ELEMENTARY MECHANICS FOR ENGINEERS

MILLS

A SHORT COURSE
IN
Elementary Mechanics
FOR
Engineers

CLIFFORD NEWTON MILLS, B.S., A.M.

Assistant Professor of Mathematics, South Dakota State College
of Agriculture and Mechanic Arts

36 ILLUSTRATIONS

NEW YORK

D. VAN NOSTRAND COMPANY

25 Park Place

1916

PRESS OF
BRAUNWORTH & CO.
BOOKBINDERS AND PRINTERS
BROOKLYN, N. Y.

PREFACE

This course in Elementary Mechanics is arranged for students who have previously studied Trigonometry. It is intended as a basis for a semester's work of three hours per week.

The subject matter is divided into three parts, namely, Kinematics, Kinetics, and Statics.

Much detailed discussion is omitted, and the course is written from the "definition viewpoint." In considering the working principles of Statics, much emphasis is placed on the importance of "moment equations." The Engineer's system of units is used.

Throughout the text are many problems, a large number of these being original. In fact, it is by solving problems that a mastery of the subject is gained. Many of the problems will not be difficult to solve, but others will require the student's best effort.

The author desires to express his thanks to Professors H. C. Solberg, Head of the Department of Mechanical Engineering, H. B. Mathews, Head of the Department of Physics, and G. L. Brown, Head of the Department of Mathematics, for numerous and valuable suggestions.

CLIFFORD N. MILLS.

SOUTH DAKOTA STATE COLLEGE,
August, 1916

CONTENTS

INTRODUCTION

CHAPTER I

KINEMATICS

Linear and Angular Motion

CHAPTER II

VECTORS

CHAPTER III

KINETICS

CHAPTER IV

STATICS

Composition and Resolution of Forces

CHAPTER V

MOTION IN A CIRCLE

CHAPTER VI

ENERGY OF ROTATION

Moments of Inertia

INTRODUCTION

1. Mechanics is a branch of Physics which treats of the motions of bodies and the causes of the changes of these motions. It is divided into two main subjects, namely, Kinematics and Dynamics. In this text the subject of Dynamics is divided into two parts called Kinetics and Statics.

Kinematics (Greek, *kinema*, motion) is the study of motion apart from matter. It considers how a body moves, and is not concerned with any of the properties of the body which moves. It does not deal with the forces which cause motion. It is strictly a geometrical science of motion.

Dynamics (Greek, *dynamis*, force) treats of the effect of forces acting upon bodies whether in a state of rest or motion.

Kinetics (Greek, *kinetikos*, to put in motion) considers the forces which produce motion. It considers why a body moves, and discusses the relations between the motion produced and the forces involved.

Statics (Greek, *statikos*, to be at rest) considers exclusively the conditions under which a body under the action of forces will remain at rest. In this case the forces acting on the body are said to be in equilibrium.

2. A physical quantity is anything that can be measured. There are three fundamental units of measurement, length, mass, and time. Any physical quantity can be expressed in terms of one or more of these so-named fundamental units.

The object that moves is called matter. It has never been defined, but we do know many properties concerning it. A limited portion of matter is called a body. A material particle is an ideal body whose dimensions, length, breadth, and thickness, are extremely small in comparison with some special unit. For instance, the dimensions of the earth and planets are very small in comparison with the distance of the earth from the sun. Therefore in the study of Astronomy the earth and planets many times are considered as material particles.

The only property inherent in matter which affects the motion of bodies is called Inertia, which is defined as the reaction of matter against a change of motion or a state of rest.

3. Motion. A displacement is a change of position without reference to time. If we take into consideration the time taken to make the change of position then it is called Motion.

If a body moves in space, its path is a line, either straight or curved. If all material particles which make up a body describe parallel lines, not circular, the motion is said to be Translatory. Rectilinear motion is a special case of translatory motion when the lines are straight.

If all particles in a moving body describe circular paths, or concentric circles, about a fixed point called the axial point, which in this case has no motion, the motion is called Angular or Rotary.

A body is said to have relative motion with respect to another body when it is continually changing its position with respect to that body. In the case of a moving train, we always consider its motion relative to some part of the earth's surface, or objects on the earth's surface. A passenger on a train may be at rest relative to the train, but he would be in motion relative to the trees, houses, etc., which the train passes.

GREEK ALPHABET

Letters.	Name.	Letters.	Name.
Α α	Alpha	Ν ν	Nu
Β β	Bēta	Ξ ξ	Xi
Γ γ	Gamma	Ο ο	Omicron
Δ δ	Delta	Π π	Pi
Ε ε	Epsīlon	Ρ ρ	Rho
Ζ ζ	Zēta	Σ σ ς	Sigma
Η η	Eta	Τ τ	Tau
Θ θ	Thēta	Υ υ	Upsīlon
Ι ι	Iōta	Φ φ	Phi
Κ κ	Kappa	Χ χ	Chi
Λ λ	Lambda	Ψ ψ	Psi
Μ μ	Mu	Ω ω	Omega

ELEMENTARY MECHANICS FOR ENGINEERS

CHAPTER I

KINEMATICS

Linear and Angular Motion

4. Velocity. The velocity of a moving body is the rate at which it changes its position. A velocity is completely determined when its direction of motion and the rate of change of position are given. If the rate of change of position is given, not mentioning the direction of motion, then the numerical value of the rate of change is called the speed of the moving body. Velocity may be uniform or variable. In this text, however, we will consider only uniform and uniformly accelerated velocity.

If a body moves in the same direction and passes over equal distances in equal intervals of time, however small these may be, its velocity is said to be uniform. The phrase " equal intervals of time " is very important in uniform motion. A body may describe equal distances in equal times and yet its motion may not be uniform, hence the clause " however small these may be " is very necessary in the definition. For instance, a train may describe 40 miles in each of two consecutive hours, and yet its motion may have varied during each hour. Therefore uniform motion implies a constant rate of change of position.

If s is the distance described with a uniform velocity of v units of length per second in t seconds, then

$$s=vt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

5. Average Velocity. The average velocity of a moving body is the number of units of length described divided by the time taken. This is the same as the constant velocity which the moving body must have in order to describe the same distance in the same time. If a body moves with different velocities in the same equal times, and the difference between consecutive velocities is constant, then

$$\text{average velocity}=\frac{v_1+v_2+v_3+\ .\ .\ .\ +v_n}{n}, \quad (2)$$

where each v denotes a different velocity. The numerator of the fraction is an arithmetical series, and its sum is $(n/2)\ (v_1+v_n)$. Hence, if the velocity is uniformly accelerated, then

$$\text{average velocity}=\frac{v_1+v_n}{2}, \quad . \quad . \quad . \quad (3)$$

where v_1 and v_n denote the initial and final velocities. Hereafter v, v_1, and v_2, will denote average velocity, initial velocity, and final velocity.

6. Acceleration. The acceleration of a moving body is the rate of change of its velocity. Acceleration may be positive or negative; if negative it is called retardation. Hereafter acceleration will be denoted by the letter a.

Acceleration is said to be uniform when equal changes of velocity take place in equal intervals of time, however small these may. In uniformly accelerated motion the average acceleration equals the increase in velocity divided by the time taken to increase, or

$$\text{average acceleration} = \frac{v_2 - v_1}{t}. \quad . \quad . \quad (4)$$

7. From the definitions of uniform and uniformly accelerated velocity one can establish the following formulas:

$$v_2 = v_1 + at, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$s = \tfrac{1}{2}(v_1 + v_2)t = vt, \quad . \quad . \quad . \quad (6)$$

$$s = v_1 t + \tfrac{1}{2}at^2, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$2as = v_2^2 - v_1^2. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the initial velocity is zero then v_1 of the formulas becomes zero.

EXAMPLES I

1. Compare the velocities of two bodies which move uniformly, one through 5 feet in half a second, and the other through 50 yards in one minute.

2. Compare the velocities of two bodies which move uniformly, one through 180 feet in one-quarter minute,

and the other through $1\frac{3}{4}$ yards in three-eighths of a second.

3. A body has described 50 feet from rest in two seconds with uniform acceleration; find the velocity acquired.

4. A railway train travels 162 miles in three hours; find the average velocity in feet per second.

5. A particle starts from rest with a constant acceleration of 20 feet per second per second. Determine the distance passed over in the fourth and sixth seconds, and the total distance passed over in twelve seconds.

6. A train attains a velocity of 54 miles per hour in $4\frac{1}{2}$ minutes after starting from rest. Find the average acceleration.

7. A car, moving at a velocity of 28 miles per hour, is subjected to a uniform retardation of $7\frac{1}{2}$ feet per second per second by the action of the brakes. How long will it take the car to come to rest, and how far will it travel during that time?

8. A particle starting from rest describes 63 feet in the fourth second. Find the acceleration.

9. A particle describes 72 feet, while its velocity increases from 16 to 20 feet per second. Find the whole space described from rest, and the acceleration.

10. A particle, in passing over 9 feet, has its velocity increased from 4 to 5 feet per second. Find the whole space described from rest, and the acceleration.

11. The velocity of a particle changes from 10 to 25 feet per second in three seconds. What is the acceleration? When will its velocity be 75 feet per second? What is the total time of motion from rest? What space will it have passed over?

12. A particle is found to be moving at the end of

ten seconds with a velocity which, if continued uniformly, would carry it through 45 miles the next hour. Find the acceleration.

13. The velocity of a particle changes from 30 to 20 feet per second in passing over 25 feet. What is the retardation? How long will it be before coming to rest, and what distance will it pass over in that time?

14. With what acceleration and how far must a particle move to have a velocity of 30 miles per hour in half a minute after starting from rest? What retardation would destroy this motion in ten seconds? How far will the particle have moved in the given time?

15. A particle moving with a velocity of $58\frac{2}{3}$ feet per second is retarded and brought to rest in 500 feet. What is the retardation?

16. A car moves with a velocity of 60 miles per hour. It is brought to rest in three minutes. Find the retardation, and the space traversed before the car comes to rest.

17. In the eleventh and fifteenth seconds from the beginning a particle moves through 24 and 32 feet respectively. Find the initial velocity, and the acceleration.

18. A particle describes 81 feet in three seconds. The acceleration then ceases, and during the next three seconds it travels 72 feet. Find the initial velocity, and the acceleration.

19. A car moves with an acceleration of 1 foot per second per second. In what time will the car acquire a velocity of 30 miles per hour, if the initial velocity is 240 feet per minute?

20. A particle starting from rest describes 171 feet in the tenth second. What is the acceleration?

8. Angular or Rotary Motion. Angular motion is of the greatest importance in the study of pulleys, shafts, and flywheels. There are two kinds of rotary motion which we will consider, namely: uniform and uniformly accelerated.

If any point in a rotating body, except points which are on the axis of rotation, describes equal central angles in equal intervals of time, however small these may be, the motion is uniform. If the point has its angular velocity increased or decreased by the same amount during consecutive intervals of time, however small these may be, the motion is uniformly accelerated.

9. Angular Measurement: Radian. In angular motion the angle described by any point in a rotating body can be measured by the number of revolutions the body makes, or by a unit called the Radian.

In circular measure, any arc equals the radius of the circle multiplied by the subtended central angle, or

$$\text{angle subtended} = \frac{\text{arc}}{\text{radius}}. \quad . \quad . \quad . \quad (9)$$

If the arc is equal in length to the radius, then the value of the subtended angle is unity, and in this case the angle is called a Radian. In Fig. 1 the angle AOB is one radian if the arc AB equals in length the radius OA or r.

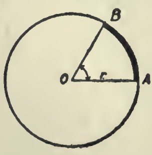


FIG. 1.

The circumference of any circle equals $2\pi r$. Hence it follows that there are 2π

radians in 360 degrees. Therefore one radian equals approximately 57.3 degrees. ($\pi = 3.1416$.)

10. Angular Velocity. Suppose a circular plate turns about an axis which is perpendicular to its plane and passes through its center O (Fig. 2). P is any point in the plate, except the axial point O. The straight line OP will turn through so many degrees in a given time. Hence the angular velocity of a particle rotating in a circle equals the angle described divided by the time required, or

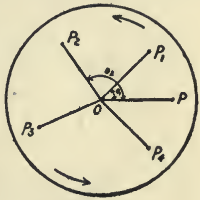


FIG. 2.

$$\text{angular velocity} = \frac{\theta}{t}, \quad . \quad . \quad . \quad . \quad (10)$$

where θ is the angle described. Since the second is usually taken as the unit of time, the angular velocity is expressed in terms of so many radians per second. For instance, if a flywheel makes 480 revolutions per minute, its angular velocity is 16π radians per second. If the wheel has uniform rotation then the 16π radians is the actual angular velocity for any second. If the wheel rotates with different speeds then the 16π radians is the average angular velocity. Angular velocity will be denoted by the letter ω. If the angular velocity is uniformly accelerated then

$$\text{average angular velocity} = \frac{\omega_1 + \omega_2}{2}. \quad . \quad (11)$$

If the angular velocity of a rotating wheel for the first second is 2π radians, for the second second 4π radians, for the third second 6π radians, and so on, then at the end of the tenth second the wheel is making 10 revolutions per second and its angular velocity is 20π radians per second. From the beginning the wheel has made a total of 55 revolutions. The average angular velocity equals $\frac{1}{2}(2\pi+20\pi)$ radians per second, or 11π radians per second. Any point in the wheel, except the axial point, has described an angular distance of 110π radians.

11. Relation between Linear and Angular Velocity. In Fig. 3, θ is the angle described in a circle of radius r in t seconds. Then the angular velocity of the particle moving from A to B is

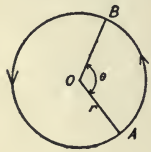


FIG. 3.

$$\omega=\frac{\theta}{t}. \quad . \quad . \quad . \quad . \quad (12)$$

But
$$\theta=\frac{\text{arc } AB}{r}=\frac{\text{linear velocity times } t}{r}. \quad . \quad (13)$$

Therefore
$$\omega=\frac{\text{linear velocity}}{r}, \quad . \quad . \quad . \quad . \quad (14)$$

or

$$\omega=\frac{v}{r}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

12. Angular Acceleration. Angular acceleration is the rate of change of angular velocity and is expressed in radians per second per second. Average angular acceleration is equal to the gain in angular velocity divided by the time taken to increase or decrease, which may be expressed as follows:

$$\text{average angular acceleration} = \frac{\omega_2 - \omega_1}{t}. \quad . \quad (16)$$

Denoting angular acceleration by the Greek letter α, formula (16) becomes

$$\alpha = \frac{\omega_2 - \omega_1}{t}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

13. Similar formulas as established in section 7 for linear motion can be established for angular motion.

$$\omega_2 = \omega_1 + \alpha t, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\theta = \tfrac{1}{2}(\omega_1 + \omega_2)t, \quad . \quad . \quad . \quad . \quad (19)$$

$$\theta = \omega_1 t + \tfrac{1}{2}\alpha t^2, \quad . \quad . \quad . \quad . \quad (20)$$

$$2\alpha\theta = \omega_2^2 - \omega_1^2. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If the initial angular velocity is zero, then ω_1 of the above formulas becomes zero.

EXAMPLES II

1. A particle describes a circle of radius 5 feet with a uniform linear velocity of 8 feet per second. Find the angular velocity.

2. A wheel turns about its center, making 200 revolutions per minute. What is its angular velocity?

3. If the minute hand of a clock is 2 feet long, find the angular velocity and the linear velocity of the end of the hand.

4. The lengths of the hour, minute and second hands of a watch are .48, .8 and .24 inch respectively. Find

the ratios of the angular velocities; also of the linear velocities of the ends of the hands.

5. A railway car moves with a speed of 45 miles per hour, and the diameter of the wheels is 3 feet. Find the angular velocity of the wheel.

6. The linear velocity of a point on a rotating body is 72 feet per second, and its distance from the axis of rotation is 6 feet. How long will it take the point to move 60 radians?

7. A flywheel rotates initially 3 times per second, and after twenty seconds it makes 140 revolutions per minute. How many revolutions will the wheel make, and what time will elapse before stopping, if the retardation is uniform?

8. A car is moving with a speed of $\frac{1}{2}$ mile per minute, and the wheels are $2\frac{1}{2}$ feet in diameter. Find the angular velocity of the wheels. If the car comes to rest in 300 feet under uniform retardation, find the angular retardation.

9. The path of the earth in going around the sun is an ellipse. Assume the path to be a circle of radius 93,000,000 miles. If the time of revolution is taken as 365 days, find the angular and linear velocity.

10. P is a point in a body turning about a fixed axis, and PN is a line drawn from P at right angles to the axis, N being the axial point. If PN describes an angle of 375 degrees in three seconds, what is the angular velocity of the point? If PN is 6 feet in length, what is the linear velocity of the point?

11. A point is describing a circle of radius 42 feet, with a uniform velocity of 12 feet per second. Find the change in its velocity after it has described one-sixth of the circumference.

12. A flywheel is making x revolutions per second, and after three-fifteenths second it is making 24 revolutions per second. If the wheel comes to rest after making a total of $158\frac{7}{16}$ revolutions, find the value of x and the retardation.

13. A flywheel is making $12\frac{1}{8}$ revolutions per second, and after seventeen seconds it is making x revolutions per second. If the wheel comes to rest after ninety-seven seconds, find the value of x, the retardation, and the total number of revolutions the wheel makes before coming to rest.

14. If a flywheel of radius 6 feet makes 30 revolutions per minute, what is its angular velocity, and what is the linear velocity of a point on its rim? What uniform retardation will bring the wheel to rest in 20 seconds?

15. A pulley 5 feet in diameter is driven by a belt travelling 500 feet a minute. Neglecting the slipping of the belt, find (*a*) the angular velocity of the pulley, (*b*) its number of revolutions per minute, and (*c*) the number of revolutions the pulley makes in 3π minutes.

14. Motion Due to Gravity. The formulas found in section 7 for linear motion may be used to determine motion due to gravity if we take the constant of acceleration as g, where g is approximately 32 feet per second per second.

If a body falls from rest the initial velocity is zero, and the space travelled equals

$$s=\frac{1}{2}gt^2. \qquad (22)$$

If a body is thrown downward with an initial velocity v_1 the space travelled equals

$$s=v_1t+\frac{1}{2}gt^2 \qquad (23)$$

If a body is thrown upward with an initial velocity v_1 the space travelled equals

$$s=v_1t-\tfrac{1}{2}gt^2. \quad \ldots \ldots \quad (24)$$

From formula (8) we get

$$2gs=v_2^2-v_1^2. \quad \ldots \ldots \quad (25)$$

The time to the greatest height equals

$$t=\frac{v_1}{g}. \quad \ldots \ldots \quad (26)$$

The greatest height equals

$$h=\frac{v_1^2}{2g}. \quad \ldots \ldots \quad (27)$$

The distance travelled during any second equals

$$d=v_1\pm\tfrac{1}{2}g(2t-1) \quad \ldots \ldots \quad (28)$$

EXAMPLES III

1. A body falls from rest. Find (1) the distance fallen in ten seconds, (2) time to fall 10 feet, and (3) the initial velocity if the body should describe 1000 feet in five seconds.

2. A body falls from rest, and during the tth second passes over 144 feet. Find the total distance fallen in the t seconds.

3. How long will it take for a stone to drop to the bottom of a well 144 feet deep?

4. A stone is dropped from a tower 128 feet high, and at the same instant another stone is projected upward from the ground. If they meet half way up the tower, find the velocity of projection of the second stone.

5. A stone is dropped into a well, and after four seconds the report of its striking the water is heard. What is the depth to the water? Take the velocity of sound as 1190 feet per second.

6. A stone is thrown downward with a velocity of 96 feet per second and reaches the bottom of a well in three seconds. What is the depth of the well?

7. The greatest height attained by a body projected upward is 400 feet. Find the initial velocity and the time the body is in the air.

8. A ball is thrown upward with a velocity of 100 feet per second. Find how high the ball will rise. At what time will the ball be 50 feet from the ground?

9. A ball is thrown upward with a velocity of 75 feet per second. When will the velocity be 22 feet per second and at what height will the ball be?

10. A stone is thrown upward and during the fifth second travels 16 feet. Find the initial velocity and the space travelled over in five seconds.

11. A bullet shot upward passes a certain point at the velocity of 400 feet per second. Find the time when the bullet will be at a point 1600 feet higher.

12. The surface of the water in a well is 256 feet below the curb. If a stone is dropped into the well, how many seconds will it be until the splash is heard?

13. A body falls from rest and during the tth second falls 176 feet. How long had it been falling before the beginning of the tth second?

14. A ball is thrown upward with a velocity of 160 feet per second. At what time will it be 256 feet high, and what is the velocity at this height?

15. A ball is thrown upward with a velocity of 100 feet per second. Find the time occupied by it in its

ascent in describing that portion of its path which lies between the heights of 60 and 120 feet above the point of projection.

16. A ball is thrown upward with a velocity of 128 feet per second. What is the greatest height attained, and the time required to reach the highest point?

