

# A solution manual for Polchinski's *String Theory*

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## Abstract

We present detailed solutions to 81 of the 202 problems in J. Polchinski's two-volume textbook *String Theory*.

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## 0 Preface

The following pages contain detailed solutions to 81 of the 202 problems in J. Polchinski's two-volume textbook *String Theory* [1, 2]. I originally wrote up these solutions while teaching myself the subject, and then later decided that they may be of some use to others doing the same. These solutions are the work of myself alone, and carry no endorsement from Polchinski.

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# 1 Chapter 1

## 1.1 Problem 1.1

(a) We work in the gauge where  $\tau = X^0$ . Non-relativistic motion means  $\dot{X}^i \equiv v^i \ll 1$ . Then

$$\begin{aligned} S_{\text{pp}} &= -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \\ &= -m \int dt \sqrt{1 - v^2} \\ &\approx \int dt \left( \frac{1}{2} m v^2 - m \right). \end{aligned} \quad (1)$$

(b) Again, we work in the gauge  $\tau = X^0$ , and assume  $\dot{X}^i \equiv v^i \ll 1$ . Defining  $u^i \equiv \partial_\sigma X^i$ , the induced metric  $h_{ab} = \partial_a X^\mu \partial_b X_\mu$  becomes:

$$\{h_{ab}\} = \begin{pmatrix} -1 + v^2 & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{u} \cdot \mathbf{v} & u^2 \end{pmatrix}. \quad (2)$$

Using the fact that the transverse velocity of the string is

$$\mathbf{v}_\text{T} = \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{u^2} \mathbf{u}, \quad (3)$$

the Nambu-Goto Lagrangian can be written:

$$\begin{aligned} L &= -\frac{1}{2\pi\alpha'} \int d\sigma (-\det\{h_{ab}\})^{1/2} \\ &= -\frac{1}{2\pi\alpha'} \int d\sigma (u^2(1 - v^2) + (\mathbf{u} \cdot \mathbf{v})^2)^{1/2} \\ &\approx -\frac{1}{2\pi\alpha'} \int d\sigma |\mathbf{u}| \left( 1 - \frac{1}{2} v^2 + \frac{\mathbf{u} \cdot \mathbf{v}^2}{2u^2} \right) \\ &= \int d\sigma |\mathbf{u}| \left[ \frac{1}{2} \rho v_\text{T}^2 - L_s T \right], \end{aligned} \quad (4)$$

where

$$\rho = \frac{1}{2\pi\alpha'} \quad (5)$$

is the mass per unit length of the string,

$$L_s = \int d\sigma |\mathbf{u}| \quad (6)$$

is its physical length, and

$$T = \frac{1}{2\pi\alpha'} = \rho \quad (7)$$

is its tension.

## 1.2 Problem 1.3

It is well known that  $\chi$ , the Euler characteristic of the surface, is a topological invariant, i.e. does not depend on the metric. We will prove by explicit computation that, in particular,  $\chi$  is invariant under Weyl transformations,

$$\gamma'_{ab} = e^{2\omega(\sigma,\tau)}\gamma_{ab}. \quad (8)$$

For this we will need the transformation law for the connection coefficients,

$$\Gamma'^a_{bc} = \Gamma^a_{bc} + \partial_b\omega\delta^a_c + \partial_c\omega\delta^a_b - \partial_d\omega\gamma^{ad}\gamma_{bc}, \quad (9)$$

and for the curvature scalar,

$$R' = e^{-2\omega}(R - 2\nabla_a\partial^a\omega). \quad (10)$$

Since the tangent and normal vectors at the boundary are normalized, they transform as

$$t'^a = e^{-\omega}t^a, \quad (11)$$

$$n'_a = e^{\omega}n_a. \quad (12)$$

The curvature of the boundary thus transforms as follows:

$$\begin{aligned} k' &= \pm t'^a n'_b (\partial_a t'^b + \Gamma'^b_{ac} t'^c) \\ &= e^{-\omega} (k \mp t^a t_a n^b \partial_a \omega), \end{aligned} \quad (13)$$

where we have used (9), (11), (12), and the fact that  $n$  and  $t$  are orthogonal. If the boundary is timelike then  $t^a t_a = -1$  and we must use the upper sign, whereas if it is spacelike then  $t^a t_a = 1$  and we must use the lower sign. Hence

$$k' = e^{-\omega} (k + n^a \partial_a \omega). \quad (14)$$

Finally, since  $ds = (-\gamma_{\tau\tau})^{1/2} d\tau$  for a timelike boundary and  $ds = \gamma_{\sigma\sigma}^{1/2} d\sigma$  for a spacelike boundary,

$$ds' = ds e^{\omega}. \quad (15)$$

Putting all of this together, and applying Stokes theorem, which says that for any vector  $v^a$ ,

$$\int_M d\tau d\sigma (-\gamma)^{1/2} \nabla_a v^a = \int_{\partial M} ds n^a v_a, \quad (16)$$

we find the transformation law for  $\chi$ :

$$\begin{aligned} \chi' &= \frac{1}{4\pi} \int_M d\tau d\sigma (-\gamma')^{1/2} R' + \frac{1}{2\pi} \int_{\partial M} ds' k' \\ &= \frac{1}{4\pi} \int_M d\tau d\sigma (-\gamma)^{1/2} (R - 2\nabla_a \partial^a \omega) + \frac{1}{2\pi} \int_{\partial M} ds (k + n^a \partial_a \omega) \\ &= \chi. \end{aligned} \quad (17)$$

### 1.3 Problem 1.5

For simplicity, let us define  $a \equiv (\pi/2p^+ \alpha' l)^{1/2}$ . Then we wish to evaluate

$$\begin{aligned}
& \sum_{n=1}^{\infty} (n - \theta) \exp[-(n - \theta)\epsilon a] \\
&= -\frac{d}{d(\epsilon a)} \sum_{n=1}^{\infty} \exp[-(n - \theta)\epsilon a] \\
&= -\frac{d}{d(\epsilon a)} \frac{e^{\theta\epsilon a}}{e^{\epsilon a} - 1} \\
&= -\frac{d}{d(\epsilon a)} \left( \frac{1}{\epsilon a} + \theta - \frac{1}{2} + \left( \frac{1}{12} - \frac{\theta}{2} + \frac{\theta^2}{2} \right) \epsilon a + \mathcal{O}(\epsilon a)^2 \right) \\
&= \frac{1}{(\epsilon a)^2} - \frac{1}{2} \left( \frac{1}{6} - \theta + \theta^2 \right) + \mathcal{O}(\epsilon a). \tag{18}
\end{aligned}$$

As expected, the cutoff dependent term is independent of  $\theta$ ; the finite result is

$$-\frac{1}{2} \left( \frac{1}{6} - \theta + \theta^2 \right). \tag{19}$$

### 1.4 Problem 1.7

The mode expansion satisfying the boundary conditions is

$$X^{25}(\tau, \sigma) = \sqrt{2\alpha'} \sum_n \frac{1}{n} \alpha_n^{25} \exp \left[ -\frac{i\pi n c \tau}{l} \right] \sin \frac{\pi n \sigma}{l}, \tag{20}$$

where the sum runs over the half-odd-integers,  $n = 1/2, -1/2, 3/2, -3/2, \dots$ . Note that there is no  $p^{25}$ . Again, Hermiticity of  $X^{25}$  implies  $\alpha_{-n}^{25} = (\alpha_n^{25})^\dagger$ . Using (1.3.18),

$$\Pi^{25}(\tau, \sigma) = -\frac{i}{\sqrt{2\alpha' l}} \sum_n \alpha_n^{25} \exp \left[ -\frac{i\pi n c \tau}{l} \right] \sin \frac{\pi n \sigma}{l}. \tag{21}$$

We will now determine the commutation relations among the  $\alpha_n^{25}$  from the equal time commutation relations (1.3.24b). Not surprisingly, they will come out the same as for the free string (1.3.25b). We have:

$$\begin{aligned}
i\delta(\sigma - \sigma') &= [X^{25}(\tau, \sigma), \Pi^{25}(\tau, \sigma')] \\
&= -\frac{i}{l} \sum_{n, n'} \frac{1}{n} [\alpha_n^{25}, \alpha_{n'}^{25}] \exp \left[ -\frac{i\pi(n + n')c\tau}{l} \right] \sin \frac{\pi n \sigma}{l} \sin \frac{\pi n' \sigma'}{l}.
\end{aligned} \tag{22}$$

Since the LHS does not depend on  $\tau$ , the coefficient of  $\exp[-i\pi m c \tau / l]$  on the RHS must vanish for  $m \neq 0$ :

$$\frac{1}{l} \sum_n \frac{1}{n} [\alpha_n^{25}, \alpha_{m-n}^{25}] \sin \frac{\pi n \sigma}{l} \sin \frac{\pi(n-m)\sigma'}{l} = \delta(\sigma - \sigma') \delta_{m,0}. \tag{23}$$



Multiplying both sides by  $\sin[\pi n'\sigma/l]$  and integrating over  $\sigma$  now yields,

$$\begin{aligned} \frac{1}{2n} \left( [\alpha_n^{25}, \alpha_{m-n}^{25}] \sin \frac{\pi(n-m)\sigma'}{l} + [\alpha_{n+m}^{25}, \alpha_{-n}^{25}] \sin \frac{\pi(n+m)\sigma'}{l} \right) \\ = \sin \frac{\pi n \sigma'}{l} \delta_{m,0}, \end{aligned} \quad (24)$$

or,

$$[\alpha_n^{25}, \alpha_{m-n}^{25}] = n \delta_{m,0}, \quad (25)$$

as advertised.

The part of the Hamiltonian (1.3.19) contributed by the  $X^{25}$  oscillators is

$$\begin{aligned} \frac{l}{4\pi\alpha'p^+} \int_0^l d\sigma \left( 2\pi\alpha' (\Pi^{25})^2 + \frac{1}{2\pi\alpha'} (\partial_\sigma X^{25})^2 \right) \\ = \frac{1}{4\alpha'p^+} \sum_{n,n'} \alpha_n^{25} \alpha_{n'}^{25} \exp \left[ -\frac{i\pi(n+n')c\tau}{l} \right] \\ \quad \times \int_0^l d\sigma \left( -\sin \frac{\pi n\sigma}{l} \sin \frac{\pi n'\sigma}{l} + \cos \frac{\pi n\sigma}{l} \cos \frac{\pi n'\sigma}{l} \right) \\ = \frac{1}{4\alpha'p^+} \sum_n \alpha_n^{25} \alpha_{-n}^{25} \\ = \frac{1}{4\alpha'p^+} \sum_{n=1/2}^{\infty} (\alpha_n^{25} \alpha_{-n}^{25} + \alpha_{-n}^{25} \alpha_n^{25}) \\ = \frac{1}{2\alpha'p^+} \sum_{n=1/2}^{\infty} \left( \alpha_{-n}^{25} \alpha_n^{25} + \frac{n}{2} \right) \\ = \frac{1}{2\alpha'p^+} \left( \sum_{n=1/2}^{\infty} \alpha_{-n}^{25} \alpha_n^{25} + \frac{1}{48} \right), \end{aligned} \quad (26)$$

where we have used (19) and (25). Thus the mass spectrum (1.3.36) becomes

$$\begin{aligned} m^2 &= 2p^+ H - p^i p^i \quad (i = 2, \dots, 24) \\ &= \frac{1}{\alpha'} \left( N - \frac{15}{16} \right), \end{aligned} \quad (27)$$

where the level spectrum is given in terms of the occupation numbers by

$$N = \sum_{i=2}^{24} \sum_{n=1}^{\infty} n N_{in} + \sum_{n=1/2}^{\infty} n N_{25,n}. \quad (28)$$

The ground state is still a tachyon,

$$m^2 = -\frac{15}{16\alpha'}. \quad (29)$$

The first excited state has the lowest  $X^{25}$  oscillator excited ( $N_{25,1/2} = 1$ ), and is also tachyonic:

$$m^2 = -\frac{7}{16\alpha'}. \quad (30)$$

There are no massless states, as the second excited state is already massive:

$$m^2 = \frac{1}{16\alpha'}. \quad (31)$$

This state is 24-fold degenerate, as it can be reached either by  $N_{i,1} = 1$  or by  $N_{25,1/2} = 2$ . Thus it is a massive vector with respect to the  $\text{SO}(24,1)$  Lorentz symmetry preserved by the D-brane. The third excited state, with

$$m^2 = \frac{9}{16\alpha'}, \quad (32)$$

is 25-fold degenerate and corresponds to a vector plus a scalar on the D-brane—it can be reached by  $N_{25,1/2} = 1$ , by  $N_{25,1/2} = 3$ , or by  $N_{i,1} = 1$ ,  $N_{25,1/2} = 1$ .

### 1.5 Problem 1.9

The mode expansion for  $X^{25}$  respecting the boundary conditions is essentially the same as the mode expansion (1.4.4), the only differences being that the first two terms are no longer allowed, and the oscillator label  $n$ , rather than running over the non-zero integers, must now run over the half-odd-integers as it did in Problem 1.7:

$$X^{25}(\tau, \sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_n \left( \frac{\alpha_n^{25}}{n} \exp\left[-\frac{2\pi i n(\sigma + c\tau)}{l}\right] + \frac{\tilde{\alpha}_n^{25}}{n} \exp\left[\frac{2\pi i n(\sigma - c\tau)}{l}\right] \right). \quad (33)$$

The canonical commutators are the same as for the untwisted closed string, (1.4.6c) and (1.4.6d),

$$[\alpha_m^{25}, \alpha_n^{25}] = m\delta_{m,-n}, \quad (34)$$

$$[\tilde{\alpha}_m^{25}, \tilde{\alpha}_n^{25}] = m\delta_{m,-n}, \quad (35)$$

as are the mass formula (1.4.8),

$$m^2 = \frac{2}{\alpha'}(N + \tilde{N} + A + \tilde{A}), \quad (36)$$

the generator of  $\sigma$ -translations (1.4.10),

$$P = -\frac{2\pi}{l}(N - \tilde{N}), \quad (37)$$

and (therefore) the level-matching condition (1.4.11),

$$N = \tilde{N}. \quad (38)$$

However, the level operator  $N$  is now slightly different,

$$N = \sum_{i=2}^{24} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{n=1/2}^{\infty} \alpha_{-n}^{25} \alpha_n^{25}; \quad (39)$$

in fact, it is the same as the level operator for the open string on a D24-brane of Problem 1.7. The left-moving level spectrum is therefore given by (28), and similarly for the right-moving level operator  $\tilde{N}$ . The zero-point constants are also the same as in Problem 1.7:

$$\begin{aligned} A = \tilde{A} &= \frac{1}{2} \left( \sum_{i=2}^{24} \sum_{n=1}^{\infty} n + \sum_{n=1/2}^{\infty} n \right) \\ &= -\frac{15}{16}. \end{aligned} \quad (40)$$

At a given level  $N = \tilde{N}$ , the occupation numbers  $N_{in}$  and  $\tilde{N}_{in}$  may be chosen independently, so long as both sets satisfy (28). Therefore the spectrum at that level will consist of the product of two copies of the D-brane open string spectrum, and the mass-squared of that level (36) will be 4 times the open string mass-squared (27). We will have tachyons at levels  $N = 0$  and  $N = 1/2$ , with

$$m^2 = -\frac{15}{4\alpha'} \quad (41)$$

and

$$m^2 = -\frac{7}{4\alpha'}, \quad (42)$$

respectively. The lowest non-tachyonic states will again be at level  $N = 1$ : a second rank SO(24) tensor with

$$m^2 = \frac{1}{4\alpha'}, \quad (43)$$

which can be decomposed into a scalar, an antisymmetric tensor, and a traceless symmetric tensor.

## 2 Chapter 2

### 2.1 Problem 2.1

(a) For holomorphic test functions  $f(z)$ ,

$$\begin{aligned} \int_R d^2z \partial \bar{\partial} \ln |z|^2 f(z) &= \int_R d^2z \bar{\partial} \frac{1}{z} f(z) \\ &= -i \oint_{\partial R} dz \frac{1}{z} f(z) \\ &= 2\pi f(0). \end{aligned} \tag{1}$$

For antiholomorphic test functions  $f(\bar{z})$ ,

$$\begin{aligned} \int_R d^2z \partial \bar{\partial} \ln |z|^2 f(\bar{z}) &= \int_R d^2z \partial \frac{1}{\bar{z}} f(\bar{z}) \\ &= i \oint_{\partial R} d\bar{z} \frac{1}{\bar{z}} f(\bar{z}) \\ &= 2\pi f(0). \end{aligned} \tag{2}$$

(b) We regulate  $\ln |z|^2$  by replacing it with  $\ln(|z|^2 + \epsilon)$ . This lead to regularizations also of  $1/\bar{z}$  and  $1/z$ :

$$\partial \bar{\partial} \ln(|z|^2 + \epsilon) = \partial \frac{z}{|z|^2 + \epsilon} = \bar{\partial} \frac{\bar{z}}{|z|^2 + \epsilon} = \frac{\epsilon}{(|z|^2 + \epsilon)^2}. \tag{3}$$

Working in polar coordinates, consider a general test function  $f(r, \theta)$ , and define  $g(r^2) \equiv \int d\theta f(r, \theta)$ . Then, assuming that  $g$  is sufficiently well behaved at zero and infinity,

$$\begin{aligned} &\int d^2z \frac{\epsilon}{(|z|^2 + \epsilon)^2} f(z, \bar{z}) \\ &= \int_0^\infty du \frac{\epsilon}{(u + \epsilon)^2} g(u) \\ &= \left( -\frac{\epsilon}{u + \epsilon} g(u) + \epsilon \ln(u + \epsilon) g'(u) \right)_0^\infty - \int_0^\infty du \epsilon \ln(u + \epsilon) g''(u). \\ &= g(0) \\ &= 2\pi f(0). \end{aligned} \tag{4}$$

## 2.2 Problem 2.3

(a) The leading behavior of the expectation value as  $z_1 \rightarrow z_2$  is

$$\begin{aligned}
& \left\langle \prod_{i=1}^n : e^{ik_i \cdot X(z_i, \bar{z}_i)} : \right\rangle \\
&= iC^X (2\pi)^D \delta^D \left( \sum_{i=1}^n k_i \right) \prod_{i,j=1}^n |z_{ij}|^{\alpha' k_i \cdot k_j} \\
&= |z_{12}|^{\alpha' k_1 \cdot k_2} iC^X (2\pi)^D \delta^D (k_1 + k_2 + \sum_{i=3}^n k_i) \\
&\quad \times \prod_{i=3}^n \left( |z_{1i}|^{\alpha' k_1 \cdot k_i} |z_{2i}|^{\alpha' k_2 \cdot k_i} \right) \prod_{i,j=3}^n |z_{ij}|^{\alpha' k_i \cdot k_j} \\
&\approx |z_{12}|^{\alpha' k_1 \cdot k_2} iC^X (2\pi)^D \delta^D (k_1 + k_2 + \sum_{i=3}^n k_i) \prod_{i=3}^n |z_{2i}|^{\alpha' (k_1+k_2) \cdot k_i} \prod_{i,j=3}^n |z_{ij}|^{\alpha' k_i \cdot k_j} \\
&= |z_{12}|^{\alpha' k_1 \cdot k_2} \left\langle : e^{i(k_1+k_2) \cdot X(z_2, \bar{z}_2)} : \prod_{i=3}^n : e^{ik_i \cdot X(z_i, \bar{z}_i)} : \right\rangle, \tag{5}
\end{aligned}$$

in agreement with (2.2.14).

(b) The  $z_i$ -dependence of the expectation value is given by

$$\begin{aligned}
& |z_{23}|^{\alpha' k_2 \cdot k_3} |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{13}|^{\alpha' k_1 \cdot k_3} \\
&= |z_{23}|^{\alpha' k_2 \cdot k_3} |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{23}|^{\alpha' k_1 \cdot k_3} \left| 1 + \frac{z_{12}}{z_{23}} \right|^{\alpha' k_1 \cdot k_3} \\
&= |z_{23}|^{\alpha' (k_1+k_2) \cdot k_3} |z_{12}|^{\alpha' k_1 \cdot k_2} \left( \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}\alpha' k_1 \cdot k_3 + 1)}{k! \Gamma(\frac{1}{2}\alpha' k_1 \cdot k_3 - k + 1)} \left( \frac{z_{12}}{z_{23}} \right)^k \right) \\
&\quad \times \left( \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}\alpha' k_1 \cdot k_3 + 1)}{k! \Gamma(\frac{1}{2}\alpha' k_1 \cdot k_3 - k + 1)} \left( \frac{\bar{z}_{12}}{\bar{z}_{23}} \right)^k \right). \tag{6}
\end{aligned}$$

The radius of convergence of a power series is given by the limit as  $k \rightarrow \infty$  of  $|a_k/a_{k+1}|$ , where the  $a_k$  are the coefficients of the series. In this case, for both of the above power series,

$$\begin{aligned}
R &= \lim_{k \rightarrow \infty} \left| \frac{(k+1)! \Gamma(\frac{1}{2}\alpha' k_1 \cdot k_3 - k)}{k! \Gamma(\frac{1}{2}\alpha' k_1 \cdot k_3 - k + 1)} z_{23} \right| \\
&= |z_{23}|. \tag{7}
\end{aligned}$$

(c) Consider the interior of the dashed line in figure 2.1, that is, the set of points  $z_1$  satisfying

$$|z_{12}| < |z_{23}|. \tag{8}$$

By equation (2.1.23), the expectation value

$$\langle : X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) : \mathcal{A}(z_3, \bar{z}_3) \mathcal{B}(z_4, \bar{z}_4) \rangle \quad (9)$$

is a harmonic function of  $z_1$  within this region. It can therefore be written as the sum of a holomorphic and an antiholomorphic function (this statement is true in any simply connected region). The Taylor expansion of a function that is holomorphic on an open disk (about the center of the disk), converges on the disk; similarly for an antiholomorphic function. Hence the two Taylor series on the RHS of (2.2.4) must converge on the disk.

### 2.3 Problem 2.5

Under the variation of the fields  $\phi_\alpha(\sigma) \rightarrow \phi_\alpha(\sigma) + \delta\phi_\alpha(\sigma)$ , the variation of the Lagrangian is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_\alpha} \delta\phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)} \partial_a\delta\phi_\alpha. \quad (10)$$

The Lagrangian equations of motion (Euler-Lagrange equations) are derived by assuming that the action is stationary under an arbitrary variation  $\delta\phi_\alpha(\sigma)$  that vanishes at infinity:

$$\begin{aligned} 0 &= \delta S \\ &= \int d^d\sigma \delta\mathcal{L} \\ &= \int d^d\sigma \left( \frac{\partial\mathcal{L}}{\partial\phi_\alpha} \delta\phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)} \partial_a\delta\phi_\alpha \right) \\ &= \int d^d\sigma \left( \frac{\partial\mathcal{L}}{\partial\phi_\alpha} - \partial_a \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)} \right) \delta\phi_\alpha \end{aligned} \quad (11)$$

implies

$$\frac{\partial\mathcal{L}}{\partial\phi_\alpha} - \partial_a \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)} = 0. \quad (12)$$

Instead of assuming that  $\delta\phi_\alpha$  vanishes at infinity, let us assume that it is a symmetry. In this case, the variation of the Lagrangian (10) must be a total derivative to insure that the action on bounded regions varies only by a surface term, thereby not affecting the equations of motion:

$$\delta\mathcal{L} = \epsilon \partial_a \mathcal{K}^a; \quad (13)$$

$\mathcal{K}^a$  is assumed to be a local function of the fields and their derivatives, although it is not obvious how to prove that this can always be arranged. Using (10), (12), and (13),

$$\begin{aligned} \partial_a j^a &= 2\pi i \partial_a \left( \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)} \epsilon^{-1} \delta\phi_\alpha - \mathcal{K}^a \right) \\ &= \frac{2\pi i}{\epsilon} \left( \frac{\partial\mathcal{L}}{\partial\phi_\alpha} \delta\phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)} \partial_a\delta\phi_\alpha - \delta\mathcal{L} \right) \\ &= 0. \end{aligned} \quad (14)$$

If we now vary the fields by  $\rho(\sigma)\delta\phi_\alpha(\sigma)$ , where  $\delta\phi_\alpha$  is a symmetry as before but  $\rho$  is an arbitrary function, then the variation of the action will be

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)}\partial_a(\delta\phi_\alpha\rho) + \frac{\partial\mathcal{L}}{\partial\phi_\alpha}\delta\phi_\alpha\rho \\ &= \left(\frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)}\partial_a\delta\phi_\alpha + \frac{\partial\mathcal{L}}{\partial\phi_\alpha}\delta\phi_\alpha\right)\rho + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)}\delta\phi_\alpha\partial_a\rho.\end{aligned}\quad (15)$$

Equation (13) must be satisfied in the case  $\rho(\sigma)$  is identically 1, so the factor in parentheses must equal  $\epsilon\partial_a\mathcal{K}^a$ :

$$\begin{aligned}\delta S &= \int d^d\sigma \left( \epsilon\partial_a\mathcal{K}^a\rho + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)}\delta\phi_\alpha\partial_a\rho \right) \\ &= \int d^d\sigma \left( -\epsilon\mathcal{K}^a + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_\alpha)}\delta\phi_\alpha \right) \partial_a\rho \\ &= \frac{\epsilon}{2\pi i} \int d^d\sigma j^a\partial_a\rho,\end{aligned}\quad (16)$$

where we have integrated by parts, assuming that  $\rho$  falls off at infinity. Since  $\delta\exp(-S) = -\exp(-S)\delta S$ , this agrees with (2.3.4) for the case of flat space, ignoring the transformation of the measure.

## 2.4 Problem 2.7

(a)  $X^\mu$ :

$$\begin{aligned}T(z)X^\mu(0,0) &= -\frac{1}{\alpha'} : \partial X^\nu(z)\partial X_\nu(z) : X^\mu(0,0) \sim \frac{1}{z}\partial X^\mu(z) \sim \frac{1}{z}\partial X^\mu(0) \\ \tilde{T}(\bar{z})X^\mu(0,0) &= -\frac{1}{\alpha'} : \bar{\partial} X^\nu(\bar{z})\bar{\partial} X_\nu(\bar{z}) : X^\mu(0,0) \sim \frac{1}{\bar{z}}\bar{\partial} X^\mu(\bar{z}) \sim \frac{1}{\bar{z}}\bar{\partial} X^\mu(0)\end{aligned}\quad (17)$$

$\partial X^\mu$ :

$$\begin{aligned}T(z)\partial X^\mu(0) &\sim \frac{1}{z^2}\partial X^\mu(z) \sim \frac{1}{z^2}\partial X^\mu(0) + \frac{1}{z}\partial^2 X^\mu(0) \\ \tilde{T}(\bar{z})\partial X^\mu(0) &\sim 0\end{aligned}\quad (18)$$

$\bar{\partial} X^\mu$ :

$$\begin{aligned}T(z)\bar{\partial} X^\mu(0) &\sim 0 \\ \tilde{T}(\bar{z})\bar{\partial} X^\mu(0) &\sim \frac{1}{\bar{z}^2}\bar{\partial} X^\mu(\bar{z}) \sim \frac{1}{\bar{z}^2}\bar{\partial} X^\mu(0) + \frac{1}{\bar{z}}\bar{\partial}^2 X^\mu(0)\end{aligned}\quad (19)$$

$\partial^2 X^\mu$ :

$$\begin{aligned}T(z)\partial^2 X^\mu(0) &\sim \frac{2}{z^3}\partial X^\mu(z) \sim \frac{2}{z^3}\partial X^\mu(0) + \frac{2}{z^2}\partial^2 X^\mu(0) + \frac{1}{z}\partial^3 X^\mu(0) \\ \tilde{T}(\bar{z})\partial^2 X^\mu(0) &\sim 0\end{aligned}\quad (20)$$

$: e^{ik \cdot X} ::$

$$\begin{aligned}
T(z) : e^{ik \cdot X(0,0)} : &\sim \frac{\alpha' k^2}{4z^2} : e^{ik \cdot X(0,0)} : + \frac{1}{z} i k_\mu : \partial X^\mu(z) e^{ik \cdot X(0,0)} : \\
&\sim \frac{\alpha' k^2}{4z^2} : e^{ik \cdot X(0,0)} : + \frac{1}{z} i k_\mu : \partial X^\mu(0) e^{ik \cdot X(0,0)} : \\
\tilde{T}(\bar{z}) : e^{ik \cdot X(0,0)} : &\sim \frac{\alpha' k^2}{4\bar{z}^2} : e^{ik \cdot X(0,0)} : + \frac{1}{\bar{z}} i k_\mu : \bar{\partial} X^\mu(\bar{z}) e^{ik \cdot X(0,0)} : \\
&\sim \frac{\alpha' k^2}{4\bar{z}^2} : e^{ik \cdot X(0,0)} : + \frac{1}{\bar{z}} i k_\mu : \partial X^\mu(0) e^{ik \cdot X(0,0)} :
\end{aligned} \tag{21}$$

(b) In the linear dilaton theory, the energy-momentum tensor is

$$\begin{aligned}
T &= -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : + V_\mu \partial^2 X^\mu, \\
\tilde{T} &= -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu : + V_\mu \bar{\partial}^2 X^\mu,
\end{aligned} \tag{22}$$

so it suffices to calculate the OPEs of the various operators with the terms  $V_\mu \partial^2 X^\mu$  and  $V_\mu \bar{\partial}^2 X^\mu$  and add them to the results found in part (a).

$X^\mu$ :

$$\begin{aligned}
V_\nu \partial^2 X^\nu(z) X^\mu(0,0) &\sim \frac{\alpha' V^\mu}{2z^2} \\
V_\nu \bar{\partial}^2 X^\nu(\bar{z}) X^\mu(0,0) &\sim \frac{\alpha' V^\mu}{2\bar{z}^2}
\end{aligned} \tag{23}$$

Not only is  $X^\mu$  is not a tensor anymore, but it does not even have well-defined weights, because it is not an eigenstate of rigid transformations.

$\partial X^\mu$ :

$$\begin{aligned}
V_\nu \partial^2 X^\nu(z) \partial X^\mu(0) &\sim \frac{\alpha' V^\mu}{z^3} \\
V_\nu \bar{\partial}^2 X^\nu(\bar{z}) \partial X^\mu(0) &\sim 0
\end{aligned} \tag{24}$$

So  $\partial X^\mu$  still has weights (1,0), but it is no longer a tensor operator.

$\bar{\partial} X^\mu$ :

$$\begin{aligned}
V_\nu \partial^2 X^\nu(z) \bar{\partial} X^\mu(0) &\sim 0 \\
V_\nu \bar{\partial}^2 X^\nu(\bar{z}) \bar{\partial} X^\mu(0) &\sim \frac{\alpha' V^\mu}{\bar{z}^3}
\end{aligned} \tag{25}$$

Similarly,  $\bar{\partial} X^\mu$  still has weights (0,1), but is no longer a tensor.

$\partial^2 X^\mu$ :

$$\begin{aligned}
V_\nu \partial^2 X^\nu(z) \partial^2 X^\mu(0) &\sim \frac{3\alpha' V^\mu}{z^4} \\
V_\nu \bar{\partial}^2 X^\nu(\bar{z}) \partial^2 X^\mu(0) &\sim 0
\end{aligned} \tag{26}$$



Nothing changes from the scalar theory: the weights are still (2,0), and  $\partial^2 X^\mu$  is still not a tensor.

$: e^{ik \cdot X} ::$

$$\begin{aligned} V_\nu \partial^2 X^\nu(z) : e^{ik \cdot X(0,0)} &: \sim \frac{i\alpha' V \cdot k}{2z^2} : e^{ik \cdot X(0,0)} : \\ V_\nu \bar{\partial}^2 X^\nu(\bar{z}) : e^{ik \cdot X(0,0)} &: \sim \frac{i\alpha' V \cdot k}{2\bar{z}^2} : e^{ik \cdot X(0,0)} : \end{aligned} \quad (27)$$

Thus  $: e^{ik \cdot X} :$  is still a tensor, but, curiously, its weights are now complex:

$$\left( \frac{\alpha'}{4}(k^2 + 2iV \cdot k), \frac{\alpha'}{4}(k^2 + 2iV \cdot k) \right). \quad (28)$$

## 2.5 Problem 2.9

Since we are interested in finding the central charges of these theories, it is only necessary to calculate the  $1/z^4$  terms in the  $TT$  OPEs, the rest of the OPE being determined by general considerations as in equation (2.4.25). In the following, we will therefore drop all terms less singular than  $1/z^4$ . For the linear dilaton CFT,

$$\begin{aligned} T(z)T(0) &= \frac{1}{\alpha'^2} : \partial X^\mu(z) \partial X_\mu(z) :: \partial X^\nu(0) \partial X_\nu(0) : \\ &\quad - \frac{2V_\nu}{\alpha'} : \partial X^\mu(z) \partial X_\mu(z) : \partial^2 X^\nu(0) \\ &\quad - \frac{2V_\mu}{\alpha'} \partial^2 X^\mu(z) : \partial X^\nu(0) \partial X_\nu(0) : + V_\mu V_\nu \partial^2 X^\mu(z) \partial^2 X^\nu(0) \\ &\sim \frac{D}{2z^4} + \frac{3\alpha' V^2}{z^4} + \mathcal{O}\left(\frac{1}{z^2}\right), \end{aligned} \quad (29)$$

so

$$c = D + 6\alpha' V^2. \quad (30)$$

Similarly,

$$\begin{aligned} \tilde{T}(\bar{z})\tilde{T}(0) &= \frac{1}{\alpha'^2} : \bar{\partial} X^\mu(\bar{z}) \bar{\partial} X_\mu(\bar{z}) :: \bar{\partial} X^\nu(0) \bar{\partial} X_\nu(0) : \\ &\quad - \frac{2V_\nu}{\alpha'} : \bar{\partial} X^\mu(\bar{z}) \bar{\partial} X_\mu(\bar{z}) : \bar{\partial}^2 X^\nu(0) \\ &\quad - \frac{2V_\mu}{\alpha'} \bar{\partial}^2 X^\mu(\bar{z}) : \bar{\partial} X^\nu(0) \bar{\partial} X_\nu(0) : + V_\mu V_\nu \bar{\partial}^2 X^\mu(\bar{z}) \bar{\partial}^2 X^\nu(0) \\ &\sim \frac{D}{2\bar{z}^4} + \frac{3\alpha' V^2}{\bar{z}^4} + \mathcal{O}\left(\frac{1}{\bar{z}^2}\right), \end{aligned} \quad (31)$$

so

$$\tilde{c} = D + 6\alpha' V^2. \quad (32)$$

For the  $bc$  system,

$$\begin{aligned}
T(z)T(0) &= (1 - \lambda)^2 : \partial b(z)c(z) :: \partial b(0)c(0) : \\
&\quad - \lambda(1 - \lambda) : \partial b(z)c(z) :: b(0)\partial c(0) : \\
&\quad - \lambda(1 - \lambda) : b(z)\partial c(z) :: \partial b(0)c(0) : \\
&\quad + \lambda^2 : b(z)\partial c(z) :: b(0)\partial c(0) : \\
&\sim \frac{-6\lambda^2 + 6\lambda - 1}{z^4} + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{33}
\end{aligned}$$

so

$$c = -12\lambda^2 + 12\lambda - 2. \tag{34}$$

Of course  $\tilde{T}(\tilde{z})\tilde{T}(0) = 0$ , so  $\tilde{c} = 0$ .

The  $\beta\gamma$  system has the same energy-momentum tensor and almost the same OPEs as the  $bc$  system. While  $\gamma(z)\beta(0) \sim 1/z$  as in the  $bc$  system, now  $\beta(z)\gamma(0) \sim -1/z$ . Each term in (33) involved one  $b(z)c(0)$  contraction and one  $c(z)b(0)$  contraction, so the central charge of the  $\beta\gamma$  system is minus that of the  $bc$  system:

$$c = 12\lambda^2 - 12\lambda + 2. \tag{35}$$

Of course  $\tilde{c} = 0$  still.

## 2.6 Problem 2.11

Assume without loss of generality that  $m > 1$ ; for  $m = 0$  and  $m = \pm 1$  the central charge term in (2.6.19) vanishes, while  $m < -1$  is equivalent to  $m > 1$ . Then  $L_m$  annihilates  $|0; 0\rangle$ , as do all but  $m - 1$  of the terms in the mode expansion (2.7.6) of  $L_{-m}$ :

$$L_{-m}|0; 0\rangle = \frac{1}{2} \sum_{n=1}^{m-1} \alpha_{n-m}^\mu \alpha_{\mu(-n)} |0; 0\rangle. \tag{36}$$

Hence the LHS of (2.6.19), when applied to  $|0; 0\rangle$ , yields,

$$\begin{aligned}
&[L_m, L_{-m}]|0; 0\rangle \\
&= L_m L_{-m}|0; 0\rangle - L_{-m} L_m |0; 0\rangle \\
&= \frac{1}{4} \sum_{n'=-\infty}^{\infty} \sum_{n=1}^{m-1} \alpha_{m-n'}^\nu \alpha_{\nu n'} \alpha_{n-m}^\mu \alpha_{\mu(-n)} |0; 0\rangle \\
&= \frac{1}{4} \sum_{n=1}^{m-1} \sum_{n'=-\infty}^{\infty} ((m-n')n' \eta^{\nu\mu} \eta_{\nu\mu} \delta_{n'n} + (m-n')n' \delta_\mu^\nu \delta_\nu^\mu \delta_{m-n',n}) |0; 0\rangle \\
&= \frac{D}{2} \sum_{n=1}^{m-1} n(m-n) |0; 0\rangle \\
&= \frac{D}{12} m(m^2 - 1) |0; 0\rangle. \tag{37}
\end{aligned}$$

Meanwhile, the RHS of (2.6.19) applied to the same state yields,

$$\left(2mL_0 + \frac{c}{12}(m^3 - m)\right) |0; 0\rangle = \frac{c}{12}(m^3 - m)|0; 0\rangle, \quad (38)$$

so

$$c = D. \quad (39)$$

## 2.7 Problem 2.13

(a) Using (2.7.16) and (2.7.17),

$$\begin{aligned} \circ b(z)c(z')\circ &= \sum_{m,m'=-\infty}^{\infty} \frac{\circ b_m c_{m'}\circ}{z^{m+\lambda} z'^{m'+1-\lambda}} \\ &= \sum_{m,m'=-\infty}^{\infty} \frac{b_m c_{m'}}{z^{m+\lambda} z'^{m'+1-\lambda}} - \sum_{m=0}^{\infty} \frac{1}{z^{m+\lambda} z'^{-m+1-\lambda}} \\ &= b(z)c(z') - \left(\frac{z}{z'}\right)^{1-\lambda} \frac{1}{z-z'}. \end{aligned} \quad (40)$$

With (2.5.7),

$$: b(z)c(z') : - \circ b(z)c(z')\circ = \frac{1}{z-z'} \left( \left(\frac{z}{z'}\right)^{1-\lambda} - 1 \right). \quad (41)$$

(b) By taking the limit of (41) as  $z' \rightarrow z$ , we find,

$$: b(z)c(z) : - \circ b(z)c(z)\circ = \frac{1-\lambda}{z}. \quad (42)$$

Using (2.8.14) we have,

$$\begin{aligned} N^g &= Q^g - \lambda + \frac{1}{2} \\ &= \frac{1}{2\pi i} \oint dz j_z - \lambda + \frac{1}{2} \\ &= -\frac{1}{2\pi i} \oint dz : b(z)c(z) : - \lambda + \frac{1}{2} \\ &= -\frac{1}{2\pi i} \oint dz \circ b(z)c(z)\circ - \frac{1}{2}. \end{aligned} \quad (43)$$

(c) If we re-write the expansion (2.7.16) of  $b(z)$  in the  $w$ -frame using the tensor transformation law (2.4.15), we find,

$$\begin{aligned} b(w) &= (\partial_z w)^{-\lambda} b(z) \\ &= (-iz)^\lambda \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+\lambda}} \\ &= e^{-\pi i \lambda / 2} \sum_{m=-\infty}^{\infty} e^{imw} b_m. \end{aligned} \quad (44)$$

Similarly,

$$c(w) = e^{-\pi i(1-\lambda)/2} \sum_{m=-\infty}^{\infty} e^{imw} c_m. \quad (45)$$

Hence, ignoring ordering,

$$\begin{aligned} j_w(w) &= -b(w)c(w) \\ &= i \sum_{m,m'=-\infty}^{\infty} e^{i(m+m')w} b_m c_{m'}, \end{aligned} \quad (46)$$

and

$$\begin{aligned} N^g &= -\frac{1}{2\pi i} \int_0^{2\pi} dw j_w \\ &= -\sum_{m=-\infty}^{\infty} b_m c_{-m} \\ &= -\sum_{m=-\infty}^{\infty} \circ b_m c_{-m} \circ - \sum_{m=0}^{\infty} 1. \end{aligned} \quad (47)$$

The ordering constant is thus determined by the value of the second infinite sum. If we write, more generally,  $\sum_{m=0}^{\infty} a$ , then we must regulate the sum in such a way that the divergent part is independent of  $a$ . For instance,

$$\begin{aligned} \sum_{m=0}^{\infty} a e^{-\epsilon a} &= \frac{a}{1 - e^{-\epsilon a}} \\ &= \frac{1}{\epsilon} + \frac{a}{2} + \mathcal{O}(\epsilon); \end{aligned} \quad (48)$$

the  $\epsilon$ -independent part is  $a/2$ , so the ordering constant in (47) equals  $-1/2$ .

