Class Notes for Modern Physics, Part 1

J. Gunion
U.C. Davis

9D, Spring Quarter

What is Modern Physics?

The study of Modern Physics is the study of the enormous revolution in our view of the physical universe that began just prior to 1900.

At that time, most physicists believed that everything in physics was completely understood. Normal intuition and all experiments fit into the context of two basic theories:

1. Newtonian Mechanics for massive bodies;

2. Maxwell's Theory for light (electromagnetic radiation).

Consistency of the two required that there be a propagating medium (and, therefore, a preferred reference frame) for light.

However, even a little thought made it clear that there was trouble on the horizon. And then came many new experimental results that made it clear that the then-existing theoretical framework was woefully inadequate to describe nature.

In a relatively short period of time, physicists were compelled to adopt:

1. the theory of special relativity based on the idea that there was no propagating medium for light (so that light traveled with the same speed regardless of the "frame" from which the light was viewed);

2. the theory of quantum mechanics, according to which the precise position and precise momentum of a particle cannot both be determined simultaneously.

   In fact, one must think of particles not as particles, but as waves, much like light.

3. At the same time, experiments made it clear that light comes in little quantum particle-like packets called photons.

4. In short, both particles and light have both a particle-like and wave-like nature.

It is useful to focus first on the inconsistencies of the "ether" picture and of the above-outlined naive picture of space and time. This will lead us to the theory of special relativity.

The latter inconsistencies are revealed by thinking carefully about Galilean transformations between coordinate systems that underpinned the pre-relativity view of space and time.

Before proceeding, let me just emphasize that in this course we will be embarking on an exploration that has been repeated in a certain sense several times now.

Indeed, the business of looking for inconsistencies in existing theories now has a long history of success, beginning with the development of special relativity, general relativity, and quantum mechanics. We have learned not to be arrogant, but rather to expect that the best theories of a given moment are imperfect and to look for difficulties (perhaps subtle ones) or extensions that are suggested by thought experiments that push the theories into a new domain.

As an example, the development of the Standard Model of fundamental interactions (that you may have heard of) began with the realization that the theory that was developed to explain the weak interactions would violate the laws of probability conservation when extended to high energies.

In fact, nowadays, we have many arguments that suggest that the Standard Model is itself little more than an "effective" theory valid at the energy scales that we have so far been able to probe. It has undesirable features when we try to extend it to higher energies (e.g. from the scale of the masses of the new W and Z bosons to the Planck mass scale that is some 16 orders of magnitude larger).

The ether picture for light propagation

- **At the end of the 19th century, light waves were an accepted fact, but all physicists were "certain" that there had to be a medium in which the light propagated (analogous to water waves, waves on a string, etc.).**

- **However, the "ether" in which light propagated had to be quite unusual.**

  The speed of light was known to be very large (the precise value we now know is $c = 3.00 \times 10^8$ m/s). A medium that supported this high speed had to be essentially incompressible (*i.e.* something vastly more incompressible than water, and even more vastly incompressible than air).

- **And yet, it was clear that light traveled over great distances from the stars, implying that this ether extended throughout a large section of the universe.**

  This means that the planets, stars, galaxies, . . . , were traveling through this ether according to Newton's laws without feeling any frictional, viscosity, . . . , type of effects.

- **Well, for anyone thinking about this nowadays, this is obviously ridiculous.**

  But, at the end of the 1900's it was impossible for physicists to accept the fact that there was no ether medium in which light traveled and it was bizarre to imagine that light could travel through "vacuum", despite the fact that Maxwell's equations were most easily understood in this context.

- **We will shortly turn to the Michelson-Morley (MM) experiment performed in 1887 in which MM set out to demonstrate the existence of the ether.**

  We will learn that they failed. To show how entrenched thinking can become, it should be noted that Michelson (who was quite a brilliant guy) never believed the result of his experiment and spent the next 20 years trying to prove his original result was wrong. He failed, but provided ever-increasing accuracy for the precise speed of light.

- **It would be natural to presume that Einstein's theory of special relativity was a response to this experiment.**

  But, in fact, he stated that when he developed his theory he was completely unaware of the MM result. He simply was thinking of

Maxwell's theory of light as a medium independent theory and asking about its consequences.

This is not totally implausible given the fact that the MM experiment was performed in the "back-woods", frontier town of Cleveland Ohio (some would say that the MM experiment is still the most important thing, other than some baseball greats, to come out of Cleveland). And communications were not so hot back in those days.

To understand the ideas behind the MM experiment and to set the stage for how we discuss space and time in an "inertial" frame, we must consider how to relate one frame to another one moving with constant velocity with respect to the first frame.

The 1900's view of this relationship is encoded in "Galilean transformations".

Galilean transformations

A coordinate system is to be thought of as:

1. a system of "meter" sticks laid down in the x, y, z directions throughout all of space.

2. a single universal clock time that applies throughout all of space ($i.e.$ is the same no matter where in space you are).

An event is thus specified by its location in the (t, x, y, z) space.

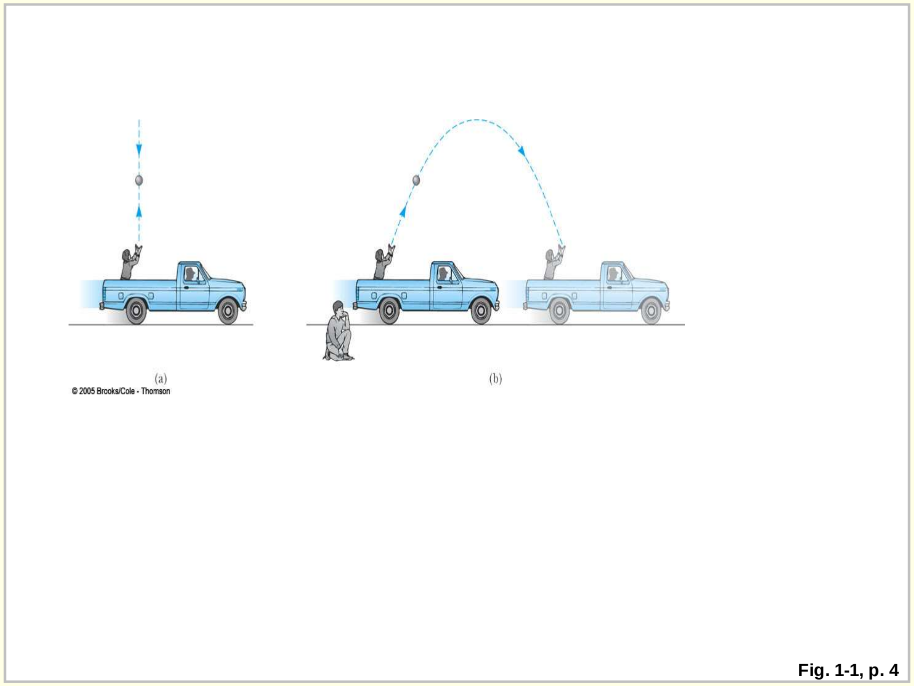
But, now suppose that there is another person moving with constant velocity in the original coordinate system in the positive x direction.

His coordinates will be related to (t, x, y, z) by:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t\,, \end{aligned} \tag{1}$$

where the last equation, according to which both observers can use the time, or set of clocks, is particularly crucial.

This relation between frames is illustrated below.



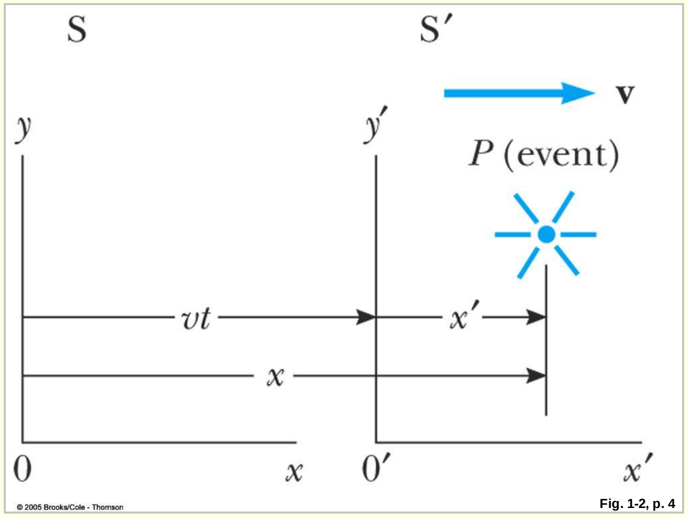


Figure 1: The Galilean Transformation.

Of course, if we are tracking an object moving in the x-direction in the two coordinate systems, we may compare its velocity and acceleration as viewed in the two coordinate systems by taking derivatives of the first equation above to obtain (using $dt = dt'$ from above)

$$
\begin{aligned}
u'_x \equiv \frac{dx'}{dt} &= \frac{dx}{dt} - v \equiv u_x - v\,, \quad \text{and} \\
a'_x \equiv \frac{du'_x}{dt} &= \frac{du_x}{dt} \equiv a_x,. \qquad (2)
\end{aligned}
$$

Note that the accelerations are the same, which is consistent with the idea that the force causing the acceleration should be the same as viewed by the two different observers (given that forces depend on separations between objects which will be the same in the two different frames):

$$F_x = ma_x = ma'_x = m'a'_x = F'_x, \qquad (3)$$

where we assumed $m = m'$ is frame independent. The fact that $F_x = ma_x$ and $F'_x = ma'_x$ have the same form in the two different reference systems is called covariance of Newton's 2nd law.

But, there is already a problem with covariance in the case of light. Light cannot be subject to the same covariance rules without conflicting with the idea of an ether in which it propagates.

In particular, imagine you are at rest in the ether and look at your reflection in the mirror. No problem – it just takes a very short time for the light to travel to the mirror and back.

However, now suppose you are on a rocket ship moving with velocity $v > c$ with respect to the ether. Then the light traveling with speed c in the ether never makes it to the mirror!!

- Thus,

1. either we are forced to give up the general concept that motion with constant velocity is indistinguishable from being at rest ($i.e.$ there must be a preferred rest frame), or
2. the Galilean transform equations eq. (1) are wrong.

It is the latter that is true.

These thoughts led to the Michelson-Morley experiment. They showed that either there is no ether or that the earth is not moving through the ether (a very geo-centric point of view by then, since it would make much more sense if the earth was moving with its orbital velocity through an ether that was a rest with respect to the galaxy or universe as a whole).

The Michelson Morley Experiment

The experimental arrangement for the MM experiment appears in the diagram below.

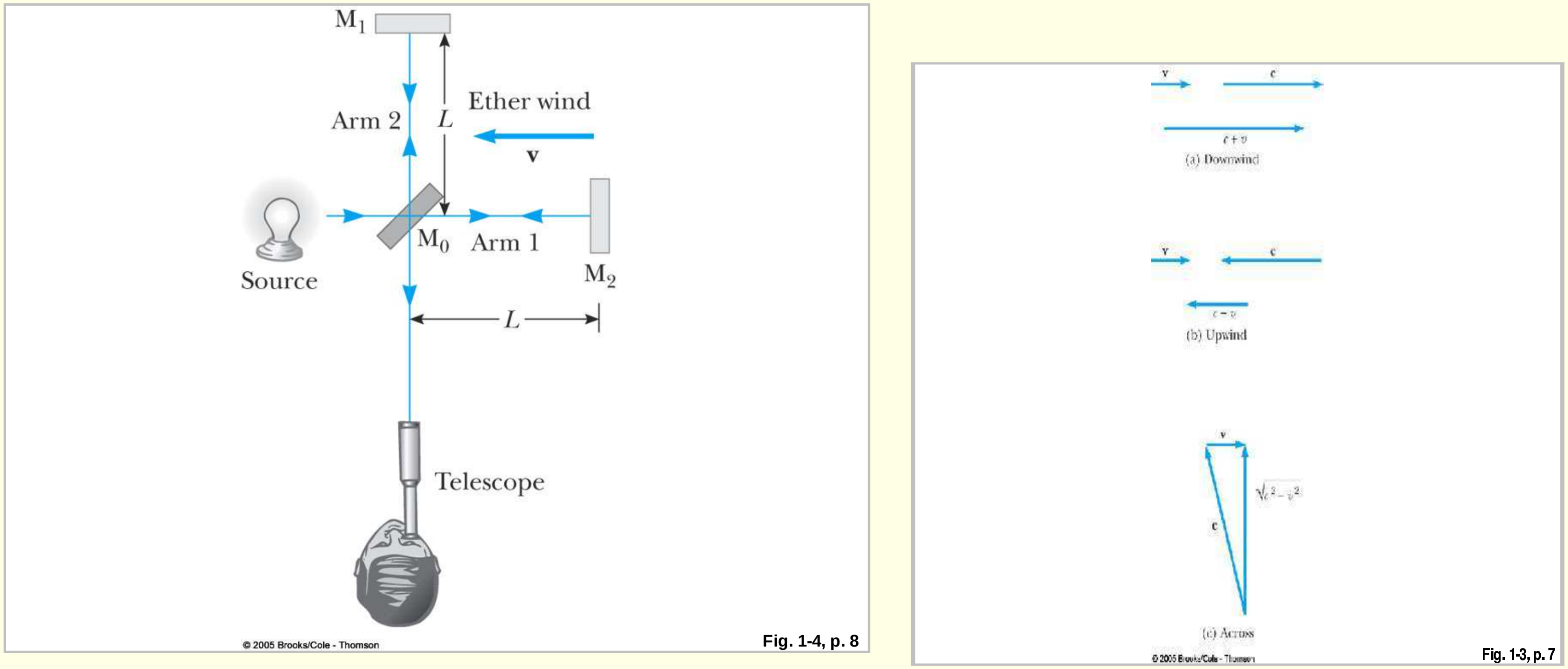


Figure 2: The Michelson Morley experimental set-up.

In the pictured arrangement, the light (wave) is split by a half-silvered mirror into two components, one traveling parallel to the earth's motion, the other traveling perpendicular to the earth's motion through the ether. The time of travel for the horizontal light to and back from the mirror will (Galilean assumed) be

$$t_{horizontal} = \frac{L}{c+v} + \frac{L}{c-v}. \tag{4}$$

The time of travel for the vertical light (which must actually be aimed "up-stream" in order to return to the splitting mirror) is given by

$$t_{vertical} = \frac{2L}{\sqrt{c^2 - v^2}}. \tag{5}$$

From this, we find (for $v = v_{earth} \sim 3 \times 10^4$ m/s and $c \simeq 3 \times 10^8$ m/s)

$$\begin{aligned} \Delta t &= t_h - t_v \simeq L\frac{v^2}{c^3} \quad \text{for } v \ll c \quad \text{or} \\ \Delta d \equiv c\Delta t &\sim 10^{-7}\, m \quad \text{for } L = 10 \text{ m and } (v/c)^2 \sim 10^{-8}. \end{aligned} \tag{6}$$

Although this is a very small number, an interferometer which measures the interference between the vertical and horizontal light waves can be sensitive to it.

We now rotate the apparatus by 90° so that the roles of the two light paths are interchanged. Since the waves are sensitive to the wave pattern oscillation

$$\sin\left(\frac{2\pi}{\lambda}(x - ct)\right) \tag{7}$$

with $t = t_h$ or $t = t_v$, one sees a shift (relative to that if $t_h = t_v$) in the constructive interference fringe corresponding to an angular amount given by

$$\frac{2\pi}{\lambda}(c2\Delta t) = \frac{2\pi}{\lambda}2\Delta d \tag{8}$$

(factor of 2 from fact that net time shift is twice the time difference between wave arrival times in any one set-up). This would be noticeable for $\lambda \sim few \times 10^{-7}m$ in a typical laboratory sized set-up using visible light.

What did they see? No shift in the interference pattern.

The time was ripe for a new idea. Enter Einstein in 1905.

Special Relativity

Postulates of special relativity:

1. **The Principle of Relativity:** All the laws of physics have the same form in all inertial reference frames.

   In other words, covariance applies to electromagnetism (there is no ether) as well as to mechanics.

2. **The Constancy of the Speed of Light:** The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

   This postulate is in fact more or less required by the first postulate. If the speed of light was different in different frames, the Maxwell equations governing the propagation of light would have to be frame-dependent.

   In fact, Einstein said he was completely unaware of the MM experiment at the time he proposed his postulates. He was just thinking about the theory of light as being absolute and frame independent.

These two apparently simple postulates imply dramatic changes in how we must visualize length, time and simultaneity.

1. **The distance between two points and the time interval between two events both depend on the frame of reference in which they are measured.**

2. **Events at different locations that occur simultaneously in one frame are not simultaneous in another frame moving uniformly with respect to the first.**

To see what exactly is true, we need to first think about how an inertial reference frame is defined. We use a coordinate grid and a set of synchronized clocks throughout all space.

As an aside, we should note that we are already questioning that such a picture actually exists when looking at very tiny distance scales where effects of $quantum\ gravity$ **are expected to enter.**

An inertial reference frame is probably only an effective description that is only valid up to the Planck mass scale.

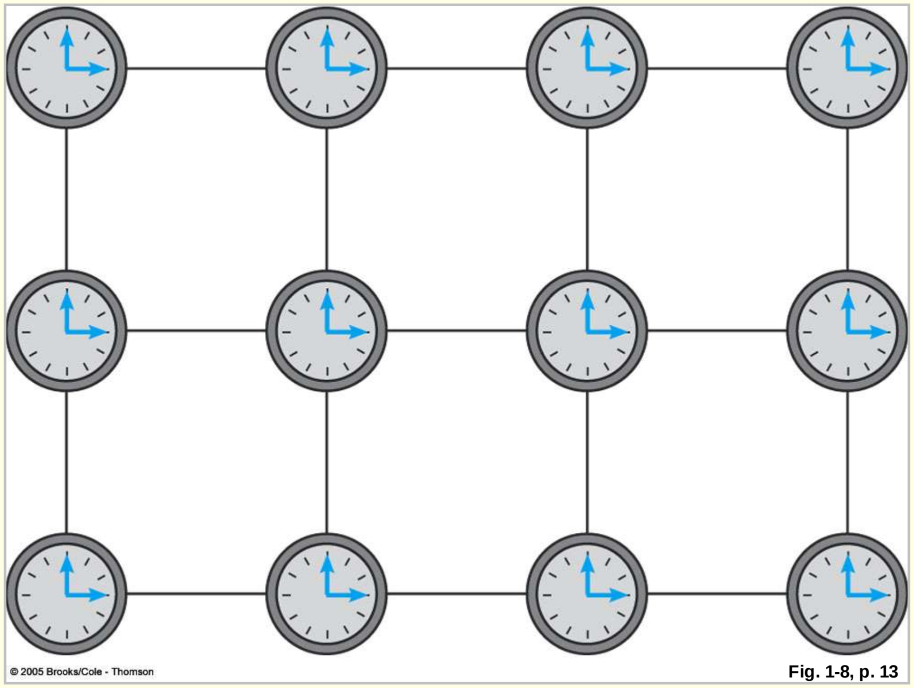


Figure 3: Picture of an inertial reference frame.

Let's return to the concept of time.

Example A

Suppose time were uniquely definable and the same in all frames.

Consider a (small) plane moving at speed v (and very close to ground) in the $+x$ direction relative to someone on the ground.