17. Acceleration of gravity on the moon is assumed to be $\frac{1}{6}$ the acceleration on the surface of the earth. How high will a body rise if it is thrown upward from the surface of the moon with a velocity of 40 feet per second?

18. The top girder of a building is x feet above the pavement. A man stands on the girder and throws a stone 144 feet upward. The stone hits the pavement seven seconds after the time of projection. Find the height of the girder above the pavement.

19. A ball is thrown downward and describes 720 feet in t seconds, and 2240 feet in $2t$ seconds. Find the time and the velocity of projection.

20. A body falls freely from the top of a tower, and during the last second it falls $\frac{144}{169}$ of the whole distance fallen. Find the height of the tower.

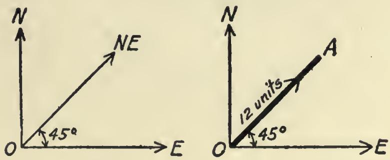
21. A ball is thrown upward with a velocity of $64\sqrt{2}$ feet per second, and when it has attained half its greatest height another ball is thrown upward with the same velocity from the same point. Determine when and where they will meet.

CHAPTER II

VECTORS

15. A vector is a straight line having definite length, direction, and sense. The "sense" of a line determines whether the direction is to the right or left. Many physical quantities can be expressed by a number denoting so many units. Such quantities are called scalar quantities. But many other quantities need to be specified more definitely than by so many units. For instance, velocity and acceleration of a moving body, action of a push or pull; each has direction as well as magnitude. Such quantities are called vector quantities, and may be represented by straight lines defined as vectors.

If a boat moves northeast with a speed of 12 miles



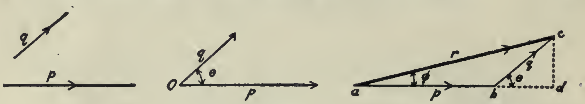
Space Diagram Vector Diagram

FIG. 4.

per hour its motion can be represented by a vector (Fig. 4).

In the space diagram the line *OB* indicates only the direction of motion, while in the vector diagram the line OA represents both the direction of motion and the magnitude of the speed.

16. Addition and Subtraction of Vectors. To find



Space Diagram Vector Diagram

FIG. 5.

the sum of two vectors p and q (Fig. 5) draw ab equal to vector p, and from the end b draw bc equal in length and parallel to vector q; join ac. Then ac is the geometric or vector sum of the vectors p and q. We may write the equation

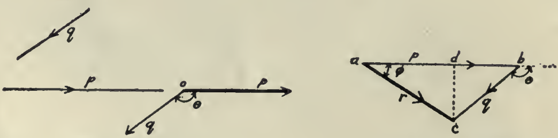
$$\text{vector } p + \text{vector } q = \text{vector } r, \quad . \quad . \quad . \quad (29)$$

or

$$p + q = r. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The value of r may be calculated as follows: from the point c drop a perpendicular cd upon ab produced. Hence $\overline{ac}^2 = \overline{ad}^2 + \overline{dc}^2$, or

$$r^2 = p^2 + q^2 + 2pq \cos \theta, \quad . \quad . \quad . \quad (31)$$



Space Diagram Vector Diagram

FIG. 6.

θ being the angle between the directions of p and q. It is evident that the formula (31) covers all possible cases arising from different values of θ. Thus, when $\theta = 0°, r = p + q$; $\theta = 90°, r^2 = p^2 + q^2$; $\theta = 180°, r = p - q$.

To subtract vector q from vector p let vector q have opposite sense (Fig. 6).

17. Resultant Velocity. A single velocity which is equivalent to two simultaneous velocities is called their resultant, and these two simultaneous velocities are called the components.

If two component velocities be represented, in magnitude and direction, by two adjacent sides, AB, AC, of a parallelogram, then their resultant velocity will be represented by the diagonal AD (Fig. 7). By the use of formula (31) we can find the value of the resultant.

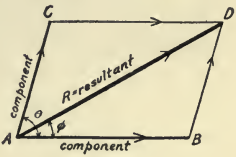
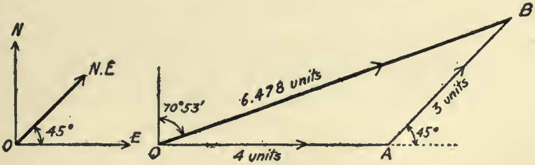


FIG. 7.

If a man walks due northeast at a speed of 3 miles per hour across the deck of a vessel going east at 4 miles per hour, the resultant velocity with which



Space Diagram

Vector Diagram

FIG. 8.

the man is moving over the water is 6.478 miles per hour, in the direction N. 70°53′ E. (see Fig. 8).

18. Rectangular Component Velocities. A velocity may be considered as the sum of any number of component velocities. Given the direction and magnitude of a velocity to find the two components of this velocity having directions at right angles to each other. Take two lines XX' and YY', at right

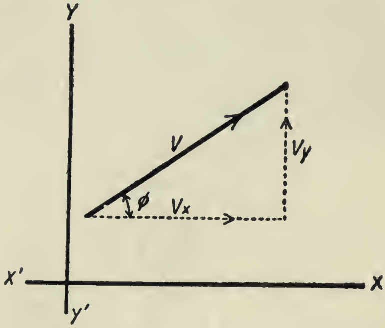


FIG. 9.

angles to each other, as the axes of reference (Fig. 9). Let V denote the given velocity, and V_x, V_y the horizontal and vertical velocities. The component velocity

$$V_x = V \cos \phi, \quad \ldots \ldots \quad (32)$$

and the component velocity

$$V_y = V \sin \phi. \quad \ldots \ldots \quad (33)$$

It is easily shown that the resultant velocity of any number of simultaneous velocities is expressed by the formula

$$R^2 = X^2 + Y^2, \quad \ldots \ldots \quad (34)$$

where

$$X = v_1 \cos \alpha + v_2 \cos \beta + \ldots + v_n \cos \gamma, \quad (35)$$

and

$$Y = v_1 \sin \alpha + v_2 \sin \beta + \ldots + v_n \sin \gamma. \quad (36)$$

Also the angle ϕ which the resultant R makes with the X axis can be found by the formula

$$\tan \phi = \frac{Y}{X}. \qquad (37)$$

EXAMPLES IV

1. Add the vectors (10°, 15); (0°, 20).

2. Subtract the vector (45°, 12) from the vector (60°, 18).

3. Add the vectors (0°, 12); (45°, 8); (135°, 10) and (225°, 4).

4. A steamer is headed due north at 12 miles per hour. The current runs northwest at 4 miles per hour. Find the true velocity of the steamer in magnitude and direction.

5. A steamer is going northeast at the rate of 24 miles per hour. Find the northern and eastern component.

6. A stone is projected at an angle of 45° to the horizontal with a velocity of 600 feet per second. Find the horizontal and vertical component velocities at point of projection.

7. In order to cross a river at right angles, the river flowing with a uniform velocity of $2\frac{1}{2}$ miles per hour, in what direction should a swimmer head if his velocity in still water is 5 miles per hour, and how long will it take him to cross, if the river is 200 yards wide?

8. A body is moving in a straight line with a velocity of 20 feet per second; find the component of its velocity in a direction at an angle of 60° to its direction of motion.

9. A vessel is steaming in a direction due north across a current running due west. At the end of two hours the

vessel has made 22 miles in a direction 35° west of north. Find the velocity of the current, and the rate at which the vessel is steaming.

10. A man wishes to cross a river to an exactly opposite point on the other bank. He can pull the boat with three times the velocity of the current. Find the inclination to the current at which he must keep the boat pointed.

11. A certain point possesses velocities represented by 3, 19 and 9 inclined at angles of 120° to one another. Find the resultant velocity in magnitude and direction.

12. A vessel is headed due north at the rate of 12 feet per second; the current flows east with a velocity of 5 feet per second, and a sailor is climbing a vertical pole at the rate of 1 foot per second. Find the velocity and the direction of the sailor in space.

13. An airship has a velocity of 40 miles per hour heading due south. The wind is blowing southeast with a velocity of 30 miles per hour. Find how far the airship travels in one hour and its direction.

14. In problem (13) suppose the wind to be blowing northeast instead of southeast at the same velocity. Find the distance travelled in one hour and its direction.

15. Add the vectors (30°, 15); (60°, −15); (135°, 10).

16. Subtract the vector (10°, 14) from the vector (225°, 14).

17. A weather vane on a ship's mast points northeast when the ship is headed due east at 16 miles per hour. If the velocity of the wind is 20 miles per hour, what is the true direction of the ship?

18. Two vessels are 10 miles apart at points *A* and *B*. The vessel at *A* is headed N. 30° E. and is steaming with a velocity of 12 miles per hour. Find in what direction the vessel at *B* must head at the same moment, if it has

a velocity of 15 miles per hour, in order to come in collision with the first vessel. Find also at what angle it will strike the first vessel and the time of travel.

19. Motion on a Smooth Inclined Plane. Let AB (Fig. 10) be the vertical section of a smooth plane making a given angle θ with the horizontal. Let P be the particle which moves on the plane.

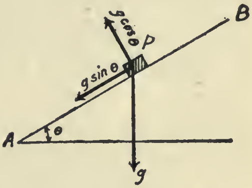


FIG. 10.

The downward acceleration of the moving particle due to gravity is g. Resolve this acceleration into its components, and it is evident that the acceleration along the plane is $g \sin \theta$. Hence in the formulas for a falling body, we need only substitute $g \sin \theta$ for g.

EXAMPLES V

1. Show that the velocity acquired by a body in sliding from rest down a smooth plane of length k is the same as that acquired by a body in falling freely through the vertical height which the body is above the horizontal.

2. A particle slides from rest down a smooth inclined plane which is 20 feet long and 14 feet high. What is its velocity when it reaches the ground, and how long does the sliding take?

3. A particle sliding down a plane 24 feet long acquires a velocity of 12 feet per second. Find the inclination of the plane.

4. A body slides from rest down a given plane. Com-

pare the times of describing the first and last thirds of its length.

5. Show that the horizontal velocity of a body moving down a plane inclined at an angle θ to the horizontal is $gt \sin\theta \cos\theta$, and the vertical velocity is $gt \sin^2\theta$.

6. A body slides from rest down a plane inclined at an angle $\theta = \tan^{-1}\frac{3}{4}$ to the horizontal. Find (1) its velocity at the end of ten seconds; (2) the horizontal and vertical components of this velocity; (3) the distance travelled.

7. A railroad track is inclined at an angle $\theta = \tan^{-1}\frac{1}{100}$ to the horizontal. A car starts down the track; find how far the car will go in four minutes, and the velocity it acquires.

8. In problem 7 suppose the car has an initial velocity of 30 miles per hour. Find the distance travelled and the velocity acquired.

9. A smooth plane is inclined at an angle of 30° to the horizontal. A body is projected up the plane with an initial velocity of 160 feet per second. Find when it is 288 feet from the starting point.

10. A plane 18 feet long is inclined 15° to the horizontal. With what velocity must a body be projected in order to just reach the top?

11. A body is projected down a plane inclined 12° to the horizontal. If the body is projected with an initial velocity of 10 feet per second, how far will it go in four seconds?

12. Find the velocity with which a body should be projected down an inclined plane of length 20 feet, so that the time along the plane shall be equal to the time of falling vertically; the angle of inclination of the plane being 30°.

13. A plane 65 feet in length is inclined at an angle $\theta = \tan^{-1} \frac{5}{12}$ to the horizontal. A body slides down the plane, and then passes without loss of velocity onto the horizontal plane; after how long a time from the beginning of the motion will it be 200 feet from the foot of the plane?

14. A plane is 288 feet long and 64 feet high. Show how to divide the plane into three parts so that a ball may slide over the portions in equal times, and find these times.

15. A body slides down an inclined plane, and during the fifth second after starting passes over a distance of 72 feet. Find the inclination of the plane.

20. Projectiles. The path of a projectile undisturbed except by the action of gravity is a parabola. The pro-

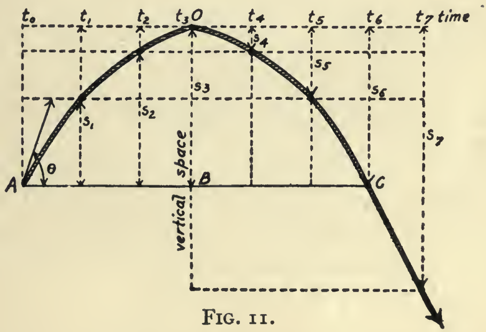


FIG. 11.

jectile is subject to two component motions, the horizontal velocity which is constant in both magnitude and direction, and the vertical uniformly accelerated velocity which is constant in direction but not in magnitude. (Fig. 11.)

In the preceding figure, θ is the angle of projection. AC is the horizontal range, O is the highest point of the path. The time it takes to travel from A to C is called the time of flight.

21. Velocity and Direction of Motion after a Given Time has Elapsed. Let u be the velocity, and B the angle which the direction of motion at the end of time t makes with the horizontal.

Now

$$u \cos B = \text{horizontal velocity at end of time } t, \quad (38)$$

$$= v_1 \cos \theta, \text{ the constant horizontal velocity.} \quad (39)$$

Again

$$u \sin B = \text{vertical velocity at end of time } t, \quad (40)$$

$$= v_1 \sin \theta - gt. \quad (41)$$

Hence,

$$u^2 = v_1^2 - 2v_1 gt \sin \theta + g^2 t^2, \quad (42)$$

and

$$\tan B = \frac{v_1 \sin \theta - gt}{v_1 \cos \theta}. \quad (43)$$

22. Velocity and Direction of Motion at a Given Height. The horizontal velocity is constant and equal to $u \cos B = v_1 \cos \theta$, and the vertical component is $u \sin B$. From (25), section 14,

$$u \sin B = \sqrt{v_1^2 \sin^2 \theta - 2gh}, \quad (44)$$

Hence

$$u^2 = v_1^2 - 2gh, \quad (45)$$

and

$$\tan B = \frac{\sqrt{v_1^2 \sin^2 \theta - 2gh}}{v_1 \cos \theta}. \quad (46)$$

$$\text{The greatest height} = \frac{v_1^2 \sin^2 \theta}{2g} \quad . \quad . \quad (47a)$$

$$\text{The horizontal range} = \frac{2v_1^2 \sin \theta \cos \theta}{g} = \frac{v_1^2 \sin 2\theta}{g}. \quad (47b)$$

$$\text{The time of flight} = \frac{2v_1 \sin \theta}{g} \quad . \quad . \quad (47c)$$

The maximum range is easily seen to be $\frac{v_1^2}{g}$, since $\sin 2\theta$ is greatest when $\theta = 45°$.

Suppose the angle of projection is $\frac{\pi}{2} - \theta$. Then

$$\text{Range} = \frac{v_1^2 \sin 2\left(\frac{\pi}{2} - \theta\right)}{g} = \frac{v_1^2 \sin (\pi - 2\theta)}{g} = \frac{v_1^2 \sin 2\theta}{g}$$

which shows that the same horizontal range is given for θ and $\frac{\pi}{2} - \theta$.

All of the preceding discussion has been entirely theoretical, and of no practical use, since the resistance of the air plays a very important part in the study of projectiles.

EXAMPLES VI

1. A bullet is projected with a velocity of 500 feet per second at an angle of 40° with the horizontal. Find (1) the greatest height attained, (2) the range and time of flight, (3) the velocity and direction of motion of the bullet when it is at a height of 400 feet.

2. A ball is thrown with a velocity of 100 feet per second; find the maximum range, and the two direc-

tions in which the ball may be thrown so as to give a range of 144 feet. Also the time of flight for the maximum range.

3. A ball is thrown horizontally from the top of a tower 100 feet high, with a velocity of 144 feet per second. Find (1) the time of flight, (2) distance from the foot of the tower of the point at which the ball hits the ground, and (3) its velocity when it hits the ground.

4. From the top of a tower 96 feet high, a ball is thrown with a velocity of $96\sqrt{3}$ feet per second, at an angle of 60° with the horizontal; find the distance from the foot of the tower of the point where the ball hits the ground.

5. What is the time of flight of a projectile, when the angle of projection is 45°, and the range is 6000 feet?

6. What is the range of a projectile when the angle of projection is 30° and the initial velocity is 200 feet per second?

7. The range of a projectile is 3250 feet and the angle of projection is 32°. What is the time of flight?

8. A bullet is projected from a gun and strikes the ground at a distance of 2000 feet. Find the initial velocity and the time of flight, if θ is 30°.

9. When the angle of elevation is 40° the range is 2449 feet. Find the range when $\theta = 29\frac{1}{2}°$.

10. If a man can throw a ball 90 yards, how long is it in the air, and to what height does it rise?

11. The velocity of a projectile when at its greatest height is $\sqrt{2/5}$ of its velocity when at half its greatest height. Find the angle of projection.

12. A ball is projected up an inclined plane of inclination 20°. The angle θ is 50°, and the initial velocity is 500 feet per second. Find the range on the plane and the time of flight.

CHAPTER III

KINETICS

23. Mass. The mass of a body is defined as the quantity of matter in the body. The mass of any body is determined by a beam scale. Such materials as coal, wood, iron, gold, etc., represent certain amounts of mass. With the exception of gases, mass is a visible and an invariable quantity.

24. Weight. All material particles of matter are attracted toward the center of the earth by a force called gravity. In any given body the lines of action of the forces acting on the particles which make up the body are apparently parallel to each other. The resultant of all the gravitational forces acting on any body is called the weight of the body. In other words, the force with which any body is attracted toward the center of the earth is the weight of the body. Weight is determined by a spring balance. Since the weight depends upon the distance of the body from the center of the earth, it is variable. It is also invisible.

25. Force. Force may be defined as that which changes or tends to change the state of rest or motion of a body. Forces are generally represented by a push or pull. If a force produces motion then there must be an acceleration of the body moved. Also the amount of acceleration produced depends upon the amount of mass to be moved. Hence the force varies as the product of the mass and the acceleration, or $F=kma$. If proper units of force, acceleration, and mass are chosen, then $k=1$ and

$$\text{Force}=\text{mass}\times\text{acceleration},$$

or

$$F=m\times a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The unit of force is the pound, which is that force which will accelerate a mass of one pound g feet per second per second.

Since acceleration is a vector quantity, and force is proportional to the acceleration, then force is a vector quantity, having both direction and magnitude.

26. Relation between Mass and Weight. Using the above definition of force one can write the following equation:

$$\text{Weight}=\text{mass}\times g, \quad . \quad . \quad . \quad . \quad . \quad (49)$$

or

$$\text{mass}=\frac{W}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

In engineering work, W/g is used to represent the mass of a body, W being given in pounds. For instance, a body weighing 100 pounds has a mass of 100/32 pounds.

27. Newton's Laws of Motion.

I. Every particle of matter continues in its state of rest or motion unless it is acted upon by some external force.

II. The acceleration of a body is proportional to the force applied, and takes place in the direction of the straight line in which the force acts.

III. To every action there is an equal and opposite reaction.

The first law refers to the property of matter, called inertia, to resist being put into motion, or to resist having its motion changed. The second law gives us a means to measure the accelerating force, as shown in the preceding section, that is, $F = \text{mass} \times \text{acceleration}$. The third law asserts that forces always occur in pairs, which is called the two-sided nature of every force. No force can be exerted unless there is some resistance to overcome. There will be no action unless there is something to act upon which will react.

The third law is quite misleading to beginners. If the law is true, let the student explain why it is that a wagon moves if pulled by a horse. In other words, explain the reaction.

28. Momentum and Impulse. The momentum of a moving body is defined as the product of the mass of the body and its velocity, or

$$\text{Momentum} = \frac{W}{g} \times v. \quad . \quad . \quad . \quad . \quad (51)$$

The impulse of a force is the product of the force and the time during which the force acts. The term "impulse" is usually considered only in cases where the force acts for a very small interval of time, as in the case of blasting with powder, collisions, and the firing of guns.

$$\text{Time average force} = \frac{\text{total impulse}}{\text{time required}}. \quad . \quad (52)$$

If a force F acts on a mass of W/g pounds for t seconds and causes its velocity to be changed from v_1 to v_2, the change in momentum during that time is $W/g\,(v_2-v_1)$. If the acceleration a is constant, then the change in velocity $(v_2-v_1)=at$. Hence

$$\frac{W}{g}(v_2-v_1)=\frac{W}{g}at \quad . \quad . \quad . \quad . \quad (53)$$

But

$$\frac{W}{g}a = \text{force } F, \quad . \quad . \quad . \quad . \quad . \quad (54)$$

therefore

$$\frac{W}{g}(v_2-v_1) = \text{force} \times \text{time}, \quad . \quad . \quad (55)$$

which means that the change in momentum is equal to the impulse.

EXAMPLE 1. A car weighing W pounds is set in motion by a uniform force of P_1 pounds and in t_1 seconds it attains a velocity of V feet per second. A retarding force of P_2 pounds brings the car to rest in t_2 seconds. Find the relation between P_1, P_2, and V.

