arXiv:hep-lat/0211036v2 16 Dec 2002

Lattice Perturbation Theory

Stefano Capitani [1]

DESY Zeuthen
John von Neumann-Institut für Computing (NIC)
Platanenallee 6, 15738 Zeuthen, Germany

Abstract

The consideration of quantum fields defined on a spacetime lattice provides computational techniques which are invaluable for studying gauge theories nonperturbatively from first principles.

Perturbation theory is an essential aspect of computations on the lattice, especially for investigating the behavior of lattice theories near the continuum limit. Particularly important is its role in connecting the outcome of Monte Carlo simulations to continuum physical results. For these matchings the calculation of the renormalization factors of lattice matrix elements is required.

In this review we explain the main methods and techniques of lattice perturbation theory, focusing on the cases of Wilson and Ginsparg-Wilson fermions. We will illustrate, among other topics, the peculiarities of perturbative techniques on the lattice, the use of computer codes for the analytic calculations and the computation of lattice integrals. Discussed are also methods for the computation of 1-loop integrals with very high precision.

The review presents in a pedagogical fashion also some of the recent developments in this kind of calculations. The coordinate method of Lüscher and Weisz is explained in detail. Also discussed are the novelties that Ginsparg-Wilson fermions have brought from the point of view of perturbation theory.

Particular emphasis is given throughout the paper to the role of chiral symmetry on the lattice and to the mixing of lattice operators under renormalization. The construction of chiral gauge theories regularized on the lattice, made possible by the recent advances in the understanding of chiral symmetry, is also discussed.

Finally, a few detailed examples of lattice perturbative calculations are presented.

---

[1] E-mail: stefano@ifh.de

Contents

1 Introduction

In a lattice field theory the quantum fields are studied and computed using a discretized version of the spacetime. The lattice spacing a, the distance between neighboring sites, induces a cutoff on the momenta of the order $1/a$.

A spacetime lattice can be viewed as a regularization which is nonperturbative. Since the other known regularizations, like dimensional regularization or Pauli-Villars, can be defined only order by order in perturbation theory, the lattice regularization has this unique advantage over them. It is a regularization which is not tied to any specific approximation method, and which allows calculations from first principles employing various numerical and analytical methods, without any need to introduce models for the physics or additional parameters.

In discretizing a continuum field theory one has to give up Lorentz invariance (and in general Poincaré invariance), but the internal symmetries can usually be preserved. In particular, gauge invariance can be kept as a symmetry of the lattice for any finite value of the lattice spacing, and this makes possible to define QCD as well as chiral gauge theories like the electroweak theory. The construction of the latter kind of theories on a lattice however presents additional issues due to chiral symmetry, which have been understood and solved only recently. The fact that one is able to maintain gauge invariance for any nonzero a is of great help in proving the renormalizability of lattice gauge theories.

Lattice gauge theories constitute a convenient regularization of QCD where its nonperturbative features, which are essential for the description of the strong interactions, can be systematically studied. The lattice can then probe the long-distance physics, which is otherwise unaccessible to investigations which use continuum QCD. Precisely to study low-energy nonperturbative phenomena the lattice was introduced by Wilson, which went on to prove in the strong coupling regime the confinement of quarks. Confinement means that quarks, the fundamental fields of the QCD Lagrangian, are not the states observed in experiments, where one can see only hadrons, and the free theory has no resemblance to the observed physical world. The quark-gluon structure of hadrons is hence intrinsically different from the structure of other composite systems. No description in terms of two-body interactions is possible in QCD. Lattice simulations of QCD show that a large part of the mass of the proton arises from the effect of strong interactions between gluons, that is from the pure energy associated with the dynamics of confinement. Only a small fraction of the proton mass is due to the quarks. Similarly, the lattice confirms that only about half of the momentum and a small part of the spin of the proton come from the momentum and spin of the constituent quarks. Computations coming from the lattice can then give important contributions to our understanding of the strong interactions.

In this review we are interested in doing lattice calculations in the weak coupling regime. This is the realm of perturbation theory, which is used to compute the renormalization of the parameters of the Lagrangian and of the matrix elements, and to study the approach of the

lattice to the continuum limit. Details of the lattice formulation that are only relevant at the nonperturbative level will not be discussed in this review. If the reader is also interested in the nonperturbative aspects of lattice field theories, they are covered at length in the books of (Creutz, 1983; Montvay and Münster, 1994; Rothe, 1997) and in the just appeared book of (Smit, 2002). The book by Rothe is the one which contains more material about lattice perturbation theory. Useful shorter reviews, which also cover many nonperturbative aspects, sometimes with a pedagogical cut, can also be found in (Kogut, 1983; Sharpe, 1994; Sharpe, 1995; DeGrand, 1996; DeGrand, 1997; Gupta, 1999; Sharpe, 1999; Wittig, 1999; Münster and Walzl, 2000; Davies, 2002; Kronfeld, 2002) and in the recent (Lüscher, 2002). Here we would like to explain the main methods and techniques of lattice perturbation theory, particularly when Wilson and Ginsparg-Wilson fermions are used. We will discuss, among other things, Feynman rules, aspects of the analytic calculations, the computer codes which are often necessary to carry them out, the mixing properties of lattice operators, and lattice integrals.

Chiral symmetry is a recurrent topic in the treatment of fermions on the lattice, and we will address some issues related to it in the course of the review. We feel that a discussion of the problems connected with the realization of chiral symmetry on the lattice is needed. The reader might otherwise wonder why one should do such involved calculations like the ones required for Ginsparg-Wilson fermions. We think that it is also interesting to see how the lattice can offer fascinating solutions to the general quantum theoretical problem of defining chiral gauge theories beyond tree level.

Also discussed is an algebraic method for the reduction of any 1-loop lattice integral (in the Wilson case) to a linear combination of a few basic constants. These constants are calculable with very high precision using in a clever way the behavior of the position space propagators at large distances. The coordinate space method, which turns out to be a very powerful tool for the computation of lattice integrals, allows then the calculation of these constants with very high precision, which is also the necessary requirement in order to be able to compute two-loop lattice integrals with many significant decimal places, as we will explain in detail. A lot of nice and interesting work has been done using these techniques in the case of bosonic integrals, which thanks to them can be easily computed with extraordinary precision at one-loop, and with good precision at two loops. A nonnegligible part of this review will discuss these calculations in detail in the bosonic case.

The focus of this review is on methods rather than on results. In fact, very few numerical results will be reported. The reader, if interested, can consult many perturbative results in the references given. Instead, our objective is different. We would like to provide computational tools which can be useful for physicists who are interested in doing this kind of calculations. Technical details will be therefore explained in a pedagogical fashion. Particular attention will be paid to certain aspects that only occur in lattice computations, and that physicists expert in continuum perturbative calculations might find curious. The main objective of this review is to show how perturbation theory works on the lattice in the more common situations. It is

hoped that one can learn from the material presented here.

A background in continuum quantum field theory is required, and an acquaintance with continuum perturbative calculations in gauge field theories, the derivation of Feynman rules in continuum QCD and the calculation of Feynman diagrams will be assumed. Familiarity with the path integral formalism, with the quantization of field theories through the functional integral and with the renormalization of continuum quantum field theories is also desired. The knowledge of elementary facts, such as the renormalization group equations, the running of the strong coupling, the β function and asymptotic freedom of QCD, will also be taken for granted.

This review is not homogeneous. I have given more space to the topics that I believe are more interesting and more likely to be of wider use in the future. Many of the choices made and of the examples reported draw from the experience of the author in doing this kind of calculations.

To keep this review into a manageable size, not all important topics or contributions will be covered. One thing that will not be discussed in detail is perturbation theory applied to Symanzik improvement, which, although very interesting and useful, would probably require a review in itself, given also the many important result that have been produced. The Schrödinger functional is also introduced only in a very general way. I will not be able to do justice to other topics like numerical perturbation theory or tadpole improvement.

Many interesting subjects had to be entirely left out because of constraints on space. Among the topics which are not covered at all are nonrelativistic theories, heavy quarks, and anisotropic lattices. I have also omitted all what concerns finite temperature perturbation theory. Many of these things are treated in detail in the reviews and books cited above, where several topics not covered here can also be found. Moreover, we will not occupy ourselves with phenomenological results, but only with how perturbation theory is useful for the extraction from the lattice of that phenomenology. In any case, there are by now so many perturbative calculations that have been made on the lattice that it would be impossible to include all of them here.

The main reason for the introduction of the lattice was to study QCD in its nonperturbative aspects, like confinement, and we will confine this review to QCD. Although very interesting, spin models, the $\lambda\phi^4$ theory and the Higgs sector, to name a few, will be left out. Even after this restriction is made, lattice QCD is still quite a broad topic by itself, and thus to contain this review into a reasonable size we have been compelled to discuss only the main actions that have been used to study QCD on the lattice. Given the great number of different actions that have been proposed for studying lattice QCD in the past 30 years, it is necessary to limit ourselves to just the few of them that are more widely used. The Wilson and staggered formulations have been the most popular ones in all this time. Recently the particular kind of chiral fermions known as Ginsparg-Wilson (like the overlap, domain wall and fixed-point fermions) have also begun to find broader application, and they present interesting challenges for lattice physicists interested in perturbation theory. These are the actions that we will cover in this paper.

The main features of the lattice construction and of lattice perturbation theory will be

discussed in detail in the context of Wilson fermions. When the other actions will be introduced the discussions will be more general, although we will try to point out the peculiarities of perturbative calculations in these particular cases. The explanations of the various lattice actions will be rather sketchy and aimed mainly at the aspects which are interesting from the point of view of perturbation theory.

1.1 Outline of the paper

The review is divided in three parts: Sections 2 and 3, which are a sort of motivation, Sections 4 to 12, which introduce the lattice, the various actions and their Feynman rules, and Sections 13 to 20, which in much more technical detail show how lattice computations are made.

We begin with two Sections which are meant to stress the importance of lattice perturbation theory and explain what is meant for renormalization of operators on the lattice. After having introduced these motivations, we start with the first technical points, defining in Section 4 a euclidean lattice and showing what the discretization of a continuum theory means in practice. We introduce typical lattice quantities and we pass to discuss in detail the Wilson action (which has not chiral invariance) in Section 5, explaining how to derive its Feynman rules in momentum space. All Feynman rules necessary for one-loop calculations are then explicitly given. In Section 6 we focus our attention on the relation between chirality and fermionic modes on the lattice, and the problems which arise when one tries to define chiral fermions on a lattice. After a brief interlude which discusses staggered fermions, which have some chiral symmetry and have been the major alternative to Wilson fermions (at least in the first two decades of the lattice), we then introduce in Section 8 Ginsparg-Wilson fermions, the long-awaited reconciliation of chirality with the lattice. We give some details about the overlap, the domain wall and the fixed-point actions, which are solutions of the Ginsparg-Wilson relation. In Section 9 we then explain how using Ginsparg-Wilson fermions it is possible to define on the lattice chiral gauge theories, where gauge invariance and chiral symmetry are maintained together at every order in perturbation theory and for any finite value of the lattice spacing.

In Section 10 we then see the approach of coupling and masses to the continuum limit and talk about the β function and the Λ parameter of the lattice theory. In Section 17 we briefly introduce the Symanzik improvement, including a short discussion about improved pure gauge actions on the lattice. We conclude the first part of the review with a brief introduction to the Schrödinger functional, which has gained a paramount place in the lattice landscape in recent years.