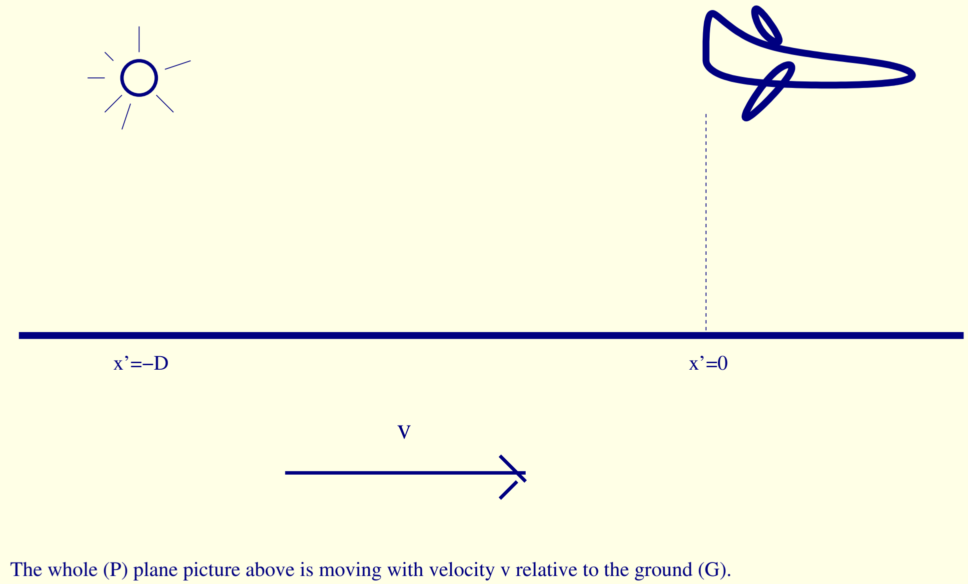


Figure 4: The frame for a plane moving relative to the earth.

When the plane is at $x' = 0$ someone at $x' = -D$ flashes a light (these are the plane's coordinates). If time is universal then both P (plane) and G (ground) agree that the light flashes at a certain time, say $t = t' = 0$.

The time at which P thinks the light arrives at $x' = 0$ (the plane never moves from $x' = 0$ – he is at rest in his coordinate system) is $t' = D/c$ (assuming light travels with velocity c).

The time at which G thinks the light arrives at the plane would also be $t = t' = D/c$ if time is universal. However, since the plane has moved by an amount

$$extra\ distance = v\frac{D}{c} \tag{9}$$

according to G while the light has been traveling, the G observer concludes that the velocity of light is

$$\frac{distance}{time} = \frac{D + v\frac{D}{c}}{\frac{D}{c}} = v + c. \tag{10}$$

Well, this contradicts Einstein's postulates of relativity. It has to be that the clocks in the G and P frames are not synchronized in the manner we assumed or that distance scales are not the same in the two frames. In fact, both apply.

Example B

- Consider 2 observers A and B that pass one another, with, say, B moving with velocity v in the x direction relative to A who we envision is at rest in "our" frame.

  A burst of light is emitted as they pass one another. Each claims that the light travels outward in spherical waves with velocity c, with the spheres centered on themselves.

A modern day application is that a terrorist dropping a bomb (that immediately detonates) from a fast moving car might hope to quickly leave behind the destruction and explosion. But, to the extent that electromagnetic radiation was the only consideration, he would always be at the center of the explosion no matter how fast he was moving in some other frame.

- **This is completely different from what one would conclude if light traveled in a medium like water.**

  Consider two boats, one (A) at rest in a pond, the other (B) moving rapidly (but without creating any wake) relative to the first boat.

  B drops a rock in the pond as he passes A.

  Because A is at rest in the pond, the ripples spread out in concentric circles from his position and B, looking back, agrees. Indeed, he could even go faster than the ripples, in which case they would never catch up to him.

Putting Einstein's visualization into mathematical language, we would say

that the expanding spheres of electromagnetic radiation should obey

$$x^2 + y^2 + z^2 = c^2t^2 \ \text{ for } A; \quad x'^2 + y'^2 + z'^2 = c^2t'^2 \ \text{ for } B\,. \qquad (11)$$

These are the equations defining how each sees the light fronts (in his own frame) emanating from the initial flash.

Demanding that the transformation from the unprime to prime system be a linear transformation[1] with coefficients determined only by the fundamental constant c and by the relative velocity of the two frames v, *and* **requiring that $x' = 0$ must correspond to $x = vt$ (see Fig. 1), there is only one solution, the so-called** *Lorentz* **transformation:**

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - \frac{v}{c^2}x)\,, \end{aligned} \qquad (12)$$

[1]Very roughly, you don't want anything else for time since otherwise you could get negative time corresponding to positive time (for a quadratic relation) or even phases would be introduced if higher powers were employed. Once linearity for the relation between t and t' is chosen, the rest must also be linear. If you are a stickler for mathematical precision, you could try looking at `http://arxiv.org/pdf/physics/0110076`.

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \,. \tag{13}$$

Note that if $v/c \ll 1$, then $\gamma \to 1$ and we get back the Galilean $approximation$.

But, if $v/c \to 1$, then $\gamma \gg 1$ and there are big changes.

Proof of above Lorentz transformation.

Write (since $y' = y$ and $z' = z$ in this situation, I ignore them)

$$x' = \gamma(x - vt) \,, \quad t' = \alpha t + \beta x \,, \tag{14}$$

where α, β, γ are all to be determined and we have inserted in the first equation the requirement that $x = vt$ gives $x' = 0$.

Now, by symmetry ($i.e.$ who says which frame should be prime and which unprime) it must also be true that the inverse transformations are obtained by simply interchanging the roles of the prime and unprime coordinates and changing the sign of v. Thus, we should also have

$$x = \gamma(x' + vt') \,. \tag{15}$$

(We will come back to the time relation.)

So, Einstein now says that for a light pulse starting at $x = 0, t = 0$ along the $+x$ axis we must have $x = ct$. However, since we have chosen the prime frame position and clock so that this corresponds to $x' = 0, t' = 0$, then we must also have $x' = ct'$.

So, substitute these two statements into $x' = \gamma(x - vt)$ to obtain

$$ct' = \gamma(ct - vt) = \gamma(1 - \frac{v}{c})ct\,. \tag{16}$$

Next, substitute these two statements into $x = \gamma(x' + vt')$ to obtain

$$ct = \gamma(ct' + vt') = \gamma(1 + \frac{v}{c})ct' = \gamma(1 + \frac{v}{c})\gamma(1 - \frac{v}{c})ct\,, \tag{17}$$

where the last step employed the equation just above. The left and final right side of the above equation are equal only if

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}\,. \tag{18}$$

Now return to the time equation. Let us take $x' = \gamma(x - vt)$ and substitute this into $x = \gamma(x' + vt')$. The result is

$$x = \gamma(\gamma(x - vt) + vt') \,. \tag{19}$$

We can easily solve this for t':

$$t' = \frac{\gamma^2 vt + (1 - \gamma^2)x}{\gamma v} = \gamma t - \frac{v}{c^2}\gamma x = \gamma(t - \frac{v}{c^2}x) \,, \tag{20}$$

as claimed. In the above, we used the simple algebra that $1 - \gamma^2 = -\frac{v^2}{c^2}\gamma^2$.

An important consequence

Using the Lorentz transform equations, we can easily show that $x^2 - c^2t^2 = x'^2 - c^2t'^2$, for any choices of x, t and the corresponding values of x', t'.

A Note on Four Vectors

We can place the quantities ct, x, y, z into an array called a 4-vector: $x_4 = (ct, x, y, z)$. The square of such a 4-vector is defined as

$x_4 \cdot x_4 \equiv c^2t^2 - x^2 - y^2 - z^2$. In the prime frame, $x'_4 = (ct', x', y', z')$ and $x'_4 \cdot x'_4 \equiv c^2t'^2 - x'^2 - y'^2 - z'^2$.

We see that a restatement of the Lorentz transformation equations, equivalently Einstein's frame-independent for the velocity of light, is to say that **the square of a 4-vector is frame-independent.**

Time Dilation

- A particular application of the Lorentz equations is *time dilation*. I will first give a mathematical derivation and then a more intuitive approach.

  Mathematical approach.

  Consider an observer S' moving with velocity v with respect to S.

  Consider a clock at rest in the S' rest frame and located at $x' = 0$, corresponding to $x = vt$.

  Suppose it ticks at $t' = 0$ and $t' = T'$. Now use

$$t' = \gamma(t - \frac{v}{c^2}x) = \gamma(t - \frac{v}{c^2}vt) = \gamma t(1 - \frac{v^2}{c^2}) = \frac{t}{\gamma}, \tag{21}$$

to conclude that the $t' = 0$ tick occurs at $t = 0$, while the $t' = T'$ tick occurs at $t = T = \gamma T'$.

So, the person at rest in S sees an interval between ticks that is larger than the person who is moving with the clock sees.

This is what we call time dilation. The time interval between ticks seen by observer S' who is a rest with respect to the clock is called *proper time*. Observer S, who sees this clock moving past him swears that it is ticking more slowly.

We could also have derived this result by using the inverse Lorentz transform equations:

$$\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma(t' + \frac{v}{c^2}x'), \end{aligned} \tag{22}$$

in particular the second one above. In this approach, we say that the clock sits at some fixed x' and ticks at $t' = t'_1$ and then at $t' = t'_2$. Using the second equation above, we find

$$T = t_1 - t_2 = \gamma(t'_1 - t'_2) \equiv \gamma T'. \tag{23}$$

Is there any more intuitive way to understand this bizarre result?

Intuitive Approach

Consider the situation depicted below. An observer sits at rest in a freight car that moves with velocity v relative to the ground. He sends a light signal to a mirror on the top of the car that bounces back to him. The time between sending and receipt is defined as the tick of the clock. He concludes that

$$T' = \Delta t' = \frac{distance\ traveled}{speed\ of\ light} = \frac{2d}{c}. \tag{24}$$

Figure 5: The "mirror" clock.

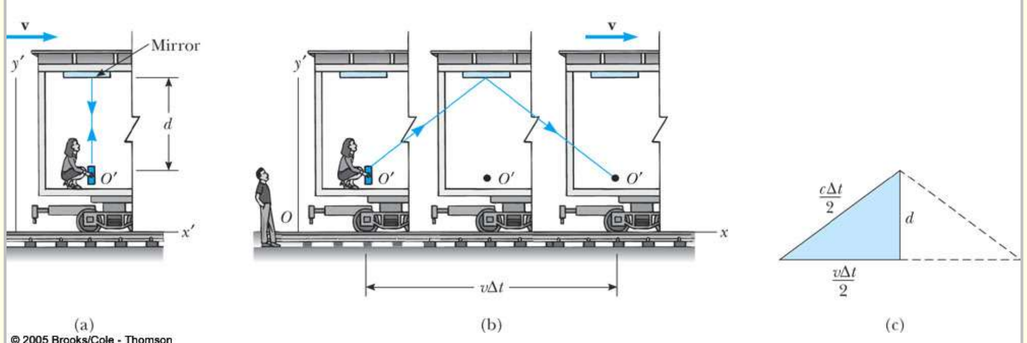


Fig. 1-10, p. 15

Meanwhile, the observer on the ground sees the freight car move a

distance $\frac{1}{2}v\Delta t = \frac{1}{2}vT$ (where T is the full tick time) between the time the signal is sent and the time the signal hits the top of the car. The geometric picture then allows us to compute $\frac{1}{2}vT$ as follows. The actual distance traveled by the light for this half of the trip is $\sqrt{(\frac{1}{2}vT)^2 + d^2}$, and this must equal the amount of distance $\frac{1}{2}cT$ that light can travel when moving with velocity c: i.e.

$$(\frac{1}{2}vT)^2 + d^2 = (\frac{1}{2}cT)^2\,, \tag{25}$$

from which we obtain

$$T^2 = \frac{4d^2}{c^2 - v^2}\,, \quad \text{or} \quad T = \frac{2d}{c}\gamma = T'\gamma\,. \tag{26}$$

Note how we employed the fact that the light is moving with velocity c according to both observers.

Now, you might ask if there is any experimental verification of this bizarre result. The answer is many!

An example: the decay of an unstable elementary particle called the muon (denoted μ).

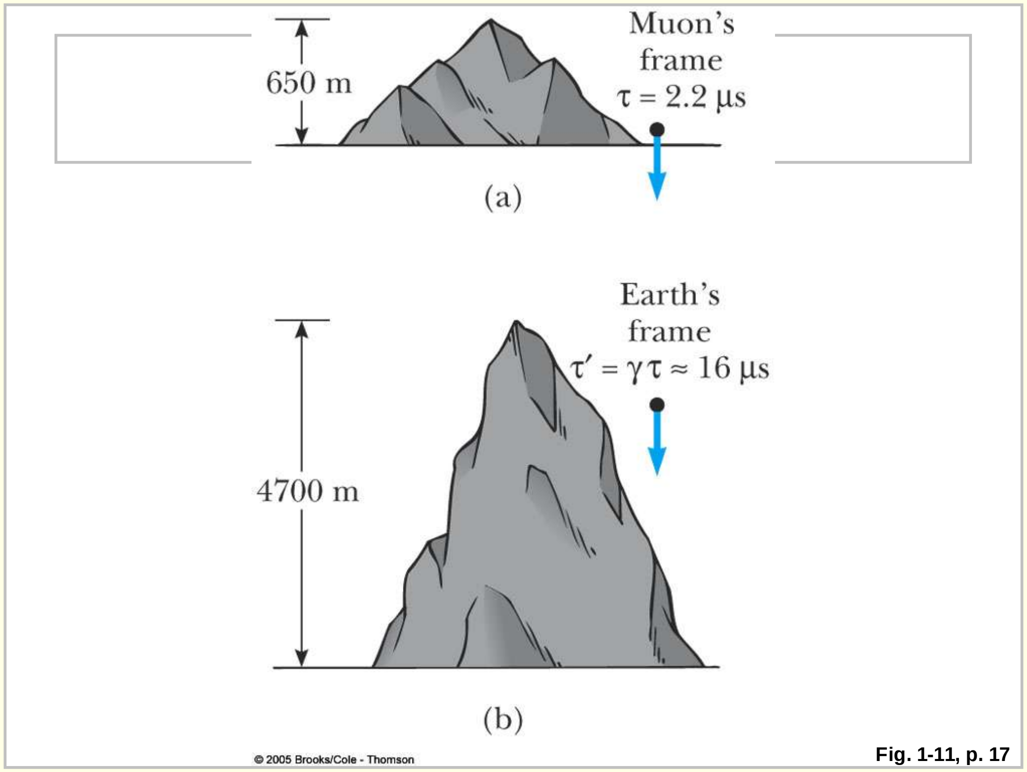


Figure 6: Muon decay in the muon (a) and the earth (b) frame.

The μ has its own internal clock that determines how fast it decays. On average, the lifetime of the μ is denoted τ (I stick to the book, even though I would rather have called this τ' — so you have to switch prime and unprime relative to above discussion.) Measurements of the exponential decay curve $e^{-t/\tau}$ (t is the muon's internal time) give $\tau = 2.2\ \mu s$ for muons at rest.

However, if the μ is moving with $v = 0.99c$ ($\gamma = 7.1$) with respect to the earth (as typical of μ's contained in cosmic rays), the earth thinks the muon clock is running much slower and has lifetime $\gamma\tau = 16\ \mu s$.

Thus, on average its travel distance in the earth rest frame is given by $v\gamma\tau = 4700\ m$, far longer than without the time dilation factor — as far as the muon observer is concerned, he only moves $v\tau = 650\ m$ on average before decaying.

Length Contraction

We imagine laying out a ruler in the moving frame S' from $x' = 0$ to $x' = L'$. How long does this ruler appear to the stationary viewer S?

Mathematical Approach

We wish to determine the length as measured by S *at some fixed time*, say $t = 0$. So go to the Lorentz transform equation

$$x' = \gamma(x - vt) \tag{27}$$

and set $t = 0$. Then, the above equation implies that $x' = 0$ corresponds to $x = 0$, while $x' = L'$ corresponds to $L' = \gamma x \equiv \gamma L$. As a result, we find that

$$L = L'/\gamma\,. \tag{28}$$

Since $\gamma > 1$, the apparent length, L, of the ruler to observer S is smaller than the length L' as seen by observer S'.

Thus, if a ruler at rest in S' is zooming past S with velocity v, S swears that the ruler is shorter by the factor $\frac{1}{\gamma}$.

The quantity L' measured by the observer S' who is at rest wrp to the ruler is called the *proper length*, sometimes denoted L_p. It is defined just like the proper time was defined as the time between ticks of a clock as measured by someone at rest wrp to the clock.

An Intuitive Approach

This approach (which follows the book) will make use of the time dilation result already derived.

We consider two stars which are separated by a proper length L_p as observed by someone on earth (which is essentially, i.e. for the purpose of this discussion, at rest wrp to the stars).

Now we consider a moving space ship. According to the observer on earth the space ship takes a time $\Delta t = L_p/v$ to travel between the stars.

Because of time dilation, the space traveler measures a smaller time of travel between the two stars: $\Delta t' = \Delta t/\gamma$. (Which is just another way of writing $\Delta t = \gamma\Delta t'$, the form written earlier when discussing time dilation.) The space traveler interprets this time of travel (which, to repeat, is the time he measures) as corresponding to a distance of travel

$$L = v\Delta t' = v\frac{\Delta t}{\gamma} = \frac{L_p}{\gamma}. \tag{29}$$

Return to the μ example

Suppose we have a mountain at rest on the earth (as in the original picture).

The height of this mountain as measured by someone on earth is the mountain's proper height, call it h_p.

Similarly, let us call the amount of time that the muon believes it takes to reach the bottom of the mountain (when moving vertically) t_p, the proper time interval in the muon rest frame.

The amount of time according to the earth observer that the muon takes to traverse the height h_p (moving vertically) is h_p/v and this must be related to t_p by the time dilation factor: i.e. $\gamma t_p = h_p/v$ or

$$t_p = \frac{1}{\gamma}\frac{h_p}{v}\,. \tag{30}$$

From the muon's point of view has he reached the bottom of the mountain after this time t_p?

For this to be true, it must be that vt_p is the height of the mountain as seen by the μ. Computing, we have

$$vt_p = v\left[\frac{1}{\gamma}\frac{h_p}{v}\right] = \frac{h_p}{\gamma}\,, \tag{31}$$

which is precisely the contracted mountain height as viewed by the μ.

So the two observers agree about when the μ has passed from the top of the mountain to the bottom of the mountain!

Note that the two observers had to agree about their relative velocity for this to work out.

Well, I suspect your head is swimming at this point with confusion about when to use γ and when to use $1/\gamma$. It takes practice, and so you must do a bunch of problems to get the hang of this.

The crucial rule is to always remember that:

1. proper time refers to the time measured at a fixed location in some frame (fixed location means the clock is at rest in that frame).

2. proper length refers to a length measured (at some given time) for an object by an observer at rest with respect to that object.

Applications

There are many other interesting applications of all this. Here, I will focus on the relativistic Doppler shift.

This is a particularly important application as it is the Doppler "red"-shift that we use to tell us that the universe is expanding from something like an initial big bang.

You are all familiar with the usual Doppler shift in which the pitch of sound for a whistle on a train headed towards you has a higher pitch than the whistle sound when the train is moving away.

This is because successive waves emitted by a source moving towards you are closer together than normal because of the advance of the source — and since their separation is the wavelength of sound, the corresponding frequency is higher.

The formula in the case of sound is probably something you have derived in an earlier course.

$$f = f_0 \left(\frac{1 + \frac{v}{c}}{1 - \frac{V}{c}} \right) , \tag{32}$$

where f_0 is the frequency of the sound as measured by the source itself, f is the frequency as measured by the observer, c is the speed of sound, v is the speed of the observer ($+$ for motion toward source), and V is the speed of the source ($+$ for motion toward the observer).

This classical Doppler effect evidently varies depending upon whether the source, the observer, or both are moving.