Solution. During the time t_1 the gain in momentum is $\frac{W}{g}V$ units, and the impulse is P_1t_1, hence

$$P_1t_1=\frac{W}{g}V. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

During the time t_2 the gain in momentum is $-\frac{W}{g}V$ units, and the impulse is $-P_2t_2$, hence

$$P_2t_2=\frac{W}{g}V \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Therefore

$$P_1t_1=P_2t_2=\frac{P_1P_2}{P_1+P_2}(t_1+t_2). \quad . \quad . \quad (c)$$

(Multiply (a) by P_2, (b) by P_1, and add.)

Example 2. A car weighing 64 tons (2000 lbs.) is acted upon by a net force of 1 ton. How long will it take to attain a velocity of 30 miles per hour starting from rest, and how far will it have moved?

Solution. Let t be the time required in seconds, then the impulse is $2000t$ units.

Thirty miles per hour is 44 feet per second, so the gain in momentum is

$$\frac{W}{g}V,$$

or

$$\frac{64\cdot 2000}{32}\cdot 44,$$

therefore

$$2000t=\frac{64\cdot 2000}{32}\cdot 44.$$

Hence

$$t=88 \text{ seconds.}$$

The average speed is half the maximum, and the distance travelled in feet is

$$22 \cdot 88 = 1936 \text{ feet.}$$

EXAMPLE 3. A weight of 20 pounds is acted upon by a constant force which in five seconds produces a velocity of 15 feet per second. Find the force, if the weight starts from rest.

Solution. The acceleration is 3 feet per second per second, hence $F = \frac{20}{32} \cdot 3 = 1\frac{7}{8}$ pounds.

EXAMPLES VII

1. Of two bodies moving with constant velocities, one describes 36 miles in one hour and twenty minutes, the other 55 feet in $1\frac{1}{4}$ seconds; the former weighs 50 pounds, and the latter 72 pounds. Compare their momenta.

2. How far will a cannon ball weighing 10 pounds travel in one minute, supposing it to possess the same momentum as a rifle bullet of 2 ounces moving with a velocity of 1000 feet per second?

3. Calculate the momentum of a hammer of 5 tons, let fall vertically half a foot.

4. A body of snow 28 pounds in weight falls from the roof of a house to the ground, a distance of 40 feet. What is the momentum?

5. Compare the amounts of momentum in a pillow of 20 pounds which has fallen vertically 1 foot and an ounce bullet moving at 200 feet per second.

6. What force will in one minute give a weight of 1 ton a velocity of 10 miles per hour?

7. How long must a force of 20 pounds act on a weight of 512 tons to give it a velocity of 1 foot per second?

8. The moving force is 3 pounds and the weight acted upon is 10 pounds. Find the distance described in ten seconds.

9. A force of 2 pounds acts on a weight of 40 pounds for 30 seconds. Find the velocity acquired and the space described.

10. A hammer weighing 3 tons falls vertically 4 feet. What is the momentum? If the force of the blow is expended in .018 second, what is the average force of the blow?

11. A car weighing 100 tons has a constant resistance of 20 pounds per ton. What force will be required to give the car a velocity of 40 miles per hour in two minutes?

12. An engine exerts a constant draw-bar pull of 14 tons on a car weighing $10\sqrt{101}$ tons up an incline of 10 per cent grade, and the resistance of the rails, etc., amounts to 12 pounds per ton. How long will it take to attain a velocity of 30 miles per hour, and what distance will be travelled?

13. How long will it take the engine and car in problem 12 to go 2048 feet up the incline, starting from rest and coming to rest at the end without use of the brakes?

14. A car travelling 60 miles per hour is brought to rest by a uniform resisting force in 4096 feet. Find the total resisting force in pounds per ton.

15. A bullet weighing 1 ounce enters a piece of wood with a velocity of 2000 feet per second and penetrates it 6 inches. What is the average resistance of the wood in pounds to the penetration of the bullet?

16. A bullet weighing 1 ounce leaves the barrel of a

gun $3\frac{1}{2}$ feet long with a velocity of 1200 feet per second. What is the impulse of the force produced by the discharge? If it takes .003 second for the bullet to traverse the barrel, what is the average force exerted on it?

29. The third law of motion expresses the conservation of energy. For instance when a projectile is fired from a cannon, the impulse or change of momentum of the shot due to the explosion is of equal amount to that of the recoiling cannon in the opposite direction.

30. Motion of two weights connected by a light inextensible string passing over a small smooth peg or pulley. In Fig. 12 suppose W_2 is the greater of the two weights. The total accelerating force is $W_2 - W_1$, and the total mass is $\frac{W_1 + W_2}{g}$; hence

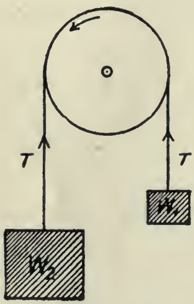


FIG. 12.

$$\text{acceleration } a = \frac{W_2 - W_1}{W_1 + W_2} g. \quad (56)$$

Consider each weight separately and the upward accelerating force on the right is $T - W_1$, T being the tension of the string throughout; on the left the downward acceleration is $W_2 - T$. Hence by (48), section 25,

$$W_2 - T = \frac{W_2}{g} a, \quad . \quad . \quad . \quad . \quad (57)$$

and

$$T-W_1=\frac{W_1}{g}a. \quad . \quad . \quad . \quad . \quad (58)$$

Hence, it follows that

$$T=\frac{2W_1W_2}{W_1+W_2}. \quad . \quad . \quad . \quad . \quad (59)$$

Also

$$\text{Velocity}=v=\frac{W_2-W_1}{W_2+W_1}gt, \quad . \quad . \quad . \quad (60)$$

and

$$\text{space}=\frac{W_2-W_1}{2(W_1+W_2)}gt^2. \quad . \quad . \quad . \quad (61)$$

31. Again, suppose W_1 to slide along a smooth horizontal plane or table. In Fig. 13 let W_2 be

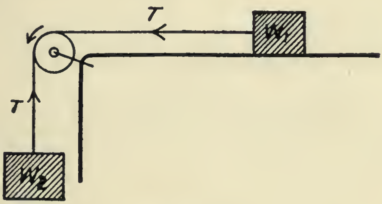


FIG. 13.

suspended by a string passing over the pulley on the edge of the table.

The accelerating force is W_2, and the total mass in motion is $\frac{W_1+W_2}{g}$, therefore

$$a=\frac{W_2}{W_1+W_2}g \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Now, since

$$a=\frac{\text{accelerating force on } W_1}{\text{mass of } W_1}=\frac{T}{W_1}g,$$

it follows that

$$T=\frac{W_1W_2}{W_1+W_2} \quad . \quad . \quad . \quad . \quad . \quad (63)$$

What would be the acceleration if W_1 were opposed by a given horizontal force F?

EXAMPLE I. A hammer weighing 10 pounds strikes a nail weighing 1 ounce with a velocity of 20 feet per second and does not rebound. The nail is driven into a block of wood which offers a uniform resistance of 400 pounds to the penetration of the nail. How far will the nail penetrate the wood?

Solution. Momentum of hammer before impact $=\frac{10}{32}\cdot 20=\frac{25}{4}$.

Momentum of hammer and nail after impact $=\frac{10+\frac{1}{16}}{32}\cdot V$, where V is the initial velocity after the impact.

Hence $\frac{25}{4}=\frac{161}{16\cdot 32}\cdot V$. Therefore V equals $\frac{3200}{161}$ feet per second. Let t be the time of penetration. Then

$$\text{Impulse } 400\, t=\frac{10}{32}\cdot 20.$$

Hence t equals $\frac{1}{64}$ second. The average velocity during the penetration is $\frac{1}{2}$ V or $\frac{1600}{161}$ feet per second.

The distance moved by the nail is

$$\frac{12}{1} \cdot \frac{1600}{161} \cdot \frac{1}{64} = 1.8 \text{ inches.}$$

EXAMPLE 2. A cannon weighing 40 tons fires a 1500-pound shell with a velocity of 1200 feet per second. With what initial velocity will the cannon recoil? If the recoil is overcome by an average force of 75 tons, how far will the cannon travel and how long will it take?

Solution. Momentum of projectile = momentum of the cannon.

$$\frac{1500}{32} \cdot 1200 = \frac{40 \cdot 2000}{32} \cdot V.$$

(V is the initial velocity of cannon.)

Therefore

$$V = \frac{1500 \cdot 1200}{40 \cdot 2000} = 22\tfrac{1}{2} \text{ feet per second.}$$

Let t be the time of the recoil.

Impulse of the retarding force = momentum of the shell.

$$75 \cdot 2000 \cdot t = \frac{1500 \cdot 1200}{32}.$$

Hence

$$t = \tfrac{3}{8} \text{ second.}$$

Therefore the distance moved by the cannon

$$= \frac{1}{2} \cdot \frac{45}{2} \cdot \frac{3}{8} = 4.22 \text{ feet.}$$

EXAMPLE 3. Two weights, W_2 of 24 pounds and W_1 of 18 pounds, are connected by a string passing over a smooth pulley. Find (1) tension of the string, (2) the acceleration and (3) the space travelled in 3 seconds.

Solution.

(1) $T = \frac{2 \cdot 24 \cdot 18}{24 + 18} = 20\frac{4}{7}$ pounds.

(2) $a = \frac{24 - 18}{24 + 18} \cdot 32 = 4\frac{4}{7}$ feet per (second)^2.

(3) space $= \frac{1}{2} \cdot \frac{24 - 18}{24 + 18} \cdot 32 \cdot 3^2 = 20\frac{4}{7}$ feet.

EXAMPLES VIII

1. A shot whose weight is 400 pounds is projected from a gun weighing 50 tons with a velocity of 1000 feet per second. Find the initial velocity of the cannon.

2. A bullet weighing 1 ounce is fired from a gun which weighs 10 pounds. The velocity of the bullet is 1000 feet per second. Find (1) the initial velocity of the gun, (2) the time of recoil, and (3) the space travelled by the gun, if the recoil is overcome by a time average force of 10 lbs.

3. A cannon weighing *a* tons fires a 1000-pound shell with a velocity of 1000 feet per second. The cannon recoiled with an initial velocity of 20 feet per second. What is the weight of the cannon?

4. A gun fires 480 bullets per minute, each bullet weighing 1 ounce and having a horizontal velocity of 1600 feet per second. Find the average force exerted on the gun.

5. A shot of 1000 pounds is fired from a 60-ton cannon with a velocity of 1800 feet per second. The recoil is

resisted by a constant pressure of 15 tons; through how many feet will the gun recoil?

6. A shot weighing 1000 pounds is fired with a velocity of 1600 feet per second from a cannon weighing 80 tons. What steady pressure will stop the cannon after a recoil of 4 feet?

7. A weight of 10 pounds falling freely a distance of 12 feet drives a nail, weighing 1 ounce, 3 inches into a block of wood. Find the uniform resistance.

8. A bullet strikes and becomes imbedded in a suspended target. The masses of the shot and target are as 1 : 100, and the target moves with a velocity of 10 feet per second as soon as the shot is imbedded. What is the initial velocity of the bullet?

9. Two weights of 3 pounds and 5 pounds are suspended by a string passing over a pulley. Find the acceleration, and the tension of the string.

10. An elevator car weighing 1 ton is made to ascend with a constant acceleration of 16 feet per second per second. What is the tension of the cable?

11. A man of weight 160 pounds stands on the platform of an elevator cage which has an upward acceleration of 2 feet per second per second. What pressure is exerted on the platform?

12. The weights at the extremities of a string passing over a pulley are 500 and 502 pounds. After 3 seconds of motion, 3 pounds are removed from the greater weight. What time will elapse before the weights are at rest again?

13. Two weights of 5 pounds and 4 pounds, together pull one weight of 7 pounds over a smooth pulley by means of a connecting string. After descending a given space the 4-pound weight is instantly taken off. Find through what space the remaining 5 pounds will descend.

14. A given weight of 500 pounds is placed on a smooth horizontal table and is pulled by a weight of 250 pounds which hangs vertically from the edge of the table. Find the acceleration, and the tension in the rope connecting them.

15. A weight of 60 pounds is resting on a smooth table and connected to a weight of 20 pounds hanging vertically from the edge of the table. What force must oppose the motion of the 60-pound weight in order to give an acceleration of $\frac{1}{8}g$?

16. A smooth inclined plane whose height is $\frac{1}{2}$ of its length has a small pulley at the top over which a string passes. To one end of the string is attached a weight of 12 pounds which rests on the plane, while from the other end, which hangs vertically, is suspended a weight of 8 pounds. Find the distance travelled by either weight in 5 seconds.

17. In example 16 what weight must be added to the 8-pound weight in order that a distance of 144 feet will be travelled in 6 seconds?

18. A body of weight 9 pounds is placed on a smooth table at a distance of 8 feet from its edge, and connected by a string passing over the edge, with a weight of 1 pound. Find (1) the acceleration, (2) the time that elapses before the body reaches the edge, and (3) its velocity upon leaving the table.

19. A body of weight 10 pounds is placed at the bottom of a plane, inclined 30° to the horizon and of length 12 feet, and is connected with a weight of 6 pounds by a string passing over the top of the plane. Find the acceleration, and the time it takes for the 10-pound weight to reach the top.

20. In problem 19 what weight attached to the vertical string will produce no motion?

32. Work. When a force acts upon a body and causes motion against resistance it is said to do work. Work is measured by the product of the force and the distance through which the body moves, or

$$W = Fs. \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

33. Unit of Work. Since work is measured by the product of the force and the distance, the unit is taken as that amount of work done by a unit force acting through a unit distance. The most generally used unit is the work done by a force of 1 pound acting through a distance of 1 foot, and it is called the foot-pound of work.

34. Power. Power is the rate of doing work. In most cases the horse-power* is the unit, and is equivalent to 33,000 foot-pounds per minute.

EXAMPLES IX

1. A man weighing 200 pounds walks to the top of a hill which is 3300 feet high. Find the work done against gravity.

2. The highest point attainable in the Washington monument is 500 feet above the base. How much work does a man do who weighs 165 pounds in going to the top?

3. A man can pump 25 gallons of water per minute to the height of 15 feet. How much work is he able to do in a 10-hour day. Assume 1 gallon of water to weigh 8 pounds.

4. 100 cubic feet of water is lifted to a height of 200

* Denoted by H.P.

feet during each minute. What must be the H.P. of an engine to do this work, if one-fourth of the power is lost through friction?

5. What is the H.P. of an engine capable of raising 10 tons of coal through a distance of 33 feet in 2 minutes?

6. The inclination of a mountain path to the horizon is 30°. How much work is done against gravity by a man of weight 150 pounds, in walking a mile along the path?

7. A tank of 22,500 gallons at an elevation of 66 feet is to be filled with water in 2 hours by a steam pump. Find the H.P. of the pump.

8. The efficiency of a given pump is 75 per cent. 52.8 cubic feet of water is raised to a height of 300 feet in one minute. What is the H.P. of the pump.

9. What power must be expended to pull at 15 miles per hour a 170-ton car up a plane inclined $\tan^{-1} \frac{13}{84}$ to the horizon?

10. What is the H.P. of an engine which keeps a train of 180 tons weight moving at a uniform rate of 60 miles per hour, the total resistance being 10 pounds per ton?

11. Two men put a car weighing 5 tons into motion by exerting on it a force of 80 pounds. The resistance being 10 pounds per ton, how far will the car have moved in one minute?

12. A train of weight 200 tons, is ascending an incline of which the inclination is $\sin^{-1} \frac{1}{200}$, and the total resistance is 20 pounds per ton. If the engine is of 240 H.P. and working at full power, find the speed at which the train is moving.

13. A stream delivers 6000 cubic feet of water per minute to the highest point of a water wheel 66 feet in diameter. If 80 per cent of the available work is use-

fully employed, what is the horse-power developed by the wheel?

14. A car of weight 200 tons is drawn up a plane, inclined at an angle $\sin^{-1} \frac{3}{500}$ to the horizon, at a rate of 30 miles per hour by an engine of 400 H.P. Find the total resistance per ton.

15. A stream of water is 5 feet wide and 3 feet deep, and flows $\frac{1}{4}$ mile per hour. The water falls 10 feet and in so doing passes over a wheel of 70 per cent efficiency. It takes 1 H.P. per minute of work to grind one bushel of corn. How much would one pay at 2 cents per bushel for grinding one hour?

16. A moving stairway moves at the rate of 2 feet per second. The maximum number of people who can ride on the stairway at any one time is 120. The length of the stairway is 50 feet, and its vertical rise is 30 feet. (*a*) Neglecting friction and assuming that the average weight of one person is 150 pounds, what is the maximum power required to operate the stairway? (*b*) If the efficiency of the system is 60 per cent, what is the maximum power required?

35. Energy is defined as the capacity for doing work. There are two kinds of energy, called potential and kinetic.

Potential Energy is called energy of position, and it is measured by the product of the weight and the vertical distance through which the weight has been lifted, or

$$\text{P.E.} = Wh. \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Kinetic Energy is called energy of motion, and it is measured by one-half the product of the mass and the square of the velocity, or

$$\text{K. E.} = \frac{1}{2}\frac{W}{g}v^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

Hence it is easily proven that the work done is equal to the change of kinetic energy; that is

$$\text{K. E.} = \text{force} \times \text{distance},$$

or

$$\text{K. E.} = \frac{1}{2}\frac{W}{g}(v_2^2 - v_1^2) = F \times s. \quad . \quad . \quad . \quad (67)$$

It is evident that problems which are solved from the consideration of changes of momentum, might often be solved by considering the change of kinetic energy.

EXAMPLE 1. A weight of 3 tons has been lifted vertically a distance of 8 feet. What is the potential energy which the weight possesses at this height?

Solution.

$$\text{P. E.} = Wh.$$

Therefore

$$\text{P. E.} = 3 \times 2000 \times 8 = 48{,}000 \text{ ft.-lbs.}$$

EXAMPLE 2. A shell of weight 1024 pounds leaves the muzzle of a cannon with a velocity of 1500 feet per second. Find the work done by the shell, and the kinetic energy of the cannon at the instant it begins to recoil if its weight is 30 tons.

Solution. The work done by the shell is equal to its kinetic energy.

Hence

$$K.\ E. = \frac{1}{2} \cdot \frac{W}{g} \cdot V^2 = \frac{1}{2} \cdot \frac{1024}{32} \cdot (1500)^2 = 36{,}000{,}000 \text{ ft.-lbs.}$$

Since the momentum of the gun equals the momentum of the projectile, the initial velocity of the gun is 25.6 feet per second, and the

$$K.\ E. = \frac{1}{2} \cdot \frac{30 \cdot 2000}{32} (25.6)^2 = 614{,}400 \text{ ft.-lbs.}$$

EXAMPLE 3. A bullet weighing $1\frac{1}{2}$ ounces moving at a velocity of 1024 feet per second overtakes a block of wood which weighs 8 pounds and having a velocity of 32 feet per second. Find the velocity of the wood after impact and how much kinetic energy has been lost, supposing the block does not rotate?

Solution. Let V = velocity of the bullet and block after impact.

$$\text{Momentum of bullet} = \frac{3}{32} \cdot \frac{1024}{32} = 3 \text{ units.}$$

$$\text{Momentum of block} = \frac{8}{32} \cdot 32 = 8 \text{ units.}$$

Hence total momentum before and after impact = 11 units.

Therefore

$$V = \frac{11 \cdot 32}{8 + \frac{3}{32}} = 43.48 \text{ feet per second.}$$

$$K.\ E. \text{ of bullet} = \frac{1}{2} \cdot \frac{3}{32} \cdot \frac{1}{32} (1024)^2 = 1536 \text{ ft.-lbs.}$$

$$\text{K. E. of block} = \frac{1}{2} \cdot \frac{8}{32} \cdot (32)^2 \qquad = 128 \text{ ft.-lbs.}$$

$$\text{Total K. E. before impact} \qquad = 1664 \text{ ft.-lbs.}$$

$$\text{Total K. E. after impact} = \frac{1}{2} \cdot \frac{8+\frac{3}{32}}{32} \cdot (43.48)^2$$
$$= 239.19 \text{ ft.-lbs.}$$

$$\text{Loss of K.E. after impact} = 1664 - 239.19$$
$$= 1424.81 \text{ ft.-lbs.}$$

EXAMPLES X

1. The hammer of a pile driver weighs 1 ton. Find its potential energy when it has been lifted 10 feet.

2. A body of weight 320 pounds is thrown up vertically with a velocity of 32 feet per second. What is its kinetic energy (1) after half a second, and (2) after one second?

3. What is the kinetic energy of a street car weighing 3.62 tons when it is moving at a speed of $7\frac{1}{2}$ miles per hour, and is laden with 20 passengers averaging 150 pounds each in weight?

4. A ball weighing 5 ounces, and moving with a velocity of 1000 feet per second, strikes a shield, and after piercing it moves on with a velocity of 400 feet per second. How much energy has been expended in piercing the shield?

5. What is the kinetic energy of a hammer of 5 tons which falls 4 feet?

6. A 32-pound ball is thrown vertically upward with a velocity of 20 feet per second. What is its kinetic energy when it has gone 5 feet?