## 2.8 Problem 2.15

To apply the doubling trick to the field  $X^\mu(z, \bar{z})$ , define for  $\Im z < 0$ ,

$$X^\mu(z, \bar{z}) \equiv X^\mu(z^*, \bar{z}^*). \quad (49)$$

Then

$$\partial^m X^\mu(z) = \bar{\partial}^m X^\mu(\bar{z}^*), \quad (50)$$

so that in particular for  $z$  on the real line,

$$\partial^m X^\mu(z) = \bar{\partial}^m X^\mu(\bar{z}), \quad (51)$$

as can also be seen from the mode expansion (2.7.26). The modes  $\alpha_m^\mu$  are defined as integrals over a semi-circle of  $\partial X^\mu(z) + \bar{\partial} X^\mu(\bar{z})$ , but with the doubling trick the integral can be extended to the full circle:

$$\alpha_m^\mu = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z) = -\sqrt{\frac{2}{\alpha'}} \oint \frac{d\bar{z}}{2\pi} \bar{z}^m \bar{\partial} X^\mu(\bar{z}). \quad (52)$$

At this point the derivation proceeds in exactly the same manner as for the closed string treated in the text. With no operator at the origin, the fields are holomorphic inside the contour, so with  $m$  positive, the contour integrals (52) vanish, and the state corresponding to the unit operator “inserted” at the origin must be the ground state  $|0;0\rangle$ :

$$1(0,0) \cong |0;0\rangle. \quad (53)$$

The state  $\alpha_{-m}^\mu |0;0\rangle$  ( $m$  positive) is given by evaluating the integrals (52), with the fields holomorphic inside the contours:

$$\alpha_{-m}^\mu |0;0\rangle \cong \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!} \partial^m X^\mu(0) = \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!} \bar{\partial}^m X^\mu(0). \quad (54)$$

Similarly, using the mode expansion (2.7.26), we see that  $X^\mu(0,0)|0;0\rangle = x^\mu|0;0\rangle$ , so

$$x^\mu|0;0\rangle \cong X^\mu(0,0). \quad (55)$$

As in the closed string case, the same correspondence applies when these operators act on states other than the ground state, as long as we normal order the resulting local operator. The result is therefore exactly the same as (2.8.7a) and (2.8.8) in the text; for example, (2.8.9) continues to hold.

## 2.9 Problem 2.17

Take the matrix element of (2.6.19) between  $\langle 1|$  and  $|1\rangle$ , with  $n = -m$  and  $m > 1$ . The LHS yields,

$$\begin{aligned} \langle 1|[L_m, L_{-m}]|1\rangle &= \langle 1|L_{-m}^\dagger L_{-m}|1\rangle \\ &= \|L_{-m}|1\rangle\|^2, \end{aligned} \quad (56)$$

using (2.9.9). Also by (2.9.9),  $L_0|1\rangle = 0$ , so on the RHS we are left with the term

$$\frac{c}{12}(m^3 - m)\langle 1|1\rangle. \quad (57)$$

Hence

$$c = \frac{12}{m^3 - m} \frac{\|L_{-m}|1\rangle\|^2}{\langle 1|1\rangle} \geq 0. \quad (58)$$

### 3 Chapter 3

#### 3.1 Problem 3.1

(a) The definition of the geodesic curvature  $k$  of a boundary given in Problem 1.3 is

$$k = -n_b t^a \nabla_a t^b, \quad (1)$$

where  $t^a$  is the unit tangent vector to the boundary and  $n^b$  is the outward directed unit normal. For a flat unit disk,  $R$  vanishes, while the geodesic curvature of the boundary is 1 (since  $t^a \nabla_a t^b = -n^b$ ). Hence

$$\chi = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1. \quad (2)$$

For the unit hemisphere, on the other hand, the boundary is a geodesic, while  $R = 2$ . Hence

$$\chi = \frac{1}{4\pi} \int d^2\sigma g^{1/2} 2 = 1, \quad (3)$$

in agreement with (2).

(b) If we cut a surface along a closed curve, the two new boundaries will have oppositely directed normals, so their contributions to the Euler number of the surface will cancel, leaving it unchanged. The Euler number of the unit sphere is

$$\chi = \frac{1}{4\pi} \int d^2\sigma g^{1/2} 2 = 2. \quad (4)$$

If we cut the sphere along  $b$  small circles, we will be left with  $b$  disks and a sphere with  $b$  holes. The Euler number of the disks is  $b$  (from part (a)), so the Euler number of the sphere with  $b$  holes is

$$\chi = 2 - b. \quad (5)$$

(c) A finite cylinder has Euler number 0, since we can put on it a globally flat metric for which the boundaries are geodesics. If we remove from a sphere  $b + 2g$  holes, and then join to  $2g$  of the holes  $g$  cylinders, the result will be a sphere with  $b$  holes and  $g$  handles; its Euler number will be

$$\chi = 2 - b - 2g. \quad (6)$$

#### 3.2 Problem 3.2

(a) This is easiest to show in complex coordinates, where  $g^{zz} = g^{\bar{z}\bar{z}} = 0$ . Contracting two indices of a symmetric tensor with lower indices by  $g^{ab}$  will pick out the components where one of the indices is  $z$  and the other  $\bar{z}$ . If the tensor is traceless then all such components must vanish. The only non-vanishing components are therefore the one with all  $z$  indices and the one with all  $\bar{z}$  indices.

(b) Let  $v_{a_1 \dots a_n}$  be a traceless symmetric tensor. Define  $P_n$  by

$$(P_n v)_{a_1 \dots a_{n+1}} \equiv \nabla_{(a_1} v_{a_2 \dots a_{n+1})} - \frac{n}{n+1} g_{a_1 a_2} \nabla_{|b|} v^b_{a_3 \dots a_{n+1}}. \quad (7)$$

This tensor is symmetric by construction, and it is easy to see that it is also traceless. Indeed, contracting with  $g^{a_1 a_2}$ , the first term becomes

$$g^{a_1 a_2} \nabla_{(a_1} v_{a_2 \dots a_{n+1})} = \frac{2}{n+1} \nabla_b v^b_{a_3 \dots a_{n+1}}, \quad (8)$$

where we have used the symmetry and tracelessness of  $v$ , and the second cancels the first:

$$\begin{aligned} & g^{a_1 a_2} g_{(a_1 a_2} \nabla_{|b|} v^b_{a_3 \dots a_{n+1})} \\ &= \frac{2}{n(n+1)} g^{a_1 a_2} g_{a_1 a_2} \nabla_b v^b_{a_3 \dots a_{n+1}} + \frac{2(n-1)}{n(n+1)} g^{a_1 a_2} g_{a_1 a_3} \nabla_b v^b_{a_2 a_4 \dots a_{n+1}} \\ &= \frac{2}{n} \nabla_b v^b_{a_3 \dots a_{n+1}}. \end{aligned} \quad (9)$$

(c) For  $u_{a_1 \dots a_{n+1}}$  a traceless symmetric tensor, define  $P_n^T$  by

$$(P_n^T u)_{a_1 \dots a_n} \equiv -\nabla_b u^b_{a_1 \dots a_n}. \quad (10)$$

This inherits the symmetry and tracelessness of  $u$ .

(d)

$$\begin{aligned} (u, P_n v) &= \int d^2\sigma g^{1/2} u^{a_1 \dots a_{n+1}} (P_n v)_{a_1 \dots a_{n+1}} \\ &= \int d^2\sigma g^{1/2} u^{a_1 \dots a_{n+1}} \left( \nabla_{a_1} v_{a_2 \dots a_{n+1}} - \frac{n}{n+1} g_{a_1 a_2} \nabla_b v^b_{a_3 \dots a_{n+1}} \right) \\ &= - \int d^2\sigma g^{1/2} \nabla_{a_1} u^{a_1 \dots a_{n+1}} v_{a_2 \dots a_{n+1}} \\ &= \int d^2\sigma g^{1/2} (P_n^T u)^{a_2 \dots a_{n+1}} v_{a_2 \dots a_{n+1}} \\ &= (P_n^T u, v) \end{aligned} \quad (11)$$

### 3.3 Problem 3.3

(a) The conformal gauge metric in complex coordinates is  $g_{z\bar{z}} = g_{\bar{z}z} = e^{2\omega}/2$ ,  $g_{zz} = g_{\bar{z}\bar{z}} = 0$ . Connection coefficients are quickly calculated:

$$\begin{aligned} \Gamma_{zz}^z &= \frac{1}{2} g^{z\bar{z}} (\partial_z g_{z\bar{z}} + \partial_z g_{\bar{z}z} - \partial_{\bar{z}} g_{zz}) \\ &= 2\partial\omega, \end{aligned} \quad (12)$$

$$\Gamma_{\bar{z}\bar{z}}^{\bar{z}} = 2\bar{\partial}\omega, \quad (13)$$

all other coefficients vanishing.

This leads to the following simplification in the formula for the covariant derivative:

$$\begin{aligned}\nabla_z T_{b_1 \dots b_n}^{a_1 \dots a_m} &= \partial_z T_{b_1 \dots b_n}^{a_1 \dots a_m} + \sum_{i=1}^m \Gamma_{zc}^{a_i} T_{b_1 \dots b_n}^{a_1 \dots c \dots a_m} - \sum_{j=1}^n \Gamma_{zb_j}^c T_{b_1 \dots c \dots b_n}^{a_1 \dots a_m} \\ &= \left( \partial + 2\partial\omega \sum_{i=1}^m \delta_z^{a_i} - 2\partial\omega \sum_{j=1}^n \delta_{b_j}^z \right) T_{b_1 \dots b_n}^{a_1 \dots a_m};\end{aligned}\quad (14)$$

in other words, it counts the difference between the number of upper  $z$  indices and lower  $z$  indices, while  $\bar{z}$  indices do not enter. Similarly,

$$\nabla_{\bar{z}} T_{b_1 \dots b_n}^{a_1 \dots a_m} = \left( \bar{\partial} + 2\bar{\partial}\omega \sum_{i=1}^m \delta_{\bar{z}}^{a_i} - 2\bar{\partial}\omega \sum_{j=1}^n \delta_{b_j}^{\bar{z}} \right) T_{b_1 \dots b_n}^{a_1 \dots a_m}.\quad (15)$$

In particular, the covariant derivative with respect to  $z$  of a tensor with only  $\bar{z}$  indices is equal to its regular derivative, and vice versa:

$$\begin{aligned}\nabla_z T_{\bar{z} \dots \bar{z}}^{\bar{z} \dots \bar{z}} &= \partial T_{\bar{z} \dots \bar{z}}^{\bar{z} \dots \bar{z}}, \\ \nabla_{\bar{z}} T_{z \dots z}^{z \dots z} &= \bar{\partial} T_{z \dots z}^{z \dots z}.\end{aligned}\quad (16)$$

**(b)** As shown in problem 3.2(a), the only non-vanishing components of a traceless symmetric tensor with lowered indices have all them  $z$  or all of them  $\bar{z}$ . If  $v$  is an  $n$ -index traceless symmetric tensor, then  $P_n v$  will be an  $(n+1)$ -index traceless symmetric tensor, and will therefore have only two non-zero components:

$$\begin{aligned}(P_n v)_{z \dots z} &= \nabla_z v_{z \dots z} \\ &= \left( \frac{1}{2} e^{2\omega} \right)^n \nabla_z v^{\bar{z} \dots \bar{z}} \\ &= \left( \frac{1}{2} e^{2\omega} \right)^n \partial v^{\bar{z} \dots \bar{z}} \\ &= (\partial - 2n\partial\omega) v_{z \dots z};\end{aligned}\quad (17)$$

$$(P_n v)_{\bar{z} \dots \bar{z}} = (\bar{\partial} - 2n\bar{\partial}\omega) v_{\bar{z} \dots \bar{z}}.\quad (18)$$

Similarly, if  $u$  is an  $(n+1)$ -index traceless symmetric tensor, then  $P_n^T u$  will be an  $n$ -index traceless symmetric tensor, and will have only two non-zero components:

$$\begin{aligned}(P_n^T u)_{z \dots z} &= -\nabla_b u^b{}_{z \dots z} \\ &= -2e^{-2\omega} \nabla_z u_{\bar{z} z \dots z} - 2e^{-2\omega} \nabla_{\bar{z}} u_{z z \dots z} \\ &= -\left( \frac{1}{2} e^{2\omega} \right)^{n-1} \partial u_{\bar{z} \dots \bar{z}} - 2e^{-2\omega} \bar{\partial} u_{z \dots z} \\ &= -2e^{-2\omega} \bar{\partial} u_{z \dots z};\end{aligned}\quad (19)$$

$$(P_n^T u)_{\bar{z} \dots \bar{z}} = -2e^{-2\omega} \partial u_{\bar{z} \dots \bar{z}}.\quad (20)$$



### 3.4 Problem 3.4

The Faddeev-Popov determinant is defined by,

$$\Delta_{\text{FP}}(\phi) \equiv \left( \int [d\zeta] \delta \left( F^A(\phi^\zeta) \right) \right)^{-1}. \quad (21)$$

By the gauge invariance of the measure  $[d\zeta]$  on the gauge group, this is a gauge-invariant function. It can be used to re-express the gauge-invariant formulation of the path integral, with arbitrary gauge-invariant insertions  $f(\phi)$ , in a gauge-fixed way:

$$\begin{aligned} \frac{1}{V} \int [d\phi] e^{-S_1(\phi)} f(\phi) &= \frac{1}{V} \int [d\phi] e^{-S_1(\phi)} \Delta_{\text{FP}}(\phi) \int [d\zeta] \delta \left( F^A(\phi^\zeta) \right) f(\phi) \\ &= \frac{1}{V} \int [d\zeta d\phi^\zeta] e^{-S_1(\phi^\zeta)} \Delta_{\text{FP}}(\phi^\zeta) \delta \left( F^A(\phi^\zeta) \right) f(\phi^\zeta) \\ &= \int [d\phi] e^{-S_1(\phi)} \Delta_{\text{FP}}(\phi) \delta \left( F^A(\phi) \right) f(\phi). \end{aligned} \quad (22)$$

In the second equality we used the gauge invariance of  $[d\phi] e^{-S_1(\phi)}$  and  $f(\phi)$ , and in the third line we renamed the variable of integration,  $\phi^\zeta \rightarrow \phi$ .

In the last line of (22),  $\Delta_{\text{FP}}$  is evaluated only for  $\phi$  on the gauge slice, so it suffices to find an expression for it that is valid there. Let  $\hat{\phi}$  be on the gauge slice (so  $F^A(\hat{\phi}) = 0$ ), parametrize the gauge group near the identity by coordinates  $\epsilon^B$ , and define

$$\delta_B F^A(\hat{\phi}) \equiv \left. \frac{\partial F^A(\hat{\phi}^\zeta)}{\partial \epsilon^B} \right|_{\epsilon=0} = \left. \frac{\partial F^A}{\partial \phi_i} \frac{\partial \hat{\phi}_i^\zeta}{\partial \epsilon^B} \right|_{\epsilon=0}. \quad (23)$$

If the  $F^A$  are properly behaved (i.e. if they have non-zero and linearly independent gradients at  $\hat{\phi}$ ), and if there are no gauge transformations that leave  $\hat{\phi}$  fixed, then  $\delta_B F^A$  will be a non-singular square matrix. If we choose the coordinates  $\epsilon^B$  such that  $[d\zeta] = [d\epsilon^B]$  locally, then the Faddeev-Popov determinant is precisely the determinant of  $\delta_B F^A$ , and can be represented as a path integral over ghost fields:

$$\begin{aligned} \Delta_{\text{FP}}(\hat{\phi}) &= \left( \int [d\epsilon^B] \delta \left( F^A(\hat{\phi}^\zeta) \right) \right)^{-1} \\ &= \left( \int [d\epsilon^B] \delta \left( \delta_B F^A(\hat{\phi}) \epsilon^B \right) \right)^{-1} \\ &= \det \left( \delta_B F^A(\hat{\phi}) \right) \\ &= \int [db_A dc^B] e^{-b_A \delta_B F^A(\hat{\phi}) c^B}. \end{aligned} \quad (24)$$

Finally, we can express the delta function appearing in the gauge-fixed path integral (22) as a path integral itself:

$$\delta \left( F^A(\phi) \right) = \int [dB_A] e^{iB_A F^A(\phi)}. \quad (25)$$

Putting it all together, we obtain (4.2.3):

$$\int [d\phi db_A dc^B dB_A] e^{-S_1(\phi) - b_A \delta_B F^A(\phi) c^B + i B_A F^A(\phi)} f(\phi). \quad (26)$$

### 3.5 Problem 3.5

For each field configuration  $\phi$ , there is a unique gauge-equivalent configuration  $\hat{\phi}_F$  in the gauge slice defined by the  $F^A$ , and a unique gauge transformation  $\zeta_F(\phi)$  that takes  $\hat{\phi}_F$  to  $\phi$ :

$$\phi = \hat{\phi}_F^{\zeta_F(\phi)}. \quad (27)$$

For  $\phi$  near  $\hat{\phi}_F$ ,  $\zeta_F(\phi)$  will be near the identity and can be parametrized by  $\epsilon_F^B(\phi)$ , the same coordinates used in the previous problem. For such  $\phi$  we have

$$F^A(\phi) = \delta_B F^A(\hat{\phi}_F) \epsilon_F^B(\phi), \quad (28)$$

and we can write the factor  $\Delta_{\text{FP}}^F(\phi) \delta(F^A(\phi))$  appearing in the gauge-fixed path integral (22) in terms of  $\epsilon_F^B(\phi)$ :

$$\begin{aligned} \Delta_{\text{FP}}^F(\phi) \delta(F^A(\phi)) &= \Delta_{\text{FP}}^F(\hat{\phi}_F) \delta(F^A(\phi)) \\ &= \det \left( \delta_B F^A(\hat{\phi}_F) \right) \delta \left( \delta_B F^A(\hat{\phi}_F) \epsilon_F^B(\phi) \right) \\ &= \delta \left( \epsilon_F^B(\phi) \right). \end{aligned} \quad (29)$$

Defining  $\zeta_G(\phi)$  in the same way, we have,

$$\zeta_G \left( \phi^{\zeta_G^{-1} \zeta_F(\phi)} \right) = \zeta_F(\phi). \quad (30)$$

Defining

$$\phi' \equiv \phi^{\zeta_G^{-1} \zeta_F(\phi)}, \quad (31)$$

it follows from (29) that

$$\Delta_{\text{FP}}^F(\phi) \delta(F^A(\phi)) = \Delta_{\text{FP}}^G(\phi') \delta(G^A(\phi')). \quad (32)$$

It is now straightforward to prove that the gauge-fixed path integral is independent of the choice of gauge:

$$\begin{aligned} \int [d\phi] e^{-S(\phi)} \Delta_{\text{FP}}^F(\phi) \delta(F^A(\phi)) f(\phi) &= \int [d\phi'] e^{-S(\phi')} \Delta_{\text{FP}}^G(\phi') \delta(G^A(\phi')) f(\phi') \\ &= \int [d\phi] e^{-S(\phi)} \Delta_{\text{FP}}^G(\phi) \delta(G^A(\phi)) f(\phi). \end{aligned} \quad (33)$$

In the first line we simultaneously used (32) and the gauge invariance of the measure  $[d\phi] e^{-S(\phi)}$  and the insertion  $f(\phi)$ ; in the second line we renamed the variable of integration from  $\phi'$  to  $\phi$ .

### 3.6 Problem 3.7

Let us begin by expressing (3.4.19) in momentum space, to know what we're aiming for. The Ricci scalar, to lowest order in the metric perturbation  $h_{ab} = g_{ab} - \delta_{ab}$ , is

$$R \approx (\partial_a \partial_b - \delta_{ab} \partial^2) h_{ab}. \quad (34)$$

In momentum space, the Green's function defined by (3.4.20) is

$$\tilde{G}(p) \approx -\frac{1}{p^2} \quad (35)$$

(again to lowest order in  $h_{ab}$ ), so the exponent of (3.4.19) is

$$-\frac{a_1}{8\pi} \int \frac{d^2 p}{(2\pi)^2} \tilde{h}_{ab}(p) \tilde{h}_{cd}(-p) \left( \frac{p_a p_b p_c p_d}{p^2} - 2\delta_{ab} p_c p_d + \delta_{ab} \delta_{cd} p^2 \right). \quad (36)$$

To first order in  $h_{ab}$ , the Polyakov action (3.2.3a) is

$$S_X = \frac{1}{2} \int d^2 \sigma \left( \partial_a X \partial_a X + \left( \frac{1}{2} h \delta_{ab} - h_{ab} \right) \partial_a X \partial_b X \right), \quad (37)$$

where  $h \equiv h_{aa}$  (we have set  $2\pi\alpha'$  to 1). We will use dimensional regularization, which breaks conformal invariance because the graviton trace couples to  $X$  when  $d \neq 2$ . The traceless part of  $h_{ab}$  in  $d$  dimensions is  $h'_{ab} = h_{ab} - h/d$ . This leaves a coupling between  $h$  and  $\partial_a X \partial_a X$  with coefficient  $1/2 - 1/d$ . The momentum-space vertex for  $h'_{ab}$  is

$$\tilde{h}'_{ab}(p) k_a (k_b + p_b), \quad (38)$$

while that for  $h$  is

$$-\frac{d-2}{2d} \tilde{h}(p) k \cdot (k+p). \quad (39)$$

There are three one-loop diagrams with two external gravitons, depending on whether the gravitons are traceless or trace.

We begin by dispensing with the  $hh$  diagram. In dimensional regularization, divergences in loop integrals show up as poles in the  $d$  plane. Arising as they do in the form of a gamma function, these are always simple poles. But the diagram is multiplied by two factors of  $d-2$  from the two  $h$  vertices, so it vanishes when we take  $d$  to 2.

The  $hh'_{ab}$  diagram is multiplied by only one factor of  $d-2$ , so part of it (the divergent part that would normally be subtracted off) might survive. It is equal to

$$-\frac{d-2}{4d} \int \frac{d^d p}{(2\pi)^d} \tilde{h}'_{ab}(p) \tilde{h}(-p) \int \frac{d^d k}{(2\pi)^d} \frac{k_a (k_b + p_b) k \cdot (k+p)}{k^2 (k+p)^2}. \quad (40)$$

The  $k$  integral can be evaluated by the usual tricks:

$$\begin{aligned} & \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{k_a (k_b + p_b) k \cdot (k+p)}{(k^2 + 2xp \cdot k + xp^2)^2} \\ &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{(q_a - xp_a)(q_b + (1-x)p_b)(q - xp) \cdot (q + (1-x)p)}{(q^2 + x(1-x)p^2)^2}. \end{aligned} \quad (41)$$

Discarding terms that vanish due to the tracelessness of  $h'_{ab}$  or that are finite in the limit  $d \rightarrow 2$  yields

$$p_a p_b \int_0^1 dx \left( \frac{1}{2} - 3x - 3x^2 \right) \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{(q^2 + x(1-x)p^2)^2}. \quad (42)$$

The divergent part of the  $q$  integral is independent of  $x$ , and the  $x$  integral vanishes, so this diagram vanishes as well.

We are left with just the  $h'_{ab} h'_{cd}$  diagram, which (including a symmetry factor of 4 for the identical vertices and identical propagators) equals

$$\frac{1}{4} \int \frac{d^d p}{(2\pi)^d} \tilde{h}'_{ab}(p) \tilde{h}'_{cd}(-p) \int \frac{d^d k}{(2\pi)^d} \frac{k_a (k_b + p_b) k_c (k_d + p_d)}{k^2 (k+p)^2}. \quad (43)$$

The usual tricks, plus the symmetry and tracelessness of  $h'_{ab}$ , allow us to write the  $k$  integral in the following way:

$$\int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{\frac{2}{d(d+2)} \delta_{ac} \delta_{bd} q^4 + \frac{1}{d} (1-2x)^2 \delta_{ac} p_b p_d q^2 + x^2 (1-x)^2 p_a p_b p_c p_d}{(q^2 + x(1-x)p^2)^2}. \quad (44)$$

The  $q^4$  and  $q^2$  terms in the numerator give rise to divergent integrals. Integrating these terms over  $q$  yields

$$\begin{aligned} \frac{1}{8\pi} \int_0^1 dx \Gamma\left(1 - \frac{d}{2}\right) \left( \frac{x(1-x)p^2}{4\pi} \right)^{d/2-1} \\ \times \left[ -\frac{2}{d} x(1-x) \delta_{ac} \delta_{bd} p^2 + (1-2x)^2 \delta_{ac} p_b p_d \right]. \end{aligned} \quad (45)$$

The divergent part of this is

$$\frac{\delta_{ac} \delta_{bd} p^2 - 2\delta_{ac} p_b p_d}{24\pi(d-2)}. \quad (46)$$

However, it is a fact that the symmetric part of the product of two symmetric, traceless,  $2 \times 2$  matrices is proportional to the identity matrix, so the two terms in the numerator are actually equal after multiplying by  $\tilde{h}'_{ab}(p) \tilde{h}'_{cd}(-p)$ —we see that dimensional regularization has already discarded the divergence for us. The finite part of (45) is (using this trick a second time)

$$\frac{\delta_{ac} \delta_{bd} p^2}{8\pi} \int_0^1 dx \left[ \left( -\gamma - \ln \left( \frac{x(1-x)p^2}{4\pi} \right) \right) \left( \frac{1}{2} - 3x + 3x^2 \right) - x(1-x) \right]. \quad (47)$$

Amazingly, this also vanishes upon performing the  $x$  integral. It remains only to perform the integral for the last term in the numerator of (44), which is convergent at  $d = 2$ :

$$\int_0^1 dx \int \frac{d^2 q}{(2\pi)^2} \frac{x^2 (1-x)^2 p_a p_b p_c p_d}{(q^2 + x(1-x)p^2)^2} = \frac{p_a p_b p_c p_d}{4\pi p^2} \int_0^1 dx x(1-x) = \frac{p_a p_b p_c p_d}{24\pi p^2}. \quad (48)$$

Plugging this back into (43), we find for the 2-graviton contribution to the vacuum amplitude,

$$\frac{1}{96\pi} \int \frac{d^2 p}{(2\pi)^2} \tilde{h}_{ab}(p) \tilde{h}_{cd}(-p) \left( \frac{p_a p_b p_c p_d}{p^2} - \delta_{ab} p_c p_d + \frac{1}{4} \delta_{ab} \delta_{cd} p^2 \right). \quad (49)$$

This result does not agree with (36), and is furthermore quite peculiar. It is Weyl invariant (since the trace  $h$  decoupled), but not diff invariant. It therefore appears that, instead of a Weyl anomaly, we have discovered a gravitational anomaly. However, just because dimensional regularization has (rather amazingly) thrown away the divergent parts of the loop integrals for us, does not mean that renormalization becomes unnecessary. We must still choose renormalization conditions, and introduce counterterms to satisfy them. In this case, we will impose diff invariance, which is more important than Weyl invariance—without it, it would be impossible to couple this CFT consistently to gravity. Locality in real space demands that the counterterms be of the same form as the last two terms in the parentheses in (49). We are therefore free to adjust the coefficients of these two terms in order to achieve diff invariance. Since (36) is manifestly diff invariant, it is clearly the desired expression, with  $a_1$  taking the value  $-1/12$ . (It is worth pointing out that there is no local counterterm quadratic in  $h_{ab}$  that one could add that is diff invariant by itself, and that would therefore have to be fixed by some additional renormalization condition. This is because diff-invariant quantities are constructed out of the Ricci scalar, and  $\int d^2\sigma R^2$  has the wrong dimension.)

### 3.7 Problem 3.9

Fix coordinates such that the boundary lies at  $\sigma_2 = 0$ . Following the prescription of problem 2.10 for normal ordering operators in the presence of a boundary, we include in the contraction the image term:

$$\Delta_b(\sigma, \sigma') = \Delta(\sigma, \sigma') + \Delta(\sigma, \sigma'^*), \quad (50)$$

where  $\sigma_1^* = \sigma_1$ ,  $\sigma_2^* = -\sigma_2$ . If  $\sigma$  and  $\sigma'$  both lie on the boundary, then the contraction is effectively doubled:

$$\Delta_b(\sigma_1, \sigma'_1) = 2\Delta((\sigma_1, \sigma_2 = 0), (\sigma'_1, \sigma'_2 = 0)). \quad (51)$$

If  $\mathcal{F}$  is a boundary operator, then the  $\sigma_2$  and  $\sigma'_2$  integrations in the definition (3.6.5) of  $[\mathcal{F}]_r$  can be done trivially:

$$[\mathcal{F}]_r = \exp\left(\frac{1}{2} \int d\sigma_1 d\sigma'_1 \Delta_b(\sigma_1, \sigma'_1) \frac{\delta}{\delta X^\nu(\sigma_1, \sigma_2 = 0)} \frac{\delta}{\delta X_\nu(\sigma'_1, \sigma'_2 = 0)}\right) \mathcal{F}. \quad (52)$$

Equation (3.6.7) becomes

$$\delta_W[\mathcal{F}]_r = [\delta_W \mathcal{F}]_r + \frac{1}{2} \int d\sigma_1 d\sigma'_1 \delta_W \Delta_b(\sigma_1, \sigma'_1) \frac{\delta}{\delta X^\nu(\sigma_1)} \frac{\delta}{\delta X_\nu(\sigma'_1)} [\mathcal{F}]_r. \quad (53)$$

The tachyon vertex operator (3.6.25) is

$$V_0 = g_o \int_{\sigma_2=0} d\sigma_1 g_{11}^{1/2}(\sigma_1) [e^{ik \cdot X(\sigma_1)}]_r, \quad (54)$$

and its Weyl variation (53) is

$$\begin{aligned}
\delta_W V_0 &= g_o \int d\sigma_1 g_{11}^{1/2}(\sigma_1) (\delta\omega(\sigma_1) + \delta_W) [e^{ik \cdot X(\sigma_1)}]_r \\
&= g_o \int d\sigma_1 g_{11}^{1/2}(\sigma_1) \left( \delta\omega(\sigma_1) - \frac{k^2}{2} \delta_W \Delta_b(\sigma_1, \sigma_1) \right) [e^{ik \cdot X(\sigma_1)}]_r \\
&= (1 - \alpha' k^2) g_o \int d\sigma_1 g_{11}^{1/2}(\sigma_1) \delta\omega(\sigma_1) [e^{ik \cdot X(\sigma_1)}]_r,
\end{aligned} \tag{55}$$

where we have used (3.6.11) in the last equality:

$$\delta_W \Delta_b(\sigma_1, \sigma_1) = 2\delta_W \Delta(\sigma_1, \sigma'_1) = 2\alpha' \delta\omega(\sigma_1). \tag{56}$$

Weyl invariance thus requires

$$k^2 = \frac{1}{\alpha'}. \tag{57}$$

The photon vertex operator (3.6.26) is

$$V_1 = -i \frac{g_o}{\sqrt{2\alpha'}} e_\mu \int_{\sigma_2=0} d\sigma_1 [\partial_1 X^\mu(\sigma_1) e^{ik \cdot X(\sigma_1)}]_r. \tag{58}$$

The spacetime gauge equivalence,

$$V_1(k, e) = V_1(k, e + \lambda k), \tag{59}$$

is clear from the fact that  $k_\mu \partial_1 X^\mu e^{ik \cdot X}$  is a total derivative. The expression (58) has no explicit metric dependence, so the variation of  $V_1$  comes entirely from the variation of the renormalization contraction:

$$\begin{aligned}
&\delta_W [\partial_1 X^\mu(\sigma_1) e^{ik \cdot X(\sigma_1)}]_r \\
&= \frac{1}{2} \int d\sigma'_1 d\sigma''_1 \delta_W \Delta_b(\sigma'_1, \sigma''_1) \frac{\delta}{\delta X^\nu(\sigma'_1)} \frac{\delta}{\delta X^\nu(\sigma''_1)} [\partial_1 X^\mu(\sigma_1) e^{ik \cdot X(\sigma_1)}]_r \\
&= ik^\mu \partial_1 \delta_W \Delta_b(\sigma_1, \sigma''_1) \Big|_{\sigma''_1=\sigma_1} [e^{ik \cdot X(\sigma_1)}]_r \\
&\quad - \frac{k^2}{2} \delta_W \Delta_b(\sigma_1, \sigma_1) [\partial_1 X^\mu(\sigma_1) e^{ik \cdot X(\sigma_1)}]_r \\
&= i\alpha' k^\mu \partial_1 \delta\omega(\sigma_1) [e^{ik \cdot X(\sigma_1)}]_r - \alpha' k^2 \delta\omega(\sigma_1) [\partial_1 X^\mu(\sigma_1) e^{ik \cdot X(\sigma_1)}]_r,
\end{aligned} \tag{60}$$

where in the last equality we have used (56) and (3.6.15a):

$$\partial_1 \delta_W \Delta_b(\sigma_1, \sigma'_1) \Big|_{\sigma'_1=\sigma_1} = 2 \partial_1 \delta_W \Delta(\sigma_1, \sigma'_1) \Big|_{\sigma'_1=\sigma_1} = \alpha' \partial_1 \delta\omega(\sigma_1). \tag{61}$$

Integration by parts yields

$$\delta_W V_1 = -i \sqrt{\frac{\alpha'}{2}} g_o (e \cdot k k_\mu - k^2 e_\mu) \int d\sigma_1 \delta\omega(\sigma_1) [\partial_1 X^\mu(\sigma_1) e^{ik \cdot X(\sigma_1)}]_r. \tag{62}$$

For this quantity to vanish for arbitrary  $\delta\omega(\sigma_1)$  requires the vector  $e \cdot k k - k^2 e$  to vanish. This will happen if  $e$  and  $k$  are collinear, but by (59)  $V_1$  vanishes in this case. The other possibility is

$$k^2 = 0, \quad e \cdot k = 0. \tag{63}$$

### 3.8 Problem 3.11

Since we are interested in the  $H^2$  term, let us assume  $G_{\mu\nu}$  to be constant,  $\Phi$  to vanish, and  $B_{\mu\nu}$  to be linear in  $X$ , implying that

$$H_{\omega\mu\nu} = 3\partial_{[\omega}B_{\mu\nu]} \quad (64)$$

is constant. With these simplifications, the sigma model action becomes

$$\begin{aligned} S_\sigma &= \frac{1}{4\pi\alpha'} \int d^2\sigma g^{1/2} \left( G_{\mu\nu} g^{ab} \partial_a X^\mu \partial_b X^\nu + i\partial_\omega B_{\mu\nu} \epsilon^{ab} X^\omega \partial_a X^\mu \partial_b X^\nu \right) \\ &= \frac{1}{4\pi\alpha'} \int d^2\sigma g^{1/2} \left( G_{\mu\nu} g^{ab} \partial_a X^\mu \partial_b X^\nu + \frac{i}{3} H_{\omega\mu\nu} \epsilon^{ab} X^\omega \partial_a X^\mu \partial_b X^\nu \right). \end{aligned} \quad (65)$$

In the second line we have used the fact that  $\epsilon^{ab} X^\omega \partial_a X^\mu \partial_b X^\nu$  is totally antisymmetric in  $\omega, \mu, \nu$  (up to integration by parts) to antisymmetrize  $\partial_\omega B_{\mu\nu}$ .