In Section 13 we begin the more technical part of the review, dedicated to how to actually carry out the perturbative computations on the lattice. We introduce at this point the symmetry group of the lattice, the hypercubic group. Since the lattice symmetries are not as restrictive as in the continuum theory, more mixings arise in general under renormalization, and we discuss some examples of them in Section 14. How to compute Feynman diagrams on the lattice is

explained in great detail in Section 15, where we talk about the lattice power counting theorem of Reisz, which is useful for the computation of divergent integrals, and we present, step by step, the complete calculation of the 1-loop renormalization constant of the operator measuring the first moment of the momentum distribution of quarks in hadrons. This example is rather simple (compared to other operators) and contains all the main interesting features one can think of: a logarithmic divergence, a covariant derivative, symmetrized indices and of course the peculiar use of Kronecker deltas in lattice calculations. Moreover, it is an example of a calculation in which the various propagators and vertices need an expansion in the lattice spacing a (in this case, at first order). Finally, it includes the computation of the quark self-energy, which is quite interesting and useful by itself. Brief discussions about overlap calculations, tadpole improvement and perturbation theory with fat link actions conclude Section 15. In Section 16 we then discuss the use of computer codes for the automated computations of lattice Feynman diagrams.

In Section 17 we explain some advanced techniques for the numerical evaluation of lattice integrals coming from Feynman diagrams (using extrapolations to infinite volume), while in Section 18 we introduce an algebraic method for the exact reduction of any Wilson 1-loop integral to a few basic constants. The bosonic case is thoroughly explained, so that the reader can learn to use it, and some applications to the exact calculations of operator tadpoles are also explicitly given. Section 18 ends with a discussion of the main points of the general fermionic case, and the expression of the 1-loop quark self-energy in terms of the basic constants.

The basic constants of the algebraic method can be computed with arbitrary precision, as explained in detail in Section 19. The values of the fundamental bosonic constants, Z_0 and Z_1, are given with a precision of about 400 significant decimal places in Appendix B. In order to be able to calculate them to this precision, we need to introduce the coordinate space method of Lüscher and Weisz, which will also be used for the computation of 2-loop integrals. The 2-loop bosonic integrals are discussed at length, and the general fermionic case is also addressed. In Section 20 we then briefly introduce numerical perturbation theory, which is promising for doing calculations at higher loops. Finally, conclusions are given in Section 21, and Appendix A summarizes some notational conventions.

2   Why lattice perturbation theory

To some readers the words "lattice" and "perturbation theory" might sound like a contradiction in terms, but we will see that this is not the case and that lattice perturbation theory has grown into a large and established subject. Although the main reason why the lattice is introduced is because it constitutes a nonperturbative regularization scheme and as such it allows nonperturbative computations, perturbative calculations on the lattice are rather important, and for many reasons.

Perturbation theory of course cannot reveal the full content of the lattice field theory, but it

can still give a lot of valuable informations. In fact, there are many applications where lattice perturbative calculations are useful and in some cases even necessary. Among them we can mention the determinations of the renormalization factors of matrix elements of operators and of the renormalization of the bare parameters of the Lagrangian, like couplings and masses. The precise knowledge of the renormalization of the strong coupling is essential for the determination of the Λ parameter of QCD in the lattice regularization (as we will see in Section 10) and of its relation to its continuum counterpart, Λ_{QCD}. In general perturbation theory is of paramount importance in order to establish the right connection of the lattice scheme which is used in practice, and of the matrix elements simulated within that scheme, to the physical continuum theory. Every lattice action defines a different regularization scheme, and thus one needs for each new action that is considered a complete set of these renormalization computations in order for the results which come out from Monte Carlo simulations to be used and understood properly.

Moreover, lattice perturbation theory is important for many other aspects, among which we can mention the study of the anomalies on the lattice, the study of the general approach to the continuum limit, including the recovery of the continuum symmetries broken by the lattice regularization (like Lorentz or chiral symmetry) in the limit $a \to 0$, and the scaling violations, i.e., the corrections to the continuum limit which are of order a^n. These lattice artifacts bring a systematic error in lattice results, which one can try to reduce by means of an "improvement", as we will see in Section 11.

Perturbative calculations are thus in many cases essential, and are the only possibility for having some analytical control over the continuum limit. As we will see in Section 10, the perturbative region is the one that must be necessarily traversed in order to reach the continuum limit. Lattice perturbation theory is in fact strictly connected to the continuum limit of the discretized versions of QCD. Because of asymptotic freedom, one has $g_0 \to 0$ when $\mu \to \infty$, which means $a \to 0$. We should also point out that one cannot underestimate the role played by perturbative calculations in proving the renormalizability of lattice gauge theories.

Finally, perturbation theory will also be important for defining chiral gauge theories on the lattice at all orders of the gauge coupling, as we will see in detail in Section 9. The lattice will thus be proven to be the only regularization that can preserve chirality and gauge invariance at the same time (without destroying basic features like locality and unitarity).

We can say in a nutshell that lattice perturbation theory is important for both conceptual and practical reasons. The phenomenological numbers that are quoted from lattice computations are very often the result of the combined effort of numerical simulations and analytic calculations, usually with some input from theoretical ideas.

In principle all known perturbative results of continuum QED and QCD can also be reproduced using a lattice regularization instead of the more popular ones. However, calculating in such a way the correction to the magnetic moment of the muon (to make an example) would be quite laborious. A lattice cutoff would not be the best choice in most cases, for which instead

regularizations like Pauli-Villars or dimensional regularization are more suited and much easier to employ. The main virtue of the lattice regularization is instead to allow nonperturbative investigations, which usually need some perturbative calculations to be interpreted properly. As we have already mentioned, the connection from Monte Carlo results of matrix elements to their corresponding physical numbers, that is the matching with the continuum physical theory, has to be realized by performing a lattice renormalization. It is in this context that lattice perturbation theory has a wide and useful range of applications, and we will discuss this important aspect of lattice computations in more detail in the next Section. In this respect, perturbative lattice renormalization is important by itself as well as in being a hint and a guide for the few cases in which one can also determine the renormalization constants nonperturbatively according to the method proposed in (Martinelli *et al.*, 1995) (for a recent review see (Sommer, 2002)). This is even more important in the cases in which operator mixings are present, which are generally more complex than in the continuum. In fact, mixing patterns on the lattice become in general more transparent when looked at using perturbative renormalization than nonperturbatively. We should also add that perturbative coefficients can be usually computed much more accurately than typical quantities in numerical simulations. Perturbative renormalization results can in any case be quite useful in checking and understanding results coming from nonperturbative methods (when available). When short-distance quantities can be calculated using such diverse techniques as lattice perturbation theory and Monte Carlo simulations, the comparison of the respective results, repeated in different situations, can give significant suggestions on the validity of perturbative and nonperturbative methods.

In some cases a nonperturbative determination of the renormalization constants can turn out to be rather difficult to achieve. For the method to work, it is necessary that there is a plateau for the signal over a substantial range of energies so that one can numerically extract the values of the renormalization factors. The nonperturbative renormalization methods can then fail because a window which is large enough cannot be found. Moreover, where mixings are present these methods could come out to be useless because the amount of mixing is too small to be disentangled with numerical techniques, although still not too small to be altogether ignored. It is then not clear whether all operators can be renormalized nonperturbatively in such a way. In these cases the only possibility to compute renormalization factors seems to be provided by the use of lattice perturbative methods. An important exception to this is given by the Schrödinger functional scheme, where using a particular procedure known as recursive finite-size scaling technique (which we will explain in Section 12) it is possible to carry out precise nonperturbative determinations of renormalized couplings, masses and operators for an extremely wide range of energies. Computations using the Schrödinger functional are however rather more involved than average and usually require much bigger computational resources.

We would also like to point out that in the case of Ginsparg-Wilson fermions the numerical computations required to extract nonperturbative renormalization factors (which come on top of the already substantial effort needed to determine the bare matrix elements) could turn
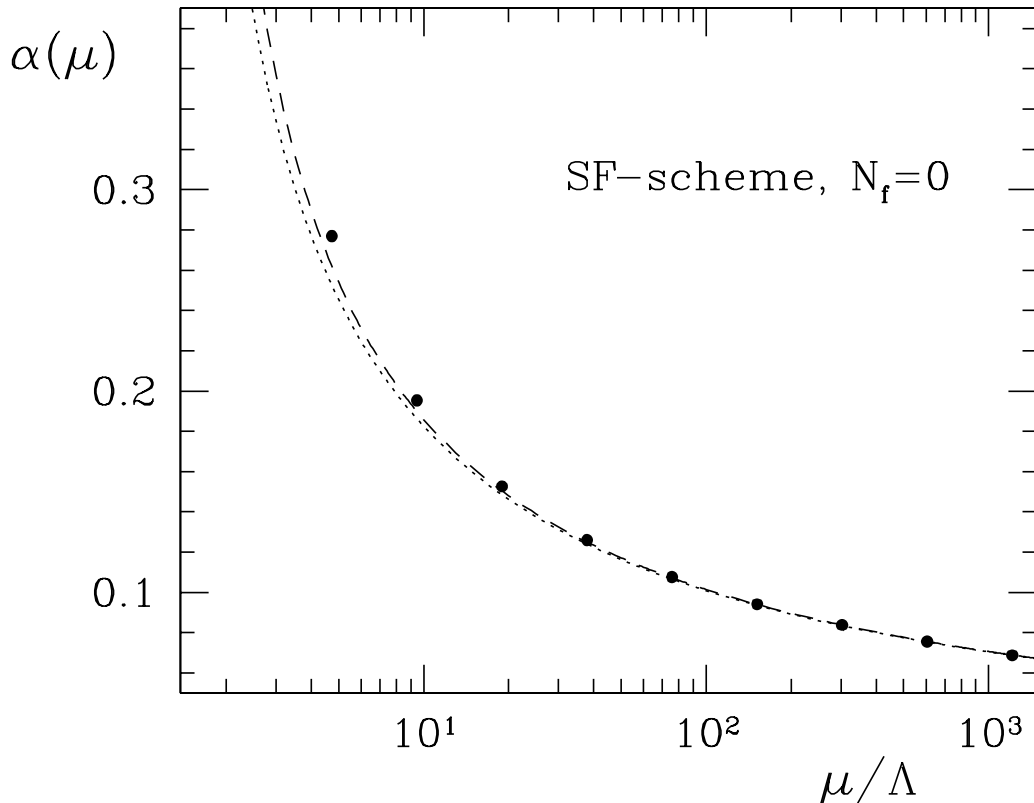
Figure 1: Perturbative and nonperturbative running of the renormalized strong coupling from the Schrödinger functional on the lattice, from (Capitani *et al.*, 1999c). In this scheme $\Lambda \sim 116$ M*eV*.

out to be quite expensive, especially in the cases of complicated operators like the ones that measure moments of parton distributions.

We can thus say, after having looked at all these different aspects, that lattice perturbation theory is important and fundamental. Of course sometimes there can be issues concerning its reliability when a 1-loop perturbative correction happens to be large, especially when the corresponding 2-loop calculation looks rather difficult to carry out. [2]

On the other hand, there are cases in which lattice perturbation theory works rather well. As an example we show in Figs. 1 and 2 the scale evolutions of the renormalized strong coupling and masses computed in the Schrödinger functional scheme (Capitani *et al.*, 1999c). [3] We can see that these scale evolutions are accurately described by perturbation theory for a wide range of energies. The perturbative and nonperturbative results are very close to each other, and

---

[2] In this case mean-field improved perturbation theory, using Parisi's boosted bare coupling, is known to reduce the magnitude of these corrections in many situations (see Section 15.6).

