Ingemar Bengtsson's notes on

# ELECTRODYNAMICS,

Fall, 2002.

This is no substitute for Jackson's book.

James Clerk Maxwell was a country gentleman and a professor at Cambridge. His equations for the electromagnetic field are

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{Gilbert's law} \tag{1}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{B} = 0 \qquad \text{Faraday's law} \tag{2}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \text{Gauss' law} \tag{3}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\partial_t \mathbf{E} = \frac{4\pi}{c}\mathbf{j} \qquad \text{Ampère-Maxwell's law .} \tag{4}$$

Here $\rho$ and $\mathbf{j}$ are the electric charge and current densities, respectively, and in a complete specification of the theory one also needs a set of dynamical variables describing electrically charged matter. It is understood that both $\rho$ and $\mathbf{j}$ are some given functions of these variables. Let us leave this matter aside for the moment, and focus on the constant $c$ that can be determined by laboratory experiments. Its value is about 300 000 km/s, and this happens to be equal—at least within the experimental uncertainty—to the velocity of light. This is not an accident. Suppose that

$$\rho = 0 \qquad \mathbf{j} = 0 \ . \tag{5}$$

This means that no electric charges are present, and we are looking at Maxwell's equations *in vacuo*. It is then easy to check that we obtain an exact solution of the equations if we take

$$E_y = f(ct - x) \qquad B_z = f(ct - x) \ , \tag{6}$$

all other components vanishing. This describes a plane electromagnetic wave propagating in the direction of the $x$-axis with the velocity $c$. The arbitrary function $f$ gives the shape of the wave, and this is not changed as the wave moves. Other solutions propagating with the same velocity are easy to find. The existence of such solutions is fairly convincing evidence that light is an electromagnetic wave described by Maxwell's equations. That electromagnetic waves with the expected properties do indeed exist was confirmed in the laboratory by Heinrich Hertz some twenty years after Maxwell's prediction.

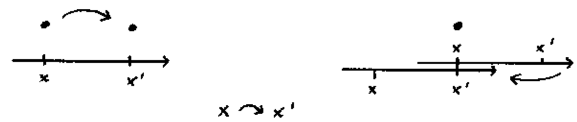


Figure 1: Active and passive transformations

1.1 INVARIANCE

It is clear that there is something unexpected about Maxwell's theory. The velocity of light is predicted to be equal to 300 000 km/s, and the question "velocity with respect to what?" presents itself. In the end Albert Einstein showed the question to be mistaken. This is not the place to go through his argument in detail; rather we will show directly that Maxwell's equations are Lorentz invariant and then sketch how this is enough to dissolve the question. First of all we have to recall what it means for an equation to be invariant under some transformation. Think of Newton's equations for the motion of two bodies under gravitational attraction. They are invariant under rotation. This does not mean that an individual solution must be invariant under rotation—indeed this is false in general—but it means that if we start with a solution and rotate it in space around an arbitrary point, the resulting configuration is again a solution of the equations. An ellipse is not rotationally symmetric, but Newton's theory is because every ellipse that can be obtained by rotating a given ellipse solves the equations provided that the original ellipse does. It is an important point—to reemerge later—that the rotated ellipse counts as a distinct solution of Newton's equations, but at the same time it is observationally indistinguishable from the original ellipse as far as measurements within the Earth-Sun system are concerned.

To see what goes on conceptually let us consider the simplest case, that of spatial translations. First we choose a set of Cartesian coordinates in space, so that a point can be uniquely labelled by a vector $\mathbf{x}$ from some chosen origin. Given a point $\mathbf{x}$ we define a new point $\mathbf{x}'$ by

$$\mathbf{x}' = \mathbf{x} + \mathbf{a} \ , \qquad (7)$$

where $\mathbf{a}$ is some constant vector. So you should read this as an assignment of a new point $\mathbf{x}'$ to every point $\mathbf{x}$. (Alternatively you could read it as a coordinate transformation that assigns a new coordinate $\mathbf{x}'$ to the original point. This would be the "passive" viewpoint, and we would then be dealing with a coordinate transformation only. But we choose the former and more interesting "active" viewpoint here.)

Suppose that the charge density $\rho$ is a given function of the points, i.e. of the coordinate $\mathbf{x}$. Then we demand that there exists a new function $\rho'$ of the points (whether labelled by $\mathbf{x}$ or by $\mathbf{x}'$ does not matter), which is such that

$$\rho'(\mathbf{x}', t) = \rho(\mathbf{x}, t) \ . \tag{8}$$

It is important to get the meaning of this equation right. At the point $\mathbf{x} + \mathbf{a}$ the function $\rho'$ takes the same value as the function $\rho$ takes at the point $\mathbf{x}$. If we like we can express the new function as a function of $\mathbf{x}$; evidently

$$\rho'(\mathbf{x} + \mathbf{a}, t) = \rho(\mathbf{x}, t) \quad \Leftrightarrow \quad \rho'(\mathbf{x}, t) = \rho(\mathbf{x} - \mathbf{a}, t) \ . \tag{9}$$

In physical terms we have moved the charge density to a new position. Adopt the same rule for the electric field, i.e.

$$\mathbf{E}'(\mathbf{x} + \mathbf{a}, t) = \mathbf{E}(\mathbf{x}, t) \ , \tag{10}$$

and again the same rule for the magnetic field and the electric current. It is then easy to see that if the functions $\rho$, $\mathbf{j}$, $\mathbf{E}$ and $\mathbf{B}$ are such that they solve Maxwell's equation, then so do the functions $\rho'$, $\mathbf{j}'$, $\mathbf{E}'$ and $\mathbf{B}'$. The conclusion is that if we have an allowed configuration—i.e. a solution of Maxwell's equations—of charges and fields with some characteristic spatial dependence, then the same configuration moved to a new position differing from the first by a constant vector $\mathbf{a}$ is also an allowed configuration. One says that Maxwell's equations are invariant under spatial translations. Similarly, they are invariant under translations in time and under spatial rotations, although the latter property can be seen only if we take into account that the electric and magnetic fields are to transform as vectors under rotations.

Now consider the transformation

$$\mathbf{x}' = \mathbf{x} + \mathbf{v}t \ . \tag{11}$$

It is known as a Galilei transformation. Newton's equations are invariant under Galilei transformations. Since its effect is to transform a stationary particle into one that moves with velocity $v$ this means—granted that the transformed solution is observationally indistinguishable from the original—that absolute velocities are unobservable. The story for Maxwell's equations is different: Replace the constant vector $\mathbf{a}$ with $\mathbf{v}t$ everywhere in the formulæ above. If we start with an allowed configuration of electric and magnetic fields the result will be a configuration that is moving with constant velocity $\mathbf{v}$ compared to the original configuration. But it is easy to see from the way the equations mix spatial and time derivatives that the new configuration is not allowed since it does not solve Maxwell's equations. This suggests that Maxwell's equations somehow single out a special state of motion that can be defined as the state of rest, and that the velocity $c$ is the velocity with respect to that state of rest. This is about as different as it can be from Newtonian gravity (say).

At this point Einstein entered the scene and observed that one can avoid this somewhat unpalatable conclusion. Since the problem evidently is that Maxwell's equations mix space and time derivatives in a characteristic fashion it is natural to try to generalize the Galilei transformation so that the new transformation also mixes space and time in a more intimate fashion. The appropriate language in which to discuss this idea is called "tensor calculus", and we will take a detour to explain tensors. Afterwards we will present Einstein's argument in the way that it was dressed up by Hermann Minkowski some years later—perhaps not the most pedagogical way one can think of, but it happens that one of the aims of this course is to give you a first exposure to tensor calculus. A moderate ability to handle "index notation" is all that you will need in succeeding chapters.

Perhaps I should add that most introductions to tensor calculus adopt the "passive" viewpoint in which the transformations considered are just changes of coordinates. This is also important but it is the active viewpoint that is needed to discuss the invariance of Maxwell's equations. (If the statement that Maxwell's equations hold in all coordinate systems were found to be false, the conclusion would be that there is something wrong about the way we handle coordinate systems. On the other hand it is a question of physics whether Maxwell's equations continue to hold after an active transformation of space and time.) Hence in the discussion below you can consider the coordinate system as having been fixed once and for all.

## 1.2 TENSORS

Suppose that space has 4 dimensions (the generalization to an arbitrary number of dimension will be trivial). A point in the space can then be uniquely described by giving the values of four coordinates $x^0, x^1, x^2$ and $x^3$. We collect them together into $x^\alpha$, where the index $\alpha$ ranges from 0 to 3. We begin counting from zero because we will eventually think of the $x^0$-direction as the time direction in "spacetime", which is four dimensional. For now this does not matter; indeed at the outset the coordinates have no other properties than that of being unique labels of points in a four dimensional space. We are interested in transformations of the space onto itself. They can be described by

$$x^\alpha \to x'^\alpha = x'^\alpha(x) \ . \tag{12}$$

Thus the point that was described by the coordinates $x^\alpha$ is moved to the point that was described by the coordinates $x'^\alpha$. We will simplify matters a little by assuming that the most general transformation that we want to describe is a linear transformation. (This excludes translations, but we can easily take care of them at the end.) Thus

$$x'^\alpha = \sum_{\beta=0}^{3} \Lambda^\alpha{}_\beta x^\beta \ . \tag{13}$$

This is simply a matrix equation. You may wonder why I write one index "upstairs" and one "downstairs". There is a reason for this that begins to emerge when we compute how the derivatives transform. Using the chain rule we get

$$\partial'_\alpha \equiv \frac{\partial}{\partial x'^\alpha} = \sum_{\beta=0}^{3} \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta} = \sum_{\beta=0}^{3} (\Lambda^{-1})^\beta{}_\alpha \frac{\partial}{\partial x^\beta} \equiv \sum_{\beta=0}^{3} (\Lambda^{-1})^\beta{}_\alpha \partial_\beta \ . \tag{14}$$

In general the gradient transforms in a different way than the coordinate vector itself. In the special case $\Lambda^{-1} = \Lambda^T$, that is to when $\Lambda$ is an orthogonal matrix, there is no difference. (You should convince yourself that this is so—you have to see through the "upstairs-downstairs" notation to do it.) Orthogonal matrices actually correspond to rotations but we will need the general case, and then we get two different kinds of vectors.

Now we want a rule that tells us how various objects on the space transform when the active transformation is carried through. This will include functions (say the density of a fluid), four-component objects (such as the velocity of a fluid on a four dimensional space), and even more complicated objects that we will come to. First, a scalar function is by definition a function that transforms according to

$$\phi'(x') = \phi(x) \ . \tag{15}$$

The new function takes the same value at the new point as the old function takes at the old point. Second, a contravariant vector is a set of four functions that transform according to

$$V'^\alpha(x') = \sum_{\beta=0}^{3} \Lambda^\alpha{}_\beta V^\beta(x) \ . \tag{16}$$

Hence a component of the new vector takes a value at the new point which is a linear combination of the values taken by the old vector at the old point. Third, a covariant vector is a set of four functions that transform according to

$$U'_\alpha(x') = \sum_{\beta=0}^{3} (\Lambda^{-1})^\beta{}_\alpha U_\beta(x) \ . \tag{17}$$

The scalar product of two vectors ought to be a scalar function. Now it is easy to see that

$$\sum_{\alpha=0}^{3} U'_\alpha V'^\alpha = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3} U_\beta (\Lambda^{-1})^\beta{}_\alpha \Lambda^\alpha{}_\gamma V^\gamma = \sum_{\alpha=0}^{3} U_\alpha V^\alpha \ , \tag{18}$$

so the obvious scalar product of a covariant and a contravariant vector is indeed a scalar, while it is not possible to define a scalar product of (say) two covariant vectors.

At this point we introduce the Einstein summation convention. The rule is that summation signs will not be written. Instead, whenever an index appears twice in a term in an equation, once in an upstairs and once in a downstairs position, summation over this index is understood. If an index appears only once it is called a free index and it is understood that it may take any value; in effect one free index means that there are four equations. Indices will never appear twice in a downstairs or upstairs position, or more than twice. Equations of the latter types simply never appear.

With the Einstein summation convention in force it is easy to define tensors of "arbitrary rank". Thus a contravariant tensor of rank four (say) is a set of $4^4$ functions that transforms according to

$$T'^{\alpha\beta\gamma\delta}(x') = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \Lambda^\gamma{}_\rho \Lambda^\delta{}_\sigma T^{\mu\nu\rho\sigma}(x) \ . \tag{19}$$

This looks a bit complicated, but it is a perfectly definite definition. Similarly a covariant tensor of rank three (say) is a set of $4^3$ functions that transforms according to

$$A'_{\alpha\beta\gamma}(x') = (\Lambda^{-1})^\mu{}_\alpha (\Lambda^{-1})^\nu{}_\beta (\Lambda^{-1})^\rho{}_\gamma A_{\mu\nu\rho}(x) \ . \tag{20}$$

"Mixed" tensors with some contravariant and some covariant indices can be defined in the obvious way.

Tensors of the same rank can be added together, that is to say that

$$T_\gamma^{\alpha\beta} \equiv U_\gamma^{\alpha\beta} + V_\gamma^{\alpha\beta} \tag{21}$$

is a mixed tensor if the terms on the right hand side are. There is also a kind of "tensor multiplication"; as an example the object

$$U_\alpha V_{\beta\gamma} = V_{\beta\gamma} U_\alpha \tag{22}$$

is a covariant tensor of rank three. (Note that it does not matter in which order we write the factors as long as they carry the correct indices.) Furthermore higher rank tensors can be "contracted" in various ways. Thus

$$M_\gamma^{\alpha\beta} \equiv T_{\gamma\mu\nu}^{\alpha\beta\mu\nu} \tag{23}$$

—an object that is constructed in accordance with the rules of the summation convention—is a mixed tensor of rank two plus one. On the other hand the object

$$N^{\alpha\beta} \equiv \sum_{\gamma=0}^{3} T'^{\alpha\beta\gamma\gamma} \tag{24}$$

is not a tensor. It is not defined according to the rules, and it does not transform like a tensor under general linear transformations.

It is easy to check that derivatives of arbitrary tensors transform as tensors with an additional covariant index so that the equation

$$M^{\gamma}_{\alpha\beta} = \partial_\alpha T^{\gamma}_{\beta} \tag{25}$$

makes sense. Note that this works essentially because the matrix $\Lambda^{\alpha}{}_{\beta}$ is independent of the coordinates. As a matter of fact this is the only point so far where our restriction to linear transformations is important.

The order between two covariant (or two contravariant) indices matters. In general therefore

$$V_{\alpha\beta} \neq V_{\beta\alpha} \ . \tag{26}$$

(On the other hand the order between a covariant and a contravariant index does not matter at this point.) It is possible to require that a certain rank two tensor obeys

$$V_{\alpha\beta} = -V_{\beta\alpha} \ . \tag{27}$$

Such a tensor is said to be anti-symmetric. The definition is meaningful because the equation is preserved by the transformations that we consider, that is to say that

$$V_{\alpha\beta} = -V_{\beta\alpha} \quad \Leftrightarrow \quad V'_{\alpha\beta} = -V'_{\beta\alpha} \ . \tag{28}$$

Clearly an anti-symmetric tensor has $4 \cdot 3/2 = 6$ rather than $4^2 = 16$ independent components (and you can easily generalize this calculation to arbitrary dimensions). More generally we can define the anti-symmetric part of a tensor as

$$T_{[\alpha\beta]} \equiv \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) \tag{29}$$

$$T_{[\alpha\beta\gamma]} \equiv \frac{1}{6}(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha} - T_{\beta\alpha\gamma} - T_{\gamma\beta\alpha} - T_{\alpha\gamma\beta}) \ , \tag{30}$$

and so on. By construction the tensor $T_{[\alpha\beta\gamma]}$ is anti-symmetric in each pair of indices separately. It has $4 \cdot 3 \cdot 2/3 \cdot 2 = 4$ independent components. Evidently we can define symmetric tensors in a similar way.

A common ingredient in many tensor arguments is the simple observation that

$$V^\alpha V^\beta T_{[\alpha\beta]} = 0 \ . \tag{31}$$

To see this, just rename $\alpha \leftrightarrow \beta$. This is allowed since $\alpha$ and $\beta$ are "dummy" indices that are being summed over. But $T_{[\alpha\beta]}$ switches sign in the process, while it does not matter in which order we write $V^\alpha$ and $V^\beta$. So the expression must be equal to minus itself. Therefore it is equal to zero.

In general the various components of a given tensor will look completely different as functions of position in the space, once an active transformation is carried through. There is one exception to this rule however. Consider the Kronecker delta, defined in terms of components by

$$\delta^\alpha_\beta = \begin{array}{l} 1 \text{ if } \alpha = \beta \\ 0 \text{ if } \alpha \neq \beta \end{array} \ . \tag{32}$$

It is easy to see that

$$\delta'^\alpha_\beta = \Lambda^\alpha{}_\gamma (\Lambda^{-1})^\delta{}_\beta \delta^\gamma_\delta = \Lambda^\alpha{}_\gamma (\Lambda^{-1})^\gamma{}_\beta = \delta^\alpha_\beta \ . \tag{33}$$

The Kronecker delta is left invariant by all the transformations that we consider, and is therefore said to be an invariant tensor. It is in fact the only invariant tensor that we can construct (except for tensor products of several deltas).

There is another tensor that almost makes it, though. Define a totally anti-symmetric tensor of rank four by

$$\epsilon^{0123} = 1 \ , \qquad \epsilon^{\alpha\beta\gamma\delta} = \epsilon^{[\alpha\beta\gamma\delta]} \ . \tag{34}$$

This works because a totally anti-symmetric rank four tensor in four dimensions has only one independent component—there is an epsilon tensor in any dimension, with rank equal to the dimension. Now if (and only if!) you remember the definition of the determinant of a matrix you will see that

$$\epsilon'^{\alpha\beta\gamma\delta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \Lambda^\gamma{}_\rho \Lambda^\delta{}_\sigma \epsilon^{\mu\nu\rho\sigma} = \det\Lambda \epsilon^{\alpha\beta\gamma\delta} \ . \tag{35}$$

The epsilon tensor does change under a general linear transformation, but it is an invariant tensor if for some reason we restrict ourselves to transformations such that

$$\det\Lambda = 1 \ . \tag{36}$$

The epsilon tensor is important in itself, and so is the idea of restricting the set of allowed transformations by the requirement that some particular tensor be invariant under allowed transformations.

And this concludes our brief introduction to tensors in general. If we were to allow arbitrary transformations rather than linear transformations only we would now have to enter a lenghty discussion of how the notion of "derivative" can be included in the tensor formalism.

## 1.2 CARTESIAN TENSORS

The idea of Cartesian tensors will be presented in three dimensional space, and so we introduce Latin indices $i, j, k$ that run from one to three. Actually the only change in the discussion above that we have to make is in the definition of the epsilon tensor, which now has three indices only. But we will inject an entirely new ingredient. Introduce the tensor

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{37}$$

and require this tensor to be invariant. This is to say that we demand

$$g'^{ij} = \Lambda^i{}_k \Lambda^j{}_l g^{kl} . \tag{38}$$

In matrix notation this becomes

$$\mathbf{1} = \Lambda \mathbf{1} \Lambda^T \qquad \Leftrightarrow \qquad \Lambda^T = \Lambda^{-1} . \tag{39}$$

Therefore the only transformations that we will allow are those that are given by orthogonal matrices. As you know (?) these are all rotations and reflections, and nothing else.

We can also introduce a covariant tensor $g_{ij}$ as the "inverse" of $g^{ij}$, in the sense that

$$g_{ik} g^{kj} = \delta_i^j . \tag{40}$$

Since both $g^{ij}$ and the Kronecker delta are represented by the unit matrix, so is $g_{ij}$.

What is the point here? One answer is that we want to define the distance between two points at $x^i$ and $x^i + \Delta x^i$ respectively. According to Pythagoras, if the coordinate axises are straight and orthogonal to each other then the square of the distance $\Delta s$ is

$$\Delta s^2 = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = g_{ij} \Delta x^i \Delta x^j . \tag{41}$$

Rotations and reflections comprise—with translations—precisely those transformations of space onto itself that leave the distance between two arbitrary points invariant. The tensor $g_{ij}$ is called the metric tensor. Note that at this point in the story our coordinates have become ordinary Cartesian coordinates in space, and the tensors are now called Cartesian tensors. They are special in the sense that they transform only under a restricted set of linear transformations.

You may wonder how we would proceed if we insist on using some curvilinear coordinate system. The answer is that it is perfectly possible to do so if some appropriate changes are made. As an example, consider spherical polars so that $(x^1, x^2, x^3) = (r, \theta, \phi)$. Then we must adopt the metric

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin\theta \end{pmatrix} . \tag{42}$$

Also it would make no sense to restrict ourselves to transformations that are linear in these coordinates. Indeed the whole discussion gets more cumbersome at various points, and for this reason we stick to Cartesian tensors here.

The distinction between covariant and contravariant tensors disappears for Cartesian tensors, since they transform in the same fashion. Moreover we can use the metric to "raise and lower indices". Thus, given a covariant tensor ($T_{ijk}$ say) we can define a corresponding contravariant tensor by

$$T^{ijk} \equiv g^{il}g^{jm}g^{kn}T_{lmn} . \tag{43}$$

This is consistent with our earlier definition of the covariant tensor $g_{ij}$ since

$$g_{ik}g^{kj} = \delta_i^j \qquad \Rightarrow g^{ij} = g^{ik}g^{jl}g_{kl} . \tag{44}$$

Anyway, because our metric is a unit matrix it evidently follows that

$$\begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix} = V^i = g^{ij}V_j = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} , \tag{45}$$

and similarly for higher rank tensors. For this reason there is no reason to keep the distinction between upstairs and downstairs indices in the formalism.

From now on we will write all indices downstairs when we work with Cartesian tensors. In particular Maxwell's equation can be written

$$\partial_i B_i = 0 \qquad \text{Gilbert's law} \tag{46}$$

$$\epsilon_{ijk}\partial_j E_k + \frac{1}{c}\partial_t B_i = 0 \qquad \text{Faraday's law} \tag{47}$$

$$\partial_i E_i = 4\pi\rho \qquad \text{Gauss' law} \tag{48}$$

$$\epsilon_{ijk}\partial_j B_k - \frac{1}{c}\partial_t E_i = \frac{4\pi}{c}j_i \qquad \text{Ampère-Maxwell's law} . \tag{49}$$

It is now an exercise to show that Maxwell's equations are invariant under rotations in space, provided that we adopt the transformation rules

$$E_i'(x') = \Lambda_{ij}E_j(x) \qquad B_i'(x') = \det\Lambda\Lambda_{ij}B_j(x) . \tag{50}$$

Note that the magnetic field is not quite a vector. It is a pseudovector, transforming with an extra sign under reflections. Thus if

$$\Lambda_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{51}$$

the electric field switches direction but the magnetic field does not. On the other hand the object

$$F_{ij} \equiv \epsilon_{ijk} B_k \tag{52}$$

is a tensor, so it would make some sense to rewrite Maxwell's equations in terms of this object. (Do it!)

You should note how scalar and cross products appear in the tensor notation:

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i \qquad \mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \ . \tag{53}$$

The epsilon tensor provides a useful way to derive the properties of the cross product. Specifically in three dimensions we have the $\epsilon$— $\delta$ identity

$$\epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \ . \tag{54}$$

(Proof: Check it component by component!) Using this identity the rule for repeated cross products follows:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \equiv \epsilon_{ijk}\epsilon_{kmn} a_j b_m c_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) a_j b_m c_n = \tag{55}$$

$$= b_i a_j c_j - c_i a_j b_j \equiv \mathbf{a} \cdot \mathbf{c} \mathbf{b} - \mathbf{a} \cdot \mathbf{b} \mathbf{c} \ . \tag{56}$$

The odds are that once you get used to it you will find the index formalism very convenient when handling cross products. Nevertheless it occasionally happens that the cross product notation is superior; the rule is that one should not be fanatic about notation.