Working in conformal gauge on the worldsheet and transforming to complex coordinates,

$$g^{1/2} g^{ab} \partial_a X^\mu \partial_b X^\nu = 4\partial X^{(\mu} \bar{\partial} X^{\nu)}, \quad (66)$$

$$g^{1/2} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu = -4i\partial X^{[\mu} \bar{\partial} X^{\nu]}, \quad (67)$$

$$d^2\sigma = \frac{1}{2} d^2z, \quad (68)$$

the action becomes

$$S_\sigma = S_f + S_i, \quad (69)$$

$$S_f = \frac{1}{2\pi\alpha'} G_{\mu\nu} \int d^2z \partial X^\mu \bar{\partial} X^\nu, \quad (70)$$

$$S_i = \frac{1}{6\pi\alpha'} H_{\omega\mu\nu} \int d^2z X^\omega \partial X^\mu \bar{\partial} X^\nu, \quad (71)$$

where we have split it into the action for a free CFT and an interaction term. The path integral is now

$$\begin{aligned} \langle \dots \rangle_\sigma &= \langle e^{-S_i} \dots \rangle_f \\ &= \langle \dots \rangle_f - \langle S_i \dots \rangle_f + \frac{1}{2} \langle S_i^2 \dots \rangle_f + \dots, \end{aligned} \quad (72)$$

where  $\langle \rangle_f$  is the path integral calculated using only the free action (70). The Weyl variation of the first term gives rise to the  $D - 26$  Weyl anomaly calculated in section 3.4, while that of the second gives rise to the term in  $\beta_{\mu\nu}^B$  that is linear in  $H$  (3.7.13b). It is the Weyl variation of the third term, quadratic in  $H$ , that we are interested in, and in particular the part proportional to

$$\int d^2z \langle : \partial X^\mu \bar{\partial} X^\nu : \dots \rangle_f, \quad (73)$$

whose coefficient gives the  $H^2$  term in  $\beta_{\mu\nu}^G$ . This third term is

$$\begin{aligned} \frac{1}{2}\langle S_i^2 \dots \rangle_f &= \frac{1}{2(6\pi\alpha')^2} H_{\omega\mu\nu} H_{\omega'\mu'\nu'} \\ &\times \int d^2z d^2z' \langle : X^\omega(z, \bar{z}) \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) :: X^{\omega'}(z', \bar{z}') \partial' X^{\mu'}(z') \bar{\partial}' X^{\nu'}(\bar{z}') : \dots \rangle_f, \end{aligned} \quad (74)$$

where we have normal-ordered the interaction vertices. The Weyl variation of this integral will come from the singular part of the OPE when  $z$  and  $z'$  approach each other. Terms in the OPE containing exactly two  $X$  fields (which will yield (73) after the  $z'$  integration is performed) are obtained by performing two cross-contractions. There are 18 different pairs of cross-contractions one can apply to the integrand of (74), but, since they can all be obtained from each other by integration by parts and permuting the indices  $\omega, \mu, \nu$ , they all give the same result. The contraction derived from the free action (70) is

$$X^\mu(z, \bar{z}) X^\nu(z', \bar{z}') =: X^\mu(z, \bar{z}) X^\nu(z', \bar{z}') : - \frac{\alpha'}{2} G^{\mu\nu} \ln |z - z'|^2, \quad (75)$$

so, picking a representative pair of cross-contractions, the part of (74) we are interested in is

$$\begin{aligned} &\frac{18}{2(6\pi\alpha')^2} H_{\omega\mu\nu} H_{\omega'\mu'\nu'} \\ &\times \int d^2z d^2z' \left( -\frac{\alpha'}{2} \right) G^{\omega\mu'} \partial' \ln |z - z'|^2 \left( -\frac{\alpha'}{2} \right) G^{\nu\omega'} \bar{\partial} \ln |z - z'|^2 \\ &\quad \times \langle : \partial X^\mu(z) \bar{\partial}' X^{\nu'}(\bar{z}') : \dots \rangle_f \\ &= -\frac{1}{16\pi^2} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} \int d^2z d^2z' \frac{1}{|z' - z|^2} \langle : \partial X^\mu(z) \bar{\partial}' X^\nu(\bar{z}') : \dots \rangle_f. \end{aligned} \quad (76)$$

The Weyl variation of this term comes from cutting off the logarithmically divergent integral of  $|z' - z|^{-2}$  near  $z' = z$ , so we can drop the less singular terms coming from the Taylor expansion of  $\bar{\partial}' X^\nu(\bar{z}')$ :

$$-\frac{1}{16\pi^2} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} \int d^2z \langle : \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) : \dots \rangle_f \int d^2z' \frac{1}{|z' - z|^2}. \quad (77)$$

The diff-invariant distance between  $z'$  and  $z$  is (for short distances)  $e^{\omega(z)}|z' - z|$ , so a diff-invariant cutoff would be at  $|z' - z| = \epsilon e^{-\omega(z)}$ . The Weyl-dependent part of the second integral of (77) is then

$$\int d^2z' \frac{1}{|z' - z|^2} \sim -2\pi \ln(\epsilon e^{-\omega(z)}) = -2\pi \ln \epsilon + 2\pi\omega(z), \quad (78)$$

and the Weyl variation of (76) is

$$-\frac{1}{8\pi} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} \int d^2z \delta\omega(z) \langle : \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) : \dots \rangle_f. \quad (79)$$

Using (66) and (68), and the fact that the difference between  $\langle \rangle_\sigma$  and  $\langle \rangle_f$  involves higher powers of  $H$  (see (72)) which we can neglect, we can write this as

$$-\frac{1}{16\pi} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} \int d^2\sigma g^{1/2} \delta\omega g^{ab} \langle : \partial_a X^\mu \partial_b X^\nu : \dots \rangle_\sigma. \quad (80)$$



This is of the form of (3.4.6), with

$$T'^a{}_a = \frac{1}{8} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} g^{ab} \partial_a X^\mu \partial_b X^\nu \quad (81)$$

being the contribution of this term to the stress tensor. According to (3.7.12),  $T'^a{}_a$  in turn contributes the following term to  $\beta_{\mu\nu}^G$ :

$$-\frac{\alpha'}{4} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega}. \quad (82)$$

### 3.9 Problem 3.13

If the dilaton  $\Phi$  is constant and  $D = d + 3$ , then the equations of motion (3.7.15) become, to leading order in  $\alpha'$ ,

$$R_{\mu\nu} - \frac{1}{4} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} = 0, \quad (83)$$

$$\nabla^\omega H_{\omega\mu\nu} = 0, \quad (84)$$

$$\frac{d-23}{\alpha'} - \frac{1}{4} H_{\mu\nu\lambda} H^{\mu\nu\lambda} = 0. \quad (85)$$

Letting  $i, j, k$  be indices on the 3-sphere and  $\alpha, \beta, \gamma$  be indices on the flat  $d$ -dimensional spacetime, we apply the ansatz

$$H_{ijk} = h \epsilon_{ijk}, \quad (86)$$

where  $h$  is a constant and  $\epsilon$  is the volume form on the sphere, with all other components vanishing. (Note that this form for  $H$  cannot be obtained as the exterior derivative of a non-singular gauge field  $B$ ;  $B$  must have a Dirac-type singularity somewhere on the sphere.) Equation (84) is then immediately satisfied, because the volume form is always covariantly constant on a manifold, so  $\nabla^i H_{ijk} = 0$ , and all other components vanish trivially. Since  $\epsilon_{ijk} \epsilon^{ijk} = 6$ , equation (85) fixes  $h$  in terms of  $d$ :

$$h^2 = \frac{2(d-23)}{3\alpha'}, \quad (87)$$

implying that there are solutions only for  $d > 23$ . The Ricci tensor on a 3-sphere of radius  $r$  is given by

$$R_{ij} = \frac{2}{r^2} G_{ij}. \quad (88)$$

Similarly,

$$\epsilon_{ikl} \epsilon_j{}^{kl} = 2G_{ij}. \quad (89)$$

Most components of equation (83) vanish trivially, but those for which both indices are on the sphere fix  $r$  in terms of  $h$ :

$$r^2 = \frac{4}{h^2} = \frac{6\alpha'}{d-23}. \quad (90)$$

## 4 Chapter 4

### 4.1 Problem 4.1

To begin, let us recall the spectrum of the open string at level  $N = 2$  in light-cone quantization. In representations of  $\text{SO}(D - 2)$ , we had a symmetric rank 2 tensor,

$$f_{ij}\alpha_{-1}^i\alpha_{-1}^j|0; k\rangle, \quad (1)$$

and a vector,

$$e_i\alpha_{-2}^i|0; k\rangle. \quad (2)$$

Together, they make up the traceless symmetric rank 2 tensor representation of  $\text{SO}(D - 1)$ , whose dimension is  $D(D - 1)/2 - 1$ . This is what we expect to find.

In the OCQ, the general state at level 2 is

$$|f, e; k\rangle = (f_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\nu + e_\mu\alpha_{-2}^\mu)|0; k\rangle, \quad (3)$$

a total of  $D(D + 1)/2 + D$  states. Its norm is

$$\begin{aligned} \langle e, f; k|e, f; k'\rangle &= \langle 0; k|(f_{\rho\sigma}^*\alpha_1^\rho\alpha_1^\sigma + e_\rho^*\alpha_2^\rho)(f_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\nu + e_\mu\alpha_{-2}^\mu)|0; k'\rangle \\ &= 2(f_{\mu\nu}^*f^{\mu\nu} + e_\mu^*e^\mu)\langle 0; k|0; k'\rangle. \end{aligned} \quad (4)$$

The terms in the mode expansion of the Virasoro generator relevant here are as follows:

$$L_0 = \alpha'p^2 + \alpha_{-1} \cdot \alpha_1 + \alpha_{-2} \cdot \alpha_2 + \cdots \quad (5)$$

$$L_1 = \sqrt{2\alpha'}p \cdot \alpha_1 + \alpha_{-1} \cdot \alpha_2 + \cdots \quad (6)$$

$$L_2 = \sqrt{2\alpha'}p \cdot \alpha_2 + \frac{1}{2}\alpha_1 \cdot \alpha_1 + \cdots \quad (7)$$

$$L_{-1} = \sqrt{2\alpha'}p \cdot \alpha_{-1} + \alpha_{-2} \cdot \alpha_1 + \cdots \quad (8)$$

$$L_{-2} = \sqrt{2\alpha'}p \cdot \alpha_{-2} + \frac{1}{2}\alpha_{-1} \cdot \alpha_{-1} + \cdots \quad (9)$$

As in the cases of the tachyon and photon, the  $L_0$  condition yields the mass-shell condition:

$$\begin{aligned} 0 &= (L_0 - 1)|f, e; k\rangle \\ &= (\alpha'k^2 + 1)|f, e; k\rangle, \end{aligned} \quad (10)$$

or  $m^2 = 1/\alpha'$ , the same as in the light-cone quantization. Since the particle is massive, we can go to its rest frame for simplicity:  $k_0 = 1/\sqrt{\alpha'}$ ,  $k_i = 0$ . The  $L_1$  condition fixes  $e$  in terms of  $f$ , removing  $D$  degrees of freedom:

$$\begin{aligned} 0 &= L_1|f, e; k\rangle \\ &= 2\left(\sqrt{2\alpha'}f_{\mu\nu}k^\nu + e_\mu\right)\alpha_{-1}^\mu|0; k\rangle, \end{aligned} \quad (11)$$

implying

$$e_\mu = \sqrt{2}f_{0\mu}. \quad (12)$$

The  $L_2$  condition adds one more constraint:

$$\begin{aligned} 0 &= L_2|f, e; k\rangle \\ &= \left(2\sqrt{2\alpha'}k_\mu e^\mu + f_\mu^\mu\right)|0; k\rangle. \end{aligned} \quad (13)$$

Using (12), this implies

$$f_{ii} = 5f_{00}, \quad (14)$$

where  $f_{ii}$  is the trace on the spacelike part of  $f$ .

There are  $D + 1$  independent spurious states at this level:

$$\begin{aligned} |g, \gamma; k\rangle &= (L_{-1}g_\mu\alpha_{-1}^\mu + L_{-2}\gamma)|0; k\rangle \\ &= \left(\sqrt{2\alpha'}g_{(\mu}k_{\nu)} + \frac{\gamma}{2}\eta_{\mu\nu}\right)\alpha_{-1}^\mu\alpha_{-1}^\nu|0; k\rangle + \left(g_\mu + \sqrt{2\alpha'}\gamma k_\mu\right)\alpha_{-2}^\mu|0; k\rangle. \end{aligned} \quad (15)$$

These states are physical and therefore null for  $g_0 = \gamma = 0$ . Removing these  $D - 1$  states from the spectrum leaves  $D(D - 1)/2$  states, the extra one with respect to the light-cone quantization being the  $\text{SO}(D - 1)$  scalar,

$$f_{ij} = f\delta_{ij}, \quad f_{00} = \frac{D - 1}{5}f, \quad e_0 = \frac{\sqrt{2}(D - 1)}{5}f, \quad (16)$$

with all other components zero. (States with vanishing  $f_{00}$  must be traceless by (14), and this is the unique state satisfying (12) and (14) that is orthogonal to all of these.) The norm of this state is proportional to

$$f_{\mu\nu}^*f^{\mu\nu} + e_\mu^*e^\mu = \frac{(D - 1)(26 - D)f^2}{25}, \quad (17)$$

positive for  $D < 26$  and negative for  $D > 26$ . In the case  $D = 26$ , this state is spurious, corresponding to (15) with  $\gamma = 2f$ ,  $g_0 = 3\sqrt{2}f$ . Removing it from the spectrum leaves us with the states  $f_{ij}$ ,  $f_{ii} = 0$ ,  $e = 0$ —precisely the traceless symmetric rank 2 tensor of  $\text{SO}(25)$  we found in the light-cone quantization.

## 5 Chapter 5

### 5.1 Problem 5.1

(a) Our starting point is the following formal expression for the path integral:

$$Z(X_0, X_1) = \int_{\substack{X(0)=X_0 \\ X(1)=X_1}} \frac{[dX de]}{V_{\text{diff}}} \exp(-S_m[X, e]), \quad (1)$$

where the action for the “matter” fields  $X^\mu$  is

$$S_m[X, e] = \frac{1}{2} \int_0^1 d\tau e (e^{-1} \partial X^\mu e^{-1} \partial X_\mu + m^2) \quad (2)$$

(where  $\partial \equiv d/d\tau$ ). We have fixed the coordinate range for  $\tau$  to be  $[0,1]$ . Coordinate diffeomorphisms  $\zeta : [0, 1] \rightarrow [0, 1]$ , under which the  $X^\mu$  are scalars,

$$X^{\mu\zeta}(\tau^\zeta) = X^\mu(\tau), \quad (3)$$

and the einbein  $e$  is a “co-vector,”

$$e^\zeta(\tau^\zeta) = e(\tau) \frac{d\tau}{d\tau^\zeta}, \quad (4)$$

leave the action (2) invariant.  $V_{\text{diff}}$  is the volume of this group of diffeomorphisms. The  $e$  integral in (1) runs over positive functions on  $[0,1]$ , and the integral

$$l \equiv \int_0^1 d\tau e \quad (5)$$

is diffeomorphism invariant and therefore a modulus; the moduli space is  $(0, \infty)$ .

In order to make sense of the functional integrals in (1) we will need to define an inner product on the space of functions on  $[0,1]$ , which will induce measures on the relevant function spaces. This inner product will depend on the einbein  $e$  in a way that is uniquely determined by the following two constraints: (1) the inner product must be diffeomorphism invariant; (2) it must depend on  $e(\tau)$  only locally, in other words, it must be of the form

$$(f, g)_e = \int_0^1 d\tau h(e(\tau)) f(\tau) g(\tau), \quad (6)$$

for some function  $h$ . As we will see, these conditions will be necessary to allow us to regularize the infinite products that will arise in carrying out the functional integrals in (1), and then to renormalize them by introducing a counter-term action, in a way that respects the symmetries of the action (2). For  $f$  and  $g$  scalars, the inner product satisfying these two conditions is

$$(f, g)_e \equiv \int_0^1 d\tau e f g. \quad (7)$$

We can express the matter action (2) using this inner product:

$$S_m[X, e] = \frac{1}{2}(e^{-1}\partial X^\mu, e^{-1}\partial X_\mu)_e + \frac{lm^2}{2}. \quad (8)$$

We now wish to express the path integral (1) in a slightly less formal way by choosing a fiducial einbein  $e_l$  for each point  $l$  in the moduli space, and replacing the integral over einbeins by an integral over the moduli space times a Faddeev-Popov determinant  $\Delta_{\text{FP}}[e_l]$ . Defining  $\Delta_{\text{FP}}$  by

$$1 = \Delta_{\text{FP}}[e] \int_0^\infty dl \int [d\zeta] \delta[e - e_l^\zeta], \quad (9)$$

we indeed have, by the usual sequence of formal manipulations,

$$Z(X_0, X_1) = \int_0^\infty dl \int_{\substack{X(0)=X_0 \\ X(1)=X_1}} [dX] \Delta_{\text{FP}}[e_l] \exp(-S_m[X, e_l]). \quad (10)$$

To calculate the Faddeev-Popov determinant (9) at the point  $e = e_l$ , we expand  $e$  about  $e_l$  for small diffeomorphisms  $\zeta$  and small changes in the modulus:

$$e_l - e_{l+\delta l}^\zeta = \partial\gamma - \frac{de_l}{dl}\delta l, \quad (11)$$

where  $\gamma$  is a scalar function parametrizing small diffeomorphisms:  $\tau^\zeta = \tau + e^{-1}\gamma$ ; to respect the fixed coordinate range,  $\gamma$  must vanish at 0 and 1. Since the change (11) is, like  $e$ , a co-vector, we will for simplicity multiply it by  $e_l^{-1}$  in order to have a scalar, and then bring into play our inner product (7) in order to express the delta functional in (9) as an integral over scalar functions  $\beta$ :

$$\Delta_{\text{FP}}^{-1}[e_l] = \int d\delta l [d\gamma d\beta] \exp\left(2\pi i(\beta, e_l^{-1}\partial\gamma - e_l^{-1}\frac{de_l}{dl}\delta l)_{e_l}\right) \quad (12)$$

The integral is inverted by replacing the bosonic variables  $\delta l$ ,  $\gamma$ , and  $\beta$  by Grassman variables  $\xi$ ,  $c$ , and  $b$ :

$$\begin{aligned} \Delta_{\text{FP}}[e_l] &= \int d\xi [dcdb] \exp\left(\frac{1}{4\pi}(b, e_l^{-1}\partial c - e_l^{-1}\frac{de_l}{dl}\xi)_{e_l}\right) \\ &= \int [dcdb] \frac{1}{4\pi}(b, e_l^{-1}\frac{de_l}{dl})_{e_l} \exp\left(\frac{1}{4\pi}(b, e_l^{-1}\partial c)_{e_l}\right). \end{aligned} \quad (13)$$

We can now write the path integral (10) in a more explicit form:

$$\begin{aligned} Z(X_0, X_1) &= \int_0^\infty dl \int_{\substack{X(0)=X_0 \\ X(1)=X_1}} [dX] \int_{c(0)=c(1)=0} [dcdb] \frac{1}{4\pi}(b, e_l^{-1}\frac{de_l}{dl})_{e_l} \\ &\quad \times \exp(-S_g[b, c, e_l] - S_m[X, e_l]), \end{aligned} \quad (14)$$

where

$$S_g[b, c, e_l] = -\frac{1}{4\pi}(b, e_l^{-1}\partial c)_{e_l}. \quad (15)$$

(b) At this point it becomes convenient to work in a specific gauge, the simplest being

$$e_l(\tau) = l. \quad (16)$$

Then the inner product (7) becomes simply

$$(f, g)_l = l \int_0^1 d\tau f g. \quad (17)$$

In order to evaluate the Faddeev-Popov determinant (13), let us decompose  $b$  and  $c$  into normalized eigenfunctions of the operator

$$\Delta = -(e_l^{-1}\partial)^2 = -l^{-2}\partial^2 : \quad (18)$$

$$b(\tau) = \frac{b_0}{\sqrt{l}} + \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} b_j \cos(\pi j \tau), \quad (19)$$

$$c(\tau) = \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} c_j \sin(\pi j \tau), \quad (20)$$

with eigenvalues

$$\nu_j = \frac{\pi^2 j^2}{l^2}. \quad (21)$$

The ghost action (15) becomes

$$S_g(b_j, c_j, l) = -\frac{1}{4l} \sum_{j=1}^{\infty} j b_j c_j. \quad (22)$$

The zero mode  $b_0$  does not enter into the action, but it is singled out by the insertion appearing in front of the exponential in (13):

$$\frac{1}{4\pi} (b, e_l^{-1} \frac{de_l}{dl})_{e_l} = \frac{b_0}{4\pi\sqrt{l}}. \quad (23)$$

The Faddeev-Popov determinant is, finally,

$$\begin{aligned} \Delta_{\text{FP}}(l) &= \int \prod_{j=0}^{\infty} db_j \prod_{j=1}^{\infty} dc_j \frac{b_0}{4\pi\sqrt{l}} \exp\left(\frac{1}{4l} \sum_{j=1}^{\infty} j b_j c_j\right) \\ &= \frac{1}{4\pi\sqrt{l}} \prod_{j=1}^{\infty} \frac{j}{4l} \\ &= \frac{1}{4\pi\sqrt{l}} \det' \left( \frac{\Delta}{16\pi^2} \right)^{1/2}, \end{aligned} \quad (24)$$

the prime on the determinant denoting omission of the zero eigenvalue.

(c) Let us decompose  $X^\mu(\tau)$  into a part which obeys the classical equations of motion,

$$X_{\text{cl}}^\mu(\tau) = X_0 + (X_1 - X_0)\tau, \quad (25)$$

plus quantum fluctuations; the fluctuations vanish at 0 and 1, and can therefore be decomposed into the same normalized eigenfunctions of  $\Delta$  as  $c$  was (20):

$$X^\mu(\tau) = X_{\text{cl}}^\mu(\tau) + \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} x_j^\mu \sin(\pi j \tau). \quad (26)$$

The matter action (8) becomes

$$S_{\text{m}}(X_0, X_1, x_j) = \frac{(X_1 - X_0)^2}{2l} + \frac{\pi^2}{l^2} \sum_{j=1}^{\infty} j^2 x_j^2 + \frac{lm^2}{2}, \quad (27)$$

and the matter part of the path integral (10)

$$\begin{aligned} & \int_{\substack{X(0)=X_0 \\ X(1)=X_1}} [dX] \exp(-S_{\text{m}}[X, e_l]) \\ &= \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{lm^2}{2}\right) \int \prod_{\mu=1}^D \prod_{j=1}^{\infty} dx_j^\mu \exp\left(-\frac{\pi^2}{l^2} \sum_{j=1}^{\infty} j^2 x_j^2\right) \\ &= \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{lm^2}{2}\right) \det' \left(\frac{\Delta}{\pi}\right)^{-D/2}, \end{aligned} \quad (28)$$

where we have conveniently chosen to work in a Euclidean spacetime in order to make all of the Gaussian integrals convergent.

(d) Putting together the results (10), (24), and (28), and dropping the irrelevant constant factors multiplying the operator  $\Delta$  in the infinite-dimensional determinants, we have:

$$Z(X_0, X_1) = \int_0^\infty dl \frac{1}{4\pi\sqrt{l}} \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{lm^2}{2}\right) (\det' \Delta)^{(1-D)/2}. \quad (29)$$

We will regularize the determinant of  $\Delta$  in the same way as it is done in Appendix A.1, by dividing by the determinant of the operator  $\Delta + \Omega^2$ :

$$\begin{aligned} \frac{\det' \Delta}{\det'(\Delta + \Omega^2)} &= \prod_{j=1}^{\infty} \frac{\pi^2 j^2}{\pi^2 j^2 + \Omega^2 l^2} \\ &= \frac{\Omega l}{\sinh \Omega l} \\ &\sim 2\Omega l \exp(-\Omega l), \end{aligned} \quad (30)$$

where the last line is the asymptotic expansion for large  $\Omega$ . The path integral (29) becomes

$$\begin{aligned} Z(X_0, X_1) & \\ &= \frac{1}{4\pi(2\Omega)^{(D-1)/2}} \int_0^\infty dl l^{-D/2} \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{l(m^2 - (D-1)\Omega)}{2}\right). \end{aligned} \quad (31)$$

The inverse divergence due to the factor of  $\Omega^{(1-D)/2}$  in front of the integral can be dealt with by a field renormalization, but since we will not concern ourselves with the overall normalization of the path integral we will simply drop all of the factors that appear in front. The divergence coming from the  $\Omega$  term in the exponent can be cancelled by a (diffeomorphism invariant) counterterm in the action,

$$S_{\text{ct}} = \int_0^1 d\tau eA = lA \quad (32)$$

The mass  $m$  is renormalized by what is left over after the cancellation of infinities,

$$m_{\text{phys}}^2 = m^2 - (D-1)\Omega - 2A, \quad (33)$$

but for simplicity we will assume that a renormalization condition has been chosen that sets  $m_{\text{phys}} = m$ .

We can now proceed to the integration over moduli space:

$$Z(X_0, X_1) = \int_0^\infty dl l^{-D/2} \exp\left(-\frac{(X_1 - X_0)^2}{2l} - \frac{lm^2}{2}\right). \quad (34)$$

The integral is most easily done after passing to momentum space:

$$\begin{aligned} \tilde{Z}(k) &\equiv \int d^D X \exp(ik \cdot X) Z(0, X) \\ &= \int_0^\infty dl l^{-D/2} \exp\left(-\frac{lm^2}{2}\right) \int d^D X \exp\left(ik \cdot X - \frac{X^2}{2l}\right) \\ &= \left(\frac{\pi}{2}\right)^{D/2} \int_0^\infty dl \exp\left(-\frac{l(k^2 + m^2)}{2}\right) \\ &= \left(\frac{\pi}{2}\right)^{D/2} \frac{2}{k^2 + m^2}; \end{aligned} \quad (35)$$

neglecting the constant factors, this is precisely the momentum space scalar propagator.



## 6 Chapter 6

### 6.1 Problem 6.1

In terms of  $u = 1/z$ , (6.2.31) is

$$\begin{aligned} \delta^d(\sum k_i) \prod_{i < j} \left| \frac{1}{u_i} - \frac{1}{u_j} \right|^{\alpha' k_i \cdot k_j} &= \delta^d(\sum k_i) \prod_{i < j} \left( |u_{ij}|^{\alpha' k_i \cdot k_j} |u_i u_j|^{-\alpha' k_i \cdot k_j} \right) \\ &= \delta^d(\sum k_i) \prod_{i < j} |u_{ij}|^{\alpha' k_i \cdot k_j} \prod_i |u_i|^{\alpha' k_i^2}. \end{aligned} \quad (1)$$

Since this is an expectation value of closed-string tachyon vertex operators,  $\alpha' k_i^2 = 4$  and the expectation value is smooth at  $u_i = 0$ .

### 6.2 Problem 6.3

For any  $n \geq 2$  numbers  $z_i$ , we have

$$\sum_{i=1}^n (-1)^i \prod_{\substack{j < k \\ j, k \neq i}} z_{jk} = \begin{vmatrix} 1 & 1 & z_1 & z_1^2 & \cdots & z_1^{n-2} \\ 1 & 1 & z_2 & z_2^2 & \cdots & z_2^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & z_n & z_n^2 & \cdots & z_n^{n-2} \end{vmatrix} = 0. \quad (2)$$

The minor of the matrix with respect to the first entry in the  $i$ th row is a Vandermonde matrix for the other  $z_j$ , so its determinant provides the  $i$ th term in the sum. Specializing to the case  $n = 5$ , relabeling  $z_5$  by  $z'_1$ , and dividing by  $z_{11'} z_{21'} z_{31'} z_{41'}$  yields

$$-\frac{z_{23} z_{24} z_{34}}{z_{11'}} + \frac{z_{13} z_{14} z_{34}}{z_{21'}} - \frac{z_{12} z_{14} z_{24}}{z_{31'}} + \frac{z_{12} z_{13} z_{23}}{z_{41'}} = \frac{z_{12} z_{13} z_{14} z_{23} z_{24} z_{34}}{z_{11'} z_{21'} z_{31'} z_{41'}}, \quad (3)$$

which is what we are required to prove.

### 6.3 Problem 6.5

(a) We have

$$I(s, t) = \frac{\Gamma(-1 - \alpha' s) \Gamma(-1 - \alpha' t)}{\Gamma(-2 - \alpha' s - \alpha' t)}, \quad (4)$$

so the pole at  $\alpha' s = J - 1$  arises from the first gamma function in the numerator. The residue of  $\Gamma(z)$  at  $z$  a non-positive integer is  $(-1)^z / \Gamma(1 - z)$ , so the residue of  $I(s, t)$  is

$$\frac{(-1)^J \Gamma(-1 - \alpha' t)}{\Gamma(J + 1) \Gamma(-1 - J - \alpha' t)} = \frac{1}{J!} (2 + \alpha' t)(3 + \alpha' t) \cdots (J + 1 + \alpha' t), \quad (5)$$

a polynomial of degree  $J$  in  $t$ . Using

$$s + t + u = -\frac{4}{\alpha'}, \quad (6)$$

once  $s$  is fixed at  $(J-1)/\alpha'$ ,  $t$  can be expressed in terms of  $t-u$ :

$$t = \frac{t-u}{2} - \frac{J+3}{2\alpha'}, \quad (7)$$

so (5) is also a polynomial of degree  $J$  in  $t-u$ .

(b) The momentum of the intermediate state in the  $s$  channel is  $k_1 + k_2 = -(k_3 + k_4)$ , so in its rest frame we have

$$k_1^i = -k_2^i, \quad k_3^i = -k_4^i, \quad k_1^0 = k_2^0 = -k_3^0 = -k_4^0 = \frac{\sqrt{s}}{2}. \quad (8)$$

Specializing to the case where all the external particles are tachyons ( $k^2 = 1/\alpha'$ ) and the intermediate state is at level 2 ( $s = 1/\alpha'$ ), we further have

$$k_1^i k_1^i = k_2^i k_2^i = k_3^i k_3^i = k_4^i k_4^i = \frac{5}{4\alpha'}. \quad (9)$$

It also determines  $t$  in terms of  $k_1^i k_3^i$ :

$$\begin{aligned} t &= -(k_1 + k_3)^2 \\ &= -\frac{5}{2\alpha'} - 2k_1^i k_3^i. \end{aligned} \quad (10)$$

Using (5), the residue of the pole in  $I(s, t)$  at  $\alpha' s = 1$  is

$$\frac{1}{2}(2 + \alpha' t)(3 + \alpha' t) = -\frac{1}{8} + 2\alpha'^2 (k_1^i k_3^i)^2. \quad (11)$$

The operator that projects matrices onto multiples of the identity matrix in  $D-1$  dimensional space is

$$P_{ij,kl}^0 = \delta_{ij} \frac{1}{D-1} \delta_{kl}, \quad (12)$$

while the one that projects them onto traceless symmetric matrices is

$$P_{ij,kl}^2 = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - P_{ij,kl}^0. \quad (13)$$

Inserting the linear combination  $\beta_0 P^0 + \beta_2 P^2$  between the matrices  $k_1^i k_1^j$  and  $k_3^k k_3^l$  yields

$$(\beta_0 - \beta_2) \frac{k_1^i k_1^i k_3^j k_3^j}{D-1} + \beta_2 (k_1^i k_3^i)^2 = (\beta_0 - \beta_2) \frac{25}{16\alpha'^2 (D-1)} + \beta_2 (k_1^i k_3^i)^2. \quad (14)$$

Comparison with (11) reveals

$$\beta_0 = \frac{2\alpha'^2 (26-D)}{25}, \quad \beta_2 = 2\alpha', \quad (15)$$

so that, as promised,  $\beta_0$  is positive, zero, or negative depending on whether  $D$  is less than, equal to, or greater than 26.

What does all this have to do with the open string spectrum at level 2? The amplitude  $I$  has a pole in  $s$  wherever  $s$  equals the mass-squared of an open string state, allowing the intermediate state in the  $s$  channel to go on shell. The residue of this pole can be written schematically as

$$\langle f|S \left( \sum_o \frac{|o\rangle\langle o|}{\langle o|o\rangle} \right) S|i\rangle, \quad (16)$$

where the sum is taken over open string states at level  $J = \alpha' s + 1$  with momentum equal to that of the initial and final states; we have not assumed that the intermediate states are normalized, to allow for the possibility that some of them might have negative norm. More specifically,

$$|i\rangle = |0; k_1\rangle|0; k_2\rangle, \quad \langle f| = \langle 0; -k_3|\langle 0; -k_4|. \quad (17)$$

The open string spectrum at level 2 was worked out as a function of  $D$  in problem 4.1. For any  $D$  it includes  $D(D-1)/2 - 1$  positive-norm states transforming in the spin 2 representation of the little group  $\text{SO}(D-1)$ . Working in the rest frame of such a state, the  $S$ -matrix elements involved in (16) are fixed by  $\text{SO}(D-1)$  invariance and (anti-)linearity in the polarization matrix  $a$ :

$$\langle a|S|0; k_1\rangle|0; k_2\rangle \propto a_{ij}^* k_1^i k_2^j, \quad \langle 0; -k_3|\langle 0; -k_4|S|a\rangle \propto a_{kl} k_3^k k_4^l. \quad (18)$$

Summing over an orthonormal basis in the space of symmetric traceless matrices yields the contribution of these states to (16):

$$\sum_a a_{ij}^* k_1^i k_2^j a_{kl} k_3^k k_4^l = k_1^i k_1^j P_{ij,kl}^2 k_3^k k_3^l. \quad (19)$$

This explains the positive value of  $\beta_2$  found in (15).

For  $D \neq 26$ , there is another state  $|b\rangle$  in the spectrum whose norm is positive for  $D < 26$  and negative for  $D > 26$ . Since this state is an  $\text{SO}(D-1)$  scalar, the  $S$ -matrix elements connecting it to the initial and final states are given by:

$$\langle b|S|0; k_1\rangle|0; k_2\rangle \propto \delta_{ij} k_1^i k_2^j, \quad \langle 0; -k_3|\langle 0; -k_4|S|b\rangle \propto \delta_{kl} k_3^k k_4^l. \quad (20)$$

Its contribution to (16) is therefore clearly a positive multiple of

$$k_1^i k_1^j P_{ij,kl}^0 k_3^k k_3^l \quad (21)$$

if  $D < 26$ , and a negative multiple if  $D > 26$ .

#### 6.4 Problem 6.7

(a) The  $X$  path integral (6.5.11) follows immediately from (6.2.36), where

$$v^\mu(y_1) = -2i\alpha' \left( \frac{k_2^\mu}{y_{12}} - \frac{k_3^\mu}{y_{13}} \right) \quad (22)$$

(we leave out the contraction between  $\dot{X}^\mu(y_1)$  and  $e^{ik_1 \cdot X(y_1)}$  because their product is already renormalized in the path integral). Momentum conservation,  $k_1 + k_2 + k_3 = 0$ , and the mass shell conditions imply

$$0 = k_1^2 = (k_2 + k_3)^2 = \frac{2}{\alpha'} + 2k_2 \cdot k_3, \quad (23)$$

$$\frac{1}{\alpha'} = k_3^2 = (k_1 + k_2)^2 = \frac{1}{\alpha'} + 2k_1 \cdot k_2, \quad (24)$$

$$\frac{1}{\alpha'} = k_2^2 = (k_1 + k_3)^2 = \frac{1}{\alpha'} + 2k_1 \cdot k_3, \quad (25)$$

so that  $2\alpha'k_2 \cdot k_3 = -2$ , while  $2\alpha'k_1 \cdot k_2 = 2\alpha'k_1 \cdot k_3 = 0$ . Equation (6.5.11) therefore simplifies to

$$\begin{aligned} & \left\langle \star \dot{X}^\mu e^{ik_1 \cdot X}(y_1) \star \star e^{ik_2 \cdot X}(y_2) \star \star e^{ik_3 \cdot X}(y_3) \star \right\rangle_{D_2} \\ & = 2\alpha' C_{D_2}^X (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3) \frac{1}{y_{23}^2} \left( \frac{k_2^\mu}{y_{12}} + \frac{k_3^\mu}{y_{13}} \right). \end{aligned} \quad (26)$$

The ghost path integral is given by (6.3.2):

$$\langle c(y_1)c(y_2)c(y_3) \rangle_{D_2} = C_{D_2}^g y_{12} y_{13} y_{23}. \quad (27)$$

Putting these together with (6.5.10) and using (6.4.14),

$$\alpha' g_0^2 e^{-\lambda} C_{D_2}^X C_{D_2}^g = 1, \quad (28)$$

yields

$$\begin{aligned} & S_{D_2}(k_1, a_1, e_1; k_2, a_2; k_3, a_3) \\ & = -2ig'_0 (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3) \frac{y_{13}e \cdot k_2 + y_{12}e \cdot k_3}{y_{23}} \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3}) + (2 \leftrightarrow 3). \end{aligned} \quad (29)$$

Momentum conservation and the physical state condition  $e_1 \cdot k_1 = 0$  imply

$$e_1 \cdot k_2 = -e_1 \cdot k_3 = \frac{1}{2} e_1 \cdot k_{23}, \quad (30)$$

so

$$\begin{aligned} & S_{D_2}(k_1, a_1, e_1; k_2, a_2; k_3, a_3) \\ & = -ig'_0 (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3) e_1 \cdot k_{23} \text{Tr}(\lambda^{a_1} [\lambda^{a_2}, \lambda^{a_3}]), \end{aligned} \quad (31)$$

in agreement with (6.5.12).

(b) Using equations (6.4.17), (6.4.20), and (6.5.6), we see that the four-tachyon amplitude near  $s = 0$  is given by

$$\begin{aligned}
& S_{D_2}(k_1, a_1; k_2, a_2; k_3, a_3; k_4, a_4) \\
&= \frac{ig_0^2}{\alpha'} (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \\
&\quad \times \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_4} \lambda^{a_3} + \lambda^{a_1} \lambda^{a_3} \lambda^{a_4} \lambda^{a_2} - \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} - \lambda^{a_1} \lambda^{a_4} \lambda^{a_3} \lambda^{a_2}) \frac{u-t}{2s} \\
&= -\frac{ig_0^2}{2\alpha'} (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \text{Tr}([\lambda^{a_1}, \lambda^{a_2}][\lambda^{a_3}, \lambda^{a_4}]) \frac{u-t}{s}. \tag{32}
\end{aligned}$$

We can calculate the same quantity using unitarity. By analogy with equation (6.4.13), the 4-tachyon amplitude near the pole at  $s = 0$  has the form

$$\begin{aligned}
& S_{D_2}(k_1, a_1; k_2, a_2; k_3, a_3; k_4, a_4) \\
&= i \int \frac{d^{26}k}{(2\pi)^{26}} \sum_{a,e} \frac{S_{D_2}(-k, a, e; k_1, a_1; k_2, a_2) S_{D_2}(k, a, e; k_3, a_3; k_4, a_4)}{-k^2 + i\epsilon} \\
&= i \int \frac{d^{26}k}{(2\pi)^{26}} \sum_{a,e} \frac{1}{-k^2 + i\epsilon} (-i) g_0' (2\pi)^{26} \delta^{26} (k_1 + k_2 - k) e \cdot k_{12} \text{Tr}(\lambda^a [\lambda^{a_1}, \lambda^{a_2}]) \\
&\quad \times (-i) g_0' (2\pi)^{26} \delta^{26} (k + k_3 + k_4) e \cdot k_{34} \text{Tr}(\lambda^a [\lambda^{a_3}, \lambda^{a_4}]) \\
&= -i g_0'^2 (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \sum_a \text{Tr}(\lambda^a [\lambda^{a_1}, \lambda^{a_2}]) \text{Tr}(\lambda^a [\lambda^{a_3}, \lambda^{a_4}]) \frac{\sum_e e \cdot k_{12} e \cdot k_{34}}{s + i\epsilon} \\
&= -i g_0'^2 (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \text{Tr}([\lambda^{a_1}, \lambda^{a_2}][\lambda^{a_3}, \lambda^{a_4}]) \frac{u-t}{s + i\epsilon}. \tag{33}
\end{aligned}$$

In the second equality we have substituted equation (31) (or (6.5.12)). The polarization vector  $e$  is summed over an orthonormal basis of (spacelike) vectors obeying the physical state condition  $e \cdot k = 0$ , which after the integration over  $k$  in the third equality becomes  $e \cdot (k_1 + k_2) = e \cdot (k_3 + k_4) = 0$ . If we choose one of the vectors in this basis to be  $e' = k_{12}/|k_{12}|$ , then none of the others will contribute to the sum in the second to last line, which becomes,

$$\sum_e e \cdot k_{12} e \cdot k_{34} = k_{12} \cdot k_{34} = u - t. \tag{34}$$

In the last equality of (33) we have also applied equation (6.5.9). Comparing (32) and (33), we see that

$$g_0' = \frac{g_0}{\sqrt{2\alpha'}}, \tag{35}$$

in agreement with (6.5.14).

