# Advanced General Relativity (CCMMS38)

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## Contents

# 1 Introduction

Special Relativity describes the motion of particles which move with a constant velocity. Its key geometrical feature is that spacetime is Lorentzian. That is, the proper length squared of a straight line is

$$\begin{aligned} (\Delta s)^2 &= -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu \end{aligned} \tag{1.1}$$

The Lorentz transformations can be identified with the linear transformations on the coordinates $x^\mu$ that preserve $\Delta s$. Thus if we have

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.2}$$

then we require that

$$\begin{aligned} (\Delta s')^2 &= \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \Delta x^\rho \Delta x^\sigma \\ &= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu \\ &= (\Delta s)^2 \end{aligned} \tag{1.3}$$

Since we take $\Delta x^\mu$ to be arbitrary we see that

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \qquad \longleftrightarrow \qquad \eta = \Lambda^T \eta \Lambda \tag{1.4}$$

where we have introduced matrix notation.

The principle of special relativity can be summed up by the statement that the laws of physics for objects moving with constant velocity should be invariant under Lorentz transformations. In other words space and time are a single aspect of one underlying object: spacetime, whose structure is determined by the tensor $\eta_{\mu\nu}$.

This was the situation about 99 years ago. However it has a glaring problem in that it cannot describe acceleration and therefore, according to Newtons laws, any force. Of course if will work in situations where the acceleration is small. Thus Einstein was led to a more general principle of relatively.

The new principle is motivated by the following thought experiment (slightly modernised): You've seen pictures of the astronauts in the space shuttle. They are weightless but why? The naive answer is that it is because they are in outer space away from the pull of earth's gravity. But then if this is the case, why do they say in orbit about the earth? Conversely if we sit in a car and it accelerates we feel a force, just like the force of gravity, only it acts to push us backwards, rather than down. The great realisation of Einstein was that a single observer cannot perform any local experiment which can tell if s/he is in free fall in a gravitational field or if there is no gravitaional force at all. Conversely they also cannot tell, using local experiments, the differences between acceleration and a gravitational field.

The important concept here is locally. Obviously a freely falling observer can tell that they are in the gravitation field of the earth if they can see the whole earth, along with their orbit. Certainly they will know if they ever hit the earth. The point is that the laws of physics should be local, *i.e.* they should only depend on the properties of spacetime and fields as evaluated at each single point independently. Therefore the laws of physics should not be able to distingish between a gravitional field and acceleration.

Thus we aim to reconcil the following observation. A particle which is freely falling in a gravitational field is physically equivalent to a particle which feels no force. Now by Newton's law no force should mean that the particle does not accelerate, that it moves in a 'straight line'. The key idea here is to realise that spacetime must therefore be curved and that 'straight lines' are not actually straight in the familiar sense of the word. The space shuttle moves in orbit around the earth because locally, that is from one instant to the next, it is following a straight path - the path of shortest length - in a spacetime that is curved. Just as an airplane travels in a great circle that passes over Greenland whenever it flies from Washington DC to London. This known as the principle of equivalance.

Mathematically we can impose these ideas by noting that a curved space is described by the concept of a manifold. Roughly speaking a manifold is a space that in a neighbourhood of each point looks like $\mathbb{R}^n$ for some $n$. Here we have $n = 4$. For infinitessimal variations $dx^\mu$ the proper distance is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.5}$$

where $g_{\mu\nu}$ is called the metric and locally determines the geometry of spacetime be determining the lengths and angles in an infinitessimal neighbourhood of each point. By definition $g_{\mu\nu}$ is symmetric: $g_{\mu\nu} = g_{\nu\mu}$. We will review manifolds and their tensors in more detail soon.

The general principle of relativity states that the laws of physics are invariant under an arbitrary - but invertable - coordinate transformation

$$x^\mu \longrightarrow x'^\mu = x'^\mu(x^\nu) \tag{1.6}$$

under which we have that $ds^2$ is invariant. The same calculation as above leads to

$$\begin{aligned} ds'^2 &= g'_{\mu\nu} dx'^\mu dx'^\nu && (1.7)\\ &= g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} dx^\rho \frac{\partial x'^\nu}{\partial x^\sigma} dx^\sigma && (1.8)\\ &= g_{\rho\sigma} dx^\rho dx^\sigma && (1.9) \end{aligned}$$

Note that the transformation needs to be invertable so that the Jacobian

$$\Lambda^\mu{}_\rho = \frac{\partial x'^\mu}{\partial x^\rho} \tag{1.10}$$

is an invertable 4 matrix whose inverse is

$$\Lambda_\sigma{}^\nu = \frac{\partial x^\nu}{\partial x'^\sigma} \tag{1.11}$$

since

$$\delta^\mu_\lambda = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda} \qquad \text{and} \qquad \delta^\mu_\lambda = \frac{\partial x'^\rho}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x'^\rho} \tag{1.12}$$

Such a change of variables is called a diffeomorhism. We now see that the invariance of the infinitessimal proper distance implies that

$$g'_{\rho\sigma} = \frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} g_{\mu\nu} \tag{1.13}$$

This is the defining property of a tensor field and we will discuss these in more detail soon.

Thus we see that we have generalised the transformation property (1.4) to local transformtions by including a general metric $g_{\mu\nu}(x)$ rather than a fixed one $\eta_{\mu\nu}$. The theory of General Relativity treats the metric $g_{\mu\nu}$ as a dynamical object and its evolution is obtained from Einstein's equation.

# 2 Manifolds and Tensors

**Warning:** This is a physicists version of a deep and beautiful mathematcal subject. You won't need to know more in the course but if you'd like to know more you can take Manifolds (CM437Z/CMMS18). No apologises will be made here for brutalising this subject.

## 2.1 Manifolds

An $n$-dimensional manifold is a space that locally looks like $\mathbb{R}^n$.

Formally the definition involves taking an open cover of a topological space $\mathcal{M}$, that is a set of pairs $(U_i, \phi_i)$ where $U_i$ is an open set of $\mathcal{M}$ and $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphsm onto its image, *i.e.* it is continuous, invertable (when restricted to its image in $\mathbb{R}^n$) and its inverse is continuous. These are subject to two key constraints:

i) $\mathcal{M} = \cup_i U_i$

ii) If $U_i \cap U_j \neq \emptyset$ then $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \subset \mathbb{R}^n \to \phi_j(U_i \cap U_j) \subset \mathbb{R}^n$ is differentiable (for our purpose we assume that all the partial derivatives exist to all orders).

What does this mean? Each point $p \in \mathcal{M}$ is contained in an open set $U_i \subset \mathcal{M}$ called a neighbourhood. The map $\phi_i$ then provides coordinates for the point $p$ and all the other points in that neighbourhood:

$$\phi_i(p) = (x^1(p), x^2(p), x^3(p), ..., x^n(p)) \in \mathbb{R}^n \tag{2.1}$$

The key point of manifolds is that there can be many possible coordinate systems for the same point and its neighbourhood, corresponding to different choices of $\phi_i$. Furthermore a particular coordinate system does not have to (and in general won't) cover the whole manifold. The second point guarantees that if two coordinate systems overlap then the transformation between one and the other is smooth.

The classic example of a manifold is the surface of a sphere, such as the earth.

Common coordinates are longitude and lattitude. However these don't cover the whole space as the north and south poles do not have a well defined longitude.

We are in addition interested in Riemannian (or technically pseudo-Riemannian) manifold which means that we also have a metric $g_{\mu\nu}$ this is an invertable matrix located at each point which determines lengths and angles of vector fields at that point, viz:

$$||V(x)||^2 = g_{\mu\nu}(x)V^\mu(x)V^\nu(x) \tag{2.2}$$

Thus if we think of $V^\mu(x)$ as the infinitessimal variation of a curve $V^\mu = dx^\mu$ then we recover the definition above for the lenght of an infinitessimally small curve passing through the point $x^\mu$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.3}$$

## 2.2 Tensors

A tensor is 'something that transforms like a vector'. Indeed it is just a generalisaton of a vector field. We saw earlier that for the notion of the proper distance to be invariant under coordinate transformations, *i.e.* diffeomorphisms, the metric had to transform as

$$g'_{\rho\sigma} = \frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}g_{\mu\nu} \tag{2.4}$$

This is an example of a $(0,2)$-tensor. We could also consider the inverse metric, that is the object $g^{\rho\sigma}$ which is the matix inverse of $g_{\mu\nu}$:

$$g_{\mu\nu}g^{\nu\sigma} = \delta^\sigma_\mu \tag{2.5}$$

**Problem:** Show that

$$g'^{\nu\sigma} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\lambda}g^{\mu\lambda} \tag{2.6}$$

Thus we can define a $(p,q)$-tensor, or rank $(p,q)$ tensor, on a manfold to be an object $T^{\mu_1\mu_2\mu_3\ldots\mu_p}{}_{\nu_1\nu_2\nu_3\ldots\nu_q}$ with $p$ upstair indices and $q$ downstairs indices that transforms under a diffeomorphsm $x^\mu \longrightarrow x'^\mu(x^\nu)$ as

$$\begin{aligned} &T'(x')^{\mu_1\mu_2\mu_3\ldots\mu_p}{}_{\nu_1\nu_2\nu_3\ldots\nu_q} \\ &= \left(\frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\frac{\partial x'^{\mu_2}}{\partial x^{\rho_2}}\frac{\partial x'^{\mu_3}}{\partial x^{\rho_3}}\cdots\frac{\partial x'^{\mu_q}}{\partial x^{\rho_q}}\right)\left(\frac{\partial x^{\lambda_1}}{\partial x'^{\nu_1}}\frac{\partial x^{\lambda_2}}{\partial x'^{\nu_2}}\frac{\partial x^{\lambda_3}}{\partial x'^{\nu_3}}\cdots\frac{\partial x^{\lambda_q}}{\partial x'^{\nu_q}}\right)T(x)^{\rho_1\rho_2\rho_3\ldots\rho_p}{}_{\lambda_1\lambda_2\lambda_3\ldots\lambda_q} \end{aligned} \tag{2.7}$$

**N.B.:** Note the positions of the primed and unprimed coordinates! So the inverse metric is an example of a $(2,0)$-tensor.

A tensor field is simply a tensor which is defined at each point on the manifold.

Thus a scalar is a $(0,0)$-tensor;

$$\phi'(x') = \phi(x) \tag{2.8}$$

a vector is a $(1,0)$-tensor;

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}(x) \tag{2.9}$$

and a covector is a $(0,1)$-tensor;

$$A'_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}(x) \tag{2.10}$$

In older books the upstairs and downstairs indices are refered to as contravariant and covariant respectively.

Given two tensors we can obtain a new one is various ways. If they have the same rank then any linear combination of them is also a tensor.

In addition a $(p,q)$-tensor can be multiplied by an $(r,s)$-tensor to produce a $(p+r, q+s)$-tensor.

Finally a $(p,q)$-tensor $T(x)^{\rho_1\rho_2\rho_3\ldots\rho_p}{}_{\lambda_1\lambda_2\lambda_3\ldots\lambda_q}$ with $p,q \geq 1$ can be contracted to form a $(p-1, q-1)$-tensor:

$$T(x)^{\rho_2\rho_3\ldots\rho_p}{}_{\lambda_2\lambda_3\ldots\lambda_q} = T(x)^{\mu\rho_2\rho_3\ldots\rho_p}{}_{\mu\lambda_2\lambda_3\ldots\lambda_q} \tag{2.11}$$

Clearly this can be done in $pq$ ways depending on which pair of indices we sum over

Also since we have a metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ we can lower and raise indices on a tensor (this doesn't really create a new tensor so it keeps the same symbol). For example if $V^{\mu}$ is a vector then

$$V_{\mu} = g_{\mu\nu} V^{\nu} \tag{2.12}$$

is a covector.

**Problem:** What are the mistakes in the following equation:

$$A_{\mu\nu}{}^{\nu\lambda} B_{\rho\nu\pi\rho}{}^{\mu\sigma} - 34 C_{\mu\rho\pi}{}^{\lambda\sigma} = D_{\rho\pi}{}^{\sigma} \tag{2.13}$$

We can also take the symmetric and anti-symmetric parts of a tensor. Consider a $(0,q)$-tensor $T_{\mu_1\mu_2\mu_3\ldots\mu_q}$ then we have

$$\begin{aligned} T_{(\mu_1\mu_2\mu_3\ldots\mu_q)} &= \frac{1}{q!}\left(T_{\mu_1\mu_2\mu_3\ldots\mu_q} + T_{\mu_2\mu_1\mu_3\ldots\mu_q} + \ldots\right) \\ T_{[\mu_1\mu_2\mu_3\ldots\mu_q]} &= \frac{1}{q!}\left(T_{\mu_1\mu_2\mu_3\ldots\mu_q} - T_{\mu_2\mu_1\mu_3\ldots\mu_q} + \ldots\right) \end{aligned} \tag{2.14}$$

where the sum is over all permuations of the indices and in the second line the plus (minus) sign occurs for even (odd) permutations.

## 2.3 Covariant Derivatives

Having introduced tensors we can consider their derivatives. However the partial derivative of a tensor is not a tensor. To see this we can consider a vector field:

$$\begin{aligned}
\frac{\partial}{\partial x'^{\nu}} V'^{\mu} &= \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial}{\partial x^{\lambda}} \left( \frac{\partial x'^{\mu}}{\partial x^{\rho}} V^{\rho} \right) \\
&= \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial}{\partial x^{\lambda}} V^{\rho} + \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\lambda}} V^{\rho}
\end{aligned} \tag{2.15}$$

The first term is fine but the second one isn't. Although we note that the derivative of a scalar is a vector, *i.e.* a $(0,1)$-tensor. To correct for this we must introduce the notion of a covariant derivative which respect the tensorial property.

The solution to this is well-known in physics. We introduce a so-called connection which modifies the derivative into a so-called covariant derivative and transforms in such a way that the covariant derivative of a tensor is again a tensor. Thus we introduce $\Gamma^{\mu}_{\lambda\rho}$ - called a connection - and define

$$D_{\nu} V^{\mu} = \partial_{\nu} V^{\mu} + \Gamma^{\mu}_{\nu\rho} V^{\rho} \tag{2.16}$$

We then require that under a diffeomorhism $\Gamma^{\mu}_{\lambda\rho}$ transforms as

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^{\tau}} \Gamma^{\tau}_{\rho\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x'^{\nu} \partial x'^{\mu}} \tag{2.17}$$

We then see that

$$\begin{aligned}
D'_{\nu} V'^{\mu} &= \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial}{\partial x^{\lambda}} V^{\rho} + \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\lambda}} V^{\rho} \\
&\quad + \left( \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\tau}} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \Gamma^{\rho}_{\lambda\sigma} + \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x'^{\nu} \partial x'^{\tau}} \right) \frac{\partial x'^{\tau}}{\partial x^{\pi}} V^{\pi} \\
&= \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\lambda}}{\partial x'^{\nu}} D_{\lambda} V^{\rho}
\end{aligned} \tag{2.18}$$

where we use the fact that

$$\frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\lambda}} = \frac{\partial}{\partial x'^{\nu}} \left( \frac{\partial x'^{\mu}}{\partial x^{\rho}} \right) = - \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x'^{\tau}}{\partial x^{\rho}} \frac{\partial}{\partial x'^{\nu}} \left( \frac{\partial x^{\sigma}}{\partial x'^{\tau}} \right) \tag{2.19}$$

For covectors we define

$$D_{\lambda} V_{\mu} = \partial_{\lambda} V_{\mu} - \Gamma^{\rho}_{\lambda\mu} V_{\rho} \tag{2.20}$$

This ensures that the scalar obtained by contracting $V^{\mu}$ and $U_{\mu}$ satisfies

$$D_{\lambda}(V^{\mu} U_{\mu}) = D_{\lambda} V^{\mu} U_{\mu} + V^{\mu} D_{\lambda} U_{\mu} = \partial_{\lambda}(V^{\mu} U_{\mu}) \tag{2.21}$$

as we expect for scalars.

The covariant derivative is then defined on a general $(p,q)$-tensor to be

$$\begin{aligned} D_\mu T^{\rho_1\rho_2\rho_3\ldots\rho_p}{}_{\lambda_1\lambda_2\lambda_3\ldots\lambda_q} &= \partial_\mu T^{\rho_1\rho_2\rho_3\ldots\rho_p}{}_{\lambda_1\lambda_2\lambda_3\ldots\lambda_q} \\ &\quad +\Gamma^{\rho_1}_{\mu\nu} T^{\nu\rho_2\rho_3\ldots\rho_p}{}_{\lambda_1\lambda_2\lambda_3\ldots\lambda_q} + \ldots \\ &\quad -\Gamma^{\nu}_{\mu\lambda_1} T^{\rho_1\rho_2\rho_3\ldots\rho_p}{}_{\nu\lambda_2\lambda_3\ldots\lambda_q} - \ldots \end{aligned} \tag{2.22}$$

where each index gets contracted with $\Gamma^\mu_{\nu\lambda}$

**Problem:** Convince yourself that the covariant derivative of a $(p,q)$-tensor is a $(p,q+1)$-tensor.

Note that the anti-symmetric part of a connection $\Gamma^\lambda_{[\mu\nu]}$ is a $(1,2)$-tensor. This is called the torsion and it is usually set to zero. In addition the difference between any two conections is a $(1,2)$-tensor.

To determine the connection $\Gamma^\lambda_{\mu\nu}$ we impose another condition, namely that the metric is covariantly constant, $D_\lambda g_{\mu\nu} = 0$. This is called the Levi-Civita connection. It is the unique connection which annihilates the metric and is torsion free.

