COMPLEX GEOMETRY OF NATURE AND GENERAL RELATIVITY

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Abstract. An attempt is made of giving a self-contained introduction to holomorphic ideas in general relativity, following work over the last thirty years by several authors. The main topics are complex manifolds, spinor and twistor methods, heaven spaces.

CHAPTER ONE

INTRODUCTION TO COMPLEX SPACE-TIME

The physical and mathematical motivations for studying complex space-times or real Riemannian four-manifolds in gravitational physics are first described. They originate from algebraic geometry, Euclidean quantum field theory, the path-integral approach to quantum gravity, and the theory of conformal gravity. The theory of complex manifolds is then briefly outlined. Here, one deals with para-compact Hausdorff spaces where local coordinates transform by complex-analytic transformations. Examples are given such as complex projective space P_m, non-singular sub-manifolds of P_m, and orientable surfaces. The plan of the whole paper is eventually presented, with emphasis on two-component spinor calculus, Penrose transform and Penrose formalism for spin-$\frac{3}{2}$ potentials.

1.1 From Lorentzian space-time to complex space-time

Although Lorentzian geometry is the mathematical framework of classical general relativity and can be seen as a good model of the world we live in (Hawking and Ellis 1973, Esposito 1992, Esposito 1994), the theoretical-physics community has developed instead many models based on a complex space-time picture. We postpone until section 3.3 the discussion of real, complexified or complex manifolds, and we here limit ourselves to say that the main motivations for studying these ideas are as follows.

(1) When one tries to make sense of quantum field theory in flat space-time, one finds it very convenient to study the Wick-rotated version of Green functions, since this leads to well defined mathematical calculations and elliptic boundary-value problems. At the end, quantities of physical interest are evaluated by analytic continuation back to *real* time in Minkowski space-time.

(2) The singularity at $r = 0$ of the Lorentzian Schwarzschild solution disappears on the real Riemannian section of the corresponding complexified space-time, since $r = 0$ no longer belongs to this manifold (Esposito 1994). Hence there are real Riemannian four-manifolds which are singularity-free, and it remains to be seen whether they are the most fundamental in modern theoretical physics.

(3) Gravitational instantons shed some light on possible boundary conditions relevant for path-integral quantum gravity and quantum cosmology (Gibbons and Hawking 1993, Esposito 1994).

(4) Unprimed and primed spin-spaces are not (anti-)isomorphic if Lorentzian space-time is replaced by a complex or real Riemannian manifold. Thus, for example, the Maxwell field strength is represented by two independent symmetric spinor fields, and the Weyl curvature is also represented by two independent symmetric spinor fields (see (2.1.35) and (2.1.36)). Since such spinor fields are no longer related by complex conjugation (i.e. the (anti-)isomorphism between the two spin-spaces), one of them may vanish without the other one having to vanish

as well. This property gives rise to the so-called self-dual or anti-self-dual gauge fields, as well as to self-dual or anti-self-dual space-times (section 4.2).

(5) The geometric study of this special class of space-time models has made substantial progress by using twistor-theory techniques. The underlying idea (Penrose 1967, Penrose 1968, Penrose and MacCallum 1973, Penrose 1975, Penrose 1977, Penrose 1980, Penrose and Ward 1980, Ward 1980a–b, Penrose 1981, Ward 1981a–b, Huggett 1985, Huggett and Tod 1985, Woodhouse 1985, Penrose 1986, Penrose 1987, Yasskin 1987, Manin 1988, Bailey and Baston 1990, Mason and Hughston 1990, Ward and Wells 1990, Mason and Woodhouse 1996) is that conformally invariant concepts such as null lines and null surfaces are the basic building blocks of the world we live in, whereas space-time points should only appear as a derived concept. By using complex-manifold theory, twistor theory provides an appropriate mathematical description of this key idea.

A possible mathematical motivation for twistors can be described as follows (papers 99 and 100 in Atiyah (1988)). In two real dimensions, many interesting problems are best tackled by using complex-variable methods. In four real dimensions, however, the introduction of two complex coordinates is not, by itself, sufficient, since no preferred choice exists. In other words, if we define the complex variables

$$z_1 \equiv x_1 + ix_2, \tag{1.1.1}$$

$$z_2 \equiv x_3 + ix_4, \tag{1.1.2}$$

we rely too much on this particular coordinate system, and a permutation of the four real coordinates x_1, x_2, x_3, x_4 would lead to new complex variables not well related to the first choice. One is thus led to introduce three complex variables $\left(u, z_1^u, z_2^u\right)$: the first variable u tells us which complex structure to use, and the next two are the complex coordinates themselves. In geometric language, we start with the complex projective three-space $P_3(C)$ (see section 1.2) with complex homogeneous coordinates (x, y, u, v), and we remove the complex projective line given by $u = v = 0$. Any line in $\Big(P_3(C) - P_1(C)\Big)$ is thus given by a pair of equations

$$x = au + bv, \tag{1.1.3}$$

$$y = cu + dv. \tag{1.1.4}$$

In particular, we are interested in those lines for which $c = -\overline{b}, d = \overline{a}$. The determinant Δ of (1.1.3) and (1.1.4) is thus given by

$$\Delta = a\overline{a} + b\overline{b} = |a|^2 + |b|^2, \tag{1.1.5}$$

which implies that the line given above never intersects the line $x = y = 0$, with the obvious exception of the case when they coincide. Moreover, no two lines intersect, and they fill out the whole of $\Big(P_3(C) - P_1(C)\Big)$. This leads to the fibration $\Big(P_3(C) - P_1(C)\Big) \longrightarrow R^4$ by assigning to each point of $\Big(P_3(C) - P_1(C)\Big)$ the four coordinates $\Big(\mathrm{Re}(a), \mathrm{Im}(a), \mathrm{Re}(b), \mathrm{Im}(b)\Big)$. Restriction of this fibration to a plane of the form

$$\alpha u + \beta v = 0, \tag{1.1.6}$$

yields an isomorphism $C^2 \cong R^4$, which depends on the ratio $(\alpha, \beta) \in P_1(C)$. This is why the picture embodies the idea of introducing complex coordinates.

Such a fibration depends on the conformal structure of R^4. Hence, it can be extended to the one-point compactification S^4 of R^4, so that we get a fibration $P_3(C) \longrightarrow S^4$ where the line $u = v = 0$, previously excluded, sits over the point at ∞ of $S^4 = R^4 \cup \Big\{\infty\Big\}$. This fibration is naturally obtained if we use the quaternions H to identify C^4 with H^2 and the four-sphere S^4 with $P_1(H)$, the quaternion projective line. We should now recall that the quaternions H are obtained from the vector space R of real numbers by adjoining three symbols i, j, k such that

$$i^2 = j^2 = k^2 = -1, \tag{1.1.7}$$

$$ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j. \tag{1.1.8}$$

Thus, a general *quaternion* $\in H$ is defined by

$$x \equiv x_1 + x_2 i + x_3 j + x_4 k, \tag{1.1.9}$$

where $\Big(x_1, x_2, x_3, x_4\Big) \in R^4$, whereas the conjugate quaternion $\overline{x}$ is given by

$$\overline{x} \equiv x_1 - x_2 i - x_3 j - x_4 k. \tag{1.1.10}$$

Note that conjugation obeys the identities

$$\overline{(xy)} = \overline{y}\,\overline{x}, \tag{1.1.11}$$

$$x\overline{x} = \overline{x}x = \sum_{\mu=1}^{4} x_\mu^2 \equiv |x|^2. \tag{1.1.12}$$

If a quaternion does not vanish, it has a unique inverse given by

$$x^{-1} \equiv \frac{\overline{x}}{|x|^2}. \tag{1.1.13}$$

Interestingly, if we identify i with $\sqrt{-1}$, we may view the complex numbers C as contained in H taking $x_3 = x_4 = 0$. Moreover, every quaternion x as in (1.1.9) has a unique decomposition

$$x = z_1 + z_2 j, \tag{1.1.14}$$

where $z_1 \equiv x_1 + x_2 i$, $z_2 \equiv x_3 + x_4 i$, by virtue of (1.1.8). This property enables one to identify H with C^2, and finally H^2 with C^4, as we said following (1.1.6).

The map $\sigma : P_3(C) \longrightarrow P_3(C)$ defined by

$$\sigma(x, y, u, v) = (-\overline{y}, \overline{x}, -\overline{v}, \overline{u}), \tag{1.1.15}$$

preserves the fibration because $c = -\overline{b}, d = \overline{a}$, and induces the antipodal map on each fibre. We can now lift problems from S^4 or R^4 to $P_3(C)$ and try to use complex methods.

1.2 Complex manifolds

Following Chern (1979), we now describe some basic ideas and properties of complex-manifold theory. The reader should thus find it easier (or, at least, less difficult) to understand the holomorphic ideas used in the rest of the paper.

We know that a manifold is a space which is locally similar to Euclidean space in that it can be covered by coordinate patches. More precisely (Hawking and Ellis 1973), we say that a *real* C^r n-dimensional manifold $\mathcal{M}$ is a set $\mathcal{M}$ together with a C^r atlas $\Big\{U_\alpha, \phi_\alpha\Big\}$, i.e. a collection of charts $\Big(U_\alpha, \phi_\alpha\Big)$, where the U_α are subsets of $\mathcal{M}$ and the ϕ_α are one-to-one maps of the corresponding U_α into open sets in R^n such that

(i) $\mathcal{M}$ is covered by the U_α, i.e. $\mathcal{M} = \bigcup_\alpha U_\alpha$

(ii) if $U_\alpha \cap U_\beta$ is non-empty, the map

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta\Big(U_\alpha \cap U_\beta\Big) \rightarrow \phi_\alpha\Big(U_\alpha \cap U_\beta\Big)$$

is a C^r map of an open subset of R^n into an open subset of R^n. In general relativity, it is of considerable importance to require that the Hausdorff separation axiom should hold. This states that if p, q are any two distinct points in $\mathcal{M}$, there exist disjoint open sets U, V in $\mathcal{M}$ such that $p \in U$, $q \in V$. The space-time manifold (M, g) is therefore taken to be a connected, four-dimensional, Hausdorff C^∞ manifold M with a Lorentz metric g on M, i.e. the assignment of a symmetric, non-degenerate bilinear form $g_{|p} : T_pM \times T_pM \rightarrow R$ with diagonal form $(-,+,+,+)$ to each tangent space. Moreover, a time orientation is given by a globally defined, timelike vector field $X : M \rightarrow TM$. This enables one to say that a timelike or null tangent vector $v \in T_pM$ is future-directed if $g(X(p), v) < 0$, or past-directed if $g(X(p), v) > 0$ (Esposito 1992, Esposito 1994).

By a complex manifold we mean a paracompact Hausdorff space covered by neighbourhoods each homeomorphic to an open set in C^m, such that where two neighbourhoods overlap, the local coordinates transform by a complex-analytic transformation. Thus, if $z^1, ..., z^m$ are local coordinates in one such neighbourhood, and if $w^1, ..., w^m$ are local coordinates in another neighbourhood, where they are both defined one has $w^i = w^i\left(z^1, ..., z^m\right)$, where each w^i is a holomorphic function of the z's, and the determinant $\partial\left(w^1, ..., w^m\right)/\partial\left(z^1, ..., z^m\right)$ does not vanish. Various examples can be given as follows (Chern 1979).

E1. The space C^m whose points are the m-tuples of complex numbers $\left(z^1, ..., z^m\right)$. In particular, C^1 is the so-called Gaussian plane.

E2. Complex projective space P_m, also denoted by $P_m(C)$ or CP^m. Denoting by $\{0\}$ the origin $(0, ..., 0)$, this is the quotient space obtained by identifying the points $\left(z^0, z^1, ..., z^m\right)$ in $C^{m+1} - \{0\}$ which differ from each other by a factor. The covering of P_m is given by $m+1$ open sets U_i defined respectively by $z^i \neq 0$, $0 \leq i \leq m$. In U_i we have the local coordinates $\zeta_i^k \equiv z^k/z^i$, $0 \leq k \leq m$, $k \neq i$. In $U_i \cap U_j$, transition of local coordinates is given by $\zeta_j^h \equiv \zeta_i^h/\zeta_i^j$, $0 \leq h \leq m$, $h \neq j$, which are holomorphic functions. A particular case is the Riemann sphere P_1.

E3. Non-singular sub-manifolds of P_m, in particular, the non-singular hyperquadric

$$\left(z^0\right)^2 + ... + \left(z^m\right)^2 = 0. \tag{1.2.1}$$

A theorem of Chow states that every compact sub-manifold embedded in P_m is the locus defined by a finite number of homogeneous polynomial equations. Compact sub-manifolds of C^m are not very important, since a connected compact sub-manifold of C^m is a point.

E4. Let Γ be the discontinuous group generated by $2m$ translations of C^m, which are linearly independent over the reals. The quotient space C^m/Γ is then called the complex torus. Moreover, let Δ be the discontinuous group generated by $z^k \rightarrow 2z^k$, $1 \leq k \leq m$. The quotient manifold $\left(C^m - \{0\}\right)/\Delta$ is the so-called Hopf manifold, and is homeomorphic to $S^1 \times S^{2m-1}$. Last but not least, we consider the group M_3 of all matrices

$$E_3 = \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.2.2}$$

and let D be the discrete group consisting of those matrices for which z_1, z_2, z_3 are Gaussian integers. This means that $z_k = m_k + in_k$, $1 \leq k \leq 3$, where m_k, n_k are rational integers. An Iwasawa manifold is then defined as the quotient space M_3/D.

E5. Orientable surfaces are particular complex manifolds. The surfaces are taken to be C^∞, and we define on them a positive-definite Riemannian metric. The Korn–Lichtenstein theorem ensures that local parameters x, y exist such that the metric locally takes the form

$$g = \lambda^2 \Big(dx \otimes dx + dy \otimes dy \Big), \ \lambda > 0, \tag{1.2.3}$$

or

$$g = \lambda^2 dz \otimes d\overline{z}, \ z \equiv x + iy. \tag{1.2.4}$$

If w is another local coordinate, we have

$$g = \lambda^2 dz \otimes d\overline{z} = \mu^2 dw \otimes d\overline{w}, \tag{1.2.5}$$

since g is globally defined. Hence dw is a multiple of dz or $d\overline{z}$. In particular, if the complex coordinates z and w define the same orientation, then dw is proportional to dz. Thus, w is a holomorphic function of z, and the surface becomes a complex manifold. Riemann surfaces are, by definition, one-dimensional complex manifolds.

Let us denote by V an m-dimensional real vector space. We say that V has a *complex structure* if there exists a linear endomorphism $J : V \rightarrow V$ such that $J^2 = -1\!\!1$, where $1\!\!1$ is the identity endomorphism. An eigenvalue of J is a complex number λ such that the equation $Jx = \lambda x$ has a non-vanishing solution $x \in V$. Applying J to both sides of this equation, one finds $-x = \lambda^2 x$. Hence $\lambda = \pm i$.

Since the complex eigenvalues occur in conjugate pairs, V is of even dimension $n = 2m$. Let us now denote by V^* the dual space of V, i.e. the space of all real-valued linear functions over V. The pairing of V and V^* is $\langle x, y^* \rangle$, $x \in V$, $y^* \in V^*$, so that this function is R-linear in each of the arguments. Following Chern 1979, we also consider $V^* \otimes C$, i.e. the space of all complex-valued R-linear functions over V. By construction, $V^* \otimes C$ is an n-complex-dimensional complex vector space. Elements $f \in V^* \otimes C$ are of type $(1,0)$ if $f(Jx) = if(x)$, and of type $(0,1)$ if $f(Jx) = -if(x)$, $x \in V$.

If V has a complex structure J, an *Hermitian structure* in V is a complex-valued function H acting on $x, y \in V$ such that

$$H\Bigl(\lambda_1 x_1 + \lambda_2 x_2, y\Bigr) = \lambda_1 H(x_1, y) + \lambda_2 H(x_2, y) \;\; x_1, x_2, y \in V \;\; \lambda_1, \lambda_2 \in R, \tag{1.2.6}$$

$$\overline{H(x,y)} = H(y,x), \tag{1.2.7}$$

$$H(Jx, y) = iH(x,y) \Longleftrightarrow H(x, Jy) = -iH(x,y). \tag{1.2.8}$$

By using the split of $H(x,y)$ into its real and imaginary parts

$$H(x,y) = F(x,y) + iG(x,y), \tag{1.2.9}$$

conditions (1.2.7) and (1.2.8) may be re-expressed as

$$F(x,y) = F(y,x), \; G(x,y) = -G(y,x), \tag{1.2.10}$$

$$F(x,y) = G(Jx,y), \; G(x,y) = -F(Jx,y). \tag{1.2.11}$$

If $\mathcal{M}$ is a C^∞ manifold of dimension n, and if T_x and T_x^* are tangent and cotangent spaces respectively at $x \in \mathcal{M}$, an *almost complex structure* on $\mathcal{M}$ is a C^∞ field of endomorphisms $J_x : T_x \rightarrow T_x$ such that $J_x^2 = -\mathbb{I}_x$, where $\mathbb{I}_x$ is the identity endomorphism in T_x. A manifold with an almost complex structure is called *almost complex*. If a manifold is almost complex, it is even-dimensional and orientable. However, this is only a necessary condition. Examples can be found (e.g. the four-sphere S^4) of even-dimensional, orientable manifolds which cannot be given an almost complex structure.

1.3 An outline of this work

Since this paper is devoted to the geometry of complex space-time in spinor form, chapter two presents the basic ideas, methods and results of two-component spinor calculus. Such a calculus is described in terms of spin-space formalism, i.e. a complex vector space endowed with a symplectic form and some fundamental isomorphisms. These mathematical properties enable one to raise and lower indices, define the conjugation of spinor fields in Lorentzian or Riemannian four-geometries, translate tensor fields into spinor fields (or the other way around). The standard two-spinor form of the Riemann curvature tensor is then obtained by relying on the (more) familiar tensor properties of the curvature. The introductory analysis ends with the Petrov classification of space-times, expressed in terms of the Weyl spinor of conformal gravity.

Since the whole of twistor theory may be viewed as a holomorphic description of space-time geometry in a conformally invariant framework, chapter three studies the key results of conformal gravity, i.e. C-spaces, Einstein spaces and complex Einstein spaces. Hence a necessary and sufficient condition for a space-time to be conformal to a complex Einstein space is obtained, following Kozameh *et al.* (1985). Such a condition involves the Bach and Eastwood–Dighton spinors, and their occurrence is derived in detail. The difference between Lorentzian space-times, Riemannian four-spaces, complexified space-times and complex space-times is also analyzed.

Chapter four is a pedagogical introduction to twistor spaces, from the point of view of mathematical physics and relativity theory. This is obtained by defining twistors as α-planes in complexified compactified Minkowski space-time, and as α-surfaces in curved space-time. In the former case, one deals with totally null two-surfaces, in that the complexified Minkowski metric vanishes on any pair of null tangent vectors to the surface. Hence such null tangent vectors have the form $\lambda^A \pi^{A'}$, where λ^A is varying and $\pi^{A'}$ is covariantly constant. This definition can be generalized to complex or real Riemannian four-manifolds, provided that the Weyl curvature is anti-self-dual. An alternative definition of twistors in Minkowski space-time is instead based on the vector space of solutions of a differential equation, which involves the symmetrized covariant derivative of an unprimed spinor field. Interestingly, a deep correspondence exists between flat space-time and twistor space. Hence complex space-time points correspond to spheres in the so-called projective twistor space, and this concept is carefully formulated. Sheaf cohomology is then presented as the mathematical tool necessary to describe a conformally invariant isomorphism between the complex vector space of holomorphic solutions of the wave equation on the forward tube of flat space-time, and the complex vector space of complex-analytic functions of three variables. These are arbitrary, in that they are not subject to any differential equation. Eventually, Ward's one-to-one correspondence between complex space-times with non-vanishing cosmological constant, and *sufficiently small* deformations of flat projective twistor space, is presented.

An example of explicit construction of anti-self-dual space-time is given in chapter five, following Ward (1978). This generalization of Penrose's non-linear graviton (Penrose 1976a-b) combines two-spinor techniques and twistor theory in a way very instructive for beginning graduate students. However, it appears necessary to go beyond anti-self-dual space-times, since they are only a particular class of (complex) space-times, and they do not enable one to recover the full physical content of (complex) general relativity. This implies going beyond the original twistor theory, since the three-complex-dimensional space of α-surfaces only exists in anti-self-dual space-times. After a brief review of alternative ideas, attention is focused on the recent attempt by Roger Penrose to define twistors as *charges* for massless spin-$\frac{3}{2}$ fields. Such an approach has been considered since a vanishing Ricci tensor provides the consistency condition for the existence and propagation of massless helicity-$\frac{3}{2}$ fields in curved space-time. Moreover, in Minkowski space-time the space of charges for such fields is naturally identified with the corresponding twistor space. The resulting geometric scheme in the presence of curvature is as follows. First, define a twistor for Ricci-flat space-time. Second, characterize the resulting twistor space. Third, reconstruct the original Ricci-flat space-time from such a twistor space. One of the main technical difficulties of the program proposed by Penrose is to obtain a *global* description of the space of potentials for massless spin-$\frac{3}{2}$ fields. The corresponding *local* theory is instead used, for other purposes, in our chapter eight (see below).

The two-spinor description of complex space-times with torsion is given in chapter six. These space-times are studied since torsion is a naturally occurring geometric property of relativistic theories of gravitation, the gauge theory of the Poincaré group leads to its presence and the occurrence of cosmological singularities can be less generic than in general relativity (Esposito 1994 and references therein). It turns out that, before studying the complex theory, many differences already arise, since the Riemann tensor has 36 independent real components at each point (Penrose 1983), rather than 20 as in general relativity. This happens since the connection is no longer symmetric. Hence the Ricci tensor acquires an anti-symmetric part, and the reality conditions for the trace-free part of Ricci and for the scalar curvature no longer hold. Hence, on taking a complex space-time with non-vanishing torsion, all components of the Riemann curvature are given by *independent* spinor fields and scalar fields, not related by any conjugation. Torsion is, itself, described by two independent spinor fields. The corresponding integrability condition for α-surfaces is shown to involve the self-dual Weyl spinor, the torsion spinor with three primed indices and one unprimed index (in a non-linear way), and covariant derivatives of such a torsion spinor. The key identities of two-spinor calculus within this framework, including in particular the spinor Ricci identities, are derived in a self-consistent way for pedagogical reasons.

Chapters seven and eight of our paper are devoted to the application of two-spinor techniques to problems motivated by supersymmetry and quantum cosmology. For this purpose, chapter seven studies spin-$\frac{1}{2}$ fields in real Riemannian four-geometries. After deriving the Dirac and Weyl equations in two-component spinor form in Riemannian backgrounds, we focus on boundary conditions for *massless* fermionic fields motivated by local supersymmetry. These involve the normal to the boundary and a pair of independent spinor fields ψ^A and $\widetilde{\psi}^{A'}$. In the case of flat Euclidean four-space bounded by a three-sphere, they eventually imply that the classical modes of the massless spin-$\frac{1}{2}$ field multiplying harmonics having positive eigenvalues for the intrinsic three-dimensional Dirac operator on S^3 should vanish on S^3. Remarkably, this coincides with the property of the classical boundary-value problem when global boundary conditions are imposed on the three-sphere in the massless case. The boundary term in the action functional is also derived. Our analysis makes it necessary to use part of the analysis in section 5.8 of Esposito (1994), to prove that the Dirac operator subject to supersymmetric boundary conditions on the three-sphere admits self-adjoint extensions. The proof relies on the Euclidean conjugation and on a result originally proved by von Neumann for complex scalar fields. Chapter seven ends with a mathematical introduction to the global theory of the total Dirac operator in Riemannian four-geometries, described as a first-order elliptic operator mapping smooth sections (i.e. the spinor fields) of a complex vector bundle into smooth sections of the same bundle. Its action on the sections is obtained by composition of Clifford multiplication with covariant differentiation, and provides an intrinsic formulation of the spinor covariant derivative frequently used in our paper.

The *local* theory of potentials for massless spin-$\frac{3}{2}$ fields is applied to the classical boundary-value problems relevant for quantum cosmology in chapter eight (cf. chapter five). For this purpose, we first study local boundary conditions involving field strengths and the normal to the boundary, originally considered in anti-de Sitter space-time, and recently applied in one-loop quantum cosmology. Following Esposito (1994) and Esposito and Pollifrone (1994), we derive the conditions under which spin-lowering and spin-raising operators preserve these local boundary conditions on a three-sphere for fields of spin $0, \frac{1}{2}, 1, \frac{3}{2}$ and 2. Second, the two-component spinor analysis of the four Dirac potentials of the totally symmetric and independent field strengths for spin $\frac{3}{2}$ is applied to the case of a three-sphere boundary. It is found that such boundary conditions can only be imposed in a flat Euclidean background, for which the gauge freedom in the choice of the massless potentials remains. Third, we study the alternative, Rarita–Schwinger form of the spin-$\frac{3}{2}$ potentials. They are no longer symmetric in the pair of unprimed or primed spinor indices, and their gauge freedom involves a spinor field which is no longer a solution of the Weyl equation. Gauge transformations on the potentials are shown to be compatible with the field equations provided that the background is Ricci-flat, in agreement with well known results in the literature. However, the preservation of boundary conditions under such gauge transformations is found to restrict the gauge freedom. The construction by Penrose of a second set of potentials which supplement the Rarita–Schwinger potentials is then applied. The equations for these potentials, jointly with the boundary conditions, imply that the background four-manifold is further restricted to be totally flat. In the last part of chapter eight, massive spin-$\frac{3}{2}$ potentials in conformally flat Einstein four-manifolds are studied. The analysis of supergauge transformations of potentials for spin $\frac{3}{2}$ shows that the gauge freedom for massive spin-$\frac{3}{2}$ potentials is generated by solutions of the supertwistor equations. Interestingly, the supercovariant form of a partial connection on a non-linear bundle is obtained, and the basic equation obeyed by the second set of potentials in the massive case is shown to be the integrability condition on super β-surfaces of a differential operator on a vector bundle of rank three.

The mathematical foundations of twistor theory are re-analyzed in chapter nine. After a review of various definitions of twistors in curved space-time, we present the Penrose transform and the ambitwistor correspondence in terms of the double-fibration picture. The Radon transform in complex geometry is also defined, and the Ward construction of massless fields as bundles is given. The latter concept has motivated the recent work by Penrose on a second set of potentials which supplement the Rarita–Schwinger potentials in curved space-time. Recent progress on quantum field theories in the presence of boundaries is then described, since the boundary conditions of chapters seven and eight are relevant for the analysis of mixed boundary conditions in quantum field theory and quantum gravity. Last, chapter ten reviews old and new ideas in complex general relativity: heaven spaces and heavenly equations, complex relativity and real solutions, multimomenta in complex general relativity.

CHAPTER TWO

TWO-COMPONENT SPINOR CALCULUS

Spinor calculus is presented by relying on spin-space formalism. Given the existence of unprimed and primed spin-space, one has the isomorphism between such vector spaces and their duals, realized by a symplectic form. Moreover, for Lorentzian metrics, complex conjugation is the (anti-)isomorphism between unprimed and primed spin-space. Finally, for any space-time point, its tangent space is isomorphic to the tensor product of unprimed and primed spin-spaces via the Infeld–van der Waerden symbols. Hence the correspondence between tensor fields and spinor fields. Euclidean conjugation in Riemannian geometries is also discussed in detail. The Maxwell field strength is written in this language, and many useful identities are given. The curvature spinors of general relativity are then constructed explicitly, and the Petrov classification of space-times is obtained in terms of the Weyl spinor for conformal gravity.

2.1 Two-component spinor calculus

Two-component spinor calculus is a powerful tool for studying classical field theories in four-dimensional space-time models. Within this framework, the basic object is spin-space, a two-dimensional complex vector space S with a symplectic form ε, i.e. an antisymmetric complex bilinear form. Unprimed spinor indices $A, B, \ldots$ take the values $0, 1$ whereas primed spinor indices $A', B', \ldots$ take the values $0', 1'$ since there are actually two such spaces: unprimed spin-space (S, ε) and primed spin-space (S', ε'). The whole two-spinor calculus in *Lorentzian* four-manifolds relies on three fundamental properties (Veblen 1933, Ruse 1937, Penrose 1960, Penrose and Rindler 1984, Esposito 1992, Esposito 1994):

(i) The isomorphism between $\left(S, \varepsilon_{AB}\right)$ and its dual $\left(S^*, \varepsilon^{AB}\right)$. This is provided by the symplectic form ε, which raises and lowers indices according to the rules

$$\varepsilon^{AB}\, \varphi_B = \varphi^A \ \in \ S, \tag{2.1.1}$$

$$\varphi^B\, \varepsilon_{BA} = \varphi_A \ \in \ S^*. \tag{2.1.2}$$

Thus, since

$$\varepsilon_{AB} = \varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.1.3}$$

one finds in components $\varphi^0 = \varphi_1, \varphi^1 = -\varphi_0$.

Similarly, one has the isomorphism $\left(S', \varepsilon_{A'B'}\right) \cong \left((S')^*, \varepsilon^{A'B'}\right)$, which implies

$$\varepsilon^{A'B'}\, \varphi_{B'} = \varphi^{A'} \ \in \ S', \tag{2.1.4}$$

$$\varphi^{B'}\, \varepsilon_{B'A'} = \varphi_{A'} \ \in \ (S')^*, \tag{2.1.5}$$

where

$$\varepsilon_{A'B'} = \varepsilon^{A'B'} = \begin{pmatrix} 0' & 1' \\ -1' & 0' \end{pmatrix}. \tag{2.1.6}$$

(ii) The (anti-)isomorphism between $\left(S, \varepsilon_{AB}\right)$ and $\left(S', \varepsilon_{A'B'}\right)$, called complex conjugation, and denoted by an overbar. According to a standard convention, one has

$$\overline{\psi^A} \equiv \overline{\psi}^{A'} \; \in \; S', \tag{2.1.7}$$

$$\overline{\psi^{A'}} \equiv \overline{\psi}^{A} \; \in \; S. \tag{2.1.8}$$

Thus, complex conjugation maps elements of a spin-space to elements of the *complementary* spin-space. Hence some authors say it is an anti-isomorphism. In components, if w^A is thought as $w^A = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, the action of (2.1.7) leads to

$$\overline{w^A} \equiv \overline{w}^{A'} \equiv \begin{pmatrix} \overline{\alpha} \\ \overline{\beta} \end{pmatrix}, \tag{2.1.9}$$

whereas, if $z^{A'} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$, then (2.1.8) leads to

$$\overline{z^{A'}} \equiv \overline{z}^{A} = \begin{pmatrix} \overline{\gamma} \\ \overline{\delta} \end{pmatrix}. \tag{2.1.10}$$

With our notation, $\overline{\alpha}$ denotes complex conjugation of the function α, and so on. Note that the symplectic structure is preserved by complex conjugation, since $\overline{\varepsilon}_{A'B'} = \varepsilon_{A'B'}$.

(iii) The isomorphism between the tangent space T at a point of space-time and the tensor product of the unprimed spin-space $\left(S, \varepsilon_{AB}\right)$ and the primed spin-space $\left(S', \varepsilon_{A'B'}\right)$:

$$T \cong \left(S, \varepsilon_{AB}\right) \otimes \left(S', \varepsilon_{A'B'}\right). \tag{2.1.11}$$

The Infeld–van der Waerden symbols $\sigma^a{}_{AA'}$ and $\sigma_a{}^{AA'}$ express this isomorphism, and the correspondence between a vector v^a and a spinor $v^{AA'}$ is given by

$$v^{AA'} \equiv v^a \; \sigma_a{}^{AA'}, \tag{2.1.12}$$

$$v^a \equiv v^{AA'} \, \sigma^a{}_{AA'}. \tag{2.1.13}$$

These mixed spinor-tensor symbols obey the identities

$$\overline{\sigma}_a{}^{AA'} = \sigma_a{}^{AA'}, \tag{2.1.14}$$

$$\sigma_a{}^{AA'} \, \sigma^b{}_{AA'} = \delta_a{}^b, \tag{2.1.15}$$

$$\sigma_a{}^{AA'} \, \sigma^a{}_{BB'} = \varepsilon_B{}^A \, \varepsilon_{B'}{}^{A'}, \tag{2.1.16}$$

$$\sigma_{[a}{}^{AA'} \, \sigma_{b]A}{}^{B'} = -\frac{i}{2} \, \varepsilon_{abcd} \, \sigma^{cAA'} \, \sigma^d{}_A{}^{B'}. \tag{2.1.17}$$

Similarly, a one-form ω_a has a spinor equivalent

$$\omega_{AA'} \equiv \omega_a \, \sigma^a{}_{AA'}, \tag{2.1.18}$$

whereas the spinor equivalent of the metric is

$$\eta_{ab} \, \sigma^a{}_{AA'} \, \sigma^b{}_{BB'} \equiv \varepsilon_{AB} \, \varepsilon_{A'B'}. \tag{2.1.19}$$

In particular, in Minkowski space-time, the above equations enable one to write down a coordinate system in 2×2 matrix form

$$x^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{2.1.20}$$

More precisely, in a (curved) space-time, one should write the following equation to obtain the spinor equivalent of a vector:

$$u^{AA'} = u^a \, e_a^{\hat{c}} \, \sigma_{\hat{c}}{}^{AA'},$$

where $e_a^{\hat{c}}$ is a standard notation for the tetrad, and $e_a^{\hat{c}} \sigma_{\hat{c}}{}^{AA'} \equiv e_a{}^{AA'}$ is called the *soldering form*. This is, by construction, a spinor-valued one-form, which encodes the relevant information about the metric g, because $g_{ab} = e_a{}^{\hat{c}} e_b{}^{\hat{d}} \eta_{\hat{c}\hat{d}}$, η being the Minkowskian metric of the so-called "internal space".

In the Lorentzian-signature case, the Maxwell two-form $F \equiv F_{ab} dx^a \wedge dx^b$ can be written spinorially (Ward and Wells 1990) as

$$F_{AA'BB'} = \frac{1}{2}\Big(F_{AA'BB'} - F_{BB'AA'}\Big) = \varphi_{AB}\, \varepsilon_{A'B'} + \varphi_{A'B'}\, \varepsilon_{AB}, \qquad (2.1.21)$$

where

$$\varphi_{AB} \equiv \frac{1}{2} F_{AC'B}{}^{C'} = \varphi_{(AB)}, \qquad (2.1.22)$$

$$\varphi_{A'B'} \equiv \frac{1}{2} F_{CB'}{}^{C}{}_{A'} = \varphi_{(A'B')}. \qquad (2.1.23)$$

These formulae are obtained by applying the identity

$$T_{AB} - T_{BA} = \varepsilon_{AB}\, T_C{}^C \qquad (2.1.24)$$

to express $\frac{1}{2}\Big(F_{AA'BB'} - F_{AB'BA'}\Big)$ and $\frac{1}{2}\Big(F_{AB'BA'} - F_{BB'AA'}\Big)$. Note also that round brackets (AB) denote (as usual) symmetrization over the spinor indices A and B, and that the antisymmetric part of φ_{AB} vanishes by virtue of the antisymmetry of F_{ab}, since (Ward and Wells 1990) $\varphi_{[AB]} = \frac{1}{4}\varepsilon_{AB}\ F_{CC'}{}^{CC'} = \frac{1}{2}\varepsilon_{AB}\, \eta^{cd}\, F_{cd} = 0$. Last but not least, in the Lorentzian case

$$\overline{\varphi_{AB}} \equiv \overline{\varphi}_{A'B'} = \varphi_{A'B'}. \qquad (2.1.25)$$

The symmetric spinor fields φ_{AB} and $\varphi_{A'B'}$ are the anti-self-dual and self-dual parts of the curvature two-form, respectively.

Similarly, the Weyl curvature $C^a{}_{bcd}$, i.e. the part of the Riemann curvature tensor invariant under conformal rescalings of the metric, may be expressed spinorially, omitting soldering forms for simplicity of notation, as

$$C_{abcd} = \psi_{ABCD}\, \varepsilon_{A'B'}\, \varepsilon_{C'D'} + \overline{\psi}_{A'B'C'D'}\, \varepsilon_{AB}\, \varepsilon_{CD}. \qquad (2.1.26)$$

In canonical gravity (Ashtekar 1988, Esposito 1994) two-component spinors lead to a considerable simplification of calculations. Denoting by n^μ the future-pointing unit timelike normal to a spacelike three-surface, its spinor version obeys the relations

$$n_{AA'}\, e^{AA'}{}_i = 0, \qquad (2.1.27)$$

$$n_{AA'}\, n^{AA'} = 1, \tag{2.1.28}$$

where $e^{AA'}_{\ \ \ \ \mu} \equiv e^{a}_{\ \mu}\, \sigma_{a}^{\ \ AA'}$ is the two-spinor version of the tetrad, i.e. the soldering form introduced before. Denoting by h the induced metric on the three-surface, other useful relations are (Esposito 1994)

$$h_{ij} = -e_{AA'i}\, e^{AA'}_{\ \ \ \ j}, \tag{2.1.29}$$

$$e^{AA'}_{\ \ \ \ 0} = N\, n^{AA'} + N^{i}\, e^{AA'}_{\ \ \ \ i}, \tag{2.1.30}$$

$$n_{AA'}\, n^{BA'} = \frac{1}{2}\varepsilon_{A}^{\ \ B}, \tag{2.1.31}$$

$$n_{AA'}\, n^{AB'} = \frac{1}{2}\varepsilon_{A'}^{\ \ \ B'}, \tag{2.1.32}$$

$$n_{[EB'}\, n_{A]A'} = \frac{1}{4}\varepsilon_{EA}\, \varepsilon_{B'A'}, \tag{2.1.33}$$

$$e_{AA'j}\, e^{AB'}_{\ \ \ \ k} = -\frac{1}{2}h_{jk}\, \varepsilon_{A'}^{\ \ \ B'} - i\varepsilon_{jkl}\sqrt{\det h}\, n_{AA'}\, e^{AB'l}. \tag{2.1.34}$$

In Eq. (2.1.30), N and N^{i} are the lapse and shift functions respectively (Esposito 1994).

To obtain the space-time curvature, we first need to define the spinor covariant derivative $\nabla_{AA'}$. If θ, ϕ, ψ are spinor fields, $\nabla_{AA'}$ is a map such that (Penrose and Rindler 1984, Stewart 1991)

(1) $\nabla_{AA'}(\theta + \phi) = \nabla_{AA'}\theta + \nabla_{AA'}\phi$ (i.e. linearity).

(2) $\nabla_{AA'}(\theta\psi) = \Bigr(\nabla_{AA'}\theta\Bigr)\psi + \theta\Bigr(\nabla_{AA'}\psi\Bigr)$ (i.e. Leibniz rule).

(3) $\psi = \nabla_{AA'}\theta$ implies $\overline{\psi} = \nabla_{AA'}\overline{\theta}$ (i.e. reality condition).

(4) $\nabla_{AA'}\varepsilon_{BC} = \nabla_{AA'}\varepsilon^{BC} = 0$, i.e. the symplectic form may be used to raise or lower indices within spinor expressions acted upon by $\nabla_{AA'}$, in addition to the usual metricity condition $\nabla g = 0$, which involves instead the product of two ε-symbols (see also section 6.3).

(5) $\nabla_{AA'}$ commutes with any index substitution not involving A, A'.

(6) For any function f, one finds $\Big(\nabla_a \nabla_b - \nabla_b \nabla_a\Big) f = 2 S_{ab}^{\;\;\;c} \; \nabla_c f$, where $S_{ab}^{\;\;\;c}$ is the torsion tensor.

(7) For any derivation D acting on spinor fields, a spinor field $\xi^{AA'}$ exists such that $D\psi = \xi^{AA'} \; \nabla_{AA'} \psi, \forall \psi$.

As proved in Penrose and Rindler (1984), such a spinor covariant derivative exists and is unique.

If Lorentzian space-time is replaced by a complex or real Riemannian four-manifold, an important modification should be made, since the (anti-)isomorphism between unprimed and primed spin-space no longer exists. This means that primed spinors can no longer be regarded as complex conjugates of unprimed spinors, or viceversa, as in (2.1.7) and (2.1.8). In particular, Eqs. (2.1.21) and (2.1.26) should be re-written as

$$F_{AA'BB'} = \varphi_{AB} \; \varepsilon_{A'B'} + \widetilde{\varphi}_{A'B'} \; \varepsilon_{AB}, \tag{2.1.35}$$

$$C_{abcd} = \psi_{ABCD} \; \varepsilon_{A'B'} \; \varepsilon_{C'D'} + \widetilde{\psi}_{A'B'C'D'} \; \varepsilon_{AB} \; \varepsilon_{CD}. \tag{2.1.36}$$

With our notation, $\varphi_{AB}, \widetilde{\varphi}_{A'B'}$, as well as $\psi_{ABCD}, \widetilde{\psi}_{A'B'C'D'}$ are *completely independent* symmetric spinor fields, not related by any conjugation.

Indeed, a conjugation can still be defined in the real Riemannian case, but it no longer relates $\Big(S, \varepsilon_{AB}\Big)$ to $\Big(S', \varepsilon_{A'B'}\Big)$. It is instead an anti-involutory operation which maps elements of a spin-space (either unprimed or primed) to elements of the *same* spin-space. By anti-involutory we mean that, when applied twice to a spinor with an odd number of indices, it yields the same spinor with the opposite sign, i.e. its square is minus the identity, whereas the square of complex conjugation as defined in (2.1.9) and (2.1.10) equals the identity. Following Woodhouse (1985) and Esposito (1994), Euclidean conjugation, denoted by a *dagger*, is defined by

$$\Big(w^A\Big)^{\dagger} \equiv \begin{pmatrix} \overline{\beta} \\ -\overline{\alpha} \end{pmatrix}, \tag{2.1.37}$$

$$\left(z^{A'}\right)^{\dagger} \equiv \begin{pmatrix} -\overline{\delta} \\ \overline{\gamma} \end{pmatrix}. \tag{2.1.38}$$

This means that, in flat Euclidean four-space, a unit 2×2 matrix $\delta_{BA'}$ exists such that

$$\left(w^{A}\right)^{\dagger} \equiv \varepsilon^{AB} \, \delta_{BA'} \, \overline{w}^{A'}. \tag{2.1.39}$$

We are here using the freedom to regard w^A either as an $SL(2,C)$ spinor for which complex conjugation can be defined, or as an $SU(2)$ spinor for which Euclidean conjugation is instead available. The soldering forms for $SU(2)$ spinors only involve spinor indices of the same spin-space, i.e. $\widetilde{e}_i^{\;AB}$ and $\widetilde{e}_i^{\;A'B'}$ (Ashtekar 1991). More precisely, denoting by E_a^i a real *triad*, where $i = 1, 2, 3$, and by $\tau^{a}{}_{A}{}^{B}$ the three Pauli matrices, the $SU(2)$ soldering forms are defined by

$$\widetilde{e}^{j}{}_{A}{}^{B} \equiv -\frac{i}{\sqrt{2}} \, E_a^j \, \tau^{a}{}_{A}{}^{B}. \tag{2.1.40}$$

Note that our conventions differ from the ones in Ashtekar (1991), i.e. we use $\widetilde{e}$ instead of σ, and a, b for Pauli-matrix indices, i, j for tangent-space indices on a three-manifold Σ, to agree with our previous notation. The soldering form in (2.1.40) provides an isomorphism between the three-real-dimensional tangent space at each point of Σ, and the three-real-dimensional vector space of 2×2 trace-free Hermitian matrices. The Riemannian three-metric on Σ is then given by

$$h^{ij} = -\widetilde{e}^{i}{}_{A}{}^{B} \, \widetilde{e}^{j}{}_{B}{}^{A}. \tag{2.1.41}$$

2.2 Curvature in general relativity

In this section, following Penrose and Rindler (1984), we want to derive the spinorial form of the Riemann curvature tensor in a Lorentzian space-time with vanishing torsion, starting from the well-known symmetries of Riemann. In agreement with the abstract-index translation of tensors into spinors, soldering forms will be omitted in the resulting equations (cf. Ashtekar (1991)).

Since $R_{abcd} = -R_{bacd}$ we may write

$$R_{abcd} = R_{AA'BB'CC'DD'} = \frac{1}{2} R_{AF'B}{}^{F'}{}_{cd} \, \varepsilon_{A'B'} + \frac{1}{2} R_{FA'}{}^{F}{}_{B'cd} \, \varepsilon_{AB}. \tag{2.2.1}$$

Moreover, on defining

$$X_{ABCD} \equiv \frac{1}{4} R_{AF'B}{}^{F'}{}_{CL'D}{}^{L'}, \tag{2.2.2}$$

$$\Phi_{ABC'D'} \equiv \frac{1}{4} R_{AF'B}{}^{F'}{}_{LC'}{}^{L}{}_{D'}, \tag{2.2.3}$$

the anti-symmetry in cd leads to

$$R_{abcd} = X_{ABCD} \, \varepsilon_{A'B'} \, \varepsilon_{C'D'} + \Phi_{ABC'D'} \, \varepsilon_{A'B'} \, \varepsilon_{CD} + \overline{\Phi}_{A'B'CD} \, \varepsilon_{AB} \, \varepsilon_{C'D'} + \overline{X}_{A'B'C'D'} \, \varepsilon_{AB} \, \varepsilon_{CD}. \tag{2.2.4}$$

According to a standard terminology, the spinors (2.2.2) and (2.2.3) are called the *curvature spinors*. In the light of the (anti-)symmetries of R_{abcd}, they have the following properties:

$$X_{ABCD} = X_{(AB)(CD)}, \tag{2.2.5}$$

$$\Phi_{ABC'D'} = \Phi_{(AB)(C'D')}, \tag{2.2.6}$$

$$X_{ABCD} = X_{CDAB}, \tag{2.2.7}$$

$$\overline{\Phi}_{ABC'D'} = \Phi_{ABC'D'}. \tag{2.2.8}$$

Remarkably, Eqs. (2.2.6) and (2.2.8) imply that $\Phi_{AA'BB'}$ corresponds to a trace-free and real tensor:

$$\Phi_{a}{}^{a} = 0, \; \Phi_{AA'BB'} = \Phi_{ab} = \overline{\Phi}_{ab}. \tag{2.2.9}$$

Moreover, from Eqs. (2.2.5) and (2.2.7) one obtains

$$X_{A(BC)}{}^{A} = 0. \tag{2.2.10}$$

Three duals of R_{abcd} exist which are very useful and are defined as follows:

$$R^{*}{}_{abcd} \equiv \frac{1}{2}\varepsilon_{cd}{}^{pq} \; R_{abpq} = i \; R_{AA'BB'CD'DC'}, \tag{2.2.11}$$

$$^{*}R_{abcd} \equiv \frac{1}{2}\varepsilon_{ab}{}^{pq} \; R_{pqcd} = i \; R_{AB'BA'CC'DD'}, \tag{2.2.12}$$

$$^{*}R^{*}{}_{abcd} \equiv \frac{1}{4}\varepsilon_{ab}{}^{pq} \; \varepsilon_{cd}{}^{rs} \; R_{pqrs} = -R_{AB'BA'CD'DC'}. \tag{2.2.13}$$

For example, in terms of the dual (2.2.11), the familiar equation $R_{a[bcd]} = 0$ reads

$$R^{*}{}_{ab}{}^{bc} = 0. \tag{2.2.14}$$

Thus, to derive the spinor form of the cyclic identity, one can apply (2.2.14) to the equation

$$R^{*}{}_{abcd} = -i \; X_{ABCD} \; \varepsilon_{A'B'} \; \varepsilon_{C'D'} + i \; \Phi_{ABC'D'} \; \varepsilon_{A'B'} \; \varepsilon_{CD}$$
$$- \, i \; \overline{\Phi}_{A'B'CD} \; \varepsilon_{AB} \; \varepsilon_{C'D'} + i \; \overline{X}_{A'B'C'D'} \; \varepsilon_{AB} \; \varepsilon_{CD}. \tag{2.2.15}$$

By virtue of (2.2.6) and (2.2.8) one thus finds

$$X_{AB}{}^{B}{}_{C} \; \varepsilon_{A'C'} = \overline{X}_{A'B'}{}^{B'}{}_{C'} \; \varepsilon_{AC}, \tag{2.2.16}$$

which implies, on defining

$$\Lambda \equiv \frac{1}{6} X_{AB}{}^{AB}, \tag{2.2.17}$$

the reality condition

$$\Lambda = \overline{\Lambda}. \tag{2.2.18}$$

Equation (2.2.1) enables one to express the Ricci tensor $R_{ab} \equiv R_{acb}{}^{c}$ in spinor form as

$$R_{ab} = 6\Lambda \; \varepsilon_{AB} \; \varepsilon_{A'B'} - 2\Phi_{ABA'B'}. \tag{2.2.19}$$

Thus, the resulting scalar curvature, trace-free part of Ricci and Einstein tensor are

$$R = 24\Lambda, \tag{2.2.20}$$

$$R_{ab} - \frac{1}{4} R\, g_{ab} = -2\Phi_{ab} = -2\Phi_{ABA'B'}, \tag{2.2.21}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R\, g_{ab} = -6\Lambda\, \varepsilon_{AB}\, \varepsilon_{A'B'} - 2\Phi_{ABA'B'}, \tag{2.2.22}$$

respectively.

We have still to obtain a more suitable form of the Riemann curvature. For this purpose, following again Penrose and Rindler (1984), we point out that the curvature spinor X_{ABCD} can be written as

$$\begin{aligned} X_{ABCD} &= \frac{1}{3}\Bigl(X_{ABCD} + X_{ACDB} + X_{ADBC}\Bigr) + \frac{1}{3}\Bigl(X_{ABCD} - X_{ACBD}\Bigr) \\ &+ \frac{1}{3}\Bigl(X_{ABCD} - X_{ADCB}\Bigr) \\ &= X_{(ABCD)} + \frac{1}{3}\varepsilon_{BC}\, X_{AF}{}^{F}{}_{D} + \frac{1}{3}\varepsilon_{BD}\, X_{AFC}{}^{F}. \end{aligned} \tag{2.2.23}$$

Since $X_{AFC}{}^{F} = 3\Lambda\, \varepsilon_{AF}$, Eq. (2.2.23) leads to

$$X_{ABCD} = \psi_{ABCD} + \Lambda\Bigl(\varepsilon_{AC}\, \varepsilon_{BD} + \varepsilon_{AD}\, \varepsilon_{BC}\Bigr), \tag{2.2.24}$$

where ψ_{ABCD} is the Weyl spinor.

Since $\Lambda = \overline{\Lambda}$ from (2.2.18), the insertion of (2.2.24) into (2.2.4), jointly with the identity

$$\varepsilon_{A'B'}\, \varepsilon_{C'D'} + \varepsilon_{A'D'}\, \varepsilon_{B'C'} - \varepsilon_{A'C'}\, \varepsilon_{B'D'} = 0, \tag{2.2.25}$$

yields the desired decomposition of the Riemann curvature as

$$\begin{aligned} R_{abcd} &= \psi_{ABCD}\, \varepsilon_{A'B'}\, \varepsilon_{C'D'} + \overline{\psi}_{A'B'C'D'}\, \varepsilon_{AB}\, \varepsilon_{CD} \\ &+ \Phi_{ABC'D'}\, \varepsilon_{A'B'}\, \varepsilon_{CD} + \overline{\Phi}_{A'B'CD}\, \varepsilon_{AB}\, \varepsilon_{C'D'} \\ &+ 2\Lambda\Bigl(\varepsilon_{AC}\, \varepsilon_{BD}\, \varepsilon_{A'C'}\, \varepsilon_{B'D'} - \varepsilon_{AD}\, \varepsilon_{BC}\, \varepsilon_{A'D'}\, \varepsilon_{B'C'}\Bigr). \end{aligned} \tag{2.2.26}$$

With this standard notation, the conformally invariant part of the curvature takes the form $C_{abcd} = {}^{(-)}C_{abcd} + {}^{(+)}C_{abcd}$, where

$${}^{(-)}C_{abcd} \equiv \psi_{ABCD}\, \varepsilon_{A'B'}\, \varepsilon_{C'D'}, \tag{2.2.27}$$

$$
{}^{(+)}C_{abcd} \equiv \overline{\psi}_{A'B'C'D'} \, \varepsilon_{AB} \, \varepsilon_{CD}, \tag{2.2.28}
$$

are the anti-self-dual and self-dual Weyl tensors, respectively.

2.3 Petrov classification

Since the Weyl spinor is totally symmetric, we may use a well known result of two-spinor calculus, according to which, if $\Omega_{AB...L}$ is totally symmetric, then there exist univalent spinors $\alpha_A, \beta_B, ..., \gamma_L$ such that (Stewart 1991)

$$
\Omega_{AB...L} = \alpha_{(A} \, \beta_B ... \gamma_{L)}, \tag{2.3.1}
$$

where $\alpha, ..., \gamma$ are called the *principal spinors* of Ω, and the corresponding real null vectors are called the *principal null directions* of Ω. In the case of the Weyl spinor, such a theorem implies that

$$
\psi_{ABCD} = \alpha_{(A} \, \beta_B \, \gamma_C \, \delta_{D)}. \tag{2.3.2}
$$

The corresponding space-times can be classified as follows (Stewart 1991).

(1) *Type I.* Four distinct principal null directions. Hence the name algebraically general.

(2) *Type II.* Two directions coincide. Hence the name algebraically special.

(3) *Type D.* Two different pairs of repeated principal null directions exist.

(4) *Type III.* Three principal null directions coincide.

(5) *Type N.* All four principal null directions coincide.

Such a classification is the Petrov classification, and it provides a relevant example of the superiority of the two-spinor formalism in four space-time dimensions, since the alternative ways to obtain it are far more complicated.

Within this framework (as well as in chapter three) we need to know that ψ_{ABCD} has two scalar invariants:

$$I \equiv \psi_{ABCD} \, \psi^{ABCD}, \tag{2.3.3}$$

$$J \equiv \psi_{AB}^{\ \ \ CD} \, \psi_{CD}^{\ \ \ EF} \, \psi_{EF}^{\ \ \ AB}. \tag{2.3.4}$$

Type-II space-times are such that $I^3 = 6J^2$, while in type-III space-times $I = J = 0$. Moreover, type-D space-times are characterized by the condition

$$\psi_{PQR(A} \, \psi_{BC}^{\ \ \ PQ} \, \psi^{R}_{\ \ DEF)} = 0, \tag{2.3.5}$$

while in type-N space-times

$$\psi_{(AB}^{\ \ \ \ EF} \, \psi_{CD)EF} = 0. \tag{2.3.6}$$

These results, despite their simplicity, are not well known to many physicists and mathematicians. Hence they have been included also in this paper, to prepare the ground for the more advanced topics of the following chapters.

CHAPTER THREE

CONFORMAL GRAVITY

Since twistor theory enables one to reconstruct the space-time geometry from conformally invariant geometric objects, it is important to know the basic tools for studying conformal gravity within the framework of general relativity. This is achieved by defining and using the Bach and Eastwood–Dighton tensors, here presented in two-spinor form (relying on previous work by Kozameh, Newman and Tod). After defining C-spaces and Einstein spaces, it is shown that a space-time is conformal to an Einstein space if and only if some equations involving the Weyl spinor, its covariant derivatives, and the trace-free part of Ricci are satisfied. Such a result is then extended to complex Einstein spaces. The conformal structure of infinity of Minkowski space-time is eventually introduced.

3.1 C-spaces

Twistor theory may be viewed as the attempt to describe fundamental physics in terms of conformally invariant geometric objects within a holomorphic framework. Space-time points are no longer of primary importance, since they only appear as derived concepts in such a scheme. To understand the following chapters, almost entirely devoted to twistor theory and its applications, it is therefore necessary to study the main results of the theory of conformal gravity. They can be understood by focusing on C-spaces, Einstein spaces, complex space-times and complex Einstein spaces, as we do from now on in this chapter.

To study C-spaces in a self-consistent way, we begin by recalling some basic properties of conformal rescalings. By definition, a *conformal rescaling* of the space-time metric g yields the metric $\widehat{g}$ as

$$\widehat{g}_{ab} \equiv e^{2\omega} \, g_{ab}, \tag{3.1.1}$$

where ω is a smooth scalar. Correspondingly, any tensor field T of type (r, s) is conformally weighted if

$$\widehat{T} \equiv e^{k\omega} \, T \tag{3.1.2}$$

for some integer k. In particular, conformal invariance of T is achieved if $k = 0$.