7. A bullet of weight 2 ounces is fired into a target with

a velocity of 1280 feet per second. The weight of the target is 10 pounds and it is free to move. Find the loss of kinetic energy by the impact.

8. A shot of 1000 pounds moving at 1600 feet per second strikes a fixed target. How far will the shot penetrate the target, exerting upon it an average force of 12,000 tons?

9. Find the initial velocity of a shell weighing 1024 pounds if the total kinetic energy is 32,000 foot-tons.

10. An ounce bullet leaves the muzzle of a rifle with a velocity of 1024 feet per second. If the barrel is 4 feet long, calculate the pressure of the powder, neglecting all friction.

11. Two weights of 5 and 10 pounds respectively impinge directly, moving with velocities of 8 and 10 feet per second. Find the common velocity after impact, and the loss of kinetic energy.

12. What space average force will bring to rest in 22 feet the laden car in example 3?

13. A bullet weighing $2\frac{1}{2}$ ounces leaves a gun with a velocity of 1550 feet per second; the length of the barrel is $2\frac{1}{2}$ feet. Find the average accelerating force upon the bullet within the barrel.

14. A locomotive of weight 15 tons acquires a speed of 20 miles per hour in one mile, under the action of a constant difference of a moving and a resisting force. Find the accelerating force.

15. A train of weight 300 tons is travelling with a speed of 60 miles per hour. Find the force which will stop it in 500 yards.

16. An engine with all brakes on and steam shut off reduced its speed from 60 to 20 miles per hour in 800 yards. At the instant the speed was 20 miles per hour the engine

collides with a car. How much additional space would have saved the collision?

17. A train of weight 100 tons, is travelling at the uniform velocity of 30 miles per hour on a level track, and the total resistance is 20 pounds per ton. Part of the train of weight 20 tons becomes detached. Find how much the train will have gained on the detached part after one minute, and the velocity of the train when the detached part comes to rest.

18. A jet of water issues in a parallel stream, at 33 feet per second, from a round nozzle of radius 2 inches. What is the horse-power cf the jet, and what thrust should a fireman give to support the jet?

19. A weight of 3 tons falls a distance of 3 feet, and in so doing lifts a weight of 168 pounds to a height of 100 feet. Find the work done by the fall of the 3-ton weight, and what per cent of the work is effective?

CHAPTER IV

STATICS

COMPOSITION AND RESOLUTION OF FORCES

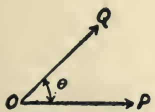
36. A force is completely determined when its direction, magnitude, and point of application are specified.

Equilibrium exists when two or more forces act upon a body, or a particle, and the body is kept in a state of rest.

Since force has magnitude and direction it can be represented by a vector. Hence the composition of forces may be found by the addition of vectors.

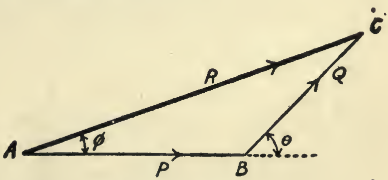
37. A **resultant force** is a single force which if applied to the particle will have the same effect as any number of concurrent forces.

38. Resultant of two forces acting at a point but inclined at an angle θ to each other.



Space Diagram

FIG. 14.



Vector Diagram

FIG. 15.

Represent the forces in direction as in Fig. 14, and in magnitude and direction as in Fig. 15.

Let AB represent the force P in magnitude and direction. Let BC represent the force Q in magnitude and direction. Now vector AB + vector BC = vector AC, the resultant. Since in the triangle ABC the two sides AB, BC, and the angle θ are given, the numerical value of AC may be found by use of trigonometry. Hence

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta. \quad . \quad . \quad . \quad (68)$$

Also, using the same scale of units as for P and Q, R may be measured in the same manner. This gives a check upon the computation.

39. Resultant of any number of forces in one plane acting upon a particle.

From trigonometry, the sum of the projections of a broken line is equal to the projection of the straight line joining the ends of the broken line.

Since a force may be broken up into its components, in the same manner as velocity, the projection upon a given line of the vector representing the resultant is the sum of the projections upon the same line of the component forces.

If three forces P, Q, and R, inclined at angles of α, β, and γ respectively, to the X-axis, act upon a particle the resultant is easily found by means of the previous theorem.

Let the forces P, Q, and S act upon a particle at O, as represented in Fig. 16.

Now

$$R^2 = \overline{OD}^2 + \overline{CD}^2,$$

but

$$OD = Oa + Ab + Bc$$
$$= P \cos\alpha + Q \cos\beta + S \cos\gamma = X, \quad (69)$$

and

$$CD = aA + bB + cC$$
$$= P \sin\alpha + Q \sin\beta + S \sin\gamma = Y. \quad (70)$$

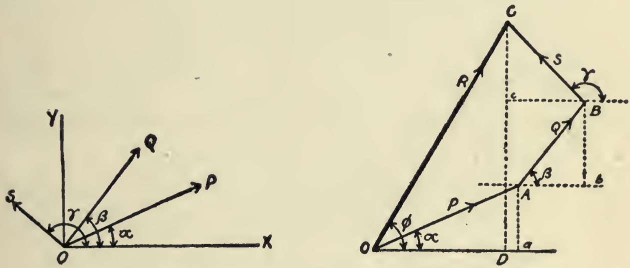
Hence

$$R^2 = X^2 + Y^2, \quad . \quad . \quad . \quad . \quad . \quad (71)$$

and

$$\tan\phi = \frac{CD}{OD} = \frac{Y}{X}, \quad . \quad . \quad . \quad . \quad . \quad (72)$$

where ϕ is the angle between the resultant and the X axis.



Space Diagram Vector Diagram

Fig. 16.

If there are n forces acting on the particle, then

$$Y = \sum_{1}^{n} P_n \sin\theta_n, \quad . \quad . \quad . \quad . \quad . \quad (70a)$$

$$X = \sum_{1}^{n} P_n \cos\theta_n, \quad . \quad . \quad . \quad . \quad . \quad (69a)$$

$$\tan\phi = \frac{Y}{X}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (72a)$$

The letter sigma, Σ, stands for "the sum of all such terms as."

EXAMPLES XI

1. Two forces, P and Q, act on a particle. What is their resultant, (1) when they act in the same direction, and (2) in opposite directions?

2. In the following examples, P and Q denote forces, θ the angle between the forces.

(a) $P=20$; $Q=15$; $\theta=90°$; find R;
(b) $P=12$; $R=13$; $\theta=90°$; find Q;
(c) $P=10$; $Q=40$; $\theta=60°$; find R;
(d) $P=8$; $Q=4$; $\theta=120°$; find R;
(e) $P=10$; $Q=6$; $R=14$; find θ.

3. Two equal forces act on a particle; find the angle between them when the square of the resultant equals their product.

4. Two forces of 84 and 187 pounds have a resultant of 205 pounds. Find the angle between the forces.

5. Let ABC be a triangle and O the middle point of the side AC. If the three forces, represented in magnitude and direction by BA, BO, and BC, act upon the point B, find the magnitude and direction of the resultant.

6. Two forces, P and Q, act on a particle. P is three times Q, and the square of the resultant is 13 times P. Find the forces if they act at an angle of 60°.

7. Three forces, equal to 10, 20, and 25 pounds, making angles of 10°, 75°, and 225° with the X-axis, act on a particle. Find the magnitude and direction of the resultant.

8. Four forces, equal to 12, 18, 20, and 30 pounds

making angles of 15°, 30°, 120°, and 210° with the X-axis, act on a particle. Find the magnitude and direction of the resultant.

9. Four men pull on ropes inclined to the X-axis at angles of 0°, 60°, 180°, and 330°, with forces of 100, 50, 200, and $50\sqrt{3}$ pounds. Find the resultant.

10. Four forces of 24, 10, 16, and 16 pounds act on a particle. The angle between the first and second is 30°; between the second and third, 90°; between the third and fourth, 120°. What is the resultant?

11. Three forces, proportional to 1, 2, and 3 respectively, act on a particle. The angle between the first and second is 60°, between the second and third is 30°. What angle does the resultant make with the first?

40. Equilibrium. If a number of forces act on a particle and keep it at rest, then the vector diagram will be a closed polygon. Therefore, formulæ (69) and (70) of Section 39 become

$$X=O, \text{ and } Y=O.$$

Therefore $$R^2=X^2+Y^2=O, \qquad (73)$$

which expresses the condition that must exist when the action of forces produces equilibrium.

41. Triangle of Forces. When three forces act on a particle and produce equilibrium, then the following relation (known as Lame's Theorem) exists.

$$\frac{\text{force } P}{\sin \angle ACB}=\frac{\text{force } Q}{\sin \angle BAC}=\frac{\text{force } R}{\sin \angle ABC}, \qquad (74)$$

where ABC is the vector triangle.

EXAMPLE 1. Four forces of P, Q, R, and S pounds acting at angles of α, β, γ, and δ, with the X-axis,

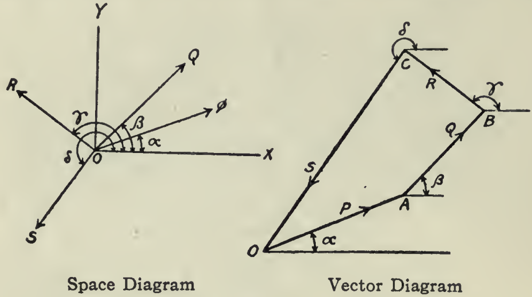


FIG. 17.

keep a particle in equilibrium. Draw the space and vector diagrams. See Fig. 17.

In the above diagram one sees at once that the figure is closed, and that the resultant is zero.

EXAMPLE 2. A body of weight W pounds is suspended by two cords of length a and b inches. The ends of the cords are fastened to two supports on

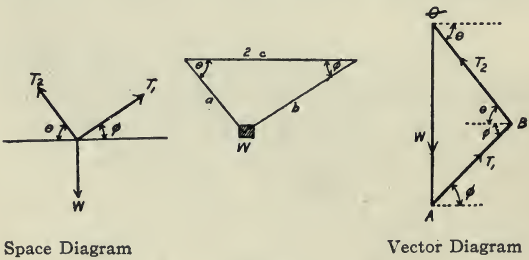


FIG. 18.

the same horizontal line, and $2c$ inches apart. Find the tensions in each cord.

Solution. Let the cord a be inclined to the horizontal $\theta°$ and the cord b be inclined $\phi°$. Draw the space and vector diagrams. Fig. 18.

There are three forces acting on the body, W pounds acting downward due to gravity, and the two tensions.

By triangle of forces

$$\frac{T_1}{W}=\frac{\sin(90°-\theta)}{\sin(\theta+\phi)}=\frac{\cos\theta}{\sin(\theta+\phi)}$$

and

$$\frac{T_2}{W}=\frac{\cos\phi}{\sin(\theta+\phi)}.$$

Now, if the values of W, a, b, and c are known, then the tensions T_1 and T_2 can be found, since

$$\sin\theta=\frac{\sqrt{a^2-c^2}}{a};\ \cos\theta=\frac{c}{a},$$

and

$$\sin\phi=\frac{\sqrt{b^2-c^2}}{b};\ \cos\phi=\frac{c}{b}.$$

EXAMPLE 3. Three forces, P, Q, and R, produce equilibrium. If $R=3$ pounds, $Q=4$ pounds, find P when the angle between R and Q is 90°.

Solution. Suppose Q to act along the horizontal, and R to act vertically. Fig. 19.

By triangle of forces

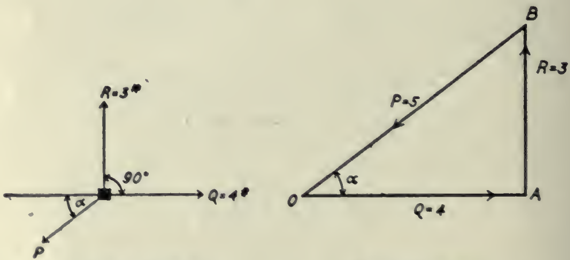
$$\frac{P}{R}=\frac{\sin 90°}{\sin\alpha}=\frac{1}{\sin\alpha},$$

but

$$\sin \alpha = \frac{AB}{OB} = \frac{3}{5}.$$

Therefore,

$$P = 3 \cdot \frac{5}{3} = 5 \text{ pounds.}$$



Space Diagram

Vector Diagram.

FIG. 19.

Also, resolving vertically,

$$P \sin \alpha = R.$$

Therefore

$$P = \frac{R}{\sin \alpha} = \frac{3}{\frac{3}{5}} = 5 \text{ pounds.}$$

EXAMPLE 4. Solve example 2 by resolution of forces.

Solution. Resolving the forces along lines of action of the forces T_1 and T_2 we get

$$T_1 \sin(\theta + \phi) = W \cos \theta,$$

and

$$T_2 \sin(\theta + \phi) = W \cos \phi.$$

Therefore,

$$T_1 = W \frac{\cos \theta}{\sin(\theta + \phi)},$$

and $$T_2 = W \frac{\cos \phi}{\sin (\theta + \phi)},$$

agreeing with results found in Example 2.

EXAMPLES XII

1. A wheel has six equally spaced radial spokes, all in tension. If the tensions of four consecutive spokes are 200, 100, 110, and 180 pounds respectively, find the tensions in the other two. Work by vector polygon, and also by resolution of forces.

2. A body of 100 pounds weight is suspended by two strings of length 3 and 4 feet attached to two points in the same horizontal line whose distance apart is 5 feet. Find the tensions of the strings.

3. Two men carry a body of weight 100 pounds between them by means of two ropes fastened to the weight. One rope is inclined 30° to the vertical and the other at 45°. Find the tensions in each rope.

4. Four forces of 3, 4, 6, and 5 pounds keep a body at rest. The first three forces make angles of 0°, 90°, and 180° with the X-axis. Find what angle the fourth force makes with the horizontal.

5. Four forces of 1, 4, $5\sqrt{3}$, and P pounds are in equilibrium. The first three forces make angles of 60°, 0°, and 150° with the X-axis. Find P and its direction.

6. A body of weight W pounds is fastened to a fixed point by means of a string of length $10\sqrt{2}$ feet. It is acted on by a horizontal force of 100 pounds. Find the value of W and the tension of the string when the weight has been pulled 10 feet from the vertical.

7. A body of weight W pounds is suspended by two strings of length 6 and 8 feet respectively, from two points in the same horizontal line 10 feet apart. If the tension of the longer string is 42 pounds, find the tension in the other and the value of W.

8. Two forces of 4 and 9 pounds respectively act on a particle, and their lines of action are inclined to each other at an angle of 60°. Find the direction and magnitude of the third force, which will produce equilibrium.

9. A pole rests vertically with its base on the ground and is held in position by three ropes all in the same horizontal plane and drawn tight. From the pole the first rope runs northeast, the second 10° west of north. Find the direction and tension of the third rope, when the pull on the first rope is 20 pounds, and the pull on the second rope is 30 pounds.

10. Two rafters, AB and AC, making an angle of 120°, support a weight of 500 pounds placed at A where they meet. What is the pressure on each rafter?

11. Three ropes are tied together, and a man pulls at each. If, when their efforts are in equilibrium, the angle between the first and second rope is 90°, between the first and the third is 150°; what are the relative strengths of each man as regards pulling?

12. On a smooth plane of length 5 and height 2, a weight of 10 pounds is kept at rest by a force along the plane. Determine the pressure on the plane and the force along the plane.

13. Three cords are tied together at a point. One of these is pulled due north with a force of 6 pounds, another due east with a force of 8 pounds. What pull on the third cord will produce equilibrium?

14. A weight of 10 tons is hanging by a chain 20 feet

long. Neglecting the weight of the chain, how much is the tension in the chain increased by the weight being pulled out by a horizontal force F to a distance of 12 feet from the vertical through the point of support? What is the horizontal force?

15. A body of weight 10 pounds, is supported on a smooth plane by a force of 2 pounds acting along the plane and a horizontal force of 5 pounds. What is the inclination of the plane?

42. Moment of a Force. The moment of a force about a point is the measure of its turning effect about that point, and is equal to the product of the force and the length of the perpendicular drawn from the given point to the line of action of the force.

The value of a moment may be represented in a vector diagram. Let the force P, Fig. 20, be represented in direction and magnitude by the line AB,

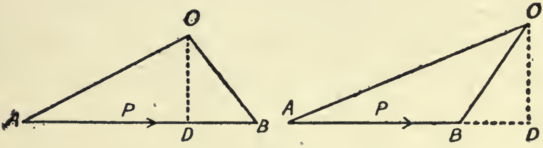


FIG. 20.

and let O be the given point, OD the perpendicular from O upon AB, or AB produced.

Join OA and OB. The moment of the force P about the point O is $P \cdot OD$. But $AB \cdot OD$ is twice the area of the triangle AOB. Hence the moment of a force about a point is represented by twice the area of the triangle whose base is the line representing the force, and whose vertex is the point about which the moment is taken. The point O is called the origin of moments.

43. Moment of the Resultant of Two Forces. Let P and Q, Fig. 21, be the two forces applied at the

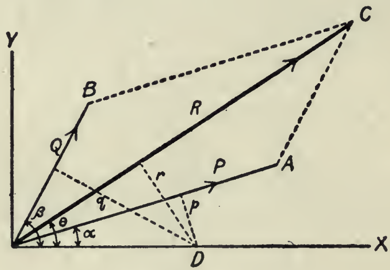


FIG. 21.

point O, and let R be their resultant. Suppose we take D as the origin of moments. Then draw perpendiculars p, r, and q to the lines of action of P, R, and Q.

Denote the angles XOA, XOC, and XOB, by α, θ, and β respectively. By the principle of projection

$$R \sin \theta = P \sin \alpha + Q \sin \beta, \quad . \quad . \quad . \quad (75)$$

or

$$R \cdot OD \sin \theta = P \cdot OD \sin \alpha + Q \cdot OD \sin \beta. \quad (76)$$

Hence

$$R \cdot r = P \cdot p + Q \cdot q. \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

Therefore the moment of the resultant of two forces is equal to the algebraic sum of the moments of the forces taken separately.

The principles used in determining the preceding theorem may be applied in finding the resultant of three or more forces.

Moments of forces which tend to cause rotation counterclockwise will be considered as positive and moments of forces which tend to cause rotation clockwise will be considered as negative.

44. A system of coplanar forces is in equilibrium when no translatory or rotary motion is produced. Hence, Section 40,

$$X=\sum_{1}^{n}F_n \cos \theta_n=O,$$

$$Y=\sum_{1}^{n}F_n \sin \theta_n=O.$$

Also the algebraic sum of the moments of the forces with respect to a given point or axis is equal to zero.

$$F_1d_1+F_2d_2+ \ldots +F_nd_n=O, \quad . \quad . \quad (78)$$

where the d represents the distance of the line of action of each force from the origin of moments. In each case the sign of the moment will be determined by the kind of rotation the force would produce.

45. Composition of Parallel Forces. Let P and Q, Fig. 22, be two forces in the same plane and applied at points A and B of a body. Let P be the greater force.

The lines of action, when prolonged, will meet in some point O. Now suppose the points of application to be transferred to O, the resultant may be determined by the parallelogram of forces. The

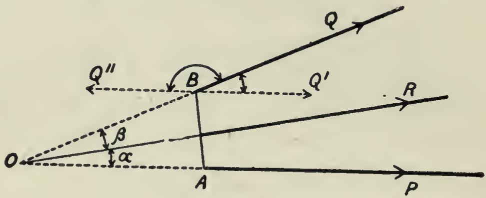


FIG. 22.

resultant R makes angles of α and β respectively, with the forces P and Q. Resolving forces along the line of action of R we get

$$R = P\cos\alpha + Q\cos\beta. \quad . \quad . \quad . \quad (79)$$

Now, suppose Q to turn around B until it becomes parallel with the force P and having the same direction. The point O recedes to the left to infinity. Hence, α and β approach zero, and

$$R = P + Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

Again, let Q turn around B until it is parallel to force P, but having opposite direction. By constructing this case of rotation we can draw the conclusion that α becomes zero, and β becomes 180°. Hence

$$R = P - Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

46. To find the resultant, and its point of application, of two parallel forces having the same direction. Let P be greater than Q, and A and B their points of application, Fig. 23.

Produce QB to P', making BP' equal to the vector length of the force P. Measure on AP the length AQ' so that AQ' is the vector length of the force Q. Draw $P'Q'$ cutting AB at O. Through O draw OR parallel to the direction of P and Q, and make OR equal to $P+Q$, which is the resultant. From the triangles $P'BO$ and $Q'AO$ we have

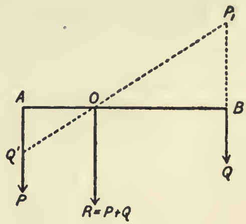


FIG. 23.

$$P'B : Q'A :: BO : OA.$$

Since $P'B$ and $Q'A$ represent the forces P and Q respectively, we have

$$P : Q :: BO : OA \quad . \quad . \quad . \quad . \quad (82)$$

If $AB=a$ and $BO=x$, we can easily show that

$$x=\frac{aP}{P+Q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

which is the distance of O from B, the point of application of force Q. Also, by the principle of moments we get

$$P \cdot AO = Q \cdot OB,$$

and, if OB equals x and AB equals a, we have

$$P \cdot (a-x) = Q \cdot x,$$

which agrees with formula (82).