- **This does not violate relativity because sound does travel in a medium — unlike light.**
- **Since light does not travel in a medium, the light wave Doppler effect will be different.**
  It can be derived using the concepts of time dilation. My derivation is an alternative approach to that given in the book. Note, in particular, that the book assumes that the source is moving towards the observer.

Imagine a light source as a clock that ticks f_0 times per second and emits a light wave peak at each tick. The proper time in the source rest frame between ticks is $t_0 = 1/f_0$.

Consider a source at rest and an observer moving away from it with velocity v. The interval between ticks as seen by this observer is given by time dilation: $t = \gamma t_0$

As viewed by the observer, he travels the distance vt away from the source between ticks.

Thus, each tick takes a time vt/c longer to reach him than the simple

time t between ticks.

The total time between the arrival of successive peaks (successive ticks) is then

$$\begin{aligned} T &= t + \frac{vt}{c} = t(1 + \frac{v}{c}) = \gamma t_0 (1 + \frac{v}{c}) \\ &= t_0 \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = t_0 \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}. \end{aligned} \tag{33}$$

The corresponding frequency of the ticks or wave peaks is just the inverse:

$$f(\text{receding case}) = \frac{1}{T} = \frac{1}{t_0} \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = f_0 \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \tag{34}$$

- Now, although I derived this for an observer moving away from the source, the same result applies if the source moves away from the observer.
- You should also note that the frequency shift depends only on the relative velocity of the source and observer. One does not need to reference any medium in which light travels.

- Since wavelength and frequency are inversely related, $f\lambda = c$, the shift in λ obeys the inverse formula.

Example:

Determining the speed of recession of the Galaxy Hydra.

A certain absorption line that would be at $\lambda_0 = 394\ nm$ were Hydra at rest, is shifted to $\lambda = 475\ nm$ according to observations on earth.

We use

$$\lambda = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\lambda_0 \tag{35}$$

to find that

$$\frac{v}{c} = \frac{\frac{\lambda^2}{\lambda_0^2} - 1}{\frac{\lambda^2}{\lambda_0^2} + 1} = 0.185\,. \tag{36}$$

Therefore, Hydra is receding from us with a velocity of $v = 0.185\ c = 5.54 \times 10^7\ m/s$.

Show tape #42 on space-time diagram starting at 20 min mark. *

Lorentz Velocity Transformation

This is obtained by taking derivatives of the Lorentz transform formulae of eq. (12), in close to analogy to what we briefly described for the Galilean case.

From eq. (12), we have

$$dx' = \gamma(dx - vdt) \quad dt' = \gamma(dt - \frac{v}{c^2}dx) \tag{37}$$

Thus,

$$u'_x \equiv \frac{dx'}{dt'} = \frac{dx - vdt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}. \tag{38}$$

Similarly, if the object we are examining has some y direction velocity (but the relative velocity of the frames is still along the x axis), we have

$$dy' = dy \tag{39}$$

and the same formula for dt' as above, from which we obtain

$$u'_y \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}, \tag{40}$$

and a similar result for u'_z.

Important notes:

1. These results reduce to Galilean forms if $v \ll c$.
2. If $u_x = c$ and $u_z = u_y = 0$, we find

$$u'_x = \frac{c - v}{1 - \frac{v}{c}} = c\,, \tag{41}$$

which is to say that if something (like light) moves with velocity c in one frame, it also moves with c in the other frame.
This was obviously required for consistency of the Lorentz transforms with Einstein's original postulate.
Another example of this is to consider the case where $u_y = c$, $u_x = 0$ and $u_z = 0$. Our velocity transform equations give

$$u'_y = \frac{c}{\gamma}\,, \quad u'_x = -v\,, \tag{42}$$

from which we compute

$${u'_x}^2 + {u'_y}^2 = v^2 + \frac{c^2}{\gamma^2} = v^2 + c^2(1 - \frac{v^2}{c^2}) = c^2\,. \tag{43}$$

Again we get the same velocity for light, c, in both frames.

An Application: the relative velocity of 2 space ships.

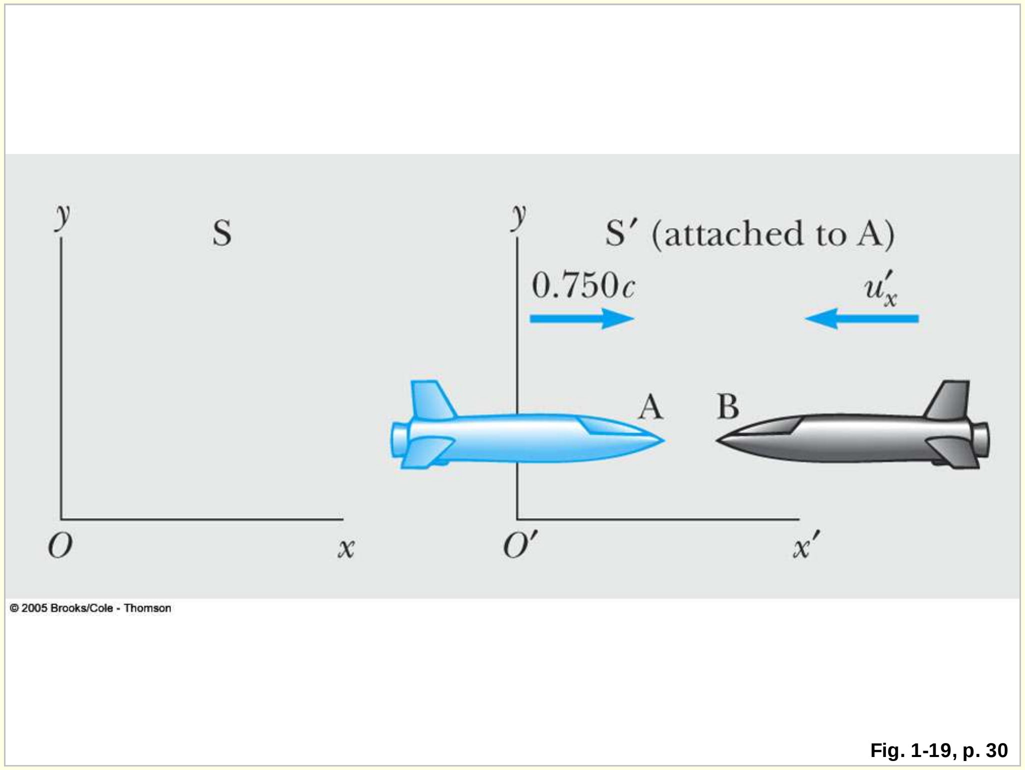


Figure 7: Two spaces ships moving towards one another relative to a stationary observer.

Consider two space ships A and B. A moves with velocity $0.75\ c$ in the $+x$ direction relative to a stationary observer. B moves with velocity

$0.85\ c$ in the $-x$ direction relative to the same stationary observer. Find the relative velocity of B with respect to A. The configuration is pictured above.

One key phrase here is "Find the relative velocity of B with respect to A." What these words mean is that A looks at B from his (A's) rest frame. What this means for solving this problem is that we must lock one frame, let us choose S' to A, *i.e.* we choose S' to be A's rest frame.

For S we choose the stationary observer. So, in applying the formulas, we identify the speed of the S' frame with the speed of A, *i.e.* $v = 0.75\ c$ wrp to the earth.

Now, we apply the formulas imagining that both S and S' are looking at B. We want the velocity of B in the rest frame of A. This is u'_x in the context of the formulas.

We have

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-0.85c - 0.75c}{1 - \frac{(-0.85c)(0.75c)}{c^2}} = -0.9771c\,. \tag{44}$$

Note that the naive Galilean result would be $-1.6c$, whereas the correct result is smaller than c in magnitude.

Space-Time and Causality

If we focus on just the x and ct coordinates, we can picture space-time as below.

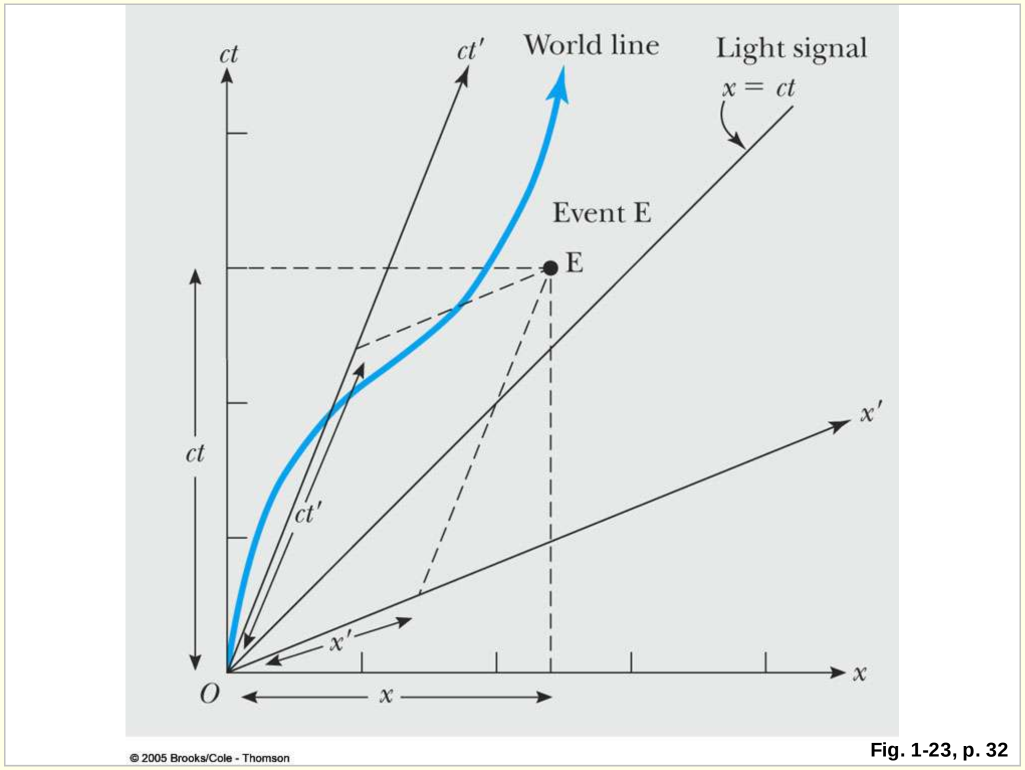


Figure 8: Two-dimensional picture of space-time.

The coordinates x and ct in frame S are drawn along perpendicular axes.

An event E is characterized by its (x, ct) location. The trajectory of a particle is characterized by its "world-line" in this space.

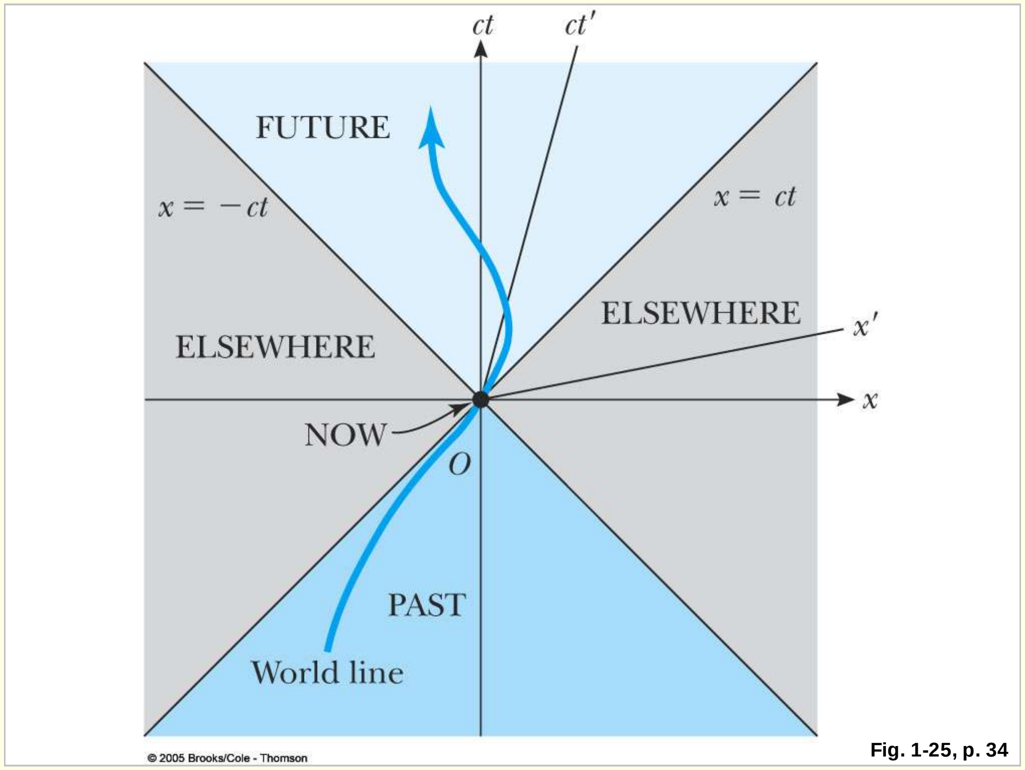


Figure 9: Causal division of space-time

Since the velocity u_x of the particle is defined by

$$u_x = c\frac{\Delta x}{\Delta ct} = \frac{c}{slope}, \quad (45)$$

and since $u_x \leq c$ (nothing moves faster than light), we see that $slope \geq 1$ is required, so that the blue "world-line" can at most be $45°$ from the vertical ct axis. At light pulse with $u_x = c$ would travel along the $45°$ line. Nothing can ever travel into the section of the diagram below the $45°$ line. Thus, the picture gets divided up into the future, the past, and unreachable regions denoted "elsewhere" in the picture above.

If you want to know how things in frame S appear to a moving observer, S', you need to draw the manner in which his coordinate axes relate to those of S. We will not go through the details, but the result appears in Figs. 8 and 9. The larger the relative velocity, the more collapsed the axes become. There is a cute Carl Sagan dvd that illustrates this. (DVD #8, Chapter 4, starting about 20:35 with motorscooter.)

We can now discuss the concept of causality. Consider two events in space time, as depicted in Fig. 10. One occurs at (ct_1, x_1) the other occurs at (ct_2, x_2). (We will assume that $y_1 = y_2$ and $z_1 = z_2$.) Compute the four-vector

$$\Delta x_4 = (ct_2 - ct_1, x_2 - x_1) \tag{46}$$

and its square:

$$(\Delta s)^2 \equiv \Delta x_4 \cdot \Delta x_4 = (ct_2 - ct_1)^2 - (x_2 - x_1)^2 \,. \tag{47}$$

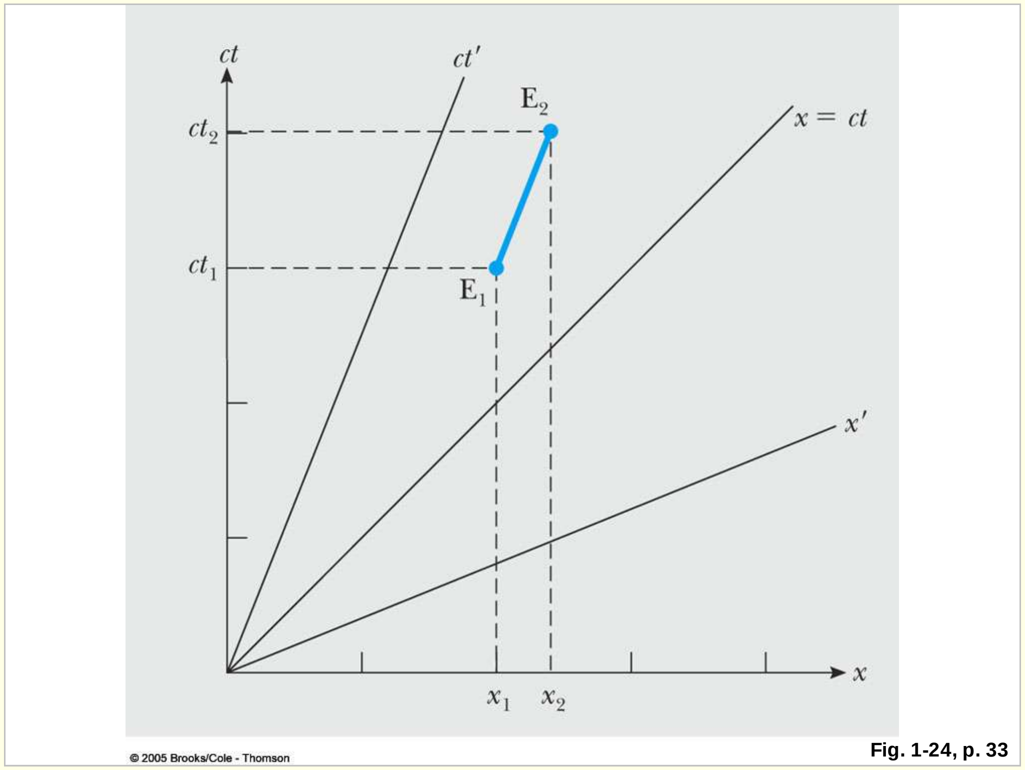


Figure 10: Depiction of two events in space-time

We know that this quantity is actually independent of frame. That we stated is a consequence of the Lorentz transformation that you can easily check. Thus,

$$(\Delta s')^2 \equiv \Delta x'_4 \cdot \Delta x'_4 = (ct'_2 - ct'_1)^2 - (x'_2 - x'_1)^2 = (\Delta s)^2 . \qquad (48)$$

This frame-independent quantity, $(\Delta s)^2$, has a very important interpretation. If $(\Delta s)^2 \geq 0$, then event E_1 can influence E_2 by virtue of the fact that a light ray emitted from x_1 at time t_1 could reach x_2 before or at time t_2. We say that "E_2 is causally related to E_1".

If $(\Delta s)^2 > 0$, we call the interval time-like. In this case, even something traveling slower than light could reach E_2 from E_1 in time to influence E_2.

If $(\Delta s)^2 = 0$, we call the interval light-like. In this case, only light emitted from E_1 could influence E_2.

If $(\Delta s)^2 < 0$, the interval is called space-like. In this case, there is no possibility for E_1 to influence E_2 since nothing can travel faster than light.

Lorentz Transformations for the 4-vector components

We can rewrite the Lorentz transform equations of eq. (12) in a form which displays a great symmetry between the roles of ct and x.

$$x' = \gamma(x - \frac{v}{c}ct) \tag{49}$$

$$ct' = \gamma(ct - \frac{v}{c}x)\,. \tag{50}$$

Example of direct use of Lorentz Transformation:

Discuss an event as seen in two different frames.

Anna and Bob are in identical spaceships $100\ m$ long each, with distances from the back labeled along the sides. Prior to taking up space travel in retirement, Bob and Anna owned a clock shop, and they glued the leftover clocks all over the walls of their ships.

As Bob's space ship is flying along, Anna passes him with relative speed $v = 0.8\ c$, headed in the same direction (say $+x$ direction). Just as the back of Anna's ship passes the back of Bob's ship, the clocks on both ships read 0. At this same instant, Bob Jr., on board Bob's space ship, is aligned with the very front edge of Anna's ship. He peers through a window in Anna's ship and looks at the clock.

(a) In relation to his own ship, where is Bob Jr., and (b) what does the clock he sees read?

Solution:

Let us choose Bob as the stationary observer (frame S). Anna will be in frame S', which then has $v = +0.8c$ relative to S (Bob).

The key here is "at the instant shown, Bob Jr. on Bob's space ship". What this means is that Bob Jr. is doing something (looking at Anna's front end) at time $t = 0$ (the time on Bob's space ship), which will not be the same as the time on some clock opposite him on Anna's space ship. (Only the backs of the two passing space ships agree about the time.) However, we do know where Bob Jr. is looking! He is looking at $x' = 100\ m$ ($i.e.$ the front of Anna's space ship).