To determine it (and show that it is unique) we consider the following

$$\begin{aligned} D_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\mu\rho} = 0 \\ D_\mu g_{\nu\lambda} &= \partial_\mu g_{\nu\lambda} - \Gamma^\rho_{\mu\nu} g_{\rho\lambda} - \Gamma^\rho_{\mu\lambda} g_{\nu\rho} = 0 \\ D_\nu g_{\lambda\mu} &= \partial_\nu g_{\lambda\mu} - \Gamma^\rho_{\nu\lambda} g_{\rho\mu} - \Gamma^\rho_{\nu\mu} g_{\lambda\rho} = 0 \end{aligned} \tag{2.23}$$

Next we take the sum of the 2nd and 3rd equation minus the 1st (and use the fact that $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$):

$$0 = \partial_\nu g_{\lambda\mu} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu} - 2\Gamma^\rho_{\mu\nu} g_{\rho\lambda} \tag{2.24}$$

Thus we find that the Levi-Civita is

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{2.25}$$

Thus we see that this is indeed symmetric in its lower two indices. Finally it is easy to verify that

$$\begin{aligned} D_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\mu\rho} \\ &= \partial_\lambda g_{\mu\nu} - \frac{1}{2}(\partial_\mu g_{\lambda\nu} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\lambda\mu}) - \frac{1}{2}(\partial_\nu g_{\lambda\mu} + \partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu}) \\ &= 0 \end{aligned} \tag{2.26}$$

**Problem:** Show that with this definition, $\Gamma^\lambda_{\mu\nu}$ indeed transforms as a connection.

## 2.4 Geodesics

We wish to find paths which minimise their proper length. Thus if $X^\mu(\tau)$ is a path in spacetime where $\tau$ parameterizes the curve and runs from $\tau = a$ to $\tau = b$ we need to minimize the functional

$$l = \int_a^b ds = \int_a^b \sqrt{|g_{\mu\nu}dx^\mu dx^\nu|} = \int_a^b \sqrt{|g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu|}d\tau \tag{2.27}$$

We do not worry about boundary terms, as we wish to find a local condition on the curve $X^\mu(\tau)$. This is simply a variational problem that you should have encountered in classical mechanics where the equations of motion are determined by extremezing the Lagrangian $\mathcal{L} = \sqrt{|g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu|}$. The Euler-Lagrange equations give

$$-\frac{d}{d\tau}\left(\frac{g_{\mu\nu}\dot{X}^\nu}{\mathcal{L}}\right) + \frac{1}{2}\frac{1}{\mathcal{L}}\frac{\partial g_{\lambda\nu}}{\partial x^\mu}\dot{X}^\lambda \dot{X}^\nu = 0 \tag{2.28}$$

Expanding things out a bit (and multipling through by $\mathcal{L}$) we find

$$\begin{aligned} 0 &= -g_{\mu\nu}\ddot{X}^\nu - \frac{1}{2}\left(2\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\nu}}{\partial x^\mu}\right)\dot{X}^\lambda \dot{X}^\nu + g_{\mu\nu}\dot{X}^\nu \frac{1}{\mathcal{L}}\frac{d\mathcal{L}}{d\tau} \\ &= -\left(g_{\mu\nu}\ddot{X}^\nu + \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{1}{2}\frac{\partial g_{\mu\lambda}}{\partial x^\nu} - \frac{1}{2}\frac{\partial g_{\lambda\nu}}{\partial x^\mu}\right)\dot{X}^\lambda \dot{X}^\nu + g_{\mu\nu}\dot{X}^\nu \frac{d}{d\tau}\ln\mathcal{L} \\ &= -g_{\mu\nu}\left(\ddot{X}^\mu + \Gamma^\mu_{\lambda\nu}\dot{X}^\lambda \dot{X}^\nu\right) + g_{\mu\nu}\dot{X}^\nu \frac{d}{d\tau}\ln\mathcal{L} \end{aligned} \tag{2.29}$$

This is equivalent to

$$\ddot{X}^\mu + \Gamma^\mu_{\lambda\nu}\dot{X}^\lambda \dot{X}^\nu = \dot{X}^\mu \frac{d}{d\tau}\ln\mathcal{L} \tag{2.30}$$

Next we note that we can clean this expression up a bit. We note that the length of a geodesic is invariant under reparameterizations of $\tau$, *i.e.* under $\tau \to \tau'(\tau)$ for any invertible function $\tau'$. If we make such a change of variables then

$$\dot{X}^\mu = \frac{d\tau'}{d\tau}\frac{dX^\mu}{d\tau'}\ , \qquad \ddot{X}^\mu = \left(\frac{d\tau'}{d\tau}\right)^2 \frac{d^2X^\mu}{d\tau'^2} + \frac{d^2\tau'}{d\tau^2}\frac{dX^\mu}{d\tau'} \tag{2.31}$$

The first equation shows that $l$ is indeed invariant since $d\tau = (d\tau/d\tau')d\tau'$. We can simplify the geodesic equation by choosing $\tau'$ such that

$$\frac{d^2\tau'}{d\tau} = \frac{d\tau'}{d\tau}\frac{d\ln\mathcal{L}}{d\tau} \iff \frac{d\tau'}{d\tau} = \mathcal{L} \tag{2.32}$$

In this case (2.30) simplifies to (dropping the prime index)

$$\ddot{X}^\mu + \Gamma^\mu_{\lambda\nu}\dot{X}^\lambda \dot{X}^\nu = 0 \tag{2.33}$$

Such a geodesic is said to be affinely parameterized and in what follows we will always assume this to be the case. Note that with this parameterization $\mathcal{L}d\tau = d\tau'$ so that $ds = d\tau'$.

Another way to think about this is to note that the tangent vector to a curve $X^\lambda(\tau)$ is

$$V^\lambda = \frac{dX^\lambda}{d\tau} \tag{2.34}$$

Thus we could write

$$\frac{d^2X^\lambda}{d\tau^2} = \frac{dV^\lambda}{d\tau} = \frac{dX^\nu}{d\tau}\frac{\partial V^\lambda}{\partial X^\nu} = V^\nu\partial_\nu V^\lambda \tag{2.35}$$

The geodesic equation becomes

$$V^\nu\left(\partial_\nu V^\lambda + \Gamma^\lambda_{\mu\nu}V^\mu\right) = V^\nu D_\nu V^\lambda = 0 \tag{2.36}$$

which states that the tangent vector to a geodesic is unchanged as as one moves along the geodesic. This called parallel transport.

More generally we see that for any vector $U^\lambda$ its derivative along a geodesic is

$$V^\mu D_\mu U^\lambda = \frac{dU^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu}V^\mu U^\nu \tag{2.37}$$

with $V^\mu = dX^\mu/d\tau$.

## 2.5 Causal Curves

We have been a but cavalier with the expression $\sqrt{|g_{\mu\nu}dx^\mu dx^\nu|}$. Since $g_{\mu\nu}$ is a symmetric matrix it has real eigenvalues. Furthermore in flat spacetime we have

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.38}$$

Note that since $g_{\mu\nu}$ must be invertable its eigenvalues can never pass through zero. Thus $g_{\mu\nu}$ always has one negative eigenvalue and three positive ones. The eigenvector associated to the negative eigenvalue is 'time'. Thus vectors can have a length-squared which is positive, negative or zero. These are called space-like, time-like or null respectively.

If $X^\mu(\tau)$ is a curve, *i.e.* a map from some interval in $\mathbb{R}$ to $\mathcal{M}$, then we can construct the 'tangent' vector at a point $X^\mu(\tau_0)$ along the curve to be

$$T^\mu = \frac{dX^\mu}{d\tau}\Big|_{\tau=\tau_0} \tag{2.39}$$

The length of this vector is determined by the metric

$$||T||^2 = g_{\mu\nu}(X(\tau_0))\frac{dX^\mu}{d\tau}\Big|_{\tau=\tau_0}\frac{dX^\nu}{d\tau}\Big|_{\tau=\tau_0} \tag{2.40}$$

An important consequence of this is that $||T||^2$ can be positive, negative or zero. Indeed for a static curve, where only the time coordinate is changing, $||T^2|| < 0$. For light rays we have $||T||^2 = 0$. Finally curves for which every point is at the same 'time' have $||T||^2 > 0$. Similarly curves are call time-like, null and space-like respectively if their tangent vectors are everywhere time-like, null or spacelike repectively.

The familar statement of Special Relativity that nothing can travel faster than the speed of light is the statment that physical observes always follow time-like curves, in particular time-like geodesics, that is curves for which (at all points)

$$g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu < 0 \tag{2.41}$$

where the derivative is with respect to the coordinate along the particles world-line. Similarly light travels along null curves:

$$g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = 0 \tag{2.42}$$

at each point.

Two events, *i.e.* two points in spacetime, are said to be causally related if there is a time-like or null curve that passes through them. In which case the earlier one (as defined by the coordinate of the worldline) can influence the later one. If no such curve exists then the two points are said to be spacelike seperated and an obeserver at one cannot know anything about the events at the other.

## 2.6 Curvature

Partial derivatives commute: $[\partial_\mu, \partial_\nu] = \partial_\mu\partial_\nu - \partial_\nu\partial_\mu = 0$. However this is not the case with covariant derivatives. Indeed

$$\begin{aligned}
[D_\mu, D_\nu]V_\lambda &= D_\mu\left(\partial_\nu V_\lambda - \Gamma^\rho_{\nu\lambda}V_\rho\right) - (\mu \leftrightarrow \nu) \\
&= \partial_\mu(\partial_\nu V_\lambda - \Gamma^\rho_{\nu\lambda}V_\rho) - \Gamma^\sigma_{\mu\nu}(\partial_\sigma V_\lambda - \Gamma^\rho_{\sigma\lambda}V_\rho) \\
&\quad -\Gamma^\sigma_{\mu\lambda}(\partial_\nu V_\sigma - \Gamma^\rho_{\nu\sigma}V_\rho) - (\mu \leftrightarrow \nu) \\
&= -\partial_\mu(\Gamma^\rho_{\nu\lambda}V_\rho) - \Gamma^\sigma_{\mu\lambda}(\partial_\nu V_\sigma - \Gamma^\rho_{\nu\sigma}V_\rho) - (\mu \leftrightarrow \nu) \\
&= -\partial_\mu\Gamma^\rho_{\nu\lambda}V_\rho + \Gamma^\sigma_{\mu\lambda}\Gamma^\rho_{\nu\sigma}V_\rho - (\mu \leftrightarrow \nu) \\
&= R_{\mu\nu\lambda}{}^\rho V_\rho
\end{aligned} \tag{2.43}$$

where

$$R_{\mu\nu\lambda}{}^\rho = -\partial_\mu\Gamma^\rho_{\nu\lambda} + \partial_\nu\Gamma^\rho_{\mu\lambda} - \Gamma^\sigma_{\nu\lambda}\Gamma^\rho_{\mu\sigma} + \Gamma^\sigma_{\mu\lambda}\Gamma^\rho_{\nu\sigma} \tag{2.44}$$

is the Riemann curvature.

**N.B.:** Different books have different conventions for $R_{\mu\nu\lambda}{}^\rho$.

For higher tensors one finds that

$$\begin{aligned}
[D_\mu, D_\nu]T^{\rho_1\rho_2\rho_3\ldots\rho_p}{}_{\lambda_1\lambda_2\lambda_3\ldots\lambda_q} &= R_{\mu\nu\lambda_1}{}^\pi T^{\rho_1\rho_2\rho_3\ldots\rho_p}{}_{\pi\lambda_2\lambda_3\ldots\lambda_q} + \ldots \\
&\quad -R_{\mu\nu\pi}{}^{\rho_1}T^{\pi\rho_2\rho_3\ldots\rho_p}{}_{\lambda_1\lambda_2\lambda_3\ldots\lambda_q} - \ldots
\end{aligned} \tag{2.45}$$

$R_{\mu\nu\lambda}{}^{\rho}$ is a tensor. To see this we note that $[D_\mu, D_\nu]V_\lambda$ is a $(0,3)$-tensor for any $V_\lambda$. Thus under a diffeomorphsim

$$R'_{\mu\nu\lambda}{}^{\rho}V'_\rho = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\pi}{\partial x'^\lambda}R_{\sigma\tau\pi}{}^{\rho}V_\rho \tag{2.46}$$

Now since

$$V_\rho = \frac{\partial x'^\theta}{\partial x^\rho}V'_\theta \tag{2.47}$$

we see that

$$R'_{\mu\nu\lambda}{}^{\rho}V'_\rho = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\pi}{\partial x'^\lambda}R_{\sigma\tau\pi}{}^{\rho}\frac{\partial x'^\theta}{\partial x^\rho}V'_\pi \tag{2.48}$$

Since $V_\rho$ is arbitrary we deduce that $R_{\sigma\tau\pi}{}^{\rho}$ is a $(1,3)$-tensor;

$$R'_{\mu\nu\lambda}{}^{\rho} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\pi}{\partial x'^\lambda}\frac{\partial x'^\rho}{\partial x^\theta}R_{\sigma\tau\pi}{}^{\theta} \tag{2.49}$$

It has some identities:

$$R_{(\mu\nu)\lambda}{}^{\rho} = 0\ , \qquad R_{[\mu\nu\lambda]}{}^{\rho} = 0\ , \qquad R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}\ , \qquad D_{[\tau}R_{\mu\nu]\lambda}{}^{\rho} = 0 \tag{2.50}$$

The first identity is obvious. The second comes from noting that

$$[D_\mu, D_\nu]D_\lambda\phi + [D_\nu, D_\lambda]D_\mu\phi + [D_\lambda, D_\mu]D_\nu\phi = 0 \tag{2.51}$$

*i.e.* the various terms all cancel since $[D_\mu, D_\nu]\phi = 0$. Thus

$$R_{\mu\nu\lambda}{}^{\rho}D_\rho + R_{\nu\lambda\mu}{}^{\rho}D_\rho + R_{\lambda\mu\nu}{}^{\rho}D_\rho = 0 \tag{2.52}$$

for any $\phi$ and hence we have $R_{[\mu\nu\lambda]}{}^{\rho} = 0$.

To establish the next identity we note it is possible to find a coordinate system such that at a given point $p$, $\partial_\lambda g_{\mu\nu} = 0$. This implies that $\Gamma^\lambda_{\mu\nu} = 0$ at $p$ (but not everywhere). Thus at this point $p$

$$R_{\mu\nu\lambda}{}^{\rho} = -\partial_\mu\Gamma^\rho_{\nu\lambda} + \partial_\nu\Gamma^\rho_{\mu\lambda} \tag{2.53}$$

and so

$$\begin{aligned}
R_{\mu\nu\lambda\rho} &= -g_{\rho\tau}\partial_\mu\Gamma^\tau_{\nu\lambda} + g_{\rho\tau}\partial_\nu\Gamma^\tau_{\mu\lambda} \\
&= \frac{1}{2}\left(-\partial_\mu\partial_\lambda g_{\rho\nu} - \partial_\mu\partial_\nu g_{\rho\lambda} + \partial_\mu\partial_\rho g_{\nu\lambda} + \partial_\nu\partial_\lambda g_{\rho\mu} + \partial_\nu\partial_\mu g_{\rho\lambda} - \partial_\nu\partial_\rho g_{\mu\lambda}\right) \\
&= \frac{1}{2}\left(-\partial_\mu\partial_\lambda g_{\rho\nu} + \partial_\mu\partial_\rho g_{\nu\lambda} + \partial_\nu\partial_\lambda g_{\rho\mu} - \partial_\nu\partial_\rho g_{\mu\lambda}\right) \\
&= R_{\lambda\rho\mu\nu}
\end{aligned} \tag{2.54}$$

Since $R_{\mu\nu\lambda\rho} - R_{\lambda\rho\mu\nu}$ is a tensor and it vanishes at $p$ in one coordinate system then it vanishes in all. Finally there was nothing special about the point $p$. Therefore $R_{\mu\nu\lambda\rho} - R_{\lambda\rho\mu\nu} = 0$ everywhere.

**Problem:** Prove the final idenity $D_{[\tau}R_{\mu\nu]\lambda}{}^{\rho} = 0$.

From the Riemann tensor we can construct the Ricci tensor by contraction

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} \tag{2.55}$$

and it follows that $R_{\mu\nu} = R_{\nu\mu}$. And lastly there is the Ricci scalar which requires us to contract using the metric

$$R = g^{\mu\nu}R_{\mu\nu} \tag{2.56}$$

**Problem:** Show that

$$D^{\mu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0 \tag{2.57}$$

**Problem:** Show that the metrics

$$ds^2 = 2dudv + dx^2 + dy^2 \qquad \text{and} \qquad ds^2 = -d\tau^2 + \tau^2 dr^2 + dx^2 + dy^2 \tag{2.58}$$

are diffeomophic to flat Minkowski space, *i.e.* find explicit coordinate transformations $x'^{\mu} = x'^{\mu}(x)$ which take the metrics given above to $g'_{\mu\nu} = \eta_{\mu\nu}$. What are the components of the curvature tensor for these metrics? Do these coordinates cover all of Minkowski space?