It is useful to know the transformation rules for covariant derivatives and Riemann curvature under the rescaling (3.1.1). For this purpose, defining

$$F^{m}{}_{ab} \equiv 2\delta^{m}{}_{a} \, \nabla_{b}\omega - g_{ab} \, g^{mn} \, \nabla_{n}\omega, \tag{3.1.3}$$

one finds

$$\widehat{\nabla}_a \, V_b = \nabla_a \, V_b - F^{m}{}_{ab} \, V_m, \tag{3.1.4}$$

where $\widehat{\nabla}_a$ denotes covariant differentiation with respect to the metric $\widehat{g}$. Hence the Weyl tensor $C_{abc}{}^{d}$, the Ricci tensor $R_{ab} \equiv R_{cab}{}^{c}$ and the Ricci scalar transform as

$$\widehat{C}_{abc}{}^{d} = C_{abc}{}^{d}, \tag{3.1.5}$$

$$\widehat{R}_{ab} = R_{ab} + 2\nabla_a \omega_b - 2\omega_a \omega_b + g_{ab}\Bigl(2\omega^c \omega_c + \nabla^c \omega_c\Bigr), \tag{3.1.6}$$

$$\widehat{R} = e^{-2\omega}\biggl[R + 6\Bigl(\nabla^c \omega_c + \omega^c \omega_c\Bigr)\biggr]. \tag{3.1.7}$$

With our notation, $\omega_c \equiv \nabla_c \omega = \omega_{,c}$.

We are here interested in space-times which are conformal to C-spaces. The latter are a class of space-times such that

$$\widehat{\nabla}^f \; \widehat{C}_{abcf} = 0. \tag{3.1.8}$$

By virtue of (3.1.3) and (3.1.4) one can see that the conformal transform of Eq. (3.1.8) is

$$\nabla^f \; C_{abcf} + \omega^f \; C_{abcf} = 0. \tag{3.1.9}$$

This is the necessary and sufficient condition for a space-time to be conformal to a C-space. Its two-spinor form is

$$\nabla^{FA'} \psi_{FBCD} + \omega^{FA'} \psi_{FBCD} = 0. \tag{3.1.10}$$

However, note that only a *real* solution $\omega^{FA'}$ of Eq. (3.1.10) satisfies Eq. (3.1.9). Hence, whenever we use Eq. (3.1.10), we are also imposing a reality condition (Kozameh *et al.* 1985).

On using the invariants defined in (2.3.3) and (2.3.4), one finds the useful identities

$$\psi_{ABCD} \; \psi^{ABCE} = \frac{1}{2} I \; \delta_D^{\;\;E}, \tag{3.1.11}$$

$$\psi_{ABCD} \; \psi^{AB}_{\;\;\;\;PQ} \; \psi^{PQCE} = \frac{1}{2} J \; \delta_D^{\;\;E}. \tag{3.1.12}$$

The idea is now to act with ψ^{ABCD} on the left-hand side of (3.1.10) and then use (3.1.11) when $I \neq 0$. This leads to

$$\omega^{AA'} = -\frac{2}{I} \; \psi^{ABCD} \; \nabla^{FA'} \; \psi_{FBCD}. \tag{3.1.13}$$

By contrast, when $I = 0$ but $J \neq 0$, we multiply twice Eq. (3.1.10) by the Weyl spinor and use (3.1.12). Hence one finds

$$\omega^{AA'} = -\frac{2}{J} \, \psi^{CD}{}_{EF} \, \psi^{EFGA} \, \nabla^{BA'} \, \psi_{BCDG}. \tag{3.1.14}$$

Thus, by virtue of (3.1.13), the reality condition $\omega^{AA'} = \overline{\omega^{AA'}} = \overline{\omega}^{AA'}$ implies

$$\overline{I} \, \psi^{ABCD} \, \nabla^{FA'} \, \psi_{FBCD} - I \, \overline{\psi}^{A'B'C'D'} \, \nabla^{AF'} \, \overline{\psi}_{F'B'C'D'} = 0. \tag{3.1.15}$$

We have thus shown that a space-time is conformally related to a C-space if and only if Eq. (3.1.10) holds for some vector $\omega^{DD'} = K^{DD'}$, and Eq. (3.1.15) holds as well.

3.2 Einstein spaces

By definition, Einstein spaces are such that their Ricci tensor is proportional to the metric: $R_{ab} = \lambda \, g_{ab}$. A space-time is conformal to an Einstein space if and only if a function ω exists (see (3.1.1)) such that (cf. (3.1.6))

$$R_{ab} + 2\nabla_a \omega_b - 2\omega_a \omega_b - \frac{1}{4} T g_{ab} = 0, \tag{3.2.1}$$

where

$$T \equiv R + 2\nabla^c \omega_c - 2\omega^c \omega_c. \tag{3.2.2}$$

Of course, Eq. (3.2.1) leads to restrictions on the metric. These are obtained by deriving the corresponding integrability conditions. For this purpose, on taking the curl of Eq. (3.2.1) and using the Bianchi identities, one finds

$$\nabla^f \, C_{abcf} + \omega^f \, C_{abcf} = 0,$$

which coincides with Eq. (3.1.9). Moreover, acting with ∇^a on Eq. (3.1.9), applying the Leibniz rule, and using again (3.1.9) to re-express $\nabla^f \, C_{abcf}$ as $-\omega^f \, C_{abcf}$, one obtains

$$\left[\nabla^a \nabla^d + \nabla^a \omega^d - \omega^a \omega^d\right] C_{abcd} = 0. \tag{3.2.3}$$

We now re-express $\nabla^a \omega^d$ from (3.2.1) as

$$\nabla^a \omega^d = \omega^a \omega^d + \frac{1}{8} T g^{ad} - \frac{1}{2} R^{ad}. \tag{3.2.4}$$

Hence Eqs. (3.2.3) and (3.2.4) lead to

$$\left[\nabla^a \nabla^d - \frac{1}{2} R^{ad}\right] C_{abcd} = 0. \tag{3.2.5}$$

This calculation only proves that the vanishing of the *Bach tensor*, defined as

$$B_{bc} \equiv \nabla^a \nabla^d C_{abcd} - \frac{1}{2} R^{ad} \, C_{abcd}, \tag{3.2.6}$$

is a *necessary* condition for a space-time to be conformal to an Einstein space (jointly with Eq. (3.1.9)). To prove *sufficiency* of the condition, we first need the following Lemma (Kozameh *et al.* 1985):

Lemma 3.2.1 Let H^{ab} be a trace-free symmetric tensor. Then, providing the scalar invariant J defined in (2.3.4) does not vanish, the only solution of the equations

$$C_{abcd} \, H^{ad} = 0, \tag{3.2.7}$$

$$C^*{}_{abcd} \, H^{ad} = 0, \tag{3.2.8}$$

is $H^{ad} = 0$. As shown in Kozameh *et al.* (1985), such a Lemma is best proved by using two-spinor methods. Hence H_{ab} corresponds to the spinor field

$$H_{AA'BB'} = \phi_{ABA'B'} = \overline{\phi}_{(A'B')(AB)}, \tag{3.2.9}$$

and Eqs. (3.2.7) and (3.2.8) imply that

$$\psi_{ABCD} \, \phi^{CD}{}_{A'B'} = 0. \tag{3.2.10}$$

Note that the extra primed spinor indices $A'B'$ are irrelevant. Hence we can focus on the simpler eigenvalue equation

$$\psi_{ABCD} \, \varphi^{CD} = \lambda \, \varphi_{AB}. \tag{3.2.11}$$

The corresponding characteristic equation for λ is

$$-\lambda^3 + \frac{1}{2} I \lambda + \det(\psi) = 0, \tag{3.2.12}$$

by virtue of (2.3.3). Moreover, the Cayley–Hamilton theorem enables one to re-write Eq. (3.2.12) as

$$\psi_{AB}{}^{PQ} \, \psi_{PQ}{}^{RS} \, \psi_{RS}{}^{CD} = \frac{1}{2} I \, \psi_{AB}{}^{CD} + \det(\psi) \delta_{(A}{}^{C} \, \delta_{B)}{}^{D}, \tag{3.2.13}$$

and contraction of AB with CD yields

$$\det(\psi) = \frac{1}{3} J. \tag{3.2.14}$$

Thus, the only solution of Eq. (3.2.10) is the trivial one unless $J = 0$ (Kozameh *et al.* 1985).

We are now in a position to prove sufficiency of the conditions (cf. Eqs. (3.1.9) and (3.2.5))

$$\nabla^f \, C_{abcf} + K^f \, C_{abcf} = 0, \tag{3.2.15}$$

$$B_{bc} = 0. \tag{3.2.16}$$

Indeed, Eq. (3.2.15) ensures that (3.1.9) is satisfied with $\omega_f = \nabla_f \omega$ for some ω. Hence Eq. (3.2.3) holds. If one now subtracts Eq. (3.2.3) from Eq. (3.2.16) one finds

$$C_{abcd} \left(R^{ad} + 2\nabla^a \omega^d - 2\omega^a \omega^d \right) = 0. \tag{3.2.17}$$

This is indeed Eq. (3.2.7) of Lemma 3.2.1. To obtain Eq. (3.2.8), we act with ∇^a on the dual of Eq. (3.1.9). This leads to

$$\nabla^a \nabla^d {C^*}_{abcd} + \Bigl(\nabla^a \omega^d - \omega^a \omega^d \Bigr) {C^*}_{abcd} = 0. \tag{3.2.18}$$

Following Kozameh *et al.* (1985), the gradient of the contracted Bianchi identity and Ricci identity is then used to derive the additional equation

$$\nabla^a \nabla^d {C^*}_{abcd} - \frac{1}{2} R^{ad} \, {C^*}_{abcd} = 0. \tag{3.2.19}$$

Subtraction of Eq. (3.2.19) from Eq. (3.2.18) now yields

$${C^*}_{abcd} \Bigl(R^{ad} + 2 \nabla^a \omega^d - 2 \omega^a \omega^d \Bigr) = 0, \tag{3.2.20}$$

which is the desired form of Eq. (3.2.8).

We have thus completed the proof that (3.2.15) and (3.2.16) are *necessary* and *sufficient* conditions for a space-time to be conformal to an Einstein space. In two-spinor language, when Einstein's equations are imposed, after a conformal rescaling the equation for the trace-free part of Ricci becomes (see section 2.2)

$$\Phi_{ABA'B'} - \nabla_{BB'} \omega_{AA'} - \nabla_{BA'} \omega_{AB'} + \omega_{AA'} \, \omega_{BB'} + \omega_{AB'} \, \omega_{BA'} = 0. \tag{3.2.21}$$

Similarly to the tensorial analysis performed so far, the spinorial analysis shows that the integrability condition for Eq. (3.2.21) is

$$\nabla^{AA'} \psi_{ABCD} + \omega^{AA'} \, \psi_{ABCD} = 0. \tag{3.2.22}$$

The fundamental theorem of conformal gravity states therefore that a space-time is conformal to an Einstein space if and only if (Kozameh *et al.* 1985)

$$\nabla^{DD'} \psi_{ABCD} + k^{DD'} \, \psi_{ABCD} = 0, \tag{3.2.23}$$

$$\overline{I} \, \psi^{ABCD} \, \nabla^{FA'} \, \psi_{FBCD} - I \, \overline{\psi}^{A'B'C'D'} \, \nabla^{AF'} \, \overline{\psi}_{F'B'C'D'} = 0, \tag{3.2.24}$$

$$B_{AFA'F'} \equiv 2\Bigl(\nabla^{C}{}_{A'}\ \nabla^{D}{}_{F'}\ \psi_{AFCD} + \Phi^{CD}{}_{A'F'}\ \psi_{AFCD}\Bigr) = 0. \qquad (3.2.25)$$

Note that reality of Eq. (3.2.25) for the Bach spinor is ensured by the Bianchi identities.

3.3 Complex space-times

Since this paper is devoted to complex general relativity and its applications, it is necessary to extend the theorem expressed by (3.2.23)–(3.2.25) to complex space-times. For this purpose, we find it appropriate to define and discuss such spaces in more detail in this section. In this respect, we should say that four distinct geometric objects are necessary to study real general relativity and complex general relativity, here defined in four-dimensions (Penrose and Rindler 1986, Esposito 1994).

(1) *Lorentzian* space-time (M, g_L). This is a Hausdorff four-manifold M jointly with a symmetric, non-degenerate bilinear form g_L to each tangent space with signature $(+,-,-,-)$ (or $(-,+,+,+)$). The latter is then called a Lorentzian four-metric g_L.

(2) *Riemannian* four-space (M, g_R), where g_R is a smooth and *positive-definite* section of the bundle of symmetric bilinear two-forms on M. Hence g_R has signature $(+,+,+,+)$.

(3) *Complexified* space-time. This manifold originates from a real-analytic space-time with real-analytic coordinates x^a and real-analytic Lorentzian metric g_L by allowing the coordinates to become complex, and by an holomorphic extension of the metric coefficients into the complex domain. In such manifolds the operation of complex conjugation, taking any point with complexified coordinates z^a into the point with coordinates $\overline{z^a}$, still exists. Note that, however, it is not possible

to define reality of tensors at *complex points*, since the conjugate tensor lies at the complex conjugate point, rather than at the original point.

(4) *Complex* space-time. This is a *four-complex-dimensional* complex-Riemannian manifold, and no four-real-dimensional subspace has been singled out to give it a reality structure (Penrose and Rindler 1986). In complex space-times no complex conjugation exists, since such a map is not invariant under holomorphic coordinate transformations.

Thus, the complex-conjugate spinors $\lambda^{A...M}$ and $\overline{\lambda}^{A'...M'}$ of a Lorentzian space-time are replaced by *independent* spinors $\lambda^{A...M}$ and $\widetilde{\lambda}^{A'...M'}$. This means that unprimed and primed spin-spaces become unrelated to one another. Moreover, the complex scalars ϕ and $\overline{\phi}$ are replaced by the pair of *independent* complex scalars ϕ and $\widetilde{\phi}$. On the other hand, quantities X that are originally *real* yield no new quantities, since the reality condition $X = \overline{X}$ becomes $X = \widetilde{X}$. For example, the covariant derivative operator ∇_a of Lorentzian space-time yields no new operator $\widetilde{\nabla}_a$, since it is originally real. One should instead regard ∇_a as a complex-holomorphic operator. The spinors $\psi_{ABCD}, \Phi_{ABC'D'}$ and the scalar Λ appearing in the Riemann curvature (see (2.2.26)) have as counterparts the spinors $\widetilde{\psi}_{A'B'C'D'}, \widetilde{\Phi}_{ABC'D'}$ and the scalar $\widetilde{\Lambda}$. However, by virtue of the *original* reality conditions in Lorentzian space-time, one has (Penrose and Rindler 1986)

$$\widetilde{\Phi}_{ABC'D'} = \Phi_{ABC'D'}, \tag{3.3.1}$$

$$\widetilde{\Lambda} = \Lambda, \tag{3.3.2}$$

while the Weyl spinors ψ_{ABCD} and $\widetilde{\psi}_{A'B'C'D'}$ remain independent of each other. Hence one Weyl spinor may vanish without the other Weyl spinor having to vanish as well. Correspondingly, a complex space-time such that $\widetilde{\psi}_{A'B'C'D'} = 0$ is called *right conformally flat* or conformally anti-self-dual, whereas if $\psi_{ABCD} = 0$, one deals with a *left conformally flat* or conformally self-dual complex space-time. Moreover, if the remaining part of the Riemann curvature vanishes as well, i.e.

$\Phi_{ABC'D'} = 0$ and $\Lambda = 0$, the word *conformally* should be omitted in the terminology described above (cf. chapter four). Interestingly, in a complex space-time the principal null directions (cf. section 2.3) of the Weyl spinors ψ_{ABCD} and $\widetilde{\psi}_{A'B'C'D'}$ are independent of each other, and one has two independent classification schemes at each point.

3.4 Complex Einstein spaces

In the light of the previous discussion, the fundamental theorem of conformal gravity in complex space-times can be stated as follows (Baston and Mason 1987).

Theorem 3.4.1 A complex space-time is conformal to a complex Einstein space if and only if

$$\nabla^{DD'} \psi_{ABCD} + k^{DD'} \psi_{ABCD} = 0, \tag{3.4.1}$$

$$\widetilde{I} \, \psi^{ABCD} \, \nabla^{FA'} \, \psi_{FBCD} - I \, \widetilde{\psi}^{A'B'C'D'} \, \nabla^{AF'} \, \widetilde{\psi}_{F'B'C'D'} = 0, \tag{3.4.2}$$

$$B_{AFA'F'} \equiv 2\Bigl(\nabla^{C}{}_{A'} \, \nabla^{D}{}_{F'} \, \psi_{AFCD} + \Phi^{CD}{}_{A'F'} \, \psi_{AFCD} \Bigr) = 0, \tag{3.4.3}$$

where I is the complex scalar invariant defined in (2.3.3), whereas $\widetilde{I}$ is the independent invariant defined as

$$\widetilde{I} \equiv \widetilde{\psi}_{A'B'C'D'} \, \widetilde{\psi}^{A'B'C'D'}. \tag{3.4.4}$$

The left-hand side of Eq. (3.4.2) is called the *Eastwood–Dighton spinor*, and the left-hand side of Eq. (3.4.3) is the *Bach spinor*.

3.5 Conformal infinity

To complete our introduction to conformal gravity, we find it helpful for the reader to outline the construction of conformal infinity for Minkowski space-time (see also an application in section 9.5). Starting from polar local coordinates in Minkowski, we first introduce (in $c = 1$ units) the retarded coordinate $w \equiv t - r$ and the advanced coordinate $v \equiv t + r$. To eliminate the resulting cross term in the local form of the metric, new coordinates p and q are defined implicitly as (Esposito 1994)

$$\tan p \equiv v, \ \tan q \equiv w, \ p - q \geq 0. \tag{3.5.1}$$

Hence one finds that a conformal-rescaling factor $\omega \equiv (\cos p)(\cos q)$ exists such that, locally, the metric of Minkowski space-time can be written as $\omega^{-2}\widetilde{g}$, where

$$\widetilde{g} \equiv -dt' \otimes dt' + \left[dr' \otimes dr' + \frac{1}{4}(\sin(2r'))^2 \, \Omega_2 \right], \tag{3.5.2}$$

where $t' \equiv \frac{(p+q)}{2}, r' \equiv \frac{(p-q)}{2}$, and Ω_2 is the metric on a unit two-sphere. Although (3.5.2) is locally identical to the metric of the Einstein static universe, it is necessary to go beyond a local analysis. This may be achieved by *analytic extension* to the whole of the Einstein static universe. The original Minkowski space-time is then found to be conformal to the following region of the Einstein static universe:

$$(t' + r') \in]-\pi, \pi[, \ (t' - r') \in]-\pi, \pi[, \ r' \geq 0. \tag{3.5.3}$$

By definition, the *boundary* of the region in (3.5.3) represents *the conformal structure of infinity* of Minkowski space-time. It consists of two null surfaces and three points, i.e. (Esposito 1994)

(i) The null surface $\text{SCRI}^- \equiv \left\{ t' - r' = q = -\frac{\pi}{2} \right\}$, i.e. the future light cone of the point $r' = 0, t' = -\frac{\pi}{2}$.

(ii) The null surface $\text{SCRI}^+ \equiv \left\{ t' + r' = p = \frac{\pi}{2} \right\}$, i.e. the past light cone of the point $r' = 0, t' = \frac{\pi}{2}$.

(iii) Past timelike infinity, i.e. the point

$$\iota^- \equiv \left\{ r' = 0, t' = -\frac{\pi}{2} \right\} \Rightarrow p = q = -\frac{\pi}{2}.$$

(iv) Future timelike infinity, defined as

$$\iota^+ \equiv \left\{ r' = 0, t' = \frac{\pi}{2} \right\} \Rightarrow p = q = \frac{\pi}{2}.$$

(v) Spacelike infinity, i.e. the point

$$\iota^0 \equiv \left\{ r' = \frac{\pi}{2}, t' = 0 \right\} \Rightarrow p = -q = \frac{\pi}{2}.$$

The extension of the SCRI formalism to curved space-times is an open research problem, but we limit ourselves to the previous definitions in this section.

CHAPTER FOUR

TWISTOR SPACES

In twistor theory, α-planes are the building blocks of classical field theory in complexified compactified Minkowski space-time. The α-planes are totally null two-surfaces S in that, if p is any point on S, and if v and w are any two null tangent vectors at $p \in S$, the complexified Minkowski metric η satisfies the identity $\eta(v,w) = v_a w^a = 0$. By definition, their null tangent vectors have the two-component spinor form $\lambda^A \pi^{A'}$, where λ^A is varying and $\pi^{A'}$ is fixed. Therefore, the induced metric vanishes identically since $\eta(v,w) = \left(\lambda^A \pi^{A'}\right)\left(\mu_A \pi_{A'}\right) = 0 = \eta(v,v) = \left(\lambda^A \pi^{A'}\right)\left(\lambda_A \pi_{A'}\right)$. One thus obtains a conformally invariant characterization of flat space-times. This definition can be generalized to complex or real Riemannian space-times with non-vanishing curvature, provided the Weyl curvature is anti-self-dual. One then finds that the curved metric g is such that $g(v,w) = 0$ on S, and the spinor field $\pi_{A'}$ is covariantly constant on S. The corresponding holomorphic two-surfaces are called α-surfaces, and they form a three-complex-dimensional family. Twistor space is the space of all α-surfaces, and depends only on the conformal structure of complex space-time.

Projective twistor space PT is isomorphic to complex projective space CP^3. The correspondence between flat space-time and twistor space shows that complex α-planes correspond to points in PT, and real null geodesics to points in PN, i.e. the space of null twistors. Moreover, a complex space-time point corresponds to a sphere in PT, and a real space-time point to a sphere in PN. Remarkably, the points x and y are null-separated if and only if the corresponding spheres in PT intersect. This is the twistor description of the light-cone structure of Minkowski space-time.

A conformally invariant isomorphism exists between the complex vector space of holomorphic solutions of $\Box \phi = 0$ on the forward tube of flat space-time, and the complex vector space of arbitrary complex-analytic functions of three variables, not subject to any differential equation. Moreover, when curvature is non-vanishing, there is a one-to-one correspondence between complex space-times with anti-self-dual Weyl curvature and scalar curvature $R = 24\Lambda$, and sufficiently small deformations of flat projective twistor space PT which preserve a one-form τ homogeneous of degree 2 and a three-form ρ homogeneous of degree 4, with $\tau \wedge d\tau = 2\Lambda \rho$. Thus, to solve the anti-self-dual Einstein equations, one has to study a geometric problem, i.e. finding the holomorphic curves in deformed projective twistor space.

4.1 α-planes in Minkowski space-time

The α-planes provide a geometric definition of twistors in Minkowski space-time. For this purpose, we first complexify flat space-time, so that real coordinates $\left(x^0, x^1, x^2, x^3\right)$ are replaced by complex coordinates $\left(z^0, z^1, z^2, z^3\right)$, and we obtain a four-dimensional complex vector space equipped with a non-degenerate complex-bilinear form (Ward and Wells 1990)

$$(z, w) \equiv z^0 w^0 - z^1 w^1 - z^2 w^2 - z^3 w^3. \tag{4.1.1}$$

The resulting matrix $z^{AA'}$, which, by construction, corresponds to the position vector $z^a = \left(z^0, z^1, z^2, z^3\right)$, is no longer Hermitian as in the real case. Moreover, we compactify such a space by identifying future null infinity with past null infinity (Penrose 1974, Penrose and Rindler 1986, Esposito 1994). The resulting manifold is here denoted by $CM^{\#}$, following Penrose and Rindler (1986).

In $CM^{\#}$ with metric η, we consider two-surfaces S whose tangent vectors have the two-component spinor form

$$v^a = \lambda^A \pi^{A'}, \tag{4.1.2}$$

where λ^A is varying and $\pi^{A'}$ is fixed. This implies that these tangent vectors are null, since $\eta(v, v) = v_a v^a = \left(\lambda^A \lambda_A\right)\left(\pi^{A'} \pi_{A'}\right) = 0$. Moreover, the induced metric on S vanishes identically since any two null tangent vectors $v^a = \lambda^A \pi^{A'}$ and $w^a = \mu^A \pi^{A'}$ at $p \in S$ are orthogonal:

$$\eta(v, w) = \left(\lambda^A \mu_A\right)\left(\pi^{A'} \pi_{A'}\right) = 0, \tag{4.1.3}$$

where we have used the property $\pi^{A'} \pi_{A'} = \varepsilon^{A'B'} \pi_{A'} \pi_{B'} = 0$. By virtue of (4.1.3), the resulting α-plane is said to be totally null. A twistor is then an α-plane with constant $\pi_{A'}$ associated to it. Note that two disjoint families of totally null two-surfaces exist in $CM^{\#}$, since one might choose null tangent vectors of the form

$$u^a = \nu^A \pi^{A'}, \tag{4.1.4}$$

where ν^A is fixed and $\pi^{A'}$ is varying. The resulting two-surfaces are called β-planes (Penrose 1986).

Theoretical physicists are sometimes more familiar with a definition involving the vector space of solutions of the differential equation

$$\mathcal{D}_{A'}^{(A} \omega^{B)} = 0, \tag{4.1.5}$$

where $\mathcal{D}$ is the flat connection, and $\mathcal{D}_{AA'}$ the corresponding spinor covariant derivative. The general solution of Eq. (4.1.5) in $CM^{\#}$ takes the form (Penrose and Rindler 1986, Esposito 1994)

$$\omega^A = \Bigr(\omega^o\Bigr)^A - i\, x^{AA'} \pi^o_{A'}, \tag{4.1.6}$$

$$\pi_{A'} = \pi^o_{A'}, \tag{4.1.7}$$

where ω^o_A and $\pi^o_{A'}$ are arbitrary constant spinors, and $x^{AA'}$ is the spinor version of the position vector with respect to some origin. A twistor is then *represented* by the pair of spinor fields $\Bigr(\omega^A, \pi_{A'}\Bigr) \Leftrightarrow Z^\alpha$ (Penrose 1975). The twistor equation (4.1.5) is conformally invariant. This is proved bearing in mind the spinor form of the flat four-metric

$$\eta_{ab} = \varepsilon_{AB}\, \varepsilon_{A'B'}, \tag{4.1.8}$$

and making the conformal rescaling

$$\widehat{\eta}_{ab} = \Omega^2 \eta_{ab}, \tag{4.1.9}$$

which implies

$$\widehat{\varepsilon}_{AB} = \Omega \varepsilon_{AB},\ \widehat{\varepsilon}_{A'B'} = \Omega \varepsilon_{A'B'},\ \widehat{\varepsilon}^{AB} = \Omega^{-1} \varepsilon^{AB},\ \widehat{\varepsilon}^{A'B'} = \Omega^{-1} \varepsilon^{A'B'}. \tag{4.1.10}$$

Thus, defining $T_a \equiv \mathcal{D}_a \Bigr(\log \Omega \Bigr)$ and choosing $\widehat{\omega}^B = \omega^B$, one finds (Penrose and Rindler 1986, Esposito 1994)

$$\widehat{\mathcal{D}}_{AA'} \widehat{\omega}^B = \mathcal{D}_{AA'} \omega^B + \varepsilon_A^{\ B} \, T_{CA'} \omega^C, \tag{4.1.11}$$

which implies

$$\widehat{\mathcal{D}}_{A'}^{(A} \widehat{\omega}^{B)} = \Omega^{-1} \mathcal{D}_{A'}^{(A} \omega^{B)}. \tag{4.1.12}$$

Note that the solutions of Eq. (4.1.5) are completely determined by the four complex components at O of ω^A and $\pi_{A'}$ in a spin-frame at O. They are a four-dimensional vector space over the complex numbers, called twistor space (Penrose and Rindler 1986, Esposito 1994).

Requiring that ν_A be constant over the β-planes implies that $\nu^A \pi^{A'} \mathcal{D}_{AA'} \nu_B = 0$, for each $\pi^{A'}$, i.e. $\nu^A \mathcal{D}_{AA'} \nu_B = 0$. Moreover, a scalar product can be defined between the ω^A field and the ν_A-scaled β-plane: $\omega^A \nu_A$. Its constancy over the β-plane implies that (Penrose 1986)

$$\nu^A \pi^{A'} \mathcal{D}_{AA'} \Bigr(\omega^B \nu_B \Bigr) = 0, \tag{4.1.13}$$

for each $\pi^{A'}$, which leads to

$$\nu_A \nu_B \Bigr(\mathcal{D}_{A'}^{(A} \omega^{B)} \Bigr) = 0, \tag{4.1.14}$$

for each β-plane and hence for each ν_A. Thus, Eq. (4.1.14) becomes the twistor equation (4.1.5). In other words, it is the twistor concept associated with a β-plane which is dual to that associated with a solution of the twistor equation (Penrose 1986).

Flat projective twistor space PT can be thought of as three-dimensional complex projective space CP^3 (cf. example E2 in section 1.2). This means that we take the space C^4 of complex numbers $\Bigr(z^0, z^1, z^2, z^3 \Bigr)$ and factor out by the proportionality relation $\Bigr(\lambda z^0, ..., \lambda z^3 \Bigr) \sim \Bigr(z^0, ..., z^3 \Bigr)$, with $\lambda \in C - \{0\}$. The homogeneous coordinates $\Bigr(z^0, ..., z^3 \Bigr)$ are, in the case of $PT \cong CP^3$, as follows:

$\left(\omega^0, \omega^1, \pi_{0'}, \pi_{1'}\right) \equiv \left(\omega^A, \pi_{A'}\right)$. The α-planes defined in this section can be obtained from the equation (cf. (4.1.6))

$$\omega^A = i \, x^{AA'} \pi_{A'}, \tag{4.1.15}$$

where $\left(\omega^A, \pi_{A'}\right)$ is regarded as fixed, with $\pi_{A'} \neq 0$. This means that Eq. (4.1.15), considered as an equation for $x^{AA'}$, has as its solution a complex two-plane in $CM^{\#}$, whose tangent vectors take the form in Eq. (4.1.2), i.e. we have found an α-plane. The α-planes are self-dual in that, if v and u are any two null tangent vectors to an α-plane, then $F \equiv v \otimes u - u \otimes v$ is a self-dual bivector since

$$F^{AA'BB'} = \varepsilon^{AB} \phi^{(A'B')}, \tag{4.1.16}$$

where $\phi^{(A'B')} = \sigma \pi^{A'} \pi^{B'}$, with $\sigma \in C - \{0\}$ (Ward 1981b). Note also that α-planes remain unchanged if we replace $\left(\omega^A, \pi_{A'}\right)$ by $\left(\lambda \omega^A, \lambda \pi_{A'}\right)$ with $\lambda \in C - \{0\}$, and that *all* α-planes arise as solutions of Eq. (4.1.15). If real solutions of such equation exist, this implies that $x^{AA'} = \overline{x}^{AA'}$. This leads to

$$\omega^A \overline{\pi}_A + \overline{\omega}^{A'} \pi_{A'} = i \, x^{AA'} \left(\pi_{A'} \overline{\pi}_A - \pi_{A'} \overline{\pi}_A\right) = 0, \tag{4.1.17}$$

where overbars denote complex conjugation in two-spinor language, defined according to the rules described in section 2.1. If (4.1.17) holds and $\pi_{A'} \neq 0$, the solution space of Eq. (4.1.15) in real Minkowski space-time is a null geodesic, and *all* null geodesics arise in this way (Ward 1981b). Moreover, if $\pi_{A'}$ vanishes, the point $\left(\omega^A, \pi_{A'}\right) = \left(\omega^A, 0\right)$ can be regarded as an α-plane at infinity in compactified Minkowski space-time. Interestingly, Eq. (4.1.15) is the two-spinor form of the equation expressing the incidence property of a point (t, x, y, z) in Minkowski space-time with the twistor Z^α, i.e. (Penrose 1981)

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}. \tag{4.1.18}$$

The left-hand side of Eq. (4.1.17) may be then re-interpreted as the twistor pseudo-norm (Penrose 1981)

$$Z^\alpha \overline{Z}_\alpha = Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1} = \omega^A \overline{\pi}_A + \pi_{A'}\overline{\omega}^{A'}, \tag{4.1.19}$$

by virtue of the property $\left(\overline{Z}_0, \overline{Z}_1, \overline{Z}_2, \overline{Z}_3\right) = \left(\overline{Z^2}, \overline{Z^3}, \overline{Z^0}, \overline{Z^1}\right)$. Such a pseudo-norm makes it possible to define the *top half* PT^+ of PT by the condition $Z^\alpha \overline{Z}_\alpha > 0$, and the *bottom half* PT^- of PT by the condition $Z^\alpha \overline{Z}_\alpha < 0$.

So far, we have seen that an α-plane corresponds to a point in PT, and null geodesics to points in PN, the space of null twistors. However, we may also interpret (4.1.15) as an equation where $x^{AA'}$ is fixed, and solve for $\left(\omega^A, \pi_{A'}\right)$. Within this framework, $\pi_{A'}$ remains arbitrary, and ω^A is thus given by $ix^{AA'}\pi_{A'}$. This yields a complex two-plane, and factorization by the proportionality relation $\left(\lambda\omega^A, \lambda\pi_{A'}\right) \sim \left(\omega^A, \pi_{A'}\right)$ leads to a complex projective one-space CP^1, with two-sphere topology. Thus, the fixed space-time point x determines a Riemann sphere $L_x \cong CP^1$ in PT. In particular, if x is real, then L_x lies entirely within PN, given by those twistors whose homogeneous coordinates satisfy Eq. (4.1.17). To sum up, a complex space-time point corresponds to a sphere in PT, whereas a real space-time point corresponds to a sphere in PN (Penrose 1981, Ward 1981b).

In Minkowski space-time, two points p and q are null-separated if and only if there is a null geodesic connecting them. In projective twistor space PT, this implies that the corresponding lines L_p and L_q intersect, since the intersection point represents the connecting null geodesic. To conclude this section it may be now instructive, following Huggett and Tod (1985), to study the relation between null twistors and null geodesics. Indeed, given the null twistors X^α, Y^α defined by

$$X^\alpha \equiv \left(i\, x_0^{AC'}\, X_{C'}, X_{A'}\right), \tag{4.1.20}$$

$$Y^\alpha \equiv \left(i\, x_1^{AC'}\, Y_{C'}, Y_{A'}\right), \tag{4.1.21}$$

the corresponding null geodesics are

$$\gamma_X : x^{AA'} \equiv x_0^{AA'} + \lambda \, \overline{X}^A \, X^{A'}, \tag{4.1.22}$$

$$\gamma_Y : x^{AA'} \equiv x_1^{AA'} + \mu \, \overline{Y}^A \, Y^{A'}. \tag{4.1.23}$$

If these intersect at some point x_2, one finds

$$x_2^{AA'} = x_0^{AA'} + \lambda \, \overline{X}^A \, X^{A'} = x_1^{AA'} + \mu \, \overline{Y}^A \, Y^{A'}, \tag{4.1.24}$$

where $\lambda, \mu \in R$. Hence

$$x_2^{AA'} \, \overline{Y}_A \, X_{A'} = x_0^{AA'} \, \overline{Y}_A \, X_{A'} = x_1^{AA'} \, \overline{Y}_A \, X_{A'}, \tag{4.1.25}$$

by virtue of the identities $X^{A'} X_{A'} = \overline{Y}^A \, \overline{Y}_A = 0$. Equation (4.1.25) leads to

$$X^\alpha \overline{Y}_\alpha = i \Bigl(x_0^{AA'} \, \overline{Y}_A \, X_{A'} - x_1^{AA'} \, \overline{Y}_A \, X_{A'} \Bigr) = 0. \tag{4.1.26}$$

Suppose instead we are given Eq. (4.1.26). This implies that some real λ and μ exist such that

$$x_0^{AA'} - x_1^{AA'} = -\lambda \, \overline{X}^A \, X^{A'} + \mu \, \overline{Y}^A \, Y^{A'}, \tag{4.1.27}$$

where signs on the right-hand side of (4.1.27) have been suggested by (4.1.24). Note that (4.1.27) only holds if $X_{A'} Y^{A'} \neq 0$, i.e. if γ_X and γ_Y are not parallel. However, the whole argument can be generalized to this case as well (our problem 4.2, Huggett and Tod 1985), and one finds that in all cases the null geodesics γ_X and γ_Y intersect if and only if $X^\alpha \, \overline{Y}_\alpha$ vanishes.

4.2 α-surfaces and twistor geometry

The α-planes defined in section 4.1 can be generalized to a suitable class of curved complex space-times. By a complex space-time (M, g) we mean a four-dimensional

Hausdorff manifold M with holomorphic metric g. Thus, with respect to a holomorphic coordinate basis x^a, g is a 4×4 matrix of holomorphic functions of x^a, and its determinant is nowhere-vanishing (Ward 1980b, Ward and Wells 1990). Remarkably, g determines a unique holomorphic connection ∇, and a holomorphic curvature tensor $R^a_{\ bcd}$. Moreover, the Ricci tensor R_{ab} becomes complex-valued, and the Weyl tensor $C^a_{\ bcd}$ may be split into *independent* holomorphic tensors, i.e. its self-dual and anti-self-dual parts, respectively. With our two-spinor notation, one has (see (2.1.36))

$$C_{abcd} = \psi_{ABCD} \, \varepsilon_{A'B'} \, \varepsilon_{C'D'} + \widetilde{\psi}_{A'B'C'D'} \, \varepsilon_{AB} \, \varepsilon_{CD}, \tag{4.2.1}$$

where $\psi_{ABCD} = \psi_{(ABCD)}, \widetilde{\psi}_{A'B'C'D'} = \widetilde{\psi}_{(A'B'C'D')}$. The spinors ψ and $\widetilde{\psi}$ are the anti-self-dual and self-dual Weyl spinors, respectively. Following Penrose (1976a,b), Ward and Wells (1990), complex vacuum space-times such that

$$\widetilde{\psi}_{A'B'C'D'} = 0, \; R_{ab} = 0, \tag{4.2.2}$$

are called *right-flat* or *anti-self-dual*, whereas complex vacuum space-times such that

$$\psi_{ABCD} = 0, \; R_{ab} = 0, \tag{4.2.3}$$

are called *left-flat* or *self-dual*. Note that this definition only makes sense if space-time is complex or real Riemannian, since in this case no complex conjugation relates primed to unprimed spinors (i.e. the corresponding spin-spaces are no longer anti-isomorphic). Hence, for example, the self-dual Weyl spinor $\widetilde{\psi}_{A'B'C'D'}$ may vanish without its anti-self-dual counterpart ψ_{ABCD} having to vanish as well, as in Eq. (4.2.2), or the converse may hold, as in Eq. (4.2.3) (see section 1.1 and problem 2.3).

By definition, α-surfaces are complex two-surfaces S in a complex space-time (M, g) whose tangent vectors v have the two-spinor form (4.1.2), where λ^A is varying, and $\pi^{A'}$ is a fixed primed spinor field on S. From this definition, the following properties can be derived (cf. section 4.1).

(i) tangent vectors to α-surfaces are null;

(ii) any two null tangent vectors v and u to an α-surface are orthogonal to one another;

(iii) the holomorphic metric g vanishes on S in that $g(v,u) = g(v,v) = 0, \forall v, u$ (cf. (4.1.3)), so that α-surfaces are totally null;

(iv) α-surfaces are self-dual, in that $F \equiv v \otimes u - u \otimes v$ takes the two-spinor form (4.1.16);

(v) α-surfaces exist in (M, g) if and only if the self-dual Weyl spinor vanishes, so that (M, g) is anti-self-dual.

Note that properties (i)–(iv), here written in a redundant form for pedagogical reasons, are the same as in the flat-space-time case, provided we replace the flat metric η with the curved metric g. Condition (v), however, is a peculiarity of curved space-times. The reader may find a detailed proof of the necessity of this condition as a particular case of the calculations appearing in chapter six, where we study a holomorphic metric-compatible connection ∇ with non-vanishing torsion. To avoid repeating ourselves, we focus instead on the sufficiency of the condition, following Ward and Wells (1990).

We want to prove that, if (M, g) is anti-self-dual, it admits a three-complex-parameter family of self-dual α-surfaces. Indeed, given any point $p \in M$ and a spinor $\mu_{A'}$ at p, one can find a spinor field $\pi_{A'}$ on M, satisfying the equation (cf. Eq. (6.2.10))

$$\pi^{A'} \Big(\nabla_{AA'} \pi_{B'} \Big) = \xi_A \pi_{B'}, \qquad (4.2.4)$$

and such that

$$\pi_{A'}(p) = \mu_{A'}(p). \qquad (4.2.5)$$

Hence $\pi_{A'}$ defines a holomorphic two-dimensional distribution, spanned by the vector fields of the form $\lambda^A \pi^{A'}$, which is integrable by virtue of (4.2.4). Thus, in particular, there exists a self-dual α-surface through p, with tangent vectors of the form $\lambda^A \mu^{A'}$ at p. Since p is arbitrary, this argument may be repeated $\forall p \in M$.

The space $\mathcal{P}$ of all self-dual α-surfaces in (M,g) is three-complex-dimensional, and is called twistor space of (M,g).

4.3 Geometric theory of partial differential equations

One of the main results of twistor theory has been a deeper understanding of the solutions of partial differential equations of classical field theory. Remarkably, a problem in analysis becomes a purely geometric problem (Ward 1981b, Ward and Wells 1990). For example, in Bateman (1904) it was shown that the general real-analytic solution of the wave equation $\Box\phi = 0$ in Minkowski space-time is

$$\phi(x,y,z,t) = \int_{-\pi}^{\pi} F(x\cos\theta + y\sin\theta + iz, y + iz\sin\theta + t\cos\theta, \theta)\, d\theta, \qquad (4.3.1)$$

where F is an arbitrary function of three variables, complex-analytic in the first two. Indeed, twistor theory tells us that F is a function on PT. More precisely, let $f\left(\omega^A, \pi_{A'}\right)$ be a complex-analytic function, homogeneous of degree -2, i.e. such that

$$f\left(\lambda\omega^A, \lambda\pi_{A'}\right) = \lambda^{-2} f\left(\omega^A, \pi_{A'}\right), \qquad (4.3.2)$$

and possibly having singularities (Ward 1981b). We now define a field $\phi(x^a)$ by

$$\phi(x^a) \equiv \frac{1}{2\pi i} \oint f\left(i\, x^{AA'}\pi_{A'}, \pi_{B'}\right) \pi_{C'}\, d\pi^{C'}, \qquad (4.3.3)$$

where the integral is taken over any closed one-dimensional contour that avoids the singularities of f. Such a field satisfies the wave equation, and every solution of $\Box\phi = 0$ can be obtained in this way. The function f has been taken to have homogeneity -2 since the corresponding one-form $f\pi_{C'}\, d\pi^{C'}$ has homogeneity zero and hence is a one-form on projective twistor space PT, or on some subregion of PT, since it may have singularities. The homogeneity is related to the property of f of being a free function of three variables. Since f is not defined on the whole of PT, and ϕ does not determine f uniquely, because we can replace f by $f+\widetilde{f}$, where $\widetilde{f}$ is any function such that

$$\oint \widetilde{f} \pi_{C'} \, d\pi^{C'} = 0, \tag{4.3.4}$$

we conclude that f is an element of the sheaf-cohomology group $H^1\Bigr(PT^+, O(-2)\Bigr)$, i.e. the complex vector space of arbitrary complex-analytic functions of three variables, not subject to any differential equations (Penrose 1980, Ward 1981b, Ward and Wells 1990). Remarkably, a conformally invariant isomorphism exists between the complex vector space of holomorphic solutions of $\Box \phi = 0$ on the forward tube CM^+ (i.e. the domain of definition of positive-frequency fields), and the sheaf-cohomology group $H^1\Bigr(PT^+, O(-2)\Bigr)$.

It is now instructive to summarize some basic ideas of sheaf-cohomology theory and its use in twistor theory, following Penrose (1980). For this purpose, let us begin by recalling how Cech cohomology is obtained. We consider a Hausdorff paracompact topological space X, covered with a locally finite system of open sets U_i. With respect to this covering, we define a *cochain* with coefficients in an additive Abelian group G (e.g. Z, R or C) in terms of elements $f_i, f_{ij}, f_{ijk} ... \in G$. These elements are assigned to the open sets U_i of the covering, and to their non-empty intersections, as follows: f_i to U_i, f_{ij} to $U_i \cap U_j$, f_{ijk} to $U_i \cap U_j \cap U_k$ and so on. The elements assigned to non-empty intersections are completely antisymmetric, so that $f_{i...p} = f_{[i...p]}$. One is thus led to define

$$\text{zero} - \text{cochain} \; \alpha \equiv \Bigr(f_1, f_2, f_3, ...\Bigr), \tag{4.3.5}$$

$$\text{one} - \text{cochain} \; \beta \equiv \Bigr(f_{12}, f_{23}, f_{13}, ...\Bigr), \tag{4.3.6}$$

$$\text{two} - \text{cochain} \; \gamma \equiv \Bigr(f_{123}, f_{124}, ...\Bigr), \tag{4.3.7}$$

and the *coboundary operator* δ:

$$\delta\alpha \equiv \Bigr(f_2 - f_1, f_3 - f_2, f_3 - f_1, ...\Bigr) \equiv \Bigr(f_{12}, f_{23}, f_{13}, ...\Bigr), \tag{4.3.8}$$

$$\delta\beta \equiv \Bigl(f_{12} - f_{13} + f_{23}, f_{12} - f_{14} + f_{24}, ...\Bigr) \equiv \Bigl(f_{123}, f_{124}, ...\Bigr). \tag{4.3.9}$$

By virtue of (4.3.8) and (4.3.9) one finds $\delta^2\alpha = \delta^2\beta = ... = 0$. *Cocycles* γ are cochains such that $\delta\gamma = 0$. *Coboundaries* are a particular set of cocycles, i.e. such that $\gamma = \delta\beta$ for some cochain β. Of course, *all* coboundaries are cocycles, whereas the converse does not hold. This enables one to define the p$^{\rm th}$ cohomology group as the quotient space

$$H^p_{\left\{U_i\right\}}(X,G) \equiv G^p_{CC}/G^p_{CB}, \tag{4.3.10}$$

where G^p_{CC} is the additive group of p-cocycles, and G^p_{CB} is the additive group of p-coboundaries. To avoid having a definition which depends on the covering $\Bigl\{U_i\Bigr\}$, one should then take finer and finer coverings of X and settle on a *sufficiently fine* covering $\Bigl\{U_i\Bigr\}^*$. Following Penrose (1980), by this we mean that all the $H^p\Bigl(U_i \cap ... \cap U_k, G\Bigr)$ vanish $\forall p > 0$. One then defines

$$H^p_{\left\{U_i\right\}^*}(X,G) \equiv H^p(X,G). \tag{4.3.11}$$

We always assume such a covering exists, is countable and locally finite. Note that, rather than thinking of f_i as an element of G assigned to U_i, of f_{ij} as assigned to U_{ij} and so on, we can think of f_i as a *function* defined on U_i and taking a constant value $\in G$. Similarly, we can think of f_{ij} as a G-valued constant function defined on $U_i \cap U_j$, and this implies it is not strictly necessary to assume that $U_i \cap U_j$ is non-empty.

The generalization to sheaf cohomology is obtained if we do not require the functions $f_i, f_{ij}, f_{ijk}...$ to be constant (there are also cases when the additive group G is allowed to vary from point to point in X). The assumption of main interest is the holomorphic nature of the f's. A sheaf is so defined that the Cech cohomology previously defined works as well as before (Penrose 1980). In other words, a sheaf S defines an additive group G_u for each open set $U \subset X$. Relevant examples are as follows.

(i) The sheaf O of germs of holomorphic functions on a complex manifold X is obtained if G_u is taken to be the additive group of all holomorphic functions on U.

(ii) Twisted holomorphic functions, i.e. functions whose values are not complex numbers, but are taken in some complex line bundle over X.

(iii) A particular class of twisted functions is obtained if X is projective twistor space PT (or PT^+, or PT^-), and the functions studied are holomorphic and homogeneous of some degree n in the twistor variable, i.e.

$$f\Big(\lambda\omega^A, \lambda\pi_{A'}\Big) = \lambda^n f\Big(\omega^A, \pi_{A'}\Big). \tag{4.3.12}$$

If G_u consists of all such twisted functions on $U \subset X$, the resulting sheaf, denoted by $O(n)$, is the sheaf of germs of holomorphic functions twisted by n on X.

(iv) We can also consider vector-bundle-valued functions, where the vector bundle B is over X, and G_u consists of the cross-sections of the portion of B lying above U.

Defining cochains and coboundary operator as before, with $f_i \in G_{U_i}$ and so on, we obtain the p$^{\text{th}}$ cohomology group of X, with coefficients in the sheaf S, as the quotient space

$$H^p(X, S) \equiv G^p(S)/G^p_{CB}(S), \tag{4.3.13}$$

where $G^p(S)$ is the group of p-cochains with coefficients in S, and $G^p_{CB}(S)$ is the group of p-coboundaries with coefficients in S. Again, we take finer and finer coverings $\Big\{U_i\Big\}$ of X, and we settle on a *sufficiently fine* covering. To understand this concept, we recall the following definitions (Penrose 1980).

Definition 4.3.1 A *coherent analytic* sheaf is locally defined by n holomorphic functions factored out by a set of s holomorphic relations.

Definition 4.3.2 A Stein manifold is a holomorphically convex open subset of C^n.

Thus, we can say that, provided S is a coherent analytic sheaf, *sufficiently fine* means that each of $U_i, U_i \cap U_j, U_i \cap U_j \cap U_k ...$ is a Stein manifold. If X is Stein and S is coherent analytic, then $H^p(X,S) = 0, \forall p > 0$.

We can now consider again the remarks following Eq. (4.3.4), i.e. the interpretation of twistor functions as elements of $H^1\Bigr(PT^+, O(-2)\Bigr)$. Let X be a part of PT, e.g. the neighbourhood of a line in PT, or the top half PT^+, or the closure $\overline{PT^+}$ of the top half. We assume X can be covered with two open sets U_1, U_2 such that every projective line L in X meets $U_1 \cap U_2$ in an annular region. For us, $U_1 \cap U_2$ corresponds to the domain of definition of a twistor function $f(Z^\alpha)$, homogeneous of degree n in the twistor Z^α (see (4.3.12)). Then $f \equiv f_{12} \equiv f_2 - f_1$ is a twisted function on $U_1 \cap U_2$, and defines a one-cochain ϵ, with coefficients in $O(n)$, for X. By construction $\delta\epsilon = 0$, hence ϵ is a cocycle. For this covering, the one-coboundaries are functions of the form $l_2 - l_1$, where l_2 is holomorphic on U_2 and l_1 on U_1. The equivalence between twistor functions is just the cohomological equivalence between one-cochains ϵ, ϵ' that their difference should be a coboundary: $\epsilon' - \epsilon = \delta\alpha$, with $\alpha = \Bigr(l_1, l_2\Bigr)$. This is why we view twistor functions as defining elements of $H^1\Bigr(X, O(n)\Bigr)$. Indeed, if we try to get finer coverings, we realize it is often impossible to make U_1 and U_2 into Stein manifolds. However, if $X = \overline{PT^+}$, the covering $\Bigr\{U_1, U_2\Bigr\}$ by two sets is sufficient for any analytic, positive-frequency field (Penrose 1980).

The most striking application of twistor theory to partial differential equations is perhaps the geometric characterization of anti-self-dual space-times with a cosmological constant. For these space-times, the Weyl tensor takes the form

$$C_{abcd}^{(A.S.D.)} = \psi_{ABCD} \, e_{A'B'} \, e_{C'D'}, \tag{4.3.14}$$

and the Ricci tensor reads

$$R_{ab} = -2\Phi_{ab} + 6\Lambda g_{ab}. \tag{4.3.15}$$

With our notation, e_{AB} and $e_{A'B'}$ are the curved-space version of the ε-symbols (denoted again by ε_{AB} and $\varepsilon_{A'B'}$ in Eqs. (2.1.36) and (4.2.1)), Φ_{ab} is the trace-free part of Ricci, 24Λ is the trace $R = R^a{}_a$ of Ricci (Ward 1980b). The local structure in projective twistor space which gives information about the metric is a pair of differential forms: a one-form τ homogeneous of degree 2 and a three-form ρ homogeneous of degree 4. Basically, τ contains relevant information about $e_{A'B'}$ and ρ tells us about e_{AB}, hence their knowledge determines $g_{ab} = e_{AB}\ e_{A'B'}$. The result proved in Ward (1980b) states that a one-to-one correspondence exists between sufficiently local anti-self-dual solutions with scalar curvature $R = 24\Lambda$ and sufficiently small deformations of flat projective twistor space which preserve the one-form τ and the three-form ρ, where $\tau \wedge d\tau = 2\Lambda\rho$. We now describe how to define the forms τ and ρ, whereas the explicit construction of a class of anti-self-dual space-times is given in chapter five.

The geometric framework is twistor space $\mathcal{P}$ defined at the end of section 4.2, i.e. the space of all α-surfaces in (M, g). We take M to be sufficiently small and convex to ensure that $\mathcal{P}$ is a complex manifold with topology $R^4 \times S^2$, since every point in an anti-self-dual space-time has such a neighbourhood (Ward 1980b). If Q, represented by the pair $\left(\alpha^A, \beta_{A'}\right)$, is any vector in $\mathcal{P}$, then τ is defined by

$$\tau(Q) \equiv e^{A'B'}\ \pi_{A'}\ \beta_{B'}. \tag{4.3.16}$$

To make sure τ is well defined, one has to check that the right-hand side of (4.3.16) remains covariantly constant over α-surfaces, i.e. is annihilated by the first-order operator $\lambda^A \pi^{A'} \nabla_{AA'}$, since otherwise τ does not correspond to a differential form on $\mathcal{P}$. It turns out that τ is well defined provided the trace-free part of Ricci vanishes. This is proved using spinor Ricci identities and the equations of local twistor transport as follows (Ward 1980b).

Let v be a vector field on the α-surface Z such that ϵv^a joins Z to the neighbouring α-surface Y. Since ϵv^a acts as a connecting vector, the Lie bracket of v^a and $\lambda^B \pi^{B'}$ vanishes for all λ^B, i.e.

$$\lambda^B\ \pi^{B'}\ \nabla_{BB'}\ v^{AA'} - v^{BB'}\ \nabla_{BB'}\ \lambda^A\ \pi^{A'} = 0. \tag{4.3.17}$$

Thus, after defining

$$\beta_{A'} \equiv v^{BB'} \; \nabla_{BB'} \; \pi_{A'}, \tag{4.3.18}$$

one finds

$$\pi_{A'} \; \lambda^B \; \pi^{B'} \; \nabla_{BB'} \; v^{AA'} = \lambda^A \; \beta^{A'} \; \pi_{A'}. \tag{4.3.19}$$

If one now applies the torsion-free spinor Ricci identities (see Eqs. (6.3.17) and (6.3.18) setting $\widetilde{\chi} = \widetilde{\Sigma} = \chi = \Sigma = 0$ therein), one finds that the spinor field $\beta_{A'}(x)$ on Z satisfies the equation

$$\lambda^B \; \pi^{B'} \; \nabla_{BB'} \; \beta_{A'} = -i \; \lambda^B \; \pi^{B'} \; P_{ABA'B'} \; \alpha^A, \tag{4.3.20}$$

where $P_{ab} = \Phi_{ab} - \Lambda g_{ab}$ and $\alpha^A = i v^{AC'} \; \pi_{C'}$. Moreover, Eq. (4.3.19) and the Leibniz rule imply that

$$\lambda^B \; \pi^{B'} \; \nabla_{BB'} \; \alpha^A = -i \; \lambda^A \; \pi^{A'} \; \beta_{A'}, \tag{4.3.21}$$

since $\pi^{B'} \nabla_{BB'} \pi_{C'} = 0$. Equations (4.3.20) and (4.3.21) are indeed the equations of *local twistor transport*, and Eq. (4.3.20) leads to

$$\begin{aligned} \lambda^C \pi^{C'} \nabla_{CC'} \Bigl(e^{A'B'} \pi_{A'} \; \beta_{B'} \Bigr) &= e^{A'B'} \pi_{A'} \Bigl(\lambda^C \pi^{C'} \nabla_{CC'} \beta_{B'} \Bigr) \\ &= -i \; \lambda^B \pi^{B'} \pi_{C'} \; e^{C'A'} \alpha^A \Bigl(\Phi_{ABA'B'} - \Lambda e_{AB} \; e_{A'B'} \Bigr) \\ &= i \; \lambda^B \pi^{A'} \pi^{B'} \alpha^A \Phi_{ABA'B'}, \end{aligned} \tag{4.3.22}$$

since $\pi^{A'} \pi^{B'} e_{A'B'} = 0$. Hence, as we said before, τ is well defined provided the trace-free part of Ricci vanishes. Note that, strictly, τ is a twisted form rather than a form on $\mathcal{P}$, since it is homogeneous of degree 2, one from $\pi_{A'}$ and one from $\beta_{B'}$. By contrast, a one-form would be independent of the scaling of $\pi_{A'}$ and $\beta_{B'}$ (Ward 1980b).