(a) So where is Bob Jr as he is doing the looking? Use $x' = \gamma(x - vt) = \gamma x$ at $t = 0$ with $\gamma = 1/0.6$ to compute $x = x'/\gamma = 60\ m$. This result is the same as you would get just thinking about Lorentz contraction of Anna's space ship as seen by Bob's space ship.

(b) What is the time on Anna's clock opposite Bob? Use

$$\begin{aligned} t' &= \gamma\left(t - \frac{v}{c^2}x\right) \\ &= -\gamma\frac{v}{c^2}x \quad \text{since } t = 0 \text{ as above} \\ &= -\gamma\frac{v}{c^2}x'/\gamma \quad \text{using our result above for } x \end{aligned}$$

$$
\begin{aligned}
&= -\frac{0.8c}{c^2}x' \\
&= -0.8\frac{1}{3\times 10^8\ m/s}100\ m \\
&= -2.66\times 10^{-7}\ s
\end{aligned}
\tag{51}
$$

Causality Example: **The death ray problem.**

At a certain instant, two identical rocket ships pass one another (headed in opposite directions) so that at a certain instant the cockpit of one is lined up with the tail of the other. (a) Can a tail gunner of number 1 fire a death ray at pilot 2? (b) Could copilot 2 send a signal to tail gunner 2 at the velocity of light enabling tail gunner 2 to kill pilot 1? Ignore separation between rockets and assume perpendicular firing of death rays.

We will examine in rest frame of 2 with 1 zipping by: $i.e.$ 1=S' and 2=S.

(a) At $t'=0$ suppose tail of 1 is at $x'=0$ and pilot and copilot of 1 are at $x'=d$. Use $x=\gamma(x'+vt')$ and $t=\gamma(t'+\frac{v}{c^2}x')$ to conclude that $x=0$ and $t=0$ is where tail 1 is, and so tail of 1 can kill pilot 2.

(b) First, we need to find out where pilot 1 is from perspective of the tail gunner of ship 2. Pilot 1 is at $x' = d$ at $t = 0$ when pilot 2 is killed. Using $x' = \gamma(x - vt)$, we find $x = x'/\gamma = d/\gamma$ (length contraction). So, it looks like there is hope. But, copilot 2 still has to get the signal to tail gunner 2. He has a certain amount of time to do so. The signal must reach tail gunner 2 by the time the front of space ship 1 passes the tail gunner 2. The time to reach this configuration (in ship 2 in S frame) is $t = (d - d/\gamma)/v$ and this must be $> d/c$ if the signal light ray from copilot 2 is to reach the tail gunner in 2 early enough. This requirement thus is:

$$\begin{aligned} \frac{d(1-\frac{1}{\gamma})}{v} &> \frac{d}{c} \quad or \\ 1 - \frac{1}{\gamma} &> \frac{v}{c} \quad or \\ 1 - \frac{v}{c} &> \sqrt{1 - \frac{v^2}{c^2}} \quad or \\ \sqrt{1 - \frac{v}{c}} &> \sqrt{1 + \frac{v}{c}}, \end{aligned} \tag{52}$$

which is impossible.

This would have been obvious from the perspective of pilot 1 as he looks at tail gunner 2. At $t' = 0$, tail gunner 2 is at $x = d$ (as always) and using $x = d = \gamma(x' + vt')$ (inverse LT) at $t' = 0$ gives $x' = \frac{d}{\gamma}$, equivalent to Lorentz contraction. The physical picture is that the tail gunner 2 at $x' = d/\gamma$ is already well behind the pilot of ship 1 located at $x' = d$ so that there was clearly no hope to begin with!

Causality interval approach.

event 1: kill at $x = x' = t = t' = 0$.

event 2: passage of tail 2 past pilot 1.

This latter passage occurs when $x' = d$ *and* $x = d$, since the pilot of 1 is at $x' = d$ and the tail gunner of 2 is at $x = d$. Thus, we must have

$$\begin{aligned} d &= \gamma(d - vt) \quad or \\ tv &= d - \frac{d}{\gamma} \quad or \\ t &= \frac{d}{v}(1 - \frac{1}{\gamma})\,. \end{aligned} \tag{53}$$

So now compute the causality interval from the tail gunner 2 point of view (*i.e.* using the unprime frame coordinates for the two events, the

2nd being hypothetical for the moment) as

$$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 = \frac{c^2 d^2}{v^2}(1 - \frac{1}{\gamma})^2 - d^2 = d^2 \left(\frac{c^2}{v^2}(1 - \frac{1}{\gamma})^2 - 1 \right) \tag{54}$$

If this is negative, $\Rightarrow$ no causal connection. We thus ask if

$$\begin{aligned} &(1 - \frac{1}{\gamma})^2 - \frac{v^2}{c^2} < 0? \quad or \\ &1 - \sqrt{1 - \frac{v^2}{c^2}} < \frac{v}{c}? \quad or \\ &1 - \frac{v}{c} < \sqrt{1 - \frac{v^2}{c^2}}? \quad or \\ &\sqrt{1 - \frac{v}{c}} < \sqrt{1 + \frac{v}{c}}? \quad \textbf{which is true.} \end{aligned} \tag{55}$$

Thus, no causal connection possible. The tail gunner of 2 cannot kill the pilot of 1 in response to tail gunner of 1 killing the pilot of 2.

Causality and Lorentz Transforms applied to the μ example

Let us define event 1 (E1) as the passage of the μ by the top of the mountain. We will define earth coordinates S and muon coordinates S' so that this occurs at $x = x' = 0$ and $t = t' = 0$. (We measure both x and x' in the vertical downwards direction.)

Let us define event 2 (E2) as the passage of the μ by the bottom of the mountain. In system S, we know that this happens at $t = h_p/v$ (h_p recall is the proper mountain height measured in the S frame). We also know that it happens a $x = h_p$.

What are the coordinates of E2 as viewed by the μ in his frame? According to LT we have $x' = \gamma(x - vt)$ and $t' = \gamma(t - \frac{v}{c^2}x)$. So for E2 we have

$$x' = \gamma(h_p - v\frac{h_p}{v}) = 0\,, \tag{56}$$

(which makes sense since the μ does not change his location in his own frame) and

$$t' = \gamma(\frac{h_p}{v} - \frac{v}{c^2}h_p) = \gamma\frac{h_p}{v}(1 - \frac{v^2}{c^2}) = \frac{h_p}{\gamma v}\,, \tag{57}$$

in agreement with the time dilation factor for S vs. S'.

So, now let us compute $(\Delta s)^2$ in the two different frames. (Recall, all coordinates $= 0$ for E1.)

$$(\Delta s)^2_S = c^2\left(\frac{h_p}{v}\right)^2 - h_p^2 = h_p^2\left(\frac{c^2}{v^2}-1\right) \quad (58)$$

$$(\Delta s)^2_{S'} = c^2\left(\frac{h_p}{\gamma v}\right)^2 - 0^2 = \frac{c^2}{v^2}h_p^2\left(1-\frac{v^2}{c^2}\right) = h_p^2\left(\frac{c^2}{v^2}-1\right). \quad (59)$$

We get the same result in both frames and, further, $(\Delta s)^2 > 0$ implies that E1 and E2 are causally connected. Well, we already knew this since the traveling μ itself was the means of establishing this causal connection!

Velocity transform equations and momentum conservation

Suppose we on earth see 2 rocket ships passing one another.

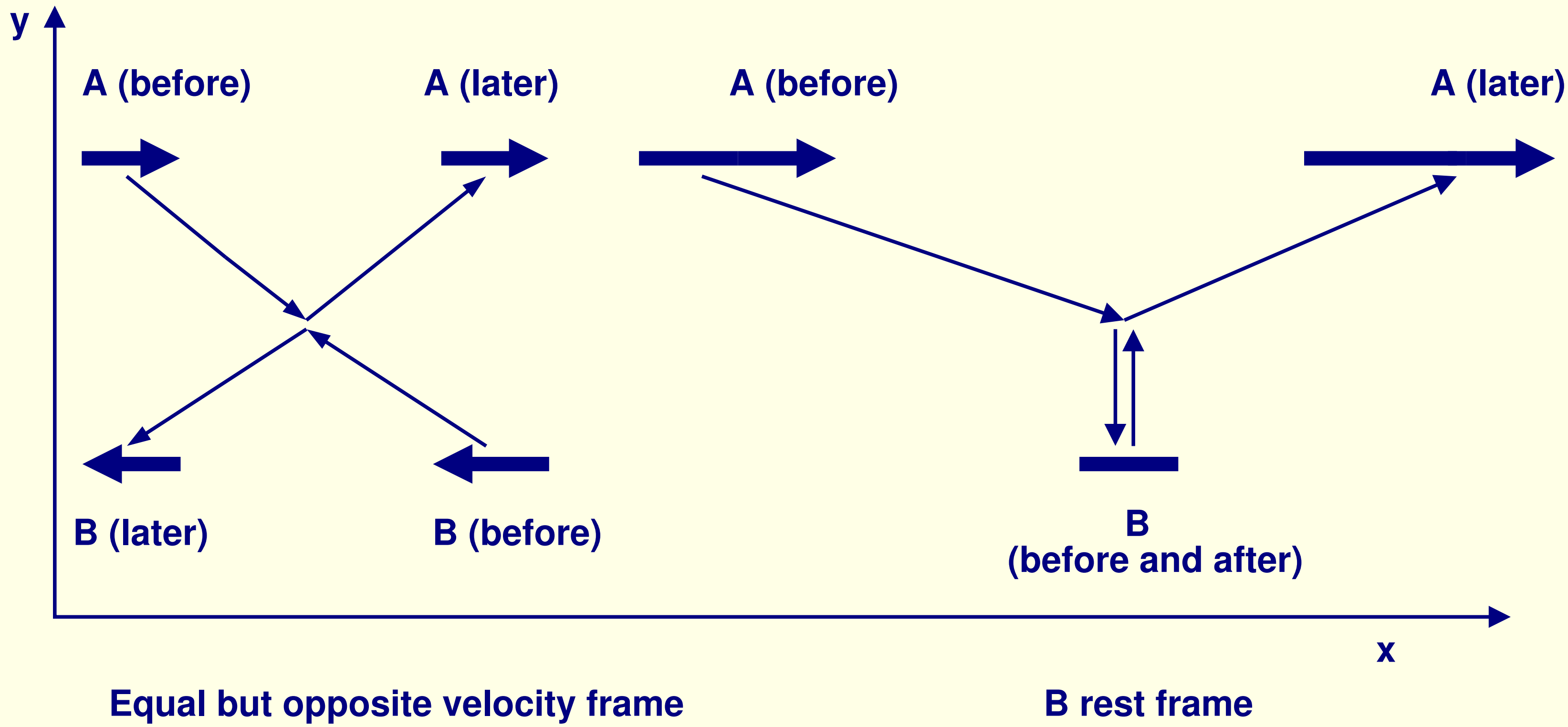


Figure 11: Depiction of two rockets passing one another: a) in equal but opposite velocity frame, each having velocity of magnitude v'; b) in B rest frame, where velocity of A appears to be v.

Consider first the equal-but-opposite velocity frame (the left-hand picture). Someone on rocket ship A throws a billiard ball of mass m downwards (in the A rest frame) and someone in B throws an identical billiard ball upwards (in the B rest frame) with exactly the same vertical velocity, $i.e.$ $u_A^y = -u_B^y$. (We will assume that these y direction velocities are $\ll c$ in magnitude.) The billiard balls collide elastically and go back to the respective starting points with exactly reversed y direction velocities. Obviously, we have

$$\Delta u_A^y = -\Delta u_B^y, \tag{60}$$

and after multiplying by m

$$m\Delta u_A^y = -m\Delta u_B^y\,, \tag{61}$$

or using the usual definition of momentum as $m \times velocity$,

$$\Delta p_A^y = -\Delta p_B^y\,, \quad \Rightarrow \quad \Delta(p_A + p_B)^y = 0\,. \tag{62}$$

Momentum is conserved and Newton is happy! Note that we do not care about the exact magnitude of the y velocity as viewed from the equal-but-opposite rocket ship velocity frame. It will not be the same as the y velocities discussed below in the B rest frame.

But, if momentum is conserved in one frame then by covariance (*i.e.* all physical laws the same in all frames), momentum should also be conserved if we view this same process in the rest frame of rocket ship B (2nd picture in Fig. 11). In this frame, we choose rocket ship A to carry the prime coordinates and rocket ship B to carry the unprime coordinates.

The situation described is that

$$u_A^{y\,\prime} = -u\,, \quad \text{and} \quad u_B^y = u\,, \tag{63}$$

which is to say that in their respective frames each thinks he has thrown the ball with velocity of magnitude u. But, if we look entirely from B's rest frame (the unprime frame), we have (using time dilation and the fact that $y_A = y'_A$ for relative frame velocity in the x direction)

$$|u_A^y| = \left|\frac{\Delta y_A}{\Delta t}\right| = \left|\frac{\Delta y'_A}{\gamma \Delta t'}\right| = \left|\frac{1}{\gamma} u_A^{y\,\prime}\right| \neq |u_B^y|\ , \tag{64}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ with v being the velocity of rocket ship A along the x direction in B's rest frame.

(Alternatively use $u_A^y = \frac{u_A^{y\,\prime}}{\gamma(1+\frac{vu_A^{x\,\prime}}{c^2})}$, the inverse velocity transform equation, with $u_A^{x\,\prime} = 0$ since A's ball only has y velocity in A's rest frame.)

As a result, we find that

$$\Delta u_A^y = 2\,|u_A^y| = -\frac{1}{\gamma}\Delta u_B^y\,. \tag{65}$$

and if we define momentum by $p_A^y = mu_A^y$ and $p_B = mu_B^y$ we do not get $\Delta p_A^y = -\Delta p_B^y$. To keep momentum conservation, observer B must claim that $m_A \neq m_B$ when looking at A's billiard ball in his B rest frame. What we need is an extra factor of γ. Sometimes, we write

$$m_A = \gamma m_B\,, \tag{66}$$

where m_B is the mass of the billiard ball when viewed from the B rest frame and m_A is the mass of an identical billiard ball that is nearly at rest in the A rest frame but being viewed from the B rest frame.

In the above discussion, we implicitly kept our vertical velocity $(\vec{u})^y \ll c$ so that in the B rest frame $m_B \simeq m_0$, where m_0 is the mass of the billiard ball as measured in its own rest frame. This approximation is fine if v is substantial (as implicitly assumed) so that $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \gg 1$. How do we generalize?

In general, we can retain momentum conservation provided we define

$$\begin{aligned}(\vec{p})^y &= m(\text{total velocity}) \times (\text{velocity of mass in } y \text{ direction}) \\ &= \gamma(u) m_0 (\vec{u})^y\,, \end{aligned} \tag{67}$$

and similarly for the x and z directions.

Here, the notation m_0 is reserved for the **proper** mass of the billiard ball, defined to be the mass of the billiard ball as measured in the billiard ball rest frame.

Hopefully, this discussion is sufficient motivation for the general result that

$$\vec{p} = \gamma(u) m_0 \vec{u} \tag{68}$$

where

$$\gamma(u) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\,, \tag{69}$$

with $u = |\vec{u}|$ being the absolute value of the total billiard ball velocity as viewed from some frame. (In the above discussion, $\vec{u} = v\hat{x} + (\vec{u})^y \hat{y} \simeq v\hat{x}$.)

If we return to our original "equal-but-opposite" rocket ship velocity frame, then the billiard balls have equal total velocities, say v',[2] in the observer's rest frame and momentum will be defined (keeping the small u^y approximation) using $m = \gamma(v')m_0$, with $\gamma(v') = \frac{1}{\sqrt{1-\frac{v'^2}{c^2}}}$, since in that frame each rocket ship (*and the billiard ball each throws with small* y *direction velocity*) is moving with velocity v' with respect to the observer. Thus, momentum will still be conserved in that frame as well.

Nowadays, we tend to back away from the use of the concept of a "frame-dependent mass", $m = \gamma(u)m_0$ for total velocity u, and simply say that momentum must include the extra factor of $\gamma(u)$ in its definition. The two points of view are algebraically equivalent.

Newton's 2nd Law

Given the above definition of the momentum that is conserved in the

[2] Compute v' from velocity addition: $v = 2v'/(1 + v'^2/c^2) \Rightarrow v' = c\left(c/v - \sqrt{c^2/v^2 - 1}\right)$.

absence of force, $\vec{p} = \gamma(u) m_0 \vec{u}$, in the presence of force it is natural to define force by

$$\vec{F} = \frac{d\vec{p}}{dt}\,. \tag{70}$$

This equation reduces to the non-relativistic Newton's law when $v \ll c$ so that $\gamma \to 1$.

Note that because $\gamma(u) m_0$ increases with the object's total velocity, u, becoming ∞ at $u = c$, it is never possible to accelerate a particle past the speed of light, or even to the speed of light.

Relativistic Energy

We start with (use one-dimensional derivation – force and motion along x axis)

$$W = \int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} \frac{dp}{dt} dx\,. \tag{71}$$

We can rewrite using

$$\frac{dp}{dt} dx = \frac{dp}{dt}\frac{dx}{dt} dt = dpu = \frac{dp}{du} duu\,. \tag{72}$$

Inserting into eq. (71), we get

$$W = \int_{u_i}^{u_f} \frac{dp}{du} u du \,, \tag{73}$$

where u_i and u_f are the initial and final velocities at locations x_1 and x_2. Now,

$$\frac{dp}{du} = \frac{d}{du}\left[\frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}}\right] = \frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{3/2}} \,. \tag{74}$$

Inserting into eq. (73) and assuming $u_i = 0$ gives

$$W = \int_0^{u_f} \frac{m_0 u}{\left(1-\frac{u^2}{c^2}\right)^{3/2}} du = \frac{m_0 c^2}{\sqrt{1-\frac{u_f^2}{c^2}}} - m_0 c^2 \,. \tag{75}$$

At this point, it is conventional to drop the subscript f and simple write W in terms of the particle velocity u. The book also drops the subscript 0 on m_0 and so m will henceforth denote the intrinsic or proper mass.

We then have,

$$W = \gamma(u)mc^2 - mc^2\,, \quad \text{with} \quad \gamma(u) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\,. \tag{76}$$

Often, $\gamma(u)$ is simply written as γ, but you must remember that it is the particle's velocity in your frame that goes into this form and not some relative velocity of two different frames.

Interpretation of W

How should we interpret W? In the non-relativistic case, W would be the kinetic energy, K. We can check this correspondence for small u/c by expanding:

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \simeq 1 + \frac{1}{2}\frac{u^2}{c^2} \tag{77}$$

from which we obtain

$$W \equiv K \simeq \frac{1}{2}mu^2 \tag{78}$$

in agreement with the usual kinetic energy formula.

Since K is a change in energy, this formula suggests that mc^2 and $\gamma(u)mc^2$ should be thought of as the rest frame energy and the energy in motion, respectively.

Thus,

$$\gamma(u)mc^2 = \text{rest energy} + \text{kinetic energy} = \text{total energy} \equiv E\,. \quad (79)$$

If this is correct then mass is a form of energy. This is dramatically confirmed in many ways, as we shall see.

An example

A proton has kinetic energy equal to half its rest mass energy. (a) What is the proton's speed? (b) What is its total energy? (c) Determine the potential difference ΔV through which the proton would have to be accelerated to attain this speed.

(a) $K = (\gamma(u) - 1)m_pc^2 = \frac{1}{2}m_pc^2 \Rightarrow \gamma(u) = 1.5$. For $\gamma(u) \equiv \frac{1}{\sqrt{1-(u/c)^2}} = 1.5, \Rightarrow u = 0.745c$.