This result confirms the normalization of the photon vertex operator as written in equation (3.6.26). The state-operator mapping gives the same normalization: in problem 2.9, we saw that



we can neglect the  $v^\mu$ . The terms we are looking for arise from the contraction of the  $q^\mu$  with each other. Specifically, the singular part of the OPE of  $q^\mu(y)$  with  $q^\nu(y')$  is

$$-2\alpha'(y-y')^{-2}\eta^{\mu\nu}, \quad (42)$$

so the combination  $e_1 \cdot e_2 e_3 \cdot e_4$  arises from the contractions of  $q^{\mu_1}(y_1)$  with  $q^{\mu_2}(y_2)$  and  $q^{\mu_3}(y_3)$  with  $q^{\mu_4}(y_4)$ :

$$\begin{aligned} & ig_0^2 \alpha'^{-1} \text{Tr}(\lambda^{a_3} \lambda^{a_4} \lambda^{a_1} \lambda^{a_2}) e_1 \cdot e_2 e_3 \cdot e_4 (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \\ & \times |y_{12}|^{2\alpha' k_1 \cdot k_2 - 1} |y_{13}|^{2\alpha' k_1 \cdot k_3 + 1} |y_{23}|^{2\alpha' k_2 \cdot k_3 + 1} \\ & \times \int_{y_3}^{y_1} dy_4 |y_{14}|^{2\alpha' k_1 \cdot k_4} |y_{24}|^{2\alpha' k_2 \cdot k_4} |y_{34}|^{2\alpha' k_3 \cdot k_4 - 2}. \end{aligned} \quad (43)$$

We can choose  $y_1$ ,  $y_2$ , and  $y_3$  as we like, so long as we obey (39), and the above expression simplifies if we take the limit  $y_2 \rightarrow \infty$  while keeping  $y_1$  and  $y_3$  fixed. Then  $|y_{12}| \sim |y_{23}| \sim y_2$  and (since  $y_3 < y_4 < y_1$ )  $|y_{24}| \sim y_2$  as well. Making these substitutions above,  $y_2$  appears with a total power of

$$\begin{aligned} 2\alpha' k_1 \cdot k_2 - 1 + 2\alpha' k_2 \cdot k_3 + 1 + 2\alpha' k_2 \cdot k_4 &= 2\alpha'(k_1 + k_3 + k_4) \cdot k_2 \\ &= -2\alpha' k_2^2 \\ &= 0. \end{aligned} \quad (44)$$

We can simplify further by setting  $y_3 = 0$  and  $y_1 = 1$ . Since  $s = -2k_3 \cdot k_4$  and  $u = -2k_1 \cdot k_4$ , the integral above reduces to:

$$\int_0^1 dy_4 (1-y_4)^{-\alpha' u} y_4^{-\alpha' s - 2} = B(-\alpha' u + 1, -\alpha' s - 1). \quad (45)$$

If we now consider a different cyclic ordering of the vertex operators, we can still fix  $y_1$ ,  $y_2$ , and  $y_3$  while integrating over  $y_4$ . Equation (43) will remain the same, with two exceptions: the order of the  $\lambda^a$  matrices appearing in the trace, and the limits of integration on  $y_4$ , will change to reflect the new order. The limits of integration will be whatever positions immediately precede and succeed  $y_4$ , while the position that is opposite  $y_4$  will be taken to infinity. It can easily be seen that the trick that allowed us to take  $y_2$  to infinity (equation (44)) works equally well for  $y_1$  or  $y_3$ . The lower and upper limits of integration can be fixed at 0 and 1 respectively as before, and the resulting integral over  $y_4$  will once again give a beta function. However, since different factors in the integrand of (43) survive for different orderings, the beta function will have different arguments in each case. Putting together the results from the six cyclic orderings, we find that the part of the

four gauge boson amplitude proportional to  $e_1 \cdot e_2 e_3 \cdot e_4$  is

$$\begin{aligned} & \frac{ig_o^2}{\alpha'} e_1 \cdot e_2 e_3 \cdot e_4 (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \\ & \times \left[ \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_4} \lambda^{a_3} + \lambda^{a_1} \lambda^{a_3} \lambda^{a_4} \lambda^{a_2}) B(-\alpha' t + 1, -\alpha' s - 1) \right. \\ & \quad + \text{Tr}(\lambda^{a_1} \lambda^{a_3} \lambda^{a_2} \lambda^{a_4} + \lambda^{a_1} \lambda^{a_4} \lambda^{a_2} \lambda^{a_3}) B(-\alpha' t + 1, -\alpha' u + 1) \\ & \quad \left. + \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} + \lambda^{a_1} \lambda^{a_4} \lambda^{a_3} \lambda^{a_2}) B(-\alpha' u + 1, -\alpha' s - 1) \right]. \end{aligned} \quad (46)$$

(b) There are four tree-level diagrams that contribute to four-boson scattering in Yang-Mills theory: the  $s$ -channel, the  $t$ -channel, the  $u$ -channel, and the four-point vertex. The four-point vertex diagram (which is independent of momenta) includes the following term proportional to  $e_1 \cdot e_2 e_3 \cdot e_4$ :

$$-ig_o'^2 e_1 \cdot e_2 e_3 \cdot e_4 (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \sum_e (f^{a_1 a_3 e} f^{a_2 a_4 e} + f^{a_1 a_4 e} f^{a_2 a_3 e}) \quad (47)$$

(see Peskin and Schroeder, equation A.12). The Yang-Mills coupling is

$$g_o' = (2\alpha')^{-1/2} g_o \quad (48)$$

(6.5.14), and the  $f^{abc}$  are the gauge group structure constants:

$$f^{abc} = \text{Tr} \left( [\lambda^a, \lambda^b] \lambda^c \right). \quad (49)$$

We can therefore re-write (47) in the following form:

$$\begin{aligned} & -\frac{ig_o^2}{\alpha'} e_1 \cdot e_2 e_3 \cdot e_4 (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \text{Tr} \left( \lambda^{a_1} \lambda^{a_3} \lambda^{a_2} \lambda^{a_4} + \lambda^{a_1} \lambda^{a_4} \lambda^{a_2} \lambda^{a_3} \right. \\ & \quad \left. - \frac{1}{2} (\lambda^{a_1} \lambda^{a_2} \lambda^{a_4} \lambda^{a_3} + \lambda^{a_1} \lambda^{a_3} \lambda^{a_4} \lambda^{a_2} + \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} + \lambda^{a_1} \lambda^{a_4} \lambda^{a_3} \lambda^{a_2}) \right). \end{aligned} \quad (50)$$

Of the three diagrams that contain three-point vertices, only the  $s$ -channel diagram contains a term proportional to  $e_1 \cdot e_2 e_3 \cdot e_4$ . It is

$$\begin{aligned} & -ig_o'^2 e_1 \cdot e_2 e_3 \cdot e_4 (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \frac{u-t}{s} \sum_e f^{a_1 a_2 e} f^{a_3 a_4 e} \\ & = -\frac{ig_o^2}{\alpha'} e_1 \cdot e_2 e_3 \cdot e_4 (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \frac{t-u}{2s} \\ & \quad \times \text{Tr} (\lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} + \lambda^{a_1} \lambda^{a_4} \lambda^{a_3} \lambda^{a_2} - \lambda^{a_1} \lambda^{a_2} \lambda^{a_4} \lambda^{a_3} - \lambda^{a_1} \lambda^{a_3} \lambda^{a_4} \lambda^{a_2}). \end{aligned} \quad (51)$$

Combining (50) and (51) and using

$$s + t + u = 0, \quad (52)$$



we obtain, for the part of the four-boson amplitude proportional to  $e_1 \cdot e_2 e_3 \cdot e_4$ , at tree level,

$$\begin{aligned}
& -\frac{ig_0^2}{\alpha'} e_1 \cdot e_2 e_3 \cdot e_4 (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \\
& \times \left[ \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_4} \lambda^{a_3} + \lambda^{a_1} \lambda^{a_3} \lambda^{a_4} \lambda^{a_2}) \left( -1 - \frac{t}{s} \right) \right. \\
& \quad + \text{Tr}(\lambda^{a_1} \lambda^{a_3} \lambda^{a_2} \lambda^{a_4} + \lambda^{a_1} \lambda^{a_4} \lambda^{a_2} \lambda^{a_3}) \\
& \quad \left. + \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} + \lambda^{a_1} \lambda^{a_4} \lambda^{a_3} \lambda^{a_2}) \left( -1 - \frac{u}{s} \right) \right].
\end{aligned} \tag{53}$$

This is intentionally written in a form suggestively similar to equation (46). It is clear that (46) reduces to (53) (up to an overall sign) if we take the limit  $\alpha' \rightarrow 0$  with  $s$ ,  $t$ , and  $u$  fixed, since in that limit

$$\begin{aligned}
B(-\alpha't + 1, -\alpha's - 1) &\approx -1 - \frac{t}{s}, \\
B(-\alpha't + 1, -\alpha'u + 1) &\approx 1, \\
B(-\alpha'u + 1, -\alpha's - 1) &\approx -1 - \frac{u}{s}.
\end{aligned} \tag{54}$$

Thus this single string theory diagram reproduces, at momenta small compared to the string scale, the sum of the field theory Feynman diagrams.

## 6.6 Problem 6.11

(a) The  $X$  path integral is given by (6.2.19):

$$\begin{aligned}
& \left\langle : \partial X^\mu \bar{\partial} X^\nu e^{ik_1 \cdot X}(z_1, \bar{z}_1) :: e^{ik_2 \cdot X}(z_2, \bar{z}_2) :: e^{ik_3 \cdot X}(z_3, \bar{z}_3) : \right\rangle_{S_2} \\
& = -i C_{S_2}^X \frac{\alpha'^2}{4} (2\pi)^{26} \delta^{26} (k_1 + k_2 + k_3) \\
& \quad \times |z_{12}|^{\alpha' k_1 \cdot k_2} |z_{13}|^{\alpha' k_1 \cdot k_3} |z_{23}|^{\alpha' k_2 \cdot k_3} \left( \frac{k_2^\mu}{z_{12}} + \frac{k_3^\mu}{z_{13}} \right) \left( \frac{k_2^\nu}{\bar{z}_{12}} + \frac{k_3^\nu}{\bar{z}_{13}} \right).
\end{aligned} \tag{55}$$

The ghost path integral is given by (6.3.4):

$$\langle : c(z_1) \tilde{c}(\bar{z}_1) :: c(z_2) \tilde{c}(\bar{z}_2) :: c(z_3) \tilde{c}(\bar{z}_3) : \rangle_{S_2} = C_{S_2}^g |z_{12}|^2 |z_{13}|^2 |z_{23}|^2. \tag{56}$$

The momentum-conserving delta function and the mass shell conditions  $k_1^2 = 0$ ,  $k_2^2 = k_3^2 = 4/\alpha'$  imply

$$k_1 \cdot k_2 = k_1 \cdot k_3 = 0, \quad k_2 \cdot k_3 = -4/\alpha'. \tag{57}$$

Using (57), the transversality of the polarization tensor,

$$e_{1\mu\nu} k_1^\mu = 0, \tag{58}$$

and the result (6.6.8),

$$C_{S_2} \equiv e^{-2\lambda} C_{S_2}^X C_{S_2}^g = \frac{8\pi}{\alpha' g_c^2}, \quad (59)$$

we can put together the full amplitude for two closed-string tachyons and one massless closed string on the sphere:

$$\begin{aligned} S_{S_2}(k_1, e_1; k_2; k_3) &= g_c^2 g_c' e^{-2\lambda} e_{1\mu\nu} \\ &\quad \times \left\langle : \tilde{c} c \partial X^\mu \bar{\partial} X^\nu e^{ik_1 \cdot X}(z_1, \bar{z}_1) :: \tilde{c} c e^{ik_2 \cdot X}(z_2, \bar{z}_3) :: \tilde{c} c e^{ik_3 \cdot X}(z_2, \bar{z}_3) : \right\rangle_{S_2} \\ &= -i C_{S_2} g_c^2 g_c' \frac{\alpha'^2}{4} (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3) \\ &\quad \times e_{1\mu\nu} \frac{|z_{12}|^2 |z_{13}|^2}{|z_{23}|^2} \left( \frac{k_2^\mu}{z_{12}} + \frac{k_3^\mu}{z_{13}} \right) \left( \frac{k_2^\nu}{\bar{z}_{12}} + \frac{k_3^\nu}{\bar{z}_{13}} \right) \\ &= -i 2\pi \alpha' g_c' (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3) \\ &\quad \times e_{1\mu\nu} \frac{|z_{12}|^2 |z_{13}|^2}{|z_{23}|^2} \left( \frac{k_{23}^\mu}{2z_{12}} - \frac{k_{23}^\mu}{2z_{13}} \right) \left( \frac{k_{23}^\nu}{2\bar{z}_{12}} - \frac{k_{23}^\nu}{2\bar{z}_{13}} \right) \\ &= -i \frac{\pi \alpha'}{2} g_c' (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3) e_{1\mu\nu} k_{23}^\mu k_{23}^\nu. \end{aligned} \quad (60)$$

**(b)** Let us calculate the amplitude for massless closed string exchange between closed string tachyons (this is a tree-level field theory calculation but for the vertices we will use the amplitude calculated above in string theory). We will restrict ourselves to the  $s$ -channel diagram, because we are interested in comparing the result with the pole at  $s = 0$  in the Virasoro-Shapiro amplitude. Here the propagator for the massless intermediate string provides the pole at  $s = 0$ :

$$-i (2\pi)^{26} \delta^{26} \left( \sum k_i \right) \frac{\pi^2 \alpha'^2 g_c'^2}{4s} \sum_e e_{\mu\nu} k_{12}^\mu k_{12}^\nu e_{\mu'\nu'} k_{34}^{\mu'} k_{34}^{\nu'}. \quad (61)$$

Here  $e$  is summed over an orthonormal basis of symmetric polarization tensors obeying the condition  $e_{\mu\nu}(k_1^\mu + k_2^\mu) = 0$ . We could choose as one element of that basis the tensor

$$e_{\mu\nu} = \frac{k_{12\mu} k_{12\nu}}{k_{12}^2}, \quad (62)$$

which obeys the transversality condition by virtue of the fact that  $k_1^2 = k_2^2$ . With this choice, none of the other elements of the basis would contribute to the sum, which reduces to

$$(k_{12} \cdot k_{34})^2 = (u - t)^2. \quad (63)$$

The amplitude (61) is thus

$$-i (2\pi)^{26} \delta^{26} \left( \sum k_i \right) \frac{\pi^2 \alpha'^2 g_c'^2}{4} \frac{(u - t)^2}{s}. \quad (64)$$

Now, the Virasoro-Shapiro amplitude is

$$i(2\pi)^{26} \delta^{26} \left( \sum k_i \right) \frac{16\pi^2 g_c^2}{\alpha'} \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)}, \quad (65)$$

where

$$a = -1 - \frac{\alpha' s}{4}, \quad b = -1 - \frac{\alpha' t}{4}, \quad c = -1 - \frac{\alpha' u}{4}. \quad (66)$$

The pole at  $s = 0$  arises from the factor of  $\Gamma(a)$ , which is, to lowest order in  $s$ ,

$$\Gamma(a) \approx \frac{4}{\alpha' s}. \quad (67)$$

To lowest order in  $s$  the other gamma functions simplify to

$$\begin{aligned} \frac{\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)} &\approx \frac{\Gamma(b)\Gamma(c)}{\Gamma(b-1)\Gamma(c-1)\Gamma(2)} \\ &= (b-1)(c-1) \\ &= -\frac{\alpha'^2}{64}(u-t)^2. \end{aligned} \quad (68)$$

Thus the part of the amplitude we are interested in is

$$-i(2\pi)^{26} \delta^{26} \left( \sum k_i \right) \pi^2 g_c^2 \frac{(u-t)^2}{s}. \quad (69)$$

Comparison with (64) shows that

$$g_c = \frac{\alpha' g_c'}{2}. \quad (70)$$

(c) In Einstein frame the tachyon kinetic term decouples from the dilaton, as the tachyon action (6.6.16) becomes

$$S_T = -\frac{1}{2} \int d^{26}x (-\tilde{G})^{1/2} \left( \tilde{G}^{\mu\nu} \partial_\mu T \partial_\nu T - \frac{4}{\alpha'} e^{\tilde{\Phi}/6} T^2 \right). \quad (71)$$

If we write a metric perturbation in the following form,

$$\tilde{G}_{\mu\nu} = \eta_{\mu\nu} + 2\kappa e_{\mu\nu} f, \quad (72)$$

where  $e_{\mu\nu} e^{\mu\nu} = 1$ , then the kinetic term for  $f$  will be canonically normalized. To lowest order in  $f$  and  $T$ , the interaction Lagrangian is

$$\mathcal{L}_{\text{int}} = \kappa e^{\mu\nu} f \partial_\mu T \partial_\nu T + \kappa e_\mu^\mu f \left( \frac{2}{\alpha'} T^2 - \frac{1}{2} \partial^\mu T \partial_\mu T \right) \quad (73)$$

(from now on all indices are raised and lowered with  $\eta_{\mu\nu}$ ). The second term, proportional to the trace of  $e$ , makes a vanishing contribution to the amplitude on-shell:

$$-i\kappa e_\mu^\mu \left( \frac{4}{\alpha'} + k_2 \cdot k_3 \right) (2\pi)^{26} \delta^{26} (k_1 + k_2 + k_3) = 0. \quad (74)$$

(The trace of the massless closed string polarization tensor  $e$  used in the string calculations of parts (a) and (b) above represents the dilaton, not the trace of the (Einstein frame) graviton, which can always be gauged away.) The amplitude from the first term of (73) is

$$2i\kappa e_{\mu\nu} k_2^\mu k_3^\nu (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3) = -i\frac{\kappa}{2} e_{\mu\nu} k_{23}^\mu k_{23}^\nu (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3), \quad (75)$$

where we have used the transversality of the graviton polarization  $e_{\mu\nu} k_1^\mu = 0$ . Comparison with the amplitude (6.6.14) shows that

$$\kappa = \pi\alpha' g'_c. \quad (76)$$

### 6.7 Problem 6.12

We can use the three CKVs of the upper half-plane to fix the position  $z$  of the closed-string vertex operator and the position  $y_1$  of one of the upper-string vertex operators. We integrate over the position  $y_2$  of the unfixed open-string vertex operator:

$$\begin{aligned} S_{D_2}(k_1, k_2, k_3) &= g_c g_0^2 e^{-\lambda} \int dy_2 \left\langle : c\tilde{c} e^{ik_1 \cdot X}(z, \bar{z}) : * c^1 e^{ik_2 \cdot X}(y_1)_{**} e^{ik_3 \cdot X}(y_2)_{*} \right\rangle_{D_1} \\ &= g_c g_0^2 e^{-\lambda} C_{D_2}^g |z - y_1| |\bar{z} - y_1| |z - \bar{z}| i C_{D_2}^X (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3) \\ &\quad \times |z - \bar{z}|^{\alpha' k_1^2/2} |z - y_1|^{2\alpha' k_1 \cdot k_2} \int dy_2 |z - y_2|^{2\alpha' k_1 \cdot k_3} |y_1 - y_2|^{2\alpha' k_2 \cdot k_3} \\ &= i C_{D_2} g_c g_0^2 (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3) \\ &\quad \times |z - y_1|^{-2} |z - \bar{z}|^3 \int dy_2 |z - y_2|^{-4} |y_1 - y_2|^2. \end{aligned} \quad (77)$$

The very last line is equal to  $4\pi$ , independent of  $z$  and  $y_1$  (as it should be), as can be calculated by contour integration in the complex plane. Taking into account (6.4.14), the result is

$$\frac{4\pi i g_c}{\alpha'} (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3). \quad (78)$$

## 7 Chapter 7

### 7.1 Problem 7.1

Equation (6.2.13) applied to the case of the torus tells us

$$\left\langle \prod_{i=1}^n : e^{ik_i \cdot X(w_i, \bar{w}_i)} : \right\rangle_{T^2} = iC_{T^2}^X(\tau)(2\pi)^d \delta^d\left(\sum k_i\right) \exp\left(-\sum_{i<j} k_i \cdot k_j G'(w_i, w_j) - \frac{1}{2} \sum_i k_i^2 G'_r(w_i, w_i)\right). \quad (1)$$

$G'$  is the Green's function (7.2.3),

$$G'(w_i, w_j) = -\frac{\alpha'}{2} \ln \left| \vartheta_1\left(\frac{w_{ij}}{2\pi} | \tau\right) \right|^2 + \alpha' \frac{(\text{Im } w_{ij})^2}{4\pi\tau_2} + k(\tau), \quad (2)$$

while  $G'_r$  is the renormalized Green's function, defined by (6.2.15),

$$G'_r(w_i, w_j) = G'(w_i, w_j) + \frac{\alpha'}{2} \ln |w_{ij}|^2, \quad (3)$$

designed to be finite in the limit  $w_j \rightarrow w_i$ :

$$\begin{aligned} \lim_{w_j \rightarrow w_i} G'_r(w_i, w_j) &= -\frac{\alpha'}{2} \ln \left| \lim_{w_{ij} \rightarrow 0} \frac{\vartheta_1\left(\frac{w_{ij}}{2\pi} | \tau\right)}{w_{ij}} \right|^2 + k(\tau) \\ &= -\frac{\alpha'}{2} \ln \left| \frac{\partial_\nu \vartheta_1(\nu=0 | \tau)}{2\pi} \right|^2 + k(\tau). \end{aligned} \quad (4)$$

The argument of the exponential in (1) is thus

$$\begin{aligned} &-\sum_{i<j} k_i \cdot k_j G'(w_i, w_j) - \frac{1}{2} \sum_i k_i^2 G'_r(w_i, w_i) \\ &= \sum_{i<j} \alpha' k_i \cdot k_j \left( \ln \left| \vartheta_1\left(\frac{w_{ij}}{2\pi} | \tau\right) \right| - \frac{(\text{Im } w_{ij})^2}{4\pi\tau_2} \right) + \frac{\alpha'}{2} \ln \left| \frac{\partial_\nu \vartheta_1(0 | \tau)}{2\pi} \right|^2 \sum_i k_i^2 \\ &\quad - \frac{1}{2} k(\tau) \sum_{i,j} k_i \cdot k_j \\ &= \sum_{i<j} \alpha' k_i \cdot k_j \left( \ln \left| \frac{2\pi}{\partial_\nu \vartheta_1(0 | \tau)} \vartheta_1\left(\frac{w_{ij}}{2\pi} | \tau\right) \right| - \frac{(\text{Im } w_{ij})^2}{4\pi\tau_2} \right), \end{aligned} \quad (5)$$

where in the second equality we have used the overall momentum-conserving delta function, which implies  $\sum_{i,j} k_i \cdot k_j = 0$ . Plugging this into (1) yields (7.2.4).

Under the modular transformation  $\tau \rightarrow \tau' = -1/\tau$  the coordinate  $w$  is mapped to  $w' = w/\tau$ . The weights of the vertex operator  $: \exp(ik_i \cdot X(w_i, \bar{w}_i)) :$  are (2.4.17)

$$h = \tilde{h} = \frac{\alpha' k_i^2}{4} \quad (6)$$

so, according to (2.4.13), the product of vertex operators on the LHS of (1) transforms to

$$\left\langle \prod_{i=1}^n : e^{ik_i \cdot X(w'_i, \bar{w}'_i)} : \right\rangle_{T^2} = |\tau|^{\sum_i \alpha' k_i^2 / 2} \left\langle \prod_{i=1}^n : e^{ik_i \cdot X(w_i, \bar{w}_i)} : \right\rangle_{T^2}. \quad (7)$$

On the RHS of (1), the vacuum amplitude

$$C_{T^2}^X = (4\pi\alpha'\tau_2)^{-d/2} |\eta(\tau)|^{-2d} \quad (8)$$

is invariant, since

$$\tau'_2 = \frac{\tau_2}{|\tau|^2}, \quad (9)$$

and (7.2.4b)

$$\eta(\tau') = (-i\tau)^{1/2} \eta(\tau). \quad (10)$$

According to (7.2.40d),

$$\vartheta_1 \left( \frac{w'_{ij}}{2\pi} | \tau' \right) = -i(-i\tau)^{1/2} \exp \left( \frac{iw_{ij}^2}{4\pi\tau} \right) \vartheta_1 \left( \frac{w_{ij}}{2\pi} | \tau \right) \quad (11)$$

and

$$\partial_\nu \vartheta_1(0 | \tau') = (-i\tau)^{3/2} \partial_\nu \vartheta_1(0 | \tau), \quad (12)$$

so

$$\ln \left| \frac{2\pi}{\partial_\nu \vartheta_1(0 | \tau')} \vartheta_1 \left( \frac{w'_{ij}}{2\pi} | \tau' \right) \right| = \ln \left| \frac{2\pi}{\tau \partial_\nu \vartheta_1(0 | \tau)} \vartheta_1 \left( \frac{w_{ij}}{2\pi} | \tau \right) \right| - \operatorname{Im} \left( \frac{w_{ij}^2}{4\pi\tau} \right). \quad (13)$$

The second term on the right is equal to

$$\operatorname{Im} \left( \frac{w_{ij}^2}{4\pi\tau} \right) = \frac{1}{4\pi|\tau|^2} (2\tau_1 \operatorname{Im} w_{ij} \operatorname{Re} w_{ij} - \tau_2 (\operatorname{Re} w_{ij})^2 + \tau_2 (\operatorname{Im} w_{ij})^2), \quad (14)$$

and it cancels the change in the last term on the RHS of (5):

$$\frac{(\operatorname{Im} w'_{ij})^2}{4\pi\tau'_2} = \frac{1}{4\pi\tau_2|\tau|^2} (-2\tau_1\tau_2 \operatorname{Im} w_{ij} \operatorname{Re} w_{ij} + \tau_2^2 (\operatorname{Re} w_{ij})^2 + \tau_1^2 (\operatorname{Im} w_{ij})^2). \quad (15)$$

The only change, then, is the new factor of  $|\tau|^{-1}$  in the logarithm on the RHS of (13), which gets taken to the power

$$\sum_{i < j} \alpha' k_i \cdot k_j = -\frac{1}{2} \sum_i \alpha' k_i^2; \quad (16)$$

we finally arrive, as expected, at the transformation law (7).

## 7.2 Problem 7.3

As usual it is convenient to solve Poisson's equation (6.2.8) in momentum space. Because the torus is compact, the momentum space is a lattice, generated by the complex numbers  $k_a = 1 - i\tau_1/\tau_2$  and  $k_b = i/\tau_2$ . The Laplacian in momentum space is

$$-|k|^2 = -|n_a k_a + n_b k_b|^2 = -\frac{|n_b - n_a \tau|^2}{\tau_2^2} \equiv -\omega_{n_a n_b}^2. \quad (17)$$

In terms of the normalized Fourier components

$$\mathbf{X}_{n_a n_b}(w) = \frac{1}{2\pi\tau_2^{1/2}} e^{i(n_a k_a + n_b k_b) \cdot w} \quad (18)$$

(where the dot product means, as usual,  $A \cdot B \equiv \text{Re } AB^*$ ), the Green's function in real space is (6.2.7)

$$G'(w, w') = \sum_{(n_a, n_b) \neq (0,0)} \frac{2\pi\alpha'}{\omega_{n_a n_b}^2} \mathbf{X}_{n_a n_b}(w)^* \mathbf{X}_{n_a n_b}(w'). \quad (19)$$

Rather than show that (19) is equal to (7.2.3) (or (2)) directly, we will show that (7.2.3) has the correct Fourier coefficients, that is,

$$\int_{T^2} d^2w \mathbf{X}_{00}(w) G'(w, w') = 0, \quad (20)$$

$$\int_{T^2} d^2w \mathbf{X}_{n_a n_b}(w) G'(w, w') = \frac{2\pi\alpha'}{\omega_{n_a n_b}^2} \mathbf{X}_{n_a n_b}(w'), \quad (n_a, n_b) \neq (0, 0). \quad (21)$$

The  $w$ -independent part of the Green's function is left as the unspecified constant  $k(\tau)$  in (7.2.3), which is adjusted (as a function of  $\tau$ ) to satisfy equation (20). To prove (21), we first divide both sides by  $\mathbf{X}_{n_a n_b}(w')$ , and use the fact that  $G'$  depends only on the difference  $w - w'$  to shift the variable of integration:

$$\int_{T^2} d^2w e^{i(n_a k_a + n_b k_b) \cdot w} G'(w, 0) = \frac{2\pi\alpha'}{\omega_{n_a n_b}^2}. \quad (22)$$

To evaluate the LHS, let us use coordinates  $x$  and  $y$  on the torus defined by  $w = 2\pi(x + y\tau)$ . The Jacobian for this change of coordinates is  $(2\pi)^2\tau_2$ , so we have

$$(2\pi)^2\tau_2 \int_0^1 dy \int_0^1 dx e^{2\pi i(n_a x + n_b y)} \left[ -\frac{\alpha'}{2} \ln |\vartheta_1(x + y\tau|\tau)|^2 + \alpha' \pi y^2 \tau_2 \right]. \quad (23)$$

Using the infinite-product representation of the theta function (7.2.38d), we can write,

$$\begin{aligned} \ln \vartheta_1(x + y\tau|\tau) &= K(\tau) - i\pi(x + y\tau) \\ &+ \sum_{m=0}^{\infty} \ln \left( 1 - e^{2\pi i(x + (y+m)\tau)} \right) + \sum_{m=1}^{\infty} \ln \left( 1 - e^{-2\pi i(x + (y-m)\tau)} \right), \end{aligned} \quad (24)$$

where we've collected the terms that are independent of  $x$  and  $y$  into the function  $K(\tau)$ , which drops out of the integral. Expression (23) thus becomes

$$-2\pi^2\alpha'\tau_2 \int_0^1 dy \int_0^1 dx e^{2\pi i(n_a x + n_b y)} \left[ 2\pi\tau_2 y(1-y) + \sum_{m=0}^{\infty} \ln \left| 1 - e^{2\pi i(x+(y+m)\tau)} \right|^2 + \sum_{m=1}^{\infty} \ln \left| 1 - e^{-2\pi i(x+(y-m)\tau)} \right|^2 \right]. \quad (25)$$

The integration of the first term in the brackets is straightforward:

$$\int_0^1 dx \int_0^1 dy e^{2\pi i(n_a x + n_b y)} 2\pi\tau_2 y(1-y) = \begin{cases} -\frac{\tau_2}{\pi n_b^2}, & n_a = 0 \\ 0, & n_a \neq 0 \end{cases}. \quad (26)$$

To integrate the logarithms, we first convert the infinite sums into infinite regions of integration in  $y$  (using the periodicity of the first factor under  $y \rightarrow y+1$ ):

$$\begin{aligned} & \int_0^1 dy e^{2\pi i(n_a x + n_b y)} \\ & \quad \times \left[ \sum_{m=0}^{\infty} \ln \left| 1 - e^{2\pi i(x+(y+m)\tau)} \right|^2 + \sum_{m=1}^{\infty} \ln \left| 1 - e^{-2\pi i(x+(y-m)\tau)} \right|^2 \right] \\ & = \int_0^{\infty} dy e^{2\pi i(n_a x + n_b y)} \ln \left| 1 - e^{2\pi i(x+y\tau)} \right|^2 \\ & \quad + \int_{-\infty}^0 dy e^{2\pi i(n_a x + n_b y)} \ln \left| 1 - e^{-2\pi i(x+y\tau)} \right|^2. \end{aligned} \quad (27)$$

The  $x$  integral can now be performed by separating the logarithms into their holomorphic and anti-holomorphic pieces and Taylor expanding. For example,

$$\begin{aligned} \int_0^1 dx e^{2\pi i(n_a x + n_b y)} \ln \left( 1 - e^{2\pi i(x+y\tau)} \right) & = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 dx e^{2\pi i((n_a+n)x + (n_b+n\tau)y)} \\ & = \begin{cases} \frac{1}{n_a} e^{2\pi i(n_b - n_a\tau)y}, & n_a < 0 \\ 0, & n_a \geq 0 \end{cases}. \end{aligned} \quad (28)$$

The  $y$  integral is now straightforward:

$$\int_0^{\infty} dy \frac{1}{n_a} e^{2\pi i(n_b - n_a\tau)y} = -\frac{1}{2\pi i n_a (n_b - n_a\tau)}. \quad (29)$$



Similarly,

$$\int_0^\infty dy \int_0^1 dx e^{2\pi i(n_a x + n_b y)} \ln \left( 1 - e^{-2\pi i(x+y\bar{\tau})} \right) = \begin{cases} \frac{1}{2\pi i n_a (n_b - n_a \bar{\tau})}, & n_a > 0 \\ 0, & n_a \leq 0 \end{cases}, \quad (30)$$

$$\int_{-\infty}^0 dy \int_0^1 dx e^{2\pi i(n_a x + n_b y)} \ln \left( 1 - e^{-2\pi i(x+y\tau)} \right) = \begin{cases} -\frac{1}{2\pi i n_a (n_b - n_a \tau)}, & n_a > 0 \\ 0, & n_a \leq 0 \end{cases}, \quad (31)$$

$$\int_{-\infty}^0 dy \int_0^1 dx e^{2\pi i(n_a x + n_b y)} \ln \left( 1 - e^{2\pi i(x+y\bar{\tau})} \right) = \begin{cases} \frac{1}{2\pi i n_a (n_b - n_a \bar{\tau})}, & n_a < 0 \\ 0, & n_a \geq 0 \end{cases}. \quad (32)$$

Expressions (26) and (29)–(32) can be added up to give a single expression valid for any sign of  $n_a$ :

$$-\frac{\tau_2}{\pi |n_b - n_a \tau|^2}; \quad (33)$$

multiplying by the prefactor  $-2\pi^2 \alpha' \tau_2$  in front of the integral in (25) indeed yields precisely the RHS of (22), which is what we were trying to prove.

### 7.3 Problem 7.5

In each case we hold  $\nu$  fixed while taking  $\tau$  to its appropriate limit.

(a) When  $\text{Im } \tau \rightarrow \infty$ ,  $q \equiv \exp(2\pi i \tau) \rightarrow 0$ , and it is clear from either the infinite sum expressions (7.2.37) or the infinite product expressions (7.2.38) that in this limit

$$\vartheta_{00}(\nu, \tau) \rightarrow 1, \quad (34)$$

$$\vartheta_{10}(\nu, \tau) \rightarrow 1, \quad (35)$$

$$\vartheta_{01}(\nu, \tau) \rightarrow 0, \quad (36)$$

$$\vartheta_{11}(\nu, \tau) \rightarrow 0. \quad (37)$$

Note that all of these limits are independent of  $\nu$ .

(b) Inverting the modular transformation (7.2.40a), we have

$$\begin{aligned} \vartheta_{00}(\nu, \tau) &= (-i\tau)^{-1/2} e^{-\pi i \nu^2 / \tau} \vartheta_{00}(\nu/\tau, -1/\tau) \\ &= (-i\tau)^{-1/2} \sum_{n=-\infty}^{\infty} e^{-\pi i (\nu-n)^2 / \tau}. \end{aligned} \quad (38)$$

When we take  $\tau$  to 0 along the imaginary axis, each term in the series will go either to 0 (if  $\text{Re}(\nu - n)^2 > 0$ ) or to infinity (if  $\text{Re}(\nu - n)^2 \leq 0$ ). Since different terms in the series cannot cancel for arbitrary  $\tau$ , the theta function can go to 0 only if every term in the series does so:

$$\forall n \in Z, \quad \text{Re}(\nu - n)^2 > 0; \quad (39)$$

otherwise it will diverge. For  $|\operatorname{Re} \nu| \leq 1/2$ , condition (39) is equivalent to

$$|\operatorname{Im} \nu| < |\operatorname{Re} \nu|; \quad (40)$$

in general, for  $|\operatorname{Re} \nu - n| \leq 1/2$ , the theta function goes to 0 if

$$|\operatorname{Im} \nu| < |\operatorname{Re} \nu - n|. \quad (41)$$

Since  $\vartheta_{01}(\nu, \tau) = \vartheta_{00}(\nu + 1/2, \tau)$ , the region in which it goes to 0 in the limit  $\tau \rightarrow 0$  is simply shifted by  $1/2$  compared to the case treated above.

For  $\vartheta_{10}$ , the story is the same as for  $\vartheta_{00}$ , since

$$\begin{aligned} \vartheta_{10}(\nu, \tau) &= (-i\tau)^{-1/2} e^{-\pi i \nu^2 / \tau} \vartheta_{01}(\nu/\tau, -1/\tau) \\ &= (-i\tau)^{-1/2} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi i (\nu - n)^2 / \tau}; \end{aligned} \quad (42)$$

the sum will again go to 0 where (39) is obeyed, and infinity elsewhere.

Finally,  $\vartheta_{11}$  goes to 0 in the same region as  $\vartheta_{01}$ , since the sum is the same as (42) except over the half-odd-integers,

$$\begin{aligned} \vartheta_{11}(\nu, \tau) &= i(-i\tau)^{-1/2} e^{-\pi i \nu^2 / \tau} \vartheta_{11}(\nu/\tau, -1/\tau) \\ &= (-i\tau)^{-1/2} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi i (\nu - n + 1/2)^2 / \tau}, \end{aligned} \quad (43)$$

so the region (39) is shifted by  $1/2$ .

(c) According to (7.2.39) and (7.2.40), under the modular transformations

$$\tau' = \tau + 1, \quad (44)$$

$$\tau' = -\frac{1}{\tau}, \quad (45)$$

the theta functions are exchanged with each other and multiplied by factors that are finite as long as  $\nu$  and  $\tau$  are finite. Also, under (45)  $\nu$  is transformed to

$$\nu' = \frac{\nu}{\tau}. \quad (46)$$

We are considering limits where  $\tau$  approaches some non-zero real value  $\tau_0$  along a path parallel to the imaginary axis, in other words, we set  $\tau = \tau_0 + i\epsilon$  and take  $\epsilon \rightarrow 0^+$ . The property of approaching the real axis along a path parallel to the imaginary axis is preserved by the modular transformations (to first order in  $\epsilon$ ):

$$\tau' = \tau_0 + 1 + i\epsilon, \quad (47)$$

$$\tau' = -\frac{1}{\tau_0} + i\frac{\epsilon}{\tau_0^2} + \mathcal{O}(\epsilon^2), \quad (48)$$

under (44) and (45) respectively. By a sequence of transformations (44) and (45) one can reach any rational limit point  $\tau_0$  starting with  $\tau_0 = 0$ , the case considered in part (b) above. During these transformations (which always begin with (44)), the region (39), in which  $\vartheta_{00}$  goes to 0, will repeatedly be shifted by  $1/2$  and rescaled by  $\tau_0$  (under (46)). (Note that the limiting value, being either 0 or infinity, is insensitive to the finite prefactors involved in the transformations (7.2.39) and (7.2.40).) It is easy to see that this cumulative sequence of rescalings will telescope into a single rescaling by a factor  $q$ , where  $p/q$  is the final value of  $\tau_0$  in reduced form.

As for the case when  $\tau_0$  is irrational, I can only conjecture that the theta functions diverge (almost) everywhere on the  $\nu$  plane in that limit.