We are now in a position to define the three-form ρ, homogeneous of degree 4. For this purpose, let us denote by Q_h, $h = 1, 2, 3$ three vectors in $\mathcal{P}$, represented by the pairs $\Bigl(\alpha_h^A, \beta_{hA'} \Bigr)$. The corresponding $\rho(Q_1, Q_2, Q_3)$ is obtained by taking

$$\rho_{123} \equiv \frac{1}{2} \Bigl(e^{A'B'} \pi_{A'} \; \beta_{1B'} \Bigr) \Bigl(e_{AB} \; \alpha_2^A \; \alpha_3^B \Bigr), \tag{4.3.23}$$

and then anti-symmetrizing ρ_{123} over $1, 2, 3$. This yields

$$\rho(Q_1, Q_2, Q_3) \equiv \frac{1}{6}\Bigl(\rho_{123} - \rho_{132} + \rho_{231} - \rho_{213} + \rho_{312} - \rho_{321}\Bigr). \tag{4.3.24}$$

The reader can check that, by virtue of Eqs. (4.3.20) and (4.3.21), ρ is well defined, since it is covariantly constant over α-surfaces:

$$\lambda^A \, \pi^{A'} \, \nabla_{AA'} \, \rho(Q_1, Q_2, Q_3) = 0. \tag{4.3.25}$$

CHAPTER FIVE

PENROSE TRANSFORM FOR GRAVITATION

Deformation theory of complex manifolds is applied to construct a class of anti-self-dual solutions of Einstein's vacuum equations, following the work of Penrose and Ward. The hard part of the analysis is to find the holomorphic cross-sections of a deformed complex manifold, and the corresponding conformal structure of an anti-self-dual space-time. This calculation is repeated in detail, using complex analysis and two-component spinor techniques.

If no assumption about anti-self-duality is made, twistor theory is by itself insufficient to characterize geometrically a solution of the full Einstein equations. After a brief review of alternative ideas based on the space of complex null geodesics of complex space-time, and Einstein-bundle constructions, attention is focused on the attempt by Penrose to define twistors as charges for massless spin-$\frac{3}{2}$ fields. This alternative definition is considered since a vanishing Ricci tensor provides the consistency condition for the existence and propagation of massless spin-$\frac{3}{2}$ fields in curved space-time, whereas in Minkowski space-time the space of charges for such fields is naturally identified with the corresponding twistor space.

The two-spinor analysis of the Dirac form of such fields in Minkowski space-time is carried out in detail by studying their two potentials with corresponding gauge freedoms. The Rarita–Schwinger form is also introduced, and self-dual vacuum Maxwell fields are obtained from massless spin-$\frac{3}{2}$ fields by spin-lowering. In curved space-time, however, the local expression of spin-$\frac{3}{2}$ field strengths in terms of the second of these potentials is no longer possible, unless one studies the self-dual Ricci-flat case. Thus, much more work is needed to characterize geometrically a Ricci-flat (complex) space-time by using this alternative concept of twistors.

5.1 Anti-self-dual space-times

Following Ward (1978), we now use twistor-space techniques to construct a family of anti-self-dual solutions of Einstein's vacuum equations. Bearing in mind the space-time twistor-space correspondence in Minkowskian geometry described in section 4.1, we take a region $\mathcal{R}$ of $CM^{\#}$, whose corresponding region in PT is $\widetilde{\mathcal{R}}$. Moreover, $\mathcal{N}$ is the non-projective version of $\widetilde{\mathcal{R}}$, which implies $\mathcal{N} \subset T \subset C^4$. In other words, as coordinates on $\mathcal{N}$ we may use $\left(\omega^o, \omega^1, \pi_{o'}, \pi_{1'}\right)$. The geometrically-oriented reader may like it to know that three important structures are associated with $\mathcal{N}$:

(i) the fibration $\left(\omega^A, \pi_{A'}\right) \rightarrow \pi_{A'}$, which implies that $\mathcal{N}$ becomes a bundle over $C^2 - \{0\}$;

(ii) the two-form $\frac{1}{2} d\omega_A \wedge d\omega^A$ on each fibre;

(iii) the projective structure $\mathcal{N} \rightarrow \widetilde{\mathcal{R}}$.

Deformations of $\mathcal{N}$ which preserve this projective structure correspond to right-flat metrics (see section 4.2) in $\mathcal{R}$. To obtain such deformations, cover $\mathcal{N}$ with two patches $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$. Coordinates on $\mathcal{Q}$ and on $\widehat{\mathcal{Q}}$ are $\left(\omega^A, \pi_{A'}\right)$ and $\left(\widehat{\omega}^A, \widehat{\pi}_{A'}\right)$ respectively. We may now *glue* $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ together according to

$$\widehat{\omega}^A = \omega^A + f^A\left(\omega^B, \pi_{B'}\right), \tag{5.1.1}$$

$$\widehat{\pi}_{A'} = \pi_{A'}, \tag{5.1.2}$$

where f^A is homogeneous of degree 1, holomorphic on $\mathcal{Q} \bigcap \widehat{\mathcal{Q}}$, and satisfies

$$\det\left({\varepsilon_A}^B + \frac{\partial f^B}{\partial \omega^A}\right) = 1. \tag{5.1.3}$$

Such a patching process yields a complex manifold $\mathcal{N}^D$ which is a deformation of $\mathcal{N}$. The corresponding right-flat space-time $\mathcal{G}$ is such that its points correspond to the holomorphic cross-sections of $\mathcal{N}^D$. The hard part of the analysis is indeed to find these cross-sections, but this can be done explicitly for a particular class of patching functions. For this purpose, we first choose a constant spinor field $p^{AA'B'} = p^{A(A'B')}$ and a homogeneous holomorphic function $g(\gamma, \pi_{A'})$ of three complex variables:

$$g\Bigr(\lambda^3\gamma, \lambda\pi_{A'}\Bigr) = \lambda^{-1}g\Bigr(\gamma, \pi_{A'}\Bigr) \ \ \forall \lambda \in C - \{0\}. \tag{5.1.4}$$

This enables one to define the spinor field

$$p^A \equiv p^{AA'B'} \ \pi_{A'} \ \pi_{B'}, \tag{5.1.5}$$

and the patching function

$$f^A \equiv p^A \ g\Bigr(p_B \omega^B, \pi_{B'}\Bigr), \tag{5.1.6}$$

and the function

$$F(x^a, \pi_{A'}) \equiv g\Bigr(i \ p_A \ x^{AC'} \ \pi_{C'}, \pi_{A'}\Bigr). \tag{5.1.7}$$

Under suitable assumptions on the singularities of g, F may turn out to be holomorphic if $x^a \in \mathcal{R}$ and if the ratio $\widetilde{\pi} \equiv \frac{\pi_{o'}}{\pi_{1'}} \in]\frac{1}{2}, \frac{5}{2}[$. It is also possible to express F as the difference of two contour integrals after defining the differential form

$$\Omega \equiv \Bigr(2\pi i \rho^{A'} \pi_{A'}\Bigr)^{-1} \ F(x^b, \rho_{B'}) \ \rho_{C'} d\rho^{C'}. \tag{5.1.8}$$

In other words, if Γ and $\widehat{\Gamma}$ are closed contours on the projective $\rho_{A'}$-sphere defined by $|\widetilde{\rho}| = 1$ and $|\widetilde{\rho}| = 2$ respectively, we may define the function

$$h \equiv \oint_{\Gamma} \Omega, \tag{5.1.9}$$

holomorphic for $\widetilde{\pi} < 2$, and the function

$$\widehat{h} \equiv \oint_{\widehat{\Gamma}} \Omega, \tag{5.1.10}$$

holomorphic for $\widetilde{\pi} > 1$. Thus, by virtue of Cauchy's integral formula, one finds (cf. Ward 1978)

$$F(x^a, \pi_{A'}) = \widehat{h}(x^a, \pi_{A'}) - h(x^a, \pi_{A'}). \tag{5.1.11}$$

The basic concepts of sheaf-cohomology presented in section 4.3 are now useful to understand the deep meaning of these formulae. For any fixed x^a, $F(x^a, \pi_{A'})$ determines an element of the sheaf-cohomology group $H^1(P_1(C), O(-1))$, where $P_1(C)$ is the Riemann sphere of projective $\pi_{A'}$ spinors and $O(-1)$ is the sheaf of germs of holomorphic functions of $\pi_{A'}$, homogeneous of degree -1. Since H^1 vanishes, F is actually a coboundary. Hence it can be split according to (5.1.11).

In the subsequent calculations, it will be useful to write a solution of the Weyl equation $\nabla^{AA'} \psi_A = 0$ in the form

$$\psi_A \equiv i \, \pi^{A'} \; \nabla_{AA'} h(x^a, \pi_{C'}). \tag{5.1.12}$$

Moreover, following again Ward (1978), we note that a spinor field ${\xi_{A'}}^{B'}(x)$ can be defined by

$${\xi_{A'}}^{B'} \pi_{B'} \equiv i \, p^{AB'C'} \; \pi_{B'} \; \pi_{C'} \; \nabla_{AA'} h(x, \pi_{D'}), \tag{5.1.13}$$

and that the following identities hold:

$$i \, p^{AA'B'} \; \pi_{B'} \; \nabla_{AA'} h(x, \pi_{C'}) = \xi \equiv \frac{1}{2} {\xi_{A'}}^{A'}, \tag{5.1.14}$$

$$\psi_A \; p^{AA'B'} = -\xi^{(A'B')}. \tag{5.1.15}$$

We may now continue the analysis of our deformed twistor space $\mathcal{N}^D$, written in the form (cf. (5.1.1) and (5.1.2))

$$\widehat{\omega}^A = \omega^A + p^A g\Bigl(p_B \omega^B, \pi_{B'}\Bigr), \tag{5.1.16a}$$

$$\widehat{\pi}_{A'} = \pi_{A'}. \tag{5.1.16b}$$

In the light of the split (5.1.11), holomorphic sections of $\mathcal{N}^D$ are given by

$$\omega^A(x^b, \pi_{B'}) = i \, x^{AA'} \; \pi_{A'} + p^A \; h(x^b, \pi_{B'}) \text{ in } \mathcal{Q}, \tag{5.1.17}$$

$$\widehat{\omega}^A(x^b, \pi_{B'}) = i\, x^{AA'}\, \pi_{A'} + p^A\, \widehat{h}(x^b, \pi_{B'}) \text{ in } \widehat{\mathcal{Q}}, \tag{5.1.18}$$

where x^b are *complex* coordinates on $\mathcal{G}$. The conformal structure of $\mathcal{G}$ can be computed as follows. A vector $U = U^{BB'} \nabla_{BB'}$ at $x^a \in \mathcal{G}$ may be represented in $\mathcal{N}^D$ by the displacement

$$\delta\omega^A = U^b\, \nabla_b\, \omega^A(x^c, \pi_{C'}). \tag{5.1.19a}$$

By virtue of (5.1.17), Eq. (5.1.19a) becomes

$$\delta\omega^A = U^{BB'} \Bigl(i\, \varepsilon_B^{\ \ A}\, \pi_{B'} + p^A\, \nabla_{BB'} h(x^c, \pi_{C'}) \Bigr). \tag{5.1.19b}$$

The vector U is null, by definition, if and only if

$$\delta\omega^A(x^b, \pi_{B'}) = 0, \tag{5.1.20}$$

for some spinor field $\pi_{B'}$. To prove that the solution of Eq. (5.1.20) exists, one defines (see (5.1.14))

$$\theta \equiv 1 - \xi, \tag{5.1.21}$$

$$\Omega^{BB'}_{\ \ \ \ AA'} \equiv \theta\, \varepsilon_A^{\ \ B}\, \varepsilon_{A'}^{\ \ B'} - \psi_A\, p_{A'}^{\ \ BB'}. \tag{5.1.22}$$

We are now aiming to show that the desired solution of Eq. (5.1.20) is given by

$$U^{BB'} = \Omega^{BB'}_{\ \ \ \ AA'}\, \lambda^A\, \pi^{A'}. \tag{5.1.23}$$

Indeed, by virtue of (5.1.21)–(5.1.23) one finds

$$U^{BB'} = (1-\xi)\lambda^B \pi^{B'} - \psi_A\, p_{A'}^{\ \ BB'}\, \lambda^A\, \pi^{A'}. \tag{5.1.24}$$

Thus, since $\pi^{B'}\pi_{B'} = 0$, the calculation of (5.1.19b) yields

$$\delta\omega^A = -\psi_C\, \lambda^C\, \pi^{A'} \Bigl[i\, p_{A'}^{\ \ AB'}\, \pi_{B'} + p_{A'}^{\ \ BB'}\, p^A\, \nabla_{BB'} h(x,\pi) \Bigr] + (1-\xi)\lambda^B\, \pi^{B'}\, p^A\, \nabla_{BB'} h(x,\pi). \tag{5.1.25}$$

Note that (5.1.12) may be used to re-express the second line of (5.1.25). This leads to

$$\delta\omega^A = -\psi_C \; \lambda^C \; \Gamma^A, \tag{5.1.26}$$

where

$$\begin{aligned}
\Gamma^A &\equiv \pi^{A'} \Big[i \, p_{A'}{}^{AB'} \; \pi_{B'} + p_{A'}{}^{BB'} \; p^A \; \nabla_{BB'} h(x,\pi) \Big] + i(1-\xi) p^A \\
&= -i \, p^{AA'B'} \; \pi_{A'} \; \pi_{B'} + i \, p^A + p^A \Big[- p^{BB'A'} \; \pi_{A'} \; \nabla_{BB'} h(x,\pi) - i\xi \Big] \\
&= \Big[-i + i + i\xi - i\xi \Big] p^A = 0,
\end{aligned} \tag{5.1.27}$$

in the light of (5.1.5) and (5.1.14). Hence the solution of Eq. (5.1.20) is given by (5.1.23).

Such null vectors determine the conformal metric of $\mathcal{G}$. For this purpose, one defines (Ward 1978)

$$\nu_{A'}{}^{B'} \equiv \varepsilon_{A'}{}^{B'} - \xi_{A'}{}^{B'}, \tag{5.1.28}$$

$$\Lambda \equiv \frac{\theta}{2} \; \nu_{A'B'} \; \nu^{A'B'}, \tag{5.1.29}$$

$$\Sigma_{BB'}{}^{CC'} \equiv \theta^{-1} \; \varepsilon_B{}^C \; \varepsilon_{B'}{}^{C'} + \Lambda^{-1} \; \psi_B \; p_{A'}{}^{CC'} \; \nu_{B'}{}^{A'}. \tag{5.1.30}$$

Interestingly, $\Sigma_b{}^c$ is the inverse of $\Omega^b{}_a$, since

$$\Omega^b{}_a \; \Sigma_b{}^c = \delta_a{}^c. \tag{5.1.31}$$

Indeed, after defining

$$H_{A'}{}^{CC'} \equiv p_{A'}{}^{CC'} - p_{D'}{}^{CC'} \; \xi_{A'}{}^{D'}, \tag{5.1.32}$$

$$\Phi_{A'}{}^{CC'} \equiv \Big[\theta \Lambda^{-1} \; H_{A'}{}^{CC'} - \Lambda^{-1} \; p_{A'}{}^{BB'} \; \psi_B \; H_{B'}{}^{CC'} - \theta^{-1} \; p_{A'}{}^{CC'} \Big], \tag{5.1.33}$$

a detailed calculation shows that

$$\Omega^{BB'}{}_{AA'} \; \Sigma_{BB'}{}^{CC'} - \varepsilon_A{}^C \; \varepsilon_{A'}{}^{C'} = \psi_A \; \Phi_{A'}{}^{CC'}. \tag{5.1.34}$$

One can now check that the right-hand side of (5.1.34) vanishes (see problem 5.1). Hence (5.1.31) holds. For our anti-self-dual space-time $\mathcal{G}$, the metric $g = g_{ab} dx^a \otimes dx^b$ is such that

$$g_{ab} = \Xi(x) \; \Sigma_a^{\ c} \; \Sigma_{bc}. \tag{5.1.35}$$

Two null vectors U and V at $x \in \mathcal{G}$ have, by definition, the form

$$U^{AA'} \equiv \Omega^{AA'}_{\ \ \ \ BB'} \; \lambda^B \; \alpha^{B'}, \tag{5.1.36}$$

$$V^{AA'} \equiv \Omega^{AA'}_{\ \ \ \ BB'} \; \chi^B \; \beta^{B'}, \tag{5.1.37}$$

for some spinors $\lambda^B, \chi^B, \alpha^{B'}, \beta^{B'}$. In the deformed space $\mathcal{N}^D$, U and V correspond to two displacements $\delta_1 \omega^A$ and $\delta_2 \omega^A$ respectively, as in Eq. (5.1.19b). If one defines the corresponding skew-symmetric form

$$\mathcal{S}_\pi(U,V) \equiv \delta_1 \omega_A \; \delta_2 \omega^A, \tag{5.1.38}$$

the metric is given by

$$g(U,V) \equiv \Bigl(\alpha^{A'} \; \beta_{A'}\Bigr) \Bigl(\alpha^{B'} \; \pi_{B'}\Bigr)^{-1} \Bigl(\beta^{C'} \; \pi_{C'}\Bigr)^{-1} \; \mathcal{S}_\pi(U,V). \tag{5.1.39}$$

However, in the light of (5.1.31), (5.1.35)–(5.1.37) one finds

$$g(U,V) \equiv g_{ab} U^a V^b = \Xi(x) \Bigl(\lambda^A \; \chi_A\Bigr) \Bigl(\alpha^{A'} \; \beta_{A'}\Bigr). \tag{5.1.40}$$

By comparison with (5.1.39) this leads to

$$\mathcal{S}_\pi(U,V) = \Xi(x) \Bigl(\lambda^A \; \chi_A\Bigr) \Bigl(\alpha^{B'} \; \pi_{B'}\Bigr) \Bigl(\beta^{C'} \; \pi_{C'}\Bigr). \tag{5.1.41}$$

If we now evaluate (5.1.41) with $\beta^{A'} = \alpha^{A'}$, comparison with the definition (5.1.38) and use of (5.1.12), (5.1.13), (5.1.19b) and (5.1.36) yield

$$\Xi = \Lambda. \tag{5.1.42}$$

The anti-self-dual solution of Einstein's equations is thus given by (5.1.30), (5.1.35) and (5.1.42).

The construction of an anti-self-dual space-time described in this section is a particular example of the so-called non-linear graviton (Penrose 1976a–b). In mathematical language, if $\mathcal{M}$ is a complex three-manifold, B is the bundle of holomorphic three-forms on $\mathcal{M}$ and H is the standard positive line bundle on P_1, a non-linear graviton is the following set of data (Hitchin 1979):

(i) $\mathcal{M}$, the total space of a holomorphic fibration $\pi : \mathcal{M} \rightarrow P_1$;

(ii) a four-parameter family of sections, each having $H \oplus H$ as normal bundle (see e.g. Huggett and Tod (1985) for the definition of normal bundle);

(iii) a non-vanishing holomorphic section s of $B \otimes \pi^* H^4$, where $H^4 \equiv H \otimes H \otimes H \otimes H$, and $\pi^* H^4$ denotes the pull-back of H^4 by π;

(iv) a real structure on $\mathcal{M}$ such that π and s are real. $\mathcal{M}$ is then fibred from the real sections of the family.

5.2 Beyond anti-self-duality

The limit of the analysis performed in section 5.1 is that it deals with a class of solutions of (complex) Einstein equations which is not sufficiently general. In Yasskin and Isenberg (1982) and Yasskin (1987) the authors have examined in detail the limits of the anti-self-dual analysis. The two main criticisms are as follows:

(a) a right-flat space-time (cf. the analysis in Law (1985)) does not represent a real Lorentzian space-time manifold. Hence it cannot be applied directly to classical gravity (Ward 1980b);

(b) there are reasons fo expecting that the equations of a quantum theory of gravity are much more complicated, and thus are not solved by right-flat space-times.

However, an alternative approach due to Le Brun has become available in the eighties (Le Brun 1985). Le Brun's approach focuses on the space G of complex null geodesics of complex space-time (M, g), called ambitwistor space. Thus, one deals with a standard rank-2 holomorphic vector bundle $E \to G$, and in the conformal class determined by the complex structure of G, a one-to-one correspondence exists between non-vanishing holomorphic sections of E and Einstein metrics on (M, g) (Le Brun 1985). The bundle E is called Einstein bundle, and has also been studied in Eastwood (1987). The work by Eastwood adds evidence in favour of the Einstein bundle being the correct generalization of the non-linear-graviton construction to the non-right-flat case (cf. Law (1985), Park (1990), Le Brun (1991), Park (1991), our section 9.6). Indeed, the theorems discussed so far provide a characterization of the vacuum Einstein equations. However, there is not yet an independent way of recognizing the Einstein bundle. Thus, this is not yet a substantial progress in solving the vacuum equations. Other relevant work on holomorphic ideas appears in Le Brun (1986), where the author proves that, in the case of four-manifolds with self-dual Weyl curvature, solutions of the Yang–Mills equations correspond to holomorphic bundles on an associated analytic space (cf. Ward (1977), Witten (1978), Ward (1981a)).

5.3 Twistors as spin-$\frac{3}{2}$ charges

In this section, we describe a proposal by Penrose to regard twistors for Ricci-flat space-times as (conserved) *charges* for massless helicity-$\frac{3}{2}$ fields (Penrose 1990, Penrose 1991a–b–c). The new approach proposed by Penrose is based on the following mathematical results (Penrose 1991b):

(i) A vanishing Ricci tensor provides the consistency condition for the existence and propagation of massless helicity-$\frac{3}{2}$ fields in curved space-time (Buchdahl 1958, Deser and Zumino 1976);

(ii) In Minkowski space-time, the space of charges for such fields is naturally identified with the corresponding twistor space.

Thus, Penrose points out that if one could find the appropriate definition of charge for massless helicity-$\frac{3}{2}$ fields in a Ricci-flat space-time, this should provide the concept of twistor appropriate for vacuum Einstein equations. The corresponding geometric program may be summarized as follows:

(1) Define a twistor for Ricci-flat space-time $(M,g)_{RF}$;

(2) Characterize the resulting twistor space $\mathcal{F}$;

(3) Reconstruct $(M,g)_{RF}$ from $\mathcal{F}$.

We now describe, following Penrose (1990), Penrose (1991a–c), properties and problems of this approach to twistor theory in flat and in curved space-times.

5.3.1 Massless spin-$\frac{3}{2}$ equations in Minkowski space-time

Let (M,η) be Minkowski space-time with flat connection $\mathcal{D}$. In (M,η) the gauge-invariant field strength for spin $\frac{3}{2}$ is represented by a totally symmetric spinor field

$$\psi_{A'B'C'} = \psi_{(A'B'C')}, \tag{5.3.1}$$

obeying a massless free-field equation

$$\mathcal{D}^{AA'} \; \psi_{A'B'C'} = 0. \tag{5.3.2}$$

With the conventions of Penrose, $\psi_{A'B'C'}$ describes spin-$\frac{3}{2}$ particles of helicity equal to $\frac{3}{2}$ (rather than -$\frac{3}{2}$). The *Dirac form* of this field strength is obtained by expressing *locally* $\psi_{A'B'C'}$ in terms of two potentials subject to gauge freedoms involving a primed and an unprimed spinor field. The first potential is a spinor field symmetric in its primed indices

$$\gamma_{B'C'}^{A} = \gamma_{(B'C')}^{A}, \tag{5.3.3}$$

subject to the differential equation

$$\mathcal{D}^{BB'} \; \gamma_{B'C'}^{A} = 0, \tag{5.3.4}$$

and such that

$$\psi_{A'B'C'} = \mathcal{D}_{AA'} \; \gamma^{A}_{B'C'}. \tag{5.3.5}$$

The second potential is a spinor field symmetric in its unprimed indices

$$\rho^{AB}_{C'} = \rho^{(AB)}_{C'}, \tag{5.3.6}$$

subject to the equation

$$\mathcal{D}^{CC'} \; \rho^{AB}_{C'} = 0, \tag{5.3.7}$$

and it yields the $\gamma^{A}_{B'C'}$ potential by means of

$$\gamma^{A}_{B'C'} = \mathcal{D}_{BB'} \; \rho^{AB}_{C'}. \tag{5.3.8}$$

If we introduce the spinor fields $\nu_{C'}$ and χ^{B} obeying the equations

$$\mathcal{D}^{AC'} \; \nu_{C'} = 0, \tag{5.3.9}$$

$$\mathcal{D}_{AC'} \; \chi^{A} = 2i \; \nu_{C'}, \tag{5.3.10}$$

the gauge freedoms for the two potentials enable one to replace them by the potentials

$$\widehat{\gamma}^{A}_{B'C'} \equiv \gamma^{A}_{B'C'} + \mathcal{D}_{B'}^{\;\;A} \; \nu_{C'}, \tag{5.3.11}$$

$$\widehat{\rho}^{AB}_{C'} \equiv \rho^{AB}_{C'} + \varepsilon^{AB} \; \nu_{C'} + i \; \mathcal{D}_{C'}^{\;\;A} \; \chi^{B}, \tag{5.3.12}$$

without affecting the theory. Note that the right-hand side of (5.3.12) does not contain antisymmetric parts since, despite the explicit occurrence of the antisymmetric ε^{AB}, one finds

$$\mathcal{D}_{C'}^{\;\;[A} \; \chi^{B]} = \frac{\varepsilon^{AB}}{2} \mathcal{D}_{LC'} \; \chi^{L} = i \; \varepsilon^{AB} \nu_{C'}, \tag{5.3.13}$$

by virtue of (5.3.10). Hence (5.3.13) leads to

$$\widehat{\rho}^{AB}_{C'} = \rho^{AB}_{C'} + i \; \mathcal{D}_{C'}^{\;\;(A} \; \chi^{B)}. \tag{5.3.14}$$

The gauge freedoms are indeed given by Eqs. (5.3.11) and (5.3.12) since in our flat space-time one finds

$$\mathcal{D}^{AA'} \; \widehat{\gamma}^{C}_{A'B'} = \mathcal{D}^{AA'} \; \mathcal{D}^{C}{}_{B'} \; \nu_{A'} = \mathcal{D}^{C}{}_{B'} \; \mathcal{D}^{AA'} \; \nu_{A'} = 0, \tag{5.3.15}$$

by virtue of (5.3.4) and (5.3.9), and

$$\begin{aligned} \mathcal{D}^{AA'} \; \widehat{\rho}^{BC}_{A'} &= \mathcal{D}^{AA'} \; \mathcal{D}^{C}{}_{A'} \; \chi^{B} = \mathcal{D}^{CA'} \; \mathcal{D}_{A'}{}^{A} \; \chi^{B} \\ &= \mathcal{D}_{A'}{}^{A} \; \mathcal{D}^{CA'} \; \chi^{B} = -\mathcal{D}^{AA'} \; \mathcal{D}^{C}{}_{A'} \; \chi^{B}, \end{aligned} \tag{5.3.16a}$$

which implies

$$\mathcal{D}^{AA'} \; \widehat{\rho}^{BC}_{A'} = 0. \tag{5.3.16b}$$

The result (5.3.16b) is a particular case of the application of spinor Ricci identities to flat space-time (cf. sections 6.3 and 8.4).

We are now in a position to show that twistors can be regarded as charges for helicity-$\frac{3}{2}$ massless fields in Minkowski space-time. For this purpose, following Penrose (1991a,c) let us suppose that the field ψ satisfying (5.3.1) and (5.3.2) exists in a region $\mathcal{R}$ of (M, η), surrounding a world-tube which contains the sources for ψ. Moreover, we consider a two-sphere $\mathcal{S}$ within $\mathcal{R}$ surrounding the world-tube. To achieve this we begin by taking a *dual* twistor, i.e. the pair of spinor fields

$$W_{\alpha} \equiv \Bigl(\lambda_{A}, \mu^{A'}\Bigr), \tag{5.3.17}$$

obeying the differential equations

$$\mathcal{D}_{AA'} \; \mu^{B'} = i \; \varepsilon_{A'}{}^{B'} \; \lambda_{A}, \tag{5.3.18}$$

$$\mathcal{D}_{AA'} \; \lambda_{B} = 0. \tag{5.3.19}$$

Hence $\mu^{B'}$ is a solution of the complex-conjugate twistor equation

$$\mathcal{D}_{A}^{(A'} \; \mu^{B')} = 0. \tag{5.3.20}$$

Thus, if one defines

$$\varphi_{A'B'} \equiv \psi_{A'B'C'} \; \mu^{C'}, \tag{5.3.21}$$

one finds, by virtue of (5.3.1), (5.3.2) and (5.3.20), that $\varphi_{A'B'}$ is a solution of the self-dual vacuum Maxwell equations

$$\mathcal{D}^{AA'} \; \varphi_{A'B'} = 0. \tag{5.3.22}$$

Note that (5.3.21) is a particular case of the spin-lowering procedure (Huggett and Tod 1985, Penrose and Rindler 1986). Moreover, $\varphi_{A'B'}$ enables one to define the self-dual two-form

$$F \equiv \varphi_{A'B'} \; dx_A^{\;\;A'} \; \wedge \; dx^{AB'}, \tag{5.3.23}$$

which leads to the following *charge* assigned to the world-tube:

$$Q \equiv \frac{i}{4\pi} \oint F. \tag{5.3.24}$$

For some twistor

$$Z^\alpha \equiv \Bigl(\omega^A, \pi_{A'}\Bigr), \tag{5.3.25}$$

the charge Q depends on the dual twistor W_α as (see problem 5.3)

$$Q = Z^\alpha \; W_\alpha = \omega^A \; \lambda_A + \pi_{A'} \; \mu^{A'}. \tag{5.3.26}$$

These equations describe the strength of the charge, for the field ψ, that should be assigned to the world-tube. Thus, a twistor Z^α arises naturally in Minkowski space-time as the charge for a helicity $+\frac{3}{2}$ massless field, whereas a dual twistor W_α is the charge for a helicity $-\frac{3}{2}$ massless field (Penrose 1991c).

Interestingly, the potentials $\gamma_{A'B'}^C$ and $\rho_{A'}^{BC}$ can be used to obtain a potential for the self-dual Maxwell field strength, since, after defining

$$\theta^C_{\;\;A'} \equiv \gamma_{A'B'}^C \; \mu^{B'} - i \; \rho_{A'}^{BC} \; \lambda_B, \tag{5.3.27}$$

one finds

$$\begin{aligned}
\mathcal{D}_{CB'} \; \theta^C_{\;\;A'} &= \Bigl(\mathcal{D}_{CB'} \; \gamma_{A'D'}^C\Bigr)\mu^{D'} + \gamma_{A'D'}^C \Bigl(\mathcal{D}_{CB'} \; \mu^{D'}\Bigr) - i\Bigl(\mathcal{D}_{CB'} \; \rho_{A'}^{BC}\Bigr)\lambda_B \\
&= \psi_{A'B'D'} \; \mu^{D'} + i \; \varepsilon_{B'}^{\;\;D'} \; \gamma_{A'D'}^C \; \lambda_C - i \; \gamma_{A'B'}^C \; \lambda_C \\
&= \psi_{A'B'D'} \; \mu^{D'} = \varphi_{A'B'},
\end{aligned} \tag{5.3.28}$$

$$
\begin{aligned}
\mathcal{D}_B^{\ \ A'} \theta^C_{\ \ A'} = & \Bigl(\mathcal{D}_B^{\ \ A'} \gamma^C_{A'B'}\Bigr)\mu^{B'} + \gamma^C_{A'B'}\Bigl(\mathcal{D}_B^{\ \ A'} \mu^{B'}\Bigr) - i\Bigl(\mathcal{D}_B^{\ \ A'} \rho^{DC}_{A'}\Bigr)\lambda_D \\
& - i\rho^{DC}_{A'}\Bigl(\mathcal{D}_B^{\ \ A'} \lambda_D\Bigr) = 0.
\end{aligned} \tag{5.3.29}
$$

Eq. (5.3.28) has been obtained by using (5.3.5), (5.3.8), (5.3.18) and (5.3.19), whereas (5.3.29) holds by virtue of (5.3.3), (5.3.4), (5.3.7), (5.3.18) and (5.3.19). The one-form corresponding to $\theta^C_{\ \ A'}$ is defined by

$$
A \equiv \theta_{BB'} \, dx^{BB'}, \tag{5.3.30}
$$

which leads to

$$
F = 2 \, dA, \tag{5.3.31}
$$

by using (5.3.23) and (5.3.28).

The *Rarita–Schwinger form* of the field strength does not require the symmetry (5.3.3) in $B'C'$ as we have done so far, and the $\gamma^A_{B'C'}$ potential is instead subject to the equations (Penrose 1991a–c) [cf. (8.6.3) and (8.6.4)]

$$
\varepsilon^{B'C'} \, \mathcal{D}_{A(A'} \, \gamma^A_{B')C'} = 0, \tag{5.3.32}
$$

$$
\mathcal{D}^{B'(B} \, \gamma^{A)}_{B'C'} = 0. \tag{5.3.33}
$$

Moreover, the spinor field $\nu_{C'}$ in (5.3.11) is no longer taken to be a solution of the Weyl equation (5.3.9).

The potentials γ and ρ may or may not be global over $\mathcal{S}$. If γ is global but ρ is not, one obtains a two-dimensional complex vector space parametrized by the spinor field $\pi_{A'}$. The corresponding subspace where $\pi_{A'} = 0$, parametrized by ω^A, is called ω-space. Thus, following Penrose (1991c), we regard π-space and ω-space as quotient spaces defined as follows:

$$
\pi - \text{space} \equiv \text{space of global } \psi'\text{s/space of global } \gamma'\text{s}, \tag{5.3.34}
$$

$$
\omega - \text{space} \equiv \text{space of global } \gamma'\text{s/space of global } \rho'\text{s}. \tag{5.3.35}
$$

5.3.2 Massless spin-$\frac{3}{2}$ field strengths in curved space-time

The conditions for the *local* existence of the $\rho_{A'}^{BC}$ potential in curved space-time are derived by requiring that, after the gauge transformation (5.3.12) (or, equivalently, (5.3.14)), also the $\widehat{\rho}_{A'}^{BC}$ potential should obey the equation

$$\nabla^{AA'} \, \widehat{\rho}_{A'}^{BC} = 0, \tag{5.3.36}$$

where ∇ is the curved connection. By virtue of the spinor Ricci identity (Ward and Wells 1990)

$$\nabla_{M'(A} \, \nabla^{M'}_{B)} \, \chi_C = \psi_{ABDC} \, \chi^D - 2\Lambda \, \chi_{(A} \, \varepsilon_{B)C}, \tag{5.3.37}$$

the insertion of (5.3.14) into (5.3.36) yields, assuming for simplicity that $\nu_{C'} = 0$ in (5.3.10), the following conditions (see (8.4.28)):

$$\psi_{ABCD} = 0, \; \Lambda = 0, \tag{5.3.38}$$

which imply we deal with a vacuum self-dual (or left-flat) space-time, since the anti-self-dual Weyl spinor has to vanish (Penrose 1991c).

Moreover, in a complex anti-self-dual vacuum space-time one finds (Penrose 1991c) that spin-$\frac{3}{2}$ field strengths $\psi_{A'B'C'}$ can be defined according to (cf. (5.3.5))

$$\psi_{A'B'C'} = \nabla_{CC'} \, \gamma_{A'B'}^{C}, \tag{5.3.39}$$

are gauge-invariant, totally symmetric, and satisfy the massless free-field equations (cf. (5.3.2))

$$\nabla^{AA'} \, \psi_{A'B'C'} = 0. \tag{5.3.40}$$

In this case there is no obstruction to defining global ψ-fields with non-vanishing π-charge, and a global π-space can be defined as in (5.3.34). It remains to be seen whether the twistor space defined by α-surfaces may then be reconstructed (section 4.2, Penrose 1976a-b, Ward and Wells 1990, Penrose 1991c).

Interestingly, in Penrose (1991b) it has been proposed to interpret the potential γ as providing a *bundle connection*. In other words, one takes the fibre coordinates to be given by a spinor $\eta_{A'}$ and a scalar μ. For a given small ϵ, one extends the ordinary Levi–Civita connection ∇ on M to bundle-valued quantities according to (Penrose 1991b)

$$\nabla_{PP'} \begin{pmatrix} \eta_{A'} \\ \mu \end{pmatrix} \equiv \begin{pmatrix} \nabla_{PP'} \; \eta_{A'} \\ \nabla_{PP'} \; \mu \end{pmatrix} - \epsilon \begin{pmatrix} 0 & \gamma_{PP'A'} \\ \gamma_{PP'}^{\; \; \; \; B'} & 0 \end{pmatrix} \begin{pmatrix} \eta_{B'} \\ \mu \end{pmatrix}, \tag{5.3.41}$$

with gauge transformations given by

$$\begin{pmatrix} \widehat{\eta}_{A'} \\ \widehat{\mu} \end{pmatrix} \equiv \begin{pmatrix} \eta_{A'} \\ \mu \end{pmatrix} + \epsilon \begin{pmatrix} 0 & \nu_{A'} \\ \nu^{B'} & 0 \end{pmatrix} \begin{pmatrix} \eta_{B'} \\ \mu \end{pmatrix}. \tag{5.3.42}$$

Note that terms of order ϵ^2 have been neglected in writing (5.3.42). However, such gauge transformations do not close under commutation, and to obtain a theory valid to all orders in ϵ one has to generalize to $SL(3,C)$ matrices before the commutators close. Writing (A) for the three-dimensional indices, so that $\eta_{(A)}$ denotes $\begin{pmatrix} \eta_{A'} \\ \mu \end{pmatrix}$, one has a connection defined by

$$\nabla_{PP'} \; \eta_{(A)} \equiv \begin{pmatrix} \nabla_{PP'} \; \eta_{A'} \\ \nabla_{PP'} \; \mu \end{pmatrix} - \gamma_{PP' \; (A)}^{\; \; \; \; \; \; \; \; (B)} \; \eta_{(B)}, \tag{5.3.43}$$

with gauge transformation

$$\widehat{\eta}_{(A)} \equiv \eta_{(A)} + \nu_{(A)}^{\; \; \; \; (B)} \; \eta_{(B)}. \tag{5.3.44}$$

With this notation, the $\nu_{(A)}^{\; \; \; \; (B)}$ are $SL(3,C)$-valued fields on M, and hence

$$\mathcal{E}^{(P) \, (Q) \, (R)} \; \nu_{(P)}^{\; \; \; \; (A)} \; \nu_{(Q)}^{\; \; \; \; (B)} \; \nu_{(R)}^{\; \; \; \; (C)} = \mathcal{E}^{(A) \, (B) \, (C)}, \tag{5.3.45}$$

where $\mathcal{E}^{(P) \, (Q) \, (R)}$ are generalized Levi–Civita symbols. The $SL(3,C)$ definition of γ-potentials takes the form (Penrose 1991b)

$$\gamma_{PP' \; (A)}^{\; \; \; \; \; \; \; \; (B)} \equiv \begin{pmatrix} \alpha_{PP'A'}^{\; \; \; \; \; \; \; \; B'} & \beta_{PP'A'} \\ \gamma_{PP'}^{\; \; \; \; B'} & \delta_{PP'} \end{pmatrix}, \tag{5.3.46}$$

while the curvature is

$$K_{pq\ (A)}^{\quad\ (B)} \equiv 2\nabla_{[p}\ \gamma_{q](A)}^{\quad\ (B)} + 2\ \gamma_{[p|(A)|}^{\quad\ (C)}\ \ \gamma_{q](C)}^{\quad\ (B)}. \tag{5.3.47}$$

Penrose has proposed this as a generalization of the Rarita–Schwinger structure in Ricci-flat space-times, and he has even speculated that a non-linear generalization of the Rarita–Schwinger equations (5.3.32) and (5.3.33) might be

$${}^{(-)}K_{PQ\ (A)}^{\quad\ (B)} = 0, \tag{5.3.48}$$

$${}^{(+)}K_{P'Q'\ (A)}^{\quad\ (B)}\ \ \mathcal{E}^{P'\ (A)\ (C)}\ \ \mathcal{E}^{Q'}_{\ \ (B)\ (D)} = 0, \tag{5.3.49}$$

where ${}^{(-)}K$ and ${}^{(+)}K$ are the anti-self-dual and self-dual parts of the curvature respectively, i.e.

$$K_{pq\ (A)}^{\quad\ (B)} = \varepsilon_{P'Q'}\ \ {}^{(-)}K_{PQ\ (A)}^{\quad\ (B)} + \varepsilon_{PQ}\ \ {}^{(+)}K_{P'Q'\ (A)}^{\quad\ (B)}. \tag{5.3.50}$$

Following Penrose (1991b), one has

$$\mathcal{E}^{P'\ (A)\ (C)} \equiv \mathcal{E}^{(P)\ (A)\ (C)}\ e_{(P)}^{\quad P'}, \tag{5.3.51}$$

$$\mathcal{E}_{Q'\ (B)\ (D)} \equiv \mathcal{E}_{(Q)\ (B)\ (D)}\ \ e_{Q'}^{\quad (Q)}, \tag{5.3.52}$$

the $e_{(P)}^{\quad P'}$ and $e_{Q'}^{\quad (Q)}$ relating the bundle directions with tangent directions in M.

CHAPTER SIX

COMPLEX SPACE-TIMES WITH TORSION

Theories of gravity with torsion are relevant since torsion is a naturally occurring geometric property of relativistic theories of gravitation, the gauge theory of the Poincaré group leads to its presence, the constraints are second-class and the occurrence of cosmological singularities can be less generic than in general relativity. In a space-time manifold with non-vanishing torsion, the Riemann tensor has 36 independent real components at each point, rather than 20 as in general relativity. The information of these 36 components is encoded in three spinor fields and in a scalar function, having 5,9,3 and 1 complex components, respectively. If space-time is complex, this means that, with respect to a holomorphic coordinate basis x^a, the metric is a 4×4 matrix of holomorphic functions of x^a, and its determinant is nowhere-vanishing. Hence the connection and Riemann are holomorphic as well, and the Ricci tensor becomes complex-valued.

After a two-component spinor analysis of the curvature and of spinor Ricci identities, the necessary condition for the existence of α-surfaces in complex space-time manifolds with non-vanishing torsion is derived. For these manifolds, Lie brackets of vector fields and spinor Ricci identities contain explicitly the effects of torsion. This leads to an integrability condition for α-surfaces which does not involve just the self-dual Weyl spinor, as in complex general relativity, but also the torsion spinor, in a non-linear way, and its covariant derivative. A similar result also holds for four-dimensional, smooth real manifolds with a positive-definite metric. Interestingly, a particular solution of the integrability condition is given by right conformally flat and right-torsion-free space-times.

6.1 Introduction

As we know from previous chapters, after the work in Penrose (1967), several efforts have been produced to understand many properties of classical and quantum field theories using twistor theory. Penrose's original idea was that the space-time picture might be inappropriate at the Planck length, whereas a more correct framework for fundamental physics should be a particular complex manifold called twistor space. In other words, concepts such as null lines and null surfaces are more fundamental than space-time concepts, and twistor space provides the precise mathematical description of this idea.

In the course of studying Minkowski space-time, twistors can be defined either via the four-complex-dimensional vector space of solutions to the differential equation (cf. Eq. (4.1.5))

$$\mathcal{D}_{A'}^{\ (A}\omega^{B)} = 0, \tag{6.1.1}$$

or via null two-surfaces in complexified compactified Minkowski space $CM^{\#}$, called α-planes. The α-planes (section 4.1) are such that the space-time metric vanishes over them, and their null tangent vectors have the two-component spinor form $\lambda^{A}\pi^{A'}$, where λ^{A} is varying and $\pi^{A'}$ is fixed (i.e. fixed by Eq. (4.2.4)). The latter definition can be generalized to complex or real Riemannian space-times provided that the Weyl curvature is anti-self-dual. This leads in turn to a powerful geometric picture, where the study of the Euclidean-time version of the partial differential equations of Einstein's theory is replaced by the problem of finding the holomorphic curves in a complex manifold called *deformed (projective) twistor space*. This finally enables one to reconstruct the space-time metric (chapter five). From the point of view of gravitational physics, this is the most relevant application of Penrose transform, which is by now a major tool for studying the differential equations of classical field theory (Ward and Wells 1990).

Note that, while in differential geometry the basic ideas of connection and curvature are local, in complex-analytic geometry there is no local information.

Any complex manifold looks locally like C^n, with no special features, and any holomorphic fibre bundle is locally an analytic product (cf. Atiyah (1988) on page 524 for a more detailed treatment of this non-trivial point). It is worth bearing in mind this difference since the Penrose transform converts problems from differential geometry into problems of complex-analytic geometry. We thus deal with a *non-local* transform, so that local curvature information is coded into *global holomorphic information*. More precisely, Penrose theory does not hold for both anti-self-dual and self-dual space-times, so that one only obtains a non-local treatment of complex space-times with anti-self-dual Weyl curvature. However, these investigations are incomplete for at least two reasons:

(a) anti-self-dual (or self-dual) space-times appear a very restricted (although quite important) class of models, and it is not clear how to generalize twistor-space definitions to general vacuum space-times;

(b) the fundamental theory of gravity at the Planck length is presumably different from Einstein's general relativity (Hawking 1979, Esposito 1994).

In this chapter we have thus tried to extend the original analysis appearing in the literature to a larger class of theories of gravity, i.e. space-time models (M, g) with torsion (we are, however, not concerned with supersymmetry). In our opinion, the main motivations for studying these space-time models are as follows.

(1) Torsion is a peculiarity of relativistic theories of gravitation, since the bundle $L(M)$ of linear frames is soldered to the base $B = M$, whereas for gauge theories other than gravitation the bundle $L(M)$ is loosely connected to M. The torsion two-form T is then defined as $T \equiv d\theta + \omega \wedge \theta$, where θ is the soldering form and ω is a connection one-form on $L(M)$. If $L(M)$ is reduced to the bundle $O(M)$ of orthonormal frames, ω is called spin-connection.

(2) The gauge theory of the Poincaré group naturally leads to theories with torsion.

(3) From the point of view of constrained Hamiltonian systems, theories with torsion are of great interest, since they are theories of gravity with second-class constraints (cf. Esposito (1994) and references therein).

(4) In space-time models with torsion, the occurrence of cosmological singularities *can* be less generic than in general relativity (Esposito 1992, Esposito 1994).

In the original work by Penrose and Ward, the first (simple) problem is to characterize curved space-time models possessing α-surfaces. As we were saying following Eq. (5.1.1), the necessary and sufficient condition is that space-time be complex, or real Riemannian (i.e. its metric is *positive-definite*), with anti-self-dual Weyl curvature. This is proved by using Frobenius' theorem, the spinor form of the Riemann curvature tensor, and spinor Ricci identities. Our chapter is thus organized as follows.

Section 6.2 describes Frobenius' theorem and its application to curved complex space-time models with non-vanishing torsion. In particular, if α-surfaces are required to exist, one finds this is equivalent to a differential equation involving two spinor fields ξ_A and $w_{AB'}$, which are completely determined by certain algebraic relations. Section 6.3 describes the spinor form of Riemann and spinor Ricci identities for theories with torsion. Section 6.4 applies the formulae of section 6.3 to obtain the integrability condition for the differential equation derived at the end of section 6.2. The integrability condition for α-surfaces is then shown to involve the self-dual Weyl spinor, the torsion spinor and covariant derivatives of torsion. Concluding remarks are presented in section 6.5.

6.2 Frobenius' theorem for theories with torsion

Frobenius' theorem is one of the main tools for studying calculus on manifolds. Following Abraham *et al.* (1983), the geometric framework and the theorem can be described as follows. Given a manifold M, let $E \subset TM$ be a sub-bundle of its tangent bundle. By definition, E is *involutive* if for any two E-valued vector fields X and Y defined on M, their Lie bracket is E-valued as well. Moreover, E is *integrable* if $\forall m_0 \in M$ there is a local submanifold $N \subset M$ through m_0, called a local integral manifold of E at m_0, whose tangent bundle coincides with E restricted to N. Frobenius' theorem ensures that a sub-bundle E of TM is involutive if and only if it is integrable.

Given a complex torsion-free space-time (M, g), it is possible to pick out in M a family of holomorphic two-surfaces, called α-surfaces, which generalize the α-planes of Minkowski space-time described in section 4.1, provided that the self-dual Weyl spinor vanishes. In the course of deriving the condition on the curvature enforced by the existence of α-surfaces, one begins by taking a totally null two-surface $\hat{S}$ in M. By definition, $\hat{S}$ is a two-dimensional complex submanifold of M such that, $\forall p \in \hat{S}$, if x and y are any two tangent vectors at p, then $g(x, x) = g(y, y) = g(x, y) = 0$. Denoting by $X = X^a e_a$ and $Y = Y^a e_a$ two vector fields tangent to $\hat{S}$, where X^a and Y^a have the two-component spinor form $X^a = \lambda^A \pi^{A'}$ and $Y^a = \mu^A \pi^{A'}$, Frobenius' theorem may be used to require that the Lie bracket of X and Y be a linear combination of X and Y, so that we write

$$[X, Y] = \varphi X + \rho Y, \tag{6.2.1}$$

where φ and ρ are scalar functions. Frobenius' theorem is indeed originally formulated for real manifolds. If the integral submanifolds of complex space-time are holomorphic, there are additional conditions which are not described here. Note also that Eq. (6.2.1) does not depend on additional structures on M (torsion, metric, etc. ...). In the torsion-free case, it turns out that the Lie bracket $[X, Y]$ can also be written as $\nabla_X Y - \nabla_Y X$, and this eventually leads to a condition which implies the vanishing of the self-dual part of the Weyl curvature, after using the spinorial formula for Riemann and spinor Ricci identities.

However, for the reasons described in section 6.1, we are here interested in models where torsion does not vanish. Even though Frobenius' theorem (cf. (6.2.1)) does not involve torsion, the Lie bracket $[X, Y]$ can be also expressed using the definition of the torsion tensor S (see comment following (6.3.3)) :

$$[X, Y] \equiv \nabla_X Y - \nabla_Y X - 2S(X, Y). \tag{6.2.2}$$

By comparison, Eqs. (6.2.1) and (6.2.2) lead to

$$X^a \nabla_a Y^b - Y^a \nabla_a X^b = \varphi X^b + \rho Y^b + 2S_{cd}^{\ \ b} X^c Y^d. \tag{6.2.3}$$

Now, the antisymmetry $S_{ab}^{\ \ c} = -S_{ba}^{\ \ c}$ of the torsion tensor can be expressed spinorially as

$$S_{ab}^{\ \ c} = \chi_{AB}^{\ \ \ CC'} \varepsilon_{A'B'} + \widetilde{\chi}_{A'B'}^{\ \ \ \ CC'} \varepsilon_{AB}, \tag{6.2.4}$$

where the spinors χ and $\widetilde{\chi}$ are symmetric in AB and $A'B'$ respectively, and from now on we use two-component spinor notation (we do not write Infeld-van der Waerden symbols for simplicity of notation). Thus, writing $X^a = \lambda^A \pi^{A'}$ and $Y^a = \mu^A \pi^{A'}$, one finds, using a technique similar to the one in section 9.1 of Ward and Wells (1990), that Eq. (6.2.3) is equivalent to

$$\pi^{A'} \Bigl(\nabla_{AA'} \pi_{B'} \Bigr) = \xi_A \pi_{B'} + w_{AB'}, \tag{6.2.5}$$

for some spinor fields ξ_A and $w_{AB'}$, if the following conditions are imposed:

$$-\mu^A \xi_A = \varphi, \tag{6.2.6}$$

$$\lambda^A \xi_A = \rho, \tag{6.2.7}$$

$$\mu_D \lambda^D w_{BB'} = -2\mu_D \lambda^D \widetilde{\chi}_{C'D'BB'} \pi^{C'} \pi^{D'}. \tag{6.2.8}$$

Note that, since our calculation involves two vector fields X and Y tangent to $\hat{S}$, its validity is only local unless the surface $\hat{S}$ is parallelizable (i.e. the bundle $L(\hat{S})$ admits a cross-section). Moreover, since $\hat{S}$ is holomorphic by hypothesis, also φ and ρ are holomorphic (cf. (6.2.1)), and this affects the unprimed spinor part of the null tangent vectors to α-surfaces in the light of (6.2.6) and (6.2.7).

By virtue of Eq. (6.2.8), one finds

$$w_{AB'} = -2\pi^{A'} \pi^{C'} \widetilde{\chi}_{A'B'AC'}, \tag{6.2.9}$$

which implies (Esposito 1993)

$$\pi^{A'} \Bigl(\nabla_{AA'} \pi_{B'} \Bigr) = \xi_A \pi_{B'} - 2\pi^{A'} \pi^{C'} \widetilde{\chi}_{A'B'AC'}. \tag{6.2.10}$$

Note that, if torsion is set to zero, Eq. (6.2.10) agrees with Eq. (9.1.2) appearing in section 9.1 of Ward and Wells (1990), where complex general relativity is studied. This is the desired necessary condition for the field $\pi_{A'}$ to define an α-surface in the presence of torsion (and it may be also shown to be sufficient, as in section 4.2). Our next task is to derive the integrability condition for Eq. (6.2.10). For this purpose, following Ward and Wells (1990), we operate with $\pi^{B'}\pi^{C'}\nabla^{A}{}_{C'}$ on both sides of Eq. (6.2.10). This leads to

$$\pi^{B'}\pi^{C'}\nabla^{A}{}_{C'}\Big[\pi^{A'}\Big(\nabla_{AA'}\pi_{B'}\Big)\Big] = \pi^{B'}\pi^{C'}\nabla^{A}{}_{C'}\Big[\xi_{A}\pi_{B'} - 2\pi^{A'}\pi^{D'}\widetilde{\chi}_{A'B'AD'}\Big]. \tag{6.2.11}$$

Using the Leibniz rule, (6.2.10) and the well known property $\pi_{A'}\pi^{A'} = \xi_{A}\xi^{A} = 0$, the two terms on the right-hand side of Eq. (6.2.11) are found to be

$$\pi^{B'}\;\pi^{C'}\Big[\nabla^{A}{}_{C'}\Big(\xi_{A}\pi_{B'}\Big)\Big] = 2\xi^{A}\;\pi^{A'}\;\pi^{B'}\;\pi^{C'}\;\widetilde{\chi}_{A'B'AC'}, \tag{6.2.12}$$

$$\begin{aligned}\pi^{B'}\;\pi^{C'}\Big[\nabla^{A}{}_{C'}\Big(-2\pi^{A'}\;\pi^{D'}\;\widetilde{\chi}_{A'B'AD'}\Big)\Big] &= -4\xi^{A}\;\pi^{A'}\;\pi^{B'}\;\pi^{D'}\;\widetilde{\chi}_{A'B'AD'}\\ &+ 8\pi^{B'}\;\pi_{F'}\;\pi_{G'}\;\widetilde{\chi}_{A'B'AD'}\;\pi^{(A'}\;\widetilde{\chi}^{F'D')AG'}\\ &- 2\pi^{A'}\;\pi^{B'}\;\pi^{C'}\;\pi^{D'}\Big(\nabla^{A}{}_{C'}\;\widetilde{\chi}_{A'B'AD'}\Big),\end{aligned} \tag{6.2.13}$$

where round brackets denote symmetrization over A' and D' on the second line of (6.2.13).

It now remains to compute the left-hand side of Eq. (6.2.11). This is given by

$$\begin{aligned}\pi^{B'}\;\pi^{C'}\;\nabla^{A}{}_{C'}\Big[\pi^{A'}\Big(\nabla_{AA'}\pi_{B'}\Big)\Big] &= \pi^{B'}\;\pi^{C'}\Big(\nabla^{A}{}_{C'}\pi^{A'}\Big)\Big(\nabla_{AA'}\pi_{B'}\Big)\\ &- \pi^{A'}\;\pi^{B'}\;\pi^{C'}\Big(\Box_{C'A'}\pi_{B'}\Big),\end{aligned} \tag{6.2.14}$$

where we have defined $\Box_{C'A'} \equiv \nabla_{A(C'}\nabla^{A}{}_{A')}$ as in section 8.4. Using Eq. (6.2.10), the first term on the right-hand side of (6.2.14) is easily found to be

$$\begin{aligned}\pi^{B'}\;\pi^{C'}\Big(\nabla^{A}{}_{C'}\pi^{A'}\Big)\Big(\nabla_{AA'}\pi_{B'}\Big) &= 4\pi^{B'}\;\pi^{C'}\;\pi_{F'}\;\pi_{G'}\;\widetilde{\chi}_{A'B'AC'}\;\widetilde{\chi}^{F'A'AG'}\\ &- 2\xi^{A}\;\pi^{A'}\;\pi^{B'}\;\pi^{C'}\;\widetilde{\chi}_{A'B'AC'}.\end{aligned} \tag{6.2.15}$$

The second term on the right-hand side of (6.2.14) can only be computed after using some fundamental identities of spinor calculus for theories with torsion, hereafter referred to as U_4-theories, as in Esposito (1992), Esposito (1994).