[3] An explanation of the way these nonperturbative evolutions are obtained is given in Section 12.
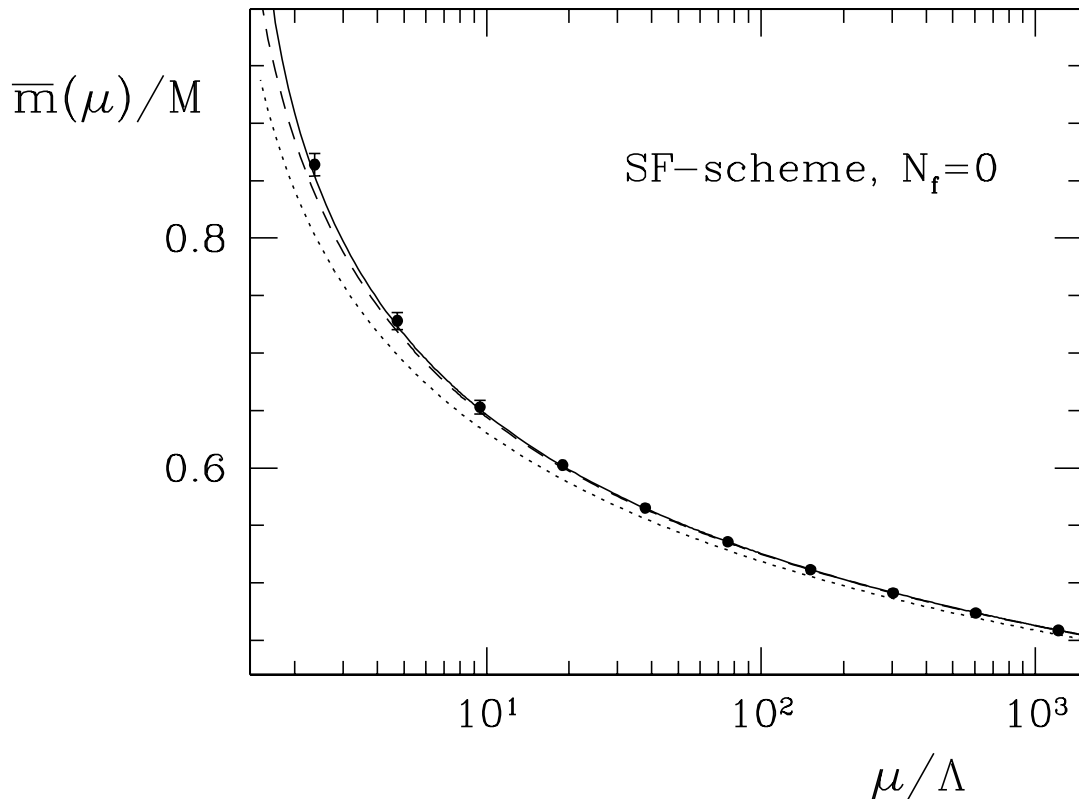
Figure 2: Perturbative and nonperturbative running of the renormalized masses from the Schrödinger functional on the lattice, from (Capitani *et al.*, 1999c). In this scheme $\Lambda \sim 116$ MeV. The renormalization group invariant mass M is defined in Eq. (10.34).

almost identical even down to energy scales which are surprising low. The best perturbative curves shown are obtained by including the $b_2 g_0^7$ term of the β function and the $d_1 g_0^4$ term of the τ function (that is, the first nonuniversal coefficients). The other curves are lower approximations. In Section 10 more details about these functions can be found.

Although the coupling and masses in the figures are computed in the Schrödinger functional scheme, and their values are different from, say, a calculation done with standard Wilson fermions, this example shows how close perturbation theory can come to nonperturbative results. The case shown is particularly interesting, because these nonperturbative results are among the best that one can at present obtain. The Schrödinger functional coupled to recursive finite-size techniques allows to control the systematic errors completely. There are fewer errors in these calculations, and they are fully understood. In other situations we cannot really exclude, when we see a discrepancy between nonperturbative and perturbative results, that at least part of this discrepancy originates from the nonperturbative side.

Another nice example of the good behavior of lattice perturbation theory is given in Fig. 3, which comes from the work of (Necco and Sommer, 2001; Necco, 2002a). These authors have computed the running coupling from the static quark force or potential in three different ways, corresponding to three different schemes: [4]

$$F(r) \;\; = \;\; \frac{dV}{dr} = C_F \frac{\alpha_{qq}(1/r)}{r^2}, \tag{2.1}$$

$$V(r) \;\; = \;\; -C_F \frac{\alpha_{\overline{V}}(1/r)}{r}, \tag{2.2}$$

$$\tilde{V}(Q) \;\; = \;\; -4\pi C_F \frac{\alpha_V(Q)}{Q^2}, \tag{2.3}$$

and showed that in first case, called qq scheme, the perturbative expansion of the coupling is rather well behaved, that is the coefficients are small and rapidly decreasing. This is what is shown in Fig. 3, which indicates that the perturbative computations can be trusted to describe with a good approximation the nonperturbative numbers for couplings up to $\alpha_{qq} \approx 0.3$. In the other two schemes, however, the coefficients are somewhat larger (especially in the last case), and the perturbative expansion of the coupling has a worse behavior. The perturbative couplings in these two schemes have a more pronounced difference with respect to the nonperturbative results.

The three schemes above differ only by kinematics, and the results show how the choice of one scheme or another can have a big influence on the perturbative behavior of the coupling. Discussions about the validity of lattice perturbation theory cannot then be complete until the dependence on the renormalization scheme (beside the dependence on the various actions used) is also taken into account and investigated. Not all schemes are suitable to be used in lattice perturbation theory to the same extent. In particular, the qq scheme is the best one among the three considered above. From this point of view the coupling in the Schrödinger functional scheme is even better behaved than α_{qq}, in other words the coefficient b_2 of the β function (the first nonuniversal coefficient) is the smallest in this scheme, and in this sense the Schrödinger functional scheme is the closest to the $\overline{\text{MS}}$ scheme.

We would also like to mention the work of (Davies *et al.*, 1997), where the QCD coupling is extracted from various Wilson loops, and which shows another instance of a good agreement between perturbation theory and simulations.

We can thus trust lattice perturbation theory, under the right conditions. It is probably not worse behaved than perturbative QCD in the continuum, which is an asymptotic expansion, and which also gives origin to large corrections in some cases. We could even say that perturbation theory can be better tested on the lattice than in the continuum, because in a lattice scheme one has also at his disposal the nonperturbative results to compare perturbation theory with. When lattice perturbation theory does not agree with nonperturbative numerical results, perhaps a look at the systematic errors coming from the numerical side can sometimes be worthwhile.

---

[4]A few different ways of defining a strong coupling constant on the lattice are discussed in (Weisz, 1996).
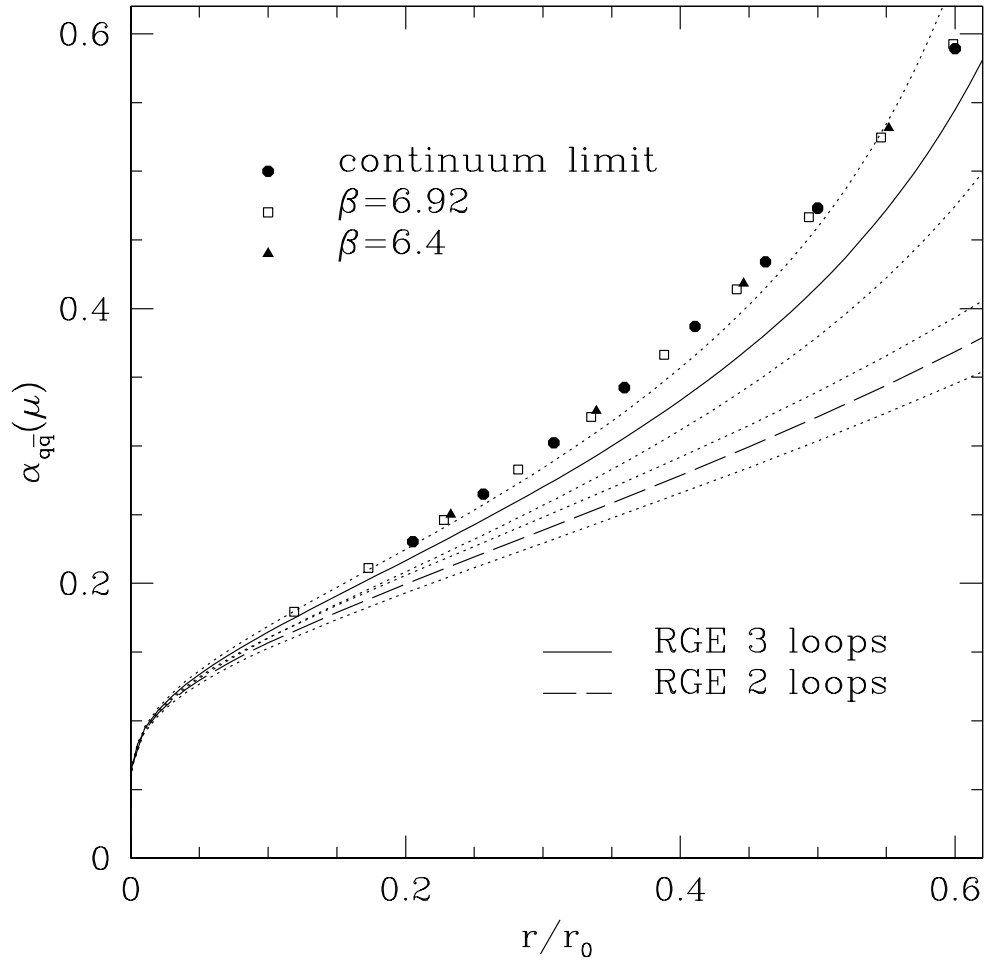
Figure 3: Perturbative and nonperturbative running of the renormalized strong coupling in the qq scheme, from (Necco and Sommer, 2001).

3   Renormalization of operators

In general, matrix elements of operators computed on the lattice using numerical simulations still require a renormalization in order to be converted into meaningful physical quantities. The Monte Carlo matrix elements can be considered as (regulated) bare numbers, and to get physical results one needs to perform a renormalization, matching the numbers to some continuum scheme, which is usually chosen to be the $\overline{\text{MS}}$ scheme of dimensional regularization.

In many physical problems one evaluates matrix elements of operators that appear in an operator product expansion. These matrix elements contain the long-distance physics of the system, and are computed numerically on the lattice, while the Wilson coefficients contain the short-distance physics, which can be gathered from perturbative calculations in the continuum. In this case the operators computed on the lattice must at the end be matched to the same continuum scheme in which the Wilson coefficient are known. Therefore one usually chooses to do the matching from the lattice to the $\overline{\text{MS}}$ scheme of dimensional regularization. A typical example is given by the moments of deep inelastic structure functions, and we will illustrate many features of perturbation theory in the course of this review using lattice operators appearing in operator product expansions which are important for the analysis of structure functions.