There are lots of tricks that can be played with the epsilon tensor. For instance

$$F_{ij} = \epsilon_{ijk} B_k \qquad \Leftrightarrow \qquad B_i = \frac{1}{2}\epsilon_{ijk} F_{jk} \ . \tag{57}$$

I will more often than not perform such calculations without comment.

## 1.3 INVARIANCE OF MAXWELL'S EQUATIONS

Now back to the problem that we had with Maxwell's equations and the constant velocity of light. The suggestion was that the transformation that creates a moving configuration from a static one must mix space and time in some clever fashion, and we will rely on the tensor formalism for four dimensional spaces to tell us how. The aim is to write Maxwell's equations as a set of equations for tensors in a four dimensional "spacetime".

First we collect the coordinates into a "four vector"

$$x^\alpha = \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} . \tag{58}$$

We use the constant $c$ to give all the four coordinates the dimension of length. It seems reasonable to use the charge density as the missing fourth component of the current four vector; hence

$$j^\alpha = \begin{pmatrix} \rho \\ \mathbf{j} \end{pmatrix} . \tag{59}$$

The electric and magnetic fields require more thought though. We cannot turn them into two four vectors since there are no candidates for their fourth components. What we can do is to form one anti-symmetric tensor with six components:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & F_{0i} \\ F_{i0} & F_{ij} \end{pmatrix} = \begin{pmatrix} 0 & -E_i \\ E_i & \epsilon_{ijk} B_k \end{pmatrix} . \tag{60}$$

This is perhaps not so surprising since we already knew that it is in some sense natural to describe the magnetic field as a Cartesian tensor $F_{ij}$ rather than as a vector. We will refer to $F_{\alpha\beta}$ as the electromagnetic tensor. Of course we cannot say whether it makes sense to talk about an electromagnetic tensor at all until we have shown that Maxwell's equations really are invariant under a set of transformations under which it is a tensor; we are coming to it.

The question is whether Maxwell's equations can be written in terms of $j^\alpha$ and $F_{\alpha\beta}$. (There are eight equations altogether, so we need two tensor equations with one free index each.) After some experimentation one finds that Gilbert's and Faraday's laws can be collected together as the single tensor equation

$$\partial_{[\alpha} F_{\beta\gamma]} = 0 \qquad \Leftrightarrow \qquad \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0 . \tag{61}$$

In four dimensions a totally anti-symmetric tensor with three indices has four components, so the number of equations is right. (When we "expand out" the anti-symmetric tensor in the second step we need only three terms because the electromagnetic tensor is anti-symmetric in itself.)

The remaining equations are not so easily disposed of. This time experimentation reveals a sign that just will not work out. The solution turns out to be to introduce by fiat a new tensor that is

$$g^{\alpha\beta} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{62}$$

We can use this tensor to "raise indices" if we like; hence

$$F^{\alpha\beta} \equiv g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu} = \begin{pmatrix} 0 & E_i \\ -E_i & \epsilon_{ijk}B_k \end{pmatrix} . \tag{63}$$

Then the remaining four Maxwell equations take the elegant form

$$\partial_\beta F^{\alpha\beta} = 4\pi j^\alpha . \tag{64}$$

It goes without saying that this is not something that you are supposed to see directly. But it is something that you ought to check. What one can see directly is that in order to write this equation it is necessary to somehow "raise the indices" on $F_{\alpha\beta}$. In the case of Cartesian tensors we did have a similar problem and we got around it by "raising indices with the Kronecker delta". The interesting thing is that this strategy does not work in spacetime—the tensor $g^{\alpha\beta}$ is not a Kronecker delta. So what has been achieved?

In fact we are now in a position to see almost without calculation what is the most general linear transformation of spacetime that leaves Maxwell's equations invariant, that is to say that transforms a solution to another solution. Since we defined $g^{\alpha\beta}$ as a tensor that takes a quite definite form the allowed transformations must leave this form invariant—the point being that $g^{\alpha\beta}$ contributes some fixed numbers to the equations, and if these numbers change when we transform a configuration we could distinguish between the original and the transformed configuration simply by checking what numbers are required for Maxwell's equations to hold. The conclusion is that we can admit any linear transformation

$$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta \tag{65}$$

that obeys

$$g'^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu g^{\mu\nu} = g^{\alpha\beta} . \tag{66}$$

Such transformations are called Lorentz transformations because they were originally discovered—without benefit of tensors—by Hendrik Lorentz. Tensors transforming under such transformations are called Lorentz tensors. The tensor $g^{\alpha\beta}$, or its covariant cousin that has the same form, is called the metric tensor.

Evidently the idea behind Lorentz tensors is similar to the idea behind Cartesian tensors, except that the metric tensor that is declared to be invariant contains a sign in the former case that is missing in the latter. Moreover all rotations in space are examples of Lorentz transformations. It is easy to see that all matrices of the form

$$\Lambda^{\alpha}{}_{\beta} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \Lambda_{ij} & \\ 0 & & & \end{array} \right) \tag{67}$$

where $\Lambda_{ij}$ is an orthogonal matrix do indeed leave $g^{\alpha\beta}$ invariant. Such a Lorentz transformation is an ordinary rotation of space that leaves the fourth dimension (i.e. time) alone. But it requires some further analysis to see precisely what goes on in general, and what happened to the velocity of light.

Before we turn to such an analysis we rewrite the first set of Maxwell's equations a little, to make them even more elegant. Given a covariant anti-symmetric tensor of rank two in four dimensions we can always define a "dual" contravariant tensor by means of the epsilon tensor (which happens to have four indices in four dimensions). Thus

$$\star F^{\alpha\beta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \left( \begin{array}{cc} 0 & \frac{1}{2}\epsilon_{ijk}F_{jk} \\ -\frac{1}{2}\epsilon_{ijk}F_{jk} & \epsilon_{ijk}F_{k0} \end{array} \right) = \left( \begin{array}{cc} 0 & B_i \\ -B_i & \epsilon_{ijk}E_k \end{array} \right) . \tag{68}$$

With this definition Maxwell's equations can be written in the form

$$\partial_\beta \star F^{\alpha\beta} = 0 \tag{69}$$

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha . \tag{70}$$

The content of this brief statement is of course exactly the same as that we started out with.

## 1.4 MINKOWSKI SPACE

Let us leave the electromagnetic field aside for the moment and concentrate on what a Lorentz transformation does to spacetime itself. As we have seen an ordinary rotation in (say) the $y-z$ plane,

$$\Lambda^{\alpha}{}_{\beta} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{array} \right) , \tag{71}$$

is a Lorentz transformation (although it is not usually called that). Due to the minus sign in the metric tensor transformations in (say) the $ct-x$ plane looks just a bit different:

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{72}$$

where $\alpha$ is an arbitrary real number. Such a transformation is called a Lorentz boost. A Lorentz boost transforms the point $(ct, x, y, z)$ to a new point $(ct', x', y', z')$ given by

$$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta , \tag{73}$$

or more explicitly by

$$ct' = \cosh\alpha\ ct + \sinh\alpha\ x \qquad x' = \sinh\alpha\ ct + \cosh\alpha\ x \tag{74}$$

$$y' = y \qquad\qquad z' = z . \tag{75}$$

This is sometimes called a hyperbolic rotation. We want to understand it.

As a first step, let us draw a map of spacetime. To simplify matters, let us pretend that space has only one dimension. Then we introduce an $x$-axis and a $t$-axis. Any point on the resulting map corresponds to "a point in space at some particular time". Now suppose that you have a particle moving with constant velocity $u$. In spacetime, this is described by the straight line

$$x = ut . \tag{76}$$

The line is called the "world line" of the particle. More generally we can consider curved world lines

$$x = x(t) . \tag{77}$$

If you think about it the particle is not moving along the world line, the world line simply is. Clearly there is nothing in this picture that could not have been imagined in the eighteenth century, and indeed the idea of the four dimensional spacetime occurred already to Joseph-Louis Lagrange. What Newton's equations (say) do for you is to predict the shape of the world line, given its location and slope at one particular value of $t$.

The next step is to draw some flowlines of the boost. By definition a flow line consists of all the points that can be reached from a given point through a Lorentz boost by varying the parameter $\alpha$. It is easy to see what the boost does to points on the $ct$-axis:

$$(ct, 0) \to (ct', x') = (\cosh\alpha\ ct, \sinh\alpha\ ct) . \tag{78}$$

This gives a flowline parametrized by the real number $\alpha$, viz.

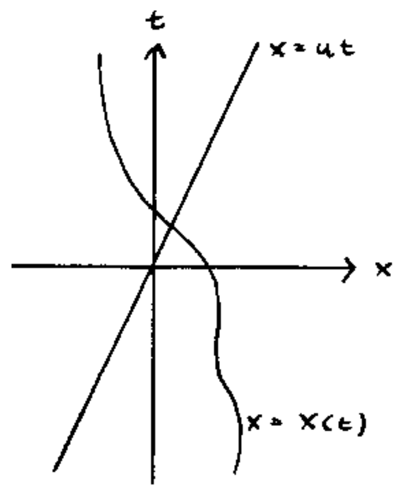


Figure 2: Spacetime, including two world lines

$$ct' = \coth\alpha \; x' \; . \tag{79}$$

Flowlines that start from the $x$-axis can also be computed. It is convenient to use $ct$ rather than $t$ on the vertical axis when we draw our picture of the flowlines. The picture should be compared with a picture that shows the flowlines of a rotation in the $y-z$-plane.

What happens to the world line of a particle moving with constant velocity if we perform a Lorentz boost? The result must be another world line, describing a particle in a different state of movement. So we choose some value of $\alpha$ and perform the boost. Making use of $x = ut$ we find that

$$ct' = (\cosh\alpha \; c + \sinh\alpha \; u)t \qquad x' = (\sinh\alpha \; c + \cosh\alpha \; u)t \; . \tag{80}$$

It follows that

$$x' = \frac{\sinh\alpha \; c + \cosh\alpha \; u}{\cosh\alpha \; c + \sinh\alpha \; u} ct' \equiv u't' \; . \tag{81}$$

This is again a straight world line, corresponding to a particle that is moving with the constant velocity $u'$. So now we know the answer to our first question.

Let us assume that the original particle was moving with zero velocity so that $u = 0$. Then it gets transformed to a particle that is moving with a velocity $v$, where

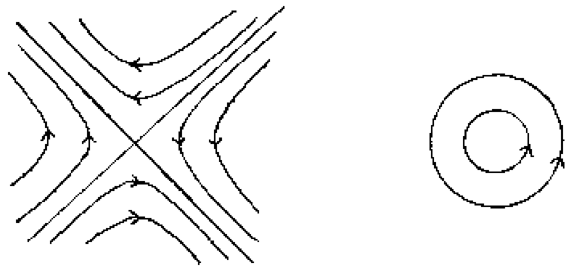

Figure 3: Flowlines of a boost and a rotation

$$v = \frac{\sinh\alpha}{\cosh\alpha}c \ . \tag{82}$$

If we like we can solve this equation for $\alpha$ as a function of $v$, and then use $v$ rather than $\alpha$ to parametrize the boost. The result of this calculation is that a Lorentz boost that transforms a stationary particle to a particle moving with velocity $v$ transforms a point $(t, x)$ to a another point $(t', x')$ according to

$$ct' = \frac{ct}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{vx}{c\sqrt{1-\frac{v^2}{c^2}}} \qquad x' = \frac{vt}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{x}{\sqrt{1-\frac{v^2}{c^2}}} \ . \tag{83}$$

This is a standard parametrization of Lorentz boosts.

We may now deduce that if we perform a boost that turns a stationary particle into a particle moving with velocity $v$ then a particle moving with velocity $u$ will turn into a one that moves with velocity

$$u' = \frac{u+v}{1+\frac{uv}{c^2}} \ . \tag{84}$$

An interesting consequence of this is that a particle moving with the velocity of light will be transformed into a particle moving with the same velocity; i.e. $c' = c$ or

$$x = ct \quad \Leftrightarrow \quad x' = ct' \ . \tag{85}$$

The worldline of such a particle is in fact left invariant by the Lorentz boost. (A rotation or a translation will change its world line, but not its velocity.) One can also prove that if the original velocity $u$ is less than that of light, then so is the transformed velocity;

$$u < c \quad \Rightarrow \quad u' < c \ . \tag{86}$$

Hence, as far as Lorentz transformations are concerned, it is consistent to say that no particle can move faster than light since such particles are not created by any Lorentz boost.

This is all very different from the Galilei transformation that we considered earlier. A Galilei transformation that transforms a stationary particle to a particle moving with velocity $v$ is given by

$$t' = t \qquad x' = x + vt \ . \tag{87}$$

If the original particle has velocity $u$ the transformed particle has velocity

$$u' = u + v \ . \tag{88}$$

Since Newton's equations are invariant under Galilei transformations it follows that there are solutions describing particles moving at arbitrarily high velocities. We may observe however that the actual value of $c$ that we want to use is very large. For many practical purposes it can be regarded as infinite. But in the limit that $c$ tends to infinity the Lorentz transformation becomes identical to the Galilei transformation, so the familiar invariance of Newtonian physics is recovered in this limit.

What are we to make of this? Since the days of Galileo Galilei it has been agreed that absolute velocities are not observable—only relative velocities can be measured. Since a Galilei transformation changes the absolute velocities another way to say this is to say that it is impossible to distinguish a certain state of the world from the state that results if the original state has been subjected to a Galilei transformation. It was Einstein's suggestion to change this statement into this:

- It is impossible to distinguish a certain state of the world from the state that results if the original state has been subjected to a Lorentz transformation.

It is also assumed that nothing can travel faster than light. Since the Lorentz transformation leaves the velocity of light unchanged the trouble we had with Maxwell's equations goes away—the fact that the velocity of light is constant is quite consistent with the idea that absolute velocities are not observable because the velocity of light with respect to some material particle is always the same, whatever the absolute velocity of the latter. This is so far so good. On the other hand it means that Newton's equations can be correct only in the limit that $c$ can be regarded as infinite. A grand conclusion, but this course will contain a considerable amount of evidence for it. In particular it will be shown that

Maxwell's equations have the property that physical effects always propagate with velocities equal to or less than that of light. We will also show that there is a consistent generalization of Newtonian particle mechanics that has this property. Hence we do not argue the case further here.

Let us think a bit more about our map of spacetime though. First of all we will change our units. Instead of measuring time in seconds and length in meters we will measure time in seconds and length in lightseconds. In these new units the numerical value of $c$ is unity. We insert this value in all formulæ, and hence we have seen the constant $c$ for the last time (almost). Next we define a distance between two arbitrary points $(t, x, y, z)$ and $(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z)$ in spacetime. We use the metric tensor for this purpose. Taking the hint from Pythagoras the distance $\Delta s$ squared will be given by

$$\Delta s^2 = g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \ . \tag{89}$$

This is something that Lagrange did not do. Moreover it looks funny, since the distance squared can be both positive, negative and zero. If we draw a new map of spacetime and fill in all the points whose distance squared from some given point is zero we find that we have drawn a cone with its apex at that point. It is called a light cone. Since we have agreed that no physical influences can propagate faster than light this means that only points on or inside the "forward" light cone can be affected by what happens at the origin. Worldlines of particles that travel slower than light are such that the distance squared between any two points on the worldline is negative. Such lines are called "timelike". If the distance squared between two points is positive we say that the distance between them is spacelike. If the distance squared between two points is negative we say that the timelike distance between them is $\Delta\tau$, where

$$\Delta\tau^2 = -\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \ . \tag{90}$$

It can easily be shown that the distance from the origin to any point within the lightcone is timelike, while the distance to any point without it is spacelike.

Four vectors can also be divided into spacelike, timelike and lightlike. We define the obvious scalar product

$$U \cdot V = U^\alpha g_{\alpha\beta} V^\beta = U^\alpha V_\alpha = U_\alpha g^{\alpha\beta} V_\beta \ . \tag{91}$$

A vector is spacelike if $U \cdot U > 0$, timelike if $U \cdot U < 0$ and lighlike (or "null") if $U \cdot U = 0$. The velocity of a massive particle is always a timelike vector, while the velocity of a photon is lightlike.

One remarkable physical consequence of our postulate must be mentioned. Suppose a clock moves from the origin to the point $(t, x)$, where it crosses another worldline. Suppose it is 12 o'clock at the origin and 4 o'clock at $(t, x)$. Then make a Lorentz transformation so that the point $(t, x)$ is moved to the point $(t', x')$. Since we have agreed that the new configuration must be indistinguishable from the first, it must still be true that the clock strikes four at $(t', x')$. But since $t \neq t'$ this implies that whatever corresponds to the clock time in our formalism, it cannot be the coordinate $t$. In fact the clock time must be a scalar function, unchanged by Lorentz transformations. There is a reasonable candidate, namely the timelike distance from the origin to $(t, x)$. Hence we decide that proper time, as measured by a good clock on a world line, is equal to $\Delta\tau$. This sounds obvious if the clock is stationary, but it has some curious consequences for moving clocks. In Euclidean geometry a straight line is the shortest possible path between two given points, but because of the minus sign in the metric it is easy to see that a straight timelike line is in fact the longest path between two given points—that is to say if the points can be connected by a timelike line at all, and if we take the proper time as a measure of length. Suppose that two twins separate and that one of them travels on a straight worldline and the other on a curved one. Assume also that the twins are robust clocks, in the sense that their aging is not affected by the acceration as such. Finally suppose that eventually they come together again. Then we conclude that at that moment the twin who did not accelerate is older than the one who did.

The conclusion is correct. Since $c$ is so large it requires fantastic precision to see this effect in everyday circumstances, but nevertheless it must be reckoned with in the commercially available GPS system which—at a modest cost—allows you to procure a gadget that can tell you within ten metres or so where on Earth you are, using electromagnetic radiation emitted by 24 satellites that are orbiting the Earth at various speeds.

**Exercises:**

**1.** Prove that

$$V_{\alpha\beta\gamma} = V_{[\alpha\beta\gamma]} \qquad \Leftrightarrow V'_{\alpha\beta\gamma} = V'_{[\alpha\beta\gamma]} \ . \tag{92}$$

**2.** Consider arbitrary transformations of space or spacetime, i.e.

$$x^\alpha \to x'^\alpha = x'^\alpha(x) \ . \tag{93}$$

Define contravariant vectors as objects that transform according to

$$V'^\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta(x) \ . \tag{94}$$

Show that in the special case of Lorentz transformations we recover the transformation of a Lorentz vector. Show how covariant vectors must transform in order to guarantee that $V^\alpha U_\alpha$ transforms like a scalar.

**3.** Investigate whether

$$\partial_\alpha \phi \qquad \text{and} \qquad \partial_\alpha U_\alpha \tag{95}$$

transform like tensors under the general transformations considered in exercise 2.

**4.** Is Kronecker's delta $\delta^\alpha_\beta$ an invariant tensor under the general transformations considered in exercise 2?

**5.** Let $g_{ij}$ be a Kronecker delta and $g_{\alpha\beta}$ a Minkowski metric (i.e. one minus sign). Check that the requirement

$$\Lambda^i{}_k \Lambda^j{}_l g^{kl} = g^{ij} \tag{96}$$

implies that $\Lambda^i{}_j$ is an orthogonal matrix. Then investigate what restrictions you must impose on $\Lambda^\alpha{}_\beta$ to ensure that

$$\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta g^{\gamma\delta} = g^{\alpha\beta} \ . \tag{97}$$

**6.** Draw a map of spacetime in which the $x$-axis and the $t$-axis appear orthogonal. Also draw a lightcone from the origin. Now perform a Lorentz boost. Draw the $x'$-axis and the $t'$-axis on the map. How do they relate to the light cone?

**7.** Consider a scalar field having the form

$$\phi(x) = \frac{1}{r^2} \, , \qquad r^2 = x^2 + y^2 + z^2 \, . \tag{98}$$

Perform a Lorentz boost in the $t - x$-plane, and express the new function $\phi'$ that you obtain in this way as a function of the coordinates $(t, x, y, z)$. What does the new function look like?

**8.** Consider the electromagnetic field from a point charge at rest at the origin,

$$E_i(x) = \frac{1}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad B_i(x) = 0 \, . \tag{99}$$

Perform a Lorentz boost in the $t - x$-plane. Compute the electromagnetic field you obtain, and express it as a function of the coordinates $(t, x, y, z)$.

**9.** Repeat exercise 8, but now for the electromagnetic field

$$E_2 = \cos(t - x) \qquad B_3 = \cos(t - x) \, , \tag{100}$$

all other components vanishing.

All of electrodynamics is contained in the equations

$$\partial_\beta \star F^{\alpha\beta} = 0 \qquad \Leftrightarrow \qquad \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} \tag{101}$$

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha \tag{102}$$

—except that these equations are almost empty unless we specify what the charge and current densities represented by the four vector $J^\alpha$ are. Maxwell's equations by themselves say nothing on this score except one important thing: They imply that

$$0 = \partial_\alpha \partial_\beta F^{\alpha\beta} = 4\pi \partial_\alpha J^\alpha \equiv 4\pi(\partial_t \rho + \partial_i j_i) \ . \tag{103}$$

This is conservation of electric charge. In a complete specification of the theory one also needs a set of dynamical variables describing electrically charged matter. Then $J^\alpha$ is a given functions of these variables, and there will be a separate set of equations that these variables must obey. For consistency they must be such that $\partial \cdot J = 0$ always. Examples of such descriptions are classical point particles, electrically conducting fluids and fields such as the electron field $\Psi_a$ used in quantum electrodynamics. The coupled set of partial differential equations that results is typically almost impossible to solve exactly in physically realistic situations, and it will be necessary to resort to various approximations in order to extract predictions from the equations. In this chapter we introduce the classical point particle and—after coupling it to the electromagnetic field—we solve the equations in the external field approximation, that is we assume that the field is unaffected by the point particle and hence that it is given once and for all. In fact we assume that the electromagnetic field is a vacuum solution of Maxwell's equations—the idea being that the charges whose motion we study are small and do not affect the field much. The complementary approximation is that the distribution of charged matter as described by $J^\alpha(x)$ is fixed, so that the equations to be solved are the ones given above. This is the topic of chapters 4 and 5. Other approximations are frequently used, such as the non-relativistic approximation employed in magnetohydrodynamics (chapter 3), or the perturbation method used in quantum electrodynamics. When such approximations are used it is necessary to derive reliable criteria for their validity, and to check whether these criteria are met in the given physical problem.

To describe a vacuum solution we take a rather picturesque view of the field, and draw it as a set of "field lines" permeating space. Remember that such pictures give information about the direction of the electric and magnetic fields throughout space; the electric field (say) is everywhere tangent to the electric field lines. There will be an electric field line through every point in space where the electric field is non-vanishing, so *a priori* the "density of field lines" is meaningless. Nevertheless in our pictures the field is stronger wherever the field lines crowd together. The reason is that the divergence of the field is zero. Suppose that we define a surface as follows: First take a compact surface that is everywhere orthogonal to the field lines, then extend the surface along the field lines from the boundary of the original surface, and finally put a cap orthogonal to the field lines so that the surface becomes closed. Gauss' law then says that

$$\int_{S_1} \mathbf{dS_1} \cdot \mathbf{E} = \int_{S_2} \mathbf{dS_2} \cdot \mathbf{E} \ . \tag{104}$$

If $< E >$ is the average strength of the field on a surface orthogonal to the field and if $A$ is the area of the surface it follows that

$$< E >_1 A_1 =< E >_2 A_2 \ . \tag{105}$$

But if we draw a finite set of field lines in our picture they will be diluted or concentrated by the change of area in exactly the same way. This is why the field line pictures are so useful.

## 2.1 POINT PARTICLES

We now wish to provide an explicit model for charged matter. Our first choice is the "point particle model", where we consider particles that are completely described by their spacetime coordinates $x^\alpha(\tau)$, where $\tau$ is some parameter along the trajectory of the particle, and which are assumed to carry a charge $\pm e$. Outside the trajectories we assume that Maxwell's vacuum equations hold. It is clear that this will lead to consistency problems, since the Coulomb field

$$E \propto \frac{1}{r^2} \tag{106}$$

diverges in the neighbourhood of a point source. These problems are discussed in chapter 6; for the moment we avoid them by adopting the external field approximation — the electromagnetic field is assumed to be fixed once and for all, independently of the location of the test charges, and the problem will be to compute the trajectories of the point particles in this external field.