6.3 Spinor Ricci identities for complex U_4 theory

Since the results we here describe play a key role in obtaining the integrability condition for α-surfaces (cf. section 6.4), we have chosen to summarize the main formulae in this separate section, following Penrose (1983), Penrose and Rindler (1984).

Using abstract-index notation, the symmetric Lorentzian metric g of real Lorentzian U_4 space-times is still expressed by (see section 2.1)

$$g_{ab} = \varepsilon_{AB} \, \varepsilon_{A'B'}. \tag{6.3.1}$$

Moreover, the full connection still obeys the metricity condition $\nabla g = 0$, and the corresponding spinor covariant derivative is assumed to satisfy the additional relations

$$\nabla_{AA'} \, \varepsilon_{BC} = 0, \; \nabla_{AA'} \, \varepsilon_{B'C'} = 0, \tag{6.3.2}$$

and is a linear, *real* operator which satisfies the Leibniz rule. However, since torsion does not vanish, the difference $\Big(\nabla_a \nabla_b - \nabla_b \nabla_a\Big)$ applied to a function f is equal to $2S_{ab}^{\ \ c} \, \nabla_c f \neq 0$. Torsion also appears explicitly in the relation defining the Riemann tensor

$$\Big(\nabla_a \nabla_b - \nabla_b \nabla_a - 2S_{ab}^{\ \ c} \, \nabla_c\Big) V^d \equiv R_{abc}^{\ \ \ d} \, V^c, \tag{6.3.3}$$

and leads to a non-symmetric Ricci tensor $R_{ab} \neq R_{ba}$, where $R_{ab} \equiv R_{acb}^{\ \ \ c}$. Note that in (6.3.3) the factor 2 multiplies $S_{ab}^{\ \ c}$ since we are using definition (6.2.2), whereas in Penrose and Rindler (1984) a definition is used where the torsion tensor is $T \equiv 2S$. The tensor R_{abcd} has now 36 independent real components at each point, rather than 20 as in general relativity. The information of these 36 components is encoded in the spinor fields

$$\psi_{ABCD},\ \Phi_{ABC'D'},\ \Sigma_{AB},$$

and in the scalar function Λ, having $5, 9, 3,$ and 1 complex components respectively, and such that

$$\psi_{ABCD} = \psi_{(ABCD)}, \tag{6.3.4}$$

$$\Phi_{ABC'D'} = \Phi_{(AB)(C'D')}, \tag{6.3.5a}$$

$$\Phi_{ABC'D'} - \overline{\Phi}_{C'D'AB} \neq 0, \tag{6.3.5b}$$

$$\Sigma_{AB} = \Sigma_{(AB)}, \tag{6.3.6a}$$

$$R_{[ab]} = \Sigma_{AB}\ \varepsilon_{A'B'} + \overline{\Sigma}_{A'B'}\ \varepsilon_{AB}, \tag{6.3.6b}$$

$$\Lambda - \overline{\Lambda} \neq 0. \tag{6.3.7}$$

In (6.3.4)–(6.3.6), round (square) brackets denote, as usual, symmetrization (antisymmetrization), and *overbars* denote complex conjugation of spinors or scalars. The spinor Σ_{AB} and the left-hand sides of (6.3.5b) and (6.3.7) are determined directly by torsion and its covariant derivative. The relations (6.3.5b), (6.3.6b) and (6.3.7) express a substantial difference with respect to general relativity, and hold in any real Lorentzian U_4 space-time.

We are, however, interested in the case of complex U_4 space-times (or real Riemannian, where the metric is positive-definite), in order to compare the necessary condition for the existence of α-surfaces with what holds for complex general relativity. In that case, it is well known that the spinor covariant derivative still obeys (6.3.2) but is now a linear, *complex-holomorphic* operator satisfying the Leibniz rule. Moreover, barred spinors are replaced by independent twiddled spinors (e.g. $\widetilde{\Sigma}_{A'B'}$) which are no longer complex conjugates of unbarred (or untwiddled) spinors, since complex conjugation is no longer available. This also holds for real

Riemannian U_4 space-times, not to be confused with real Lorentzian U_4 space-times, but of course, in the positive-definite case the spinor covariant derivative is a real, rather than complex-holomorphic operator.

For the sake of clarity, we hereafter write CU_4, RU_4, LU_4 to denote complex, real Riemannian or real Lorentzian U_4-theory, respectively. In the light of our previous discussion, the spinorial form of Riemann for CU_4 and RU_4 theories is

$$
\begin{aligned}
R_{abcd} = & \psi_{ABCD}\, \varepsilon_{A'B'}\, \varepsilon_{C'D'} + \widetilde{\psi}_{A'B'C'D'}\, \varepsilon_{AB}\, \varepsilon_{CD} \\
& + \Phi_{ABC'D'}\, \varepsilon_{A'B'}\, \varepsilon_{CD} + \widetilde{\Phi}_{A'B'CD}\, \varepsilon_{AB}\, \varepsilon_{C'D'} \\
& + \Sigma_{AB}\, \varepsilon_{A'B'}\, \varepsilon_{CD}\, \varepsilon_{C'D'} + \widetilde{\Sigma}_{A'B'}\, \varepsilon_{AB}\, \varepsilon_{CD}\, \varepsilon_{C'D'} \\
& + \Lambda \Big(\varepsilon_{AC}\, \varepsilon_{BD} + \varepsilon_{AD}\, \varepsilon_{BC} \Big) \varepsilon_{A'B'}\, \varepsilon_{C'D'} \\
& + \widetilde{\Lambda} \Big(\varepsilon_{A'C'}\, \varepsilon_{B'D'} + \varepsilon_{A'D'}\, \varepsilon_{B'C'} \Big) \varepsilon_{AB}\, \varepsilon_{CD}. \qquad (6.3.8)
\end{aligned}
$$

The spinors ψ_{ABCD} and $\widetilde{\psi}_{A'B'C'D'}$ appearing in (6.3.8) are called anti-self-dual and self-dual Weyl spinors respectively as in general relativity, and they represent the part of Riemann invariant under conformal rescalings of the metric. This property is proved at the end of section 4 of Penrose (1983), following Eq. (49) therein. Note that in Penrose (1983) a class of conformal rescalings is studied such that $\hat{g} = \Omega \overline{\Omega}\ g$ (where Ω is a smooth, nowhere-vanishing, complex-valued function), and leading to the presence of torsion. We are, however, not interested in this method for generating torsion, and we only study models where torsion *already* exists before any conformal rescaling of the metric.

We are now in a position to compute $\square_{C'A'} \pi_{B'}$ appearing in (6.2.14). For this purpose, following the method in section 4.9 of Penrose and Rindler (1984), we define the operator

$$
\square_{ab} \equiv 2 \nabla_{[a} \nabla_{b]} - 2 S_{ab}{}^{c}\, \nabla_c, \qquad (6.3.9)
$$

and the self-dual null bivector

$$
k^{ab} \equiv \kappa^A\, \kappa^B\, \varepsilon^{A'B'}. \qquad (6.3.10)
$$

The Ricci identity for U_4 theories

$$\square_{ab} k^{cd} = R_{abe}{}^{c}\, k^{ed} + R_{abe}{}^{d}\, k^{ce}, \tag{6.3.11}$$

then yields

$$2\varepsilon^{E'F'} \kappa^{(E} \square_{ab} \kappa^{F)} = \Big(\varepsilon^{ED} \varepsilon^{E'D'} \varepsilon^{C'F'} \kappa^{C} \kappa^{F} + \varepsilon^{FD} \varepsilon^{F'D'} \varepsilon^{E'C'} \kappa^{E} \kappa^{C} \Big) R_{abcd}. \tag{6.3.12}$$

This is why, using (6.3.8) and the identity

$$2\nabla_{[a} \nabla_{b]} = \varepsilon_{A'B'} \square_{AB} + \varepsilon_{AB} \square_{A'B'}, \tag{6.3.13}$$

a lengthy calculation of the 16 terms occurring in (6.3.12) yields

$$\begin{aligned} \kappa^{(C} \Big[\varepsilon_{A'B'} \square_{AB} + \varepsilon_{AB} \square_{A'B'} - 2S_{AA'BB'}{}^{HH'}\, \nabla_{HH'} \Big] \kappa^{D)} & \\ = \varepsilon_{AB} \Big[\widetilde{\Phi}_{A'B'E}{}^{(C} \kappa^{D)} \kappa^{E} + \widetilde{\Sigma}_{A'B'} \kappa^{(C} \kappa^{D)} \Big] & \\ + \varepsilon_{A'B'} \Big[\psi_{ABE}{}^{(C} \kappa^{D)} \kappa^{E} & \\ - 2\Lambda \kappa^{(C} \kappa_{(B} \varepsilon_{A)}{}^{D)} + \Sigma_{AB} \kappa^{(C} \kappa^{D)} \Big]. & \end{aligned} \tag{6.3.14}$$

We now write explicitly the symmetrizations over C and D occurring in (6.3.14). Thus, using (6.2.4) and comparing left- and right-hand side of (6.3.14), one finds the equations

$$\Big[\square_{AB} - 2\chi_{AB}{}^{HH'} \nabla_{HH'} \Big] \kappa^{C} = \psi_{ABE}{}^{C}\, \kappa^{E} - 2\Lambda \kappa_{(A} \varepsilon_{B)}{}^{C} + \Sigma_{AB}\, \kappa^{C}, \tag{6.3.15}$$

$$\Big[\square_{A'B'} - 2\widetilde{\chi}_{A'B'}{}^{HH'} \nabla_{HH'} \Big] \kappa^{C} = \widetilde{\Phi}_{A'B'E}{}^{C}\, \kappa^{E} + \widetilde{\Sigma}_{A'B'}\, \kappa^{C}. \tag{6.3.16}$$

Equations (6.3.15) and (6.3.16) are two of the four spinor Ricci identities for CU_4 or RU_4 theories. The remaining spinor Ricci identities are

$$\Big[\square_{A'B'} - 2\widetilde{\chi}_{A'B'}{}^{HH'} \nabla_{HH'} \Big] \pi^{C'} = \widetilde{\psi}_{A'B'E'}{}^{C'}\, \pi^{E'} - 2\widetilde{\Lambda} \pi_{(A'} \varepsilon_{B')}{}^{C'} + \widetilde{\Sigma}_{A'B'}\, \pi^{C'}, \tag{6.3.17}$$

$$\Big[\Box_{AB} - 2\chi_{AB}{}^{HH'}\nabla_{HH'}\Big]\pi^{C'} = \Phi_{ABE'}{}^{C'}\,\pi^{E'} + \Sigma_{AB}\,\pi^{C'}. \qquad (6.3.18)$$

6.4 Integrability condition for α-surfaces

Since $\pi^{A'}\pi_{A'} = 0$, insertion of (6.3.17) into (6.2.14) and careful use of Eq. (6.2.10) yield

$$-\pi^{A'}\pi^{B'}\pi^{C'}\Big(\Box_{C'A'}\pi_{B'}\Big) = -\pi^{A'}\pi^{B'}\pi^{C'}\pi^{D'}\widetilde{\psi}_{A'B'C'D'} + 4\pi^{B'}\pi^{C'}\pi_{F'}\,\pi_{G'}\,\widetilde{\chi}_{A'B'AC'}\,\widetilde{\chi}^{F'G'AA'}. \qquad (6.4.1)$$

In the light of (6.2.11)–(6.2.15) and (6.4.1), one thus finds the following integrability condition for Eq. (6.2.10) in the case of CU_4 or RU_4 theories (Esposito 1993):

$$\widetilde{\psi}_{A'B'C'D'} = -4\widetilde{\chi}_{A'B'AL'}\,\widetilde{\chi}_{C'}{}^{L'A}{}_{D'} + 4\widetilde{\chi}_{L'B'AC'}\,\widetilde{\chi}_{A'D'}{}^{AL'} + 2\nabla^{A}{}_{D'}\Big(\widetilde{\chi}_{A'B'AC'}\Big). \qquad (6.4.2)$$

Note that contributions involving ξ^A add up to zero.

6.5 Concluding remarks

We have studied complex or real Riemannian space-times with non-vanishing torsion. By analogy with complex general relativity, α-surfaces have been defined as totally null two-surfaces whose null tangent vectors have the two-component spinor form $\lambda^A\pi^{A'}$, with λ^A varying and $\pi^{A'}$ fixed (cf. section 6.1, Ward and Wells 1990). Using Frobenius' theorem, this leads to Eq. (6.2.10), which differs from the equation corresponding to general relativity by the term involving the torsion spinor. The integrability condition for Eq. (6.2.10) is then given by Eq.

(6.4.2), which involves the self-dual Weyl spinor (as in complex general relativity), terms quadratic in the torsion spinor, and the covariant derivative of the torsion spinor. Our results (6.2.10) and (6.4.2) are quite generic, in that they do not make use of any field equations. We only assumed we were not studying supersymmetric theories of gravity.

A naturally occurring question is whether an alternative way exists to derive our results (6.2.10) and (6.4.2). This is indeed possible, since in terms of the Levi–Civita connection the necessary and sufficient condition for the existence of α-surfaces is the vanishing of the self-dual torsion-free Weyl spinor; one has then to translate this condition into a property of the Weyl spinor and torsion of the full U_4-connection. One then finds that the integrability condition for α-surfaces, at first expressed using the self-dual Weyl spinor of the Levi–Civita connection, coincides with Eq. (6.4.2).

We believe, however, that the more fundamental geometric object is the full U_4-connection with torsion. This point of view is especially relevant when one studies the Hamiltonian form of these theories, and is along the lines of previous work by the author, where other properties of U_4-theories have been studied working with the complete U_4-connection (Esposito 1992, Esposito 1994). It was thus our aim to derive Eq. (6.4.2) in a way independent of the use of formulae relating curvature spinors of the Levi–Civita connection to torsion and curvature spinors of the U_4-connection. We hope our chapter shows that this program can be consistently developed.

Interestingly, a *particular* solution of Eq. (6.4.2) is given by

$$\widetilde{\psi}_{A'B'C'D'} = 0, \tag{6.5.1}$$

$$\widetilde{\chi}_{A'C'AB'} = 0. \tag{6.5.2}$$

This means that the surviving part of torsion is ${\chi_{AB}}^{CC'} \varepsilon_{A'B'}$ (cf. (6.2.4)), which does not affect the integrability condition for α-surfaces, and that the U_4 Weyl curvature is anti-self-dual. Note that this is only possible for CU_4 and RU_4 models of gravity, since only for these theories Eqs. (6.5.1) and (6.5.2) do not imply the vanishing of $\chi_{ACBA'}$ and ψ_{ABCD} (cf. section 6.3). By analogy with complex general relativity, those particular CU_4 and RU_4 space-times satisfying Eqs. (6.5.1) and (6.5.2) are here called right conformally flat (in the light of Eq. (6.5.1)) and *right-torsion-free* (in the light of Eq. (6.5.2)). Note that our definition does not involve the Ricci tensor, and is therefore different from Eq. (6.2.1) of Ward and Wells (1990) (see (4.2.2)).

CHAPTER SEVEN

SPIN-$\frac{1}{2}$ FIELDS IN RIEMANNIAN GEOMETRIES

Local supersymmetry leads to boundary conditions for fermionic fields in one-loop quantum cosmology involving the Euclidean normal ${ }_{e}n_{A}^{\ \ A'}$ to the boundary and a pair of independent spinor fields ψ^{A} and $\widetilde{\psi}^{A'}$. This chapter studies the corresponding classical properties, i.e. the classical boundary-value problem and boundary terms in the variational problem. If $\sqrt{2}\ { }_{e}n_{A}^{\ \ A'}\ \psi^{A} \mp \widetilde{\psi}^{A'} \equiv \Phi^{A'}$ is set to zero on a three-sphere bounding flat Euclidean four-space, the modes of the massless spin-$\frac{1}{2}$ field multiplying harmonics having positive eigenvalues for the intrinsic three-dimensional Dirac operator on S^{3} should vanish on S^{3}. Remarkably, this coincides with the property of the classical boundary-value problem when spectral boundary conditions are imposed on S^{3} in the massless case. Moreover, the boundary term in the action functional is proportional to the integral on the boundary of $\Phi^{A'}\ { }_{e}n_{AA'}\ \psi^{A}$. The existence of self-adjoint extensions of the Dirac operator subject to supersymmetric boundary conditions is then proved. The global theory of the Dirac operator in compact Riemannian manifolds is eventually described.

7.1 Dirac and Weyl equations in two-component spinor form

Dirac's theory of massive and massless spin-$\frac{1}{2}$ particles is still a key element of modern particle physics and field theory. From the point of view of theoretical physics, the description of such particles motivates indeed the whole theory of Dirac operators. We are here concerned with a two-component spinor analysis of the corresponding spin-$\frac{1}{2}$ fields in Riemannian four-geometries (M, g) with boundary. A massive spin-$\frac{1}{2}$ Dirac field is then described by the four independent spinor fields $\phi^A, \chi^A, \widetilde{\phi}^{A'}, \widetilde{\chi}^{A'}$, and the action functional takes the form

$$I \equiv I_V + I_B, \tag{7.1.1}$$

where

$$\begin{aligned} I_V \equiv & \frac{i}{2} \int_M \Big[\widetilde{\phi}^{A'} \Big(\nabla_{AA'} \, \phi^A \Big) - \Big(\nabla_{AA'} \, \widetilde{\phi}^{A'} \Big) \phi^A \Big] \sqrt{\det g} \; d^4x \\ & + \frac{i}{2} \int_M \Big[\widetilde{\chi}^{A'} \Big(\nabla_{AA'} \, \chi^A \Big) - \Big(\nabla_{AA'} \, \widetilde{\chi}^{A'} \Big) \chi^A \Big] \sqrt{\det g} \; d^4x \\ & + \frac{m}{\sqrt{2}} \int_M \Big[\chi_A \phi^A + \widetilde{\phi}^{A'} \widetilde{\chi}_{A'} \Big] \sqrt{\det g} \; d^4x, \end{aligned} \tag{7.1.2}$$

and I_B is a suitable boundary term, necessary to obtain a well posed variational problem. Its form is determined once one knows which spinor fields are fixed on the boundary (e.g. section 7.2). With our notation, the occurrence of i depends on conventions for Infeld–van der Waerden symbols (see section 7.2). One thus finds the field equations

$$\nabla_{AA'} \, \phi^A = \frac{im}{\sqrt{2}} \, \widetilde{\chi}_{A'}, \tag{7.1.3}$$

$$\nabla_{AA'} \, \chi^A = \frac{im}{\sqrt{2}} \, \widetilde{\phi}_{A'}, \tag{7.1.4}$$

$$\nabla_{AA'} \, \widetilde{\phi}^{A'} = -\frac{im}{\sqrt{2}} \, \chi_A, \tag{7.1.5}$$

$$\nabla_{AA'} \, \widetilde{\chi}^{A'} = -\frac{im}{\sqrt{2}} \, \phi_A. \tag{7.1.6}$$

Note that this is a coupled system of first-order differential equations, obtained after applying differentiation rules for anti-commuting spinor fields. This means the spinor field acted upon by the $\nabla_{AA'}$ operator should be always brought to the left, hence leading to a minus sign if such a field was not already on the left. Integration by parts and careful use of boundary terms are also necessary. The equations (7.1.3)–(7.1.6) reproduce the familiar form of the Dirac equation expressed in terms of γ-matrices. In particular, for massless fermionic fields one obtains the independent Weyl equations

$$\nabla^{AA'} \, \phi_A = 0, \tag{7.1.7}$$

$$\nabla^{AA'} \, \widetilde{\phi}_{A'} = 0, \tag{7.1.8}$$

not related by any conjugation.

7.2 Boundary terms for massless fermionic fields

Locally supersymmetric boundary conditions have been recently studied in quantum cosmology to understand its one-loop properties. They involve the normal to the boundary and the field for spin $\frac{1}{2}$, the normal to the boundary and the spin-$\frac{3}{2}$ potential for gravitinos, Dirichlet conditions for real scalar fields, magnetic or electric field for electromagnetism, mixed boundary conditions for the four-metric of the gravitational field (and in particular Dirichlet conditions on the perturbed three-metric). The aim of this section is to describe the corresponding classical properties in the case of massless spin-$\frac{1}{2}$ fields.

For this purpose, we consider flat Euclidean four-space bounded by a three-sphere of radius a. The alternative possibility is a more involved boundary-value problem, where flat Euclidean four-space is bounded by two concentric three-spheres of radii r_1 and r_2. The spin-$\frac{1}{2}$ field, represented by a pair of independent spinor fields ψ^A and $\widetilde{\psi}^{A'}$, is expanded on a family of three-spheres centred on the origin as (D'Eath and Halliwell 1987, D'Eath and Esposito 1991a, Esposito 1994)

$$\psi^A = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq} \Big[m_{np}(\tau)\rho^{nqA} + \widetilde{r}_{np}(\tau)\overline{\sigma}^{nqA} \Big], \tag{7.2.1}$$

$$\widetilde{\psi}^{A'} = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq} \Big[\widetilde{m}_{np}(\tau)\overline{\rho}^{nqA'} + r_{np}(\tau)\sigma^{nqA'} \Big]. \tag{7.2.2}$$

With our notation, τ is the Euclidean-time coordinate, the α_n^{pq} are block-diagonal matrices with blocks $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, the $\rho-$ and σ-harmonics obey the identities described in D'Eath and Halliwell (1987), Esposito (1994). Last but not least, the modes m_{np} and r_{np} are regular at $\tau = 0$, whereas the modes $\widetilde{m}_{np}$ and $\widetilde{r}_{np}$ are singular at $\tau = 0$ if the spin-$\frac{1}{2}$ field is massless. Bearing in mind that the harmonics ρ^{nqA} and $\sigma^{nqA'}$ have positive eigenvalues $\frac{1}{2}\Big(n + \frac{3}{2}\Big)$ for the three-dimensional Dirac operator on the bounding S^3 (Esposito 1994), the decomposition (7.2.1) and (7.2.2) can be re-expressed as

$$\psi^A = \psi^A_{(+)} + \psi^A_{(-)}, \tag{7.2.3}$$

$$\widetilde{\psi}^{A'} = \widetilde{\psi}^{A'}_{(+)} + \widetilde{\psi}^{A'}_{(-)}. \tag{7.2.4}$$

In (7.2.3) and (7.2.4), the $(+)$ parts correspond to the modes m_{np} and r_{np}, whereas the $(-)$ parts correspond to the singular modes $\widetilde{m}_{np}$ and $\widetilde{r}_{np}$, which multiply harmonics having negative eigenvalues $-\frac{1}{2}\Big(n+\frac{3}{2}\Big)$ for the three-dimensional Dirac operator on S^3. If one wants to find a classical solution of the Weyl equation which is regular $\forall \tau \in [0, a]$, one is thus forced to set to zero the modes $\widetilde{m}_{np}$ and $\widetilde{r}_{np}$ $\forall \tau \in [0, a]$ (D'Eath and Halliwell 1987). This is why, if one requires the local boundary conditions (Esposito 1994)

$$\sqrt{2}\; {}_e n_A{}^{A'} \; \psi^A \mp \widetilde{\psi}^{A'} = \Phi^{A'} \text{ on } S^3, \tag{7.2.5}$$

such a condition can be expressed as

$$\sqrt{2}\ {}_e n_A^{\ \ A'}\ \psi_{(+)}^A = \Phi_1^{A'} \text{ on } S^3, \tag{7.2.6}$$

$$\mp \widetilde{\psi}_{(+)}^{A'} = \Phi_2^{A'} \text{ on } S^3, \tag{7.2.7}$$

where $\Phi_1^{A'}$ and $\Phi_2^{A'}$ are the parts of the spinor field $\Phi^{A'}$ related to the $\overline{\rho}$- and σ-harmonics, respectively. In particular, if $\Phi_1^{A'} = \Phi_2^{A'} = 0$ on S^3, one finds

$$\sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq}\ m_{np}(a)\ {}_e n_A^{\ \ A'}\ \rho_{nq}^A = 0, \tag{7.2.8}$$

$$\sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq}\ r_{np}(a)\ \sigma_{nq}^{A'} = 0, \tag{7.2.9}$$

where a is the three-sphere radius. Since the harmonics appearing in (7.2.8) and (7.2.9) are linearly independent, these relations lead to $m_{np}(a) = r_{np}(a) = 0$ $\forall n, p$. Remarkably, this simple calculation shows that the classical boundary-value problems for regular solutions of the Weyl equation subject to local or spectral conditions on S^3 share the same property provided that $\Phi^{A'}$ is set to zero in (7.2.5): the regular modes m_{np} and r_{np} should vanish on the bounding S^3.

To study the corresponding variational problem for a massless fermionic field, we should now bear in mind that the spin-$\frac{1}{2}$ action functional in a Riemannian four-geometry takes the form (D'Eath and Esposito 1991a, Esposito 1994)

$$I_E \equiv \frac{i}{2} \int_M \left[\widetilde{\psi}^{A'} \left(\nabla_{AA'}\ \psi^A \right) - \left(\nabla_{AA'}\ \widetilde{\psi}^{A'} \right) \psi^A \right] \sqrt{\det g}\ d^4x + \widehat{I}_B. \tag{7.2.10}$$

This action is *real*, and the factor i occurs by virtue of the convention for Infeld–van der Waerden symbols used in D'Eath and Esposito (1991a), Esposito (1994). In (7.2.10) $\widehat{I}_B$ is a suitable boundary term, to be added to ensure that I_E is stationary under the boundary conditions chosen at the various components of the boundary (e.g. initial and final surfaces, as in D'Eath and Halliwell (1987)). Of course, the variation δI_E of I_E is linear in the variations $\delta\psi^A$ and $\delta\widetilde{\psi}^{A'}$. Defining $\kappa \equiv \frac{2}{i}$ and $\kappa\widehat{I}_B \equiv I_B$, variational rules for anticommuting spinor fields lead to

$$\kappa\Bigl(\delta I_E\Bigr) = \int_M \Bigl[2\delta\widetilde{\psi}^{A'}\Bigl(\nabla_{AA'}\psi^A\Bigr)\Bigr]\sqrt{\det g}\; d^4x - \int_M \Bigl[\Bigl(\nabla_{AA'}\widetilde{\psi}^{A'}\Bigr)2\delta\psi^A\Bigr]\sqrt{\det g}\; d^4x$$
$$- \int_{\partial M}\Bigl[{}_e n_{AA'}\Bigl(\delta\widetilde{\psi}^{A'}\Bigr)\psi^A\Bigr]\sqrt{\det h}\; d^3x + \int_{\partial M}\Bigl[{}_e n_{AA'}\widetilde{\psi}^{A'}\Bigl(\delta\psi^A\Bigr)\Bigr]\sqrt{\det h}\; d^3x$$
$$+ \delta I_B, \tag{7.2.11}$$

where I_B should be chosen in such a way that its variation δI_B combines with the sum of the two terms on the second line of (7.2.11) so as to specify what is fixed on the boundary (see below). Indeed, setting $\epsilon \equiv \pm 1$ and using the boundary conditions (7.2.5) one finds

$${}_e n_{AA'}\;\widetilde{\psi}^{A'} = \frac{\epsilon}{\sqrt{2}}\psi_A - \epsilon\;{}_e n_{AA'}\;\Phi^{A'} \text{ on } S^3. \tag{7.2.12}$$

Thus, anticommutation rules for spinor fields (D'Eath and Halliwell 1987) show that the second line of Eq. (7.2.11) reads

$$\delta I_{\partial M} \equiv -\int_{\partial M}\Bigl[\Bigl(\delta\widetilde{\psi}^{A'}\Bigr){}_e n_{AA'}\psi^A\Bigr]\sqrt{\det h}\; d^3x + \int_{\partial M}\Bigl[{}_e n_{AA'}\widetilde{\psi}^{A'}\Bigl(\delta\psi^A\Bigr)\Bigr]\sqrt{\det h}\; d^3x$$
$$= \epsilon\int_{\partial M} {}_e n_{AA'}\Bigl[\Bigl(\delta\Phi^{A'}\Bigr)\psi^A - \Phi^{A'}\Bigl(\delta\psi^A\Bigr)\Bigr]\sqrt{\det h}\; d^3x. \tag{7.2.13}$$

Now it is clear that setting

$$I_B \equiv \epsilon\int_{\partial M} \Phi^{A'}\,{}_e n_{AA'}\;\psi^A\sqrt{\det h}\; d^3x, \tag{7.2.14}$$

enables one to specify $\Phi^{A'}$ on the boundary, since

$$\delta\Bigl[I_{\partial M} + I_B\Bigr] = 2\epsilon\int_{\partial M} {}_e n_{AA'}\Bigl(\delta\Phi^{A'}\Bigr)\psi^A\sqrt{\det h}\; d^3x. \tag{7.2.15}$$

Hence the action integral (7.2.10) appropriate for our boundary-value problem is (Esposito *et al.* 1994)

$$I_E = \frac{i}{2} \int_M \left[\widetilde{\psi}^{A'} \left(\nabla_{AA'} \, \psi^A \right) - \left(\nabla_{AA'} \, \widetilde{\psi}^{A'} \right) \psi^A \right] \sqrt{\det g} \, d^4x$$

$$+ \frac{i\epsilon}{2} \int_{\partial M} \Phi^{A'} \; {}_e n_{AA'} \, \psi^A \sqrt{\det h} \, d^3x. \tag{7.2.16}$$

Note that, by virtue of (7.2.5), Eq. (7.2.13) may also be cast in the form

$$\delta I_{\partial M} = \frac{1}{\sqrt{2}} \int_{\partial M} \left[\widetilde{\psi}^{A'} \left(\delta \Phi_{A'} \right) - \left(\delta \widetilde{\psi}^{A'} \right) \Phi_{A'} \right] \sqrt{\det h} \, d^3x, \tag{7.2.17}$$

which implies that an equivalent form of I_B is

$$I_B \equiv \frac{1}{\sqrt{2}} \int_{\partial M} \widetilde{\psi}^{A'} \; \Phi_{A'} \sqrt{\det h} \, d^3x. \tag{7.2.18}$$

The local boundary conditions studied at the classical level in this section, have been applied to one-loop quantum cosmology in D'Eath and Esposito (1991a), Kamenshchik and Mishakov (1993), Esposito (1994). Interestingly, our work seems to add evidence in favour of quantum amplitudes having to respect the properties of the classical boundary-value problem. In other words, if fermionic fields are massless, their one-loop properties in the presence of boundaries coincide in the case of spectral (D'Eath and Halliwell 1987, D'Eath and Esposito 1991b, Esposito 1994) or local boundary conditions (D'Eath and Esposito 1991a, Kamenshchik and Mishakov 1993, Esposito 1994), while we find that classical modes for a regular solution of the Weyl equation obey the same conditions on a three-sphere boundary with spectral or local boundary conditions, provided that the spinor field $\Phi^{A'}$ of (7.2.5) is set to zero on S^3. We also hope that the analysis presented in Eqs. (7.2.10)–(7.2.18) may clarify the spin-$\frac{1}{2}$ variational problem in the case of local boundary conditions on a three-sphere (cf. the analysis in Charap and Nelson (1983), York (1986), Hayward (1993) for pure gravity).

7.3 Self-adjointness of the boundary-value problem

So far we have seen that the framework for the formulation of local boundary conditions involving normals and field strengths or fields is the Euclidean regime, where one deals with Riemannian metrics. Thus, we will pay special attention to the conjugation of $SU(2)$ spinors in Euclidean four-space. In fact such a conjugation will play a key role in proving self-adjointness. For this purpose, it can be useful to recall at first some basic results about $SU(2)$ spinors on an abstract Riemannian three-manifold (Σ, h). In that case, one considers a bundle over the three-manifold, each fibre of which is isomorphic to a two-dimensional complex vector space W. It is then possible to define a nowhere vanishing antisymmetric ε_{AB} (the usual one of section 2.1) so as to raise and lower internal indices, and a positive-definite Hermitian inner product on each fibre: $(\psi, \phi) = \overline{\psi}^{A'} G_{A'A} \phi^A$. The requirements of Hermiticity and positivity imply respectively that $\overline{G}_{A'A} = G_{A'A}$, $\overline{\psi}^{A'} G_{A'A} \psi^A > 0, \forall \; \psi^A \neq 0$. This $G_{A'A}$ converts primed indices to unprimed ones, and it is given by $i\sqrt{2} \; n_{AA'}$. Given the space H of all objects $\alpha^A{}_B$ such that $\alpha^A{}_A = 0$ and $\left(\alpha^\dagger\right)^A{}_B = -\alpha^A{}_B$, one finds there always exists a $SU(2)$ soldering form $\sigma^a{}_A{}^B$ (i.e. a global isomorphism) between H and the tangent space on (Σ, h) such that $h^{ab} = -\sigma^a{}_A{}^B \; \sigma^b{}_B{}^A$. Therefore one also finds $\sigma^a{}_A{}^A = 0$ and $\left(\sigma^a{}_A{}^B\right)^\dagger = -\sigma^a{}_A{}^B$. One then defines ψ^A an $SU(2)$ spinor on (Σ, h). A basic remark is that $SU(2)$ transformations are those $SL(2, C)$ transformations which preserve $n^{AA'} = n^a \sigma_a{}^{AA'}$, where $n^a = (1, 0, 0, 0)$ is the normal to Σ. The Euclidean conjugation used here (not to be confused with complex conjugation in Minkowski space-time) is such that (see now section 2.1)

$$(\psi_A + \lambda \phi_A)^\dagger = \psi_A{}^\dagger + \lambda^* \phi_A{}^\dagger, \; \left(\psi_A{}^\dagger\right)^\dagger = -\psi_A, \tag{7.3.1}$$

$$\varepsilon_{AB}{}^\dagger = \varepsilon_{AB}, \; (\psi_A \; \phi_B)^\dagger = \psi_A{}^\dagger \; \phi_B{}^\dagger, \tag{7.3.2}$$

$$(\psi_A)^\dagger \psi^A > 0, \; \forall \; \psi_A \neq 0. \tag{7.3.3}$$

In (7.3.1) and in the following pages, the symbol $*$ denotes complex conjugation of scalars. How to generalize this picture to Euclidean four-space? For this purpose, let us now focus our attention on states that are pairs of spinor fields, defining

$$w \equiv \left(\psi^A, \, \widetilde{\psi}^{A'}\right), \; z \equiv \left(\phi^A, \, \widetilde{\phi}^{A'}\right), \tag{7.3.4}$$

on the ball of radius a in Euclidean four-space, subject always to the boundary conditions (7.2.5). Our w and z are subject also to suitable differentiability conditions, to be specified later. Let us also define the operator C

$$C \, : \left(\psi^A, \, \widetilde{\psi}^{A'}\right) \rightarrow \left(\nabla^A{}_{B'} \, \widetilde{\psi}^{B'}, \nabla_B{}^{A'} \, \psi^B\right), \tag{7.3.5}$$

and the *dagger* operation

$$\left(\psi^A\right)^\dagger \equiv \varepsilon^{AB} \; \delta_{BA'} \; \overline{\psi}^{A'}, \; \left(\widetilde{\psi}^{A'}\right)^\dagger \equiv \varepsilon^{A'B'} \; \delta_{B'A} \; \overline{\widetilde{\psi}}^A . \tag{7.3.6}$$

The consideration of C is suggested of course by the action (7.2.10). In (7.3.6), $\delta_{BA'}$ is an identity matrix playing the same role of $G_{AA'}$ for $SU(2)$ spinors on (Σ, h), so that $\delta_{BA'}$ is preserved by $SU(2)$ transformations. Moreover, the *bar* symbol $\overline{\psi^A} = \overline{\psi}^{A'}$ denotes the usual complex conjugation of $SL(2, C)$ spinors. Hence one finds

$$\left(\left(\psi^A\right)^\dagger\right)^\dagger = \varepsilon^{AC} \; \delta_{CB'} \; \overline{\left(\psi^{B\dagger}\right)'} = \varepsilon^{AC} \; \delta_{CB'} \; \varepsilon^{B'D'} \; \delta_{D'F} \; \psi^F = -\psi^A, \tag{7.3.7}$$

in view of the definition of ε^{AB}. Thus, the *dagger* operation defined in (7.3.6) is anti-involutory, because, when applied twice to ψ^A, it yields $-\psi^A$.

From now on we study commuting spinors, for simplicity of exposition of the self-adjointness. It is easy to check that the *dagger*, also called in the literature Euclidean conjugation (section 2.1), satisfies all properties (7.3.1)–(7.3.3). We can now define the scalar product

$$(w, z) \equiv \int_M \left[\psi^\dagger_A \phi^A + \widetilde{\psi}^\dagger_{A'} \widetilde{\phi}^{A'}\right] \sqrt{g} \; d^4x. \tag{7.3.8}$$

This is indeed a scalar product, because it satisfies all following properties for all vectors u, v, w and $\forall \lambda \in \; C$:

$$(u,u) > 0, \; \forall u \neq 0, \tag{7.3.9}$$

$$(u, v+w) = (u,v) + (u,w), \tag{7.3.10}$$

$$(u, \lambda v) = \lambda (u,v), \; (\lambda u, v) = \lambda^* (u,v), \tag{7.3.11}$$

$$(v,u) = (u,v)^*. \tag{7.3.12}$$

We are now aiming to check that C or iC is a symmetric operator, i.e. that $(Cz, w) = (z, Cw)$ or $(iCz, w) = (z, iCw)\,, \forall z, w$. This will be used in the course of proving further that the symmetric operator has self-adjoint extensions. In order to prove this result it is clear, in view of (7.3.8), we need to know the properties of the spinor covariant derivative acting on $SU(2)$ spinors. In the case of $SL(2,C)$ spinors this derivative is a linear, torsion-free map $\nabla_{AA'}$ which satisfies the Leibniz rule, annihilates ε_{AB} and is real (i.e. $\psi = \nabla_{AA'}\theta \Rightarrow \overline{\psi} = \nabla_{AA'}\overline{\theta}$). Moreover, we know that

$$\nabla^{AA'} = e^{AA'}{}_{\mu} \, \nabla^{\mu} = e^{a}{}_{\mu} \, \sigma_{a}{}^{AA'} \, \nabla^{\mu}. \tag{7.3.13}$$

In Euclidean four-space, we use both (7.3.13) and the relation

$$\sigma_{\mu AC'} \, \sigma_{\nu B}{}^{C'} + \sigma_{\nu BC'} \, \sigma_{\mu A}{}^{C'} = \delta_{\mu\nu} \varepsilon_{AB}, \tag{7.3.14}$$

where $\delta_{\mu\nu}$ has signature $(+,+,+,+)$. This implies that $\sigma_0 = -\frac{i}{\sqrt{2}} I$, $\sigma_i = \frac{\Sigma_i}{\sqrt{2}}$, $\forall i = 1,2,3$, where Σ_i are the Pauli matrices. Now, in view of (7.3.5) and (7.3.8) one finds

$$(Cz, w) = \int_M \left(\nabla_{AB'} \, \phi^A\right)^{\dagger} \widetilde{\psi}^{B'} \sqrt{g} \; d^4x + \int_M \left(\nabla_{BA'} \, \widetilde{\phi}^{A'}\right)^{\dagger} \psi^B \sqrt{g} \; d^4x, \tag{7.3.15}$$

whereas, using the Leibniz rule to evaluate

$$\nabla^{A}{}_{B'} \left(\phi^{\dagger}_{A} \widetilde{\psi}^{B'}\right)$$

and

$$\nabla_B^{\ A'} \left(\left(\widetilde{\phi}_{A'} \right)^{\dagger} \psi^B \right),$$

and integrating by parts, one finds

$$\begin{aligned}
(z, Cw) = &\int_M \left(\nabla_{AB'} \ \phi^{A\dagger} \right) \widetilde{\psi}^{B'} \sqrt{g} \ d^4x + \int_M \left(\nabla_{BA'} \left(\widetilde{\phi}^{A'} \right)^{\dagger} \right) \psi^B \sqrt{g} \ d^4x \\
&- \int_{\partial M} ({}_e n_{AB'}) \phi^{A\dagger} \ \widetilde{\psi}^{B'} \sqrt{h} \ d^3x \\
&- \int_{\partial M} ({}_e n_{BA'}) \left(\widetilde{\phi}^{A'} \right)^{\dagger} \psi^B \sqrt{h} \ d^3x. \qquad (7.3.16)
\end{aligned}$$

Now we use (7.3.6), section 2.1, the identity

$$\left({}_e n^{AA'} \phi_A \right)^{\dagger} = \varepsilon^{A'B'} \ \delta_{B'C} \ \overline{{}_e n^{DC'}} \ \overline{\phi_D} = -\varepsilon^{A'B'} \ \delta_{B'C} \ \left({}_e n^{CD'} \right) \overline{\phi}_{D'}, \qquad (7.3.17)$$

and the boundary conditions on S^3: $\sqrt{2} \ {}_e n^{CB'} \psi_C = \widetilde{\psi}^{B'}$, $\sqrt{2} \ {}_e n^{AA'} \phi_A = \widetilde{\phi}^{A'}$. In so doing, the sum of the boundary terms in (7.3.16) is found to vanish. This implies in turn that equality of the volume integrands is sufficient to show that (Cz, w) and (z, Cw) are equal. For example, one finds in flat space, using also (7.3.6): $\left(\nabla_{BA'} \widetilde{\phi}^{A'} \right)^{\dagger} = \delta_{BF'} \ \overline{\sigma}^{F'}_{\ \ C}{}^{\ a} \ \partial_a \left(\overline{\widetilde{\phi}}^C \right)$, whereas:

$$\left(\nabla_{BA'} \left(\widetilde{\phi}^{A'} \right)^{\dagger} \right) = -\delta_{CF'} \ \sigma_B^{\ F'a} \ \partial_a \left(\overline{\widetilde{\phi}}^C \right).$$

In other words, we are led to study the condition

$$\delta_{BF'} \ \overline{\sigma}^{F'}_{\ \ C}{}^{\ a} = \pm \delta_{BF'} \ \sigma_C^{\ F'a}, \qquad (7.3.18)$$

$\forall \ a = 0, 1, 2, 3$. Now, using the relations

$$\sqrt{2} \ \sigma_{AA'}^{\ \ \ 0} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \ \sqrt{2} \ \sigma_{AA'}^{\ \ \ 1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (7.3.19)$$

$$\sqrt{2} \ \sigma_{AA'}^{\ \ \ 2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sqrt{2} \ \sigma_{AA'}^{\ \ \ 3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (7.3.20)$$

$$\sigma^{A}{}_{A'}{}^{a} = \varepsilon^{AB}\, \sigma_{BA'}{}^{a},\ \sigma_{A}{}^{A'a} = -\sigma_{AB'}{}^{a}\, \varepsilon^{B'A'}, \tag{7.3.21}$$

one finds that the complex conjugate of $\sigma^{A}{}_{A'}{}^{a}$ is always equal to $\sigma_{A}{}^{A'a}$, which is not in agreement with the choice of the $(-)$ sign on the right-hand side of (7.3.18). This implies in turn that the symmetric operator we are looking for is iC, where C has been defined in (7.3.5). The generalization to a curved four-dimensional Riemannian space is obtained via the relation $e^{AA'}{}_{\mu} = e^{a}{}_{\mu}\, \sigma_{a}{}^{AA'}$.

Now, it is known that every symmetric operator has a closure, and the operator and its closure have the same closed extensions. Moreover, a closed symmetric operator on a Hilbert space is self-adjoint if and only if its spectrum is a subset of the real axis. To prove self-adjointness for our boundary-value problem, we may recall an important result due to von Neumann (Reed and Simon 1975). This theorem states that, given a symmetric operator A with domain $D(A)$, if a map $F : D(A) \rightarrow D(A)$ exists such that

$$F(\alpha w + \beta z) = \alpha^{*} F(w) + \beta^{*} F(z), \tag{7.3.22}$$

$$(w, w) = (Fw, Fw), \tag{7.3.23}$$

$$F^{2} = \pm I, \tag{7.3.24}$$

$$FA = AF, \tag{7.3.25}$$

then A has self-adjoint extensions. In our case, denoting by D the operator (cf. (7.3.6))

$$D : \left(\psi^{A},\ \widetilde{\psi}^{A'}\right) \rightarrow \left(\left(\psi^{A}\right)^{\dagger}, \left(\widetilde{\psi}^{A'}\right)^{\dagger}\right), \tag{7.3.26}$$

let us focus our attention on the operators $F \equiv iD$ and $A \equiv iC$. The operator F maps indeed $D(A)$ into $D(A)$. In fact, bearing in mind the definitions

$$G \equiv \left\{\varphi = \left(\phi^{A},\ \widetilde{\phi}^{A'}\right) : \varphi \text{ is at least } C^{1}\right\}, \tag{7.3.27}$$

$$D(A) \equiv \left\{\varphi \in G : \sqrt{2}\ {}_{e}n^{AA'} \phi_{A} = \epsilon\, \widetilde{\phi}^{A'} \text{ on } S^{3}\right\}, \tag{7.3.28}$$

one finds that F maps $\left(\phi^A, \ \widetilde{\phi}^{A'}\right)$ into $\left(\beta^A, \ \widetilde{\beta}^{A'}\right) = \left(i\left(\phi^A\right)^\dagger, i\left(\widetilde{\phi}^{A'}\right)^\dagger\right)$ with

$$\sqrt{2} \ {}_e n^{AA'} \beta_A = \gamma \ \widetilde{\beta}^{A'} \ \text{on} \ S^3, \tag{7.3.29}$$

where $\gamma = \epsilon^*$. The boundary condition (7.3.29) is clearly of the type which occurs in (7.3.28) provided that ϵ is real, and the differentiability of $\left(\beta^A, \ \widetilde{\beta}^{A'}\right)$ is not affected by the action of F (cf. (7.3.26)). In deriving (7.3.29), we have used the result for $\left({}_e n^{AA'} \phi_A\right)^\dagger$ obtained in (7.3.17). It is worth emphasizing that the requirement of self-adjointness enforces the choice of a real function ϵ, which is a constant in our case. Moreover, in view of (7.3.7), one immediately sees that (7.3.22) and (7.3.24) hold when $F = iD$, if we write (7.3.24) as $F^2 = -I$. This is indeed a crucial point which deserves special attention. Condition (7.3.24) is written in Reed and Simon (1975) as $F^2 = I$, and examples are later given (see page 144 therein) where F is complex conjugation. But we are formulating our problem in the Euclidean regime, where we have seen that the only possible conjugation is the *dagger* operation, which is anti-involutory on spinors with an odd number of indices. Thus, we are here generalizing von Neumann's theorem in the following way. If F is a map $D(A) \rightarrow D(A)$ which satisfies (7.3.22)–(7.3.25), then the same is clearly true of $\widetilde{F} \equiv -iD = -F$. Hence

$$-F \ D(A) \subseteq D(A), \tag{7.3.30}$$

$$F \ D(A) \subseteq D(A). \tag{7.3.31}$$

Acting with F on both sides of (7.3.30), one finds

$$D(A) \subseteq F \ D(A), \tag{7.3.32}$$

using the property $F^2 = -I$. But then the relations (7.3.31) and (7.3.32) imply that $F \ D(A) = D(A)$, so that F takes $D(A)$ into $D(A)$ also in the case of the anti-involutory Euclidean conjugation that we called *dagger*. Comparison with the proof presented at the beginning of page 144 in Reed and Simon (1975) shows that this is all what we need so as to generalize von Neumann's theorem to the Dirac operator acting on $SU(2)$ spinors in Euclidean four-space (one later uses properties (7.3.25), (7.3.22) and (7.3.23) as well to complete the proof).

It remains to verify conditions (7.3.23) and (7.3.25). First, note that

$$\begin{aligned}(Fw, Fw) &= (iDw, iDw) \\ &= \int_M \Bigl(i\,\psi_A^{\dagger}\Bigr)^{\dagger} i\Bigl(\psi^A\Bigr)^{\dagger} \sqrt{g}\, d^4x + \int_M \Bigl(i\,\widetilde{\psi}_{A'}^{\dagger}\Bigr)^{\dagger} i\Bigl(\widetilde{\psi}^{A'}\Bigr)^{\dagger} \sqrt{g}\, d^4x \\ &= (w, w)\,, \end{aligned} \tag{7.3.33}$$

where we have used (7.3.7), (7.3.8) and the commutation property of our spinors. Second, one finds

$$FAw = (iD)\,(iC)\,w = i\Bigl[i\left(\nabla^{A}{}_{B'}\,\widetilde{\psi}^{B'}, \nabla_{B}{}^{A'}\psi^{B}\right)\Bigr]^{\dagger} = \left(\nabla^{A}{}_{B'}\,\widetilde{\psi}^{B'}, \nabla_{B}{}^{A'}\psi^{B}\right)^{\dagger}, \tag{7.3.34}$$

$$AFw = (iC)\,(iD)\,w = iCi\left(\psi^{A\dagger}, \Bigl(\widetilde{\psi}^{A'}\Bigr)^{\dagger}\right) = -\left(\nabla^{A}{}_{B'}\Bigl(\widetilde{\psi}^{B'}\Bigr)^{\dagger}, \nabla_{B}{}^{A'}\psi^{B\dagger}\right), \tag{7.3.35}$$

which in turn implies that also (7.3.25) holds in view of what we found just before (7.3.18) and after (7.3.21). To sum up, we have proved that the operator iC arising in our boundary-value problem is symmetric and has self-adjoint extensions. Hence the eigenvalues of iC are real, and the eigenvalues λ_n of C are purely imaginary. This is what we mean by self-adjointness of our boundary-value problem, although it remains to be seen whether there is a unique self-adjoint extension of our first-order operator.

7.4 Global theory of the Dirac operator

In this chapter and in other sections of our paper there are many applications of the Dirac operator relying on two-component spinor formalism. Hence it appears necessary to describe some general properties of such an operator, frequently studied in theoretical and mathematical physics.

In Riemannian four-geometries, the *total* Dirac operator may be defined as a first-order elliptic operator mapping smooth sections of a complex vector bundle into smooth sections of the same bundle. Its action on the sections (i.e. the spinor fields) is given by composition of Clifford multiplication (see appendix A) with covariant differentiation. To understand these statements, we first summarize the properties of connections on complex vector bundles, and we then introduce the basic properties of spin-structures which enable one to understand how to construct the vector bundle relevant for the theory of the Dirac operator.

A complex vector bundle (e.g. Chern (1979)) is a bundle whose fibres are isomorphic to complex vector spaces. Denoting by E the total space, by M the base space, one has the projection map $\pi : E \rightarrow M$ and the sections $s : M \rightarrow E$ such that the composition of π with s yields the identity on the base space: $\pi \cdot s = \mathrm{id}_M$. The sections s represent the physical fields in our applications. Moreover, denoting by T and T^* the tangent and cotangent bundles of M respectively, a connection ∇ is a map from the space $\Gamma(E)$ of smooth sections of E into the space of smooth sections of the tensor-product bundle $T^* \otimes E$:

$$\nabla : \Gamma(E) \rightarrow \Gamma(T^* \otimes E),$$

such that the following properties hold:

$$\nabla(s_1 + s_2) = \nabla s_1 + \nabla s_2, \tag{7.4.1}$$

$$\nabla(fs) = df \otimes s + f \nabla s, \tag{7.4.2}$$

where $s_1, s_2, s \in \Gamma(E)$ and f is any C^∞ function. The action of the connection ∇ is expressed in terms of the connection matrix θ as

$$\nabla s = \theta \otimes s. \tag{7.4.3}$$

If one takes a section s' related to s by

$$s' = h\ s, \tag{7.4.4}$$

in the light of (7.4.2)–(7.4.4) one finds by comparison that

$$\theta' h = d\, h + h\, \theta. \tag{7.4.5}$$

Moreover, the transformation law of the curvature matrix

$$\Omega \equiv d\theta - \theta \wedge \theta, \tag{7.4.6}$$

is found to be

$$\Omega' = h\, \Omega\, h^{-1}. \tag{7.4.7}$$

We can now introduce spin-structures and the corresponding complex vector bundle acted upon by the total Dirac operator. Let X be a compact oriented differentiable n-dimensional manifold (without boundary) on which a Riemannian metric is introduced. Let Q be the principal tangential $SO(n)$-bundle of X. A spin-structure of X is a principal Spin(n)-bundle P over X together with a covering map $\widetilde{\pi} : P \rightarrow Q$ such that the following commutative structure exists. Given the Cartesian product $P \times \text{Spin}(n)$, one first reaches P by the right action of Spin(n) on P, and one eventually arrives at Q by the projection map $\widetilde{\pi}$. This is equivalent to first reaching the Cartesian product $Q \times SO(n)$ by the map $\widetilde{\pi} \times \rho$, and eventually arriving at Q by the right action of $SO(n)$ on Q. Of course, by ρ we denote the double covering Spin$(n) \rightarrow SO(n)$. In other words, P and Q as above are principal fibre bundles over X, and one has a commutative diagram with $P \times \text{Spin}(n)$ and P on the top, and $Q \times SO(n)$ and Q on the bottom. The projection map from $P \times \text{Spin}(n)$ into $Q \times SO(n)$ is $\widetilde{\pi} \times \rho$, and the projection map from P into Q is $\widetilde{\pi}$. Horizontal arrows should be drawn to denote the right action of Spin(n) on P on the top, and of $SO(n)$ on Q on the bottom.

The group Spin(n) has a complex representation space Σ of dimension 2^n called the spin-representation. If $G \in \text{Spin}(n), x \in R^n, u \in \Sigma$, one has therefore

$$G(xu) = GxG^{-1} \cdot G(u) = \rho(G)x \cdot G(u), \tag{7.4.8}$$

where $\rho : \text{Spin}(n) \rightarrow SO(n)$ is the covering map as we said before. If X is even-dimensional, i.e. $n = 2l$, the representation is the direct sum of two irreducible representations $\Sigma^{\pm}$ of dimension 2^{n-1}. If X is a Spin$(2l)$ manifold with principal bundle P, we can form the associated complex vector bundles

$$E^{+} \equiv P \times \Sigma^{+}, \tag{7.4.9a}$$

$$E^{-} \equiv P \times \Sigma^{-}, \tag{7.4.9b}$$

$$E \equiv E^{+} \oplus E^{-}. \tag{7.4.10}$$

Sections of these vector bundles are spinor fields on X.

The total Dirac operator is a first-order elliptic differential operator D : $\Gamma(E) \rightarrow \Gamma(E)$ defined as follows. Recall first that the Riemannian metric defines a natural $SO(2l)$ connection, and this may be used to give a connection for P. One may therefore consider the connection ∇ at the beginning of this section, i.e. a linear map from $\Gamma(E)$ into $\Gamma(T^{*} \otimes E)$. On the other hand, the tangent and cotangent bundles of X are isomorphic, and one has the map $\Gamma(T \otimes E) \rightarrow \Gamma(E)$ induced by *Clifford multiplication* (see Ward and Wells (1990) and our appendix A on Clifford algebras and Clifford multiplication). The total Dirac operator D is defined to be the *composition* of these two maps. Thus, in terms of an orthonormal base e_i of T, one has *locally*

$$Ds = \sum_{i} e_i(\nabla_i s), \tag{7.4.11}$$

where $\nabla_i s$ is the covariant derivative of $s \in \Gamma(E)$ in the direction e_i, and $e_i(\ )$ denotes Clifford multiplication (cf. (7.3.13)). Moreover, the total Dirac operator D induces two operators

$$D^{+} : \Gamma(E^{+}) \rightarrow \Gamma(E^{-}), \tag{7.4.12}$$

$$D^{-} : \Gamma(E^{-}) \rightarrow \Gamma(E^{+}), \tag{7.4.13}$$

each of which is elliptic. It should be emphasized that ellipticity of the total and partial Dirac operators only holds in Riemannian manifolds, whereas it does not apply to the Lorentzian manifolds of general relativity and of the original

Dirac theory of spin-$\frac{1}{2}$ particles. This description of the Dirac operator should be compared with the mathematical structures presented in section 2.1.

CHAPTER EIGHT

SPIN-$\frac{3}{2}$ POTENTIALS

Local boundary conditions involving field strengths and the normal to the boundary, originally studied in anti-de Sitter space-time, have been considered in one-loop quantum cosmology. This chapter derives the conditions under which spin-lowering and spin-raising operators preserve these local boundary conditions on a three-sphere for fields of spin $0, \frac{1}{2}, 1, \frac{3}{2}$ and 2. Moreover, the two-component spinor analysis of the four potentials of the totally symmetric and independent field strengths for spin $\frac{3}{2}$ is applied to the case of a three-sphere boundary. It is shown that such boundary conditions can only be imposed in a flat Euclidean background, for which the gauge freedom in the choice of the massless potentials remains.

