + \text{h.c.} \tag{2.142}$$

where $f(\tau)$ controls the switching on and off of the interaction. In order for $f(\tau)$ not to induce spurious switch on and off effects it should have a sufficiently long plateau $\Delta\tau$ that the golden rule can establish itself ($\Delta\tau >> \Delta M^{-1}$ and $\Delta\tau >> a^{-1}$) (see discussion after eq. (2.8)). Furtherore in [63] it is shown that if $f(\tau)$ obeys the condition

$$\int d\tau e^{a|\tau|} f(\tau) < \infty \quad \Leftrightarrow \quad \int dt f(\tau(t)) < \infty \tag{2.143}$$

then the energy fluxes are regular and no singularities appear on the horizons.

When $f(\tau)$ does not obey eq. (2.143) the transients do not appear in finite Rindler time. Rather they are found on the horizons $U = 0$ and $V = 0$ where they give rise to singular energy fluxes. This situation is analyzed by regulating the Bogoljubov coefficients eq. (2.28) which amounts to replacing U by $U - i\epsilon$ as in eq. (2.41). When this is done one finds that the two terms eq. (2.130) and eq. (2.131) take the form

$$\begin{aligned}
\langle\psi_-|\Pi_+ T_{UU}|\psi_-\rangle &= \frac{g^2}{2a^2}\beta_{\Delta M}^2\alpha_{\Delta M}^2 (U - i\epsilon)^{-i\Delta M/a-1}(U + i\epsilon)^{i\Delta M/a-1} \\
&= \frac{g^2}{2a^2}\beta_{\Delta M}^2 \frac{1}{U^2+\epsilon^2}\left(\alpha_{\Delta M}^2\theta(U) + \beta_{\Delta M}^2\theta(-U)\right)
\end{aligned} \tag{2.144}$$

and

$$\begin{aligned}
\langle\psi_-|\Pi_- T_{UU}|\psi_-\rangle &= -\frac{g^2}{2a^2}\beta_{\Delta M}^2\alpha_{\Delta M}^2 \text{Re}\left[(U + i\epsilon)^{-i\Delta M/a-1}(U + i\epsilon)^{i\Delta M/a-1}\right] \\
&= -\frac{g^2}{2a^2}\beta_{\Delta M}^2\alpha_{\Delta M}^2 \text{Re}\left[\frac{1}{(U+i\epsilon)^2}\right]
\end{aligned} \tag{2.145}$$

Thus for $|U| > \epsilon$, eq. (2.134) remains valid whereas for $|U| < \epsilon$ all terms are positive. The total energy emitted is obtained by integrating over U (by contour integration). The contribution from eq. (2.145) vanishes as required whereas eq. (2.144) gives a positive contribution equal to $(g^2/2a^2)(\pi/\epsilon)\beta_{\Delta M}^2\alpha_{\Delta M}^2$ ie. total positive energy is radiated, one part of which is concentrated along the horizon and is positive, and the rest is in the detectors quadrant. In this way both causality and the positivity of energy of Minkowski excitations are respected.

A similar analysis is possible for $\langle\psi_+|T_{UU}|\psi_+\rangle$. In the equilibrium situation the terms from $\langle\psi_-|T_{UU}|\psi_-\rangle$ and $\langle\psi_+|T_{UU}|\psi_+\rangle$ combine in such a way that they give zero everywhere except for a singular positive energy flux on the horizon $-\epsilon < U < \epsilon$.

This completes the formal proof of the qualitative discussion presented in the preceding subsection. We thus have shown that as in the accelerated charge problem considered by Boulware this singular flux can be interpreted as infinitely blueshifted transients which occurred when the atom was set into acceleration at $\tau = a^{-1}\ln a\epsilon$. When eq. (2.143) is satisfied complicated expressions arise wherein the role of ϵ is played by $\Delta\tau^{-1}$. As these functions are not very interesting to display here, we reserve a more interesting spot in Section 2.6.3 to put them on exhibition (see Fig. (2.5)).

It is now clear how eq. (2.120) makes sense. The total contribution to $\int_{-\infty}^{+\infty} dU\ \langle\psi_-|T_{UU}|\psi_-\rangle$ is from the Π_+ contribution only and hence can be expressed as coming almost entirely from the wrong quadrant. Thus for $\beta^2 << 1$ one may write the Π_+ contribution as an integral over u_L (see eq. (2.24)) thereby obtaining the absorption part of eq. (2.120). The emission part (ie. the contribution of $|\psi_+\rangle$) is obtained in similar manner to yield the sum eq. (2.120). Causality is verified when one analyzes the complete distribution of energy, taking into account the Π_- interference term. The mean energy density radiated is then found only in the transients.

2.6.3 The Vacuum Fluctuations Correlated to the Excitations of the Atom

In the previous subsection, we have analyzed the mean energy radiated by the atom and we saw how it can be decomposed into two contributions. These correspond to the energy emitted by the atom when it is found excited or not excited at $t = \infty$. We now address the question: what configurations of energy-momentum were present in Minkowski vacuum which give rise to spontaneous excitation of the two level atom? They certainly carry zero total energy but locally should have a positive Rindler energy density to excite the atom.

We shall use the weak-value formalism since it is shown in Appendix C that it provides a framework to investigate on $\mathcal{I}^+$ or on $\mathcal{I}^-$ the nature of the field configurations correlated to the excitation of the atom.

Before the atom interacts with the radiation, we have that in the mean

$$\langle -|\langle 0_M|T_{UU}(\mathcal{I}^-)|0_M\rangle|-\rangle = 0 \tag{2.146}$$

where $T_{UU}(\mathcal{I}^-)$ is the Heisenberg operator, T_{UU}, evaluated on the surface $\mathcal{I}^-$, ie. on a surface which is temporally situated before the time τ_i when the atom begins to interact with the field.

In this mean appear two cancelling contributions according to the class of final states considered: contributions for which the atom excites in the period (τ_i, τ_f) where both $\tau_i > t$ and $\tau_f > t$ and that for which the atom does not excite in this same time interval. Thus we must project $|0_M\rangle|-\rangle$, the Schrödinger state at times $\tau < \tau_i$, into the various outcomes that are realized at later times. We carry this out by inserting at $t = +\infty$ the projectors Π_+ and Π_- (see eq. (2.124))

$$\begin{aligned} 0 &= \langle -|\langle 0_M|T_{UU}(\mathcal{I}^-)|0_M\rangle|-\rangle = \langle -|\langle 0_M|\mathrm{T}e^{i\int_{\tau_i}^{\tau_f} d\tau H_{int}}\mathrm{T}e^{-i\int_{\tau_i}^{\tau_f} d\tau H_{int}}T_{UU}|0_M\rangle|-\rangle \\ &= \langle -|\langle 0_M|\mathrm{T}e^{i\int_{\tau_i}^{\tau_f} d\tau H_{int}}(\Pi_+ + \Pi_-)\mathrm{T}e^{-i\int_{\tau_i}^{\tau_f} d\tau H_{int}}T_{UU}|0_M\rangle|-\rangle \end{aligned} \tag{2.147}$$

Let us examine the Π_+ contribution

$$\langle -|\langle 0_M|\mathrm{T}e^{i\int_{\tau_i}^{\tau_f} d\tau H_{int}}\Pi_+\mathrm{T}e^{-i\int_{\tau_i}^{\tau_f} d\tau H_{int}}T_{UU}|0_M\rangle|-\rangle \tag{2.148}$$

where Π_+ is the Heisenberg operator which projects onto the excited state at $\tau = \tau_f$. This contribution is anticipated to be positive since it is the contribution to the mean $\langle 0_M|T_{UU}(\mathcal{I}^-)|0_M\rangle$ that results in excitation. The correct normalization which appears in the weak value formalism (see eqs. (1.52) and (2.100)) is to rewrite eq. (2.148) as $P_+\langle T_{UU}(\mathcal{I}^-)\rangle_{w+}$ where P_+ is the probability for excitation in the interval (τ_i, τ_f):

$$\begin{aligned} P_+ &= \langle \psi_-|\Pi_+|\psi_-\rangle \\ \langle T_{UU}(\mathcal{I}^-)\rangle_{w+} &= \frac{1}{P_+}\langle \psi_-|\Pi_+ T_{UU}(\mathcal{I}^-)|\psi_-\rangle \end{aligned} \tag{2.149}$$

In this rewriting $\langle T_{UU}(\mathcal{I}^-)\rangle_{w+}$ is what has been identified in Section 1.4 and 2.5 and Appendix C with the energy of the field configuration which gives rise to excitation of the atom.

To order g^2 we find

$$\begin{aligned}
\langle T_{UU}(\mathcal{I}^-)\rangle_{w+} &= \frac{\langle -|\langle 0_M| \left(1 + ig\int_{\tau_i}^{\tau_f} e^{i\Delta M\tau}\phi(\tau)\sigma_+\right)\Pi_+\,(\text{h.c.})\,T_{UU}|0_M\rangle|-\rangle}{\langle -|\langle 0_M| \left(1 + ig\int_{\tau_i}^{\tau_f} d\tau\; e^{i\Delta M\tau}\phi(\tau)\sigma_+\right)\Pi_+\,(\text{h.c.})\,|0_M\rangle|-\rangle} \\
&= \frac{\langle 0_M|\int_{\tau_i}^{\tau_f} d\tau\; e^{i\Delta M\tau}\phi(\tau)\int_{\tau_i}^{\tau_f} d\tau'\; e^{-i\Delta M\tau'}\phi(\tau')T_{UU}|0_M\rangle}{\langle 0_M|\int_{\tau_i}^{\tau_f} d\tau\; e^{i\Delta M\tau}\phi(\tau)\int_{\tau_i}^{\tau_f} d\tau'\; e^{-i\Delta M\tau'}\phi(\tau')|0_M\rangle} \qquad (2.150)
\end{aligned}$$

As in eq. (2.125) the g^2 term in the wave function does not contribute to matrix elements when Π_+ is inserted.

It is instructive once more to consider the resonant piece (ie. $\tau_f - \tau_i \to \infty$) of $\langle T_{UU}(\mathcal{I}^-)\rangle_{w+}$ even though we know from the previous subsection that the transients are not correctly described in this approximation. Thus we replace the double integral in T_{UU} by a simple integral and take all Rindler frequencies to be equal to the resonant frequency ΔM and obtain

$$\begin{aligned}
\langle T_{UU}(\mathcal{I}^-)\rangle_{w+} &= 2\frac{\alpha_{\Delta M}}{\beta_{\Delta M}}\partial_U\hat{\varphi}^*_{-\Delta M}\partial_U\hat{\varphi}^*_{\Delta M} \\
&= \frac{\Delta M}{2\pi}\frac{1}{(aU)^2}\alpha^2_{\Delta M} \qquad (2.151)
\end{aligned}$$

Where we have used eq. (2.38). Notice that $\langle T_{UU}(\mathcal{I}^-)\rangle_{w+}$ is non vanishing both for $U > 0$ and $U < 0$. It can be interpreted in R by appealing to the isomorphism with the thermal bath. Since we have post-selected that the atom will get excited necessarily there was a Rindler quanta in R which will excite the atom. The weak value therefore contains the Rindler energy of this particle. The factor $\alpha^2_{\Delta M} = n(\Delta M) + 1$ rather than 1 takes into account that in a thermal bath their may be more than one quantum. The two level atom is sensitive to this since it responds to the mean number of quanta.

In L, the partner of the Rindleron in R appears with the same Rindler energy ΔM because Minkowski vacuum is filled with correlated Rindlerons in R and L (eq. (2.44)) of zero total Rindler energy eq. (2.47). But since we are in Minkowski vacuum and the interaction has not yet occurred, we necessarily have

$$\int_{-\infty}^{+\infty} dU \langle T_{UU}(\mathcal{I}^-)\rangle_{w+} = 0 \qquad (2.152)$$

because the operator $\int_{-\infty}^{+\infty} dUT_{UU}$ annihilates Minkowski vacuum. Hence $\langle T_{UU}(\mathcal{I}^-)\rangle_{w+}$ carries no integrated energy as behooves a vacuum fluctuation (see eqs. (2.116) and (2.117)).

Upon integrating eq. (2.151) a contradiction seems to arise as in the previous section. The transients have been incorrectly taken into account. A correct handling of the ultraviolet Minkowski frequencies (eq. (2.41)) solves this problem and in the limit $\tau_f - \tau_i \to \infty$, the weak value is the distribution

$$\langle T_{UU}(\mathcal{I}^-)\rangle_{w+} = \frac{\Delta M}{2\pi}\frac{1}{a^2(U+i\epsilon)^2}\alpha_{\Delta M}^2 \tag{2.153}$$

with a regularized energy content on the horizon $U = 0$ which restores the property eq. (2.152) (verified by closing the contour in the complex plane whereupon the double pole at $U = -i\epsilon$ has zero residue).

We now consider the model eq. (2.142) so as to have smooth T_{UU}'s (functions rather than distributions). The manner in which eq. (2.152) is realized is a subtle interplay of several effects which we now summarize.

We first obtain that for $U > 0$ the weak value is real and positive since

$$\begin{aligned}\langle T_{UU}((\mathcal{I}^-; U>0)\rangle_{w+} &= \frac{\langle 0_M|\int d\tau H_{int}(\tau)\int d\tau' H_{int}(\tau')T_{UU}|0_M\rangle}{\langle 0_M|\int d\tau H_{int}(\tau)\int d\tau' H_{int}(\tau')|0_M\rangle}\\ &= \frac{\langle 0_M|\int d\tau H_{int}(\tau)T_{UU}\int d\tau' H_{int}(\tau')|0_M\rangle}{\langle 0_M|\int d\tau H_{int}(\tau)\int d\tau' H_{int}(\tau')|0_M\rangle}\end{aligned} \tag{2.154}$$

which is manifestly real. We have used the causality development of eq. (2.133) to make the necessary commutation. Moreover eq. (2.154) is positive since it is the expectation value of T_{UU} in a one particle state.

Since the integral over all U vanishes, the integral over $U < 0$ must yield a negative result which exactly compensates the integral over $U > 0$. On the other hand the Rindler energy $\int_{-\infty}^{+\infty} du\ T_{uu}(\mathcal{I}^-; U < 0)$ is positive (although the integrand contains small end effects) since it describes the energy of the Rindleron which shall be absorbed by the atom. The net negative Minkowski energy on the right ((U < 0) once more finds its origin in the transients which are small in the Rindler description but large in the Minkowski description due to the enormous Doppler shift $du/dU = e^{au}$. Furthermore for $U < 0$, the weak value is not real. Nevertheless the complexity appears only in the transients (in eq. (2.150) at $U = 0$). This imaginary part also integrates to zero by virtue of eq. (2.152). In Section 3.5 we shall dwell somewhat further on the physical content of the imaginary part of $\langle T_{\mu\nu}(\mathcal{I}^-)\rangle_{w+}$ (see eq. (3.109)).

The reader may check that the energy fluctuations correlated to the transitions of the atom posess all the properties of the fluctuations correlated to the production of a specific photon by a mirror as studied in the previous section. However whereas in the mirror the post selection was an abstract construction, we have here shown that it is realized operationally by coupling an external system to the radiation.

Putting together the analysis of this subsection and the preceding one we have obtained a description in terms of energy density of the field configurations in Minkowski vacuum which make an accelerated two level atom excite (eq. (2.150)) and the resulting field configuration after excitation (eq. (2.130)). This is depicted shematicaly in Fig. (2.5a) for an atom which begins in the ground state around time τ_i and excites in the interval between τ_i and τ_f. If no excitation occurs one has the complement of eq. (2.150) and (2.130). This is depicted in Fig. (2.5b). The mean energy momentum is recovered when summing over excitation and no excitation and is depicted in Fig. (2.5c).

For simplicity of drawing it is the Rindler energy density T_{uu} which is drawn in every case. This is because the rapid fluctuations of the Minkowski energy in the vicinity of the horizon makes it virtualy impossible to represent in a comprehensible diagram. In all three figures the trajectory of the atom is drawn as a hyperbola between the points τ_i and τ_f. Energy fluxes for U–modes are represented by dotted lines. These flow from the past energy configuration (drawn in the SO corner of the diagram) to the future energy configuration (drawn in the NE corner).

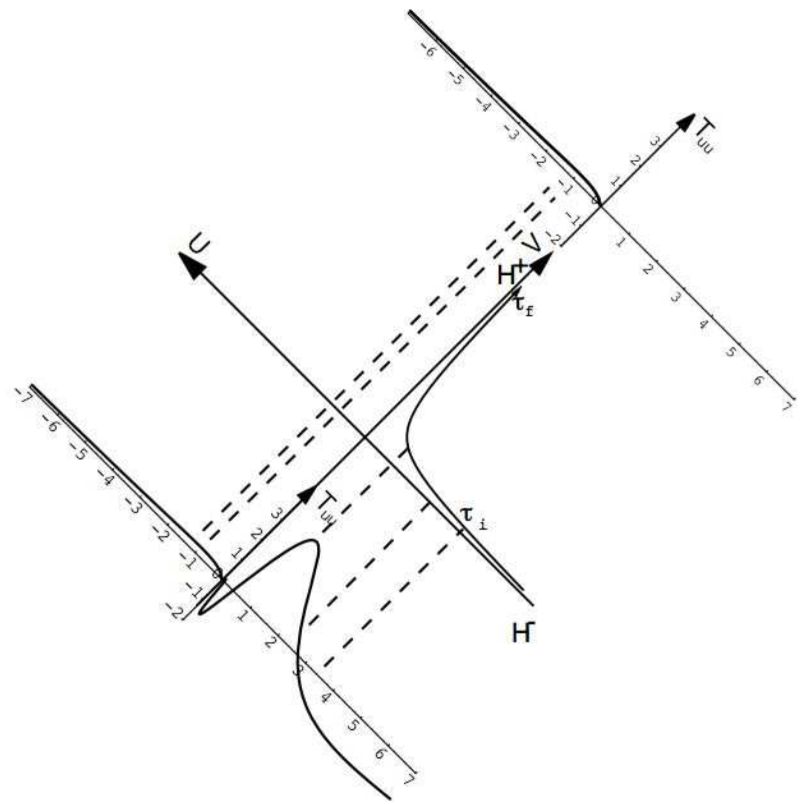

Fig. 2.5 *Birds'eye view of the energy of the field configurations both in the past and in the future of a two level atom which begins in the ground state around time τ_i and which either does (Fig.* **a***) or does not (Fig.* **b***) excite in the interval between τ_i and τ_f. Fig.* **c** *represents the mean effect.*

In Fig (a) the atom has become excited as is witnessed by the absorption of the positive Rindler peak in the middle of the configuration for $U < 0$. In consequence this peak has disappeared in the future and there only remains the partner in the region $U > 0$. (In fact the energy in the future for $U < 0$ is not quite zero but is order $e^{-\beta\Delta M}$ taken to be $e^{-\beta\Delta M} \simeq 10^{-5}$ in this case so that this is in fact too small to draw).
Fig (b) is the complementary case in which the atom has not become excited. In order to make the drawing visible the scale of the axes has been changed by a factor $\simeq 10^{-5}$.

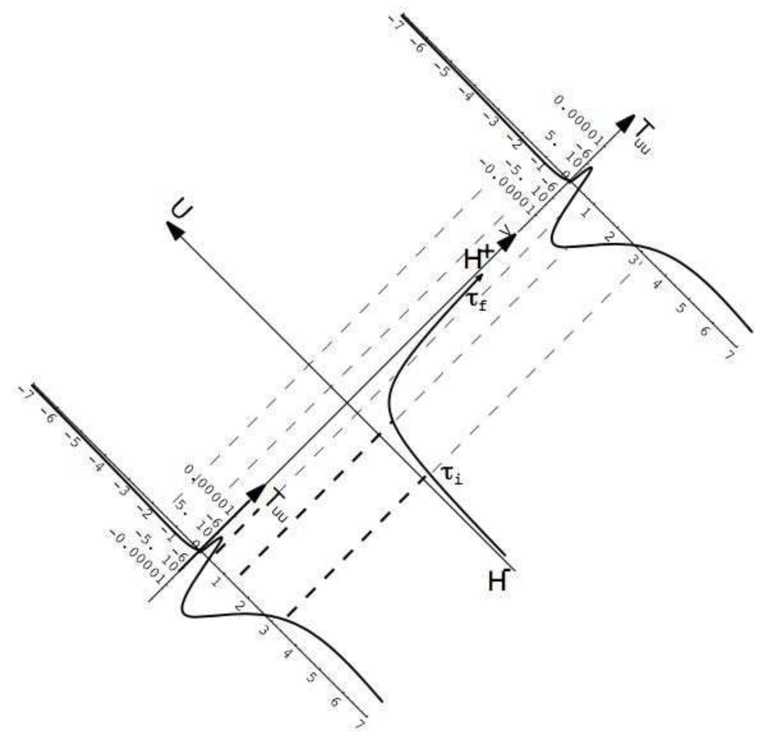

Fig. 2.5 b

The mean effect (fig (c)) therefore having mean energy zero in the past gives rise to a negative Rindler energy in the future (illustrated by the central negative dip in the future). As emphasized in the text there are transient effects which carry positive energy and which contrive to render the mean emitted Minkowski energy positive. It is further to be noted that causality is respected in that the partner contributions ($U > 0$) have cancelled between fig (a) and (b).

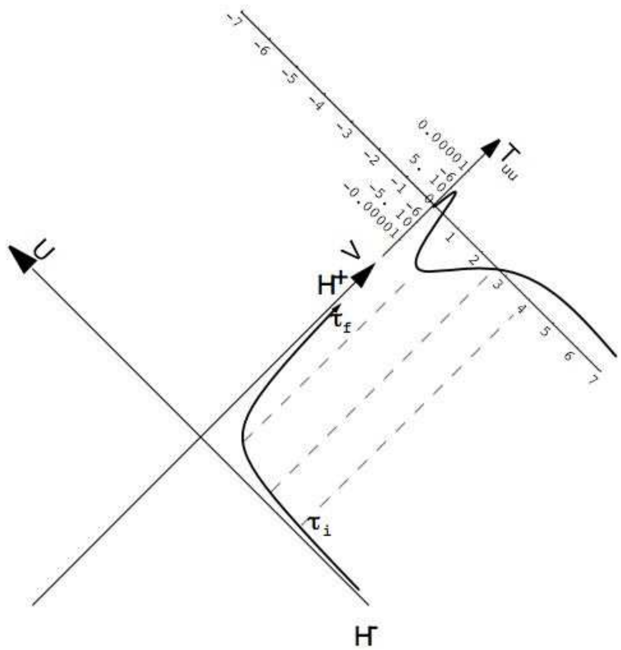

Fig. 2.5 c

Chapitre 3

Black Hole Evaporation

3.1 Kinematics

The reader is referred (for instance) to references [40],[1],[2],[24],[60] for an introduction to classical black hole physics. For the conceptual issues raised by quantum mechanics in the presence of horizons it suffices to work with the Schwarzschild black hole. Not that the Kerr hole does not give rise to interesting effects, but its complications appear out of context in the present review. By Schwarzschild black hole we include, and indeed mostly discuss, the incipient black hole wherein the star's matter has fallen into a region which is asymptotically close to its Schwarzschild radius, its future event horizon at $r_S = 2M$. Throughout we shall take the Planck mass equal to unity. Thus r is measured in Planckian distances and M in Planck masses. For a star the size of the sun where $M = 1.1\ 10^{57}$ proton masses $= 0.9\ 10^{38}$ Planck masses, we have $r_S = 1.8\ 10^{38}$ Planck distances.

Outside the star the metric is Schwarzschild

$$\begin{aligned} ds^2 &= -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \qquad (3.1) \\ &= (1-\frac{2M}{r})(-dt^2 + dr^{*2}) + r^2 d\Omega^2 \\ &= -(1-\frac{2M}{r})du\ dv + r^2 d\Omega^2 \qquad (3.2) \end{aligned}$$

where r^* is the tortoise coordinate

$$r^* = r + 2M \ln|\frac{r-2M}{2M}| \qquad (3.3)$$

and

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 \ . \tag{3.4}$$

Thus radial light rays follow u or $v = constant$ where

$$\left.\begin{array}{c} u \\ v \end{array}\right\} = t \mp r^* \ . \tag{3.5}$$

In addition to the Schwarzschild coordinates displayed in eqs (3.1, 3.2) we shall refer to both Kruskal [57],[40], and advanced Eddington Finkelstein [32],[34],[40] coordinates. In the first Schwarzschild quadrant (R), in which $r > 2M$, Kruskal coordinates are defined by

$$\begin{array}{rcl} V & = & 4Me^{v/4M} \\ U & = & -4Me^{-u/4M} \end{array} \quad \text{in R} \tag{3.6}$$

The relation between the Kruskal U, V and the Schwarzschild u, v is thus identical to the relation, eq. (2.20), between Minkowski and Rindler light-like coordinates. This isomorphism will play a crucial role in the understanding of the various properties of the Hawking radiation. In Kruskal coordinates the metric reads

$$ds^2 = -\frac{2M}{r} e^{-r/2M} dU dV + r^2 d\Omega^2 \tag{3.7}$$

where r is given implicitly by $UV = (4M)^2(1 - r/2M)e^{r/2M}$, and radial light cones are surfaces of constant U or V.

Were the complete space given by the analytic extension of the geometry whose metric is eq. (3.7) there would be four quadrants (see Fig. (3.1)) separated by horizons $U = 0$ and $V = 0$ the first of which, R, is coordinatized by the Schwarzschild coordinates u, v of eq. (3.5). The other three are coordinatized by Schwarzschild local coordinates in a manner analogous to the coordinatization Minkowski space into four Rindler quadrants.