(b) $E = \gamma(u)m_pc^2 = 1.5(1.67 \times 10^{-27}\ kg)(3 \times 10^8\ m/s)^2 = 2.25 \times 10^{-10}\ J = 1409\ \mathrm{MeV}$.

(c)

$$
\begin{aligned}
K &= (\gamma(u)-1)m_pc^2 = 0.5m_pc^2 \\
&= 0.5(1.67\times 10^{-27}\ kg)(3\times 10^8\ m/s)^2 \\
&= |q\Delta V| = (1.6\times 10^{-19}\ C)\Delta V
\end{aligned}
\tag{80}
$$

$\Rightarrow \Delta V = 470\ MV$.

Classically, much less ΔV would have sufficed ($291\ MV$), but that increasing $\gamma(u)$ factor implies more and more is needed as u gets closer to c.

The relationship between energy, momentum and mass

We have $E^2 = \gamma^2(u)m^2c^4$ and $p^2c^2 = \gamma^2(u)m^2c^2u^2$. Using these inputs we find

$$
E^2 - p^2c^2 = \gamma^2(u)m^2c^4\left(1-\frac{u^2}{c^2}\right) = m^2c^4, . \tag{81}
$$

Since m is an intrinsic property of the particle, the right-hand side is always the same, no matter what frame we examine E and p in.

If we were to set $m=0$, we get $E=pc$. This is in fact the relationship between energy and momentum that applies to the particles of light called photons and to other massless particles.

In any case, the above invariance implies that

$$p_4 = (E/c, p_x, p_y, p_z) \tag{82}$$

is a 4-vector in the sense we defined earlier since its square

$$p_4 \cdot p_4 = E^2/c^2 - \vec{p} \cdot \vec{p} = m^2 c^2 \,, \tag{83}$$

is an invariant independent of frame, just like $x_4 \cdot x_4 = c^2 t^2 - \vec{x} \cdot \vec{x}$ is. Based on this fact, a reasonable guess would be that E/c and $\vec{p}$ transform from frame to frame using the same Lorentz transform type of equations as apply to ct and $\vec{x}$! This is indeed the case.

Proof We need one basic identity that you can check:

$$\gamma(u') \equiv \frac{1}{\sqrt{1 - \frac{{u'_x}^2 + {u'_y}^2 + {u'_z}^2}{c^2}}} = (1 - \frac{u_x v}{c^2}) \gamma(v) \gamma(u) \,. \tag{84}$$

This follows algebraically from the velocity transform equations, eqs. (38) and (40). Using the above identity, we compute, again using the velocity transform equations, eqs. (38) and (40),

$$\begin{aligned} p'_x &\equiv \gamma(u')mu'_x \\ &= \left(1-\frac{u_x v}{c^2}\right)\gamma(v)\gamma(u)m\frac{u_x - v}{\left(1-\frac{u_x v}{c^2}\right)} \\ &= \gamma(v)\left(\gamma(u)mu_x - \frac{v}{c}\gamma(u)mc\right) \\ &= \gamma(v)\left(p_x - \frac{v}{c}\frac{E}{c}\right), \end{aligned} \tag{85}$$

which has the same form as eq. (49) with x replaced by p_x and ct replaced by E/c! Similarly, we find

$$\begin{aligned} \frac{E'}{c} &\equiv \frac{\gamma(u')mc^2}{c} \\ &= \left(1-\frac{u_x v}{c^2}\right)\gamma(v)\gamma(u)mc \\ &= \gamma(v)\left(\gamma(u)mc - \frac{v}{c}\gamma(u)mu_x\right) \\ &= \gamma(v)\left(\frac{E}{c} - \frac{v}{c}p_x\right) \end{aligned} \tag{86}$$

which has the same form as eq. (50) with x replaced by p_x and ct

replaced by E/c! For the y and z component momenta, we expect no change. Let us check.

$$\begin{aligned} p'_y &\equiv \gamma(u')mu'_y \\ &= \left(1 - \frac{u_x v}{c^2}\right)\gamma(v)\gamma(u)m\frac{u_y}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)} \\ &= \gamma(u)mu_y \\ &= p_y\,. \end{aligned} \quad (87)$$

Note: If E and p are constant in frame S in the absence of applied forces, then the above transformation equations imply that they are also constant in frame S'. (Just take a time derivative of the two sides of the equations.) The physical law that says "energy and momentum are constant in the absence of applied force" is thus frame independent, provided E and p are defined as we have done.

In other words, the fundamental postulate of relativity is obeyed by the physical laws pertaining to momentum and energy.

For a group of particles, we would sum over their individual momenta and energies, but that would not change the above statements. If the sum of energies and the sum of momenta remain unchanged in one frame due to the absence of force, these sums will remain unchanged in all frames.

Energy Conservation

To repeat, it is natural that E as defined is the fundamental quantity, as opposed to the kinetic energy, K, since it is E conservation that is independent of frame. In contrast, K may be conserved in one frame and not in another frame (or not conserved in any frame).

A really simple example of why we must deal with the total energy E is provided in elementary particle physics. Experimentally, it is possible to collide an electron and a positron (coming together with equal magnitude but oppositely directed velocities — the center-of-mass frame) to make a proton and an antiproton at rest:

$$e^+e^- \to p\overline{p} \,. \tag{88}$$

In the initial state, most of the energy resides in the kinetic energies of the e^+ and e^-, whereas in the final state all of the energy is contained in the mass of the p and $\overline{p}$. Kinetic energy is thus converted to rest mass (energy). And, the opposite is also true! — think nuclear fission bomb.

Another example is illustrated below.

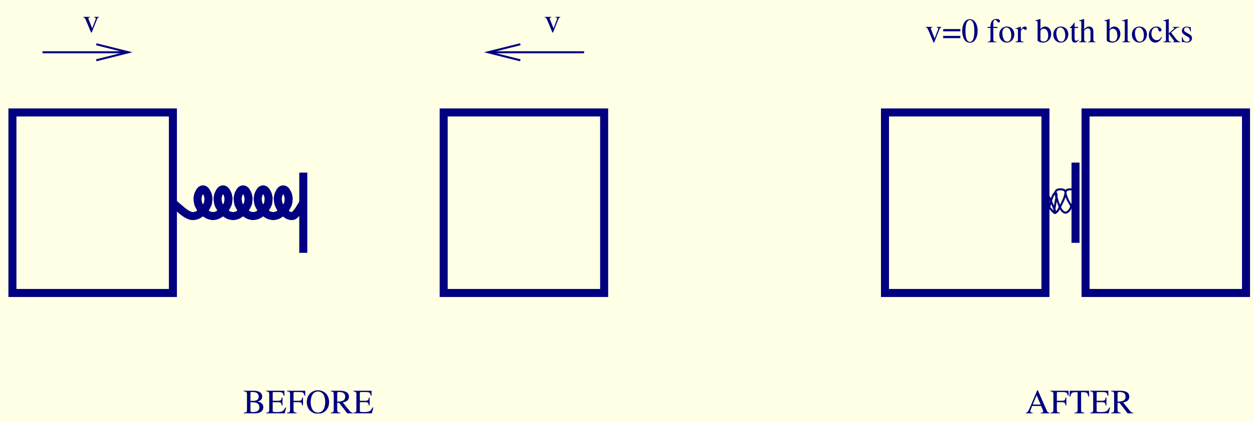


Figure 12: Depiction of two blocks of equal mass with spring attached one, colliding to create a single object with compressed (massless) spring and motionless blocks. (Imagine there are little latches that catch and hold the blocks together when the spring becomes compressed.)

Where did the kinetic energy go? Obviously, it went into the compression of the spring. Einstein says that initially $E_i = 2mc^2 + 2K$. If we view the two boxes plus compressed spring as one total object, with zero kinetic energy, then kinetic energy is clearly not conserved, but energy would be conserved provided we simply define

$$E_f = Mc^2 \,, \tag{89}$$

where M must account for not only the initial block masses, but also the kinetic energy that was converted to spring compression energy. That is, $M > 2m$. In fact,

$$M = 2m + \frac{2K}{c^2} \tag{90}$$

is the appropriate definition.

Sometimes, the combined object can have mass smaller than the objects of which it is composed. This happens if there is some force that attracts the objects towards one another. We write

$$M_{\text{bound system}} c^2 + E_B = \sum_i m_i c^2 \tag{91}$$

where E_B is the *binding energy*. An example of this situation is provided by the *deuteron* which is composed of a proton and a neutron. It is a stable object precisely because $E_B > 0$ which means that

$$M_d < m_p + m_n \tag{92}$$

and energy cannot be conserved by the process in which

$$d \to p + n\,, \tag{93}$$

since kinetic energy can only be positive, whereas for the above process we would need to have negative kinetic energies for the p and n.

The magnitude of the deuteron's binding energy is (using the MeV type units that you are all familiar with and writing m_p and m_n in these units, where a factor of c^2 is implicitly supplied)

$$\begin{aligned} E_B(deuteron) &= m_p + m_n - M_d \\ &= 938.27 \text{ MeV} + 939.57 \text{ MeV} - 1875.61 \text{ MeV} \\ &= 2.23 \text{ MeV} . \end{aligned} \quad (94)$$

So, it is only sensible to suppose that E is conserved and that this conservation is independent of frame. Then, for any process in any frame we would have

$$\sum_{i=\text{initial particles}} E_i = \sum_{f=\text{final particles}} E_f . \quad (95)$$

If we combine this with three-momentum conservation, and put all the energies together with their three momenta into $\mathbf{4}$-vectors, we get

$$\sum_{i=\text{initial particles}} p_4^i = \sum_{f=\text{final particles}} p_4^f \,. \tag{96}$$

We have seen that going from one frame to another mixes up the components of the p_4 vectors (just like spatial rotation would mix up the $\mathbf{3}$-vector components), but the equality between the two sides of the equation would not be altered.

At this point, please read the 4vector.pdf file on my web page and do the extra problem at the end of this file.

The Quantum Theory of Light

Maxwell's equations imply that E&M waves are possible:

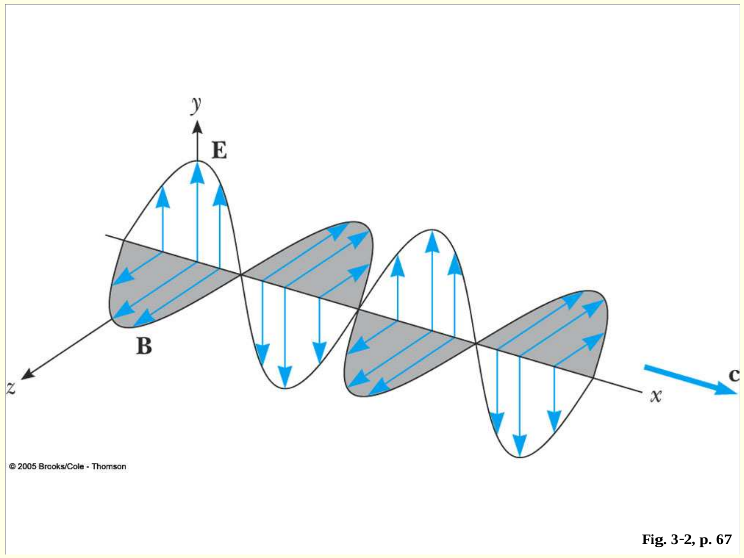


Figure 13: Depiction of electromagnetic wave.

We know:

1. $c = 1/\sqrt{\mu_0 \epsilon_0}$, **where** μ_0 **and** ϵ_0 **were determined by magnetic induction and static Gauss law experiments.** $\Rightarrow c = 3 \times 10^8\ m/s$, **which was known to be speed of light from MM experiment.**

   $\Rightarrow$ **light**$= E\&M$ **wave.**

2. Wave nature of light verified by interference phenomena.

3. Maxwell Eqs. $\Rightarrow$ same things should hold for $E\&M$ waves of much lower f (larger λ since $c = \lambda f$).

4. Hertz verified interference and velocity c for radio waves. $\Rightarrow$ confirmation.

Blackbody Radiation

Please read the book on this subject, the bottom line being the famous formula due to Planck for e_f, the power emitted by a perfect black body per unit area, per unit frequency at temperature T:

$$e_f = \frac{c}{4}\frac{8\pi h f^3}{c^3}\left(\frac{1}{e^{hf/k_BT} - 1}\right) , \tag{97}$$

where Planck had to introduce a new fundamental constant $h = 6.626 \times 10^{-34}\ J \cdot s$ and k_B is Boltzmann's constant $k_B = 1.38 \times 10^{-23}\ J/K$.

A now-famous example of a blackbody spectrum is the cosmic microwave background. Following the big-bang, the universe expanded very rapidly and came to near thermal equilibrium, so that the spectrum of the residual electromagnetic radiation should follow rather closely a perfect blackbody spectrum. The spectrum, along with the Planck curve, is shown below.

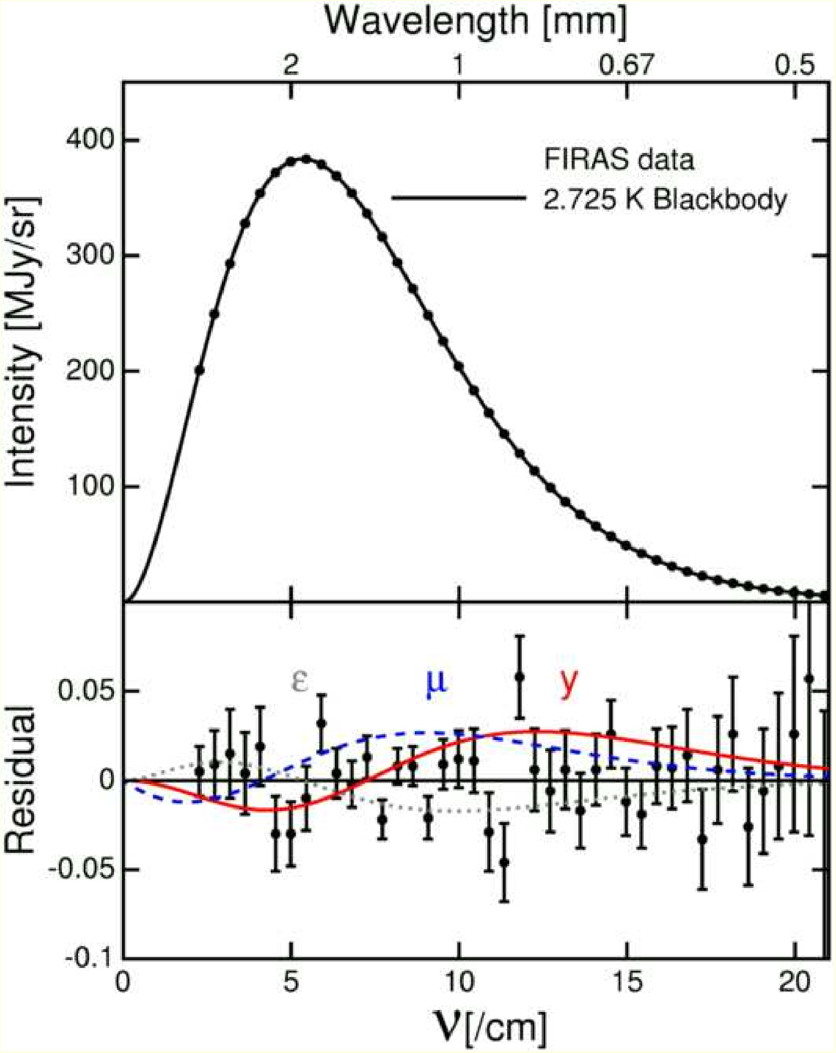


Figure 14: The CMB spectrum compared to a perfect Planck blackbody spectrum, assuming the current blackbody temperature of $T = 2.725\ K$.

The "residuals" contain important information. In particular, they tell us about the early history of structure (*e.g.* galaxy cluster) formation and the like. In particular, they tell us something about dark matter and dark energy. But these residuals are extremely small perturbations about the overall blackbody spectrum.

Planck was able to explain his formula by assuming:

1. **The walls of the glowing cavity were composed of billions of "resonators", all vibrating at different frequencies.**

2. **However, Maxwellian theory says that an oscillator of frequency f could have any value of energy and could change its amplitude continuously as it radiated any fraction of its energy.**

   This would not have allowed Planck to derive his formula.

   Instead, Planck had to assume that the total energy of a resonator with mechanical frequency f could only be an integral multiple of hf, $E = nhf$ ($n = 1, 2, 3, \ldots$), where h was his new fundamental constant.

   Further, the resonator had to only be able to change its energy by dropping to the next lowest state, implying $\Delta E = hf$ always.

The derivation of the Planck formula is basically the computation of the average energy of an oscillator using the standard statistical e^{-E/k_BT} weighting for different energy values. According to Planck this is done by assuming that E comes in units of hf so that one sums with above weighting over $E = nhf$, $n = 1, 2, \ldots, \infty$:

$$\overline{E} = \frac{\sum_{n=0}^{\infty}(nhf)e^{-nhf/k_BT}}{\sum_{n=0}^{\infty} e^{-nhf/k_BT}} = \frac{hf}{e^{hf/k_BT} - 1} . \quad (98)$$

The other factors in Planck's formula eq. (97) have a trivial origin that you can read about.

How much energy for a resonator are we talking about? For green light, we have

$$hf = \frac{hc}{\lambda} = \frac{(6.63 \times 10^{-34}\ J \cdot s)(3 \times 10^8\ m/s)}{540 \times 10^{-9}\ m} = 3.68 \times 10^{-19} J = 2.3\, eV . \quad (99)$$

The Photoelectric Effect

The 1900 Planck formula and explanation would have remained an obscure and esoteric idea except for the Photoelectric Effect, which was in

turn explained by (who else) Einstein. Einstein's explanation required that light come in little packets (photons) each with energy hf.

What is the photoelectric effect?

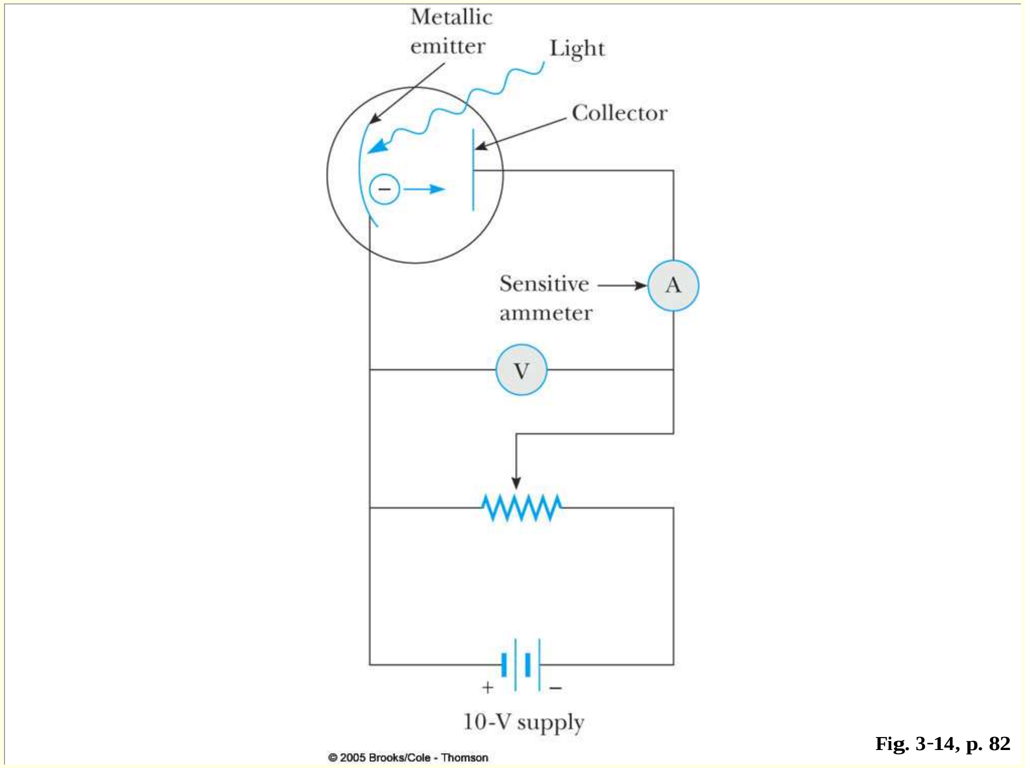


Figure 15: Photoelectric Effect apparatus.