There is a nice geometrical picture of curvature. Let us imagine the parallel transport of a vector $U_\lambda$ around an infinitessimal parallelogram. That is we consider first an infinitessimal step along a direction with tangent $V_1^{\mu}$, followed by a step along a direction with tangent $V_2^{\mu}$ and then we go back the first curve and finally back along the second curve. After the first step $U_\lambda \to U_\lambda + \delta\tau_1\dot{U}_\lambda$ where $\delta\tau_1$ denotes the infinitessimal affine parameter of the curve whose tangent is $V_1^{\mu}$. Parallel transport of $U_\lambda$ along a curve with tangent $V^{\mu}$ means that

$$0 = V^{\mu}D_{\mu}U_{\lambda} = \frac{dU_{\lambda}}{d\tau} + \Gamma^{\rho}_{\mu\lambda}V^{\mu}U_{\rho} \tag{2.59}$$

where we have identified $\dot{U}_\lambda = V^{\mu}\partial_{\mu}U_{\lambda}$. Thus after the first step we find

$$U'_{\lambda} = U_{\lambda} - \delta\tau_1 V_1^{\mu}\Gamma^{\rho}_{\mu\lambda}U_{\rho} \tag{2.60}$$

Here we use a prime to denote the value of a quantity at the point reached from the initial point by one infinitessimal step. Note this is a new and temporary notation: $U'_\lambda$ is not $U_\lambda$ as measured in another, primed coordinate system.

Let us consider a second step along a curve with tangent $V'^{\mu}_2$ and infinitessimal affine parameter $\delta\tau_2$. Now we denote the quantities after this second step by a double-prime. We have

$$\begin{aligned} U''_{\lambda} &= U'_{\lambda} - \delta\tau_2 V'^{\mu}_2\Gamma'^{\rho}_{\mu\lambda}U'_{\rho} \\ &= (U_{\lambda} - \delta\tau_1 V_1^{\mu}\Gamma^{\rho}_{\mu\lambda}U_{\rho}) - \delta\tau_2 V'^{\mu}_2\Gamma'^{\rho}_{\mu\lambda}(U_{\rho} - \delta\tau_1\Gamma^{\sigma}_{\rho\pi}V_1^{\pi}U_{\sigma}) \end{aligned} \tag{2.61}$$

where we have substituted in our expression for $U'_\lambda$. Next we need to expand $V'^\mu_2$ and $\Gamma'^\rho_{\mu\lambda}$ in terms of $V^\mu_2$ and $\Gamma^\rho_{\mu\lambda}$ and their derivatives. In particular we have

$$\begin{aligned} \Gamma'^\rho_{\mu\lambda} &= \Gamma^\rho_{\mu\lambda} + \delta\tau_1 \dot\Gamma^\rho_{\mu\lambda} = \Gamma^\rho_{\mu\lambda} + \delta\tau_1 V_1^\nu \partial_\nu \Gamma^\rho_{\mu\lambda} \\ V'^\mu_2 &= V^\mu_2 + \delta\tau_1 \dot V^\mu_2 = V^\mu_2 + \delta\tau_1 V_1^\nu \partial_\nu V^\mu_2 \end{aligned} \tag{2.62}$$

It is helpful to use Riemann normal coordinates at the initial point so that $\Gamma^\rho_{\mu\lambda} = 0$ there. We then simply have

$$U''_\lambda = U_\lambda - \delta\tau_1 \delta\tau_2 V_1^\nu V_2^\mu \partial_\nu \Gamma^\rho_{\mu\lambda} U_\rho \tag{2.63}$$

plus higher order terms in the infinitessimal generators $\delta\tau_1$ and $\delta\tau_2$.

Finally to compute the change in $U_\lambda$ after transporting it the entire way around the parallelogram we simply take the difference between $U''_\lambda$ calculated by first going along the curve 1 and then the curve 2 and $U''_\lambda$ that is obtained by first going along curve 2 and then curve 1. In this way we find

$$\begin{aligned} \delta U_\lambda &= -\delta\tau_1 \delta\tau_2 V_1^\nu V_2^\mu \partial_\nu \Gamma^\rho_{\mu\lambda} U_\rho + \delta\tau_2 \delta\tau_1 V_2^\nu V_1^\mu \partial_\nu \Gamma^\rho_{\mu\lambda} U_\rho \\ &= (\partial_\nu \Gamma^\rho_{\mu\lambda} - \partial_\mu \Gamma^\rho_{\nu\lambda}) \delta\tau_1 \delta\tau_2 V_1^\mu V_2^\nu U_\rho \\ &= R_{\mu\nu\lambda}{}^\rho \delta\tau_1 \delta\tau_2 V_1^\mu V_2^\nu U_\rho \end{aligned} \tag{2.64}$$

Having obtained a covariant expression we can now say that this holds in any coordinate system. Thus the curvature measures how much a vector is rotated as we parallel transport it around a closed curve, *i.e.* even though we keep the vector pointing in the same direction along the parallelogram at each step, when we return to the origin we find that it has rotated.

# 3 General Relativity

We are finally in a position to write down Einstein's equation that determines the dynamics of the metric field $g_{\mu\nu}$ and examine some physical consequences.

## 3.1 "Derivation"

Einsteins idea was that matter causes spacetime to become curved so that geodesics in the presence of large masses can explain the motion of bodies in a gravitational field. The next important step is to postulate an equation for the metric in the presence of matter (or energy since they are interchangable in relativity). In addition since gravity is universal the coupling of geometry to matter should only depend on the mass and energy present and not what kind of matter it is.

Thus we need to look for an equation of the form

$$\text{Geometry} = \text{Matter} \tag{3.1}$$

The bulk properties of matter are described by the energy-momentum tensor $T_{\mu\nu}$. Furthermore we want an equation that is second order in derivatives of the metric tensor (since we want to mimic a Newtonian style force law). Another hint comes from the fact that in flat space the energy-momentum tensor is conserved $\partial^\mu T_{\mu\nu} = 0$. Since this is not covariant we postulate that the general expression is $D^\mu T_{\mu\nu} = 0$ Given the identity

$$D^\mu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \tag{3.2}$$

an obvious choice is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{3.3}$$

where $\Lambda$ and $\kappa$ are constants. Here we have included an additional term which is obviously covariantly conserved: $D^\mu g_{\mu\nu} = 0$.

There is a long story about $\Lambda$ - the so-called cosmological constant. It is in fact the biggest mystery in the exact sciences. The problem is that we don't know why the cosmological constant is so small, by a factor of $10^{-120}$ from what one would expect. Furthermore recent observations strongly imply that it is just slightly greater than zero.

Nowadays one usually absorbs the cosmological constant term into the energy momentum tensor $T_{\mu\nu} \to T_{\mu\nu} + \Lambda\kappa^{-2} g_{\mu\nu}$ where it is identified with the energy density of the vacuum. We will do the same. Thus we take Einstein's equation to be

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \tag{3.4}$$

It is often said that this equations says "matter tells space how to bend and space tells matter how to move". It turns out that this guess is at least mathematically good: it is not overdetermined, *i.e.* as a set of differential equations it is well posed with a suitable st of initial conditions. This would not be the case if we hadn't choosen the left hand side to be covariantly conserved.

**Problem:** Show that Einsteins equation is equivalent to

$$R_{\mu\nu} = \kappa^2 T_{\mu\nu} - \frac{1}{2} \kappa^2 g_{\mu\nu} T \tag{3.5}$$

## 3.2 Diffeomorphisms as Gauge Symmetries

We have constructed a theory of spacetime which is covariant under choices of coordinate system. Thus there is a built in symmetry. Namely if we have one coordinate system $x^\mu$ then we are free to change to another $x'^\mu = x'^\mu(x)$ which is any invertable, function of $x^\mu$. Thus there is a huge redundency of the system. You may be used to symmetries of a system which are global, whereby the fields can all by changed by some constant transformation. For example in spherically symmetric situations there is an $SO(3)$ symmetry whereby we can rotate the coordinate system by a constant rotation and leave the physical problem unchanged. A symmetry is said to be local if the transformation can be taken to be different at each spacetime point. The symmetry is said to be gauged.

This the case in General Relativity. In Special Relativity one has the freedom to change coordinates by only by constant Lorentz transformations $\Lambda^\mu{}_\nu$. Now we allow for arbitrary spacetime dependent transformations. Thus we can say that coordinate transformations are gauge transformations.

An important point about gauge symmetries is that they are not so much symmetries of a system but redundencies in the description. For example in a situation with rotational symmetry, such as a planet moving about the sun, although there is a symmetry that changes the angle in the plane of motion there is still physically a difference between two different values of the angular variable. In a gauge theory this is not the case. If two field configurations are related by a gauge transformation then they are physically identical.

Let us consider an infinitessimal coordinate transformation

$$x'^\mu = x^\mu + \xi^\mu(x) \tag{3.6}$$

Now we have from (3.6) that

$$\begin{aligned} \delta^\mu_\nu &= \frac{\partial x^\mu}{\partial x'^\nu} + \frac{\partial \xi^\mu}{\partial x'^\nu} \\ &= \frac{\partial x^\mu}{\partial x'^\nu} + \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial \xi^\mu}{\partial x^\lambda} \\ &= \frac{\partial x^\mu}{\partial x'^\nu} + \frac{\partial \xi^\mu}{\partial x^\nu} + \ldots \end{aligned} \tag{3.7}$$

where we drop higher order terms in $\xi^\mu$. Thus

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \frac{\partial \xi^\mu}{\partial x^\nu} \tag{3.8}$$

The metric therefore changes according to

$$\begin{aligned} g'(x')_{\mu\nu} &= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g(x)_{\rho\sigma} \\ &= g(x)_{\mu\nu} - \frac{\partial \xi^\lambda}{\partial x^\mu} g_{\lambda\nu} - \frac{\partial \xi^\lambda}{\partial x^\nu} g_{\mu\lambda} + \ldots \end{aligned} \tag{3.9}$$

On the other hand we have that

$$\begin{aligned} g'(x')_{\mu\nu} &= g'(x)_{\mu\nu} + \partial_\lambda g'(x)_{\mu\nu}\xi^\lambda \\ &= g'(x)_{\mu\nu} + \partial_\lambda g(x)_{\mu\nu}\xi^\lambda + \ldots \end{aligned} \tag{3.10}$$

So that we have

$$\begin{aligned} \delta g(x)_{\mu\nu} &= g'(x)_{\mu\nu} - g(x)_{\mu\nu} \\ &= -\partial_\lambda g(x)_{\mu\nu}\xi^\lambda - \frac{\partial \xi^\lambda}{\partial x^\mu} g_{\lambda\nu} - \frac{\partial \xi^\lambda}{\partial x^\nu} g_{\mu\lambda} \\ &= -\partial_\mu(\xi^\lambda g_{\lambda\nu}) - \partial_\nu(\xi^\lambda g_{\lambda\mu}) + (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})\xi^\lambda \\ &= -D_\mu \xi_\nu - D_\nu \xi_\mu \end{aligned} \tag{3.11}$$

Since we are free to choose coordinate systems we see that there is a gauge symmetry which acts on the metric as

$$\delta g_{\mu\nu} = -2D_{(\mu}\xi_{\nu)} \tag{3.12}$$

for any vector field $\xi^\mu$. That is to say a spacetime described by the metric $g(x)_{\mu\nu}$ and that described by $g(x)_{\mu\nu} - 2D_{(\mu}\xi_{\nu)}$ are the same.

**Definition:** A vector field $\xi^\mu$ which satisfies $D_{(\mu}\xi_{\nu)} = 0$ is called a Killing vector

A Killing vector generates a symmetry of a particular metric $g_{\mu\nu}$ since the metric is invariant under coordinate transformations generated by $\xi^\mu$.

**Problem:** Solve for the Killing vectors of flat Minkowski space and identify them physically, *i.e.* find the general solution to $\partial_{(\mu}\xi_{\nu)} = 0$.

## 3.3 Weak Field Limit

Let us now consider the linearised theory. That is we consider spacetimes which are nearly flat

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ , \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{3.13}$$

where indices are now raised with simply $\eta^{\mu\nu}$. We expand to first order in $h_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \tag{3.14}$$

This leads to

$$\begin{aligned} R_{\mu\nu} &= \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\lambda\nu} \\ &= \frac{1}{2}\eta^{\lambda\rho}\left(\partial_\lambda\partial_\mu h_{\rho\nu} + \partial_\lambda\partial_\nu h_{\mu\rho} - \partial_\lambda\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\lambda h_{\rho\nu} - \partial_\mu\partial_\nu h_{\lambda\rho} + \partial_\mu\partial_\rho h_{\lambda\nu}\right) \\ &= \frac{1}{2}\partial_\mu\partial^\rho h_{\nu\rho} + \frac{1}{2}\partial_\nu\partial^\rho h_{\mu\rho} - \frac{1}{2}\partial^2 h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h \end{aligned} \tag{3.15}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. We also find that

$$\begin{aligned} R &= \eta^{\mu\nu}R_{\mu\nu} \\ &= \partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h \end{aligned} \tag{3.16}$$

Thus

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\left(\partial_\mu\partial^\rho h_{\nu\rho} + \partial_\nu\partial^\rho h_{\mu\rho} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\rho\partial^\lambda h_{\rho\lambda} + \eta_{\mu\nu}\partial^2 h\right) \tag{3.17}$$

Next we let $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ so that Einsteins equation is

$$-\frac{1}{2}\partial^2\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\lambda\bar{h}_{\rho\lambda} + \frac{1}{2}\partial_\mu\partial^\lambda\bar{h}_{\lambda\nu} + \frac{1}{2}\partial_\nu\partial^\lambda\bar{h}_{\mu\lambda} = \kappa^2 T_{\mu\nu} \tag{3.18}$$

This is still too complicated so we can 'fix a gauge' by using a diffeomorhism $\delta x^\mu = \xi^\mu$ such that

$$\partial^2 \xi_\mu = \partial^\nu \bar{h}_{\mu\nu} \tag{3.19}$$

then

$$\partial^\nu \bar{h}'_{\mu\nu} = \partial^\nu(\bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial^\lambda \xi_\lambda) = 0 \tag{3.20}$$

In this gauge Einstein's equation is simply

$$\partial^2 \bar{h}'_{\mu\nu} = -2\kappa^2 T_{\mu\nu} \tag{3.21}$$

with the gauge fixing condition being that

$$\partial^\nu \bar{h}'_{\mu\nu} = 0 \tag{3.22}$$

Note that we have not complelely fixed the gauge. If we further transform $x^\mu \rightarrow x^\mu + \zeta^\mu$ with $\partial^2 \zeta^\mu = 0$ then

$$\bar{h}''_{\mu\nu} = \bar{h}'_{\mu\nu} - \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu + \eta_{\mu\nu}\partial^\lambda \zeta_\lambda \tag{3.23}$$

will still satisfy (3.21) and (3.22)

## 3.4 Newton's law: Fixing $\kappa$

Let us now examine a slowly moving body in a weak field. The energy momentum tensor will then be dominated by the time components (as the spatial pieces are suppressed by factors of the velocity). Thus we take

$$T_{00} = \rho \ , \qquad T_{0i} = T_{ij} = 0 \tag{3.24}$$

where $\rho$ is the energy density and $i = 1, 2, 3$. Using the weak field equations we see that $\bar{h}_{ij} = \bar{h}_{0i} = 0$ and

$$\partial^2 \bar{h}_{00} = -2\kappa^2 \rho \tag{3.25}$$

Note that the gauge fixing condition $\partial^\mu h_{\mu\nu} = 0$ will be satisfied so long as the energy-momentum tensor is conserved $\partial^\mu T_{\mu\nu}$ (to lowest order in the perturbation we can replace covariant derivatives on $T_{\mu\nu}$ with partial derivatives since $T_{\mu\nu}$ will be the same order as $h_{\mu\nu}$).

Now $\bar{h} = h - 2h = -h$ so that we can invert to find

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\bar{h}_{00} \tag{3.26}$$

or

$$h_{00} = \frac{1}{2}\bar{h}_{00} \ , \qquad h_{ij} = \frac{1}{2}\bar{h}_{00}\delta_{ij} \tag{3.27}$$

In other words the metric has the form

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \tag{3.28}$$

where $\Phi = -\bar{h}_{00}/4$ satisfies $\partial^2\Phi = \kappa^2\rho/2$.

Next we consider Geodesics. We can assume that, to lowest order in the velocity, $dX^0/d\tau = 1$ and neglect $dX^i/d\tau$. Thus the geodesic equation reduces to

$$\frac{d^2X^0}{d\tau^2} + \Gamma^0_{00} = 0\ , \qquad \frac{d^2X^i}{d\tau^2} + \Gamma^i_{00} = 0 \tag{3.29}$$

We may further neglect time derivatives of the energy density $\rho$ and therefore of $h_{\mu\nu}$ too, as these will be suppressed by the velocity. Thus we find

$$\Gamma^i_{00} = -\frac{1}{2}\partial_i h_{00} = \partial_i\Phi\ , \qquad \Gamma^0_{00} = 0 \tag{3.30}$$

The second equation shows that we may take $X^0 = \tau$, which is a consistency check on our assumption that $dX^0/d\tau = 1$. The first equation then becomes the familar Newtonian force law for a particle in a potential given by $\Phi$

$$\frac{d^2X^i}{d\tau^2} = -\partial_i\Phi \tag{3.31}$$

Thus we can identify $\Phi$ with the potential energy of a gravitational field generated by the density $\rho$.