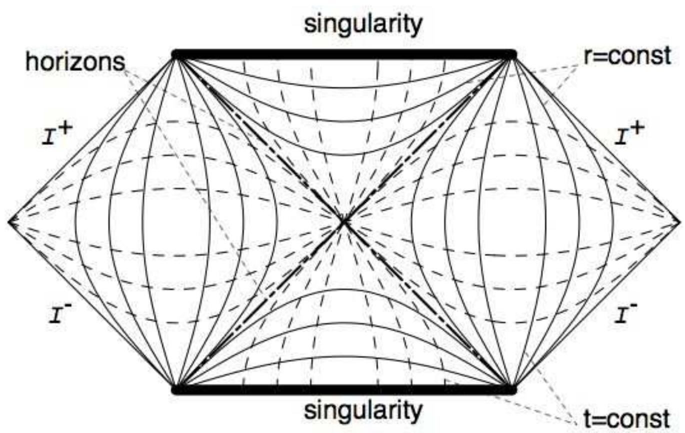

Fig. 3.1 *Penrose diagram of the maximal analytic extension of Schwarzschild space. The horizons, the singularities as well as $r = const$ and $t = const$ lines have been represented.*

For the collapsing black hole the outside space is confined to two of these quadrants R and F (see Fig. (3.2)).
Whereas Kruskal coordinates can be used to describe both of them, the equations which relate t and r to U and V through eq. (3.3) to eq. (3.6) are good only for the quadrant R. In the F quadrant, one may introduce u_F given by $U = 4Me^{u_F/4M}$. The relation between V and v is still given by eq. (3.6) since v is finite on the future horizon $U = 0$. Then, with t, r given in terms of u_F, v as in eq. (3.5) the metric is once more eq. (3.1). Note that in R, t and r are time-like and space-like variables respectively whereas in F, where $r - 2M < 0$, t is space-like and r time like. Finally, we introduce the advanced Eddington-Finkelstein set v, r that covers both R and F and will be found convenient when we study the back reaction. In regions exterior to the star one has

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2 d\Omega^2 \tag{3.8}$$

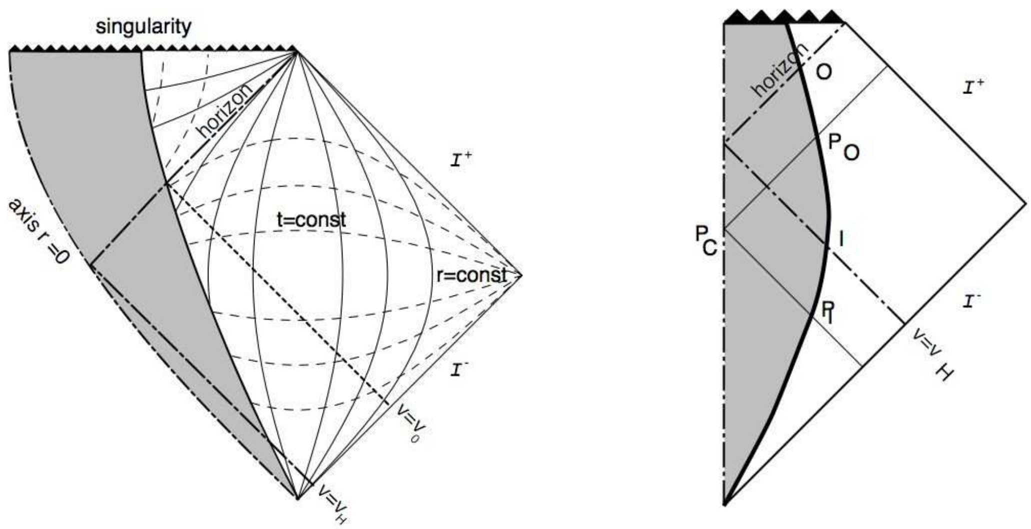

Fig. 3.2 a*The exterior of the star (in white) is a recopy of the relevant part of the complete Penrose diagram Fig. (3.1) whereas the grey part representing the inside of the star is obtained by extending ingoing and outgoing light cones through the two regions. The locus of the apexes of these cones being the line $r = 0$. As in all Penrose diagrams radial light rays are represented by straight lines at 45° degrees In te to the vertical.* **b** *The same Penrose diagram as in* **a** *redrawn by availing ourselves of further reparametrisation freedom of radial null rays inherent to the construction in such manner that the line $r = 0$ is drawn as a straight line. This is the Penrose diagram which is usualy found in standard texts. The points labeled P_I, P_C, P_O, I and O refer to the discussion of Hawking radiation in Appendix D*

In the relevant quadrants the domain of variation of the variables are:

- Schwarzschild in R

$$-\infty < u < +\infty \ , \ -\infty < v < +\infty$$
$$\text{also } -\infty < t < +\infty \ , \ 2M < r < \infty \text{ (or } -\infty < r^* < +\infty) \quad (3.9)$$

- Kruskal in R

$$-\infty < U < 0 \ , \ 0 < V < \infty \tag{3.10}$$

- Kruskal in F

$$0 < U < \infty \ , \ 0 < V < \infty \text{ and } UV < 1 \tag{3.11}$$

- Eddington Finkelstein in R and F

$$0 < r < \infty \ , \ -\infty < v < \infty \tag{3.12}$$

So much for the geometry of Schwarzschild space, i.e. the exterior of the star. We must now describe the interior of the star. A convenient idealization, first used by Unruh [91], is the model of a collapsing shell wherein the interior is empty, hence described by flat space. We shall pursue this case explicitly and show how, in Appendix D the essential result of the analysis emerges from the general case.

Inside the space is Minkowski so that

$$ds^2 = -dT^2 + dr^2 + r^2 d\Omega^2 = -d\mathcal{U} d\mathcal{V} + r^2 d\Omega^2 \tag{3.13}$$

where $\mathcal{U}, \mathcal{V} = T \mp r$. We have set the spatial coordinate r to be the same in both regions so that the area of spheres (SO(3) orbits) is $4\pi r^2$ everywhere and for all times.

In Fig. (3.2) we have drawn the Penrose diagram which shows the salient features of the collapsing geometry. In this diagram we do not take into account the loss of mass due to the Hawking radiation, hence it is only useful to describe the early stages of evaporation. The heavy line traces out the trajectory of the surface of the star $R_{st}(v)$. We parametrize the trajectory in terms of the Eddington Finkelstein time v rather than the Schwarzschild time t, so as to cover its whole history. At $v = v_0$ the shell reaches its horizon H, the light like surface given by $r = 2M$. One may extend this light like surface H into the interior of the star as indicated.

At still later times the shell reaches the space-like singularity $r = 0$. This is a 3-surface not to be confused with the 1-dimensional time-like line $r = 0$ which is the axis of SO(3) symmetry of the whole collapsing geometry. This is brought out in Fig. (3.3). This line of symmetry is the locus of the vertices of light cones which trace out the paths of spherical waves in the journey from $\mathcal{I}^-$ to $\mathcal{I}^+$ (light like past and future infinity). In Fig. (3.2) these cones are represented by lines reflected off $r = 0$. This picture is blown up into 3 dimensions in Fig. (3.3) wherein spheres are represented by circles.

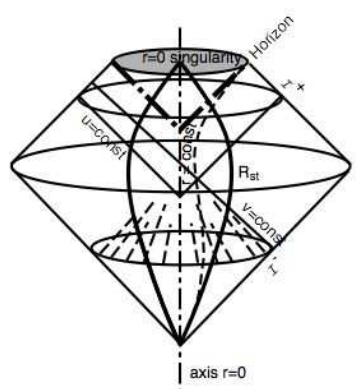

Fig. 3.3 *The Penrose diagram Fig. (3. 2 b) for a collapsing star blown up to* $2 + 1$ *dimensions. We have represented the horizon, the singularity, the surface of the star, a typical light cone. The dotted line represents a* $r = const$ *line.*

We have also represented the horizon light cone, H, its extension into the star and its backward history as a light cone that terminates on $\mathcal{I}^-$. A spherical flash of light on this cone is the last flash that can be emitted from $\mathcal{I}^-$ so as to arrive at $\mathcal{I}^+$. It will be shown that Hawking radiation is concerned with the combined Doppler and gravitational red shifts which spherical light cones, traveling just before H, experience in their trip from $\mathcal{I}^-$ to $\mathcal{I}^+$.

Therefore, in order to compute those combined effects, we need to match the internal coordinate system $\mathcal{U}, \mathcal{V}$ to the external Schwarzschild set u, v. We now turn to this task.

Since u =const. defines a future half light cone, it can be extended back in time and enter the star with apex at $r = 0$; similarly for v. Thus the light cones labeled outside the star by the Schwarzschild coordinates u, v can be used to coordinatize points inside the star. The same future half light cone is described either by a constant value of $\mathcal{U}$ or u (and similarly for the past half null cones in terms of $\mathcal{V}$ and v). In consequence $\mathcal{U}$ is a function of u only and $\mathcal{V}$ a function of v only.

The radial coordinate r has the same meaning inside and outside the star. Thus on the shell the displacement in r is the same in both systems:

$$2dr = d\mathcal{V} - d\mathcal{U} = (1 - \frac{2M}{R_{st}})(dv - du) \tag{3.14}$$

where differentials are along the trajectory. A further condition is that intervals of proper time on the shell be the same in both systems. Thus

$$d\mathcal{V}d\mathcal{U} = (1 - \frac{2M}{R_{st}})dvdu \tag{3.15}$$

These equations together with a trajectory for the star's surface $R_{st}(v)$ are sufficient to solve for $\mathcal{U}(u)$ and $\mathcal{V}(v)$. We rewrite them as

$$\frac{d\mathcal{V}}{dv} - \frac{d\mathcal{U}}{du}\frac{du}{dv}|_{st} = (1 - \frac{2M}{R_{st}})(1 - \frac{du}{dv}|_{st}) \tag{3.16}$$

$$\frac{d\mathcal{V}}{dv}\frac{d\mathcal{U}}{du} = (1 - \frac{2M}{R_{st}}) \tag{3.17}$$

When the shell is far from its horizon ($R_{st} >> 2M$) $\mathcal{U}(u)$ and $\mathcal{V}(v)$ are slowly varying functions the exact form of which is irrelevant for our purpose. On the other hand when the shell approaches its horizon asymptotically these functions acquire a universal behavior characterized by M only. As it is precisely this domain of the variables which is relevant to describe the steady state Hawking evaporation we shall consider this case.

To characterize the shell's trajectory near the horizon is very simple. Suppose R_{st} crosses the horizon at some Eddington Finkelstein advanced time v_0 with finite acceleration (one verifies that this is indeed the case when the shell follows a geodesic). Then to first order we have

$$2M - R_{st}(v) = k(v - v_0) \quad (k > 0) \tag{3.18}$$

Here $v(= t + r^*)$ is equal to $t + R^*_{st}$ on the surface so this same equation gives $R_{st}(t)$

$$-\frac{R_{st}(t) - 2M}{k} = R^*_{st}(t) + t - v_0 \tag{3.19}$$

Near the horizon $R_{st}^* \simeq 2M + 2M \ln(R_{st}/2M - 1)$ so we can solve iteratively to give

$$R_{st}(t) - 2M = 2Me^{(v_0-2M-t)/2M} + O(e^{-t/M}) = Ae^{-t/2M} + O(e^{-t/M}) \quad (3.20)$$

where A is a positive constant. Moreover to the same approximation we may replace t by $(u + v_0)/2$ so as to yield

$$R_{st}(u) - 2M = A'e^{-u/4M} + O(e^{-u/2M}) \quad (3.21)$$

where A' is another positive constant. The relation between u and v on the surface is thus

$$v - v_0 \;\; = \;\; -\frac{A'}{k} e^{-u/4M} + O(e^{-u/2M}) \quad (3.22)$$

whence asymptotically

$$\frac{dv}{du}\Big|_{st} \;\; = \;\; \frac{A'}{4Mk} e^{-u/4M} = \frac{R_{st}(u) - 2M}{4Mk} \quad (3.23)$$

Inserting eq. (3.23) into eq. (3.16) and eq. (3.17) one sees that the first term on the r.h.s. of eq. (3.16) is negligible with respect to du/dv. Further we have in the limit $v \to v_0$, $d\mathcal{V}/dv|_{v_0} = \lambda$ where λ is some positive constant. It then follows that

$$\frac{d\mathcal{U}}{du} = Be^{-u/4M} + O(e^{-u/2M}) = -\frac{\mathcal{U}}{4M} + O(\mathcal{U}^2) \quad (3.24)$$

where B is another positive constant. These constants will be discussed in Appendix D, see also [33]. Their value is irrelevant for the calculation of the asymptotic Hawking radiation. The upshot is that near the horizon

$$\begin{aligned} \mathcal{V} - 4M \;&=\; \lambda(v - v_0) \\ \mathcal{U} \;&=\; B(-4Me^{-u/4M}) + O(e^{-u/2M}) \end{aligned} \quad (3.25)$$

Very important is the fact that $\mathcal{U}$ tends exponentially fast to Kruskal U (defined in eq. (3.6)) and that $\mathcal{V}$ does not but rather is lineary related to the Schwarzschild v. Note also that

$$\frac{d\mathcal{U}}{du} = (B/A')(R_{st} - 2M) \quad (3.26)$$

i.e. is proportional to g_{00}, rather than the usual $\sqrt{g_{00}}$. A direct calculation reveals that the factor $R_{st} - 2M$ is a composite of the static red shift and a Doppler shift due to the retreating surface (see Appendix D).

In eqs (3.24, 3.25) we have set $\mathcal{V} = 4M$ and $\mathcal{U} = 0$ at the point where the star crosses the horizon H. The light cone H which shall generate the horizon is given by $\mathcal{U} = 0$ after it reaches $r = 0$ and therefore by $\mathcal{V} = 0$ before (since $2r = \mathcal{V} - \mathcal{U}$ and $r = 0$ at the apex of H).

Fig. (3.4) gives a sketch of the exterior R and interior I of the star in the orthogonal coordinate grid u, v which have their usual meaning in R as functions of t, r eq. (3.5) and where in I we have

$$
\begin{aligned}
2r &= \mathcal{V} - \mathcal{U} = \lambda v + \frac{B}{4M} e^{-u/4M} + constant \\
2T &= \mathcal{V} + \mathcal{U} = \lambda v - \frac{B}{4M} e^{-u/4M} + constant'
\end{aligned} \tag{3.27}
$$

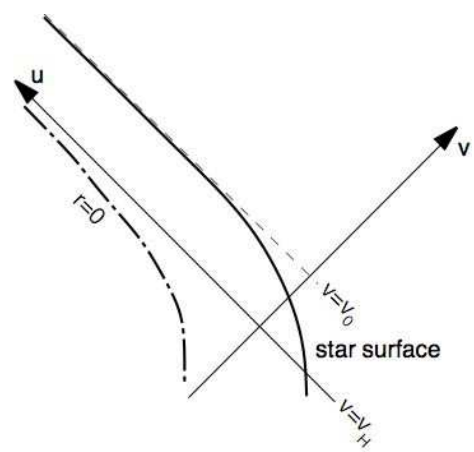

Fig. 3.4 *The geometry of a collapsing star in u, v coordinates*

One should compare this drawing with the Penrose diagram Fig.(3.1) and compare the trajectory of light rays in the two drawings. One should also compare the shell's trajectory eq. (3.22) with the mirror trajectory eq. (2.69) and remark that the Doppler shifts of eq. (3.24) and eq. (2.68) are identical.

Therefore, we shall find also a steady thermal flux (Hawking radiation) in the collapsing situation.

3.2 Hawking Radiation

As in the preceding chapters, quantization proceeds through the construction of modes, here solutions of $\Box\psi = 0$. In the simple shell model the d'Alembertian is a simple operator in the inside and outside domains. Begin with the outside (Schwarzschild space) wherein from eq. (3.1)

$$\begin{aligned}\Box\psi &= \frac{1}{\sqrt{g}}\partial_\mu\, g^{\mu\nu}\sqrt{g}\partial_\nu\, \psi \\ &= \frac{1}{(r-2M)}\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^{*2}}+(1-\frac{2M}{r})(\frac{2M}{r^3}-\frac{\vec{L}^2(\theta,\varphi)}{r^2})\right]\, r\psi\end{aligned} \tag{3.28}$$

Here $\vec{L}$ is the angular momentum operator. Since it commutes with the d'Alembertian we pass over immediately to states of fixed l, i.e. modes are written as $\psi_l = (\sqrt{4\pi}r)^{-1}\varphi_l(r)Y_l^m(\theta,\varphi)$ where

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^{*2}}+V_l(r)\right)\varphi_l = 0 \tag{3.29}$$

with

$$V_l(r) = (1-\frac{2M}{r})(\frac{2M}{r^3}+\frac{l(l+1)}{r^2}) \tag{3.30}$$

Similarly in the inside region one has

$$\left(-\frac{\partial^2}{\partial T^2}+\frac{\partial^2}{\partial r^2}+\frac{l(l+1)}{r^2}\right)\varphi_l = 0 \tag{3.31}$$

$V_l(r)$ plays the role of a centrifugal barrier, present even for s-waves, where it gives a positive potential energy bump at $r = 8M/3$. It has been shown by numerical calculation that about 90% of Hawking radiation is in s-waves [68],[83].

Therefore, in this section, we shall restrict ourselves to s-waves. As in Minkowski space, s-wave modes vanish at the origin $\varphi_{l=0}(r=0) = 0$ (since

$\psi = \varphi/r$) and define a one dimensional problem. Hawking radiation is concerned with the outgoing reflected part. Because of the existence of $V_{l=0}(r)$ in eq. (3.29), low energy modes will be reflected back and the problem becomes a usual quantum problem of finding the transmission coefficient. Once more for conceptual purposes this complication is irrelevant and we shall put $V_{l=0}(r) = 0$. Nevertheless, at the appropriate place we shall mention the modifications that ensue when its effects are included. In fact, in the last analysis though this barrier is a nuisance for mathematics it will prove to be of benefit since one will not have to confront the infra-red problem associated with myriads of soft Hawking photons. Only frequencies $\lambda \geq O(1/2M)$ are passed. So the scrupulous reader may waive his qualms in the knowledge that what follows is quite rigorous for sufficently large λ.

With the above mentioned simplifications the s-waves obey

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^{*2}}\right)\varphi = \frac{\partial}{\partial u}\frac{\partial}{\partial v}\varphi = 0 \qquad \text{in R} \tag{3.32}$$

$$\left(\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial r^{2}}\right)\varphi = \frac{\partial}{\partial \mathcal{U}}\frac{\partial}{\partial \mathcal{V}}\varphi = 0 \qquad \text{in I} \tag{3.33}$$

From eq. (3.32) and eq. (3.33) the solutions are the sum of an ingoing and an outgoing piece

$$\begin{array}{rclc} \varphi & = & X(v) + \Xi(u) & \text{in R} \\ \varphi & = & \chi(\mathcal{V}) + \xi(\mathcal{U}) & \text{in I} \end{array} \tag{3.34}$$

where continuity on the star's surface imposes[1] that $\chi(\mathcal{V}) = X(v(\mathcal{V}))$ and $\xi(\mathcal{U}) = \Xi(u(\mathcal{U}))$. Moreover, since φ must vanish at $r = \mathcal{V} - \mathcal{U} = 0$, one finds that everywhere φ takes the form

$$\varphi = \chi(\mathcal{V}) - \chi(\mathcal{U}) \tag{3.35}$$

Before discussing in detail the form of the modes, we note that eq. (3.35) already contains information about the division of modes into the producing (class I) and non producing classes (class II) as in chapters 1 and 2. In the distant past, before collapse begins, we set up vacuum on $\mathcal{I}^-$; this is in fact Minkowski vacuum since the space is flat on $\mathcal{I}^-$. The modes propagate

[1]In a realistic four dimensional model there will be some scattering in the star (mixing U and V modes) at low frequencies. But at the exponentially large frequencies encountered in Hawking radiation the star is completely transparent

from $\mathcal{I}^-$; hence they are of the form $e^{-i\omega v}$ (once more we are thinking of a broad wave packet). They diminish in radius, and either penetrate or do not penetrate the star's surface in there inward journey according to the value of v (See Fig.(3.2)). Hawking radiation comes about from modes in the class that penetrates into the star, but not all of them. Only those that reflect in a finite Schwarzschild time ($v < v_H$ in Fig.(3.4)) thereby picking up a dependence on u which gives rise to the radiation.

The asymptotic requirement that there be vacuum on $\mathcal{I}^-$ fixes that the incoming modes be of the form $\varphi_\omega^{in} = \chi_\omega(v) = e^{-i\omega v}/\sqrt{4\pi\omega}$. We shall be concerned with the behavior of the modes in the vicinity of the light cone H that generates the horizon when $\mathcal{U}(u)$ reaches its asymptotic behavior (eq. (3.25)). Hence we need the relation between v, u and $\mathcal{V}, \mathcal{U}$ in the vicinity of H only. More precisely we need only the function $v(\mathcal{V})$ in the vicinity of $\mathcal{V} = 0$ when H is an infalling light cone since we have already the function $\mathcal{U}(u)$. Near $\mathcal{V} = 0$ the function $\mathcal{V}(v)$ can be approximated linearly $v - v_H = \kappa\mathcal{V} + O(\mathcal{V}^2)$ where κ is an irrelevant constant which depends on the shell's speed at $\mathcal{V} = 0$. Hence in the region where H is an infalling light cone the modes take the forms

$$\varphi_\omega^{in} = \frac{e^{-i\omega v}}{\sqrt{4\pi\omega}} = \frac{e^{-i\omega(\kappa\mathcal{V} + v_H)}}{\sqrt{4\pi\omega}} \tag{3.36}$$

We have used the fact that one may coordinatize the whole space with either v or $\mathcal{V}$ and used the continuity at the star's surface. After reflection at $r = 0$, H is an outgoing light cone and the in modes read

$$\varphi_\omega^{in} = -\frac{e^{-i\omega(\kappa\mathcal{U} + v_H)}}{\sqrt{4\pi\omega}} = -\frac{e^{i\omega(Ke^{-u/4M} + v_H)}}{\sqrt{4\pi\omega}} \tag{3.37}$$

where $K(= \kappa B4M$, see eq. (3.25)) is a positive constant. Note that the approximation

$$\mathcal{U} = -B4Me^{-u/4M} = BU \tag{3.38}$$

is rapidly excellent since the corrections are of the order $e^{-u/2M}$. Therefore the asymptotic form eq. (3.37) is also rapidly valid. In this asymptotic regime, the Hawking radiation is independent of K because K can be reabsorbed into a redefinition of the origin of u. For convenience we choose $K = 4M$ and drop the phase $-e^{-i\omega v_H}$.

The rest is copying out the results of Chapter 2. We repeat the salient facts. Once more because the Bogoljubov transformation does not mix U'ness and V'ness, each sector may be handled independently. Concerning the V modes, at this point, nothing need be said, the modes being of the form $e^{-i\omega v}$ throughout. The in-vacuum stays equal to the out-vacuum i.e. the vacuum defined by single particle states in R.

The physics under present scrutiny is encoded in the overlap between the u part of the scattered in-modes given in eq. (3.37) and the out-modes $\varphi_\lambda^{out} = e^{-i\lambda u}/\sqrt{4\pi\lambda}$:

$$\varphi_\omega^{in} = \int_0^\infty d\omega \left[\alpha_{\lambda\omega}\varphi_\lambda^{out} + \beta_{\lambda\omega}\varphi_\lambda^{out*}\right] \tag{3.39}$$

$$\begin{aligned}
\alpha_{\lambda\omega} &= \int_{-\infty}^{+\infty} du\ \varphi_\lambda^{out*}(u)(-i\overleftrightarrow{\partial_u})\varphi_\omega^{in}(\mathcal{U}(u)) \\
\beta_{\lambda\omega} &= \int_{-\infty}^{+\infty} du\ \varphi_\lambda^{out}(u)(-i\overleftrightarrow{\partial_u})\varphi_\omega^{in}(\mathcal{U}(u))
\end{aligned} \tag{3.40}$$

Since $\mathcal{U}(u)$ is correctly approximated by BU only when the exponential corrections $e^{-u/2M}$ are negligible, there is no universal form for $\alpha_{\lambda\omega}$ and $\beta_{\lambda\omega}$ when the Doppler shift is small i.e. when $\omega \simeq \lambda$, hence there is no steady flux in this early stage. On the contrary, when the Doppler shift becomes exponential, the corrections fade out, and one does find a universal behavior for $\alpha_{\lambda\omega}$ and $\beta_{\lambda\omega}$ when $\omega >> \lambda$. This is due to the finite interval in u in which these overlaps acquire their value. Hence, for $\omega >> \lambda$, one can replace the expressions for $\alpha_{\lambda\omega}$ and $\beta_{\lambda\omega}$ by the following expressions, thereby making contact with the Rindler–Minkowski Bogoljubov transformation,

$$\begin{aligned}
\alpha_{\lambda\omega} &= \frac{1}{2\pi}\frac{\sqrt{\lambda}}{\sqrt{\omega}}\int_{-\infty}^{+\infty} du e^{i\lambda u} e^{i\omega 4M e^{-u/4M}} \\
&= \frac{4M}{2\pi}\frac{\sqrt{\lambda}}{\sqrt{\omega}}\int_{-\infty}^{0} dU(-4MU)^{-i\lambda-1}e^{-i\omega U} \\
&= \frac{4M}{2\pi}\sqrt{\frac{\lambda}{\omega}}\Gamma(-i4M\lambda)\left(\frac{a}{\omega}\right)^{-i4M\lambda} e^{\pi 2M\lambda} \\
\beta_{\lambda\omega} &= e^{-4M\lambda}\alpha_{\lambda\omega}^*
\end{aligned} \tag{3.41}$$

where we have used eqs (2.28). Hence one finds a steady thermal flux at the Hawking temperature $T_H = 1/8\pi M = 1/\beta_H$ since, for all $\omega >> \lambda$, one has

$$|\frac{\beta_{\lambda\omega}}{\alpha_{\lambda\omega}}|^2 = e^{-8\pi M\lambda} \tag{3.42}$$

We note that the transitory regime can nevertheless be handled analytically in some specific cases, see for instance Appendix D and reference [63], and one explicitly verifies that there is an early stage without significant flux which, after a certain u-time, tends exponentially fast to the asymptotic behavior. One can then verify that modes with $l > 0$ do not contribute significantly to Hawking radiation. This is seen by comparing the Hawking temperature with the height of the centrifugal barrier in eq. (3.30).