How do we choose our equations of motion? We must require that they shall be relativistically invariant and consistent with Maxwell's equations (which means that the electric current $J^\alpha(x)$ must be divergence free). They should also agree with experiments. The easiest way to ensure the first two properties is to derive the equations from the principle of least action. The two requirements will be met automatically if the Lagrangian is a gauge invariant scalar. Apart from this the action is not to be "derived" from first principles, instead we are trying to suggest a reasonable candidate. The most reasonable candidate is usually the simplest one, so it should not be too difficult to do this.

How do we choose the action? We consider a free particle first, and add the interaction to an external electromagnetic field afterwards. As in non-relativistic physics, the degrees of freedom of a particle are its three position coordinates $x_i$. But the question how to handle its time coordinate $t$ is not so immediately answered in the relativistic case. The only quantity that we can form from $x^\alpha$ which is invariant under both Lorentz boosts and translations is

$$d\tau^2 = -dx^\alpha dx_\alpha = dtdt - dx_i dx_i \ . \tag{107}$$

Now we know that the parameter $\tau$ measures the timelike length of the worldline, and that this corresponds to time as measured by a clock carried by the particle. The calibration of this clock is left arbitrary for the time being. Since the Lagrangian of our particle has to be a scalar it is natural to try

$$S = -\text{mass} \cdot \text{length of worldline} = -m \int d\tau = -m \int d\tau \sqrt{-\frac{dx^\alpha}{d\tau}\frac{dx_\alpha}{d\tau}} \ , \tag{108}$$

which looks a bit peculiar but which will turn out to be suitable precisely because of the square root. We can now investigate the effects of a recalibration of the internal clock:

$$\tau' = \tau'(\tau) \ . \tag{109}$$

The point is that

$$S = -m \int d\tau' \frac{d\tau}{d\tau'} \sqrt{-\frac{d\tau'}{d\tau}\frac{dx^\alpha}{d\tau'}\frac{d\tau'}{d\tau}\frac{dx_\alpha}{d\tau'}} = -m \int d\tau' \sqrt{-\frac{dx'^\alpha}{d\tau'}\frac{dx'_\alpha}{d\tau'}} \ . \tag{110}$$

In words, the action is unchanged by recalibrations of the internal time $\tau$. Since we do not know how the internal clock is calibrated, this is a very satisfactory result. If we want to, we can use the coordinate time $t$ as our time parameter:

$$x^0 = t = t(\tau) \ . \tag{111}$$

We just proved that the value of the action is unchanged by this manœuvre. But it will look different, indeed

$$S = -m \int dt \sqrt{-\frac{dx^\alpha}{dt}\frac{dx_\alpha}{dt}} = -m \int dt \sqrt{1 - \dot{x}^2} \ , \tag{112}$$

where the dot denotes differentiation with respect to $t$, as usual. We have now solved the problem of finding a relativistically invariant action which is a function of $x_i$ only. If we insert the velocity of light $c$, and take the limit $c \to \infty$, we recover the action for a non-relativistic particle, as indeed we must if our chosen action is intended to describe physics.

We can now derive a relativistically invariant set of equations of motion by varying our action. Before doing so we will include an extra term that describes the coupling of our test particle to an external electromagnetic field. The simplest way to do this — which will lead to the experimentally correct Lorentz equation of motion — is to use the vector potential for this purpose. We suggest

$$S = -m \int d\tau + e \int d\tau \frac{dx^\alpha}{d\tau} A_\alpha \ . \tag{113}$$

The form of the interaction term is dictated by Lorentz invariance and gauge invariance;

$$\delta A_\alpha = \partial_\alpha \Lambda \qquad \Rightarrow \qquad \delta S = e \int d\tau \frac{dx^\alpha}{d\tau} \partial_\alpha \Lambda = e \int d\tau \frac{d\Lambda}{d\tau} = 0 \ . \tag{114}$$

(We are restricting ourselves to gauge transformations that are zero at the end-points.) The interaction term also has the nice feature that it is invariant under arbitrary reparametrizations of $\tau$, so again we can use the coordinate time $t$ as the time parameter if we want to. It must be kept in mind that the vector potential is a function of both the parameter $t$ and the dynamical variables $x_i$.

We can now derive the equations of motion. There are two ways to do this; the first way is to reparametrize the action so that $t$ is being used as the time parameter, and then vary it with respect to $x_i(t)$. In this way we obtain Lorentz' equation in its original 3 + 1 form:

$$\frac{d}{dt} \left( \frac{m \dot{x}_i}{\sqrt{1 - \dot{x}_i \dot{x}_i}} \right) = e(E_i + \epsilon_{ijk} \dot{x}_j B_k) \ . \tag{115}$$

It would not have been so easy to guess this directly.

It is worth while dwelling a little on the second way to derive the equations of motion, because at first sight it seems that we have a problem: If we vary $x^\alpha(\tau)$ in the action as it stands it would appear that we will obtain four rather than three equations. So how can this be consistent with what we had? The answer is that the reparametrization invariance of the action is a gauge invariance—not the same gauge invariance as the gauge invariance of the Maxwell equations, but a different realization of the same idea.

To see how this works, consider an arbitrary variation of the action (and let us agree to let the dot stand for differentiation with respect to $\tau$):

$$\delta S = - \int d\tau \delta x^\alpha \left( \frac{d}{d\tau} \left( \frac{m \dot{x}_\alpha}{\sqrt{-\dot{x} \cdot \dot{x}}} \right) - e \dot{x}^\beta F_{\alpha\beta} \right) \ . \tag{116}$$

The equations of motion are therefore

$$V_\alpha \equiv \frac{d}{d\tau}\left(\frac{m\dot{x}_\alpha}{\sqrt{-\dot{x}\cdot\dot{x}}}\right) - e\dot{x}^\beta F_{\alpha\beta} = 0 \ . \tag{117}$$

However, this is not four equations. Only three of them are independent since (as a short calculation verifies)

$$\dot{x}\cdot V = 0 \ . \tag{118}$$

This holds whatever functions $x^\alpha(\tau)$ we consider—and therefore the action is automatically invariant under transformations of the special form

$$\delta x^\alpha(\tau) = \xi(\tau)\dot{x}^\alpha(\tau) \ . \tag{119}$$

This special form contains one arbitrary function while a general variation contains four. It follows that only three of the latter can be used to derive non-trivial equations of motion.

The "gauge transformation" leaving the action invariant that we found is exactly the infinitesimal form of an arbitrary reparametrization of the parameter;

$$\tau \to \tau' = \tau - \xi(\tau) \ . \tag{120}$$

Since we think of $x$ as a scalar function of $\tau$ it follows for infinitesimal reparametrizations that

$$x'(\tau') = x(\tau) \qquad \Rightarrow \qquad \delta x(\tau) = x'(\tau) - x(\tau) = \xi(\tau)\dot{x}(\tau) \ . \tag{121}$$

The entire calculation illustrates the important fact that whenever an action has a gauge invariance—that is to say whenever it is invariant under a variation of the dynamical variables that contains an arbitrary function like $\Lambda(x,t)$ or $\xi(\tau)$—then the number of independent equations of motion is smaller than it would appear at first sight. There is another remarkable thing to notice. Let us look at the free particle for simplicity. Imagine transforming from the Lagrangian (108) to the Hamiltonian formulation. The first step is to derive the momenta

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = \frac{m\dot{x}_\alpha}{\sqrt{-\dot{x}^\beta\dot{x}_\beta}} \ . \tag{122}$$

The remarkable thing is that

$$p_\alpha p^\alpha + m^2 = 0 \ . \tag{123}$$

This is an identity. It is clear that we do not have four pairs of freely specifiable positions and momenta. Physically this was expected (since a free particle in 3 dimensional space should have 3 degrees of freedom only). This is a theme that can be developed quite far—in effect we have found a gauge theory where the number of independent equations of motion is fewer than expected, and where there are constraints on the kind of initial conditions that are allowed. This turns out to be true for all gauge theories, including electrodynamics itself as we will see later. (The analogy between eqs. (103) and (118) may be clear already?)

What is perhaps less clear is what I mean by "gauge theory". A precise and general definition is a little hard to give at this stage, but the following is precise and general enough: Let a theory with dynamical variables $q_i$ and evolution parameter $t$ be described by the Lagrangian $L(q, \dot{q})$. Then

- The theory has a symmetry if there exists a transformation such that

$$\delta q_i = \epsilon V_i(q, \dot{q}) \quad \Rightarrow \quad \delta S = 0 \ , \tag{124}$$

  where $\epsilon$ is a small but otherwise arbitary number.

- The theory has a gauge symmetry if there exists a transformation such that

$$\delta q_i = \epsilon(t) V_i(q, \dot{q}) \quad \Rightarrow \quad \delta S = 0 \ , \tag{125}$$

  where $\epsilon(t)$ is a small but otherwise arbitary function.

A gauge theory is a theory that has a gauge symmetry—and yes, there is a huge difference between symmetry and gauge symmetry even though the definitions look similar. We will return to this later on.

## 2.2 MOTION IN EXTERNAL FIELDS

We will now use Lorentz' equation

$$\dot{p}_i = eE_i + e\epsilon_{ijk}\dot{x}_j B_k + F_i \tag{126}$$

(where $F_i$ is any non-electromagnetic force that may be present) to study the motion of an electrically charged particle in an external electromagnetic field, regarded as a fixed function of $x$. The momentum is given by

$$p_i = m\gamma\dot{x}_i \ ; \qquad \gamma = \frac{1}{\sqrt{1 - \dot{x}_i\dot{x}_i}} \ . \tag{127}$$

We will be able to obtain exact solutions only in simple cases. The importance of these solutions nevertheless transcends these simple cases, since they can be used as building blocks to understand more complicated cases by means of perturbation theory.

The first case to be considered is a constant electric field along the $z$-axis. The particle is then subject to constant acceleration, and you probably know already (if not, it is easy to show, especially if you exercise a little ingenuity) that the solution is given by a hyperboloid in Minkowski space,

$$z^2 - t^2 = \frac{1}{\alpha^2} \ , \tag{128}$$

where

$$\alpha = \frac{eE}{m} \tag{129}$$

and we adjusted the initial data suitably. When $t \to \infty$ the velocity tends to $c$.

Motion in a magnetic field is more interesting. It is also easy, since

$$\frac{d}{dt}(m\gamma v_i) = e\epsilon_{ijk}v_jB_k \qquad \Rightarrow \qquad v^2 = \text{constant} \ . \tag{130}$$

Therefore the energy of the particle is conserved. Also $\gamma$ is conserved, so the relativistic problem is no more difficult than the non-relativistic one. If the field is constant and directed along the $z$-axis, the equations reduce to

$$\dot{v}_x = \frac{eB}{m\gamma}v_y \qquad \dot{v}_y = -\frac{eB}{m\gamma}v_x \qquad \dot{v}_z = 0 \ . \tag{131}$$

The solution is immediate:

$$v_x = v_\perp \cos(\omega_B t) \qquad v_y = \mp v_\perp \sin(\omega_B t) \qquad v_z = \text{constant} \ . \tag{132}$$

Here the transverse velocity $v_\perp$ is constant and the cyclotron frequency $\omega_B$ is given by

$$\omega_B \equiv \frac{|e|B}{m\gamma} \ . \tag{133}$$

The motion has two components; the particle is moving with constant speed along a field line while at the same time it is gyrating around it with a frequency given by $\omega_B$. Another integration yields the trajectory, which is a spiral wound around a field line. The radius of the spiral — known as the Larmor radius — is

$$a = \frac{v_\perp}{\omega_B} = \frac{m\gamma v_\perp}{|e|B} \ . \tag{134}$$

Hence the radius increases with $m$ and decreases with $B$. The spiral is a left hand or a right hand screw depending on the sign of the charge. To remember which is which we observe that the motion is such that the magnetic field generated by the particle itself opposes the external field.

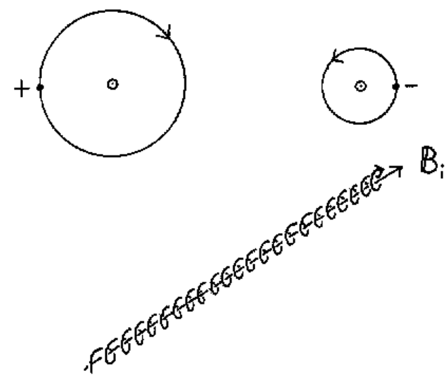


Figure 4: Motion in a constant magnetic field (directed towards you).

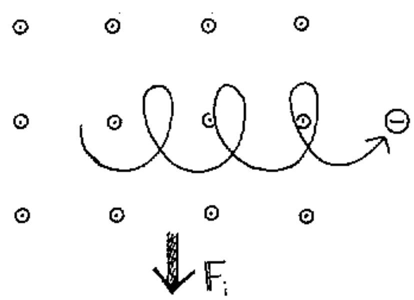


Figure 5: $F \times B$ drift.

This is so far so good. To discuss more complicated cases we adopt the non-relativistic approximation

$$p_i = m\dot{x}_i \ . \tag{135}$$

The first case to be discussed is that of a constant magnetic field and an orthogonal external force $F_i$. Suppose that $B_i$ points along the $z$-axis and that we pull the particle in the $y$-direction. As the particle speeds up the velocity dependent Lorentz force becomes stronger, and bends the trajectory. Since the Larmor radius grows with $v_\perp$ the circle gets deformed so that one segment grows and another shrinks. When we draw a smooth curve composed of such segments we see that on the average it goes off in the $x$-direction, with a speed that we may call the drift velocity. Note that positive and negative charges drift in opposite directions, which means that an electric current will be generated in a plasma subject to both magnetic and gravitational fields.

We want an analytic expression for the drift velocity. We will derive it for the case of crossed electric and magnetic fields (and call it "$E \times B$ drift"). Suppose that a constant magnetic field is directed along the $z$-axis and a constant electric field along the $y$-axis. The sign of the electric force acting on a particle now depends on its charge, which means that ions and electrons are pulled in opposite directions and therefore they will drift in the same direction. We introduce a vector $v_{Di}$ defined by

$$\dot{v}_{Di} = 0 \qquad E_i + \epsilon_{ijk} v_{Dj} B_k = 0 \ . \tag{136}$$

(It will turn out that $v_{Di}$ is precisely the drift velocity.) Then

$$m(\dot{v}_i - \dot{v}_{Di}) = m\dot{v}_i = e(E_i + \epsilon_{ijk} v_j B_k) = e\epsilon_{ijk}(v_j - v_{Dj})B_k \ . \tag{137}$$

Hence $v_i - v_{Di}$ is a solution to the problem of a charged particle in a constant magnetic field only. We have this solution already, and it follows that

$$v_i(t) = v_{Di} + \text{a spiral around a field line} \ . \tag{138}$$

Evidently $v_{Di}$ is the drift velocity, and it only remains to solve for it. If $E \cdot B = 0$ (which we assume) it is easy to check that a solution is

$$v_{Dx} = \frac{E_y}{B_z} \ , \tag{139}$$

all other components zero. This agrees with our qualitative solution. The solution for $v_{Di}$ is not unique since we can add a component parallell to the magnetic field, but without loss of essential generality we can make the solution unique by stipulating that the drift velocity is perpendicular to the field lines—motion along the field lines is taken care of by the spiralling motion that we are superposing. Note that our non-relativistic approximation assumes that $v_D << 1$, therefore it can apply only provided that

$$E << B \ . \tag{140}$$

This will be so in many physically interesting cases, as will be further discussed in the chapter on magnetohydrodynamics.

For later reference, we also give the solution for a magnetic field crossed with an external force:

$$v_{Di} = \frac{1}{eB^2} \epsilon_{ijk} F_j B_k \ . \tag{141}$$

This is as far as we will get with exact solutions. We still have to deal with inhomogeneous magnetic fields, and to do so we will make use of the solutions we have already and try to make some kind of Taylor expansion in $x/L$, where

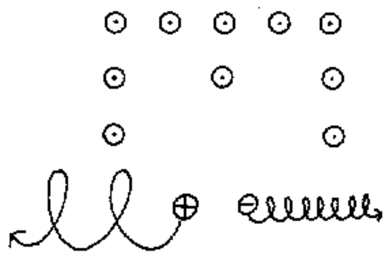

Figure 6: Grad–$B$ drift.

$L$ is the scale length of the inhomogeneity. Our basic assumption is that such an expansion is valid, which means that the inhomogeneities have to be modest.

One case of interest is that of a magnetic field directed along the $z$-axis but increasing in strength in (say) the $y$-direction. This gives rise to an effect called grad-$B$ drift. The point is again that the Larmor radius $a$ varies with $y$ since

$$a \propto \frac{1}{B} . \tag{142}$$

When we apply our qualitative argument this implies that the particle will drift along the $x$-axis, with a direction which depends on the sign of the charge. To get an analytic expression for this drift velocity we regard it as caused by an "effective" force in the $y$-direction due to the fact that as the particle spirals around its field line it meets a stronger magnetic field, and hence a stronger Lorentz force, along a part of its trajectory. The effective force can then be computed as a time average of the Lorentz force $F_y$ along the unperturbed trajectory, using the formula

$$< F_y > \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T F_y(t, y) dt . \tag{143}$$

Since the inhomogeneity is modest the Lorentz force itself can be approximated by

$$F_y = -ev_x B_z(y) \approx -ev_x (B_z(0) + y \partial_y B_z(0)) . \tag{144}$$

When this expression is inserted into the integral along with our previous soluṫiuon $y$ and $v_x$ for the unperturbed motion we find that

$$< F_y > = -e < v_\perp \cos(\omega_B t)(B_z^0 \pm a \cos(\omega_B t) \partial_y B_z^0) > = \mp \frac{aev_\perp}{2} \partial_y B_z^0 . \tag{145}$$

(For a constant field the time averaged Lorentz force vanishes, as it should.) Finally we use this effective force in our previous solution for a particle subjected

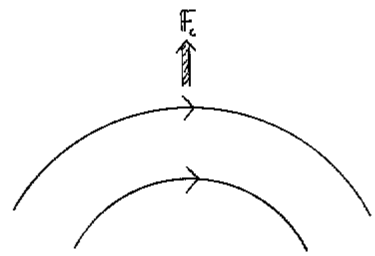

Figure 7: Curvature drift.

to an external force crossed with a magnetic field, and conclude that the grad-$B$ drift is given by

$$v_{Di} = \pm \frac{av_{\perp}}{2B^2} \epsilon_{ijk} B_j \partial_k B \ , \tag{146}$$

where $B$ is the strength of the magnetic field.

Another inhomogeneity occurs if the magnetic field lines are curved. If the area considered is sufficiently small this can always be approximated by field lines that are circles of some radius $r$. In order to be consistent with Ampère's law the strength of the magnetic field must decrease with $r$;

$$B \propto \frac{1}{r} \ . \tag{147}$$

The unperturbed motion is given by spirals around the field lines. There are now two effects. There is a grad-$B$ drift out of the plane in which the field lines lie, but there is also a curvature drift due to the fact that the particles experience a centrifugal force

$$F \propto \frac{v_{||}^2}{r} \ , \tag{148}$$

where $v_{||}$ is the velocity along the field lines. This drift will also be directed out of the plane, and unfortunately the two effects add. The reason why this is unfortunate is that if one tries to confine a plasma by bending magnetic field lines into a torus then the grad-$B$ drift and the curvature drift will conspire to help the plasma escape. And for this reason fusion reactors are difficult to build.

Other ways to confine electrically charged particles can be envisaged. Consider an axisymmetric magnetic field whose strength increases in the $z$-direction:

$$\mathbf{B} = B_r(r,z)\mathbf{e}_r + B_z(r,z)\mathbf{e}_z \ . \tag{149}$$

Gilbert's law must be satisfied. In our coordinates:

$$\partial_i B_i = 0 \qquad \Leftrightarrow \qquad \frac{\partial_r(rB_r)}{r} + \partial_z B_z = 0 \ . \tag{150}$$

In the approximation that

$$\partial_z B_z(r,z) \approx \partial_z B_z(z) = \text{some freely specified function} \tag{151}$$

Maxwell's equation can then be solved by

$$B_r = -\frac{1}{2} r \partial_z B_z \ . \tag{152}$$

So much for Maxwell's equations.

In the situation that we are describing there is a non-vanishing Lorentz force along the $z$-axis:

$$F_z = e(v_r B_\phi - v_\phi B_r) = \frac{1}{2} e r v_\phi \partial_z B_z \ . \tag{153}$$

($B_\phi$ is zero.) Since we are looking for a small perturbation of the gyration around the field lines we approximate

$$v_\phi \approx -v_\perp \qquad r \approx a \qquad \Rightarrow \qquad F_z = -\mu \partial_z B_z \ , \tag{154}$$

where

$$\mu \equiv \frac{m v_\perp^2}{2B} = \frac{e v_\perp a}{2} \ . \tag{155}$$

In fact $\mu$ equals the magnetic moment for a loop around the field line:

$$\mu = \text{current} \cdot \text{area} = e\frac{\omega_B}{2\pi} \cdot \pi a^2 = \frac{m v_\perp^2}{2B} \ . \tag{156}$$

Anyway the conclusion is that the particle is subject to a force directed away from the direction in which the magnetic field is increasing. The crucial question is whether this force is strong enough to cause a particle that moves in this direction to turn around.

This question can be answered elegantly. The argument assumes that the effect of the inhomogeneity in the magnetic field can be regarded as an "adiabatic" perturbation, which means that the particle feels a magnetic field which changes slowly compared to unperturbed gyration. More generally, suppose that a mechanical system admits some constants of the motion, and then suppose that the parameters that define the system (in our case the magnetic field) are changing with time at a rate $\epsilon$. Then we are interested in those constants of the motion that change at a rate $\epsilon^2$; in the limit when $\epsilon$ is very small they can be regarded as constants of the motion of the perturbed system as well, and they are called adiabatic invariants. To be concrete, consider

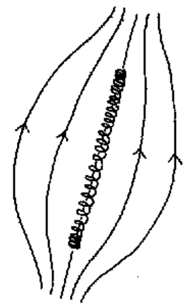

Figure 8: A particle in a magnetic bottle.

$$J = \int pdq \ . \tag{157}$$

Here $q$ is some periodic angle variable, $p$ is its canonically conjugated momentum, and the integral is evaluated over an entire period of the unperturbed motion. Using rather advanced techniques from analytical mechanics it can be shown that $J$ is an adiabatic invariant.

In our problem the momentum $p$ is the angular momentum and $q$ is the angular coordinate $\phi$; then

$$J = \int pdq = \int mv_\perp a d\phi = 2\pi m v_\perp a = 4\pi \frac{m}{e}\mu \ . \tag{158}$$

It follows that $\mu$ is a constant. (A nice result in itself.) On the other hand the kinetic energy is preserved by motion in a magnetic field, so we also have that

$$\text{constant} = \frac{1}{2}mv_{||}^2 + \frac{1}{2}mv_\perp^2 = \frac{1}{2}mv_{||}^2 + \mu B \ , \tag{159}$$

where $v_{||}$ is the velocity along the field lines. Combining these results we find that when the particle moves into a region with increasing $B$ then $v_{||}^2$ must decrease. If $B$ becomes strong enough $v_{||}^2$ goes to zero—the particle turns around! The adiabatic argument evidently breaks down for particles whose velocity is precisely aligned with the field lines (when $\mu = 0$) and by continuity for particles for which the angle between the velocity and the field lines is small. Indeed for such particles the perturbation is not adiabatic. The upshot of all this is that we have shown that an electrically charged particle can be confined in a magnetic "bottle". This is the mechanism that traps electrically charged particles in the Earth's magnetic field, for instance. Note that if we have a plasma of many particles in such a bottle the distribution of velocities is non-isotropic, since particles whose velocities are closely aligned with the field do escape.

There is much more to be said about the motion of charged particles in electromagnetic fields, but the above constitutes a good beginning.

**Exercises:**

**1.** Fix $\tau = t$ directly in the action, and then derive Lorentz' equation for a particle in an electromagnetic field.