Shine light on a metal surface and out come electrons.

How would you view this in Maxwell's eyes?

- $\vec{E}$ and $\vec{B}$ come in and shake the e^- and get it oscillating and it gets out of metal.
- In this picture, the bigger $\vec{E}$ and $\vec{B}$ (higher intensity) the bigger the shaking and the more energy you can give to the e^-. In fact, the energy density of an $E\&M$ wave is

$$u = \frac{Energy}{Volume} = \frac{1}{2}\epsilon_0|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2 \tag{100}$$

  which clearly gets bigger and bigger as $\vec{E}$ and $\vec{B}$ get larger.
  Conversely, by decreasing $\vec{E}$ and $\vec{B}$ it should be possible to have the light wave carry only a very tiny amount of energy.

Einstein looked at the experimental data and concluded this could not be correct.

Instead, he proposed that light comes out in "photon" bundles of definite energy $E = hf$.

An e^- is very unlikely to be "hit" by more than one photon,

$\Rightarrow$ e^- can only get so much energy — **no more** *and* **no less** (even for tiny classical light intensity).

In contrast, and to repeat, the classical expectation is that the maximum kinetic energy of the ejected e^-, K_{max}, should be a function of **intensity**, which means it should increase with increasing $|\vec{E}|$ and $|\vec{B}|$ since more energy would be deliverable to individual electrons.

Further, by decreasing the intensity, the classical expectation is that we would eventually reach a point at which too little energy is delivered to the e^- to release it from the metal.

Neither is true experimentally.

We know for sure now that Einstein was right (of course). (This is what he got his Nobel for and not for relativity!)

Back to the experiment.

Two classically unexpected results were seen:

1. K^e_{max} (as measured by finding the "stopping" voltage, $V = -V_s$, that repels the electron emitted from the emitter from reaching the collector given by $K^e_{max} = eV_s$) depends on f and **only** f ($i.e.$ not on intensity at all).

2. For $f < f_0$, where f_0 depends upon the metal on which the light shines, no e^-'s come out.

3. In fact, one finds that K^e_{max} is directly proportional to f with slope given by h.

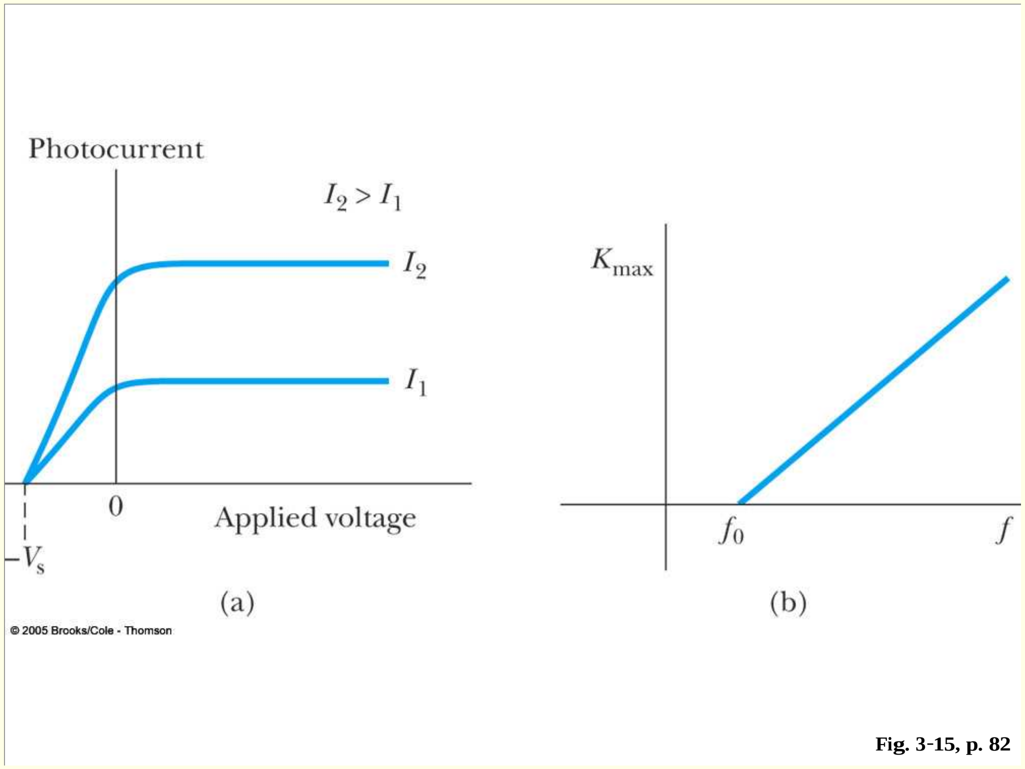


Figure 16: Photoelectric Effect plots of (a) photocurrent vs. applied V for two different light intensities and (b) of K^e_{max} vs. f.

Enter Einstein

1. Assume that light comes in discrete photons, each carrying energy hf.

2. **Higher intensity is to be interpreted as more photons.**
3. **Only one photon light bundle is absorbed by any one single e^- in the metal.**
4. **If the energy of the photon can overcome the metal binding, then the e^- gets out. The picture is (our notation for a photon is γ)**

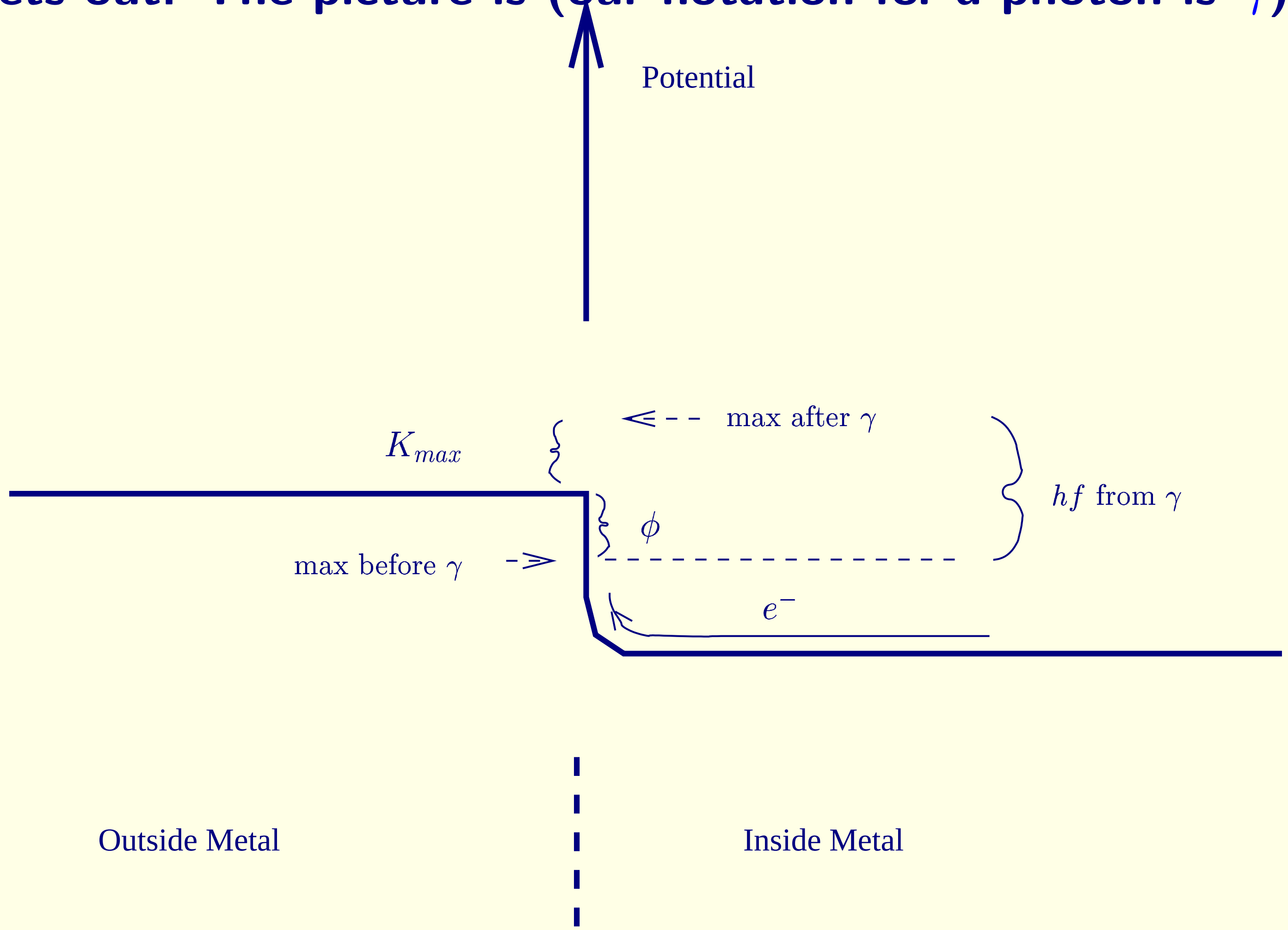


Figure 17: Einstein's picture of photoelectric excitation of e^- in a metal.

The maximum energy of the ejected e^- would then be $K^e_{max} = hf - \phi$, where $\phi \equiv$ work function is the metal potential the e^- must overcome to get out — ϕ depends on the metal.

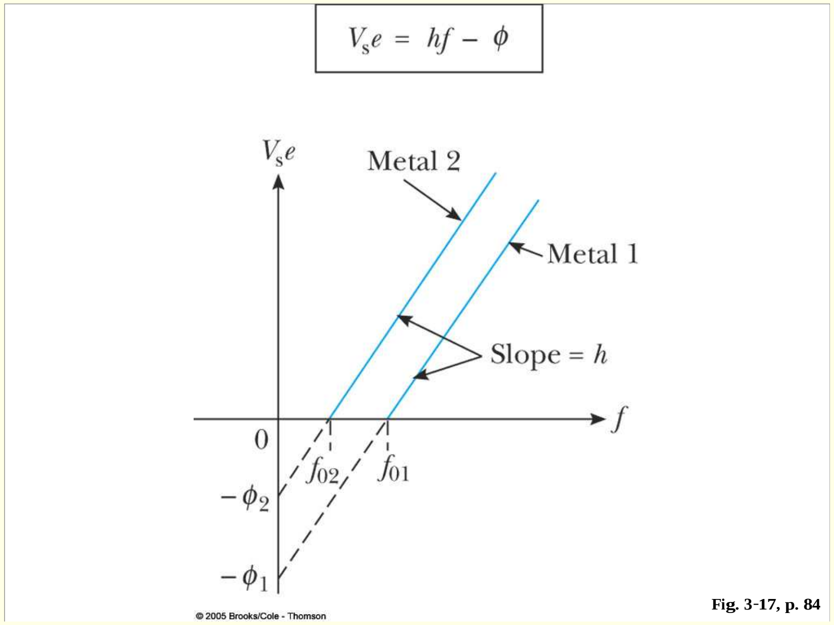


Figure 18: Dependence of V_s on the metal and on f.

Note that some electrons get energy $< K^e_{max}$ because the photon does not give all its energy to the e^- (energy imparted depends on scattering angle, ...) and also the e^- coming out may collide with other stuff before trying to exit from the metal surface where it encounters ϕ.

5. This picture predicts that there is a certain minimum frequency $f_0 = \phi/h$ such that $K^e_{max} = 0$, as observed experimentally.

An example:

Suppose for iron we observe that the frequency f_0 for which $K^e_{max} = eV_s = 0$ is $f_0 = 1.1 \times 10^{15}\ Hz$. What is the stopping potential for light with $\lambda = 250\ nm$?

Two steps are required:

I) $K^e_{max} = 0 \Rightarrow hf_0 = \phi$ or

$$\phi = (4.14 \times 10^{-15}\ eV \cdot s)(1.1 \times 10^{15}\ s^{-1}) = 4.5\ eV\,, \tag{101}$$

where we have written $h = 6.63 \times 10^{-34}\ J \cdot s$ in eV units using $1\ eV = 1.602 \times 10^{-19}\ J$.

II) Given a λ, we know $f = c/\lambda$ and then we also know that

$$\begin{aligned} K^e_{max} &= eV_s = hf - \phi = (4.14 \times 10^{-15}\ eV \cdot s)\frac{3 \times 10^8\ m/s}{250 \times 10^{-9}\ m} - 4.5\ eV \\ &= 4.96\ eV - 4.5\ eV = 0.46\ eV\,. \end{aligned} \tag{102}$$

Since electrons have charge e, we conclude that $V_s = 0.46\ V$.

To summarize, our new picture of light is:

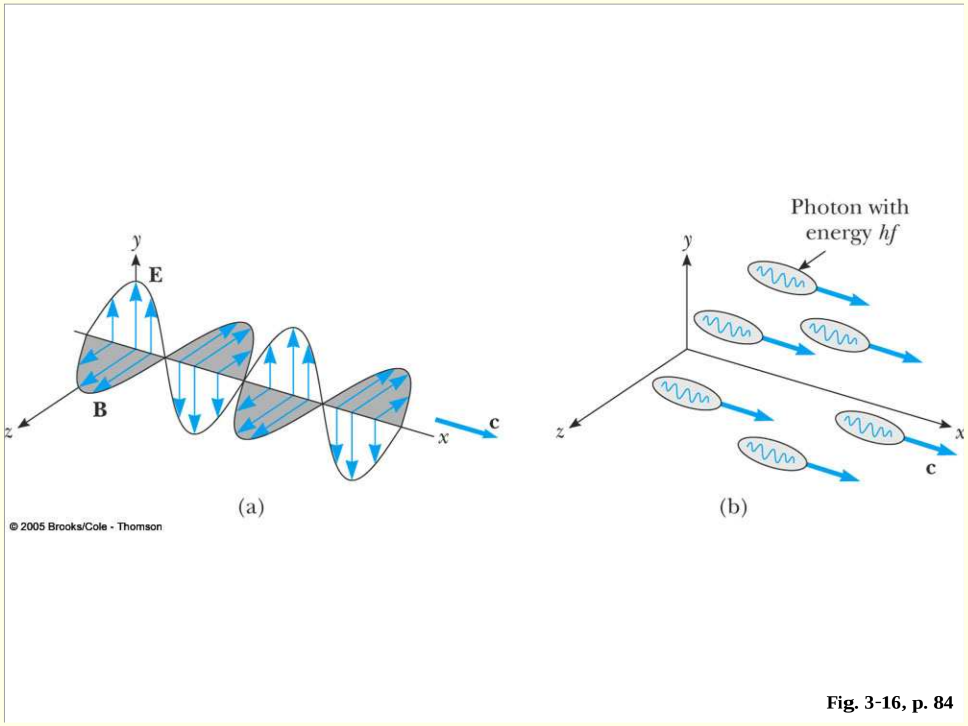


Figure 19: The classical Maxwell vs. Einstein picture of light.

In this picture, we can ask the following question:

How many photons per second emanate from a $10\ mW$, $633\ nm$ laser?

Answer: For each photon,

$$E = hf = h\frac{c}{\lambda} = (6.63\times10^{-34}\,J\cdot s)\left(\frac{3\times10^{8}\ m/s}{633\times10^{-9}\ m}\right) = 3.14\times10^{-19}\,J\,. \tag{103}$$

To find the number of particle per unit time, we divide energy per unit time by energy per particle:

$$\frac{\text{number of particles}}{\text{time}} = \frac{10\times10^{-3}J/s}{3.14\times10^{-19}J/photon} = 3.18\times10^{16}\,photons/s\,. \tag{104}$$

X-rays

We use the term $X\text{-}ray$ **for** $E\&M$ **radiation with** $\lambda\in[10^{-2},10]\ nm$ **region of the spectrum.**

We want to use X-rays to demonstrate the particle nature of $E\&M$ **radiation.**

X-rays are produced by smashing high-speed electrons into a metal target.

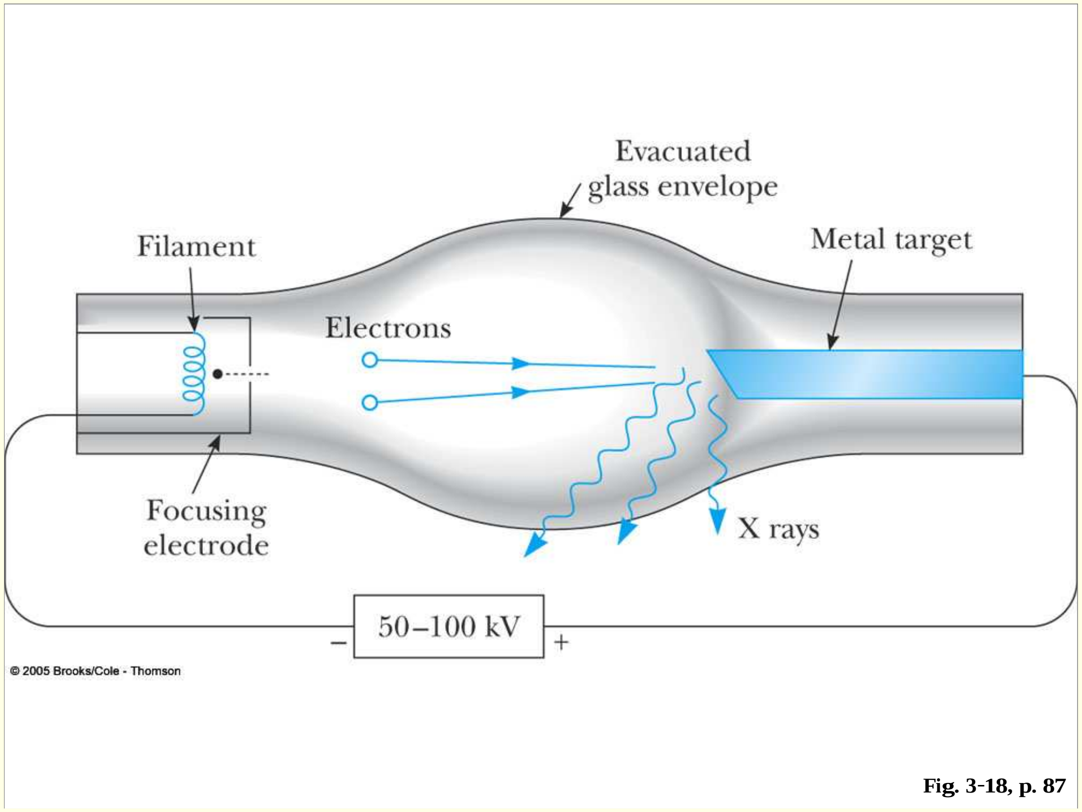


Figure 20: Smashing an electron into a target to produce X-rays.

Any charged particle radiates $E\&M$ energy when it accelerates, and the smashing gives violent acceleration so that much radiation is produced.

Classically (a la Maxwell) one expects to see radiation produced over the entire spectrum of wavelengths.

Experiment differs. There is always a *cutoff* wavelength, λ_{min} below which there is no radiation.