Let us recall how a flux arises. As in section 2.4, for a given ω and λ, the integrals eq. (3.41) arise from the regions around the saddle u^* at (see eq. (2.52))

$$\mathrm{Re}(u^*(\omega,\lambda)) = 4M \ln \omega/\lambda \tag{3.43}$$

With Gerlach [38], we call this saddle time the "resonance" time. The width of the significant region around u^* is independent of ω and given by $\simeq \sqrt{4M/\lambda}$ thereby justifying the replacement of eq. (3.40) by eq. (3.41) for $\omega >> \lambda$. This independence of the width allows for the derivation of an exact asymptotic formula for the rate i.e. for the flux. First pick an interval $u_1 \leq u \leq u_1 + \Delta u$ with u_1 sufficiently large so that the correction term in eq. (3.25) can be neglected and $\Delta u >> \sqrt{4M/\lambda}$. We wish to calculate the flux in R due to quanta of frequency λ. In this interval the frequencies ω which contribute to $< a^\dagger_\lambda a_\lambda >$ (i.e. the number of quanta of frequency λ) satisfy

$$\lambda e^{u_1/4M} \leq \omega \leq \lambda e^{(u_1+\Delta u)/4M} \tag{3.44}$$

Whence the rate is given, following eqs (2.55, 2.54),

$$\begin{aligned}
\lim_{\Delta u\to\infty} < a^\dagger_\lambda a_\lambda > \frac{1}{\Delta u} &= \lim_{\Delta u\to\infty} \frac{1}{\Delta u}\int_{\lambda e^{u_1/4M}}^{\lambda e^{(u_1+\Delta u)/4M}} d\omega |\beta_{\lambda\omega}|^2 \\
&= \lim_{\Delta u\to\infty} \frac{4M}{\Delta u}\int_{\lambda e^{u_1/4M}}^{\lambda e^{(u_1+\Delta u)/4M}} \frac{d\omega}{\omega}\frac{1}{e^{\beta_H\lambda}-1}\frac{1}{2\pi} \\
&= (e^{\beta_H\lambda}-1)^{-1} 1/2\pi
\end{aligned} \tag{3.45}$$

Since at fixed r, one has $\Delta u = \Delta t$, $< a^\dagger_\lambda a_\lambda > /\Delta u$ indeed represents a particle flux per unit time. In this way the $\delta(0)$ which would come up by applying naively eq. (2.80) is replaced by $\Delta u/2\pi$. (This point has been the subject of some misunderstanding in the literature. There is no infra-red divergence in the number flux at finite λ! To define the total number flux one should take into account the cutoff provided at small λ by the potential barrier.)

Similarly the total energy flux is

$$< T_{uu} >= \int_0^\infty \frac{\lambda d\lambda}{2\pi}(e^{\beta_H \lambda} - 1)^{-1} = \frac{\pi}{12}\beta_H^{-2} \tag{3.46}$$

These results overestimate the flux since they do not take into account the suppression of the emission of low frequency quanta $\lambda \leq \beta_H^{-1}$ due to the s-wave repulsive barrier.

Unruh [91] has given a nice mnemonic device to represent these results. Namely the steady state of Hawking evaporation (in the early stages when the decrease in M due to the radiation is neglected) is simulated by a vacuum state - U(nruh) vacuum. This state is a cross between Schwarzschild vacuum (also called Boulware vacuum [13]) and Kruskal vacuum (also called Hartle-Hawking vacuum [43]). Outside the star, U-vacuum is vacuum of Schwarzschild v modes (i.e. $e^{-i\omega v}/\sqrt{4\pi\omega}$) but the Kruskal vacuum of U modes (i.e. $e^{-i\omega U}/\sqrt{4\pi\omega}$). This is simply because in the collapsing situation the v modes are $e^{-i\omega v}$ and the u modes $e^{i\omega \mathcal{U}}$ (up to phases and irrelevant constants). The matching conditions on the surface of the star have universally related $\mathcal{U}$ to U, eq. (3.38), when the shell is close to the horizon H. This is the crux of Unruh's beautiful isomorphism.

One must nevertheless bear in mind the following restriction of the Unruh mnemonic. The reference to the Kruskal character of the u part of the in modes is after their reflection from $r = 0$. There is an absolute frame of reference in this problem which is given by the movement of the star's surface. This precludes the possibility of making arbitrary boosts (in Kruskal coordinates eq. (3.6)) so as to change the character of ω, λ mixing in eq. (3.43). Such boosts correspond to translations in t, so that an event at $t = 0$ with $\omega = \lambda$ at resonance in the boosted frame could result say in $\omega = \lambda e^{t_1/4M}$. The Unruh isomorphism is carried out in a fixed frame e.g. the star at rest until $t = t_0$. The Gerlach resonance condition $\omega/\lambda = e^{u^*/4M}$ which describes the Doppler shift at and around u^* cannot be boosted away. This restriction will play an important role upon confronting the consequences of the very high frequencies ω entering into the Bogoljubov coeficients eq. (3.41) as seen from eq. (3.43).

A contemporary and elegant derivation of the concept of U-vacuum is due to Hawking [46] and Damour and Ruffini [27]. Their method is the black hole analog of the technique illustrated in eqs. (2.41, 2.42). It is of some interest to sketch their method since it makes contact with the tunneling methods of Section 1.2.

In Eddington Finkelstein coordinates eq. (3.8) the d'Alembertian near the horizon oustide the star is $-\partial_x(\frac{x}{2M}\partial_x + 2\partial_v)$ where $x = r - 2M$. The mode equation is

$$-\frac{\partial}{\partial x}[\frac{x}{2M}\frac{\partial}{\partial x} - 2i\lambda]\chi_\lambda(x)e^{-i\lambda v} = 0 \tag{3.47}$$

having solutions $\chi_\lambda(x) = \text{const.}$ (v-modes) and the u-modes

$$\chi_\lambda(x) = A\left[\theta(x)\frac{(x)^{4iM\lambda}}{\sqrt{4\pi\lambda}}\right] + B\left[\theta(-x)\frac{(-x)^{4iM\lambda}}{\sqrt{4\pi\lambda}}\right] \tag{3.48}$$

One checks that for $x > 0$, $e^{i\lambda v}x^{-4iM\lambda}(= e^{i\lambda u})$ is indeed a u mode in R. Completeness in the complete space spanned by the Eddington Finkelstein coordinates (collapsing star) requires the term in $\theta(-x)$ as well. We now require that in this complete space the modes be positive Kruskal frequency. Since at v fixed, ∂_x is pure light like (it is in fact $-2\partial_U$) $\chi_\lambda(x)$ must enjoy upper half analyticity in x. The normed modes are thus

$$\chi_\lambda = \frac{e^{+\beta_H\lambda/2}}{\sqrt{2\text{sinh}\beta_H\lambda}}\left[\theta(x)\frac{(x)^{4iM\lambda}}{\sqrt{4\pi\lambda}}\right] + \frac{e^{-\beta_H\lambda/2}}{\sqrt{2\text{sinh}\beta_H\lambda}}\left[\theta(-x)\frac{(-x)^{4iM\lambda}}{\sqrt{4\pi\lambda}}\right] \quad \lambda > 0 \tag{3.49}$$

as in eq. (2.41). The modes χ_λ are simply the rewriting of the in-modes φ_ω^{in} which diagonalize the Bogoljubov transformation (3.39). Indeed eq. (3.49) shows that the modes $\hat{\varphi}_\lambda = e^{-i\lambda v}\chi_\lambda$ ($-\infty < \lambda < +\infty$) (hereafter called Unruh modes) can be written in terms of the Schwarzschild modes φ_λ^{out} as

$$\begin{aligned}\hat{\varphi}_\lambda &= \alpha_\lambda\varphi_\lambda^{out} + \beta_\lambda\varphi_\lambda^{outF*} \quad \lambda > 0\\ \hat{\varphi}_{-\lambda} &= \beta_\lambda\varphi_\lambda^{out*} + \alpha_\lambda\varphi_\lambda^{outF} \quad \lambda > 0\end{aligned} \tag{3.50}$$

where we have introduced the modes

$$\varphi_\lambda^{outF} = (1/\sqrt{4\pi\lambda})e^{-i\lambda v}(-x)^{-4iM\lambda}\theta(-x) \tag{3.51}$$

To make contact with Section 1.2, we note the similarity between the differential equation for $\chi_\lambda(x)$: $(x\partial_x - 2i4M\lambda)\chi_\lambda(x) = 0$ and eq. (1.7). The analogy goes quite far. For example we take the Fourier transform of $\chi_\lambda(x)(= \tilde{\xi}_\lambda(p))$ we get $(\partial_p p - 2i\lambda)\tilde{\xi}_\lambda(p) = 0$. The modes $\tilde{\xi}_\lambda(p)$ having been obtained by integration over all x are complete and in fact can serve as a complete set of in-basis states [74]. Their Fourier transform then gives them as a linear combination of out states. Each of these latter live on one side or the other of H, the horizon line $x = 0$.

This is then precisely the same mathematical mechanism used in eq. (1.15) to go from the in-state (proportional to $\theta(u)$) to the linear combination of states defined in terms of the conjugate momentum to $u(= v)$, a combination of $\theta(v)$ and $\theta(-v)$. The classical exponential approach in a given region of space-time to a horizon is translated in each case to a quantum formalism of this type. However, in the black hole case, the transcription of the modes to a manifest tunneling is ambiguous. We may write the operator $ix\partial_x$ as $[\Pi^2 - \xi^2]$ where $\Pi = \frac{1}{\sqrt{2}}(iA\partial_x + A^{-1}x)$ and $\xi = \frac{1}{\sqrt{2}}(iA\partial_x - A^{-1}x)$ with $[\Pi, \xi] = -i$. But the constant A is arbitrary so tunneling in ξ is not only non local in x but occurs at arbitrary scales. Nevertheless, the above highlights that the collapse has produced pairs living on both sides of x and that these pairs are the realization of vacuum fluctuation which have "tunneled" into reality. The pair formation is illustrated in Figs (3.5) and (3.6) in a Penrose diagram and in Eddington Finkelstein coordinates. To this end we take into account the fact when extrapolated backward in time they bounce off $r = 0$. Of course the $\theta(U)$ piece is never seen at finite Schwarzschild time, but only its backward reflected piece. As the wave progresses from $\mathcal{I}_-$, this partner of the Hawking photon never manages to reflect, but, at $t \to \infty$, simply crowds into the horizon line extended into the origin.

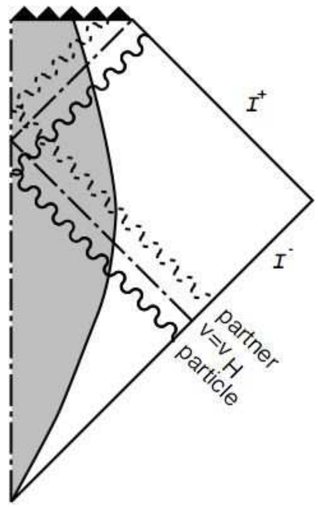

Fig. 3.5 *The classical trajectory (stationnary phase) of a pair produced in the geometry of a collapsing black hole represented on a Penrose diagram. Only the member of the pair which reaches $\mathcal{I}^+$ is on shell, the partner falls into the singularity.*

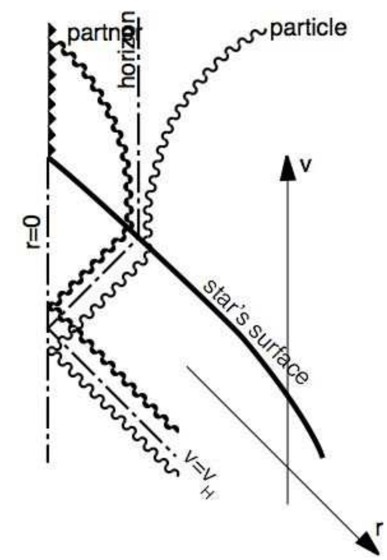

Fig. 3.6 *The same as in the previous figure (3.5) but in Eddington Finkelstein coordinates.*

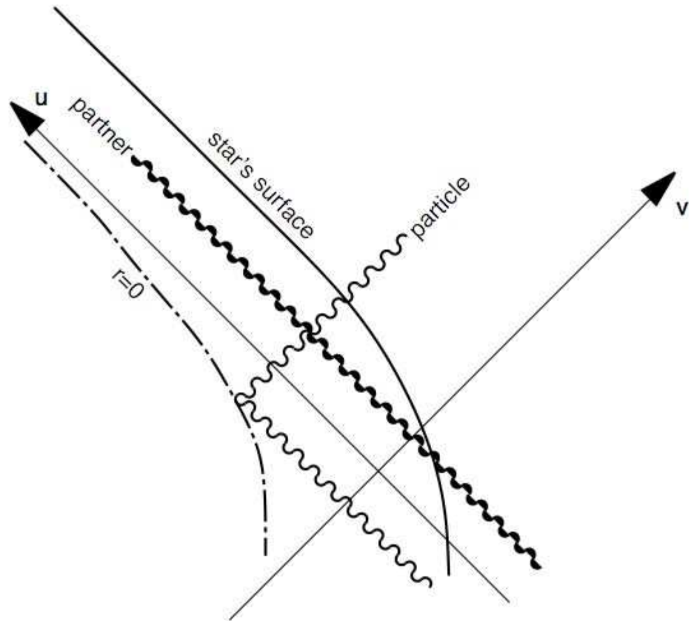

Fig. 3.7 *Pair formation drawn in u, v coordinates.*

To compare with the accelerated mirror of Section 2.5 we also have drawn in Fig.(3.7) the two pieces of $\hat{\varphi}_\lambda(U)$ in the extended u, v system.

From these pictures and the above analysis it is seen that the conversion of fluctuations into particles, unlike in Chapter 1 (Fig. (1.2)), does not occur over a well defined space-time domain. This is due to the absence of a scale, and it is reflected in the arbitrariness of the parameter A which appeared in the definition of Π and ξ in the above paragraph (we note in passing that the introduction of mass does not help since the frequencies of the modes within the star rapidly become large compared to any known mass). Nevertheless, though tunneling seems inappropriate as a tool in black hole physics (as it appears in the present formulation) one should not lose sight of its conceptual content. In chapter 1, particle production could have been understood either in terms of tunneling of modes or of their backward scattering in time according to the gauge choice. It is therefore useful to define a generalized tunneling concept, the conversion of a mode from a domain of virtuality (i.e. as a vacuum fluctuation) to a domain of realization (i.e. measurable as a quantum of excitation, in a counter for example). The process, in general, is caused by the imposition of an external field. In black hole evaporation the process induced by the gravitational field of the collapsing star is summarized succintly: virtuality on $\mathcal{I}^-$ induces reality on $\mathcal{I}^+$. The precise formulation of this aspect of the problem is the subject of Section 3.5.

We also note here that we have come upon a rather unexpected fragility of the theoretical foundations which have been set to work thus far. Is it reasonable to expect modes to remain free and unmixed in their infinite excursion from $\mathcal{I}^-$ to $\mathcal{I}^+$? Indeed, eq. (3.43) tells us that after a time of the order of $\Delta u = M \ln M$ the frequencies ω of the vacuum fluctuations converted upon resonance into Hawking quanta exceed the Planck frequency. More of this in Section 3.7.

3.3 Renormalized Energy Momentum Tensor in Unruh Vacuum.

One of the approaches used to address the back reaction to black hole evaporation consists in solving the semi-classical equations

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle \tag{3.52}$$

wherein the mean value of the energy momentum tensor is taken as source of Einstein's equations. The fundamental assumption is that the source terms are $\langle T_{\mu\nu} \rangle$ i.e. that the average solution is the solution of the equations using the average source. When the fluctuations are large, or when one wishes to evaluate the importance and the consequences of the fluctuations, one has to analyze other matrix elements than the mean appearing in eq. (3.52). This is why we shall study in Section 3.5 non diagonal matrix elements of the operator $T_{\mu\nu}$.

The mean energy momentum tensor which appears on the r.h.s. of eq. (3.52) is formally infinite and must be regularized and renormalized before starting to solve eq. (3.52). In this section we deal with this part only, in the next section we shall treat the gravitational back reaction in this semi-classical approximation.

Until this time, the complete four dimensional computation of $\langle T_{\mu\nu} \rangle$ has not yet been carried out. What has been achieved is a good approximation to $\langle T_{\mu\nu} \rangle$ in the static spherical symmetric case [49]. In the evaporating situation no analytical expression for the mean energy momentum tensor has been obtained. However we recall that most of Hawking radiation is due to s-waves and that, for the s-waves, the problem simplifies a lot if one drops the residual centrifugal barrier. (Indeed one ends up with a conformally invariant two dimensional theory.) Hence one expects that the essential properties of the four dimensional theory are recovered from the effective two dimensional one.

We shall therefore continue with this truncated theory and calculate analytically $\langle T_{\mu\nu} \rangle$ in the Unruh vacuum following ref.[30]. Since the modes in the two dimensional model are rescaled by a factor of $\sqrt{4\pi}r$ as compared with the original field (see eq. (3.29)), the fluxes should be rescaled by $4\pi r^2$

$$\langle T^{4D}_{\mu\nu} \rangle = \frac{1}{4\pi r^2} \langle T^{2D}_{\mu\nu} \rangle \tag{3.53}$$

This equation applies for $\mu, \nu = U, V$ and we shall see that Bianchi indentities imply $\langle T_{\theta\theta} \rangle = 0$.

The renormalization of the energy momentum tensor has already been carried out, in two dimensions, in the case of a flat background geometry in Section 2.5 and we recall the result, eq. (2.86) (extended to the V modes as well)

$$\begin{aligned} \langle T_{UU} \rangle_{fg} - \langle T_{UU} \rangle_0 &= \frac{1}{12\pi} f'^{1/2} \partial_U^2 f'^{-1/2} \\ \langle T_{VV} \rangle_{fg} - \langle T_{VV} \rangle_0 &= \frac{1}{12\pi} g'^{1/2} \partial_V^2 g'^{-1/2} \end{aligned} \tag{3.54}$$

Here $\langle T_{UU} \rangle_0$ means Minskowski vacuum expectation value and U means inertial Minkowski coordinate. $\langle \quad \rangle_{fg}$ means the average with respect to the vacuum defined by the modes $e^{-i\omega f(U)}/\sqrt{4\pi\omega}$ and $e^{-i\omega g(V)}/\sqrt{4\pi\omega}$. An important property of eqs (3.54) is that they can be inverted so as to express $\langle T_{ff} \rangle$ and $\langle T_{gg} \rangle$ in terms of the inverse functions $U(f)$ and $V(g)$: all one does is flip the sign and replace f, g by their inverse functions, as is required by the reciprocity of these relations.

In a curved background we shall generalize the subtraction (3.54) by subtracting from $\langle T_{\mu\nu}(x) \rangle$ the value $\langle I(x)|T_{\mu\nu}(x)|I(x) \rangle$ calculated from the inertial modes at x [65], i.e. those modes which most resemble Minkowski modes at x. This is taken on the principle that flat space is a solution, i.e. that the expectation value of $T_{\mu\nu}$ in Minkowski vacuum is zero. The natural generalization is to pose that in the local vacuum $|I(x)\rangle$ of local inertial modes, $\langle I(x)|T_{\mu\nu}(x)|I(x)\rangle_{ren}=0$. For an arbitrary state one then postulates that $\langle T_{\mu\nu}(x)\rangle_{ren}/4\pi r^2$ is the gravitational source where

$$\langle T_{\mu\nu}(x)\rangle_{ren}=\langle T_{\mu\nu}(x)\rangle-\langle I(x)|T_{\mu\nu}(x)|I(x)\rangle \tag{3.55}$$

Furthermore, one must implement this difference of infinities with a regularization scheme which as in the previous chapters is taken to be the split point method e.g.

$$\langle T_{UU}\rangle=\lim_{U\to U'}\partial_U\partial_{U'}\langle\phi(U)\phi(U')\rangle \tag{3.56}$$

To compute $\langle T_{\mu\nu}\rangle_{ren}$ we first need to construct inertial coordinates about x. To this end, we express the spherically symmetric geometry, in a "conformal" gauge ($g_{UV}=C, g_{UU}=g_{VV}=0$ everywhere)

$$ds^2=-C(U,V)dUdV+r^2(U,V)d\Omega^2 \tag{3.57}$$

The residual reparametrization invariance $U\to U'(U), V\to V'(V)$ is fixed by requiring that the state of the field, in which one wants to compute $\langle T_{\mu\nu}\rangle_{ren}$ is vacuum with respect to the modes $e^{-i\omega U}, e^{-i\omega V}$. In this way the conformal factor $C(U,V)$ encodes the vacuum state of the ϕ field as in the flat exemple eq. (2.85).

The inertial set $\hat{U},\hat{V}$ based on the U,V set is

$$\begin{aligned}\hat{U}-\hat{U}_0 &= \int_{U_0}^{U}\frac{C(U',V_0)}{C(U_0,V_0)}dU'\\ \hat{V}-\hat{V}_0 &= \int_{V_0}^{V}\frac{C(U_0,V')}{C(U_0,V_0)}\,dV'\end{aligned} \tag{3.58}$$

whereupon

$$ds^2=-\frac{C(U,V)C^2(U_0,V_0)}{C(U_0,V)C(U,V_0)}d\hat{U}d\hat{V}+r^2(U,V)d\Omega^2 \tag{3.59}$$

with U,V in eq. (3.59) functions of $\hat{U},\hat{V}$ obtained by inversion of eqs (3.58). The coordinates $\hat{U},\hat{V}$ are affine parameters along the radial light–like geodesics $U=U_0$ and $V=V_0$ which pass through $x=(U_0,V_0)$. Hence they are the inertial set at x which we shall use to define the local vacuum $|I(x)\rangle$ since they constitute the most inertial parametrization of the light like geodesics

$U = U_0$, $V = V_0$ (in particular the Christoffel symbols vanish at x in coordinates $\hat{U}, \hat{V}$). One may now apply eqs (3.54) in terms of the function $U(\hat{U})$. Lest the reader have no misunderstanding of the approximate status of the following computation, it comes from mode analysis, not from the form of the metric. We consider only the contribution of the s-waves and in addition drop the residual barrier so as to make the effective field theory conformal in the U, V sector.

One identifies the Minkowski coordinates of eqs (3.54) which defined the subtraction, with the coordinates $\hat{U}, \hat{V}$ of eq. (3.58) and similarly one identifies the functions f, g, which defined the state of the field, with the functions $U(\hat{U}), V(\hat{V})$ whereupon

$$\begin{aligned} \langle T_{UU} \rangle_{ren} &= -\frac{1}{12\pi} C^{1/2} \partial_U^2 C^{-1/2} \\ \langle T_{VV} \rangle_{ren} &= -\frac{1}{12\pi} C^{1/2} \partial_V^2 C^{-1/2} \end{aligned} \tag{3.60}$$

where we have used the reciprocity mentioned after eqs (3.54).

Moreover, in addition to eqs (3.60), the trace of the energy momentum tensor tr $T_{\mu\nu} = 4T_{UV}/C = m^2\phi^2$ no longer vanishes even though classically it vanishes for a massless scalar field. This is because C is a function of both U and V contrary to f in eqs (3.54). Indeed, requiring that the renormalized energy momentum tensor be conserved

$$T^V_{U;V} = \frac{1}{C} T_{UU,V} + (C^{-1} T_{UV})_{,U} = 0 \tag{3.61}$$

and the similar equation for $T^U_{U;V}$ and requiring also that T_{UU} and T_{VV} be given by eqs (3.60) imposes a trace given by

$$\langle T \rangle_{ren} = 2g^{UV} \langle T_{UV} \rangle_{ren} = \frac{1}{24\pi} R \tag{3.62}$$

where $R = \Box \ln C = 4C^{-1}\partial_U \partial_V \ln C$ is the two dimensional curvature. Equation (3.61) is the energy conservation in two dimensions but we emphasize that it is also valid in four dimensions if one divides the two dimensional fluxes by $4\pi r^2$ and if one sets $T_{\theta\theta}$ to zero. Then our $\langle T_{\mu\nu} \rangle_{ren}$, divided by $4\pi r^2$, is a legitimate source for the Einstein equations in four dimensions.

One sees that the result $\langle T \rangle_{ren} = 0$ when $m^2 = 0$ is erroneous because one has not taken into account the change in the local vacuum upon changing the point $x = (U_0, V_0)$ which leads to the non-vanishing of the first term of the r.h.s. of eq. (3.61). More refined regularizations leads to the result directly [30] [10]. One can also interpret the origin of a non vanishing trace by remarking that the presence of a curvature within the Compton length of a vacuum fluctuation causes the existence of a term $O(R/m^2)$ which must be subtracted to keep the one loop approximation to the effective gravitation action finite.

Note also that the quantum trace given in eq. (3.62) is a pure geometric quantity, the same for all states[2]. Indeed the difference in energy between two states (of which eqs (3.54) is a particular example) is given by conserved ΔT_{UU} and ΔT_{VV}, hence with $\Delta T_{UV} = 0$.

The above procedure of subtracting $\langle I|T_{\mu\nu}|I\rangle$ can in principle be applied in the original four dimentional theory and be generalized to fields with mass (see ref.[61]). A number of alternative procedures have been devised for renormalizing the energy momentum tensor (see ref.[10] for a review). Happily they are all equivalent: indeed Wald [98] has shown that if the renormalized energy momentum tensor satisfies some simple and natural conditions it is completely fixed (see nevertheless ref.[17]).

We now apply eqs (3.60) to our problem. As a warm-up let us start with Boulware vacuum. This is the case one would have if the star were eternally static at some fixed radius ($= R_0$) greater that its Schwarzschild radius ($= 2M$) and we shall have in mind the case $R_0 - 2M << 2M$.

In the Schwarzschild region, the modes $e^{-i\lambda u}/\sqrt{4\pi\lambda}$ and $e^{-i\lambda v}/\sqrt{4\pi\lambda}$ define the Boulware (B) vacuum (e.g. usual vacuum at $r = \infty$) . The conformal factor C is $(1 - 2M/r)$ in the Schwarzschild coordinates u, v defined in eq. (3.5). Since $C(u, v) = C(v - u)$ we see immediately from eqs (3.60) that $\langle T_{uu}\rangle_B = \langle T_{vv}\rangle_B$ thereby implying no flux, $\langle T_{rt}\rangle_B = 0$, as it should be for a static situation. Moreover since $\lim_{r\to\infty} \partial_r C(r) = 0$ we have $\langle T_{uu}\rangle_B = \langle T_{vv}\rangle_B = 0$ at $r = \infty$. This is as it should be because the Schwarzschild metric is asymptotically flat and because the Schwarzschild modes are identical to the usual Minkowski modes for large r.

It is also very easy to obtain $\langle T_{uu}\rangle_B$ in the vicinity of $r = 2M$. Indeed, near $r = 2M$,

$$C(r) \simeq -e^{r^*/2M} = -e^{(v-u)/4M} \tag{3.63}$$

[2] This is true even in the full four dimensional theory wherein the trace anomaly is a quadratic form of the Riemann 4-tensor.

so that from eqs (3.60)

$$\lim_{r\to 2M}\langle T_{uu}\rangle_B = \langle T_{vv}\rangle_B = -(1/12\pi)(1/64M^2) = -(\pi/12)T_H^2 \tag{3.64}$$

i.e. minus the asymptotic Hawking flux eq. (3.46). The origin of this negative energy density lies on the Rindler character of the Schwarzschild geometry expressed in the Schwarzschild coordinates u, v in eq. (3.63). Indeed, the Minkowski metric expressed in Rindler coordinates (eq. (2.19)) reads

$$ds^2 = -dUdV = -e^{a(v-u)}dudv. \tag{3.65}$$

Thus the specification of being in Boulware vacuum becomes, near $r = 2M$, equivalent to being in Rindler vacuum in Minkowski space (see eqs. (2.44, 2.109)).