**2.** The field of a (hypothetical!) magnetic monopole is

$$B_i = b \frac{x_i}{r^3} \ , \tag{160}$$

where $b$ is the magnetic charge (if any). Discuss the motion of an electrically charged particle in this field. To get the qualitative behaviour right, note that in the gravitational two body problem conservation of angular momentum is used to show that the motion is confined to a plane. Now try to find an analogous conserved quantity for the present problem.

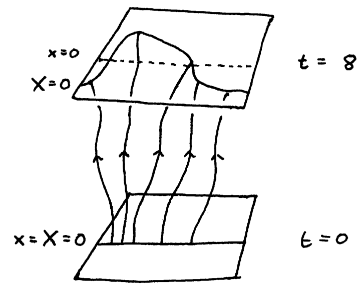


Figure 9: Lagrangian and Eulerian coordinates at two different times

# 3 — MAGNETOHYDRODYNAMICS

For the time being we abandon the point particle description of charged matter, and replace it with a hydromechanical one. Fluids are described by a field theory, and historically this was the first field theory to be studied—the discovery of the electromagnetic field came later. As we did for point particles, we will restrict ourselves to a purely classical description, but now with the advantage that the concept of an electrically conducting fluid does not lead to any obvious absurdities like divergent field strengths. The equations are consistent, even though we will consider the coupled problem in which both the fluid and the electromagnetic field are dynamical. We will use the non-relativistic approximation throughout this chapter.

## 3.1 FLUID DYNAMICS

How do we describe a fluid? You are invited to think of a fluid as a substance made up of little fluid elements that can be characterised by their density $\rho$ and their velocity $v_i$; the values of these quantities vary from fluid element to fluid element. This means that we assume that local thermodynamic equilibrium holds, and that the atomic structure of matter is so far beneath us that it can be completely ignored. The following is intended as a brief sketch of the resulting theory of fluid mechanics.

There are two kinds of coordinates in use, Lagrangian and Eulerian—both are useful and both were introduced by Léonard Euler, who thought of everything. The Lagrangian coordinates $X^i$ label the individual fluid elements. We can carry through this labelling at some particular moment in time $t = 0$ in such a way that the Lagrangian coordinates form a Cartesian system then, but as time goes the fluid flows and the Lagrangian coordinates will become highly curvilinear regarded as spatial coordinates. This is inconvenient, and therefore we introduce the Eulerian coordinates $x^i$ which are labels for the spatial points. They can be taken to be Cartesian by construction; their drawback is that the Eulerian coordinates of a particular fluid element become functions of time $x_i(\tau)$. Here I am being very careful and I denote time in the Lagrangian picture by $\tau$. This is the proper time along the wordline followed by some fluid element. Since the discussion is non–relativistic it is true that $\tau = t$, where $t$ is the usual (Eulerian) coordinate time.

It is assumed that we can transform between the coordinates $(\tau, X_i)$ and $(t, x_i)$, that is to say that

$$x^i = x^i(X, \tau) \qquad \Leftrightarrow \qquad X^i = X^i(x, t) \ . \tag{161}$$

If we knew the explicit form of these functions we would know everything there is to know about the time development of the fluid. Since $\tau = t$ it seems that one kind of label for time would be enough, but according to the rules for how derivatives transform it will be true that

$$\frac{\partial}{\partial \tau} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} + \frac{\partial x_i}{\partial \tau}\frac{\partial}{\partial x_i} = \partial_t + v_i \partial_i \equiv D_t \ , \tag{162}$$

where a new piece of notation was introduced in the last step. This is often (and confusingly) called the Eulerian time derivative.

The velocity of the fluid is defined by

$$v_i = \frac{dx_i}{d\tau} \ . \tag{163}$$

It is important to realize that $\rho$ and $v_i$ are the density and the velocity of the individual fluid elements that make up the fluid. Therefore, if they are expressed as functions of the Eulerian coordinates they are time dependent for two reasons, explicitly and also because the coordinates are time dependent:

$$\frac{d}{d\tau}\rho(x,t) = \frac{d}{d\tau}\rho(x(X,\tau),\tau) = \partial_t \rho(x,t) + v_i \partial_i \rho(x,t) \ . \tag{164}$$

It is the Eulerian time derivative that appears here.

To understand the uses of both Eulerian and Lagrangian coordinates it is instructive to derive the equation of mass conservation in some detail. Consider a volume $V$ that moves with the fluid so that by construction fluid elements cannot enter or leave $V$. This means that the size of $V$ becomes time dependent. Conservation of mass within this comoving volume is something we understand in comoving, that is Lagrangian, coordinates—it becomes the statement that

$$0 = \frac{d}{d\tau}\int_V \rho(x,t) dV(t) = \int_V \left( D_t \rho dV + \rho \frac{d}{d\tau} dV(t) \right) \ . \tag{165}$$

In order to differentiate the volume element we use the formula

$$\epsilon_{ijk} M_{im} M_{jn} M_{kp} = \epsilon_{mnr} \det M \ , \tag{166}$$

which holds for an arbitrary matrix **M**. Now

$$dV(t) \equiv d^3x = J d^3X \ , \tag{167}$$

where the Lagrangian volume element $d^3X$ is time independent and

$$J = \det \frac{\partial x^i}{\partial X^j} \ . \tag{168}$$

Using our formula for the determinant it is not hard to show that

$$\frac{dJ}{d\tau} = J \partial_i v_i \ . \tag{169}$$

Collecting things together we see that

$$\frac{d}{d\tau} \int_V \rho(x,t) dV(t) = \int_V (D_t \rho + \rho \partial_i v_i) dV(t) = 0 \ . \tag{170}$$

Since this equation holds for arbitrary comoving volumes the integrand must be zero, so that we obtain the equation for mass conservation in the form

$$D_t \rho + \rho \partial_i v_i = \partial_t \rho + \partial_i (\rho v_i) = 0 \ . \tag{171}$$

If you found the derivation a little sketchy, consult the exercises.

We now want an equation of motion for the fluid. According to Newton's laws a particular fluid element will accelerate for two reasons—because there may be an external force (say gravity) acting on the entire little volume, and because neighbouring fluid elements exert a force on the surface $S$ of the fluid element. Therefore the equation of motion for the fluid can be written as

$$\rho D_t v_i = F_i + \partial_j T_{ij} \ , \tag{172}$$

where $T_{ij}$ is the stress tensor and $F_i$ the external force. To see what the stress tensor has to do with surface forces we take the integral of this equation over a small fluid element (so small that its density is constant) and using Gauss' theorem we see that

$$M a_i = \int_V F_i dV + \int_S T_{ij} dS_j \ . \tag{173}$$

The stress tensor indeed describes surface forces. An object that associates a vector to another vector (in this case a force vector to a normal vector of a surface) must be a second rank tensor. To be physically reasonable it must be a symmetric tensor (examination shows that if it were not then the fluid element would start to rotate like mad!). And finally it must be given as an explicit

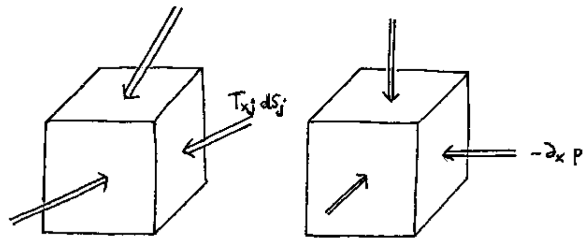


Figure 10: Surface forces, in general and for an isotropic perfect fluid

function of the dynamical variables before the theory can be said to be fully specified. We will use the particularly simple choice

$$T_{ij} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} , \tag{174}$$

where $p$ is the pressure. This means that the surface forces from neighbouring fluid elements act orthogonally to the surface—indeed it is the absence of tangential forces that characterizes a fluid. We have also assumed that the fluid itself does not single out any special direction in space—it is said to be isotropic. Finally we need an equation of state which gives $p$ as a function of the dynamical variables. A useful choice is

$$p = p(\rho) , \tag{175}$$

for some specified function $p$. This equation of state defines a "barotropic" fluid; a common and interesting choice of function is

$$P = C\rho^\gamma \tag{176}$$

where the exponent $\gamma$ is some number. More generally we might wish the fluid to be described by (say) its local temperature, and then a more complicated equation of state would have to be adopted.

We can now write down a complete set of equations for the fluid:

$$\partial_t \rho + \partial_i(\rho v_i) = 0 \tag{177}$$

$$\rho D_t v_i = -\partial_i p + F_i + \eta \triangle v_i . \tag{178}$$

$$p = p(\rho) \ . \tag{179}$$

One extra term was added to the equation of motion; it is a kind of friction term that describes the viscosity of the fluid. The coefficient of viscosity is $\eta$ and $\triangle$ is the Laplace operator. The equation as a whole is known as the Navier-Stokes equation.

To get a small idea about what is in these equations, let us consider them in the approximation that the viscosity is zero and that no external forces are present. Then the equations to be solved are

$$\partial_t \rho + \partial_i(\rho v_i) = 0 \tag{180}$$

$$\rho \partial_t v_i + \rho v_j \partial_j v_i + \frac{dp}{d\rho} \partial_i \rho = 0 \ . \tag{181}$$

A solution is evidently $\rho = \rho_0 =$ constant and $v_i = 0$. If we linearize around this solution we find that

$$\partial_t \rho_1 + \rho_0 \partial_i v_i = 0 \tag{182}$$

$$\rho_0 \partial_t v_i + s^2 \partial_i \rho_1 = 0 \ , \tag{183}$$

where

$$\rho = \rho_0 + \rho_1 \qquad\qquad s^2 \equiv \left( \frac{dp}{d\rho} \right)_0 \ . \tag{184}$$

It follows that

$$\partial_t^2 \rho_1 = s^2 \partial_i \partial_i \rho_1 \ . \tag{185}$$

This is the wave equation and $s$ is the velocity of the propagating wave. It is easy to find a plane wave solution of the equations:

$$\rho_1 = K e^{i(k_i x_i - \omega t)} \qquad\qquad v_i = \frac{K}{\rho_0} s \hat{k}_i e^{i(k_i x_i - \omega t)} \ , \tag{186}$$

where $\omega \equiv s\sqrt{k_i k_i}$ and $\hat{k}_i$ is a unit vector in the direction of $k_i$. The wave travels in the direction of $k_i$. Because the movement of the fluid, given by $v_i$, is in the same direction the wave is said to be longitudinal. These are sound waves, and $s$ is the velocity of sound.

## 3.2 THE MHD EQUATIONS

We have yet to bring in electromagnetism. Our fluid is supposed to be electrically neutral — the charges of ions and the electrons cancel each other — but to conduct electric currents, and the electric and magnetic fields will both determine and be determined by the fluid. Magnetic fields are then far more important than electric fields, because when the charge density is zero electric fields arise only through Faraday's law as a consequence of changes in the magnetic field. About the fluid, one might at first sight suppose that two fluids will be needed, one for the ions and one for the electrons, but in fact there is a range of physically important phenomena for which a one fluid description suffices. The basic reason is that electromagnetic forces are very strong; although a relative velocity between ions and electrons is needed for an electric current to exist, this relative velocity can be very small, in fact utterly ignorable compared to the bulk motion of the fluid.

Let us see what the equations for magnetohydrodynamics are, in the relevant approximation. As far as the (electrically neutral) fluid is concerned, all that we will do is to set

$$F_i = \epsilon_{ijk} j_j B_k \tag{187}$$

in the equation of motion given above. The current density $j_i$ is caused solely by the motion of the electrons relative to the ions. We are going to treat the fluid as isotropic, using eq. (174). Actually an electrically conducting fluid in a magnetic field is not necessarily isotropic — we argued previously that a plasma in a magnetic bottle has an anisotropic velocity distribution — but we will ignore this point.

The equations for the magnetic and electric fields require more thought. Our approximations will be that $v = v/c$ is small, that changes over time are small compared to changes over space (in our units time is measured in seconds and space in lightseconds, so this is the non-relativistic approximation again) and that the magnetic field is more important than the electric field. To be precise about it, we will assume that

$$o(E) = o(vB) \qquad o(\partial_t) = o(v\partial_x) \ . \tag{188}$$

In this approximation Maxwell's equations can be written

$$\partial_i B_i = 0 \qquad \text{Gilbert's law} \tag{189}$$

$$\epsilon_{ijk}\partial_j E_k + \partial_t B_i = 0 \qquad \text{Faraday's law} \tag{190}$$

$$\epsilon_{ijk}\partial_j B_k = 4\pi j_i \qquad \text{Ampère's law .} \tag{191}$$

We dropped Gauss' law since the charge density is zero anyway. We see immediately that we can use Ampère's law to define the current in terms of $B_i$. The result can be inserted in the equation of motion for the fluid. We would like to eliminate the electric field as well, and use only $\rho$, $v_i$ and $B_i$ as our variables. The way to do this is to use Ohm's law to express the electric field in terms of the current (and hence in terms of $B_i$).

Now Ohm's law has a chance to be true only in the rest frame of the fluid or, which is practically the same thing, in the rest frame of the ions where the equation of motion for the electrons is

$$m_e \dot{v}_i'^e = -e(E_i' + \epsilon_{ijk} v_j'^e B_k') - \gamma v_i'^e + f_i \ . \tag{192}$$

The friction term is caused by collisions and $f_i$ denotes inertial forces. We assume that the collision frequency is high enough so that $v_i'^e$ quickly reaches a constant value, in which case the left hand side is zero and $f_i$ goes away. We will also ignore the term involving the magnetic field, which can be justified either if the collision frequency is much larger than the cyclotron frequency or if this term vanishes when we take spatial averages (that is to say, there may be a problem here if the magnetic field has strong spatial gradients). Under these assumptions Ohm's law follows:

$$j_i' = \sigma E_i' \ . \tag{193}$$

Here $\sigma$ is the electrical conductivity of the fluid (and for simplicity we assume it to be constant).

Now we must transform Ohm's law from the rest frame of the fluid to the frame that we will actually use to describe things. In our approximation the Lorentz transformation becomes

$$B_i' = B_i \qquad\qquad E_i' = E_i + \epsilon_{ijk} v_j B_k \ . \tag{194}$$

The current is caused by the relative motion of positive and negative charges, and relative velocities are invariant under such changes of reference frames, so we can supplement these equations with

$$j_i' = j_i \ . \tag{195}$$

Hence both the magnetic field and the current are invariant under boosts in this approximation. The electric field on the other hand depends on the frame of reference. We can now translate Ohm's law to the frame we are using; if we also use Ampere's law to eliminate $j_i$ we get

$$E_i = \frac{1}{4\pi\sigma} \epsilon_{ijk} \partial_j B_k - \epsilon_{ijk} v_j B_k \ . \tag{196}$$

The evolution equation for $B_i$ can now be obtained by inserting this expression in Faraday's law.

We are ready to state a complete set of equations for magnetohydrodynamics:

$$\partial_t \rho + \partial_i(\rho v_i) = 0 \tag{197}$$

$$p = C\rho^\gamma \tag{198}$$

$$\rho D_t v_i = -\partial_i p - \frac{1}{8\pi}\partial_i B^2 + \frac{1}{4\pi} B_j \partial_j B_i + \eta \triangle v_i \ . \tag{199}$$

$$\partial_t B_i = -\epsilon_{ijk}\partial_j E_k(\sigma, v, B) \ . \tag{200}$$

These equations must be supplemented with the equation $\partial_i B_i = 0$ as an initial condition.

Inspection shows that the full set of equations for magnetohydrodynamics is much more complicated than the Navier-Stokes equation, which in itself is sufficiently non-linear so that its general solution is quite out of reach. Therefore we have to aim for qualitative understanding. We begin with the equation for the magnetic field, which I will write out (for once) in cross product notation (because I want to keep the repeated cross products, and then the index notation is clumsy). If we make use of Gilbert's law and assume that $\sigma$ is constant in space we get

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{v}) = \frac{1}{4\pi\sigma}\triangle \mathbf{B} \ . \tag{201}$$

If we ignore the second term on the left hand side, this is just the diffusion equation

$$\partial_t \mathbf{B} = \chi \triangle \mathbf{B} \ . \tag{202}$$

The effect is therefore that a concentration of magnetic field lines will diffuse through the fluid, and disappear in the characteristic diffusion time

$$t_D \sim \frac{L^2}{\chi} \sim \sigma L^2 \ , \tag{203}$$

where $L$ is a typical length in the problem. Diffusion can be ignored if the time scale of the processes that we are interested in is smaller than $t_D$, that is to say if either the conductivity or the length scale is very large. The latter is often the case in astrophysical applications. Let us therefore consider the equation

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{v}) = 0 \ . \tag{204}$$

This equation has an interesting interpretation (called Kelvin's circulation theorem in honour of a famous friend of Maxwell's). We define the magnetic flux through a surface that moves with the fluid, and take its time derivative:

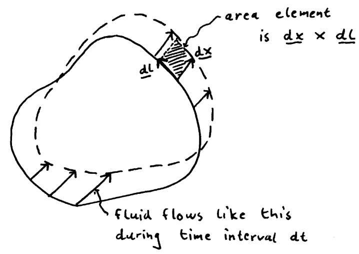


Figure 11: Change of an area enclosed by a loop

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = \int_S \partial_t\mathbf{B}\cdot d\mathbf{S} + \int_S \mathbf{B}\cdot(\text{rate of change of area}) \ . \tag{205}$$

To figure out how the area element changes with time, consult fig. 11. When we integrate the change of area elements around the loop and apply Stokes' theorem (named for a teacher of Maxwell's) the second integral becomes

$$\int_{\partial S}\mathbf{B}\cdot(\mathbf{v}\times d\mathbf{l}) = \int_{\partial S} d\mathbf{l}\cdot(\mathbf{B}\times\mathbf{v}) = \int_S \nabla\times(\mathbf{B}\times\mathbf{v})\cdot d\mathbf{S} \ . \tag{206}$$

Putting things together and using our equation for $B_i$ we get

$$\frac{d\Phi}{dt} = \int_S(\partial_t\mathbf{B} + \nabla\times(\mathbf{B}\times\mathbf{v}))\cdot d\mathbf{S} = 0 \ . \tag{207}$$

The conclusion is that when the diffusion of the field lines can be ignored then the magnetic flux through a loop moving with the fluid remains constant. Another way to say this is that the magnetic field lines are frozen into the fluid and are carried along with it.

What can we say about the equation of motion for the fluid itself? If its viscosity can be ignored it is enough to understand

$$\rho D_t v_i = -\partial_i(p + \frac{1}{8\pi}B^2) + \frac{1}{4\pi}B_j\partial_j B_i \ . \tag{208}$$

There are two contributions from the magnetic field. One works just like an addition to the pressure. The second term has a picturesque interpretation; it tells us that the field lines carry tension and resist bending. To see this, we observe that if the field lines are straight and the magnetic field strength constant then there is no variation along them, and hence our term vanishes. To see it in more detail, remember that the vector $B_i$ is a tangent vector of its field line. Suppose that the field line is given in parametric form as

$$x_i(\sigma) \ : \qquad B_i = B\dot{x}_i(\sigma) \ , \tag{209}$$

where the dot stands for differentiation with respect to the parameter $\sigma$. Here we have chosen a parameter along the curve such that

$$\dot{x}_i\dot{x}_i = 1 \ . \tag{210}$$

(This can always be arranged.) In this way we get a clean separation between the geometry of the field line, considered as a curve, and the strength of the magnetic field. A small calculation follows:

$$B_j\partial_j B_i = B\frac{dx_i}{d\sigma}\frac{\partial}{\partial x_j}(B\dot{x}_i) = B\frac{d}{d\sigma}(B\dot{x}_i) = B^2\ddot{x}_i + B\dot{B}\dot{x}_i \ . \tag{211}$$

If the field line is bent, the vector $\ddot{x}_i$ is directed inwards—in fact this part of the expression is analogous to the "centripetal force" in mechanics. Hence the equation of motion for the fluid says that the fluid is drawn inwards, and since the magnetic field lines are frozen into the fluid the field line will follow. There is also a component of the force acting along the field line. Indeed the field line behaves like a tensionful string, that resists stretching and bending.

To see how magnetic pressure and tension work we first take up a simple problem which arises when one tries to confine a plasma, as in fusion research. We drive a current in the direction of (say) the $z$-axis, and observe that Ampère's law implies that a magnetic field in the $\phi$ direction will result. Can the pressure from this magnetic field confine the plasma in a cylinder? To answer this question we look for a steady state solution in which the velocity of the fluid is zero. The only non-vanishing component of $B_i$ is $B_\phi = B(r)$. The precise form of this function depends on the radial dependence of the current $I(r)$, and we leave it open. A steady state solution for the fluid will exist if we can arrange matters so that the $r$-component of the equation

$$0 = -\partial_i(p + \frac{B^2}{8\pi}) + \frac{1}{4\pi}B_j\partial_j B_i \tag{212}$$

holds. Working it out (or remembering the analogy to the centripetal force) we find that

$$0 = -\partial_r(p + \frac{B^2}{8\pi}) - \frac{B^2}{4\pi r} \ . \tag{213}$$

The second term is directed inwards, as promised. It follows that the hydrodynamical pressure of the fluid in a steady state is given in terms of $B(r)$ — and hence in terms of the current — by

$$\frac{dp}{dr} = -\frac{1}{8\pi r^2}\frac{d}{dr}(r^2B^2) \ . \tag{214}$$

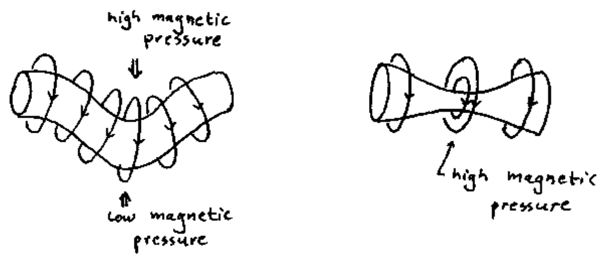


Figure 12: The kink and neck instabilities

To achieve thermonuclear fusion plasmas of very high density and temperature, hence high pressure, are needed. Our equation shows that this requires very strong magnetic fields, hence high currents. Actually the situation is worse than this, because our solution is unstable. If the plasma cylinder develops a small kink, the magnetic pressure will cause the kink to grow. Moreover a "neck" along the cylinder will be "strangled" by the magnetic field. But if we thread a magnetic field through the cylinder the kink instability is stabilized by magnetic tension and the neck instability by magnetic pressure. Indeed fusion researchers believe that a fusion reactor can be built in 20 years time.

## 3.3 ALFVÉN WAVES

It was observed by Hannes Alfvén that magnetohydrodynamics gives rise to more than just sound waves. Indeed if there is tension in the field lines it should be possible to pluck them and make them vibrate like the strings of a violin. To verify that this is so we assume that viscosity and magnetic diffusion can be ignored, and then we linearize the equations of MHD around the solution $\rho = \rho_0 = \text{constant}$, $B_i = B_i^0 = \text{constant}$, $v_i = 0$. We define

$$\rho = \rho_0 + \rho_1 \qquad\qquad B_i = B_i^0 + B_i^1 \ . \tag{215}$$

The linearized equations are

$$\partial_t \rho_1 + \rho_0 \partial_i v_i = 0 \tag{216}$$

$$\rho_0 \partial_t v_i + s^2 \partial_i \rho_1 + \frac{1}{4\pi} B_j^0 (\partial_i B_j^1 - \partial_j B_i^1) = 0 \tag{217}$$

$$\partial_t B_i^1 + B_i^0 \partial_j v_j - B_j^0 \partial_j v_i = 0 \ , \tag{218}$$

where $s^2$ was defined in eq. (184).