For a spherical mass distribution

$$\rho = \begin{cases} \rho_0 & r < R \\ 0 & r \geq R \end{cases} \tag{3.32}$$

with $\rho_0$ a constant located at the orgin we have

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) &= \kappa^2\rho_0/2 \quad r < R \\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) &= 0 \quad r \geq R \end{aligned} \tag{3.33}$$

The solution is

$$\Phi = \begin{cases} \kappa^2\rho_0 r^2/12 & r < R \\ -\kappa^2\rho_0 R^3/6r + \kappa^2\rho_0 R^2/4 & r \geq R \end{cases} \tag{3.34}$$

Finally we identify

$$\frac{4}{3}\pi R^3\rho_0 = M \tag{3.35}$$

with the mass of the distribution. Thus for $r > R$ we find the gravitational potential

$$\Phi = -\frac{\kappa^2}{8\pi}\frac{M}{r} + const \tag{3.36}$$

Thus if we identify

$$\kappa^2 = 8\pi G_N \tag{3.37}$$

where $G_N$ is Newton's constant, then we recover the Newtonian gravitational force law as the equation for a geodesic in a weak gravitational field!

## 3.5 Tidal Forces: Geodesic Deviation

Since General Relativity postulates that point like object in free fall experience no force, one might wonder if it is possible to 'feel' gravity at all - which would seem to contradict experience. Note that we feel gravity's pull on earth because we are not (usually) in a freely falling frame. However surely there must be a way for the astronaughts in the spaceshuttle to realise that they are orbiting the earth without waiting to come crashing down.

A key point is that the equivalance principle is only valid locally, at each point. Two neighbouring points which are each in free fall will fall differently. Hence if two such points are physically connected, they will feel a force coming from difference in the way that they free fall. These forces are known as tidal forces. This is what will get you as you fall into a black hole singularity - it will turn you into spagetti as the force on your feet will be so much strong than the force on your head (assuming that you jump in feet first).

Thus we want to consider how the difference of two nearby geodesics, $\delta X^\lambda(\tau) = X'^\lambda(\tau) - X^\lambda(\tau)$, changes as the two bodies are in free fall. We are interested in the rate of change of the displacement between the two curves along the geodesic, *i.e.* the acceleration of the seperation. Thus if $U^\lambda = V^\nu D_\nu \delta X^\lambda$ is the velocity of the displacement then from the discussion of geodesics we have that

$$\begin{aligned} U^\lambda = V^\nu D_\nu \delta X^\lambda &= \frac{d\delta X^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} V^\nu \delta X^\mu \\ V^\nu D_\nu (U^\lambda) &= \frac{dU^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} U^\nu V^\mu \end{aligned} \tag{3.38}$$

This leads to the acceleration

$$\begin{aligned} a^\lambda = V^\nu D_\nu (U^\lambda) &= \frac{d}{d\tau}\left(\frac{d\delta X^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} V^\nu \delta X^\mu\right) + \Gamma^\lambda_{\mu\nu} U^\nu V^\mu \\ &= \frac{d^2\delta X^\lambda}{d\tau^2} + \partial_\rho \Gamma^\lambda_{\mu\nu} V^\nu V^\rho \delta X^\mu - \Gamma^\mu_{\rho\sigma}\Gamma^\lambda_{\mu\nu}\delta X^\mu V^\rho V^\sigma + \Gamma^\lambda_{\mu\nu} V^\nu \frac{d\delta X^\mu}{d\tau} \\ &\quad + \Gamma^\lambda_{\mu\nu}\left(\frac{d\delta X^\nu}{d\tau} + \Gamma^\nu_{\rho\sigma} V^\rho \delta X^\sigma\right) V^\mu \end{aligned} \tag{3.39}$$

note that we used the geodesic equation $dV^\mu/d\tau = -\Gamma^\mu_{\rho\sigma} V^\rho V^\sigma$ in the third term on the second line. We next must expand the geodesic equations

$$\begin{aligned} \frac{d^2 X^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}(X)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} &= 0 \\ \frac{d^2 X'^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}(X')\frac{dX'^\mu}{d\tau}\frac{dX'^\nu}{d\tau} &= 0 \end{aligned} \tag{3.40}$$

to lowest order in $X'^\nu - X^\nu = \delta X^\nu$ to find an equation for $\delta X^\nu$;

$$\frac{d^2\delta X^\lambda}{d\tau^2} + 2\Gamma^\lambda_{\nu\rho}V^\nu\frac{d\delta X^\rho}{d\tau} + \partial_\rho\Gamma^\lambda_{\nu\sigma}\delta X^\rho V^\nu V^\sigma = 0 \tag{3.41}$$

Substiting into (3.39) produces

$$\begin{aligned} a^\lambda &= -\partial_\rho\Gamma^\lambda_{\nu\sigma}\delta X^\rho V^\nu V^\sigma - \Gamma^\mu_{\rho\sigma}\Gamma^\lambda_{\mu\nu}\delta X^\mu V^\rho V^\sigma \\ &\quad +\partial_\rho\Gamma^\lambda_{\mu\nu}V^\nu V^\rho\delta X^\mu + \Gamma^\lambda_{\mu\nu}\Gamma^\nu_{\rho\sigma}V^\rho V^\mu\delta X^\sigma \\ &= -R_{\rho\mu\nu}{}^\lambda V^\mu V^\nu\delta X^\rho \end{aligned} \tag{3.42}$$

Thus one can measure the curvature by examining the proper accerleration of the seperation of two nearby objects in free fall.

## 3.6 Gravitational Waves

Let us consider solutions to the linearised equations of motion in a vacuum. Recall that the equations are

$$\partial^2\bar{h}_{\mu\nu} = 0 \tag{3.43}$$

with the gauge fixing condition being that

$$\partial^\nu\bar{h}_{\mu\nu} = 0 \tag{3.44}$$

Thus we write

$$\bar{h}_{\mu\nu} = \Psi_{\mu\nu}e^{ik_\lambda x^\lambda} \tag{3.45}$$

then we find

$$k_\lambda k^\lambda = 0 \qquad \text{and} \qquad k^\nu\Psi_{\mu\nu} = 0\ . \tag{3.46}$$

Of course at the end of the day we will have to take either the real of the imaginary part of $h_{\mu\nu}$ to get a real solution.

Next we use the residual gauge freedom (3.23) with $\zeta^\mu = B^\mu e^{ik_\lambda x^\lambda}$ this shifts

$$\Psi_{\mu\nu} \to \Psi'_{\mu\nu} = \Psi_{\mu\nu} - ik_\mu B_\nu - ik_\nu B_\mu + i\eta_{\mu\nu}k^\rho B_\rho \tag{3.47}$$

We may choose $B^\mu$ so that

$$\Psi'_{\mu\nu}\eta^{\mu\nu} = \Psi_{\mu\nu}\eta^{\mu\nu} + 2ik^\rho B_\rho = 0 \qquad \text{“Traceless''} \tag{3.48}$$

This is only one constraint on $B^\lambda$ therefore we can impose three more constraints. One choice is

$$\Psi'_{0i} = \Psi_{0i} - ik_0B_i - ik_iB_0 = 0 \qquad \text{“Transverse''} \tag{3.49}$$

For a fix null momentum $k^\lambda$ we can rotate coordinates so that

$$k^\lambda = (k, 0, 0, k) \tag{3.50}$$

The conditions $k^\mu \Psi'_{\mu\nu} = \Psi'_{i0} = \eta^{\mu\nu}\Psi'_{\mu\nu} = 0$ mean that

$$\Psi'_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & \beta & -\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.51}$$

Feeding this all back leads to the metric perturbation

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha e^{ik(t-z)} & \beta e^{ik(t-z)} & 0 \\ 0 & \beta e^{ik(t-z)} & -\alpha e^{ik(t-z)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.52}$$

which represents a gravitational wave propagating along the $z$-axis at the speed of light. Note that there are two different polarisation states corresponding to $\alpha$ and $\beta$. Note that these are complex but at the end of the day we must take the real part which leads to two states (with independent phases). And of course we can superimpose solutions corresponding to different values of $k$. This also shows that on-shell, that is when the equations of motion are satisfied, the gravitational field has two degrees of freemdom.

The effect of these waves is to locally distort spacetime. Such waves should be produced by large cosmic events such as supernovae explosions. Since gravity is a very weak force gravity waves have yet to be observed. However there is great hope that they will be observed in the near future but the LIGO experiment. A gravitational wave passing through a spatial region will cause a small variation in the curvature. This in turn will lead to small tidal force. The LIGO experiments are built to measure such tiny forces over distance scales of the order of the entire earth.

**Problem:** Show that

$$ds^2 = 2dudv + H(x,y)du^2 + dx^2 + dy^2 \tag{3.53}$$

is an exact solution to Einsteins equation with $T_{\mu\nu} = 0$ and $(\partial_x^2 + \partial_y^2)H = 0$.

## 3.7 Lagrangian Description

We'd like to find a Lagrangian whose equation of motion gives Einsteins equation. Thus we seek an action of the form

$$S = \int d^4x \mathcal{L}(g_{\mu\nu}, \partial_\lambda g_{\mu\nu}) \tag{3.54}$$

so that the Euler-Lagrange equations

$$-\partial_\lambda \left( \frac{\delta \mathcal{L}}{\delta \partial_\lambda g_{\mu\nu}} \right) + \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} = 0 \tag{3.55}$$

give Einstein's equation.

We need the action to be well defined, *i.e.* independent of coordinate transformations. We first note that, under a diffeomorphism, $x^\mu \to x'^\mu(x)$

$$d^4x' = \det\left(\frac{\partial x'}{\partial x}\right) d^4x \tag{3.56}$$

Thus we require that

$$\mathcal{L}'(g'_{\mu\nu}, \partial'_\lambda g'_{\mu\nu}) = \det\left(\frac{\partial x}{\partial x'}\right) \mathcal{L}(g_{\mu\nu}, \partial'_\lambda g_{\mu\nu}) \tag{3.57}$$

so that

$$d^4x'\mathcal{L}'(g'_{\mu\nu}, \partial'_\lambda g'_{\mu\nu}) = d^4x\mathcal{L}(g_{\mu\nu}, \partial_\lambda g_{\mu\nu}) \tag{3.58}$$

This is acomplished by noting that

$$\begin{aligned}
\sqrt{-\det(g'_{\mu\nu})} &= \sqrt{-\det\left(\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} g_{\lambda\rho}\right)} \\
&= \sqrt{\det\left(\frac{\partial x^\lambda}{\partial x'^\mu}\right)}\sqrt{\det\left(\frac{\partial x^\rho}{\partial x'^\nu}\right)}\sqrt{-\det(g'_{\lambda\rho})} \\
&= \det\left(\frac{\partial x}{\partial x'}\right)\sqrt{-\det(g')}
\end{aligned} \tag{3.59}$$

Here we have neglected indices and viewed the various two-index expressions as matrices. Thus we have that

$$d^4x'\sqrt{-\det(g')} = d^4x\sqrt{-\det(g)} \tag{3.60}$$

so that if

$$\mathcal{L} = \sqrt{-\det(g)}L \tag{3.61}$$

then we need only ensure that $L$ is a scalar.

**N.B.:** Quantities that transform like $\mathcal{L}$ and $\det(g)$, that is with a factor of the determinant of the coordinate transformation, are called scalar densities.

**N.B.:** Normally one simply writes $g = \det(g)$

Thus our action is of the form

$$S = \int d^4x\sqrt{-g}L \tag{3.62}$$

We expect that the Largangian $L$ is second order in derivatives, as is usually the case for field theories. A natural guess for $L$ is then simply $L = cR + L_m$, where $c$ is a constant, $R$ is the scalar curvature and $L_m$ is the Lagranian for the matter fields.

To see that this works we first need to know how to evaluate

$$\frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}} = \frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\delta g}{\delta g_{\mu\nu}} \tag{3.63}$$

To this end we recall the matrix identity

$$\begin{aligned} \det(g+\delta g) &= \det(g(1+g^{-1}\delta g)) \\ &= \det(g)\det(1+g^{-1}\delta g) \\ &= \det(g)(1+\mathrm{Tr}(g^{-1}\delta g)) \end{aligned} \tag{3.64}$$

Thus

$$\delta\det(g) = \det(g)g^{\mu\nu}\delta g_{\mu\nu} \tag{3.65}$$

so that

$$\frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}} = \frac{1}{2}\sqrt{-g}g^{\mu\nu} \tag{3.66}$$

Finally, from the fact that $g_{\mu\nu}g^{\nu\lambda} = \delta_\nu^\lambda$ we learn that $\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\lambda}\delta g^{\rho\lambda}$ and hence we also have that

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu} \tag{3.67}$$

Next we look at

$$\begin{aligned} \delta R &= \delta\left(g^{\mu\nu}R_{\mu\nu}\right) \\ &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \end{aligned} \tag{3.68}$$

In particular we need $\delta R_{\mu\nu}$. This in turn requires knowing $\delta\Gamma^\lambda_{\mu\nu} = \Gamma(g+\delta g)^\lambda_{\mu\nu} - \Gamma(g)^\lambda_{\mu\nu}$. here we recall the fact that the difference between two connections is a tensor. Thus $\delta\Gamma^\lambda_{\mu\nu}$ is a tensor. To calculate $g^{\mu\nu}\delta R_{\mu\nu}$ we can use our trick to go to a frame where $\Gamma^\lambda_{\mu\nu} = 0$ at some point $p$. Thus

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\left(-\partial_\mu\delta\Gamma^\lambda_{\nu\lambda} + \partial_\lambda\delta\Gamma^\lambda_{\mu\nu}\right) + .. \\ &= g^{\mu\nu}\left(-D_\mu\delta\Gamma^\lambda_{\nu\lambda} + D_\lambda\delta\Gamma^\lambda_{\mu\nu}\right) \\ &= D_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\lambda\rho}\delta\Gamma^\nu_{\rho\nu}\right) \end{aligned} \tag{3.69}$$

In the first line the dots denote terms proportional to $\Gamma^\lambda_{\mu\nu}$ which vanish. In the second line we have written derivatives as covariant derivatives since again $\Gamma^\lambda_{\mu\nu} = 0$. Thus, putting aside the matter Lagrangian, we see that

$$\delta(c\sqrt{-g}R) = c\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + D_\rho V^\rho\right) \tag{3.70}$$

where

$$V^\lambda = g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\lambda\rho}\delta\Gamma^\nu_{\rho\nu} \tag{3.71}$$

It is useful to note that

$$\begin{aligned}
\delta\Gamma^\lambda_{\mu\nu} &= \delta g^{\lambda\rho}g_{\rho\sigma}\Gamma^\sigma_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}(\partial_\mu\delta g_{\rho\nu} + \partial_\nu\delta g_{\mu\rho} - \partial_\rho\delta g_{\mu\nu}) \\
&= +\frac{1}{2}g^{\lambda\rho}(D_\mu\delta g_{\rho\nu} + D_\nu\delta g_{\mu\rho} - D_\rho\delta g_{\mu\nu})
\end{aligned} \tag{3.72}$$

Hence

$$\begin{aligned}
V^\lambda &= \frac{1}{2}\left(2g^{\mu\nu}g^{\lambda\rho}D_\mu\delta g_{\rho\nu} - g^{\mu\nu}g^{\lambda\rho}D_\rho g_{\mu\nu} - g^{\lambda\rho}g^{\nu\sigma}D_\rho\delta g_{\sigma\nu}\right) \\
&= g^{\lambda\mu}D^\nu\delta g_{\mu\nu} - g^{\rho\sigma}D^\lambda\delta g_{\rho\sigma}
\end{aligned} \tag{3.73}$$

**Problem:** Show that, for vectors $V^\mu$,

$$D_\rho V^\rho = \frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}V^\rho\right) \tag{3.74}$$

Thus, dropping the total derivative term whose contribution to $S$ vanishes (with suitable boundary conditions), we see that the action is extremised for

$$c\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{\delta L_m}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}L_m = 0 \tag{3.75}$$

Therefore if we take

$$c = \frac{1}{2\kappa^2} = \frac{1}{16\pi G_N} \tag{3.76}$$

and identify

$$T_{\mu\nu} = -2\frac{\delta L_m}{\delta g^{\mu\nu}} + g_{\mu\nu}L_m = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} \tag{3.77}$$

where $\mathcal{L}_m = \sqrt{-g}L_m$, then we arrive at Einstein's equations. Therefore we find that the action we require is

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + L_m\right) \tag{3.78}$$

An important example of a gravity-matter Lagrangian is that of electromagnetism. In addition to gravity, *i.e.* a metric we introduce a vector potential $A_\mu$ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Lagrangian is

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{4}F^2\right) \tag{3.79}$$

A crucial feature of this Lagrangian is that, in addition to diffeomorhisms, there is an additional local (or gauge) symmetry

$$A_\mu \to A_\mu + \partial_\mu \lambda \tag{3.80}$$

where $\lambda$ is an arbitrary function.

Noting that $F^2 = g^{\mu\nu} g^{\rho\lambda} F_{\mu\rho} F_{\nu\lambda}$ we construct the energy momentum tensor

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} \\ &= F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F^2 \end{aligned} \tag{3.81}$$

On the other hand we have the $A_\mu$ equation of motion

$$\begin{aligned} 0 &= \frac{1}{2}\partial_\mu \left( \sqrt{-g} (\partial^\mu A^\nu - \partial^\nu A^\nu) \right) \\ &= \frac{1}{2}\partial_\mu \left( \sqrt{-g} F^{\mu\nu} \right) \\ &= \frac{\sqrt{-g}}{2} \left( \partial_\mu F^{\mu\nu} + \partial_\mu \ln \sqrt{-g} F^{\mu\nu} \right) \\ &= \frac{\sqrt{-g}}{2} \left( \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\rho\mu} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho} \right) \\ &= \frac{\sqrt{-g}}{2} D_\mu F^{\mu\nu} \end{aligned} \tag{3.82}$$

where we used the fact that $\Gamma^\mu_{\mu\nu} = \partial_\nu \ln \sqrt{-g}$ and $\Gamma^\nu_{\mu\rho} F^{\mu\rho} = 0$.