It is to be observed that $\langle T_{uu}\rangle_B$ being finite on the future horizon $u \to \infty$ ($U \to 0$) leads to singular values of $\langle T_{UU}\rangle_B$ on H: $\langle T_{UU}\rangle_B = (du/dU)^2\langle T_{uu}\rangle_B = (4M/U)^2\langle T_{uu}\rangle_B$. Similarly $\langle T_{VV}\rangle_B$ blows up on the past horizon ($V = 0$) which exists in the complete Schwarzshild space. If this situation would pertain to the collapsing case (i.e. to the true physical state of affairs) one would arrive at a catastrophic situation in that as $U \to 0$, the values of $\langle T_{UU}\rangle$ would tend to $-\infty$; and it is $\langle T_{UU}\rangle$ which is close to that measured by an inertial observer near the horizon (since $ds^2 = -e^{-1}dUdV$ on the horizon, see eq. (3.7)). The accomodation to this singular behavior is a remarkable feature of black hole evaporation.

Indeed, in the collapsing case, in the Unruh vacuum, whereas the v-modes remain $e^{-i\lambda v}/\sqrt{4\pi\lambda}$ in the outer Schwarzschild region the u-modes rapidly behave like $e^{-i\omega U}/\sqrt{4\pi\omega}$, see eqs. (3.37, 3.38) and the discussion after eq. (3.44). Without calculation we see from eqs (3.60) that near the horizon $\langle T_{uu}\rangle_U$ vanishes (where the subscript U refers to Unruh vacuum). Indeed space is regular near the horizon and the inertial modes differ only slightly from the Kruskal modes. Therefore $\langle T_{UU}(r = 2M)\rangle_U$ is finite and $\langle T_{uu}\rangle_U = (U/4M)^2\langle T_{UU}\rangle_U$ vanishes quadratically on the horizon $U = 0$. The former singularity is thus obliterated. The quadratic vanishing is necessary for having no singularity for a free falling observer since the combined effect of the gravitational and Doppler shifts already encountered in eq. (3.26) is always present for any inertial trajectory crossing the future horizon. In refering once more to the isomorphism between this situation and the Minkowski-Rindler case, we see that the u part of the state in the U-vacuum behaves like a regular Minkowski state on the horizon $U = 0$.

One can now either proceed by calculation to find $\langle T_{uu}\rangle_U$, using eq. (3.52) to go from Boulware to Unruh vacuum (with $g(v) = v$; $f(u) = U(u) = -4Me^{-u/4M}$ see eqs. (3.6, 3.38)), or easier yet (and perhaps more physical) appeal to the equation of conservation (3.61). Indeed $\langle T_{uu}\rangle_U$ differs from $\langle T_{uu}\rangle_B$ by a function of u only. But this function of u must be a constant because we are in a steady state characterized by a rate eqs. (3.43, 3.44). Whence

$$\langle T_{uu}(r)\rangle_U = \langle T_{uu}(r)\rangle_B - \langle T_{uu}(2M)\rangle_B \tag{3.66}$$

i.e. the constant is fixed by $\langle T_{uu}\rangle_U = 0$ at $r = 2M$. Hence, from eq. (3.64), the constant flux $T^r_{\ t} = T_{uu} - T_{vv}$ is given, in Unruh vacuum, by $\langle T_{uu}(r = \infty)\rangle_U = (\pi/12)T_H^2$ which is the thermal flux at the Hawking temperature as in the mode analysis of Section 3.2.

The behavior of $\langle T_{\mu\nu}\rangle$ for finite r is obtained from eqs (3.60) with $C = (1 - 2M/r)$ and $\partial_{r^*} = (dr/dr^*)\partial_r$:

$$\begin{aligned} \langle T_{vv}\rangle_U &= \langle T_{vv}\rangle_B = \langle T_{uu}\rangle_B \\ &= \frac{\pi}{12}T_H^2\left(\frac{48M^4}{r^4} - \frac{32M^3}{r^3}\right) \end{aligned} \tag{3.67}$$

and

$$\langle T_{uu}\rangle_U = \frac{\pi}{12}T_H^2\left(1 - \frac{2M}{r}\right)^2\left(1 + \frac{4M}{r} + \frac{12M}{r^2}\right) \tag{3.68}$$

wherein the quadratic vanishing at $r = 2M$ and the asymptotic behavior are displayed.

All this is without back reaction i.e. without taking into account the decrease of M in time. But the energy conservation (the Einstein equations at large r) dictates that there is a necessary backreaction wherein

$$\frac{dM}{dt} = \langle T_{rt}\rangle_U = -\frac{\pi}{12}\left(\frac{1}{8\pi M}\right)^2 \tag{3.69}$$

We recall that the flux T_{rt} should be integrated over the sphere and that our $T_{\mu\nu}$ has been rescaled by $4\pi r^2$, therefore eq. (3.69) is the usual four dimensional expression for the mass loss. Upon correcting for the existence of the potential barrier and adding the contribution of the higher angular momentum modes $l \geq 1$ one finds

$$\frac{dM}{dt} = -\xi\frac{1}{M^2} \tag{3.70}$$

where ξ depends on the spin of the radiated field [68]. Then, if one assumes that at later times the rate of evaporation is given by the same equation, the decay time for complete evaporation is $\tau_{decay} = M^3/3\xi$ in Planckian units.

In eq. (3.69) the sphere over which one calculates can be situated at any value of r since the flux is conserved. But the interpretation of the integral changes with r. At large r one has $\langle T_t^r \rangle_U = \langle T_{uu} \rangle_U$. This is a traditional positive energy outflow. Instead, near the horizon $\langle T_t^r \rangle_U = -\langle T_{vv} \rangle_B$ since $\langle T_{uu} \rangle_U$ vanishes there. How these two properties contrive to modify the metric and describe a black hole with a slowly varying mass parameter is the subject of the next section.

3.4 The Semi-Classical Back Reaction

Up to this point in our primer we have concentrated on the response of matter to a fixed background geometry, that of the collapsing star. The dynamical response to this time dependent geometry is the emission of energy to infinity accompanied by an accommodation of the vacuum near the black hole's horizon. Indeed at the horizon the mean flux is carried by $\langle T_{vv} \rangle$ only. As emphasized at the end of the previous section, the rate of change of mass is $\langle T_{rt} \rangle|_{fixed\ r}$ where the value of r is arbitrary. Hence near the horizon the description of the mass lost by the star is given in terms of a halo of negative energy which accumulates around the horizon.

The above discussion then suggests the shrinking of the area of the horizon. But how does a significant reduction of the area, hence a large change in the metric, affect $T_{\mu\nu}$, and how in turn do these affect the metric? The complete answer to this question (i.e. wherein one takes into account the full quantum properties of the operator $T_{\mu\nu}$) is the subject of present research and is far from resolution -very far indeed in that its ultimate elucidation may well entail (or lead to) the quantum theory of gravity (see Section 3.7 in this regard).

Until the present time the only quantitative treatment available is in the context of the semi-classical theory i.e. gravity is classical and Einstein's equations are driven by the mean value of $T_{\mu\nu}$ (see eq. (3.52)). Since the latter is a function of $g_{\mu\nu}$ and its derivatives, one has highly non trivial equations to solve. In what follows we shall report on what is known concerning this enterprise. We first give a semi quantitative description of what happens. It is then followed by more rigorous considerations and the presentation of the properties of the evaporating geometry obtained by numerical integration of a simplified model.

What are the features that one wishes to display? The first concerns the validity of eq. (3.69) at later times. Is the mass loss at time t determined by $M(t)$, the mass at that time, through Hawking's equation? In other words is the system Markoffian? One finds that the answer is yes, and that it is due to the continued negative energy flux across the horizon. This results in an outside metric which is determined by $M(t)$ and not by history. The second question concerns the distribution of this negative energy. Is it a halo, or is it distributed homogeneously within the collapsing star so as to give rise to a sort of "effective" star of mass $M(t)$? One finds the former. There is a halo which accumulates in a region enclosed between the surface of the star and the horizon, which we now explain on qualitative grounds.

In order to describe the geometry near the horizon it is necessary to choose a coordinate system. A convenient choice is the Eddington-Finkelstein set (r, v) introduced in eq. (3.8). The advantages of these coordinates are:

1) they cover both sides of the future horizon;

2) in the case of a purely ingoing light like flux, the metric

$$ds^2 = -(1 - \frac{2M(v)}{r})dv^2 + 2dvdr + r^2 d\Omega^2 \tag{3.71}$$

where the only parameter is

$$M(v) = \int^v dv T_{vv} \tag{3.72}$$

is an exact solution of Einstein's equations (the Vaidya solution). We remind the reader that we are working with energy momentum tensor rescaled by $4\pi r^2$, see eq. (3.53). In the evaporating situation near the horizon the flux is entirely carried by T_{vv} hence 3.71 is valid in the vicinity of the horizon and can be used to describe the physics there.

We are interested in the behavior of outgoing light rays because the modes which give rise to Hawking radiation flow along these light rays. Let us first recall what values of r a light ray, reflected from $r = 0$, visits without back reaction (i.e. with $M = M_0$). Photons which emerge from the star at $r = 2M_0 + \epsilon$ (with $\epsilon > 0$) proceed to infinity after adhering to the horizon $r = 2M_0$ for a while (a v lapse of order $M_0 \ln(\epsilon/2M_0)$). Those which emerge at $r = 2M_0 - \epsilon$ also adhere to the horizon for a while before falling into the singularity. These trajectories describe the locus

$$u = const = v - 2r^* = v - 2r - 4M_0 \ln |r/2M_0 - 1| \tag{3.73}$$

for r near the horizon. The main point is the existence of a horizon at $r = 2M_0$ which separates the out's from the in's, those which escape from those which are trapped and crash into the singularity.

Now suppose that the mass decreases due to the absorbtion of negative energy: $\langle T_{vv}\rangle \simeq -1/M^2$. Then the locus which separates the out's from the in's is expected to shrink. Thus outgoing light rays feel a slowly diminishing gravitational potential and some of those which previously were trapped may now escape after having been sucked in for a while. The locus where they cease to fall in and start to increase in radius is called the apparent horizon. To describe this quantitatively we consider the equation for outgoing light rays when the mass is varying, in the metric 3.71

$$\frac{dr}{dv} = \frac{1}{2}\frac{r - 2M(v)}{r} \tag{3.74}$$

The apparent horizon is the locus where $dr/dv = 0$, i.e.

$$r_{ah} = 2M(v) \tag{3.75}$$

And it shrinks according to eq. (3.72) i.e. $dr_{ah}/dv \simeq -1/r_{ah}^2$. However the apparent horizon does not separate the geodesics which will ultimately fall into the singularity from the escaping ones since it is not an outgoing geodesic (for an evaporating black hole, when T_{vv} is negative, it is time like [47]). It is nevertheless very close to the event horizon (the light ray which does separate the two classes). An estimate of the radius of the event horizon is the inflexion point of the outgoing geodesics $d^2r/d^2v = 0$:

$$r_{eh}(v) = 2M(v) + 8MdM/dv = r_{ah}(v) - O(1/M) \tag{3.76}$$

Therefore all the degrees of freedom which are in the whole region $r < 2M(v) - O(1/M)$ remain, in this semi-classical treatment, inaccessible to the outside observer. This inaccessibility implies that the loss of mass does not come from the evaporation of degrees of freedom from inside to outside. Rather, it is due to the accumulation of negative energy.

The slow rate of change of r_{ah} suggests that the processes which occur in this geometry are the same as in the case with no backreaction and with

M equal to $M(v)$. For instance an important property of the outgoing light rays in the geometry 3.71 is that they are given, in good approximation, by the usual formula, eq. (3.73), with M replaced by $M(v)$

$$v - 2r - 4M(v)\ln(r - 2M(v)) = u \tag{3.77}$$

provided

$$M >> r - 2M(v) >> 1/M \tag{3.78}$$

where the restrictions come from the rate of change: $dM/dv \simeq 1/M^2$. Hence when this is valid the out-going geodesics are a scaled replica of what happens without back reaction with M replaced by $M(v)$.

Based on the above one may conjecture that the radiation emitted at time v is also a scaled replica of the radiation emitted in the absence of back reaction with $M = M(v)$. To understand why the evaporation is only controlled by $M(v)$ (and therefore why the past history of the black hole plays no role) and to have a full appreciation of the processes which occur around the apparent horizon (where eq. (3.77) is not valid) one must resort to a more detailed analysis. This is done in the following paragraphs. The upshot is that all that has been discussed qualitatively is correct to $O(1/M)$.

[Following Bardeen[6] and York[96] who followed up the ideas of Hajicek and Israel [42] we shall use the metric

$$ds^2 = -e^{2\psi}(1 - 2m(v,r)/r)dv^2 + 2e^{\psi}dvdr + r^2 d\Omega^2 \tag{3.79}$$

which describes a general spherically symmetric space-time. In these coordinates Einstein's equations are

$$\begin{aligned} \frac{\partial m}{\partial v} &= T^r{}_v \\ \frac{\partial m}{\partial r} &= -T^v{}_v \\ \frac{\partial \psi}{\partial r} &= T_{rr}/r \end{aligned} \tag{3.80}$$

Note that ψ is defined only up to the addition of an arbitrary function of v corresponding to a reparametrization of the v coordinate. For simplicity we shall suppose that the right hand side of (3.80) is given by the two dimensional renormalized energy momentum tensor discussed in section 3.3 (see eq. 3.60 and 3.62). However the proof is general provided $M >> 1$. The r.h.s. of 3.80

can be taken to be the full 4 dimensional renormalized energy momentum tensor.

We shall proceed in three steps following [62]: first we shall suppose that the renormalized energy momentum tensor resembles the renormalized tensor in the absence of back reaction. We shall then show that under this hypothesis the metric coefficients in (3.79) are slowly varying functions of r and v. Finally we shall solve adiabatically the Klein Gordon equation in this slowly varying metric and show that the renormalized tensor indeed possesses the properties supposed at the outset thereby proving that the calculation is consistent.

Our first task is to obtain estimates for $T_{\mu\nu}$ both far and near the black hole. We begin with the former. We suppose that when r is equal to a few times $2m$ (say $r = O(6m)$) there is only an outgoing flux $T_{uu}(r >> 2m) = L_H(u)$ where L_H is the luminosity of the black hole. This is justified since in the absence of back reaction the other components of $T_{\mu\nu}$ decrease as large inverse powers of r. For instance the trace anomaly, in our 2 dimensional problem decreases as M/r^3 in the static Schwarzschild geometry. We shall also suppose that L_H is small (i.e. $L_H M^2 = O(1)$) and varies slowly. Hence when $r > O(6M)$ an outgoing Vaidya metric is an exact solution of Einstein's equations

$$\begin{aligned} ds^2 = -(1 - 2M(u)/r)du^2 - 2dudr + r^2 d\Omega^2 \\ M(u) = \textstyle\int^u du' \; L_H(u') \end{aligned} \tag{3.81}$$

The change of coordinates from (3.80) to (3.81) is obtained by writing the equation for infalling radial null geodesics in the metric (3.81) as

$$F dv = du + \frac{2dr}{1 - 2M(u)/r} \tag{3.82}$$

where F is an integration factor. Upon using (3.82) to change coordinates from the set (u, r) to (v, r) one finds that $e^\psi = F$ and $m(r, v) = M(u)$. Hence when $r > O(6M)$ the r.h.s. of (3.80) is given by

$$\begin{aligned} T^r_{\ v}(r > O(6M)) &= -e^\psi L_H \\ -T^v_{\ v}(r > O(6M)) &= 2L_H/(1 - 2M/r) \\ T_{rr}(r > O(6M))/r &= 4L_H/r(1 - 2M/r)^2 \end{aligned} \tag{3.83}$$

We now estimate $T_{\mu\nu}$ near the horizon (i.e. near $r_{ah} = 2m(r_{ah}, v)$) by assuming that the energy momentum tensor measured by an inertial observer falling across the horizon is finite and of order L_H. Near $r = r_{ah}$ one may neglect $g_{vv} = -e^{2\psi}(1 - 2m/r)$ and use the reparametrization invariance of v to choose $\psi(r_{ah}, v) = 0$ whereupon the metric becomes $ds^2 \simeq 2dvdr + r^2 d\Omega^2$. Hence near r_{ah}, r and v behave like inertial light like coordinates (ie. the proper time of an inertial infalling observer near the apparent horizon is $cv + c^{-1}r$ where c is a constant which depends on the precise trajectory of the observer). That the energy momentum is of order L_H near the horizon is reexpressed as (for the components T_v^v and T_{rr})

$$\begin{aligned} T_v^v(r \simeq r_{ah}) &= O(L_H) \\ T_{rr}(r \simeq r_{ah}) &= O(L_H) \end{aligned} \tag{3.84}$$

Where we have used the inertial character of the set v, r near r_{ah}. In addition we shall determine $T_v^r(r \simeq r_{ah})$ by making appeal to the conservation of energy

$$T^r{}_{v,r} + T^v{}_{v,v} = 0 \tag{3.85}$$

where $T^v{}_{v,v} = O(L_{H,v})$. Integrating the conservation equation from $r = 2m$ to $r = O(6m)$ yields T_v^r near the horizon in terms of its value where (3.83) is valid. Putting everything together, near the horizon we have

$$\begin{aligned} \frac{\partial m}{\partial v} &= -L_H e^{\psi} + O(mL_{H,v}) \\ \frac{\partial m}{\partial r} &= O(L_H) \\ \frac{\partial \psi}{\partial r} &= O(L_H/r) \end{aligned} \tag{3.86}$$

As announced all metric coefficients vary slowly if L_H is small and varies slowly. Integrating the equation for ψ yields $e^{\psi} \simeq r^{L_H}$ for all $r \geq r_{ah}(v)$. Hence ψ can safely be neglected up to distances $r = O(e^{1/L_H})$. From now on we suppose for simplicity of the algebra that $\psi = 0$.

In order to calculate the modes and $\langle T_{\mu\nu}\rangle_{ren}$ we must first investigate the outgoing radial nul geodesics in the metric (3.79) with $\psi = 0$. As mentioned in 3.77, away from the horizon these geodesics are a scaled replica of the geodesics in the absence of back reaction. Near the horizon their structure is complicated by the distinction that has to be made between the apparent and event horizon. We recall (equation 3.75) that the apparent horizon is the locus where outgoing geodesics obey $dr/dv = 0$, therefore solution of $r_{ah}(v) = 2m(v, r_{ah}(v))$.

The event horizon $r_{eh}(v)$ is the last light ray which reaches $\mathcal{I}_+$. It satisfies the equation of outgoing nul geodesics

$$\frac{dr_{eh}}{dv} = \frac{1}{2}\frac{r_{eh} - 2m(r_{eh}, v)}{r_H} \tag{3.87}$$

Setting $r_{eh}(v) = r_{ah}(v) + \Delta(v)$ one can rewrite 3.87 in the form $\Delta(v) = 2(r_{ah}(v) + \Delta)(r_{ah})_{,v} + \Delta_{,v}) + 2m(v, r_{eh}(v)) - r_{ah}(v)$. Solving recursively one obtains an asymptotic expansion for Δ the first term of which is $\Delta = r_{ah}(v)(r_{ah})_{,v} \simeq -1/M$ (see equation 3.76).

To obtain the trajectory of the outgoing nul geodesics we change variables to the set $(v, x = r - r_{eh}(v))$ i.e. x is the comoving distance from the event horizon. In these coordinates the metric eq. (3.79) becomes (using 3.87)

$$ds^2 = -\frac{2m(v, r_{eh} + x)x}{r_{eh}(r_{eh} + x)}dv^2 + 2dvdx + r^2 d\Omega^2 \tag{3.88}$$

This metric resembles the Edington-Finkelstein metric in the absence of back reaction in a crucial way. To wit $g_{vv}(x, v)$ vanishes on an outgoing null geodesic, the event horizon $x = 0$. Using this form for the metric it is now straightforward to obtain the outgoing null geodesics. Indeed when $x << r_{eh}$ the equation for radial outgoing nul geodesics can be solved exactly to yield an exponential approach to the horizon of the form $\tilde{v} - 2\ln x = f(u)$ where

$$\tilde{v} = \int^v dv\ \frac{2m(v, r_{eh}(v))}{r_{eh}^2(v)} \tag{3.89}$$

This motivates the following ansatz for the outgoing radial null geodesics valid in all space time outside the collapsing star

$$\tilde{v} - 2\frac{x}{r_{eh}(v)} - 2\ln x + \delta = \int^u \frac{du'}{2\tilde{m}(u')} + D \tag{3.90}$$

with D a constant of integration. This should be compared with the solution in the absence of backreaction given in eq. (3.73). The function of u on the r.h.s. of (3.90) has been written as $\int^u du'/\tilde{m}(u')$ for dimensional reasons. The quantity δ is of order $O(L_H(Mx + x^2)/M^2)$ for all x (this is shown by substitution of (3.90) into the equation for radial nul geodesics and integrating the equation for δ along the geodesics $u = const$). The function $\tilde{m}(u)$ is determined by requiring that the variable u in equation (3.90) be the same as in the Vaidya metric equation (3.81). The difference $M(u) - \tilde{m}(u)$ is then of order $O(ML_H)$. This is found by using (3.90) to change coordinates from the set (v, r) to the set (u, r) at the radius $r = O(6M)$ where (3.81) is valid.

Equation (3.90) is sufficient to prove our hypothesis, to wit that the flux emitted is $O(M^{-2})$ and that the energy momentum tensor is regular at the horizon. Indeed, in our model, the flux emitted is given by eq. (3.54):

$$\langle T_{uu}(u, \mathcal{I}^+)\rangle = (1/12\pi)(d\mathcal{U}/du)^{1/2}\partial_u^2(d\mathcal{U}/du)^{-1/2} \tag{3.91}$$

where the derivatives are taken at fixed v. The variable $\mathcal{U}$, defined in eq. (3.13), labels the outgoing geodesics as measured by an inertial observer inside the star. The jacobian $du/d\mathcal{U}$ is calculated by remarking that at the surface of the star the derivative at fixed v is $d\mathcal{U}/dr|_{v=vstar} = -2$. Hence differentiating 3.90 yields

$$\begin{aligned} du/d\mathcal{U}|_v &= (dr/d\mathcal{U})|_{v=vstar}(du/dr)|_{v=vstar} \\ &= (1/2)4\tilde{m}(u)/x|_{v=vstar} = -4\tilde{m}(u)/(\mathcal{U} - \mathcal{U}_{eh}) \end{aligned} \tag{3.92}$$

Hence $\langle T_{uu}(u, \mathcal{I}^+)\rangle$ is equal to $(\pi/12)T_H^2(u)$ where

$$T_H(u) \simeq \frac{1}{8\pi\tilde{m}(u)} = \frac{1}{8\pi M(u)}(1 + O(L_H)) \tag{3.93}$$

is the Hawking temperature at time u when the residual mass is $M(u)$.

The calculation of $\langle T_{uu}(v, x)\rangle_{ren}$ every place (and not only on $\mathcal{I}^+$) is a slight generalization of the above calculation. The renormalized energy momentum tensor is given by

$$\langle T_{uu}\rangle_{ren} = (d\hat{U}/du)^2(1/12\pi)(d\mathcal{U}/d\hat{U})^{1/2}\partial_u^2(d\mathcal{U}/d\hat{U})^{-1/2} \tag{3.94}$$

where $\hat{U}(u, v)$ is the inertial coordinate introduced in equation 3.58. Since $\hat{U}(u, v)$ is an affine parameter along radial nul geodesics $v = const$ one obtains that $\hat{U}(u, v) = x(u, v)$. Hence

$$d\hat{U}/d\mathcal{U}|_v = dx/du|_v du/d\mathcal{U}|_v \tag{3.95}$$

which upon differentiating eq. 3.91 and using 3.92 is found to be finite on the horizon. Hence as in section 3.3 $\langle T_{uu}\rangle_{ren}$ vanishes quadratically at the event horizon $\mathcal{U} = \mathcal{U}_{eh}$. Thus the mean $\langle T_{\mu\nu}(r, v)\rangle$ is the one computed without backreaction, in the collapsing geometry of a star whose mass is $M(v)$ and this is valid as long as $\partial_v M(v) << 1$.]

These conclusions have been verified numerically in a model [75] wherein the sources of the Einstein equations are a classical infalling flux of spherical light-like dust (i.e. falling along $v = constant$) and the energy momentum tensor given once more by eqs. (3.60 and 3.62) properly rescaled see eq. (3.53). Since this quantum energy momentum tensor is a local function of derivatives of the conformal factor C, one is lead to a new set of differential equations of the second order which describe dynamically the evolution of space-time. The main advantage of this numerical integration is to provide the whole geometry from the distortion of Minkowski space-time due to the infalling shell up to the complete evaporation of the collapsed object.

There is little point here in going through the algebraic formulation for this model in view of the general considerations which have just been set forth. We do wish however to point out one interesting feature of this work. In view of the $T_{\mu\nu}$ used, the system of coordinates is that of eq. 3.57. It is relevant to remark that there has been an independent mathematical approach to the black hole problem inspired by some elements of string theory called the dilatonic black hole in 1+1 dimensions. It resembles in form the abovementioned calculation. Indeed the dilatonic scalar field $\phi(U, V)$ is played by $\ln r(U, V)$ [19][82][84][77]. The original hope was that the dilatonic black hole model could be developed into a complete quantum mechanical theory. Unfortunately these hopes have not been realized and at the present time our knowledge is restricted to the semiclassical theory. Since we are now possessed of a more physical semiclassical theory we shall not present this work in this review.

We now present the numerical results in a series of figures.