The next step is to have a look at these equations. It is clear that we can obtain an equation that depends on $v_i$ only. To be precise about it, it is

$$\partial_t^2 v_i - (s^2 + v_A^2)\partial_i \partial_j v_j + v_j^A v_k^A (\partial_i \partial_k v_j - \partial_j \partial_k v_i) + v_i^A v_j^A \partial_j \partial_k v_k = 0 \ , \tag{219}$$

where we made the convenient definition

$$v_i^A \equiv \frac{B_i^0}{\sqrt{4\pi\rho_0}} \ . \tag{220}$$

The dimension of this object is that of velocity, and the superscript is in honour of Alfvén. There is no striking simplicity about the equation though. To make progress we see if there is a plane wave solution, so we try the Ansatz

$$v_i(x,t) = u_i e^{i(k_i x_i - \omega t)} \ . \tag{221}$$

(Here $u_i$ is some constant vector, and the wave propagates in the direction of $k_i$.) Our equation becomes

$$-\omega^2 u_i + (s^2 + v_A^2) k \cdot u k_i - v^A \cdot k (v^A \cdot u k_i - v^A \cdot k u_i + k \cdot u v_i^A) = 0 \ . \tag{222}$$

It is more illuminating to write it as

$$\left( (s^2 + v_A^2) k_i k_j - v^A \cdot k (k_i v_j^A + v_i^A k_j) + (v^A \cdot k)^2 \delta_{ij} \right) u_j = \omega^2 u_i \ . \tag{223}$$

This is illuminating because it can be written as

$$M_{ij} u_j = \omega^2 u_i \ , \tag{224}$$

where $M_{ij}$ is a known matrix, because we are going to make a choice for the direction $k_i$. So this is an Eigenvalue problem where the unknowns are the vector $u_i$ and the Eigenvalue $\omega^2$.

We investigate two special choices for the propagation direction $k_i$. First we assume that

$$v^A \cdot k = 0 \ . \tag{225}$$

This corresponds to a wave that travels perpendicularly to the field lines. It is a longitudinal wave ($u_i \propto k_i$) with the (phase) velocity

$$\sqrt{s^2 + v_A^2} \ . \tag{226}$$

This is unsurprising: It just says that the speed of sound is determined by the combined effect of the hydrostatic and magnetic pressures. This wave is called magnetosonic.

A more interesting case is a wave that travels in the direction of the field, that is to say

$$v_i^A = v^A \hat{k}_i \ . \tag{227}$$

(The "hat" always denotes a vector normalized to a unit vector.) Then our equation reduces to

$$(k^2 v_A^2 - \omega^2) u_i + (s^2 - v_A^2) k_i k \cdot u = 0 \ . \tag{228}$$

At this point the second case splits into two subcases. We can assume that the wave is longitudinal, i.e. that

$$u_i = u \hat{k}_i \ . \tag{229}$$

The equation becomes

$$(k^2 s^2 - \omega^2) u_i = 0 \ . \tag{230}$$

The wave propagates (along the field lines) at the speed $s$. Or else we may assume that the wave is transverse,

$$k \cdot u = 0 \ . \tag{231}$$

Then the equation becomes

$$(k^2 v_A^2 - \omega^2) u_i = 0 \ . \tag{232}$$

The wave propagates (along the field lines) at the Alfvén velocity $v^A$. Let us study this transverse Alfvén wave in a little more detail, to make sure that it really corresponds to vibrating field lines. This means that we should solve the equation for $B_i^1$ as well. This is easy to do. The solution is

$$B_i^1 = -\frac{B^0 \cdot k}{\omega} v_i \tag{233}$$

Just in case you forgot, $B_i^0 = B^0 \hat{k}_i$, and $v_i$ is orthogonal to $\hat{k}_i$, so we can conclude that the field lines really do vibrate in the transverse direction, like strings. You should not allow the complicated algebra to obscure the way it works. It is quite simple really. The over all conclusion is that the spectrum of oscillations in a plasma is much more interesting than it is in an ordinary fluid. Taking viscosity and magnetic diffusion (which we ignored) into account will complicate the analysis still further, but physically it is evident that this will cause damping of the waves.

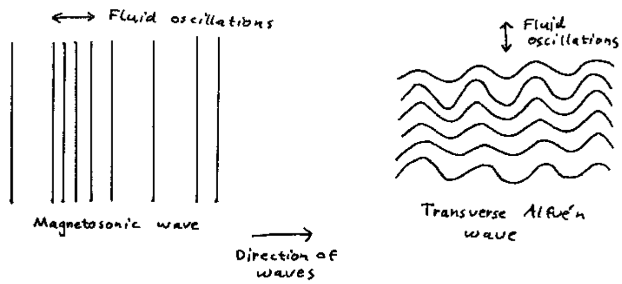


Figure 13: Two kinds of MHD waves

To wind up this story, we should say that the fluid description that we have studied has its limitations. We can try to characterize a real plasma through its Debye length, which is the distance at which electric charges are shielded, and its plasma oscillations, which consist of electric charges oscillating around their equilibrium positions. For the fluid description to be valid it is necessary that the phenomena that we wish to describe take place on longer length and time scales than the Debye length and the period of the plasma oscillations. Also the interesting scales have to be larger than the scales that characterize the gyration of the charges around the field lines. If any of these requirements fails one has to fall back on kinetic theory, the Boltzmann equation, and the like. But this is not for us, now.

In conclusion, the set of equations for magnetohydrodynamics that we derived is highly non-linear and there is no way in which their general solution can be found. On the other hand it was easy to extract non-trivial insights about the behaviour of ionized matter from these equations. We found that the magnetic field lines are frozen into the fluid, although they also have a tendency to diffuse away. A concentration of field lines exerts a pressure on the fluid, and moreover the field lines carry tension and can vibrate like strings.

**Exercises:**

**1.** Perform the derivation of the differential equation that describes conservation of mass in full detail.

**2.** In our equations for MHD we neglected the relative velocity between the electrons and the ions. In the outer layer of the Sun the electron density is about $n_e = 10^{29}$ per cubic meter, the magnetic field is about $B = 10^3$ Gauss, and a typical length scale is $L = 10^8$ meters. Make an order of magnitude estimate of the relative velocity $v_{rel}$ to see whether it can be neglected compared to the bulk velocity.

**3.** Consider two magnetic field lines, one that goes around in a circle and has constant magnetic field there, and one that goes along the $x$-axis with a magnetic field strength that varies in magnitude in the direction of the axis. Compute the vector field $B_j \partial_j B_i$ and sketch what it looks like. Under the assumption that the field lines are frozen into a fluid that evolves according to eq. (208), verify that the field lines behave similarly to violin strings.

# 4 — MAXWELL'S EQUATIONS

We change subject again. We go back to Maxwell's equations in their full glory. Again we make an approximation though namely the opposite approximation compared to chapter 2: We assume that the (conserved) current $J^\alpha$ is given once and for all, and investigate the dynamics of the electromagnetic field coupled to this fixed current. The key player is really the wave equation—now viewed as the equation that determines the propagation of electromagnetic waves, not sound waves, so their speed is always equal to $c$.

## 4.1 SOME REMARKS ON THE WAVE EQUATION

A quick manipulation of Maxwell's equations *in vacuo* (when $J^\alpha = 0$) shows that

$$\partial_\gamma \partial^\gamma F_{\alpha\beta} = -\partial^\gamma (\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha}) = 0 \ . \tag{234}$$

When there are no charges around all the components of the field strength have to satisfy this equation. Before we try to understand Maxwell's equations it seems advisable to understand a scalar field that obeys

$$\Box \phi = 0 \tag{235}$$

where we have defined the d'Alembert operator

$$\Box \equiv \partial_\alpha \partial^\alpha \equiv \partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2 \ . \tag{236}$$

The equation is known as the wave equation. In view of the special theory of relativity one of the things that we want to convince ourselves of is that nothing goes faster than light in this model.

First I want to stress that the wave equation can be regarded as an ordinary dynamical system—with the complication that there is an infinite number of "degrees of freedom". Those of you who have some familiarity with analytical mechanics will enjoy looking at the action integral

$$S[\phi] = -\frac{1}{2}\int d^4x (\partial_\alpha \phi \partial^\alpha \phi + m^2\phi^2) = \frac{1}{2}\int d^4x (\partial_t \phi \partial_t \phi - \partial_i \phi \partial_i \phi - m^2 \phi^2) \ , \tag{237}$$

where $m$ is a real number called the "mass" of the field. The field equation follows when we apply the Principle of Least Action. We subject the dynamical variable $\phi(x)$ to arbitrary small variations and require that the variation of the action should vanish. If we allow ourselves to drop some total derivatives this is a very simple calculation:

$$0 = \delta S = \int d^4x \delta\phi(\partial_\alpha \partial^\alpha \phi - m^2\phi) \quad \Rightarrow \quad (\Box - m^2)\phi = 0 \ . \tag{238}$$

This is the Klein-Gordon equation; we call it the wave equation if the mass vanishes.

I assume that you know the Principle of Least Action, but perhaps you need to be convinced that it really applies to field theories. Recall (from analytical mechanics) how the variation of an action that depends on a large but finite number of degrees of freedom is carried out:

$$\delta \left( \frac{1}{2} \int dt \sum_{n=1}^{N} (\dot{\phi}_n \dot{\phi}_n - \omega^2 \phi_n \phi_n) \right) = - \int dt \sum_{n=1}^{N} \delta\phi_n (\ddot{\phi}_n + \omega^2 \phi_n) \ . \tag{239}$$

It is clear that a field theory can be loosely described as going to the limit where $N$ is infinite;

$$\phi_n(t) \ \rightarrow \ \phi(t,x) \ , \qquad \sum_n \ \rightarrow \ \int d^3x \ . \tag{240}$$

(To get the spatial derivative terms in the limit we would have to introduce some kind of "nearest neighbour interaction" in our model.) In a field theory we are dealing with an infinite set of degrees of freedom—in the present case there is one per point in space. The space coordinates play the role of the index $n$. At a formal level it is a quite straightforward generalization. Actually there are some subtleties; as usual in analytical mechanics we ignored a total time derivative in the calculation, but ignoring a total spatial derivative—we did that also—requires some justification. We will not go into this here, since the action principle plays a peripheral role in the course.

Another way to convince ourselves that the space coordinates should be regarded as a kind of continuous "indices" is as follows: We can rewrite the wave equation in the form

$$\ddot{\phi}(\mathbf{x},t) = \triangle\phi(\mathbf{x},t) \ . \tag{241}$$

This is not too different from the Newtonian equations

$$\ddot{x}_i(t) = F_i(x(t)) \ . \tag{242}$$

In the latter case a unique solution is obtained by specifying the values of $x_i$ and $\dot{x}_i$ for all values of $i$ at some particular time $t_0$; the equation then allows us to solve for all the higher order time derivatives and to construct the solution by power series. We can imagine that the same procedure can be carried out for the wave equation, provided that we specify $\phi$ and $\dot{\phi}$ for all values of $\mathbf{x}$ at some particular time $t_0$. In fact—although we will not proceed by power series—we will see that this intuition is correct.

In addition to the wave equation we are also interested in the related inhomogeneous equation that includes a fixed function $\rho(x)$ as a "source",

$$\Box\phi = \rho \ . \tag{243}$$

We will come to it soon.

The wave equation itself describes a particularly simple kind of waves. We have set an arbitrary constant $c = 1$ in the definition of $\Box$ and will therefore get waves propagating "at the speed of light", while other physicists might prefer waves propagating at the speed of sound. But what do we mean by a "wave"? An attempt at a general definition might be a disturbance that propagates into an undisturbed region in such a way that there is a discontinuity in the field—or perhaps only in its first or second derivatives—at the boundary between the disturbed and the undisturbed regions. In this sense the wave equation does describe waves, as we will see.

Apart from the minus sign the d'Alembert operator looks like a four dimensional version of the Laplace operator, but the minus sign is absolutely crucial. The Klein-Gordon equation is the archetype of a hyperbolic differential equation, while the Laplace operator is the archetype of an elliptic differential operator. Precisely, a differential operator

$$g^{\alpha\beta}\partial_\alpha\partial_\beta \tag{244}$$

is called elliptic if all of the eigenvalues of $g^{\alpha\beta}$ have the same sign, it is called hyperbolic if one eigenvalue has a different sign, and the remaining cases are not so important. (If we make the definition slightly more general so that terms linear in the derivatives are allowed as well then it becomes interesting to consider the case where one eigenvalue is zero—these are the parabolic equations, and the diffusion equation is an important example.) To illustrate the difference between the elliptic and the hyperbolic case, let us see if there can be discontinuous solutions of the equation

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = 0 \ . \tag{245}$$

We will be quite drastic about it and look for discontinuities in the field itself (rather than in its first or second derivatives). Thus we make the Ansatz

$$\phi(x) = \Theta(f(x))\varphi(x) \ , \tag{246}$$

where $\Theta$ is Heaviside's step function, whose derivative is Dirac's delta function. According to our Ansatz the equation

$$f(x) = 0 \tag{247}$$

defines a hypersurface in spacetime where the solution is discontinuous. To see whether there are such solutions we insert the Ansatz in the equation and find that

$$\begin{aligned} g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = \delta'(f)\varphi g^{\alpha\beta}\partial_\alpha f\partial_\beta f + \\ +\delta(f)g^{\alpha\beta}(\varphi\partial_\alpha\partial_\beta f + 2\partial_\alpha f\partial_\beta\varphi) + \Theta(f)g^{\alpha\beta}\partial_\alpha\partial_\beta\varphi) = 0 \ . \end{aligned} \tag{248}$$

There are three sets of terms here, and they have to vanish separately if the equation is to hold. The terms that multiply the step function vanish only if $\varphi(x)$ obeys the wave equation, so we see that this equation is obeyed on both sides of the putative discontinuity. The terms that multiply the delta function give a relation between the discontinuity and the form of the hypersurface. The terms that multiply the derivative of the delta function give a condition on the hypersurface itself. They vanish if and only if

$$g^{\alpha\beta}k_\alpha k_\beta = 0 \ , \tag{249}$$

where

$$k_\alpha \equiv \partial_\alpha f \tag{250}$$

is the normal vector of our hypersurface. But if we diagonalize $g^{\alpha\beta}$ we see that this can never happen if the equation is elliptic (so that all the eigenvalues are positive). In the hyperbolic case it is perfectly possible, though; surfaces having this property do exist; if the matrix $g^{\alpha\beta}$ is the metric in Minkowski spacetime they are called lightlike surfaces. An example (regular everywhere except at the origin) is the light cone

$$t^2 - x^2 - y^2 - z^2 = 0 \ . \tag{251}$$

Naturally lightlike surfaces can have a considerably more general form.

The conclusion is that solutions of hyperbolic differential equations may be discontinuous, but only across quite special hypersurfaces (called characteristic surfaces). These are wave fronts moving with some characteristic speed—in our case with the speed of light $c = 1$. Elliptic differential equation do not admit such solutions. Another important difference between elliptic and hyperbolic equations emerges when we try to write down their general solution. A general solution of a differential equation always contains undetermined constants; we have already argued that for the wave equation we will need two numbers for each spatial point (since we can think of the spatial coordinates as indices labelling the degrees of freedom). It is time to verify this expectation and to

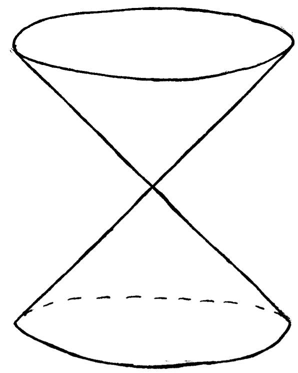

Figure 14: The light cone of the origin

make it precise; the argument that we will give actually fails for the Laplace equation.

## 4.2 GREEN FUNCTIONS

Let us shift our attention to the inhomogeneous equation that includes a source term. We will try to formulate conditions under which the solution is unique. Roughly speaking we want to make sense of the implication that

$$\Box\phi = \rho \qquad \Leftrightarrow \qquad \phi = \frac{1}{\Box}\rho \ . \tag{252}$$

The problem is somehow to "invert" a differential operator. The "inverse" is also known as a Green function. It may be helpful to make an analogy to the inverse of an ordinary matrix equation:

$$\sum_j M_{ij}V_j = W_i \qquad \Leftrightarrow \qquad V_i = \sum_j M^{-1}_{ij}W_j \ . \tag{253}$$

There are two features of this that are of interest to us: An inverse will exist at all only if certain conditions are obeyed (namely that the matrix has no zero eigenvalues) and the calculation is very much simplified if the matrix is given in diagonal form. What we want to do can be regarded as the problem of inverting an infinite dimensional matrix, with rows and columns labelled by $x$ and $x'$. We perform the replacements

$$\Box \to \Box_x\delta(x,x') \qquad \frac{1}{\Box} \to D(x,x') \ . \tag{254}$$

"Matrix multiplication" now involves an integral rather than a sum, as in

$$\Box\frac{1}{\Box} = 1 \quad \rightarrow \quad \int d^4x'' \ \Box_x\delta(x,x'')D(x'',x') = \delta(x,x') \ . \tag{255}$$

The "matrix" equation that we consider is

$$\int d^4x' \ \Box_x\delta(x,x')\phi(x') = \rho(x) \quad \Leftrightarrow \quad \phi(x) = \int d^4x' \ D(x,x')\rho(x') \ . \tag{256}$$

In this sense $D(x,x')$ is the sought for inverse of $\Box$. In a little less detail, the equation that defines the inverse is

$$\Box D(x,x') = \delta(x,x') \ . \tag{257}$$

The inverse is a function—or a "distribution"—that depends on two spacetime points. It is called the Green function.

It is clear that the existence of the Green function can not be a quite trivial matter. If we pursue the analogy to the finite dimensional case we see that the d'Alembertian "matrix" does have zero eigenvalues, since it is easy to find solutions to the homogeneous equation

$$\Box\phi = 0 \ . \tag{258}$$

All is not lost, however; this only means that we will have to add some extra conditions to our problem which are such that the homogeneous solutions are excluded. It should come as no surprise that the inversion of an infinite dimensional "matrix" requires extra care. There will exist an inverse, but it will not be unique, rather it will depend on the extra conditions that we have to add in order to make it well defined. This comment applies to both the Laplace and the d'Alembert operators, although the details differ somewhat—in the former case the extra conditions are the boundary conditions that are needed to specify a unique solution of the Laplace equation, and in the latter it will turn out to be initial conditions. In the elliptic case the boundary surrounds the region in which the solution is sought, but in the hyperbolic case it does not.

Now we turn to the actual calculation of Green functions for the d'Alembertian. We will refer to $x$ as the "observation point" and to $x'$ as the "source point", for reasons which will emerge at the end. Taking the hint from the matrix analogy, we begin by "diagonalizing" the operator. This is done by means of a Fourier transformation;

$$\tilde{\phi}(p) = \int d^4x \ e^{ip\cdot x}\phi(x) \qquad \phi(x) = \int \frac{d^4p}{(2\pi)^4} \ e^{-ip\cdot x}\tilde{\phi}(p) \ . \tag{259}$$

Then the solution of the differential equation becomes trivial:

$$-p^2\tilde{\phi}(p) = \tilde{\rho}(p) \qquad \Leftrightarrow \qquad \tilde{\phi}(p) = -\frac{\tilde{\rho}(p)}{p^2} \ . \tag{260}$$

The difficulty now resides in the transformation back to $x$-space:

$$\phi(x) = -\int d^4x' \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{p^2}\rho(x') \ . \tag{261}$$

If we compare this expression to the formal expression involving the Green function that we are trying to reproduce, we can conclude that

$$D(x,x') = -\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{(\omega - p_0)(\omega + p_0)} \ , \tag{262}$$

where we made use of the definition

$$\omega \equiv \sqrt{p_i p_i} \qquad \Rightarrow \qquad p^2 \equiv p_i p_i - p_0 p_0 = (\omega + p_0)(\omega - p_0) \ . \tag{263}$$

It remains to do the momentum integrals explicitly. The way to proceed is to perform the integral over $p_0$ first, using the calculus of residues. (If you do not know the calculus of residues, too bad, since you will miss the point here — however, this is the only place where I will use complex analysis, and you can take the final result on trust. Even better, look up the recipe for how to do integrals this way in a book. It is simple.)

A subtlety appears immediately, since the integrand has poles on the real axis at

$$p_0 = \pm\omega \ . \tag{264}$$

The integral is in fact ill defined unless a prescription for how to handle these is supplied. So we supply one. At first sight this looks like cheating, but it is not so at all. Recall that the original equation does not have a unique solution until some extra conditions are specified. The choice of these conditions depends on the physical situation being considered, and it turns out that any consistent set of conditions corresponds to a definite pole presciption for our integral, and conversely. Any possible pole prescription defines a Green function, and all of them are of interest in their own right.

So we make the our problem more specific. We will look for a solution of the equation

$$\Box D_{ret}(x,x') = \delta(x,x') \tag{265}$$

—whatever the pole prescription our integral will satisfy this—and the extra condition

$$t' > t \qquad \Rightarrow \qquad D_{ret}(x,x') = 0 \ . \tag{266}$$

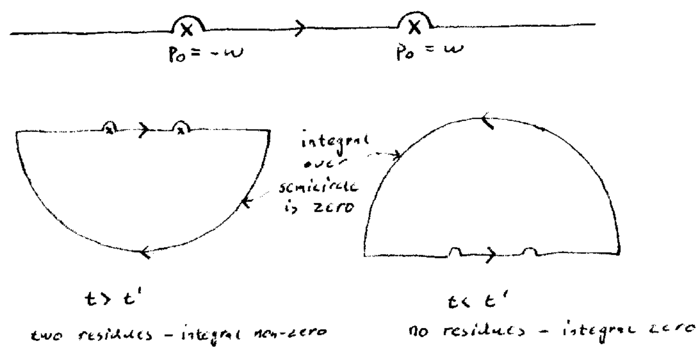


Figure 15: How the retarded Green function is defined by pole prescriptions

These conditions define the retarded Green function. There is also an advanced Green function that obeys

$$t' < t \qquad \Rightarrow \qquad D_{adv}(x, x') = 0 \ . \tag{267}$$

We will soon see why the condition that defines the retarded Green function is an interesting one. For the moment the point is that our problem has become an unambiguous one because a definite pole prescription will now be forced upon us. To see this, note that if $t - t' > 0$ then we can close the contour in the lower half plane, while if $t - t' < 0$ we can close it in the upper half plane. This means that if we deform the contour so that it passes above the singularities we find that there are no poles within the contour for $t - t' < 0$, and hence the resulting Green function vanishes when the observation point $x$ is earlier than the source point $x'$. The Green function defined by this condition is precisely the retarded Green function.

Let us compute the retarded Green function: For the $p_0$-integral the calculus of residues gives us

$$\begin{aligned} \int_{-\infty}^{\infty} dp_0 \ \frac{e^{-ip_0(t-t')}}{(p_0 - \omega)(p_0 + \omega)} &= -\Theta(t - t')2\pi i \sum \mathrm{R}es = \\ &= -\Theta(t - t')\frac{2\pi}{\omega} \sin\left(\omega(t - t')\right) \ , \end{aligned} \tag{268}$$

where $\Theta$ is the step function again. Plugging this result into the expression for $D(x, x')$ we can continue the integration. We switch to spherical polars in momentum space:

Figure 16: Pole prescriptions for the advanced Green function

$$D_{ret}(x,x') = -\Theta(t-t') \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip_i(x_i - x_i')}}{\omega} \sin\left(\omega(t-t')\right) =$$
$$= -\Theta(t-t') \int \frac{d\omega d\theta d\phi}{(2\pi)^3} \omega \sin\theta e^{-i\omega|x_i - x_i'|\cos\theta} \sin\left(\omega(t-t')\right) \ . \tag{269}$$

This is trivial to do if we perform the angular integrations first, collect terms, and make use of a well known representation of the delta function afterwards. The final result is

$$D_{ret}(x,x') = -\frac{\Theta(t-t')}{4\pi|x_i - x_i'|}\delta(t - t' - |x_i - x_i'|) \ . \tag{270}$$

If we had chosen to deform the contour so that it passes below the singularities we would have obtained the advanced Green function

$$D_{adv}(x,x') = -\frac{\Theta(t'-t)}{4\pi|x_i - x_i'|}\delta(t - t' + |x_i - x_i'|) \ . \tag{271}$$

This vanishes if the source point lies to the past of the observation point.