#### 7.4 Problem 7.7

The expectation value for fixed open string tachyon vertex operators on the boundary of the cylinder is very similar to the corresponding formula (7.2.4) for closed string tachyon vertex operators on the torus. The major difference comes from the fact that the Green's function is doubled. The method of images gives the Green's function for the cylinder in terms of that for the torus (7.2.3):

$$G'_{C_2}(w, w') = G'_{T^2}(w, w') + G'_{T^2}(w, -\bar{w}'). \quad (49)$$

However, since the boundary of  $C_2$  is given by those points that are invariant under the involution  $w \rightarrow -\bar{w}$ , the two terms on the RHS above are equal if either  $w$  or  $w'$  is on the boundary. The renormalized self-contractions are also doubled, so we have

$$\left\langle \prod_{i=1}^n \star e^{ik_i \cdot X(w_i, \bar{w}_i) \star} \right\rangle_{C_2} = iC_{C_2}^X(t) (2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) \prod_{i < j} \left| \eta(it)^{-3} \vartheta_1 \left( \frac{w_{ij}}{2\pi} | it \right) \exp \left[ -\frac{(\text{Im } w_{ij})^2}{4\pi t} \right] \right|^{2\alpha' k_i \cdot k_j}. \quad (50)$$

The boundary of the cylinder breaks into two connected components, and the vertex operator positions  $w_i$  must be integrated over both components. If there are Chan-Paton factors, then the integrand will include two traces, one for each component of the boundary, and the order of the factors in each trace will be the order of the operators on the corresponding component. We will denote the product of these two traces  $T(w_1, \dots, w_n)$ , and of course it will also depend implicitly on the Chan-Paton factors themselves  $\lambda^{a_i}$ . Borrowing from the cylinder vacuum amplitude given

in (7.4.1), we can write the  $n$ -tachyon amplitude as

$$\begin{aligned}
S_{C_2}(k_1, a_1; \dots; k_n, a_n) = & \\
i(2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) g_0^n \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-13} \eta(it)^{-24} \prod_{i=1}^n \left( \int_{\partial C_2} dw_i \right) & \\
\times T(w_1, \dots, w_n) \prod_{i < j} \left| \eta(it)^{-3} \vartheta_1 \left( \frac{w_{ij}}{2\pi} | it \right) \exp \left[ -\frac{(\text{Im } w_{ij})^2}{4\pi t} \right] \right|^{2\alpha' k_i \cdot k_j} . & \quad (51)
\end{aligned}$$

### 7.5 Problem 7.8

In this problem we consider the part of the amplitude (51) in which the first  $m \geq 2$  vertex operators are on one of the cylinder's boundaries, and the other  $n - m \geq 2$  are on the other one. For concreteness let us put the first set on the boundary at  $\text{Re } w = 0$  and the second set on the boundary at  $\text{Re } w = \pi$ , and then double the amplitude (51). Since we will be focusing on the region of the moduli space where  $t$  is very small, it is convenient to scale the vertex operator positions with  $t$ , so

$$w_i = \begin{cases} 2\pi i x_i t, & i = 1, \dots, m \\ \pi + 2\pi i x_i t, & i = m + 1, \dots, n \end{cases} . \quad (52)$$

The  $x_i$  run from 0 to 1 independent of  $t$ , allowing us to change of the order of integration in (51). Using (16) and the mass shell condition  $\alpha' k_i^2 = 1$  we can write the part of the amplitude we're interested in as follows:

$$\begin{aligned}
S'(k_1, a_1; \dots; k_n, a_n) = & i(2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) g_0^n 2^{-13} (2\pi)^{n-26} \alpha'^{-13} \\
\times \prod_i \left( \int_0^1 dx_i \right) T_1(x_1, \dots, x_m) T_2(x_{m+1}, \dots, x_n) & \\
\times \int_0^\infty dt t^{n-14} \eta(it)^{3n-24} \prod_{i < j} |\vartheta_{1,2}(ix_{ij}t | it) \exp(-\pi x_{ij}^2 t)|^{2\alpha' k_i \cdot k_j} . & \quad (53)
\end{aligned}$$

In the last product the type of theta function to use depends on whether the vertex operators  $i$  and  $j$  are on the same boundary (in which case  $\vartheta_1$ ) or on opposite boundaries (in which case  $\vartheta_2(ix_{ij}t | it) = -\vartheta_1(ix_{ij}t - 1/2 | it)$ ). Concentrating now on the last line of (53), let us apply the modular transformations (7.2.40b), (7.2.40d), and (7.2.44b), and change the variable of integration to  $u = 1/t$ :

$$\int_0^\infty du \eta(iu)^{3n-24} \prod_{i < j} |\vartheta_{1,4}(x_{ij} | iu)|^{2\alpha' k_i \cdot k_j} . \quad (54)$$

For large  $u$ , each of the factors in the integrand of (54) can be written as a fractional power of  $q \equiv e^{-2\pi u}$  times a power series in  $q$  (with coefficients that may depend on the  $x_{ij}$ ; see (7.2.37),

(7.2.38), and (7.2.43)):

$$\eta(iu) = q^{1/24}(1 - q + \dots); \quad (55)$$

$$\vartheta_1(x|iu) = 2q^{1/8} \sin(\pi x) (1 - (1 + 2 \cos(2\pi x))q + \dots); \quad (56)$$

$$\vartheta_4(x|iu) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2/2} e^{2\pi i k x} = 1 - 2 \cos(2\pi x) q^{1/2} + \dots \quad (57)$$

For  $\eta$  and  $\vartheta_1$  this power series involves only integer powers of  $q$ , whereas for  $\vartheta_4$  it mixes integer and half-integer powers. Substituting (55)-(57) into (54) yields

$$\begin{aligned} \prod_{\substack{i < j \\ i \simeq j}} |2 \sin \pi x_{ij}|^{2\alpha' k_i \cdot k_j} \int_0^\infty du q^{n/8 - 1 + \alpha' \sum_{\substack{i < j \\ i \simeq j}} k_i \cdot k_j / 4} (1 + \dots) \\ = \prod_{\substack{i < j \\ i \simeq j}} |2 \sin \pi x_{ij}|^{2\alpha' k_i \cdot k_j} \int_0^\infty du q^{-1 - \alpha' s / 4} (1 + \dots) \end{aligned} \quad (58)$$

The symbol  $i \simeq j$  means that the sum or product is only over pairs of vertex operators on the same boundary of the cylinder. We obtain the second line from the first by using

$$\begin{aligned} \sum_{\substack{i < j \\ i \simeq j}} k_i \cdot k_j &= \sum_{i < j} k_i \cdot k_j - \sum_{i=1}^m \sum_{j=m+1}^n k_i \cdot k_j \\ &= -\frac{1}{2} \sum_{i=1}^n k_i^2 - \left( \sum_{i=1}^m k_i \right) \cdot \left( \sum_{j=m+1}^n k_j \right) \\ &= -\frac{n}{2\alpha'} - s, \end{aligned} \quad (59)$$

where  $s = -(\sum_{i=1}^n k_i)^2$  is the mass squared of the intermediate state propagating along the long cylinder. (The two  $n$ 's we have cancelled against each other in (58) both come from  $\alpha' \sum_i k_i^2$ , so in fact we could have done without the mass shell condition in the derivation.) Each power of  $q$  appearing in the power series  $(1 + \dots)$  in (58) will produce, upon performing the  $u$  integration, a pole in  $s$ :

$$\int_0^\infty du q^{-1 - \alpha' s / 4 + k} = \frac{2}{\pi(4k - 4 - \alpha' s)}. \quad (60)$$

Since every (non-negative) integer power  $k$  appears in the expansion of (54), we have the entire sequence of closed string masses at  $s = 4(k - 1)/\alpha'$ .

What about the half-integer powers of  $q$  that appear in the expansion of  $\vartheta_4$ , (57)? The poles from these terms in fact vanish, but only after integrating over the vertex operator positions. To see this, consider the effect of uniformly translating all the vertex operator positions on just one of the boundaries by an amount  $y$ :  $x_i \rightarrow x_i + y$ ,  $i = 1, \dots, m$ . This translation changes the relative

position  $x_{ij}$  only if the vertex operators  $i$  and  $j$  are on opposite boundaries; it thus leaves the Chan-Paton factors  $T_1$  and  $T_2$  in (53) invariant, as well as all the factors in the integrand of (54) except those involving  $\vartheta_4$ ; those become

$$\prod_{i=1}^m \prod_{j=m+1}^n |\vartheta_4(x_{ij} + y|iu)|^{2\alpha' k_i \cdot k_j}. \quad (61)$$

Expanding this out using (57), each term will be of the form

$$c q^{\sum_l k_l^2/2} e^{2\pi i y \sum_l k_l}, \quad (62)$$

where the  $k_l$  are integers and the coefficient  $c$  depends on the  $x_i$  and the momenta  $k_i$ . Since  $y$  is effectively integrated over when one integrates over the vertex operator positions, only the terms for which  $\sum_l k_l = 0$  will survive. This condition implies that  $\sum_l k_l^2$  must be even, so only integer powers of  $q$  produce poles in the amplitude.

## 7.6 Problem 7.9

We wish to consider the result of the previous problem in the simplest case, when  $n = 4$ ,  $m = 2$ , and there are no Chan-Paton indices. We are interested in particular in the first pole, at  $s = -4/\alpha'$ , where the intermediate closed string goes on shell as a tachyon. Neglecting the “...” and approximating the exponents  $2\alpha' k_1 \cdot k_2 = 2\alpha' k_3 \cdot k_4 = -\alpha' s - 2$  by 2, (58) becomes

$$-\sin^2(\pi x_{12}) \sin^2(\pi x_{34}) \frac{32}{\pi(\alpha' s + 4)}. \quad (63)$$

After integrating over the positions  $x_i$ , the amplitude (53) is

$$S'(k_1, \dots, k_4) = -i(2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) g_o^4 2^{-9} (2\pi)^{-23} \alpha'^{-14} \frac{1}{s + 4/\alpha'}. \quad (64)$$

In Problem 6.12, we calculated the three-point vertex for two open-string tachyons to go to a closed-string tachyon:

$$\frac{4\pi i g_c}{\alpha'} (2\pi)^{26} \delta^{26} (k_1 + k_2 + k_3). \quad (65)$$

We can reproduce the  $s$ -channel pole of (64) by simply writing down the Feynman diagram using this vertex and the closed-string tachyon propagator,  $i/(s + 4/\alpha')$ :

$$-i(2\pi)^{26} \delta^{26} \left( \sum_i k_i \right) g_c^2 4(2\pi)^2 \alpha'^{-2} \frac{1}{s + 4/\alpha'}. \quad (66)$$

We can now compare the residues of the poles in (64) and (66) to obtain the following relation between  $g_c$  and  $g_o$  (note that, since the vertex (65) is valid only on-shell, it is only appropriate to compare the residues of the poles at  $s = -4/\alpha'$  in (64) and (66), not the detailed dependence on  $s$  away from the pole):

$$\frac{g_o^2}{g_c} = 2^{11/2} (2\pi)^{25/2} \alpha'^6. \quad (67)$$

This is in agreement with (8.7.28).

### 7.7 Problem 7.10

When the gauge group is a product of  $U(n_i)$  factors, the generators are block diagonal. A tree-level diagram is proportional to  $\text{Tr}(\lambda^{a_1} \cdots \lambda^{a_l})$ , and vanishes unless all the Chan-Paton factors are in the same block. Unitarity requires

$$\text{Tr}(\lambda^{a_1} \cdots \lambda^{a_i} \lambda^{a_{i+1}} \cdots \lambda^{a_l}) = \text{Tr}(\lambda^{a_1} \cdots \lambda^{a_i} \lambda^e) \text{Tr}(\lambda^e \lambda^{a_{i+1}} \cdots \lambda^{a_l}), \quad (68)$$

which is an identity for any product of  $U(n_i)$ s. (The LHS vanishes if all the  $\lambda^a$  are not in the same block, and equals the usual  $U(n)$  value if they are. The RHS has the same property, because the  $\lambda^e$  must be in the same block both with the first group of  $\lambda^a$ s and with the second group for a non-zero result.)

Now consider the cylinder with two vertex operators on each boundary, with Chan-Paton factors  $\lambda^a$  and  $\lambda^b$  on one boundary and  $\lambda^c$  and  $\lambda^d$  on the other. This amplitude is proportional to  $\text{Tr}(\lambda^a \lambda^b) \text{Tr}(\lambda^c \lambda^d)$ , i.e. we only need the two Chan-Paton factors in each pair to be in the same block. However, if we make a unitary cut in the open string channel, then the cylinder becomes a disk with an open string propagator connecting two points on the boundary. If on the intermediate state we sum only over block-diagonal generators, then this amplitude will vanish unless  $\lambda^a, \lambda^b, \lambda^c, \lambda^d$  are all in the same block. For this to be consistent there can only be one block, i.e. the gauge group must be simple.

### 7.8 Problem 7.11

We will be using the expression (7.3.9) for the point-particle vacuum amplitude to obtain a generalized version of the cylinder vacuum amplitude (7.4.1). Since (7.3.9) is an integral over the circle modulus  $l$ , whereas (7.4.1) is an integral over the cylinder modulus  $t$ , we need to know the relationship between these quantities. The modulus  $l$  is defined with respect to the point-particle action (5.1.1), which (after choosing the analog of unit gauge for the einbein  $e$ ) is

$$\frac{1}{2} \int_0^l d\tau ((\partial_\tau x)^2 + m^2). \quad (69)$$

The Polyakov action in unit gauge (2.1.1), on a cylinder with modulus  $t$ , is

$$\frac{1}{4\pi\alpha'} \int_0^\pi dw_1 \int_0^{2\pi t} dw_2 ((\partial_1 X)^2 + (\partial_2 X)^2). \quad (70)$$

Decomposing  $X(w_1, w_2)$  into its center-of-mass motion  $x(w_2)$  and its internal state  $y(w_1, w_2)$  (with  $\int dw_1 y = 0$ ), the Polyakov action becomes

$$\frac{1}{4\alpha'} \int_0^{2\pi t} dw_2 \left( (\partial_2 x)^2 + \frac{1}{\pi} \int_0^\pi dw_1 ((\partial_1 y)^2 + (\partial_2 y)^2) \right). \quad (71)$$

We can equate (69) and (71) by making the identifications  $\tau = 2\alpha'w_2$ ,  $l = 4\pi\alpha't$ , and

$$m^2 = \frac{1}{4\pi\alpha'^2} \int_0^\pi dw_1 ((\partial_1 y)^2 + (\partial_2 y)^2). \quad (72)$$

Using the relation  $l = 4\pi\alpha't$  to translate between the circle and the cylinder moduli, we can now sum the circle vacuum amplitude (7.3.9) over the open string spectrum, obtaining the second line of (7.4.1):

$$Z_{C_2} = iV_D \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-D/2} \sum_{i \in \mathcal{H}_0^\perp} e^{-2\pi t \alpha' m_i^2}. \quad (73)$$

Taking a spectrum with  $D'$  net sets of oscillators and a ground state at  $m^2 = -1/\alpha'$ , the sum is evaluated in the usual way (with  $q = e^{-2\pi t}$ ):

$$\begin{aligned} \sum_{i \in \mathcal{H}_0^\perp} q^{\alpha' m_i^2} &= \left( \prod_{i=1}^{D'} \prod_{n=1}^{\infty} \sum_{N_{in}=0}^{\infty} \right) q^{-1 + \sum_{n=1}^{\infty} n N_{in}} \\ &= q^{-1} \prod_{i=1}^{D'} \prod_{n=1}^{\infty} \left( \sum_{N_{in}=0}^{\infty} q^{n N_{in}} \right) \\ &= q^{-1} \prod_{i=1}^{D'} \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \\ &= q^{D'/24-1} \eta(it)^{-D'}. \end{aligned} \quad (74)$$

As in Problem 7.8 above, we are interested in studying the propagation of closed string modes along long, thin cylinders, which corresponds to the region  $t \ll 1$ . So let us change the variable of integration to  $u = 1/t$ ,

$$\begin{aligned} Z_{C_2} &= \frac{iV_D}{2(8\pi^2 \alpha')^{D/2}} \int_0^\infty dt t^{-D/2-1} e^{-2\pi t (D'/24-1)} \eta(it)^{-D'} \\ &= \frac{iV_D}{2(8\pi^2 \alpha')^{D/2}} \int_0^\infty du u^{(D-D'-2)/2} e^{-2\pi (D'/24-1)/u} \eta(iu)^{-D'}, \end{aligned} \quad (75)$$

and add a factor of  $e^{-\pi u \alpha' k^2/2}$  to the integrand, where  $k$  is the momentum flowing along the cylinder:

$$\int_0^\infty du u^{(D-D'-2)/2} e^{-2\pi (D'/24-1)/u - \pi u \alpha' k^2/2} \eta(iu)^{-D'}. \quad (76)$$

Now we can expand the eta-function for large  $u$  using the product representation (7.2.43). Each term will be of the form

$$\int_0^\infty du u^{(D-D'-2)/2} e^{-2\pi (D'/24-1)/u - 2\pi u (\alpha' k^2/4 + m - D'24)}, \quad (77)$$

where  $m$  is a non-negative integer.



In the case  $D = 26$ ,  $D' = 24$ , (77) reveals the expected series of closed-string poles, as found in Problem 7.8. If  $D' \neq 24$ , the integral yields a modified Bessel function,

$$\int_0^\infty du u^{c-1} e^{-a/u-b/u} = 2b^{-c/2} a^{c/2} K_c \left( 2\sqrt{ab} \right), \quad (78)$$

which has a branch cut along the negative real axis, hence the constraint  $D' = 24$ . When  $D' = 24$ , (77) simplifies to

$$\int_0^\infty du u^{(D-26)/2} e^{-2\pi u(\alpha' k^2/4+m-1)} = \frac{\Gamma\left(\frac{D-24}{2}\right)}{(2\pi(\alpha' k^2/4+m-1))^{(D-24)/2}}. \quad (79)$$

For  $D$  odd there is a branch cut. For  $D$  even but less than 26, the gamma function is infinite; even if one employs a “minimal subtraction” scheme to remove the infinity, the remainder has a logarithmic branch cut in  $k^2$ . For even  $D \geq 26$  there is indeed a pole, but this pole is simple (as one expects for a particle propagator) only for  $D = 26$ .

## 7.9 Problem 7.13

We follow the same steps in calculating the vacuum Klein bottle amplitude as Polchinski does in calculating the vacuum torus amplitude, starting with finding the scalar partition function:

$$\begin{aligned} Z_X(t) &= \langle 1 \rangle_{X, K_2} \\ &= \text{Tr}(\Omega e^{-2\pi t H}) \\ &= q^{-13/6} \text{Tr}(\Omega q^{L_0 + \tilde{L}_0}). \end{aligned} \quad (80)$$

The operator  $\Omega$  implements the orientation-reversing boundary condition in the Euclidean time ( $\sigma^2$ ) direction of the Klein bottle. Since  $\Omega$  switches left-movers and right-movers, it is convenient to work with states and operators that have definite properties under orientation reversal. We therefore define the raising and lowering operators  $\alpha_m^\pm = (\alpha_m \pm \tilde{\alpha}_m)/\sqrt{2}$ , so that  $\alpha_m^+$  commutes with  $\Omega$ , while  $\alpha_m^-$  anti-commutes with it (we have suppressed the spacetime index  $\mu$ ). These are normalized to have the usual commutation relations, and can be used to build up the spectrum of the closed string in the usual way. The ground states  $|0; k\rangle$  of the closed string are invariant under orientation reversal. The  $\Omega q^{L_0 + \tilde{L}_0}$  eigenvalue of a ground state is  $q^{\alpha' k^2/2}$ ; each raising operator  $\alpha_m^+$  multiplies that eigenvalue by  $q^m$ , while each raising operator  $\alpha_m^-$  multiplies it by  $-q^m$ . Summing over all combinations of such operators, the partition function is

$$\begin{aligned} Z_X(t) &= q^{-13/6} V_{26} \int \frac{d^{26} k}{(2\pi)^{26}} q^{\alpha' k^2/2} \prod_{m=1}^{\infty} (1 - q^m)^{-1} (1 + q^m)^{-1} \\ &= i V_{26} (4\pi^2 \alpha' t)^{-13} \eta(2it)^{-26}. \end{aligned} \quad (81)$$

For  $bc$  path integrals on the Klein bottle, it is again convenient to introduce raising and lowering operators with definite properties under orientation reversal:  $b_m^\pm = (b_m \pm \tilde{b}_m)/\sqrt{2}$ ,  $c_m^\pm = (c_m \pm \tilde{c}_m)/\sqrt{2}$ . The particular path integral we will be interested in is

$$\langle c_m^\pm b_0^+ \rangle_{K_2} = q^{13/6} \text{Tr} \left( (-1)^F \Omega c_m^\pm b_0^+ q^{L_0 + \tilde{L}_0} \right). \quad (82)$$

Build up the  $bc$  spectrum by starting with a ground state  $|\downarrow\downarrow\rangle$  that is annihilated by  $b_m^\pm$  for  $m \geq 0$  and  $c_m^\pm$  for  $m > 0$ , and acting on it with the raising operators  $b_m^\pm$  ( $m < 0$ ) and  $c_m^\pm$  ( $m \leq 0$ ). The operator  $(-1)^F \Omega q^{L_0 + \tilde{L}_0}$  is diagonal in this basis, whereas the operator  $c_m^\pm b_0^+$  takes basis states to other basis states (if it does not annihilate them). Therefore the trace (81) vanishes. The only exception is the case of the operator  $c_0^+ b_0^+$ , which is diagonal in this basis; it projects onto the subspace of states that are built up from  $c_0^+ |\downarrow\downarrow\rangle$ . This state has eigenvalue  $-q^{-2}$  under  $(-1)^F \Omega c_0^+ b_0^+ q^{L_0 + \tilde{L}_0}$ . Acting with  $c_0^-$  does not change this eigenvalue; acting with  $b_{-m}^+$  or  $c_{-m}^+$  ( $m > 0$ ) multiplies it by  $-q^m$ , and with  $b_{-m}^-$  or  $c_{-m}^-$  by  $q^m$ . We thus have

$$\langle c_0^+ b_0^+ \rangle_{K_2} = 2q^{1/6} \prod_{m=1}^{\infty} (1 - q^m)^2 (1 + q^m)^2 = 2\eta(2it)^2. \quad (83)$$

(We have taken the absolute value of the result.)

The Klein bottle has only one CKV, which translates in the  $\sigma^2$  direction. Let us temporarily include an arbitrary number of vertex operators in the path integral, and fix the  $\sigma^2$  coordinate of the first one  $\mathcal{V}_1$ . According to (5.3.9), the amplitude is

$$S = \int_0^\infty \frac{dt}{4} \left\langle \int d\sigma_1^1 c^2 \mathcal{V}_1(\sigma_1^1, \sigma_1^2) \frac{1}{4\pi} (b, \partial_t \hat{g}) \prod_{i=2}^n \int \frac{d^2 w_i}{2} \mathcal{V}_i(w_i, \bar{w}_i) \right\rangle_{K_2}. \quad (84)$$

The overall factor of  $1/4$  is from the discrete symmetries of the Klein bottle, with  $1/2$  from  $w \rightarrow \bar{w}$  and  $1/2$  from  $w \rightarrow -w$ .

To evaluate the  $b$  insertion, let us temporarily fix the coordinate region at that for  $t = t_0$  and let the metric vary with  $t$ :

$$\hat{g}(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^2/t_0^2 \end{pmatrix}. \quad (85)$$

Then

$$\partial_t \hat{g}(t_0) = \begin{pmatrix} 0 & 0 \\ 0 & 2/t_0 \end{pmatrix} \quad (86)$$

and

$$\begin{aligned} \frac{1}{4\pi} (b, \partial_t \hat{g}) &= \frac{1}{4\pi} \int d\sigma^1 d\sigma^2 \frac{2}{t} b_{22} \\ &= - \int d\sigma^1 (b_{ww} + b_{\bar{w}\bar{w}}) \\ &= 2\pi (b_0 + \tilde{b}_0) \\ &= 2\sqrt{2}\pi b_0^+. \end{aligned} \quad (87)$$

If we expand the  $c$  insertion in the path integral in terms of the  $c_m$  and  $\tilde{c}_m$ ,

$$c^2(\sigma^1, \sigma^2) = \frac{1}{2} \left( \frac{c(z)}{z} + \frac{\tilde{c}(\bar{z})}{\bar{z}} \right) = \frac{1}{2} \sum_m \left( \frac{c_m}{z^m} + \frac{\tilde{c}_m}{\bar{z}^m} \right), \quad (88)$$

then, as we saw above, only the  $m = 0$  term, which is  $c_0^+/\sqrt{2}$ , will contribute to the ghost path integral. This allows us to factor the  $c$  ghost out of the integral over the first vertex operator position in (83), and put all the vertex operators on the same footing:

$$S = \int_0^\infty \frac{dt}{4t} \left\langle c_0^+ b_0^+ \prod_{i=1}^n \int \frac{d^2 w_i}{2} \mathcal{V}_i(w_i, \bar{w}_i) \right\rangle_{K_2}. \quad (89)$$

We can now extrapolate to the case where there are no vertex operators simply by setting  $n = 0$  above. Using (80) and (82), this gives

$$Z_{K_2} = iV_{26} \int_0^\infty \frac{dt}{2t} (4\pi^2 \alpha' t)^{-13} \eta(2it)^{-24}. \quad (90)$$

This is off from Polchinski's result (7.4.15) by a factor of 2.

### 7.10 Problem 7.15

(a) If the  $\sigma^2$  coordinate is periodically identified (with period  $2\pi$ ), then a cross-cap at  $\sigma^1 = 0$  implies the identification

$$(\sigma^1, \sigma^2) \cong (-\sigma^1, \sigma^2 + \pi). \quad (91)$$

This means the following boundary conditions on the scalar and ghost fields:

$$\partial_1 X^\mu(0, \sigma^2) = -\partial_1 X^\mu(0, \sigma^2 + \pi), \quad \partial_2 X^\mu(0, \sigma^2) = \partial_2 X^\mu(0, \sigma^2 + \pi), \quad (92)$$

$$c^1(0, \sigma^2) = -c^1(0, \sigma^2 + \pi), \quad c^2(0, \sigma^2) = c^2(0, \sigma^2 + \pi), \quad (93)$$

$$b_{12}(0, \sigma^2) = -b_{12}(0, \sigma^2 + \pi), \quad b_{11}(0, \sigma^2) = b_{11}(0, \sigma^2 + \pi). \quad (94)$$

These imply the following conditions on the modes at  $\sigma^1 = 0$ :

$$\alpha_n^\mu + (-1)^n \tilde{\alpha}_{-n}^\mu = c_n + (-1)^n \tilde{c}_{-n} = b_n - (-1)^n \tilde{b}_{-n} = 0 \quad (95)$$

(for all  $n$ ). The state corresponding to the cross-cap is then

$$|C\rangle \propto \exp \left[ - \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{n} \alpha_{-n} \cdot \tilde{\alpha}_{-n} + b_{-n} \tilde{c}_{-n} + \tilde{b}_{-n} c_{-n} \right) \right] (c_0 + \tilde{c}_0) |0; 0; \downarrow \downarrow\rangle. \quad (96)$$

(b) The Klein bottle vacuum amplitude is

$$\int_0^\infty ds \langle C | c_0 b_0 e^{-s(L_0 + \tilde{L}_0)} | C \rangle. \quad (97)$$

Since the raising and lowering operators for different oscillators commute with each other (or, in the case of the ghost oscillators, commute in pairs), we can factorize the integrand into a separate amplitude for each oscillator:

$$\begin{aligned} & e^{2s} \langle \downarrow \downarrow | (b_0 + \tilde{b}_0) c_0 b_0 (c_0 + \tilde{c}_0) | \downarrow \downarrow \rangle \langle 0 | e^{-s\alpha' p^2/2} | 0 \rangle \\ & \times \prod_{n=1}^{\infty} \left( \langle 0 | e^{-(1)^n c_n \tilde{b}_n} e^{-sn(b_{-n} c_n + \tilde{c}_{-n} \tilde{b}_n)} e^{-(1)^n b_{-n} \tilde{c}_{-n}} | 0 \rangle \right. \\ & \quad \times \langle 0 | e^{-(1)^n \tilde{c}_n b_n} e^{-sn(\tilde{b}_{-n} \tilde{c}_n + c_{-n} b_n)} e^{-(1)^n \tilde{b}_{-n} c_{-n}} | 0 \rangle \\ & \quad \left. \times \prod_{\mu=0}^{25} \langle 0 | e^{-(1)^n \alpha_{n\mu}^\mu / n} e^{-s(\alpha_{-n}^\mu \alpha_{n\mu} + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\mu})} e^{-(1)^n \alpha_{-n}^\mu \tilde{\alpha}_{-n\mu} / n} | 0 \rangle \right) \quad (98) \end{aligned}$$

(no summation over  $\mu$  in the last line). We have used the expressions (4.3.17) for the adjoints of the raising and lowering operators. The zero-mode amplitudes are independent of  $s$ , so we won't bother with them. The exponentials of the ghost raising operators truncate after the second term:

$$e^{-(1)^n b_{-n} \tilde{c}_{-n}} | 0 \rangle = | 0 \rangle - (-1)^n b_{-n} \tilde{c}_{-n} | 0 \rangle, \quad (99)$$

$$\langle 0 | e^{-(1)^n c_n \tilde{b}_n} = \langle 0 | - (-1)^n \langle 0 | c_n \tilde{b}_n. \quad (100)$$

Both terms in (99) are eigenstates of  $b_{-n} c_n + \tilde{c}_{-n} \tilde{b}_n$ , with eigenvalues of 0 and 2 respectively. The first ghost amplitude is thus:

$$\left( \langle 0 | - (-1)^n \langle 0 | c_n \tilde{b}_n \right) \left( | 0 \rangle - e^{-2sn} (-1)^n b_{-n} \tilde{c}_{-n} | 0 \rangle \right) = 1 - e^{-2sn}. \quad (101)$$

The second ghost amplitude gives the same result. To evaluate the scalar amplitudes, we must expand out the  $|C\rangle$  exponential:

$$e^{-(1)^n \alpha_{-n}^\mu \tilde{\alpha}_{n\mu} / n} | 0 \rangle = \sum_{m=0}^{\infty} \frac{1}{m! n^m} (-1)^{(n+1)m} (\alpha_{-n}^\mu \tilde{\alpha}_{n\mu})^m | 0 \rangle. \quad (102)$$

Each term in the series is an eigenstate of  $\alpha_{-n}^\mu \alpha_{n\mu} + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\mu}$ , with eigenvalue  $2nm$ , and each term has unit norm, so the amplitude is

$$\sum_{m=0}^{\infty} e^{-2snm} = \frac{1}{1 - e^{-2sn}}. \quad (103)$$

Finally, we find that the total amplitude (98) is proportional to

$$e^{2s} \prod_{n=1}^{\infty} (1 - e^{-2sn})^{-24} = \eta(is/\pi)^{-24}. \quad (104)$$

This is the  $s$ -dependent part of the Klein bottle vacuum amplitude, and agrees with the integrand of (7.4.19).

The vacuum amplitude for the Möbius strip is

$$\int_0^\infty ds \langle B | c_0 b_0 e^{-s(L_0 + \bar{L}_0)} | C \rangle. \quad (105)$$

The only difference from the above analysis is the absence of the factor  $(-1)^n$  multiplying the bras. Thus the ghost amplitude (101) becomes instead

$$\left( \langle 0 | - \langle 0 | c_n \tilde{b}_n \right) \left( | 0 \rangle - e^{-2sn} (-1)^n b_{-n} \tilde{c}_{-n} | 0 \rangle \right) = 1 - (-1)^n e^{-2sn}, \quad (106)$$

while the scalar amplitude (103) becomes

$$\sum_{m=0}^{\infty} (-1)^{nm} e^{-2snm} = \frac{1}{1 - (-1)^n e^{-2sn}}. \quad (107)$$

The total amplitude is then

$$\begin{aligned} e^{2s} \prod_{n=1}^{\infty} (1 - (-1)^n e^{-2sn})^{-24} &= e^{2s} \prod_{n=1}^{\infty} (1 + e^{-4s(n-1/2)})^{-24} \prod_{n=1}^{\infty} (1 - e^{-4sn})^{-24} \\ &= \vartheta_{00}(0, 2is/\pi)^{-12} \eta(2is/\pi)^{-12}, \end{aligned} \quad (108)$$

in agreement with the integrand of (7.4.23).

## 8 Chapter 8

### 8.1 Problem 8.1

(a) The spatial world-sheet coordinate  $\sigma^1$  should be chosen in the range  $-\pi < \sigma^1 < \pi$  for (8.2.21) to work. In other words, define  $\sigma^1 = -\text{Im} \ln z$  (with the branch cut for the logarithm on the negative real axis). The only non-zero commutators involved are

$$[x_L, p_L] = i, \quad [\alpha_m, \alpha_n] = m\delta_{m,-n}. \quad (1)$$

Hence

$$\begin{aligned} [X_L(z_1), X_L(z_2)] &= -i\frac{\alpha'}{2} \ln z_2 [x_L, p_L] - i\frac{\alpha'}{2} \ln z_1 [p_L, x_L] - \frac{\alpha'}{2} \sum_{m,n \neq 0} \frac{[\alpha_m, \alpha_n]}{mnz_1^m z_2^n} \\ &= \frac{\alpha'}{2} \left( \ln z_2 - \ln z_1 - \sum_{n \neq 0} \frac{1}{n} \left( \frac{z_1}{z_2} \right)^n \right) \\ &= \frac{\alpha'}{2} \left( \ln z_2 - \ln z_1 + \ln \left( 1 - \frac{z_1}{z_2} \right) - \ln \left( 1 - \frac{z_2}{z_1} \right) \right) \\ &= \frac{\alpha'}{2} \left( \ln z_2 - \ln z_1 + \ln \left( \frac{1 - \frac{z_1}{z_2}}{1 - \frac{z_2}{z_1}} \right) \right) \\ &= \frac{\alpha'}{2} \left( \ln z_2 - \ln z_1 + \ln \left( -\frac{z_1}{z_2} \right) \right) \\ &= \frac{i\alpha'}{2} (\sigma_1^1 - \sigma_2^1 + (\sigma_2^1 - \sigma_1^1 \pm \pi)). \end{aligned} \quad (2)$$

Because of where we have chosen to put the branch cut for the logarithm, the quantity in the inner parentheses must be between  $-\pi$  and  $\pi$ . The upper sign is therefore chosen if  $\sigma_1^1 > \sigma_2^1$ , and the lower otherwise. (Note that the fourth equality is legitimate because the arguments of both  $1 - \frac{z_1}{z_2}$  and  $1 - \frac{z_2}{z_1}$  are in the range  $(-\pi/2, \pi/2)$ .)

(b) Inspection of the above derivation shows that

$$[X_R(z_1), X_R(z_2)] = -\frac{\pi i \alpha'}{2} \text{sign}(\sigma_1^1 - \sigma_2^1). \quad (3)$$

The CBH formula tells us that, for two operators  $A$  and  $B$  whose commutator is a scalar,

$$e^A e^B = e^{[A,B]} e^B e^A. \quad (4)$$

In passing  $\mathcal{V}_{k_L k_R}(z, \bar{z})$  through  $\mathcal{V}_{k'_L k'_R}(z', \bar{z}')$ , (4) will give signs from several sources. The factors from the cocycles commuting past the operators  $e^{ik_L x_L + ik_R x_R}$  and  $e^{ik'_L x_L + ik'_R x_R}$  are given in (8.2.23). This is cancelled by the factor from commuting the normal ordered exponentials past each other:

$$e^{-(k_L k'_L - k_R k'_R) \pi i \alpha' \text{sign}(\sigma^1 - \sigma'^1)/2} = (-1)^{nw' + n'w}. \quad (5)$$

### 8.2 Problem 8.3

(a) In the sigma-model action, we can separate  $X^{25}$  from the other scalars, which we call  $X^\mu$ :

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} \left( (g^{ab}G_{\mu\nu} + i\epsilon^{ab}B_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu \right. \\ \left. + 2(g^{ab}G_{25\mu} + i\epsilon^{ab}B_{25\mu}) \partial_a X^{25} \partial_b X^\mu + g^{ab}G_{25,25} \partial_a X^{25} \partial_b X^{25} \right). \quad (6)$$

(Since we won't calculate the shift in the dilaton, we're setting aside the relevant term in the action.)

(b) We can gauge the  $X^{25}$  translational symmetry by introducing a worldsheet gauge field  $A_a$ :

$$S'_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} \left( (g^{ab}G_{\mu\nu} + i\epsilon^{ab}B_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu \right. \\ \left. + 2(g^{ab}G_{25\mu} + i\epsilon^{ab}B_{25\mu}) (\partial_a X^{25} + A_a) \partial_b X^\mu \right. \\ \left. + g^{ab}G_{25,25} (\partial_a X^{25} + A_a) (\partial_b X^{25} + A_b) \right). \quad (7)$$

This action is invariant under  $X^{25} \rightarrow X^{25} + \lambda$ ,  $A_a \rightarrow A_a - \partial_a \lambda$ . In fact, it's consistent to allow the gauge parameter  $\lambda(\sigma)$  to be periodic with the same periodicity as  $X^{25}$  (making the gauge group a compact  $U(1)$ ). This periodicity will be necessary later, to allow us to unwind the string.

(c) We now add a Lagrange multiplier term to the action,

$$S''_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} \left( (g^{ab}G_{\mu\nu} + i\epsilon^{ab}B_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu \right. \\ \left. + 2(g^{ab}G_{25\mu} + i\epsilon^{ab}B_{25\mu}) (\partial_a X^{25} + A_a) \partial_b X^\mu \right. \\ \left. + g^{ab}G_{25,25} (\partial_a X^{25} + A_a) (\partial_b X^{25} + A_b) + i\phi \epsilon^{ab} (\partial_a A_b - \partial_b A_a) \right). \quad (8)$$

Integrating over  $\phi$  forces  $\epsilon^{ab} \partial_a A_b$  to vanish, which on a topologically trivial worldsheet means that  $A_a$  is gauge-equivalent to 0, bringing us back to the action (6). Of course, there's not much point in making  $X^{25}$  periodic on a topologically trivial worldsheet, and on a non-trivial worldsheet the gauge field may have non-zero holonomies around closed loops. In order for these to be multiples of  $2\pi R$  (and therefore removable by a gauge transformation),  $\phi$  must also be periodic (with period  $2\pi/R$ ). For details, see Rocek and Verlinde (1992).

(d) Any  $X^{25}$  configuration is gauge equivalent to  $X^{25} = 0$ , this condition leaving no additional gauge degrees of freedom. The action, after performing an integration by parts (and ignoring the

holonomy issue) is

$$S_\sigma''' = \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} \left( (g^{ab}G_{\mu\nu} + i\epsilon^{ab}B_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu + G_{25,25} g^{ab} A_a A_b \right. \\ \left. + 2 \left( G_{25\mu} g^{ab} \partial_b X^\mu + iB_{25\mu} \epsilon^{ab} \partial_b X^\mu + i\epsilon^{ab} \partial_b \phi \right) A_a \right). \quad (9)$$

We can complete the square on  $A_a$ ,

$$S_\sigma''' = \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} \left( (g^{ab}G_{\mu\nu} + i\epsilon^{ab}B_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu \right. \\ \left. + G_{25,25} g^{ab} (A_a + G_{25,25}^{-1} B_a) (A_b + G_{25,25}^{-1} B_b) - G_{25,25}^{-1} g^{ab} B_a B_b \right), \quad (10)$$

where  $B^a = G_{25\mu} g^{ab} \partial_b X^\mu + iB_{25\mu} \epsilon^{ab} \partial_b X^\mu + i\epsilon^{ab} \partial_b \phi$ . Integrating over  $A_a$ , and ignoring the result, and using the fact that in two dimensions  $g_{ac} \epsilon^{ab} \epsilon^{cd} = g_{bd}$ , we have

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} \left( (g^{ab}G'_{\mu\nu} + i\epsilon^{ab}B'_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu \right. \\ \left. + 2 \left( g^{ab}G'_{25\mu} + i\epsilon^{ab}B'_{25\mu} \right) \partial_a \phi \partial_b X^\mu + g^{ab}G'_{25,25} \partial_a \phi \partial_b \phi \right), \quad (11)$$

where

$$\begin{aligned} G'_{\mu\nu} &= G_{\mu\nu} - G_{25,25}^{-1} G_{25\mu} G_{25\nu} + G_{25,25}^{-1} B_{25\mu} B_{25\nu}, \\ B'_{\mu\nu} &= B_{\mu\nu} - G_{25,25}^{-1} G_{25\mu} B_{25\nu} + G_{25,25}^{-1} B_{25\mu} G_{25\nu}, \\ G'_{25\mu} &= G_{25,25}^{-1} B_{25\mu}, \\ B'_{25\mu} &= G_{25,25}^{-1} G_{25\mu}, \\ G'_{25,25} &= G_{25,25}^{-1}. \end{aligned} \quad (12)$$

Two features are clearly what we expect to occur upon T-duality: the inversion of  $G_{25,25}$ , and the exchange of  $B_{25\mu}$  with  $G_{25\mu}$ , reflecting the fact that winding states, which couple to the former, are exchanged with compact momentum states, which couple to the latter.

### 8.3 Problem 8.4

The generalization to  $k$  dimensions of the Poisson resummation formula (8.2.10) is

$$\sum_{n \in Z^k} \exp(-\pi a^{mn} n_m n_n + 2\pi i b^n n_n) = (\det a_{mn})^{1/2} \sum_{m \in Z^k} \exp(-\pi a_{mn} (m^m - b^m) (m^n - b^n)), \quad (13)$$

where  $a^{mn}$  is a symmetric matrix and  $a_{mn}$  is its inverse. This can be proven by induction using (8.2.10). The Virasoro generators for the compactified  $X$ s are

$$L_0 = \frac{1}{4\alpha'} v_L^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n, \quad (14)$$

$$\tilde{L}_0 = \frac{1}{4\alpha'} v_R^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n, \quad (15)$$



where products of vectors are taken with the metric  $G_{mn}$ . The partition function is

$$(q\bar{q})^{-1/24} \text{Tr} \left( q^{L_0} \bar{q}^{\bar{L}_0} \right) = |\eta(\tau)|^{-2k} \sum_{n, w_1 \in Z^k} \exp \left[ -\frac{\pi\tau_2}{\alpha'} (v^2 + R^2 w_1^2) + 2\pi i \tau_1 n \cdot w_1 \right]. \quad (16)$$

Using (13) we now get

$$V_k Z_X(\tau)^k \sum_{w_1, w_2 \in Z^k} \exp \left[ -\frac{\pi R^2}{\alpha' \tau_2} |w_2 - \tau w_1|^2 - 2\pi i b_{mn} w_1^n w_2^m \right]. \quad (17)$$

This includes the expected phase factor (8.2.12)—but unfortunately with the wrong sign! The volume factor  $V_k = R^k (\det G_{mn})^{1/2}$  comes from the integral over the zero-mode.