The second part of the chapter studies the two-spinor form of the Rarita–Schwinger potentials subject to local boundary conditions compatible with local supersymmetry. The massless Rarita–Schwinger field equations are studied in four-real-dimensional Riemannian backgrounds with boundary. Gauge transformations on the potentials are shown to be compatible with the field equations provided that the background is Ricci-flat, in agreement with previous results in the literature. However, the preservation of boundary conditions under such gauge transformations leads to a restriction of the gauge freedom. The construction by Penrose of a second set of potentials which supplement the Rarita–Schwinger potentials is then applied. The equations for these potentials, jointly with the boundary conditions, imply that the background four-geometry is further restricted to be totally flat. The analysis of other gauge transformations confirms that, in the massless case, the only admissible class of Riemannian backgrounds with boundary is totally flat.

In the third part of the chapter, the two-component spinor form of massive spin-$\frac{3}{2}$ potentials in conformally flat Einstein four-manifolds is studied. Following earlier work in the literature, a non-vanishing cosmological constant makes it necessary to introduce a supercovariant derivative operator. The analysis of supergauge transformations of potentials for spin $\frac{3}{2}$ shows that the gauge freedom for massive spin-$\frac{3}{2}$ potentials is generated by solutions of the supertwistor equations. The supercovariant form of a partial connection on a non-linear bundle is then obtained, and the basic equation obeyed by the second set of potentials in the massive case is shown to be the integrability condition on super β-surfaces of a differential operator on a vector bundle of rank three.

8.1 Introduction

Much work in the literature has studied the quantization of gauge theories and supersymmetric field theories in the presence of boundaries, with application to one-loop quantum cosmology (Moss and Poletti 1990, Poletti 1990, D'Eath and Esposito 1991a,b, Barvinsky *et al.* 1992a,b, Kamenshchik and Mishakov 1992, 1993, 1994, Esposito 1994). In particular, in the work described in Esposito (1994), two possible sets of local boundary conditions were studied. One of these, first proposed in anti-de Sitter space-time (Breitenlohner and Freedman 1982, Hawking 1983), involves the normal to the boundary and Dirichlet or Neumann conditions for spin 0, the normal and the field for massless spin-$\frac{1}{2}$ fermions, and the normal and totally symmetric field strengths for spins 1, $\frac{3}{2}$ and 2. Although more attention has been paid to alternative local boundary conditions motivated by supersymmetry (Poletti 1990, D'Eath and Esposito 1991a, Kamenshchik and Mishakov 1993-94, Esposito 1994), and studied in our sections 8.5-8.9, the analysis of the former boundary conditions remains of mathematical and physical interest by virtue of its links with twistor theory. The aim of the first part of this chapter is to derive the mathematical properties of the corresponding boundary-value problems, since these are relevant for quantum cosmology and twistor theory.

For this purpose, sections 8.2 and 8.3 derive the conditions under which spin-lowering and spin-raising operators preserve local boundary conditions involving field strengths and normals. Section 8.4 applies the two-spinor form of Dirac spin-$\frac{3}{2}$ potentials to Riemannian four-geometries with a three-sphere boundary. Boundary conditions on spinor-valued one-forms describing gravitino fields are studied in sections 8.5-8.9 for the massless Rarita–Schwinger equations. Massive spin-$\frac{3}{2}$ potentials are instead investigated in sections 8.10–8.15. Concluding remarks and open problems are presented in section 8.16.

8.2 Spin-lowering operators in cosmology

In section 5.7 of Esposito (1994), a flat Euclidean background bounded by a three-sphere was studied. On the bounding S^3, the following boundary conditions for a spin-s field were required:

$$2^s \ {}_e n^{AA'} ... \ {}_e n^{LL'} \ \phi_{A...L} = \epsilon \ \widetilde{\phi}^{A'...L'}. \tag{8.2.1}$$

With our notation, ${}_e n^{AA'}$ is the Euclidean normal to S^3 (D'Eath and Halliwell 1987, Esposito 1994), $\phi_{A...L} = \phi_{(A...L)}$ and $\widetilde{\phi}_{A'...L'} = \widetilde{\phi}_{(A'...L')}$ are totally symmetric and independent (i.e. not related by any conjugation) field strengths, which reduce to the massless spin-$\frac{1}{2}$ field for $s = \frac{1}{2}$. Moreover, the complex scalar field ϕ is such that its real part obeys Dirichlet conditions on S^3 and its imaginary part obeys Neumann conditions on S^3, or the other way around, according to the value of the parameter $\epsilon \equiv \pm 1$ occurring in (8.2.1).

In flat Euclidean four-space, we write the solutions of the twistor equations

$$D_{A'}^{\ \ (A} \ \omega^{B)} = 0, \tag{8.2.2}$$

$$D_{A}^{\ \ (A'} \ \widetilde{\omega}^{B')} = 0, \tag{8.2.3}$$

as (cf. section 4.1)

$$\omega^A = (\omega^o)^A - i\Bigr({}_e x^{AA'}\Bigr)\pi^o_{A'}, \tag{8.2.4}$$

$$\widetilde{\omega}^{A'} = (\widetilde{\omega}^o)^{A'} - i\Bigr({}_e x^{AA'}\Bigr)\widetilde{\pi}^o_{A}. \tag{8.2.5}$$

Note that, since unprimed and primed spin-spaces are no longer anti-isomorphic in the case of Riemannian four-metrics, Eq. (8.2.3) is not obtained by complex conjugation of Eq. (8.2.2). Hence the spinor field $\widetilde{\omega}^{B'}$ is independent of ω^B. This leads to distinct solutions (8.2.4) and (8.2.5), where the spinor fields $\omega^o_A, \widetilde{\omega}^o_{A'}, \widetilde{\pi}^o_A, \pi^o_{A'}$ are covariantly constant with respect to the flat connection D, whose corresponding spinor covariant derivative is here denoted by $D_{AB'}$. The following theorem can be now proved:

Theorem 8.2.1 Let ω^D be a solution of the twistor equation (8.2.2) in flat Euclidean space with a three-sphere boundary, and let $\widetilde{\omega}^{D'}$ be the solution of the independent equation (8.2.3) in the same four-geometry with boundary. Then a form exists of the spin-lowering operator which preserves the local boundary conditions on S^3:

$$4 \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; {}_e n^{DD'} \; \phi_{ABCD} = \epsilon \; \widetilde{\phi}^{A'B'C'D'}, \tag{8.2.6}$$

$$2^{\frac{3}{2}} \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; \phi_{ABC} = \epsilon \; \widetilde{\phi}^{A'B'C'}. \tag{8.2.7}$$

Of course, the independent field strengths appearing in (8.2.6) and (8.2.7) are assumed to satisfy the corresponding massless free-field equations.

Proof. Multiplying both sides of (8.2.6) by ${}_e n_{FD'}$ one gets

$$-2 \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; \phi_{ABCF} = \epsilon \; \widetilde{\phi}^{A'B'C'D'} \; {}_e n_{FD'}. \tag{8.2.8}$$

Taking into account the total symmetry of the field strengths, putting $F = D$ and multiplying both sides of (8.2.8) by $\sqrt{2}\,\omega^D$ one eventually gets

$$-2^{\frac{3}{2}} \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; \phi_{ABCD} \; \omega^D = \epsilon \; \sqrt{2} \; \widetilde{\phi}^{A'B'C'D'} \; {}_e n_{DD'} \; \omega^D, \tag{8.2.9}$$

$$2^{\frac{3}{2}} \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; \phi_{ABCD} \; \omega^D = \epsilon \; \widetilde{\phi}^{A'B'C'D'} \; \widetilde{\omega}_{D'}, \tag{8.2.10}$$

where (8.2.10) is obtained by inserting into (8.2.7) the definition of the spin-lowering operator. The comparison of (8.2.9) and (8.2.10) yields the preservation condition

$$\sqrt{2} \; {}_e n_{DA'} \; \omega^D = -\widetilde{\omega}_{A'}. \tag{8.2.11}$$

In the light of (8.2.4) and (8.2.5), Eq. (8.2.11) is found to imply

$$\sqrt{2} \; {}_e n_{DA'} \; (\omega^o)^D - i\sqrt{2} \; {}_e n_{DA'} \; {}_e x^{DD'} \; \pi^o_{D'} = -\widetilde{\omega}^o_{A'} - i \; {}_e x_{DA'} \; (\widetilde{\pi}^o)^D. \tag{8.2.12}$$

Requiring that (8.2.12) should be identically satisfied, and using the identity ${}_e n^{AA'} = \frac{1}{r} \; {}_e x^{AA'}$ on a three-sphere of radius r, one finds

$$\widetilde{\omega}^o_{A'} = i\sqrt{2} \, r \; {}_e n_{DA'} \; {}_e n^{DD'} \; \pi^o_{D'} = -\frac{ir}{\sqrt{2}} \, \pi^o_{A'}, \tag{8.2.13}$$

$$-\sqrt{2}\ {}_{e}n_{DA'}\ (\omega^{o})^{D} = ir\ {}_{e}n_{DA'}\ (\widetilde{\pi}^{o})^{D}. \tag{8.2.14}$$

Multiplying both sides of (8.2.14) by ${}_{e}n^{BA'}$, and then acting with ε_{BA} on both sides of the resulting relation, one gets

$$\omega_{A}^{o} = -\frac{ir}{\sqrt{2}}\ \widetilde{\pi}_{A}^{o}. \tag{8.2.15}$$

The equations (8.2.11), (8.2.13) and (8.2.15) completely solve the problem of finding a spin-lowering operator which preserves the boundary conditions (8.2.6) and (8.2.7) on S^3. Q.E.D.

If one requires local boundary conditions on S^3 involving field strengths and normals also for lower spins (i.e. spin $\frac{3}{2}$ vs spin 1, spin 1 vs spin $\frac{1}{2}$, spin $\frac{1}{2}$ vs spin 0), then by using the same technique of the theorem just proved, one finds that the preservation condition obeyed by the spin-lowering operator is still expressed by (8.2.13) and (8.2.15).

8.3 Spin-raising operators in cosmology

To derive the corresponding preservation condition for spin-raising operators, we begin by studying the relation between spin-$\frac{1}{2}$ and spin-1 fields. In this case, the independent spin-1 field strengths take the form (Penrose and Rindler 1986)

$$\psi_{AB} = i\ \widetilde{\omega}^{L'} \Bigr(D_{BL'}\ \chi_{A}\Bigr) - 2\chi_{(A}\ \widetilde{\pi}_{B)}^{o}, \tag{8.3.1}$$

$$\widetilde{\psi}_{A'B'} = -i\ \omega^{L} \Bigr(D_{LB'}\ \widetilde{\chi}_{A'}\Bigr) - 2\widetilde{\chi}_{(A'}\ \pi_{B')}^{o}, \tag{8.3.2}$$

where the independent spinor fields $\Bigr(\chi_{A}, \widetilde{\chi}_{A'}\Bigr)$ represent a massless spin-$\frac{1}{2}$ field obeying the Weyl equations on flat Euclidean four-space and subject to the boundary conditions

$$\sqrt{2}\ {}_{e}n^{AA'}\ \chi_{A} = \epsilon\ \widetilde{\chi}^{A'} \tag{8.3.3}$$

on a three-sphere of radius r. Thus, by requiring that (8.3.1) and (8.3.2) should obey (8.2.1) on S^3 with $s=1$, and bearing in mind (8.3.3), one finds

$$2\epsilon\left[\sqrt{2}\,\widetilde{\pi}_A^o\,\widetilde{\chi}^{(A'}\;{}_e n^{AB')} - \widetilde{\chi}^{(A'}\;\pi^{o\;B')}\right] = i\left[2\;{}_e n^{AA'}\;{}_e n^{BB'}\;\widetilde{\omega}^{L'}\;D_{L'(B}\,\chi_{A)} + \epsilon\,\omega^L\;{D_L}^{(B'}\;\widetilde{\chi}^{A')}\right] \tag{8.3.4}$$

on the bounding S^3. It is now clear how to carry out the calculation for higher spins. Denoting by s the spin obtained by spin-raising, and defining $n \equiv 2s$, one finds

$$n\epsilon\left[\sqrt{2}\,\widetilde{\pi}_A^o\;{}_e n^{A(A'}\;\widetilde{\chi}^{B'...K')} - \widetilde{\chi}^{(A'...D'}\;\pi^{o\;K')}\right]$$
$$= i\left[2^{\frac{n}{2}}\;{}_e n^{AA'}...{}_e n^{KK'}\;\widetilde{\omega}^{L'}\;D_{L'(K}\,\chi_{A...D)} + \epsilon\,\omega^L\;{D_L}^{(K'}\;\widetilde{\chi}^{A'...D')}\right] \tag{8.3.5}$$

on the three-sphere boundary. In the comparison spin-0 vs spin-$\frac{1}{2}$, the preservation condition is not obviously obtained from (8.3.5). The desired result is here found by applying the spin-raising operators to the independent scalar fields ϕ and $\widetilde{\phi}$ (see below) and bearing in mind (8.2.4), (8.2.5) and the boundary conditions

$$\phi = \epsilon\;\widetilde{\phi} \text{ on } S^3, \tag{8.3.6}$$

$${}_e n^{AA'} D_{AA'}\phi = -\epsilon\;{}_e n^{BB'} D_{BB'}\widetilde{\phi} \text{ on } S^3. \tag{8.3.7}$$

This leads to the following condition on S^3 (cf. Eq. (5.7.23) of Esposito (1994)):

$$0 = i\phi\left[\frac{\widetilde{\pi}_A^o}{\sqrt{2}} - \pi_{A'}^o\;{}_e {n_A}^{A'}\right] - \left[\frac{\widetilde{\omega}^{K'}}{\sqrt{2}}\Big(D_{AK'}\phi\Big) - \frac{\omega_A}{2}\;{}_e {n_C}^{K'}\Big({D^C}_{K'}\phi\Big)\right]$$
$$+\,\epsilon\;{}_e {n_{(A}}^{A'}\;\omega^B\;D_{B)A'}\;\widetilde{\phi}. \tag{8.3.8}$$

Note that, while the preservation conditions (8.2.13) and (8.2.15) for spin-lowering operators are purely algebraic, the preservation conditions (8.3.5) and (8.3.8) for spin-raising operators are more complicated, since they also involve the value at the boundary of four-dimensional covariant derivatives of spinor fields or scalar fields. Two independent scalar fields have been introduced, since the spinor fields obtained by applying the spin-raising operators to ϕ and $\widetilde{\phi}$ respectively are independent as well in our case.

8.4 Dirac's spin-$\frac{3}{2}$ potentials in cosmology

In this section we focus on the totally symmetric field strengths ϕ_{ABC} and $\widetilde{\phi}_{A'B'C'}$ for spin-$\frac{3}{2}$ fields, and we express them in terms of their potentials, rather than using spin-raising (or spin-lowering) operators. The corresponding theory in Minkowski space-time (and curved space-time) is described in Penrose (1990), Penrose (1991a–c), and adapted here to the case of flat Euclidean four-space with flat connection D. It turns out that $\widetilde{\phi}_{A'B'C'}$ can then be obtained locally from two potentials defined as follows. The first potential satisfies the properties (section 5.3, Penrose 1990, Penrose 1991a–c, Esposito and Pollifrone 1994)

$$\gamma_{A'B'}^{C} = \gamma_{(A'B')}^{C}, \tag{8.4.1}$$

$$D^{AA'} \gamma_{A'B'}^{C} = 0, \tag{8.4.2}$$

$$\widetilde{\phi}_{A'B'C'} = D_{CC'} \gamma_{A'B'}^{C}, \tag{8.4.3}$$

with the gauge freedom of replacing it by

$$\widehat{\gamma}_{A'B'}^{C} \equiv \gamma_{A'B'}^{C} + D^{C}{}_{B'} \widetilde{\nu}_{A'}, \tag{8.4.4}$$

where $\widetilde{\nu}_{A'}$ satisfies the positive-helicity Weyl equation

$$D^{AA'} \widetilde{\nu}_{A'} = 0. \tag{8.4.5}$$

The second potential is defined by the conditions

$$\rho_{A'}^{BC} = \rho_{A'}^{(BC)}, \tag{8.4.6}$$

$$D^{AA'} \; \rho_{A'}^{BC} = 0, \tag{8.4.7}$$

$$\gamma_{A'B'}^{C} = D_{BB'} \; \rho_{A'}^{BC}, \tag{8.4.8}$$

with the gauge freedom of being replaced by

$$\widehat{\rho}_{A'}^{BC} \equiv \rho_{A'}^{BC} + D^{C}{}_{A'} \; \chi^{B}, \tag{8.4.9}$$

where χ^B satisfies the negative-helicity Weyl equation

$$D_{BB'} \; \chi^{B} = 0. \tag{8.4.10}$$

Moreover, in flat Euclidean four-space the field strength ϕ_{ABC} is expressed locally in terms of the potential $\Gamma_{AB}^{C'} = \Gamma_{(AB)}^{C'}$, independent of $\gamma_{A'B'}^{C}$, as

$$\phi_{ABC} = D_{CC'} \; \Gamma_{AB}^{C'}, \tag{8.4.11}$$

with gauge freedom

$$\widehat{\Gamma}_{AB}^{C'} \equiv \Gamma_{AB}^{C'} + D^{C'}{}_{B} \; \nu_{A}. \tag{8.4.12}$$

Thus, if we insert (8.4.3) and (8.4.11) into the boundary conditions (8.2.1) with $s = \frac{3}{2}$, and require that also the gauge-equivalent potentials (8.4.4) and (8.4.12) should obey such boundary conditions on S^3, we find that

$$2^{\frac{3}{2}} \; {}_{e}n^{A}{}_{A'} \; {}_{e}n^{B}{}_{B'} \; {}_{e}n^{C}{}_{C'} \; D_{CL'} \; D^{L'}{}_{B} \; \nu_{A} = \epsilon \; D_{LC'} \; D^{L}{}_{B'} \; \widetilde{\nu}_{A'} \tag{8.4.13}$$

on the three-sphere. Note that, from now on (as already done in (8.3.5) and (8.3.8)), covariant derivatives appearing in boundary conditions are first taken on the background and then evaluated on S^3. In the case of our flat background, (8.4.13) is identically satisfied since $D_{CL'} \; D^{L'}{}_{B} \; \nu_A$ and $D_{LC'} \; D^{L}{}_{B'} \; \widetilde{\nu}_{A'}$ vanish by virtue of spinor Ricci identities. In a curved background, however, denoting by ∇ its curved connection, and defining $\square_{AB} \equiv \nabla_{M'(A} \nabla^{M'}{}_{B)}$, $\square_{A'B'} \equiv \nabla_{X(A'} \; \nabla^{X}{}_{B')}$, since the spinor Ricci identities we need are (Ward and Wells 1990)

$$\square_{AB} \; \nu_{C} = \psi_{ABDC} \; \nu^{D} - 2\Lambda \; \nu_{(A} \; \varepsilon_{B)C}, \tag{8.4.14}$$

$$\square_{A'B'} \, \widetilde{\nu}_{C'} = \widetilde{\psi}_{A'B'D'C'} \, \widetilde{\nu}^{D'} - 2\widetilde{\Lambda} \, \widetilde{\nu}_{(A'} \, \varepsilon_{B')C'}, \tag{8.4.15}$$

one finds that the corresponding boundary conditions

$$2^{\frac{3}{2}} \, {}_{e}n^{A}{}_{A'} \, {}_{e}n^{B}{}_{B'} \, {}_{e}n^{C}{}_{C'} \, \nabla_{CL'} \, \nabla^{L'}{}_{B} \, \nu_{A} = \epsilon \, \nabla_{LC'} \, \nabla^{L}{}_{B'} \, \widetilde{\nu}_{A'} \tag{8.4.16}$$

are identically satisfied if and only if one of the following conditions holds: (i) $\nu_A = \widetilde{\nu}_{A'} = 0$; (ii) the Weyl spinors $\psi_{ABCD}, \widetilde{\psi}_{A'B'C'D'}$ and the scalars $\Lambda, \widetilde{\Lambda}$ vanish everywhere. However, since in a curved space-time with vanishing $\Lambda, \widetilde{\Lambda}$, the potentials with the gauge freedoms (8.4.4) and (8.4.12) only exist provided that D is replaced by ∇ and the trace-free part Φ_{ab} of the Ricci tensor vanishes as well (Buchdahl 1958), the background four-geometry is actually flat Euclidean four-space. We require that (8.4.16) should be identically satisfied to avoid, after a gauge transformation, obtaining more boundary conditions than the ones originally imposed. The curvature of the background should not, itself, be subject to a boundary condition.

The same result can be derived by using the potential $\rho_{A'}^{BC}$ and its independent counterpart $\Lambda_{A}^{B'C'}$. This spinor field yields the $\Gamma_{AB}^{C'}$ potential by means of

$$\Gamma_{AB}^{C'} = D_{BB'} \, \Lambda_{A}^{B'C'}, \tag{8.4.17}$$

and has the gauge freedom

$$\widehat{\Lambda}_{A}^{B'C'} \equiv \Lambda_{A}^{B'C'} + D^{C'}{}_{A} \, \widetilde{\chi}^{B'}, \tag{8.4.18}$$

where $\widetilde{\chi}^{B'}$ satisfies the positive-helicity Weyl equation

$$D_{BF'} \, \widetilde{\chi}^{F'} = 0. \tag{8.4.19}$$

Thus, if also the gauge-equivalent potentials (8.4.9) and (8.4.18) have to satisfy the boundary conditions (8.2.1) on S^3, one finds

$$2^{\frac{3}{2}} \, {}_{e}n^{A}{}_{A'} \, {}_{e}n^{B}{}_{B'} \, {}_{e}n^{C}{}_{C'} \, D_{CL'} \, D_{BF'} \, D^{L'}{}_{A} \, \widetilde{\chi}^{F'} = \epsilon \, D_{LC'} \, D_{MB'} \, D^{L}{}_{A'} \, \chi^{M} \tag{8.4.20}$$

on the three-sphere. In our flat background, covariant derivatives commute, hence (8.4.20) is identically satisfied by virtue of (8.4.10) and (8.4.19). However, in the curved case the boundary conditions (8.4.20) are replaced by

$$2^{\frac{3}{2}} \; {}_{e}n^{A}{}_{A'} \; {}_{e}n^{B}{}_{B'} \; {}_{e}n^{C}{}_{C'} \; \nabla_{CL'} \; \nabla_{BF'} \; \nabla^{L'}{}_{A} \; \widetilde{\chi}^{F'} = \epsilon \; \nabla_{LC'} \; \nabla_{MB'} \; \nabla^{L}{}_{A'} \; \chi^{M} \quad (8.4.21)$$

on S^3, if the *local* expressions of ϕ_{ABC} and $\widetilde{\phi}_{A'B'C'}$ in terms of potentials still hold (Penrose 1990, Penrose 1991a–c). By virtue of (8.4.14) and (8.4.15), where ν_C is replaced by χ_C and $\widetilde{\nu}_{C'}$ is replaced by $\widetilde{\chi}_{C'}$, this means that the Weyl spinors $\psi_{ABCD}, \widetilde{\psi}_{A'B'C'D'}$ and the scalars $\Lambda, \widetilde{\Lambda}$ should vanish, since one should find

$$\nabla^{AA'} \; \widehat{\rho}_{A'}^{BC} = 0, \; \nabla^{AA'} \; \widehat{\Lambda}_{A}^{B'C'} = 0. \quad (8.4.22)$$

If we assume that $\nabla_{BF'} \; \widetilde{\chi}^{F'} = 0$ and $\nabla_{MB'} \; \chi^{M} = 0$, we have to show that (8.4.21) differs from (8.4.20) by terms involving a part of the curvature that is vanishing everywhere. This is proved by using the basic rules of two-spinor calculus and spinor Ricci identities. Thus, bearing in mind that

$$\square^{AB} \; \widetilde{\chi}_{B'} = \Phi^{AB}{}_{L'B'} \; \widetilde{\chi}^{L'}, \quad (8.4.23)$$

$$\square^{A'B'} \; \chi_{B} = \widetilde{\Phi}^{A'B'}{}_{LB} \; \chi^{L}, \quad (8.4.24)$$

one finds (see (8.4.29))

$$\begin{aligned} \nabla^{BB'} \; \nabla^{CA'} \; \chi_{B} &= \nabla^{(BB'} \; \nabla^{C)A'} \; \chi_{B} + \nabla^{[BB'} \; \nabla^{C]A'} \; \chi_{B} \\ &= -\frac{1}{2} \nabla_{B}{}^{B'} \; \nabla^{CA'} \; \chi^{B} + \frac{1}{2} \widetilde{\Phi}^{A'B'LC} \; \chi_{L}. \end{aligned} \quad (8.4.25)$$

Thus, if $\widetilde{\Phi}^{A'B'LC}$ vanishes, also the left-hand side of (8.4.25) has to vanish since this leads to the equation $\nabla^{BB'} \; \nabla^{CA'} \; \chi_{B} = \frac{1}{2} \nabla^{BB'} \; \nabla^{CA'} \; \chi_{B}$. Hence (8.4.25) is identically satisfied. Similarly, the left-hand side of (8.4.21) can be made to vanish identically if the additional condition $\Phi^{CDF'M'} = 0$ holds. The conditions

$$\Phi^{CDF'M'} = 0, \; \widetilde{\Phi}^{A'B'CL} = 0, \quad (8.4.26)$$

when combined with the conditions

$$\psi_{ABCD} = \widetilde{\psi}_{A'B'C'D'} = 0, \ \Lambda = \widetilde{\Lambda} = 0, \tag{8.4.27}$$

arising from (8.4.22) for the local existence of $\rho_{A'}^{BC}$ and $\Lambda_{A}^{B'C'}$ potentials, imply that the whole Riemann curvature should vanish. Hence, in the boundary-value problems we are interested in, the only admissible background four-geometry (of the Einstein type (Besse 1987)) is flat Euclidean four-space.

Note that (8.4.25) is *not* an identity, since we have already set Λ to zero by requiring that

$$\nabla^{AA'} \widehat{\rho}_{A'}^{BC} = -\psi^{ABC}{}_{F} \, \chi^{F} + \Lambda \Big(\chi^{A} \, \varepsilon^{CB} + 3\chi^{B} \, \varepsilon^{AC} + \chi^{C} \, \varepsilon^{AB} \Big) \tag{8.4.28}$$

should vanish. In general, for any solution χ_B of the Weyl equation, by virtue of the corresponding identity $\Box \chi_B = -6\Lambda \, \chi_B$ (see problem 2.7), one finds

$$\nabla^{BB'} \nabla^{CA'} \chi_B = \frac{1}{2} \nabla^{BB'} \nabla^{CA'} \chi_B + \frac{1}{2} \widetilde{\Phi}^{A'B'LC} \, \chi_L + \frac{3}{2} \Lambda \varepsilon^{B'A'} \, \chi^{C}. \tag{8.4.29}$$

As the reader may check, the action of the $\Box \equiv \nabla_{CA'} \nabla^{CA'}$ operator on χ_B is obtained by acting with the spinor covariant derivative $\nabla_{AA'}$ on the Weyl equation ${\nabla_B}^{A'} \chi^B = 0$.

8.5 Boundary conditions in supergravity

The boundary conditions studied in the previous sections are not appropriate if one studies supergravity multiplets and supersymmetry transformations at the boundary (Esposito 1994). By contrast, it turns out one has to impose another set of locally supersymmetric boundary conditions, first proposed in Luckock and Moss (1989). These are in general mixed, and involve in particular Dirichlet conditions for the transverse modes of the vector potential of electromagnetism, a mixture of Dirichlet and Neumann conditions for scalar fields, and local boundary conditions for the spin-$\frac{1}{2}$ field and the spin-$\frac{3}{2}$ potential. Using two-component spinor notation for supergravity (D'Eath 1984), the spin-$\frac{3}{2}$ boundary conditions take the form

$$\sqrt{2}\ {}_{e}n_{A}^{\ \ A'}\ \psi^{A}_{\ \ i} = \epsilon\ \widetilde{\psi}^{A'}_{\ \ i} \ \text{on}\ S^3. \tag{8.5.1}$$

With our notation, $\epsilon \equiv \pm 1$, ${}_{e}n_{A}^{\ \ A'}$ is the Euclidean normal to S^3, and $\left(\psi^{A}_{\ \ i}, \widetilde{\psi}^{A'}_{\ \ i}\right)$ are the *independent* (i.e. not related by any conjugation) spatial components (hence $i = 1, 2, 3$) of the spinor-valued one-forms appearing in the action functional of Euclidean supergravity (D'Eath 1984, Esposito 1994).

It appears necessary to understand whether the analysis in the previous section and in Esposito and Pollifrone (1994) can be used to derive restrictions on the classical boundary-value problem corresponding to (8.5.1). For this purpose, we study a Riemannian background four-geometry, and we use the decompositions of the spinor-valued one-forms in such a background, i.e.

$$\psi^{A}_{\ \ i} = h^{-\frac{1}{4}} \left[\chi^{(AB)B'} + \varepsilon^{AB}\ \widetilde{\phi}^{B'}\right] e_{BB'i}, \tag{8.5.2}$$

$$\widetilde{\psi}^{A'}_{\ \ i} = h^{-\frac{1}{4}} \left[\widetilde{\chi}^{(A'B')B} + \varepsilon^{A'B'}\ \phi^{B}\right] e_{BB'i}, \tag{8.5.3}$$

where h is the determinant of the three-metric on S^3, and $e_{BB'i}$ is the spatial component of the tetrad, written in two-spinor language. If we now reduce the classical theory of simple supergravity to its physical degrees of freedom by imposing the gauge conditions (Esposito 1994)

$$e_{AA'}^{\ \ \ \ i}\ \psi^{A}_{\ \ i} = 0, \tag{8.5.4}$$

$$e_{AA'}^{\ \ \ \ i}\ \widetilde{\psi}^{A'}_{\ \ i} = 0, \tag{8.5.5}$$

we find that the expansions of (8.5.2) and (8.5.3) on a family of three-spheres centred on the origin take the forms (Esposito 1994)

$$\psi^{A}_{\ \ i} = \frac{h^{-\frac{1}{4}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+4)} \alpha_{n}^{pq} \left[m_{np}^{(\beta)}(\tau)\ \beta^{nqABB'} + \widetilde{r}_{np}^{(\mu)}(\tau)\ \overline{\mu}^{nqABB'}\right] e_{BB'i}, \tag{8.5.6}$$

$$\widetilde{\psi}^{A'}_{\ \ i} = \frac{h^{-\frac{1}{4}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+4)} \alpha_n^{pq} \Big[\widetilde{m}_{np}^{(\beta)}(\tau) \, \overline{\beta}^{nqA'B'B} + r_{np}^{(\mu)}(\tau) \, \mu^{nqA'B'B} \Big] e_{BB'i}. \tag{8.5.7}$$

With our notation, α_n^{pq} are block-diagonal matrices with blocks $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and the β- and μ-harmonics on S^3 are given by (Esposito 1994)

$$\beta^{nq}_{\ \ \ ACC'} = \rho^{nq}_{\ \ \ (ACD)} \, n^{D}_{\ \ C'}, \tag{8.5.8}$$

$$\mu^{nq}_{\ \ \ A'B'B} = \sigma^{nq}_{\ \ \ (A'B'C')} \, n_B^{\ \ C'}. \tag{8.5.9}$$

In the light of (8.5.6)–(8.5.9), one gets the following physical-degrees-of-freedom form of the spinor-valued one-forms of supergravity (cf. D'Eath (1984)):

$$\psi^A_{\ \ i} = h^{-\frac{1}{4}} \, \phi^{(ABC)} \, {}_e n_C^{\ \ B'} \, e_{BB'i}, \tag{8.5.10}$$

$$\widetilde{\psi}^{A'}_{\ \ i} = h^{-\frac{1}{4}} \, \widetilde{\phi}^{(A'B'C')} \, {}_e n^B_{\ \ C'} \, e_{BB'i}, \tag{8.5.11}$$

where $\phi^{(ABC)}$ and $\widetilde{\phi}^{(A'B'C')}$ are totally symmetric and independent spinor fields.

Within this framework, a *sufficient* condition for the validity of the boundary conditions (8.5.1) on S^3 is

$$\sqrt{2} \, {}_e n_A^{\ \ A'} \, {}_e n_C^{\ \ B'} \, \phi^{(ABC)} = \epsilon \, {}_e n^B_{\ \ C'} \, \widetilde{\phi}^{(A'B'C')}. \tag{8.5.12}$$

However, our construction does not prove that such $\phi^{(ABC)}$ and $\widetilde{\phi}^{(A'B'C')}$ can be expressed in terms of four potentials as in Esposito and Pollifrone (1994).

It should be emphasized that our analysis, although motivated by quantum cosmology, is entirely classical. Hence we have not discussed ghost modes. The theory has been reduced to its physical degrees of freedom to make a comparison with the results in Esposito and Pollifrone (1994), but totally symmetric field strengths do not enable one to recover the full physical content of simple supergravity. Hence the four-sphere background studied in Poletti (1990) is not ruled out by the work in this section, and a more careful analysis is in order (see sections 8.10–8.15).

8.6 Rarita–Schwinger potentials

We are here interested in the independent spatial components $\left(\psi^{A}_{\ i}, \widetilde{\psi}^{A'}_{\ i}\right)$ of the gravitino field in Riemannian backgrounds. In terms of the spatial components $e_{AB'i}$ of the tetrad, and of spinor fields, they can be expressed as (Aichelburg and Urbantke 1981, D'Eath 1984, Penrose 1991)

$$\psi_{A\ i} = \Gamma^{C'}_{\ \ AB} \ e^{B}_{\ \ C'i}, \tag{8.6.1}$$

$$\widetilde{\psi}_{A'\ i} = \gamma^{C}_{\ \ A'B'} \ e_{C}^{\ \ B'}{}_{i}. \tag{8.6.2}$$

A first important difference with respect to the Dirac form of the potentials studied in Esposito and Pollifrone (1994) is that the spinor fields $\Gamma^{C'}_{\ \ AB}$ and $\gamma^{C}_{\ \ A'B'}$ are no longer symmetric in the second and third index. From now on, they will be referred to as spin-$\frac{3}{2}$ potentials. They obey the differential equations (see appendix B and cf. Rarita and Schwinger (1941), Aichelburg and Urbantke (1981), Penrose (1991))

$$\varepsilon^{B'C'} \ \nabla_{A(A'} \ \gamma^{A}_{\ \ B')C'} = -3\Lambda \ \widetilde{\alpha}_{A'}, \tag{8.6.3}$$

$$\nabla^{B'(B} \ \gamma^{A)}_{\ \ B'C'} = \Phi^{ABL'}_{\ \ \ \ \ \ C'} \ \widetilde{\alpha}_{L'}, \tag{8.6.4}$$

$$\varepsilon^{BC} \ \nabla_{A'(A} \ \Gamma^{A'}_{\ \ B)C} = -3\Lambda \ \alpha_{A}, \tag{8.6.5}$$

$$\nabla^{B(B'} \ \Gamma^{A')}_{\ \ BC} = \widetilde{\Phi}^{A'B'L}_{\ \ \ \ \ \ \ C} \ \alpha_{L}, \tag{8.6.6}$$

where $\nabla_{AB'}$ is the spinor covariant derivative corresponding to the curved connection ∇ of the background, the spinors $\Phi^{AB}_{\ \ \ C'D'}$ and $\widetilde{\Phi}^{A'B'}_{\ \ \ \ CD}$ correspond to the trace-free part of the Ricci tensor, the scalar Λ corresponds to the scalar curvature $R = 24\Lambda$ of the background, and $\alpha_{A}, \widetilde{\alpha}_{A'}$ are a pair of independent spinor fields, corresponding to the Majorana field in the Lorentzian regime. Moreover, the potentials are subject to the gauge transformations (cf. section 8.9)

$$\widehat{\gamma}^{A}_{\ \ B'C'} \equiv \gamma^{A}_{\ \ B'C'} + \nabla^{A}_{\ \ B'} \ \lambda_{C'}, \tag{8.6.7}$$

$$\widehat{\Gamma}^{A'}{}_{BC} \equiv \Gamma^{A'}{}_{BC} + \nabla^{A'}{}_{B} \, \nu_C. \tag{8.6.8}$$

A second important difference with respect to the Dirac potentials is that the spinor fields ν_B and $\lambda_{B'}$ are no longer taken to be solutions of the Weyl equation. They should be freely specifiable (see section 8.7).

8.7 Compatibility conditions

Our task is now to derive compatibility conditions, by requiring that the field equations (8.6.3)–(8.6.6) should also be satisfied by the gauge-transformed potentials appearing on the left-hand side of (8.6.7) and (8.6.8). For this purpose, after defining the operators

$$\square_{AB} \equiv \nabla_{M'(A} \, \nabla_{B)}{}^{M'}, \tag{8.7.1}$$

$$\square_{A'B'} \equiv \nabla_{F(A'} \, \nabla_{B')}{}^{F}, \tag{8.7.2}$$

we need the standard identity $\Omega_{[AB]} = \frac{1}{2}\varepsilon_{AB} \, \Omega_C{}^C$ and the spinor Ricci identities

$$\square_{AB} \, \nu_C = \psi_{ABCD} \, \nu^D - 2\Lambda \, \nu_{(A} \, \varepsilon_{B)C}, \tag{8.7.3}$$

$$\square_{A'B'} \lambda_{C'} = \widetilde{\psi}_{A'B'C'D'} \, \lambda^{D'} - 2\Lambda \, \lambda_{(A'} \, \varepsilon_{B')C'}, \tag{8.7.4}$$

$$\square^{AB} \, \lambda_{B'} = \Phi^{AB}{}_{M'B'} \, \lambda^{M'}, \tag{8.7.5}$$

$$\square^{A'B'} \, \nu_B = \widetilde{\Phi}^{A'B'}{}_{MB} \, \nu^M. \tag{8.7.6}$$

Of course, $\widetilde{\psi}_{A'B'C'D'}$ and ψ_{ABCD} are the self-dual and anti-self-dual Weyl spinors, respectively.

Thus, on using the Eqs. (8.6.3)–(8.6.8) and (8.7.1)–(8.7.6), the basic rules of two-spinor calculus (Penrose and Rindler 1986, Ward and Wells 1990, Stewart 1991) lead to the compatibility equations

$$3\Lambda \, \lambda_{A'} = 0, \tag{8.7.7}$$

$$\Phi^{AB}{}_{M'}{}^{C'} \; \lambda^{M'} = 0, \tag{8.7.8}$$

$$3\Lambda \; \nu_A = 0, \tag{8.7.9}$$

$$\widetilde{\Phi}^{A'B'}{}_{M}{}^{C} \; \nu^{M} = 0. \tag{8.7.10}$$

Non-trivial solutions of (8.7.7)–(8.7.10) only exist if the scalar curvature and the trace-free part of the Ricci tensor vanish. Hence the gauge transformations (8.6.7) and (8.6.8) lead to spinor fields ν_A and $\lambda_{A'}$ which are freely specifiable *inside* Ricci-flat backgrounds, while the boundary conditions (8.5.1) are preserved under the action of (8.6.7) and (8.6.8) provided that the following conditions hold at the boundary:

$$\sqrt{2} \; {}_e n_A{}^{A'} \; \Bigl(\nabla^{AC'} \; \nu^B\Bigr) e_{BC'i} = \pm \Bigl(\nabla^{CA'} \lambda^{B'}\Bigr) e_{CB'i} \text{ at } \partial M. \tag{8.7.11}$$

8.8 Second set of potentials in Ricci-flat backgrounds

As shown by Penrose (1994), in a Ricci-flat manifold the Rarita–Schwinger potentials may be supplemented by a second set of potentials. Here we use such a construction in its local form. For this purpose, we introduce the second set of potentials for spin $\frac{3}{2}$ by requiring that locally (Penrose 1994)

$$\gamma_{A'B'}{}^{C} \equiv \nabla_{BB'} \; \rho_{A'}{}^{CB}. \tag{8.8.1}$$

Of course, special attention should be payed to the index ordering in (8.8.1), since the spin-$\frac{3}{2}$ potentials are not symmetric. On inserting (8.8.1) into (8.6.3), a repeated use of symmetrizations and anti-symmetrizations leads to the equation (hereafter $\Box \equiv \nabla_{CF'}\nabla^{CF'}$)

$$\begin{aligned} &\varepsilon_{FL} \; \nabla_{AA'} \; \nabla^{B'(F} \; \rho_{B'}{}^{A)L} + \frac{1}{2}\nabla^{A}{}_{A'} \; \nabla^{B'M} \; \rho_{B'(AM)} \\ &+ \Box_{AM} \; \rho_{A'}{}^{(AM)} + \frac{3}{8}\Box \rho_{A'} = 0, \end{aligned} \tag{8.8.2}$$

where, following Penrose (1994), we have defined

$$\rho_{A'} \equiv \rho_{A'C}^{\ \ \ \ C}, \tag{8.8.3}$$

and we bear in mind that our background has to be Ricci-flat. Thus, if the following equation holds (Penrose 1994):

$$\nabla^{B'(F} \ \rho_{B'}^{\ \ A)L} = 0, \tag{8.8.4}$$

one finds

$$\nabla^{B'M} \ \rho_{B'(AM)} = \frac{3}{2} \nabla_{A}^{\ \ F'} \ \rho_{F'}, \tag{8.8.5}$$

and hence Eq. (8.8.2) may be cast in the form

$$\Box_{AM} \ \rho_{A'}^{\ \ (AM)} = 0. \tag{8.8.6}$$

On the other hand, a very useful identity resulting from Eq. (4.9.13) of Penrose and Rindler (1984) enables one to show that

$$\Box_{AM} \ \rho_{A'}^{\ \ (AM)} = -\Phi_{AMA'}^{\ \ \ \ \ \ L'} \ \rho_{L'}^{\ \ (AM)}. \tag{8.8.7}$$

Hence Eq. (8.8.6) reduces to an identity by virtue of Ricci-flatness. Moreover, we have to insert (8.8.1) into the field equation (8.6.4) for γ-potentials. By virtue of Eq. (8.8.4) and of the identities (cf. Penrose and Rindler (1984))

$$\Box^{BM} \ \rho_{B'\ \ M}^{\ \ A} = -\psi^{ABLM} \ \rho_{(LM)B'} - \Phi^{BM}_{\ \ \ \ B'}{}^{D'} \ \rho^{A}_{\ \ MD'} + 4\Lambda \ \rho^{(AB)}_{\ \ \ \ \ \ B'}, \tag{8.8.8}$$

$$\Box^{B'F'} \ \rho_{B'}^{\ \ (AB)} = 3\Lambda \ \rho^{(AB)F'} + \widetilde{\Phi}^{B'F'}_{\ \ \ \ \ \ L}{}^{A} \ \rho^{(LB)}_{\ \ \ \ \ \ B'} + \widetilde{\Phi}^{B'F'B}_{\ \ \ \ \ \ \ \ L} \ \rho^{(AL)}_{\ \ \ \ \ \ B'}, \tag{8.8.9}$$

this leads to the equation

$$\psi^{ABLM} \ \rho_{(LM)C'} = 0, \tag{8.8.10}$$

where we have again used the Ricci-flatness condition.

Of course, potentials supplementing the Γ-potentials may also be constructed locally. On defining (cf. (8.8.1))

$$\Gamma_{AB}^{\ \ \ \ C'} \equiv \nabla_{B'B} \ \theta_{A}^{\ \ C'B'}, \tag{8.8.11}$$

$$\theta_A \equiv \theta_{AC'}{}^{C'}, \qquad (8.8.12)$$

and requiring that (Penrose 1994, Esposito 1995)

$$\nabla^{B(F'} \; \theta_B{}^{A')L'} = 0, \qquad (8.8.13)$$

one finds

$$\nabla^{BM'} \; \theta_{B(A'M')} = \frac{3}{2} \nabla_{A'}{}^F \; \theta_F, \qquad (8.8.14)$$

and a similar calculation yields an identity and the equation

$$\widetilde{\psi}^{A'B'L'M'} \; \theta_{(L'M')C} = 0. \qquad (8.8.15)$$

Note that Eqs. (8.8.10) and (8.8.15) relate explicitly the second set of potentials to the curvature of the background. This inconsistency is avoided if one of the following conditions holds (Esposito, Gionti *et al.* 1995):

(i) The whole conformal curvature of the background vanishes.

(ii) ψ^{ABLM} and $\theta_{(L'M')C}$, or $\widetilde{\psi}^{A'B'L'M'}$ and $\rho_{(LM)C'}$, vanish.

(iii) The symmetric parts of the ρ- and θ-potentials vanish.

In the first case one finds that the only admissible background is again flat Euclidean four-space with boundary, as in Esposito and Pollifrone (1994). By contrast, in the other cases, left-flat, right-flat or Ricci-flat backgrounds are still admissible, provided that the ρ- and θ-potentials take the form

$$\rho_{A'}{}^{CB} = \varepsilon^{CB} \; \widetilde{\alpha}_{A'}, \qquad (8.8.16)$$

$$\theta_A{}^{C'B'} = \varepsilon^{C'B'} \; \alpha_A, \qquad (8.8.17)$$

where α_A and $\widetilde{\alpha}_{A'}$ solve the Weyl equations

$$\nabla^{AA'} \; \alpha_A = 0, \qquad (8.8.18)$$

$$\nabla^{AA'} \; \widetilde{\alpha}_{A'} = 0. \qquad (8.8.19)$$

Eqs. (8.8.16)–(8.8.19) ensure also the validity of Eqs. (8.8.4) and (8.8.13).

However, if one requires the preservation of Eqs. (8.8.4) and (8.8.13) under the following gauge transformations for ρ- and θ-potentials (the order of the indices AL, $A'L'$ is of crucial importance):

$$\widehat{\rho}_{B'}^{\ \ AL} \equiv \rho_{B'}^{\ \ AL} + \nabla_{B'}^{\ \ A} \ \mu^{L}, \tag{8.8.20}$$

$$\widehat{\theta}_{B}^{\ \ A'L'} \equiv \theta_{B}^{\ \ A'L'} + \nabla_{B}^{\ \ A'} \ \sigma^{L'}, \tag{8.8.21}$$

one finds compatibility conditions in Ricci-flat backgrounds of the form

$$\psi_{AFLD} \ \mu^{D} = 0, \tag{8.8.22}$$

$$\widetilde{\psi}_{A'F'L'D'} \ \sigma^{D'} = 0. \tag{8.8.23}$$

Thus, to ensure *unrestricted* gauge freedom (except at the boundary) for the second set of potentials, one is forced to work with flat Euclidean backgrounds. The boundary conditions (8.5.1) play a role in this respect, since they make it necessary to consider both ψ_i^A and $\widetilde{\psi}_i^{A'}$, and hence both $\rho_{B'}^{\ \ AL}$ and $\theta_{B}^{\ \ A'L'}$. Otherwise, one might use Eq. (8.8.22) to set to zero the anti-self-dual Weyl spinor only, *or* Eq. (8.8.23) to set to zero the self-dual Weyl spinor only, so that self-dual (left-flat) or anti-self-dual (right-flat) Riemannian backgrounds with boundary would survive.

8.9 Other gauge transformations

In the massless case, flat Euclidean backgrounds with boundary are really the only possible choice for spin-$\frac{3}{2}$ potentials with a gauge freedom. To prove this, we have also investigated an alternative set of gauge transformations for spin-$\frac{3}{2}$ potentials, written in the form (cf. (8.6.7) and (8.6.8))

$$\widehat{\gamma}^{A}_{\ \ B'C'} \equiv \gamma^{A}_{\ \ B'C'} + \nabla^{A}_{\ \ C'} \ \lambda_{B'}, \tag{8.9.1}$$

$$\widehat{\Gamma}^{A'}_{\ \ BC} \equiv \Gamma^{A'}_{\ \ BC} + \nabla^{A'}_{\ \ C} \ \nu_{B}. \tag{8.9.2}$$

These gauge transformations *do not* correspond to the usual formulation of the Rarita–Schwinger system, but we will see that they can be interpreted in terms of familiar physical concepts.

On imposing that the field equations (8.6.3)–(8.6.6) should be preserved under the action of (8.9.1) and (8.9.2), and setting to zero the trace-free part of the Ricci spinor (since it is inconsistent to have gauge fields $\lambda_{B'}$ and ν_B which depend explicitly on the curvature of the background) one finds compatibility conditions in the form of differential equations, i.e. (cf. Esposito (1995))

$$\Box \lambda_{B'} = -2\Lambda \, \lambda_{B'}, \tag{8.9.3}$$

$$\nabla^{(A(B'} \, \nabla^{C')B)} \lambda_{B'} = 0, \tag{8.9.4}$$

$$\Box \nu_B = -2\Lambda \, \nu_B, \tag{8.9.5}$$

$$\nabla^{(A'(B} \, \nabla^{C)B')} \, \nu_B = 0. \tag{8.9.6}$$

In a flat Riemannian four-manifold with flat connection D, covariant derivatives commute and $\Lambda = 0$. Hence it is possible to express $\lambda_{B'}$ and ν_B as solutions of the Weyl equations

$$D^{AB'} \, \lambda_{B'} = 0, \tag{8.9.7}$$

$$D^{BA'} \, \nu_B = 0, \tag{8.9.8}$$

which agree with the flat-space version of (8.9.3)–(8.9.6). The boundary conditions (8.5.1) are then preserved under the action of (8.9.1) and (8.9.2) if ν_B and $\lambda_{B'}$ obey the boundary conditions (cf. (8.7.11))

$$\sqrt{2} \, {}_e n_A^{\ A'} \Big(D^{BC'} \, \nu^A \Big) e_{BC'i} = \pm \Big(D^{CB'} \, \lambda^{A'} \Big) e_{CB'i} \text{ at } \partial M. \tag{8.9.9}$$

In the curved case, on defining

$$\phi^A \equiv \nabla^{AA'} \, \lambda_{A'}, \tag{8.9.10}$$

$$\widetilde{\phi}^{A'} \equiv \nabla^{AA'} \, \nu_A, \tag{8.9.11}$$

equations (8.9.4) and (8.9.6) imply that these spinor fields solve the equations (cf. Esposito (1995))

$$\nabla_{C'}^{\ (A} \phi^{B)} = 0, \tag{8.9.12}$$

$$\nabla_{C}^{\ (A'} \widetilde{\phi}^{B')} = 0. \tag{8.9.13}$$

Moreover, Eqs. (8.9.3), (8.9.5) and the spinor Ricci identities imply that

$$\nabla_{AB'} \phi^{A} = 2\Lambda \lambda_{B'}, \tag{8.9.14}$$

$$\nabla_{BA'} \widetilde{\phi}^{A'} = 2\Lambda \nu_{B}. \tag{8.9.15}$$

Equations (8.9.12) and (8.9.13) are the twistor equations (Penrose and Rindler 1986) in Riemannian four-geometries. The consistency conditions for the existence of non-trivial solutions of such equations in curved Riemannian four-manifolds are given by (Penrose and Rindler 1986)

$$\psi_{ABCD} = 0, \tag{8.9.16}$$

and

$$\widetilde{\psi}_{A'B'C'D'} = 0, \tag{8.9.17}$$

respectively.

Further consistency conditions for our problem are derived by acting with covariant differentiation on the twistor equation, i.e.

$$\nabla_{A'}^{\ \ C} \nabla^{AA'} \phi^{B} + \nabla_{A'}^{\ \ C} \nabla^{BA'} \phi^{A} = 0. \tag{8.9.18}$$

While the complete symmetrization in ABC yields Eq. (8.9.16), the use of Eq. (8.9.18), jointly with the spinor Ricci identities of section 8.7, yields

$$\Box \phi^{B} = 2\Lambda \phi^{B}, \tag{8.9.19}$$

and an analogous equation is found for $\widetilde{\phi}^{B'}$. Thus, since Eq. (8.9.12) implies

$$\nabla_{C'}^{\ \ A} \phi^{B} = \varepsilon^{AB} \pi_{C'}, \tag{8.9.20}$$

we may obtain from (8.9.20) the equation

$$\nabla^{BA'} \pi_{A'} = 2\Lambda \, \phi^B, \tag{8.9.21}$$

by virtue of the spinor Ricci identities and of Eq. (8.9.19). On the other hand, in the light of (8.9.20), Eq. (8.9.14) leads to

$$\nabla_{AB'} \phi^A = 2\pi_{B'} = 2\Lambda \, \lambda_{B'}. \tag{8.9.22}$$

Hence $\pi_{A'} = \Lambda \, \lambda_{A'}$, and the definition (8.9.10) yields

$$\nabla^{BA'} \pi_{A'} = \Lambda \, \phi^B. \tag{8.9.23}$$

By comparison of Eqs. (8.9.21) and (8.9.23), one gets the equation $\Lambda \, \phi^B = 0$. If $\Lambda \neq 0$, this implies that ϕ^B, $\pi_{B'}$ and $\lambda_{B'}$ have to vanish, and there is no gauge freedom fou our model. This inconsistency is avoided if and only if $\Lambda = 0$, and the corresponding background is forced to be totally flat, since we have already set to zero the trace-free part of the Ricci spinor and the whole conformal curvature. The same argument applies to $\widetilde{\phi}^{B'}$ and to the gauge field ν_B. The present analysis corrects the statements made in section 8.8 of Esposito (1995), where it was not realized that, in our *massless* model, a non-vanishing cosmological constant is incompatible with a gauge freedom for the spin-$\frac{3}{2}$ potential. More precisely, if one sets $\Lambda = 0$ from the beginning in Eqs. (8.9.3) and (8.9.5), the system (8.9.3)–(8.9.6) admits solutions of the Weyl equation in Ricci-flat manifolds. These backgrounds are further restricted to be totally flat on considering the Eqs. (8.8.10) and (8.8.15) for an arbitrary form of the ρ- and θ-potentials. As already pointed out at the end of section 8.8, the boundary conditions (8.5.1) play a role, since otherwise one might focus on right-flat or left-flat Riemannian backgrounds with boundary.

Yet other gauge transformations can be studied (e.g. the ones involving gauge fields $\lambda_{B'}$ and ν_B which solve the twistor equations), but they are all incompatible with a non-vanishing cosmological constant in the massless case.

8.10 The superconnection

In the massless case, the two-spinor form of the Rarita–Schwinger equations is the one given in Eqs. (8.6.3)–(8.6.6) with vanishing right-hand sides, where $\nabla_{AA'}$ is the spinor covariant derivative corresponding to the connection ∇ of the background. In the massive case, however, the appropriate connection, hereafter denoted by S, has an additional term which couples to the cosmological constant $\lambda = 6\Lambda$. In the language of γ-matrices, the new covariant derivative S_μ to be inserted *in the field equations* (Townsend 1977) takes the form

$$S_\mu \equiv \nabla_\mu + f(\Lambda)\gamma_\mu, \tag{8.10.1}$$

where $f(\Lambda)$ vanishes at $\Lambda = 0$, and γ_μ are the curved-space γ-matrices. Since, following Esposito and Pollifrone (1996), we are interested in the two-spinor formulation of the problem, we have to bear in mind the action of γ-matrices on any spinor $\varphi \equiv \left(\beta^C, \widetilde{\beta}_{C'}\right)$. Note that unprimed and primed spin-spaces are no longer (anti-)isomorphic in the case of positive-definite four-metrics, since there is no complex conjugation which turns primed spinors into unprimed spinors, or the other way around (Penrose and Rindler 1986). Hence β^C and $\widetilde{\beta}_{C'}$ are totally unrelated. With this understanding, we write the supergauge transformations for massive spin-$\frac{3}{2}$ potentials in the form (cf. (8.6.7) and (8.6.8))

$$\widehat{\gamma}^A{}_{B'C'} \equiv \gamma^A{}_{B'C'} + S^A{}_{B'}\, \lambda_{C'}, \tag{8.10.2}$$

$$\widehat{\Gamma}^{A'}{}_{BC} \equiv \Gamma^{A'}{}_{BC} + S^{A'}{}_B\, \nu_C, \tag{8.10.3}$$

where the action of $S_{AA'}$ on the gauge fields $\left(\nu^B, \lambda_{B'}\right)$ is defined by (cf. (8.10.1))

$$S_{AA'}\, \nu_B \equiv \nabla_{AA'}\, \nu_B + f_1(\Lambda)\varepsilon_{AB}\, \lambda_{A'}, \tag{8.10.4}$$

$$S_{AA'}\, \lambda_{B'} \equiv \nabla_{AA'}\, \lambda_{B'} + f_2(\Lambda)\varepsilon_{A'B'}\, \nu_A. \tag{8.10.5}$$

With our notation, $R = 24\Lambda$ is the scalar curvature, f_1 and f_2 are two functions which vanish at $\Lambda = 0$, whose form will be determined later by a geometric analysis.