Note that the retarded Green function has support (that is to say, it is non-zero) on the forward light cone from the source point $x'$, while the support of the advanced Green function is on the backward light cone. To emphasize this fact we rewrite our Green functions, using the delta function identity

$$\delta(x^2) \equiv \delta(|x|^2 - t^2) = \frac{\delta(t - |x|)}{2|x|} + \frac{\delta(t + |x|)}{2|x|} \ . \tag{272}$$

You may need a quick reminder about how to prove things about the delta function: A delta function is defined by what it does inside an integrand. Thus

$$\int dx \ F(x)\delta(ax) = \int \frac{dy}{|a|} \ F(\frac{y}{a})\delta(y) = \frac{F(0)}{|a|} \qquad \Rightarrow \qquad \delta(ax) = \frac{\delta(x)}{|a|} \ . \tag{273}$$

A similar but more involved calculation shows that

$$\delta(f(x)) = \sum_a \frac{\delta(x - x_a)}{|\frac{df}{dx}(x_a)|} \ , \tag{274}$$

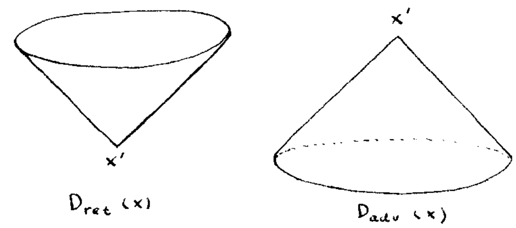


Figure 17: The support of some Green functions

where we are summing over those values $x_a$ for which $f(x_a)$ is zero. Our delta function identity is a special case of this relation. We can use it to verify that

$$D_{ret}(x, x') = -\frac{\Theta(t - t')}{2\pi}\delta((x - x')^2) \ , \tag{275}$$

and similarly for the advanced Green function. The fact that their support is confined to the light cone $(x - x')^2 = 0$ is now manifestly displayed.

Note by the way that we had a little bit of luck in the calculation. (Or perhaps it was more than that?) If the dimension of spacetime had been 2 + 1 rather than 3 + 1 then the "space" part of the momentum integral that we performed to get $D_{ret}$ would not have been so easy to do. In fact in 2 + 1 dimensions the support of these Green functions is not confined to the light cone, but extends to its interior. Running ahead a bit, the retarded Green function describes how a small disturbance affects the future. In 3 + 1 dimensions the effects of an event governed by the wave equation propagate into the future with the speed of light (this is a strong form of Huygen's principle), but in 2 + 1 dimensions the effects "ring on" for a considerable amount of time. The latter type of behaviour is in fact generic for wave motion (think of the rings left behind by a stone that is thrown into a lake) but fortunately not for the kind of waves that we use to obtain information about our environment—light and sound waves, both of which are described by the wave equation in 3 + 1 dimensions.

If we now go back to the homogeneous wave equation we find that the Green functions provide us with a general solution of the initial value problem. To see this we define a new object which is called the commutator Green function (because of the way it occurs in quantum field theory):

$$D(x, x') \equiv D_{ret}(x, x') - D_{adv}(x, x') = -\int \frac{d^3p}{(2\pi)^3}\frac{e^{-ip_i(x_i - y_i)}}{\omega}\sin\left(\omega(t - t')\right) \ . \tag{276}$$

It has the following easily verified properties:

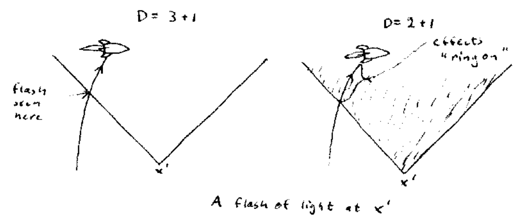


Figure 18: Huygen's principle versus effects that "ring on"

$$\Box D(x,x') = 0 \qquad D(x,x')_{t=t'} = 0 \qquad \partial_t D(x,x')_{t=t'} = -\delta^{(3)}(x,x') \ . \tag{277}$$

Now consider the expression

$$\phi(x) = \int d^3x' \ (\partial_{t'} D(x,x')u(x') - D(x,x')v(x')) \ , \tag{278}$$

where the integral is taken over the hypersurface $t' =$ constant and $u$ and $v$ are two arbitrary functions defined on this hypersurface. It is easy to check that, by construction, $\phi(x)$ is a solution to the homogeneous wave equation. Moreover this solution obeys

$$\phi(x,t') = u(x) \qquad \partial_t \phi(x,t') = v(x) \ . \tag{279}$$

In words, we have constructed a solution of the wave equation by specifying the form of the function and its first time derivative on an arbitrarily chosen spacelike hypersurface in an arbitrary way. More precisely, such data given on a spacelike hypersurface will determine the solution within the characteristic cone whose base is the hypersurface; the boundaries of this cone are the lightlike characteristic surfaces that we found earlier. Conversely, any solution can be uniquely specified in this manner. And this is the solution of the initial data problem. The story for an elliptic differential equation is rather different; in that case the required data are either a function or its normal derivative given on a surface that surrounds the region where the solution is to be determined.

With the initial value problem of the homogeneous equation under control we can rephrase our work on the inhomogeneous equation as follows: Suppose that the source is "localized in time" so that

$$t < t_0 \qquad \Rightarrow \qquad \rho(t,x) = 0 \ . \tag{280}$$

Then the appropriate solution of the inhomogeneous equation is

$$\phi(t,x) = \phi_{in}(t,x) + \int d^4x' D_{ret}(x,x')\rho(x') \ , \tag{281}$$

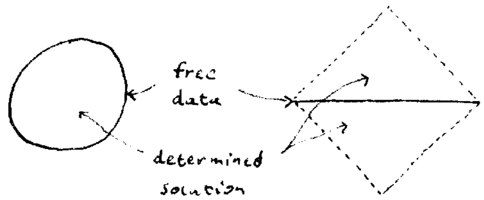


Figure 19: Boundary value problems and initial value problems

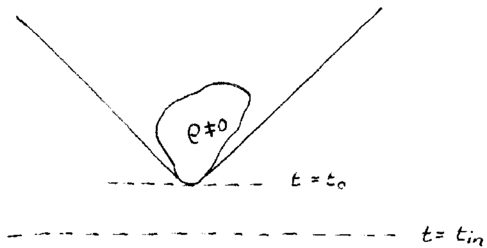


Figure 20: How effects propagate into the future

where the "in" field is a solution of the homogeneous wave equation such that

$$\phi(t_{in}, x) = \phi_{in}(t_{in}, x) \tag{282}$$

at some time $t_{in} < t_0$ at which we are specifying the initial conditions of the problem. Note that we have a solution of the homogeneous wave equation wherever $\rho(x) = 0$, and that this solution will be discontinuous across a lightlike characteristic surface "starting out" from the region where $\rho \neq 0$.

If you think this through you will see that the retarded Green function is used to describe the later effects of an earlier cause. We could replace the retarded Green function with the advanced Green function and we would still have a solution of the equation, but it would not obey the stated initial condition. (If you do not believe that causes have to precede effects then you can play with the idea that the advanced Green function should be used in physical problems.)

## 4.3 SOME REMARKS ON MAXWELL'S EQUATIONS

Given that we understand the wave equation, how far can we carry a similar analysis of Maxwell's equations? The answer turns out to be that we already have all the knowledge we need. Let us begin by looking for characteristic surfaces, that is for wavefronts. We make the discontinuous Ansatz

$$F_{\alpha\beta}(x) = \Theta(f(x)) f_{\alpha\beta}(x) \ . \tag{283}$$

Maxwell's equations *in vacuo* then imply that $f_{\alpha\beta}$ is a solution of Maxwell's equations and that on the hypersurface $f(x) = 0$ itself it must obey

$$f_{\alpha\beta}k^\beta = 0 \qquad \star f_{\alpha\beta}k^\beta = 0 \ , \tag{284}$$

where $k_\alpha$ is the normal vector of the surface $f(x) = 0$. But this implies that $f_{\alpha\beta}$ must take a very special form at every point on the discontinuity itself, namely

$$f_{\alpha\beta} = a k_{[\alpha} l_{\beta]} \ ; \qquad k^2 = k \cdot l = 0 \ . \tag{285}$$

Hence the normal vector $k_\alpha$ is a lightlike vector. If you think about it you also see that $l_\alpha$ must be spacelike.

The proof goes as follows: Choose an arbitrary basis $(k_\alpha, p_\alpha, m_\alpha, n_\alpha)$ in Minkowski space. In terms of this basis we can expand

$$f_{\alpha\beta} = a_1 k_{[\alpha}p_{\beta]} + a_2 k_{[\alpha}m_{\beta]} + a_3 k_{[\alpha}n_{\beta]} + a_4 p_{[\alpha}m_{\beta]} + a_5 p_{[\alpha}n_{\beta]} + a_6 m_{[\alpha}n_{\beta]} \ . \tag{286}$$

Then the second of the Maxwell equations implies that

$$\epsilon^{\alpha\beta\gamma\delta}k_\beta f_{\gamma\delta} = 0 \qquad \Rightarrow \qquad a_4 = a_5 = a_6 = 0 \ . \tag{287}$$

In the next step we define a new vector by

$$a l_\alpha = a_1 p_\alpha + a_2 m_\alpha + a_3 n_\alpha \ . \tag{288}$$

We are almost there. It only remains to observe that

$$f_{\alpha\beta}k^\beta = k_\alpha k \cdot l - l_\alpha k^2 = 0 \qquad \Rightarrow \qquad k^2 = k \cdot l = 0 \tag{289}$$

(since $k_\alpha$ and $l_\alpha$ are linearly independent). This is it.

So there are two important conclusions: The characteristic surfaces have a lightlike normal, and the field strength takes a very special form close to the discontinuity. To see how special it is we observe that there are two Lorentz invariant quantities that we may use to classify electromagnetic fields, and both of them vanish on both sides of the discontinuity. Indeed

$$F_{\alpha\beta}F^{\alpha\beta} = F_{\alpha\beta} \star F^{\alpha\beta} = 0 \qquad \Leftrightarrow \qquad E^2 - B^2 = E \cdot B = 0 \ . \tag{290}$$

Any field for which this happens is called a radiation field, and we will study such fields in considerable detail later on.

It is clear that we are studying hyperbolic equations. On the other hand it is also clear that there is something a little "non-local" about Maxwell's equations, with the flavour of elliptic boundary value problems. In fact they can be written as

$$\int_{\partial V} \mathbf{dS} \cdot \mathbf{B} = 0 \qquad \text{Gilbert's law} \tag{291}$$

$$\int_{\partial S} \mathbf{dl} \cdot \mathbf{E} + \partial_t \int_S \mathbf{dS} \cdot \mathbf{B} = 0 \qquad \text{Faraday's law} \tag{292}$$

$$\int_{\partial V} \mathbf{dS} \cdot \mathbf{E} = 4\pi \int_V dV \rho \qquad \text{Gauss' law} \tag{293}$$

$$\int_{\partial S} \mathbf{dl} \cdot B = \int_S \mathbf{dS} \cdot (4\pi \mathbf{j} + \partial_t \mathbf{E}) \qquad \text{Ampère-Maxwell's law .} \tag{294}$$

If these equations hold for arbitrary volumes $V$ bounded by surfaces $\partial V$ and for arbitrary surfaces $S$ bounded by curves $\partial S$ then the above obviously rather "non-local" equations are equivalent to Maxwell's equations. Hence we can draw important conclusions about the situation in a volume or on a surface by studying the situation at their boundaries. This vaguely non-local flavour of otherwise local equations is in fact typical of an important class of theories called gauge theories, and it is to gauge invariance that we turn next.

## 4.4 GAUGE SYMMETRY

There are a number of problems with the formulation of Maxwell's theory in terms of field strengths. Not only its non-local flavour but also the fact that as they stand the field equations cannot be derived from a local action principle. These problems go away once the theory is formulated in terms of a vector potential. This idea turns out to be very deep and a rather straightforward modification leads directly to the field equations governing the strong and weak interactions. Electrodynamics remains the simplest possible case—in particular once the equations have been reformulated in this way it will be evident that the dynamics is precisely that of the wave equation that we have solved already.

We first concentrate on the half of Maxwell's equation that can be written in the equivalent forms

$$\partial_\beta \star F^{\alpha\beta} = 0 \quad \Leftrightarrow \quad \partial_{[\alpha} F_{\beta\gamma]} = 0 \quad \Leftrightarrow \quad \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0 \; . \tag{295}$$

In 3+1 notation this is Faraday's and Gilbert's laws. It is easy to see that a solution is given by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \ , \tag{296}$$

where the—quite unrestricted !—four vector $A_\alpha$ is known as the vector potential. It is not quite so easy to see that the converse holds, but it does (the proof proceeds through explicit integrations and is messy rather than difficult). The precise statement is known as Poincaré's lemma, and says that for any antisymmetric covariant tensors $F_\alpha$, $F_{\alpha\beta}$, and so on (the pattern should be obvious) defined on a simply connected space ($\mathbf{R}^n$ is simply connected) one can find antisymmetric covariant tensors $A$, $A_\alpha$, ... such that

$$\partial_{[\alpha} F_{\beta]} = 0 \qquad \Leftrightarrow \qquad F_\alpha = \partial_\alpha A \tag{297}$$

$$\partial_{[\alpha} F_{\beta\gamma]} = 0 \qquad \Leftrightarrow \qquad F_{[\alpha\beta]} = \partial_{[\alpha} A_{\beta]} \tag{298}$$

$$\partial_{[\alpha} F_{\beta\gamma\delta]} = 0 \qquad \Leftrightarrow \qquad F_{[\alpha\beta\gamma]} = \partial_{[\alpha} A_{\beta\gamma]} \ , \tag{299}$$

and so on. The solution is not unique. Given any solution the general solution is given by

$$A' = A + \text{constant} \qquad A'_\alpha = A_\alpha + \partial_\alpha \lambda \qquad A'_{\alpha\beta} = A_{\alpha\beta} + \partial_{[\alpha}\lambda_{\beta]} \ , \tag{300}$$

where $\lambda$ (and so on) are completely unrestricted.

Poincaré's lemma is very important, so let us state very clearly what it says about electromagnetism: The entire content of Gilbert's and Faraday's laws is that the field strength can be expressed as the four dimensional "curl" of a vector potential; moreover if the field strength has been fully specified then the vector potential is determined only up to a gradient, in other words the field strength is unchanged by the gauge transformation

$$A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha \Lambda \qquad \Rightarrow \qquad F_{\alpha\beta} \to F'_{\alpha\beta} = F_{\alpha\beta} \ , \tag{301}$$

where $\Lambda(x)$ is an arbitrary function. To avoid any misunderstanding of the notation, there is no coordinate transformation involved here. Only a gauge transformation. The field strength is gauge invariant, and so are the electric and magnetic fields. A striking way to state the mathematical result is the following: Suppose we are given a covariant vector field $A_\alpha(x)$, and we want to know whether it can be written as a gradient of a scalar. Poincaré's lemma says that this happens if and only if its field strength vanishes.

Poincaré's lemma holds if space is simply connected, which means that any closed curve can be deformed to a point without leaving the space. This is true for $\mathbf{R}^n$ but not for the surface of a cylinder (say). In the latter case it can happen that there is a vector potential that "points" around the cylinder. Such a vector potential can give rise to a vanishing field strength even though it cannot be set to zero by means of a gauge transformation. This loophole in Poincaré's lemma is used in the Aharonov-Bohm effect, which you may have heard of. If not, forget it for now and assume that we are on $\mathbf{R}^n$.

Gauge symmetry is a most important property. It dictates the form of the electromagnetic interactions and in suitably modified form it also dictates the form of the other known force fields in Nature. Therefore it is important to be clear about what it means. First of all it is not a symmetry in the sense that Lorentz transformations are symmetries (of Maxwell's equations, say). The idea behind a symmetry is that we can transform a given solution and obtain another solution—another solution that counts as distinct from the first, even though it is observationally indistinguishable from the first. A gauge transformation on the other hand transforms a description of a certain solution into another description of the same solution. Let us jump ahead a bit and suppose that we have solved the other Maxwell equation as well. Now try to imagine the infinite dimensional space of all solutions of Maxwell's equations for the vector potential. From any given solution we can reach other solutions be means of a gauge transformation

$$A_\alpha \rightarrow A'_\alpha = A_\alpha + \partial_\alpha \Lambda \ . \tag{302}$$

Consider the set of all vector potentials which can be reached from a given vector potential $A_\alpha$ by means of a gauge transformation. We call this the gauge orbit through the point $A_\alpha$ (which is a point in the space of solutions). It is clear that not every solution lies on the same gauge orbit (if the field strengths are different the gauge orbits must be different since the field strength is gauge invariant). It is also clear that different gauge orbits cannot intersect. Hence we get a picture of the space of solutions as a space "foliated" by gauge orbits. The fundamental idea is now that a physical state of the field corresponds to a gauge orbit, that is to say to an equivalence class of solutions, and not to just a point in the space of solutions as would be the case for the wave equation.

If you do not like this picture you can simplify it through a choice of gauge. The idea then is to choose one point from every gauge orbit, and to discard all the other solutions. Technically what one does is to impose some condition on the vector potential, such as the Coulomb gauge

$$\partial_i A_i = 0 \ . \tag{303}$$

The point is that even if we start with a vector potential $A_i$ which does not obey this condition, we can always find another — let us say $A'_i$ — which does. If $A'_i$ is related to $A_i$ by a gauge transformation as above, then

$$\partial_i A'_i = \partial_i A_i + \partial_i \partial_i \Lambda \ . \tag{304}$$

Hence

$$\partial_i A'_i = 0 \qquad \Leftrightarrow \qquad \triangle \Lambda = -\partial_i A_i \ . \tag{305}$$

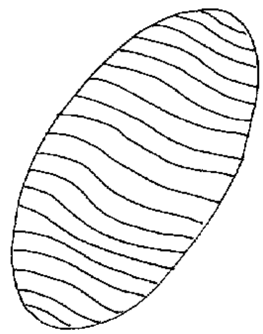

Figure 21: The space of solutions of Maxwell's equations, with gauge orbits.

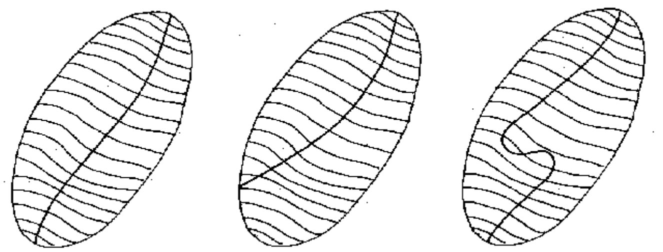

Figure 22: Gauge fixing: Choose one point on every gauge orbit. (Two of these gauge choices are bad ones.)

Now we solve Poisson's equation for $\Lambda$ so that this equation holds. Then the Coulomb gauge condition holds for $A'_i$. (Usually we do not even have to actually do this—we are just using this argument to show that no generality is lost by assuming that the vector potential obeys the Coulomb condition in the first place.) Moreover there is no ambiguity left in the vector potential once the Coulomb gauge condition is imposed; any gauge transformation would result in a violation of the gauge condition. This follows from the fact that if we assume that $\Lambda$ goes to zero at infinity then

$$\triangle \Lambda = 0 \quad \Rightarrow \quad \Lambda = 0 \; . \tag{306}$$

Hence the Coulomb gauge does what we advertized: It selects one special point from every gauge orbit, and there is a one-to-one correspondence between physical states and solutions of Maxwell's equations that also obey the Coulomb gauge condition.

A similar argument can be used to show that we can always impose the Lorenz gauge

$$\partial_\alpha A^\alpha = 0 \ , \tag{307}$$

provided that we can always solve the equation

$$\Box\Lambda = -\partial \cdot A \tag{308}$$

for an arbitrary function $\partial \cdot A(x)$. This is actually a little more sophisticated than the Coulomb gauge since it entails solving an initial value problem rather than just solving the Laplace equation at fixed time, but we know how to do it. It is also more sophisticated because the Lorenz gauge condition does not fully specify a point on the gauge orbit; we have the freedom to perform gauge transformations with gauge functions that obey

$$\Box\Lambda = 0 \ . \tag{309}$$

This equation does not imply that $\Lambda$ vanishes. This is a little regrettable from a pedagogical point of view: The Coulomb gauge is easier to explain in a satisfactory way, but—as we will see—the Lorenz gauge is particularly easy to use. If you find my treatment of these matters a little vague I can only refer you to a course on constrained Hamiltonian systems, where they are explained fully.

The Coulomb and Lorenz gauges are the only ones that we will use, but—in particular in quantum field theory — many other gauge choices have been found useful for special purposes. The main conclusion so far is that we have found the general exact solution of the equation

$$\partial_\beta \star F^{\alpha\beta} = 0 \ , \tag{310}$$

and that we have introduced the idea of gauge symmetry to deal with the peculiarities of the solution.

## 4.5 THE DYNAMICS OF MAXWELL'S EQUATIONS

The next equation,

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha \ , \tag{311}$$

contains true dynamical information and is not so easily disposed of. If—and only if !—we make use of the vector potential we can give an elegant derivation of this equation from the principle of least action. This is an important point because—as far as we know—all fundamental equations can be so derived. The action is

$$S[A] = -\int d^4x \ (\frac{1}{16\pi} F_{\alpha\beta}F^{\alpha\beta} + A_\alpha J^\alpha) \ . \tag{312}$$

Here it is understood that the vector potential is the dynamical variable and that

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha \ . \tag{313}$$

(We hold the source term $J^\alpha(x)$ fixed throughout this chapter.) Hence one half of Maxwell's equations,

$$\partial_{[\alpha} F_{\beta\gamma]} = 0 \ , \tag{314}$$

hold per definition. The variation of the action is simple to perform and indeed leads to the desired equation.

The equation is

$$\partial^\beta F_{\beta\alpha} = \Box A_\alpha - \partial_\alpha \partial \cdot A = -4\pi J_\alpha \ . \tag{315}$$

One's first reaction is that since the d'Alembert operator appears the way to solve it is to impose initial data on the vector potential and its first time derivative at some special time and then solve for the vector potential at arbitrary times. There is a problem with this however. Indeed it is obvious that there is no way that initial conditions can help us to specify a unique solution for the vector potential, since we can always perform a gauge transformation to a different vector potential

$$A'_\alpha = A_\alpha + \partial_\alpha \Lambda \tag{316}$$

that also obeys the equation and the initial conditions. All that we have to do is to ensure that the arbitrary function $\Lambda(x)$ vanishes in a neighbourhood of the initial data surface. For this reason it is clear that the equation does not admit a Green function as it stands.

There is another way to see that there is a problem. Let us make a "3+1 split" of the equation. Then

$$\Box A_i - \partial_i(\partial_j A_j - \partial_t A_0) = -4\pi j_i \tag{317}$$

$$\triangle A_0 - \partial_t \partial_i A_i = -4\pi\rho \ . \tag{318}$$

The problem is that the second order time derivatives of $A_0$ have dropped out—the zeroth component of the equation does not give a clue to the time development of $A_0$, rather it gives a constraint that the putative initial data have to satisfy. Hence two problems appear: The problem is underdetermined in the sense that the time development is partly arbitrary, and it is overdetermined in the sense that initial data cannot be chosen freely. This situation is in fact typical of gauge theories in general, all the way up to Einstein's theory of gravitation which is also a gauge theory.

Fortunately it is not too difficult to wiggle out of the problem. The basic difficulty is the ambiguity in the vector potential and we can get rid of this ambiguity through a gauge fixing condition. The simplest choice for our purposes is the Lorenz gauge

$$\partial \cdot A = 0 \ . \tag{319}$$

The point is that we can choose to impose this condition without any loss of essential generality. Now the field equation collapses to

$$\Box A_\alpha = -4\pi J_\alpha \ . \tag{320}$$

This is simply the inhomogeneous wave equation, and hence we have the general solution of Maxwell's equation with fixed sources and the Lorenz gauge imposed under complete control. There is no problem with keeping the gauge condition in force, since it follows from the gauge fixed equation together with conservation of electric charge that

$$\Box \partial \cdot A = -4\pi \partial \cdot J = 0 \ . \tag{321}$$

If we make sure that the gauge condition holds on the initial data surface (and that its time derivative is zero as well) then it will hold for all times. In particular, a conserved current will not cause trouble with the Lorenz gauge.