Thus, on the lattice one has to perform a matching to a continuum scheme, that is one looks for numbers which connect the bare lattice results to physical continuum renormalized numbers. We will now discuss how a perturbative matching can be done at one loop. Some good introductory material on the matching between lattice and continuum and on the basic concepts of lattice perturbation theory can be found in (Sachrajda, 1990; Sharpe, 1994; Sharpe, 1995). A short review of the situation of perturbative calculations at around 1995 is given in (Morningstar, 1996).

It turns out that to obtain physical continuum matrix elements from Monte Carlo simulations one needs to compute 1-loop matrix elements on the lattice as well as in the continuum. Lattice operators are chosen so that at tree level they have the same matrix elements as the original continuum operators, at least for momenta much lower than the lattice cutoff, $p \ll \pi/a$. Then at one loop one has

$$\langle q|O_i^{lat}|q\rangle \;\; = \;\; \sum_j \left(\delta_{ij} + \frac{g_0^2}{16\pi^2}\left(-\gamma_{ij}^{(0)} \log a^2 p^2 + R_{ij}^{lat} \right) \right) \cdot \langle q|O_j^{tree}|q\rangle \qquad (3.1)$$

$$\langle q|O_i^{\overline{\text{MS}}}|q\rangle \;\; = \;\; \sum_j \left(\delta_{ij} + \frac{g_{\overline{\text{MS}}}^2}{16\pi^2}\left(-\gamma_{ij}^{(0)} \log \frac{p^2}{\mu^2} + R_{ij}^{\overline{\text{MS}}} \right) \right) \cdot \langle q|O_j^{tree}|q\rangle. \qquad (3.2)$$

The lattice and continuum finite constants, R_{ij}^{lat} and $R_{ij}^{\overline{\text{MS}}}$, and therefore the lattice and continuum 1-loop renormalization constants, do not have the same value. [5] This happens because

---

[5]We note that while R_{ij}^{lat} is the whole momentum-independent 1-loop correction, $R_{ij}^{\overline{\text{MS}}}$ does not contain the pole in ϵ and the factors proportional to γ_E and $\log 4\pi$.

lattice propagators and vertices, as will be seen in detail later on, are quite different from their continuum counterparts, especially when the loop momentum is of order $1/a$. Therefore the 1-loop renormalization factors on the lattice and in the continuum are in general not equal. Note however that the 1-loop anomalous dimensions are the same.

The connection between the original lattice numbers and the final continuum physical results is given, looking at Eqs. (3.1) and (3.2), by

$$\langle q|O_i^{\overline{\text{MS}}}|q\rangle = \sum_j \left(\delta_{ij} - \frac{g_0^2}{16\pi^2}\left(-\gamma_{ij}^{(0)}\log a^2\mu^2 + R_{ij}^{lat} - R_{ij}^{\overline{\text{MS}}} \right) \right) \cdot \langle q|O_j^{lat}|q\rangle. \qquad (3.3)$$

The differences $\Delta R_{ij} = R_{ij}^{lat} - R_{ij}^{\overline{\text{MS}}}$ enter then in the matching factors

$$Z_{ij}(a\mu, g_0) = \delta_{ij} - \frac{g_0^2}{16\pi^2}\left(-\gamma_{ij}^{(0)}\log a^2\mu^2 + \Delta R_{ij} \right), \qquad (3.4)$$

which are the main objective when one performs computations of renormalization constants of operators on the lattice. [6] Lattice operators have more mixing options than continuum ones, due to the lower symmetry of the lattice theory. There is no Lorentz invariance (as we will see in more detail in Section 13), and in many cases other symmetries, like chiral symmetry, are also broken. Thus, the matching factors are not in general square matrices, that is one has $i \leq j$ in Eq. (3.3). To include all relevant operators one should look at the tree-level structures which appear in the calculation of lattice radiative corrections.

While R^{lat} and $R^{\overline{\text{MS}}}$ depend on the state $|q\rangle$, ΔR is independent of it and the Z depends only on $a\mu$. This is as it should be, since the renormalization factors are a property of the operators and are independent of the particular states with which a given matrix element is constructed. This is the reason why we have not specified $|q\rangle$. Furthermore, the matching

---

[6]The coupling that appears in Eq. (3.3) is usually chosen to be the lattice one, g_0, as advocated in (Sachrajda, 1990). Of course choosing one coupling or the other makes only a 2-loop difference, but these terms could still be numerically important. The validity of this procedure should be checked by looking at the size of higher-order corrections. Unfortunately on the lattice these terms are known only in a very few cases and no definite conclusions can then be reached with regard to this point.

In the work of (Ji, 1995) a generalization to higher loops was proposed, which gives an exact matching condition to all orders. This is done using the lattice and continuum renormalization group evolutions (see Section 10). Using the lattice evolution, one goes to very high energies, which means very small couplings because of asymptotic freedom, and there he does the matching to $\overline{\text{MS}}$. It thus essentially becomes a matching at the tree level. After this, one goes back to the original scale μ, using the continuum renormalization group evolution backwards. For the matching at the scale μ one then obtains a factor

$$\exp\left\{ -\int_0^{g_{\overline{\text{MS}}}(\mu)} du\, \frac{\gamma_{\overline{\text{MS}}}(u)}{\beta_{\overline{\text{MS}}}(u)} \right\} \cdot \exp\left\{ -\int_{g_0(a)}^0 dv\, \frac{\gamma_{lat}(v)}{\beta_{lat}(v)} \right\}, \qquad (3.5)$$

where the γ function governs the evolution of the renormalized operator. Effectively this formula uses the high-order coefficients of the β and γ functions. This approach has been used in (Gupta, Bhattacharya and Sharpe, 1997), where a discussion on these issues is made.

factors between the lattice and the $\overline{\text{MS}}$ scheme are gauge invariant, and this property can be exploited to make important checks of lattice perturbative calculations.

At the end of the process that we have just explained one has computed, using both lattice and continuum perturbative techniques, the renormalization factor $Z_O(a\mu)$ which converts the lattice operator $O(a)$ into the physical renormalized operator $\widehat{O}(\mu)$:

$$\widehat{O}(\mu) = Z_O(a\mu)\, O(a). \tag{3.6}$$

The reader should always keep in mind that in this way one does the matching of the bare Monte Carlo results (obtained using a lattice regulator) to the physical renormalized results in the $\overline{\text{MS}}$ scheme.

As for any general quantum field theory, the process at the end of which one obtains physical numbers is accomplished in two steps. First one regularizes the ultraviolet divergences, and in this case the regulator is given by the lattice itself. Afterwards one renormalizes the theory so regulated, and on the lattice this results in a matching to a continuum scheme. In the end, after renormalization the lattice cutoff must be removed, which means that one has to go to the continuum limit $a \to 0$ of the lattice theory, and that it is safe to do so at this point (if the renormalization procedure is consistent). What remains after all this is only the scale μ brought in by the renormalization. In our case the scale μ at which the matrix elements are renormalized should be in the range

$$\Lambda_{QCD} < \mu < \frac{\pi}{a}. \tag{3.7}$$

The lower bound ensures that perturbation theory is valid, while the upper bound ensures that the cutoff effects, proportional to positive powers of the lattice spacing, are small. One usually sets

$$\mu = \frac{1}{a}, \tag{3.8}$$

and since the 1-loop anomalous dimensions are the same on the lattice and in the continuum, only a finite renormalization connects the lattice to the $\overline{\text{MS}}$ scheme:

$$\langle q|O_i^{\overline{\text{MS}}}|q\rangle = \sum_j \left(\delta_{ij} - \frac{g_0^2}{16\pi^2}\left(R_{ij}^{lat} - R_{ij}^{\overline{\text{MS}}}\right)\right) \cdot \langle q|O_j^{lat}|q\rangle. \tag{3.9}$$

Every lattice action defines a different regularization scheme, and therefore these finite renormalization factors are in principle different for different actions. The bare numbers, that is the Monte Carlo results for a given matrix element, are also different, and so everything adjusts to give the same physical result.

We conclude mentioning that Sharpe (1994) has observed that when the operators come from an operator product expansion one should multiply the 1-loop matching factors introduced above with the *2-loop* Wilson coefficients, in order to be consistent. This can be seen by looking

at the 2-loop renormalization group evolution for the Wilson coefficients,

$$\frac{c(\mu_1)}{c(\mu_2)} = \left(\frac{g_0^2(\mu_1)}{g_0^2(\mu_2)}\right)^{-\gamma^{(0)}/2\beta_0} \left[1 + \frac{g_0^2(\mu_2) - g_0^2(\mu_1)}{16\pi^2}\left(\frac{\gamma^{(1)}}{2\beta_0} - \frac{\gamma^{(0)}\beta_1}{2\beta_0^2}\right) + O(g_0^4)\right]. \tag{3.10}$$

The term proportional to $g_0^2(\mu_2) - g_0^2(\mu_1)$ is analogous to a 1-loop matching factor, but this term contains β_1 and $\gamma^{(1)}$, which are 2-loop coefficients. So, it is only combining 1-loop renormalization matching with 2-loop Wilson coefficients that one is doing calculations in a consistent way.

In this Section we have learned that we need perturbative lattice calculations in order to extract a physical number from Monte Carlo simulations of matrix elements of operators (unless one opts for nonperturbative renormalization, when this is possible). We will try to explain how to do this kind of calculations in the rest of the review.

4   Discretization

Lattice calculations are done in euclidean space. A new time coordinate is introduced by doing a Wick rotation from Minkowski space to imaginary times:

$$x_0^E = \mathrm{i}x_0^M. \tag{4.1}$$

In momentum space this corresponds to $k_0^E = -\mathrm{i}k_0^M$, so that the Fourier transforms continue in euclidean space to be defined through the same phase factor. The reason for working with imaginary times is that the imaginary factor in front of the Minkowski-space action becomes a minus sign in the euclidean functional integral,

$$\mathrm{e}^{\mathrm{i}S_M} \longrightarrow \mathrm{e}^{-S_E}, \tag{4.2}$$

and the lattice field theory in euclidean space acquires many analogies with a statistical system. The path integral of the particular quantum field theory under study becomes the partition function of the corresponding statistical system. The transition to imaginary times brings a close connection between field theory and statistical physics which has many interesting facets. In particular, when the euclidean action is real and bounded from below one can see the functional integral as a probability system weighted by a Boltzmann distribution e^{-S_E}. It is this feature that allows Monte Carlo methods to be used. [7] Furthermore, on a euclidean lattice of finite volume the path integral is naturally well defined, since the measure contains only a finite number of variables and the exponential factor, which has a negative exponent, gives an absolutely convergent multi-dimensional integral. One then can generate configurations with

---

[7]However, when the action is complex, like in the case of QCD with a finite baryon number density, this is not possible. It is this circumstance that has hampered progress in the lattice studies of finite density QCD.

the appropriate probability distribution and in this way sample the functional integral with Monte Carlo techniques. This is the practical basis of Monte Carlo simulations.