Thus we indeed see that

$$\begin{aligned} D^\mu T_{\mu\nu} &= D^\mu F_{\mu\rho} F_\nu{}^\rho + F_{\mu\rho} D^\mu F_\nu{}^\rho - \frac{1}{2} g_{\mu\nu} F_{\rho\lambda} D^\mu F^{\rho\lambda} \\ &= F_{\mu\rho} D^\mu F_\nu{}^\rho + \frac{1}{2} g_{\mu\nu} F_{\rho\lambda} (D^\rho F^{\lambda\mu} + D^\lambda F^{\mu\rho}) \\ &= 0 \end{aligned} \tag{3.83}$$

where we have used the equation of motion as well as the Bianchi identity

$$D^\mu F^{\rho\lambda} + D^\rho F^{\lambda\mu} + D^\lambda F^{\mu\rho} = 0 \tag{3.84}$$

**Problem:** For a scalar field theory one has

$$L_m = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \tag{3.85}$$

Determine the resulting energy momentum tensor (as given by (3.77)) and show that it is indeed covariantly conserved when the scalar field satisfies its equation of motion.

# 4 The Schwarzchild Black Hole

**Definition:** A spacetime is stationary if it has a timelike Killing vector.

If we choose a coordinate system where the time-like Killing vector is just $K^\mu = \delta^\mu_0$ then the infinitessimal coordinate transformation induced by $K^\mu$ is just

$$x^0 \to x^0 + \epsilon\ , \qquad x^i \to x^i \tag{4.1}$$

where $i = 1, 2, 3$. Thus the metric is invariant under time translations, *i.e.* $\partial_0 g_{\mu\nu} = 0$

**Definition:** A surface in spacetime is called spacelike if all its tangent vectors, at every point, are spacelike.

**Definition:** A spacetime is static if it is stationary and there exists a spacelike hypersurface whose normal vector is the timelike Killing vector (a normal vector to a hypersurface is vector which is orthogonal to all the tangent vectors).

Going to the coordinate system where $K^\mu = \delta^\mu_0$ we see that a metric is static if

$$g_{\mu\nu}K^\mu T^\nu = g_{0\nu}T^\nu = 0 \tag{4.2}$$

for all tangent vectors $T^\nu$. Since these are all spacelike we must have that $g_0 i = 0$.

The difference between static and stationary is like the difference between a lake and a river, *i.e.* the former is completely still and the later, while remaining constant, is not still.

## 4.1 Schwarschild Solution

Let us look for an exact static and spherically symmetric solution to Einstein's equation with $T_{\mu\nu} = 0$. Therefore we take the anstaz:

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.3}$$

**Problem:** Show that the solution to Einstein's equation is

$$ds^2 = -\left(1 - \frac{R}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{R}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.4}$$

for any constant $R$.

Hint: Show that if

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.5}$$

then $R_{\mu\nu} = 0$ is equivalant to

$$
\begin{aligned}
\Big((rB' - rA' - 1)e^{-2B} + 1\Big)\sin^2\theta &= 0 \\
(rB' - rA' - 1)e^{-2B} + 1 &= 0 \\
-A'' - A'^2 + A'B' + \frac{2}{r}B' &= 0 \\
\left(A'' + A'^2 - A'B' + \frac{2}{r}A'\right)e^{2(A-B)} &= 0
\end{aligned}
\tag{4.6}
$$

and then solve these equations.

To identify the parameter $R$ we need to look at the weak field limit. To do this we must transform to the coordinates we used above in section 3.4.

**Problem:** Show that one can change coordinates so that the Schwarzchild solution is

$$
ds^2 = -\left(\frac{1 - \frac{R}{4\rho}}{1 + \frac{R}{4\rho}}\right)^2 dt^2 + \left(1 + \frac{R}{4\rho}\right)^4 \left(dx^2 + dy^2 + dz^2\right) \tag{4.7}
$$

These are called isotropic coordinates.

**Problem:** Alternatively show that the Schwarzchild metric in isotropic coordinates solves the vacuum Einstein and then change coordinates to find (4.4).

In the weak field limit $r >> R$ we find

$$
ds^2 = -(1 - \frac{R}{\rho})dt^2 + (1 + \frac{R}{\rho})(dx^2 + dy^2 + dz^2) \tag{4.8}
$$

We can therefore identify

$$
\Phi = -\frac{R}{2\rho} \tag{4.9}
$$

as the gravitational potential. We should compare this with the gravitational potential about a point source from Newton:

$$
\Phi = -\frac{G_N M}{\rho} \tag{4.10}
$$

Thus $R = 2G_N M$.

The Schwarzchild metric is in fact it is unique, given our assumptions:

**Theorem:** (Birkhoff) The Schwarzchild metric is the unique, up to diffeomorphisms, stationary and spherically symmetric solution to the vacuum Einstrien equations.

We will not prove this theorem here, it simply follows from the Einstein equations. It is an elementary version of a 'no-hair' theorem. The surprising thing about it is that the solution contains no information about what makes up the matter. The result remains true if more complicated matter terms are added to the Lagrangian.

In both these coordinates it seems as if there is a problem ar $r = 2G_N M$ and also at $r = 0$. Certainly we can't use the metric at $r = 2G_N M$. If you calculate some curvature invariant, such as $R_{\mu\nu\lambda}{}^{\rho} R^{\mu\nu\lambda}{}_{\rho}$ then there is no apparent problem at $r = 2G_N M$ but there is a divergence at $r = 0$. For $r < 2G_N M$ we can still use the Schwarzchild solution. But note that the role of 'time' is played by $r$. In particular, for $r < 2G_N M$, the metric is no longer static. To understand more we must look at geodesics, and then the so-called Kruscal extension.

It is important to note that we do not have to consider the Schwarzchild metric to be valid everywhere. For example it is also the unique solution outside a static spherically symmetry distribution of matter, whose total mass is $M$. In particular it describes the spacetime geometry outside a star such as the sun or a planet such as the earth (ignoring their rotation).

## 4.2 Geodesics

We need to consider the geodesics in the Schwarzchild solution. These are extrema of the action

$$\begin{aligned} S &= \int d\tau \sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} \\ &= \int d\tau \sqrt{\left(1 - \frac{2G_N M}{r}\right)\dot{t}^2 - \left(1 - \frac{2G_N M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\varphi}^2} \end{aligned} \tag{4.11}$$

If we think of this as a Lagrangian for four fields $t(\tau), r(\tau), \theta(\tau)$ and $\varphi(\tau)$, then $t$ and $\theta$ do not have and 'potential' terms. Thus it follows that

$$E = \frac{\left(1 - \frac{2G_N M}{r}\right)\dot{t}}{\sqrt{\left(1 - \frac{2G_N M}{r}\right)\dot{t}^2 - \left(1 - \frac{2G_N M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\varphi}^2}} \tag{4.12}$$

and

$$l = \frac{r^2\sin^2\theta\dot{\varphi}}{\sqrt{\left(1 - \frac{2G_N M}{r}\right)\dot{t}^2 - \left(1 - \frac{2G_N M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\varphi}^2}} \tag{4.13}$$

are constant along any geodesic. The square root terms in the demonemator are a pain, they are simply $\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}$. However we have the following (recall our discussion on affinely parameterized geodesics)

**Theorem:** Along an (affine) geodesic $g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$ is constant

**Proof:** We simply differentiate

$$
\begin{aligned}
\frac{d}{d\tau}\left(g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu\right) &= \partial_\lambda g_{\mu\nu}\dot{X}^\lambda\dot{X}^\mu\dot{X}^\nu + 2g_{\mu\nu}\dot{X}^\mu\ddot{X}^\nu \\
&= \partial_\lambda g_{\mu\nu}\dot{X}^\lambda\dot{X}^\mu\dot{X}^\nu - 2g_{\mu\nu}\dot{X}^\mu\Gamma^\nu_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho \\
&= \left(\partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu}\right)\dot{X}^\lambda\dot{X}^\mu\dot{X}^\nu \\
&= 0
\end{aligned}
\tag{4.14}
$$

Thus we can absorb the nasty square root terms into the definition of the constants $E$ and $l$ and take

$$
E = \left(1 - \frac{2G_N M}{r}\right)\dot{t}\ , \qquad l = r^2\sin^2\theta\dot{\varphi} \tag{4.15}
$$

In fact this is a special case of the following theorem:

**Theorem:** If $k_\mu$ is a Killing vector then $k_\mu dX^\mu/ds$ is constant along a geodesic.

**Proof:** We simply differentiate:

$$
\begin{aligned}
\frac{d}{ds}\left(k_\mu\frac{dX^\mu}{ds}\right) &= \partial_\nu k_\mu X^\nu\frac{dX^\mu}{ds} + k_\mu\frac{d^2X^\mu}{ds^2} \\
&= \partial_\nu k_\mu\frac{dX^\nu}{ds}\frac{dX^\mu}{ds} - \Gamma^\mu_{\lambda\rho}\frac{dX^\lambda}{ds}\frac{dX^\rho}{ds} \\
&= D_\nu k_\mu\frac{dX^\mu}{ds}\frac{dX^\nu}{ds} \\
&= 0
\end{aligned}
\tag{4.16}
$$

where we used the geodesic equation and the fact that $D_{(\nu}k_{\mu)} = 0$.

Indeed there are three Killing vectors of Schwarzchild, time translations and two independent rotations of the two-sphere $d\theta^2 + \sin^2\theta d\varphi^2$. The third Killing vector is not so apparent in these coordinates but it allows us to fix the geodesic to lie in the $\theta = \pi/2$ plane. This is a familiar result in classical mechanics: conservation of angular momentum in a spherically symmetric potential implies that the motion is restricted to a two dimensional plane in space.

We have not identified what $\tau$ is but from our discussion of affinely parameterized geodesic we saw that $ds = d\tau$ so that $s = \tau$ without loss of generality. In this case we find

$$
\epsilon = -\left(\frac{ds}{d\tau}\right)^2 = \left(1 - \frac{2G_N M}{r}\right)^{-1}E^2 - \left(1 - \frac{2G_N M}{r}\right)^{-1}\dot{r}^2 - \frac{l^2}{r^2} \tag{4.17}
$$

where $\epsilon = 1$ for timelike geodesic and $\epsilon = 0$ for null geodesics. A little rearranging leads us to

$$
\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(\epsilon + \frac{l^2}{r^2}\right)\left(1 - \frac{2G_N M}{r}\right) = \frac{1}{2}E^2 \tag{4.18}
$$

This is the equation for a particle with position $r$ in a potential

$$
\begin{aligned}
V &= \frac{1}{2}\left(\epsilon+\frac{l^2}{r^2}\right)\left(1-\frac{2G_NM}{r}\right) \\
&= \frac{1}{2}\epsilon-\frac{G_NM\epsilon}{r}+\frac{l^2}{2r^2}-\frac{G_NMl^2}{r^3}
\end{aligned}
\tag{4.19}
$$

The $1/r$ and $1/r^2$ terms are the usual Newtonian potential (with $\epsilon = 1$ for a timelike observer), but we see that there are relativistic corrections due to the additional $1/r^3$ term. These leads to two classic predictions of General Relativity

The first is that a light ray passing close by the sun will bend. It is not clear in the Newtonian theory how to calculate the bending of light since it has no mass. You might try to interpret it as a particle and assign some effective mass for it due to its energy, *i.e.* by including effects of Special Relativity. However the amount predicted is only half that observed.

The second effect is that planets no longer move in ellipical orbits about the sun. This deviation is extremely small for most planets but it was already observed prior to Einstein that the closest planet to the sun, namely Mercury, does not follow an exactly elliptical path. Rather the 'ellipise' slowly rotates. This is called the perihelion shift of Mercury.

These were the first tests of General Relativity (and have been observational confirmed). We do not have time for a detailed discussion of them here - they should be in any introductory course of General Relativity and any text book.

Let us look for circular geodesics $\dot{r} = 0$. These occur at

$$
0=\frac{dV}{dr}=\frac{l^2}{r^3}\left(1-\frac{2G_NM}{r}\right)-\left(\epsilon+\frac{l^2}{r^2}\right)\frac{G_NM}{r^2} \tag{4.20}
$$

which leads to

$$
(r-2G_NM)-\left(\frac{\epsilon}{l^2}r^2+1\right)G_NM=0 \tag{4.21}
$$

If $\epsilon = 0$ there is a single unstable solution at $r = 3G_NM$. For timelike geodesics $\epsilon = 1$ and we find

$$
r_{\pm}=\frac{l^2\pm\sqrt{l^4-12l^2G_NM^2}}{2G_NM} \tag{4.22}
$$

Clearly these solutions only exist for a large enough value of $l^2$, one will be stable and the other unstable.

It follows from general considerations that one will pass through $r = 2G_NM$ in a finite affine time. In addition one will hit $r = 0$ too. However these geodesics are parameterised by the proper time $s$, which is the time that the infalling observer feels. Let us consider what an observer at a safe distance from $r = 2G_NM$ sees.

Thus we want to consider $dr/dt$ as opposed to $dr/ds$. We simply note that

$$
\begin{aligned}
\frac{dr}{ds} &= \frac{dt}{ds}\frac{dr}{dt} \\
&= \dot{t}\frac{dr}{dt} \\
&= E\left(1-\frac{2G_NM}{r}\right)^{-1}\frac{dr}{dt}
\end{aligned}
\tag{4.23}
$$

From here we see that the geodesic equation is

$$
\frac{1}{2}E^2\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{2G_NM}{r}\right)^2\left(\frac{1}{2}E^2-\frac{1}{2}\left(\epsilon+\frac{l^2}{r^2}\right)\left(1-\frac{2G_NM}{r}\right)\right) \tag{4.24}
$$

Near $r \to \infty$ this modification does not do much as $t \sim Es$. However as we approach $r = 2G_NM$ we see that

$$
\begin{aligned}
\frac{1}{2}E^2\left(\frac{d\delta r}{dt}\right)^2 &= \frac{1}{2}E^2\left(1-\frac{2G_NM}{2G_NM+\delta r}\right)^2+\ldots \\
&= \frac{1}{2}\frac{E^2}{4G_N^2M^2}(\delta r)^2+\ldots
\end{aligned}
\tag{4.25}
$$

where $r = 2G_NM + \delta r$. Thus we see that

$$
\frac{d\delta r}{dt} = -\frac{\delta r}{2G_NM}+\ldots \tag{4.26}
$$

and hence, near $r = 2G_NM$ we have

$$
\delta r = e^{-\frac{t-t_0}{2G_NM}} \tag{4.27}
$$

This shows that $r$ never reaches $r = 2G_NM$ for any finite value of $t$. Thus an observer at infinity, for whom $t$ is the time variable, will never see an infalling observer reach $r = 2G_NM$. Whereas we saw that the infalling observer will pass smoothly through $r = 2G_NM$ in a finite proper time.

Thus to an outside observer the region $r \leq 2G_NM$ is causally disconnected, they cannot send in any probe, say a light beam or an astronaught, which will be able to go into this region and return. The surface $r = 2G_NM$ is called the horizon because observers outside the horizon will never be able to probe what is beyond $r = 2G_NM$, whereas a freely falling observer will smoothly pass though in a finite time. Of course it also can be shown that, as is well known, no signal inside $r = 2G_NM$ can reach the outside. This is the classic example of a black hole.

## 4.3 Kruskal Extention

Note that for a positive mass, which is what we assume to be the case, the metric does somehing funny at $r = 2M$, namely $g_{00}$ vanishes while $g_{rr}$ diverges. This is breakdown of the coordinates and it turns out that it does not represent a break down of spacetime. The opposite occurs at $r = 0$ but this is a real singularity.

First let us discuss a simpler, two-dimensional example, Rindler space:

$$ds^2 = -r^2dt^2 + dr^2 \tag{4.28}$$

If we look for null geodesics we must solve

$$rdt = \pm dr \tag{4.29}$$

or

$$t \mp \ln(r) = c \tag{4.30}$$

where $c$ is a constant. This suggests that we introduce new coordinates that are natural for such a free falling observer:

$$u = t - \ln(r) \ , \qquad v = t + \ln(r) \tag{4.31}$$

or

$$t = \frac{u+v}{2} \ , \qquad r = e^{(v-u)/2} \tag{4.32}$$

In these coordinates the null geodesics are described by

$$u = const \qquad \text{or} \qquad v = const \tag{4.33}$$

Hence we find

$$\begin{aligned} ds^2 &= -\frac{1}{4}e^{(v-u)}(du^2 + dv^2 + 2dudv) + \frac{1}{4}e^{(v-u)}(dv^2 + du^2 - dvdu) \\ &= -e^{(v-u)}dudv \end{aligned} \tag{4.34}$$

Finally we can recognise this spacetime by taking

$$U = -e^{-u} \ , \qquad V = e^{v} \tag{4.35}$$

so that

$$ds^2 = -dUdV \tag{4.36}$$

Which is just Minkowski space, once we write $U = T + X$, $V = T - X$ so that

$$ds^2 = -dT^2 + dX^2 \tag{4.37}$$

In particular the point $r = 0$ which looks problematic in the orginal coordinate system corresonds to $U = 0$ or $V = 0$ which is the light cone through the origin. And the region

$r > 0$ which one is tempted to think of as the physical region is infact just a quadrant of Minskowski space with $UV > 0$.

There are two lesson from this example. Firstly that one shouldn't be fooled by what the metric looks like in any coordinate system. Secondly a way to see the true geometry (and hence the real physics of a spacetime) is to follow the path of a light ray. As long as this is smooth then the metric should look senseible in the coordinates adapted to the observer.