The action of $S_{AA'}$ on a many-index spinor $T^{A...L}_{B'...F'}$ can be obtained by expanding such a T as a sum of products of spin-vectors, i.e. (Penrose and Rindler 1984)

$$T^{A...L}_{B'...F'} = \sum_i \alpha^A_{(i)} ... \beta^L_{(i)} \, \gamma^{(i)}_{B'} ... \delta^{(i)}_{F'}, \tag{8.10.6}$$

and then applying the Leibniz rule and the definitions (8.10.4) and (8.10.5), where $\alpha^A_{(i)}$ has an independent partner $\widetilde{\alpha}^{A'}_{(i)}$, ... , $\gamma^{(i)}_{B'}$ has an independent partner $\widetilde{\gamma}^{(i)}_B$, ... , and so on. Thus, one has for example

$$\Bigl(S_{AA'} - \nabla_{AA'}\Bigr) \, T_{BCE'} = \sum_i \Bigl[f_1 \varepsilon_{AB} \, \widetilde{\alpha}^{(i)}_{A'} \, \beta^{(i)}_C \, \gamma^{(i)}_{E'} + f_1 \varepsilon_{AC} \, \alpha^{(i)}_B \, \widetilde{\beta}^{(i)}_{A'} \, \gamma^{(i)}_{E'}$$

$$+ f_2 \varepsilon_{A'E'} \, \alpha^{(i)}_B \, \beta^{(i)}_C \, \widetilde{\gamma}^{(i)}_A \Bigr]. \tag{8.10.7}$$

A further requirement is that $S_{AA'}$ should annihilate the curved ε-spinors. Hence in our analysis we always assume that

$$S_{AA'} \, \varepsilon_{BC} = 0, \tag{8.10.8}$$

$$S_{AA'} \, \varepsilon_{B'C'} = 0. \tag{8.10.9}$$

In the light of the definitions and assumptions presented so far, one can write the Rarita–Schwinger equations with non-vanishing cosmological constant $\lambda = 6\Lambda$, i.e.

$$\varepsilon^{B'C'} \, S_{A(A'} \, \gamma^A{}_{B')C'} = \Lambda \, \widetilde{F}_{A'}, \tag{8.10.10}$$

$$S^{B'(B} \, \gamma^{A)}{}_{B'C'} = 0, \tag{8.10.11}$$

$$\varepsilon^{BC} \, S_{A'(A} \, \Gamma^{A'}{}_{B)C} = \Lambda \, F_A, \tag{8.10.12}$$

$$S^{B(B'} \, \Gamma^{A')}{}_{BC} = 0. \tag{8.10.13}$$

With our notation, F_A and $\widetilde{F}_{A'}$ are spinor fields proportional to the traces of the second set of potentials for spin $\frac{3}{2}$. These will be studied in section 8.13.

8.11 Gauge freedom of the second kind

The gauge freedom of the second kind is the one which does not affect the potentials after a gauge transformation. This requirement corresponds to the case analyzed in Siklos (1985), where it is pointed out that, while the Lagrangian of $N = 1$ supergravity is invariant under gauge transformations with arbitrary spinor fields $\left(\nu^A, \lambda_{A'}\right)$, the actual *solutions* are only invariant if the supercovariant derivatives (8.10.4) and (8.10.5) vanish.

On setting to zero $S_{AA'} \, \nu_B$ and $S_{AA'} \, \lambda_{B'}$, one gets a coupled set of equations which are the Euclidean version of the Killing-spinor equation (Siklos 1985), i.e.

$$\nabla^{A'}{}_B \, \nu_C = -f_1(\Lambda)\lambda^{A'} \, \varepsilon_{BC}, \tag{8.11.1}$$

$$\nabla^{A}{}_{B'} \, \lambda_{C'} = -f_2(\Lambda)\nu^{A} \, \varepsilon_{B'C'}. \tag{8.11.2}$$

What is peculiar of Eqs. (8.11.1) and (8.11.2) is that their right-hand sides involve spinor fields which are, themselves, solutions of the twistor equation. Hence one deals with a special type of twistors, which do not exist in a generic curved manifold. Equation (8.11.1) can be solved for $\lambda^{A'}$ as

$$\lambda_{C'} = \frac{1}{2f_1(\Lambda)} \nabla_{C'}{}^{B} \, \nu_B. \tag{8.11.3}$$

The insertion of (8.11.3) into Eq. (8.11.2) and the use of spinor Ricci identities (see (8.7.3)–(8.7.6)) yields the second-order equation

$$\Box \nu_A + (6\Lambda + 8 f_1 f_2)\nu_A = 0. \tag{8.11.4}$$

On the other hand, Eq. (8.11.1) implies the twistor equation

$$\nabla^{A'}{}_{(B} \, \nu_{C)} = 0. \tag{8.11.5}$$

Covariant differentiation of Eq. (8.11.5), jointly with spinor Ricci identities, leads to (see Eq. (8.9.19))

$$\Box \nu_A - 2\Lambda \nu_A = 0. \tag{8.11.6}$$

By comparison of Eqs. (8.11.4) and (8.11.6) one finds the condition $f_1 f_2 = -\Lambda$. The integrability condition of Eq. (8.11.5) is given by (Penrose and Rindler 1986)

$$\psi_{ABCD} \, \nu^D = 0, \tag{8.11.7}$$

which implies that our manifold is conformally left-flat.

The condition $f_1 f_2 = -\Lambda$ is also obtained by comparison of first-order equations, since for example

$$\nabla^{AA'} \, \nu_A = 2 f_1 \lambda^{A'} = -2 \frac{\Lambda}{f_2} \lambda^{A'}. \tag{8.11.8}$$

The first equality in (8.11.8) results from Eq. (8.11.1), while the second one is obtained since the twistor equations also imply that (see Eq. (8.11.2))

$$\nabla^{AA'} \Bigl(- f_2 \nu_A \Bigr) = 2\Lambda \, \lambda^{A'}. \tag{8.11.9}$$

Analogous results are obtained on considering the twistor equation resulting from Eq. (8.11.2), i.e.

$$\nabla^{A}{}_{(B'} \, \lambda_{C')} = 0. \tag{8.11.10}$$

The integrability condition of Eq. (8.11.10) is

$$\widetilde{\psi}_{A'B'C'D'} \, \lambda^{D'} = 0. \tag{8.11.11}$$

Since our gauge fields cannot be four-fold principal spinors of the Weyl spinor (cf. Lewandowski (1991)), Eqs. (8.11.7) and (8.11.11) imply that our background geometry is conformally flat.

8.12 Compatibility conditions

We now require that the field equations (8.10.10)–(8.10.13) should be preserved under the action of the supergauge transformations (8.10.2) and (8.10.3). This is the procedure one follows in the massless case, and is a milder requirement with respect to the analysis of section 8.11.

If ν^B and $\lambda_{B'}$ are twistors, but not necessarily Killing spinors, they obey the Eqs. (8.11.5) and (8.11.10), which imply that, for some independent spinor fields π^A and $\widetilde{\pi}^{A'}$, one has

$$\nabla^{A'}{}_B \, \nu_C = \varepsilon_{BC} \, \widetilde{\pi}^{A'}, \tag{8.12.1}$$

$$\nabla^A{}_{B'} \, \lambda_{C'} = \varepsilon_{B'C'} \, \pi^A. \tag{8.12.2}$$

In the compatibility equations, whenever one has terms of the kind $S_{AA'} \, \nabla^A{}_{B'} \, \lambda_{C'}$, it is therefore more convenient to symmetrize and anti-symmetrize over B' and C'. A repeated use of this algorithm leads to a considerable simplification of the lengthy calculations. For example, the preservation condition of Eq. (8.10.10) has the general form

$$3f_2\Bigl(\nabla_{AA'} \, \nu^A + 2f_1\lambda_{A'}\Bigr) + \varepsilon^{B'C'}\Bigl[S_{AA'}\Bigl(\nabla^A{}_{B'} \, \lambda_{C'}\Bigr) + S_{AB'}\Bigl(\nabla^A{}_{A'} \, \lambda_{C'}\Bigr)\Bigr] = 0. \tag{8.12.3}$$

By virtue of Eq. (8.12.2), Eq. (8.12.3) becomes

$$f_2\Bigl(\nabla_{AA'} \, \nu^A + 2f_1\lambda_{A'}\Bigr) + S_{AA'} \, \pi^A = 0. \tag{8.12.4}$$

Following (8.10.4) and (8.10.5), the action of the supercovariant derivative on $\pi_A, \widetilde{\pi}_{A'}$ yields

$$S_{AA'} \, \pi_B \equiv \nabla_{AA'} \, \pi_B + f_1(\Lambda)\varepsilon_{AB} \, \widetilde{\pi}_{A'}, \tag{8.12.5}$$

$$S_{AA'} \, \widetilde{\pi}_{B'} \equiv \nabla_{AA'} \, \widetilde{\pi}_{B'} + f_2(\Lambda)\varepsilon_{A'B'} \, \pi_A. \tag{8.12.6}$$

Equations (8.12.4) and (8.12.5), jointly with the equations

$$\Box\lambda_{A'} - 2\Lambda \, \lambda_{A'} = 0, \tag{8.12.7}$$

$$\nabla^{AA'} \pi_A = 2\Lambda \lambda^{A'}, \tag{8.12.8}$$

which result from Eq. (8.12.2), lead to

$$(f_1 + f_2)\widetilde{\pi}_{A'} + (f_1 f_2 - \Lambda)\lambda_{A'} = 0. \tag{8.12.9}$$

Moreover, the preservation of Eq. (8.10.11) under (8.10.2) leads to the equation

$$S^{B'(A} \pi^{B)} + f_2 \nabla^{B'(A} \nu^{B)} = 0, \tag{8.12.10}$$

which reduces to

$$\nabla^{B'(A} \pi^{B)} = 0, \tag{8.12.11}$$

by virtue of (8.12.1) and (8.12.5). Note that a supertwistor is also a twistor, since

$$S^{B'(A} \pi^{B)} = \nabla^{B'(A} \pi^{B)}, \tag{8.12.12}$$

by virtue of the definition (8.12.5). It is now clear that, for a gauge freedom generated by twistors, the preservation of Eqs. (8.10.12) and (8.10.13) under (8.10.3) leads to the compatibility equations

$$(f_1 + f_2)\pi_A + (f_1 f_2 - \Lambda)\nu_A = 0, \tag{8.12.13}$$

$$\nabla^{B(A'} \widetilde{\pi}^{B')} = 0, \tag{8.12.14}$$

where we have also used the equation (see Eqs. (8.11.6) and (8.12.1))

$$\nabla^{AA'} \widetilde{\pi}_{A'} = 2\Lambda \nu^A. \tag{8.12.15}$$

Note that, if $f_1 + f_2 \neq 0$, one can solve Eqs. (8.12.9) and (8.12.13) as

$$\widetilde{\pi}_{A'} = \frac{(\Lambda - f_1 f_2)}{(f_1 + f_2)} \lambda_{A'}, \tag{8.12.16}$$

$$\pi_A = \frac{(\Lambda - f_1 f_2)}{(f_1 + f_2)} \nu_A, \tag{8.12.17}$$

and hence one deals again with Euclidean Killing spinors as in section 8.11. However, if

$$f_1 + f_2 = 0, \tag{8.12.18}$$

$$f_1 f_2 - \Lambda = 0, \tag{8.12.19}$$

the spinor fields $\widetilde{\pi}_{A'}$ and $\lambda_{A'}$ become *unrelated*, as well as π_A and ν_A. This is a crucial point. Hence one may have $f_1 = \pm\sqrt{-\Lambda}$, $f_2 = \mp\sqrt{-\Lambda}$, and one finds a more general structure (Esposito and Pollifrone 1996).

In the generic case, we do not assume that ν^B and $\lambda_{B'}$ obey any equation. This means that, on the second line of Eq. (8.12.3), it is more convenient to express the term in square brackets as $2S_{A(A'} \nabla^{A}{}_{B')} \lambda_{C'}$. The rule (8.10.7) for the action of $S_{AA'}$ on spinors with many indices leads therefore to the compatibility conditions

$$3f_2\Big(\nabla_{AA'}\,\nu^A + 2f_1\lambda_{A'}\Big) - 6\Lambda\,\lambda_{A'} + 4f_1\widetilde{P}_{(A'B')}{}^{B'} + 3f_2\widetilde{Q}_{A'} = 0, \tag{8.12.20}$$

$$3f_1\Big(\nabla_{AA'}\,\lambda^{A'} + 2f_2\nu_A\Big) - 6\Lambda\,\nu_A + 4f_2P_{(AB)}{}^{B} + 3f_1Q_A = 0, \tag{8.12.21}$$

$$\Phi^{AB}{}_{C'D'}\,\lambda^{D'} + f_2U^{(AB)}{}_{C'} - f_2\nabla_{C'}{}^{(A}\,\nu^{B)} = 0, \tag{8.12.22}$$

$$\widetilde{\Phi}^{A'B'}{}_{CD}\,\nu^{D} + f_1\widetilde{U}^{(A'B')}{}_{C} - f_1\nabla_{C}{}^{(A'}\,\lambda^{B')} = 0, \tag{8.12.23}$$

where the detailed form of $P, \widetilde{P}, Q, \widetilde{Q}, U, \widetilde{U}$ is not strictly necessary, but we can say that they do not depend explicitly on the trace-free part of the Ricci spinor, or on the Weyl spinors. Note that, in the massless limit $f_1 = f_2 = 0$, the Eqs. (8.12.20)–(8.12.23) reduce to the familiar form of compatibility equations which admit non-trivial solutions only in Ricci-flat backgrounds.

Our consistency analysis still makes it necessary to set to zero $\Phi^{AB}{}_{C'D'}$ (and hence $\widetilde{\Phi}^{A'B'}{}_{CD}$ by reality (Penrose and Rindler 1984)). The remaining contributions to (8.12.20)–(8.12.23) should then become algebraic relations by virtue of the twistor equation. This is confirmed by the analysis of gauge freedom for the second set of potentials in section 8.13.

8.13 Second set of potentials

According to the prescription of section 8.10, which replaces $\nabla_{AA'}$ by $S_{AA'}$ in the field equations (Townsend 1977), we now *assume* that the super Rarita–Schwinger equations corresponding to (8.8.4) and (8.8.13) are (see section 8.15)

$$S^{B'(F} \; \rho_{B'}^{\;\;A)L} = 0, \tag{8.13.1}$$

$$S^{B(F'} \; \theta_{B}^{A')L'} = 0, \tag{8.13.2}$$

where the second set of potentials are subject locally to the supergauge transformations

$$\widehat{\rho}_{B'}^{\;\;AL} \equiv \rho_{B'}^{\;\;AL} + S_{B'}^{\;\;A} \; \mu^{L}, \tag{8.13.3}$$

$$\widehat{\theta}_{B}^{\;\;A'L'} \equiv \theta_{B}^{\;\;A'L'} + S_{B}^{\;\;A'} \; \zeta^{L'}. \tag{8.13.4}$$

The analysis of the gauge freedom of the second kind is analogous to the one in section 8.11, since equations like (8.10.4) and (8.10.5) now apply to μ_L and $\zeta_{L'}$. Hence we do not repeat this investigation.

A more general gauge freedom of the twistor type relies on the supertwistor equations (see Eq. (8.12.12))

$$S_{B'}^{\;\;(A} \; \mu^{L)} = \nabla_{B'}^{\;\;(A} \; \mu^{L)} = 0, \tag{8.13.5}$$

$$S_{B}^{\;\;(A'} \; \zeta^{L')} = \nabla_{B}^{\;\;(A'} \; \zeta^{L')} = 0. \tag{8.13.6}$$

Thus, if one requires preservation of the super Rarita–Schwinger equations (8.13.1) and (8.13.2) under the supergauge transformations (8.13.3) and (8.13.4), one finds the preservation conditions

$$S^{B'(F} \; S_{B'}^{\;\;A)} \; \mu^{L} = 0, \tag{8.13.7}$$

$$S^{B(F'} \; S_{B}^{\;\;A')} \; \zeta^{L'} = 0, \tag{8.13.8}$$

which lead to

$$(f_1+f_2)\pi_F+(f_1f_2-\Lambda)\mu_F=0, \tag{8.13.9}$$

$$(f_1+f_2)\widetilde{\pi}_{F'}+(f_1f_2-\Lambda)\zeta_{F'}=0. \tag{8.13.10}$$

Hence we can repeat the remarks following Eqs. (8.12.16)–(8.12.19). Again, it is essential that π_F,μ_F and $\widetilde{\pi}_{F'},\zeta_{F'}$ may be unrelated if (8.12.18) and (8.12.19) hold. In the massless case this is impossible, and hence there is no gauge freedom compatible with a non-vanishing cosmological constant.

If one does not assume the validity of Eqs. (8.13.5) and (8.13.6), the general preservation equations (8.13.7) and (8.13.8) lead instead to the compatibility conditions

$$\begin{aligned}&\psi^{AFL}_{\quad\;\; D}\,\mu^D-2\Lambda\,\mu^{(A}\,\varepsilon^{F)L}+2f_2\omega^{(AF)L}+f_1\varepsilon^{L(A}\,T^{F)}\\&+f_1\varepsilon^{L(A}\,S^{F)B'}\,\zeta_{B'}=0,\end{aligned} \tag{8.13.11}$$

$$\begin{aligned}&\widetilde{\psi}^{A'F'L'}_{\quad\;\;\;\; D'}\,\zeta^{D'}-2\Lambda\,\zeta^{(A'}\,\varepsilon^{F')L'}+2f_1\widetilde{\omega}^{(A'F')L'}+f_2\varepsilon^{L'(A'}\,\widetilde{T}^{F')}\\&+f_2\varepsilon^{L'(A'}\,\widetilde{S}^{F')B}\,\mu_B=0.\end{aligned} \tag{8.13.12}$$

If we now combine the compatibility equations (8.12.20)–(8.12.23) with (8.13.11) and (8.13.12), and require that the gauge fields $\nu_A,\lambda_{A'},\mu_A,\zeta_{A'}$ should not depend explicitly on the curvature of the background, we find that the trace-free part of the Ricci spinor has to vanish, and the Riemannian four-geometry is forced to be conformally flat, since under our assumptions the equations

$$\psi_{AFLD}\,\mu^D=0, \tag{8.13.13}$$

$$\widetilde{\psi}_{A'F'L'D'}\,\zeta^{D'}=0, \tag{8.13.14}$$

force the anti-self-dual and self-dual Weyl spinors to vanish. Equations (8.13.13) and (8.13.14) are just the integrability conditions for the existence of non-trivial solutions of the supertwistor equations (8.13.5) and (8.13.6). Hence the spinor fields $\omega,S,T,\widetilde{\omega},\widetilde{S}$ and $\widetilde{T}$ in (8.13.11) and (8.13.12) are such that these equations reduce to (8.13.9) and (8.13.10). In other words, for massive spin-$\frac{3}{2}$ potentials, the gauge freedom is indeed generated by solutions of the twistor equations in conformally flat Einstein four-manifolds.

Last, on inserting the local equations (8.8.1) and (8.8.11) into the second half of the Rarita–Schwinger equations, and then replacing $\nabla_{AA'}$ by $S_{AA'}$, one finds equations whose preservation under the supergauge transformations (8.13.3) and (8.13.4) is again guaranteed if the supertwistor equations (8.13.5) and (8.13.6) hold.

8.14 Non-linear superconnection

As a first step in the proof that Eqs. (8.13.1) and (8.13.2) arise naturally as integrability conditions of a suitable connection, we introduce a partial superconnection $W_{A'}$ (cf. Penrose (1994)) acting on unprimed spinor fields η_D defined on the Riemannian background.

With our notation

$$W_{A'} \; \eta_D \equiv \eta^A \; S_{AA'} \; \eta_D - \eta_B \; \eta_C \; \rho_{A'}{}^{BC} \; \eta_D. \tag{8.14.1}$$

Writing

$$W_{A'} = \eta^A \; \Omega_{AA'}, \tag{8.14.2}$$

where the operator $\Omega_{AA'}$ acts on spinor fields η_D, we obtain

$$\eta^A \; \Omega_{AA'} = \eta^A \; S_{AA'} - \eta_B \; \eta_C \; \rho_{A'}{}^{BC}. \tag{8.14.3}$$

Following Penrose (1994), we require that $\Omega_{AA'}$ should provide a genuine superconnection on the spin-bundle, so that it acts in any direction. Thus, from (8.14.3) one can take (cf. Penrose (1994))

$$\Omega_{AA'} \equiv S_{AA'} - \eta^C \; \rho_{A'AC} = S_{AA'} - \eta^C \; \rho_{A'(AC)} + \frac{1}{2} \eta_A \; \rho_{A'}. \tag{8.14.4}$$

Note that (8.14.4) makes it necessary to know the trace $\rho_{A'}$, while in (8.14.1) only the symmetric part of ${\rho_{A'}}^{BC}$ survives. Thus we can see that, independently of the analysis in the previous sections, the definition of $\Omega_{AA'}$ picks out a potential of the Rarita–Schwinger type (Penrose 1994).

8.15 Integrability condition

In section 8.14 we have introduced a superconnection $\Omega_{AA'}$ which acts on a bundle with non-linear fibres, where the term $-\eta^C \ \rho_{A'AC}$ is responsible for the non-linear nature of $\Omega_{AA'}$ (see (8.14.4)). Following Penrose (1994), we now pass to a description in terms of a vector bundle of rank three. On introducing the local coordinates (u_A, ξ), where

$$u_A = \xi \ \eta_A, \tag{8.15.1}$$

the action of the new operator $\widetilde{\Omega}_{AA'}$ reads (cf. Penrose (1994))

$$\widetilde{\Omega}_{AA'}(u_B, \xi) \equiv \Bigl(S_{AA'} \ u_B, S_{AA'} \ \xi - u^C \ \rho_{A'AC}\Bigr). \tag{8.15.2}$$

Now we are able to prove that Eqs. (8.13.1) and (8.13.2) are integrability conditions.

The super β-surfaces are totally null two-surfaces whose tangent vector has the form $u^A \ \pi^{A'}$, where $\pi^{A'}$ is varying and u^A obeys the equation

$$u^A \ S_{AA'} \ u_B = 0, \tag{8.15.3}$$

which means that u^A is supercovariantly constant over the surface. On defining

$$\tau_{A'} \equiv u_B \ u_C \ {\rho_{A'}}^{BC}, \tag{8.15.4}$$

the condition for $\widetilde{\Omega}_{AA'}$ to be integrable on super β-surfaces is (cf. Penrose (1994))

$$u^A \ \widetilde{\Omega}_{AA'} \ \tau^{A'} = u_A \ u_B \ u_C \ S^{A'(A} \ {\rho_{A'}}^{B)C} = 0, \tag{8.15.5}$$

by virtue of the Leibniz rule and of (8.15.2)–(8.15.4). Equation (8.15.5) implies

$$S^{A'(A} \, \rho_{A'}^{\;\;\;B)C} = 0, \tag{8.15.6}$$

which is indeed Eq. (8.13.1). Similarly, on studying super α-surfaces defined by the equation

$$\widetilde{u}^{A'} \, S_{AA'} \, \widetilde{u}_{B'} = 0, \tag{8.15.7}$$

one obtains Eq. (8.13.2). Thus, although Eqs. (8.13.1) and (8.13.2) are naturally suggested by the local theory of spin-$\frac{3}{2}$ potentials, they have a deeper geometric origin, as shown.

8.16 Results and open problems

The consideration of boundary conditions is essential if one wants to obtain a well-defined formulation of physical theories in quantum cosmology (Hartle and Hawking 1983, Hawking 1984). In particular, one-loop quantum cosmology (Esposito 1994a, Esposito *et al.* 1997) makes it necessary to study spin-$\frac{3}{2}$ potentials about four-dimensional Riemannian backgrounds with boundary. Following Esposito (1994), Esposito and Pollifrone (1994), we have first derived the conditions (8.2.13), (8.2.15), (8.3.5) and (8.3.8) under which spin-lowering and spin-raising operators preserve the local boundary conditions studied in Breitenlohner and Freedman (1982), Hawking (1983), Esposito (1994). Note that, for spin 0, we have introduced a pair of independent scalar fields on the real Riemannian section of a complex space-time, following Hawking (1979), rather than a single scalar field, as done in Esposito (1994). Setting $\phi \equiv \phi_1 + i\phi_2, \widetilde{\phi} \equiv \phi_3 + i\phi_4$, this choice leads to the boundary conditions

$$\phi_1 = \epsilon \, \phi_3 \text{ on } S^3, \tag{8.16.1}$$

$$\phi_2 = \epsilon \, \phi_4 \text{ on } S^3, \tag{8.16.2}$$

$$ {}_e n^{AA'} D_{AA'} \phi_1 = -\epsilon \; {}_e n^{AA'} D_{AA'} \phi_3 \text{ on } S^3, \tag{8.16.3} $$

$$ {}_e n^{AA'} D_{AA'} \phi_2 = -\epsilon \; {}_e n^{AA'} D_{AA'} \phi_4 \text{ on } S^3, \tag{8.16.4} $$

and it deserves further study.

We have then focused on the Dirac potentials for spin-$\frac{3}{2}$ field strengths in flat or curved Riemannian four-space bounded by a three-sphere. Remarkably, it turns out that local boundary conditions involving field strengths and normals can only be imposed in a flat Euclidean background, for which the gauge freedom in the choice of the potentials remains. In Penrose (1991c) it was found that ρ potentials exist *locally* only in the self-dual Ricci-flat case, whereas γ potentials may be introduced in the anti-self-dual case. Our result may be interpreted as a further restriction provided by (quantum) cosmology. What happens is that the boundary conditions (8.2.1) fix at the boundary a spinor field involving *both* the field strength ϕ_{ABC} and the field strength $\widetilde{\phi}_{A'B'C'}$. The local existence of potentials for the field strength ϕ_{ABC}, jointly with the occurrence of a boundary, forces half of the Riemann curvature of the background to vanish. Similarly, the remaining half of such Riemann curvature has to vanish on considering the field strength $\widetilde{\phi}_{A'B'C'}$. Hence the background four-geometry can only be flat Euclidean space. This is different from the analysis in Penrose (1990), Penrose (1991a,b), since in that case one is not dealing with boundary conditions forcing us to consider both ϕ_{ABC} and $\widetilde{\phi}_{A'B'C'}$.

A naturally occurring question is whether the Dirac potentials can be used to perform one-loop calculations for spin-$\frac{3}{2}$ field strengths subject to (8.2.1) on S^3. This problem may provide another example of the fertile interplay between twistor theory and quantum cosmology (Esposito 1994), and its solution might shed new light on one-loop quantum cosmology and on the quantization program for gauge theories in the presence of boundaries. For this purpose, it is necessary to study Riemannian background four-geometries bounded by two three-surfaces (cf. Kamenshchik and Mishakov (1994)). Moreover, the consideration of non-physical degrees of freedom of gauge fields, set to zero in our classical analysis, is necessary to achieve a covariant quantization scheme.

Sections 8.6–8.9 have studied Rarita–Schwinger potentials in four-dimensional Riemannian backgrounds with boundary, to complement the analysis of Dirac's potentials appearing in section 8.4. Our results are as follows. First, the gauge transformations (8.6.7) and (8.6.8) are compatible with the massless Rarita–Schwinger equations provided that the background four-geometry is Ricci-flat (Deser and Zumino 1976). However, the presence of a boundary restricts the gauge freedom, since the boundary conditions (8.5.1) are preserved under the action of (8.6.7) and (8.6.8) only if the boundary conditions (8.7.11) hold.

Second, the Penrose construction of a second set of potentials in Ricci-flat four-manifolds shows that the admissible backgrounds may be further restricted to be totally flat, or left-flat, or right-flat, unless these potentials take the special form (8.8.16) and (8.8.17). Hence the potentials supplementing the Rarita–Schwinger potentials have a very clear physical meaning in Ricci-flat four-geometries with boundary: they are related to the spinor fields $\left(\alpha_A, \widetilde{\alpha}_{A'}\right)$ corresponding to the Majorana field in the Lorentzian version of Eqs. (8.6.3)–(8.6.6). [One should bear in mind that, in real Riemannian four-manifolds, the only admissible spinor conjugation is Euclidean conjugation, which is anti-involutory on spinor fields with an odd number of indices (Woodhouse 1985). Hence no Majorana field can be defined in real Riemannian four-geometries.]

Third, to ensure unrestricted gauge freedom for the ρ- and θ-potentials, one is forced to work with flat Euclidean backgrounds, when the boundary conditions (8.5.1) are imposed. Thus, the very restrictive results obtained in Esposito and Pollifrone (1994) for massless Dirac potentials with the boundary conditions (8.2.7) are indeed confirmed also for massless Rarita–Schwinger potentials subject to the supersymmetric boundary conditions (8.5.1). Interestingly, a formalism originally motivated by twistor theory has been applied to classical boundary-value problems relevant for one-loop quantum cosmology.

Fourth, the gauge transformations (8.9.1) and (8.9.2) with non-trivial gauge fields are compatible with the field equations (8.6.3)–(8.6.6) if and only if the background is totally flat. The corresponding gauge fields solve the Weyl equations (8.9.7) and (8.9.8), subject to the boundary conditions (8.9.9). Indeed, it is well known that the Rarita–Schwinger description of a massless spin-$\frac{3}{2}$ field is equivalent to the Dirac description in a special choice of gauge (Penrose 1994). In such a gauge, the spinor fields $\lambda_{B'}$ and ν_B solve the Weyl equations, and this is exactly what we find in section 8.9 on choosing the gauge transformations (8.9.1) and (8.9.2).

Moreover, some interesting problems are found to arise:

(i) Can one relate Eqs. (8.8.4) and (8.8.13) to the theory of integrability conditions relevant for massless fields in curved backgrounds (see Penrose (1994))? What happens when such equations do not hold?

(ii) Is there an underlying global theory of Rarita–Schwinger potentials? In the affirmative case, what are the key features of the global theory?

(iii) Can one reconstruct the Riemannian four-geometry from the twistor space in Ricci-flat or conformally flat backgrounds with boundary, or from whatever is going to replace twistor space?

Thus, the results and problems presented in our chapter seem to add evidence in favour of a deep link existing between twistor geometry, quantum cosmology and modern field theory.

In the sections 8.10–8.15, we have given an entirely two-spinor description of massive spin-$\frac{3}{2}$ potentials in Einstein four-geometries. Although the supercovariant derivative (8.10.1) was well known in the literature, following the work in Townsend (1977), and its Lorentzian version was already applied in Perry (1984) and Siklos (1985), the systematic analysis of spin-$\frac{3}{2}$ potentials with the local form of their supergauge transformations was not yet available in the literature, to the best of our knowledge, before the work in Esposito and Pollifrone (1996).

Our first result is the two-spinor proof that, for massive spin-$\frac{3}{2}$ potentials, the gauge freedom is generated by solutions of the supertwistor equations in conformally flat Einstein four-manifolds. Moreover, we have shown that the first-order equations (8.13.1) and (8.13.2), whose consideration is suggested by the local theory of massive spin-$\frac{3}{2}$ potentials, admit a deeper geometric interpretation as integrability conditions on super β- and super α-surfaces of a connection on a rank-three vector bundle. One now has to find explicit solutions of Eqs. (8.10.10)–(8.10.13), and the supercovariant form of β-surfaces studied in our chapter deserves a more careful consideration. Hence we hope that our work can lead to a better understanding of twistor geometry and consistent supergravity theories in four dimensions. For other work on spin-$\frac{3}{2}$ potentials and supercovariant derivatives, the reader is referred to Tod (1983), Torres del Castillo (1989), Torres del Castillo (1990), Torres del Castillo (1992), Frauendiener (1995), Izquierdo and Townsend (1995), Tod (1995), Frauendiener *et al.* (1996), Tod (1996).

CHAPTER NINE

UNDERLYING MATHEMATICAL STRUCTURES

This chapter begins with a review of four definitions of twistors in curved space-time proposed by Penrose in the seventies, i.e. local twistors, global null twistors, hypersurface twistors and asymptotic twistors. The Penrose transform for gravitation is then re-analyzed, with emphasis on the double-fibration picture. Double fibrations are also used to introduce the ambitwistor correspondence, and the Radon transform in complex analysis is mentioned. Attention is then focused on the Ward picture of massless fields as bundles, which has motivated the analysis by Penrose of a second set of potentials which supplement the Rarita–Schwinger potentials in curved space-time (chapter eight). The boundary conditions studied in chapters seven and eight have been recently applied in the quantization program of field theories. Hence the chapter ends with a review of progress made in studying bosonic fields subject to boundary conditions respecting BRST invariance and local supersymmetry. Interestingly, it remains to be seen whether the methods of spectral geometry can be applied to obtain an explicit proof of gauge independence of quantum amplitudes.

9.1 Introduction

This review chapter is written for those readers who are more interested in the mathematical foundations of twistor theory (see appendices C and D). In Minkowski space-time, twistors are defined as the elements of the vector space of solutions of the differential equation (4.1.5), or as α-planes. The latter concept, more geometric, has been extended to curved space-time through the totally null surfaces called α-surfaces, whose integrability condition (in the absence of torsion) is the vanishing of the self-dual Weyl spinor. To avoid having to set to zero half of the conformal curvature of complex space-time, yet another definition of twistors, i.e. charges for massless spin-$\frac{3}{2}$ fields in Ricci-flat space-times, has been proposed by Penrose.

The first part of this chapter supplements these efforts by describing various definitions of twistors in curved space-time. Each of these ideas has its merits and its drawbacks. To compare local twistors at different points of space-time one is led to introduce local twistor transport (cf. section 4.3) along a curve, which moves the point with respect to which the twistor is defined, but not the twistor itself.

On studying the space of null twistors, a closed two-form and a one-form are naturally obtained, but their definition cannot be extended to non-null twistors unless one studies Minkowski space-time. In other words, one deals with a symplectic structure which remains invariant, since a non-rotating congruence of null geodesics remains non-rotating in the presence of curvature. However, the attempt to obtain an invariant complex structure fails, since a shear-free congruence of null geodesics acquires shear in the presence of conformal curvature.

If an analytic space-time with analytic hypersurface $\mathcal{S}$ in it are given, one can, however, construct an hypersurface twistor space relative to $\mathcal{S}$. The differential equations describing the geometry of hypersurface twistors encode, by construction, the information on the complex structure, which here retains a key role. The differential forms introduced in the theory of global null twistors can also be expressed in the language of hypersurface twistors. However, the whole construction relies on the choice of some analytic (spacelike) hypersurface in curved space-time.

To overcome this difficulty, asymptotic twistors are introduced in asymptotically flat space-times. One is thus led to combine the geometry of future and past null infinity, which are null hypersurfaces, with the differential equations of hypersurface twistors and with the local twistor description. Unfortunately, it is unclear how to achieve such a synthesis in a generic space-time.

In the second part, attention is focused on the geometry of conformally invariant operators, and on the description of the Penrose transform in a more abstract mathematical language, i.e. in terms of a double fibration of the projective primed spin-bundle over twistor space and space-time, respectively. The ambitwistor correspondence of Le Brun is then introduced, in terms of a holomorphic double fibration, and a mention is made of the Radon transform, i.e. an integral transform which associates to a real-valued function on R^2 its integral along a straight line in R^2. Such a mathematical construction is very important for modern twistor theory, by virtue of its links with the abstract theory of the Penrose transform.

Ward's construction of twisted photons and massless fields as bundles is described in section 9.9, since it enables one to understand the geometric structures underlying the theory of spin-$\frac{3}{2}$ potentials used in section 8.8. In particular, Eq. (8.8.4) is related to a class of integrability conditions arising from the generalization of Ward's construction, as is shown in Penrose (1994). Remarkably, this sheds new light on the differential equations describing the local theory of spin-$\frac{3}{2}$ potentials (cf. section 8.15).

Since the boundary conditions of chapters seven and eight are relevant for the elliptic boundary-value problems occurring in modern attempts to obtain a mathematically consistent formulation of quantum field theories in the presence of boundaries, recent progress on these problems is summarized in section 9.10. While the conformal anomalies for gauge fields in Riemannian manifolds with boundary have been correctly evaluated after many years of dedicated work by several authors, it remains to be seen whether the *explicit* (i.e. not formal) proof of gauge independence of quantum amplitudes can be obtained. It appears exciting that gauge independence of quantum amplitudes might be related to the invariance under homotopy of the residue of a meromorphic function, obtained from the eigenvalues of the elliptic operators of the problem.

9.2 Local twistors

A *local* twistor Z^α at $P \in \mathcal{M}$ is represented by a pair of spinors $\omega^A, \pi_{A'}$ at P:

$$Z^\alpha \longleftrightarrow \Bigl(\omega^A, \pi_{A'}\Bigr), \tag{9.2.1}$$

with respect to the metric g on $\mathcal{M}$. After a conformal rescaling $\widehat{g} \equiv \Omega^2 g$ of the metric, the representation of Z^α changes according to the rule

$$\Bigl(\widehat{\omega}^A, \widehat{\pi}_{A'}\Bigr) = \Bigl(\omega^A, \pi_{A'} + i\, T_{AA'}\, \omega^A\Bigr), \tag{9.2.2}$$

where $T_{AA'} \equiv \nabla_{AA'} \log(\Omega)$. The comparison of local twistors at *different points* of $\mathcal{M}$ makes it necessary to introduce the *local twistor transport* along a curve τ in $\mathcal{M}$ with tangent vector t. This does not lead to a displacement of the twistor along τ, but moves the *point* with respect to which the twistor is defined. On defining the spinor

$$P_{AA'BB'} \equiv \frac{1}{12} R\, g_{AA'BB'} - \frac{1}{2} R_{AA'BB'}, \tag{9.2.3}$$

the equations of local twistor transport are (cf. Eqs. (4.3.20) and (4.3.21))

$$t^{BB'}\, \nabla_{BB'}\, \omega^A = -i\, t^{AB'}\, \pi_{B'}, \tag{9.2.4}$$

$$t^{BB'}\, \nabla_{BB'}\, \pi_{A'} = -i\, P_{BB'AA'}\, t^{BB'}\, \omega^A. \tag{9.2.5}$$

A more general concept is the one of covariant derivative in the t-direction of a *local twistor field* on $\mathcal{M}$ according to the rule

$$t^{BB'} \nabla_{BB'} Z^{\alpha} \longleftrightarrow \Bigl(t^{BB'} \nabla_{BB'} \omega^{A} + i \, t^{AB'} \pi_{B'},$$

$$t^{BB'} \nabla_{BB'} \pi_{A'} + i \, P_{BB'AA'} \, t^{BB'} \omega^{A} \Bigr). \qquad (9.2.6)$$

After a conformal rescaling of the metric, both Z^{α} and its covariant derivative change according to (9.2.2). In particular, this implies that local twistor transport is conformally invariant.

The presence of conformal curvature is responsible for a local twistor not returning to its original state after being carried around a small loop by local twistor transport. In fact, as shown in Penrose (1975), denoting by $[t,u]$ the Lie bracket of t and u, one finds

$$\Bigl[t^{p} \nabla_{p}, u^{q} \nabla_{q} \Bigr] Z^{\beta} - [t,u]^{p} \nabla_{p} Z^{\beta} \longleftrightarrow t^{PP'} \, u^{QQ'} \left\{ S^{B}_{PP'QQ'}, V_{PP'QQ'B'} \right\}, \quad (9.2.7)$$

where

$$S^{B}_{PP'QQ'} \equiv \varepsilon_{P'Q'} \, \psi_{PQA}{}^{B} \, \omega^{A}, \qquad (9.2.8)$$

$$V_{PP'QQ'B'} \equiv -i \Bigl(\varepsilon_{PQ} \, \nabla_{AA'} \, \widetilde{\psi}_{B'P'Q'}{}^{A'} + \varepsilon_{P'Q'} \, \nabla_{BB'} \, \psi_{APQ}{}^{B} \Bigr) \omega^{A}$$

$$- \, \varepsilon_{PQ} \, \widetilde{\psi}_{P'Q'B'}{}^{A'} \, \pi_{A'}. \qquad (9.2.9)$$

Equation (9.2.7) implies that, for these twistors to be defined globally on space-time, our $(\mathcal{M}, g)$ should be conformally flat.

In a Lorentzian space-time $(\mathcal{M}, g)_{L}$, one can define local twistor transport of dual twistors W_{α} by complex conjugation of Eqs. (9.2.4) and (9.2.5). On re-interpreting the complex conjugate of ω^{A} (resp. $\pi_{A'}$) as some spinor $\pi^{A'}$ (resp. ω_{A}), this leads to

$$t^{BB'} \nabla_{BB'} \pi^{A'} = i \, t^{BA'} \omega_{B}, \qquad (9.2.10)$$

$$t^{BB'} \nabla_{BB'} \omega_{A} = i \, P_{BB'AA'} \, t^{BB'} \pi^{A'}. \qquad (9.2.11)$$

Moreover, in $(\mathcal{M}, g)_L$ the covariant derivative in the t-direction of a *local dual twistor field* is also obtained by complex conjugation of (9.2.6), and leads to

$$t^{BB'} \nabla_{BB'} W_\alpha \longleftrightarrow \Bigl(t^{BB'} \nabla_{BB'} \omega_A - i \, P_{BB'AA'} \, t^{BB'} \, \pi^{A'},$$
$$t^{BB'} \nabla_{BB'} \pi^{A'} - i \, t^{BA'} \, \omega_B\Bigr). \tag{9.2.12}$$

One thus finds

$$t^b \nabla_b \Bigl(Z^\alpha W_\alpha\Bigr) = Z^\alpha \, t^b \nabla_b \, W_\alpha + W_\alpha \, t^b \nabla_b \, Z^\alpha, \tag{9.2.13}$$

where the left-hand side denotes the ordinary derivative of the scalar $Z^\alpha W_\alpha$ along τ. This implies that, if local twistor transport of Z^α and W_α is preserved along τ, their scalar product is covariantly constant along τ.

9.3 Global null twistors

To define global null twistors one is led to consider null geodesics Z in curved space-time, and the $\pi_{A'}$ spinor parallelly propagated along Z. The corresponding momentum vector $p_{AA'} = \overline{\pi}_A \, \pi_{A'}$ is then tangent to Z. Of course, we want the resulting space $\mathcal{N}$ of null twistors to be physically meaningful. Following Penrose (1975), the space-time $(\mathcal{M}, g)$ is taken to be globally hyperbolic to ensure that $\mathcal{N}$ is a Hausdorff manifold (see section 1.2). Since the space of unscaled null geodesics is five-dimensional, and the freedom for $\pi_{A'}$ is just a complex multiplying factor, the space of null twistors turns out to be seven-dimensional. Global hyperbolicity of $\mathcal{M}$ is indeed the strongest causality assumption, and it ensures that Cauchy surfaces exist in $\mathcal{M}$ (Hawking and Ellis 1973, Esposito 1994, and references therein).

On $\mathcal{N}$ a closed two-form ω exists, i.e.

$$\omega \equiv dp_a \wedge dx^a. \tag{9.3.1}$$

Although ω is initially defined on the cotangent bundle $T^*\mathcal{M}$, it actually yields a two-form on $\mathcal{N}$ if it is taken to be constant under the rescaling

$$\pi_{A'} \rightarrow e^{i\theta} \, \pi_{A'}, \tag{9.3.2}$$

with real parameter θ. Such a two-form may be viewed as the rotation of a congruence, since it can be written as

$$\omega = \nabla_{[b} \, p_{c]} \, dx^b \wedge dx^c, \tag{9.3.3}$$

where $\nabla_{[b} \, p_{c]}$ yields the rotation of the field p on $\mathcal{M}$, for a congruence of geodesics. Our two-form ω may be obtained by exterior differentiation of the one-form

$$\phi \equiv p_a \, dx^a, \tag{9.3.4}$$

i.e.

$$\omega = d\phi. \tag{9.3.5}$$

Note that ϕ is defined on the space of null twistors and is constant under the rescaling (9.3.2). Penrose has proposed an interpretation of ϕ as measuring the time-delay in a family of scaled null geodesics (Penrose 1975).

The main problem is how to extend these definitions to non-null twistors. Indeed, this is possible in Minkowski space-time, where

$$\omega = i \, dZ^\alpha \wedge d\overline{Z}_\alpha, \tag{9.3.6}$$

$$\phi = i \, Z^\alpha \, d\overline{Z}_\alpha. \tag{9.3.7}$$

It is clear that Eqs. (9.3.6) and (9.3.7), if viewed as definitions, do not depend on the twistor Z^α being null (in Minkowski). Alternative choices for ϕ are

$$\phi_1 \equiv -i \, \overline{Z}_\alpha \, dZ^\alpha, \tag{9.3.8}$$

$$\phi_2 \equiv \frac{i}{2}\Big(Z^\alpha \, d\overline{Z}_\alpha - \overline{Z}_\alpha \, dZ^\alpha\Big). \tag{9.3.9}$$

The *invariant structure* of (flat) twistor space is then given by the one-form ϕ, the two-form ω, and the scalar $s \equiv \frac{1}{2} \; Z^{\alpha} \; \overline{Z}_{\alpha}$. Although one might be tempted to consider only ϕ and s as basic structures, since exterior differentiation yields ω as in (9.3.5), the two-form ω is very important since it provides a symplectic structure for flat twistor space (cf. Tod (1977)). However, if one restricts ω to the space of null twistors, one first has to factor out the phase circles

$$Z^{\beta} \rightarrow e^{i\theta} \; Z^{\beta}, \tag{9.3.10}$$

θ being real, to obtain again a symplectic structure. On restriction to $\mathcal{N}$, the triple (ω, ϕ, s) has an invariant meaning also in curved space-time, hence its name.

Suppose now that there are two regions M_1 and M_2 of Minkowski space-time separated by a region of curved space-time (Penrose 1975). In each flat region, one can define ω and ϕ on twistor space according to (9.3.6) and (9.3.7), and then re-express them as in (9.3.1), (9.3.4) on the space $\mathcal{N}$ of null twistors in curved space-time. If there are regions of $\mathcal{N}$ where *both* definitions are valid, the flat-twistor-space definitions should agree with the curved ones in these regions of $\mathcal{N}$. However, it is unclear how to carry a *non-null* twistor from M_1 to M_2, if in between them there is a region of curved space-time.

It should be emphasized that, although one has a good definition of *invariant structure* on the space $\mathcal{N}$ of null twistors in curved space-time, with the corresponding symplectic structure, such a construction of global null twistors does not enable one to introduce a complex structure. The underlying reason is that a *non-rotating* congruence of null geodesics remains non-rotating on passing through a region of curved space-time. By contrast, a *shear-free* congruence of null geodesics acquires shear on passing through a region of conformal curvature. This is why the symplectic structure is invariant, while *the complex structure is not invariant and is actually affected by the conformal curvature.*

Since twistor theory relies instead on holomorphic ideas and complex structures in a conformally invariant framework, it is necessary to introduce yet another definition of twistors in curved space-time, where the complex structure retains its key role. This problem is studied in the following section.

9.4 Hypersurface twistors

Given some hypersurface $\mathcal{S}$ in space-time, we are going to construct a twistor space $T(\mathcal{S})$, relative to $\mathcal{S}$, with an associated complex structure. On going from $\mathcal{S}$ to a different hypersurface $\mathcal{S}'$, the corresponding twistor space $T(\mathcal{S}')$ turns out to be a complex manifold different from $T(\mathcal{S})$. For any $T(\mathcal{S})$, its elements are the hypersurface twistors. To construct these mathematical structures, we follow again Penrose (1975) and we focus on an analytic space-time $\mathcal{M}$, with analytic hypersurface $\mathcal{S}$ in $\mathcal{M}$. These assumptions enable one to consider the corresponding complexifications $C\mathcal{M}$ and $C\mathcal{S}$. We know from chapter four that any twistor Z^α in $\mathcal{M}$ defines a totally null plane CZ and a spinor $\pi_{A'}$ such that the tangent vector to CZ takes the form $\xi^A \; \pi^{A'}$. Since $\pi_{A'}$ is constant on CZ, it is also constant along the complex curve γ giving the intersection $CZ \cap C\mathcal{S}$. The geometric objects we are interested in are the normal n to $C\mathcal{S}$ and the tangent t to γ. Since, by construction, t has to be orthogonal to n:

$$n_{AA'} \; t^{AA'} = 0, \tag{9.4.1}$$

it can be written in the form

$$t^{AA'} = n^{AB'} \; \pi_{B'} \; \pi^{A'}, \tag{9.4.2}$$

which clearly satisfies (9.4.1) by virtue of the identity $\pi_{B'} \; \pi^{B'} = 0$. Thus, for $\pi_{A'}$ to be constant along γ, the following equation should hold:

$$t^{AA'} \; \nabla_{AA'} \; \pi_{C'} = n^{AB'} \; \pi_{B'} \; \pi^{A'} \; \nabla_{AA'} \; \pi_{C'} = 0. \tag{9.4.3}$$

Note that Eq. (9.4.3) also provides a differential equation for γ (i.e., for a given normal, the direction of γ is fixed by (9.4.2)), and the solutions of (9.4.3) on

$C\mathcal{S}$ are the elements of the hypersurface twistor space $T(\mathcal{S})$. Since no complex conjugation is involved in deriving Eq. (9.4.3), the resulting $T(\mathcal{S})$ is a complex manifold (see section 3.3).

It is now helpful to introduce some notation. We write $Z^{(h)}$ for any element of $T(\mathcal{S})$, and we remark that if $Z^{(h)} \in T(\mathcal{S})$ corresponds to $\pi_{A'}$ along γ satisfying (9.4.3), then $\rho Z^{(h)} \in T(\mathcal{S})$ corresponds to $\rho \pi_{A'}$ along the same curve γ, $\forall \rho \in C$ (Penrose 1975). This means one may consider the space $PT(\mathcal{S})$ of *equivalence classes* of proportional hypersurface twistors, and regard it as the space of curves γ defined above. The zero-element $0^{(h)} \in T(\mathcal{S})$, however, does not correspond to any element of $PT(\mathcal{S})$. For each $Z^{(h)} \in T(\mathcal{S})$, $0Z^{(h)}$ is defined as $0^{(h)} \in T(\mathcal{S})$. If the curve γ contains a real point of $\mathcal{S}$, the corresponding hypersurface twistor $Z^{(h)} \in T(\mathcal{S})$ is said to be *null*. Of course, one may well ask how many real points of $\mathcal{S}$ can be found on γ. It turns out that, if the complexification $C\mathcal{S}$ of $\mathcal{S}$ is suitably chosen, only one real point of $\mathcal{S}$ can lie on each of the curves γ. The set $P\mathcal{N}(\mathcal{S})$ of such curves is five-real-dimensional, and the corresponding set $\mathcal{N}(\mathcal{S})$, i.e. the γ-curves with $\pi_{A'}$ spinor, is seven-real-dimensional. Moreover, the hypersurface twistor space is four-complex-dimensional, and the space $PT(\mathcal{S})$ of equivalence classes defined above is three-complex-dimensional.

The space $\mathcal{N}(\mathcal{S})$ of null hypersurface twistors has two remarkable properties:

(i) $\mathcal{N}(\mathcal{S})$ may be identified with the space $\mathcal{N}$ of global null twistors defined in section 9.3. To prove this one points out that the spinor $\pi_{A'}$ at the real point of γ (for $Z^{(h)} \in \mathcal{N}(\mathcal{S})$) defines a null geodesic in $\mathcal{M}$. Such a null geodesic passes through that point in the real null direction given by $v^{AA'} \equiv \overline{\pi}^{A} \ \pi^{A'}$. Parallel propagation of $\pi_{A'}$ along this null geodesic yields a unique element of $\mathcal{N}$. On the other hand, each global null twistor in $\mathcal{N}$ defines a null geodesic and a $\pi_{A'}$. Such a null geodesic intersects $\mathcal{S}$ at a unique point. A unique γ-curve in $C\mathcal{S}$ exists, passing through this point x and defined uniquely by $\pi_{A'}$ at x.

(ii) The hypersurface $\mathcal{S}$ enables one to supplement the elements of $\mathcal{N}(\mathcal{S})$ by some non-null twistors, giving rise to the four-complex-dimensional manifold $T(\mathcal{S})$. Unfortunately, the whole construction depends on the particular choice of (spacelike Cauchy) hypersurface in $(\mathcal{M}, g)$.

The holomorphic operation

$$Z^{(h)} \rightarrow \rho \, Z^{(h)}, \; Z^{(h)} \in T(\mathcal{S}),$$

enables one to introduce homogeneous holomorphic functions on $T(\mathcal{S})$. Setting to zero these functions gives rise to regions of $CT(\mathcal{S})$ corresponding to congruences of γ-curves on $\mathcal{S}$. A congruence of null geodesics in $\mathcal{M}$ is defined by γ-curves on $\mathcal{S}$ having real points. Consider now $\pi_{A'}$ as a spinor field on $C(\mathcal{S})$, subject to the scaling $\pi_{A'} \rightarrow \rho \, \pi_{A'}$. On making this scaling, the new field $\beta_{A'} \equiv \rho \, \pi_{A'}$ no longer solves Eq. (9.4.3), since the following term survives on the left-hand side:

$$E_{C'} \equiv n^{AB'} \; \pi_{B'} \; \pi_{C'} \; \pi^{A'} \; \nabla_{AA'} \rho. \tag{9.4.4}$$

This suggests to consider the weaker condition

$$n^{AB'} \; \pi_{B'} \left(\pi^{A'} \; \pi^{C'} \; \nabla_{AA'} \; \pi_{C'} \right) = 0 \text{ on } \mathcal{S}, \tag{9.4.5}$$

since $\pi^{C'}$ has a vanishing contraction with $E_{C'}$. Equation (9.4.5) should be regarded as an equation for the spinor field $\pi_{A'}$ restricted to $\mathcal{S}$. Following Penrose (1975), round brackets have been used to emphasize the role of the spinor field

$$B_A \equiv \pi^{A'} \; \pi^{C'} \; \nabla_{AA'} \; \pi_{C'},$$

whose vanishing leads to a shear-free congruence of null geodesics with tangent vector $v^{AA'} \equiv \overline{\pi}^A \; \pi^{A'}$.

A careful consideration of extensions and restrictions of spinor fields enables one to write an equivalent form of Eq. (9.4.5). In other words, if we extend $\pi_{A'}$ to a spinor field on the whole of $\mathcal{M}$, Eq. (9.4.5) holds if we replace $n^{AB'} \; \pi_{B'}$ by

$\overline{\pi}^A$. This implies that the same equation holds on $\mathcal{S}$ if we omit $n^{AB'}\ \pi_{B'}$. Hence one eventually deals with the equation

$$\pi^{A'}\ \pi^{C'}\ \nabla_{AA'}\ \pi_{C'} = 0. \tag{9.4.6}$$

Since it is well known in general relativity that conformal curvature is responsible for a shear-free congruence of null geodesics to acquire shear, the previous analysis proves that the complex structure of hypersurface twistor space is affected by the particular choice of $\mathcal{S}$ unless the space-time is conformally flat.

The *dual hypersurface twistor space* $T^*(\mathcal{S})$ may be defined by interchanging primed and unprimed indices in Eq. (9.4.3), i.e.

$$n^{BA'}\ \widetilde{\pi}_B\ \widetilde{\pi}^A\ \nabla_{AA'}\ \widetilde{\pi}_C = 0. \tag{9.4.7}$$

In agreement with the notation used in our paper and proposed by Penrose, the *tilde* symbol denotes spinor fields not obtained by complex conjugation of the spinor fields living in the complementary spin-space, since, in a complex manifold, complex conjugation is not invariant under holomorphic coordinate transformations. Hence the complex nature of $T(\mathcal{S})$ and $T^*(\mathcal{S})$ is responsible for the spinor fields in (9.4.3) and (9.4.7) being totally independent. Equation (9.4.7) defines a unique complex curve $\widetilde{\gamma}$ in $C\mathcal{S}$ through each point of $C\mathcal{S}$. The geometric interpretation of $n^{BA'}\ \widetilde{\pi}_B\ \widetilde{\pi}^A$ is in terms of the tangent direction to the curve $\widetilde{\gamma}$ for any choice of $\widetilde{\pi}_A$. The curve $\widetilde{\gamma}$ and the spinor field $\widetilde{\pi}_A$ solving Eq. (9.4.7) define a dual hypersurface twistor $\widetilde{Z}_{(h)} \in T^*(\mathcal{S})$. Indeed, the *complex conjugate* $\overline{Z}_{(h)}$ of the hypersurface twistor $Z^{(h)} \in T(\mathcal{S})$ may also be defined if the following conditions hold:

$$\widetilde{\pi}_A = \overline{\pi}_A,\ \widetilde{\gamma} = \gamma. \tag{9.4.8}$$

The *incidence* between $Z^{(h)} \in T(\mathcal{S})$ and $\widetilde{Z}_{(h)} \in T^*(S)$ is instead defined by the condition

$$Z^{(h)}\ \widetilde{Z}_{(h)} = 0, \tag{9.4.9}$$

where (h) is not an index, but a label to denote *hypersurface* twistors (instead of the dot used in Penrose (1975)). Thus, γ and $\widetilde{\gamma}$ have a point of $C\mathcal{S}$ in common. Null hypersurface twistors are then defined by the condition

$$Z^{(h)} \, \overline{Z}_{(h)} = 0. \tag{9.4.10}$$

However, it is hard to make sense of the (scalar) product $Z^{(h)} \, \widetilde{Z}_{(h)}$ for arbitrary elements of $T(\mathcal{S})$ and $T^*(\mathcal{S})$, respectively.