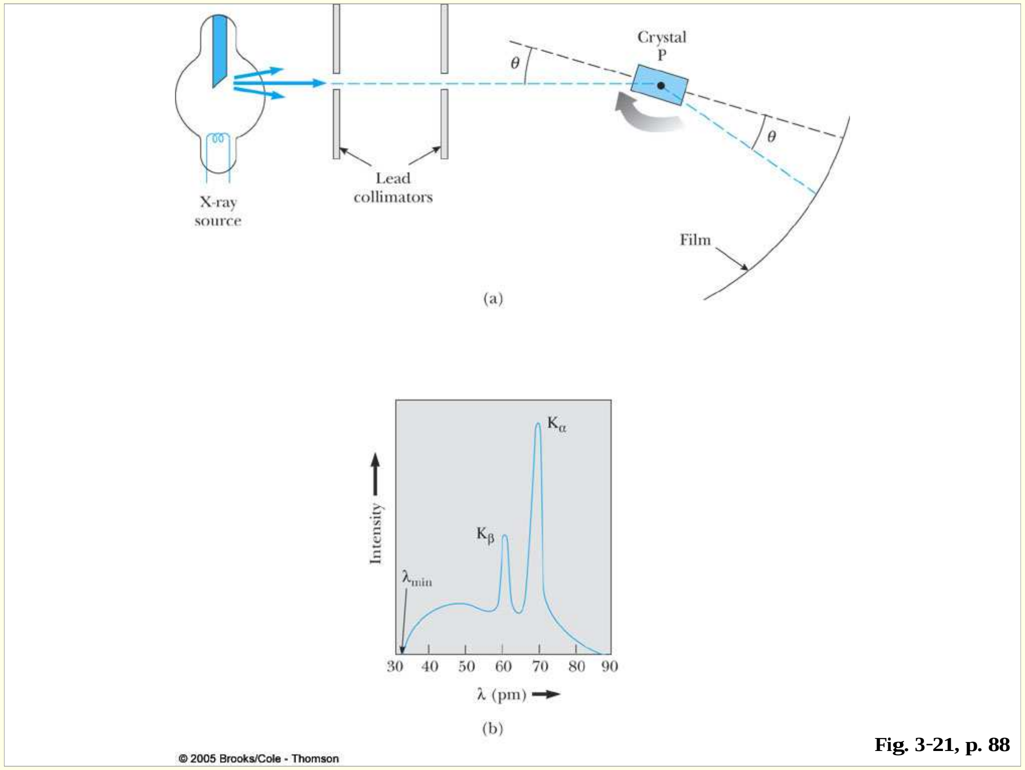


Figure 21: Typical x-ray spectrum showing λ_{min} cutoff. (λ is determined by using crystal diffraction to spread different λ's onto different parts of a film (in old days).

The broad part of the spectrum agrees with Maxwellian theory, but the cutoff requires the photon concept.

The explanation is as follows:

1. **The minimum λ_{min} arises when all of the energy of the incoming e^- is converted to energy of the outgoing x-ray photon.**
2. **In such a case, and assuming an accelerating voltage of V, we would have**

$$eV = hf = \frac{hc}{\lambda_{min}}, \quad \text{or} \quad \lambda_{min} = \frac{hc}{eV}. \tag{105}$$

   This is precisely what is observed. Of course, if the collision is less than "perfect", then less energetic, and therefore larger λ, photons can come out. Also, the electron energy could be spread out over many photons. In either case, the photon energy (or energies) will be smaller and their wavelength longer. We can never do better than to put all the incoming e^- energy into a single outgoing photon.
3. **In contrast, if there were no minimum energy (analogous to hf of the photon) associated with $E\&M$ radiation of a given frequency, then still large f's (and smaller λ's) would have been possible by setting $eV =$ arbitrarily small number $\times f$.**

The case was becoming stronger and stronger that this photon concept was really correct.

Enter Compton (1922). (The above described X-ray wavelength studies were by Bragg around 1912.)

The Compton Effect

Even more important than understanding X-rays themselves was what Compton did with them.

He took energetic photons from a peak in the spectrum and collided them with a stationary e^-.

Classically (wave picture) we expect

1. e^- oscillates with frequency of incoming $E\&M$ radiation and reradiates with $f_{out} \leq f_{in}$ over broad range of f_{out}.
2. The exact spectrum should depend on details of intensity and length of exposure to the X-rays.

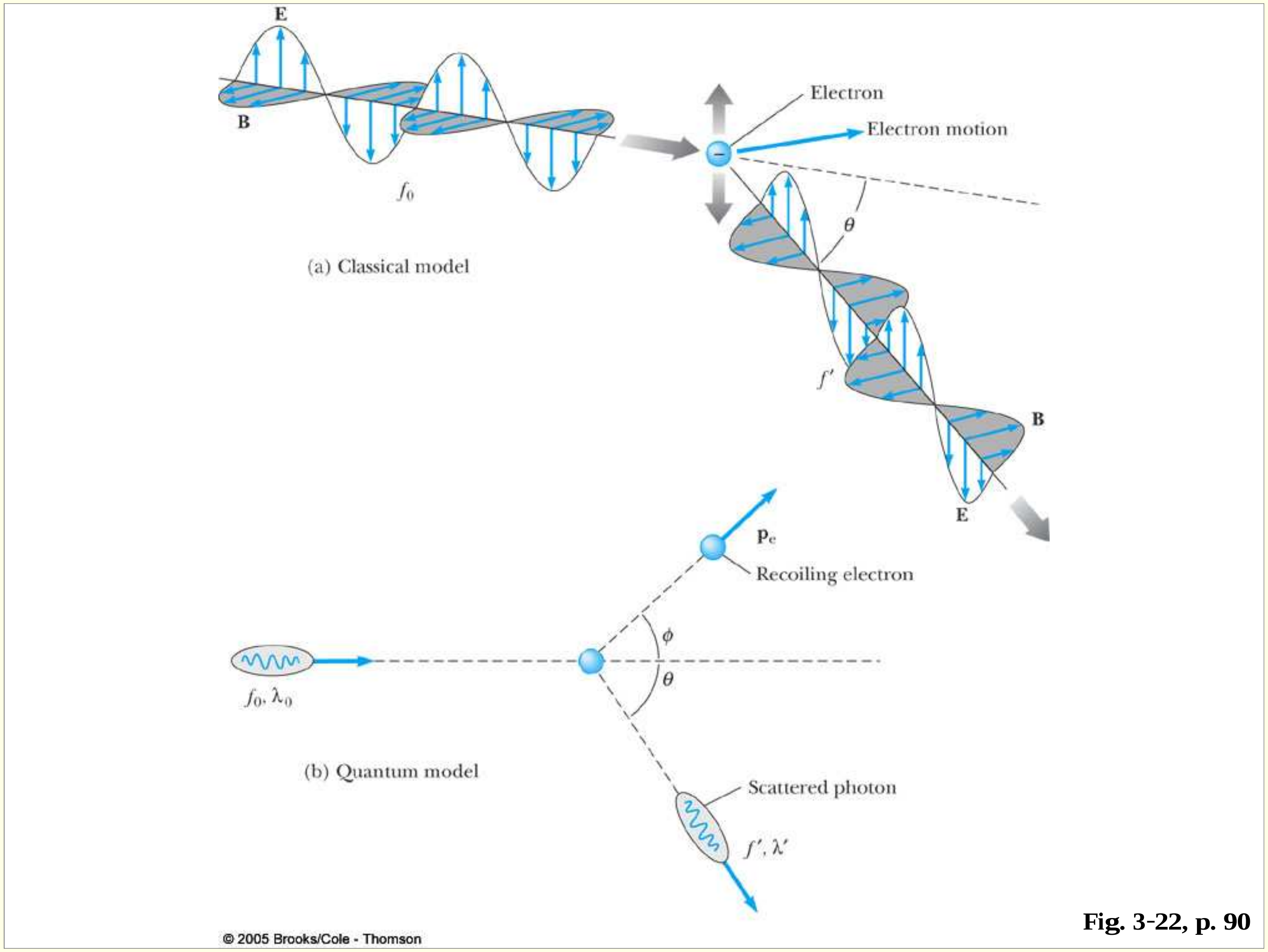


Figure 22: Pictures of Compton scattering: wave point of view vs. particle point of view.

What Compton found was very different:

1. f_{out} (equivalently, λ_{out}) depends only on f_{in} (*i.e.* λ_{in}) and the angle of scattering of the X-ray, θ.
   f_{out} does not depend on intensity or exposure time.
2. Further, $f_{out} < f_{in}$ unless $\theta = 0$.

The exact result found was (using λ for λ_{in} and λ' for λ_{out})

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)\,, \qquad (106)$$

where $\frac{h}{m_e c} = 0.00243\ nm = 2.43 \times 10^{-12}\ m$ is called the *Compton Wavelength*. In a very real sense, it turns out to be the size of the e^- as seen by a X-ray photon.

This result can be understood in one and only one way:

1. Light must act as a particle with $E = hf$ and $p = E/c$ (as consistent with zero mass limit of $E^2 - p^2c^2 = m^2c^4 = 0$).
2. The collision must be like relativistic billiard balls conserving energy and momentum in the relativistic sense.

This result of eq. (106) is a slight approximation in that it neglects the binding work function energy of the target e^- which is actually in a graphite or similar target.

However, this is a good approximation for work functions in the eV range given that X-rays have energies in the $keV = 10^3\ eV$ range.

Derivation of Compton Formula

We repeat the picture:

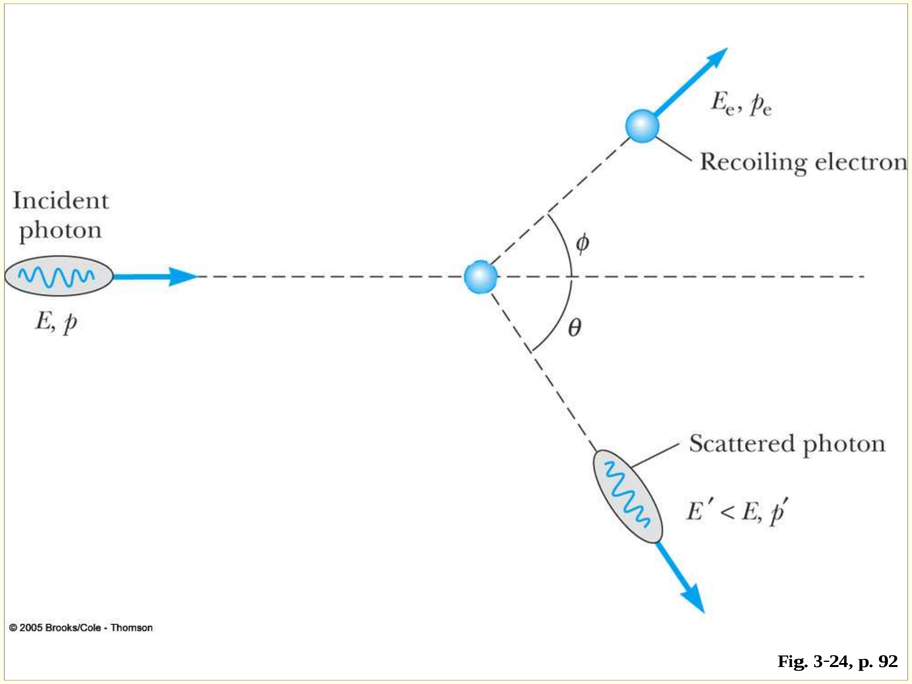


Figure 23: Compton scattering from the particle point of view.

Before the collision, we have a photon with $p_x = p$ and energy $E = pc$ hitting an electron at rest with $\vec{p} = 0$ and $E = m_e c^2$.

After the collision, we have an electron with $p_e^{y\prime} = p_e' \sin\phi$, $p_e^{x\prime} = p_e' \cos\phi$, where $p_e' = |\vec{p}_e'|$, and $E_e' = \sqrt{{p_e'}^2 c^2 + m_e^2 c^4}$.

We also have the scattered photon with $p_\gamma^{y\prime} = -p' \sin\theta$, $p_\gamma^{x\prime} = p' \cos\theta$ and $E' = p'c$. (I have defined $p' = |\vec{p}_\gamma|$.)

Energy conservation thus gives:

$$E + m_e c^2 = E_e' + E' \tag{107}$$

Momentum conservation in the x and y directions gives:

$$p = p_e' \cos\phi + p' \cos\theta \tag{108}$$

$$0 = p_e' \sin\phi - p' \sin\theta \, , \tag{109}$$

and we need to keep in mind the energy relations: $p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$, $p' = \frac{hf'}{c} = \frac{h}{\lambda'}$ and $E_e' = \sqrt{{p_e'}^2 c^2 + m_c^2 c^4}$.

Inputing the latter, and choosing θ to be some particular value defined by the point at which I look at the scattered photon, we end up with 3 equations in 3 unknowns: ϕ, p'_e and p'. We can solve these as follows:

1. Rewrite eq. (108) in the form

$$p - p' \cos\theta = p'_e \cos\phi \,. \tag{110}$$

2. Rewrite eq. (109) in the form

$$p' \sin\theta = p'_e \sin\phi \,. \tag{111}$$

3. Square these two equations and add $\Rightarrow$ ϕ disappears and we end up with

$${p'_e}^2 = (p')^2 + p^2 - 2pp' \cos\theta \,. \tag{112}$$

4. Using the photon energy relations, this is rewritten as

$${p'_e}^2 = \left(\frac{hf'}{c}\right)^2 + \left(\frac{hf}{c}\right)^2 - \frac{2h^2 f f'}{c^2} \cos\theta \tag{113}$$

while the energy conservation equation eq. (107) becomes

$$E'_e = hf - hf' + m_e c^2 \quad \Rightarrow \quad {E'_e}^2 = (hf - hf' + m_e c^2)^2. \quad (114)$$

5. Now use ${E'_e}^2 - {p'_e}^2 c^2 = m_e^2 c^4$ and the above expression for ${E'_e}^2$ to solve (Problem 33, not assigned but would be good for you to do) and obtain

$$\lambda' - \lambda = \frac{h}{m_e c^2}(1 - \cos\theta)\,. \quad (115)$$

Note the importance of using $E = pc$ for the X-ray photon.

This is in fact something you know holds for Maxwell equation waves (I hope you did this in the $E\&M$ course).

It is also what is required for the $m = 0$ limit of $E^2 = p^2c^2 + m^2c^4$.

Note that the reason that $\lambda' > \lambda$ (except at $\theta = 0$ is that the e^- takes away some kinetic energy in this relativistic elastic collision so that the final photon must have less energy and momentum and therefore smaller f', implying larger λ'.

We also note that $p = \frac{h}{\lambda}$ and $p = \frac{hf}{c}$ are equivalent for light. However, only the first turns out to be correct when we consider the wave nature of a particle with mass.

An example:

An X-ray photon of $0.0500\ nm$ wavelength strikes a free, stationary electron. The photon scatters at $90°$. Determine the momenta of the incident photon, the scattered photon, and the electron.

Answer:

For the incident photon we have

$$p = \frac{h}{\lambda} = \frac{6.63 \times 10^{-34}\ J \cdot s}{0.05 \times 10^{-9}\ m} = 1.33 \times 10^{-23}\ kg \cdot m/s\,. \tag{116}$$

Inserting the other specified inputs, we solve eq. (115) for the scattered photon's wavelength:

$$\lambda' - 0.0500 \times 10^{9}\ m = \frac{(6.63 \times 10^{-34})}{(9.11 \times 10^{-31}\ kg)(3 \times 10^{8}\ m/s)}(1 - \cos 90°)$$
$$\Rightarrow \quad \lambda' = 0.0524\ nm\,. \tag{117}$$

From this, we compute

$$p' = \frac{h}{\lambda'} = \frac{6.63 \times 10^{-34}\ J \cdot s}{0.0524 \times 10^{-9}\ m} = 1.26 \times 10^{-23}\ kg \cdot m/s\,. \qquad (118)$$

Finally, we may use eq. (112) to compute (in units of $10^{-23}\ kg \cdot m/s$)

$$(p'_e)^2 = (1.33)^2 + (1.26)^2 - 2(1.33)(1.26)\cos 90^\circ = 3.36 \qquad (119)$$

or $p'_e = 1.83 \times 10^{-23}\ kg \cdot m/s$. The angle at which this electron is traveling can then be obtained using eq. (111)

$$\sin\phi = \frac{p' \sin 90^\circ}{p'_e} = \frac{p'}{p'_e} = \frac{1.26}{1.83} = 0.69 \quad \Rightarrow \quad \phi = 0.76\,rad = 43.55^\circ\,. \qquad (120)$$

What is an elementary particle?

Consider the mechanical analogy below of two blocks held together by a very stiff spring.

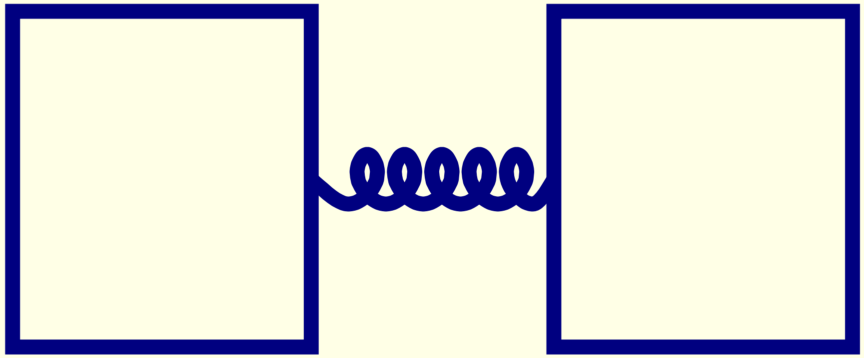


Figure 24: A composite system consisting of two blocks held together by a very stiff spring.

If a billiard ball comes in and collides with this system in a fairly “soft” low energy manner, then the two-block system will simply recoil as though it were a single unified object (albeit, one with a non-spherically symmetric nature). Energy and momentum conservation can be applied to the process as if the two-block system had no internal structure.

However, if the billiard ball is sufficiently energetic, then the stiff spring will start to oscillate in some way and the two-block system will not act as a single unit. Some of the incoming billiard ball energy will be absorbed into the energy of the oscillation, which will amount to an increase in mass of the two-block system. A Compton-like calculation would no longer be applicable.

Similarly, the billiard ball itself could have internal structure (maybe it has an interior that might heat up under the collision, . . .).

In Compton scattering, no matter how energetic a photon we employ (even photons with $E \sim 10^{11}\ eV$), there is no sign of any such internal structure. The Compton formula always works.

We call particles that have no internal structure *elementary particles*. All evidence to date is consistent with both the photon and the electron being truly elementary (also termed "point-like") particles.

In contrast, the proton we now know (by bombarding it with very energetic photons, for example) is a composite object made of what are called *quarks*. We will return to this picture late in the quarter.

Is $E\&M$ radiation a wave or a particle?

Answer: it is both!

Which kind of nature it displays depends upon the experimental situation considered, that is how the $E\&M$ wave or particle interacts with its environment. This is partly determined by its own wavelength λ and partly by the nature of the object it interacts with, in particular the size D of the experimental apparatus or probe that "looks" at the photon.

We will find that a similar statement applies also to particles with mass such as the electron. This will be the next subject we come to. But, for now, let's explore this issue in a bit more depth for $E\&M$ radiation.

An analogy

Consider a boat of length D in a lake with, let us say, 5 waves of substantial amplitude but rather long wavelength, $\lambda \gg D$, moving toward and eventually past the boat. The boat goes up one side of each wave and rides down the other. Observers in the boat would easily know that a wave consisting of several troughs and several peaks had passed by, even if they were blindfolded.