Let us return then to Schwarzchild:

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2G_N M}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.38}$$

The angular variables will just go along for the ride. So we start by considering a radial null geodesic

$$dt = \pm \frac{dr}{1 - \frac{2G_N M}{r}} \tag{4.39}$$

so that along such a geodesic

$$t \mp \int^r \frac{r'}{r' - 2G_N M} = t \mp r \mp 2G_N M \ln\left(\frac{r}{2G_N M} - 1\right) \tag{4.40}$$

is constant. Thus we let

$$\begin{aligned} u &= t - r - 2G_N M \ln\left(\frac{r}{2G_N M} - 1\right) \\ v &= t + r + 2G_N M \ln\left(\frac{r}{2G_N M} - 1\right) \end{aligned} \tag{4.41}$$

These are called Eddington-Finklestein coordinates (or perhaps more correctly just $(u, r)$ or $(v, r)$). Here we already see that $r = 2G_N M$ posses no problems as

$$\begin{aligned} -\left(1 - \frac{2G_N M}{r}\right) dudv &= -\left(1 - \frac{2G_N M}{r}\right)\left(dt - \frac{dr}{1 - \frac{2G_N M}{r}}\right)\left(dt + \frac{dr}{1 - \frac{2G_N M}{r}}\right) \\ &= -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2G_N M}{r}} \end{aligned} \tag{4.42}$$

but we also have that

$$\begin{aligned} \frac{2G_N M}{r} e^{-r/2G_N M} e^{(v-u)/4G_N M} &= \frac{2G_N M}{r} e^{-r/2G_N M} e^{r/2G_N M + \ln\left(\frac{r}{2G_N M} - 1\right)} \\ &= \left(1 - \frac{2G_N M}{r}\right) \end{aligned} \tag{4.43}$$

hence

$$ds^2 = -2G_N M \frac{e^{-r/2G_N M}}{r} e^{(v-u)/4G_N M} du dv + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.44}$$

and here there is no problem at $r = 2G_N M$.

We have one more step to go. Next we let

$$U = -e^{-u/4G_N M} \ , \qquad V = e^{v/4G_N M} \tag{4.45}$$

so that we find

$$ds^2 = -32G_N^3 M^3 \frac{e^{-r/2G_N M}}{r} dU dV + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.46}$$

Finally one can write $U = T + X$, $V = T - X$ to find

$$ds^2 = 32G_N^3 M^3 \frac{e^{-r/2G_N M}}{r}(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.47}$$

This the Kruskal extention and it shows that spacetime is perfectly well behaved at $r = 2G_N M$. Indeed we would find something sick if we were to stop at $r = 0$.

The importance of this is captured by the following definition

**Definition:** A spacetime is geodesically complete if any of its geodesics can be extended for an arbitrary long period of its proper length, or ends on a singularity.

What this means is that in a geodesically complete spacetime any free falling observer will either go on falling forever or hit some kind of singularity. Otherwise one might literally find oneself at the "edge of the universe", and then what would you do?

## 4.4 Causal Structure and Spacetime Diagrams

It is helpful to draw a so-called spacetime (or Carter or Penrose or Carter-Penrose) diagram. These are most useful when spacetime has rotational symmetry and we need only keep track of the $(t, r)$ coordinates. Each point on a spacetime diagram then represents a two-sphere with radius $r$. We also always draw things so that light rays travel at 45 degrees and 'time' runs upwards.

Technically a spacetime diagram is a conformal map from a two-dimensional spacetime (*i.e.* just the $(t, r)$ coordinates) to a closed and connected subset of $\mathbb{R}^2$. Here conformal means that the metric is allowed to be multiplied by an overall spacetime dependent function. This ensures that the null geodesics of the two spacetimes are the same. So that the causal structure of the diagram gives a faithful representation of the true causal structure. The conformal transformation is also designed to map infinity in spacetime onto the boundary of the diagram at a finite distance (in term of the metric on the piece of paper where you draw the diagram). But it is best to see what is going on by drawing some spacetime diagrams.

Let us consider Minkowski space again, suppressing the angular coordinates:

$$ds^2 = -dt^2 + dr^2 \tag{4.48}$$

and consider light cone coodinates $u = r + t$, $v = r - t$:

$$ds^2 = dudv \tag{4.49}$$

Next we consider a change of variables that maps $u, v = \infty$ to a finite value, say,

$$u = \tan u' \ , \qquad v = \tan v' \tag{4.50}$$

with $u', v' \in (-\pi/2, \pi/2)$ and

$$ds^2 = \frac{1}{\cos^2 u' \cos^2 v'} du'dv' \tag{4.51}$$

The conformal transformation above means that we switch to the new metric

$$ds'^2 = \cos^2 u' \cos^2 v' ds^2 = du'dv' \tag{4.52}$$

In this spacetime $u', v'$ have a finite range and so the whole spacetime fits inside a square $-\pi/2 < u', v' < \pi/2$. However we also have that $r \geq 0$ so that $u' \geq -v'$. This reduces the spacetime to a triangle. Conformal compactification also means that we include the boundaries and hence extend the range to $u', v' \in [-\pi/2, \pi/2]$ with $u' \geq -v'$.

The key point of this is that the null geodesics remain unchanged and hence the causal structure is unchanged. We can identify several different components to the boundaries of spacetime (note that the left side of the triangle corresponds to $r = 0$ is not a boundary of spacetime since here the 2-spheres that we have been ignoring shrink to zero size):

i) $v' = \pi/2$ past null infinity. This is where all past directed light rays end up. This is usually called Scri-minus: $\mathcal{I}^-$.

ii) $u' = \pi/2$ future null infinity. This is where all future directed light rays end up. This is usually called Scri-plus: $\mathcal{I}^+$.

iii) $u' = v' = \pi/2$ spatial infinity. This simply corrsponds to $r \to \infty$ for any fixed $t$ and it denoted by $i^0$.

iv) $u' = -v' = \pi/2$ future timelike infinity, corresponding to $t \to +\infty$. It is the end point of any timelike future directed geodesic and is denoted by $i^+$.

iv) $-u' = v' = \pi/2$ past timelike infinity, corresponding to $t \to -\infty$. It is the end point of any timelike future directed geodesic and is denoted by $i^-$.

We can now draw spacetime diagram for Schwarzchild, or more acurately the geodesically complete spacetime for which the original Schwarzchild solution is a portion of. If use the Kruskal coordinates (and drop the angular variables)

$$ds^2 = -32G_N^3M^3\frac{e^{-r/2G_NM}}{r}dUdV \tag{4.53}$$

To construct the conformally rescaled metric we again let $U = \tan U'$ and $V = \tan V'$ so that

$$ds^2 = -\frac{32G_N^3M^3}{\cos^2 U' \cos^2 V'}\frac{e^{-r/2G_NM}}{r}dU'dV' \tag{4.54}$$

and hence the conformally rescaled metric is again just $ds'^2 = dU'dV'$. Thus spacetime is contained inside a square only now there are more interesting features.

We start with the region outside $r = 2G_NM$ this is remanicent of Minkowski space, except that at $r = 2G_NM$ the two-spheres do not shrink to zero size. Indeed we have seen that null geodesics will just pass through $r = 2G_NM$ without notice. Thus we find a new region corresponding to $r < 2G_NM$. In many way this just the same as the region $r > 2G_NM$ except that $r$ and $t$ have changed roles *i.e.* $r$ is now timelike.

We should pause here to explain our logic. It comes down to this: which coordinates are we using and when? For $r > 2G_NM$ the natural coordinates are those we used above in (4.4). In this region there will be $\mathcal{I}^\pm$ and $i^\pm$ , $i^0$ just like Minkowski space. However for $r = 2G_NM$ these coorsinates break down and so the the edge corrsponding to $r = 0$ in the Minskowski spacetime diagram is absent. What we have seen is that the Kruskal extension allows us to pass through $r \leq 2G_NM$. In other words if we cut off the original solution at $r = 2G_NM$ then we would find a geodesically incomplete spacetime. Once we are in the region $r < 2G_NM$ then we can again use the more familiar coordinates $(t, r, \theta, \varphi)$, since a similar transformation to these coordinates will work again, however we see that $r$ is timelike and $t$ spacelike.

Thus in the region $r < 2G_NM$ the singularity at $r = 0$ is a spaceelike singularity. This means that the singularity at $r = 0$ is not in space but in time. Thus the top of the square is cut off by the singularity. You are doomed to hit the singularity not so much because it is so srtong and compelling, although in an obvious sense it is, but rather because once you pass through the horizon it becomes a point located in the future at all points in space. Rather than the reverse outside $r = 2G_NM$ where the singularity lies at a fix point in space for all time. It is your fate, just like death and taxes.

But we shouldn't stop here, there are infact two more regions of space that the Kruskal extention reveals. Namley things are still time and space symmetric. Hence there are mirror portions of the the complete spacetime.

There is another asymptotically flat spatial region that mirrors the region outside $r = 2G_NM$ with its own $\mathcal{I}^\pm$ and $i^\pm$ , $i^0$. However one can't get there from here, *i.e.* no time-like or null geodesic can go from one asymptotic region to the other. Perhaps though, if you do fall into a black hole you might meet someone from there and compare notes.

There is also the time reverse of the interior region $r < 2G_N M$. This too has a timelike singularity but it lies in the past. In this region all observers must have come from the singularity and they will eventually leave this region into one of the two $r > 2G_N M$ spatially asmptotically flat regions. It is known as a white hole.

We don't trust white holes, or indeed this new spatially infinite reigion realistically because we don't really trust Scharwzchild for all time. In particular it represents a black holes that has, to the outside observer existed for all time. In reality we expect that black holes form by some process of gravitational collapse. Therefore the Schwarzshild soltuion represents the final state but shouldn't be treated as eternal and time symmetric. Therefore there is no reason to believe in white holes as physically relevant.

We also obtain a solution to Einstein's equations if $M < 0$, although there is no clear interpretation for this. In this case one will hit the curvature singularity at $r = 0$ before the horizon at $r = 2G_N M$. This is known as a naked singularity since there is no horizon to prevent an outside observer from seeing the singularity. The so-called cosmic sensorship conjecture of Penrose asserts that there are no naked singularities in a physically sensible theory.

**Problem:** What is the spacetime diagram for a negative mass Schwarzchild solution?

# 5 More Black Holes

## 5.1 Gravitational Collapse

A key point of black holes that they are not bizzare artifacts of some strange assumption such as perfect spherical symmetry but rather are enevitable, as shown by the so-called singularity theorems of Hawking and Penrose, which we won't have time to go into the details of here (roughly speaking they assert that under reasonble physical conditions black holes should be formed, according Einstein's equation). Indeed black holes are common in astronomy, there even seems to be one at the centre of our galaxy (as there seems to be at the centre at of every galaxy).

The first discussion of the notion of a black hole dates back hundreds of years when it was noted that for a sufficiently dense star the escape velocity from the surface of a star is faster than the speed of light. The escape velocity is the minmum speed required to eject a body from the surface to infinity. Thus for a body of mass $m$ and a star of mass $M$ the energy is

$$E = \frac{1}{2}mv^2 - \frac{G_N mM}{r} \tag{5.55}$$

A body that makes it to $r \to \infty$ will have $E \geq 0$ since the kinetic term will dominate at large $r$. Thus by energy conservation we have that, at the surface,

$$v^2 \geq \frac{2G_N M}{R} \tag{5.56}$$

where $R$ is the radius of the star. Setting $v = 1$, the speed of light, we find the Schwarzchild radius as a bound on the size of a star that can be seen!

The Schwarzchild solution is the unique stationary and sphereically symmetric vacuum solution to Einstein's equation. If we imagine some kind of physical process of gravitational collapse (such as the formation of a star) which preserves rorational symmetry (at least at late time) then the end result will be described by Schwarzchild for $r > R$, the radius of the star. Therefore if $R < 2G_N M$ we predict the existance of a black hole.

How does this happen the in real universe? If we start with an ordinary star it survives by burning up its nuclear fuel. The pressure produced by the star's nuclear reaction stabilises it against the gravitational attraction of the matter that makes it up. However eventually the star will burn up all if fuel and the presure that it creates will drop. Thus the star will become unstable and collapse. The question is how massive does it have to be to collapse to a black hole. The answer is determined by the Chandrekhar bound.

If a star is not too massive, less than about 1.5 times the mass of our Sun, then the star will be protected against complete collapse by the Pauli exclusion principle since the electons in the atoms of the star cannot be compressed without being squeezed into similar quantum states. Thus there is a certain electron degeneracy pressure that will stabilise the star and it will simply become a so-called white dwarf.

However if the star is more massive then even the electron degeneracy cannot keep it up. The star will collapse further until the neutrons in the nuclei become too close. Normally neutrons are unstable and will decay into electrons, protons and a neutrino ($\beta$-decay). But if the density of neutrons is sufficiently great the inverse reaction becomes energically favourable and a stable state will emerge. This is a neutron star and there is a neutron degeneracy pressure, which is analogous to the electron degeneracy presure, that protects the star against gravitational collapse (but at a smaller radius as compared to a white dwarf since the neutron is much more massive than an electron the degeneracy pressure is larger).

But the star can be so massive that even this will not stabilise it. In particular for a star that is about 2 times the mass of our Sun we expect that it will collapse into a point-like singularity. (One might postulate that there are quark stars where the degeneracy pressure of the quarks that make up the nucleons prevents the star from collapse but even these will collapse for sufficiently large masses.)

## 5.2 Null Surfaces and Killing Horizons

So let us pause for a moment and think about what we really mean by a black hole.

First let us define a black hole. It should be clear from the example of Schwarzchild that it is not so much the singularity that it is important but rather the horizon.

**Definition:** A black hole is a spacetime that has an event horizon.

**Definition:** An event horizon is the boundary of the causal past of future null infinity.

Ya wot? Future null infinity is the set of end points of future directed (*i.e.* those light rays travelling towards the future) null geodesics which reach asymptotic spatial infinity. The causal past of some set $X$ is simply the set of all points in the spacetime that can be connected to $X$ by a future directed timelike or null geodesic. In other words those points in the spacetime that could affect the physics of $X$ (assuming causality). So this definition says that the event horizon is the boundary to the set of all spacetime events that an observer at spatial infinity could ever see.

This is the exact definition of a black hole but in practise it is too difficult to impose since it requires knowning the full spacetime, for all time, which is likely to be beyond human capabilities.

From the Schwarzchild example we saw that the key effect that seperates the two regions of spacetime arose because the metric component $g_{rr} \to \infty$ at $r = 2G_N M$. What is happening is that the normal vectors to the surface $r = constant$ are changing from being spacelike for $r > 2G_N M$, to null at $r = 2G_N M$ to timelike for $r < 2G_N M$. Thus the light cones are tipping over so that, at $r = 2G_N M$ the entire forward lightcone lies within $r < 2G_N M$.

This leads to the notion of a null surface:

**Definition:** A null surface is a surface whose normal vector is null.

**Example:** If the a surface is descibed by an equation of the form

$$S(x^\mu) = 0 \tag{5.57}$$

then the normal vector is $n_\mu = \partial_\mu S$. To see this note that for any curve $x^\mu(\tau)$ located on the surface we have

$$S(x^\mu(\tau)) = 0 \tag{5.58}$$

thus near any given point on the surface we can expand near $\tau = 0$ $x^\mu(\epsilon) = x_0^\mu + \epsilon T^\mu + ...$ where $T^\mu$ is the tangent vector to the curve at $x^\mu = x_0^\mu$. Thus we must have that

$$0 = S(x_0^\mu + \epsilon T^\mu + ...) = S(x_0^\mu) + \epsilon T^\mu \partial_\mu S(x_0^\mu) + ... = \epsilon T^\mu \partial_\mu S(x_0^\mu) + ... \tag{5.59}$$

Thus it follows that

$$n_\mu T^\mu = 0 \tag{5.60}$$

for any tangent vector to the surface, *i.e.* $n^\mu$ is indeed the normal vector.

**Theorem:** A null surface satisfies the condition that its normal vector is also tangent to it.

**Proof:** To see this we note that if $x^\mu(\epsilon) = x_0^\mu + \epsilon n^\mu + ...$ is a curve near a point $x^\mu = x_0^\mu$ lying on $S = 0$ with the tangent vector $n^\mu = \partial^\mu S$ (*i.e.* the normal to the null surface) then

$$S(x_0^\mu + \epsilon n^\mu + ...) = S(x_0^\mu) + \epsilon n^\mu \partial_\mu S = \epsilon n^\mu n_\mu + ... = 0 \tag{5.61}$$

since $n_\mu n^\mu = 0$ for a null surface.

**Definition:** A Killing horizon is a null surface $\mathcal{S}$ whose normal vector coincides with a Killing vector field on $\mathcal{S}$.