We are now interested in holomorphic maps

$$F : T^*(\mathcal{S}) \times T(\mathcal{S}) \rightarrow C. \tag{9.4.11}$$

Since $T(\mathcal{S})$ and $T^*(\mathcal{S})$ are both four-complex-dimensional, the space $T^*(\mathcal{S}) \times T(\mathcal{S})$ is eight-complex-dimensional. A seven-complex dimensional subspace $\widetilde{N}(\mathcal{S})$ can be singled out in $T^*(\mathcal{S}) \times T(\mathcal{S})$, on considering those pairs $\left(\widetilde{Z}_{(h)}, Z^{(h)}\right)$ such that Eq. (9.4.9) holds. One may want to study these holomorphic maps in the course of writing contour-integral formulae for solutions of the massless free-field equations, where the integrand involves a homogeneous function F acting on twistors and dual twistors. Omitting the details (Penrose 1975), we only say that, when the space-time point y under consideration does not lie on $C\mathcal{S}$, one has to reinterpret F as a function of $U_{(h)} \in T^*(\mathcal{S}')$, $X^{(h)} \in T(\mathcal{S}')$, where the hypersurface $\mathcal{S}'$, or $C\mathcal{S}'$, is chosen to pass through the point y.

A naturally occurring question is how to deal with the one-form ϕ and the two-form ω introduced in section 9.3. Indeed, if the space-time is analytic, such forms ϕ and ω can be complexified. On making a complexification, two one-forms ϕ and $\widetilde{\phi}$ are obtained, which take the same values on $C\mathcal{N}$, but whose functional forms are different. For $Z^{(h)} \in T(\mathcal{S})$, $W_{(h)} \in T^*(\mathcal{S})$, $X^{(h)} \in T(\mathcal{S}')$, $U_{(h)} \in T^*(\mathcal{S}')$, $\mathcal{S}$ and $\mathcal{S}'$ being two different hypersurfaces in $\mathcal{M}$, one has (Penrose 1975)

$$\omega = i \, dZ^{(h)} \wedge dW_{(h)} = i \, dX^{(h)} \wedge dU_{(h)}, \tag{9.4.12}$$

$$\phi = i \, Z^{(h)} \, dW_{(h)} = i \, X^{(h)} \, dU_{(h)}, \tag{9.4.13}$$

$$\widetilde{\phi} = -i \, W_{(h)} \, dZ^{(h)} = -i \, U_{(h)} \, dX^{(h)}. \qquad (9.4.14)$$

Hence one is led to ask wether the passage from a $\Bigr(W_{(h)}, Z^{(h)}\Bigr)$ description on $\mathcal{S}$ to a $\Bigr(U_{(h)}, X^{(h)}\Bigr)$ description on $\mathcal{S}'$ can be regarded as a canonical transformation. This is achieved on introducing the equivalence relations (Penrose 1975)

$$\Bigr(W_{(h)}, Z^{(h)}\Bigr) \equiv \Bigr(\rho^{-1} \, W_{(h)}, \rho \, Z^{(h)}\Bigr), \qquad (9.4.15)$$

$$\Bigr(U_{(h)}, X^{(h)}\Bigr) \equiv \Bigr(\sigma^{-1} \, U_{(h)}, \sigma \, X^{(h)}\Bigr), \qquad (9.4.16)$$

which yield a six-complex-dimensional space S_6 (see problem 9.2).

9.5 Asymptotic twistors

Although in the theory of hypersurface twistors the complex structure plays a key role, their definition depends on an arbitrary hypersurface $\mathcal{S}$, and the attempt to define the scalar product $Z^{(h)} \, W_{(h)}$ faces great difficulties. The concept of asymptotic twistor tries to overcome these limitations by focusing on asymptotically flat space-times. Hence the emphasis is on null hypersurfaces, i.e. SCRI$^+$ and SCRI$^-$ (cf. section 3.5), rather than on spacelike hypersurfaces. Since the construction of hypersurface twistors is independent of conformal rescalings of the metric, while future and past null infinity have well known properties (Hawking and Ellis 1973), the theory of asymptotic twistors appears well defined. Its key features are as follows.

First, one complexifies future null infinity $\mathcal{I}^+$ to get $C\mathcal{I}^+$. Hence its complexified metric is described by complexified coordinates $\eta, \widetilde{\eta}, u$, where η and $\widetilde{\eta}$ are totally independent (cf. section 3.5). The corresponding planes $\eta =$ constant, $\widetilde{\eta} =$ constant, are totally null planes (in that the complexified metric of $C\mathcal{I}^+$ vanishes over them) with a topological twist (Penrose 1975).

Second, note that for any null hypersurface, its normal has the spinor form

$$n^{AA'} = \iota^A \, \widetilde{\iota}^{A'}. \tag{9.5.1}$$

Thus, if $\widetilde{\iota}^{B'} \, \pi_{B'} \neq 0$, the insertion of (9.5.1) into Eq. (9.4.3) yields

$$\iota^A \, \pi^{A'} \, \nabla_{AA'} \, \pi_{C'} = 0. \tag{9.5.2}$$

Similarly, if $\iota^B \, \widetilde{\pi}_B \neq 0$, the insertion of (9.5.1) into the Eq. (9.4.7) for dual hypersurface twistors leads to

$$\widetilde{\pi}^A \, \widetilde{\iota}^{A'} \, \nabla_{AA'} \, \widetilde{\pi}_C = 0. \tag{9.5.3}$$

These equations tell us that the γ-curves are null geodesics on $C\mathcal{I}^+$, lying entirely in the $\widetilde{\eta}$ = constant planes, while the $\widetilde{\gamma}$ curves are null geodesics lying in the $\eta =$ constant planes.

By definition, an *asymptotic twistor* is an element $Z^{(a)} \in T(\mathcal{I}^+)$, and corresponds to a null geodesic γ in $C\mathcal{I}^+$ with tangent vector $\iota^A \, \pi^{A'}$, where $\pi_{A'}$ undergoes parallel propagation along γ. By contrast, a *dual asymptotic twistor* is an element $\widetilde{Z}_{(a)} \in T^*(\mathcal{I}^+)$, and corresponds to a null geodesic $\widetilde{\gamma}$ in $C\mathcal{I}^+$ with tangent vector $\widetilde{\pi}^A \, \widetilde{\iota}^{A'}$, where $\widetilde{\pi}_A$ undergoes parallel propagation along $\widetilde{\gamma}$.

It now remains to be seen how to define the scalar product $Z^{(a)} \, \widetilde{Z}_{(a)}$. For this purpose, denoting by λ the intersection of the $\widetilde{\eta} =$ constant plane containing γ with the $\eta =$ constant plane containing $\widetilde{\gamma}$, we assume for simplicity that λ intersects $C\mathcal{I}^+$ in such a way that a continuous path β exists in $\gamma \cup \lambda \cup \widetilde{\gamma}$, unique up to homotopy, connecting $Q \in \gamma$ to $\widetilde{Q} \in \widetilde{\gamma}$. One then gives a local twistor description of $Z^{(a)}$ as $\left(0, \pi_{A'}\right)$ at Q, and one carries this along β by local twistor transport (section 9.2) to $\widetilde{Q}$. At the point $\widetilde{Q}$, the local twistor obtained in this way has the usual scalar product with the local twistor description $\left(\widetilde{\pi}_A, 0\right)$ at $\widetilde{Q}$ of $\widetilde{Z}_{(a)}$. By virtue of Eqs. (9.2.4), (9.2.5) and (9.2.13), such a definition of scalar product is independent of the choice made to locate Q and $\widetilde{Q}$, and it also applies on going from $\widetilde{Q}$ to Q. Thus, the theory of asymptotic twistors combines in an essential way the asymptotic structure of space-time with the properties of local twistors and hypersurface twistors. Note also that $Z^{(a)} \widetilde{Z}_{(a)}$ has been defined as a holomorphic function on some open subset of $T(\mathcal{I}^+) \times T^*(\mathcal{I}^+)$ containing $C\mathcal{N}(\mathcal{I}^+)$. Hence one can take derivatives with respect to $Z^{(a)}$ and $\widetilde{Z}_{(a)}$ so as to obtain the differential forms in (9.4.12)–(9.4.14). If $W_{(a)} \in T^*(\mathcal{I}^+)$, $Z^{(a)} \in T(\mathcal{I}^+)$, $U_{(a)} \in T^*(\mathcal{I}^-)$, $X^{(a)} \in T(\mathcal{I}^-)$, one can write

$$\omega = i \, dZ^{(a)} \wedge dW_{(a)} = i \, dX^{(a)} \wedge dU_{(a)}, \tag{9.5.4}$$

$$\phi = i \, Z^{(a)} \, dW_{(a)} = i \, X^{(a)} \, dU_{(a)}, \tag{9.5.5}$$

$$\widetilde{\phi} = -i \, W_{(a)} \, dZ^{(a)} = -i \, U_{(a)} \, dX^{(a)}. \tag{9.5.6}$$

The asymptotic twistor space at future null infinity is also very useful in that its global complex structure enables one to study the outgoing radiation field arising from gravitation (Penrose 1975).

9.6 Penrose transform

As we know from chapter four, on studying the massless free-field equations in Minkowski space-time, the Penrose transform provides the homomorphism (Eastwood 1990)

$$\mathcal{P} : H^1(V, \mathcal{O}(-n-2)) \rightarrow \Gamma(U, Z_n). \tag{9.6.1}$$

With the notation in (9.6.1), U is an open subset of compactified complexified Minkowski space-time, V is the corresponding open subset of projective twistor space, $O(-n-2)$ is the sheaf of germs (appendix D) of holomorphic functions homogeneous of degree $-n-2$, Z_n is the sheaf of germs of holomorphic solutions of the massless free-field equations of helicity $\frac{n}{2}$. Although the Penrose transform may be viewed as a geometric way of studying the partial differential equations of mathematical physics, the main problem is to go beyond flat space-time and reconstruct a generic curved space-time from its twistor space or from some more general structures. Here, following Eastwood (1990), we study a four-complex-dimensional conformal manifold M, which is assumed to be geodesically convex. For a given choice of spin-structure on M, let F be the projective primed spin-bundle over M with local coordinates $x^a, \pi_{A'}$. After choosing a metric in the conformal class, the corresponding metric connection is lifted horizontally to a differential operator $\nabla_{AL'}$ on spinor fields on F.

Denoting by ϕ_B a spinor field on M of conformal weight w, a conformal rescaling $\widehat{g} = \Omega^2 g$ of the metric leads to a change of the operator according to the rule

$$\widehat{\nabla}_{AL'} \; \phi_B = \nabla_{AL'} \; \phi_B - Y_{BL'} \; \phi_A + w \; Y_{AL'} \; \phi_B + \pi_{L'} \; Y_{AB'} \; \frac{\partial \phi_B}{\partial \pi_{B'}}, \tag{9.6.2}$$

where $Y_{AL'} \equiv \Omega^{-1} \nabla_{AL'} \Omega$. In particular, on *functions* of weight w one finds

$$\widehat{\nabla}_{AL'} \phi = \nabla_{AL'} \phi + w \; Y_{AL'} \phi + \pi_{L'} \; Y_{AB'} \; \frac{\partial \phi}{\partial \pi_{B'}}. \tag{9.6.3}$$

Thus, if the conformal weight vanishes, acting with $\pi^{A'}$ on both sides of (9.6.3) and defining

$$\nabla_A \equiv \pi^{A'} \; \nabla_{AA'}, \tag{9.6.4}$$

one obtains

$$\widehat{\nabla}_A \phi = \nabla_A \phi. \tag{9.6.5}$$

This means that ∇_A is a conformally invariant operator on ordinary functions and hence may be regarded as an invariant distribution on the projective spin-bundle F (Eastwood 1990). From chapters four and six we know that such a distribution is integrable if and only if the self-dual Weyl spinor $\widetilde{\psi}_{A'B'C'D'}$ vanishes. One can then integrate the distribution on F to give a new space P as the space of leaves. This leads to the double fibration familiar to the mathematicians working on twistor theory:

$$P \xleftarrow{\mu} F \xrightarrow{\nu} M. \tag{9.6.6}$$

In (9.6.6) P is the twistor space of M, and the submanifolds $\nu(\mu^{-1}(z))$ of M, for $z \in P$, are the α-surfaces in M (cf. chapter four). Each point $x \in M$ is known to give rise to a line $L_x \equiv \mu(\nu^{-1}(x))$ in P, whose points *correspond* to the α-surfaces through x as described in chapter four. The conformally anti-self-dual complex space-time M with its conformal structure is then recovered from its twistor space P, and an explicit construction has been given in section 5.1.

To get a deeper understanding of this non-linear-graviton construction, we now introduce the Einstein bundle E. For this purpose, let us consider a function ϕ of conformal weight 1. Equation (9.6.3) implies that, under a conformal rescaling of the metric, $\nabla_A \phi$ rescales as

$$\widehat{\nabla}_A \phi = \nabla_A \phi + Y_A \phi. \tag{9.6.7}$$

Thus, the transformation rule for $\nabla_A \nabla_B \phi$ is

$$\widehat{\nabla}_A \widehat{\nabla}_B \phi = \nabla_A \widehat{\nabla}_B \phi - Y_B \widehat{\nabla}_A \phi = \nabla_A \nabla_B \phi + \Bigr[(\nabla_A Y_B) - Y_B Y_A\Bigr]\phi. \tag{9.6.8}$$

Although $\nabla_A \nabla_B \phi$ is not conformally invariant, Eq. (9.6.8) suggests how to modify our operator to make it into a conformally invariant operator. For this purpose, denoting by $\Phi_{ABA'B'}$ the trace-free part of the Ricci spinor, and defining

$$\Phi_{AB} \equiv \pi^{A'} \; \pi^{B'} \; \Phi_{ABA'B'}. \tag{9.6.9}$$

we point out that, under a conformal rescaling, Φ_{AB} transforms as

$$\widehat{\Phi}_{AB} = \Phi_{AB} - \nabla_A Y_B + Y_A Y_B. \tag{9.6.10}$$

Equations (9.6.8) and (9.6.10) imply that the conformally invariant operator we are looking for is (Eastwood 1990)

$$D_{AB} \equiv \nabla_A \nabla_B + \Phi_{AB}, \tag{9.6.11}$$

acting on functions of weight 1. In geometric language, ∇_A and D_{AB} act along the fibres of μ. A vector bundle E over P is then obtained by considering the vector space of functions defined on $\mu^{-1}Z$ such that $D_{AB}\phi = 0$ and having conformal weight 1. Such a space is indeed three-dimensional, since α-surfaces inherit from the conformal structure on M a flat projective structure, and D_{AB} in (9.6.11) is a projectively invariant differential operator (Eastwood 1990, and earlier analysis by Bailey cited therein).

Remarkably, the Penrose transform establishes an isomorphism between the space of smooth sections $\Gamma(P,E)$ (E being our Einstein bundle on P) and the space of functions ϕ of conformal weight 1 on M such that

$$\nabla_{(A}^{(A'} \; \nabla_{B)}^{B')} \phi + \Phi_{AB}^{A'B'} \phi = 0. \tag{9.6.12}$$

The proof is obtained by first pointing out that, in the light of the definition of E, $\Gamma(P,E)$ is isomorphic to the space of functions ϕ of conformal weight 1 on the spin-bundle F such that

$$\nabla_A \nabla_B \phi + \Phi_{AB}\phi = 0. \tag{9.6.13}$$

The next step is the remark that the fibres of $\nu : F \rightarrow M$ are Riemann spheres and hence are compact, which implies that $\phi(x^a, \pi_{A'})$ is a function of x^a only. The resulting equation on the spin-bundle F is

$$\pi^{A'} \; \pi^{B'} \; \nabla_{AA'} \; \nabla_{BB'}\phi + \pi^{A'} \; \pi^{B'} \; \Phi_{ABA'B'}\phi = 0. \tag{9.6.14}$$

At this stage, the contribution of $\pi^{A'} \; \pi^{B'}$ has been factorized, which implies we are left with Eq. (9.6.12). Conformal invariance of the equation on M is guaranteed by the use of the conformally invariant operator D_{AB}.

From the point of view of gravitational physics, what is important is the resulting isomorphism between nowhere vanishing sections of E over P and Einstein metrics in the conformal class on M. Of course, the Einstein condition means that the Ricci tensor is proportional to the metric, and hence the trace-free part of Ricci vanishes: $\Phi_{ab} = 0$. To prove this basic property one points out that, since ϕ may be chosen to be nowhere vanishing, $\widehat{\phi}$ can be set to 1, so that Eq. (9.6.13) implies $\widehat{\Phi}_{ab} = 0$, which is indeed the Einstein condition. The converse also holds (Eastwood 1990).

Moreover, a pairing between solutions of differential equations can be established. To achieve this, note first that the tangent bundle of P corresponds to solutions of the differential equation (Eastwood 1990)

$$\nabla_{(A}\, \omega_{B)} = 0, \tag{9.6.15}$$

where ω_B is homogeneous of degree 1 in $\pi_{A'}$ and has conformal weight 1. Now the desired pairing is between solutions of Eq. (9.6.15) where $\omega_B \in \mathcal{O}_B(-1)[1]$ as above, and solutions of

$$\nabla_A \nabla_B \phi + \Phi_{AB} \phi = 0 \;\; \phi \in \mathcal{O}[1]. \tag{9.6.16}$$

Following again Eastwood (1990), we now consider a function f which is conformally invariant, and constant along the fibres of $\mu : F \rightarrow P$. Since f is defined as

$$f \equiv 2\omega^A \nabla_A \phi - \phi \nabla_A \omega^A, \tag{9.6.17}$$

its conformal invariance is proved by inserting (9.6.7) into the transformation rule

$$\widehat{f} = 2\omega^A \widehat{\nabla}_A \phi - \phi \widehat{\nabla}_A \omega^A. \tag{9.6.18}$$

The constancy of f along the fibres of μ is proved in two steps. First, the Leibniz rule, Eq. (8.7.3) and Eq. (9.6.15) imply that

$$\nabla_B f = 2\omega^A \nabla_B \nabla_A \phi - \phi \nabla_B \nabla_A \omega^A. \tag{9.6.19}$$

Second, using an identity for $\nabla_B \nabla_A \omega^A$ and then applying again Eq. (9.6.15) one finds (Eastwood 1990)

$$\nabla_B \nabla_A \omega^A = \varepsilon_{BA}\; \Phi_C^A\; \omega^C + \frac{1}{2} \nabla_A\; \delta_B^A\; \nabla_C \omega^C, \tag{9.6.20}$$

which implies

$$\nabla_B \nabla_A \omega^A = -2\Phi_{AB}\; \omega^A. \tag{9.6.21}$$

Thus, Eqs. (9.6.16), (9.6.19) and (9.6.21) lead to

$$\nabla_B f = 2\omega^A \Bigl(\nabla_A \nabla_B + \Phi_{AB}\Bigr)\phi = 0. \tag{9.6.22}$$

Q.E.D.

The results presented so far may be combined to show that an Einstein metric in the given conformal class on M corresponds to a nowhere vanishing one-form τ on twistor space P, homogeneous of degree two (cf. section 4.3). One then considers $\tau \wedge d\tau$, which can be written as $2\Lambda\rho$ for some function Λ. This Λ is indeed the cosmological constant, since the holomorphic functions in P are necessarily constant.

9.7 Ambitwistor correspondence

In this section we consider again a complex space-time $(\mathcal{M}, g)$, where $\mathcal{M}$ is a four-complex-dimensional complex manifold, and g is a holomorphic non-degenerate symmetric two-tensor on $\mathcal{M}$ (i.e. a complex-Riemannian metric). A family of null geodesics can be associated to $(\mathcal{M}, g)$ by considering those inextendible, connected, one-dimensional complex submanifolds $\gamma \subset \mathcal{M}$ such that any tangent vector field $v \in \Gamma(\gamma, O(T\gamma))$ satisfies (Le Brun 1990)

$$\nabla_v v = \sigma v, \tag{9.7.1}$$

$$g(v, v) = 0, \tag{9.7.2}$$

where σ is a proportionality parameter and ∇ is the Levi-Civita connection of g. These curves determine completely the conformal class of the complex metric g, since a vector is null if and only if it is tangent to some null geodesic γ. Conversely, the conformal class determines the set of null geodesics (Le Brun 1990). We now denote by $\mathcal{N}$ the set of null geodesics of $(\mathcal{M}, g)$, and by $\mathcal{Q}$ the hypersurface of null covectors defined by

$$\mathcal{Q} \equiv \left\{ [\phi] \in PT^*\mathcal{M} : g^{-1}(\phi, \phi) = 0 \right\}. \tag{9.7.3}$$

A *quotient* map $q : \mathcal{Q} \rightarrow \mathcal{N}$ can be given as the map assigning, to each point of $\mathcal{Q}$, the leaf through it. If $\mathcal{N}$ is equipped with the quotient topology, and if $(\mathcal{M}, g)$ is geodesically convex, $\mathcal{N}$ is then Hausdorff and has a *unique* complex structure making q into a holomorphic map of maximal rank. The corresponding complex manifold $\mathcal{N}$ is, by definition, the *ambitwistor space* of $(\mathcal{M}, g)$.

Denoting by $p : \mathcal{Q} \rightarrow \mathcal{M}$ the restriction to $\mathcal{Q}$ of the canonical projection $\pi : PT^*\mathcal{M} \rightarrow \mathcal{M}$, one has a holomorphic double fibration

$$\mathcal{N} \xleftarrow{q} \mathcal{Q} \xrightarrow{p} \mathcal{M}, \tag{9.7.4}$$

the *ambitwistor correspondence*, which relates complex space-time to its space of null geodesics. For example, in the case of the four-quadric $Q_4 \subset P_5$, obtained by conformal compactification of

$$\left(C^4, \sum_{j=1}^{4} (dz^j)^{\otimes 2} \right),$$

the corresponding ambitwistor space is (Le Brun 1990)

$$\mathcal{A} \equiv \left\{ \left([Z^\alpha], [W_\alpha] \right) \in P_3 \times P_3 : \sum_{\alpha=1}^{4} Z^\alpha W_\alpha = 0 \right\}. \tag{9.7.5}$$

Ambitwistor space has been used as an attempt to go beyond the space of α-surfaces, i.e. twistor space (chapter four). However, we prefer to limit ourselves to a description of the main ideas, to avoid becoming too technical. Hence the reader is referred to Le Brun's original papers appearing in the bibliography for a thorough analysis of ambitwistor geometry.

9.8 Radon transform

In the mathematical literature, the analysis of the Penrose transform is frequently supplemented by the study of the Radon transform, and the former is sometimes referred to as the Radon–Penrose transform. Indeed, the transform introduced in Radon (1917) associates to a real-valued function f on R^2 the following integral:

$$(Rf)(L) \equiv \int_L f, \tag{9.8.1}$$

where L is a straight line in R^2. On inverting the Radon and Penrose transforms, however, one appreciates there is a substantial difference between them (Bailey *et al.* 1994). In other words, (9.8.1) is invertible in that the value of the original function at a particular point may be recovered from its integrals along all cycles passing near that point. By contrast, in the Penrose transform, the original data in a neighbourhood of a particular cycle can be recovered from the transform restricted to that neighbourhood. Hence the Radon transform is globally invertible, while *the Penrose transform may be inverted locally.* [I am grateful to Mike Eastwood for making it possible for me to study the work appearing in Bailey et al. (1994). No original result obtained in Bailey *et al.* (1994) has been even mentioned in this section]

9.9 Massless fields as bundles

In the last part of chapter eight, motivated by our early work on one-loop quantum cosmology, we have studied a second set of potentials for gravitino fields in curved Riemannian backgrounds with non-vanishing cosmological constant. Our analysis is a direct generalization of the work in Penrose (1994), where the author studies the Ricci-flat case and relies on the analysis of twisted photons appearing in Ward

(1979). Thus, we here review the mathematical foundations of these potentials in the simpler case of Maxwell theory.

With the notation in Ward (1979), B is the primed spin-bundle over space-time, and $\left(x^a, \pi_{A'}\right)$ are coordinates on B. Of course, x^a are space-time coordinates and $\pi_{A'}$ are coordinates on primed spin-space. Moreover, we introduce the Euler vector field on B:

$$T \equiv \pi^{A'} \frac{\partial}{\partial \pi_{A'}}. \tag{9.9.1}$$

A function f on B such that $Tf = 0$ is homogeneous of degree zero in $\pi_{A'}$ and hence is defined on the projective spin-bundle. We are now interested in the two-dimensional distribution spanned by the two vector fields $\pi^{A'}\nabla_{AA'}$. The integral surfaces of such a distribution are the elements of non-projective twistor space T. To deform T without changing PT, Ward replaced $\pi^{A'}\nabla_{AA'}$ by $\pi^{A'}\nabla_{AA'} - \psi_A T$, ψ_0 and ψ_1 being two functions on B. By virtue of Frobenius' theorem (cf. section 6.2), the necessary and sufficient condition for the integrability of the new distribution is the validity of the equation

$$\pi^{A'} \; \nabla_{AA'} \; \psi^A - \psi_A \; T \; \psi^A = 0, \tag{9.9.2}$$

for *all* values of $\pi^{A'}$. In geometric language, if Eq. (9.9.2) holds $\forall \pi^{A'}$, a four-dimensional space T' of integral surfaces exists, and T' is a holomorphic bundle over projective twistor space PT. One can also say that T' is a *deformation* of flat twistor space T. If ψ_A takes the form

$$\psi_A(x, \pi_{A'}) = i \; \Phi_A^{\;\;A'...L'}(x) \; \pi_{A'}...\pi_{L'}, \tag{9.9.3}$$

then Eq. (9.9.2) becomes

$$\nabla^{A(A'} \; \Phi_A^{\;\;B'...M')} = 0. \tag{9.9.4}$$

Thus, the spinor field $\Phi_A^{\;\;A'...L'}$ is a potential for a massless free field

$$\phi_{AB...M} \equiv \nabla_{(B}^{B'}...\nabla_M^{M'} \; \Phi_{A)B'...M'}, \tag{9.9.5}$$

since the massless free-field equations

$$\nabla^{AA'} \phi_{AB...M} = 0 \tag{9.9.6}$$

result from Eq. (9.9.4).

In the particular case of Maxwell theory, suppose that

$$\psi_A = i \, \Phi_A^{\ A'}(x) \, \pi_{A'}, \tag{9.9.7}$$

with (cf. Eq. (8.8.4))

$$\nabla^{A(A'} \, \Phi_A^{\ B')} = 0. \tag{9.9.8}$$

Note that here the space T' of integral surfaces is a principal fibre bundle over PT with group the non-vanishing complex numbers. Following Ward (1979), here PT is just the *neighbourhood* of a line in CP^3, but not the whole of CP^3.

For the mathematically-oriented reader, we should say that, in the language of sheaf cohomology, one has the exact sequence

$$... \rightarrow H^1(PT, Z) \rightarrow H^1(PT, \mathcal{O}) \rightarrow H^1(PT, \mathcal{O}^*) \rightarrow H^2(PT, Z) \rightarrow ... \, . \tag{9.9.9}$$

If PT has $R^4 \times S^2$ topology (see section 4.3), then $H^1(PT, Z) = 0$. Moreover, $H^1(PT, \mathcal{O})$ is isomorphic to the space of left-handed Maxwell fields ϕ_{AB} satisfying the massless free-field equations

$$\nabla^{AA'} \phi_{AB} = 0. \tag{9.9.10}$$

$H^1(PT, \mathcal{O}^*)$ is the space of line bundles over PT, and $H^2(PT, Z) \cong Z$ is the space of possible Chern classes of such bundles. Thus, the space of left-handed Maxwell fields is isomorphic to the space of deformed line bundles T'. To realize this correspondence, we should bear in mind that a twistor determines an α-surface, jointly with a primed spinor field $\pi_{A'}$ propagated over the α-surface (chapter four). The usual propagation is parallel transport:

$$\pi^{A'} \, \nabla_{AA'} \, \pi_{B'} = 0. \tag{9.9.11}$$

However, in the deformed case, the propagation equation is taken to be

$$\pi^{A'}\Bigr(\nabla_{AA'} + i\,\Phi_{AA'}\Bigr)\pi_{B'} = 0. \tag{9.9.12}$$

Remarkably, the integrability condition for Eq. (9.9.12) is Eq. (9.9.8) (Ward 1979 and our problem 9.4). This property suggests that also Eq. (8.8.4) may be viewed as an integrability condition. In Penrose (1994), this geometric interpretation has been investigated for spin-$\frac{3}{2}$ fields. It appears striking that the equations of the local theory of spin-$\frac{3}{2}$ potentials lead naturally to equations which can be related to integrability conditions. Conversely, from some suitable integrability conditions, one may hope of constructing a local theory of potentials for gauge fields. The interplay between these two points of view deserves further consideration.

9.10 Quantization of field theories

The boundary conditions studied in chapters seven and eight are a part of the general set which should be imposed on bosonic and fermionic fields to respect BRST invariance and local supersymmetry. In this chapter devoted to mathematical foundations we describe some recent progress on these issues, but we do not repeat our early analysis appearing in Esposito (1994).

The way in which quantum fields respond to the presence of boundaries is responsible for many interesting physical effects such as, for example, the Casimir effect, and the quantization program of spinor fields, gauge fields and gravitation in the presence of boundaries is currently leading to a better understanding of modern quantum field theories (Esposito *et al.* 1997). The motivations for this investigation come from at least three areas of physics and mathematics, i.e.

(i) *Cosmology*. One wants to understand what is the quantum state of the universe, and how to formulate boundary conditions for the universe (Esposito 1994 and references therein).

(ii) *Field Theory.* It appears necessary to get a deeper understanding of different quantization techniques in field theory, i.e. the reduction to physical degrees of freedom before quantization, or the Faddeev–Popov Lagrangian method, or the Batalin–Fradkin–Vilkovisky extended phase space. Moreover, perturbative properties of supergravity theories and conformal anomalies in field theory deserve further investigation, especially within the framework of semiclassical evaluation of path integrals in field theory via zeta-function regularization.

(iii) *Mathematics.* A (pure) mathematician may regard quantum cosmology as a problem in cobordism theory (i.e. when a compact manifold may be regarded as the boundary of another compact manifold), and one-loop quantum cosmology as a relevant application of the theory of eigenvalues in Riemannian geometry, of self-adjointness theory, and of the analysis of asymptotic heat kernels for manifolds with boundary.

On using zeta-function regularization (Esposito 1994), the $\zeta(0)$ value yields the scaling of quantum amplitudes and the one-loop divergences of physical theories. The choices to be made concern the quantization technique, the background four-geometry, the boundary three-geometry, the boundary conditions respecting Becchi–Rouet–Stora–Tyutin invariance and local supersymmetry, the gauge condition, the regularization algorithm. We are here interested in the mode-by-mode analysis of BRST-covariant Faddeev–Popov amplitudes for Euclidean Maxwell theory, which relies on the expansion of the electromagnetic potential in harmonics on the boundary three-geometry. In the case of three-sphere boundaries, one has (Esposito 1994)

$$A_0(x,\tau) = \sum_{n=1}^{\infty} R_n(\tau) Q^{(n)}(x), \tag{9.10.1}$$

$$A_k(x,\tau) = \sum_{n=2}^{\infty} \Bigl[f_n(\tau) S_k^{(n)}(x) + g_n(\tau) P_k^{(n)}(x) \Bigr], \tag{9.10.2}$$

where $Q^{(n)}(x), S_k^{(n)}(x)$ and $P_k^{(n)}(x)$ are scalar, transverse and longitudinal vector harmonics on S^3, respectively.

Magnetic conditions set to zero at the boundary the gauge-averaging functional, the tangential components of the potential, and the ghost field, i.e.

$$[\Phi(A)]_{\partial M} = 0, \; [A_k]_{\partial M} = 0, \; [\epsilon]_{\partial M} = 0. \tag{9.10.3}$$

Alternatively, electric conditions set to zero at the boundary the normal component of the potential, the normal derivative of tangential components of the potential, and the normal derivative of the ghost field, i.e.

$$[A_0]_{\partial M} = 0, \; \left[\frac{\partial A_k}{\partial \tau} \right]_{\partial M} = 0, \; \left[\frac{\partial \epsilon}{\partial \tau} \right]_{\partial M} = 0. \tag{9.10.4}$$

One may check that these boundary conditions are compatible with BRST transformations, and do not give rise to additional boundary conditions after a gauge transformation (Esposito *et al.* 1997).

By using zeta-function regularization and flat Euclidean backgrounds, the effects of relativistic gauges are as follows (Esposito and Kamenshchik 1994, Esposito *et al.* 1997, and references therein).

(i) In the Lorenz gauge, the mode-by-mode analysis of one-loop amplitudes agrees with the results of the Schwinger–DeWitt technique, both in the one-boundary case (i.e. the disk) and in the two-boundary case (i.e. the ring).

(ii) In the presence of boundaries, the effects of gauge modes and ghost modes *do not* cancel each other.

(iii) When combined with the contribution of physical degrees of freedom, i.e. the transverse part of the potential, this lack of cancellation is exactly what one needs to achieve agreement with the results of the Schwinger–DeWitt technique.

(iv) Thus, physical degrees of freedom are, by themselves, insufficient to recover the full information about one-loop amplitudes.

(v) Moreover, even on taking into account physical, non-physical and ghost modes, the analysis of relativistic gauges different from the Lorenz gauge yields gauge-independent amplitudes only in the two-boundary case.

(vi) Gauge modes obey a coupled set of second-order eigenvalue equations. For some particular choices of gauge conditions it is possible to decouple such a set of differential equations, by means of two functional matrices which diagonalize the original operator matrix.

(vii) For arbitrary choices of relativistic gauges, gauge modes remain coupled. The explicit proof of gauge independence of quantum amplitudes becomes a problem in homotopy theory. Hence there seems to be a deep relation between the Atiyah–Patodi–Singer theory of Riemannian four-manifolds with boundary (Atiyah *et al.* 1976), the zeta-function, and the BKKM function (Barvinsky *et al.* 1992b):

$$I(M^2,s) \equiv \sum_{n=n_0}^{\infty} d(n)\, n^{-2s}\, \log\Bigr[f_n(M^2)\Bigr]. \tag{9.10.5}$$

In (9.10.5), $d(n)$ is the degeneracy of the eigenvalues parametrized by the integer n, and $f_n(M^2)$ is the function occurring in the equation obeyed by the eigenvalues by virtue of the boundary conditions, after taking out false roots. The analytic continuation of (9.10.5) to the whole complex-s plane is given by

$$\text{“}I(M^2,s)\text{”} = \frac{I_{\rm pole}(M^2)}{s} + I^R(M^2) + O(s), \tag{9.10.6}$$

and enables one to evaluate $\zeta(0)$ as

$$\zeta(0) = I_{\rm log} + I_{\rm pole}(\infty) - I_{\rm pole}(0), \tag{9.10.7}$$

$I_{\rm log}$ being the coefficient of $\log(M)$ appearing in I^R as $M \rightarrow \infty$.

A detailed mode-by-mode study of perturbative quantum gravity about a flat Euclidean background bounded by two concentric three-spheres, including non-physical degrees of freedom and ghost modes, leads to one-loop amplitudes in agreement with the covariant Schwinger–DeWitt method (Esposito, Kamenshchik *et al.* 1994). This calculation provides the generalization of the previous analysis of fermionic fields and electromagnetic fields (Esposito 1994). The basic idea is to expand the metric perturbations h_{00}, h_{0i} and h_{ij} on a family of three-spheres centred on the origin, and then use the de Donder gauge-averaging functional in the Faddeev–Popov Euclidean action. The resulting eigenvalue equation for metric perturbations about a flat Euclidean background:

$$\Box h_{\mu\nu}^{(\lambda)} + \lambda \, h_{\mu\nu}^{(\lambda)} = 0, \tag{9.10.8}$$

gives rise to seven coupled eigenvalue equations for metric perturbations. On considering also the ghost one-form φ_μ, and imposing the mixed boundary conditions of Luckock, Moss and Poletti,

$$[h_{ij}]_{\partial M} = 0, \tag{9.10.9a}$$

$$[h_{i0}]_{\partial M} = 0, \tag{9.10.9b}$$

$$[\varphi_0]_{\partial M} = 0, \tag{9.10.9c}$$

$$\left[\frac{\partial h_{00}}{\partial \tau} + \frac{6}{\tau} h_{00} - \frac{\partial}{\partial \tau}\Bigr(g^{ij} h_{ij}\Bigr)\right]_{\partial M} = 0, \tag{9.10.10}$$

$$\left[\frac{\partial \varphi_i}{\partial \tau} - \frac{2}{\tau}\varphi_i\right]_{\partial M} = 0, \tag{9.10.11}$$

the analysis in Esposito, Kamenshchik *et al.* 1994 has shown that the full $\zeta(0)$ vanishes in the two-boundary problem, while the contributions of ghost modes and gauge modes *do not* cancel each other, as it already happens for Euclidean Maxwell theory.

The main open problem seems to be the explicit proof of gauge independence of one-loop amplitudes for relativistic gauges, in the case of flat Euclidean space bounded by two concentric three-spheres. For this purpose, one may have to show that, for coupled gauge modes, $I_{\log}$ and the difference $I_{\rm pole}(\infty) - I_{\rm pole}(0)$ are not affected by a change in the gauge parameters. Three steps are in order:

(i) To relate the regularization at large x used in Esposito (1994) to the BKKM regularization relying on the function (9.10.5).

(ii) To evaluate $I_{\log}$ from an asymptotic analysis of coupled eigenvalue equations.

(iii) To evaluate $I_{\rm pole}(\infty) - I_{\rm pole}(0)$ by relating the analytic continuation to the whole complex-s plane of the difference $I(\infty, s) - I(0, s)$, to the analytic continuation of the zeta-function.

The last step might involve a non-local, integral transform relating the BKKM function to the zeta-function, and a non-trivial application of the Atiyah–Patodi–Singer spectral analysis of Riemannian four-manifolds with boundary (Atiyah *et al.* 1976). In other words, one might have to prove that, *in the two-boundary problem only*, $I_{\rm pole}(\infty) - I_{\rm pole}(0)$ resulting from coupled gauge modes is the residue of a meromorphic function, invariant under a smooth variation in the gauge parameters of the matrix of elliptic self-adjoint operators appearing in the system

$$\widehat{\mathcal{A}}_n g_n + \widehat{\mathcal{B}}_n R_n = 0, \ \forall n \geq 2, \tag{9.10.12}$$

$$\widehat{\mathcal{C}}_n g_n + \widehat{\mathcal{D}}_n R_n = 0, \ \forall n \geq 2, \tag{9.10.13}$$

where one has

$$\widehat{\mathcal{A}}_n \equiv \frac{d^2}{d\tau^2} + \frac{1}{\tau}\frac{d}{d\tau} - \frac{\gamma_3^2}{\alpha}\frac{(n^2-1)}{\tau^2} + \lambda_n, \tag{9.10.14}$$

$$\widehat{\mathcal{B}}_n \equiv -\Bigl(1 + \frac{\gamma_1\gamma_3}{\alpha}\Bigr)(n^2-1)\frac{d}{d\tau} - \Bigl(1 + \frac{\gamma_2\gamma_3}{\alpha}\Bigr)\frac{(n^2-1)}{\tau}, \tag{9.10.15}$$

$$\widehat{\mathcal{C}}_n \equiv \Bigl(1 + \frac{\gamma_1\gamma_3}{\alpha}\Bigr)\frac{1}{\tau^2}\frac{d}{d\tau} + \frac{\gamma_3}{\alpha}(\gamma_1 - \gamma_2)\frac{1}{\tau^3}, \tag{9.10.16}$$

$$\widehat{\mathcal{D}}_n \equiv \frac{\gamma_1^2}{\alpha}\frac{d^2}{d\tau^2} + \frac{3\gamma_1^2}{\alpha}\frac{1}{\tau}\frac{d}{d\tau} + \Bigl[\frac{\gamma_2}{\alpha}(2\gamma_1 - \gamma_2) - (n^2-1)\Bigr]\frac{1}{\tau^2} + \lambda_n. \tag{9.10.17}$$

With our notation, γ_1, γ_2 and γ_3 are dimensionless parameters which enable one to study the most general gauge-averaging functional. This may be written in the form (the boundary being given by three-spheres)

$$\Phi(A) \equiv \gamma_1 {}^{(4)}\nabla^0 A_0 + \frac{\gamma_2}{3} A_0 \ {\rm Tr}(K) - \gamma_3 {}^{(3)}\nabla^i A_i, \tag{9.10.18}$$

where K is the extrinsic-curvature tensor of the boundary.

Other relevant research problems are the mode-by-mode analysis of one-loop amplitudes for gravitinos, including gauge modes and ghost modes studied within the Faddeev–Popov formalism. Last, but not least, the mode-by-mode analysis of linearized gravity in the unitary gauge in the one-boundary case, and the mode-by-mode analysis of one-loop amplitudes in the case of curved backgrounds, appear to be necessary to complete the picture outlined so far. The recent progress on problems with boundaries, however, seems to strengthen the evidence in favour of new perspectives being in sight in quantum field theory (Avramidi and Esposito 1998a,b, 1999).

CHAPTER TEN

OLD AND NEW IDEAS IN COMPLEX GENERAL RELATIVITY

The analysis of (conformally) right-flat space-times of the previous chapters has its counterpart in the theory of heaven spaces developed by Plebanski. This chapter begins with a review of weak heaven spaces, strong heaven spaces, heavenly tetrads and heavenly equations. An outline is also presented of the work by McIntosh, Hickman and other authors on complex relativity and real solutions. The last section is instead devoted to modern developments in complex general relativity. In particular, the analysis of real general relativity based on multisymplectic techniques has shown that boundary terms may occur in the constraint equations, unless some boundary conditions are imposed. The corresponding form of such boundary terms in complex general relativity is here studied. A complex Ricci-flat space-time is recovered provided that some boundary conditions are imposed on two-complex-dimensional surfaces. One then finds that the holomorphic multimomenta should vanish on an arbitrary three-complex-dimensional surface, to avoid having restrictions at this surface on the spinor fields expressing the invariance of the theory under holomorphic coordinate transformations. The Hamiltonian constraint of real general relativity is then replaced by a geometric structure linear in the holomorphic multimomenta, and a link with twistor theory is found. Moreover, a deep relation emerges between complex space-times which are not anti-self-dual and two-complex-dimensional surfaces which are not totally null.

10.1 Introduction

One of the most recurring themes of this paper is the analysis of complex or real Riemannian manifolds where half of the conformal curvature vanishes and the vacuum Einstein equations hold. Chapter five has provided an explicit construction of such anti-self-dual space-times, and the underlying Penrose-transform theory has been presented in chapters four and nine. However, alternative ways exist to construct these solutions of the Einstein equations, and hence this chapter supplements the previous chapters by describing the work in Plebanski (1975). By using the tetrad formalism and some basic results in the theory of partial differential equations, the so-called *heaven spaces* and *heavenly tetrads* are defined and constructed in detail. A brief review is then presented of the work by Hickman, McIntosh *et al.* on complex relativity and real solutions.

The last section of this chapter is instead devoted to new ideas in complex general relativity. First, the multisymplectic form of such a theory is outlined. Hence one deals with jet bundles described, locally, by a holomorphic coordinate system with holomorphic tetrad, holomorphic connection one-form, multivelocities corresponding to the tetrad and multivelocities corresponding to the connection, both of holomorphic nature (Esposito and Stornaiolo 1995). Remarkably, the equations of complex general relativity are all linear in the holomorphic multimomenta, and the anti-self-dual space-times relevant for twistor theory turn out to be a particular case of this more general structure. Moreover, the analysis of two-complex-dimensional surfaces in the generic case is shown to maintain a key role in complex general relativity.

10.2 Heaven spaces

In his theory of heaven spaces, Plebanski studies a four-dimensional *analytic* manifold M_4 with metric given in terms of tetrad vectors as (Plebanski 1975)

$$g = 2e^1 e^2 + 2e^3 e^4 = g_{ab} \, e^a e^b \ \in \Lambda^1 \otimes \Lambda^1. \tag{10.2.1}$$

The definition of the 2×2 matrices

$$\tau^{AB'} \equiv \sqrt{2} \begin{pmatrix} e^4 & e^2 \\ e^1 & -e^3 \end{pmatrix} \tag{10.2.2}$$

enables one to re-express the metric as

$$g = -\det \tau^{AB'} = \frac{1}{2} \, \varepsilon_{AB} \, \varepsilon_{C'D'} \, \tau^{AC'} \, \tau^{BD'}. \tag{10.2.3}$$

Moreover, since the manifold is analytic, there exist two *independent* sets of 2×2 complex matrices with unit determinant: $L^{A'}{}_A \in \ SL(2,C)$ and $\widetilde{L}^{B'}{}_B \in \ \widetilde{SL}(2,C)$. On defining a new set of tetrad vectors such that

$$\sqrt{2} \begin{pmatrix} e^{4'} & e^{2'} \\ e^{1'} & -e^{3'} \end{pmatrix} = L^{A'}{}_A \, \widetilde{L}^B{}_{B'} \, \tau^{AB'}, \tag{10.2.4}$$

the metric is still obtained as $2e^{1'} e^{2'} + 2e^{3'} e^{4'}$. Hence the tetrad gauge group may be viewed as

$$\mathcal{G} \equiv SL(2,C) \times \widetilde{SL}(2,C). \tag{10.2.5}$$

A key role in the following analysis is played by a pair of differential forms whose spinorial version is obtained from the wedge product of the matrices in (10.2.2), i.e.

$$\tau^{AB'} \wedge \tau^{CD'} = S^{AC} \, \varepsilon^{B'D'} + \varepsilon^{AC} \, \widetilde{S}^{B'D'}, \tag{10.2.6}$$

where

$$S^{AB} \equiv \frac{1}{2} \, \varepsilon_{R'S'} \, \tau^{AR'} \wedge \tau^{BS'} = \frac{1}{2} \, e^a \wedge e^b \, S_{ab}{}^{AB}, \tag{10.2.7}$$

$$\widetilde{S}^{A'B'} \equiv \frac{1}{2} \, \varepsilon_{RS} \, \tau^{RA'} \wedge \tau^{SB'} = \frac{1}{2} \, e^a \wedge e^b \, \widetilde{S}_{ab}{}^{A'B'}. \tag{10.2.8}$$

The forms S^{AB} and $\widetilde{S}^{A'B'}$ are self-dual and anti-self-dual respectively, in that the action of the Hodge-star operator on them leads to (Plebanski 1975)

$$^*S^{AB} = S^{AB}, \tag{10.2.9}$$

$$^*\widetilde{S}^{A'B'} = -\widetilde{S}^{A'B'}. \tag{10.2.10}$$

To obtain the desired spinor description of the curvature, we introduce the anti-symmetric connection forms $\Gamma_{ab} = \Gamma_{[ab]}$ through the first structure equations

$$de^a = e^b \wedge \Gamma^a{}_b. \tag{10.2.11}$$

The spinorial counterpart of Γ_{ab} is given by

$$\Gamma_{AB} \equiv -\frac{1}{4} \Gamma_{ab} \, S^{ab}{}_{AB}, \tag{10.2.12}$$

$$\widetilde{\Gamma}_{A'B'} \equiv -\frac{1}{4} \Gamma_{ab} \, \widetilde{S}^{ab}{}_{A'B'}, \tag{10.2.13}$$

which implies

$$\Gamma_{ab} = -\frac{1}{2} S_{ab}{}^{AB} \, \Gamma_{AB} - \frac{1}{2} \widetilde{S}_{ab}{}^{A'B'} \, \widetilde{\Gamma}_{A'B'}. \tag{10.2.14}$$

To appreciate that Γ_{AB} and $\widetilde{\Gamma}_{A'B'}$ are actually independent, the reader may find it useful to check that (Plebanski 1975)

$$\Gamma_{AB} = -\frac{1}{2} \begin{pmatrix} 2\Gamma_{42} & \Gamma_{12} + \Gamma_{34} \\ \Gamma_{12} + \Gamma_{34} & 2\Gamma_{31} \end{pmatrix}, \tag{10.2.15}$$

$$\widetilde{\Gamma}_{A'B'} = -\frac{1}{2} \begin{pmatrix} 2\Gamma_{41} & -\Gamma_{12} + \Gamma_{34} \\ -\Gamma_{12} + \Gamma_{34} & 2\Gamma_{32} \end{pmatrix}. \tag{10.2.16}$$

The action of exterior differentiation on $\tau^{AB'}, S^{AB}, \widetilde{S}^{A'B'}$ shows that

$$d\tau^{AB'} = \tau^{AL'} \wedge \widetilde{\Gamma}^{B'}{}_{L'} + \tau^{LB'} \wedge \Gamma^A{}_L, \tag{10.2.17}$$

$$dS^{AB} = -3S^{(AB}\ \Gamma^{C)}{}_{C}, \tag{10.2.18}$$

$$d\widetilde{S}^{A'B'} = -3\widetilde{S}^{(A'B'}\ \widetilde{\Gamma}^{C')}{}_{C'}, \tag{10.2.19}$$

and two *independent* curvature forms are obtained as

$$\begin{aligned} R^{A}{}_{B} &\equiv d\Gamma^{A}{}_{B} + \Gamma^{A}{}_{L} \wedge \Gamma^{L}{}_{B} \\ &= -\frac{1}{2}\ \psi^{A}{}_{BCD}\ S^{CD} + \frac{R}{24}\ S^{A}{}_{B} + \frac{1}{2}\ \Phi^{A}{}_{BC'D'}\ \widetilde{S}^{C'D'}, \end{aligned} \tag{10.2.20}$$

$$\begin{aligned} \widetilde{R}^{A'}{}_{B'} &\equiv d\widetilde{\Gamma}^{A'}{}_{B'} + \widetilde{\Gamma}^{A'}{}_{L'} \wedge \widetilde{\Gamma}^{L'}{}_{B'} \\ &= -\frac{1}{2}\ \widetilde{\psi}^{A'}{}_{B'C'D'}\ \widetilde{S}^{C'D'} + \frac{R}{24}\ \widetilde{S}^{A'}{}_{B'} + \frac{1}{2}\ \Phi_{CD}{}^{A'}{}_{B'}\ S^{CD}. \end{aligned} \tag{10.2.21}$$

The spinors and scalars in (10.2.20) and (10.2.21) have the same meaning as in the previous chapters. With the conventions in Plebanski (1975), the Weyl spinors are obtained as

$$\psi_{ABCD} = \frac{1}{16}\ S^{ab}{}_{AB}\ C_{abcd}\ S^{cd}{}_{CD} = \psi_{(ABCD)}, \tag{10.2.22}$$

$$\widetilde{\psi}_{A'B'C'D'} = \frac{1}{16}\ \widetilde{S}^{ab}{}_{A'B'}\ C_{abcd}\ \widetilde{S}^{cd}{}_{C'D'} = \widetilde{\psi}_{(A'B'C'D')}, \tag{10.2.23}$$

and conversely the Weyl tensor is

$$C_{abcd} = \frac{1}{4}\ S_{ab}{}^{AB}\ \psi_{ABCD}\ S_{cd}{}^{CD} + \frac{1}{4}\ \widetilde{S}_{ab}{}^{A'B'}\ \widetilde{\psi}_{A'B'C'D'}\ \widetilde{S}_{cd}{}^{C'D'}. \tag{10.2.24}$$

The spinor version of the Petrov classification (section 2.3) is hence obtained by stating that k^A and $\omega^{A'}$ are the two types of P-spinors if and only if the *independent* conditions hold:

$$\psi_{ABCD}\ k^A\ k^B\ k^C\ k^D = 0, \tag{10.2.25}$$

$$\widetilde{\psi}_{A'B'C'D'}\ \omega^{A'}\ \omega^{B'}\ \omega^{C'}\ \omega^{D'} = 0. \tag{10.2.26}$$

For our purposes, we can omit the details about the principal null directions, and focus instead on the classification of spinor fields and analytic manifolds under consideration. Indeed, Plebanski proposed to call all objects which are $\widetilde{SL}(2,C)$

scalars and are geometric objects with respect to $SL(2,C)$, the *heavenly objects* (e.g. $S^{AB}, \Gamma_{AB}, \psi_{ABCD}$). Similarly, objects which are $SL(2,C)$ scalars and behave like geometric objects with respect to $\widetilde{SL}(2,C)$ belong to the complementary world, i.e. the set of *hellish objects* (e.g. $\widetilde{S}^{A'B'}, \widetilde{\Gamma}_{A'B'}, \widetilde{\psi}_{A'B'C'D'}$). Last, spinor fields with (abstract) indices belonging to both primed and unprimed spin-spaces are the *earthly objects*.

With the terminology of Plebanski, a *weak heaven* space is defined by the condition

$$\widetilde{\psi}_{A'B'C'D'} = 0, \tag{10.2.27}$$

and corresponds to the *conformally right-flat* space of chapter three. Moreover, a *strong heaven* space is a four-dimensional analytic manifold where a choice of null tetrad exists such that

$$\widetilde{\Gamma}_{A'B'} = 0. \tag{10.2.28}$$

One then has *a forteriori*, by virtue of (10.2.21), the conditions (Plebanski 1975)

$$\widetilde{\psi}_{A'B'C'D'} = 0, \ \Phi_{ABC'D'} = 0, \ R = 0. \tag{10.2.29}$$

The vacuum Einstein equations are then automatically fulfilled in a strong heaven space, which turns out to be a right-flat space-time in modern language. Of course, strong heaven spaces are non-trivial if and only if the anti-self-dual Weyl spinor ψ_{ABCD} does not vanish, otherwise they reduce to flat four-dimensional space-time.