#### 8.4 Problem 8.5

(a) With  $l \equiv (n/r + mr/2, n/r - mr/2)$ , we have

$$l \circ l' = nm' + n'm. \quad (18)$$

Evenness of the lattice is obvious. The dual lattice is generated by the vectors  $(n, m) = (1, 0)$  and  $(0, 1)$ , which also generate the original lattice; hence it is self-dual.

(b) With

$$l \equiv \frac{1}{\sqrt{2\alpha'}} (v + wR, v - wR), \quad (19)$$

one can easily calculate

$$l \circ l' = n \cdot w' + n' \cdot w \quad (20)$$

(in particular,  $B_{mn}$  drops out). Again, at this point it is more or less obvious that the lattice is even and self-dual.

#### 8.5 Problem 8.6

The metric (8.4.37) can be written

$$G_{mn} = \frac{\alpha' \rho_2}{R^2} M_{mn}(\tau), \quad M(\tau) = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}. \quad (21)$$

Thus

$$G^{mn} \partial_\mu G_{np} = \rho_2^{-1} \partial_\mu \rho_2 \delta^m_p + (M^{-1} \partial_\mu M)^m_p. \quad (22)$$

The decoupling between  $\tau$  and  $\rho$  is due to the fact that the determinant of  $M$  is constant (in fact it's 1), so that  $M^{-1} \partial_\mu M$  is traceless:

$$G^{mn} G^{pq} \partial_\mu G_{mp} \partial^\mu G_{nq} = \frac{2 \partial_\mu \rho_2 \partial^\mu \rho_2}{\rho_2^2} + \text{Tr}(M^{-1} \partial_\mu M)^2. \quad (23)$$

With a little algebra the second term can be shown to equal  $2\partial_\mu\tau\partial^\mu\bar{\tau}/\tau_2^2$ . Meanwhile, the antisymmetry of  $B$  implies that  $G^{mn}G^{pq}\partial_\mu B_{mp}\partial^\mu B_{nq}$  essentially calculates the determinant of the inverse metric  $\det G^{mn} = (R^2/\alpha'\rho_2)^2$ , multiplying it by  $2\partial_\mu B_{24,25}\partial^\mu B_{24,25} = 2(\alpha'/R^2)^2\partial_\mu\rho_1\partial^\mu\rho_1$ . Adding this to (23) we arrive at (twice) (8.4.39).

If  $\tau$  and  $\rho$  are both imaginary, then  $B = 0$  and the torus is rectangular with proper radii

$$R_{24} = \sqrt{G_{24,24}}R = \sqrt{\frac{\alpha'\rho_2}{\tau_2}}, \quad (24)$$

$$R_{25} = \sqrt{G_{25,25}}R = \sqrt{\alpha'\rho_2\tau_2}. \quad (25)$$

Clearly switching  $\rho$  and  $\tau$  is a T-duality on  $X^{24}$ , while  $\rho \rightarrow -1/\rho$  is T-duality on both  $X^{24}$  and  $X^{25}$  combined with  $X^{24} \leftrightarrow X^{25}$ .

## 8.6 Problem 8.7

(a) Since we have already done this problem for the case  $p = 25$  in problem 6.9(a), we can simply adapt the result from that problem (equation (46) in the solutions to chapter 6) to general  $p$ . (In this case, the Chan-Paton factors are trivial, and we must include contributions from all three combinations of polarizations.) The open string coupling  $g_{o,p}$  depends on  $p$ , and we can compute it either by T-duality or by comparing to the low-energy action (8.7.2).

Due to the Dirichlet boundary conditions, there is no zero mode in the path integral and therefore no momentum-conserving delta function in those directions. Except for this fact, the three-tachyon and Veneziano amplitudes calculated in section 6.4 go through unchanged, so we have

$$C_{D2,p} = \frac{1}{\alpha'g_{o,p}^2}. \quad (26)$$

The four-ripple amplitude is

$$S = \frac{2ig_{o,p}^2}{\alpha'}(2\pi)^{p+1}\delta^{p+1}\left(\sum_i k_i\right) \times (e_1 \cdot e_2 e_3 \cdot e_4 F(t, u) + e_1 \cdot e_3 e_2 \cdot e_4 F(s, u) + e_1 \cdot e_4 e_2 \cdot e_3 F(s, t)), \quad (27)$$

where

$$F(x, y) \equiv B(-\alpha'x + 1, \alpha'x + \alpha'y - 1) + B(-\alpha'y + 1, \alpha'x + \alpha'y - 1) + B(-\alpha'x + 1, -\alpha'y + 1). \quad (28)$$

If we had instead obtained this amplitude by T-dualizing the answer to problem 6.9(a), using the fact that  $\kappa$ , and therefore  $g_{o,25}^2$ , transform according to (8.3.30), we would have found the same result with  $g_{o,p}^2$  replaced with  $g_{o,25}^2/(2\pi\sqrt{\alpha'})^{25-p}$ , so we find

$$g_{o,p} = \frac{g_{o,25}}{(2\pi\sqrt{\alpha'})^{(25-p)/2}}. \quad (29)$$

(b) To examine the Regge limit, let us re-write the amplitude (27) in the following way:

$$\begin{aligned}
S = & \frac{2ig_{0,p}^2}{\alpha'} (2\pi)^{p+1} \delta^{p+1} \left( \sum_i k_i \right) \left( 1 - \cos \pi \alpha' t + \tan \frac{\pi \alpha' s}{2} \sin \pi \alpha' t \right) \\
& \times \left( e_1 \cdot e_2 e_3 \cdot e_4 \frac{\Gamma(\alpha' s + \alpha' t + 1) \Gamma(-\alpha' t + 1)}{\Gamma(\alpha' s + 2)} + e_1 \cdot e_3 e_2 \cdot e_4 \frac{\Gamma(\alpha' s + \alpha' t + 1) \Gamma(-\alpha' t - 1)}{\Gamma(\alpha' s)} \right. \\
& \left. + e_1 \cdot e_4 e_2 \cdot e_3 \frac{\Gamma(\alpha' s + \alpha' t - 1) \Gamma(-\alpha' t + 1)}{\Gamma(\alpha' s)} \right). \quad (30)
\end{aligned}$$

The factor with the sines and cosines gives a pole wherever  $\alpha' s$  is an odd integer, while the last factor gives the overall behavior in the limit  $s \rightarrow \infty$ . In that limit the coefficients of  $e_1 \cdot e_2 e_3 \cdot e_4$  and  $e_1 \cdot e_4 e_2 \cdot e_3$  go like  $s^{\alpha' t - 1} \Gamma(-\alpha' t + 1)$ , while the coefficient of  $e_1 \cdot e_3 e_2 \cdot e_4$  goes like  $s^{\alpha' t + 1} \Gamma(-\alpha' t - 1)$ , and therefore dominates (unless  $e_1 \cdot e_3$  or  $e_2 \cdot e_4$  vanishes). We thus have Regge behavior.

For hard scattering, the amplitude (27) has the same exponential falloff (6.4.19) as the Veneziano amplitude, since the only differences are shifts of 2 in the arguments of some of the gamma functions, which will not affect their asymptotic behavior.

(c) Expanding  $F(x, y)$  for small  $\alpha'$ , fixing  $x$  and  $y$ , we find (with some assistance from *Mathematica*) that the leading term is quadratic:

$$F(x, y) = -\frac{\pi^2 \alpha'^2}{2} xy + O(\alpha'^3). \quad (31)$$

Hence the low energy limit of the amplitude (27) is

$$S \approx -i\pi^2 \alpha' g_{0,p}^2 (2\pi)^{p+1} \delta^{p+1} \left( \sum_i k_i \right) (e_1 \cdot e_2 e_3 \cdot e_4 tu + e_1 \cdot e_3 e_2 \cdot e_4 su + e_1 \cdot e_4 e_2 \cdot e_3 st). \quad (32)$$

The D-brane is embedded in a flat spacetime,  $G_{\mu\nu} = \eta_{\mu\nu}$ , with vanishing  $B$  and  $F$  fields and constant dilaton. We use a coordinate system on the brane  $\xi^a = X^a$ ,  $a = 0, \dots, p$ , so the induced metric is

$$G_{ab} = \eta_{ab} + \partial_a X^m \partial_b X^m, \quad (33)$$

where the fields  $X^m$ ,  $m = p+1, \dots, 25$ , are the fluctuations in the transverse position of the brane, whose scattering amplitude we wish to find. Expanding the action (8.7.2) to quartic order in the fluctuations, we can use the formula

$$\det(I + A) = 1 + \text{Tr } A + \frac{1}{2} (\text{Tr } A)^2 - \frac{1}{2} \text{Tr } A^2 + O(A^3), \quad (34)$$

to find

$$\mathbf{S}_p = -\tau_p \int d^{p+1} \xi \left( 1 + \frac{1}{2} \eta^{ab} \partial_a X^m \partial_b X^m + \frac{1}{8} (\eta^{ab} \eta^{cd} - 2\eta^{ac} \eta^{bd}) \partial_a X^m \partial_b X^m \partial_c X^n \partial_d X^n \right). \quad (35)$$

The fields  $X^m$  are not canonically normalized, and the coupling constant is in fact

$$\frac{1}{8\tau_p} = \frac{\pi^2 \alpha' g_{0,p}^2}{4} \quad (36)$$

where we have used (6.6.18), (8.7.26), and (8.7.28), and (24). All the ways of contracting four  $X^m$ s with the interaction term yield

$$e_1 \cdot e_2 e_3 \cdot e_4 (8k_1 \cdot k_2 k_3 \cdot k_4 - 8k_1 \cdot k_3 k_2 \cdot k_4 - 8k_1 \cdot k_4 k_2 \cdot k_3) = 4e_1 \cdot e_2 e_3 \cdot e_4 t u \quad (37)$$

plus similar terms for  $e_1 \cdot e_3 e_2 \cdot e_4$  and  $e_1 \cdot e_4 e_2 \cdot e_3$ . Multiplying this by the coupling constant (30), and a factor  $-i(2\pi)^{p+1} \delta^{p+1}(\sum_i k_i)$ , yields precisely the amplitude (32), showing that the two ways of calculating  $g_{0,p}$  agree.

## 8.7 Problem 8.9

(a) There are two principal changes in the case of Dirichlet boundary conditions from the derivation of the disk expectation value (6.2.33): First, there is no zero mode, so there is no momentum-space delta function (this corresponds to the fact that the D-brane breaks translation invariance in the transverse directions and therefore does not conserve momentum). Second, the image charge in the Green's function has the opposite sign:

$$G'_D(\sigma_1, \sigma_2) = -\frac{\alpha'}{2} \ln |z_1 - z_2|^2 + \frac{\alpha'}{2} \ln |z_1 - \bar{z}_2|^2. \quad (38)$$

Denoting the parts of the momenta parallel and perpendicular to the D-brane by  $k$  and  $q$  respectively, the expectation value becomes

$$\left\langle \prod_{i=1}^n : e^{i(k_i + q_i) \cdot X(z_i, \bar{z}_i)} : \right\rangle_{D_{2,p}} = i C_{D_{2,p}}^X (2\pi)^{p+1} \delta^{p+1} \left( \sum_i k_i \right) \prod_{i=1}^n |z_i - \bar{z}_i|^{\alpha'(k_i^2 - q_i^2)/2} \prod_{i < j} |z_i - z_j|^{\alpha'(k_i \cdot k_j + q_i \cdot q_j)} |z_i - \bar{z}_j|^{\alpha'(k_i \cdot k_j - q_i \cdot q_j)}. \quad (39)$$

(b) For expectation values including operators  $\partial_a X^M$  in the interior, one performs the usual contractions, but using the Green's function (38) rather than (6.2.32) for the Dirichlet directions.

## 8.8 Problem 8.11

(a) The disk admits three real CKVs. Fixing the position of one of the closed-string vertex operators eliminates two of these, leaving the one which generates rotations about the fixed operator. We can eliminate this last CKV by integrating the second vertex operator along a line connecting the fixed vertex operator to the edge of the disk. The simplest way to implement this on the

upper half-plane is by fixing  $z_2$  on the positive imaginary axis and integrating  $z_1$  from 0 to  $z_2$ . The amplitude is

$$S = g_c^2 e^{-\lambda} \int_0^{z_2} dz_1 \left\langle : c^1 e^{i(k_1+q_1)\cdot X}(z_1, \bar{z}_1) :: c\tilde{c} e^{i(k_2+q_2)\cdot X}(z_2, \bar{z}_2) : \right\rangle_{D_{2,p}}, \quad (40)$$

where, as in problem 8.9,  $k$  and  $q$  represent the momenta parallel and perpendicular to the D-brane respectively. The ghost path integral is

$$\begin{aligned} \left\langle \frac{1}{2} (c(z_1) + \tilde{c}(\bar{z}_1)) c(z_2) \tilde{c}(\bar{z}_2) \right\rangle_{D_2} &= \frac{C_{D_2}^g}{2} ((z_1 - z_2)(z_1 - \bar{z}_2) + (\bar{z}_1 - z_2)(\bar{z}_1 - \bar{z}_2)) (z_2 - \bar{z}_2) \\ &= 2C_{D_2}^g (z_1 - z_2)(z_1 + z_2)z_2. \end{aligned} \quad (41)$$

To evaluate the  $X$  path integral we use the result of problem 8.9(a):

$$\begin{aligned} &\left\langle : e^{i(k_1+q_1)\cdot X}(z_1, \bar{z}_1) :: e^{i(k_2+q_2)\cdot X}(z_2, \bar{z}_2) : \right\rangle_{D_{2,p}} \\ &= iC_{D_{2,p}}^X (2\pi)^{p+1} \delta^{p+1}(k_1 + k_2) \\ &\quad \times |z_1 - \bar{z}_1|^{\alpha'(k_1^2 - q_1^2)/2} |z_2 - \bar{z}_2|^{\alpha'(k_2^2 - q_2^2)/2} |z_1 - z_2|^{\alpha'(k_1 \cdot k_2 + q_1 \cdot q_2)} |z_1 - \bar{z}_2|^{\alpha'(k_1 \cdot k_2 - q_1 \cdot q_2)} \\ &= iC_{D_{2,p}}^X (2\pi)^{p+1} \delta^{p+1}(k_1 + k_2) \\ &\quad \times 2^{2\alpha'k^2 - 4} |z_1|^{\alpha'k^2 - 2} |z_2|^{\alpha'k^2 - 2} |z_1 - z_2|^{-\alpha's/2 - 4} |z_1 + z_2|^{\alpha's/2 - 2k^2 + 4}. \end{aligned} \quad (42)$$

We have used the kinematic relations  $k_1 + k_2 = 0$  and  $(k_1 + q_1)^2 = (k_2 + q_2)^2 = 4/\alpha'$ , and defined the parameters

$$\begin{aligned} k^2 &\equiv k_1^2 = k_2^2 = 4 - q_1^2 = 4 - q_2^2, \\ s &\equiv -(q_1 + q_2)^2. \end{aligned} \quad (43)$$

So we have

$$\begin{aligned} S &= -g_c^2 C_{D_{2,p}} (2\pi)^{p+1} \delta^{p+1}(k_1 + k_2) \\ &\quad \times 2^{2\alpha'k^2 - 3} |z_2|^{\alpha'k^2 - 1} \int_0^{z_2} dz_1 |z_1|^{\alpha'k^2 - 2} |z_1 - z_2|^{-\alpha's/2 - 3} |z_1 + z_2|^{\alpha's/2 - 2k^2 + 5} \\ &= -ig_c^2 C_{D_{2,p}} (2\pi)^{p+1} \delta^{p+1}(k_1 + k_2) 2^{2\alpha'k^2 - 3} \int_0^1 dx x^{\alpha'k^2 - 2} (1-x)^{-\alpha's/2 - 3} (1+x)^{\alpha's/2 - 2k^2 + 5} \\ &= -i \frac{\pi^{3/2} (2\pi\sqrt{\alpha'})^{11-p} g_c}{32} (2\pi)^{p+1} \delta^{p+1}(k_1 + k_2) B(\alpha'k^2 - 1, -\alpha's/4 - 1). \end{aligned} \quad (44)$$

In the last line we have used (26), (29), and (8.7.28) to calculate  $g_c^2 C_{D_{2,p}}$ .

**(b)** In the Regge limit we increase the scattering energy,  $k^2 \rightarrow -\infty$ , while holding fixed the momentum transfer  $s$ . As usual, the beta function in the amplitude give us Regge behavior:

$$S \sim (-k^2)^{\alpha's/4 + 1} \Gamma(-\alpha's/4 - 1). \quad (45)$$

In the hard scattering limit we again take  $k^2 \rightarrow -\infty$ , but this time fixing  $k^2/s$ . As in the Veneziano amplitude, the beta function gives exponential behavior in this limit.

(c) The beta function in the amplitude has poles at  $\alpha'k^2 = 1, 0, -1, \dots$ , representing on-shell intermediate open strings on the D-brane: the closed string is absorbed and then later re-emitted by the D-brane. These poles come from the region of the integral in (44) where  $z_1$  approaches the boundary.

Note that there are also poles at  $s = 0$  and  $s = -4$  (the poles at positive  $s$  are kinematically forbidden), representing an on-shell intermediate closed string: the tachyon “decays” into another tachyon and either a massless or a tachyonic closed string, and the latter is then absorbed (completely, without producing open strings) by the D-brane. These poles come from the region of the integral in (44) where  $z_1$  approaches  $z_2$ .

## 9 Appendix A

### 9.1 Problem A.1

(a) We proceed by the same method as in the example on pages 339-341. Our orthonormal basis for the periodic functions on  $[0, U]$  will be:

$$\begin{aligned} f_0(u) &= \frac{1}{\sqrt{U}}, \\ f_j(u) &= \sqrt{\frac{2}{U}} \cos \frac{2\pi j u}{U}, \\ g_j(u) &= \sqrt{\frac{2}{U}} \sin \frac{2\pi j u}{U}, \end{aligned} \quad (1)$$

where  $j$  runs over the positive integers. These are eigenfunctions of  $\Delta = -\partial_u^2 + \omega^2$  with eigenvalues

$$\lambda_j = \left(\frac{2\pi j}{U}\right)^2 + \omega^2. \quad (2)$$

Hence (neglecting the counter-term action)

$$\begin{aligned} \text{Tr exp}(-\hat{H}U) &= \int [dq]_{\text{P}} \exp(-S_{\text{E}}) \\ &= \left(\det_{\text{P}} \frac{\Delta}{2\pi}\right)^{-1/2} \\ &= \sqrt{\frac{2\pi}{\lambda_0}} \prod_{j=1}^{\infty} \frac{2\pi}{\lambda_j} \\ &= \frac{\sqrt{2\pi}}{\omega} \prod_{j=1}^{\infty} \frac{2\pi U^2}{4\pi^2 j^2 + \omega^2 U^2}. \end{aligned} \quad (3)$$

This infinite product vanishes, so we regulate it by dividing by the same determinant with  $\omega \rightarrow \Omega$ :

$$\frac{\Omega}{\omega} \prod_{j=1}^{\infty} \frac{1 + \left(\frac{\Omega U}{2\pi j}\right)^2}{1 + \left(\frac{\omega U}{2\pi j}\right)^2} = \frac{\sinh \frac{1}{2}\Omega U}{\sinh \frac{1}{2}\omega U}. \quad (4)$$

For large  $\Omega$  this becomes

$$\frac{e^{\Omega U/2}}{2 \sinh \frac{1}{2}\omega U}; \quad (5)$$

the divergence can easily be cancelled with a counter-term Lagrangian  $L_{\text{ct}} = \Omega/2$ , giving

$$\text{Tr exp}(-\hat{H}U) = \frac{1}{2 \sinh \frac{1}{2}\omega U}. \quad (6)$$

The eigenvalues of  $\hat{H}$  are simply

$$E_i = \left(i + \frac{1}{2}\right)\omega, \quad (7)$$

for non-negative integer  $i$ , so using (A.1.32) gives

$$\begin{aligned} \text{Tr} \exp(-\hat{H}U) &= \sum_{i=0}^{\infty} \exp(-E_i U) \\ &= e^{-\omega U/2} \sum_{i=0}^{\infty} e^{-i\omega U} \\ &= \frac{1}{2 \sinh \frac{1}{2}\omega U}. \end{aligned} \quad (8)$$

To be honest, the overall normalization of (6) must be obtained by comparison with this result.

(b) Our basis for the anti-periodic functions on  $[0, U]$  consists of the eigenfunctions of  $\Delta$ ,

$$\begin{aligned} f_j(u) &= \sqrt{\frac{2}{U}} \cos \frac{2\pi(j + \frac{1}{2})u}{U}, \\ g_j(u) &= \sqrt{\frac{2}{U}} \sin \frac{2\pi(j + \frac{1}{2})u}{U}, \end{aligned} \quad (9)$$

where the  $j$  are non-negative integers, with eigenvalues

$$\lambda_j = \left( \frac{2\pi(j + \frac{1}{2})}{U} \right)^2 + \omega^2. \quad (10)$$

Before including the counter-term and regulating, we have

$$\text{Tr} \left[ \exp(-\hat{H}U) \hat{R} \right] = \prod_{j=0}^{\infty} \frac{2\pi U^2}{(2\pi(j + \frac{1}{2}))^2 + (\omega U)^2}. \quad (11)$$

After including the counter-term action and dividing by the regulator, this becomes

$$\begin{aligned} e^{-L_{\text{ct}}U} \prod_{j=0}^{\infty} \frac{1 + \left( \frac{\Omega U}{2\pi(j + \frac{1}{2})} \right)^2}{1 + \left( \frac{\omega U}{2\pi(j + \frac{1}{2})} \right)^2} &= e^{-L_{\text{ct}}U} \frac{\cosh \frac{1}{2}\Omega U}{\cosh \frac{1}{2}\omega U} \\ &\sim \frac{e^{(\Omega/2 - L_{\text{ct}})U}}{2 \cosh \frac{1}{2}\omega U}, \end{aligned} \quad (12)$$

so the answer is

$$\text{Tr} \left[ \exp(-\hat{H}U) \hat{R} \right] = \frac{1}{2 \cosh \frac{1}{2}\omega U}. \quad (13)$$

This result can easily be reproduced by summing over eigenstates of  $\hat{H}$  with weight  $(-1)^R$ , since the even  $i$  eigenstates are also even under reflection, and odd  $i$  eigenstates odd under reflection:

$$\begin{aligned} \text{Tr} \left[ \exp(-\hat{H}U) \hat{R} \right] &= \sum_{i=0}^{\infty} (-1)^i e^{-(j+1/2)\omega U} \\ &= \frac{1}{2 \cosh \frac{1}{2}\omega U}. \end{aligned} \quad (14)$$



### 9.2 Problem A.3

The action can be written

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int d^2\sigma X(-\partial_1^2 - \partial_2^2 + m^2)X \\ &= \frac{1}{2} \int d^2\sigma X \Delta X, \end{aligned} \quad (15)$$

$$\Delta = \frac{1}{2\pi\alpha'}(-\partial_1^2 - \partial_2^2 + m^2). \quad (16)$$

The periodic eigenfunctions of  $\Delta$  can be given in a basis of products of periodic eigenfunction of  $-\partial_1^2$  on  $\sigma_1$  with periodic eigenfunctions of  $-\partial_2^2$  on  $\sigma_2$ :

$$\begin{aligned} F_{jk}(\sigma_1, \sigma_2) &= f_j(\sigma_1)g_k(\sigma_2), \quad (17) \\ f_0(\sigma_1) &= \frac{1}{\sqrt{2\pi}}, \\ f_j(\sigma_1) &= \frac{1}{\sqrt{\pi}} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} j\sigma_1, \quad j = 1, 2, \dots, \\ g_0(\sigma_2) &= \frac{1}{\sqrt{T}}, \\ g_k(\sigma_2) &= \frac{1}{\sqrt{2T}} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} k\sigma_2, \quad k = 1, 2, \dots \end{aligned}$$

The eigenvalues are

$$\lambda_{jk} = \frac{1}{2\pi\alpha'} \left( j^2 + \left( \frac{2\pi k}{T} \right)^2 + m^2 \right), \quad (18)$$

with multiplicity  $n_j n_k$ , where  $n_0 = 1$  and  $n_i = 2$  ( $i = 1, 2, \dots$ ). The path integral is

$$\begin{aligned} \left( \det_{\text{P}} \frac{\Delta}{2\pi} \right)^{-1/2} &= \prod_{j,k=0}^{\infty} \left( \frac{2\pi}{\lambda_{jk}} \right)^{n_j n_k / 2} \\ &= \prod_{j=0}^{\infty} \left( \prod_{k=0}^{\infty} \left( \frac{2\pi}{\lambda_{jk}} \right)^{n_k / 2} \right)^{n_j} \\ &= \prod_{j=0}^{\infty} \left( \frac{1}{2 \sinh \frac{1}{2} \sqrt{j^2 + m^2 T}} \right)^{n_j}, \end{aligned} \quad (19)$$

where in the last step we have used the result of problem A.1. The infinite product vanishes; it would have to be regulated and a counter-term introduced to extract the finite part.

### 9.3 Problem A.5

We assume that the Hamiltonian for this system is

$$H = m\chi\psi. \quad (20)$$

The periodic trace is

$$\begin{aligned}
\text{Tr} \left[ (-1)^{\hat{F}} \exp(-\hat{H}U) \right] &= \int d\psi \langle \psi, U | \psi, 0 \rangle_{\text{E}} \\
&= \int [d\chi d\psi] \exp \left[ \int_0^U du (-\chi \partial_u \psi - H) \right] \\
&= \int [d\chi d\psi] \exp \left[ \int_0^U du \chi \Delta \psi \right], \tag{21}
\end{aligned}$$

where

$$\Delta = -\partial_u - m. \tag{22}$$

The periodic eigenfunctions of  $\Delta$  on  $[0, U]$  are

$$f_j(u) = \frac{1}{\sqrt{U}} e^{2\pi i j u / U}, \tag{23}$$

while the eigenfunctions of  $\Delta^T = \partial_u - m$  are

$$g_j(u) = \frac{1}{\sqrt{U}} e^{-2\pi i j u / U}, \tag{24}$$

with  $j$  running over the integers. Their eigenvalues are

$$\lambda_j = -\left( \frac{2\pi i j}{U} - m \right), \tag{25}$$

so the trace becomes

$$\begin{aligned}
\int [d\chi d\psi] \exp \left[ \int_0^U du \chi \Delta \psi \right] &= \prod_{j=-\infty}^{\infty} \lambda_j \\
&= \prod_{j=-\infty}^{\infty} -\left( \frac{2\pi i j}{U} + m \right) \\
&= -m \prod_{j=1}^{\infty} \left( \left( \frac{2\pi j}{U} \right)^2 + m^2 \right). \tag{26}
\end{aligned}$$

This is essentially the inverse of the infinite product that was considered in problem A.1(a) (eq. (3)). Regulating and renormalizing in the same manner as in that problem yields:

$$\text{Tr} \left[ (-1)^{\hat{F}} \exp(-\hat{H}U) \right] = 2 \sinh \frac{1}{2} mU. \tag{27}$$

This answer can very easily be checked by explicit calculation of the LHS. The Hamiltonian operator can be obtained from the classical Hamiltonian by first antisymmetrizing on  $\chi$  and  $\psi$ :

$$H = m\chi\psi = \frac{1}{2}m(\chi\psi - \psi\chi) \tag{28}$$

$$\longrightarrow \hat{H} = \frac{1}{2}m(\hat{\chi}\hat{\psi} - \hat{\psi}\hat{\chi}). \quad (29)$$

Now

$$\begin{aligned} \hat{H}|\uparrow\rangle &= \frac{1}{2}|\uparrow\rangle, \\ \hat{H}|\downarrow\rangle &= -\frac{1}{2}|\downarrow\rangle, \end{aligned} \quad (30)$$

so, using A.2.22,

$$\text{Tr} \left[ (-1)^{\hat{F}} \exp(-\hat{H}U) \right] = e^{mU/2} - e^{-mU/2}, \quad (31)$$

in agreement with (27).

The anti-periodic trace is calculated in the same way, the only difference being that the index  $j$  runs over the half-integers rather than the integers in order to make the eigenfunctions (23) and (24) anti-periodic. Eq. (26) becomes

$$\int [d\chi d\psi] \exp \left[ \int_0^U du \chi \Delta \psi \right] = \prod_{j=1/2, 3/2, \dots}^{\infty} \left( \left( \frac{2\pi j}{U} \right)^2 + m^2 \right). \quad (32)$$

When we regulate the product, it becomes

$$\begin{aligned} \prod_j \frac{\left( \frac{2\pi j}{U} \right)^2 + m^2}{\left( \frac{2\pi j}{U} \right)^2 + M^2} &= \frac{\prod_j \left( 1 + \left( \frac{mU}{2\pi j} \right)^2 \right)}{\prod_j \left( 1 + \left( \frac{MU}{2\pi j} \right)^2 \right)} \\ &= \frac{\cosh \frac{1}{2}mU}{\cosh \frac{1}{2}MU}. \end{aligned} \quad (33)$$

With the same counter-term Lagrangian as before to cancel the divergence in the denominator as  $M \rightarrow \infty$ , we are simply left with

$$\text{Tr} \exp(-\hat{H}U) = 2 \cosh \frac{1}{2}mU, \quad (34)$$

the same as would be found using (30).

## 10 Chapter 10

### 10.1 Problem 10.1

(a) The OPEs are:

$$T_F(z)X^\mu(0) \sim -i\sqrt{\frac{\alpha'}{2}}\frac{\psi^\mu(0)}{z}, \quad T_F(z)\psi^\mu(0) \sim i\sqrt{\frac{\alpha'}{2}}\frac{\partial X^\mu(0)}{z}. \quad (1)$$

(b) The result follows trivially from (2.3.11) and the above OPEs.

### 10.2 Problem 10.2

(a) We have

$$\delta_{\eta_1}\delta_{\eta_2}X = \delta_{\eta_1}(\eta_2\psi + \eta_2^*\tilde{\psi}) = -\eta_2\eta_1\partial X - \eta_2^*\eta_1^*\bar{\partial}X, \quad (2)$$

so, using the anti-commutativity of the  $\eta_i$ ,

$$[\delta_{\eta_1}, \delta_{\eta_2}]X = 2\eta_1\eta_2\partial X + 2\eta_1^*\eta_2^*\bar{\partial}X = \delta_v X \quad (3)$$

(see (2.4.7)). For  $\psi$  we must apply the equation of motion  $\partial\tilde{\psi} = 0$ :

$$\delta_{\eta_1}\delta_{\eta_2}\psi = \delta_{\eta_1}(-\eta_2\partial X) = -\eta_2\partial(\eta_1\psi + \eta_1^*\tilde{\psi}) = -\eta_2\eta_1\partial\psi - \eta_2\partial\eta_1\psi, \quad (4)$$

so

$$[\delta_{\eta_1}, \delta_{\eta_2}]\psi = -v\partial\psi - \frac{1}{2}\partial v\psi = \delta_v\psi, \quad (5)$$

the second term correctly reproducing the weight of  $\psi$ . The  $\tilde{\psi}$  transformation works out similarly.

(b) For  $X$ :

$$\delta_\eta\delta_v X = -v\eta\partial\psi - v\partial\eta\psi - v^*\eta^*\bar{\partial}\tilde{\psi} - v^*(\partial\eta)^*\tilde{\psi}, \quad (6)$$

$$\delta_v\delta_\eta X = -v\eta\partial\psi - \frac{1}{2}\eta\partial v\psi - v^*\eta^*\bar{\partial}\tilde{\psi} - \frac{1}{2}\eta^*(\partial v)^*\tilde{\psi}, \quad (7)$$

so

$$[\delta_\eta, \delta_v]X = \delta_{\eta'} X, \quad (8)$$

where

$$\eta' = -v\partial\eta + \frac{1}{2}\partial v\eta. \quad (9)$$

For  $\psi$ ,

$$\delta_\eta\delta_v\psi = v\partial\eta\partial X + v\eta\partial^2 X + \frac{1}{2}\eta\partial v\partial X, \quad (10)$$

$$\delta_v\delta_\eta\psi = \eta v\partial^2 X + \eta\partial v\partial X, \quad (11)$$

so

$$[\delta_\eta, \delta_v]\psi = \delta_{\eta'}\psi, \quad (12)$$

and similarly for  $\tilde{\psi}$ .

### 10.3 Problem 10.3

(a) Since the OPE of  $T_B^X = -(1/\alpha')\partial X^\mu\partial X_\mu$  and  $T_B^\psi = -(1/2)\psi^\mu\partial\psi_\mu$  is non-singular,

$$T_B(z)T_B(0) \sim T_B^X(z)T_B^X(0) + T_B^\psi(z)T_B^\psi(0), \quad (13)$$

which does indeed reproduce (10.1.13a). We then have

$$\begin{aligned} \frac{T_B(z)T_F(0)}{i\sqrt{2/\alpha'}} &\sim \frac{1}{z^2}\partial X^\mu(z)\psi_\mu(0) + \frac{1}{2z}\partial^\mu(z)\partial\psi^\mu(z)\partial X_\mu(0) + \frac{1}{2z^2}\psi^\mu(z)\partial X_\mu(0) \\ &\sim \frac{3}{2z^2}\psi^\mu\partial X_\mu(0) + \psi^\mu\partial^2 X_\mu(0) + \frac{1}{z}\partial\psi^\mu\partial X_\mu(0), \\ T_F(z)T_F(0) &\sim \frac{D}{z^3} - \frac{2}{\alpha'z}\partial X_\mu(z)\partial X^\mu(0) + \frac{1}{z^2}\psi^\mu(z)\psi_\mu(0). \end{aligned} \quad (14)$$

(b) Again, there is no need to check the  $T_B T_B$  OPE, since that is simply the sum of the  $X$  part and the  $\psi$  part. The new terms in  $T_B$  and  $T_F$  add two singular terms to their OPE, namely

$$\begin{aligned} V_\mu\partial^2 X^\mu(z)i\sqrt{\frac{2}{\alpha'}}\psi^\nu\partial X_\nu(0) + \frac{1}{2}\psi^\mu\partial\psi_\mu(z)i\sqrt{2\alpha'}V_\nu\partial\psi^\nu(0) \\ \sim \frac{i\sqrt{2\alpha'}}{z^3}V_\mu\psi^\mu(0) - i\sqrt{\frac{2}{\alpha'}}\frac{1}{z^2}V_\mu\partial\psi^\mu(z) - \frac{i\sqrt{2\alpha'}}{z^3}V_\mu\psi^\mu(z) \\ \sim -3i\sqrt{\frac{\alpha'}{2}}\frac{1}{z^2}V_\mu\partial\psi^\mu(0) - \frac{i\sqrt{2\alpha'}}{z}V_\mu\partial^2\psi^\mu(0), \end{aligned} \quad (15)$$

which are precisely the extra terms we expect on the right hand side of (10.1.1b). The new terms in the  $T_F T_F$  OPE are

$$\begin{aligned} 2\psi^\mu\partial X_\mu(z)V_\nu\partial\psi^\nu(0) + 2V_\mu\partial\psi^\mu(z)\psi^\nu\partial X_\nu(0) - 2\alpha'V_\mu\partial\psi^\mu(z)V_\nu\partial\psi^\nu(0) \\ \sim \frac{2}{z^2}V_\mu\partial X^\mu(z) - \frac{2}{z^2}V_\mu\partial X^\mu(0) + \frac{4\alpha'V^2}{z^3} \\ \sim \frac{2}{z}V_\mu\partial^2 X^\mu(0) + \frac{4\alpha'V^2}{z^3}. \end{aligned} \quad (16)$$

### 10.4 Problem 10.4

The  $[L_m, L_n]$  commutator (10.2.11a) is as in the bosonic case. The current associated with the charge  $\{G_r, G_s\}$  is, according to (2.6.14),

$$\begin{aligned} \text{Res}_{z_1 \rightarrow z_2} z_1^{r+1/2} T_F(z_1) z_2^{s+1/2} T_F(z_2) &= \text{Res}_{z_{12} \rightarrow 0} z_2^{r+s+1} \left(1 + \frac{z_{12}}{z_2}\right)^{r+1/2} \left(\frac{2c}{3z_{12}^3} + \frac{2}{z_{12}} T_B(z_2)\right) \\ &= \frac{(4r^2 - 1)c}{12} z_2^{r+s-1} + 2z_2^{r+s+1} T_B(z_2). \end{aligned} \quad (17)$$

$\{G_r, G_s\}$  is in turn the residue of this expression in  $z_2$ , which is easily seen to equal the RHS of (10.2.11b). Similarly, for  $[L_m, G_r]$  we have

$$\begin{aligned} \text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} T_B(z_1) z_2^{r+1/2} T_F(z_2) \\ &= \text{Res}_{z_{12} \rightarrow 0} z_2^{r+m+3/2} \left(1 + \frac{z_{12}}{z_2}\right)^{m+1} \left(\frac{3}{2z_{12}^2} T_F(z_2) + \frac{1}{z_{12}} \partial T_F(z_2)\right) \\ &= \frac{3(m+1)}{2} z_2^{r+m+1/2} T_F(z_2) + z_2^{r+m+3/2} \partial T_F(z_2). \end{aligned} \quad (18)$$

The residue in  $z_2$  of this is

$$\frac{3(m+1)}{2} G_{r+m} - \left(r + m + \frac{3}{2}\right) G_{r+m}, \quad (19)$$

in agreement with (10.2.11c).

## 10.5 Problem 10.5

Let us denote by  $c_B$  the central charge appearing in the  $T_B T_B$  OPE, and by  $c_F$  that appearing in the  $T_F T_F$  OPE. One of the Jacobi identities for the superconformal generators is, using (10.2.11),

$$\begin{aligned} 0 &= [L_m, \{G_r, G_s\}] + \{G_r, [G_s, L_m]\} - \{G_s, [L_m, G_r]\} \\ &= \frac{1}{6} \left( c_B(m^3 - m) + \frac{c_F}{4} ((2s - m)(4r^2 - 1) + (2r - m)(4s^2 - 1)) \right) \delta_{m+r+s,0} \\ &= \frac{1}{6} (c_B - c_F)(m^3 - m) \delta_{m+r+s,0}. \end{aligned} \quad (20)$$

Hence  $c_B = c_F$ .

## 10.6 Problem 10.7

Taking (for example)  $z_1$  to infinity while holding the other  $z_i$  fixed at finite values, the expectation value (10.3.7) goes like  $z_1^{-1}$ . Since  $e^{i\epsilon_1 H(z_1)}$  is a tensor of weight  $1/2$ , transforming to the  $u = 1/z_1$  frame this expectation value becomes constant,  $O(1)$ , in the limit  $u \rightarrow 0$ . This is the correct behavior—the only poles and zeroes of the expectation value should be at the positions of the other operators. If we now consider some other function, with exactly the same poles and zeroes and behavior as  $z_1 \rightarrow \infty$ , the ratio between this function and the one given in (10.3.7) would have to be an entire function which approaches a constant as  $z_1 \rightarrow \infty$ . But the only such function is a constant, so (applying the same argument to the dependence on all the  $z_i$ ) the expression in (10.3.7) is unique up to a constant.