From now on we will work in the euclidean space in four dimensions, with metric (1,1,1,1), and we will drop all euclidean subscripts from lattice quantities, so that x_0 is for example the time component after the Wick rotation. The Dirac matrices in euclidean space satisfy an anticommutation relation with $g_{\mu\nu}$ replaced by $\delta_{\mu\nu}$:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \tag{4.3}$$

and they are all hermitian:

$$(\gamma_\mu)^\dagger = \gamma_\mu. \tag{4.4}$$

The euclidean γ_5 matrix is defined by

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \tag{4.5}$$

is also hermitian, and satisfies $(\gamma_5)^2 = 1$. The relation between Dirac matrices in Minkowski and euclidean space is

$$\gamma_0^E = \gamma_0^M, \quad \gamma_i^E = -i\gamma_i^M. \tag{4.6}$$

This can be inferred from the kinetic term of the Dirac action in the functional integral:

$$\exp\left\{i\overline{\psi}\gamma_\mu^M \partial_\mu^M \psi\right\} \longrightarrow \exp\left\{-\overline{\psi}\gamma_\mu^E \partial_\mu^E \psi\right\}. \tag{4.7}$$

The explicit euclidean Dirac matrices in the chiral representation are given in Appendix A.

We want to construct field theories on a hypercubic lattice. This is a discrete subset of the euclidean spacetime, where the sites are denoted by $x_\mu = an_\mu$ (with n_μ integers). We will work in this review only with hypercubic lattices, where the lattice spacing is the same in all directions. A 2-dimensional projection of such a (finite) lattice is given in Fig. 4. For convenience we will sometimes omit to indicate the lattice spacing a, that is we will use $a = 1$. The missing factors of a can always be reinstated by doing a naive dimensional counting.

In going from continuum to lattice actions one replaces integrals with sums,

$$\int d^4x \to a^4 \sum_x, \tag{4.8}$$

where on the right-hand side x means now sites: $x = an$. [8] Lattice actions are then written in terms of sums over lattice sites. The distance between neighboring sites is a, and this minimum distance induces a cutoff on the modes in momentum space, so that the lattice spacing a acts as an ultraviolet regulator. The range of momenta is thus restricted to an interval of range $2\pi/a$, called the first Brillouin zone, and which can be chosen to be

$$B_Z = \left\{k : -\frac{\pi}{a} < k_\mu \leq \frac{\pi}{a}\right\}. \tag{4.9}$$

---

[8]We use in general the same symbols for continuum and lattice quantities, hoping that this does not cause confusion in the reader. An exception is given by the lattice derivatives, for which we will use special symbols.
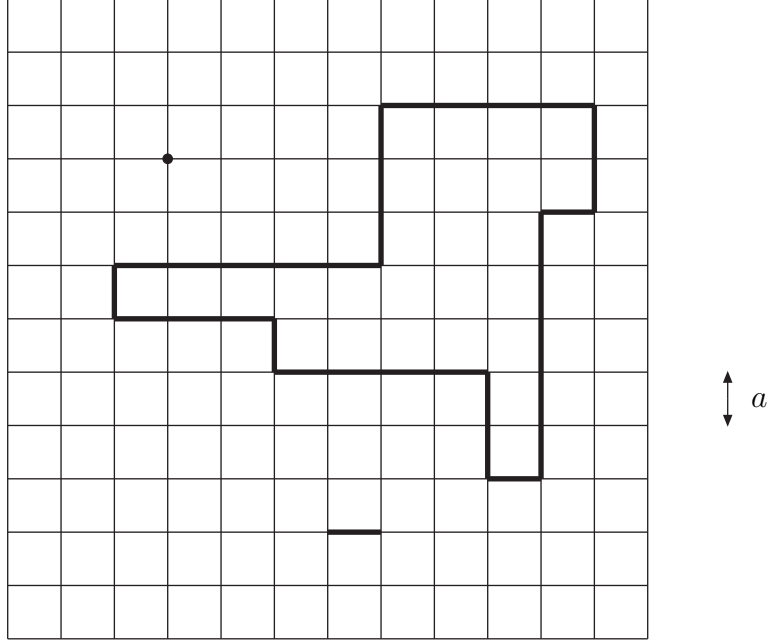
Figure 4: A 2-dimensional projection of a lattice. A site, a link and a closed loop are also shown.

This is region of the allowed values of k, and is the domain of integration when lattice calculations are made in momentum space. For a lattice of finite volume $V = L_0 L_1 L_2 L_3$ the allowed momenta in the first Brillouin zone become a discrete set, given by

$$(k_n)_\mu = \frac{2\pi}{a} \frac{n_\mu}{L_\mu} \qquad n_\mu = -L_\mu/2 + 1, \ldots, 0, 1, \ldots, L_\mu/2, \tag{4.10}$$

and so in principle one deals with sums also in momentum space. However, as we will see later, when doing perturbation theory one assumes to do the calculations in infinite volume and so the sums over the modes of the first Brillouin zone become integrals:

$$\frac{1}{V} \sum_k \longrightarrow \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_0}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_2}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_3}{2\pi}. \tag{4.11}$$

The one-sided forward and backward lattice derivatives (also known as right and left derivatives) can be respectively written as

$$\nabla_\mu \psi(x) = \frac{\psi(x + a\hat{\mu}) - \psi(x)}{a}, \tag{4.12}$$

$$\nabla_\mu^\star \psi(x) = \frac{\psi(x) - \psi(x - a\hat{\mu})}{a}, \tag{4.13}$$

where $\hat{\mu}$ denotes the unit vector in the μ direction. It is easy to see that

$$(\nabla_\mu)^\dagger = -\nabla_\mu^\star, \tag{4.14}$$

$$(\nabla_\mu^\star)^\dagger = -\nabla_\mu, \tag{4.15}$$

21

that is they are (up to a sign) conjugate to each other. Therefore ∇_μ or $\nabla_\mu^\star$ alone cannot be chosen in a lattice theory that is supposed to have a hermitian Hamiltonian. In this case one needs their sum, $\nabla_\mu + \nabla_\mu^\star$, which is anti-hermitian and gives a lattice derivative operator extending over the length of two lattice spacings:

$$\frac{1}{2}\left(\nabla + \nabla^\star\right)_\mu \psi(x) = \frac{\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})}{2a}. \tag{4.16}$$

Note that the second-order differential operator $\nabla_\mu \nabla_\mu^\star = \nabla_\mu^\star \nabla_\mu$ is hermitian, and when μ is summed corresponds to the 4-dimensional lattice Laplacian,

$$\Delta \psi(x) = \sum_\mu \nabla_\mu^\star \nabla_\mu \psi(x) = \sum_\mu \frac{\psi(x + a\hat{\mu}) + \psi(x - a\hat{\mu}) - 2\psi(x)}{a^2}. \tag{4.17}$$

It is also useful to know that

$$\sum_x (\nabla_\mu f(x)) g(x) = -\sum_x f(x)(\nabla_\mu^\star g(x)), \tag{4.18}$$

that is,

$$\sum_x \Big(f(x + a\hat{\mu}) g(x) - f(x) g(x) \Big) = \sum_x \Big(f(x) g(x - a\hat{\mu}) - f(x) g(x) \Big), \tag{4.19}$$

which is valid for an infinite lattice, and also for a finite one if f and g are periodic (or their support is smaller than the lattice). The formula above corresponds to an integration by parts on the lattice, which in practical terms amounts to a shift of the summation variable.

There is in general some freedom in the construction of lattice actions. For the discretization of continuum actions and operators and the practical setting of the corresponding lattice theory many choices are possible. Since the lattice symmetries are less restrictive than the continuum ones, there is more than one possibility in formulating a gauge theory starting from a given continuum gauge theory. In particular, one has quite a few choices for the exact form of the QCD action on the lattice, depending on what features are of interest in the studies that one wants to carry out using the discretized version. There is not an optimal lattice action to use in all cases, and each action has some advantages and disadvantages which weigh differently in different contexts. This means that deciding for one action instead of another depends on whether chiral symmetry, flavor symmetry, locality, or unitarity are more or less relevant to the physical system under study. There is a special emphasis on the symmetry properties. One also evaluates the convenience of lattice actions by considering some balance between costs and gains from the point of view of numerical simulations and of perturbation theory. Therefore, the final choice of a lattice action depends also on the problem that one wants to study.

There are many lattice actions which fall in the same universality class, that is they have the same naive continuum limit, and each of them constitutes a different regularization, for finite a, of the same physical theory. Since every lattice action defines a different regularization scheme, one needs for each action that is used a new complete set of renormalization computations

of the type discussed in Section 3, in order for the results which come out from Monte Carlo simulations to be used, interpreted and understood properly. Using different actions leads to different numerical results for the matrix elements computed in Monte Carlo simulations, and also the values of the renormalization factors, and of the Λ parameter, depend in general on the lattice action chosen. Even the number and type of counterterms required in the renormalization of operators can be different in each case. For example, a weak operator which is computed on the lattice using the Wilson action, where chiral symmetry is broken, needs in general more counterterms to be renormalized than when is computed with the overlap action, which is chiral invariant. Of course all the various differences among lattice actions arise only at the level of finite lattice spacing. The final extrapolations to the continuum limit must lead, within errors, to the same physical results.

In the case of QCD there seems to be a lot of room in choosing an action for the fermion part, although also the pure gauge action has some popular variants (but the plaquette, or Wilson, action has a clear predominance over the other ones here, except in particular situations, where for example improved gauge actions may be more convenient). The main features of perturbation theory will be first introduced in the context of Wilson fermions, which are one of the most widely used lattice formulations. Then a few other fermion actions will be discussed along the way, pointing out the differences with the standard perturbation theory made with Wilson fermions, which is generally simpler.

5   Wilson's formulation of lattice QCD

One of the most popular lattice formulations of QCD is the one invented by Wilson (1974; 1977), which was also the first formulation ever for a lattice gauge theory. Its remarkable feature is that it maintains exact gauge invariance also at any nonzero values of the lattice spacing. The discretization of the (euclidean) QCD action for one quark flavor

$$S = \int d^4x \left[\overline{\psi}(x)\left(\slashed{D} + m_f \right)\psi(x) + \frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x) \right] \right] \tag{5.1}$$

that Wilson proposed is the following:

$$
\begin{aligned}
S_W \;=\; & a^4 \sum_x \left[-\frac{1}{2a} \sum_\mu \left[\overline{\psi}(x)(r - \gamma_\mu)U_\mu(x)\psi(x + a\hat{\mu}) \right. \right. \\
& \left. + \overline{\psi}(x + a\hat{\mu})(r + \gamma_\mu)U_\mu^\dagger(x)\psi(x) \right] + \overline{\psi}(x)\left(m_0 + \frac{4r}{a} \right)\psi(x) \right] \\
& + \frac{1}{g_0^2}\, a^4 \sum_{x,\mu\nu} \left[N_c - \mathrm{Re}\,\mathrm{Tr}\left[U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\nu^\dagger(x) \right] \right],
\end{aligned} \tag{5.2}
$$

where $x = an$ and $0 < r \leq 1$. This action has only nearest-neighbor interactions. [9] The derivative in the Dirac operator is the symmetric one, Eq. (4.16) (with an integration by parts). The fields $U_\mu(x)$ live on the links which connect two neighboring lattice sites, and these variables are naturally defined in the middle point of a link. Each link carries a direction, so that

$$U_\mu^{-1}(x) = U_\mu^\dagger(x) = U_{-\mu}(x + a\hat{\mu}). \tag{5.3}$$

These link variables, which are unitary, are not linear in the gauge potential $A_\mu(x)$. The fact is that they belong to the group $SU(N_c)$ rather than to the corresponding Lie algebra, as would be the case in the continuum. The relation of the $U_\mu(x)$ matrices with the gauge fields $A_\mu(x)$, the variables which have a direct correspondence with the continuum, is then given by

$$U_\mu(x) = e^{ig_0 a T^a A_\mu^a(x)} \quad (a = 1, \ldots, N_c^2 - 1), \tag{5.4}$$

where the T^a are $SU(N_c)$ matrices in the fundamental representation.