10.3 First heavenly equation

A space which is a strong heaven according to (10.2.28) is characterized by a key function Ω which obeys the so-called first heavenly equation. The basic ideas are as follows. In the light of (10.2.19) and (10.2.28), $d\widetilde{S}^{A'B'}$ vanishes, and hence, *in* *a simply connected region*, an element $U^{A'B'}$ of the bundle Λ^1 exists such that locally

$$\widetilde{S}^{A'B'} = dU^{A'B'}. \tag{10.3.1}$$

Thus, since

$$\widetilde{S}^{1'1'} = 2e^4 \wedge e^1, \tag{10.3.2}$$

$$\widetilde{S}^{2'2'} = 2e^3 \wedge e^2, \tag{10.3.3}$$

$$\widetilde{S}^{1'2'} = -e^1 \wedge e^2 + e^3 \wedge e^4, \tag{10.3.4}$$

Eq. (10.3.1) leads to

$$2e^4 \wedge e^1 = dU^{1'1'}, \tag{10.3.5}$$

$$2e^3 \wedge e^2 = dU^{2'2'}. \tag{10.3.6}$$

Now the Darboux theorem holds in our complex manifold, and hence scalar functions p, q, r, s exist such that

$$2e^4 \wedge e^1 = 2dp \wedge dq = 2d(p\, dq + d\tau), \tag{10.3.7}$$

$$2e^3 \wedge e^2 = 2dr \wedge ds = 2d(r\, ds + d\sigma), \tag{10.3.8}$$

$$e^1 \wedge e^2 \wedge e^3 \wedge e^4 = dp \wedge dq \wedge dr \wedge ds. \tag{10.3.9}$$

The form of the *heavenly tetrad* in these coordinates is

$$e^1 = A\, dp + B\, dq, \tag{10.3.10}$$

$$e^2 = G\, dr + H\, ds, \tag{10.3.11}$$

$$e^3 = E\, dr + F\, ds, \tag{10.3.12}$$

$$e^4 = -C\, dp - D\, dq. \tag{10.3.13}$$

If one now inserts (10.3.10)–(10.3.13) into (10.3.7)–(10.3.9), one finds that

$$AD - BC = EH - FG = 1, \tag{10.3.14}$$

which is supplemented by a set of equations resulting from the condition $d\widetilde{S}^{1'2'}=0$. These equations imply the existence of a function, the *first key function*, such that (Plebanski 1975)

$$AG-CE=\Omega_{pr}, \tag{10.3.15}$$

$$BG-DE=\Omega_{qr}, \tag{10.3.16}$$

$$AH-CF=\Omega_{ps}, \tag{10.3.17}$$

$$BH-DF=\Omega_{qs}. \tag{10.3.18}$$

Thus, E,F,G,H are given by

$$E=B\,\Omega_{pr}-A\,\Omega_{qr}, \tag{10.3.19}$$

$$F=B\,\Omega_{ps}-A\,\Omega_{qs}, \tag{10.3.20}$$

$$G=D\,\Omega_{pr}-C\,\Omega_{qr}, \tag{10.3.21}$$

$$H=D\,\Omega_{ps}-C\,\Omega_{qs}. \tag{10.3.22}$$

The request of compatibility of (10.3.19)–(10.3.22) with (10.3.14) leads to the *first heavenly equation*

$$\det\begin{pmatrix}\Omega_{pr} & \Omega_{ps}\\ \Omega_{qr} & \Omega_{qs}\end{pmatrix}=1. \tag{10.3.23}$$

10.4 Second heavenly equation

A more convenient description of the heavenly tetrad is obtained by introducing the coordinates

$$x\equiv\Omega_p,\ y\equiv\Omega_q, \tag{10.4.1}$$

and then defining

$$A\equiv-\Omega_{pp},\ B\equiv-\Omega_{pq},\ C\equiv-\Omega_{qq}. \tag{10.4.2}$$

The corresponding heavenly tetrad reads (Plebanski 1975)

$$e^1 = dp, \tag{10.4.3}$$

$$e^2 = dx + A\, dp + B\, dq, \tag{10.4.4}$$

$$e^3 = -dy - B\, dp - C\, dq, \tag{10.4.5}$$

$$e^4 = -dq. \tag{10.4.6}$$

Now the closure condition for $\widetilde{S}^{2'2'} : d\widetilde{S}^{2'2'} = 0$, leads to the equations

$$A_x + B_y = 0, \tag{10.4.7}$$

$$B_x + C_y = 0, \tag{10.4.8}$$

$$\Bigl(AC - B^2\Bigr)_x + B_q - C_p = 0, \tag{10.4.9}$$

$$\Bigl(AC - B^2\Bigr)_y - A_q + B_p = 0. \tag{10.4.10}$$

By virtue of (10.4.7) and (10.4.8), a function θ exists such that

$$A = -\theta_{yy},\ B = \theta_{xy},\ C = -\theta_{xx}. \tag{10.4.11}$$

On inserting (10.4.11) into (10.4.9) and (10.4.10) one finds

$$\partial_w \Bigl(\theta_{xx}\ \theta_{yy} - \theta_{xy}^2 + \theta_{xp} + \theta_{yq}\Bigr) = 0, \tag{10.4.12}$$

where $w = x, y$. Thus, one can write that

$$\theta_{xx}\theta_{yy} - \theta_{xy}^2 + \theta_{xp} + \theta_{yq} = f_p(p, q), \tag{10.4.13}$$

where f is an arbitrary function of p and q. This suggests defining the function

$$\Theta \equiv \theta - xf, \tag{10.4.14}$$

which implies

$$f_p = \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 + \Theta_{xp} + \Theta_{yq} + f_p,$$

and hence

$$\Theta_{xx}\,\Theta_{yy}-\Theta_{xy}^2+\Theta_{xp}+\Theta_{yq}=0. \tag{10.4.15}$$

Equation (10.4.15) ensures that all forms $\widetilde{S}^{A'B'}$ are closed, and is called the *second heavenly equation*. Plebanski was able to find heavenly metrics of all possible algebraically degenerate types. An example is given by the function

$$\Theta \equiv \frac{\beta}{2\alpha(\alpha-1)} x^{\alpha}\, y^{1-\alpha}. \tag{10.4.16}$$

The reader may check that such a solution is of the type $[2-2]\otimes[-]$ if $\alpha=-1,2$, and is of the type $[2-1-1]\otimes[-]$ whenever $\alpha \neq -1,2$ (Plebanski 1975). More work on related topics and on yet other ideas in complex general relativity can be found in Plebanski and Hacyan (1975), Finley and Plebanski (1976), Newman (1976), Plebanski and Schild (1976), Ko *et al.* (1977), Boyer *et al.* (1978), Hansen *et al.* (1978), Tod (1980), Tod and Winicour (1980), Finley and Plebanski (1981), Ko *et al.* (1981), Sparling and Tod (1981), Bergmann and Smith (1991), Plebanski and Przanowski (1994), Plebanski and Garcia–Compean (1995a,b).

10.5 Complex relativity and real solutions

Another research line has dealt with real solutions of Einstein's field equations as seen from the viewpoint of complex relativity (Hall *et al.* 1985, McIntosh and Hickman 1985, Hickman and McIntosh 1986a,b, McIntosh *et al.* 1988). In particular, Hickman and McIntosh (1986a) integrated Einstein's vacuum equations in complex relativity in a number of cases when the Weyl tensor is of type $N \otimes N$, i.e. the left and right Weyl spinors are each of type N. Three of the five metrics obtained were found to be complexified versions of Robinson–Trautman and two families of plane-fronted wave real-type N vacuum metrics, whereas the other two metrics were shown to have no real slices. Moreover, in Hickman and McIntosh (1986b) the authors integrated the vacuum Einstein equations for integrable double

Kerr–Schild (hereafter, IDKS) spaces, and were able to show that the vacuum equations can be reduced to a single hyperheavenly equation (cf. section 10.4) in terms of two potentials.

This section is devoted to a review of the fifth paper in the series, by McIntosh *et al.* (1988). To begin, recall that the metric of IDKS spaces can be written as

$$g = g_0 + P\theta^2 \otimes \theta^2 + 2R\theta^2 \otimes \theta^4 + Q\theta^4 \otimes \theta^4, \tag{10.5.1}$$

where P, Q, R are complex parameters, g_0 is a Minkowski metric, θ^2 and θ^4 span an integrable codistribution and are null with respect to both g and g_0. When the condition

$$PQ - R^2 = 0 \tag{10.5.2}$$

is fulfilled, the IDKS metric (10.5.1) reduces to an integrable single Kerr–Schild (hereafter, ISKS) metric with a null vector $\mathbf{l}$, and the tetrad can be aligned so that g can be written in the form

$$g = g_0 + P\theta^2 \otimes \theta^2, \tag{10.5.3}$$

where P is complex and $\mathbf{l} \cdot \theta^2 = 0$.

Interestingly, a metric which is of the form (10.5.1) and hence is IDKS with respect to g_0, may be ISKS with respect to some other flat-space background metric, and hence may be expressed in the form (10.5.3) *for some other* g_0. An intriguing problem is the freedom of transformations which keep a particular metric in the form (10.5.1) or (10.5.3). There is indeed a combined problem of coordinate freedom and tetrad freedom in choosing θ^2 and θ^4, or θ^2.

A generalized form of the IDKS metric can be written, in local coordinates (u, v, x, y), with the help of the following tetrad:

$$\theta^1 \equiv dx + (G_y + y^{-1}\mathcal{G}_y)du + (F_y + y^{-1}\mathcal{F}_y)dv, \tag{10.5.4}$$

$$\theta^2 \equiv y du, \tag{10.5.5}$$

$$\theta^3 \equiv dy - (G_x + y^{-1}\mathcal{G}_x)du - (F_x + y^{-1}\mathcal{F}_x)dv, \tag{10.5.6}$$

$$\theta^4 \equiv dv + x du, \tag{10.5.7}$$

where, denoting by H and Ω two functions of the variables (u, v, x, y), one has

$$F \equiv H_x, \tag{10.5.8}$$

$$G \equiv x H_x + y H_y - 3H, \tag{10.5.9}$$

$$\mathcal{F} \equiv \Omega_x, \tag{10.5.10}$$

$$\mathcal{G} \equiv x\Omega_x + y\Omega_y - \Omega, \tag{10.5.11}$$

with the understanding that subscripts denote partial derivatives of the function with respect to the variable occurring in the subscript, e.g. $\Omega_x \equiv \frac{\partial \Omega}{\partial x}$. The basis dual to (10.5.4)–(10.5.7) is

$$D \equiv \partial_x, \tag{10.5.12}$$

$$\delta \equiv \partial_y, \tag{10.5.13}$$

$$\begin{aligned} \triangle \equiv y^{-1} \Big\{ \partial_u - x\partial_v - \Big[G_y - xF_y + y^{-1}(\mathcal{G}_y - x\mathcal{F}_y) \Big] \partial_x \\ + \Big[G_x - xF_x + y^{-1}(\mathcal{G}_x - x\mathcal{F}_x) \Big] \partial_y \Big\}, \end{aligned} \tag{10.5.14}$$

$$\widetilde{\delta} \equiv \partial_v - (F_y + y^{-1}\mathcal{F}_y)\partial_x + (F_x + xy^{-1}\mathcal{F}_x)\partial_y. \tag{10.5.15}$$

The non-vacuum IDKS metric can then be written as

$$\begin{aligned} g = g_0 + 2\Big[xG_x + yG_y + y^{-1}(x\mathcal{G}_x + y\mathcal{G}_y) \Big] du \otimes du \\ + 4(G_x + F_x + y^{-1}\mathcal{G}_x) du \otimes dv + 2(F_x + y^{-1}\mathcal{F}_x) dv \otimes dv, \end{aligned} \tag{10.5.16}$$

where

$$g_0 = 2\Big[y dx \otimes du - dy \otimes (dv + x du) \Big]. \tag{10.5.17}$$

On evaluating the left connection one-forms for the tetrad (10.5.4)–(10.5.7), one finds that the non-vanishing tetrad components of the left Weyl tensor are

$$\Psi_2 = 2y^{-3}\mathcal{F}_x, \tag{10.5.18}$$

$$\Psi_3 = (\widetilde{\delta} + y^{-1}F_x)\gamma - (\triangle + y^{-1}F_y)\alpha - \lambda y^{-1}, \tag{10.5.19}$$

$$\Psi_4 = (\widetilde{\delta} + 4y^{-2}\mathcal{F}_x + y^{-1}F_x)\nu - [\triangle + y^{-1}F_y + 2y^{-2}\mathcal{F}_y]\lambda, \tag{10.5.20}$$

where

$$\gamma \equiv y^{-2}\mathcal{F}_y, \ \alpha \equiv y^{-2}\mathcal{F}_x, \tag{10.5.21}$$

$$\lambda \equiv y^{-1}\Big[\Sigma_x + y^{-2}(\mathcal{F}_x G_x - F_x \mathcal{G}_x)\Big], \tag{10.5.22}$$

$$\nu \equiv y^{-1}\Big\{\Sigma_y + y^{-2}\Big[\mathcal{F}_x(G_y + y^{-1}\mathcal{G}_y) - \mathcal{G}_x(F_y + y^{-1}\mathcal{F}_y) + (\mathcal{G}_v - \mathcal{F}_u)\Big]\Big\}, \tag{10.5.23}$$

having denoted by Σ the function

$$\Sigma \equiv (F_x + y^{-1}\mathcal{F}_x)(G_y + \mathcal{G}_y) - (F_y + y^{-1}\mathcal{F}_y)(G_x + y^{-1}\mathcal{G}_x) + (G + y^{-1}\mathcal{G})_v - (F + y^{-1}\mathcal{F})_u. \tag{10.5.24}$$

Moreover, from the evaluation of the right connection one-forms, one finds that the right Weyl tensor components are given by

$$\widetilde{\Psi}_0 = H_{xxxx} + y^{-1}\Omega_{xxxx}, \tag{10.5.25}$$

$$\widetilde{\Psi}_1 = H_{xxxy} + y^{-1}\Omega_{xxxy}, \tag{10.5.26}$$

$$\widetilde{\Psi}_2 = H_{xxyy} + y^{-1}\Omega_{xxyy}, \tag{10.5.27}$$

$$\widetilde{\Psi}_3 = H_{xyyy} + y^{-1}\Omega_{xyyy}, \tag{10.5.28}$$

$$\widetilde{\Psi}_4 = H_{yyyy} + y^{-1}\Omega_{yyyy}. \tag{10.5.29}$$

The vacuum field equations are obtained for the following form of $\Omega, \mathcal{F}$ and $\mathcal{G}$:

$$\Omega = -\frac{1}{2}Lx^2, \tag{10.5.30}$$

$$\mathcal{F} = -Lx, \tag{10.5.31}$$

$$\mathcal{G} = -\frac{1}{2}Lx^2, \tag{10.5.32}$$

where L is an arbitrary function of u and v. The field equations reduce then to the *Plebanski–Robinson equation*

$$\Sigma = S - LH_{yy} = \lambda^0(u,v)x + \nu^0(u,v)y, \tag{10.5.33}$$

with the function S given by

$$S \equiv G_y F_x - G_x F_y + G_v - F_u, \tag{10.5.34}$$

whereas λ^0 and ν^0 are arbitrary functions of u and v.

Following McIntosh *et al.* (1988) one should stress that, for a given metric and for a particular coordinate and tetrad frame, H is not unique. Both the metric described by (10.5.16) and (10.5.17), and the Plebanski–Robinson equation (10.5.33), are invariant under the transformation

$$H \rightarrow H + f(u,v)y^3 + g(u,v). \tag{10.5.35}$$

Moreover, the metric (10.5.16) is linear in H and Ω. This implies that, for some known vacuum metrics (e.g. Schwarzschild) H can be written in the form

$$H = H_m + H_0, \tag{10.5.36}$$

where H_0 is the H function for a form of the flat-space metric and is proportional to the curvature constant, whereas H_m is proportional to the mass constant.

Interestingly, different coordinate versions of flat-space metrics are obtained when dealing with various forms of both complex and complexified metrics. In McIntosh *et al.* (1988), three forms of H are derived which generate flat space and are hence denoted by H_0. They are as follows.

(i) **First form of H_0.**

$$H_0 = \frac{k}{4}(x^2 - 2y^2), \tag{10.5.37}$$

where k is a real parameter. The resulting metric can be written as

$$g = g_0 + k(2y^2 - x^2)du \otimes du + kdv \otimes dv, \tag{10.5.38}$$

where the metric g_0 reads

$$g_0 = 2\Big[ydx \otimes du - xdu \otimes dy - dy \otimes dv\Big]. \tag{10.5.39}$$

The metric g is an IDKS metric with respect to g_0, and du and dv span an integrable codistribution.

(ii) **Second form of H_0**

$$H_0 = 0. \tag{10.5.40}$$

The corresponding metric can be written in the form

$$g = g_0 = 2\Big[d\xi \otimes d\eta - d\zeta \otimes d\widetilde{\zeta}\Big], \tag{10.5.41}$$

with coordinate transformation

$$\xi\sqrt{2k} = -\left(\frac{x}{\sqrt{2}} - y + kv\right), \tag{10.5.42}$$

$$\eta\sqrt{2k} = \left(\frac{x}{\sqrt{2}} + y - kv\right), \tag{10.5.43}$$

$$\zeta\sqrt{2k} = \left(\frac{x}{\sqrt{2}} + y\right) e^{ku\sqrt{2}}, \tag{10.5.44}$$

$$\widetilde{\zeta}\sqrt{2k} = -\left(\frac{x}{\sqrt{2}} - y\right) e^{-ku\sqrt{2}}. \tag{10.5.45}$$

(iii) **Third form of H_0**

$$H_0 = \frac{k}{2}\frac{(UX + V)^2}{U^4}, \tag{10.5.46}$$

where the coordinates (X, Y, U, V) replace (x, y, u, v). One then finds that

$$g = g_0 + 2k\Big[d(V/U) \otimes d(V/U) - 2H_0 dU \otimes dU\Big], \tag{10.5.47}$$

with the metric g_0 having the form

$$g_0 = 2\Big[Y dX \otimes dU - X dU \otimes dY - dY \otimes dV\Big]. \tag{10.5.48}$$

The coordinate transformation which relates X, Y, U, V and ξ, η, ζ and $\widetilde{\zeta}$ used in (10.5.42)–(10.5.45) can be shown to be

$$\xi = X, \tag{10.5.49}$$

$$\eta = UY - k\frac{(2V + UX)}{U}, \tag{10.5.50}$$

$$\zeta = Y - k\frac{(V + UX)}{U^2}, \tag{10.5.51}$$

$$\widetilde{\zeta} = V + UX. \tag{10.5.52}$$

10.6 Multimomenta in complex general relativity

Among the various approaches to the quantization of the gravitational field, much insight has been gained by the use of twistor theory and Hamiltonian techniques. For example, it is by now well known how to reconstruct an anti-self-dual space-time from deformations of flat projective twistor space (chapter five), and the various definitions of twistors in curved space-time enable one to obtain relevant information about complex space-time geometry within a holomorphic, conformally invariant framework (chapter nine). Moreover, the recent approaches to canonical gravity described in Ashtekar (1991) have led to many exact solutions of the quantum constraint equations of general relativity, although their physical relevance for the quantization program remains unclear. A basic difference between the Penrose formalism and the Ashtekar formalism is as follows. The twistor program refers to a four-complex-dimensional complex-Riemannian manifold with holomorphic metric, holomorphic connection and holomorphic curvature tensor, where the complex

Einstein equations are imposed. By contrast, in the recent approaches to canonical gravity, one studies complex tetrads on a four-real-dimensional Lorentzian manifold, and real general relativity may be recovered provided that one is able to impose suitable reality conditions. The aim of this section is to describe a new property of complex general relativity within the holomorphic framework relevant for twistor theory, whose derivation results from recent attempts to obtain a manifestly covariant formulation of Ashtekar's program (Esposito *et al.* 1995, Esposito and Stornaiolo 1995).

Indeed, it has been recently shown in Esposito *et al.* (1995) that the constraint analysis of general relativity may be performed by using multisymplectic techniques, without relying on a 3+1 split of the space-time four-geometry. The constraint equations have been derived while paying attention to boundary terms, and the Hamiltonian constraint turns out to be linear in the *multimomenta* (see below). While the latter property is more relevant for the (as yet unknown) quantum theory of gravitation, the former result on boundary terms deserves further thinking already at the classical level, and is the object of our investigation.

We here write the Lorentzian space-time four-metric as

$$g_{ab} = {e_a}^{\hat{c}} \, {e_b}^{\hat{d}} \, \eta_{\hat{c}\hat{d}}, \tag{10.6.1}$$

where ${e_a}^{\hat{c}}$ is the tetrad and η is the Minkowski metric. In first-order formalism, the tetrad ${e_a}^{\hat{c}}$ and the connection one-form ${\omega_a}^{\hat{b}\hat{c}}$ are regarded as independent variables. In Esposito *et al.* (1995) it has been shown that, on using jet-bundle formalism and covariant multimomentum maps, the constraint equations of real general relativity hold on an *arbitrary* three-real-dimensional hypersurface Σ provided that one of the following three conditions holds:

(i) Σ has no boundary;

(ii) the multimomenta

$$\tilde{p}^{ab}{}_{\hat{c}\hat{d}} \equiv e\Big({e^a}_{\hat{c}} \, {e^b}_{\hat{d}} - {e^b}_{\hat{c}} \, {e^a}_{\hat{d}}\Big)$$

vanish at $\partial\Sigma$, e being the determinant of the tetrad;

(iii) an element of the algebra $o(3,1)$ corresponding to the gauge group, represented by the antisymmetric $\lambda^{\hat{a}\hat{b}}$, vanishes at $\partial\Sigma$, and the connection one-form ${\omega_a}^{\hat{b}\hat{c}}$ or ξ^b vanishes at $\partial\Sigma$, ξ being a vector field describing diffeomorphisms on the base-space.

In other words, boundary terms may occur in the constraint equations of real general relativity, and they result from the total divergences of

$$\sigma^{ab} \equiv {\tilde{p}^{ab}}_{\hat{c}\hat{d}} \, \lambda^{\hat{c}\hat{d}}, \tag{10.6.2}$$

$$\rho^{ab} \equiv {\tilde{p}^{ab}}_{\hat{c}\hat{d}} \, {\omega_f}^{\hat{c}\hat{d}} \, \xi^f, \tag{10.6.3}$$

integrated over Σ.

In two-component spinor language, denoting by ${\tau^{\hat{a}}}_{BB'}$ the Infeld–van der Waerden symbols, the two-spinor version of the tetrad reads

$${e^a}_{BB'} \equiv {e^a}_{\hat{a}} \, {\tau^{\hat{a}}}_{BB'}, \tag{10.6.4}$$

which implies that σ^{ab} defined in (10.6.2) takes the form

$$\sigma^{ab} = e\Big({e^a}_{CC'} \, {e^b}_{DD'} - {e^a}_{DD'} \, {e^b}_{CC'}\Big){\tau_{\hat{a}}}^{CC'} \, {\tau_{\hat{b}}}^{DD'} \, \lambda^{\hat{a}\hat{b}}. \tag{10.6.5}$$

Thus, on defining the spinor field

$$\lambda^{CC'DD'} \equiv {\tau_{\hat{a}}}^{CC'} \, \tau_{\hat{b}}^{DD'} \, \lambda^{\hat{a}\hat{b}} \equiv \Lambda_1^{(CD)} \, \varepsilon^{C'D'} + \Lambda_2^{(C'D')} \, \varepsilon^{CD}, \tag{10.6.6}$$

the first of the boundary conditions in (iii) is satisfied provided that

$$\Lambda_1^{(CD)} = 0$$

at $\partial\Sigma$ in real general relativity, since then $\Lambda_2^{(C'D')}$ is obtained by complex conjugation of $\Lambda_1^{(CD)}$, and hence the condition $\Lambda_2^{(C'D')} = 0$ at $\partial\Sigma$ leads to no further information.

In the *holomorphic* framework, however, no complex conjugation relating primed to unprimed spin-space can be defined, since such a map is not invariant under holomorphic coordinate transformations (chapter three). Hence spinor fields belonging to unprimed or primed spin-space are *totally independent*, and the first of the boundary conditions in (iii) reads

$$\Lambda^{(CD)} = 0 \text{ at } \partial\Sigma_c, \tag{10.6.7}$$

$$\widetilde{\Lambda}^{(C'D')} = 0 \text{ at } \partial\Sigma_c, \tag{10.6.8}$$

where $\partial\Sigma_c$ is a two-complex-dimensional complex surface, bounding the three-complex-dimensional surface Σ_c, and the *tilde* is used to denote *independent* spinor fields, not related by any conjugation.

Similarly, ρ^{ab} defined in (10.6.3) takes the form

$$\rho^{ab} = e\Bigl(e^a{}_{CC'}\, e^b{}_{DD'} - e^a{}_{DD'}\, e^b{}_{CC'}\Bigr)\Bigl(\Omega_f^{(CD)}\, \varepsilon^{C'D'} + \widetilde{\Omega}_f^{(C'D')}\, \varepsilon^{CD}\Bigr)\xi^f, \tag{10.6.9}$$

and hence the second of the boundary conditions in (iii) leads to the *independent* boundary conditions

$$\Omega_f^{(CD)} = 0 \text{ at } \partial\Sigma_c, \tag{10.6.10}$$

$$\widetilde{\Omega}_f^{(C'D')} = 0 \text{ at } \partial\Sigma_c, \tag{10.6.11}$$

in complex general relativity.

The resulting picture of complex general relativity is highly non-trivial. One starts from a one-jet bundle J^1 which, in local coordinates, is described by a holomorphic coordinate system, with holomorphic tetrad, holomorphic connection one-form $\omega_a{}^{\hat{b}\hat{c}}$, multivelocities corresponding to the tetrad and multivelocities corresponding to $\omega_a{}^{\hat{b}\hat{c}}$, both of holomorphic nature. The intrinsic form of the field equations, which is a generalization of a mathematical structure already existing in classical mechanics, leads to the complex vacuum Einstein equations $R_{ab} = 0$, and to a condition on the covariant divergence of the multimomenta. Moreover, the covariant multimomentum map, evaluated on a section of J^1 and integrated on an arbitrary three-complex-dimensional surface Σ_c, reflects the invariance of complex general relativity under all holomorphic coordinate transformations. Since space-time is now a complex manifold, one deals with holomorphic coordinates which are all on the same footing, and hence no time coordinate can be defined. Thus, the counterpart of the constraint equations results from the holomorphic version of the covariant multimomentum map, but cannot be related to a Cauchy problem as in the Lorentzian theory. In particular, the Hamiltonian constraint of Lorentzian general relativity is replaced by a geometric structure which is linear in the holomorphic multimomenta, provided that two boundary terms can be set to zero (of course, our multimomenta are holomorphic by construction, since in complex general relativity the tetrad is holomorphic). For this purpose, one of the following three conditions should hold:

(i) Σ_c has no boundary;

(ii) the holomorphic multimomenta vanish at $\partial\Sigma_c$;

(iii) the equations (10.6.7) and (10.6.8) hold at $\partial\Sigma_c$, as well as the equations (10.6.10) and (10.6.11). The latter equations may be replaced by the condition $u^{AA'} = 0$ at $\partial\Sigma_c$, where u is a holomorphic vector field describing holomorphic coordinate transformations on the base-space, i.e. on complex space-time.

Note that it is not *a priori* obvious that the three-complex-dimensional surface Σ_c has no boundary. Hence one really has to consider the boundary conditions (ii) or (iii) in the holomorphic framework. They imply that the holomorphic multimomenta have to vanish everywhere on Σ_c (by virtue of a well known result in complex analysis), or the elements of $o(4, C)$ have to vanish everywhere on Σ_c, jointly with the self-dual and anti-self-dual parts of the connection one-form. The latter of these conditions may be replaced by the vanishing of the holomorphic vector field u on Σ_c. In other words, if Σ_c has a boundary, unless the holomorphic multimomenta vanish on the whole of Σ_c, there are restrictions at Σ_c on the spinor fields expressing the holomorphic nature of the theory and its invariance under all holomorphic coordinate transformations. Indeed, already in real Lorentzian four-manifolds one faces a choice between boundary conditions on the multimomenta and restrictions on the invariance group resulting from boundary effects. We choose the former, following Esposito and Stornaiolo (1995), and emphasize their role in complex general relativity. Of course, the spinor fields involved in the boundary conditions are instead non-vanishing on the four-complex-dimensional space-time.

Remarkably, to ensure that the holomorphic multimomenta $\tilde{p}^{ab}{}_{\hat{c}\hat{d}}$ vanish at $\partial\Sigma_c$, and hence on Σ_c as well, the determinant e of the tetrad should vanish at $\partial\Sigma_c$, or $e^{-1}\ \tilde{p}^{ab}{}_{\hat{c}\hat{d}}$ should vanish at $\partial\Sigma_c$. The former case admits as a subset the totally null two-complex-dimensional surfaces known as α-surfaces and β-surfaces (chapter four). Since the integrability condition for α-surfaces is expressed by the vanishing of the self-dual Weyl spinor, our formalism enables one to recover the anti-self-dual (also called right-flat) space-time relevant for twistor theory, where both the Ricci spinor and the self-dual Weyl spinor vanish. However, if $\partial\Sigma_c$ is not totally null, the resulting theory does not correspond to twistor theory. The latter case implies that the tetrad vectors are turned into holomorphic vectors u_1, u_2, u_3, u_4 such that one of the following conditions holds at $\partial\Sigma_c$, and hence on Σ_c as well: (i) $u_1 = u_2 = u_3 = u_4 = 0$; (ii) $u_1 = u_2 = u_3 = 0, u_4 \neq 0$; (iii) $u_1 = u_2 = 0, u_3 = \gamma u_4, \gamma \in \mathcal{C}$; (iv) $u_1 = 0, \gamma_2 u_2 = \gamma_3 u_3 = \gamma_4 u_4, \gamma_i \in \mathcal{C}, i = 2, 3, 4$; (v) $\gamma_1 u_1 = \gamma_2 u_2 = \gamma_3 u_3 = \gamma_4 u_4, \gamma_i \in \mathcal{C}, i = 1, 2, 3, 4$.

It now appears important to understand the relation between complex general relativity derived from jet-bundle theory and complex general relativity as in the Penrose twistor program. For this purpose, one has to study the topology and the geometry of the space of two-complex-dimensional surfaces $\partial\Sigma_c$ in the generic case. This leads to a deep link between complex space-times which are not anti-self-dual and two-complex-dimensional surfaces which are not totally null. In other words, on going beyond twistor theory, one finds that the analysis of two-complex-dimensional surfaces still plays a key role. Last, but not least, one has to solve equations which are now linear in the *holomorphic multimomenta*, both in classical and in quantum gravity (these equations correspond to the constraint equations of the Lorentzian theory). Hence this analysis seems to add evidence in favour of new perspectives being in sight in relativistic theories of gravitation.

For other recent developments in complex, spinor and twistor geometry, we refer the reader to the work in Lewandowski *et al.* (1990, 1991), Dunajski and Mason (1997), Nurowski (1997), Tod and Dunajski (1997), Penrose (1997), Dunajski (1999), Frauendiener and Sparling (1999).

APPENDIX A: Clifford algebras

In section 7.4 we have defined the total Dirac operator in Riemannian geometries as the first-order elliptic operator whose action on the sections is given by composition of Clifford multiplication with covariant differentiation. Following Ward and Wells (1990), this appendix presents a self-contained description of Clifford algebras and Clifford multiplication.

Let V be a real vector space equipped with an inner product $\langle \ , \ \rangle$, defined by a non-degenerate quadratic form Q of signature (p, q). Let $T(V)$ be the tensor algebra of V and consider the ideal $\mathcal{I}$ in $T(V)$ generated by $x \otimes x + Q(x)$. By definition, $\mathcal{I}$ consists of sums of terms of the kind $a \otimes \Big\{ x \otimes x + Q(x) \Big\} \otimes b$, $x \in V, a, b \in T(V)$. The quotient space

$$Cl(V) \equiv Cl(V, Q) \equiv T(V)/\mathcal{I} \tag{A.1}$$

is the Clifford algebra of the vector space V equipped with the quadratic form Q. The product induced by the tensor product in $T(V)$ is known as Clifford multiplication or the Clifford product and is denoted by $x \cdot y$, for $x, y \in Cl(V)$. The dimension of $Cl(V)$ is 2^n if $\dim(V) = n$. A basis for $Cl(V)$ is given by the scalar 1 and the products

$$e_{i_1} \cdot e_{i_2} \cdot e_{i_n} \qquad i_1 < ... < i_n,$$

where $\Big\{ e_1, ..., e_n \Big\}$ is an orthonormal basis for V. Moreover, the products satisfy

$$e_i \cdot e_j + e_j \cdot e_i = 0 \ i \neq j, \tag{A.2}$$

$$e_i \cdot e_i = -2 \langle e_i, e_i \rangle \ i = 1, ..., n. \tag{A.3}$$

As a vector space, $Cl(V)$ is isomorphic to $\Lambda^*(V)$, the Grassmann algebra, with

$$e_{i_1} ... e_{i_n} \longrightarrow e_{i_1} \wedge ... \wedge e_{i_n}.$$

There are two natural *involutions* on $Cl(V)$. The first, denoted by $\alpha : Cl(V) \rightarrow Cl(V)$, is induced by the involution $x \rightarrow -x$ defined on V, which extends to an automorphism of $Cl(V)$. The eigenspace of α with eigenvalue $+1$ consists of the even elements of $Cl(V)$, and the eigenspace of α of eigenvalue -1 consists of the odd elements of $Cl(V)$.

The second involution is a mapping $x \rightarrow x^t$, induced on generators by

$$\left(e_{i_1}...e_{i_p}\right)^t = e_{i_p}...e_{i_1},$$

where e_i are basis elements of V. Moreover, we define $x \rightarrow \overline{x}$, a third involution of $Cl(V)$, by $\overline{x} \equiv \alpha(x^t)$.

One then defines $Cl^*(V)$ to be the group of invertible elements of $Cl(V)$, and the Clifford group $\Gamma(V)$ is the subgroup of $Cl^*(V)$ defined by

$$\Gamma(V) \equiv \left\{ x \in Cl^*(V) : y \in V \Rightarrow \alpha(x)yx^{-1} \in V \right\}. \tag{A.4}$$

One can show that the map $\rho : V \rightarrow V$ given by $\rho(x)y = \alpha(x)yx^{-1}$ is an isometry of V with respect to the quadratic form Q. The map $x \rightarrow \|x\| \equiv x\overline{x}$ is the square-norm map, and enables one to define a remarkable subgroup of the Clifford group, i.e.

$$\text{Pin}(V) \equiv \left\{ x \in \Gamma(V) : \|x\| = 1 \right\}. \tag{A.5}$$

APPENDIX B: Rarita–Schwinger equations

Following Aichelburg and Urbantke (1981), one can express the Γ-potentials of (8.6.1) as

$$\Gamma^{A}{}_{BB'} = \nabla_{BB'} \, \alpha^{A}. \tag{B.1}$$

Thus, acting with $\nabla_{CC'}$ on both sides of (B.1), symmetrizing over $C'B'$ and using the spinor Ricci identity (8.7.6), one finds

$$\nabla_{C(C'} \, \Gamma^{AC}{}_{B')} = \widetilde{\Phi}_{B'C'L}{}^{A} \, \alpha^{L}. \tag{B.2}$$

Moreover, acting with $\nabla_{C}{}^{C'}$ on both sides of (B.1), putting $B' = C'$ (with contraction over this index), and using the spinor Ricci identity (8.7.4) leads to

$$\varepsilon^{AB} \, \nabla_{(C}{}^{C'} \, \Gamma_{|A|B)C'} = -3\Lambda \, \alpha_{C}. \tag{B.3}$$

Equations (B.1)–(B.3) rely on the conventions in Aichelburg and Urbantke (1981). However, to achieve agreement with the conventions in Penrose (1994) and in our paper, the equations (8.6.3)–(8.6.6) are obtained by defining (cf. (B.1))

$$\Gamma_{B}{}^{A}{}_{B'} \equiv \nabla_{BB'} \, \alpha^{A}, \tag{B.4}$$

and similarly for the γ-potentials of (8.6.2) (for the effect of torsion terms, see comments following equation (21) in Aichelburg and Urbantke (1981)).

APPENDIX C: Fibre bundles

The basic idea in fibre-bundle theory is to deal with topological spaces which are locally, but not necessarily globally, a product of two spaces. This appendix begins with the definition of fibre bundles and the reconstruction theorem for bundles, jointly with a number of examples, following Nash and Sen (1983). A more formal presentation of some related topics is then given, for completeness.

A fibre bundle may be defined as the collection of the following five mathematical objects:

(1) A topological space E called the total space.

(2) A topological space X, i.e. the base space, and a projection $\pi : E \rightarrow X$ of E onto X.

(3) A third topological space F, i.e. the fibre.

(4) A group G of homeomorphisms of F, called the structure group.

(5) A set $\{U_\alpha\}$ of open coordinate neighbourhoods which cover X. These reflect the *local* product structure of E. Thus, a homeomorphism ϕ_α is given

$$\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F, \tag{C.1}$$

such that the composition of the projection map π with the inverse of ϕ_α yields points of U_α, i.e.

$$\pi\, \phi_\alpha^{-1}(x, f) = x \; x \in U_\alpha, \; f \in F. \tag{C.2}$$

To see how this abstract definition works, let us focus on the Möbius strip, which can be obtained by twisting ends of a rectangular strip before joining them. In this case, the base space X is the circle S^1, while the fibre F is a line segment. For any $x \in X$, the action of π^{-1} on x yields the fibre over x. The structure group G appears on going from local coordinates $\Big(U_\alpha, \phi_\alpha\Big)$ to local coordinates $\Big(U_\beta, \phi_\beta\Big)$.

If U_α and U_β have a non-empty intersection, then $\phi_\alpha \circ \phi_\beta^{-1}$ is a continuous invertible map

$$\phi_\alpha \circ \phi_\beta^{-1} : \Big(U_\alpha \cap U_\beta \Big) \times F \rightarrow \Big(U_\alpha \cap U_\beta \Big) \times F. \tag{C.3}$$

For fixed $x \in U_\alpha \cap U_\beta$, such a map becomes a map $h_{\alpha\beta}$ from F to F. This is, by definition, the transition function, and yields a homeomorphism of the fibre F. The structure group G of E is then defined as the set of all these maps $h_{\alpha\beta}$ for *all* choices of local coordinates $\Big(U_\alpha, \phi_\alpha \Big)$. Here, it consists of just two elements $\{e, h\}$. This is best seen on considering the covering $\{U_\alpha\}$ which is given by two open arcs of S^1 denoted by U_1 and U_2. Their intersection consists of two disjoint open arcs A and B, and hence the transition functions $h_{\alpha\beta}$ are found to be

$$h_{12}(x) = e \text{ if } x \in A, \ h \text{ if } x \in B, \tag{C.4}$$

$$h_{12}(x) = h_{21}^{-1}(x), \tag{C.5}$$

$$h_{11}(x) = h_{22}(x) = e. \tag{C.6}$$

To detect the group $G = \{e, h\}$ it is enough to move the fibre once round the Möbius strip. By virtue of this operation, F is reflected in its midpoint, which implies that the group element h is responsible for such a reflection. Moreover, on squaring up the reflection one obtains the identity e, and hence G has indeed just two elements.

So far, our definition of a bundle involves the total space, the base space, the fibre, the structure group and the set of open coordinate neighbourhoods covering the base space. However, the essential information about a fibre bundle can be obtained from a smaller set of mathematical objects, i.e. the base space, the fibre, the structure group and the transition functions $h_{\alpha\beta}$. Following again Nash and Sen (1983) we now prove the reconstruction theorem for bundles, which tells us how to obtain the total space E, the projection map π and the homeomorphisms ϕ_α from $\Big(X, F, G, \{h_{\alpha\beta}\} \Big)$.

First, E is obtained from an equivalence relation, as follows. One considers the set $\widetilde{E}$ defined as the union of all products of the form $U_\alpha \times F$, i.e.

$$\widetilde{E} \equiv \bigcup_\alpha U_\alpha \times F. \tag{C.7}$$

One here writes (x, f) for an element of $\widetilde{E}$, where $x \in U_\alpha$. An equivalence relation $\sim$ is then introduced by requiring that, given $(x, f) \in U_\alpha \times F$ and $(x', f') \in U_\beta \times F$, these elements are equivalent,

$$(x, f) \sim (x', f'), \tag{C.8}$$

if

$$x = x' \quad \text{and} \quad h_{\alpha\beta}(x) f = f'. \tag{C.9}$$

This means that the transition functions enable one to pass from f to f', while the points x and x' coincide. The desired total space E is hence given as

$$E \equiv \widetilde{E}/\sim, \tag{C.10}$$

i.e. E is the set of all equivalence classes under $\sim$.

Second, denoting by $[(x, f)]$ the equivalence class containing the element (x, f) of $U_\alpha \times F$, the projection $\pi : E \to X$ is defined as the map

$$\pi : [(x, f)] \to x. \tag{C.11}$$

In other words, π maps the equivalence class $[(x, f)]$ into $x \in U_\alpha$.

Third, the function ϕ_α is defined (indirectly) by giving its inverse

$$\phi_\alpha^{-1} : U_\alpha \times F \to \pi^{-1}(U_\alpha). \tag{C.12}$$

Note that, by construction, ϕ_α^{-1} satisfies the condition

$$\pi\, \phi_\alpha^{-1}(x, f) = x \in U_\alpha, \tag{C.13}$$

and this is what we actually need, despite one might be tempted to think in terms of ϕ_α rather than its inverse.

The readers who are not familiar with fibre-bundle theory may find it helpful to see an application of this reconstruction theorem. For this purpose, we focus again on the Möbius strip. Thus, our data are the base $X = S^1$, a line segment representing the fibre, the structure group $\{e, h\}$, where h is responsible for F being reflected in its midpoint, and the transition functions $h_{\alpha\beta}$ in (C.4)–(C.6). Following the definition (C.8) and (C.9) of equivalence relation, and bearing in mind that $h_{12} = h$, one finds

$$f = f' \text{ if } x \in A, \tag{C.14}$$

$$hf = f' \text{ if } x \in B, \tag{C.15}$$

where A and B are the two open arcs whose disjoint union gives the intersection of the covering arcs U_1 and U_2. In the light of (C.14) and (C.15), if $x \in A$ then the equivalence class $[(x, f)]$ consists of (x, f) only, whereas, if $x \in B$, $[(x, f)]$ consists of two elements, i.e. (x, f) and (x, hf). Hence it should be clear how to construct the total space E by using equivalence classes, according to (C.10). What happens can be divided into three steps (Nash and Sen 1983):

(i) The base space splits into two, and one has the covering arcs U_1, U_2 and the intersection regions A and B.

(ii) The space $\widetilde{E}$ defined in (C.7) splits into two. The regions $A \cap F$ are glued together without a twist, since the equivalence class $[(x, f)]$ has only the element (x, f) if $x \in A$. By contrast, a twist is necessary to glue together the regions $B \cap F$, since $[(x, f)]$ consists of two elements if $x \in B$. The identification of (x, f) and (x, hf) under the action of $\sim$, makes it necessary to glue with twist the regions $B \cap F$.

(iii) The bundle $E \equiv \widetilde{E}/\sim$ has been obtained. Shaded regions may be drawn, which are isomorphic to $A \cap F$ and $B \cap F$, respectively.

If we now come back to the general theory of fibre bundles, we should mention some important properties of the transition functions $h_{\alpha\beta}$. They obey a set of compatibility conditions, where repeated indices are not summed over, i.e.

$$h_{\alpha\alpha}(x) = e, \ x \in U_\alpha, \tag{C.16}$$

$$h_{\alpha\beta}(x) = (h_{\beta\alpha}(x))^{-1}, \ x \in U_\alpha \cap U_\beta, \tag{C.17}$$

$$h_{\alpha\beta}(x) \ h_{\beta\gamma}(x) = h_{\alpha\gamma}(x), \ x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{C.18}$$

A simple calculation can be now made which shows that any bundle can be actually seen as an equivalence class of bundles. The underlying argument is as follows. Suppose two bundles E and E' are given, with the same base space, fibre, and group. Moreover, let $\{\phi_\alpha, U_\alpha\}$ and $\{\psi_\alpha, U_\alpha\}$ be the sets of coordinates and coverings for E and E', respectively. The map

$$\lambda_\alpha \equiv \phi_\alpha \circ \psi_\alpha^{-1} : U_\alpha \times F \rightarrow U_\alpha \times F$$

is now required to be a homeomorphism of F belonging to the structure group G. Thus, if one combines the definitions

$$\lambda_\alpha(x) \equiv \phi_\alpha \circ \psi_\alpha^{-1}(x), \tag{C.19}$$

$$h_{\alpha\beta}(x) \equiv \phi_\alpha \circ \phi_\beta^{-1}(x), \tag{C.20}$$

$$h'_{\alpha\beta}(x) \equiv \psi_\alpha \circ \psi_\beta^{-1}(x), \tag{C.21}$$

one finds

$$\lambda_\alpha^{-1}(x) h_{\alpha\beta}(x) \lambda_\beta(x) = \psi_\alpha \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1} \circ \phi_\beta \circ \psi_\beta^{-1}(x) = h'_{\alpha\beta}(x). \tag{C.22}$$

Thus, since λ_α belongs to the structure group G by hypothesis, as the transition function $h_{\alpha\beta}$ varies, both $\lambda_\alpha^{-1} \ h_{\alpha\beta} \ \lambda_\beta$ and $h'_{\alpha\beta}$ generate all elements of G. The only difference between the bundles E and E' lies in the assignment of coordinates, and the equivalence of such bundles is expressed by (C.22). The careful reader may have noticed that in our argument the coverings of the base space for E and

E' have been taken to coincide. However, this restriction is unnecessary. One may instead consider coordinates and coverings given by $\{\phi_\alpha, U_\alpha\}$ for E, and by $\{\psi_\alpha, V_\alpha\}$ for E'. The equivalence of E and E' is then defined by requiring that the homeomorphism $\phi_\alpha \circ \psi_\beta^{-1}(x)$ should coincide with an element of the structure group G for $x \in U_\alpha \cap V_\beta$ (Nash and Sen 1983).

Besides the Möbius strip, the naturally occurring examples of bundles are the tangent and cotangent bundles and the frame bundle. The tangent bundle $T(M)$ is defined as the collection of all tangent spaces $T_p(M)$, for all points p in the manifold M, i.e.

$$T(M) \equiv \bigcup_{p \in M} (p, T_p(M)). \tag{C.23}$$

By construction, the base space is M itself, and the fibre at $p \in M$ is the tangent space $T_p(M)$. Moreover, the projection map $\pi : T(M) \rightarrow M$ associates to any tangent vector $\in T_p(M)$ the point $p \in M$. Note that, if M is n-dimensional, the fibre at p is an n-dimensional vector space isomorphic to R^n. The *local* product structure of $T(M)$ becomes evident if one can construct a homeomorphism $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times R^n$. Thus, we are expressing $T(M)$ in terms of points of M and tangent vectors at such points. This is indeed the case since, for a tangent vector V at p, its expression in local coordinates is

$$V = b^i(p) \frac{\partial}{\partial x^i}\bigg|_p . \tag{C.24}$$

Hence the desired ϕ_α has to map V into the pair $\Big(p, b^i(p)\Big)$. Moreover, the structure group is the general linear group $GL(n, R)$, whose action on elements of the fibre should be viewed as the action of a matrix on a vector.

The *frame* bundle of M requires taking a total space $B(M)$ as the set of all frames at all points in M. Such (linear) frames b at $x \in M$ are, of course, an ordered set $\Big(b_1, b_2, ..., b_n\Big)$ of basis vectors for the tangent space $T_x(M)$. The projection $\pi : B(M) \rightarrow M$ acts by mapping a base b into the point of M to which

b is attached. Denoting by u an element of $GL(n, R)$, the $GL(n, R)$ action on $B(M)$ is defined by

$$\Big(b_1, ..., b_n\Big)u \equiv \Big(b_j u_{j1}, ..., b_j u_{jn}\Big). \tag{C.25}$$

The coordinates for a differentiable structure on $B(M)$ are $\Big(x^1, ..., x^n; u_i^j\Big)$, where $x^1, ..., x^n$ are coordinate functions in a coordinate neighbourhood $V \subset M$, while u_i^j appear in the representation of the map

$$\gamma : V \times GL(n, R) \rightarrow \pi^{-1}(V), \tag{C.26}$$

by means of the rule (Isham 1989)

$$(x, u) \rightarrow \left(u_1^j (\partial_j)_x, ..., u_n^j (\partial_j)_x \right).$$

To complete our introduction to fibre bundles, we now define cross-sections, sub-bundles, vector bundles, and connections on principal bundles, following Isham (1989).

(i) Cross-sections are very important from the point of view of physical applications, since in classical field theory the physical fields may be viewed as sections of a suitable class of bundles. The idea is to deal with functions defined on the base space and taking values in the fibre of the bundle. Thus, given a bundle (E, π, M), a *cross-section* is a map $s : M \rightarrow E$ such that the image of each point $x \in M$ lies in the fibre $\pi^{-1}(x)$ over x:

$$\pi \circ s = \mathrm{id}_M. \tag{C.27}$$

In other words, one has the projection map from E to M, and the cross-section from M to E, and their composition yields the identity on the base space. In the particular case of a product bundle, a cross-section defines a unique function $\widehat{s} : M \rightarrow F$ given by

$$s(x) = (x, \widehat{s}(x)), \ \forall x \in M. \tag{C.28}$$

(ii) The advantage of introducing the sub-bundle E' of a given bundle E lies in the possibility to refer to a mathematical structure less complicated than the original. Let (E, π, M) be a fibre bundle with fibre F. A sub-bundle of (E, π, M) is a subspace of E with the extra property that it always contains complete fibres of E, and hence is itself a fibre bundle. The formal definition demands that the following conditions on (E', π', M') should hold:

$$E' \subset E, \tag{C.29}$$

$$M' \subset M, \tag{C.30}$$

$$\pi' = \pi \mid_E . \tag{C.31}$$

In particular, if $T \equiv (E, \pi, M)$ is a sub-bundle of the product bundle $(M \times F, \mathrm{pr}_1, M)$, then cross-sections of T have the form $s(x) = (x, \widehat{s}(x))$, where $\widehat{s} : M \rightarrow F$ is a function such that, $\forall x \in M$, $(x, \widehat{s}(x)) \in E$. For example, the tangent bundle TS^n of the n-sphere S^n may be viewed as the sub-bundle of $S^n \times R^{n+1}$ (Isham 1989)

$$E(TS^n) \approx \left\{ (x, y) \in S^n \times R^{n+1} : x \cdot y = 0 \right\} . \tag{C.32}$$

Cross-sections of TS^n are vector fields on the n-sphere. It is also instructive to introduce the normal bundle $\nu(S^n)$ of S^n, i.e. the set of all vectors in R^{n+1} which are normal to points on S^n (Isham 1989):

$$E(\nu(S^n)) \equiv \left\{ (x, y) \in S^n \times R^{n+1} : \exists k \in R : y = kx \right\} . \tag{C.33}$$

(iii) In the case of vector bundles, the fibres are isomorphic to a vector space, and the space of cross-sections has the structure of a vector space. Vector bundles are relevant for theoretical physics, since gauge theory may be formulated in terms of vector bundles (Ward and Wells 1990), and the space of cross-sections can replace the space of functions on a manifold (although, in this respect, the opposite point of view may be taken). By definition, a n-dimensional real (resp. complex) vector bundle (E, π, M) is a fibre bundle in which each fibre is isomorphic to a

n-dimensional real (resp. complex) vector space. Moreover, $\forall x \in M$, a neighbourhood $U \subset M$ of x exists, jointly with a *local* trivialization $\rho : U \times R^n \rightarrow \pi^{-1}(U)$ such that, $\forall y \in U$, $\rho : \{y\} \times R^n \rightarrow \pi^{-1}(y)$ is a *linear map.*

The simplest examples are the product space $M \times R^n$, and the tangent and cotangent bundles of a manifold M. A less trivial example is given by the normal bundle (cf. (C.33)). If M is a m-dimensional sub-manifold of R^n, its *normal bundle* is a $(n-m)$-dimensional vector bundle $\nu(M)$ over M, with total space (Isham 1989)

$$E(\nu(M)) \equiv \{(x,v) \in M \times R^n : v \cdot w = 0, \forall w \in T_x(M)\}, \tag{C.34}$$

and projection map $\pi : E(\nu(M)) \rightarrow M$ defined by $\pi(x,v) \equiv x$. Last, but not least, we mention the *canonical real line bundle* γ_n over the real projective space RP^n, with total space

$$E(\gamma_n) \equiv \left\{([x],v) \in RP^n \times R^{n+1} : v = \lambda\, x, \lambda \in R\right\}, \tag{C.35}$$

where $[x]$ denotes the line passing through $x \in R^{n+1}$. The projection map $\pi : E(\gamma_n) \rightarrow RP^n$ is defined by the condition

$$\pi([x],v) \equiv [x]. \tag{C.36}$$

Its inverse is therefore the line in R^{n+1} passing through x. Note that γ_n is a one-dimensional vector bundle.

(iv) In Nash and Sen (1983), principal bundles are defined by requiring that the fibre F should be (isomorphic to) the structure group. However, a more precise definition, such as the one given in Isham (1989), relies on the theory of Lie groups. Since it is impossible to describe such a theory in a short appendix, we refer the reader to Isham (1989) and references therein for the theory of Lie groups, and we limit ourselves to the following definitions.

A bundle (E,π,M) is a G-bundle if E is a right G-space and if (E,π,M) is isomorphic to the bundle $(E,\sigma,E/G)$, where E/G is the orbit space of the G-action on E, and σ is the usual projection map. Moreover, if G acts freely on E, then (E, π, M) is said to be a *principal* G-bundle, and G is the structure group of the bundle. Since G acts freely on E by hypothesis, each orbit is homeomorphic to G, and hence one has a fibre bundle with fibre G (see earlier remarks).

To define connections in a principal bundle, with the associated covariant differentiation, one has to look for vector fields on the bundle space P that point from one fibre to another. The first basic remark is that the tangent space $T_p(P)$ at a point $p \in P$ admits a natural direct-sum decomposition into two sub-spaces $V_p(P)$ and $H_p(P)$, and the connection enables one to obtain such a split of $T_p(P)$. Hence the elements of $T_p(P)$ are uniquely decomposed into a sum of components lying in $V_p(P)$ and $H_p(P)$ by virtue of the connection. The first sub-space, $V_p(P)$, is defined as

$$V_p(P) \equiv \{t \in T_p(P) : \pi_* t = 0\}, \tag{C.37}$$

where $\pi : P \rightarrow M$ is the projection map from the total space to the base space. The elements of $V_p(P)$ are, by construction, *vertical* vectors in that they point along the fibre. The desired vectors, which point away from the fibres, lie instead in the horizontal sub-space $H_p(P)$. By definition, a *connection* in a principal bundle $P \rightarrow M$ with group G is a smooth assignment, to each $p \in P$, of a *horizontal* sub-space $H_p(P)$ of $T_p(P)$ such that

$$T_p(P) \approx V_p(P) \oplus H_p(P). \tag{C.38}$$

By virtue of (C.38), a connection is also called, within this framework, a *distribution*. Moreover, the decomposition (C.38) is required to be compatible with the right action of G on P.

The constructions outlined in this appendix are the first step towards a geometric and intrinsic formulation of gauge theories, and they are frequently applied also in twistor theory (sections 5.1–5.3, 9.6 and 9.7).

APPENDIX D: Sheaf theory

In chapter four we have given an elementary introduction to sheaf cohomology. However, to understand the language of section 9.6, it may be helpful to supplement our early treatment by some more precise definitions. This is here achieved by relying on Chern (1979).

The definition of a sheaf of Abelian groups involves two topological spaces $\mathcal{S}$ and M, jointly with a map $\pi : \mathcal{S} \rightarrow M$. The sheaf of Abelian groups is then the pair $(\mathcal{S}, \pi)$ such that:

(i) π is a local homeomorphism;

(ii) $\forall x \in M$, the set $\pi^{-1}(x)$, i.e. the *stalk* over x, is an Abelian group;

(iii) the group operations are continuous in the topology of $\mathcal{S}$.

Denoting by U an open set of M, a *section* of the sheaf $\mathcal{S}$ over U is a continuous map $f : U \rightarrow S$ such that its composition with π yields the identity (cf. appendix C). The set $\Gamma(U, \mathcal{S})$ of all (smooth) sections over U is an Abelian group, since if $f, g \in \Gamma(U, \mathcal{S})$, one can define $f - g$ by the condition $(f - g)(x) \equiv f(x) - g(x), x \in U$. The *zero* of $\Gamma(U, \mathcal{S})$ is the zero section assigning the zero of the stalk $\pi^{-1}(x)$ to every $x \in U$.

The next step is the definition of *presheaf* of Abelian groups over M. This is obtained on considering the homomorphism between sections over U and sections over V, for V an open subset of U. More precisely, by a presheaf of Abelian groups over M we mean (Chern 1979):

(i) a basis for the open sets of M;

(ii) an Abelian group S_U assigned to each open set U of the basis;

(iii) a homomorphism $\rho_{VU} : S_U \rightarrow S_V$ associated to each inclusion $V \subset U$, such that

$$\rho_{WV} \, \rho_{VU} = \rho_{WU} \text{ whenever } W \subset V \subset U.$$

The sheaf is then obtained from the presheaf by a limiting procedure (cf. chapter four). For a given complex manifold M, the following sheaves play a very important role (cf. section 9.6):

(i) The sheaf $\mathcal{A}_{pq}$ of germs of complex-valued C^∞ forms of type (p, q). In particular, the sheaf of germs of complex-valued C^∞ functions is denoted by $\mathcal{A}_{00}$.

(ii) The sheaf C_{pq} of germs of complex-valued C^∞ forms of type (p, q), closed under the operator $\overline{\partial}$. The sheaf of germs of holomorphic functions (i.e. zero-forms) is denoted by $\mathcal{O} = C_{00}$. This is the most important sheaf in twistor theory (as well as in the theory of complex manifolds, cf. Chern (1979)).

(iii) The sheaf $\mathcal{O}^*$ of germs of nowhere-vanishing holomorphic functions. The group operation is the multiplication of germs of holomorphic functions.

Following again Chern (1979), we complete this brief review by introducing *fine sheaves*. They are fine in that they admit a partition of unity subordinate to *any* locally finite open covering, and play a fundamental role in cohomology, since the corresponding cohomology groups $H^q(M, \mathcal{S})$ vanish $\forall q \geq 1$. Partitions of unity of a sheaf of Abelian groups, subordinate to the locally finite open covering $\mathcal{U}$ of M, are a collection of sheaf homomorphisms $\eta_i : \mathcal{S} \rightarrow \mathcal{S}$ such that:

(i) η_i is the zero map in an open neighbourhood of $M - U_i$;

(ii) $\sum_i \eta_i$ equals the identity map of the sheaf $(\mathcal{S}, \pi)$.

The sheaf of germs of complex-valued C^∞ forms is indeed fine, while $\mathcal{C}_{pq}$ and the constant sheaf are not fine.

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