Let us remind ourselves about the nature of the general solution. In particular, suppose that the source is localized in time so that

$$\lim_{t\rightarrow -\infty} J^\alpha(x) = 0 \ , \tag{322}$$

and that the incoming radiation is known to be

$$\lim_{t\rightarrow -\infty} A^\alpha(x) = A^\alpha_{in}(x) \ , \tag{323}$$

where the "in" field is some solution of the wave equation. Under these conditions the unique solution for the vector potential is

$$A^\alpha(x) = A^\alpha_{in}(x) - 4\pi \int d^4x' \ D_{ret}(x,x')J^\alpha(x') \ . \tag{324}$$

Let us be a little bit more explicit about this. Assume for simplicity that the "in" field vanishes. Then the electrostatic potential (say) is

$$\varphi \equiv A^0 = \int d^3x' \frac{\rho(x')}{|x_i - x'_i|} \ |_{ret} \ , \tag{325}$$

where the integral is over all space and the subscript reminds us that the function $\rho(t',x')$ must be evaluated at the retarded time $t' = t - |x_i - x'_i|$. If $\rho$ is time independent the expression reduces to the Coulomb potential that is familiar from electrostatics. The integral may be quite difficult to evaluate in closed form; the case when the source is a point particle moving in an arbitrary way will occupy us later on.

The situation that we will consider in chapter 5 is indeed that when the in field vanishes and there is electromagnetic radiation produced by a source $J^\alpha(x)$ that is localized in space and time. For the moment the nice thing about this situation is that the initial data then are such that the Lorenz gauge is manifestly trouble free. There are still some puzzles concerning gauge invariance though. Let us try to count the "number of degrees of freedom" in the electromagnetic field. Recall that in a quite precise sense a scalar field has one degree of freedom per point in space. (The amount of information that we have to specify in order to get a well posed initial value problem is therefore two numbers per spatial point.) When working in the Lorenz gauge we found that Maxwell's equations reduced to the wave equation for the vector potential—if you like four copies of the wave equation, which would suggest that the electromagnetic field carries four times as many degrees of freedom per point. But this cannot be quite right, since as a matter of fact we do not need all the information that is specified in this way; the point being that—as we have seen—even when the Lorenz gauge is imposed the vector potential remains arbitrary to some extent and can be changed by a gauge transformation without changing the physics. It is more transparent to use the Coulomb gauge here, since the Coulomb condition really selects one point from every gauge orbit. I refer you to the exercises for this, and simply state the answer: There are two degrees of freedom per spatial point, or in other words exactly two times as much freedom as there is in the wave equation.

The correct framework for addressing this problem is the Hamiltonian formulation of the equations; this can be obtained from the action but only provided that one knows how to handle gauge invariance within the Hamiltonian formalism. It would carry us too far to go through this topic here.

## 4.6 GAUGE THEORIES AND MATHEMATICS

The ideas behind Maxwell's equations, and gauge theories in general, are far reaching. An important strand in mathematical physics is the application of gauge theories to mathematics, and especially to topology. We will give one example of this—indeed an old example, known to Gauss. Suppose that we have two closed curves in space, given in parametric form as

$$C \ : \quad x_i(s) \qquad\qquad C' \ : \quad x'_i(s') \ . \tag{326}$$

Suppose that the explicit form is very complicated, so that it is hard to visualize the curves. Now there is an important property of a pair of closed curves called the linking number. This is an integer that is preserved under arbitrary

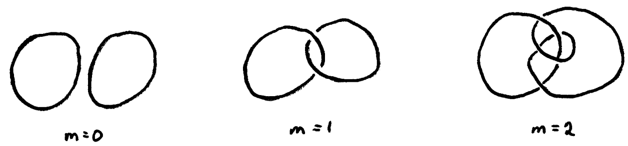


Figure 23: Linking numbers

deformations of the curves, as long as they are not allowed to pass through each other. To define it, deform one of the closed loops to a circle. This assumes that it does not form a knot—as a matter of fact ideas from gauge theories really show their strength when it comes to classification of knots, but we have to begin somewhere. Then define the linking number as the number of times the other curve passes through the disk spanned by the circle, counting intersections as positive or negative depending on the whether the parameter $s'$ increases or decreases as we pass through the disk. It is intuitively obvious that the linking number is well defined, but it can clearly be difficult to compute if we are just given the equations for the curves.

Fortunately the calculation can be reduced to a problem in elementary electrostatics. Suppose that there is a current running through the second loop, such that

$$j_i(x')d^3x' = Idl'_i \ . \tag{327}$$

This is to say that the current is confined to the loop $C'$. Then it follows from Ampère's law (without the displacement current, since we are doing electrostatics) that

$$\int_C dl_iB_i = 4\pi\int_S dS_ij_i = 4\pi Im \ , \tag{328}$$

where $m$ is the linking number—the equation being a simple consequence of the way the linking number is defined. On the other hand we know that

$$\triangle B_i = -4\pi\epsilon_{ijk}\partial_j j_k \qquad \Rightarrow \qquad B_i(x) = \epsilon_{ijk}\partial_j\int d^3x'\frac{j_k(x')}{|x-x'|} \ . \tag{329}$$

Here we made use of the Green function for the Laplacian. If we use the fact that the current has support only on the loop $C'$ and carry through the differentiation we get

$$B_i(x) = -I\int_{C'}\frac{\epsilon_{ijk}(x_j-x'_j)dl'_k}{|x-x'|^3} \ . \tag{330}$$

We already have a formula for the linking number in terms of a line integral of the magnetic field around the first loop $C$, so it follows that

$$m = \frac{1}{4\pi}\epsilon_{ijk}\int_C\int_{C'}\frac{(x_i - x'_i)dl_j dl'_k}{|x - x'|^3} \ . \tag{331}$$

This is Gauss' formula for the linking number. The question whether two given loops are linked can now be settled by means of a straightforward calculation.

Many more examples of the application of gauge theories to solve mathematical problems can be given. For instance we have already observed that the theory is somewhat sensitive to the topology of the space on which it is defined since the validity of Poincaré's lemma depends on it. This theme can be developed quite far, but we let go of mathematics at this point.

**Exercises:**

**1.** Can a solution of the wave equation be discontinuous across the surface $t = z$? If so, what can you say about the nature of the discontinuity across this surface? How would you interpret such a solution physically?

**2.** Find a solution of the wave equation that describes a spherical wave going out from a source at the origin; i.e. a solution that depends on $r$ and $t$ only and that is singular at $r = 0$. Do this in two ways:

a) Look at the wave equation in spherical polars and guess the answer.

b) Assume the source $\rho(x) = \Theta(t)F(t)\delta^{(3)}(x)$ and use $D_{ret}(x, x')$.

In b) Alice sits at the origin and starts talking at $t = 0$.

**3.** Use a Fourier transformation and the calculus of residues to compute the Green function of the operator

$$\triangle - m^2 \ ,$$

where $\triangle$ is the Laplacian. Assume that space is infinite and, as a boundary condition, that the field $\phi$ that occurs in the equation

$$(\triangle - m^2)\phi(x) = \rho(x)$$

tends to zero at infinity (sufficiently fast). How do you handle the case $m = 0$?

Alternatively, find the Green function $G(x) = G(x - 0)$ by assuming that it is spherically symmetric (a function of $r$ only) and observing that it must solve the homogeneous equation everywhere except at $r = 0$. Then invent a way of fixing the overall factor.

**4.** Show that at any given point one can find four vectors such that the field strength tensor at that point takes the form

$$F_{\alpha\beta} = p_{[\alpha}q_{\beta]} + m_{[\alpha}n_{\beta]} \ . \tag{332}$$

(Hint: Think of $F_{\alpha\beta}$ as an anti-symmetric matrix. A symmetric matrix can always be diagonalized. What is the corresponding statement for an anti-symmetric matrix?) Note that eq. (285) is more special in two ways. Which?

**5.** Consider the action

$$S = \int d^4x \ (-\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - A_\alpha J^\alpha) \ ,$$

where $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$ och $J^\alpha(x)$ is a fixed external current. Show that the action is invariant under the gauge transformation

$$A_\alpha(x) \to A'_\alpha(x) = A_\alpha(x) + \partial_\alpha \Lambda(x) \qquad J^\alpha \to J'^\alpha = J^\alpha$$

(where $\Lambda$ is an arbitrary scalar function), if and only if $J^\alpha$ obeys a suitable condition. What is the physical interpretation of this condition?

**6.** Suppose space has a hole in it. Specifically, remove all points with $x^2 + y^2 < 1$ from space. Define a vector potential on what is left, which is everywhere regular, gives a vanishing magnetic field, and which is such that it cannot be gauge transformed to zero. What went wrong with Poincaré's lemma?

**7.** By definition a vector field that obeys

$$\partial_i V_i = 0$$

is called a transverse vector field, while a vector field that can be written as the gradient of a scalar field is called a longitudinal vector field. Introduce the operators

$$P^T_{ij} = \delta_{ij} - \frac{\partial_i\partial_j}{\triangle} \qquad P^L_{ij} = \frac{\partial_i\partial_j}{\triangle} \ . \tag{333}$$

(As usual "$1/\triangle$" is a formal way of writing the Green function of the Laplacian, which is unique once boundary conditions have been specified.) Show that these operators are projection operators, and use them to show that an arbitrary vector field can be written as the sum of a transverse and a longitudinal vector field.

**8.** Consider Maxwell's equations

$$\Box A_\alpha - \partial_\alpha \partial \cdot A = -4\pi J_\alpha$$

in the Coulomb gauge ($\partial_i A_i = 0$). Make a 3+1 split and follow this up by splitting the vector equation into transverse and longitudinal parts. Think about it and show that the number of degrees of freedom of the theory is twice the number of degrees of freedom in the wave equation.

**9.** Consider a symmetric tensor field $h_{\alpha\beta}$ that obeys the equation

$$\Box h_{\alpha\beta} - \partial_\alpha \partial^\gamma h_{\gamma\beta} - \partial_\beta \partial^\gamma h_{\gamma\alpha} + \partial_\alpha \partial_\beta h_\gamma{}^\gamma = 0 \ . \tag{334}$$

(This is a linearization of Einstein's equations for the gravitational field.) Show that this equation has a gauge symmetry, that is to say that there is an analogue of the transformation

$$A_\alpha \to A_\alpha + \partial_\alpha \Lambda \ . \tag{335}$$

Given any solution $h_{\alpha\beta}$, the gauge transformed field must automatically solve the equation as well.

To study the radiation emitted from moving charges we adopt an approximation which is the opposite of the external field approximation: We assume that the trajectories $X^\alpha(\tau)$ of the charged particles are given, and use Maxwell's equation

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha \tag{336}$$

to compute the resulting electromagnetic fields. Before going into details, let us recall why radiation takes places in the first place. A stationary electric charge gives rise to an electric field

$$\mathbf{E} = \frac{e}{r^2}\mathbf{e}_r \tag{337}$$

directed radially outwards from the particle. The electric field from a charge moving with constant speed can be computed from this by means of a Lorentz transformation. After some manipulations, we obtain

$$\mathbf{E} = \frac{e}{r^2}\frac{1-v^2}{(1-v^2\sin^2\psi)^{\frac{3}{2}}}\mathbf{e}_r \ , \tag{338}$$

where $r$ is the distance from the point where the field is measured to the point where the particle is situated at the moment when the field is measured, $\mathbf{e}_r$ points in the direction between these two points and $\psi$ is the angle between $\mathbf{e}_r$ and the velocity of the charge. This is a kind of squashed Coulomb field, and the point to notice is that it still falls off as one over $r$ squared —- nothing very interesting happens.

But let us now consider a particle initially at rest at the origin, submitted to a brief period of acceleration $a$ at time $t = 0$, and then moving with constant speed $v$. The news that the particle has accelerated travels with finite speed, which means that sufficiently far away from the particle the field that is observed will remain a Coulomb field directed away from the origin. Where $r << ct$ it will have a different direction, since it points away from the instantaneous location of the charge, which at $ct \approx r$ may be quite some distance away from the origin. So, at $r \approx ct$ the field "jumps", and after that it changes (perhaps slowly) only because of the movement of the charge. The most interesting thing about the field in the region where the "jump" occurs is that it falls off as one over $r$ rather than as one over $r$ squared. This conclusion follows if we assume that the field is linear both in $e$ (which is obvious) and in $a$ (as will follow from Maxwell's equation). Because then we can use dimensional analysis to conclude that

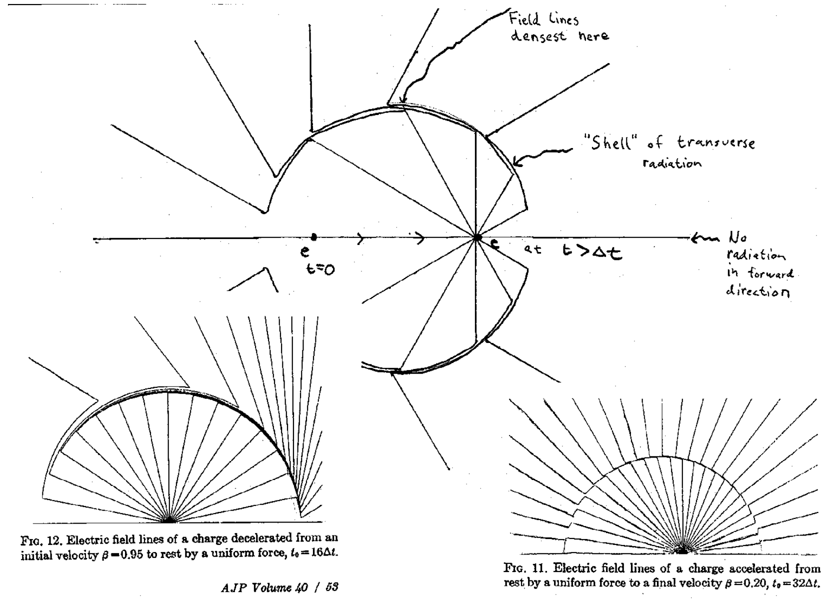


Figure 24: An electric charge has been moved (during $\Delta t$ seconds) to the right. A shell of radiation is moving outwards. Some exact calculations by others (for related situations) are shown too.

$$E \propto \frac{ea}{rc^2} \ . \tag{339}$$

This is the reason why electromagnetic radiation can be detected from far away!

## 5.1 A RELATIVISTIC REMINDER

Since our discussion will involve a certain amount of passing back and forth between the four vector description of particles on the one hand and the 3+1 description on the other it is as well to collect a few relevant formulæ and arguments here. Suppose that we have a worldline $X^\alpha(\tau)$ and that a 3+1 split yields

$$X^\alpha = (x_i, t) \ . \tag{340}$$

We want to compute the four-velocity

$$\dot{X}^\alpha \equiv \frac{dX^\alpha}{d\tau} \ . \tag{341}$$

Since it is understood that the parameter $\tau$ denotes proper time we have

$$d\tau^2 = dt^2 - dx_i^2 \quad \Rightarrow \quad \frac{d\tau}{dt} = \sqrt{1-v^2} \equiv \frac{1}{\gamma} \ , \tag{342}$$

where

$$v_i \equiv \frac{dx_i}{dt} \tag{343}$$

is the ordinary spatial velocity of the particle described by the world line and the "relativistic factor" $\gamma$ will recur. With these definitions it is straightforward to see that

$$\dot{X}^\alpha = \frac{dt}{d\tau}\frac{dX^\alpha}{dt} = (\gamma v_i, \gamma) \ . \tag{344}$$

We can go on to compute the four-acceleration; when doing this we will have to differentiate $\gamma$ and the result is

$$\frac{d\gamma}{dt} = \gamma^3 a \cdot v \ . \tag{345}$$

We get in this way that the four-acceleration is

$$\ddot{X}^\alpha = (\gamma^2 a_i + \gamma^4 a \cdot v v_i, \gamma^4 a \cdot v) \ , \tag{346}$$

where $a_i$ denotes the ordinary acceration in three-space.

While this is not a course in relativistic kinematics it is perhaps as well to remind you that the energy-momentum four-vector of the particle is

$$P_\alpha = m\dot{X}_\alpha = (m\gamma v_i, m\gamma) \ . \tag{347}$$

Newton's second law (in three-space) must be modified into

$$F_i = \frac{d}{dt}(m\gamma v_i) = m\gamma(a_i + \gamma^2 a \cdot v v_i) \ . \tag{348}$$

Note that the particle makes much more resistance to acceration in the direction of its velocity (in which case the relativistic factor $\gamma$ changes) than to perpendicular acceleration (in which $\gamma$ remains constant). Let $F$ denote the magnitude of the force. For perpendicular acceleration we obtain

$$F \cdot v = 0 \quad \Rightarrow \quad F = m\gamma a_\perp \ . \tag{349}$$

For acceleration in the direction of motion we get

$$F_i = F\hat{v}_i \quad \Rightarrow \quad F = m\gamma^3 a_{||} \ . \tag{350}$$

When $\gamma$ is large this makes a huge difference.

There is also a conceptual point that is worth clarifying. The energy of a particle is the fourth component of a four-vector. In the rest frame of the particle (where the momentum of the particle vanishes) this four vector is

$$P^{\alpha} = (0, 0, 0, E) = (0, 0, 0, mc^2) \ , \tag{351}$$

where $E$ is the energy in the rest frame of the particle and we restored an explicit factor of $c^2$ so as to recover a famous formula. On the other hand the four-velocity in the rest frame is

$$V^{\alpha} = \dot{X}^{\alpha} = (0, 0, 0, 1) \ . \tag{352}$$

It follows that

$$E = -V^{\alpha} P_{\alpha} \ . \tag{353}$$

This is a Lorentz scalar and does not depend on the choice of the coordinate system.

This is as it should be—the notion of "energy of a particle as measured in its own rest frame" is independent of the velocity with which the particle moves relative to the chosen frame of reference. Therefore it should be a scalar. It is instructive to see why the "energy of a particle as measured in an arbitrary frame of reference" cannot be regarded as a scalar in a similar useful manner. Whatever frame of reference we use there will be a vector

$$n'^{\alpha} = (0, 0, 0, 1) \ . \tag{354}$$

The energy $E'$ of the particle as measured in this frame of reference is the fourth component of $P^{\alpha}$ in this frame, and hence it can be written as a Lorentz scalar

$$E' = -n'^{\alpha} P_{\alpha} \ . \tag{355}$$

While this is true it is not useful unless there is something in the physics of the problem that singles out the vector $n'^{\alpha}$ for attention. If we change our frame of reference to another arbitrarily chosen one we have to define a new four vector $n''^{\alpha}$ and we obtain a new Lorentz scalar

$$E'' = -n''^{\alpha} P_{\alpha} \neq -n'^{\alpha} P_{\alpha} = E' \ . \tag{356}$$

Therefore the choice of scalar ($E'$ or $E''$) that describes the energy of the particle depends on the frame of reference chosen. The scalar is not interesting unless the frame of reference itself is interesting.

## 5.2 LIÉNARD-WICHERT FIELDS

We now turn to a more detailed analysis. The first task is to give an exact solution of Maxwell's equations given that the current is that produced by a charge moving on a fixed trajectory $X^\alpha(\tau)$, namely

$$J^\alpha(x) = e \int d\tau \; \dot{X}^\alpha \delta(x, X(\tau)) \; . \tag{357}$$

In the Lorenz gauge the solution for the vector potential can be written down immediatiately using the retarded Green function:

$$A^\alpha(x) = -4\pi \int d^4y \; D_{ret}(x,y) J^\alpha(y) \; . \tag{358}$$

This is the Liénard—Wiechert potential, and it only remains to make it more explicit. Making it so will turn out to be a rather lengthy affair.

Calculating the Liénard-Wiechert fields

We already have the Green function, namely

$$D_{ret}(x,y) = -\frac{\Theta(x^0 - y^0)}{2\pi}\delta((x-y)^2) \; . \tag{359}$$

If we plug this into our expression for $A^\alpha$ and use the delta function in the current to perform the $y$-integral we obtain

$$A^\alpha(x) = 2e \int d\tau \; \Theta(x^0 - X^0(\tau))\delta((x-X)^2)\dot{X}^\alpha \; . \tag{360}$$

To do the $\tau$-integral we observe that there is a unique solution of

$$(x-X)^2 = 0 \qquad \Leftrightarrow \qquad \tau = \tau_0 \; . \tag{361}$$

(Actually there are two solutions, but one of them is killed by the $\Theta$ function. The point $x^\alpha = X^\alpha(\tau_0)$ is called the retarded point. It is where the charge was located at the time when it emitted the radiation which is felt "now" by the observer.) Therefore—using the delta function identity that we know already—

$$\delta((x-X)^2) = \frac{\delta(\tau - \tau_0)}{\frac{d}{d\tau}(x-X)^2} = -\frac{\delta(\tau-\tau_0)}{2(x-X)\cdot\dot{X}} \; . \tag{362}$$

Now we can do the remaining integral and arrive at the Liénard-Wiechert potential

$$A^\alpha(x) = -e\frac{\dot{X}^\alpha}{\dot{X}\cdot(x-X)}|_{ret} \ , \tag{363}$$

where the subscript reminds us that $X^\alpha$ is to be evaluated at the retarded proper time $\tau_0$. Now the retarded time is actually a function of $x$ and $t$ since

$$t = \tau_0 + r(\tau_0) \ , \tag{364}$$

where $r = |x_i - X_i(\tau_0)|$. Inverting this equation we obtain $\tau_0 = \tau_0(x,t)$. Since we have not specified the worldline $X^\alpha(\tau)$ we can not be more specific than that, but it should be clear from the picture what is going on.

We are more interested in the resulting field strength though. It is not straightforward to differentiate the Liénard-Wiechert potential because the location of the retarded point depends on $x$, so we may as well backtrack and consider

$$F^{\alpha\beta} = 2e\int d\tau \ \Theta(x^0 - X^0(\tau))\partial^\alpha\delta((x-X)^2)\dot{X}^\beta - (\alpha \leftrightarrow \beta) \ . \tag{365}$$

(The derivative of the $\Theta$ function is a delta function. Therefore it contributes only at the retarded point, and we ignore it. The same argument is used in the partial integration below.) We want to remove the derivative from the delta function so that we can do the $\tau$ integral. To achieve this we observe that

$$\partial^\alpha\delta(f(\tau)) = \frac{\partial f}{\partial x_\alpha}\frac{d\tau}{\partial f}\frac{d}{d\tau}\delta(f(\tau)) \ . \tag{366}$$

Now at least the derivative is with respect to $\tau$, so we can perform a partial integration. Then, just as when we calculated the potential, the delta function is rewritten as a definite thing times $\delta(\tau - \tau_0)$. Finally we do the $\tau$ integral and end up with

$$F^{\alpha\beta} = -\frac{e}{\dot{X}\cdot(x-X)}\frac{d}{d\tau}\left(\frac{(x^\alpha - X^\alpha)\dot{X}^\beta - (x^\beta - X^\beta)\dot{X}^\alpha}{\dot{X}\cdot(x-X)}\right)|_{ret} \ . \tag{367}$$

The whole thing to be evaluated at the retarded proper time. The expression may not be very transparent, but we see by inspection that the field has a term proportional to the acceleration so that our intuitive argument about the fall off of the radiation field is valid.

Finally we carry out the differentiation and then rewrite everything in 3 + 1 notation, using

$$(x-X)^\alpha = (rn_i, r) \tag{368}$$

$$\dot{X}^\alpha = (\gamma v_i, \gamma) \tag{369}$$

$$\ddot{X}^\alpha = (\gamma^2 a_i + \gamma^4 a \cdot v v_i, \gamma^4 a \cdot v) \ . \tag{370}$$

Here $v_i$ and $a_i$ denote the ordinary velocity and acceleration of the particle. After a certain amount of calculation we can conclude that

$$B_i = \epsilon_{ijk} n_j E_k \ . \tag{371}$$

Moreover the electric field itself can be conveniently split into two parts,

$$E_i = E_i^{near} + E_i^{far} \ , \tag{372}$$

where the near field is given by

$$E_i^{near} = \frac{e}{\gamma^2 r^2 (1 - n \cdot v)^3} (n_i - v_i)_{|ret} \tag{373}$$

and the far field is

$$E_i^{far} = \frac{e}{r(1 - n \cdot v)^3} (n \cdot a(n_i - v_i) - (1 - n \cdot v) a_i)_{|ret} \ . \tag{374}$$

It only remains to understand these expressions.