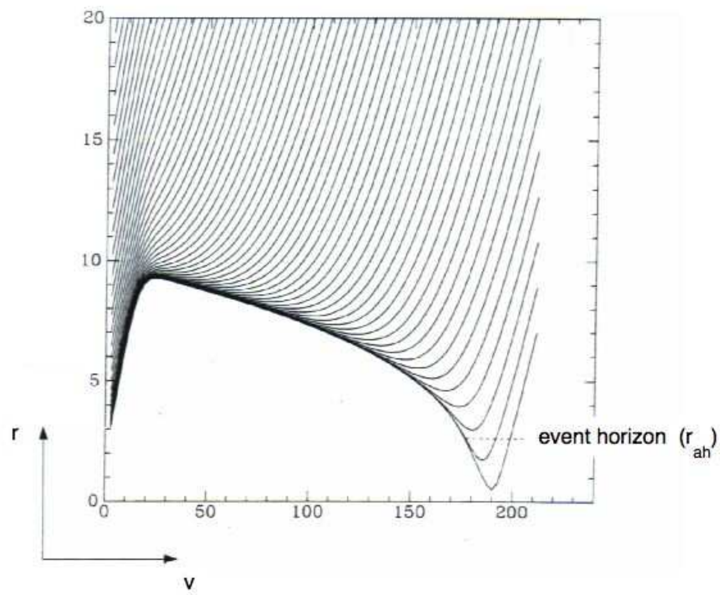

Fig. 3.8 *The outgoing nul geodesics ($u = const$) in the geometry of an evaporating black hole, depicted in r, v coordinates.*

Figure (3.8) shows what happens in terms of the coordinates r and v (see eq. 3.71 and 3.79) wherein the axis v is drawn horizontally and the axis r vertically. The units are Planckian and M_0 is equal to eight. The classical trajectory of infall is not designated but it is a packet centered around the line $v = 25$. The contour lines which are shown are equally spaced $u = const$ outgoing light rays (that is equally spaced on $\mathcal{I}^+$, here represented essentially by $r = 20$). The event horizon appears as a thick line because it is where all these outgoing geodesics accumulate. The geodesics which fall into the singularity are not represented. They would lie in the white zone which lies beneath the event horizon. All of these geodesics emanate from the point of reflection $r = 0$ (that is well within the star). The first ones continue to increase in r on their voyage to $\mathcal{I}^+$ whereas the later ones first increase in r, then decrease in the evaporating geometry, then increase once more. The locus where they reexpand is the apparent horizon. It is interesting to remark that the paths of photons in the white zone which begin their trajectory at $r = 0$ and fall back towards $r = 0$ resemble those of a closed Robertson Walker universe. Indeed the maximum radius encountered by the outgoing light rays in this zone is $2M_0$ and then the radius diminishes according to the evaporation rate. At the apparent horizon $r_{ah} = 2M(v)$ one finds the throat which connects the interior region to the external one. Thus at the end of the evaporation, one has two macroscopic (smooth i.e. wherein the mean geometry is far from the Planckian regime) regions connected by a throat of Planckian dimensions. This is the situation that precedes the splitting of the geometry into two disjoined regions (universes) with a change of topology [48]. From the semi-classical scenario this seems unavoidable and confirms that information is forever lost to the outside universe.

In this theory there is a singular space like line $r = r_\alpha = O(1)$ (which is not represented in figure 3.8). This line is singular in that the dynamical equations loose meaning at this radius. When the apparent horizon reaches r_α (i.e. when the residual mass of the hole is $r_\alpha/2$), one has to stop the numerical integration. Thus the semiclassical model cannot describe or give any hint about the endpoint of the black hole evaporation. This is as it should be since at the end of the evaporation, the mean curvature reaches the Planckian domain where the semiclassical treatment has no justification whatever.

Two other figures of interest are Figs (3.9) and (3.10) . These represent the same geometry in other coordinates.

In Fig (3.9)

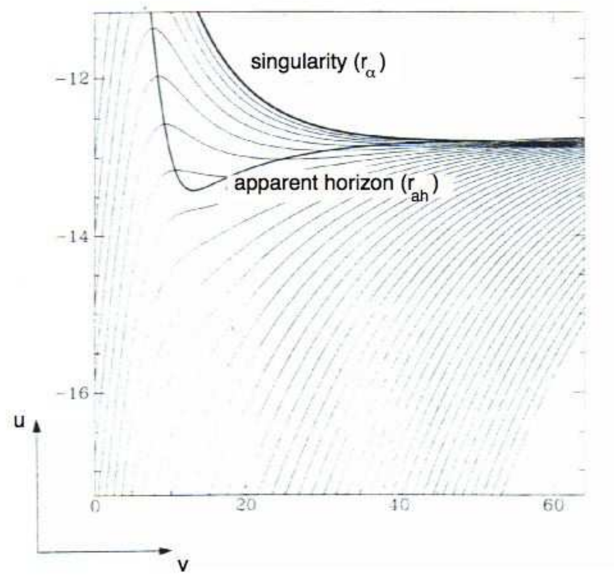

Fig. 3.9 *The $r = const$ lines in the geometry of an evaporating black hole, depicted in $\mathcal{U}$,$\mathcal{V}$ coordinates.*

we present the geometry in the $\mathcal{U}, \mathcal{V}$ coordinates defined in eqs. (3.13). These are the inertial coordinates in the Minkowski region inside the spherical infalling shell. We have represented the contour lines of $r = constant$ since they offer a convenient visualization of the evaporation process. Indeed, inside the shell, where there is no matter one has Minkowski space time in which $r = constant$ are always time like straight lines. Outside the shell, in the absence of back reaction, one would have Schwarzschild space with mass M_0 and the apparent horizon coincides with the static event horizon at $r = 2M_0$ which separates time like from space like $r = constant$ lines.

As we have explained, in the presence of back reaction, the evaporation process is accompanied by the shrinking of the apparent horizon $r_{ah}(v)$. This horizon is the exterior boundary of the trapped region wherein $r = constant$ lines are spacelike. The inner boundary of the trapped region (the other solution of $\partial_v r = 0$ at fixed $\mathcal{U}$) lies within the shell and is space like. [Indeed because the flux T_{vv} is positive here, due to classical matter falling into the black hole, the apparent horizon is space like. In the evaporating situation T_{vv} is negative and the associated apparent horizon is time like.] In this figure we have also drawn the singular space-like line $r = r_\alpha$ which meets the apparent horizon $r_{ah}(v)$ at $\mathcal{V} \simeq 52$.

In the $\mathcal{U}, \mathcal{V}$ coordinates, the whole evaporation period is contained in a tiny $\mathcal{U}$ lapse $(-14 < \mathcal{U} < -12.8)$. This is why we have presented in Fig (3.10) the evaporating geometry in the u, v coordinates.

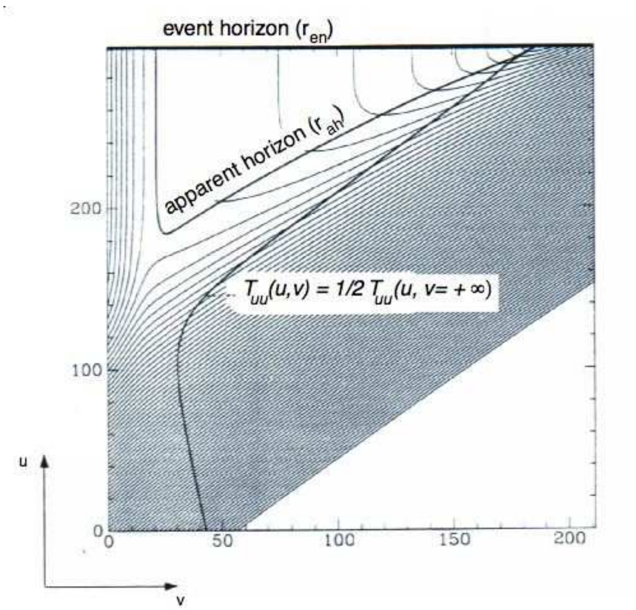

Fig. 3.10 *The $r = const$ lines in the geometry of an evaporating black hole, depicted in u, v coordinates.*

These later are the inertial ones at $r = \infty$ where they are normalized by $v - u = 2r$. One sees the dramatic effect of the exponential jacobian eq. (3.92) relating u to $\mathcal{U}$ which blows up the region between r_{ah} and r_{eh}. In the u, v coordinates, the apparent horizon appears as an almost static line [where static is defined as follows : if a mirror is put along it, two infalling light rays separated by Δv (i.e. by Δt at fixed r) will be reflected with the same Δu (i.e. by the same Δt).] The property of staticity is obviously satisfied by the $r = constant$ lines in a static geometry but strictly speaking no longer in the evaporating situation where, as we have just described, $r = constant$ passes from space-likeness back to time-likeness. We also note that the singular line at $r = r_\alpha$ is not present in this figure since it is beyond the last u-line (i.e. the event horizon $r_{eh}(v)$) at which the apparent horizon meets this singular line.

In Fig (3.10) we have drawn an extra line which sits outside the trapped region. This line designates the locus where $\langle T_{uu}(r,u)\rangle$ reaches half of its asymptotic value $(\langle T_{uu}(\infty,u)\rangle)$. Being well outside the trapped region, this proves that the flux is concerned with the external geometry only and is characterized by the time dependent mass $M(u)$ and not by the whole interior geometry. This shows also that the infalling matter is not at all affected by the evaporation process since the infalling matter is in the causally inaccessible *past* of the places where the mean fluxes build up.

The semiclassical theory which we have presented in this section is a mathematically consistent and well understood theory which predicts that black holes evaporate following the law $dM/dt = -\xi M^{-2}$. However the validity of the semiclassical theory even when the curvature is far from the Planck scale is far from obvious, owing to the fact that very small distance scales are invoked in order to derive the solution. Indeed the jacobian $d\mathcal{U}/du = e^{-u/4M(u)}$ used in obtaining the flux at infinity in eq. 3.91 becomes exponentially small, ie one makes appeal to the structure of the vacuum inside the star on exponentially small scales in order to derive the radiation at later times. Another related problem is that the distance between the event horizon and the apparent horizon is $O(1/M(v))$ which for macroscopic black holes is much smaller than the Planck length. [This distance is an invariant: it is the maximum proper time to go from one horizon to the other. This is seen by writing the metric eq. (3.79) near the horizon in the form $ds^2 = 2drdv + r^2 d\Omega^2$ since we can drop the term in $(r-2M)/2Mdv^2$]. We shall discuss in more detail this problematic aspect of the theory of black holes in section 3.7.

3.5 From Vacuum Fluctuations to Hawking Radiation

In the previous sections we have described the mean value of the energy momentum tensor and its effect on the background geometry. In this section we consider fluctuations. More precisely we describe the field configurations which evolve into a particular Hawking photon using the weak value formalism. This will provide us with the explicit history of the creation of the particle out of vacuum. It will also provide us with the matrix elements of $T_{\mu\nu}$ which play the rôle of the mean when calculating backreaction effects to S matrix elements wherein the final state contains this specific pair.

As in the case of the accelerating mirror, we follow the history of the vacuum fluctuation associated with the emission of this photon. What follows is a resumé of Sections 2.5, 2.6.3 and the forthcoming paragraphs.

Consider a created particle in a wave packet emitted at retarded time u_0, of frequency λ (where $\lambda = O(M^{-1})$) with width $\Delta u = O(\lambda^{-1})$. From the Gerlach resonance condition $u^*(\omega, \lambda) = 4M \ln(\omega/\lambda)$, the frequency ω of the mode comprising the fluctuation within the star that is converted into this Hawking photon is $\omega = O(\lambda e^{u_0/4M})$. This fluctuation is set up on $\mathcal{I}^-$ and is composed of three parts. Let $v = v_H$ be the backward reflected light cone of the horizon $\mathcal{U} = 0$ (see Fig. 3.2). Then, as in Section 2.5, the fluctuation, represented as a packet, straddles $v = v_H$. It is spread out on either side of $v = v_H$ with a spread $\Delta v = O(\omega^{-1}) = O(\Delta u e^{-u_0/4M})$. The part with $v > v_H$ has positive energy density. The fluctuation with $v < v_H$ also has a positive energy hump, as well as an oscillating broader distribution of energy. This latter is net negative and the total energy of all these contributions vanishes as behooves a vacuum fluctuation.

As in Section 2.5, the two different contributions, $\theta(\pm(v - v_H))$ possess very different future destinies. The piece for $v - v_H < 0$, reflects off $r = 0$, travels through the star and gets converted into a Hawking photon. However unlike the mirror it is only at large radius that space gets flat and that one gets a true on mass shell quantum. Once the photon gets out of the star, the oscillating net negative piece of the fluctuation becomes negligible because of the time dependent Doppler shift $(d\mathcal{U}/du)^2$ encountered in converting $T_{\mathcal{U}\mathcal{U}}$ to T_{uu}. The emerging photon is a net lump of positive energy, a propagating outgoing photon. On the contrary, the piece of the fluctuation for $v - v_H > 0$ cannot get to the horizon in finite Schwarzschild time. It approaches the horizon exponentially near the center of the star and there it sits carrying net positive energy of $O(M^{-1} e^{u_0/4M})$. However, as in the case of the accelerating mirror, the average energy carried by all fluctuations is zero. Thus an observation of the absence of a photon emitted at $u = u_0$ is associated with an "anti–partner" fluctuation carrying negative energy near the center of the star. So on the average the fluctuations carry no energy. The problems raised by the gravitational back reaction to the exponentially large fluctuations near the horizon will be discussed in Section 3.7.

The detailed evaluation of the weak values of $T_{\mu\nu}$ for the effective two dimensional model proceeds exactly as in the accelerated mirror problem. Indeed the two dimensional part of the black hole geometry depicted in the u, v coordinate system (see Fig. 3.4) is almost identical to the accelerated mirror problem (see Fig. 2.3). The role of the mirror is played by the center of the star $r = 0$ (see the reflection condition eq. (3.35). The only difference is that the conformal factor of the metric is trivial in the mirror problem ($ds^2 = -dudv$) whereas it is nontrivial in the black hole problem. This introduces some complications when computing the renormalized energy momentum tensor which we now address. But it does not affect the fluctuating part of $T_{\mu\nu}$ since we are considering the simplified s-waves which are conformally invariant.

In Section 3.3 the mean energy momentum tensor of the truncated model was computed. In order to calculate any matrix element (and not only the mean) of the renormalized energy momentum operator we need a slightly more general formalism wherein the renormalized energy momentum operator is written as

$$T_{\mu\nu}(x)_{ren} = T_{\mu\nu}(x) - \langle I(x)|T_{\mu\nu}(x)|I(x)\rangle I \tag{3.96}$$

Here $T_{\mu\nu}(x)$ is the bare energy momentum operator, $|I(x)\rangle$ is the inertial vacuum, I is the identity operator, $\langle I(x)|T_{\mu\nu}(x)|I(x)\rangle$is the expectation value of $T_{\mu\nu}$ in the inertial vacuum which is conserved upon including the trace anomaly. Note that in this expression the trace anomaly is entirely included in the second term so it is state independent as required. A convenient reexpression of eq. (3.96) which isolates the mean of $T_{\mu\nu}$ from its fluctuating part is

$$\begin{aligned} T_{\mu\nu\ ren} &= :T_{\mu\nu}: + \left(\langle 0_{in}|T_{\mu\nu}|0_{in}\rangle - \langle I(x)|T_{\mu\nu}|I(x)\rangle\right) I \\ &= :T_{\mu\nu}: + \langle 0_{in}|T_{\mu\nu}|0_{in}\rangle_{ren} \end{aligned} \tag{3.97}$$

where $\langle 0_{in}|T_{\mu\nu}|0_{in}\rangle$ is the expectation value of $T_{\mu\nu}$ in the Heisenberg vacuum $|0_{in}\rangle$ and $:T_{\mu\nu}:$ is the energy momentum operator normal ordered with respect to this Heisenberg vacuum.

We are now in position to recopy the results of Section 2.5. We first recall (see eqs (3.49,3.50,3.51) that to each out Schwarzschild mode

$$\varphi_\lambda^{out} = \frac{1}{\sqrt{4\pi\lambda}} \left(|\frac{v - v_H}{4M}|^{-i\lambda 4M} \theta(v_H - v) - e^{-i\lambda u} \right) \tag{3.98}$$

there corresponds a partner mode

$$\varphi_\lambda^{outF} = \frac{1}{\sqrt{4\pi\lambda}} \left((\frac{v - v_H}{4M})^{-i\lambda 4M} \theta(v - v_H) - e^{-i\lambda u_F} \right) \tag{3.99}$$

where we have set for simplicity the constants which appear in eq. (3.37) to $K = 4M$, $B = 1$, $\kappa = 1$. The Unruh modes which are positive frequency on $\mathcal{I}^-$ are given by

$$\begin{aligned} \hat{\varphi}_\lambda &= \alpha_\lambda \varphi_\lambda^{out} + \beta_\lambda \varphi_\lambda^{outF*} \quad \lambda > 0 \\ \hat{\varphi}_{-\lambda} &= \beta_\lambda \varphi_\lambda^{out*} + \alpha_\lambda \varphi_\lambda^{outF} \quad \lambda > 0 \end{aligned} \tag{3.100}$$

To these modes are associated the operators a_λ^{out}, a_λ^{outF}, $\hat{a}_\lambda$.

The Heisenberg state $|0_{in}\rangle$ is that anihilated by the $\hat{a}_\lambda$ operators. It can be expressed in terms of out states as

$$|0_{in}\rangle = \frac{1}{\sqrt{Z}} \prod_\lambda e^{\frac{\beta_\lambda}{\alpha_\lambda} a_\lambda^{out\dagger} a_\lambda^{outF\dagger}} |0_{out}\rangle \tag{3.101}$$

thereby exhibiting the correlations between the produced Hawking quanta ($a_\lambda^{out\dagger}$) and the partners ($a_\lambda^{outF\dagger}$). Following the development of eqs (2.102) et seq. one shows that to each produced hawking photon in a wave packet

$$\psi_i = \int_0^\infty d\lambda \ \gamma_{i\lambda} \varphi_\lambda^{out} \tag{3.102}$$

there corresponds a partner in the packet $N_i^{-1} \int_0^\infty d\lambda \ \gamma_{i\lambda} \left(\beta_\lambda/\alpha_\lambda\right) \varphi_\lambda^{outF}$ where N_i is a normalization factor given by $N_i^2 = \int d\lambda |\gamma_{i\lambda}(\beta_\lambda/\alpha_\lambda)|^2$. Note that the wave function of the partner does not have the same mode decomposition as ψ_i.

We can now compute the weak value of the energy momentum tensor, i.e. the energy correlated to the creation of a photon in mode ψ_i.

$$\langle T_{\mu\nu} \rangle_w = \frac{\langle 0_{in} | \Pi T_{\mu\nu} | 0_{in} \rangle}{\langle 0_{in} | \Pi | 0_{in} \rangle} \tag{3.103}$$

As in Section 2.5, the projector Π selects the final state wherein one photon is produced in the mode ψ_i:

$$\Pi = I_{partners} \otimes \int_0^\infty d\lambda \gamma_{i\lambda} a_\lambda^\dagger |0_{out}\rangle\langle 0_{out}| \int_0^\infty d\lambda \gamma_{i\lambda}^* a_\lambda \tag{3.104}$$

A calculation similar to that leading to eq. (2.105) yields for $T_{vv} = \partial_v\phi\partial_v\phi$ (a similar expression obtains for T_{uu})

$$\begin{aligned}
\langle T_{vv}\rangle_w &= 2\frac{\left(\int_0^\infty d\lambda\ (\gamma^*_{i\lambda}/\alpha_\lambda)\partial_v\hat{\varphi}^*_\lambda\right)\left(\int_0^\infty d\lambda'\ \gamma_{i\lambda'}(\beta_{\lambda'}/\alpha^2_{\lambda'})\partial_v\hat{\varphi}^*_{-\lambda'}\right)}{\int_0^\infty d\lambda\ |\gamma_{i\lambda}|^2(\beta_\lambda/\alpha_\lambda)^2} \\
&\quad + \frac{\langle 0_{out}|:T_{vv}:|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} + \langle 0_{in}|T_{vv}|0_{in}\rangle_{ren}
\end{aligned} \tag{3.105}$$

The second and third terms of eq. (3.105) are background (they are independent of ψ_i). The third term was the subject of Section 3.3 and the second is the difference between Unruh and Boulware vacuum. Therefore in computing these weak values the background is that of Boulware vacuum. This is the precise analogue of Section 2.5 where the background was Rindler vacuum.

The first term (hereafter referred to as $\langle T_{\mu\nu}\rangle_{\psi_i}$) describes the energy momentum of the vacuum fluctuation which will become the Hawking photon ψ_i and its partner. From eq. (3.105) it is apparent that the energy of this vacuum fluctuation vanishes. Indeed the annihilation of the vacuum by the total energy operator $\int_{-\infty}^{+\infty} dv :T_{vv}: |0_{in}\rangle = 0$ implies that the integral of the first two terms on the r.h.s. of eq. (3.105) vanish as in eq. (2.116).

It is convenient to rewrite $\langle T_{vv}\rangle_{\psi_i}$ in terms of out modes to obtain

$$\begin{aligned}
\langle T_{vv}\rangle_{\psi_i} &= 2\theta(v_H - v)\frac{\left(\int_0^\infty d\lambda\ \gamma_{i\lambda}(\beta^2_\lambda/\alpha^2_\lambda)\partial_v\varphi^{out}_\lambda\right)\left(\int_0^\infty d\lambda'\ \gamma^*_{i\lambda'}\partial_v\varphi^{out*}_{\lambda'}\right)}{\int_0^\infty d\lambda\ |\gamma_{i\lambda}|^2(\beta^2_\lambda/\alpha^2_\lambda)} + \\
&\quad 2\theta(v - v_H)\frac{\left(\int_0^\infty d\lambda\ \gamma_{i\lambda}(\beta_\lambda/\alpha_\lambda)\partial_v\varphi^{outF}_\lambda\right)\left(\int_0^\infty d\lambda'\ \gamma^*_{i\lambda'}(\beta_{\lambda'}/\alpha_{\lambda'})\partial_v\varphi^{outF*}_{\lambda'}\right)}{\int_0^\infty d\lambda\ |\gamma_{i\lambda}|^2(\beta^2_\lambda/\alpha^2_\lambda)}
\end{aligned} \tag{3.106}$$

In this form it is apparent that the $\theta(v-v_H)$ piece is real and positive whereas the $\theta(v_H - v)$ piece is not. It is complex and oscillates in such a way that the total energy $\int_{-\infty}^{\infty} dvT_{vv}$ vanishes.

Only the piece $\theta(v_H - v)$ is reflected in finite Schwarzschild time and reaches $\mathcal{I}^+$. There it takes the form

$$\langle T_{uu}\rangle_{\psi_i} = 2\frac{\left(\int_0^\infty d\lambda\ \gamma_{i\lambda}(\beta^2_\lambda/\alpha^2_\lambda)\partial_u\varphi^{out}_\lambda\right)\left(\int_0^\infty d\lambda'\ \gamma^*_{i\lambda'}\partial_u\varphi^{out*}_{\lambda'}\right)}{\int_0^\infty d\lambda\ |\gamma_{i\lambda}|^2(\beta^2_\lambda/\alpha^2_\lambda)} \tag{3.107}$$

It carries the Schwarzschild energy λ_0 of the post selected photon. Indeed one has

$$\int_{-\infty}^{+\infty} du\langle T_{uu}\rangle_{\psi_i} = \frac{\int_0^\infty d\lambda\ \lambda|\gamma_{i\lambda}|^2|\beta_\lambda/\alpha_\lambda|^2}{\int_0^\infty d\lambda\ |\gamma_{i\lambda}|^2|\beta_\lambda/\alpha_\lambda|^2} \simeq \lambda_0 \tag{3.108}$$

where λ_0 is the energy of the produced hawking photon.

We mention that one may in similar fashion postselect the presence of any number Hawking photons. In this case one will find a decomposition of the weak value very similar to eq. (3.105). There will be a term corresponding to the Boulware energy (the second line of eq. (3.105)) and a sum of terms of the form $\langle T_{\mu\nu}\rangle_{\psi_i}$, one for each post selected photon. Thus the background contribution can be naturally separated from the fluctuating contributions.

As in the electric field (see eq. (1.59)), the physical relevance of the imaginary part of $\langle T_{uu}\rangle_{\psi_i}$ can be seen by modifying slightly the background geometry: $g_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$. Then the change in the probability of finding on $\mathcal{I}^+$ the Hawking photon selected by Π (eq. (3.104)) is given by

$$P_{g_{\mu\nu}+\delta g_{\mu\nu}} = P_{g_{\mu\nu}} \left(1 - \int d^4x\sqrt{g}\delta g_{\mu\nu} 2\mathrm{Im}\left[\langle T_{\mu\nu}\rangle_w\right]\right) \tag{3.109}$$

3.6 Thermodynamics of Black Holes

In order not to disrupt the continuity of the text we shall introduce the notion of black hole entropy by giving a thermodynamic interpretation to the evaporation phenomenon which has been the subject of this review until this point. The argument pursued will be heuristic and based on analogy. Subsequently a more rigorous derivation based on a true equilibrium situation, the eternal black hole will be presented.

A convenient pedagogical crutch to start with is the idealization used in the preceding chapters, the 1+1 dimensional problem with unit transmission coefficient. We emphasize at the outset that the notion (and the value) of the entropy of the black hole itself which is deduced from this model, is by no means contingent on the idealization used to get it.

The central result of the idealized model is that at a certain time a thermal flux emerges as the collapsing star approaches its horizon. Then the mean energy emitted during Δv (or Δt at fixed r) is given in eq. (3.46): $< T_{uu} > \Delta t = \frac{\pi}{12}\beta_H^{-2}\Delta t$ and it is distributed thermally (with $\beta_H = 8\pi M$. The probability for the simultaneous occurrence of quanta in a state whose occupation numbers for frequency ω_i are n_i is

$$P_{\{n_i\}} = Z^{-1} e^{-\beta_H \sum n_i\omega_i} \tag{3.110}$$

We remind the reader that the density matrix eq. (3.110) is a consequence of the Bogoljubov transformation eq. (3.39). The normalizing factor Z is likened to a partition function of a one dimensional gas (c.f. eq. (2.45)) whose volume is Δt, i.e.

$$\ln Z \;\; = \;\; -\Delta t \int \frac{d\omega}{2\pi} \ln(1 - e^{-\beta_H \omega}){-1} \tag{3.111}$$

and the mean number of quanta with frequency ω_i is

$$\langle n_i \rangle = \frac{\partial \ln Z}{\partial \beta \omega_i} = (e^{\beta_H \omega_i} - 1)^{-1} \tag{3.112}$$

The fact that the photons are all outgoing does no injustice to the application of usual statistical mechanics and thermodynamics defined by the canonical ensemble.

The conditions under which evaporation is taking place are isothermal to a very good approximation. By that we mean there is a Δt sufficiently small so that even when a macroscopic number of photons evaporate in this time interval, β_H does not change. For example for $M \simeq 1$ solar mass ($= 10^{38}$ Planck masses) during the time interval that 10^{20} photons are emitted one has $\Delta M = O(10^{20}/M)$ and $\Delta\beta_H/\beta_H = O(\Delta M/M) = O(10^{-56})$.