47. The following method of finding the resultant of parallel forces having the same direction is interesting, and easily applied. Let P and Q (Fig. 24)

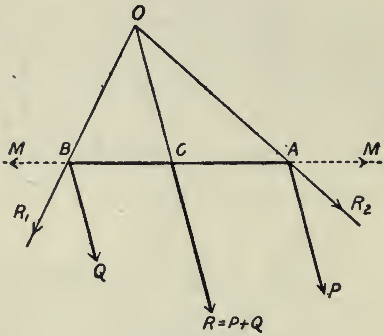


FIG. 24.

be two parallel forces having the same direction and equal in magnitude to AP and BQ.

At A and B, introduce two equal and opposite forces, M, acting in the line AB, one acting at A and the other acting at B. The resultant of M and Q is R_1, and the resultant of M and P is R_2. Now R_1 and R_2 intersect at some point O. By constructing the parallelogram of the forces R_1 and R_2, their resultant is R and its line of action cuts AB at the point C. The distance of the point C from either A or B agrees with the results of section 46.

48. To find the resultant of two parallel forces acting in opposite directions. $P > Q$. Let P and Q be the forces; A and B their points of application.

Fig. 25. Prolong QB until BP' equals P, and draw AQ' equal to Q. Draw $P'Q'$ and produce it until

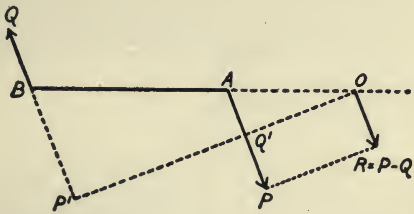


FIG. 25.

it cuts AB produced at point O; draw OR parallel to AP and equal to $Q'P$. Now $R=P-Q$. From the triangles $OP'B$ and $OQ'A$ we have

$$BP' : AQ' :: BO : AO,$$

or

$$P : Q :: BO : AO. \quad . \quad . \quad . \quad . \quad (84)$$

If $AB=a$ and $BO=x$, then

$$x=\frac{aP}{P-Q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

which is the distance of O from B, the point of application of force Q. Also by principle of moments we get

$$Q\cdot BO=P\cdot AO \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Hence

$$Q\cdot x=P\cdot(x-a).$$

Therefore

$$x=\frac{aP}{P-Q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

By the same method as used in section (47) one can find the resultant of two parallel forces having opposite direction, and its line of action.

EXAMPLES XIII

1. A light horizontal beam AB of 12-foot span carries loads of 700, 600, and 900 pounds at distances of 1 foot, 5 feet, and 10 feet respectively from A. Find the supporting forces.

2. Two forces, $P=100$ pounds and $Q=60$ pounds, are applied at the points A and B, and act in the same direction. Find the magnitude and line of action of the resultant if AB is 16 feet.

3. Find the magnitude and line of action of the resultant when (the forces acting in the same direction)

(1) $P=9$; $Q=27$; $AB=12$ inches;
(2) $P=18$; $Q=24$; $AB=28$ inches;
(3) $P=8$; $Q=10$; $AB=9$ inches.

4. Find the magnitude and line of action of the resultant when (the forces acting in opposite directions)

(1) $P=20$; $Q=10$; $AB=12$ inches;
(2) $P=10$; $Q=30$; $AB=12$ inches;
(3) $P=18$; $Q=12$; $AB=15$ inches.

5. The forces having the same direction, if $Q=11$, $AO=7$ inches, $AB=8\frac{3}{4}$ inches; find P and R.

6. The forces having opposite directions, if $Q=11$, O is $15\frac{3}{4}$ inches to the left of B, AB is $8\frac{3}{4}$ inches; find P and R.

7. Two men carry a stone which weighs 240 pounds,

by means of a plank 5 feet long. One man carries double what the other carries. How must the stone be placed on the plank?

8. A uniform rod, AB, 20 feet long weighing 2 pounds per foot, can turn freely about a point in it, and the rod is in equilibrium when a weight of 10 pounds is placed at the end A. How far from the end A is the point about which it can turn?

N. B. The weight of a uniform rod may be taken to act at its middle point.

9. A valve 4 inches in diameter is held down by a lever and a weight, the length of the lever being 20 inches and the valve spindle being 4 inches from the fulcrum. Find the pressure per square inch which will lift the valve when the weight hung on the end of the lever is 40 pounds and the lever itself weighs 10 pounds.

10. Three men carry a log which is uniform and has a length of 8 feet, and weighs 240 pounds. If one of the men lifts at an end, and the other two lift by means of a cross bar, how ought the cross bar be applied in order that each man may carry one-third the weight, neglecting the weight of the cross bar?

11. A rod, AB, of length 5 feet and weight 10 pounds, is found to balance itself if supported on a fulcrum 3 feet from A. If this rod were placed horizontally on two points, one under A and the other under B, what pressure would it exert on each point?

12. A uniform bar, AB, 18 feet long rests in equilibrium upon a fulcrum 2 feet from A. A weight of 5 pounds is hung at B and 110 pounds at A. What is the weight of the bar?

13. A horizontal uniform beam, AB, of length 21 feet, weighs 10 tons, and rests on two supports at the ends.

What is the pressure on each support when a weight of 4 tons is (1) placed at the middle and (2) 7 feet from A?

14. A horizontal beam, AB, of length 16 feet, weighs 8 tons and rests on two supports. At points 2, 4, 6, 8, 10, 12, and 14 feet from A, weights of 1, 2, 3, 4, 5, 6, and 7 tons respectively, are placed. Find the supporting forces.

15. Forces equal to the weights of 1 pound, 2 pounds, 3 pounds, and 4 pounds act along the sides of length a of a square, $ABCD$, taken in order. Determine the resultant.

16. A ladder whose weight is 168 pounds, and whose length is 25 feet, rests with one end against a smooth vertical wall, and with the other end upon a smooth horizontal floor; if it is prevented from slipping by a peg at its lower end, and if the upper end is 24 feet above the floor, find the reaction of the peg, ground, and the wall.

17. A bar, AB which weighs 20 pounds, 30 inches long, is hinged at A so as to be free to move in a vertical plane. The end B is supported by a cord, BC, so placed that the angle ABC is 150° and AB is horizontal. A weight of 20 pounds is hung on the bar at a point D, between A and B and 12 inches from A. Find the tension in the cord and the pressure of the rod on the hinge. Also find what angle the line of action of the pressure on the hinge makes with AB.

49. Center of a System of Parallel Forces, Center of Gravity. Let A, B, and C (Fig. 26) be the points at which parallel forces F_1, F_2, and F_3, respectively, act. The line of action of the resultant force R may be found by applying successively the results of Section 46.

Now, F_1 and F_2 may be replaced by a force F_1+F_2, at a point H in AB, such that $AH/BH=F_2/F_1$. Also the force acting at H, and the force F_3 acting at C may be replaced by a force $F_1+F_2+F_3$ at the point K in HC, such that

$$HG/GC=F_3/(F_1+F_2).$$

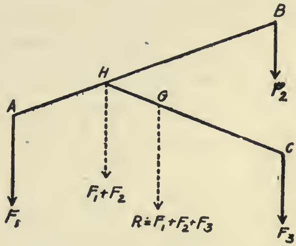


FIG. 26.

Any system of parallel forces, acting in the same direction, may be replaced by a force equal to the algebraic sum of the given forces. If G is the point where the resultant force acts, then this point is called the center of the parallel forces.

Center of Gravity. Gravitational forces have a tendency to pull all bodies toward the center of the earth. Since the bodies are small in comparison with the size of the earth, the lines of action of the forces acting on the particles which make up the bodies may be assumed to be parallel. The point of application of the resultant of the parallel forces which act on the particles which make up a body is called the center of gravity of the body.

The resultant of these gravitational forces acting on a body is called its weight, and the line of action of the resultant passes through the center of gravity.

50. Center of gravity of two particles of given weights at a given distance apart.

Let A and B, Fig. 27, be the positions of the centers of gravity of the two bodies of weights w_1 and w_2. Let G be the center of gravity of the system. Taking

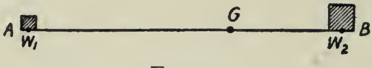


FIG. 27.

moments about the point B we get

$$(w_1+w_2)\cdot GB=w_1\cdot AB.$$

Taking moments about the point A we get

$$(w_1+w_2)\cdot AG=w_2\cdot AB.$$

Hence

$$GB=\frac{w_1}{w_1+w_2}\cdot AB, \quad . \quad . \quad . \quad . \quad (88)$$

and

$$AG=\frac{w_2}{w_1+w_2}\cdot AB. \quad . \quad . \quad . \quad . \quad (89)$$

If $w_1=w_2$ the center of gravity is midway between them, i.e.,

$$AG=GB=\tfrac{1}{2}AB.$$

51. Plane and Line Moments. The product of any element by its distance from a given plane, or line, is called a plane, or line, moment, where element means a unit length, unit area, or unit volume.

The products a_1d_1, a_2d_2, etc., may be called moments of areas, where a is the area of the plane figure, and d is the distance of the center of gravity of the plane figure from a point, line, or plane.

The products V_1d_1, V_2d_2, etc., may be called moments of volumes, where V is the volume of the geometric solid, and d is the distance of the center of gravity of the solid from a point, line, or plane.

52. Theorems on Center of Gravity:

THEOREM I. If a system of particles, whose weights are $w_1, w_2, \ldots w_n$, be in the same plane, and if OX and OY be two fixed straight lines in the plane at right angles, and if the distances of the particles from OX and OY be $y_1, y_2, \ldots y_n$, and $x_1, x_2, \ldots x_n$, respectively, the distance of their center of gravity from OX and OY being $\overline{X}$ and $\overline{Y}$, then

$$\overline{X}=\frac{w_1x_1+w_2x_2+\ \ldots\ +w_nx_n}{w_1+w_2+\ \ldots\ +w_n}, \quad . \quad . \quad (90)$$

and

$$\overline{Y}=\frac{w_1y_1+w_2y_2+\ \ldots\ +w_ny_n}{w_1+w_2+\ \ldots\ +w_n} \quad . \quad . \quad (91)$$

THEOREM II. The moment of an entire line, area, or volume, is equal to the algebraic sum of the moments of its parts.

The lever arm of a line, area, or volume, is the distance of the center of gravity from the given point, line, or plane, about which the moments are taken.

THEOREM III. The center of gravity of a uniform straight rod is at its middle point.

THEOREM IV. The center of gravity of a uniform triangular plate is at the point of intersection of the medians.

THEOREM V. The center of gravity of a uniform plate having the shape of a parallelogram is at the point of intersection of the diagonals.

THEOREM VI. The center of gravity of a uniform plate having the shape of a polygon may be found by dividing the polygon into triangles and applying Theorem II.

THEOREM VII. The center of gravity of a right cone, or tetrahedron, of uniform material is at a distance of $\frac{1}{4}$ the altitude from the base, and on the line joining the center of the base to the vertex.

THEOREM VIII. The center of gravity of a circular arc of length l, which subtends an angle 2θ at the center, is at a distance of $r \cdot \frac{\sin \theta}{\theta}$ from the center of the circle and on the bisector of the angle subtended.

THEOREM IX. The center of gravity of a circular sector which subtends an angle 2θ at the center is on the bisector of the subtended angle at a distance of $\frac{2}{3}r \cdot \frac{\sin \theta}{\theta}$ from the center. Hence the center of gravity of a semicircular area is at a distance of $\frac{4r}{3\pi}$ from its straight-line boundary.

The center of gravity of a circular segment may be found by applying Theorem II.

THEOREM X. The distance of the center of gravity of a hemispherical shell from the plane of its rim is half the radius of the shell.

THEOREM XI. The center of gravity of a sector of a sphere is at a distance of $\frac{3}{4} r \cos^2 \frac{\theta}{2}$ from the center of the sphere on the bisector of the angle subtended at the center.

The center of gravity of a solid hemisphere is at a distance of $\frac{3}{8} r$ from the plane boundary and on the perpendicular through the center of the plane base.

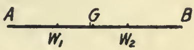
The proofs of these theorems are easily found by employing methods of Calculus, but elementary proofs will be given here.

Theorem I can be developed by using the principles employed in Section 50.

Theorem II is deduced readily from Theorem I.

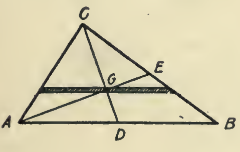
Proof of Theorem III.

Let AB be a uniform rod, and G its mid-point. Divide the rod into pairs of particles of equal weight situated at equal distances from G. The center of gravity of each pair, such as the particles w_1 and w_2, is midway between them, Section 50, that is, at the mid-point of the rod.



Proof of Theorem IV.

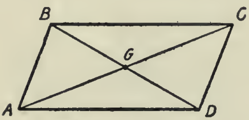
Given a triangular plate ABC. Divide the plate into very narrow strips parallel to AB. Each of the strips may be considered as a rod whose center of gravity is at its mid-point. But the mid-points of all these parallel strips are on the



median *CD*, of the triangle. If we had divided the plate into narrow strips parallel to *BC*, the mid-points of the strips would be on the median *AE*. Therefore the center of gravity of a uniform triangular plate must be at the point of intersection of the medians.

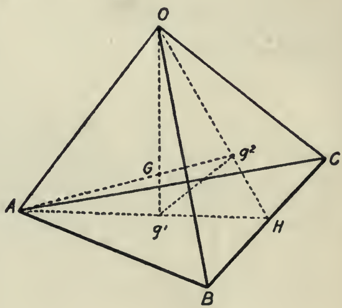
Proof of Theorem V.

Given a plate *ABCD* having the shape of a parallelogram. The diagonals of any parallelogram bisect each other. The center of gravity of the triangle *ABC* is in the line *BD*, and the center of gravity of the triangle *ADC* is in the line *BD*. Likewise the c.g. (center of gravity) of the triangles *ABD* and *BDC* are in the line *AC*. Hence the c.g. of the plate is at the point *G*, the intersection of the diagonals.



Proof of Theorem VII.

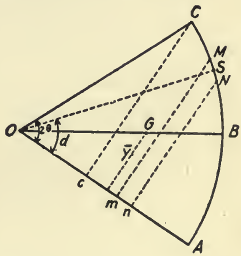
Given a tetrahedron *O–ABC*. The c.g. of the base *ABC* is g_1, and the c.g. of the face *BOC* is g_2. Let *G* be the c.g. of the pyramid, or tetrahedron. Divide the pyramid into a large number of very thin plates parallel to the base *ABC*. The c.g. of the pyra-



mid is on the line Og_1. If we divide the pyramid into plates parallel to the face BOC, then the c.g. of the pyramid is on the line Ag_2. Hence Ag_2 and Og_1 must intersect at some point which is G. We know that $Hg_1=\frac{1}{3}AH$, and $Hg_2=\frac{1}{3}OH$. By similar triangles $g_1g_2=\frac{1}{3}OA$, and $Gg_1=\frac{1}{3}OG$. Therefore $Gg_1=\frac{1}{4}Og_1$. Extending this analysis it is readily proved that the c.g. of any pyramid, or cone, is $\frac{1}{4}$ the distance from the c.g. of the base to the vertex.

Proof of Theorem VIII.

Let ABC be a portion of a circular arc which subtends an angle 2θ at the center, r being the radius, and B being the mid-point of the arc. Divide the arc into a large number of very short lengths and let MN be one of these parts. Let α be the angle which OS makes with the radius OA, S being the mid-point of the arc MN. Since MN is very short then the distance of MN from OA is $r\sin\alpha$. Applying Theorem I, the distance of the c.g. of the arc ABC from OA, $\bar{y}$, is



$$\bar{y}=\frac{\Sigma(r\cdot\sin\alpha\cdot\overline{MN})}{\Sigma\overline{MN}},$$

$$=r\frac{\Sigma(\overline{MN}\cdot\sin\alpha)}{\Sigma\overline{MN}}.$$

$$\overline{MN}\cdot\sin\alpha=\overline{mn},$$

the projection of $\overline{MN}$ on OA. Therefore

$$\Sigma(\overline{MN} \cdot \sin \alpha) = Ac,$$

and

$$\Sigma \overline{MN} = \text{arc } AC.$$

Hence

$$\bar{y} = \frac{r \cdot Ac}{AC} = \frac{r(r - r \cos 2\theta)}{AC} = \frac{r(1 - \cos 2\theta)}{\frac{AC}{r}}$$

$$= \frac{r(1 - \cos 2\theta)}{2\theta} = \frac{r \cdot \sin^2 \theta}{\theta}.$$

By symmetry it is evident that G, the c.g. of the arc ABC, must be on the radius OB. Hence

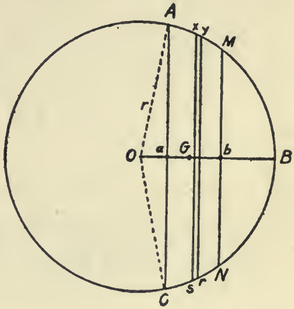
$$OG = \frac{\bar{y}}{\sin \theta} = \frac{r \sin \theta}{\theta}.$$

Proof of Theorem IX.

In the figure for the proof of Theorem VIII we may consider OMN as a triangle, since MN is so short that it is assumed as a straight line. The c.g. of any triangle is on the median and $\frac{2}{3}$ the length of the median from the vertex. Hence the c.g. of each triangle which is a part of the sector O–ABC is on the arc of a circle of radius $\frac{2}{3}r$. Therefore, by Theorem VIII, the c.g. of the sector O–ABC is on the radius OB and at a distance of $\frac{2}{3}r \cdot (\sin \theta)/\theta$ from the center.

Proof of Theorem X.

Given ABC a cross-section of hemispherical shell. Let $ACNM$ be a cross-section of a zone. Divide the zone $ACNM$ into a large number of equal zones, such as $xyrs$. The c.g. of each zone is on the axial line OB. Therefore, if G is the c.g. of the zone $ACNM$,



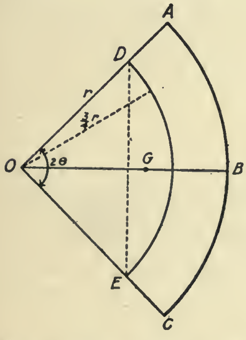
$$OG = \tfrac{1}{2}(Oa + Ob).$$

Hence for a hemispherical shell

$$OG = \tfrac{1}{2}(0 + OB) = \tfrac{1}{2}r.$$

Proof of Theorem XI.

Given $O - ABC$ a cross-section of a spherical sector.



Divide the sector into equal small pyramids having a common vertex O, the bases of the pyramids being the curved surface of the sphere. The c.g. of each pyramid is at a distance of $\tfrac{3}{4}r$ from the center O. Hence the c.g. of the sector is the same as the c.g. of a shell of radius $\tfrac{3}{4}r$. Let 2θ be the angle subtended by the arc ABC.

Then

$$OG=\frac{\frac{3}{4}r\cos\theta+\frac{3}{4}r}{2}=\frac{3}{8}r(1+\cos\theta),$$

$$=\frac{3}{4}r\cos^2\frac{\theta}{2}.$$

EXAMPLES XIV

1. Two bodies are attached to the end of a rod 5 feet long. One weighs 8 pounds and the other 14 pounds. Find the center of gravity of the system.

2. Three bodies are attached to the corners of a horizontal equilateral triangle 14 inches on a side. The bodies weigh 10, 15, and 20 pounds. Find the position of the center of gravity of the system.

3. A horizontal rod, AB, is 4 feet in length, and of weight 10 pounds. It is trisected in the points C and D, and at the points A, B, C, and D, are placed bodies of 2, 4, 6, and 8 pounds weight respectively; find the center of gravity of the system.

4. If, in (3) the body at A is removed, and another body is substituted, find the weight of this unknown body so that the new center of gravity may be at the middle point of the rod.

5. To the end of a horizontal rod 4 feet long, and of weight 8 pounds, is attached a weight of 12 pounds. Find the center of gravity of the system.

6. A wheel of weight 100 pounds is attached to the end of a straight uniform axle 4 feet long and of weight 50 pounds. Find the center of gravity of the system.

7. If a horizontal square tin plate weighs 5 ounces, and a small body weighing 2 ounces is placed at one corner of the plate, where is the center of gravity?

8. Find the center of gravity of the figure A when the

three pieces are of uniform material, and the central piece is half a side in length, and is joined at the mid-points of the sides.

9. Find the center of gravity of a T-square, the two pieces being of the same dimensions, 2 feet in length and 2 inches wide.

10. A board consists of a square of side 2 feet, and another square of side 6 inches, having the center of its side the same as the center of the side of the larger square. Find the distance of the center of gravity from the center of the base of the larger square.