**Example:** In the Schwarzchild solution we can consider the surface defined by $S = UV = 0$. This clearly has two distinct parts, $U = 0$ and $V = 0$. These are both mapped to $r = 2G_N M$ in the original coordinates. The normal vector is

$$n_U = V \ , \qquad n_V = U \tag{5.62}$$

or

$$n^U = -\frac{r}{32G_N^3 M^3} e^{r/2G_N M} U \qquad n^V = -\frac{r}{32G_N^3 M^3} r e^{r/2G_N M} V \tag{5.63}$$

and indeed we have that

$$g_{\mu\nu} n^\mu n^\nu = -\frac{r}{16G_N^3 M^3} e^{r/2G_N M} UV = 0 \tag{5.64}$$

so that this is a null surface. On the other hand the timelike Killing vector $K^\mu = \delta_0^\mu$ in the original coordinates becomes, in the Kruskal coodinates,

$$K^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \delta_0^\nu = \frac{\partial x'^\mu}{\partial t} \tag{5.65}$$

so that

$$K^U = -\frac{1}{4G_N M} U \ , \qquad K'^V = \frac{1}{4G_N M} V \tag{5.66}$$

thus

$$g_{\mu\nu} K^\mu K^\nu = \frac{4G_N M}{r} e^{-r/2G_N M} UV \tag{5.67}$$

which is timelike for $UV < 0$ (*i.e.* $r > 2G_N M$), spacelike for $UV > 0$ (*i.e.* $r < 2G_N M$) and null at the surface $UV = 0$. Furthermore at $U = 0$ or $V = 0$ we see that $K^\mu$ is proportional to $n^\mu$. Thus $UV = 0$ is a Killing horizon. Indeed the fact that it has two components means that it is a bifuricate Killing horizon.

So let us discuss some other important black hole solutions.

## 5.3 Reisner-Nordstrom

If we couple electromagnetism to gravity, as we did above. Then we can look for static and rotationally invariant solutions which carry electric charge. The generalisation of the Schwarzchild solution is the Reisner-Nordstrom metric:

$$ds^2 = -\left(1 - \frac{2G_N M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2G_N M}{r} + \frac{Q^2}{r^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{5.68}$$

So this is not vacuum solution to Einstein's equation as there is a non-vanishing $U(1)$ gauge potential $A_0 = Q/r$.

One one level this looks very simililar to Schwarzchild. However there are some key differences. Clearly the differeneces are summed up by the shift

$$1 - \frac{2G_N M}{r} \to 1 - \frac{2G_N M}{r} + \frac{Q^2}{r^2} \tag{5.69}$$

This has one major consequence: there are now two Killing horizons located at

$$g_{00} = 0 \qquad \leftrightarrow \qquad r = r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - Q^2} \tag{5.70}$$

provided that $|Q| < G_N M$. That these are Killing horizons follows from the fact that near $r = r_\pm$ the metric will look the same as Schwarzchild near the Schwarzchild radius.

**Problem:** Construct the Kruskal extention for Reisner-Nordstrom and show that $r = r_\pm$ are indeed Killing horizons.

An important consquence of having two horizons is that the singularity at $r = 0$ is timelike. Recall that in Schwarzchild the singularity at $r = 0$ is spacelike, *i.e.* $r$ is a timelike coordinate inside the horizon. Here $r$ is spacelike for large $r$ then timelike for $r_- < r < r_+$ and then spacelike again for $r < r_-$. In particular this means that you are not doomed to hit the singularity should you fall in. Indeed the spacetime diagram is periodic, continaing repeated regions, including repeating regions containing an asymptotic spatial infinity.

A very interesting special case is the extreme Reisner-Nordstrom $|Q| = G_N M$. In this case the two Killing horizons degenerate into a single Killing horizon at $r = G_N M$. This solution has many amazing features. To exhibit them let us change to isotropic coordinates.

$$ds^2 = -f^2(\rho)dt^2 + g^2(\rho)\left(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) \tag{5.71}$$

To determine $f$ and $g$ (see a previous problem) we compare these two forms of the metric

$$g d\rho = \frac{dr}{\left(1 - \frac{Q}{r}\right)} , \qquad g\rho = r \tag{5.72}$$

The second equation tells us that

$$dr = dg\rho + g d\rho = \frac{dg}{g} r + g d\rho \tag{5.73}$$

and substituting this into the first equation gives

$$dr - \frac{dg}{g} r = \frac{dr}{\left(1 - \frac{Q}{r}\right)} \qquad \text{or} \qquad \frac{d\ln g}{dr} = \frac{1}{r} - \frac{1}{r - Q} \tag{5.74}$$

so, fixing an integration constant, gives

$$\ln g = \ln(r) - \ln(r - Q) = \ln\left(\frac{1}{1 - \frac{Q}{r}}\right) \tag{5.75}$$

thus we find

$$g = \frac{1}{1 - \frac{Q}{r}} \tag{5.76}$$

and

$$\rho = \frac{r}{g} = r\left(1 - \frac{Q}{r}\right) = r - Q \tag{5.77}$$

So we finally arrive at

$$ds^2 = -\left(\frac{1}{1 + \frac{Q}{\rho}}\right)^2 dt^2 + \left(1 + \frac{Q}{\rho}\right)^2 \left(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) \tag{5.78}$$

Note that the horizon $r = Q$ is mapped to $\rho = 0$. Thus $\rho = 0$ is nonsingular despite its highly singular looking form. We can also consider the proper spatial distance to $\rho = 0$, from some finite value of $\rho$. For small $\rho/Q$ we find

$$ds = Q\frac{d\rho}{\rho} = Qd\ln\rho \tag{5.79}$$

and hence

$$s = \int_{\rho_0}^{\epsilon} ds \sim Q\ln\epsilon \to \infty \tag{5.80}$$

Thus there is an internal spatial infinity as one approaches the horizon. If we expand the metric near $\rho = 0$ we find

$$ds^2 = -\frac{\rho^2}{Q^2}dt^2 + \frac{Q^2}{\rho^2}d\rho^2 + Q^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{5.81}$$

Next we rewite this as

$$ds^2 = -e^{-2z}dt^2 + dz^2 + Q^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{5.82}$$

where $z = Q\ln\rho$. This is the near horizon geometry and has been studied a great deal in recent years. Primarily because the metric factorises into $adS_2 \times S^2$. The $S^2$ is just the metric on a 2-sphere of fixed radius $Q$ and $adS_2$, two-dimensional anti-de Sitter space, is the metric

$$ds^2 = -e^{-2z}dt^2 + dz^2 \tag{5.83}$$

We will have more to say about anti-de Sitter space later.

The point of this is that we may rewrite the metric as

$$ds^2 = -H^{-2}dt^2 + H^2(dx^2 + dy^2 + dz^2)\ , \qquad A_0 = -H^{-1} \tag{5.84}$$

where we have explicitly written the electomagnetic potential in these new coordinates (and taken the liberty of adding an irrelevent constant). If you were to now look at the Einstein equations you would find they collapsed to the single equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = 0 \tag{5.85}$$

This is just ordinary flat space Laplace equation. In particular it is a linear equation. It has many more solutions than a simple $1/\rho$ function. We can sum over various such sources and find

$$H = 1 + \sum_{i=1}^{N} \frac{Q_i}{|\vec{x} - \vec{x}_i|} \tag{5.86}$$

These solutions are known as the Papapetrou-Majumbdar metrics. They represent a collection of $N$ extreme Reisner-Nordstrom black holes located at $\vec{x} = \vec{x}_i$. This a static metric and hence the black holes are all in equilibrium, their attractive gravitational forces exactly cancel the repulsive electrostatic forces. These types of solutions have been studied a great deal in recent years. They have another property which is that they are supersymmetric. That means that if they arise as solutions to a supergravity theory then they will preserve some fraction of the supersymmetries of Minkowski space.

Finally one can consider the case $|Q| > G_N M$. This is similar to Schwarzchild for $M < 0$. Here there are no Horizons and the metric is good up to $r = 0$ which is a real singularity.

**Problem:** Consider a particle with mass $m$ and charge $q$ moving in a Papapetrou-Majumdar spacetime. The action for such a particle is

$$S = m \int d\tau \sqrt{-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}} - q \int d\tau A_{\mu}\dot{X}^{\mu} \tag{5.87}$$

The first term should be familar as giving the length of its worldline. The second term describes its coupling to the electromagnetic field (for example there is a velocity coupling to the spatial components of $A_\mu$ that will give the Lorentz force law). How are the geodesic equations modified by the electromagetic interactions (it is simplest to consider a parameterization where $\tau$ is the proper time)?

We can fix $\tau = t$, *i.e.* $\dot{X}^0 = 1$, the time coordinate in Reisner-Nordstrom. This is called static gauge. Show that if the particle is also extremal, *i.e.* if $q = m$, then this action has no potential, *i.e.* the Lagrangian only contains velocity dependent terms. Hence the paricle can be placed anywhere in space and will not be attacted or repelled by one of the black holes.

## 5.4 Kerr

Finally the most general (four-dimnsional) black hole solution includes a rotational parameter $a$ corresonding to a non-vanishing angular momentum. This is the Kerr metric

$$\begin{aligned} ds^2 &= -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\varphi \\ &\quad + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\varphi^2 \end{aligned} \tag{5.88}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\ , \qquad \Delta = r^2 + a^2 + Q^2 - 2G_NMr \tag{5.89}$$

and

$$A_0 = -\frac{Qr}{\Sigma}\ , \qquad A_\varphi = \frac{aQr\sin^2\theta}{\Sigma} \tag{5.90}$$

Note that here we no-longer have $SO(3)$ rotational symmetry but rather just an asymuthal $U(1)$.

The first step to understanding this solution is to turn off $M = Q = 0$ and just keep the rotation parameter $a \neq 0$

$$ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\varphi^2 \tag{5.91}$$

This is just Minkowski space written in spheroidal coordinates. The region $r = 0$ is in fact a sphere.

Turning $M$ and $Q$ back on we see that there is a singularity at $\Sigma = 0$ which is at $r = 0$ and $\theta = \pi/2$. Thus the singularity is a ring, *i.e.* the equitorial plane of the sphere at $r = 0$. Indeed this is a real curvature singularity.

To see things more clearly let us hold $\theta$ and $\varphi$ fixed.

$$ds^2 = -\left(\frac{r^2 + a^2\cos^2\theta + Q^2 - 2G_NMr}{r^2 + a^2\cos^2\theta}\right)dt^2 + \left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2 + Q^2 - 2G_NMr}\right)dr^2 \tag{5.92}$$

We see that there are two null hypersurfaces at $\Delta = 0$ or

$$r = G_NM \pm \sqrt{G_N^2M^2 - a^2 - Q^2} \tag{5.93}$$

That is the normal to the surfaces $r = constant$ are $n_\mu = \delta^r_\mu$ and these satisfy $n_\mu n^\mu = 0$ at $r = r_\pm$. That is so long as

$$G_N^2M^2 \geq a^2 + Q^2 \tag{5.94}$$

otherwise the singularity at $r = 0, \theta = \pi/2$ will be naked. Thus, as with Reisner-Nordstrom, we find and inner and an outer horizon which is null surface. Indeed one can find Kruskal like extension through $r = r_\pm$. These surfaces behave just like the horizons in Reisner-Nordsrom and Schwarzchild. It turns out that these are Killing Horizons but not for $K^\mu = \delta^\mu_0$, which generates time translations, or $\psi^\mu = \delta^\mu_\varphi$ which generates $U(1)$ rotations, but rather

$$\chi^\mu = \delta^\mu_0 + \frac{a}{r_+^2 + a^2}\delta^\mu_\varphi \tag{5.95}$$

which generates time translations followed by a rotaton. In other words the Killing horizons of Kerr rotate.

Even without spherical symmetry on can still consider spacetime diagrams, only now you need to take different slices as $\theta$ is varied. For $\theta = \pi/2$ the spacetime diagram looks the same as Resiner-Nordstrom with its two event horizons and a time-like singularity. However for $\theta \neq \pi/2$ one does not find a singularity. Instead one can fly out into another asymptotically flat region, corresponding to $r \rightarrow -\infty$, by passing through the ring singularity.

There is also a novel effect. The timelike killing vector $\xi^\mu = \delta^\mu_0$ is in fact spacelike if

$$r^2 + a^2\cos^2\theta + Q^2 - 2G_NMr < 0 \tag{5.96}$$

Since $a^2\cos^2\theta < a^2$ this surface is

$$(r - r_+)(r - r_-) < a^2\sin^2\theta \tag{5.97}$$

and lies outside $r = r_+$ except at $\theta = 0, \pi$ where it touches $r = r_+$. There will also be another such surface at $r < r_-$. The region outside $r = r_+$ but inside (5.97) is known as the ergosphere. It is interesting because it lies outside the horizon and so can be probed by an observer at infinity. If you fly into this region then it becomes impossible to remain stationary; to you spacetime will look wildly time dependent. Indeed inside the ergosphere the only non-positive constribution to $ds^2$ comes from the mixed term $dtd\varphi$. Thus one is forced to travel in the $\varphi$ direction, *i.e.* you will get whipped around no matter what you do.

There is an interesting effect associated to the ergosphere known as the Penrose process. This caused by the change in norm of the timelike Killing vector from negative to positive. Consider a particle moving in the Kerr spacetime. The energy of the particle is a conserved quantity given by

$$E = -K_\mu\frac{dX^\mu}{d\tau} = -g_{0\mu}\frac{dX^\mu}{d\tau} \tag{5.98}$$

The point is that $E$ can be negative in the erosphere even if $dX^\mu/d\tau$ is future directed. Thus we can extract energy from the black hole by throwing in a normal postive evergy object, say a dump truck, so that it passes through the ergosphere and then returns back to its starting point. However while in the ergo sphere the dump truck dumps its contents into the black hole. Consevation of momentum implies that

$$p^\mu_{\text{initial}} = p^\mu_{\text{truck}} + p^\mu_{\text{rubbish}} \tag{5.99}$$

and therefore, by contraction with $K^\mu$

$$E_{\text{initial}} = E_{\text{truck}} + E_{\text{rubbish}} \tag{5.100}$$

All we have to do is choose the set up so that the rubbish has negative energy inside the ergosphere and falls into the black hole. Note that we can do this because, inside the ergo region, $K^\mu$ is spacelike and hence the associated conserved quantity $E$ will have the interpration as a component of the spatial momentum. Therefore we can choose it to be either positive or negative by aiming the rubbish in the correct direction - inwards. Thus when the dump truck returns to us it will have energy $E_{truck} > E_{\text{initial}}$.

Of course what has really happened is that we have extracted energy from the black hole. It turns out that you can only extract the rotational energy of the black hole. As a result it will start spinning more slowly. Furthermore you can't keep on with this process to reduce the total energy of the black to zero or even negative.

# 6 Non-asymptotically Flat Solutions

The spacetimes we have described so far are all asymptotically flat. Meaning that at large spatial infinity the metric components approach that of flat Minkowski spacetime. This is occurs because at spatial infinity we are imagining that all the physical fields go to their vacuum values and the cosomologcal constant (which we view as the vacuum energy) vanishes. This may not be true for several reasons. Firstly there is no reason to think that if you travel far enough away from here that you will stop encountering stars and planets. Certainly you may go far from any particular star but it is not reasonable to assume that you can go infinitely far from all stars. In cosmology one assumes that the distribution of galaxies is even over large distances. Alternatively it could be that the cosmological constant, *i.e.* the vacuum energy, is not zero. Indeed this seems to be that, in our Universe, the cosmological constant is just so slightly positive. Finally there are important theoretical developements that have invloved the study of spacetimes with a negative cosmological constant.

## 6.1 dS

We can consider case that $V(vacuum) > 0$ so that Einsteins equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\lambda^2 g_{\mu\nu} \tag{6.101}$$

for some $\lambda$. There is not a unique solution to this equation, just as Minkowski space is not the unique solution to Einstein's equation with $\lambda = 0$. However there is a 'most symmetric' solution, *i.e.* a solution with the largest number of Killing vectors. This is one of the first solutions to Einstein's equation and is used in Cosmology. Indeed oberservations now strongly suggest that our universe looks like this on the largest scales. It is also used in so-called inflationary senarios of the early universe. Finally it is associated with very strange quantum features, such as only admitting a finite number of states.

In Euclidean signature the most symmetric space with positive curvature is a sphere. We can construct it by considering a five-dimensional space

$$ds_5^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2 \tag{6.102}$$

and imposing the constraint

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 = R^2 \tag{6.103}$$

This will give the familiar result

$$ds^2 = R^2 \left(d\theta^2 + \cos^2\theta d\Omega_3^2\right) \tag{6.104}$$

where $d\Omega_3^2$ is the metric on a 3-sphere. It is clear that the 4-sphere has an $SO(5)$ symmetry which is preserved by the constraint (6.103). This leads to

$$\dim(SO(5)) = \frac{5 \times 4}{2} = 10 \tag{6.105}$$

Killing vectors. On the other hand we saw that Killing vectors satisfy $D_{(\mu}K_{\nu)} = 0$ and there are therefore at most $\frac{1}{2}4\times 5 = 10$ independent Killing vectors on a four dimensional spacetime. Thus the sphere is maximally symmetric.

In Minkowskian signature we need to change the sign of the $(dx^1)^2$ term in $ds_5^2$. This can be done by Wick rotating $x^1 = ix^0$ which leads to $\theta = it$. Thus we look for a surface in

$$ds_5^2 = -(dx^0)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2 \tag{6.106}$$

obtained from the constraint

$$-(x^0)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 = R^2 \tag{6.107}$$

This can be solved by writing

$$x^0 = R\sinh t\ , \qquad x^{2,3,4,5} = R\cosh t\psi^{2,3,4,5} \tag{6.108}$$

where $(\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 = 1$. In this way we find de Sitter space

$$ds_{dS}^2 = R^2\left(-dt^2 + \cosh^2 t d\Omega_3^2\right) \tag{6.109}$$

**Problem:** Show this.

This spacetime has an $SO(1,4)$ symmetry comming from the symmetries of $ds_5^2$ which are preserved by the contraint (6.107). Since the dimension of $SO(1,4)$ is 10 we find a maximal number of Killing vectors. If you calculate the curvature (try this!) then you will see that the Einstein equation is satisfied so long as

$$\lambda^2 = \frac{3}{R^2} \tag{6.110}$$

It should be noted that there are an infinite number of coordinate systems one can use for de Sitter space. The above choice are the so-called global coordinates because they describe the entirety of de Sitter space. Unlike the usual coordinates for Schwarzchild which only cover regions away from the hoizon. In the literature there are other popular forms for the metric.