One may liken this quantum evaporation to the irreversible process in which a large mass of liquid evaporates into a tiny amount of vapor in time Δt. The whole system is insulated from its surroundings (i.e. $E_{total} = constant$) with a little vapor occupying a space above the liquid. Then increase the volume a little bit. To make the illustration more cogent we neglect terms proportional to the chemical potential (after all it doesn't cost anything to make a vacuum fluctuation). Then the only increase in entropy is due to the increase of volume of the vapor phase. The liquid loses a "little" energy ΔE and the vapor increases in energy ΔE where "little" means $\Delta E/E_{total} << 1$ and we envision that almost all the energy is localized in the liquid. Then the temperature change is negligible ($\Delta T/T = O(\Delta E/E)$). One says that the liquid is a "reservoir". The "reservoir limit" for the change in its entropy is $\Delta S = \Delta E/T$. [In thermodynamics one introduces a reservoir to convert entropy considerations of a system to its free energy, the relevant function to describe isothermal processes undergone by a system in contact with the reservoir].

From the above interpretation we are led to ascribe to the black hole an entropy. In point of fact it was this remarkable insight of Bekenstein [8] that must have incited Hawking to attribute a temperature to the black hole and hence evaporation. During the time Δt wherein a mass ΔM_{BH}

($= -\Delta M_{gas} = -\sum_i \langle n_i \rangle \omega_i$) is evaporated (with $|\Delta M_{BH}|/M << 1$), the process is effectively isothermal and the black hole acts as a reservoir. Its change in entropy is thus

$$\Delta S_{BH} = \beta_H \Delta M_{BH} = 8\pi M \Delta M_{BH} \simeq \Delta(A/4) \tag{3.113}$$

where we have introduced the area of the horizon surface ($A = 4\pi(2M)^2$).

A noteworthy feature of black hole evaporation as compared to the liquid vapor analogy is that no time is required to get the "vapor" into an equilibrium state after it evaporates. To complete this discussion, one may calculate the increase in entropy occasioned by the evaporation. Using eq. (3.110) one has

$$\begin{aligned} \Delta S_{total} &= \Delta S_{BH} + \Delta S_{gas} = \beta_H(\Delta M_{BH} + \Delta M_{gas}) + \frac{\partial \ln Z}{\partial \Delta t}\Delta t \\ &= \frac{\partial \ln Z}{\partial \Delta t}\Delta t = p\Delta t \end{aligned} \tag{3.114}$$

Recall that the pressure is given by $\partial \ln Z/\partial V$ and in our case ΔV is the one dimensional volume Δt. Thus the entropy increase is $p\Delta V$ as for the analog liquid-vapor example (with zero chemical potential).

But how is that? We started with one state and we are now calculating the increase in the total number of states. The answer of course is that we are describing the gas by a density matrix and have completely forgotten the correlation to the degrees of freedom left within the black hole. It is therefore the extra trace over these degrees of freedom which is responsible for the increase of entropy ΔS_{total} and this is a perfectly legitimate description for the outside observer.

It is thus possible to develop a phenomenological thermodynamics for the outside observer which ascribes to the black hole an entropy change which is $\beta_H(M)\Delta M$. We may then calculate the entropy of the black hole by integrating this change from 0 to M as to obtain

$$S_{BH}(M) - S_{BH}(0) = \int_0^M \beta_H(M')dM' = \pi(2M)^2 = A/4 \tag{3.115}$$

One may ask why $S_{BH}(M)$ is intrinsic to the black hole i.e. equal to $log\Omega(M)$, where Ω is the number of degrees of freedom of the black hole. After all the black hole was used as a reservoir only. The answer is that the black hole even if it is a reservoir for the radiation is at the same time in very good approximation a microcanonical ensemble unto itself. This is because M changes so slowly ($dM/dv \simeq -1/M^2$). Thus $\beta_H = \partial_M log\Omega(M)$ is a valid microcanonical expression for how the density of states varies with the mass. In usual statistical mechanics, one turns the argument the other way around since one starts with the density of states. The (temperature)$^{-1}$ is then introduced as the logarithmic derivative of Ω.

A possible constant of integration in eq. (3.115) is a subject of much debate [23]. If there is a remnant at the end of evaporation then $S_{BH}(0)$ is its entropy. Since we have as yet no way to think about this remnant it is the better part of valor not to commit oneself as to its value. If the remnant leaks away then this leakage should be accompanied by further entropy increase, in a model dependent way and in the sense of a density matrix. The situation is analogous to that sketched in Fig. (2.4) for a decelerating mirror. In principle one could measure the correlations between these late leakage photons and the earlier radiated ones. Then one would be able to check whether or not there is a pure state with no change in entropy. Thus the former entropy increase would have resulted from neglecting the correlations. If otherwise the remnant sinks through a singularity these degrees of freedom get lost and the entropy increase is given by eq. (3.114), i.e. we are stuck with the density matrix description and quantum mechanics applied to this problem has become non unitarity as initially suggested by Hawking [46]. Nothing could be more interesting-or exasperating. For further discussion, see Section 3.7.

We mention that when one takes into account the transmission coefficient, much of what has been said still survives. Eq. (3.110) must be changed in that each factor $e^{-\beta\omega_i}$ must be multiplied by a transmission factor Γ_i, and one must include all the angular momentum modes. The value of Z changes, but thermodynamics is retained in a modified sense since for $N >> 1$ (where $N = \sum_i n_i$), relative energy fluctuations are still $\mathcal{O}(1/\sqrt{N})$ because the distribution remains Poisson, and the process is still isothermal. All that is modified is the numerical value of ΔS_{total}. In particular eq. (3.114) is retained since $-\partial \ln Z/\partial\beta|_{\beta_H} = \Delta M_{gas}$ for the modified ensemble as well. The entropy ascribed to the black hole and its physical interpretation as described after eq. (3.114), is independent of the mechanical details. And one still has $\Delta S_{total} = p\Delta t > 0$.

The idea of black hole entropy in terms of a reservoir is reinforced by the analysis of the equilibrium situation which is characteristic of the eternal black hole. This idea is once more a seminal discovery of Hawking [45]. We shall follow Hawking's idea, but, unlike him, we shall also deduce the black hole entropy itself.

In a closed box of volume V, it has been shown by Hawking that for a sufficiently large energy, the energy becomes partitioned into a black hole surrounded by radiation in thermal equilibrium. We shall see below that there is a limit wherein essentially all the energy is in the black hole whose radius is nevertheless much smaller than the radius of the box. Then, the only rôle of the radiation is to furnish a temperature. In the limit envisaged, this latter is the Hawking temperature, T_H, which value is then used to compute the entropy of the hole.

The relevant geometry which describes the static situation characteristic of the eternal black hole is the full Schwarzschild quadrant R ($r > 2M$) of Kruskal space. This geometry is depicted in Fig. (3.1). In the case where almost all the mass is in the black hole the geometry in R approximates to that of empty Schwarzschild space. This space has both past and future horizons. For the case envisioned of a black hole formed from a collapsing shell the past horizon is not present (see Fig. (3.2)). Nevertheless, the use of the eternal black hole geometry is a correct mathematical idealization to be understood as follows. Outgoing photons that issue from the black hole, in the course of its formation at late stages are Kruskal in character (i.e. the modes used to describe Unruh vacuum). Because the black hole is enclosed in a finite volume, these modes get reflected off the walls and come back as Kruskal modes. So after some time a stationary state gets established in which both the incoming and outgoing modes (or pieces thereof) are of Kruskal character. Note that all angular momenta participate in thermal equilibrium since considerations of reversibility render useless any appeal to the smallness of the transmission coefficient for the higher angular momentum waves. In this equilibrium case, the hole is surrounded by a gas of energy density proportional to T_H^4. So the s-wave truncation makes no sense in this case. The state is then set up in terms of the modes which traverse the whole space. But the part of the space which is physically relevant is restricted to that part of R which is bordered by the surface of infalling matter.

Every stationary state of the photon gas in the eternal geometry except the Hartle-Hawking vacuum gives rise to a singular energy momentum tensor on at least one of the horizons. The Hartle-Hawking vacuum [43] is that constructed from the quanta of Kruskal modes. It is the analog of Minkowski vacuum in Minkowski space and R is the analog of the right Rindler quadrant. One can understand the origin of the theorem on singular energy density near the horizon by reference to eq. (3.64) et seq. where it was shown that in Boulware vacuum $< T_{UU} >$ is singular. To undo this singularity clearly requires a very special condition (which, for the collapsing case, was shown to be $\langle T_{uu}(r \simeq 2M)\rangle = O[(r-2M)^2]$).

In order to prove this point, to see the thermal character of Hartle-Hawking vacuum in R, and in fact to construct the corresponding density matrix, it is useful to make the euclidian continuation of the Schwarzschild geometry $t \to it$ (a construction which is possible owing to the staticity of the geometry). Forgetting angles the line element is $(1-2M/r)(d(it))^2 + (1-2M/r)^{-1}dr^2$. This has the form of a cigar which terminates at $r = 2M$ and extends as a cylinder of radius $4M$ out to infinity. To see this note that near $r = 2M$, the line element takes the form

$$ds^2 \underset{r\to 2M}{\longrightarrow} \rho^2 d\theta^2 + d\rho^2 \tag{3.116}$$

where $\rho = 2\sqrt{2M(r-2M)}$; $\theta = it/4M$. Equation (3.116) is the line element in the neighborhood of $r = 2M$ written in local euclidean polar coordinates. No conical singularity at that point requires periodicity of θ equal to 2π. On the other hand as $r \to \infty$ the metric goes over to the metric of a cylinder ($= (4M)^2 d\theta^2 + dr^2$), the periodicity in θ remains equal to 2π.

Thus regular functions defined on this space are periodic in it with period equal to $2\pi(4M) = \beta_H$. For large r, Schwarzschild t coincides with proper time, hence β_H^{-1} is temperature at large r. And Green's functions defined as Hartle-Hawking expectation values are regular. A discussion of these Green's functions, in general, is given in ref. [37]. For the purpose at hand it suffices to note that near the horizon the geometry is regular (eq. (3.116)). In this region Hartle-Hawking Green's functions are related to Boulware Green's functions in the same way that in flat space Minkowski Green's functions are related to Rindler Green's functions. Near the horizon the analogy is exact with the acceleration replaced by ρ^{-1}.

Having displayed the properties at equilibrium, let us now construct the necessary conditions to derive the black hole entropy. Our condition of negligible energy in the gas is

$$M >> VM^{-4} \quad \text{thus} \quad M >> V^{1/5} \tag{3.117}$$

On the other hand we require that the volume occupied by the gas be much greater than that occupied by the hole in order to validate the estimate of its energy (i.e. $E_{gas} = VT^4$ as in absence of gravity)

$$V >> M^3 \tag{3.118}$$

Equations (3.117) and (3.118) are compatible if $M^5 >> M^3$ i.e. $M >> 1$. In that case the total energy at equilibrium is in very good approximation

$$E = M + V\beta_H^{-4} \tag{3.119}$$

We now use the fact that this equilibrium configuration should be derivable from the variational principle of entropy $\delta S_{total}|_{E,V} = 0$. Indeed, by attributing an entropy to the black hole (S_{BH}) and by taking the variation with the energy repartition ($\delta M = -\delta E_{gas}$), one finds

$$\begin{aligned}\delta S_{total}|_{E,V} &= \delta M(\frac{\partial S_{BH}}{\partial M} - \frac{\partial S_{gas}}{\partial E_{gas}}) \\ &= \delta M(\frac{\partial S_{BH}}{\partial M} - \beta_H) = 0 \end{aligned}\tag{3.120}$$

whence the equality of the temperatures gives

$$\frac{\partial S_{BH}}{\partial M} = 8\pi M \tag{3.121}$$

In eq. (3.120), we have used conventional canonical thermodynamics for the radiation (i.e. $dE_{gas} = T_H dS_{gas}$) and ascribed to it the temperature T_H albeit that the total system is microcanonical in which one phase (the black hole) acts as a reservoir. The above considerations are quite general and the use of the photon gas was by way of illustration. Equation (3.120) will be true for any model of matter. In this sense eq. (3.121) is a very powerful result. Whatever happens in the ultimate destiny of black hole physics one would be loath to give it up.

A critique (see ref. [76]) of the notion of black hole entropy follows from a closer inspection of the distribution of energy density in the Hartle-Hawking vacuum [49]. Vacuum polarization effects in $< T^{\nu}_{\ \mu} >$ (which lead to negative energy density near the horizon) prevent a clean split of total entropy into its matter and black hole components. Therefore the value $S_{BH} = A/4$ is to be considered as valid only in the reservoir limit as discussed above. This does not mean that S_{total} does not exist but rather that in the general case it cannot indeed be unambiguously divided into two terms (matter and black hole) because gravitation is a long range force.

That the entropy is proportional to the area of the horizon has been based on the fact that the Hawking temperature is proportional to M^{-1}. This in turn is related to the imaginary period of the relevant Green's functions of quantum fields in Schwarzschild space. A transfer of dimensions has occurred from mass^{-1} to length (or time since $c = 1$) owing to the existence of $\hbar$. Thus it would seem that the identification of entropy with area is essentially quantum in character. But quite surprisingly, a purely classical development already foretells a good bit. On one hand, Hawking and collaborators have shown that in the classical evolution of matter-gravitation configurations, the area of the horizon always increases. [This review is not the place to prove this important classical topological theorem. The reader is referred to the important monograph of Ellis and Hawking [47].] This can be related to more general results that the total entropy of black holes and matter always increases (see ref. [9]). On the other hand, Bardeen, Carter and Hawking [7] have used a tool called the Killing identity to investigate the role of the horizon's area as an entropy, in purely classical terms. The point of departure is an exact geometrical identity satisfied by Killing vectors, upon which one grafts Einstein's equations to relate the curvature, which appears in the identity, to the energy momentum tensor. The version we shall present below is applicable to static spherically symmetric systems (See also ref. [23] for a similar derivation in the framework of the hamiltonian formalism). In addition we refer to efforts [39], [18] which identify the black hole entropy as an action integral for the pure gravitational sector. The value $S_{BH} = A/4$ is then obtained from the classical Einstein action taking due care of boundary terms. The bearing of this result on the characterization of the degrees of freedom locked within the black hole (but coupled to the external world) remains a subject of debate.

Spherically symmetric static systems can be described by the line element

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2 \tag{3.122}$$

We assume that at large r there is no matter present, so Birkhoff's theorem applies and one has

$$e^{-2\Lambda} = (1 - \frac{2M}{r}) \quad r \to \infty \tag{3.123}$$

Further we fix completely the coordinates by imposing $\phi(r = \infty) = 0$. M is the total Keplerian mass measured from infinity. It is the sum of the black hole and matter mass. Indeed, the equation $R^0_0 - R/2 = 8\pi T^0_0$ can be integrated exactly to yield

$$\begin{aligned} e^{-2\Lambda} &= (1 - \tfrac{2m(r)}{r}) \\ m(r) &= M - \textstyle\int_r^\infty dr 4\pi r^2 \rho(r) \end{aligned} \tag{3.124}$$

The horizon of the black hole is the radius $r = 2m_B$ at which $e^{-2\Lambda}$ vanishes. Regularity of the geometry then implies that in the vicinity of the horizon one has

$$e^{-2\Lambda} \simeq \frac{r - 2m_B}{2m_B} \quad ; \quad e^{2\phi} \simeq k^2 \frac{r - 2m_B}{2m_B} \tag{3.125}$$

The value of k differs from unity owing to the presence of matter. It plays an important conceptual role in what follows.

The relevant geometrical (Killing) identity in this approach is

$$R^0_0 r^2 e^{\phi+\Lambda} = -\left(e^{\phi-\Lambda} r^2 \phi'\right)' \quad , \tag{3.126}$$

a consequence of the staticity of the geometry . One then uses Einstein's equation $R^0_0 = 8\pi(T^0_0 - \frac{1}{2}T)$. We shall suppose that the matter is a perfect fluid $T^\nu_\mu = (\rho + p)\delta^\nu_0 \delta^0_\mu - p\delta^\nu_\mu$ (introducing a difference between radial and tangential pressure does not change the final answer), to give

$$\frac{d}{dr}(e^{\phi-\Lambda} r^2 \frac{d\phi}{dr}) = 4\pi(\rho + 3p)e^{\phi+\Lambda} \tag{3.127}$$

whereupon integration from $r = 2m_B$ to $r = \infty$ yields

$$M - km_B = 4\pi \int_{2m_B}^{\infty} (\rho + 3p) r^2 e^{\phi+\Lambda} dr \tag{3.128}$$

(This is an alternative way of writing M as compared to $M = m_B + 4\pi \int_{2m_B}^\infty \rho r^2 dr$; see eq. (3.124).)

The term km_B will be expressed henceforward as a term proportional to the black hole area ($A = 4\pi(2m_B)^2$)

$$km_B = \frac{\kappa}{4\pi} A \quad ; \quad \kappa = k/4m_B \tag{3.129}$$

The constant κ, called the surface gravity, has very important physical significance. It is the gravitational acceleration at radius r measured at infinity, for example from the tension in a string attached to a test mass located at r [70]. To see this, suppose the test mass, μ, initially at rest, is dropped by a static observer at r. It picks up kinetic energy T (as measured by a static observer at $r+\delta r$) in an infinitesimal distance δr (ie. proper distance $\delta l = e^{\Lambda}\delta r$) equal to $\mu(d\phi/dr)\delta r$. When measured at $r=\infty$, this energy is redshifted to $\mu e^{\phi}\phi'\delta r$. Equating it to a work term $F_\infty \delta l$ one finds $F_\infty = \mu e^{\phi-\Lambda}\phi'$. For r at the horizon this gives $F_\infty/\mu = \kappa$. It is noteworthy that in the absence of matter (with $e^{2\phi} = e^{-2\Lambda} = (1-2m_B/r)$) one has $F_\infty = \mu m_B/r^2$. Consistency with eq. (3.127) when $\rho = p = 0$ is an essential point. Were Newton's law for the force other than r^{-2}, the repercussions would be serious indeed.

The reason why we have belabored the physical interpretation of κ is that in the euclidean continuation of the metric near the horizon, eq. (3.125) gives rise to the polar coordinate representation of flat space which for Schwarzschild space is given by eq. (3.116) with $\theta = it/(4M)$. The only difference when $k \neq 1$ is $\theta = k(it)/(4m_B) = \kappa(it)$. When transcribed into quantum mechanics this period in proper euclidean time is transformed into the inverse Hawking temperature $\beta_H = 8\pi m_B/k = 2\pi/\kappa$.

To see that the area has to do with an entropy one compares two static solutions by varying external parameters ρ, p, m_B, etc in such manner as to be consistent with Einstein's equations. The steps require a bit of algebra and is relegated to a parenthesis. We first quote the result

$$\delta M = \frac{\kappa}{8\pi}\delta A + \int_{2m_B}^{\infty} (\tilde{T} d\delta S_{matter} + \tilde{\mu} d\delta N_{matter}) + p_B k A \delta(2m_B) \qquad (3.130)$$

Here $\tilde{T}$ and $\tilde{\mu}$ are the local temperature and chemical potential, scaled correctly by the red shift (Tolman scaling) $\tilde{T} = e^{\phi}T(r)$, $\tilde{\mu} = e^{\phi}\mu(r)$. δS_{matter} and δN_{matter} are the local entropy and particle number of matter. There are no pressure terms at infinity since the space is asymptotically flat but we have retained the work term due to the pressure p_B occurring near the horizon. This term doesn't appear in the original formula of Bardeen, Carter, Hawking who assumed that ρ and p vanish at the horizon. However it becomes relevant for instance when we compare geometries of black holes in a box surrounded by radiation in a Hartle-Hawking state. If instead of the model considered here above one considers a black hole and surrounding matter enclosed in a finite volume having walls, the energy of neighboring configurations have to differ by a term like $-p\delta V$ but also by a term (model dependent) taking into account the stress in the wall.

[The difference between two neighboring solutions are characterized by δM, δm_B, $\delta\kappa$, $\delta\rho$, δp, $\delta\phi$ and $\delta\Lambda$, quantities that cannot be all independent because of the Einstein field equations. Starting from eq. (3.128) we obtain

$$\delta M = \frac{1}{4\pi}\delta(\kappa A) + 4\pi\delta\int_{2m_B}^{\infty}(\rho+3p)r^2 e^{\phi+\Lambda}dr \tag{3.131}$$

The variation of this last integral is the main task of the calculation. This is most easy done by splitting it as a sum of two parts using $-R = 8\pi(-\rho+3p)$

$$\begin{aligned} 4\pi\int_{2m_B}^{\infty}(\rho+3p)r^2 e^{\phi+\Lambda}dr &= \int_{2m_B}^{\infty} -\frac{R}{2}e^{\phi+\Lambda}r^2 dr \\ &+8\pi\int_{2m_B}^{\infty}\rho e^{\phi+\Lambda}r^2 dr \end{aligned} \tag{3.132}$$

The first term, hereafter denoted I_1, on the right hand side is the gravitational action. Its variation must take into account boundary terms which arise both because m_B varies and by integration by parts to yield

$$\begin{aligned} \delta I_1 &= e^{\phi+\Lambda}r^2 R|_{2m_B}\delta m_B + (\lim_{r\to\infty} - \lim_{r\to 2m_B})e^{\phi-\Lambda}r^2(\delta\phi' + \phi'\delta\phi - (\frac{2}{r}+\phi')\delta\Lambda) \\ &+ \int_{2m_B}^{\infty}(G_0^0\delta\phi + G_r^r\delta\Lambda)e^{\phi+\Lambda}r^2 dr \end{aligned} \tag{3.133}$$

To prepare the evaluation of the variation of the second term, let us recall the thermodynamic relation

$$\delta E = -p\delta V + T\delta S + \mu\delta N \tag{3.134}$$

which in terms of local densities on 3-surfaces $t = \mathit{constant}$ becomes

$$\begin{aligned} \delta(\rho 4\pi e^{\Lambda}r^2 dr) &= -p\delta(4\pi e^{\Lambda}r^2 dr) + T\delta(s4\pi e^{\Lambda}r^2 dr) + \mu\delta(n4\pi e^{\Lambda}r^2 dr) \\ &= -p4\pi e^{\Lambda}r^2\delta\Lambda dr + Td\delta S + \mu d\delta N \end{aligned} \tag{3.135}$$

Acordingly we obtain

$$\begin{aligned} \delta I_2 &= -8\pi\rho e^{\phi+\Lambda}r^2|_{2m_B}\delta(2m_B) \\ &+2\int_{2m_B}^{\infty} e^{\phi+\Lambda}4\pi\rho r^2\delta\phi dr \\ &+2\int_{2m_B}^{\infty} e^{\phi}\left[Td\delta S + \mu d\delta N) - p4\pi r^2 e^{\Lambda}\delta\Lambda dr\right] \end{aligned} \tag{3.136}$$

The variation of κA results from the variation of both m_B and ϕ and Λ:

$$\begin{aligned}\frac{1}{4\pi}\delta(\kappa A) &= \delta(e^{\phi-\Lambda}r^2\phi') \\ &= (e^{\phi-\Lambda)}r^2\phi')'|_{2m_B}\delta(2m_B) \\ &\quad +e^{\phi-\Lambda}r^2(\delta\phi'+\phi'(\delta\phi-\delta\Lambda))|_{2m_B} \end{aligned} \tag{3.137}$$

Putting all together and using the Einstein equations $G^0_0 = 8\pi\rho$, $G^r_r = 8\pi p$ and the expression of $R/2 = 1/r^2 - e^{-2\Lambda}(\frac{1}{r^2}+\phi''+\phi'^2-\phi'\Lambda'+2\phi'/r-2\Lambda'/r)$ and the asymptotic behavior of the metric components $e^{\phi-\Lambda}=1$, $\delta\phi' = \delta M/r^2$, $\delta\phi = -\delta\Lambda = -\delta M/r$ and the condition defining the radial coordinate at the horizon $e^{-2\Lambda}(\delta\Lambda + \Lambda'\delta(2m_B)) = 0$ we obtain

$$\delta M = k\delta m_B + \int_{2m_B}^{\infty}(\tilde{T}d\delta S + \tilde{\mu}d\delta N) - 4\pi e^{\phi+\Lambda}\rho r^2|_{2m_B}\delta(2m_B) \tag{3.138}$$

where $\tilde{T} = e^{\phi}T$ and $\tilde{\mu} = e^{\phi}\mu$ are the Tolman temperature and chemical potential. On the horizon, staticity implies $R_{00}=0$, ie. $\rho+p=0$ (the same can be deduced from the regularity of $\sqrt{-g}|_{2m_B} = k4m_B^2$), so the last term can be translated into a term giving the work (measured at infinity) due to the pressure at the horizon when the volume of the exterior of the black hole varies.]

In eq. (3.130) it is difficult to resist setting the black hole entropy equal to $A/4$ since $\kappa/2\pi$ is the periodicity of the euclidianized time hence the temperature. How is it that the classical theory anticipates quantum mechanics? To this end, we cite the efforts of Gibbons and Hawking and Brown and York referred to above, based as they are on the classical euclidean gravitational action with imaginary time of period β. According to the tenets of quantum statistical theory, one converts action to entropy universally through division by $\hbar$. It is only through quantum mechanics that entropy acquires an absolute sense. The term κdA in eq. (3.130) is derivable from the difference between the classical actions of gravity which arises owing to different matter configurations. From the above identification of $\kappa/2\pi$ with $T_H/\hbar$, one rewrites $\kappa dA/8\pi$ as $T_H dS_{BH}$ hence $dS_{BH} = dA/4\hbar$. Thus it seems that the classical Killing indentity "derivation" of entropy has no thermodynamical interpretation without quantum mechanics (See ref. [33]).

3.7 Problems and Perspectives

As elegant as is the semi-classical theory of black hole radiation, it is fraught with severe conceptual problems.

The most essential is concerned with the consequences of the exponential increase of the energy densities of the vacuum fluctuations inside the star and about the horizon which are converted into Hawking photons [51], [53]. This is due to the Doppler shift, eq. (3.43)

$$\omega = \lambda e^{u/4M} \tag{3.139}$$

as the surface of the star approaches the horizon ($u \to \infty$). Therefore the Planck scale $\omega = O(1)$ is reached very early in the history of the evaporation for a characteristic value of λ of $O(M^{-1})$) after a time $u = O(M \ln M)$. This should be compared to the lifetime of the black hole $O(M^3)$. Thus after the emission of a few photons ($\Delta n = O(\ln M)$), Hawking radiation is concerned with the conversion of "transplanckian" vacuum fluctuations (of frequencies ω) into "cisplanckian" photons (of frequencies λ). The point to emphasize is the mixing of radically different energy scales which is occasioned by the exponential growth of the Doppler shift. This is in contradistinction to more conventional physics wherein different scales remain separate (e.g. atomic versus nuclear).