11. In example 10 suppose an equilateral triangle is described on one side of the square, instead of the smaller square. Find the distance of the center of gravity from the center of the square.

12. A piece of cardboard consists of a rectangle of length a and width b, and a triangle of altitude h described on the side a. Find the distance of the center of gravity from the base.

13. A trapezoid with upper base b and lower base a has an altitude h. Find the position of the center of gravity and its distance from the base a.

14. From a triangular board is cut off, by a line parallel to its base, $\frac{1}{9}$ of its area. Find the position of the center of gravity of the remainder, if the base of the triangle is 40 inches and its altitude is 36 inches.

15. An isosceles right triangle ABC has squares described externally on its sides. The lengths of AB and BC are each $27\sqrt{2}$ inches, the right angle being at B. Show that the center of gravity is on the line joining B to the middle of AC, and 26 inches from B.

16. A quadrilateral consists of two isosceles triangles on opposite sides of a base 12 inches long. The side of

the larger triangle is 10 inches and the side of the smaller triangle is 8 inches. Find the distance of the center of gravity from the common base.

17. The middle points of two adjacent sides of a square are joined, and the triangle formed by this straight line and the edges is cut off. Find the distance of the center of gravity of the remainder from the center of the square.

18. From a circular disc of radius 12 inches is cut out a circular area whose diameter is 3 inches and center $1\frac{1}{2}$ inches from the circumference of the disc. Find the position of the center of gravity of the remainder, and its distance from the center of the disc.

19. From an equilateral triangle, of side 20 inches, is cut out a circular area of radius 5 inches, the circle being tangent to the middle of a side. Find the position of the center of gravity of the remainder, and its distance from the center of the triangle.

20. The lever arm of a testing machine weighs 2000 pounds, and is placed horizontally on a knife edge. It sustains a downward pull of 8 tons placed 5 inches from the knife edge, and carries a load of 1 ton on the opposite side of the knife edge, and placed 30 inches from it. Find the distance of the center of gravity of the lever arm from the knife edge.

21. The radius of a circle is 20 inches. Find the center of the circular arc which subtends an angle of 50 degrees at the center.

22. Find the center of gravity of a semicircular arc of a circle of radius 25 inches.

23. The radius of a circle is 20 inches and the central angle of a sector is 60 degrees. Find the center of gravity of the sector. Find center of gravity of a semicircular area.

24. In example 23 find the center of gravity of the segment bounded by the chord and the arc.

25. Let AOB be a given sector of a circle of radius 42 inches, angle AOB being 120 degrees. Through O', the center of the triangle AOB, draw lines $O'H$ and $O'K$ parallel to OA and OB, H and K being on the chord AB. If the area $O'HK$ is removed, find the distance of the center of gravity of the remainder from the center of the circle.

26. A cast-iron eccentric consists of a short cylinder 12 inches in diameter, and has through it a cylindrical hole 3 inches in diameter, the axis of the hole being parallel to that of the eccentric and 4 inches from it. Find the distance of the center of gravity from the center of the eccentric.

27. Out of a square plate, of side 12 inches, show how to cut a triangle having one side of the square for a base, so that the remainder may have its center of gravity at the vertex of the triangle.

28. Out of a square plate, of side 12 inches, is cut an equilateral triangle of side 6 inches, the center of the base of the triangle coinciding with the center of the square, and the base parallel to the side of the square. Find the distance of the center of gravity from the center of the square plate.

29. Where must a circular hole, of 2-foot radius, be punched out of a circular disc, of 5 foot radius, so that the center of gravity of the remainder may be 5 inches from the center of the disc.

30. A square hole is punched out of a circular disc, the diagonal of the square being the radius of the circle. If the radius of the circle is 20 feet, find the distance of the center of gravity of the remainder from the center of the circle.

31. An I-section of a girder is made up of three rectangles, two flanges having their sides horizontal, and one web connecting them having its long side vertical. The top flange section is 8 inches by 2 inches, and the bottom flange is 16 inches by 4 inches. The web section is 10 inches by 2 inches. Find the distance of the center of gravity of the area of the cross-section from the bottom of the flange.

32. A piece of iron consists of an isosceles trapezoid with semicircular areas described upon the ends. The altitude of the trapezoid is 4 feet and the ends measure 2 feet and $2\frac{1}{2}$ feet. Find the distance of the center of gravity from the center of the $2\frac{1}{2}$-foot base.

33. A circular board 10 inches in diameter has two square holes cut out of it, one of side 2 inches, and the other of side 1 inch, the centers of the squares being on diameters mutually perpendicular, and 2 inches from the center of the circular plate. Find the center of gravity of the remainder.

34. A counterpoise on a wheel of an engine has the shape of a crescent and fits inside the rim of the wheel 8 feet in diameter, and the crescent subtends an angle of 120° at the center of the wheel. The center of the circle determined by the inner rim of the crescent is on the rim of the wheel. Find the distance of the center of gravity of the counterpoise from the center of the wheel.

35. A solid consists of a right cylinder of radius r and 4 feet long and a right cone of altitude $2\frac{1}{2}$ feet, the base coinciding with one end of the cylinder. If the cylinder and cone are made of the same uniform material, find the distance of the center of gravity of the solid from the vertex of the cone.

36. A solid consists of a right triangular prism having an equilateral base of side 12 inches and a triangular pyramid with its base coinciding with the base of the prism. The altitude of the prism is 2 feet, and the height of the pyramid is 18 inches. The axis of the prism and pyramid being in the same line, find the distance of the center of gravity of the solid from the common base.

37. A solid consists of a right cylinder and a hemisphere described on one end. If the altitude of the cylinder is 4 feet, and its radius is 1 foot, find the distance of the center of gravity from the center of the cylinder.

38. A piece of solid iron consists of two cylinders, one placed on the other and their axes being in the same line, the altitude and radius of the lower cylinder being 10 inches and 4 inches, and the upper being 3 inches and 1 inch. Find the distance of the center of gravity from the bottom of the solid.

39. The upper and lower bases of a frustum of a right cone have diameters 5 and 20 inches respectively. The altitude is 15 inches. Find the distance of the center of gravity from the lower base.

40. The longest element of a cone makes an angle of 60° with the diameter of the base. If this element is 20 feet in length, find the distance of the center of gravity from the plane of the base.

41. A concrete dam is 10 feet high, 2 feet thick at the top, and 6 feet thick at the bottom. The front face being vertical, find the distance of the center of gravity from the vertical face.

42. A solid right cone stands on a base 24 inches in diameter, and its altitude is 15 inches. The upper part of the cone is removed by passing a plane parallel to the

base and 5 inches from the vertex. Find the distance of the center of gravity from the lower base.

43. The barrel of a cannon is shaped like a frustum of a cone, being 12 feet long, 18 inches in diameter at one end and 12 inches at the other. Through this frustum is bored a cylindrical hole 6 inches in diameter, the axis of the cylinder and frustum being the same. Find the distance of the center of gravity from the larger end of the barrel.

44. A solid right cone and a solid hemisphere have a common base. The altitude of the cone is 12 inches, and the diameter of the base is 12 inches. Find the distance of the center of gravity from the center of the common base.

45. A hemispherical shell of uniform material has an internal radius of 12 inches, and a thickness of 2 inches. Find the distance of the center of gravity from the center of the sphere.

46. A solid monument consists of three parts, the lowest part is a cylindrical stone of height 10 feet, and radius of base 3 feet; the middle part is a stone the shape of a frustum of a cone, the altitude is 3 feet, the lower base coinciding with the upper base of the cylinder, and the upper base has a radius of 2 feet; the third part is a stone hemisphere with its base the same as the upper base of the frustum. Find the distance of the center of gravity from the base of the monument.

47. A solid right cone is 12 feet in height and the radius of the base is 2 feet. The greatest possible sphere is cut out from the cone. Find the distance of the center of gravity of the remaining volume from the vertex of the cone.

48. A solid sphere of radius 18 inches consists of two

parts; the density of one half is three times that of the other. How far is the center of gravity from the center of the sphere?

49. From a solid sphere of radius 13 feet is cut off a segment by passing a plane 5 feet from the center of the sphere, and perpendicular to a diameter. Find the distance of the center of gravity of the smaller segment from the center of the sphere, and the distance of the center of gravity of the remaining volume from the center of the sphere, if the smaller segment is removed.

53. Stable, Unstable, and Neutral Equilibrium. When a body is moved from its position of support and left, it will in some cases return to it, pass by and return again, and continue thus to vibrate until it settles in its place of support by friction and other resistances. This condition is called stable equilibrium.

In other cases, when a body is moved from its position of support and left, it will depart further from it, and never recover that position again. This is called unstable equilibrium.

In other cases still, the body, when moved from its place of support and left, will remain, neither returning to it nor departing further from it. This is called neutral equilibrium.

In general, there is stable equilibrium when the center of gravity, upon being disturbed in either direction, begins to rise; unstable when if disturbed either way, it begins to descend, and neutral when the disturbance neither raises nor lowers the center of gravity. A body will remain at rest on a plane surface if the vertical line through its center of gravity falls within the base of support.

EXAMPLES XV

1. A rectangular block 20 feet high, 4 feet wide and 3 feet thick, is racked into an oblique form until it is on the point of falling. What is the inclination to the horizontal?

2. If the block in example 1 were tilted instead of racked, until it is about to fall, what is its inclination to the horizontal?

3. A cube lies on an inclined plane and is prevented by friction from sliding down; to what inclination must the plane be tipped so that the cube will be ready to turn over the lower edge?

4. What is the greatest height which a right cylinder of 10 inches diameter may have in order that it may rest with one end on a plane inclined $45°$ to the horizontal?

5. A cylinder of height a and radius of base r, is placed on an inclined plane and prevented from sliding; if the inclination θ of the plane is gradually increased, find when the cylinder will topple over.

6. A body, consisting of a cylinder and a hemisphere joined at their bases, is placed with the spherical end upon a horizontal plane. If the radius of the sphere is 10 inches, what must be the length of the cylinder in order to have neutral equilibrium?

7. A cone and a hemisphere have a common base of radius 2 feet. Find the height of the cone when neutral equilibrium exists, when the body rests with the hemispherical surface on a horizontal plane.

8. The altitude of a cone is 24 inches and the diameter of the base is 15 inches. What is the greatest inclination on which it may stand in equilibrium on its base?

9. A right cone contains $288\sqrt{3}\,\pi$ cubic inches of material. What is the greatest height it may have in order to rest with its base on a plane inclined 30° to the horizontal, and what is the radius of the base?

10. A right pyramid of stone is 12 feet high and has a pentagonal base of side 8 feet. How much may the pyramid be turned about an edge of the base before it will topple?

11. A rectangular stone 8 feet long, 4 feet wide and 3 feet in thickness, is laid with one-quarter of its length projecting over the edge of a concrete base, and resting on the 4 by 8 face; a stone and a quarter is then laid on the first stone with one-quarter projecting over the first stone and so on. How many courses will cause the pile to topple?

12.* A cubical block of material resting on a horizontal plane weighs 1000 pounds and its edge is 2 feet in length. Find the work in foot-pounds necessary to turn the block over one edge.

13. How much work is expended in turning over the cone of example 8, if it rests on a horizontal plane, and its weight is 60 pounds per cubic foot?

14. How much work is expended in turning over the pyramid of example 10, if it rests on a horizontal plane, and its weight is 4 tons?

15. What amount of work is expended in turning over a right cone, with a circular base, whose altitude

* The total work done in lifting a body is equal to the weight of the body multiplied by the vertical distance through which its center of gravity has been raised.

is 12 feet and radius of base 4 feet, the weight of the material being 100 pounds per cubic foot?

16. A cylindrical well is 150 feet deep and 10 feet in diameter. Supposing the well to be filled with water to a depth of 50 feet, how much work is expended in raising the water to the top?

17. A rectangular tank 7.854 feet long, 6 feet wide, and 4 feet deep is filled with water from a cylindrical tank 78.54 square feet horizontal cross-sectional area. The level of the water, before filling begins, stands 16.8 feet below the bottom of the rectangular tank. How much work is expended in filling the rectangular tank?

18. A hollow cylindrical boiler shell, 8 feet internal diameter and 20 feet long, is placed with its axis horizontal. It is to be half filled with water from a reservoir, the level of which remains constantly 6 feet below the axis of the boiler. How much work is required to lift the water?

19. A uniform chain, 60 feet long and weighing 15 pounds per foot, hangs vertically. How much work is expended in winding up the chain?

20. A chain 144 feet long hangs vertically. Its weight at the top end is 20 pounds per foot and at the lower end 10 pounds per foot, the weight per foot varying uniformly from top to the lower end. How much work is expended in winding up the chain?

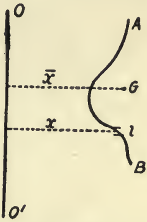
21. A regular polygon just tumbles down an inclined plane whose inclination is 10°. How many sides has the polygon?

54. Theorems of Guldinus or Pappus. (*a*) If a plane curve revolves around any axis in its plane, the area of the surface generated is equal to the length of the revolving curve multiplied by the length of the path described by its center of gravity.

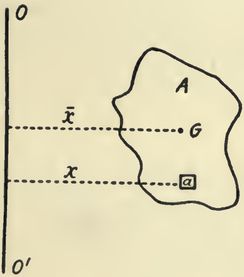
(*b*) If a plane area revolves around any axis in its plane the volume generated is equal to the revolving area multiplied by the length of the path described by its center of gravity.

Proof of Theorem (*a*).

Let AB be a part of a curve which revolves about the axis OO'. Let L be the length of the curve between A and B. Divide L into a large number of small parts having lengths of l_1, l_2, l_3, . . . l_n, each part being x_1, x_2, x_3, . . . x_n, from the axis of revolution. Any part l of the curve generates a surface of area $2\pi x \times l$. Hence the total area generated is



$$\sum_1^n (2\pi x_n l_n) = 2\pi \sum_1^n (x_n l_n).$$



But $\sum_1^n (x_n l_n) = L \cdot \bar{x}$ (Theorem II, Section 52). Therefore the total surface generated is $2\pi\bar{x} \cdot L$.

Proof of Theorem (*b*).

Let A be the area which revolves about the axis OO'. Divide the surface A into a large number of small parts, a_1, a_2, a_3, . . . a_n, each being x_1, x_2, x_3, . . . x_n, from the axis of revolution.

Any part a will generate a volume of $2\pi x \times a$. Hence the total volume is

$$\sum_1^n (2\pi x_n a_n) = 2\pi \sum_1^n (x_n a_n).$$

But $\sum_1^n (x_n a_n) = A \cdot \bar{x}$ (Theorem II, Section 52). Therefore the total volume generated is $2\pi \bar{x} \cdot A$.

EXAMPLES XVI

1. Find the volume of a sphere by revolving a semicircle.

2. Show that the volume of a cone may be found by revolving a right triangle.

3. Show that the surface of a sphere may be found by revolving a semi-circumference.

4. A circle, 10 inches in radius, has its center on the Y-axis and 18 inches from the origin. Find the surface and volume of the torus ring by revolving the circle about the X-axis.

5. Find the volume generated by revolving a circle 10 feet in diameter about a tangent. Also the surface.

6. Find the volume generated by revolving a semicircle 20 inches in diameter about a tangent parallel to its diameter.

7. A circle of 10 inches radius with an inscribed regular hexagon, revolves about an axis of rotation 20 inches distant from its center and parallel to a side of the hexagon. Find the difference in area of the generated surfaces. Also the difference in volumes generated.

8. An equilateral triangle rotates about an axis without it, parallel to, and at a distance of 10 inches from one

of its sides. Find the surface and volume thus generated, a side of the triangle being 4 inches.

9. Find the surface and volume of a square ring described by a square of side 1 foot revolving around an axis parallel to one of its sides, and 4 feet distant from the side.

10. The center of gravity of a semi-ellipse is at a distance of $4b/3\pi$ from the center of its straight line boundary. Its area is $\frac{1}{2}\pi ab$, where a is half the longer axis and b is half the shorter axis. Find the volume generated by revolving the semi-ellipse about the axis a, if $a=3$ feet and $b=2$ feet.

11. Find the surface and volume of a cylindric ring, the diameter of the inner circumference being 12 inches and the diameter of the cross-section 16 inches.

12. A groove of semicircular section 3 inches in radius is cut in a cylinder whose radius is 12 inches. Find (1) the area of the curved surface of the groove, and (2) the volume of the material removed.

13. The arc of a circle of 30 inches radius subtends an angle of 120° at the center. Find the area of the surface generated when this arc revolves about its chord; also find the volume of the solid generated by the revolution of the segment about its chord.

14. A groove is cut in a cylinder casting 48 inches in diameter. The cross-section of the groove is an isosceles trapezoid with outer base 10 inches, the inner base 6 inches, and 6 inches deep. Find the volume of the material removed.

15. Find (1) the surface and (2) the volume of a ring with a circular section whose internal diameter is 12 inches and thickness is 3 inches.

CHAPTER V

MOTION IN A CIRCLE

55. In Section 11 we considered the uniform circular motion of a particle, and found that the linear velocity is equal to the radius multiplied by the angular velocity. Since the average acceleration is equal to the change in velocity divided by the time, we can find by the vector diagram the direc-

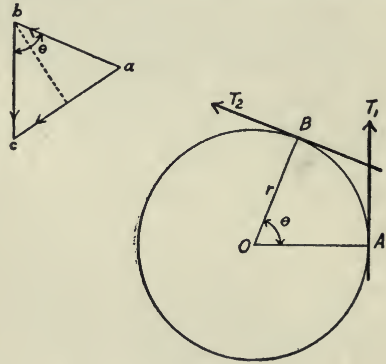


FIG. 28.

tion of the acceleration. In Fig. 28, let the vector *ab*, parallel to the tangent BT_2, represent the linear velocity at *B*, and the vector *bc*, parallel to the tangent AT_1, represent the linear velocity at *A*. Since the motion is uniform, vector *ab* equals vector *bc*. Then to find the change of velocity between *A* and *B*, we must subtract the vector representing the velocity at *A* from the vector representing the velocity at *B*. Therefore vector *ac* represents the change of velocity between *A* and *B*. But $ac = 2ab \sin\frac{\theta}{2}$. Hence $ac = 2v \sin\frac{\theta}{2}$. The time equals θ/ω (Section 11, formula 12). Therefore the average acceleration equals

$$\frac{2v \sin\frac{\theta}{2}}{\frac{\theta}{\omega}} = \omega v \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}.$$

The limiting value of $\frac{\sin \theta/2}{\theta/2}$ is unity, and the average acceleration becomes ωv, $\omega^2 r$, or v^2/r.

Now, as the angle θ decreases, the angle *bac* approaches a right angle, and the acceleration is toward the center of the circle.

56. Centripetal and Centrifugal Forces. We have just seen that if a particle is describing a circle of radius r about a center O with angular velocity of ω radians per second, it has an acceleration of $\omega^2 r$ toward the center of the circle. The force is equal to the mass times the acceleration; and in this case it is $W/g\ \omega^2 r$ pounds. This force which tends to pull an object inward is called the centripetal force. According to the third law of motion, there must be a reaction in the opposite direction to the centripetal force. This opposite force is called the centrifugal force. In other words, the particle exerts a pull of $W/g\ \omega^2 r$ pounds away from the center of the circle. In some cases the centripetal force may be a thrust instead of a tension, as in the case of a car going around a curve.

57. Motion on a Circular Inclined Track. Suppose a particle P (Fig. 29) moves with uniform velocity V, around a smooth circular track of radius OP equal to R feet. To find the inclination of the track in order that the particle keeps its circular path. Now, there are only two forces acting on the particle, its own weight, and the reaction Q of the track which is perpendicular to the plane of the track. The resultant of these forces, $W/g \cdot V^2/R$, is directed toward the center O of the horizontal circle. From the vector diagram we get at once

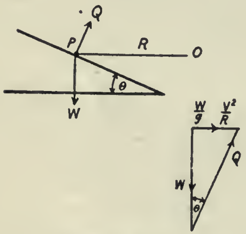


FIG. 29.

$$\tan\theta = \frac{\frac{W}{g}\cdot\frac{V^2}{R}}{W} = \frac{V^2}{gR}, \qquad (92)$$

which is the required angle.

58. Motion on Railway Curves. The track must be inclined so that the reaction Q is normal to the plane of the track. Suppose G is the gauge of the track in inches, V the velocity of the car in feet per second, and R the radius of the curve which the car is rounding. Let AB (Fig. 30) represent the gauge. AC is the height in inches which the outer rail is above the inner rail, and θ is the inclination of the plane of the track. Now, $AC = AB \sin\theta$. When θ is very small, $\sin\theta = \tan\theta$. Hence $AC = AB \tan\theta$. From Section 57, formula (92),

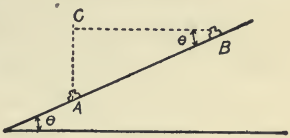


FIG. 30.

$$AC = G \tan\theta = G \cdot \frac{V^2}{gR} \quad . \quad . \quad . \quad . \quad (93)$$

59. The Conical Pendulum. If a particle is tied by a string to a fixed point O, and moves so that it describes a circle in a horizontal plane, the string describing a cone whose axis is the vertical line through O, then the string and particle together are called the conical pendulum. Consider the case when the circular motion is uniform.