It should be clear already that de Sitter space describes a universe which consists of a 3-sphere which is infinity big in the far past, collapses down to a minimal radius $R$ at $t = 0$ and then expands again. At early and late times this expansion is exponential. Thus if there is a positive cosmological constant as there appears to be, then, no matter how small it is, it will eventually dominate over any other contributions to the geometry of the universe. In addition to this a rapid exponential growth is now strongly believed to have happened in the early universe, proir to nucleosynthesis. This is known as cosmological inflation.

To understand things better we change coordinates to

$$dt = \cosh t d\tau \qquad \longrightarrow \qquad \tau = 2\arctan(e^t) \tag{6.111}$$

so that

$$\cosh(t) = \frac{\tan(\tau/2) + \cot(\tau/2)}{2} = \frac{1}{2}\frac{1}{\cos(\tau/2)\sin(\tau/2)} = \frac{1}{\sin(\tau)} \tag{6.112}$$

and hence

$$ds^2_{dS} = \frac{R^2}{\sin^2(\tau)}\left(-d\tau^2 + d\Omega_3^2\right) \tag{6.113}$$

This metric is conformally just the cylinder $\mathbf{R} \times S^3$. Thus we can construct a spacetime diagram as before by chopping off the conformal factor, leaving a square. Along the $x$-axis we plot one of the angular coordinates of the $S^3$, say $\psi$ which therefore shrinks to zero at the end points $\psi = 0, \pi$. The $y$-axis is time $\tau$ which now runs between $\tau = 0$ and $\tau = \pi$.

This also implies that a light ray sent out from $\psi = 0$ at the beginging of time, $\tau = -\pi/2$, will only just make it to $\psi = \pi$ (the antipode to $\psi = 0$) at $\tau = \pi/2$ - the end of time. Thus at most half of the space is unobservable to any one observer, even if if they were to live for the entirety of time. Therefore there is an observer dependent horizon.

To study the horizon more clearly consider an alternative coordinate system for a part of de Sitter space. Let us solve the constraints by

$$x^0 = \sqrt{R^2 - r^2}\sinh t\ , \qquad x^5 = \sqrt{R^2 - r^2}\cosh t\ , \qquad x^{2,3,4} = r\psi^{2,3,4} \tag{6.114}$$

This only covers $x^5 \geq 0$, half of the spatial 3-sphere. Using this one finds

$$ds^2 = -(R^2 - r^2)dt^2 + \frac{R^2 dr^2}{R^2 - r^2} + r^2 d\Omega_2^2 \tag{6.115}$$

**Problem:** Show this.

Therefore this observer sees a static universe! Although we are only seeing half of it - the half that is in causal contact. The point $r = 0$ is nothing special, all points are the same in de Sitter space because of the maximal symmetry. This coordinate system is what a stationary observer located at $r = 0$ would see around him. We see that there is a Killing horizon at $r = R$ just as in the black hole solutions. This is the 'boundary' between the the two halves of global de Sitter space. Thus this horizon is remincient of Schwarzchild, except that you see yourself as inside it and spacetime appears static. Furthermore this horizon is not fixed in spacetime but sits at a different place for each observer.

## 6.2 adS

Next we consider the case that $V(vacuum) = -\lambda^2 > 0$. Einsteins equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \lambda^2 g_{\mu\nu} \tag{6.116}$$

for some $\lambda$. Again there is not a unique solution to this equation but rather a 'most symmetric' solution, this is known as anti-de Sitter Space. It has gained a huge amount of attention recently due to the Maldacena or adS/CFT correspondence which relates quantum gravity in this background to quantum gauge in one less dimension.

To construct anti-de Sitter space one starts with five-dimensional space but with two times:

$$ds_5^2 = -(dx^0)^2 - (dx^5)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{6.117}$$

Next one considers the hyperbloid

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^0)^2 - (x^5)^2 = -R^2 \tag{6.118}$$

We can solve this contraint by writing

$$x^0 = R\cosh\rho\cos t\ , \qquad x^5 = R\cosh\rho\sin t, \qquad x^{1,2,3} = R\sinh\rho\psi^{1,2,3} \tag{6.119}$$

where $\psi^{1,2,3}$ define a 2-sphere $(\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 = 1$. Substituting this constraint in $ds_5^2$ leads to

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2) \tag{6.120}$$

**Problem:** Show this.

By construction this spacetime has an $SO(3,2)$ symmetry - the symmetries of the metric $ds_5^2$ that are left in variant by the constriant (6.118). Just as for the sphere and de Sitter space the number of Killing vectors is given by the dimension of $SO(2,3)$ which is 10. Once again one finds that this is a solution to Einstein's equation if

$$\lambda^2 = \frac{3}{R^2} \tag{6.121}$$

Anti-de Sitter space is a little strange. The first thing to note is that entire hyperbloid is cover if we take $\rho \geq 0$ and $0 \leq t \leq 2\pi$. Hence 'time' is naturally a periodic variable. This is physically unacceptable and so people these days mean the universal covering space of adS when they talk of anti-de Sitter space. This means that we simply unwrap time and view it as running from minus infinty to plus infinity.

To understand its causal structure we can change variables to

$$\theta = \arctan(\sinh\rho) \tag{6.122}$$

so now $0 \leq \theta < \pi/2$. Thus we have

$$\begin{aligned}\sinh^2\rho &= \tan^2\theta\ , \qquad \cosh^2\rho = 1 + \tan^2\theta = \sec^2\rho \\ d\theta &= \frac{\cosh\rho}{1+\sinh^2\rho}d\rho = \frac{d\rho}{\cosh\rho}\end{aligned} \tag{6.123}$$

hence $d\rho = d\theta/\cos\theta$. Therefore the metric is

$$ds^2 = \frac{R^2}{\cos^2\theta}\left(-dt^2 + d\theta^2 + \sin^2\theta d\Omega_2^2\right) \tag{6.124}$$

This is conformal to $\mathbf{R} \times S^3$, a cylinder. Space is conformal to a 3-sphere and time is simply linear. It follows that a light ray starting from some point in the interior and heading outwards will hit the boundary of the cylinder in a finite proper time since the boundary is a a finite value of $\theta$. If we assign reflecting boundary conditions to the boundary then light rays will bounce back to where they came from within a finite proper time.

However the story is rather different for timelike observers. Consider timelike observer moving radially outwards. This means that

$$-1 = R^2(-\cosh^2 \rho \dot{t}^2 + \dot{\rho}^2) \tag{6.125}$$

If we note that $\cosh^2 \rho \dot{t} = E$ is a constant along a geodesic then we see that

$$-R^{-2} = -\frac{E^2}{\cosh^2 \rho} + \dot{\rho}^2 \longrightarrow \dot{\rho} = \sqrt{\frac{E^2}{\cosh^2 \rho} - R^{-2}} \tag{6.126}$$

For large $\rho$ we see that the first term in the square root gets small. Therefore there is a bound on how far out you can go for a given $E$

$$\rho \leq \rho_{max} = \text{arccosh} ER \tag{6.127}$$

so you'll never make it the edge of spacetime. Rather it is as if you were in a potential well in that you require more and more energy to go further and further out but you will always be forced to turn around and come back.

Thus anti-de Sitter space is as if the sky where a big mirror that reflected back to you all light signals that you sent out. However if you got in a space ship you could never reach this big mirror in the sky and would always fall back to earth.

After years of neglect anti-de Sitter space is perhaps now the best understood spacetime, even more so than Minkowski space. There is a conjecture, the adS/CFT or Maldacena conjecture, which asserts that a theory of quantum gravity in a spacetime which is asymptotically $D$-dimensional anti-de Sitter is exactly equivalent to a $(D-1)$-dimensional conformal field theory which one can think of as residing on the boundary. This is remarkable - unbelieveable even, although all the evidence suggests that it is true. Quantum gravity in an anti-de Sitter background will include such effects as black hole formation (and evaporation) and the claim is that there is a dual description of all this in terms of a non-gravitational quantum gauge theory in one less dimension. In particular the quantum gauge theory is, as far as anyone knows, a perfectly okay unitary theory. Whereas it is far from clear that quantum gravity, including black holes, is unitary. Indeed we don't even really know how to defined quantum gravity.

## 6.3 FRW

The next solution we discuss is actually a family of solutions that includes both de Sitter space and anti-de Sitter Space. The idea here is model the cosmic history of our universe. In particular we assume that the spacetime has spacelike hyper-surfaces with the maximum symmetry. Therefore the 3-dimensional spatial cross sections are either hyperbolic space $H^3$, flat space $\mathbf{R}^3$ or spheres $S^3$. Let us endow this space with local coordinates $x^i$ and metric $\gamma_{ij}$. Without loss of generality we can take the four-dimensional metric to be of the form

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^idx^j \tag{6.128}$$

Thus the only parameter is the scale factor $a(t)$ which gives the physical size of the spatial hypersurfaces.

We also consider a more general energy-momentum tensor. To be consistent with the symmetries we assume that

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & pa^2\gamma_{ij} \end{pmatrix} \tag{6.129}$$

*i.e.* we assume that the only non-vanishing components consist of an energy density $\rho$ as well as an isotropic pressure $p_i$. Note that we do not assume that $\rho$ and $p$ are constant. Thus the general assumption of the model is that, on a large cosmological scale, the universe is isotropic and homegeneous. The former means that there is no preferred direction whereas the latter means that there are no preferred points. These seem like very reasonable assumptions. Certainly as far as we can tell, on the largest scales that we can observe, the universe consists of an even but sparse distribution of galaxies and hence can be thought of as homogeneous. As far as we can tell the universe is also isotropic however since we can only look out from where we are it could be the universe is not isotropic, so that it has some kind of centre, in which case we must be relatively near the centre. However such an earth-centric view has been out of fashion in cosmology for hundreds of years, *i.e.* since Copernicus.

Next we must calculate the Levi-Civita connection coefficients

$$\begin{aligned} \Gamma^0_{ij} &= a\dot{a}\gamma_{ij} \\ \Gamma^i_{0j} &= \frac{\dot{a}}{a}\delta^i_j \\ \Gamma^i_{jk} &= \gamma^i_{jk} \end{aligned} \tag{6.130}$$

where $\gamma^i_{jk}$ are the Levi-Civita connection coefficients of the spatial metric $\gamma_{ij}$. From these we can calculate the Ricci tensor

$$R_{\mu\nu} = R_{\mu\lambda\nu}{}^{\lambda} = -\partial_\mu\Gamma^\lambda_{\nu\lambda} + \partial_\lambda\Gamma^\lambda_{\mu\nu} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\sigma\lambda} \tag{6.131}$$

We find the non-zero components are

$$\begin{aligned} R_{00} &= -\partial_0(3\frac{\dot{a}}{a}) - 3\frac{\dot{a}^2}{a^2} \\ &= -3\frac{\ddot{a}}{a} \end{aligned}$$

$$
\begin{aligned}
R_{ij} &= r_{ij} + \partial_0(a\dot{a}\gamma_{ij}) - 2\dot{a}^2\gamma_{ij} + 3\dot{a}^2\gamma_{ij} \\
&= r_{ij} + a\ddot{a}\gamma_{ij} + 2\dot{a}^2\gamma_{ij}
\end{aligned} \qquad (6.132)\ (6.133)
$$

where $r_{ij}$ is the Ricci tensor for the spatial manifold. Since it is maximally symmetric we have that

$$
r_{ij} = 2k\gamma_{ij} \qquad (6.134)
$$

for some $k$. In particular $k$ is positive, zero or negative for the cases of $S^3$, $\mathbf{R}^3$ and $H^3$ respectively. Furthermore by rescaling the $x^i$ coordinates we can, without loss of generality, take $k = 1, 0, -1$. Experiments indicate that, in our universe, $k = 0$.

Continuing we see that the Ricci scalar is

$$
\begin{aligned}
R &= -R_{00} + a^{-1}\gamma^{ij}R_{ij} \\
&= 6\frac{\ddot{a}}{a} + 6\frac{k}{a^2} + 6\frac{\dot{a}^2}{a^2}
\end{aligned} \qquad (6.135)
$$

Putting these together we find that the Einstein equation is

$$
\begin{aligned}
R_{00} + \frac{1}{2}R &= 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} \\
&= 8\pi G_N T_{00} \\
R_{ij} - \frac{1}{2}a^2\gamma_{ij}R &= (-2a\ddot{a} - \dot{a}^2 - k)\gamma_{ij} \\
&= 8\pi G_N T_{ij}
\end{aligned} \qquad (6.136)
$$

Thus we find the equations

$$
3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = 8\pi G_N\rho \qquad (6.137)
$$

$$
2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G_N p \qquad (6.138)
$$

The first equation is known as the Friedman equation. Note that it is first order in time derivatives. This is a consequence of the Bianchi identity. Quite often the second equation is rewritten, using the Friedman equation, so that

$$
\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8}{3}\pi G_N\rho \qquad (6.139)
$$

$$
\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G_N(3p + \rho) \qquad (6.140)
$$

$$
(6.141)
$$

Let us look at some special cases. When $p = -\rho$ we see that $T_{\mu\nu} = -\rho g_{\mu\nu}$ and this corresponds to a cosmological constant $\Lambda = \rho$. Thus we should recover de Sitter space for $\rho > 0$ and anti-de Sitter space for $\rho < 0$. Indeed one can check that the solutions are

$$\begin{aligned} a &= \frac{1}{\lambda}\cosh(\lambda t) \qquad k = 1, \qquad \lambda^2 = \frac{8}{3}\pi G_N\rho \\ a &= \frac{1}{\lambda}\cos(\lambda t) \qquad k = -1, \qquad \lambda^2 = -\frac{8}{3}\pi G_N\rho \end{aligned} \tag{6.142}$$

for $\rho > 0$ and $\rho < 0$ respectively. It is needless to say that when $\rho = 0$ we can take $k = 0$ and $a = 1$ to recover Minkowski space.

Often one introduces the Hubble 'constant' $H = \dot{a}/a$ however, except for the case of exponential expansion, $H$ is not constant (although its time variation is over cosmic scales).

Note that at late times, meaning large $a$, one can drop the $k$ term from the equations.

In addition the matter 'equation of state' is often taken to be $p = w\rho$ where $w$ is a constant. For any known type of matter one has that $w \geq -1$ and so this is generally assumed to be the case. In this case one can solve the equations. Let us assume for simplicity that $k = 0$ and try

$$a = a_0 t^\gamma \tag{6.143}$$

for some constant $\gamma$. From the Friedman equation we see that

$$\frac{8}{3}\pi G_N\rho = \gamma^2 t^{-2} \tag{6.144}$$

and hence substitution into the remaining equation gives

$$\begin{aligned} \gamma(\gamma-1)t^{-2} &= -\frac{4}{3}\pi G_N(3w+1)\rho \\ &= -\frac{3w+1}{2}\gamma^2 t^{-2} \end{aligned}$$

This implies that

$$\frac{3w+3}{2}\gamma^2 = \gamma \tag{6.145}$$

and hence we find

$$a(t) = a_1 t^{\frac{2}{3w+3}} \tag{6.146}$$

where $a_1 = a(1)$. In the limit that $w \to -1$ one recovers the exponential growth of de Sitter space.

It is clear that these metrics are conformal to $\mathbf{R} \times \Sigma$, where $\Sigma = S^3, \mathbf{R}^3, H^3$:

$$ds^2 = a^2(\tau)(-d\tau^2 + \gamma_{ij}dx^i dx^j) \tag{6.147}$$

with $d\tau = a^{-1}(t)dt$. Thus the causal structure is just that of $\mathbf{R} \times \Sigma$. However it should be kept in mind that generically (but not always, *e.g.* consider de Sitter space) one finds that $a = 0$ in the past. This corresponds to a singularity - the Big Bang - and this is in the causal past of every observer. In addition there can a zero of $a$ in the future - the Big Crunch - although current observations would seem to rule that out.

Another concept that arises is that of a particle horizon. Consider a massless particle moving such that, along $\Sigma$, $\gamma_{ij}dx^i dx^j = dr^2$ for some variable $r$. It follows a null geodesic

$$0 = -dt^2 + a^2 dr^2 \tag{6.148}$$

Thus in the time after $t = t_0$ the particle will travel a distance

$$R = \int_{t_0}^{t} \frac{dt'}{a(t')} \tag{6.149}$$

The point is that this can be bounded. For example if the Hubble parameter $H = \dot{a}/a$ is constant, *i.e.* $a = a_0 e^{Ht}$ where $a_0 = a(0)$, then

$$R = -a_0^{-1} e^{-Ht}|_{t_0}^{t} = a_0^{-1}(e^{-Ht_0} - e^{-Ht}) \tag{6.150}$$

Thus if we send the particle out today so that $t_0 = 0$, even if we wait until $t \to \infty$ we see that $r_{12}$ will be bounded, *i.e.* the particle can only make it out a finite distance due to the expansion of the universe.

# References

[1] J. Hartle, *Gravity*, Addison Wesley, 2003.

[2] R. Wald, *General Relativity*, University of Chicago Press, 1984.

[3] S. Hawking and J. Ellis, *The Large Scale Structure of Spacetime*, CUP, 1973.

[4] S. Weinberg, *Gravitation and Cosmology*, Wiley, 1972.