Now suppose the situation is that we have 5 waves, but the wavelength is such that $\lambda \ll D$. The observers in the boat would feel a sudden impact from the peaks of the waves, but would certainly not be able to resolve the separation between the peaks. It would feel like some object had collided with them. The boat responds to all parts of a $\lambda \ll D$ wave pattern at once, and not to different parts of the wave pattern at different times.

Back to $E\&M$ radiation and photons

To see the particle nature of light, Compton scattering works very nicely, but only if $\lambda' - \lambda$ is a significant fraction of λ is it experimentally easy to see the change. In terms of the Compton size of the the electron $\lambda_e \equiv \frac{h}{m_e c}$, the shift from λ to λ' would be an immeasurably tiny fraction of the original λ if $\lambda \gg \lambda_e$. This is why Compton needed keV photons to really reveal the particle nature of light — he needed large enough energy so that $\lambda \sim \lambda_e$. Thus, λ_e is the equivalent of the boat size D in the earlier analogy.

If the change $\lambda' - \lambda \equiv \Delta\lambda$ is very small compared to λ, $\Delta\lambda \ll \lambda$, then there is hardly any difference between the Compton result and the classical result which says that λ would not change. This is an example of the Correspondence Principle, according to which there is a limit of the quantum theory in which it becomes indistinguishable from the classical theory (in this case, Maxwell's wave theory of light).

Another aspect of taking λ large, is that as the wavelength increases, the energy per photon decreases so that a given light intensity would have to be produced by more and more photons, which would kind of become continuous and everywhere in the wave. Intuitively, this implies that the Maxwell wave description would become more and more accurate.

Back to wave vs. particle

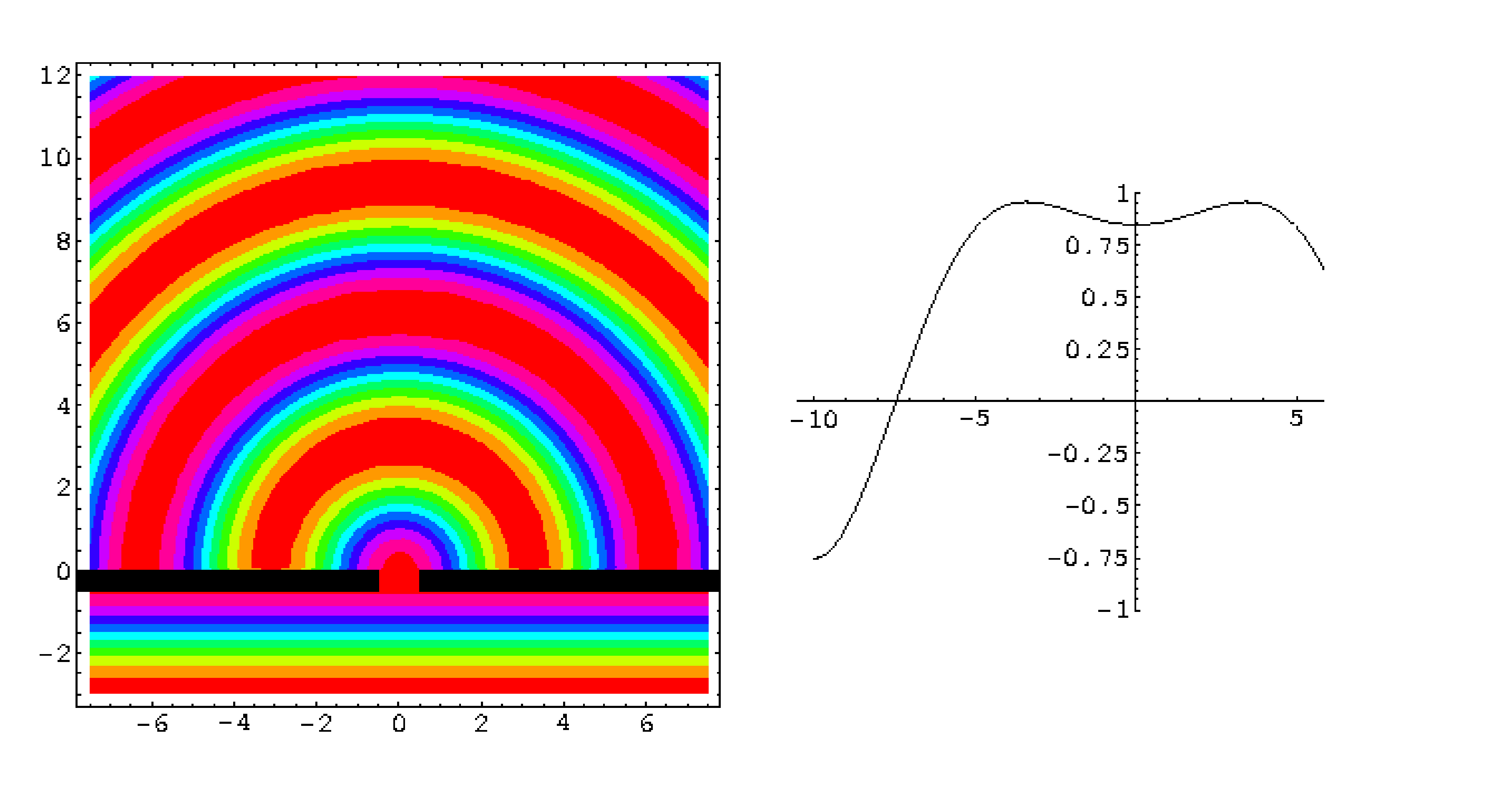
Passing light through a single slit is a good way to reveal its wave nature. But to see the diffraction, the wavelength has to be big compared to the slit size, D. (If $D \gg \lambda$, then the wave, just like a particle, would just pass straight through the slit without being modified significantly.)

In contrast, small λ compared to the electron Compton wavelength was what Compton needed to reveal the particle nature. These two natures of the light are not incompatible, but rather complementary.

To emphasize the wave limit, let us consider in a bit more detail light passing through a single slit. We know that one gets a diffraction pattern with a first minimum from the central peak at an angle of $\theta \sim \lambda/D$. Below, you will see an example with $\lambda \gg D$ so that the incoming plane wave is completely diffracted to very large angles (so large that you don't even see the 1st diffraction minimum).

In contrast, if $D \gg \lambda$, the wave behavior is indistinguishable from what would be expected for a particle — as stated earlier both would just pass straight through.

singleslit1



So, now you should ask yourself how this hugely wave-light diffraction phenomenon can be consistent with light traveling in little photon packets. First, recall that there must be a lot of them to achieve a given intensity of light in the large λ limit where each photon carries only a small amount of energy.

What has to happen is that some of the photons go in one direction after passing through the slit, and some in other directions.

The obvious question is how is this possible. You would be tempted to say that each photon acts like a particle, goes through the slit and hits the screen more or less opposite the slit.

This becomes even more confounding if you keep $\lambda \gg D$ and decrease the intensity of the light to the point where only one photon is traveling through the slit and to the screen at a time. If you develop the film on the screen after just a few photons have arrived, you would in fact find a few dots on the screen, seemingly sort of randomly distributed.

If you continue operating at low intensity long enough, you would eventually get the wide distribution (or if $\lambda \sim D$, a diffraction pattern) showing on the exposed film. This is not just some crazy thought experiment, but is actually what happens.

Momentum Conservation?

If you think a moment about the above photon-by-photon limit, you will realize that each photon that is sent off in a direction perpendicular to the original wave direction must have a non-zero momentum perpendicular to that original direction.

Where is this momentum coming from? Well, momentum is still conserved, it is just that the apparatus (the slit) was implicitly presumed to be extremely massive and although it is recoiling in such a way as to conserve overall momentum, that recoil is totally negligible and unobservable.

What is important to note is that for small D, there is a big spread in possible photon momenta in the direction $\perp$ to the wave direction. Conversely, if $D \gg \lambda$, then there is little spreading and redirection of the incoming wave, corresponding to not very much $\perp$ momentum.

This is an example of what we will call the uncertainty principle, according to which: the better you try to define the location of a wave in a particular direction, the more uncertain is the momentum (in the same direction) of the particles that are actually contained in that wave

The 2-slit case

We can push all this further by going to the case of two identical slits separated by some distance S. (For this discussion, λ will be much larger than the size of the slits, called D earlier, but not necessarily large compared to the separation S.)

Using the above discussion, we would expect that if $\lambda \gg S$ then the result would be a very spread out interference pattern. In fact, in the limit where the two slits lie on top of one another, we get (for slit width $D \ll \lambda$) the same very spread out pattern shown earlier. Of course, for $\lambda \sim S$ there should be more and more interference minima visible.

And, you should ask, what is the photon by photon picture in this case?

In fact, somehow the individual photons know that they should only go to where the locations of constructive interference are and that they should go with much less probability to the places where there is destructive interference!

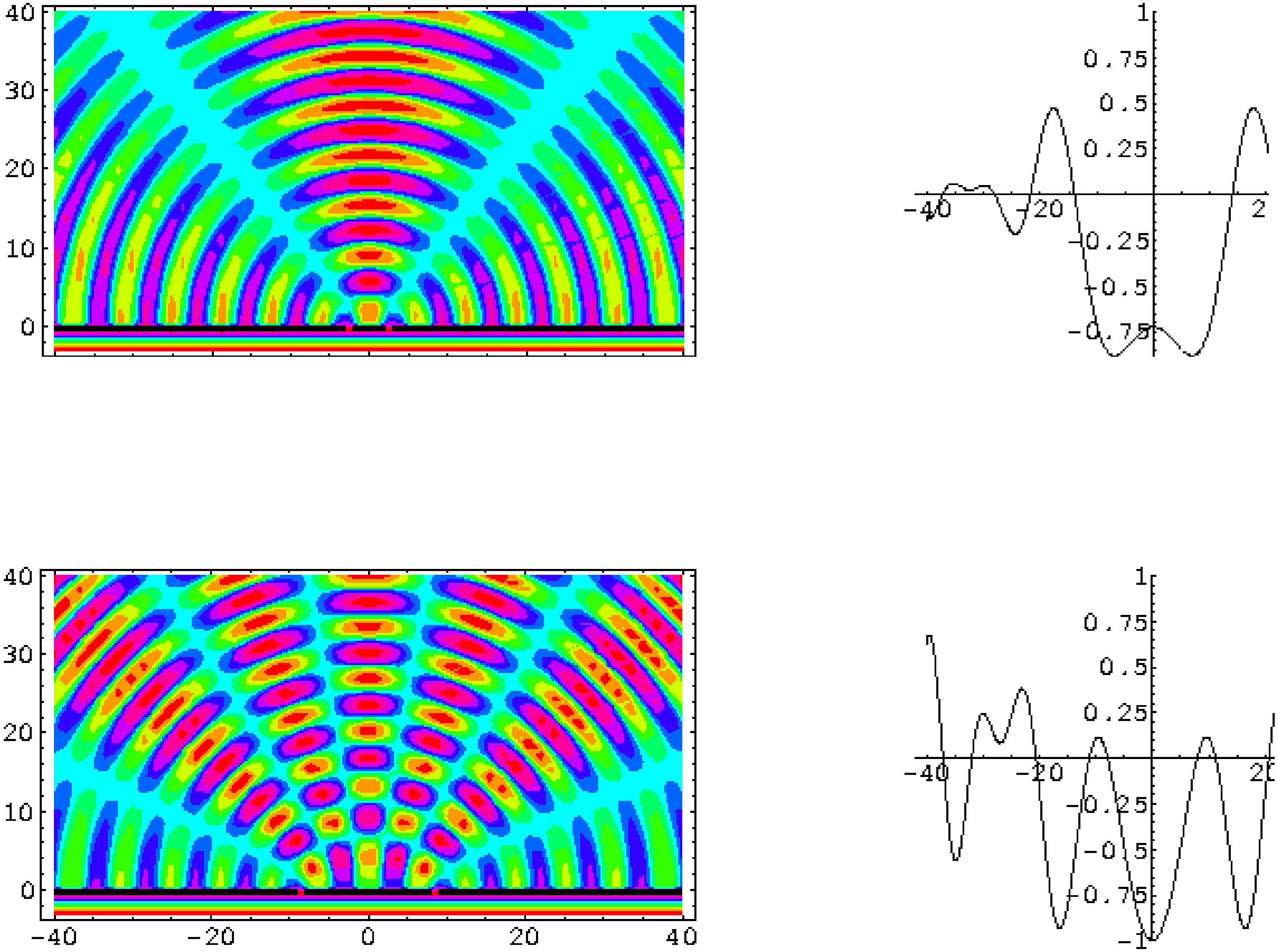
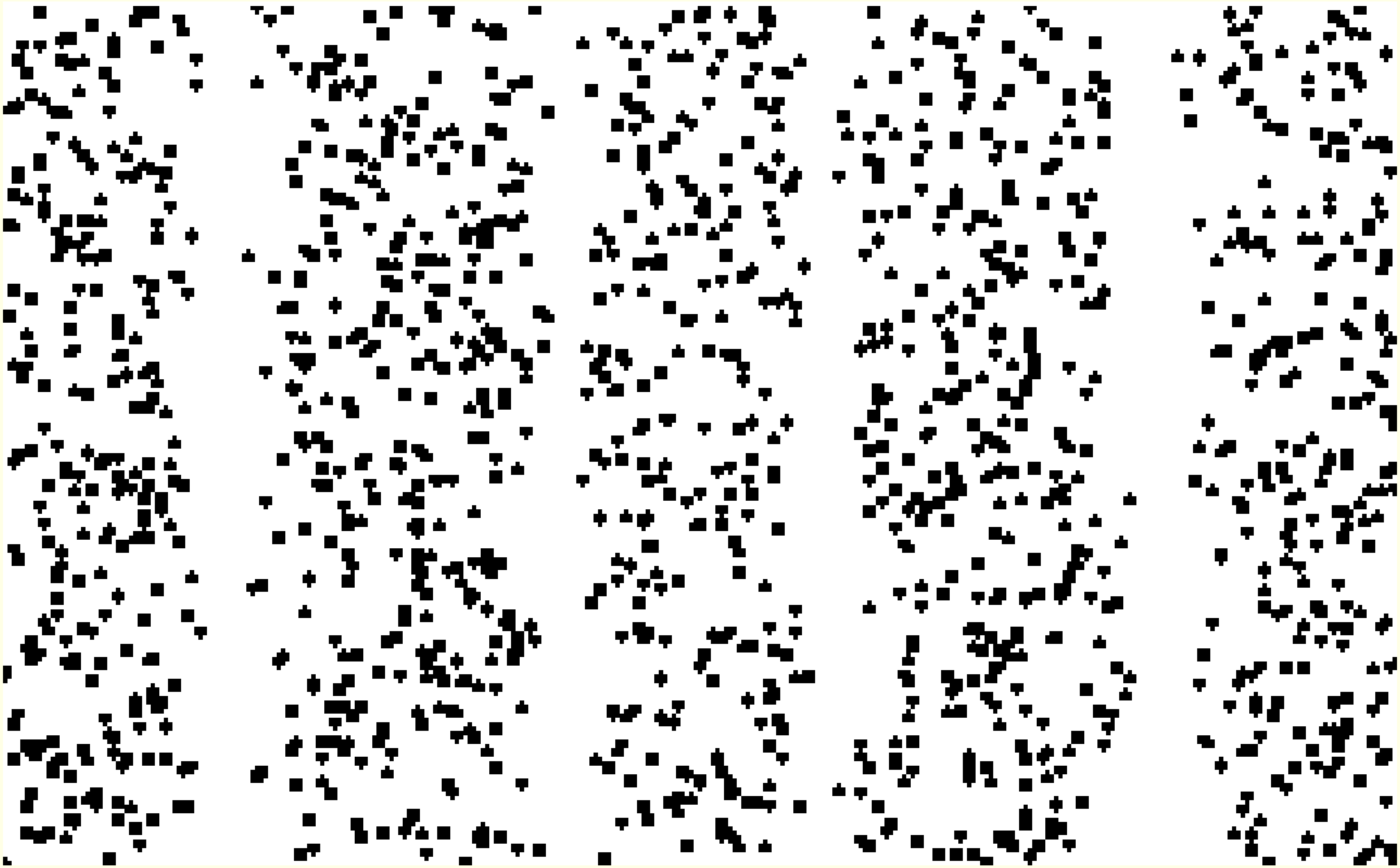


Figure 25: Top: $\lambda \sim S$. Bottom: $\lambda < S$.

dots



Probability and Wave-Particle Duality

The only way of mathematically realizing the above situation is in terms of a probability amplitude.

The idea, to which we will return again and again, is that the particles (photons) arrive at a given location on the screen (or other detector) with a *probability* determined by the square of the wave amplitude (that is the intensity of the wave) at that particular location!

To reemphasize, we write this in equation form as

$$\textbf{Probability (density) of finding a } \underline{particle} \propto |\textbf{amplitude of } \underline{wave}|^2 \quad (121)$$

where we had to use the absolute square since a typical wave amplitude that is the solution of the Maxwell wave propagation equations is a complex exponential of some sort. (Be sure to review about complex exponentials if you do not remember that mathematics. Also very useful to remind yourself about Fourier transforms at this time.)

At this point, you might ask yourself the following question: can we determine the slit through which a given photon passed?

This is, in fact, not possible. In fact, if you temporarily closed one slit or the other in between the infrequent photons of the low-luminosity limit, the interference pattern would disappear! By changing the apparatus, we alter the phenomena that depends upon both slits being simultaneously present. We must "allow" each photon the opportunity to pass through either slit, even though on a photon-by-photon basis it will always have passed through one or the other. We just can't say which.

An Example

Light of wavelength $633\ nm$ is directed at a double slit, and the interference pattern is viewed on a screen. The intensity at the center of the screen is $4.0\ W/m^2$.

(a) At what rate are photons detected at the center of the screen?

(b) At what rate are photons detected at the first interference minimum?

(c) At what rate are photons detected at a point on the screen where the waves from the two sources are out of phase by $1/3$ of a cycle?

(Note: you should recall from physical optics that the double-slit intensity varies according to $I = I_0 \cos^2(\frac{1}{2}\phi)$, where ϕ is the phase difference

between the waves from the two slits and I_0 is the intensity when $\phi = 0$, $i.e.$ at the center of the screen.)

Answer:

(a) Each photon has energy

$$E = h\frac{c}{\lambda} = (6.63\times 10^{-34}\ J\cdot s)\frac{3\times 10^{8}\ m/s}{633\times 10^{-9}\ m} = 3.14\times 10^{-19}\ J\,. \quad (122)$$

We must therefore have

$$\frac{4.0\ J/(s\cdot m^2)}{3.14\times 10^{-19}\ J/photon} = 1.27\times 10^{19}\ photons/(s\cdot m^2)\,. \quad (123)$$

(b) The first interference minimum is a point at which destructive interference is complete by virtue of the phase difference between the waves from the two slit sources being out of phase by π. This means 0 intensity and therefore no photons are arriving.

(c) For out of phase by $\phi = \frac{1}{3}(2\pi)$ (here, 2π corresponds to a complete

cycle), one has

$$I = (4.0\ W/m^2)\cos^2\left[\frac{1}{2}\left(\tfrac{1}{3}2\pi\right)\right] = 1.0\ W/m^2\,. \qquad (124)$$

Since this is one-fourth of the intensity at the center, the number of photons per second per unit area is $1/4$ of the value at the center of the screen, or $3.18 \times 10^{18}\ photons/(s \cdot m^2)$.

To repeat: Although it is photon hits that are being recorded on the screen, the probability of their detection is governed by the behavior of the associated electromagnetic wave.