We recall that the observation of a photon near $\mathcal{I}^+$ implies the existence of a particular localized vacuum fluctuation all the way back on $\mathcal{I}^-$. The energies involved in this vacuum fluctuation are of the order of ω. Thus there is an implicit assumption of infinite mean free path of modes. This is implied ab initio through the use of free field theory ($\Box\phi = 0$). Whilst there is nothing wrong with that insofar as elementary particle interactions are concerned (their scale being of O(Gev or Tev) and presumably being asymptotically free), it is most probably incorrect since the gravitational interactions really get in the way. In fact, if anything, the existence of the Planckian mass scale would inevitably tend to stronger forces at higher energies (in local field theory at least, but maybe not for string theory). Be that as it may, if one uses the Newtonian law of gravitation as a guide, two spherically symmetric shells of mass m at radius r will have a gravitational interaction energy which exceeds the mass m, for $m = O(1)$ and $r = O(1)$. From the above considerations, it is seen that these scales are attained at the threshold of entry into the transplanckian region ($\omega = O(1)$, $u = O(M \ln M)$) as the fluctuation approaches the center of the star. The assumption of free field theory is thus à priori completely inadequate.

Whatever is the accommodation of the transplanckian fluctuations within the star, there is another vexing short distance problem which comes about from the fluctuations of the position of the apparent horizon due to the emission or non emission of a Hawking photon. In the semi-classical theory the change in radius of the apparent horizon is due to $\langle T_{vv}\rangle_{ren}$ only, since $\langle T_{uu}\rangle_{ren} = 0$ on the horizon. However the post-selected presence (absence) of a photon around u_0 is contiguous with the contraction (expansion) of the apparent horizon with respect to the mean evolution. More precisely, when the Hawking photon of frequency $\lambda = M^{-1}$ (measured on $\mathcal{I}^+$) is within M^{-1} of the horizon (i.e. $r - 2M$ at fixed v is less than M^{-1}) the classical geometric concept of the apparent horizon (the locus where $\partial_v r|_u = 0$) loses meaning since its radius fluctuates by $O(M^{-1})$. So the approximation of a free field in a fixed background seems to break down near the horizon. Thus how can we be sure that the result of the semiclassical theory wherein there is no coupling between u and v modes corresponds to the true physics near the horizon ? Might there not arise crucial correlations which are absent in the semi-classical theory ($\langle T_{uu}T_{vv}\rangle - \langle T_{uu}\rangle\langle T_{vv}\rangle \neq 0$) ? Then putting the whole blame on $\langle T_{vv}\rangle$ for the change of the geometry near the horizon could be quite misleading. The mechanism of evaporation might then be closer to pair creation near the horizon.

Does this mean that everything contained in the semi classical approximation is irrelevant[54]? Probably not. For one thing it is very unlikely that Hawking radiation does not occur. One must distinguish between the theory of vacuum fluctuations based on free field theory (given in Section 3.5) and Hawking radiation as a mean theory (see Section 3.4). In the first, expectation values of $\langle T_{\mu\nu}\rangle$ in the in-vacuum wash out the fluctuations and provide a dynamical origin of the Hawking flux. So one can refer to the derivation of Hawking radiation in a strong or weak sense, i.e. as derived from free field theory or as some effective theory giving rise to a similar $\langle T_{\mu\nu}\rangle$. In support of the existence of such a effective theory, one needs but appeal to the regularity of $\langle T_{\mu\nu}\rangle$ on the horizon(s). Indeed both in the collapsing and eternal situations (Sections 3.3 and 3.6) it has been emphasized that one can envisage Hawking radiation as a response to an incipient singularity of the Boulware vacuum at the horizon(s) in such a manner as to erase that singularity. (To see the connection between the eternal and collapsing case imagine punching a small hole in the surface of a recipient containing the "eternal" black hole. It then becomes "ephemeral". Radiation leaks out and this is neither more nor less than Hawking radiation albeit with a small transmission coefficient. In fact the potential barrier which stops s-waves with energy smaller than $O(1/M)$ and almost all higher angular momentum modes plays already the role of the small hole since one has almost a thermal equilibrium behind the barrier[20]). Such a general consideration might well be inherent in the complete theory wherein one accounts for the gravitational quantum back reaction. This latter could give rise to violent fluctuations at Planckian distances which nevertheless leave a regular mean to drive a classical background geometry at larger scales.

How might we then envisage the fluctuations ? Almost certainly one would expect that the reduction of the problem to free s-wave modes is incorrect. Rather modes will interact mixing angular momenta out to distances where the notion of a free field is legitimate -sufficiently outside the horizon so that the Doppler shift does not entail transplanckian frequencies when the Hawking photon is extrapolated backwards in time. Within the interior region one would expect a "soup". And it is to be noted that this soup is present in Minkowski space, since it is inside the star.

It is interesting to speculate on how one might describe this situation. After all, even if there is an ungainly soup, it nevertheless fluctuates in a systematic way if one imposes spherical symmetry and translational symmetry in time. The fluctuations can then be sorted out according to angular momentum and frequency; perhaps they must be endowed with a lifetime as well. These things are quasi particles. If Hawking radiation exists then we know for sure that as a quasi particle passes through the horizon region it gets converted from "quasi" to the free field fluctuations which we have treated in this review. So it should be possible to come to grips with this problem somewhere in the middle region. It is not impossible that one will be able to prove that the Hawking radiation develops out of this Planckian nether nether land and that the true transplanckian fluctuations are irrelevant [3]. After all these latter don't seem to bother us very much here and now so there is some hope that they are no nuisance there either. They may even result, as suggested above, in a Planckian spread of the region between the apparent and real horizon since they incorporate gravity within them.

[3] In a recent numerical calculation in a model which is analogous to the black hole, Unruh [**?**] has shown that a severe modification of the dispersion relation $\omega(k)$ for k greater than some threshold value in no way affects the thermal spectrum of Hawking emission.

Another very disturbing problem arises in the semi classical theory. This is the so called unitarity issue[46]. In usual evaporation of an isolated system into vacuum, correlations are always present. At the early stages an evaporated molecule gives a kick back to the unevaporated mass causing correlations between what has and what has not evaporated. At later stages these correlations get transferred to correlations among the evaporated molecules[69]. If one were in a pure quantum state, it is these correlations which encode its purity. Of course one says evaporation is accompanied by entropy increase, hence increase in the number of states. But we attribute this to the coarse graining that is implicit in the definition of entropy.

In black hole evaporation, in the semi classical theory, the correlations are in nature similar to that encountered in usual evaporation in its early stages. Reference to Section 3.5 shows that each Hawking photon leaves behind a "partner", a field configuration in vacuum of a specified local character within the star. As the evaporation proceeds these correlating configurations build up. However in the semi classical theory these configurations never get out (they are shut up in the closed geometry which devellops during the evaporation see Section 3.4) and the Hawking radiation contains no information on the quantum state of the star. One does not recover unitarity at $r = \infty$.

A few options (see for instance the review article of Preskill[78]) seem available on how to confront this situation:

- As suggested by the semi classical theory, these inside configurations are forever lost to the outside observer (they could end up in the singularity, or in an infinitely long lived remnant). Unitarity is truly violated as originally claimed Hawking[46].

- The semi classical theory fails at Planckian size black hole, evaporation stops and a finite long living remnant forms whereupon the correlations between the Hawking quanta, their partners and the star's matter are recovered at $r = \infty$ to reconstruct purity. We recall that for the accelerating mirror (Section 2.5) the correlations to the partners are completely recovered upon decelerating the mirror. However these considerations cannot be applied directly for the black hole since, for purity, the degrees of freedom of the star should be recovered as well.

- The semi-classical picture is all wrong in this regard and the correlations occur outside the horizon. All the information about the state of the star leaks out in the radiation[51][52][84]. This option however necessitates either a violation of causality or a fundamental revision of the concept of background geometry at scales large compared to the Planck scale. Some authors entertain the thought that different backgrounds are appropriate for different observers[89], e.g. Schwarzschildian or free falling observer. The backgrounds would be "post-selected" by the observer.

In the opinion of the authors it is futile to confront the unitarity issue without some clear ideas about the transplanckian issue i.e. quantum gravity. Indeed the unitary problem cannot be settled without a deep understanding of the dynamical origin of black hole radiation.

We also wish to point out that in this quest the fact that the entropy of a black hole is proportional to its area may play a vital role. How is it that the entropy of a black hole is equal to the number of Planckian cells that are necessary to pave its surface? The horizon seems to block out the cells which lie deeper than a Planck length within the hole. Is this related to the expected scenario that emission will occur at the surface outside the apparent horizon i.e. where a quantum fluctuation begins to belie its presence?

Such are the problems that one must face. Whether their solution will lead to the quantum theory of gravity or the inverse is a moot point. And this primer is certainly not the place to speculate any further on the question.

No doubt there will turn up further stormy weather to stir up the already troubled waters that must be traversed on this journey to terra incognita. Nevertheless we wish the reader at least some fair weather. Good Luck and Bon Voyage.

Appendix A

Bogoljubov Transformation.

In superfluid helium at rest a macroscopic number of particles occupy the zero momentum state $< a_0^\dagger a_0 >= N_0$ and N_0/N is a finite fraction as well as N/V where V is the volume, N the total number. The thermodynamic limit is $N \to \infty, V \to \infty$ with N_0/N and N/V fixed. The commutation relation $[a_0, a_0^+] = 1$ is then a negligible consideration when considering operators containing a_0 and a_0^+ as products that multiply the typical unperturbed states which make up the vacuum (ground state), these unperturbed states being eigenfunctions of $< n_k >$.

The hamiltonian of interacting particles is (with $\xi_k = k^2/2m$)

$$\begin{aligned}
H &= H_0 + V \\
H_0 &= \sum_k n_k(\xi_k - \mu) \\
V &= \frac{1}{2} \sum_{k_1,k_2,k_3,k_4} v(k_1, k_2; k_3, k_4) \quad \delta_{k_1+k_2,k_3+k_4} a_{k_1}^+ a_{k_2}^+ a_{k_3} a_{k_4}
\end{aligned} \tag{A.1}$$

The interaction potential has matrix elements of $O(1/V^2)$ since the unperturbed states in a box are $e^{ikr}/\sqrt{V}$.

Each $\sum_k$ is of $O(V)$ so each of the terms in eq. (A.1) is $O(N)$, μ is a chemical potential put in for convenience so that one can allow N to fluctuate albeit such that $< \Delta N^2 > / < N >^2 = O(1/N)$.

The unperturbed ground state has $N_0 = N$, $\mu = 0$, and $E = 0$. The idea of Bogoljubov [12] was to develop a perturbation theory in the small number

$[(N_0/N)-1]$. So the technique is to keep terms in leading order in $\sqrt{N_0}$ in eq. (A.1) where one counts $a_0 = \sqrt{N_0}e^{i\varphi}$, $a_0^+ = \sqrt{N_0}e^{-i\varphi}$ in accord with neglect of the commutator. The leading orders are then $O(N_0^2)$ and $O(N_0)$. One returns to terms of $O(\sqrt{N_0})$ and $O(1)$ in a standard perturbative procedure as a subsequent step.

Thus to $O(N_0)$, the perturbation V becomes

$$V = \frac{N_0}{2}\sum_k \Big[v(k,-k;0,0)(e^{2i\varphi}a_k^+a_{-k}^+ + e^{-2i\varphi}a_k a_{-k}) + v(0,k;0,k)a_k^+a_k + v(k,0;0,k)a_k a_k^+\Big] \tag{A.2}$$

The third and fourth terms of eq. (A.2) are standard Hartree Fock single particle energies and may be absorbed into H_0

$$\begin{aligned} H_0 &= \sum_k n_k(E_k - \mu) \\ E_k &= \xi_k + [v(0,k;0,k) + v(k,0;0,k)]N_0/2 \end{aligned} \tag{A.3}$$

The result is a quadratic hamiltonian. This is diagonalized in the following (Bogoljubov) transformation

$$\begin{aligned} b_k &= \alpha_k a_k + \beta_k a_{-k}^\dagger \\ b_k^\dagger &= \alpha_k^* a_{-k}^\dagger + \beta_k^* a_k \end{aligned} \tag{A.4}$$

where the phase of α is $e^{-i\varphi}$ and of β is $e^{+i\varphi}$ and

$$E_k|\alpha_k\beta_k| = (|\alpha_k|^2 + |\beta_k|^2)|V_k| \tag{A.5}$$

with $|V_k| = |v(k,-k;0,0)|$. Canonical commutation relations for b_k gives $|\alpha_k|^2 - |\beta_k|^2 = 1$.

Thus the unperturbed ground state is not an eigenstate of the total hamiltonian H but is unstable. In particular, had we chosen this state as the initial state it would evolve to the true ground state through emission of $k, -k$ pairs. We leave to the reader the pleasure to confirm the Nambu Goldstone theorem for this case

$$[H- <\Omega|H|\Omega>] = \sum \omega_k b_k^\dagger b_k \tag{A.6}$$

where $\lim_{k\to 0}\omega_k = C|k| + O(k^2)$. He may also show that $b_k^\dagger|\Omega\rangle$ corresponds to the creation of a longitudinal density fluctuation in this approximation, i.e. a phonon.

Appendix B

Functional Integral Technique.

We use standard field theoretical techniques. The uninformed reader will find an account in refs. [86][10][73]. The in vacuum to out vacuum amplitude is

$$e^{iW} = \int \mathcal{D}\phi \; e^{iS(\phi)} = \langle out, 0, |0, in\rangle \tag{B.1}$$

where ϕ is a complex scalar field, $S(\phi)$ is the action to go from initial to final configurations [which we take to be a quadratic form: free field theory in the presence of an electromagnetic and/or gravitational field] in time t (which tends to ∞) and the mass^2 has a small negative imaginary part. The mass dependence of S is of the form $\int -m^2\phi\phi^* d^dx$ (with $\sqrt{g} = 1$) so that

$$\begin{aligned} \frac{\partial W}{\partial m^2} &= -\int d^dx \langle out, 0, |\phi^2|0, in\rangle = -\int G_F(x,x) d^dx \\ &= -\mathrm{tr} G_F \quad , \end{aligned} \tag{B.2}$$

where $G_F(x, x')$ being the Feynman propagator to go from x' to x. In terms of the heat kernel K one has

$$G_F(x, x') = \int_0^\infty ds \; e^{-im^2 s} K(x, x'; s) \tag{B.3}$$

where

$$(\Box + i\frac{\partial}{\partial s})K = i\delta(s)\delta(x - x') \tag{B.4}$$

In eq. (B.2) we have written $\partial W/\partial m^2$ as a trace. Clearly this is formal and one has to watch one's step on the measure. The following steps are valid because the passage from x representation to u representation is unitary [73]. Putting it all together we have upon integrating eq. (B.2) over m^2

$$W = -\int_0^\infty \frac{ds}{s} e^{-im^2 s} \mathrm{tr} K(s) \tag{B.5}$$

We now specialize to the case of constant electric field in the gauge $A_x = 0$, $A_t = Ex$. As in Chapter 1, we can label the modes by ω and work in the u representation, whereupon

$$W = -i \sum_\omega \int_0^\infty \frac{ds}{s} e^{-im^2 s} \int K_\omega(u, u; s) du \tag{B.6}$$

where $K_\omega(u, u_0, s)$ obeys

$$[u\partial_u + \frac{1}{2} - \partial_\tau] K_\omega = \delta(\tau)\delta(u - u_0) \tag{B.7}$$

and $\tau = 2E\ s$, and we used eq. (1.7) with $i\varepsilon$ replaced by $\partial/\partial\tau$. The solution is

$$K_\omega(u, u_0; \tau) = \theta(\tau)\delta(ue^\tau - u_0)e^{\tau/2} \tag{B.8}$$

to give from eq. (B.5)

$$\begin{aligned}
\mathrm{Im}W &= -\mathrm{Im}\ i \sum_\omega \int_0^\infty \frac{d\tau}{\tau} e^{-im^2\tau/2E} \int du \delta(ue^\tau - u)e^{\tau/2} \\
&= -\mathrm{Im} \frac{i}{2} \sum_\omega \int_0^\infty \frac{ds}{s} \frac{e^{-im^2 s}}{\sinh Es} \\
&= \frac{1}{2} \sum_\omega \ln(1 + e^{-\pi m^2/E})
\end{aligned} \tag{B.9}$$

where the last equality is obtained by picking up the poles (with $m^2 = m^2 - i\epsilon$) recovering therefore the Schwinger formula eq. (1.44) since $\sum_\omega = ELT/2\pi$.

When the path parameter τ is expressed in terms of proper time these poles on the imaginary axis are related to multiple excursions through the tunneling region. For example, Born approximation (WKB) for the tunneling amplitude corresponds to the pole at $Es = i\pi$ and represents one excursion back and forth ($\Delta\tau_{proper} = 2\pi/a$ in the movement of the wave packets of Section 2.3).

One may perform similar tricks to put into evidence the instability of Schwarzschild vacuum in the complete space spanned by the Eddington Finkelstein coordinates. Here the Schwinger counting parameter s is once more proportional to proper time. The complete analysis is considerately more tedious and complicated than the above but the essential features are the same. We refer the reader to ref. [73] for details.

In this case it turns out that the convenient variable that describes the effective motion of a packet is p, the momentum conjugate to the Eddington Finkelstein coordinate $x = r - 2M$ at fixed v. There are three classes of paths which contribute to W, those in which initial and final momenta have the same sign and that in which it goes from negative to positive momentum. Upon taking the trace initial and final momenta are set equal, so this operation requires a careful limiting procedure. One finds that it is the third class that encodes the instability and that the time to execute this movement is $\Delta v = i8\pi M = i\beta_H$ i.e. β_H is the imaginary time to go from p to $-p$ and back. One finds

$$\mathrm{Im}W_{BH} = -\frac{T}{4\pi}\int_0^\infty d\omega\ \ln[1 - e^{-\beta_H\omega}] \tag{B.10}$$

precisely the one dimensional partition function. In this manner $\beta_H\omega$ is indeed interpretable as the action for a Hawking photon to tunnel out into existence.

Appendix C

Pre- and Post-Selection, Weak Measurements.

Pre- and post-selection consists in specifying both the initial and the final state of a system (denoted by S in the sequel). Pre and post selection is not an unusual procedure in physics. For instance when dealing with transition amplitudes, scattering amplitudes, etc... one is performing pre and post selection.

In the first part of this appendix we shall implement post-selection in a rather formal way by acting on the state with projection operators which select the desired final state(s) following the treatment of [63]. This generalizes the approach of [3].

In the second part of this appendix we show how post-selection may be realized operationally following the rules of quantum mechanics by coupling to S an additional system in a metastable state (the "post selector" PS) which will make a transition only if the system is in the required final state(s). The weak value of an operator obtained in this manner changes as time goes by from an asymmetric form to an expectation value, thereby making contact with more familiar physics. This extended formalism finds important application when considering the physics of the accelerated detector since the accelerated detector itself plays the role of post selector. In this way one can study the EPR correlations between the state of a uniformly accelerated detector and the radiation field, thereby clarifying and generalizing the results of [94], [4], [41].

C.1 Weak Values

The approach developed by Aharonov et al.[3] for studying pre- and post-selected ensembles consists in performing at an intermediate time a "weak measurement" on S. In essence one studies the first order effect of S (ie. the back reaction) onto an additional system taken by Aharonov et al. to be the measuring device. But the formalism is more general. Indeed when the first order (or weak-coupling) approximation is valid, the backreaction takes a simple and universal form governed by a c-number, the "weak value" of the operator which controls the interaction.

The system to be studied is in the state $|\psi_i\rangle$ at time t_i (or more generally is described by a density matrix ρ_i). The unperturbed time evolution of this pre-selected state can be described by the following density matrix

$$\rho_S(t) = U_S(t, t_i)|\psi_i\rangle\langle\psi_i|U_S(t_i, t) \tag{C.1}$$

where $U_S = \exp(-iH_S t)$ is the time evolution operator for the system S. The post-selection at time t_f consists in specifying that the system belongs to a certain subspace, $\mathcal{H}_S^0$, of $\mathcal{H}_S$. Then the probability to find the system in this subspace at time t_f is

$$P_{\Pi_S^0} = Tr_S\Big[\Pi_S^0 \rho(t_f)\Big] = Tr_S\Big[\Pi_S^0 U_S(t_f, t_i)|\psi_i\rangle\langle\psi_i|U_S(t_i, t_f)\Big] \tag{C.2}$$

where Π_S^0 is the projection operator onto $\mathcal{H}_S^0$ and Tr_S is the trace over the states of system S. In the special cases wherein the specification of the final state is to be in a pure state $|\psi_f\rangle$ (ie. $\Pi_S^0 = |\psi_f\rangle\langle\psi_f|$) then the probability is simply given by the overlap

$$P_f = |\langle\psi_f|U_S(t_f, t_i)|\psi_i\rangle|^2 \tag{C.3}$$

Following Aharonov et al. we introduce an additional system, called the "weak detector" (WD), coupled to S. The interaction hamiltonian between S and WD is taken to be of the form $H_{S-WD}(t) = \epsilon f(t) A_S B_{WD}$ where ϵ is a coupling constant, $f(t)$ is a c-number function, A_S and B_{WD} are hermitian operators acting on S and WD respectively.

Then to first order in ϵ (the coupling is weak), the evolution of the coupled system S and WD is given by

$$\rho(t_f) \;\; = \;\; |\Psi(t_f)\rangle\langle\Psi(t_f)|$$

where

$$
\begin{aligned}
|\Psi(t_f)\rangle &= \Big[U_S(t_f,t_i)U_{WD}(t_f,t_i) - i\epsilon\int_{t_i}^{t_f} dt\ U_S(t_f,t)U_{WD}(t_f,t)f(t)A_S B_{WD}\times \\
&\quad U_S(t,t_i)U_{WD}(t,t_i)\Big]|\psi_i\rangle|WD\rangle \qquad \text{(C.4)}
\end{aligned}
$$

where U_S and U_{WD} are the free evolution operators for S and WD and $|WD\rangle$ is the initial state of WD. Upon post-selecting at $t = t_f$ that S belongs to the subspace $\mathcal{H}_S^0$ and tracing over the states of the system S, the reduced density matrix describing the WD is obtained

$$
\rho_{WD}(t_f) = Tr_S\Big[\Pi_S^0 \rho(t_f)\Big] \qquad \text{(C.5)}
$$

In the first order approximation in which we are working it takes a very simple form

$$
\begin{aligned}
\rho_{WD}(t_f) &\simeq P_{\Pi_S^0}|\Psi_{WD}(t_f)\rangle\langle\Psi_{WD}(t_f)| \\
&\text{where} \\
|\Psi_{WD}(t_f)\rangle &= \Big[U_{WD}(t_f,t_i) - i\epsilon\int_{t_i}^{t_f} dt\ U_{WD}(t_f,t)f(t)A_{Sweak}(t)B_{WD}U_{WD}(t,t_i)\Big]|WD\rangle
\end{aligned}
\qquad \text{(C.6)}
$$

where $P_{\Pi_S^0}$ is the probability to be in subspace $\mathcal{H}_S^0$ and

$$
A_{Sweak}(t) = \frac{Tr_S\Big[\Pi_S^0 U_S(t_f,t)A_S U_S(t,t_i)|\psi_i\rangle\langle\psi_i|U_S(t_i,t_f)\Big]}{Tr_S\Big[\Pi_S^0 U_S(t_f,t_i)|\psi_i\rangle\langle\psi_i|U_S(t_i,t_f)\Big]} \qquad \text{(C.7)}
$$

is a c-number called the weak value of A. If one specifies completely the final state, $\Pi_S^0 = |\psi_f\rangle\langle\psi_f|$ then the result of Aharonov et al. obtains:

$$
A_{Sweak}(t) = \frac{\langle\psi_f|U_S(t_f,t)A_S U_S(t,t_i)|\psi_i\rangle}{\langle\psi_f|U_S(t_f,t_i)|\psi_i\rangle} \qquad \text{(C.8)}
$$

The principal feature of the above formalism is its independence on the internal structure of the WD. The first order backreaction of S onto WD is universal: it is always controlled by the c-number $A_{Sweak}(t)$, the "weak value of A". Therefore if S is coupled to itself by an interaction hamiltonian, the backreaction will be controlled by the weak value of $H_{\rm int}$ in first order perturbation theory. For instance the modification of the probability that the final state belongs to $\mathcal{H}_S^0$ is given by the imaginary part of $H_{\text{int } weak}$. Indeed

$$
\begin{aligned}
P'_{\Pi_S^0} &= Tr_S\left[\Pi_S^0(1-i\int dt\ H_{\text{int}})\rho_i(1-i\int dt\ H_{\text{int}})\right] \\
&= P_{\Pi_S^0}(1-2\mathrm{Im}H_{\text{int } weak})
\end{aligned} \tag{C.9}
$$

The weak value of A is complex. By performing a series of measurements on WD and by varying the coupling function $f(t)$, the real and imaginary part of A_{Sweak} could in principle be determined. Here the word "measurement" must be understood in its usual quantum sense: the average over repeated realizations of the same situation. This means that the weak value of A_S should also be understood as an average. The fluctuations around A_{Sweak} are encoded in the second order terms of eq. (C.4) which have been neglected in eq. (C.4).

To illustrate the role of the real and imaginary parts of A_{Sweak}, we recall the example of Aharonov et al consisting of a weak detector which has one degree of freedom q, with a gaussian initial state $< q|WD >= e^{-q^2/2\Delta^2}$,$-\infty < q < +\infty$. The unperturbed hamiltonian of WD is taken to vanish (hence $U_{WD}(t_1,t_2)=1$) and the interaction hamiltonian is $H_{S-WD}(t)=\epsilon\delta(t-t_0)pA_S$ where p is the momentum conjugate to q. Then after the post-selection the state of the WD is given to first order by

$$
\begin{aligned}
< q|WD(t_f)> &= (1-i\epsilon pA_{Sweak}(t_0))\,e^{-q^2/2\Delta^2} \\
&\simeq e^{-i\epsilon pA_{Sweak}(t_0)}e^{-q^2/2\Delta^2} \\
&= e^{-(q-\epsilon A_{Sweak}(t_0))^2/2\Delta^2} \\
&= e^{-(q-\epsilon \mathrm{Re}A_{Sweak}(t_0))^2/2\Delta^2}e^{+i\epsilon q\mathrm{Im}A_{Sweak}(t_0)/\Delta^2}
\end{aligned} \tag{C.10}
$$

The real part of A_{Sweak} induces a translation of the center of the gaussian, the imaginary part a change in the momentum. Their effect on the WD is therefore measurable. The validity of the first order approximation requires $\epsilon A_{Sweak}/\Delta << 1$.