### 10.7 Problem 10.10

On the bosonic side, the energy eigenvalue in terms of the momentum  $k_L$  and oscillator occupation numbers  $N_n$  is

$$L_0 = \frac{1}{2}k_L^2 + \sum_{n=1}^{\infty} nN_n. \quad (21)$$

On the fermionic side, there are two sets of oscillators, generated by the fields  $\psi$  and  $\bar{\psi}$ , and the energy is

$$L_0 = \sum_{n=1}^{\infty} \left( n - \frac{1}{2} \right) (N_n + \bar{N}_n). \quad (22)$$

We will denote states on the bosonic side by  $(k_L, N_1, N_2)$ , and on the fermionic side by  $(N_1 + \bar{N}_1, N_2 + \bar{N}_2, N_3 + \bar{N}_3)$ . We won't need any higher oscillators for this problem. On the fermionic side  $N_n + \bar{N}_n$  can take the values 0, 1, or 2, with degeneracy 1, 2, and 1 respectively. Here are the states with  $L_0 = 0, 1/2, \dots, 5/2$  on each side:

$L_0$	$(k_L, N_1, N_2)$	$(N_1 + \bar{N}_1, N_2 + \bar{N}_2, N_3 + \bar{N}_3)$
0	(0, 0, 0)	(0, 0, 0)
1/2	$(\pm 1, 0, 0)$	(1, 0, 0)
1	(0, 1, 0)	(2, 0, 0)
3/2	$(\pm 1, 1, 0)$	(0, 1, 0)
2	$(\pm 2, 0, 0)$	(1, 1, 0)
	(0, 2, 0)	
	(0, 0, 1)	
5/2	$(\pm 1, 2, 0)$	(2, 1, 0)
	$(\pm 1, 0, 1)$	(0, 0, 1)

(23)

### 10.8 Problem 10.11

We will use the first of the suggested methods. The OPE we need is

$$: e^{iH(z)} :: e^{inH(0)} := z^n : e^{i(n+1)H(0)} : + O(z^{n+1}). \quad (24)$$

Hence

$$\psi(z)F_n(0) = z^n F_{n+1}(0) + O(z^{n+1}). \quad (25)$$

It's easy to see that this is satisfied by

$$F_n =: \prod_{i=0}^{n-1} \frac{1}{i!} \partial^i \psi :. \quad (26)$$

The OPE of  $\psi(z)$  with  $F_n(0)$  is non-singular, but as we Taylor expand  $\psi(z)$ , all the terms vanish until the  $n$ th one because  $\psi$  is fermionic.  $F_n$  and  $e^{inH}$  obviously have the same fermion number

$n$ . Their dimensions work out nicely: for  $e^{inH}$  we have  $n^2/2$ ; for  $F_n$  we have  $n$   $\psi$ 's and  $n(n-1)/2$  derivatives, for a total of  $n/2 + n(n-1)/2 = n^2/2$ .

For  $e^{-inH}$  we obviously just replace  $\psi$  with  $\bar{\psi}$ .

### 10.9 Problem 10.14

We start with the NS sector. The most general massless state is

$$|\psi\rangle = (e \cdot \psi_{-1/2} + f\beta_{-1/2} + g\gamma_{-1/2})|0; k\rangle_{\text{NS}}, \quad (27)$$

where  $k^2 = 0$  (by the  $L_0$  condition) and  $|0; k\rangle_{\text{NS}}$  is annihilated by  $b_0$ . The BRST charge acting on  $|\psi\rangle$  is

$$\begin{aligned} Q_{\text{B}}|\psi\rangle &= (c_0L_0 + \gamma_{-1/2}G_{1/2}^{\text{m}} + \gamma_{1/2}G_{-1/2}^{\text{m}})|\psi\rangle \\ &= \sqrt{2\alpha'}(e \cdot k\gamma_{-1/2} + fk \cdot \psi_{-1/2})|0; k\rangle_{\text{NS}}. \end{aligned} \quad (28)$$

The  $L_0$  term of course vanishes, along with many others we have not indicated. For  $|\psi\rangle$  to be closed requires  $e \cdot k = f = 0$ . Furthermore exactness of (28) implies  $g \cong g + \sqrt{2\alpha'}e' \cdot k$  for any  $e'$  (so we might as well set  $g = 0$ ), while  $e \cong e + \sqrt{2\alpha'}f'k$  for any  $f'$ . We are left with the 8 transverse polarizations of a massless vector.

The R case is even easier: all of the work is done by the constraint (10.5.26), and none by the BRST operator. The general massless state is

$$|\psi\rangle = |u; k\rangle_{\text{R}}, \quad (29)$$

where  $u$  is a 10-dimensional Dirac spinor, and  $k^2 = 0$ .  $|\psi\rangle$  is defined to be annihilated by  $b_0$  and  $\beta_0$ , which implies that it is annihilated by  $G_0^g$ . According to (10.5.26), this in turn implies that it is annihilated by  $G_0^{\text{m}}$ , which is exactly the OCQ condition. The only thing left to check is that all the states satisfying these conditions are BRST closed, which follows more or less trivially from all the above conditions. Finally, since they are all closed, none of them can be exact.



## 11 Chapter 11

### 11.1 Problem 11.1

In order to establish the normalizations, we first calculate the  $T_F^+ T_F^-$  OPE:

$$e^{+i\sqrt{3}H(z)} e^{-i\sqrt{3}H(0)} \sim \frac{1}{z^3} + \frac{i\sqrt{3}\partial H(0)}{z^2} - \frac{3\partial H(0)\partial H(0)}{2z} + \frac{i2\sqrt{3}\partial^2 H(0)}{z}. \quad (1)$$

Since the third term is supposed to be  $2T_B(0)/z$ , where

$$T_B = -\frac{1}{2}\partial H\partial H, \quad (2)$$

we need

$$T_F^\pm = \sqrt{\frac{2}{3}} e^{\pm i\sqrt{3}H}. \quad (3)$$

It follows from the first term that  $c = 1$  (as we already knew), and from the second that

$$j = \frac{i}{\sqrt{3}}\partial H. \quad (4)$$

It is now straightforward to verify each of the OPEs in turn. The  $T_B T_F^\pm$  and  $T_B j$  OPEs are from Chapter 2, and show that  $T_F^\pm$  and  $j$  have weight  $\frac{3}{2}$  and 1 respectively. The fact that the  $T_F^\pm T_F^\pm$  OPE is non-singular was also shown in Chapter 2. For the  $j T_F^\pm$  OPE we have

$$j(z) T_F^\pm(0) = \pm \sqrt{\frac{2}{3}} \frac{e^{\pm i\sqrt{3}H(0)}}{z}. \quad (5)$$

Finally,

$$j(z)j(0) = \frac{1}{3z^2}. \quad (6)$$

### 11.2 Problem 11.3

See the last paragraph of section 11.2.

### 11.3 Problem 11.4

(a) Dividing the 32 left-moving fermions into two groups of 16, the untwisted theory contains 8 sectors:

$$\{(+++), (--+), (+--), (-+-)\} \times \{(\text{NS}, \text{NS}, \text{NS}), (\text{R}, \text{R}, \text{R})\}, \quad (7)$$

where the first symbol in each triplet corresponds to the first 16 left-moving fermions, the second to the second 16, and the third to the 8 right-moving fermions. If we now twist by  $\exp(\pi i F_1)$ , we project out  $(--+)$  and  $(-+-)$ , but project in the twisted sectors  $(\text{R}, \text{NS}, \text{NS})$  and  $(\text{NS}, \text{R}, \text{R})$ .

We again have 8 sectors:

$$\{(+++), (+--)\} \times \{(\text{NS}, \text{NS}, \text{NS}), (\text{R}, \text{R}, \text{R}), (\text{R}, \text{NS}, \text{NS}), (\text{NS}, \text{R}, \text{R})\}, \quad (8)$$

Let us find the massless spacetime bosons first, to establish the gauge group. These will have right-movers in the NS+ sector, so there are two possibilities, (NS+, NS+, NS+) and (R+, NS+, NS+), and the states are easily enumerated (primes refer to the second set of left-moving fermions):

$$\begin{aligned}
\alpha_{-1}^i \tilde{\psi}_{-1/2}^j |0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle & \quad \text{graviton, dilaton, antisymmetric tensor;} \\
\lambda_{-1/2}^{A'} \lambda_{-1/2}^{B'} \tilde{\psi}_{-1/2}^i |0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle & \quad \text{adjoint of } SO(16)'; \\
\lambda_{-1/2}^A \lambda_{-1/2}^B \tilde{\psi}_{-1/2}^i |0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle & \quad \text{adjoint of } SO(16); \\
\tilde{\psi}_{-1/2}^i |u_{\text{R}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle & \quad \mathbf{128} \text{ of } SO(16).
\end{aligned} \tag{9}$$

The last two sets combine to form an adjoint of  $E_8$ , so the gauge group is  $E_8 \times SO(16)$ . There are similarly two sectors containing massless spacetime fermions, (NS+, R+, R+) and (NS+, R-, R-) (the (R, R, R) states are all massive due to the positive normal-ordering constant for the left-moving R fermions); these will give respectively  $(\mathbf{8}, \mathbf{1}, \mathbf{128})$  and  $(\mathbf{8}', \mathbf{1}, \mathbf{128}')$  of  $SO(16)_{\text{spin}} \times E_8 \times SO(16)$ . Finally, the tachyon must be in (NS+, NS-, NS-); the only states are  $\lambda_{-1/2}^A |0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle$ , which transform as  $(\mathbf{1}, \mathbf{1}, \mathbf{8}_v)$ .

(b) Dividing the left-moving fermions into four groups of 8, if we twist the above theory by the total fermion number of the first and third groups, we get a total of 32 sectors:

$$\begin{aligned}
& \{(++++) , (----) , (+++-) , (----+)\} \times \\
& \{(\text{NS}, \text{NS}, \text{NS}, \text{NS}, \text{NS}), (\text{NS}, \text{R}, \text{R}, \text{NS}, \text{NS}), (\text{R}, \text{NS}, \text{R}, \text{NS}, \text{NS}), (\text{R}, \text{R}, \text{NS}, \text{NS}, \text{NS}), \\
& (\text{NS}, \text{NS}, \text{R}, \text{R}, \text{R}), (\text{NS}, \text{R}, \text{NS}, \text{R}, \text{R}), (\text{R}, \text{NS}, \text{NS}, \text{R}, \text{R}), (\text{R}, \text{R}, \text{R}, \text{R}, \text{R})\}. \tag{10}
\end{aligned}$$

Again we begin by listing the massless spacetime bosons, together with their  $SO(8)_{\text{spin}} \times SO(8)_1 \times SO(8)_2 \times SO(8)_3 \times SO(8)_4$  quantum numbers:

$$\begin{aligned}
(\text{NS+}, \text{NS+}, \text{NS+}, \text{NS+}, \text{NS+}) : & \\
\alpha_{-1}^i \tilde{\psi}_{-1/2}^j |0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle & \quad (\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) \\
\lambda_{-1/2}^{A_4} \lambda_{-1/2}^{B_4} \tilde{\psi}_{-1/2}^i |0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle & \quad (\mathbf{8}_v, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{28}) \\
\lambda_{-1/2}^{A_1} \lambda_{-1/2}^{B_1} \tilde{\psi}_{-1/2}^i |0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle & \quad (\mathbf{8}_v, \mathbf{28}, \mathbf{1}, \mathbf{1}, \mathbf{1}) \\
\lambda_{-1/2}^{A_2} \lambda_{-1/2}^{B_2} \tilde{\psi}_{-1/2}^i |0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle & \quad (\mathbf{8}_v, \mathbf{1}, \mathbf{28}, \mathbf{1}, \mathbf{1}) \\
\lambda_{-1/2}^{A_3} \lambda_{-1/2}^{B_3} \tilde{\psi}_{-1/2}^i |0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle & \quad (\mathbf{8}_v, \mathbf{1}, \mathbf{1}, \mathbf{28}, \mathbf{1}) \\
(\text{NS+}, \text{R+}, \text{R+}, \text{NS+}, \text{NS+}) : & \quad \tilde{\psi}_{-1/2}^i |0_{\text{NS}}, u_{\text{R}}, v_{\text{R}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle \quad (\mathbf{8}_v, \mathbf{1}, \mathbf{8}, \mathbf{8}, \mathbf{1}) \\
(\text{R+}, \text{NS+}, \text{R+}, \text{NS+}, \text{NS+}) : & \quad \tilde{\psi}_{-1/2}^i |u_{\text{R}}, 0_{\text{NS}}, v_{\text{R}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle \quad (\mathbf{8}_v, \mathbf{8}, \mathbf{1}, \mathbf{8}, \mathbf{1}) \\
(\text{R+}, \text{R+}, \text{NS+}, \text{NS+}, \text{NS+}) : & \quad \tilde{\psi}_{-1/2}^i |u_{\text{R}}, v_{\text{R}}, 0_{\text{NS}}, 0_{\text{NS}}, 0_{\text{NS}}\rangle \quad (\mathbf{8}_v, \mathbf{8}, \mathbf{8}, \mathbf{1}, \mathbf{1}). \tag{11}
\end{aligned}$$

The first set of states gives the dilaton, antisymmetric tensor, and graviton. The next set gives the  $SO(8)_4$  gauge bosons. The rest combine to give gauge bosons of  $SO(24)$ , once we perform triality rotations on  $SO(8)_{1,2,3}$  to turn the bispinors into bivectors.

The massless spacetime fermions are:

$$\begin{aligned}
(\mathbf{R}+, \mathbf{NS}+, \mathbf{NS}+, \mathbf{R}+, \mathbf{R}+) &: |u_{\mathbf{R}}, 0_{\mathbf{NS}}, 0_{\mathbf{NS}}, v_{\mathbf{R}}, w_{\mathbf{R}}\rangle & (\mathbf{8}, \mathbf{8}, \mathbf{1}, \mathbf{1}, \mathbf{8}) \\
(\mathbf{NS}+, \mathbf{R}+, \mathbf{NS}+, \mathbf{R}+, \mathbf{R}+) &: |0_{\mathbf{NS}}, u_{\mathbf{R}}, 0_{\mathbf{NS}}, v_{\mathbf{R}}, w_{\mathbf{R}}\rangle & (\mathbf{8}, \mathbf{1}, \mathbf{8}, \mathbf{1}, \mathbf{8}) \\
(\mathbf{NS}+, \mathbf{NS}+, \mathbf{R}+, \mathbf{R}+, \mathbf{R}+) &: |0_{\mathbf{NS}}, 0_{\mathbf{NS}}, u_{\mathbf{R}}, v_{\mathbf{R}}, w_{\mathbf{R}}\rangle & (\mathbf{8}, \mathbf{1}, \mathbf{1}, \mathbf{8}, \mathbf{8}).
\end{aligned} \tag{12}$$

The triality rotation again turns the  $SO(8)_{1,2,3}$  spinors into vectors, which combine into an  $SO(24)$  vector.

Finally, the tachyon is an  $SO(8)_4$  vector but is neutral under  $SO(24)$ :

$$(\mathbf{NS}+, \mathbf{NS}+, \mathbf{NS}+, \mathbf{NS}-, \mathbf{NS}-) : \lambda_{-1/2}^{A_4} |0_{\mathbf{NS}}, 0_{\mathbf{NS}}, 0_{\mathbf{NS}}, 0_{\mathbf{NS}}\rangle \quad (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8}_v). \tag{13}$$

#### 11.4 Problem 11.7

We wish to show that the state  $j_{-1}^a j_{-1}^a |0\rangle$  corresponds to the operator  $:jj(0):$ . Since  $j_{-1}^a |0\rangle$  clearly corresponds to  $j^a(0)$ , and  $j_{-1}^a = \oint dz/(2\pi i) j^a(z)/z$ , we have

$$:jj(0): = \oint \frac{dz}{2\pi i} \frac{j^a(z)j^a(0)}{z}, \tag{14}$$

where the contour goes around the origin. The contour integral picks out the  $z^0$  term in the Laurent expansion of  $j^a(z)j^a(0)$ , which is precisely (11.5.18).

We will now use this to prove the first line of (11.5.20), with  $z_1 = 0$  and  $z_3 = z$ . Using the Laurent expansion (11.5.2) we have

$$:jj(0): j^c(z) \cong \sum_{m=-\infty}^{\infty} \frac{1}{z^{m+1}} j_m^c j_{-1}^a j_{-1}^a |0\rangle. \tag{15}$$

Only three terms in this sum are potentially interesting; the rest are either non-singular (for  $m < 0$ ) or zero (for  $m > 2$ , since the total level of the state would be negative). In fact, the term with  $m = 2$  must also vanish, since (being at level 0) it can only be proportional to the ground state  $|0\rangle$ , leaving no room for the free Lie algebra index on (15); this can also be checked explicitly. For the other two terms we have:

$$j_0^c j_{-1}^a j_{-1}^a |0\rangle = i f^{cab} (j_{-1}^a j_{-1}^b + j_{-1}^b j_{-1}^a) |0\rangle = 0, \tag{16}$$

$$\begin{aligned}
j_1^c j_{-1}^a j_{-1}^a |0\rangle &= (\hat{k} \delta^{ca} + i f^{cab} j_0^b + j_{-1}^a j_1^c) j_{-1}^a |0\rangle \\
&= (2\hat{k} j_{-1}^c - f^{cab} f^{bad} j_{-1}^d) |0\rangle \\
&= (k + h(g)) \psi^2 j_{-1}^c |0\rangle.
\end{aligned} \tag{17}$$

The RHS of (17) clearly corresponds to the RHS of (11.5.20).

To check the  $TT$  OPE (11.5.24), we employ the same strategy, using the Laurent coefficients (11.5.26). The OPE will be the operator corresponding to

$$\frac{1}{(k+h(g))\psi^2} \left( \frac{1}{z^4}L_2 + \frac{1}{z^3}L_1 + \frac{1}{z^2}L_0 + \frac{1}{z}L_{-1} \right) j_{-1}^a j_{-1}^a |0\rangle; \quad (18)$$

terms with  $L_m$  are non-singular for  $m < -1$  and vanish for  $m > 2$ . Life is made much easier by the fact that  $j_m^b j_{-1}^a j_{-1}^a |0\rangle = 0$  for  $m = 0$  and  $m > 1$ , and the value for  $m = 1$  is given by (17) above. Thus:

$$\begin{aligned} L_2 j_{-1}^a j_{-1}^a |0\rangle &= \frac{1}{(k+h(g))\psi^2} j_1^b j_1^b j_{-1}^a j_{-1}^a |0\rangle = j_1^b j_{-1}^b |0\rangle = \hat{k} \dim(g) |0\rangle, \\ L_1 j_{-1}^a j_{-1}^a |0\rangle &= 0, \\ L_0 j_{-1}^a j_{-1}^a |0\rangle &= \frac{2}{(k+h(g))\psi^2} j_{-1}^b j_1^b j_{-1}^a j_{-1}^a |0\rangle = 2 j_{-1}^b j_{-1}^b |0\rangle, \\ L_{-1} j_{-1}^a j_{-1}^a |0\rangle &= \frac{2}{(k+h(g))\psi^2} j_{-2}^b j_1^b j_{-1}^a j_{-1}^a |0\rangle = 2 j_{-2}^b j_{-1}^b |0\rangle. \end{aligned} \quad (19)$$

All of these states are easily translated back into operators, the only slightly non-trivial one being the last. From the Laurent expansion we see that the state corresponding to  $\partial T_B^s(0)$  is indeed  $L_{-3}|0\rangle = 2/((k+h(g))\psi^2)j_{-2}^b j_{-1}^b |0\rangle$ . Thus we have precisely the OPE (11.5.24).

### 11.5 Problem 11.8

The operator  $:jj(0):$  is defined to be the  $z^0$  term in the Laurent expansion of  $j^a(z)j^a(0)$ . First let us calculate the contribution from a single current  $i\lambda^A\lambda^B$  ( $A \neq B$ ):

$$\begin{aligned} i\lambda^A(z)\lambda^B(z)i\lambda^A(0)\lambda^B(0) &= \lambda^A(z)\lambda^A(0)\lambda^B(z)\lambda^B(0) \quad (\text{no sum}) \\ &=: \lambda^A(z)\lambda^A(0)\lambda^B(z)\lambda^B(0) : + \frac{1}{z} : \lambda^A(z)\lambda^A(0) : + \frac{1}{z} : \lambda^B(z)\lambda^B(0) : + \frac{1}{z^2}. \end{aligned} \quad (20)$$

Clearly the order  $z^0$  term is  $:\partial\lambda^A\lambda^A: + :\partial\lambda^B\lambda^B:$ . Summing over all  $A$  and  $B$  with  $B \neq A$  double counts the currents, so we divide by 2:

$$:jj: = (n-1) : \partial\lambda^A\lambda^A : \quad (\text{sum}). \quad (21)$$

Finally, we have  $k = 1$ ,  $h(SO(n)) = n - 2$ , and  $\psi^2 = 2$ , so

$$T_B^s = \frac{1}{2} : \partial\lambda^A\lambda^A : \quad (\text{sum}). \quad (22)$$

### 11.6 Problem 11.9

In the notation of (11.6.5) and (11.6.6), the lattice  $\Gamma$  is  $\Gamma_{22,6}$ , i.e. the set of points of the form

$$(n_1, \dots, n_{28}) \quad \text{or} \quad (n_1 + \frac{1}{2}, \dots, n_{28} + \frac{1}{2}),$$

$$\sum_i n_i \in 2Z \tag{23}$$

for any integers  $n_i$ . It will be convenient to divide  $\Gamma$  into two sublattices,  $\Gamma = \Gamma_1 \cup \Gamma_2$  where

$$\Gamma_1 = \{(n_1, \dots, n_{28}) : \sum n_i \in 2Z\},$$

$$\Gamma_2 = \Gamma_1 + l_0, \quad l_0 \equiv (\frac{1}{2}, \dots, \frac{1}{2}). \tag{24}$$

Evenness of  $l \in \Gamma_1$  follows from the fact that the number of odd  $n_i$  must be even, implying

$$l \circ l = \sum_{i=1}^{22} n_i^2 - \sum_{i=23}^{28} n_i^2 \in 2Z. \tag{25}$$

Evenness of  $l + l_0 \in \Gamma_2$  follows from the same fact:

$$(l + l_0) \circ (l + l_0) = l \circ l + 2l_0 \circ l + l_0 \circ l_0 = l \circ l + \sum_{i=1}^{22} n_i - \sum_{i=23}^{28} n_i + 4 \in 2Z. \tag{26}$$

Evenness implies integrality, so  $\Gamma \subset \Gamma^*$ . It's easy to see that the dual lattice to  $\Gamma_1$  is  $\Gamma_1^* = Z^{28} \cup (Z^{28} + l_0) \supset \Gamma$ . But to be dual for example to  $l_0$  requires a vector to have an even number of odd  $n_i$ s, so  $\Gamma^* = \Gamma$ .

To find the gauge bosons we need to find the lattice vectors satisfying (11.6.15). But these are obviously the root vectors of  $SO(44)$ . In addition there are the 22 gauge bosons with vertex operators  $\partial X^m \tilde{\psi}^\mu$ , providing the Cartan generators to fill out the adjoint representation of  $SO(44)$ , and the 6 gauge bosons with vertex operators  $\partial X^\mu \tilde{\psi}^m$ , generating  $U(1)^6$ .

### 11.7 Problem 11.12

It seems to me that both the statement of the problem and the derivation of the Hagedorn temperature for the bosonic string (Vol. I, pp. 320-21) are misleading. Equation (7.3.20) is not the correct one to use to find the asymptotic density of states of a string theory, since it does not take into account the level matching constraint in the physical spectrum. It's essentially a matter of luck that Polchinski ends up with the right Hagedorn temperature, (9.8.13).

Taking into account level matching we have  $n(m) = n_L(m)n_R(m)$ . To find  $n_L$  and  $n_R$  we treat the left-moving and right-moving CFTs separately, and include only the physical parts of the spectrum, that is, neglect the ghosts and the timelike and longitudinal oscillators. Then we have, as in (9.8.11),

$$\sum_{m^2} n_L(m) e^{-\pi\alpha' m^2 l/2} \sim e^{\pi c/(12l)}, \tag{27}$$

implying

$$n_L(m) \sim e^{\pi m \sqrt{\alpha' c/6}}. \quad (28)$$

Similarly for the right-movers, giving

$$n(m) \sim e^{\pi m \sqrt{\alpha'/6}(\sqrt{c} + \sqrt{\tilde{c}})}. \quad (29)$$

The Hagedorn temperature is then given by

$$T_H^{-1} = \pi \sqrt{\alpha'} \left( \sqrt{\frac{c}{6}} + \sqrt{\frac{\tilde{c}}{6}} \right). \quad (30)$$

For the type I and II strings, this gives

$$T_H = \frac{1}{2\pi\sqrt{2\alpha'}}, \quad (31)$$

while for the heterotic theories we have

$$T_H = \frac{1}{\pi\sqrt{\alpha'}} \left( 1 - \frac{1}{\sqrt{2}} \right). \quad (32)$$

The Hagedorn temperature for an open type I string is the same, (31), as for the closed string, since  $n_{\text{open}}(m) = n_L(2m)$ .

There is an interesting point that we glossed over above. The RHS of (27) is obtained by doing a modular transformation  $l \rightarrow 1/l$  on the torus partition function with  $\tau = il$ . Then for small  $l$  only the lowest-lying state in the theory contributes. We implicitly took the lowest-lying state to be the vacuum, corresponding to the unit operator. However, that state is projected out by the GSO projection in all of the above theories, else it would give rise to a tachyon. So should we really consider it? To see that we should, let us derive (27) more carefully, for example in the case of the left-movers of the type II string. The GSO projection there is  $(-1)^F = -1$ , where  $F$  is the worldsheet fermion number of the transverse fermions (*not* including the ghosts). The partition function from which we will extract  $n_L(m)$ , the number of projected-in states, is

$$Z(il) = \sum_{i \in \text{R,NS}} q^{h_i - c/24} \frac{1}{2} (1 - (-1)^{F_i}) = \sum_{m^2} n_L(m) e^{-\pi \alpha' m^2 l/2}. \quad (33)$$

This partition function corresponds to a path integral on the torus in which we sum over all four spin structures, with minus signs when the fermions are periodic in the  $\sigma_2$  (“time”) direction. Upon doing the modular integral, this minus sign corresponds to giving a minus sign to R sector states. But the sum on R and NS sectors in (33) means that we now project onto states with  $(-1)^F = 1$ :

$$Z(il) = \sum_{i \in \text{R,NS}} e^{-2\pi/l(h_i - c/24)} (-1)^{\alpha_i} \frac{1}{2} (1 + (-1)^{F_i}) \sim e^{\pi c/(12l)}. \quad (34)$$

We see that the ground state, with  $h = 0$ , is indeed projected in, and therefore dominates in the limit  $l \rightarrow 0$ .

## 12 Chapter 13

### 12.1 Problem 13.2

An open string with ends attached to D $p$ -branes is T-dual to an open type I string. An open string with both ends attached to the same D $p$ -brane and zero winding number is T-dual to an open type I string with zero momentum in the  $9 - p$  dualized directions, and Chan-Paton factor of the form

$$t^a = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \otimes \text{diag}(1, 0, \dots, 0). \quad (1)$$

Four open strings attached to the same D $p$ -brane are T-dual to four open type I strings with zero momentum in the  $9 - p$  dualized directions and the same Chan-Paton factor (1). The scattering amplitude for four gauge boson open string states was calculated in section 12.4. Using

$$\text{Tr}(t^a)^4 = \frac{1}{2}, \quad (2)$$

the result (12.4.22) becomes in this case

$$S(k_i, e_i) = -8ig_{\text{YM}}^2 \alpha'^2 (2\pi)^{10} \delta^{10} \left( \sum_i k_i \right) K(k_i, e_i) \left( \frac{\Gamma(-\alpha' s) \Gamma(-\alpha' u)}{\Gamma(1 - \alpha' s - \alpha' u)} + 2 \text{ permutations} \right). \quad (3)$$

The kinematic factor  $K$  is written in three different ways in (12.4.25) and (12.4.26), and we won't bother to reproduce it here. Since the momenta  $k_i$  all have vanishing components in the  $9 - p$  dualized directions, the amplitude becomes,

$$-8ig_{(p+1), \text{YM}}^2 \alpha'^2 V_{9-p}^2 (2\pi)^{p+1} \delta^{p+1} \left( \sum_i k_i \right) \times K(k_i, e_i) \left( \frac{\Gamma(-\alpha' s) \Gamma(-\alpha' u)}{\Gamma(1 - \alpha' s - \alpha' u)} + 2 \text{ permutations} \right), \quad (4)$$

where  $V_{9-p}$  is the volume of the transverse space, and we have used (13.3.29). But in order to get the proper  $(p + 1)$ -dimensional scattering amplitude, we must renormalize the wave function of each string (which is spread out uniformly in the transverse space) by a factor of  $\sqrt{V_{9-p}}$ :

$$S'(k_i, e_i) = -8ig_{(p+1), \text{YM}}^2 \alpha'^2 (2\pi)^{p+1} \delta^{p+1} \left( \sum_i k_i \right) \times K(k_i, e_i) \left( \frac{\Gamma(-\alpha' s) \Gamma(-\alpha' u)}{\Gamma(1 - \alpha' s - \alpha' u)} + 2 \text{ permutations} \right). \quad (5)$$

Finally, using (13.3.30) and (13.3.28), we can write the dimensionally reduced type I Yang-Mills coupling  $g_{(p+1), \text{YM}}$ , in terms of the coupling  $g_{\text{D}p}$  on the brane:

$$g_{(p+1), \text{YM}}^2 = g_{\text{D}p, \text{SO}(32)}^2 = 2g_{\text{D}p}^2, \quad (6)$$

so

$$S'(k_i, e_i) = -16ig_{\text{D}p}^2 \alpha'^2 (2\pi)^{p+1} \delta^{p+1} \left( \sum_i k_i \right) \times K(k_i, e_i) \left( \frac{\Gamma(-\alpha' s) \Gamma(-\alpha' u)}{\Gamma(1 - \alpha' s - \alpha' u)} + 2 \text{ permutations} \right). \quad (7)$$

(One can also use (13.3.25) to write  $g_{\text{D}p}$  in terms of the string coupling  $g$ .)

## 12.2 Problem 13.3

(a) By equations (B.1.8) and (B.1.10),

$$\Gamma^{2a} \Gamma^{2a+1} = -2i S_a, \quad (8)$$

where  $a = 1, 2, 3, 4$ , so

$$\beta^{2a} \beta^{2a+1} = 2i S_a. \quad (9)$$

If we define

$$\beta \equiv \beta^1 \beta^2 \beta^3, \quad (10)$$

and label the D4-branes extended in the (6,7,8,9), (4,5,8,9), and (4,5,6,7) directions by the subscripts 2, 3, and 4 respectively, then

$$\beta_2^\perp = \beta^1 \beta^2 \beta^3 \beta^4 \beta^5 = 2i S_2 \beta \quad (11)$$

and similarly

$$\beta_3^\perp = 2i S_3 \beta, \quad \beta_4^\perp = 2i S_4 \beta. \quad (12)$$

The supersymmetries preserved by brane  $a$  ( $a = 2, 3, 4$ ) are

$$Q_s + (\beta_a^\perp \tilde{Q})_s = Q_s + 2i s_a (\beta \tilde{Q})_s, \quad (13)$$

so for a supersymmetry to be unbroken by all three branes simply requires  $s_2 = s_3 = s_4$ . Taking into account the chirality condition  $\Gamma = +1$  on  $Q_s$ , there are four unbroken supersymmetries:

$$\begin{aligned} & Q_{(+++++)} + i(\beta \tilde{Q})_{(+++++)}, \\ & Q_{(---++)} + i(\beta \tilde{Q})_{(---++)}, \\ & Q_{(+-----)} - i(\beta \tilde{Q})_{(+-----)}, \\ & Q_{(-+----)} - i(\beta \tilde{Q})_{(-+----)}. \end{aligned} \quad (14)$$



(b) For the D0-brane,

$$\beta_{D0}^\perp = \beta^1 \beta^2 \beta^3 \beta^4 \beta^5 \beta^6 \beta^7 \beta^8 \beta^9 = -8i S_2 S_3 S_4 \beta, \tag{15}$$

so the unbroken supersymmetries are of the form,

$$Q_s + (\beta_{D0}^\perp \tilde{Q})_s = Q_s - 8i s_2 s_3 s_4 (\beta \tilde{Q})_s. \tag{16}$$

The signs in all four previously unbroken supersymmetries (14) are just wrong to remain unbroken by the D0-brane, so that this configuration preserves no supersymmetry.

(c) Let us make a brane scan of the original configuration:

	1	2	3	4	5	6	7	8	9
D4 <sub>2</sub>	D	D	D	D	D	N	N	N	N
D4 <sub>3</sub>	D	D	D	N	N	D	D	N	N
D4 <sub>4</sub>	D	D	D	N	N	N	N	D	D
D0	D	D	D	D	D	D	D	D	D

There are nine distinct T-dualities that can be performed in the 4, 5, 6, 7, 8, and 9 directions, up to the symmetries  $4 \leftrightarrow 5$ ,  $6 \leftrightarrow 7$ ,  $8 \leftrightarrow 9$ , and  $(45) \leftrightarrow (67) \leftrightarrow (89)$ . They result in the following brane content:

T-dualized directions	$(p_1, p_2, p_3, p_4)$
4	(5, 3, 3, 1)
4, 5	(6, 2, 2, 2)
4, 6	(4, 4, 2, 2)
4, 5, 6	(5, 3, 1, 3)
4, 5, 6, 7	(4, 4, 0, 4)
4, 6, 8	(3, 3, 3, 3)
4, 5, 6, 8	(4, 2, 2, 4)
4, 5, 6, 7, 8	(3, 3, 1, 5)
4, 5, 6, 7, 8, 9	(2, 2, 2, 6)

Further T-duality in one, two, or all three of the 1, 2, and 3 directions will turn any of these configurations into  $(p_1 + 1, p_2 + 1, p_3 + 1, p_4 + 1)$ ,  $(p_1 + 2, p_2 + 2, p_3 + 2, p_4 + 2)$ , and  $(p_1 + 3, p_2 + 3, p_3 + 3, p_4 + 3)$  respectively.

T-dualizing at general angles to the coordinate axes will result in combinations of the above configurations for the directions involved, with the smaller-dimensional brane in each column being replaced by a magnetic field on the larger-dimensional brane.

### 12.3 Problem 13.4

(a) Let the D2-brane be extended in the 8 and 9 directions, and let it be separated from the D0-brane in the 1 direction by a distance  $y$ . T-dualizing this configuration in the 2, 4, 6, and 8

directions yields 2 D4-branes, the first (from the D2-brane) extended in the 2, 4, 6, and 9 directions, and the second (from the D0-brane) in the 2, 4, 6, and 8 directions. This is in the class of D4-brane configurations studied in section 13.4; in our case the angles defined there take the values

$$\phi_1 = \phi_2 = \phi_3 = 0, \quad \phi_4 = \frac{\pi}{2}, \quad (17)$$

and therefore (according to (13.4.22)),

$$\phi'_1 = \phi'_4 = \frac{\pi}{4}, \quad \phi'_2 = \phi'_3 = -\frac{\pi}{4}. \quad (18)$$

For the three directions in which the D4-branes are parallel, we must make the substitution (13.4.25) (without the exponential factor, since we have chosen the separation between the D0- and D2-branes to vanish in those directions, but with a factor  $-i$ , to make the potential real and attractive). The result is

$$V(y) = - \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{-1/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{i\vartheta_{11}^4(it/4, it)}{\vartheta_{11}(it/2, it)\eta^9(it)}. \quad (19)$$

Alternatively, using the modular transformations (7.4.44b) and (13.4.18b),

$$V(y) = -\frac{1}{\sqrt{8\pi^2\alpha'}} \int_0^\infty dt t^{3/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \frac{\vartheta_{11}^4(1/4, i/t)}{\vartheta_{11}(1/2, i/t)\eta^9(i/t)}. \quad (20)$$

**(b)** For the field theory calculation we lean heavily on the similar calculation done in section 8.7, adapting it to  $D = 10$ . Polchinski employs the shifted dilaton

$$\tilde{\Phi} = \Phi - \Phi_0, \quad (21)$$

whose expectation value vanishes, and the Einstein metric

$$\tilde{G} = e^{-\tilde{\Phi}/2} G; \quad (22)$$

their propagators are given in (8.7.23):

$$\langle \tilde{\Phi}\tilde{\Phi}(k) \rangle = -\frac{2i\kappa^2}{k^2}, \quad (23)$$

$$\langle h_{\mu\nu}h_{\sigma\rho}(k) \rangle = -\frac{2i\kappa^2}{k^2} \left( \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{4}\eta_{\mu\nu}\eta_{\sigma\rho} \right), \quad (24)$$

where  $h = \tilde{G} - \eta$ . The D-brane action (13.3.14) expanded for small values of  $\tilde{\Phi}$  and  $h$  is

$$S_p = -\tau_p \int d^{p+1}\xi \left( \frac{p-3}{4}\tilde{\Phi} + \frac{1}{2}h_a{}^a \right), \quad (25)$$

where the trace on  $h$  is taken over directions tangent to the brane. From the point of view of the supergravity, the D-brane is thus a source for  $\tilde{\Phi}$ ,

$$J_{\tilde{\Phi},p}(X) = \frac{3-p}{4}\tau_p \delta^{9-p}(X_\perp - X'_\perp), \quad (26)$$

and for  $h$ ,

$$J_{h,p}^{\mu\nu}(X) = -\frac{1}{2}\tau_p e_p^{\mu\nu} \delta^{9-p}(X_\perp - X'_\perp), \quad (27)$$

where  $X'_\perp$  is the position of the brane in the transverse coordinates, and  $e_p^{\mu\nu}$  is  $\eta^{\mu\nu}$  in the directions parallel to the brane and 0 otherwise. In momentum space the sources are

$$\tilde{J}_{\tilde{\Phi},p}(k) = \frac{3-p}{4}\tau_p(2\pi)^{p+1}\delta^{p+1}(k_\parallel)e^{ik_\perp \cdot X'_\perp}, \quad (28)$$

$$\tilde{J}_{h,p}^{\mu\nu}(k) = -\frac{1}{2}\tau_p e_p^{\mu\nu}(2\pi)^{p+1}\delta^{p+1}(k_\parallel)e^{ik_\perp \cdot X'_\perp}. \quad (29)$$

Between the D0-brane, located at the origin of space, and the D2-brane, extended in the 8 and 9 directions and located in the other directions at the point

$$(X'_1, X'_2, X'_3, X'_4, X'_5, X'_6, X'_7) = (y, 0, 0, 0, 0, 0, 0), \quad (30)$$

the amplitude for dilaton exchange is

$$\begin{aligned} \mathcal{A}_{\tilde{\Phi}} &= -\int \frac{d^{10}k}{(2\pi)^{10}} \tilde{J}_{\tilde{\Phi},0}(k) \langle \tilde{\Phi} \tilde{\Phi}(k) \rangle \tilde{J}_{\tilde{\Phi},2}(-k) \\ &= i\frac{3}{8}\tau_0\tau_2\kappa^2 \int \frac{d^{10}k}{(2\pi)^{10}} 2\pi\delta(k_0) \frac{1}{k^2} (2\pi)^3 \delta^3(k_0, k_8, k_9) e^{ik_1 y} \\ &= iT\frac{3}{8}\tau_0\tau_2\kappa^2 \int \frac{d^7k}{(2\pi)^7} \frac{e^{iky}}{k^2} \\ &= iT\frac{3}{8}\tau_0\tau_2\kappa^2 G_7(y), \end{aligned} \quad (31)$$

where  $G_7$  is the 7-dimensional massless scalar Green function. We divide the amplitude by  $-iT$  to obtain the static potential due to dilaton exchange:

$$V_{\tilde{\Phi}}(y) = -\frac{3}{8}\tau_0\tau_2\kappa^2 G_7(y). \quad (32)$$

The calculation for the graviton exchange is similar, the only difference being that the numerical factor  $(1/4)2(3/4)$  is replaced by

$$\frac{1}{2}e_0^{\mu\nu} 2 \left( \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{4}\eta_{\mu\nu}\eta_{\sigma\rho} \right) \frac{1}{2}e_2^{\sigma\rho} = \frac{5}{8}. \quad (33)$$

The total potential between the D0-brane and D2-brane is therefore

$$V(y) = \tau_0\tau_2\kappa^2 G_7(y) = -\pi(4\pi^2\alpha')^2 G_7(y), \quad (34)$$

where we have applied (13.3.4). As expected, gravitation and dilaton exchange are both attractive forces.