The Wilson action is a gauge-invariant regularization of QCD, and it has exact local gauge invariance on the lattice at any finite a. The gauge-invariant construction is done directly on the lattice, extending a discretized version of the *free* continuum fermionic action. It is not therefore a trivial straightforward discretization of the whole gauge-invariant continuum QCD action, which would recover the gauge invariance only in the continuum limit. A naive lattice discretization of the minimal substitution rule $\partial_\mu \to D_\mu$ would in fact result in an action that violates gauge invariance on the lattice, whereas with the choice made by Wilson gauge invariance is kept as a symmetry of the theory for any a. It is this requirement that causes the group variables U_μ to appear in the action instead of the algebra variables A_μ. The lattice gauge transformations are given by

$$
\begin{aligned}
U_\mu(x) &\rightarrow \Omega(x)\, U_\mu(x)\, \Omega^{-1}(x + a\hat{\mu}) \\
\psi(x) &\rightarrow \Omega(x)\, \psi(x) \\
\overline{\psi}(x) &\rightarrow \overline{\psi}(x)\, \Omega^{-1}(x),
\end{aligned} \tag{5.5}
$$

with $\Omega \in SU(N_c)$, and it is easy to see that they leave the quark-gluon interaction term in the Wilson action invariant. Note that also in the lattice theory the local character of the invariance group is maintained.

This form of local gauge invariance imposes strong constraints on the form of the gauge field-strength tensor $F_{\mu\nu}$. Given the above formula for the lattice gauge transformations of the U_μ's, it is easy to see that the simplest gauge-invariant object that one can build from the link variables involves their path-ordered product. In particular, one obtains a gauge-invariant quantity by taking the trace of the product of U_μ's on adjoining links forming a closed path, thanks to the unitarity of the U_μ's and the cyclic property of the trace.

---

[9]Other actions can have more complicated interactions, like for example overlap fermions (see Section 8).

The physical theory is a local one, and so in constructing the pure gauge action we should direct our attention toward small loops. The simplest lattice approximation of $F_{\mu\nu}$ is then the product of the links of an elementary square, called "plaquette":

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\nu^\dagger(x), \tag{5.6}$$

which is shown in Fig. 5. This form is not surprising, given that the gauge field-strength tensor is in differential geometry the curvature of the metric tensor. One could also take larger closed loops, but this minimal choice gives better signal-to-noise ratios, and for the standard Wilson action the trace of the plaquette is then used. [10] This is the expression appearing in the last line of Eq. (5.2). The factor N_c can be understood by looking at the expansion of the plaquette Eq. (5.6) in powers of a, which is

$$P_{\mu\nu}(x) = 1 + \mathrm{i}g_0 a^2 F_{\mu\nu}(x) - \frac{1}{2}g_0^2 a^4 F_{\mu\nu}^2(x) + \mathrm{i}a^3 G_{\mu\nu}(x) + \mathrm{i}a^4 H_{\mu\nu}(x), \tag{5.7}$$

with G and H hermitian fields. [11] We have then

$$\mathrm{Re}\,\mathrm{Tr}P_{\mu\nu}(x) = N_c - \frac{1}{2}g_0^2 a^4\,\mathrm{Tr}F_{\mu\nu}^2(x) + O(a^6), \tag{5.9}$$

where we have used $\mathrm{Tr}F_{\mu\nu} = 0$, because the trace of the generators is zero. The plaquette action then has the right continuum limit, and the first corrections to the continuum pure gauge action are of order a^2. These are irrelevant terms, which are zero in the continuum limit, but they are important for determining the rate of convergence to the continuum physics. It can also be shown that in the fermionic part of the action the corrections with respect to the continuum limit are of order a. In Section 11 we will see how to modify the fermion action in order to decrease the error on the fermionic part to order a^2.

The plaquette action is also often written as

$$\beta \cdot a^4 \sum_P \left(1 - \frac{1}{N_c}\mathrm{Re}\,\mathrm{Tr}U_P\right), \tag{5.10}$$

where U_P is given in Eq. (5.6), and in numerical simulations of QCD the coefficient in front of the action is

$$\beta = \frac{2N_c}{g_0^2} = \frac{6}{g_0^2}. \tag{5.11}$$

---

[10]Other actions which use different approximations for $F_{\mu\nu}$, and which have the aim of reducing the discretization errors, are discussed in Section 11.2.

[11]This expansion can be derived using

$$A_\mu(x + a\hat{\nu}) = A_\mu(x) + a\partial_\nu A_\mu(x) + \cdots. \tag{5.8}$$
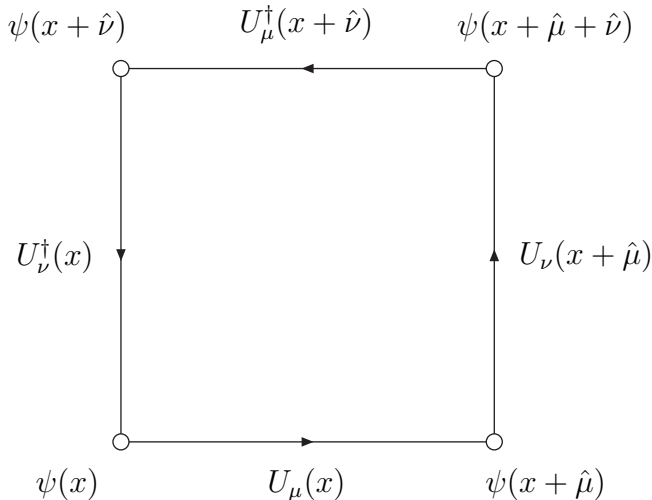
Figure 5: The plaquette.

The factor two comes out because here one takes the sum over the oriented plaquettes, that is a sum over ordered indices (for example, $\mu > \nu$), while in Eq. (5.2) the sum over μ and ν is free.

In the weak coupling regime, where g_0 is small, the functional integral is dominated by the configurations which are near the trivial field configuration $U_\mu(x) = 1$. Perturbation theory is then a saddle-point expansion around the classical vacuum configurations, and the relevant degrees of freedom are given by the components of the gauge potential, $A_\mu^a(x)$. Thus, while the fundamental gauge variables for the Monte Carlo simulations are the U_μ's and the action is relatively simple when expressed in terms of these variables, in perturbation theory the true dynamical variables are the A_μ's. This mismatch causes a good part of the complications of lattice perturbation theory. In fact, when the Wilson action is written in terms of the A_μ's, using $U_\mu = 1 + ig_0 a A_\mu - g_0^2 a^2 A_\mu^2 + \cdots$, it becomes much more complicated. Moreover, it consists of an infinite number of terms, which give rise to an infinite number of interaction vertices. Fortunately, only a finite number of vertices is needed at any given order in g_0.

All vertices except a few of them are "irrelevant", that is they are proportional to some positive power of the lattice spacing a and so they vanish in the naive continuum limit. However, this does not mean that they can be thrown away when doing perturbation theory. Quite on the contrary, they usually contribute to correlation functions in the continuum limit through divergent ($\sim 1/a^n$) loop corrections. These irrelevant vertices are indeed important in many cases, contributing to the mass, coupling and wave-function renormalizations (Sharatchandra, 1978). All these vertices are in fact necessary to ensure the gauge invariance of the physical amplitudes. Only when they are included can gauge-invariant Ward Identities be constructed, and the renormalizability of the lattice theory can be proven.

Figure 6: Diagrams for the self-energy of the gluon on the lattice. The diagrams on the upper row have a continuum analog, while the diagrams on the lower row are a 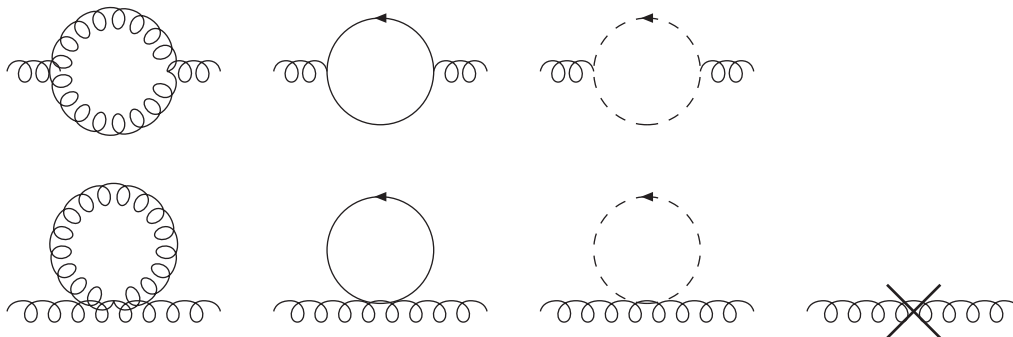pure lattice artifact. They are however necessary to maintain the gauge invariance of the lattice theory, and are important for its renormalizability.

An example of this fact is given by the diagrams contributing to the gluon self-energy at one loop (Fig. 6). If one would only consider the diagrams on the upper row, that is the ones that would also exist in the continuum, the lattice results would contain an unphysical $1/(am)^2$ divergence. This divergence is canceled away only when the results of the diagrams on the lower row are added, that is only when gauge invariance is fully restored. Notice that for this to happen also the measure counterterm is needed (see Section 5.2.1). In a similar way, terms of the type $p_\mu^2 \delta_{\mu\nu}$, which are not Lorentz covariant and are often present in the individual diagrams, disappear only after all diagrams have been considered and summed.

From what we have seen so far, we can understand that a lattice regularization does not just amount to introducing in the theory a momentum cutoff. In fact, it is a more complicated regularization than just setting a nonzero lattice spacing, because one has also to provide a lattice action. Different actions define different lattice regularizations. Because of the particular form of lattice actions, the Feynman rules are much more complicated that in the continuum, and in the case of gauge theories new interaction vertices appear which have no analog in the continuum. The structure of lattice integrals is also completely different, due to the overall periodicity which causes the appearance of trigonometric functions. The lattice integrands are then given by rational functions of trigonometric expressions.

At the end of the day, lattice perturbation theory is much more complicated than continuum perturbation theory: there are more fundamental vertices and more diagrams, and these propagators and vertices, with which one builds the Feynman diagrams, are more complicated on the lattice than they are in the continuum, which can lead to expressions containing a huge number of terms. Finally, one has also to evaluate more complicated integrals. Lattice perturbative calculations are thus rather involved. As a consequence, for the calculation of all but

the simplest matrix elements computer codes have to be used (see Section 16).

Matrix elements computed in euclidean space do not always correspond to the analytic continuation of matrix elements of a physical theory in Minkowski space. For this to happen, the lattice action has to satisfy a property known as reflection positivity, which involves time reflections and complex conjugations (roughly speaking is the analog of hermitian conjugation in Minkowski space). In this case the reconstruction theorem of Osterwalder and Schrader (1973; 1975) says that it is possible to reconstruct a Hilbert space in Minkowski space in the usual way starting from the lattice theory.

The Wilson action with $r = 1$ is reflection positive, and therefore corresponds to a well-defined physical theory in Minkowski space (Lüscher, 1977; Creutz, 1987). For $r \neq 1$ instead the lattice theory contains additional time doublers, which disappear in the continuum limit. [12] In the following we will only work with $r = 1$. This is what is usually meant for Wilson action.