We see that near the particle there is a field falling off like $1/r^2$ that becomes an ordinary Coulomb field in the non-relativistic limit. In fact if we evaluate it as a function of the present position of the charge, rather than as a function of its retarded position, it becomes precisely the squashed Coulomb field that we discussed above. If the charge moves with constant velocity the near field is all there is to it.

The far field, also known as the radiation field, is indeed linear in the acceleration and falls off like $1/r$. It has the properties that we were led to expect from our qualitative discussion. In particular

$$n \cdot E^{far} = n \cdot B^{far} = 0 \ . \tag{375}$$

The far fields are transverse. Moreover both the Lorentz invariant quantities that we can form from an electromagnetic field, namely

$$I_1 = E^2 - B^2 \qquad\qquad I_2 = E \cdot B \ , \tag{376}$$

are zero for the far field. This is in agreement with our discussion of characteristic surfaces for Maxwell's equations (in chapter 4), when we showed that this has to be so at any surface in spacetime where the field is discontinuous, and perhaps vanishing on one side of the surface. Hence we do indeed expect these invariants to vanish for any propagating wave in Maxwell's theory.

## 5.3 THE NON-RELATIVISTIC APPROXIMATION

Ultimately our interest is not in the electric and magnetic fields. What we really want to know is how much power the particle radiates, and how this power is distributed over angles. The instantaneous flux of energy is described by the Poynting vector

$$S_i = \frac{1}{4\pi}\epsilon_{ijk}E_jB_k \approx \frac{1}{4\pi}E^2 n_i \ , \tag{377}$$

where the second equality is valid for the radiation field. Since we are mostly interested in looking at the accelerating particle from a distance we will from now on make the approximation

$$E_i \approx E_i^{far} \ . \tag{378}$$

What we are after is the power radiated per solid angle, which is given by $S \cdot n$.

For a first study we adopt the non-relativistic approximation, in which

$$E_i \approx \frac{e}{r}(n \cdot a n_i - a_i) \ . \tag{379}$$

We want to know the total amount of power radiated per unit solid angle as a function of time. A quick calculation reveals that

$$\frac{dP}{d\Omega} \equiv r^2 S_i n_i = \frac{e^2}{4\pi}(a^2 - (n \cdot a)^2) = \frac{e^2}{4\pi}a^2 \sin^2\theta \ , \tag{380}$$

where $\theta$ is the angle between $\mathbf{n}$ and $\mathbf{a}$. The result is due to Larmor. The total radiated power is

$$P = \int d\Omega \frac{dP}{d\Omega} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{dP}{d\Omega} = \frac{2}{3}e^2 a^2 \ . \tag{381}$$

So much for the non-relativistic result.

## 5.4 TOTAL RADIATED POWER

In the relativistic case we have to be a little careful about the definition of "radiated power". This happens because retarded time $t'$ differs from time as measured by the observer. Therefore there will be a difference between power emitted and power received. To see how, suppose that the distance between the charge and the observer is $r$ and use $t'$ to denote retarded time. Then

$$t = t' + r \quad \Rightarrow \quad \Delta t = \Delta t' + \Delta r \quad \Rightarrow \frac{\Delta t}{\Delta t'} = 1 + \frac{\Delta r}{\Delta t'} \approx 1 - n \cdot v \ . \tag{382}$$

This quantity is important enough to deserve a symbol of its own;

$$\kappa \equiv 1 - n \cdot v = \frac{dt}{dt'} \ . \tag{383}$$

Consider a certain amount of energy $E$ emitted by the particle into a given solid angle, received by the observer between $t_1$ and $t_2$. The total received energy is defined by

$$E = \int_{t_1}^{t_2} dt \frac{dP_{rec}}{d\Omega} \ . \tag{384}$$

The total emitted energy on the other hand is defined by

$$E = \int_{t_1'}^{t_2'} dt' \frac{dP_{em}}{d\Omega} \ . \tag{385}$$

Finally we can be explicit about it all. The total energy radiated into a unit solid angle between two arbitrary times is

$$E = \int_{t_1}^{t_2} dt r^2 S \cdot n = \int_{t_1'}^{t_2'} dt' \frac{dt}{dt'} r^2 S \cdot n \ . \tag{386}$$

From this formula we can read off the total emitted and received powers, and observe that

$$\frac{dP_{em}}{d\Omega} = \kappa \frac{dP_{rec}}{d\Omega} \ . \tag{387}$$

Whether the one or the other of these are of primary interest depends on the circumstances.

The factor $\kappa$

The total emitted power is the power emitted by a charge in its own rest frame. This is an invariant description, not dependent on our choice of laboratory (or observatory) frame, and hence the total emitted power is a spacetime scalar. A precise argument is that total emitted power is the time derivative of the fourth component of a four vector $P_\alpha$, and hence it can be written in terms of four vectors as

$$P_{em} = \frac{dE}{dt'} = -V^\beta \partial_\beta (V^\alpha P_\alpha) \ . \tag{388}$$

It is also clear that in the reference frame of the charge the total emitted power must reduce to the non-relativistic Larmor result. There is then only one reasonable conjecture that one can make for the relativistic formula, namely

$$P_{em} = \frac{2}{3}\frac{e^2}{m^2}\ddot{X}^\alpha \ddot{X}_\alpha \ . \tag{389}$$

This result is in fact correct. Using eq. (346) we find

$$P_{em} = \frac{2}{3}e^2\gamma^4(a^2 + \gamma^2(a\cdot v)^2) = \frac{2}{3}e^2\gamma^4(a_\perp^2 + \gamma^2 a_{||}^2) \ , \tag{390}$$

where we decomposed the acceleration into parts which are perpendicular and parallell to the velocity in the last step. An obvious first comment is that the power goes up rapidly when the velocity of the particle becomes relativistic, due to the factor $\gamma^4$.

We can use our formula to explain why particle physics is so expensive. To do experiments one uses magnetic fields cleverly contrived to make charged particles run around in circles. Hence, if the radius of the accelerator is $R$,

$$a\cdot v = 0 \qquad\qquad a = \frac{v^2}{R} \ . \tag{391}$$

It follows that the total amount of energy emitted per revolution is

$$\Delta E = \frac{2\pi R}{v}P_{em} = \frac{4\pi}{3}\frac{e^2}{R}\gamma^4 v^3 \ . \tag{392}$$

Since $\gamma$ occurs raised to the power four, we conclude that there is considerable energy loss when relativistic particles are running around in circles, which is what they do at CERN.

Radiative losses are less serious for linear accelerations. To make a fair comparison of linear and perpendicular accelerations we must remember that it is easiest to accelerate a relativistic particle perpendicularly to its direction of motion. Suppose that we have a perpendicular and a parallel force of equal magnitude $F$. Then for perpendicular acceleration we have

$$F = \frac{d}{dt}(m\gamma v) = m\gamma a_\perp \ , \tag{393}$$

while for acceleration in the direction of motion we get (since then we have to differentiate $\gamma$ as well)

$$F = \frac{d}{dt}(m\gamma v) = m\gamma^3 a_{||} \ . \tag{394}$$

The power emitted in the perpendicular case is

$$P_{em} = \frac{2}{3}\frac{e^2}{m^2}\gamma^2(m\gamma a_\perp)^2 = \frac{2}{3}\frac{e^2}{m^2}\gamma^2 F^2 \ , \tag{395}$$

and in the parallel case

$$P_{em} = \frac{2}{3}\frac{e^2}{m^2}(m\gamma^3 a_{||})^2 = \frac{2}{3}\frac{e^2}{m^2}F^2 \ . \tag{396}$$

For given $F$ the power in the perpendicular case is greater by a factor $\gamma^2$. Hence if we consider radiation from relativistic particles moving in an arbitrary fashion it may be a good approximation to ignore the acceleration parallel to the velocity.

## 5.5 ANGULAR DISTRIBUTION OF RADIATED POWER

Let us now look at the radiation from a relativistic particle in somewhat more detail. What is the angular distribution like? The total received power per unit solid angle is given by

$$\frac{dP_{rec}}{d\Omega} = r^2 S_i n_i \ , \tag{397}$$

evaluated at the retarded time as usual. A minor calculation, using eq. (387), then shows that the total emitted power is

$$\frac{dP_{em}}{d\Omega} = \frac{e^2}{4\pi\kappa^5}\left(n\cdot a(n_i - v_i) - \kappa a_i\right)^2 \ . \tag{398}$$

After a certain amount of additional calculation this becomes

$$\frac{dP_{em}}{d\Omega} = \frac{e^2}{4\pi}\left(\frac{a^2}{\kappa^3} + \frac{2n\cdot av\cdot a}{\kappa^4} - \frac{(n\cdot a)^2(1-v^2)}{\kappa^5}\right) \ . \tag{399}$$

A more instructive form is

$$\frac{dP_{em}}{d\Omega} = \frac{e^2}{\kappa^5} \times \text{something} \ , \tag{400}$$

where "something" depends on the details of the motion. In the non-relativistic case $\kappa \approx 1$, and the prefactor does not matter. In the extreme relativistic case

$$\kappa = 1 - n\cdot v \approx 1 - \cos\psi \ , \tag{401}$$

where $\psi$ is the angle between the velocity and the direction of the emitted radiation. Examination of the expression for emitted power then shows that for relativistic motion the radiation will be very strongly peaked in the forward direction.

It is instructive to make a rough estimate of this beaming effect. We are looking for the range of angles for which

$$o(\kappa) = o(1-v) \ . \tag{402}$$

Then we have for small angles, while at the same time $v$ is very close to being one, that

$$\kappa = 1 - v\cos\psi \approx 1 - v + \frac{\psi^2}{2} \ . \tag{403}$$

Roughly speaking this means that we require

$$o(\psi^2) = o(2(1-v)) \quad \Rightarrow \quad \psi \sim \sqrt{(1+v)(1-v)} = \frac{1}{\gamma} \ . \tag{404}$$

The conclusion of this exercise is that for highly relativistic motion the radiation will be concentrated within a cone centered around the forwards direction and having an opening angle of about

$$\psi \sim \frac{1}{\gamma} \tag{405}$$

—and this statement is independent of the details of the motion.

The details of the motion cannot be completely ignored. Let us look at two special cases. First the case when the acceleration and the velocity of the charge are parallel. Then we obtain (if we use $\theta$ to denote the angle between $n_i$ and $a_i$ and perform a modest calculation)

$$\frac{dP_{em}}{d\Omega} = \frac{e^2a^2}{4\pi\kappa^5}\sin^2\theta \ . \tag{406}$$

This is a factor $1/\kappa^5$ times the Larmor result; the radiation will be beamed around the forwards direction but it will still be zero in precisely that direction. In the case when the acceleration is perpendicular to the velocity we get

$$\frac{dP_{em}}{d\Omega} = \frac{e^2a^2}{4\pi\kappa^3}\left(1 - \frac{\cos^2\theta}{\gamma^2\kappa^2}\right) \ . \tag{407}$$

As noted already, this case is the most important one.

## 5.6 SYNCHROTRON RADIATION

The case of relativistic charges running around in circles is of great importance in various branches of physics. Particle accelerators have been mentioned. But circular motion also occurs naturally whenever there are magnetic fields around.

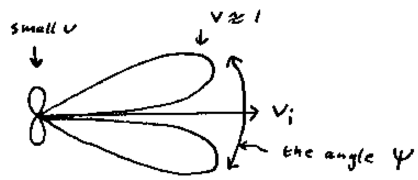


Figure 25: Angular distribution of emitted power (acceleration and velocity parallel)

Moreover (as we have seen) arbitrary motion may be approximated as instantaneously cicular motion when radiation from relativistic particles is considered. The received frequency spectrum has a surprising feature that was first understood by Julian Schwinger, who analyzed the large radiative losses in accelerators. His results were then applied by Shklovskii and Hoyle to the radio emission from galaxies such as Cygnus A. The detailed theory is quite involved, but a rough understanding of the surprising feature is easily achieved.

Let us consider an electron moving in a magnetic field. In the relativistic case the cyclotron frequency is about

$$\omega_B = \frac{|e|B}{m\gamma} \sim \frac{1}{R} , \tag{408}$$

where $R$ is the radius of the circle. Due to the beaming effect, radiation will be emitted into a cone with opening angle

$$\psi \sim \frac{1}{\gamma} . \tag{409}$$

It follows that the radiation will reach the observer in sharp pulses. How sharp? Suppose that the width of the pulse (in time) is $\Delta L$. If we Fourier decompose a sharp pulse of this width we can use the relation

$$\Delta\omega\Delta L \sim 1 \tag{410}$$

to estimate the highest frequency observed. The obvious guess is that one will see frequencies up to the cyclotron frequency that characterizes the motion, and then nothing. However, this is wrong.

To see why we observe that during the time that the observer is illuminated by the beam the particle itself has moved a distance $d$, so that the illumination occurs during a time interval

$$\Delta t = \frac{d}{v} = \frac{R\psi}{v} \sim \frac{R}{\gamma v} \ . \tag{411}$$

During this time interval the forwards edge of the pulse (that moves with the velocity of light) will move a distance $\Delta t$, and hence the width of the pulse that reaches the observer is

$$L = D - d = \frac{R}{\gamma v} - \frac{R}{\gamma} = (\frac{1}{v} - 1)\frac{R}{\gamma} \approx \frac{R}{2\gamma^3} \ , \tag{412}$$

where the last step follows from the fact that

$$\lim_{v \to 1} \frac{\frac{1}{v} - 1}{1 - v^2} = \frac{1}{2} \ . \tag{413}$$

The conclusion of this (admittedly rough) argument is that the width of the observed pulse is

$$\Delta L \sim \frac{R}{\gamma^3} \qquad \Rightarrow \qquad \Delta\omega \sim \frac{\gamma^3}{R} \sim \gamma^3 \omega_B \ . \tag{414}$$

Hence we will actually see very high harmonics of the fundamental frequency.

Sharpening of the pulse in synchrotron radiation

**Exercises:**

**1.** Compute the electric field from a charge moving with constant velocity in two ways: By Lorentz transforming a Coulomb field, and by evaluating the Liénard-Wiechert near field in terms of the present position of the charge. Check that the results agree.

**2.** Argue using conservation of energy that the radiation field must fall off like $1/r$.

**3.** Draw the angular distribution of the emitted power for the case when the acceleration and velocity are perpendicular.

**4.** The "searchlight effect", namely that the emitted radiation from a fast particle is concentrated in the forwards direction, can be understood from the geometry of Minkowski space. Draw the forward light cone of the point of emission in the instantaneous rest frame of the particle (where the non–relativistic Larmor result holds). Then cut this light cone with a plane of simultaneity for a fast moving observer and observe where most of the radiation goes from her point of view. What is the connection to the appearance of the sky from a relativistic space ship, as given in good sf films?

# 6 — RADIATION DAMPING

Our discussion of electrodynamics with point particles representing matter is clearly incomplete—we have considered the external field and external current approximations, but we have not tried to consider the full non-linear problem where fields and point particles are in mutual interaction. We will take some steps to remedy this defect, along the same path that we often tread to take the interaction of a mechanical system with its environment into account. The idea is that energy "leaks" from the mechanical system to its environment, and that this can be modelled by adding a friction term to the equations. For a particle sliding on a table, we have

$$ma = F_{ext} - kv \ , \tag{415}$$

where $F^{ext}$ is some mechanical force acting on the particle. This equation is not invariant under time reversal, nor should it be since the leakage of energy to the environment is irreversible. Can we do something similar for the energy that leaks out in the form of radiation? It turns out that we can, but the scheme that we will come up with suffers from a certain amount of pathological behaviour. In our treatment of charged particles we have closed our eyes for the obvious difficulty that the energy in the Coulomb field of a pointlike charge is—on the face of it—infinite. It seems reasonable that it is this difficulty that comes back to haunt our attempt to handle radiation damping through energy balance. As a matter of fact there are consistency problems here that have never been completely resolved. Our analysis will show why: The coupling strength of electrodynamics being what it is the difficulties would show up in a region where quantum electrodynamics must take over anyway.

## 6.1 THE NON-RELATIVISTIC CASE

We restrict ourselves to the non-relativistic approximation. Then the leakage of energy is given by

$$P(t) = \frac{2}{3}e^2a^2 \ . \tag{416}$$

Suppose that we can indeed describe the motion as

$$ma = F_{ext} + F_{rad} \ . \tag{417}$$

We must arrange that the work done by the radiative reaction force $F_{rad}$ is equal to the total energy radiated, which means that it must be true that

$$0 = \int_{t_1}^{t_2} F^{rad} \cdot v dt + \int_{t_1}^{t_2} \frac{2}{3} e^2 a^2 dt = \int_{t_1}^{t_2} (F_{rad} - \frac{2}{3} e^2 \dot{a}) \cdot v dt + \frac{2}{3} e^2 [a \cdot v]_{t_1}^{t_2} \ . \quad (418)$$

Let us assume that the motion is such that the boundary terms can be ignored—for instance because the motion is periodic and we are integrating over one period. Then it is consistent with our result to assume that the equation we are after is

$$ma = F_{ext} + m\tau\dot{a} \ , \quad (419)$$

where we have defined a constant $\tau$ by

$$\tau = \frac{2e^2}{3m} \ . \quad (420)$$

The dimension of $\tau$ is that of time. For an electron, we find that $\tau = 6.26 \cdot 10^{-24}$ seconds. The equation is due to Lorentz.

Our equation for radiation damping does have the expected property that it is not invariant under time reversal. However, what is surprising is that it contains third order time derivatives. This means that the set of solutions is uncomfortably large. In particular, for vanishing external force it is physically clear that the only solution should be $a = 0$, but in fact there is a one parameter family of solutions

$$a = \text{constant} \times e^{t/\tau} \ . \quad (421)$$

There are "runaway" solutions of our equation. It will therefore be necessary to supplement the equation with some additional restriction on the space of solutions before we can accept it.

The following suggestion was made by Dirac (whose point of view was much deeper than ours—he was trying to make sense of the relativistic theory, in spite of the bothersome fact that a point particle has a divergent electromagnetic self energy, while an extended particle is hard to define in a relativistic way). The general solution of the equation with an external force $F$ included is given by

$$a = e^{t/\tau}(a_0 - \frac{1}{m\tau} \int_0^t e^{-t'/\tau} F(t') dt') \ . \quad (422)$$

The constant $a_0$ is at our disposal. However, unlike the integration constants that occur when we compute the velocity and the position we will not try to fix $a_0$ by specifying its initial value. Instead we insist that the acceleration must tend to zero in the remote future; hence we choose

$$a_0 = \frac{1}{m\tau}\int_0^\infty e^{-t'/\tau}F(t')dt' \ . \tag{423}$$

Adding the two terms together we get

$$a = \frac{1}{m\tau}e^{t/\tau}\int_t^\infty e^{-t'/\tau}F(t')dt' \ . \tag{424}$$

This equation for the trajectory of the particle is the one that we will adopt. It is important to realize that any solution of Dirac's equation is also a solution to the original Lorentz' equation, but that the converse does not hold—runaway solutions have been eliminated.

Unfortunately our equation is no longer a local differential equation in the time variable, but an integrodifferential equation. It says that the acceleration at any given time is determined by the force that will act on the particle in the future, weighted by the (rapidly falling) factor $e^{t/\tau}$. Clearly an unusual equation! To get an idea of how it works we consider a simple situation where the particle is acted on by a force which is localized in time,

$$F(t) = k\Theta(t-t_1)\Theta(t_2-t) \ . \tag{425}$$

The acceleration of the particle according to Dirac's equation is now readily computed:

$$t < t_1 \ : \ a = \frac{k}{m}e^{t/\tau}(e^{-t_1/\tau} - e^{-t_2/\tau}) \quad t_1 < t < t_2 \ : \ a = \frac{k}{m}(1 - e^{(t-t_2)/\tau}) \ . \tag{426}$$

The acceleration is zero for $t_2 < t$. On the other hand "preacceleration" occurs—the particle starts to accelerate before the force is turned on. This behaviour seems on the face of it to be manifestly unphysical, and presumably its presence means that our equation can not be quite correct, but we can always argue that since the timescale $\tau$ is so short this does not matter. Indeed because of the uncertainty relation $\Delta t \Delta E \sim \hbar$ the uncertainty in energy becomes equal to the rest energy $m$ of the particle when we consider a time resolution $\tau_q \sim \hbar/m$ which, for an electron, is about $137 \times \tau$. The conclusion is that the classical theory breaks down completely before we can even begin to actually measure the preacceleration (but only because the fine structure constant $e^2/\hbar$ takes the small value $1/137$).

Preacceleration

In spite of the somewhat dubious character of the Lorentz-Dirac equation it will give reasonable results when we apply it to the case of periodic motion. Let us consider a harmonic oscillator and use our original differential equation

$$\ddot{x} + \omega_0^2 x = \tau \dddot{x} \ . \tag{427}$$

To solve it we try the Ansatz

$$x(t) = x_0 e^{-\alpha t} \qquad \Rightarrow \qquad \tau\alpha^3 + \alpha^2 + \omega_0^2 = 0 \ . \tag{428}$$

This can be solved for $\alpha$. It is convenient to rewrite the equation according to

$$z \equiv \tau\alpha \quad \& \quad \delta \equiv \tau\omega_0 \qquad \Rightarrow \qquad z^2(z+1) + \delta^2 = 0 \ . \tag{429}$$

Since $\tau$ is a small quantity so is $\delta$, and we work out the answer to lowest non-trivial order in $\delta$. When $\delta = 0$ the three roots are

$$\delta = 0 \qquad \Rightarrow \qquad z \in \{0, 0, -1\} \ . \tag{430}$$

For $\delta$ non-zero but small there will be one real root close to $-1$ and a pair of complex conjugate roots close to $\pm i\delta$. The real root is unacceptable, since it leads to a runaway solution. The complex roots on the other hand correspond to almost oscillatory motion. A slightly more careful treatment reveals that to the lowest relevant order in $\delta$ the root close to $i\delta$ is

$$z \approx \frac{1}{2}\delta^2 + i\delta - \frac{3}{4}i\delta^3 \ . \tag{431}$$

Hence we find

$$\alpha = \Gamma/2 \pm i(\omega_0 + \Delta\omega) \ , \qquad \Gamma \approx \tau\omega_0^2 \qquad \Delta\omega \approx -\frac{3}{4}\tau^2\omega_0^3 \ , \tag{432}$$

where $\Gamma$ is known as the decay constant and $\Delta\omega$ as the level shift. The important point is that

$$\tau\omega_0 << 1 \qquad \Rightarrow \qquad \omega_0 >> \Gamma >> \Delta\omega \ . \tag{433}$$

The decay constant is larger than the level shift and both are small corrections to the oscillatory motion.

As usual in radiation problems it is the Fourier transform of the acceleration that we are mainly interested in. The upshot of the discussion above is that

$$a(t) \sim e^{-\frac{\Gamma}{2} - i(\omega_0 + \Delta\omega)} + \text{complex conjugate} \tag{434}$$

$$a(\omega) \sim \int_0^\infty e^{-\frac{\Gamma}{2} - i(\omega_0 + \Delta\omega)} e^{i\omega t} dt \sim \frac{1}{\Gamma/2 + i(\omega - \omega_0 - \Delta\omega)} \ . \tag{435}$$

The total energy radiated is

$$I \sim \int a^2(t)dt \sim \int |a(\omega)|^2 d\omega \ , \tag{436}$$

and hence the energy radiated per unit frequency is

$$\frac{dI}{d\omega} \sim |a(\omega)|^2 \sim \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0 - \Delta\omega)^2} \ . \tag{437}$$

This curve defines a spectral line, and it has a shape which is typical for decaying systems. The width of the spectral line can be defined as the width of the distribution at half maximum intensity, and is equal to $\Gamma$. The classical level shift is much smaller than the line width whereas the level shift in quantum electrodynamics, which is called the Lamb shift, turns out to be comparable to the line width. This was a famous mystery once upon a time.

A spectral line including radiation damping