In the large- $y$  limit of (20), the integrand becomes very small except where  $t$  is very small. The ratio of modular functions involved in the integrand is in fact finite in the limit  $t \rightarrow 0$ ,

$$\lim_{t \rightarrow 0} \frac{\vartheta_{11}^4(1/4, i/t)}{\vartheta_{11}(1/2, i/t)\eta^9(i/t)} = 2, \quad (35)$$

(according to *Mathematica*), so that the first term in the asymptotic expansion of the potential in  $1/y$  is

$$\begin{aligned} V(r) &\approx -\frac{1}{\sqrt{2\pi^2\alpha'}} \int_0^\infty dt t^{3/2} \exp\left(-\frac{ty^2}{2\pi\alpha'}\right) \\ &= -\pi^{-1/2}(2\pi\alpha')^2 \Gamma\left(\frac{5}{2}\right) y^{-5} \\ &= -\pi(4\pi^2\alpha')^2 G_7(y), \end{aligned} \quad (36)$$

in agreement with (34).

## 12.4 Problem 13.12

The tension of the  $(p_i, q_i)$ -string is (13.6.3)

$$\tau_{(p_i, q_i)} = \frac{\sqrt{p_i^2 + q_i^2/g^2}}{2\pi\alpha'}. \quad (37)$$

Let the three strings sit in the  $(X^1, X^2)$  plane. If the angle string  $i$  makes with the  $X^1$  axis is  $\theta_i$ , then the force it exerts on the junction point is

$$(F_i^1, F_i^2) = \frac{1}{2\pi\alpha'} (\cos \theta_i \sqrt{p_i^2 + q_i^2/g^2}, \sin \theta_i \sqrt{p_i^2 + q_i^2/g^2}). \quad (38)$$

If we orient each string at the angle

$$\cos \theta_i = \frac{p_i}{\sqrt{p_i^2 + q_i^2/g^2}}, \quad \sin \theta_i = \frac{q_i/g}{\sqrt{p_i^2 + q_i^2/g^2}}, \quad (39)$$

then the total force exerted on the junction point is

$$\frac{1}{2\pi\alpha'} \sum_{i=1}^3 (p_i, q_i/g), \quad (40)$$

which vanishes if  $\sum p_i = \sum q_i = 0$ . This is the unique stable configuration, up to rotations and reflections.

The supersymmetry algebra for a static  $(p, q)$  string extended in the  $X^i$  direction is (13.6.1)

$$\frac{1}{2L} \left\{ \begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{bmatrix}, \begin{bmatrix} Q_\beta^\dagger \\ \tilde{Q}_\beta^\dagger \end{bmatrix} \right\} = \tau_{(p,q)} \delta_{\alpha\beta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{(\Gamma^0 \Gamma^i)_{\alpha\beta}}{2\pi\alpha'} \begin{bmatrix} p & q/g \\ q/g & -p \end{bmatrix}. \quad (41)$$

Defining

$$u \equiv \frac{p}{\sqrt{p^2 + q^2/g^2}}, \quad U \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+u} & \sqrt{1-u} \\ -\sqrt{1-u} & \sqrt{1+u} \end{bmatrix}, \quad (42)$$

we can use  $U$  to diagonalize the matrix on the RHS of (41):

$$\frac{1}{2L\tau_{(p,q)}} \left\{ U \begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{bmatrix}, \begin{bmatrix} Q_\beta^\dagger & \tilde{Q}_\beta^\dagger \end{bmatrix} U^T \right\} = \begin{bmatrix} (I_{16} + \Gamma^0 \Gamma^i)_{\alpha\beta} & 0 \\ 0 & (I_{16} - \Gamma^0 \Gamma^i)_{\alpha\beta} \end{bmatrix}. \quad (43)$$

The top row of this  $2 \times 2$  matrix equation tells us that, in a basis in spinor space in which  $\Gamma^0 \Gamma^i$  is diagonal, the supersymmetry generator

$$\sqrt{1+u}Q_\alpha + \sqrt{1-u}\tilde{Q}_\alpha \quad (44)$$

annihilates this state if  $(I_{16} + \Gamma^0 \Gamma^i)_{\alpha\alpha} = 0$ . We can use  $(I_{16} - \Gamma^0 \Gamma^i)$  to project onto this eight-dimensional subspace, yielding eight supersymmetries that leave this state invariant:

$$\left[ (I_{16} - \Gamma^0 \Gamma^i)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q}) \right]_\alpha. \quad (45)$$

The other eight unbroken supersymmetries are given by the bottom row of (43), after projecting onto the subspace annihilated by  $(I_{16} - \Gamma^0 \Gamma^i)$ :

$$\left[ (I_{16} + \Gamma^0 \Gamma^i)(-\sqrt{1-u}Q + \sqrt{1+u}\tilde{Q}) \right]_\alpha. \quad (46)$$

Now let us suppose that the string is aligned in the direction (39), which depends on  $p$  and  $q$ . We will show that eight of the sixteen unbroken supersymmetries do not depend on  $p$  or  $q$ , and therefore any configuration of  $(p, q)$  strings that all obey (39) will leave these eight unbroken. If the string is aligned in the direction (39), then

$$\Gamma^i = \frac{p}{\sqrt{p^2 + q^2/g^2}} \Gamma^1 + \frac{q/g}{\sqrt{p^2 + q^2/g^2}} \Gamma^2 = u\Gamma^1 + \sqrt{1-u^2}\Gamma^2. \quad (47)$$

Our first set of unbroken supersymmetries (45) becomes

$$\left[ (I_{16} - u\Gamma^0 \Gamma^1 - \sqrt{1-u^2}\Gamma^0 \Gamma^2)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q}) \right]_\alpha. \quad (48)$$

We work in a basis of eigenspinors of the operators  $S_a$  defined in (B.1.10). In this basis  $\Gamma^0 \Gamma^1 = 2S_0$ , while

$$\Gamma^0 \Gamma^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes I_2 \otimes I_2 \otimes I_2. \quad (49)$$

We can divide the sixteen values of the spinor index  $\alpha$  into four groups of four according to the

eigenvalues of  $S_0$  and  $S_1$ :

$$\left[ (I_{16} - u\Gamma^0\Gamma^1 - \sqrt{1-u^2}\Gamma^0\Gamma^2)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q}) \right]_{(++s_2s_3s_4)} \quad (50)$$

$$= (1-u)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q})_{(++s_2s_3s_4)} \\ + \sqrt{1-u^2}(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q})_{(--s_2s_3s_4)},$$

$$\left[ (I_{16} - u\Gamma^0\Gamma^1 - \sqrt{1-u^2}\Gamma^0\Gamma^2)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q}) \right]_{(++s_2s_3s_4)} \quad (51)$$

$$= (1+u)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q})_{(--s_2s_3s_4)} \\ + \sqrt{1-u^2}(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q})_{(++s_2s_3s_4)},$$

$$\left[ (I_{16} - u\Gamma^0\Gamma^1 - \sqrt{1-u^2}\Gamma^0\Gamma^2)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q}) \right]_{(+s_2s_3s_4)} \quad (52)$$

$$= (1-u)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q})_{(+s_2s_3s_4)} \\ - \sqrt{1-u^2}(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q})_{(+s_2s_3s_4)},$$

$$\left[ (I_{16} - u\Gamma^0\Gamma^1 - \sqrt{1-u^2}\Gamma^0\Gamma^2)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q}) \right]_{(+s_2s_3s_4)} \quad (53)$$

$$= (1+u)(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q})_{(+s_2s_3s_4)} \\ - \sqrt{1-u^2}(\sqrt{1+u}Q + \sqrt{1-u}\tilde{Q})_{(+s_2s_3s_4)}.$$

(The indexing by  $s_2, s_3, s_4$  is somewhat redundant, since the chirality condition on both  $Q$  and  $\tilde{Q}$  implies the restriction  $8s_2s_3s_4 = 1$  in the case of (50) and (51), and  $8s_2s_3s_4 = -1$  in the case of (52) and (53).) It easy to see that (50) and (51) differ only by a factor of  $\sqrt{(1-u)/(1+u)}$ , and (52) and (53) similarly by a factor of  $-\sqrt{(1-u)/(1+u)}$ , so (50) and (52) alone are sufficient to describe the eight independent supersymmetry generators in this sector. In the other sector, given by (46), there is a similar repetition of generators, and the eight independent generators are

$$\left[ (I_{16} + u\Gamma^0\Gamma^1 + \sqrt{1-u^2}\Gamma^0\Gamma^2)(-\sqrt{1-u}Q + \sqrt{1+u}\tilde{Q}) \right]_{(++s_2s_3s_4)} \quad (54)$$

$$= (1+u)(-\sqrt{1-u}Q + \sqrt{1+u}\tilde{Q})_{(++s_2s_3s_4)} \\ - \sqrt{1-u^2}(-\sqrt{1-u}Q + \sqrt{1+u}\tilde{Q})_{(--s_2s_3s_4)},$$

$$\left[ (I_{16} + u\Gamma^0\Gamma^1 + \sqrt{1-u^2}\Gamma^0\Gamma^2)(-\sqrt{1-u}Q + \sqrt{1+u}\tilde{Q}) \right]_{(+s_2s_3s_4)} \quad (55)$$

$$= (1+u)(-\sqrt{1-u}Q + \sqrt{1+u}\tilde{Q})_{(+s_2s_3s_4)} \\ + \sqrt{1-u^2}(-\sqrt{1-u}Q + \sqrt{1+u}\tilde{Q})_{(+s_2s_3s_4)}.$$

Are there linear combinations of the generators (50), (52), (54), and (55) that are independent of  $u$ , and therefore unbroken no matter what the values of  $p$  and  $q$ ? Indeed, by dividing (50) by  $2\sqrt{1-u}$  and (54) by  $2\sqrt{1+u}$  and adding them, we come up with four such generators:

$$\tilde{Q}_{(++s_2s_3s_4)} + Q_{(--s_2s_3s_4)}. \quad (56)$$

Four more are found by dividing (52) by  $2\sqrt{1-u}$  and (55) by  $\sqrt{1+u}$ :

$$\tilde{Q}_{(+s_2s_3s_4)} - Q_{(-s_2s_3s_4)}. \quad (57)$$

As promised, one quarter of the original supersymmetries leave the entire configuration described in the first paragraph of this solution invariant.

## 13 Chapter 14

### 13.1 Problem 14.1

The excitation on the F-string will carry some energy (per unit length)  $p_0$ , and momentum (per unit length) in the 1-direction  $p_1$ . Since the string excitations move at the speed of light, left-moving excitations have  $p_0 = -p_1$ , while right-moving excitations have  $p_0 = p_1$ . The supersymmetry algebra (13.2.9) for this string is similar to (13.6.1), with additional terms for the excitation:

$$\begin{aligned} & \frac{1}{2L} \left\{ \left[ \begin{array}{c} Q_\alpha \\ \tilde{Q}_\alpha \end{array} \right], \left[ Q_\beta^\dagger \quad \tilde{Q}_\beta^\dagger \right] \right\} \\ & = \frac{1}{2\pi\alpha'} \begin{bmatrix} (\delta + \Gamma^0\Gamma^1)_{\alpha\beta} & 0 \\ 0 & (\delta - \Gamma^0\Gamma^1)_{\alpha\beta} \end{bmatrix} + (p_0\delta_{\alpha\beta} + p_1(\Gamma^0\Gamma^1)_{\alpha\beta}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \quad (1)$$

The first term on the RHS vanishes for those supersymmetries preserved by the unexcited F-string, namely  $Q$ s for which  $\Gamma^0\Gamma^1 = -1$  and  $\tilde{Q}$ s for which  $\Gamma^0\Gamma^1 = 1$ . The second term thus also vanishes (making the state BPS) for the  $Q$ s if the excitation is left-moving, and for the  $\tilde{Q}$ s if the excitation is right-moving.

For the D-string the story is almost the same, except that the first term above is different:

$$\begin{aligned} & \frac{1}{2L} \left\{ \left[ \begin{array}{c} Q_\alpha \\ \tilde{Q}_\alpha \end{array} \right], \left[ Q_\beta^\dagger \quad \tilde{Q}_\beta^\dagger \right] \right\} \\ & = \frac{1}{2\pi\alpha'g} \begin{bmatrix} \delta_{\alpha\beta} & (\Gamma^0\Gamma^1)_{\alpha\beta} \\ (\Gamma^0\Gamma^1)_{\alpha\beta} & \delta_{\alpha\beta} \end{bmatrix} + (p_0\delta_{\alpha\beta} + p_1(\Gamma^0\Gamma^1)_{\alpha\beta}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \quad (2)$$

When diagonalized, the first term yields the usual preserved supersymmetries, of the form  $Q_\alpha + (\beta^\perp\tilde{Q})_\alpha$ . When  $1 - (\Gamma^0\Gamma^1)_{\alpha\alpha} = 0$  this supersymmetry is also preserved by the second term if the excitation is left-moving; when  $1 + (\Gamma^0\Gamma^1)_{\alpha\alpha} = 0$  it is preserved if the excitation is right-moving. Either way, the state is BPS.

### 13.2 Problem 14.2

The supergravity solution for two static parallel NS5-branes is given in (14.1.15) and (14.1.17):

$$\begin{aligned} e^{2\Phi} &= g^2 + \frac{Q_1}{2\pi^2(x^m - x_1^m)^2} + \frac{Q_2}{2\pi^2(x^m - x_2^m)^2}, \\ G_{mn} &= g^{-1}e^{2\Phi}\delta_{mn}, \quad G_{\mu\nu} = g\eta_{\mu\nu}, \\ H_{mnp} &= -\epsilon_{mnp}{}^q\partial_q\Phi, \end{aligned} \quad (3)$$

where  $\mu, \nu = 0, \dots, 5$  and  $m, n = 6, \dots, 9$  are the parallel and transverse directions respectively, and the branes are located in the transverse space at  $x_1^m$  and  $x_2^m$ . (We have altered (14.1.15a) slightly in order to make (3) S-dual to the D-brane solution (14.8.1).) A D-string stretched between the



two branes at any given excitation level is a point particle with respect to the 5+1 dimensional Poincaré symmetry of the parallel dimensions. In other words, if we make an ansatz for the solution of the form

$$X^\mu = X^\mu(\tau), \quad X^m = X^m(\sigma), \quad (4)$$

then, after performing the integral over  $\sigma$  in the D-string action, we should obtain the point-particle action (1.2.2) in 5+1 dimensions,

$$S_{\text{pp}} = -m \int d\tau \sqrt{-\partial_\tau X^\mu \partial_\tau X_\mu}, \quad (5)$$

where  $m$  is the mass of the solution  $X^m(\sigma)$  with respect to the 5+1 dimensional Poincaré symmetry.

Assuming that the gauge field is not excited, with this ansatz the D-string action (13.3.14) factorizes:

$$\begin{aligned} S_{\text{D1}} &= -\frac{1}{2\pi\alpha'} \int d\tau d\sigma e^{-\Phi} \sqrt{-\det(G_{ab} + B_{ab})} \\ &= -\frac{1}{2\pi\alpha'} \int d\tau d\sigma e^{-\Phi} \\ &\quad \times \sqrt{-\begin{vmatrix} G_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu & (G_{\mu n} + B_{\mu n}) \partial_\tau X^\mu \partial_\sigma X^n \\ (G_{m\nu} + B_{m\nu}) \partial_\sigma X^m \partial_\tau X^\nu & G_{mn} \partial_\sigma X^m \partial_\sigma X^n \end{vmatrix}} \\ &= -\frac{1}{2\pi\alpha'} \int d\sigma e^{-\Phi} \sqrt{\partial_\sigma X^m \partial_\sigma X_m} \int d\tau \sqrt{-\partial_\tau X^\mu \partial_\tau X_\mu}. \end{aligned} \quad (6)$$

In the last equality we have used the fact that neither the metric nor the two-form potential in the solution (3) have mixed  $\mu n$  components. Comparison with (5) shows that

$$m = \frac{g^{-1/2}}{2\pi\alpha'} \int d\sigma |\partial_\sigma X^m|, \quad (7)$$

where the integrand is the coordinate (not the proper) line element in this coordinate system. The ground state is therefore a straight line connecting the two branes:

$$m = \frac{g^{-1/2} |x_2^m - x_1^m|}{2\pi\alpha'}. \quad (8)$$

As explained above, this mass is defined with respect the geometry of the parallel directions, and it is only in string frame that the parallel metric  $G_{\mu\nu}$  is independent of the transverse position. We can nonetheless define an Einstein-frame mass  $m_E$  with respect to  $G_{\mu\nu}$  at  $|x^m| = \infty$ , and it is this mass that transforms simply under S-duality. (Here we are using the definition (14.1.7) of the Einstein frame,  $G_E = e^{-\Phi/2} G$ , which is slightly different from the one used in volume I and in Problem 14.6 below, where  $G_E = e^{-\tilde{\Phi}/2} G$ .) From the definition (5) of the mass,

$$m_E = g^{1/4} m, \quad (9)$$

so (7) becomes

$$m_E = \frac{g^{-1/4}}{2\pi\alpha'} \int d\sigma |\partial_\sigma X^m|. \quad (10)$$

We can calculate the mass of an F-string stretched between two D5-branes in two different pictures: we can use the black 5-brane supergravity solution (14.8.1) and do a calculation similar to the one above, or we can consider the F-string to be stretched between two elementary D5-branes embedded in flat spacetime. In the first calculation, the S-duality is manifest at every step, since the NS5-brane and the black 5-brane solutions are related by S-duality, as are the D-string and F-string actions. The second calculation yields the same answer, and is much easier: since the tension of the F-string is  $1/2\pi\alpha'$ , and in flat spacetime ( $G_{\mu\nu} = \eta_{\mu\nu}$ ) its proper length and coordinate length are the same, its total energy is

$$m = \frac{1}{2\pi\alpha'} \int d\sigma |\partial_\sigma X^m|. \quad (11)$$

Its Einstein-frame mass is then

$$m_E = \frac{g^{1/4}}{2\pi\alpha'} \int d\sigma |\partial_\sigma X^m|, \quad (12)$$

which indeed agrees with (10) under  $g \rightarrow 1/g$ .

### 13.3 Problem 14.6

To find the expectation values of the dilaton and graviton in the low energy field theory, we add to the action a source term

$$S' = \int d^{10}X \left( K_{\tilde{\Phi}} \tilde{\Phi} + K_h^{\mu\nu} h_{\mu\nu} \right), \quad (13)$$

and take functional derivatives of the partition function  $Z[K_{\tilde{\Phi}}, K_h]$  with respect to  $K_{\tilde{\Phi}}$  and  $K_h$ . For the D-brane, which is a real, physical source for the fields, we also include the sources  $J_{\tilde{\Phi}}$  and  $J_h$ , calculated in problem 13.4(b) (see (26) and (27) of that solution):

$$J_{\tilde{\Phi}}(X) = \frac{3-p}{4} \tau_p \delta^{9-p}(X_\perp), \quad (14)$$

$$J_h^{\mu\nu}(X) = -\frac{1}{2} \tau_p e_p^{\mu\nu} \delta^{9-p}(X_\perp). \quad (15)$$

Recall that  $h$  is the perturbation in the Einstein-frame metric,  $\tilde{G} = \eta + h$ , and that  $e^{\mu\nu}$  equals  $\eta^{\mu\nu}$  for  $\mu, \nu$  parallel to the brane and zero otherwise. (We have put the brane at the origin, so that  $X'_\perp = 0$ .) Since the dilaton decouples from the Einstein-frame graviton, we can calculate the partition functions  $Z[K_{\tilde{\Phi}}]$  and  $Z[K_h]$  separately. Using the propagator (23),

$$\begin{aligned} Z[K_{\tilde{\Phi}}] &= -Z[0] \int \frac{d^{10}k}{(2\pi)^{10}} \tilde{J}_{\tilde{\Phi}}(-k) \langle \tilde{\Phi} \tilde{\Phi}(k) \rangle \tilde{K}_{\tilde{\Phi}}(k) \\ &= Z[0] \frac{3-p}{2} i\kappa^2 \tau_p \int \frac{d^{9-p}k_\perp}{(2\pi)^{9-p}} \frac{1}{k_\perp^2} \tilde{K}_{\tilde{\Phi}}(k_\perp, k_\parallel = 0) \\ &= Z[0] \frac{3-p}{2} i\kappa^2 \tau_p \int d^{9-p}X_\perp G_{9-p}(X_\perp) \int d^{p+1}X_\parallel K_{\tilde{\Phi}}(X), \end{aligned} \quad (16)$$

so that

$$\begin{aligned}\langle \tilde{\Phi}(X) \rangle &= \frac{1}{iZ[0]} \frac{\delta Z[K_{\tilde{\Phi}}]}{\delta K_{\tilde{\Phi}}(X)} \\ &= \frac{3-p}{2} \kappa^2 \tau_p G_{9-p}(X_{\perp}).\end{aligned}\quad (17)$$

Using (13.3.22), (13.3.23), and the position-space expression for  $G_d$ , this becomes

$$\begin{aligned}\langle \tilde{\Phi}(X) \rangle &= \frac{3-p}{4} (4\pi)^{(5-p)/2} \Gamma\left(\frac{7-p}{2}\right) g \alpha^{(7-p)/2} r^{p-7} \\ &= \frac{3-p}{4} \frac{\rho^{7-p}}{r^{7-p}},\end{aligned}\quad (18)$$

where  $\rho^{7-p}$  is as defined in (14.8.2b) with  $Q = 1$ . Hence

$$\begin{aligned}e^{2\langle \Phi \rangle} &\approx g^2 (1 + 2\langle \tilde{\Phi} \rangle) \\ &\approx g^2 \left(1 + \frac{\rho^{7-p}}{r^{7-p}}\right)^{(3-p)/2},\end{aligned}\quad (19)$$

in agreement with (14.8.1b) (corrected by a factor of  $g^2$ ).

The graviton calculation is very similar. Using the propagator (24),

$$\begin{aligned}Z[K_h] &= -Z[0] \int \frac{d^{10}X}{(2\pi)^{10}} \tilde{J}_h^{\mu\nu}(-k) \langle h_{\mu\nu} h_{\rho\sigma}(k) \rangle \tilde{K}_h^{\rho\sigma}(k) \\ &= Z[0] \left(\frac{p+1}{8} \eta_{\mu\nu} - e_{\mu\nu}\right) 2i\kappa^2 \tau_p \int \frac{d^{9-p}k_{\perp}}{(2\pi)^{10}} \frac{1}{k_{\perp}^2} \tilde{K}_h^{\mu\nu}(k_{\perp}, k_{\parallel} = 0) \\ &= Z[0] \left(\frac{p+1}{8} \eta_{\mu\nu} - e_{\mu\nu}\right) 2i\kappa^2 \tau_p \\ &\quad \times \int d^{9-p}X_{\perp} G_{9-p}(X_{\perp}) \int d^{p+1}X_{\parallel} K_h^{\mu\nu}(X),\end{aligned}\quad (20)$$

so

$$\begin{aligned}\langle h_{\mu\nu}(X) \rangle &= \left(\frac{p+1}{8} \eta_{\mu\nu} - e_{\mu\nu}\right) 2\kappa^2 \tau_p G_{9-p}(X_{\perp}) \\ &= \left(\frac{p+1}{8} \eta_{\mu\nu} - e_{\mu\nu}\right) \frac{\rho^{7-p}}{r^{7-p}}.\end{aligned}\quad (21)$$

Hence for  $\mu, \nu$  aligned along the brane,

$$\langle \tilde{G}_{\mu\nu} \rangle \approx \left(1 + \frac{\rho^{7-p}}{r^{7-p}}\right)^{(p-7)/8} \eta_{\mu\nu},\quad (22)$$

while for  $m, n$  transverse to the brane,

$$\langle \tilde{G}_{mn} \rangle \approx \left(1 + \frac{\rho^{7-p}}{r^{7-p}}\right)^{(p+1)/8} \delta_{mn}.\quad (23)$$

The string frame metric,  $G = e^{\tilde{\Phi}/2} \tilde{G}$ , is therefore

$$\langle G_{\mu\nu} \rangle \approx \left( 1 + \frac{\rho^{7-p}}{r^{7-p}} \right)^{-1/2} \eta_{\mu\nu}, \quad (24)$$

$$\langle G_{mn} \rangle \approx \left( 1 + \frac{\rho^{7-p}}{r^{7-p}} \right)^{1/2} \delta_{mn}, \quad (25)$$

in agreement with (14.8.1).

## 14 Chapter 15

### 14.1 Problem 15.1

The matrix of inner products is

$$\mathcal{M}^3 = \langle h | \begin{bmatrix} L_1^3 \\ L_1 L_2 \\ L_3 \end{bmatrix} \begin{bmatrix} L_{-1}^3 & L_{-2} L_{-1} & L_{-3} \end{bmatrix} | h \rangle \quad (1)$$

$$= \begin{bmatrix} 24h(h+1)(2h+1) & 12h(3h+1) & 24h \\ 12h(3h+1) & h(8h+8+c) & 10h \\ 24h & 10h & 6h+2c \end{bmatrix}. \quad (2)$$

This matches the Kac formula, with

$$K_3 = 2304. \quad (3)$$

### 14.2 Problem 15.3

Let's begin by recording some useful symmetry relations of the operator product coefficient with lower indices, derived from the definition (6.7.13) and (6.7.14),

$$c_{ijk} = \langle \mathcal{A}'_i(\infty, \infty) \mathcal{A}_j(1, 1) \mathcal{A}_k(0, 0) \rangle_{S_2}. \quad (4)$$

The following relations then hold, with the sign of the coefficient depending on the statistics of the operators:

$$c_{ijk} = \pm (-1)^{h_j + \tilde{h}_j} c_{kji} \quad \text{if } \mathcal{A}_j \text{ is primary} \quad (5)$$

$$c_{ijk} = \pm (-1)^{h_i + h_j + h_k + \tilde{h}_i + \tilde{h}_j + \tilde{h}_k} c_{ikj} \quad \text{if } \mathcal{A}_i \text{ is primary} \quad (6)$$

$$c_{ijk} = \pm (-1)^{h_k + \tilde{h}_k} c_{jik} \quad \text{if } \mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k \text{ are primary.} \quad (7)$$

Actually, in the above “primary” may be weakened to “quasi-primary” (meaning annihilated by  $L_1$ , rather than  $L_n$  for all  $n > 0$ , and therefore transforming as a tensor under  $PSL(2, C)$  rather than general local conformal transformations), but Polchinski does not seem to find the notion of quasi-primary operator interesting or useful.

Armed with these symmetries (and in particular relation (5)), but glibly ignoring phase factors as Polchinski does, we would like to claim that the correct form for (15.2.7) should be as follows (we haven't bothered to raise the index):

$$c_{i\{k, \bar{k}\}, mn} = \lim_{\substack{z_n \rightarrow \infty \\ z_m \rightarrow 1}} z_n^{2h_n} \bar{z}_n^{2\tilde{h}_n} \mathcal{L}_{-\{k\}} \tilde{\mathcal{L}}_{-\{\bar{k}\}} \langle \mathcal{O}_n(z_n, \bar{z}_n) \mathcal{O}_m(z_m, \bar{z}_m) \mathcal{O}_i(0, 0) \rangle_{S_2}. \quad (8)$$

We would also like to claim that the LHS of (15.2.9) should read

$$\langle \mathcal{O}'_i(\infty, \infty) \mathcal{O}_j(1, 1) \mathcal{O}_m(z, \bar{z}) \mathcal{O}_n(0, 0) \rangle_{S_2}. \quad (9)$$

We are now ready to solve the problem. The case  $N = 0$  is trivial, since by the definition (15.2.8),

$$\beta_{mn}^{i\{\}} = 1, \quad (10)$$

so the coefficient of  $z^{-h_m-h_n+h_i}$  in  $\mathcal{F}_{mn}^{jl}(i|z)$  is 1. For  $N = 1$  there is again only one operator,  $L_{-1} \cdot \mathcal{O}_i$ . We have

$$\beta_{mn}^{i\{1\}} = \frac{1}{2h_i}(h_i + h_m - h_n). \quad (11)$$

Thus the coefficient of  $z^{-h_m-h_n+h_i+1}$  is

$$\frac{1}{2h_i}(h_i + h_m - h_n)(h_i + h_j - h_l). \quad (12)$$

## 15 Appendix B

### 15.1 Problem B.1

Under a change of spinor representation basis,  $\Gamma^\mu \rightarrow U\Gamma^\mu U^{-1}$ ,  $B_1$ ,  $B_2$ , and  $C$ , all transform the same way:

$$B_1 \rightarrow U^* B_1 U^{-1}, \quad B_2 \rightarrow U^* B_2 U^{-1}, \quad C \rightarrow U^* C U^{-1}. \quad (1)$$

The invariance of the following equations under this change of basis is more or less trivial:

(B.1.17) (the definition of  $B_1$  and  $B_2$ ):

$$\begin{aligned} U^* B_1 U^{-1} U \Gamma^\mu U^{-1} (U^* B_1 U^{-1})^{-1} &= U^* B_1 \Gamma^\mu B_1^{-1} U^T \\ &= (-1)^k U^* \Gamma^{\mu*} U^T \\ &= (-1)^k (U \Gamma^\mu U^{-1})^*, \end{aligned} \quad (2)$$

$$\begin{aligned} U^* B_2 U^{-1} U \Gamma^\mu U^{-1} (U^* B_2 U^{-1})^{-1} &= U^* B_2 \Gamma^\mu B_2^{-1} U^T \\ &= (-1)^{k+1} U^* \Gamma^{\mu*} U^T \\ &= (-1)^{k+1} (U \Gamma^\mu U^{-1})^*. \end{aligned} \quad (3)$$

(B.1.18), using the fact that  $\Sigma^{\mu\nu}$  transforms the same way as  $\Gamma^\mu$ :

$$\begin{aligned} U^* B U^{-1} U \Sigma^{\mu\nu} U^{-1} U B^{-1} U^T &= U^* B \Sigma^{\mu\nu} B^{-1} U^T \\ &= -U^* \Sigma^{\mu\nu*} U^T \\ &= -(U \Sigma^{\mu\nu} U^{-1})^*. \end{aligned} \quad (4)$$

The invariance of (B.1.19) is the same as that of (B.1.17).

(B.1.21):

$$U B_1^* U^T U^* B_1 U^{-1} = U B_1^* B_1 U^{-1} = (-1)^{k(k+1)/2} U U^{-1} = (-1)^{k(k+1)/2}, \quad (5)$$

$$U B_2^* U^T U^* B_2 U^{-1} = U B_2^* B_2 U^{-1} = (-1)^{k(k-1)/2} U U^{-1} = (-1)^{k(k-1)/2}. \quad (6)$$

(B.1.24) (the definition of  $C$ ):

$$\begin{aligned} U^* C U^{-1} U \Gamma^\mu U^{-1} U C^{-1} U^T &= U^* C \Gamma^\mu C^{-1} U^T \\ &= -U^* \Gamma^{\mu T} U^T \\ &= -(U \Gamma^\mu U^{-1})^T. \end{aligned} \quad (7)$$

(B.1.25): all three sides clearly transform by multiplying on the left by  $U$  and on the right by  $U^{-1}$ .

(B.1.27):

$$U^* B U^{-1} U \Gamma^0 U^{-1} = U^* B \Gamma^0 U^{-1} = U^* C U^{-1}. \quad (8)$$

We will determine the relation between  $B$  and  $B^T$  in the  $\zeta^{(s)}$  basis, where (for  $d = 2k + 2$ )  $B_1$  and  $B_2$  are defined by equation (B.1.16):

$$B_1 = \Gamma^3 \Gamma^5 \dots \Gamma^{d-1}, \quad B_2 = \Gamma B_1. \quad (9)$$

Since all of the  $\Gamma$ 's that enter into this product are antisymmetric in this basis (since they are Hermitian and imaginary), we have

$$\begin{aligned} B_1^T &= (\Gamma^3 \Gamma^5 \dots \Gamma^{d-1})^T \\ &= (-1)^k \Gamma^{d-1} \Gamma^{d-3} \dots \Gamma^3 \\ &= (-1)^{k(k+1)/2} \Gamma^3 \Gamma^5 \dots \Gamma^{d-1} \\ &= (-1)^{k(k+1)/2} B_1. \end{aligned} \quad (10)$$

Using (10), the fact that  $\Gamma$  is real and symmetric in this basis, and (B.1.19), we find

$$\begin{aligned} B_2^T &= (\Gamma B_1)^T \\ &= B_1^T \Gamma^T \\ &= (-1)^{k(k+1)/2} B_1 \Gamma \\ &= (-1)^{k(k+1)/2+k} \Gamma^* B_1 \\ &= (-1)^{k(k-1)/2} B_2. \end{aligned} \quad (11)$$

Since  $B_1$  is used when  $k = 0, 3 \pmod{4}$ , and  $B_2$  is used when  $k = 0, 1 \pmod{4}$ , so in any dimension in which one can impose a Majorana condition we have

$$B^T = B. \quad (12)$$

When  $k$  is even,  $C = B_1 \Gamma^0$ , so, using the fact that in this basis  $\Gamma^0$  is real and antisymmetric, and (B.1.17), we find

$$\begin{aligned} C^T &= \Gamma^{0T} B_1^T \\ &= (-1)^{k(k+1)/2+1} \Gamma^{0*} B_1 \\ &= (-1)^{k/2+1} B_1 \Gamma^0 \\ &= (-1)^{k/2+1} C. \end{aligned} \quad (13)$$

On the other hand, if  $k$  is odd, then  $C = B_2 \Gamma^0$ , and

$$\begin{aligned} C^T &= \Gamma^{0T} B_2^T \\ &= (-1)^{k(k-1)/2+1} \Gamma^{0*} B_2 \\ &= (-1)^{(k+1)/2} B_2 \Gamma^0 \\ &= (-1)^{(k+1)/2} C. \end{aligned} \quad (14)$$

That all of these relations are invariant under change of basis follows directly from the transformation law (1), since  $B^T$  and  $C^T$  transform the same way as  $B$  and  $C$ .



### 15.2 Problem B.3

The decomposition of  $\text{SO}(1,3)$  spinor representations under the subgroup  $\text{SO}(1,1) \times \text{SO}(2)$  is described most simply in terms of Weyl representations: one positive chirality spinor,  $\zeta^{++}$ , transforms as a positive chirality Weyl spinor under both  $\text{SO}(1,1)$  and  $\text{SO}(2)$ , while the other,  $\zeta^{--}$ , transforms as a negative chirality Weyl spinor under both (see (B.1.44a)). Let us therefore use the Weyl-spinor description of the 4 real supercharges of  $d = 4$ ,  $\mathcal{N} = 1$  supersymmetry. The supersymmetry algebra is (B.2.1a):

$$\begin{aligned} \{Q_{++}, Q_{++}^\dagger\} &= 2(P^0 - P^1), \\ \{Q_{--}, Q_{--}^\dagger\} &= 2(P^0 + P^1), \\ \{Q_{++}, Q_{--}^\dagger\} &= 2(P^2 + iP^3). \end{aligned} \quad (15)$$

We must now decompose these Weyl spinors into Majorana-Weyl spinors. Define

$$Q_L^1 = \frac{1}{2}(Q_{++} + Q_{++}^\dagger), \quad Q_L^2 = \frac{1}{2i}(Q_{++} - Q_{++}^\dagger), \quad (16)$$

$$Q_R^1 = \frac{1}{2}(Q_{--} + Q_{--}^\dagger), \quad Q_R^2 = \frac{1}{2i}(Q_{--} - Q_{--}^\dagger). \quad (17)$$

(The subscripts  $L$  and  $R$  signify that the respective supercharges have positive and negative  $\text{SO}(1,1)$  chirality.) Hence, for instance,

$$\{Q_L^1, Q_L^1\} = \frac{1}{4} \left( \{Q_{++}, Q_{++}\} + \{Q_{++}^\dagger, Q_{++}^\dagger\} + 2\{Q_{++}, Q_{++}^\dagger\} \right). \quad (18)$$

The last term is given by the algebra (14), but what do we do with the first two terms? The  $d = 4$  algebra has a  $\text{U}(1)$  R-symmetry under which  $Q_{++}$  and  $Q_{--}$  are both multiplied by the same phase. In order for the  $d = 2$  algebra to inherit that symmetry, we must assume that

$$Q_{++}^2 = Q_{--}^2 = \{Q_{++}, Q_{--}\} = 0. \quad (19)$$

The  $d = 2$  algebra is then

$$\{Q_L^A, Q_L^B\} = \delta^{AB}(P^0 - P^1), \quad (20)$$

$$\{Q_R^A, Q_R^B\} = \delta^{AB}(P^0 + P^1), \quad (21)$$

$$\{Q_L^A, Q_R^B\} = Z^{AB}, \quad (22)$$

where

$$Z = \begin{bmatrix} P^2 & -P^3 \\ P^3 & P^2 \end{bmatrix}. \quad (23)$$

The central charges are thus the Kaluza-Klein momenta associated with the reduced dimensions. If these momenta are 0 (as in dimensional reduction in the strict sense), then the algebra possesses a further R-symmetry, namely the  $\text{SO}(2)$  of rotations in the 2-3 plane. We saw at the beginning that  $Q_{++}$  is positively charged and  $Q_{--}$  negatively charged under this symmetry. This symmetry and the original  $\text{U}(1)$  R-symmetry of the  $d = 4$  algebra can be recombined into two independent  $\text{SO}(2)$  R-symmetry groups of the  $Q_L^A$  and  $Q_R^A$  pairs of supercharges.

### 15.3 Problem B.5

Unfortunately, it appears that some kind of fudge will be necessary to get this to work out correctly. There may be an error lurking in the book. The candidate fudges are: (1) The vector multiplet is  $\mathbf{8}_v + \mathbf{8}'$ , not  $\mathbf{8}_v + \mathbf{8}$  (this is suggested by the second sentence of Section B.6). (2) The supercharges of the  $N = 1$  theory are in the  $\mathbf{16}'$ , not  $\mathbf{16}$ , of  $\text{SO}(9,1)$ . (3) The frame is one in which  $k^0 = k^1$ , not  $k_0 = k_1$  as purportedly used in the book. We will arbitrarily choose fudge #1, although it's hard to see how this can fit into the analysis of the type I spectrum in Chapter 10.

The  $\mathbf{8}_v$  states have helicities  $(\pm 1, 0, 0, 0)$ ,  $(0, \pm 1, 0, 0)$ ,  $(0, 0, \pm 1, 0)$ , and  $(0, 0, 0, \pm 1)$ . The  $\mathbf{8}'$  states have helicities  $\pm(-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})$ ,  $\pm(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})$ ,  $\pm(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2})$ , and  $\pm(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2})$ .

In a frame in which  $k_0 = k_1$ , the supersymmetry algebra is

$$\{Q_\alpha, Q_\beta^\dagger\} = 2P_\mu(\Gamma^\mu\Gamma^0)_{\alpha\beta} = -2k_0(1 + 2S_0)_{\alpha\beta}, \quad (24)$$

so that supercharges with  $s_0 = -\frac{1}{2}$  annihilate all states. The supercharges with  $s_0 = +\frac{1}{2}$  form an  $\mathbf{8}$  representation of the  $\text{SO}(8)$  little group. Since the operator  $B$  switches the sign of all the helicities  $s_1, \dots, s_4$ , the Majorana condition pairs these eight supercharges into four independent sets of fermionic raising and lowering operators. Let  $Q_{(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})}$ ,  $Q_{(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2})}$ ,  $Q_{(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})}$ , and  $Q_{(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2})}$  be the raising operators. We can obtain all sixteen of the states in  $\mathbf{8}_v + \mathbf{8}'$  by starting with the state  $(-1, 0, 0, 0)$  and acting on it with all possible combinations of these four operators. The four states in the  $\mathbf{8}'$  with  $s_1 = -\frac{1}{2}$  are obtained by acting with a single operator. Acting with a second operator yields the six states in the  $\mathbf{8}_v$  with  $s_1 = 0$ . A third operator gives  $s_1 = +\frac{1}{2}$ , the other four states of the  $\mathbf{8}'$ . Finally, acting with all four operators yields the last state of the  $\mathbf{8}_v$ ,  $(+1, 0, 0, 0)$ .

## References

- [1] J. Polchinski, *String Theory, Vol. 1: Introduction to the Bosonic String*. Cambridge University Press, 1998.
- [2] J. Polchinski, *String Theory, Vol. 2: Superstring Theory and Beyond*. Cambridge University Press, 1998.