5.1 Fourier transforms

To perform calculations of Feynman diagrams in momentum space (the main topic of this review) we need to define the Fourier transforms on the lattice. They are given in infinite volume (which is the standard setting in perturbation theory) by

$$
\begin{aligned}
\psi(x) &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p}{(2\pi)^4} \, e^{ixp} \, \psi(p), \\
\overline{\psi}(x) &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p}{(2\pi)^4} \, e^{-ixp} \, \overline{\psi}(p), \\
A_\mu(x) &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \, e^{i(x+a\hat{\mu}/2)k} \, A_\mu(k),
\end{aligned}
\tag{5.12}
$$

where with abuse of notation we indicate $\psi(x)$ and its Fourier transformed function with the same symbol. The inverse Fourier transforms are given by

$$
\begin{aligned}
\psi(p) &= a^4 \sum_x e^{-ixp} \, \psi(x) \\
\overline{\psi}(p) &= a^4 \sum_x e^{ixp} \, \overline{\psi}(x) \\
A_\mu(k) &= a^4 \sum_x e^{-i(x+a\hat{\mu}/2)k} \, A_\mu(x),
\end{aligned}
\tag{5.13}
$$

and this means that

$$
\delta^{(4)}(p) = \frac{a^4}{(2\pi)^4} \sum_x e^{-ixp}.
\tag{5.14}
$$

---

[12] This is at variance with the doublers which appear for $r = 0$ (naive fermion action), which do not disappear in the continuum limit, as we will see in Section 6.

This lattice delta function is zero except at $p = 2\pi n$. The Kronecker delta in position space is

$$\delta_{xy} = a^4 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p}{(2\pi)^4} \, e^{i(x-y)p}. \tag{5.15}$$

Of course on a lattice of finite volume the allowed momenta are a discrete set. However, in perturbation theory we will always consider the limit of infinite volume.

Notice that the Fourier transform of $A_\mu(x)$ is taken at the point $x + a\hat\mu/2$, that is in the middle of the link. This comes out naturally from its definition. This choice turns out to be quite important also for the general economy of the calculations, as we can see from the following example.

Let us consider the gauge interaction of the quarks in the Wilson action at first order in the gauge coupling, which gives rise to the quark-quark-gluon vertex. Going to momentum space we have

$$
\begin{aligned}
S_{qqg} &= -\frac{ig_0}{2} a^4 \sum_{x,\mu} \left(\overline\psi(x)(r - \gamma_\mu)A_\mu(x)\psi(x + a\hat\mu) - \overline\psi(x + a\hat\mu)(r + \gamma_\mu)A_\mu(x)\psi(x) \right) \\
&= -\frac{ig_0}{2} a^4 \sum_{x,\mu} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p'}{(2\pi)^4} \, e^{ix(p+k-p')} \, e^{iak_\mu/2} \\
&\quad \times \left(\overline\psi(p')(r - \gamma_\mu)A_\mu(k)\psi(p)e^{iap_\mu} - \overline\psi(p')e^{-iap'_\mu}(r + \gamma_\mu)A_\mu(k)\psi(p) \right) \\
&= \frac{ig_0}{2} \sum_\mu \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p'}{(2\pi)^4} \, (2\pi)^4 \delta^{(4)}(p + k - p') \, e^{iak_\mu/2} \\
&\quad \times \left(\overline\psi(p')\gamma_\mu A_\mu(k)\psi(p)(e^{iap_\mu} + e^{-iap'_\mu}) + r\overline\psi(p')A_\mu(k)\psi(p)(-e^{iap_\mu} + e^{-iap'_\mu}) \right) \\
&= \frac{ig_0}{2} \sum_\mu \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p'}{(2\pi)^4} \, (2\pi)^4 \delta^{(4)}(p + k - p') \, e^{iak_\mu/2} \\
&\quad \times \left(\overline\psi(p')\gamma_\mu A_\mu(k)\psi(p) \, e^{iap_\mu/2}e^{-iap'_\mu/2} \cdot 2\cos\frac{a(p+p')_\mu}{2} \right. \\
&\quad \left. + r\overline\psi(p')A_\mu(k)\psi(p) \, e^{iap_\mu/2}e^{-iap'_\mu/2} \cdot (-2i)\sin\frac{a(p+p')_\mu}{2} \right). \tag{5.17}
\end{aligned}
$$

We can notice at this point that all exponential phases cancel with each other, because of the δ function expressing the momentum conservation at the vertex (where $p' = p + k$). We are then left with

$$
\begin{aligned}
S_{qqg} &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p'}{(2\pi)^4} \, (2\pi)^4 \delta^{(4)}(p + k - p') \\
&\quad \times ig_0 \sum_\mu \overline\psi(p')\left(\gamma_\mu \cos\frac{a(p+p')_\mu}{2} - ir\sin\frac{a(p+p')_\mu}{2} \right) A_\mu(k)\psi(p), \tag{5.18}
\end{aligned}
$$

which gives us the lattice Feynman rule for this vertex. It is easy to see that in the continuum limit this Wilson vertex reduces to the familiar vertex of QCD,

$$\int_{-\infty}^{\infty} \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d^4 p'}{(2\pi)^4} \, (2\pi)^4 \delta^{(4)}(p + k - p') \cdot ig_0 \sum_\mu \overline\psi(p')\gamma_\mu A_\mu(k)\psi(p). \tag{5.19}$$

Notice that had we chosen for the Fourier transform of the gauge potential the expression

$$A_\mu(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \, e^{ixk} \, A_\mu(k) \tag{5.20}$$

the exponential phases would not have canceled, and the $e^{iap_\mu/2} e^{-iap'_\mu/2}$ terms would still be present in the final expression of the vertex. This is a general feature of lattice perturbation theory: if one uses the Fourier transforms in Eq. (5.12), all terms of the type $e^{iak_\mu/2}$ coming from the various gluons exactly combine to cancel all other phases floating around, and then only sine and cosine functions remain in the Feynman rules in momentum space. This cancellation becomes rather convenient in the case of complicated vertices containing a large number of gluons.

We are now going to give the explicit expressions for the propagators and for the vertices of order g_0 and g_0^2 of the Wilson action, which is all what is needed for 1-loop calculations. In the following we will not explicitly write the δ function of the momenta present in each vertex and propagator. In our conventions for the vertices, all gluon lines are entering, and when there are two quarks or ghosts one of them is entering and the other one is exiting.

5.2 Pure gauge action

As we have seen, in the Wilson formulation gauge invariance is imposed directly on the free fermion lattice action, so that the group elements $U_\mu(x)$ appear instead of the algebra elements $A_\mu(x)$, which are the fundamental perturbative variables. To derive the gluon vertices from the pure gauge action, one has then to expand the U_μ's in the plaquette in terms of the A_μ's. As a consequence, an infinite number of interaction vertices are generated, which express the self-interaction of n gluons, with any n. Since the power of the coupling which appears in these vertices grows with the number of gluons, only a finite number of them is needed at any given order in g_0.

In lattice QCD the A_μ's are also nontrivial color matrices, and therefore they do not commute with each other. The expansion of the plaquette in terms of the A_μ's can be carried out using the Baker-Campbell-Hausdorff formula

$$e^A e^B = \exp\left\{ A + B + \frac{1}{2}\big[A, B\big] + \frac{1}{12}\big[A - B, \big[A, B\big]\big] + \frac{1}{24}\big[\big[A, \big[A, B\big], B\big]\big] + \cdots \right\}. \tag{5.21}$$

Since the color matrices T^a are traceless and are closed under commutation, the exponent in the expansion of the plaquette obtained using the Baker-Campbell-Hausdorff formula is also traceless, so that the knowledge of the terms of this expansion which are cubic in A_μ^a is sufficient to calculate all vertices with a maximum of four gluons, that is to order g_0^2, which is all that is needed for 1-loop calculations. [13]

---

[13]For computing vertices of higher order it is useful to know that the Baker-Campbell-Hausdorff formula can

Figure 7: Propagators and vertices sufficient for 1-loop calculations in lattice QCD.

be written as

$$\mathrm{e}^A \mathrm{e}^B = \exp\Big\{ \sum_{n=1}^{\infty} C_n(A,B) \Big\}, \tag{5.22}$$

where the C_n's can be determined recursively:

$$\begin{aligned} C_{n+1}(A,B) &= \frac{1}{2(n+1)} \Big[A - B, C_n(A,B) \Big] \\ &+ \sum_{\substack{p \geq 1 \\ 2p \leq n}} \frac{B_{2p}}{(2p)!(n+1)} \sum_{\substack{m_1,\dots,m_{2p}>0 \\ m_1+\dots+m_{2p}=n}} \Big[C_{m_1}(A,B), \Big[\dots, \Big[C_{m_{2p}}(A,B), A+B \Big] \cdots \Big] \Big], \end{aligned} \tag{5.23}$$

with $C_1(A,B) = A + B$, and B_{2p} a Bernoulli number. The Bernoulli numbers B_i are defined by

$$\frac{x}{\mathrm{e}^x - 1} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_2 x^4}{4!} + \frac{B_3 x^6}{6!} - \cdots, \qquad |x| < 2\pi. \tag{5.24}$$

Looking at Eq. (5.4), we see that the entries of the matrices $ag_0 A_\mu(x)$ are angular variables, which thus assume values between zero and 2π. In the functional integral the range of integration of the fields $A_\mu^a(x)$ is extended to infinity, so that it is possible to work with Gaussian integrals in the zeroth-order computations. It is only after $A_\mu^a(x)$ has been decompactified that the propagators and the correlation functions of operators can then be computed.

The propagators come as usual from the inverse of the quadratic part of the action. For the full expression of the gluon propagator we have to wait until gauge fixing is implemented, and we will report it later. The 3-gluon vertex is:

$$
W_{\mu\nu\lambda}^{abc}(p,q,r) = -ig_0 f^{abc} \frac{2}{a} \left\{ \delta_{\mu\nu} \sin \frac{a(p-q)_\lambda}{2} \cos \frac{ar_\mu}{2} \right. \tag{5.25}
$$
$$
\left. + \delta_{\nu\lambda} \sin \frac{a(q-r)_\mu}{2} \cos \frac{ap_\nu}{2} + \delta_{\lambda\mu} \sin \frac{a(r-p)_\nu}{2} \cos \frac{aq_\lambda}{2} \right\},
$$

where $p + q + r = 0$, all momenta are entering and are assigned clockwise (see Fig. 7). This lattice vertex for $a \to 0$ reduces to the continuum vertex

$$
-ig_0 f^{abc} \left\{ \delta_{\mu\nu}(p-q)_\lambda + \delta_{\nu\lambda}(q-r)_\mu + \delta_{\lambda\mu}(r-p)_\nu \right\}. \tag{5.26}
$$