It is instructive to see how unitarity is realised in the above formalism. Take Π_S^j to be a complete orthogonal set of projectors acting on the Hilbert space of S. Denote by P_j the probability that the final state of the system belong to the subspace spanned by Π_S^j and by A_{Sweak}^j the corresponding weak value of A. Then the mean value of A_S is

$$
\langle\psi_i|A_S|\psi_i\rangle = \sum_j P_j A_{Sweak}^j \tag{C.11}
$$

Thus the mean backreaction if no post-selection is performed is the average over the post-selected backreactions (in the linear response approximation). Notice that the imaginary parts of the weak values necessarily cancel since the l.h.s. of eq. (C.11) is real. Equation (C.11) is the short cut used in the main text to obtain with minimum effort the weak values.

C.2 Physical Implementation of Post Selection

Up to now the postselection has been implemented by projecting by hand the state of the system onto a certain subspace $\mathcal{H}_S^0$. Such a projection may be realised operationally by introducing an additional quantum system, a "post-selector" (PS), coupled in such a way that it will make a transition if and only if the system S is in the required final state. Then by considering only that subspace of the final states in which PS has made the transition, a post-selected state is specified. This quantum description of the post-selection is similar in spirit to the measurement theory developed in ref. [95]: by introducing explicitly the measuring device in the hamiltonian the collapse of the wave function ceases to be a necessary concomitant of measurement theory. As we have mentioned, this formalism is the basis for a general treatment of the energy density correlated to transitions of an accelerated detector.

We shall consider the very simple model of a PS having two states, initially in the ground state, and coupled to the system by an interaction of the form

$$H_{S-PS} = \lambda g(t)(a^\dagger Q_S + aQ_S^\dagger) \tag{C.12}$$

where λ is a coupling constant, $g(t)$ a time dependent function, $a^\dagger$ the operator that induce transitions from the ground state to the exited state of the PS, Q_S an operator acting on the system S. The postselection is performed at $t = t_f$ and consists in finding the PS in the exited state.

For simplicity we shall work to second order in λ (although in principle the interaction of PS with S need not be weak). The wave function of the combined system $S + WD + PS$ is in interaction representation to order ϵ and order λ^2

$$\mathcal{T}e^{-i\int dt\ H_{S-WD}(t)+H_{S-PS}(t)}|\psi_i\rangle|WD\rangle|0_{PS}\rangle =$$

$$\Big[1 - i\int dt\,(H_{S-WD}(t) + H_{S-PS}(t))$$
$$-\frac{1}{2}\int dt\int dt'\mathcal{T}\left[H_{S-PS}(t)H_{S-PS}(t')\right] - \int dt\int dt'\mathcal{T}\left[H_{S-WD}(t)H_{S-PS}(t')\right]$$
$$+\frac{i}{2}\int dt\int dt'\int dt''\mathcal{T}\left[H_{S-PS}(t)H_{S-PS}(t')H_{S-WD}(t'')\right]\Big]|\psi_i\rangle|WD\rangle|0_{PS}\rangle \tag{C.13}$$

where $|0_{PS}\rangle$ is the ground state of PS and $\mathcal{T}$ is the time ordering operator. The probability of being in the excited state at $t = t_f$ at order λ^2 is

$$P_{excited} = \lambda^2\langle\psi_i|\int dtg(t)Q_S^\dagger\int dt'g(t')Q_S|\psi_i\rangle \tag{C.14}$$

Upon imposing that the PS be in its excited state at $t = t_f$ the resulting wave function is, to order ϵ and λ^2,

$$\Big[-i\int dt\lambda g(t)Q_S(t)$$
$$-\int dt\int dt'\mathcal{T}\left[\epsilon f(t)A_S(t)B_{WD}(t)\lambda g(t')Q_S(t')\right]\Big]|\psi_i\rangle|WD\rangle a^\dagger|0_{PS}\rangle \tag{C.15}$$

Making a density matrix out of the state (C.15), tracing over the states of S and PS yields the reduced density matrix $|\Psi_{WD}\rangle\langle\Psi_{WD}|$ of WD to order ϵ where

$$|\Psi_{WD}\rangle = \left[1 - i\epsilon\int dt_0 f(t_0)B_{WD}(t_0)A^{excited}_{Sweak}(t_0)\right]|WD\rangle \tag{C.16}$$

and

$$A^{excited}_{Sweak}(t_0) = \frac{\langle\psi_i|\int dtg(t)Q_S^\dagger(t)\int dt'g(t')\mathcal{T}\left[A_S(t_0)Q_S(t')\right]|\psi_i\rangle}{\langle\psi_i|\int dtg(t)Q_S^\dagger(t)\int dt'g(t')Q_S(t')|\psi_i\rangle} \tag{C.17}$$

Note how the weak value of A_S results from the quantum mechanical interference of the two terms in eq. (C.15).

There are several important cases when the time ordering in eq. (C.17) simplifies. If $g(t)$ is non vanishing only after $t = t_0$ then A_W takes a typical (for a weak value) asymmetric form

$$A^{excited}_{Sweak}(t_0) = \frac{\langle\psi_i|\int dtg(t)Q_S^\dagger(t)\int dt'g(t')Q_S(t')A_S(t_0)|\psi_i\rangle}{\langle\psi_i|\int dtg(t)Q_S^\dagger(t)\int dt'g(t')Q_S(t')|\psi_i\rangle} \tag{C.18}$$

If in addition $g(t) = \delta(t - t_f)$, $t_f > t_0$ and $Q_S = \Pi^0_S$, eq. (C.7) is recovered using $(\Pi^0_S)^2 = \Pi^0_S$. This is expected since in this case the post–selector has simply gotten correlated to the system in the subspace $\mathcal{H}^0_S$

If on the other hand $g(t)$ is non vanishing only before $t = t_0$ then the time ordering operator becomes trivial once more and eq. (C.17) takes the form

$$A^{excited}_{Sweak}(t_0) = \frac{\langle\psi_i| \int dt g(t) Q^\dagger_S(t) A_S(t_0) \int dt' g(t') Q_S(t')|\psi_i\rangle}{\langle\psi_i| \int dt g(t) Q^\dagger_S(t) \int dt' g(t') Q_S(t')|\psi_i\rangle} \tag{C.19}$$

This is by construction the expectation value of A_S if the PS has made a transition. It is necessarily real contrary to eq. (C.18) when the weak measurement is performed before the 'collapse' induced by the post-selection.

Finally, the weak value of A_S if the PS has not made a transition can also be computed. Once more the two cases discussed in eqs (C.18) and (C.19) are particularly simple: if $g(t)$ is non vanishing only after $t = t_0$ one finds

$$\begin{aligned} A^{deexcited}_{Sweak}(t_0) \quad = \quad & \frac{1}{1 - P_{excited}}\Bigg(\langle\psi_i|A_S|\psi_i\rangle \\ & -\lambda^2 \mathrm{Re}\left[\langle\psi_i| \int dt g(t) Q^\dagger_S(t) \int dt' g(t') Q_S(t') A_S(t_0)|\psi_i\rangle\right]\Bigg) \end{aligned} \tag{C.20}$$

On the other hand if $g(t)$ is non vanishing only before $t = t_0$ one finds

$$\begin{aligned} A^{deexcited}_{Sweak}(t_0) \quad = \quad & \frac{1}{1 - P_{excited}}\Bigg(\langle\psi_i|A_S|\psi_i\rangle \\ & -\frac{1}{2}\lambda^2 \mathrm{Re}\left[\langle\psi_i|A_S(t_0) \int dt \int dt' \mathcal{T} g(t) Q^\dagger_S(t) g(t') Q_S(t')|\psi_i\rangle\right]\Bigg) \end{aligned} \tag{C.21}$$

These are related to the mean value of A_S and to eq. (C.17) through the unitary relation eq. (C.11): if $g(t)$ is non vanishing only after $t = t_0$

$$P_{excited} A^{excited}_{Sweak}(t_0) + (1 - P_{excited}) A^{deexcited}_{Sweak} = \langle\psi_i|A_S|\psi_i\rangle \tag{C.22}$$

if $g(t)$ is non vanishing only before $t = t_0$

$$P_{excited} A^{excited}_{Sweak}(t_0) + (1 - P_{excited}) A^{deexcited}_{Sweak} \quad =$$

$$
\begin{aligned}
&\langle\psi_i|\mathcal{T}e^{i\int dt\ H_{S-PS}}A_S\mathcal{T}e^{-i\int dt\ H_{S-PS}}|\psi_i\rangle \ = \\
&\langle\psi_i|A_S|\psi_i\rangle + \lambda^2\langle\psi_i|\int dt g(t)Q_S^\dagger(t)A_S(t_0)\int dt' g(t')Q_S^\dagger(t')|\psi_i\rangle \ - \\
&-\lambda^2\mathrm{Re}\langle\psi_i|A_S(t_0)\int dt\int dt'\mathcal{T}g(t)Q_S^\dagger(t)g(t')Q_S^\dagger(t')|\psi_i\rangle
\end{aligned}
\tag{C.23}
$$

where the right hand side is the average value of A_S before eq. (C.22) and after eq. (C.23) the detector has interacted with S.

Appendix D

S-wave Hawking Radiation for a General Collapsing Spherical Star Without Back Reaction

We shall consider a collapsing sphere of matter with a well defined surface. Exterior to the surface the geometry is Schwarzschild, parametrized by a fixed mass M. This part of the space shall be coordinatized by advanced Eddington-Finkelstein coordinates (v, r, θ, φ) defined in eq. (3.8). Interior to the star we shall use a set (T, X, θ, φ) in terms of which the length interval for a general dynamic spherically symmetric space is

$$ds^2 = -a^2(T, X)\ [dT^2 - dX^2] + b^2(T, X)\ d\Omega^2 \tag{D.1}$$

with $d\Omega^2 = d\theta^2 + \sin^2\ \theta d\varphi^2$. This system is used in the numerical calculation of $\langle T_{\mu\nu}\rangle$ that was reported in Section 3.3 (with $U, V = T \mp X$). In that case it was used to cover the whole space so as to account for the back reaction outside the star as well. Since we are not taking into account the back reaction in the present case, we use eq.(D.1) in the inside region only. The curvature stemming from the metric components a, b is driven by classical sources.

The coordinate X is a radial coordinate, but areas of spheres are $4\pi b^2$. Thus at a given point of coordinates $(T,\ X)$ on the star's surface one has the identification

$$r = b(T, X) \tag{D.2}$$

We shall choose the origin of X so that $r = 0$ coincides with $X = 0$.

Hawking radiation follows from the combined gravitational and Doppler shifts that a photon experiences on its voyage from $\mathcal{I}^-$to $\mathcal{I}^+$. It begins on $\mathcal{I}^-$ as a packet (superposition) of modes $e^{-i\omega v}$ and end up on $\mathcal{I}^+$ as a packet of modes $e^{i\omega f(u)}$. We consider a particular radial packet that emanates from a point on $\mathcal{I}^-$ denoted by $v = \hat{u}$. The function $f(u)$ varies according to the value of $\hat{u}$ so we will call it $\hat{u}(u)$. The shift of frequency on $\mathcal{I}^+$ is

$$\omega(u)|_{\mathcal{I}^+} = \omega(d\hat{u}/du) \qquad . \tag{D.3}$$

One may deduce this, for example, by the conservation of the number of oscillations in a given segment of wavefront $(= \omega dv = \omega d\hat{u} = \omega|_{\mathcal{I}^+} du)$

There are three important points which the ray visits. These are:

1. $P_I \equiv (\overline{X}, \overline{T})$ where the ray penetrates the star. This labels the sphere which is the intersection between the incoming lightcone $v = \hat{u}$ and the star's surface.

2. P_C where reflection on the axis $X = 0$ occurs.

3. $P_O \equiv (\tilde{X}, \tilde{T})$, where the ray leaves the star.

One has

$$\tilde{T} - \tilde{X} = \overline{T} + \overline{X} \tag{D.4}$$

since the coordinate time interval $(= \tilde{T} - \overline{T})$ is equal to the coordinate distance toward in the star $(= \tilde{X} + \bar{X})$.

We now must specify the surface of the trajectory. In any given model this specification is, of course, correlated to the metric coefficients a, b along the trajectory, but there is no need to keep track of this in the general analysis. We parametrize the trajectory of the surface in each of the systems according to

$$\begin{aligned} v &= V_{\mathrm{S}}(r) \quad , \\ X &= \Xi_{\mathrm{S}}(T) \quad . \end{aligned} \tag{D.5}$$

Then the first intersection point P_I is given in terms of the ray $\hat{u}$ by the solution of

$$
\begin{aligned}
\hat{u} &= V_{\mathrm{S}}(\overline{r}) \\
\overline{r} &= b(\overline{T}, \Xi_{\mathrm{S}}(\overline{T}) \\
\overline{X} &= \Xi_{\mathrm{S}}(\overline{T})
\end{aligned} \tag{D.6}
$$

The second intersection point is obtained from eq.(D.4)

$$
\tilde{T} - \Xi_{\mathrm{S}}(\tilde{T}) = \overline{T} + \Xi_{\mathrm{S}}(\tilde{T}) \tag{D.7}
$$

The wave then propagates out to $\mathcal{I}^+$ according to $e^{i\omega\tilde{U}}$ where $\tilde{U} = \tilde{T} - \tilde{X}$. The exterior coordinate u is given by

$$
u = V_{\mathrm{S}}(\tilde{r}) - 2r^\star(\tilde{r}) \tag{D.8}
$$

where $\tilde{r} = b(\tilde{T}, \Xi_{\mathrm{S}}(\tilde{T}))$. Inverting this last relation gives $\tilde{T}(\tilde{r})$. One then computes $\tilde{X}(\tilde{r})$ from eq.(D.2) hence $\tilde{U}(\tilde{r})$. Together with eq.(D.8) this chain then gives the required relation between u and $\tilde{U}$. This matter is the sought function $\tilde{U} = \hat{u}(u)$. In this way it is seen that the phase of the outgoing wave gives a sort of " X-ray picture " of the interior of the star.

To perform these various inversions is an arduous task. At the end of the section we shall present the results of a calculation wherein the collapsing star consists of dust. As announced in Section 3.4 almost all of Hawking radiation occurs in a small interval where the point $P_O \equiv (\tilde{X}, \tilde{T})$ is near the horizon, $\mathcal{H}$. As in Section 3.2 we shall linearize the equation of motion of the star's trajectory for such points. Similarly, the point $P_I \equiv (\overline{X}, \overline{T})$ is near the extension of $\mathcal{H}$ into a past lightcone (the last null rays that are reflected into a future lightcone expanding up to the asymptotically flat infinity). See Fig. (3.2b). We shall denote the intersection of the star's surface with $\mathcal{H}$ as $O \equiv (X_O, T_O)$ and the intersection with the backward extension of $\mathcal{H}$ as $I \equiv (X_I, T_I)$. The linearized forms of the trajectory of the surface of the star for points near O and I are

$$
\begin{aligned}
v - v_O &= k_O(r - 2M) \quad , \\
v - v_I &= k_I(r - r_I) \quad ,
\end{aligned} \tag{D.9}
$$

which in interior coordinates will be written

$$
\begin{aligned}
X - X_O &= \beta_O(T - T_O) \quad , \\
X - X_I &= \beta_I(T - T_I) \quad .
\end{aligned} \tag{D.10}
$$

The relation between β_O and k_O (or β_I and k_I) is obtained by equating the expressions of proper time intervals ($= ds^2$) on the star's surface at the intersection points O and I, when expressed in both coordinate systems. Thus

$$(1-\frac{2M}{r})\, dv^2 - 2drdv = a^2[dT^2 - dX^2] \tag{D.11}$$

with all differentials taken along the trajectory of the star's surface. In such differentials we also have

$$dr = b'\, dX + \dot{b}\, dT \qquad . \tag{D.12}$$

Dots are derivatives with respect to T and primes derivatives with respect to X. Using eqs.(D.9, D.10, D.11, D.12) gives at $X = X_O$

$$[(1-\frac{2M}{r_I})\, k_I^2 - 2k_I][\dot{b}_I + b'_I\beta_I]^2 \;=\; a_I(1-\beta_I^2) \tag{D.13}$$

$$-2k_O[\dot{b}_O + b'_O\beta_O]^2 \;=\; a_O(1-\beta_O^2) \tag{D.14}$$

Here $a_O, b_O, (a_I, b_I)$ denotes the values of the metric components a and b at points O and I respectively.

Let us now track the various chain of variables by following the ray backwards in time. Near O we have on the suface point $(\tilde{X}, \tilde{T})$ the value of u given by

$$u \simeq v_0 - 4M - 4Mln\frac{\tilde{r}-2M}{2M} \tag{D.15}$$

where

$$\tilde{r} = b(\tilde{T}, \Xi_{\rm S}(\tilde{T}) \cong 2M + \dot{b}_O(\tilde{T}-T_O) + b'_O\dot{\Xi}_{\rm S}(\tilde{T}-T_O)$$

so that

$$\tilde{r} - 2M \cong (\dot{b}_O + b'_O\ \beta_O)(\tilde{T}-T_O) \tag{D.16}$$

And from eq.(D.7) we have

$$\tilde{T} - T_O = \overline{T} - T_I + (\Xi_{\rm S}(\overline{T}) - \Xi_{\rm S}(T_I)) + (\Xi_{\rm S}(\tilde{T}) - \Xi_{\rm S}(T_O)) \tag{D.17}$$

to give after linearization

$$(\tilde{T} - T_O) \cong \frac{1+\beta_I}{I-\beta_O}(\overline{T} - T_I) \tag{D.18}$$

Equations (D.15, D.16, D.18) then permit one to express u in terms of $(\overline{T} - \overline{T}_I)$. We must go one step further back and express $(\overline{T} - T_I)$ in terms of $v - v_I$. For this we use

$$\frac{v - v_I}{k_I} = r - r_I = b_I'(\overline{X} - X_I) + \dot{b}_I(\overline{T} - T_I) = (b_I'\beta_I + \dot{b}_I)(\overline{T} - T_I) \quad . \tag{D.19}$$

Substituting for $(\overline{T} - T_I)$ the value $(v - v_I)$ of eq. (D.19) and proceeding the chainwise through eqs. (D.18, D.16, D.15) and using the relationships eqs. (D.13) and (D.14) yields the final result

$$u = v_O - 4M - 4M \ln \left| \frac{\sqrt{(1 - \frac{2M}{r_1})k_I^2 - 2k_I}}{k_I\sqrt{-2k_O}} \, \frac{a_O\sqrt{\frac{1+\beta_O}{1-\beta_O}}}{a_I\sqrt{\frac{1-\beta_I}{1+\beta_I}}} \, \frac{\hat{u} - v_I}{2M} \right| \tag{D.20}$$

for the coordinate u of the outgoing ray in the exterior space which began on $\mathcal{I}_-$ at the point $v = \hat{u}$.

Equation (D.20) has a physical interpretation when one writes the total shift as a product of three shifts (denoted D_1, D_2 and D_3).The factor D_1 is the shift produced in the voyage from $\mathcal{I}^-$ to I, D_2 from I to O and D_3 from O to $\mathcal{I}^+$. The identifications are

$$\begin{aligned} D_1 &= \frac{k_I}{\sqrt{(1 - \frac{2M}{r_I})k_I^2 - 2k_I}} \\ D_2 &= \frac{a_I}{a_O}\sqrt{\frac{1+\beta_O}{1-\beta_O}}\sqrt{\frac{1+\beta_I}{1-\beta_I}} \\ D_3 &= \sqrt{-2k_O}\,\frac{(r_O - 2M)}{4M} \end{aligned}$$

It is of course D_3 that is the important shift that gives rise to the steady state Hawking radiation.

To illustrate this discussion and to exhibit how the Hawking flux reaches its asymptotic value we consider a star consisting a cloud of dust in parabolic collapse. The trajectory of the surface of the star is

$$v = 2M\left[\frac{5}{3} - \frac{2}{3}(\frac{r}{2M})^{3/2} + \frac{r}{2M} - 2\sqrt{\frac{r}{2M}} + 2\ln\left(\frac{1 + \sqrt{\frac{r}{2M}}}{2}\right)\right] \tag{D.21}$$

so that $v = 0$ at $r = 2M$. The interior metric is a Roberston-Walker one, given by

$$ds^2 = \left(\frac{M}{2X_\star^3}\right)^2 T^4 \left[-dT^2 + dX^2\right] + X^2 \left(\frac{M}{2X_\star^3}\right)^2 T^4 \, d\Omega^2 \qquad \text{(D.22)}$$

where M is the mass of the star, and $X = X_\star$ the equation of motion of its surface. (The value of $X_\star$ fixes the density in the star).

The figures (D.1) and (D.2) show the outgoing flux (times $24\pi\ (2M^2)$) measured at infinity (Φ_∞) and by a free falling observer on the surface of the star (Φ_s). The fluxes are calculated using the two dimensional energy momentum tensor discussed in Section 3.3. Their analytic expressions (where we have choosen $X_\star = 10M$) are:

$$\begin{aligned}
\Phi_\infty &= \frac{1}{24\pi\ 4M^2}\frac{18\rho^{3/2} + 27\rho - 21\rho^{1/2} - 15}{\rho^4(8 + 24\rho^{1/2} + 26\rho + 12\rho^{3/2} + 2\rho^2} && \text{(D.23)}\\
\frac{t_\infty}{2M} &= \frac{5}{3} - \frac{2}{3}(\rho)^{3/2} + \rho - 2\sqrt{\rho} + 2\ln\left(1 + \sqrt{\rho}\right) \\
&\quad -2\left[\rho + \ln(\rho - 1)\right] && \text{(D.24)}\\
\Phi_s &= \frac{1}{24\pi\ 4M^2}\frac{-(4\rho^{3/2} + 24\rho + 39\rho^{1/2} + 15)}{\rho^4(8 + 24\rho^{1/2} + 26\rho + 12\rho^{3/2} + 2\rho^2} && \text{(D.25)}\\
\frac{\tau_s}{2M} &= \frac{2}{3}(1 - \rho^{3/2}) && \text{(D.26)}
\end{aligned}$$

where the parameter $\rho = r/2M$ represents the radial Eddington-Flinkenstein coordinate of the surface of the star and τ_s the proper time measured along it, t_∞ being the minkowskian time at infinity. The Hawking flux attains its asymptotic value at infinity when the last term in eq. (D.24) becomes dominant, ie. when the approximation eq. (D.20) is valid.

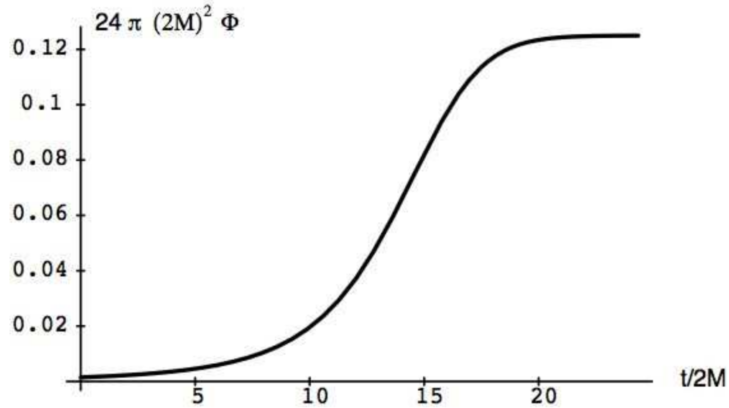

Fig. D.1 *Hawking flux at infinity as a function of time. The background geometry is that of a collapsing cloud of dust.*

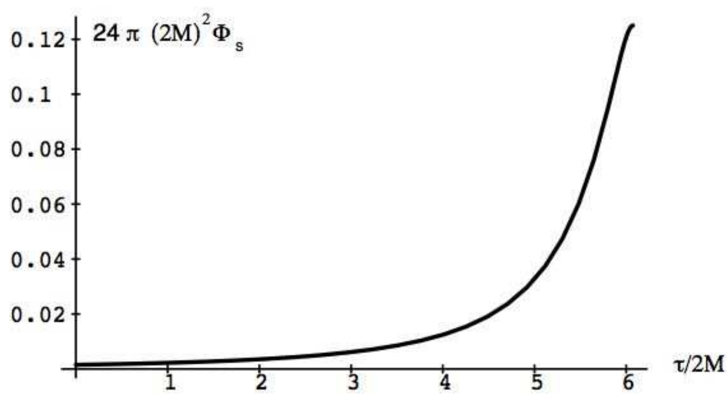

Fig. D.2 *Outgoing flux seen by an observer on the surface of the star as a function of his proper time.*

Note Added in Proof

In connexion with the remark at the end of Section 1.3 concerning the thermal distribution of pairs produced in an external field and in connexion with the analysis of the mean energy emitted by a uniformly accelerated detector we would like to mention the work of Nikishov[Ni] (see also Myrhvold[My]) who anticipated the Unruh effect by an analysis of the photons emitted by accelerated electrons.

As mentioned in footnote 10 (pages 153 and 154), upon taking into account recoil effects the properties of the emitted fluxes are drastically modified. One can show that the correlations which encode the weak values are also modified. The result is that due to recoil effects the weak value Eq. (2.149) vanishes on $\mathcal{I}^-$. Upon taking into account quantum gravitational effects in the black hole problem (see the end of Section 3.5), such recoil effects may play an important rôle and could for instance modify the properties of the weak values. Indeed we emphasize that the peculiar properties of $\langle T_{vv} \rangle_{\psi_i}$ (Eq. (3.106)) result from the (unjustified) assumptions of a free field theory evolving in a given classical background geometry. We hope that the study of the weak value $\langle T_{vv} \rangle_{\psi_i}$ can be used to investigate the validity of both assumptions.

In order to understand the rôle of the transplanckian frequencies that are involved in the emergence of Hawking quanta, Unruh showed in a recent paper[93] (see footnote 13, page 201), through a numerical analysis that Hawking radiation is unaffected by a truncation of the free field spectrum at the Planck scale. We have investigated and extended his result in Ref. [BMPS95] where we show analytically how the appeal to transplanckian frequencies can be avoided whilst retaining the thermal spectrum of emitted particles.

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