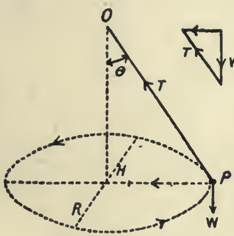


FIG. 31.

Fig. 31 represents a conical pendulum, where a particle P, tied by a string to the fixed point O, describes the horizontal circle with uniform motion about the center H vertically under O.

Let T = the tension of the string OP in pounds;
ω = angular velocity of P about H in radians per second;

R = radius HP of the horizontal circle;
W = weight of the particle P in pounds;
a = length of the string OP in feet;
θ = angle which string makes with a vertical line through O;
h = height of OH in feet;
v = linear velocity of the particle in feet per second.

The two forces, the tension T of the string, and W, the weight of the particle, have a resultant $W/g\ \omega^2 R$ toward the center of the horizontal circle. In the vector diagram we see that

$$\tan\theta = \frac{\frac{W}{g}\omega^2 R}{W} = \frac{\omega^2 R}{g}, \quad . \quad . \quad . \quad (94)$$

Hence

$$h = \frac{R}{\tan\theta} = \frac{g}{\omega^2} \quad . \quad . \quad . \quad . \quad . \quad (95)$$

This shows that h, the height of the conical pendulum, is independent of the length of the string, and is dependent upon the angular velocity about H, the center of the horizontal circle.

Resolving vertically

$$T\cos\theta = W, \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

which will give the value of T after θ has been found.

Resolving horizontally

$$T\sin\theta = \frac{Wv^2}{gR}, \quad . \quad . \quad . \quad . \quad . \quad (97)$$

since $T \sin \theta$ is the only force in the direction PH. From (96) and (97) we get

$$\frac{v^2}{a \sin^2 \theta} = \frac{g}{\cos \theta},$$

and

$$\cos \theta = \frac{g}{4\pi^2 n^2 a}, \quad . \quad . \quad . \quad . \quad . \quad (98)$$

as

$$v = 2\pi n a \sin \theta, \quad . \quad . \quad . \quad . \quad . \quad (99)$$

where n is the number of revolutions per second.

Hence by (96) and (98), we find that

$$T = 4\frac{W}{g}\pi^2 n^2 a \text{ pounds.} \quad . \quad . \quad . \quad (100)$$

Since

$$h = \frac{g}{\omega^2}, \quad \omega = \sqrt{\frac{g}{h}}. \quad . \quad . \quad . \quad . \quad . \quad (101)$$

The time of one complete revolution of the particle is

$$\text{time} = \frac{2\pi \text{ radians}}{\text{angular velocity}} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}} \quad . \quad (102)$$

Hence the time is dependent upon h only.

60. Motion in a Vertical Circle. Suppose a particle P moves in a vertical circular path and is kept in the circular path by a string fixed at some point, or by an inward pressure of a circular track. To find the tension of the string, or inward pressure of the track, and the velocity at any point, and the condition that the particle will make a complete revolution. In Fig. 32 let

R = the normal pressure of the track or the tension of a string;

W = the weight of the rotating particle;

v = its velocity in feet per second in any position, B, such that OB makes an angle θ to the vertical OA, A being the lowest point on the circumference;

V_A = velocity at A;

r = radius of the circle in feet.

Suppose the particle is projected from the position A with an initial velocity of V_A, then the velocity at any point B is (Section 14, formula (25)),

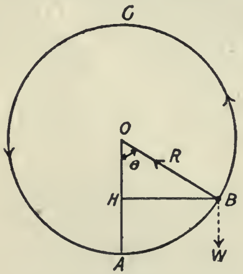


FIG. 32.

$$v^2 = V_A^2 - 2g \cdot AH. \quad (103)$$

Also, if the particle is moving clockwise, the kinetic energy at B is $\frac{1}{2} W/g \cdot v^2$. At A the kinetic energy is $\frac{1}{2} W/g \cdot V_A^2$, and the potential energy at B is $W \cdot AH$. Since, by the principle of work, no energy is lost, the total mechanical energy at B is equal to that at A. Hence

$$\frac{1}{2}\frac{W}{g} \cdot v^2 + W \cdot AH = \frac{1}{2}\frac{W}{g} \cdot V_A^2, \quad . \quad . \quad . \quad (104)$$

or

$$v^2 + 2g \cdot AH = V_A^2, \quad . \quad . \quad . \quad . \quad (105)$$

which is the same as (103). It is evident that the normal reaction at B is

$$R=\frac{W}{g}\cdot\frac{v^2}{r}+W\cos\theta. \quad . \quad . \quad . \quad . \quad (106)$$

The least value of R will occur when the particle is at C, the highest point of the circle. One can readily find the condition which must exist so that the particle describes a complete circle, which condition is

$$V_A>\sqrt{5\,gr}. \quad . \quad . \quad . \quad . \quad . \quad (107)$$

EXAMPLES XVII

1. A ball weighing 10 pounds is whirled around on a smooth horizontal circular track of radius 10 feet, with a velocity of 30 feet per second. What is the tension in the cord?

2. In example (1) suppose the string will only sustain a tension of $1600\,\pi^2/g$ pounds. Find the greatest number of revolutions per second that the ball can make without breaking the string.

3. With what velocity must a body revolve on a smooth horizontal circular track of 5 feet radius in order that the centrifugal force may equal the weight of the body?

4. A weight of 5 pounds is attached to the end of a cord 3 feet long just capable of sustaining a pull of 100 pounds. How many revolutions per second must the body make in order that the cord may be on the point of breaking?

5. A railway car, weighing 7 tons, moving at the velocity of 30 miles per hour, describes an arc whose

radius is 400 yards. What is the outward pressure on the track?

6. At what speed will an engine going around a curve of 960 yards radius, exert a horizontal thrust on the rails equal to $\frac{1}{90}$ of its own weight?

7. What is the least radius of the curve around which a car may run on a level at 30 miles per hour without producing a side thrust of more than $\frac{1}{100}$ of its own weight?

8. What is the tension of a string 4 feet long, which has a weight of 10 pounds attached to it, describing a horizontal circle once in half a second.

9. A car is going around a curve of 500 feet radius at 30 miles per hour. Find the centrifugal force if the car weighs 6 tons.

10. A skater describes a circle of 100 feet radius with a velocity of 18 feet per second. What is his inclination to the ice?

11. Calculate the tension of an endless chain, of 1 pound per foot, and 30 feet long, when made to rotate as a horizontal circle once a second.

12. What weight will exert a tension of 16 π^2 pounds, when rotated twice per second around a circle of radius 10 feet?

13. A uniform disc rotates 300 times per minute about an axis through its center and perpendicular to its plane. It has attached to it two weights, one of 6 pounds and the other of 8 pounds, at an angular distance of 90° apart, the first being 2 feet from the axis and the second 4 feet from the axis. Find the magnitude and direction of the resultant centrifugal force on the axis. Find also where a weight of 16 pounds must be placed on the disc to make the resultant centrifugal force zero.

14. A railway car is going around a curve of 500 feet radius at 30 miles per hour. Find how much a plumb bob, hung from the roof of the car by a thread 6 feet long, would be deflected from the vertical.

15. At what angle should the plane of the rails of a car track be inclined to the horizontal, when the gauge is $56\frac{1}{2}$ inches, the radius of the curve is 300 yards, and the highest velocity attainable is 45 miles per hour?

16. How much must the outer rail of a car track of 4 feet $8\frac{1}{2}$ inches gauge be elevated on a curve of 750 feet radius in order that a train may exert a thrust normal to the track when travelling at 45 miles per hour?

17. What is the maximum speed of a car rounding a curve of 1000 feet radius, when the outer rail is 3 inches above the inner one, and the gauge is 5 feet?

18. To what angle should a circular auto course of radius $\frac{1}{2}$ mile be banked in order to allow a speed of 90 miles per hour, supposing no friction?

19. In a high school gymnasium is a circular track of 20 laps to the mile. How much must the track be banked in order to allow a maximum speed of 15 miles per hour?

20. A string 4 feet long, fixed at one end has attached, to its other end a ball of weight 1 pound which describes a horizontal circle, making 60 circuits per minute. What is the inclination of the string to the vertical and the tension in the string?

21. Find in inches the change in height of the conical pendulum of example 20, if its speed is increased to 100 revolutions per minute.

22. What percentage change of angular speed in a conical pendulum will correspond to the decrease in height of 2 per cent?

23. A string of length 4 feet and having one end attached to a fixed point, and the other end to a weight of 40 pounds, revolves as a conical pendulum 30 times per minute. Find the tension in the string and its inclination to the vertical.

24. The revolving ball of a conical pendulum weighs 10 pounds, and the height of the pendulum is 6 inches. What is the speed?

25. A ball of weight 2 pounds, attached to a string, is moving in a vertical circle of 4 feet diameter. If its velocity when passing through the lowest point is 30 feet per second, find its velocity and the tension of the string when it is 2 feet and 3 feet above the lowest point, and at the highest point.

26. A ball weighing 2 pounds is whirling in a vertical circle at the extremity of a string 4 feet long. Find the velocity of the ball and tension of the string (1) at the highest position, (2) at the lowest, (3) at a point 2 feet above the lowest point, (4) at a point 5 feet above the lowest point, if the velocity at the lowest point is just sufficient to cause the ball to make a complete revolution.

CHAPTER VI

ENERGY OF ROTATION

Moment of Inertia

61. Energy of Rotation. If a particle of mass w/g is moving in a circle of radius r, with angular velocity ω, then the linear velocity of the particle is ωr; hence the kinetic energy is

$$\text{K. E.} = \frac{1}{2}\frac{w}{g}(\omega r)^2 \quad . \quad . \quad . \quad . \quad (108)$$

We may suppose that a rotating body is made up of a number of particles of mass w_1/g, w_2/g, w_3/g, etc., at distances r_1, r_2, r_3, etc., from the axis of rotation. If ω is the angular velocity of the body, then the linear velocities of the particles are ωr_1, ωr_2, ωr_3, etc., since ω is the same for all. Hence the kinetic energy of a rotating body, being the sum of the energies of all the particles, is expressed by the equation

$$\text{K. E.} = \frac{1}{2}\frac{w_1}{g}(\omega r_1)^2 + \frac{1}{2}\frac{w_2}{g}(\omega r_2)^2 + \frac{1}{2}\frac{w_3}{g}(\omega r_3)^2 +, \text{etc.},$$

or

$$\text{K. E.} = \tfrac{1}{2}\,\omega^2 \Sigma\left(\frac{w}{g}r^2\right) \quad . \quad . \quad . \quad . \quad . \quad (109)$$

62. Moment of Inertia. If the mass of each particle which makes up a body, is multiplied by the square of its distance from a given axis, the sum of the products is called the moment of inertia of the body about that axis.

Let I = the moment of inertia of the body about a given axis;
w/g = the mass of any particle of the body;
r = distance of the particle from the given axis.

Then

$$I = \Sigma\left(\frac{w}{g}r^2\right) \quad . \quad . \quad . \quad . \quad . \quad (110)$$

The value of the kinetic energy is given by the equation

$$\text{K. E.} = \tfrac{1}{2}\, I\omega^2. \quad . \quad . \quad . \quad . \quad (111)$$

63. Radius of Gyration. If the whole mass W/g of a body is concentrated at a single point which moves in a circle of radius k with the same angular velocity as the body, then the kinetic energy is $\frac{1}{2}\, W/g\ \omega^2 k^2$. If k is so chosen that the energy is same as that of the rotating body, then

$$\frac{1}{2}\, I\omega^2 = \frac{1}{2}\,\frac{W}{g}\,\omega^2 k^2.$$

Hence

$$I = \frac{W}{g}k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

or

$$k^2 = \frac{\text{Moment of inertia}}{\text{Mass}}. \quad . \quad . \quad (113)$$

Then (109) may be written

$$K.\ E. = \frac{1}{2}\frac{W}{g}\omega^2 k^2 \quad . \quad . \quad . \quad . \quad (114)$$

64. Moment of Inertia of a Thin Plate about an Axis Perpendicular to Its Plane. Suppose two lines, OX and OY, perpendicular to each other and lying in the plane of the plate, meet at the point O where the axis pierces the plate (Fig. 33).

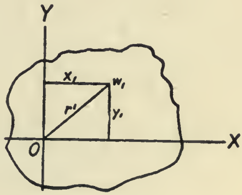


FIG. 33.

Any constituent particle w_1 is at a distance of x_1 from the Y axis, and at a distance of y_1 from the X axis. The moments of inertia of the plate about the X and Y axes are

$$I_x = \Sigma\left(\frac{w}{g}y^2\right)$$

and

$$I_y = \Sigma\left(\frac{w}{g}x^2\right).$$

Adding these two equations we get

$$I_x + I_y = \Sigma\frac{w}{g}(x^2 + y^2),$$

$$= \Sigma\frac{w}{g}r^2.$$

Denoting the value of $\Sigma w/g\ r^2$ by I_0 we have

$$I_0 = I_x + I_y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

Also since $x^2 + y^2 = r^2$,

$$k_0^2 = k_x^2 + k_y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

65. Moments of Inertia of a Thin Plate about Parallel Axes in Its Plane. Given the moment of inertia of the plate about an axis in the plane of the plate which passes through the center of gravity of the plate (Fig. 34), to find the moment of inertia about a new axis ZZ' in the plane of the plate, which is parallel to the axis OO' and at a distance a from it.

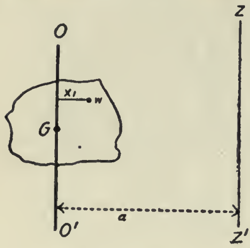


FIG. 34.

Let $I_{OO'}$ be the moment of inertia of the plate about OO'; and let $I_{zz'}$ be the moment of inertia of the plate about ZZ'.

Then

$$I_{zz'}=\Sigma\frac{w}{g}(a-x)^2,$$

$$=a^2\Sigma\frac{w}{g}+\Sigma\frac{w}{g}\,x^2-2a\Sigma wx.$$

The sum Σwx is, by Theorem I, Section 52, equal to Σw times the distance of the center of gravity of the plate from OO'. But $\bar{x}$ is zero, hence

$$I_{zz'}=I_{OO'}+\frac{W}{g}a^2, \quad . \quad . \quad . \quad . \quad . \quad (117)$$

where W is the total weight of the plate. Dividing each term of this equation by W/g we get

$$k^2_{zz'}=k^2_{OO'}+a^2. \quad . \quad . \quad . \quad . \quad . \quad (118)$$

66. Table of Moments of Inertia

Name of Body and Dimensions	Position of Axis about which Rotation Takes Place	Moment of Inertia
Uniform circular rod, length l, radius r	Traverse through end	$\frac{W}{g}\left(\frac{r^2}{4}+\frac{l^2}{3}\right)$
Uniform circular rod, length l, radius r	Traverse through c.g.	$\frac{W}{g}\left(\frac{r^2}{4}+\frac{l^2}{12}\right)$
Solid circular cylinder, radius r	About its own axis	$\frac{W}{g}\frac{r^2}{2}$
Hollow circular cylinder, radii, R *and* r	About its own axis	$\frac{W}{g}\left(\frac{R^2+r^2}{2}\right)$
Solid sphere of radius r	About a diameter	$\frac{W}{g}\frac{2}{5}r^2$
Solid cone, radius of base r	About its own axis	$\frac{W}{g}\frac{3}{10}r^2$
Thin rectangular plate, length a, width b	About an axis coinciding with side b	$\frac{W}{g}\frac{1}{3}a^2$
Thin rectangular plate, length a, width b	Axis through c.g. parallel to side b	$\frac{W}{g}\frac{1}{12}a^2$
Thin rectangular plate, length a, width b	Axis through c.g. perpendicular to plane of plate	$\frac{W}{g}\frac{a^2+b^2}{12}$
Thin triangular plate, altitude h, base b	Through c.g. parallel to base	$\frac{W}{g}\frac{1}{18}h^2$
Thin circular disc, radius r	Through center, perpendicular to disc	$\frac{W}{g}\frac{1}{2}r^2$
Thin circular disc, radius r	About a diameter	$\frac{W}{g}\frac{1}{4}r^2$

If we consider the cross-section of a uniform circular rod as being very small, then W/g becomes the length of the rod. Also if the thickness of a uniform plate is very small, then W/g represents the area of the plate.

EXAMPLE 1. A solid disc of iron is 20 inches in diameter and 4 inches thick. If the disc rotates 300 times per minute, what is its kinetic energy? Iron weighs .32 pound per cubic inch.

Solution. The weight of the disc is 128π pounds. Therefore the kinetic energy equals

$$\frac{128\pi \times 100 \times 100\pi^2}{2 \times 32 \times 2 \times 144} = 2153.2 \text{ ft.-lbs.}$$

EXAMPLE 2. A solid circular disc of radius 2 feet and thickness 4 inches, weighs 3200 pounds. How much work is required to increase its speed from 90 to 120 rotations per minute?

Solution. Change of kinetic energy = Work done (Section 35). Therefore the work required equals

$$\frac{3200 \times 4 \times (16\pi^2 - 9\pi^2)}{2 \times 2 \times 32} = 700\pi^2 \text{ ft.-lbs.}$$

EXAMPLE 3. A thin rectangular plate 12 inches by 8 inches has a rectangular part 8 inches by 5 inches removed, the diagonals of the two parts being in the same straight line. Find the moment of inertia of the part remaining about one of the shorter outer sides.

Solution. Let I_{AB} be the moment of inertia of the part remaining about the side AB (Fig. 35). The moment of inertia of the area *abcd* about AB equals $\frac{1}{12}\times 40\times 8^2+40\times 6^2$. The moment of inertia of $ABCD$ about AB equals $\frac{1}{3}\times 8\times 12\times 12^2$. The difference between these two moments of inertia equals I_{AB}. Hence

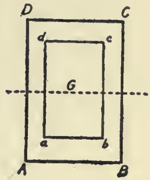


FIG. 35.

$$I_{AB}=4608-\frac{4960}{3}=2961\tfrac{1}{3}\text{ (inches)}^4.$$

EXAMPLES XVIII

1. A circular disc of radius 4 feet and weight 100 pounds makes 60 revolutions per minute. Find the kinetic energy of the disc.

2. Find the moment of inertia of a cylinder of weight 12 pounds and 4 inches radius, about an axis parallel to its own, and at a distance of 10 feet from it.

3. The radius of gyration of a flywheel is $4\sqrt{2}$ feet. The wheel stores 20,000 π^2 foot-pounds of kinetic energy when rotating 100 times per minute. Find the weight of the wheel.

4. A flywheel requires 20,000 foot-pounds of work to be done upon it to increase its speed from 68 to 70 rotations per minute. What is the moment of inertia of the wheel? If the wheel weighs 4096 pounds, what is its radius?

5. A flywheel has a weight of 30 tons, which may be supposed to be distributed along the circumference of a circle 8 feet in radius; the wheel rotates 20 times per minute. Find its kinetic energy.

6. What is the kinetic energy of a circular saw having a diameter of 2 feet, and $\frac{1}{8}$ inch thick, if a point on the circumference has a linear velocity of 6000 feet per minute? Density of steel being considered as 500 pounds per cubic foot.

7. A girder of I-shaped cross-section has two horizontal flanges 10 inches broad and 2 inches thick, connected by a vertical web 12 inches high and 2 inches thick. Find the moment of inertia of the area of the section (1) about a horizontal axis through its center of gravity, (2) about the lower edge as an axis.

8. Find the moment of inertia of the area enclosed between two concentric circles of 20 inches and 12 inches diameter, about a diameter as an axis.

9. Find the radius of gyration of a flywheel rim 6 feet in external diameter and 6 inches thick, about its axis. If the rim is 10 inches broad, and of cast iron, what is its moment of inertia about its axis? Cast iron weighs .26 pound per cubic inch. If the wheel rotates 200 times per minute, what is its kinetic energy?

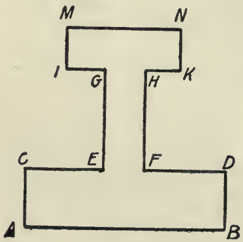


FIG. 36.

10. Fig. 36 represents the cross-section of a girder. $MN=6$ inches, $KN=2$ inches, $GH=2$ inches, $GE=8$ inches, $AC=2\frac{1}{2}$ inches, and $AB=10$ inches. Find the moment of inertia of the section (1) about AB (2) about an axis parallel to AB and passing through the center of gravity of section.

LOGARITHMS OF NUMBERS

No.	0	1	2	3	4	5	6	7	8	9
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059
13	11392	11727	12057	12385	12710	13033	13354	13672	13988	14301
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140
16	20412	20683	20952	21219	21484	21748	22011	22272	22531	22789
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225
45	65321	65418	65514	65610	65706	65801	65896	65992	66086	66181
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957
No.	0	1	2	3	4	5	6	7	8	9

LOGARITHMS OF NUMBERS

No.	0	1	2	3	4	5	6	7	8	9
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154
78	89209	89265	89321	89376	89432	89[illegible]87	89542	89597	89653	89708
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795
81	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855
83	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376
84	92428	92480	92531	92583	93634	92686	92737	92788	92840	92891
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890
89	94939	94988	95036	95085	95134	95182	95231	95279	95329	95376
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957
No.	0	1	2	3	4	5	6	7	8	9

TABLE OF NATURAL VALUES OF THE TRIGONOMETRIC FUNCTIONS

Angle in Degrees.	sin	tan	cot	cos	Angle in Degrees.
0	0000	0000	Infinite	1.0000	90
1	0175	0175	57.2900	9998	89
2	0349	0349	28.6363	9994	88
3	0523	0524	19.0811	9986	87
4	0698	0699	14.3007	9976	86
5	0872	0875	11.4301	9962	85
6	1045	1051	9.5144	9945	84
7	1219	1228	8.1443	9925	83
8	1392	1405	7.1154	9903	82
9	1564	1584	6.3138	9877	81
10	1736	1763	5.6713	9848	80
11	1908	1944	5.1446	9816	79
12	2079	2126	4.7046	9781	78
13	2250	2309	4.3315	9744	77
14	2419	2493	4.0108	9703	76
15	2588	2679	3.7321	9659	75
16	2756	2867	3.4874	9613	74
17	2924	3057	3.2709	9563	73
18	3090	3249	3.0777	9511	72
19	3256	3443	2.9042	9455	71
20	3420	3640	2.7475	9397	70
21	3584	3839	2.6051	9336	69
22	3746	4040	2.4751	9272	68
23	3907	4245	2.3559	9205	67
24	4067	4452	2.2460	9135	66
25	4226	4663	2.1445	9063	65
26	4384	4877	2.0503	8988	64
27	4540	5095	1.9626	8910	63
28	4695	5317	1.8807	8829	62
29	4848	5543	1.8040	8746	61
30	5000	5774	1.7321	8660	60
31	5150	6009	1.6643	8572	59
32	5299	6249	1.6003	8480	58
33	5446	6494	1.5399	8387	57
34	5592	6745	1.4826	8290	56
35	5736	7002	1.4281	8192	55
36	5878	7265	1.3764	8090	54
37	6018	7539	1.3270	7986	53
38	6157	7813	1.2799	7880	52
39	6293	8098	1.2349	7771	51
40	6428	8391	1.1918	7660	50
41	6561	8693	1.1504	7547	49
42	6691	9004	1.1106	7431	48
43	6820	9325	1.0724	7314	47
44	6947	9657	1.0355	7193	46
45	7071	1.0000	1.0000	7071	45
Angle in Degrees.	cos	cot	tan	sin	Angle in Degrees.

ANSWERS

EXAMPLES I

1. 4 : 1.
2. 6 : 7.
3. 50 ft. per sec.
4. $79\frac{1}{5}$ ft. per sec.
5. 70 ft.; 110 ft.; 1440 ft.
6. .293 ft. per sec. per sec.
7. $5\frac{107}{225}$ sec.; 112.44 ft.
8. 18 ft. per sec. per sec.
9. 200 ft.; 1 ft. per sec. per sec.
10. 25 ft.; $\frac{1}{2}$ ft. per sec. per sec.
11. 5 ft. per $(\text{sec.})^2$; 10 sec.; 15 sec.; 562.5 ft.
12. $6\frac{2}{5}$ ft. per $(\text{sec.})^2$.
13. Retardation 10 ft. per $(\text{sec.})^2$; 2 sec.; 20 ft.
14. 1.467 ft. per $(\text{sec.})^2$; $\frac{1}{8}$ mile; 4.4 ft. per $(\text{sec.})^2$; $\frac{1}{24}$ mile.
15. 3.44 ft. per $(\text{sec.})^2$.
16. $\frac{22}{45}$ ft. per $(\text{sec.})^2$; 7920 ft.
17. 3 ft. per sec.; 2 ft. per $(\text{sec.})^2$.
18. 30 ft. per sec.; 2 ft. per $(\text{sec.})^2$.
19. 40 sec.
20. 18 ft. per $(\text{sec.})^2$.

EXAMPLES II

1. $\omega=\frac{8}{5}$ radians per sec.
2. $20\,\pi/3$ radians per sec.
3. $\pi/1800$ radians per sec.
4. Linear, 1 : 20 : 360; Angular, 1 : 12 : 720
5. 44 radians per sec.
6. 5 sec.
7. 135 revolutions; 90 sec.
8. 35.2 radians per sec.; 2.581 radians per sec.

9. $\frac{\pi}{365.24 \cdot 1800}$ radians per sec.; 18.52 miles per sec.
10. 2.19 radians per sec.; 13.14 ft. per sec.
11. 42 ft. per sec., at 120° with first direction.
12. $x = 24\frac{3}{8}$ rev. per sec.; $\frac{15}{8}$ rev. per sec.
13. $x = 10$ rev. per sec.; $\frac{1}{8}$ rev. per sec.; $588\frac{1}{16}$ rev.
14. π radians; 18.84 ft. per sec.; .157 radian per sec. per sec.
15. (*a*) $3\frac{1}{3}$ radians per sec.; (*b*) 31.83 rev.; (*c*) 300 rev.

EXAMPLES III

1. 1600 ft.; .79 sec.; 120 ft. per sec.
2. 400 ft.
3. 3.06 sec.
4. 64 ft. per sec.
5. 231.72 ft.
6. 432 ft.
7. $V_1 = 160$ ft. per sec.; total time is 10 sec.
8. $156\frac{1}{4}$ ft.; .548 sec. and 5.7 sec. after time of the projection.
9. $80\frac{21}{64}$ ft. above point of projection; $\frac{53}{32}$ sec. and $\frac{97}{32}$ sec. after projection.
10. 160 ft. per sec.; 400 ft.
11. 5 sec. and 20 sec. later.
12. $4\frac{256}{1190}$ sec.
13. 5 sec.
14. 2 sec. or 8 sec.; 96 ft. per sec.
15. .9463 sec.
16. 256 ft.; 4 sec.
17. 150 ft.
18. 112 ft.
19. $t = 5$ sec.; 64 ft. per sec.
20. $42\frac{1}{4}$ ft.

EXAMPLES IV

1. (4° 17′, 34.87).
2. (85° 50′, 7.12).
3. (51° 55′, 12.55).
4. 15.095 miles per hour directed N. 10° 48′ W.
5. North $12\sqrt{2}$ miles per hour; East $12\sqrt{2}$ miles per hour.
6. Horizontal Velocity = Vertical Velocity = $300\sqrt{2}$ ft. per sec.
7. Upstream 30°; 1.57 minutes.

8. 10 ft. per sec.

9. Current 6.3096 miles per hour; steamer 9.0112 miles per hour.

10. $70° 32'$.

11. 14 ft. per sec.; $\cos^{-1} \frac{13}{14}$ with greatest velocity.

12. $\sqrt{170}$ ft. per sec.; $\tan^{-1} \frac{1}{13}$ with line making $\tan^{-1} \frac{12}{5}$ with the east direction.

13. 64.78 miles directed S. $19° 7'$ E.

14. 28.3 miles directed S. $48° 28'$ E.

15. $(135°, 2.234)$.

16. $(207° 30', 26.7)$.

17. N. $10° 33'$ E.

18. N. $46° 9'$ W.; $76° 9'$; 35 min. 40 sec.

EXAMPLES V

2. 29.93 ft. per sec.; 1.336 sec.

3. $5° 23'$.

4. $(\sqrt{3}+\sqrt{2}) : 1$.

6. (1) 192 ft. per sec.; (2) $153\frac{3}{5}$ ft. per sec. and $115\frac{1}{5}$ ft. per sec.; (3) 960 ft.

7. 9211 ft.; 76.8 ft. per sec.

8. 19,771 ft.; 120.8 ft. per sec.

9. 2 sec. or 18 sec.

10. 17.26 ft. per sec.

11. 93.22 ft.

12. 18.97 ft. per sec.

13. $8\frac{1}{4}$ sec.

14. 32 ft., 96 ft., 160 ft. respectively from top; 3 sec.

15. $30°$.

EXAMPLES VI

1. (1) 1614 ft.; (2) 7694 ft. and 20 sec.; (3) 473.7 ft. per sec. and $B=36° 3'$ or $143° 57'$.

2. $312\frac{1}{2}$ ft.; $13° 43'$ or $76° 17'$; 4.42 sec.

3. $t=\frac{5}{2}$ sec.; 360 ft.; 165 ft. per sec.

4. 799.8 ft.

5. 19.35 sec.

6. 1082.5 ft.

7. 11.2 sec.

8. 271.2 ft. per sec.; 8.47 sec.

9. 2131.5 ft.
10. $67\frac{1}{2}$ ft.; 4.11 sec.
11. 60°.
12. 8189 ft.; 19.95 sec.

EXAMPLES VII

1. 5 : 8.
2. 750 ft.
3. 1768 units.
4. 44.2 units.
5. 64 : 5.
6. $15\frac{5}{18}$ lbs.
7. 1600 seconds.
8. 480 ft.
9. 48 ft. per sec.; 720 ft.
10. 3000 units; $166,666\frac{2}{3}$ lbs.
11. $5055\frac{5}{9}$ lbs.
12. 41 sec.; 902 ft.
13. 71 sec.
14. 59.8 lbs. per ton.
15. $7812\frac{1}{2}$ lbs.
16. 2.34 units; 780 lbs.

EXAMPLES VIII

1. 4 ft. per sec.
2. (1) $6\frac{1}{4}$ ft. per sec.; (2) $\frac{25}{128}$ sec.; (3) $\frac{625}{1024}$ ft.
3. 25 tons.
4. 25 lbs.
5. $14\frac{1}{16}$ ft.
6. $31\frac{1}{4}$ tons.
7. 477 lbs.
8. 1010 ft.
9. 8 ft. per sec.; $3\frac{3}{4}$ lbs.
10. 3000 lbs.
11. 170 lbs.
12. 5.982 sec.
13. $\frac{3}{4}$ given distance.
14. $\frac{1}{3}$ g.; $166\frac{2}{3}$ lbs.
15. 10 lbs.

16. 40 ft.
17. 4 lbs.
18. $\frac{1}{10}$ g.; $\sqrt{5}$ sec.; $\frac{16}{\sqrt{5}}$ ft. per sec.
19. $\frac{1}{16}$ g.; $2\sqrt{3}$ sec.
20. 5 lbs.

EXAMPLES IX

1. 660,000 ft.-lbs.
2. 82,500 ft.-lbs.
3. 1,800,000 ft.-lbs.
4. 51 H.P.
5. 10 H.P.
6. 396,000 ft.-lbs.
7. 3 H.P.
8. 40 H.P.
9. 2080 H.P.
10. 288 H.P.
11. $172\frac{4}{5}$ ft.
12. 15 miles per hour.
13. 600 H.P.
14. 13 lbs.
15. $5.25.
16. (*a*) 39.27 H.P.; (*b*) 65.45 H.P.

EXAMPLES X

1. 20,000 ft.-lbs.
2. (1) 1280 ft.-lbs.; (2) 0 ft.-lbs.
3. 19,360 ft.-lbs.
4. 4102 ft.-lbs.
5. 40,000 ft.-lbs.
6. 40 ft.-lbs.
7. $3160\frac{1}{2}$ ft.-lbs.
8. $1\frac{2}{3}$ ft.
9. 2000 ft. per sec.
10. 256 lbs.
11. $9\frac{1}{3}$ ft. per sec.; $\frac{5}{24}$ ft.-lbs.
12. 880 lbs.
13. 2346 lbs.
14. 76.4 lbs.
15. 48,400 lbs.

16. 100 yards.
17. Gain 500 ft.; $37\frac{1}{2}$ mi. per hour.
18. 5.57 H.P. and 185.6 lbs.
19. 18,000 ft.-lbs.; $93\frac{1}{3}\%$.

EXAMPLES XI

1. $(P+Q)$ in same direction.
2. (*a*) 25; (*b*) 5; (*c*) $10\sqrt{21}$; (*d*) $4\sqrt{3}$; (*e*) 60°.
3. 120°.
4. 90°.
5. Direction is along *BO* and resultant equals 3*BO*.
6. $P=9$, $Q=3$.
7. 4.294 lbs., making an angle of 128° 9′, with *X*-axis.
8. 16.87 lbs. making an angle of 121° 23′ with *X*-axis.
9. Zero lbs.; hence, in equilibrium.
10. 17.4 lbs.
11. 67° nearly.

EXAMPLES XII

1. 120 lbs. and 90 lbs.
2. 80 lbs. and 60 lbs.
3. 52 lbs. and 73 lbs.
4. $\cos^{-1}\frac{3}{5}$.
5. 6 lbs.; 120° with *X*-axis.
6. $W=100$ lbs; $T=141.4$ lbs.
7. $T=56$ lbs.; $W=70$ lbs.
8. 162° 31′ inclined to the force of 9 lbs.; $\sqrt{133}$ lbs.
9. 44.58 lbs. directed S. 11° 44′ W.
10. $P=500$ lbs.
11. $\sqrt{3} : 1 : 2$.
12. $P=9.17$ lbs. and $F=4$ lbs.
13. 10 lbs.
14. $2\frac{1}{2}$ tons; $7\frac{1}{2}$ tons.
15. $\cos^{-1}\frac{4}{5}$.

EXAMPLES XIII

1. $R_B=1058\frac{1}{3}$ lbs.; $R_A=1141\frac{2}{3}$ lbs.
2. 160 lbs. at a distance of 6 ft. from *A*.
3. (1) 36 lbs., 3 inches from *B*; (2) 42 lbs., 12 inches from *B*; (3) 18 lbs., 4 inches from *B*.

4. (1) 10 lbs., having the same direction as P and applied at a distance of 12 inches to the left of A; (2) 20 lbs., having the same direction as Q and applied 6 inches to the right of B; (3) 6 lbs., having the same direction as P and applied 30 inches to the left of A.

5. $P = 2\frac{3}{4}$ lbs. and $R = 13\frac{3}{4}$ lbs.

6. $P = 24\frac{3}{4}$ lbs. and $R = 13\frac{3}{4}$ lbs.

7. $3\frac{1}{3}$ ft. from the lesser force.

8. 8 ft.

9. 17.9 lbs.

10. 6 ft. from end man.

11. 4 lbs. and 6 lbs.

12. 20 lbs.

13. (1) 7 tons; (2) $7\frac{2}{3}$ tons and $6\frac{1}{3}$ tons.

14. $14\frac{1}{2}$ tons and $21\frac{1}{2}$ tons.

15. $2\sqrt{2}$ lbs. inclined 45° to side AB, and cutting AB produced $\frac{5}{2}a$ to the left of A.

16. $24\frac{1}{2}$ lbs.; 168 lbs.; $24\frac{1}{2}$ lbs.

17. Tension 36 lbs.; pressure on hinge $4\sqrt{91}$ lbs.; $\theta = \tan^{-1}\frac{11}{9\sqrt{3}}$ downward.

EXAMPLES XIV

1. $3\frac{2}{11}$ ft. from 8 lbs. weight.

2. 4.04 in. from side joining the 10 and 20 lb. weights.

3. $2\frac{4}{9}$ ft. from A.

4. $8\frac{2}{3}$ lbs.

5. $1\frac{1}{5}$ ft. from center of the rod.

6. $\frac{2}{3}$ ft. from center of the wheel.

7. $\frac{1}{7}a\sqrt{2}$ from center of the square.

8. At the center of the cross-bar.

9. $18\frac{1}{2}$ in. from end of the long arm.

10. 12.88 in.

11. 5.72 in.

12. $\dfrac{b^2+bh+\frac{1}{3}h^2}{2b+h}$.

13. $\dfrac{h}{3}\cdot\dfrac{a+2b}{a+b}$.

14. 10. in. from the base.

16. .9 in. from the common base.
17. $\frac{1}{21} a\sqrt{2}$.
18. .166 in.
19. .637 in.
20. 10 in.
21. 19.37 in.
22. 15.91 in.
23. 12.73 in. from center; 8.486 in. from center.
24. 18.39 in.
25. 23.37 in.
26. .267 in.
27. $h=6(3-\sqrt{3})$ in.
28. .21 in.
29. $26\frac{1}{4}$ in.
30. 1.88 ft
31. 5.48 in.
32. 1.76 ft.
33. .113 in.
34. 38.22 in.
35. 4.05 ft.
36. 8.7 in.
37. 4.08 in.
38. 5.119 in.
39. 4.82 in.
40. 4.33 ft.
41. $2\frac{1}{6}$ ft.
42. $3\frac{6}{13}$ in.
43. $5\frac{1}{16}$ ft.
44. $\frac{3}{8}$ in.
45. 6.52 in.
46. 6.44 ft.
47. 8.11 ft.
48. $3\frac{3}{8}$ in.
49. 7.838 ft.; 2.286 ft.

EXAMPLES XV

1. $78° 28'$.
2. $78° 41'$.
3. $45°$.
4. 10 inches.

5. $\tan\theta = \frac{2r}{a}$.

6. $h = 5\sqrt{2}$ inches.

7. $h = 2\sqrt{3}$ ft.

8. 51° 21′.

9. $h = 24\sqrt{3}$ inches and radius is 6 inches.

10. 61° 25′.

11. Equilibrium when there are 4 courses; will topple when there are 5 courses.

12. 414 ft.-lbs.

13. 97,335π ft.-lbs.

14. 26,158 ft.-lbs.

15. 40,212.48 ft.-lbs.

16. 30,679,687.5 ft.-lbs.

17. 235,620 ft.-lbs.

18. 43,030π ft.-lbs.

19. 27,000 ft.-lbs.

20. 138,240 ft.-lbs.

21. 18.

EXAMPLES XVI

4. 720π^2 sq. in.; 3600π^2 cu. in.

5. 250π^2 cu. in.; 100π^2 sq. ft.

6. 5680 cu. in.

7. 355.85 sq. in.; 6830.66 cu. in.

8. 841.06 sq. in.; 485.5 cu. in.

9. 113.1 sq. in.; 28.27 cu. in.

10. 16π cu. ft.

11. 4421.6 sq. in.; 17,686.4 cu. in.

12. 597.5 sq. in.; 952.8 cu. in.

13. 3873 sq. in.; 21,360.5 cu. in.

14. 6408.8 cu. in.

15. 444.1 sq. in.; 333.1 cu. in.

EXAMPLES XVII

1. $28\frac{1}{8}$ lbs.

2. $n = 2$.

3. $v = 12.6$ ft. per sec.

4. $n = 2.32$.

5. 706 lbs.
6. $v = 21\frac{9}{11}$ mi. per hour.
7. $R = 6050$ ft.
8. 197.36 lbs.
9. 1452 lbs.
10. About 84°.
11. $60\pi/g$ lbs. per ft.
12. 3.2 lbs.
13. $106.7\pi^2$ lbs. inclined 20° 33′ to the 8-lb. force; $x = 2.134$ ft. from center on opposite side of resultant.
14. 6° 54′.
15. 8° 36′.
16. $10\frac{1}{4}$ inches.
17. $27\frac{3}{11}$ mi. per hour.
18. 11° 39′.
19. 19° 48′.
20. 78° 19′; 4.934 lbs.
21. 6.23 inches.
22. $\omega_2 = 101\%\omega_1$.
23. $T = 49.3$ lbs.; $\theta = 35° 51′$.
24. 76.4 rev. per min.
25. $V_1 = 27.78$ ft. per sec.; $V_2 = 26.61$ ft. per sec.; $V_3 = 25.38$ ft. per sec.; $T_1 = 24\frac{1}{8}$ lbs.; $T_2 = 21\frac{1}{8}$ lbs.; $T_3 = 18\frac{1}{8}$ lbs.
26. $V_1 = 11.31$ ft. per sec.; $V_2 = 25.3$ ft. per sec.; $V_3 = 22.11$ ft. per sec.; $V_4 = 17.84$ ft. per sec.; $T_1 = 0$ lbs.; $T_2 = 12$ lbs.; $T_3 = 9$ lbs.; $T_4 = 4\frac{1}{2}$ lbs.

EXAMPLES XVIII

1. 493.48 ft.-lbs.
2. 37.5 units.
3. 3600 lbs.
4. 13,216 units; 14.37 ft.
5. 263,188 ft.-lbs.
6. 1278 ft.-lbs.
7. (1) $2261\frac{1}{3}$ (inches)^4; (2) $6357\frac{1}{3}$ (inches)^4.
8. $34{,}816\pi$ (inches)^4.
9. 770.7 units; 169,040 ft.-lbs.
10. (1) $2404\frac{5}{12}$ (inches)^4; (2) 995.63 (inches)^4.

INDEX

The numbers refer to sections