It is useful from now on to introduce the shorthand notation

$$
\widehat{ak}_\mu = \frac{2}{a} \sin \frac{ak_\mu}{2}, \tag{5.27}
$$

especially for writing the 4-gluon vertex, which is quite complicated. It is given by (Rothe, 1997):

$$
W_{\mu\nu\lambda\rho}^{abcd}(p,q,r,s) = -g_0^2 \left\{ \sum_e f_{abe} f_{cde} \left[\delta_{\mu\lambda}\delta_{\nu\rho} \left(\cos \frac{a(q-s)_\mu}{2} \cos \frac{a(k-r)_\nu}{2} - \frac{a^4}{12} \widehat{k}_\nu \widehat{q}_\mu \widehat{r}_\nu \widehat{s}_\mu \right) \right. \right.
$$
$$
- \delta_{\mu\rho}\delta_{\nu\lambda} \left(\cos \frac{a(q-r)_\mu}{2} \cos \frac{a(k-s)_\nu}{2} - \frac{a^4}{12} \widehat{k}_\nu \widehat{q}_\mu \widehat{r}_\mu \widehat{s}_\nu \right)
$$
$$
+ \frac{1}{6} \delta_{\nu\lambda}\delta_{\nu\rho} a^2 (\widehat{s-r})_\mu \widehat{k}_\nu \cos \frac{aq_\mu}{2} - \frac{1}{6} \delta_{\mu\lambda}\delta_{\mu\rho} a^2 (\widehat{s-r})_\nu \widehat{q}_\mu \cos \frac{ak_\nu}{2}
$$
$$
+ \frac{1}{6} \delta_{\mu\nu}\delta_{\mu\rho} a^2 (\widehat{q-k})_\lambda \widehat{r}_\rho \cos \frac{as_\lambda}{2} - \frac{1}{6} \delta_{\mu\nu}\delta_{\mu\lambda} a^2 (\widehat{q-k})_\rho \widehat{s}_\lambda \cos \frac{ar_\rho}{2}
$$
$$
\left. + \frac{1}{12} \delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho} a^2 \sum_\sigma (\widehat{q-k})_\sigma (\widehat{s-r})_\sigma \right]
$$
$$
+ (b \leftrightarrow c, \nu \leftrightarrow \lambda, q \leftrightarrow r) + (b \leftrightarrow d, \nu \leftrightarrow \rho, q \leftrightarrow s) \right\}
$$
$$
+ \frac{g_0^2}{12} a^4 \left\{ \frac{2}{3} (\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \sum_e (\delta_{abe}\delta_{cde} + \delta_{ace}\delta_{bde} + \delta_{ade}\delta_{bce}) \right\}
$$

The first few Bernoulli numbers are: $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$, $B_6 = 691/2730$, $B_7 = 7/6$.

$$\times \left\{ \delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho} \sum_\sigma \widehat{k}_\sigma \widehat{q}_\sigma \widehat{r}_\sigma \widehat{s}_\sigma - \delta_{\mu\nu}\delta_{\mu\lambda} \, \widehat{k}_\rho \widehat{q}_\rho \widehat{r}_\rho \widehat{s}_\mu \right.$$

$$-\delta_{\mu\nu}\delta_{\mu\rho} \, \widehat{k}_\lambda \widehat{q}_\lambda \widehat{s}_\lambda \widehat{r}_\mu - \delta_{\mu\lambda}\delta_{\mu\rho} \, \widehat{k}_\nu \widehat{r}_\nu \widehat{s}_\nu \widehat{q}_\mu - \delta_{\nu\lambda}\delta_{\nu\rho} \, \widehat{q}_\mu \widehat{r}_\mu \widehat{s}_\mu \widehat{k}_\nu$$

$$\left. +\delta_{\mu\nu}\delta_{\lambda\rho} \, \widehat{k}_\lambda \widehat{q}_\lambda \widehat{r}_\mu \widehat{s}_\mu + \delta_{\mu\lambda}\delta_{\nu\rho} \, \widehat{k}_\nu \widehat{r}_\nu \widehat{q}_\mu \widehat{s}_\mu + \delta_{\mu\rho}\delta_{\nu\lambda} \, \widehat{k}_\nu \widehat{s}_\nu \widehat{q}_\mu \widehat{r}_\mu \right\}. \tag{5.28}$$

In the continuum limit this expression becomes the four-gluon vertex of continuum QCD. [14]

The vertices containing five or more gluons are at least of order g_0^3, and thus they are not necessary for calculations at one loop. To my knowledge, an explicit expression for the five-gluon vertex has not yet been given in the literature. General algorithms for the automated calculation of higher-order vertices (for a given configuration of external momenta) have been reported in (Lüscher and Weisz, 1986).

For nonabelian gauge theories, the calculation of the pure gauge part of course does not end here. One has still to consider the gauge integration measure, which gives a $1/a^2$ mass counterterm at order g_0^2 and an infinite number of measure vertices of order g_0^3 and higher, as well as the Faddeev-Popov procedure on the lattice, with which the Feynman rules for the ghost propagator and the various ghost vertices can be derived. We anticipate that the effective ghost-gauge field interaction, at variance with the continuum, is not linear in the gauge potential A_μ, and thus also in this sector we find an infinite number of new vertices that have no continuum analog, like for example the vertex involving two ghosts and two gluons.

5.2.1   Measure

The definition of the gauge-invariant integration measure on the lattice turns out for nonabelian gauge groups to be nontrivial, and again generates an infinite number of vertices.

Let us at first consider only the gauge potential, A_μ, at a certain point. We start with the 2-form

$$d^2 s = \text{Tr}(dU_\mu^\dagger dU_\mu), \tag{5.31}$$

where $dU_\mu = U_\mu(A_\mu + dA_\mu) - U_\mu(A_\mu)$, which is invariant under left or right multiplication of U_μ with an $SU(3)$ matrix. When rewritten in terms of A_μ, this 2-form defines a metric g on

---

[14]It is interesting to note that also in lattice QED there are vertices in which the gauge particles are self-interacting, but of course they have to be irrelevant in the continuum limit. The lowest-order vertex in pure gauge contains four lattice photons coming from the $a^4 g_0^2 F_{\mu\nu}^4$ term of the expansion of the lattice QED action

$$\frac{1}{2a^4 g_0^2} \cdot a^4 \sum_x \sum_{\mu\nu} \left(1 - \cos(a^2 g_0 F_{\mu\nu}(x)) \right), \tag{5.29}$$

where

$$F_{\mu\nu}(x) = \nabla_\mu A_\nu(x) - \nabla_\nu A_\mu(x). \tag{5.30}$$

This vertex contains four derivatives, it is of order a^4 and it disappears in the continuum limit.

the space of the A_μ's,

$$d^2s = g_{ab}(A)\, dA_\mu^a dA_\mu^b, \tag{5.32}$$

which gives the gauge-invariant measure (the Haar measure) [15]

$$d\mu(A) = \sqrt{\det g(A)} \prod_{a,\mu} dA_\mu^a. \tag{5.35}$$

The calculation of $g(A)$ can be done using the properties of the $SU(3)$ group. It turns out that the infinitesimal transformation can be written as (Boulware, 1970)

$$U_\mu(A_\mu + dA_\mu) = U_\mu(A_\mu)(1 + iag_0\, dA_\mu^a\, M_{ab}(A_\mu)\, T^b), \tag{5.36}$$

where the matrix M is given by

$$M(A_\mu) = \left(\frac{e^{iag_0\widetilde{A}_\mu} - 1}{iag_0\widetilde{A}_\mu}\right), \tag{5.37}$$

and $\widetilde{A}$ denotes the gauge potential in the *adjoint* representation:

$$\widetilde{A}_\mu = A_\mu^a t^a, \qquad (t^a)^{bc} = -if^{abc}, \qquad \text{Tr}(t^a t^b) = 3\delta^{ab}. \tag{5.38}$$

One can then write (Kawai, Nakayama and Seo, 1981)

$$\begin{aligned}
g_{ab}dA_\mu^a dA_\mu^b &= \text{Tr}(dU_\mu^\dagger dU_\mu) \\
&= \text{Tr}\Big\{T^c\, M_{ca}^\dagger(A_\mu)\, (-i)d(ag_0 A_\mu^a)\, U_\mu^\dagger(A_\mu)\cdot U_\mu(A_\mu)id(ag_0 A_\mu^b)\, M_{bd}(A_\mu)\, T^d\Big\} \\
&= a^2 g_0^2 \cdot \frac{1}{2}\, M_{ca}^\dagger(A_\mu)M^{bc}(A_\mu)\, dA_\mu^a dA_\mu^b. \tag{5.39}
\end{aligned}$$

The metric is thus given, leaving aside a factor that will simplify in the ratios expressing the expectation values of operators, by

$$g(A) = \frac{1}{2}\Big(M^\dagger(A)M(A)\Big), \tag{5.40}$$

which explicitly is

$$g(A) = \frac{1 - \cos(ag_0 A_\mu^a t^a)}{(ag_0 A_\mu^a t^a)^2} = \frac{1}{2} + \sum_{l=1}^{\infty} \frac{(-1)^l}{(2l+2)!}\Big(iag_0 A_\mu^a t^a\Big)^{2l}. \tag{5.41}$$

---

[15] The gauge-invariant Haar measure on the group has the properties:

$$\int [dU] = 1, \tag{5.33}$$

$$\int [dU]f(U) = \int [dU]f(U_0 U), \tag{5.34}$$

for any arbitrary but sensible function f, and for any arbitrary element U_0 of the gauge group.

The measure for the Wilson action can then be written as the product over all sites of the above expression, and is given by

$$\mathcal{D}U = \prod_{x,\mu} \sqrt{\det\left(\frac{1}{2}M^\dagger(A_\mu(x))M(A_\mu(x))\right)} \, \mathcal{D}A, \qquad \mathcal{D}A = \prod_{x,\mu,a} dA_\mu^a(x). \tag{5.42}$$

It is convenient to write this measure term in the form

$$\mathcal{D}U = \mathrm{e}^{-S_{\mathrm{meas}}[A]}\, \mathcal{D}A. \tag{5.43}$$

This can be done by using the identity $\det g = \exp(\mathrm{Tr}\log g)$, so that at the end we obtain, using Eq. (5.41),

$$S_{\mathrm{meas}}[A] = -\frac{1}{2}\sum_{x,\mu} \mathrm{Tr}\,\log\frac{2(1-\cos(ag_0 A_\mu^a t^a))}{(ag_0 A_\mu^a t^a)^2} = -\frac{1}{2}\sum_{x,\mu} \mathrm{Tr}\,\log\left[1 + 2\sum_{l=1}^{\infty}\frac{(-1)^l}{(2l+2)!}\left(ag_0 A_\mu^a t^a\right)^{2l}\right]. \tag{5.44}$$

This formula contains the infinite number of vertices coming from the measure term. These measure vertices do not contribute at the tree level. The lowest order is just

$$S_{\mathrm{meas}}[A] = \frac{g_0^2}{8a^2}\sum_{x,a,\mu}(A_\mu^a)^2, \tag{5.45}$$

and this term, which is quadratic in A_μ, is nonetheless part of the interaction and not a kinetic term. It acts like a mass counterterm of order g_0^2, and is needed to restore gauge invariance in lattice Feynman amplitudes. It cancels, for example, the quadratic divergence in the 1-loop gluon self-energy (see Fig. 6). In momentum space this mass counterterm is (see Fig. 7)

$$-\frac{g_0^2}{4a^2}\delta_{\mu\nu}\delta_{ab}. \tag{5.46}$$

The higher orders in Eq. (5.44), which give self-interaction vertices of the gluons, are at least of order g_0^3 and thus only relevant for calculations with two or more loops.

As a last comment, we mention that in lattice QED things are much simpler: the abelian measure is just given by

$$\mathcal{D}U = \prod_{x,\mu,a} dA_\mu^a(x), \tag{5.47}$$

and thus there are no measure counterterms.

5.2.2 Gauge fixing and the Faddeev-Popov procedure

Although in some situations can be convenient, gauge fixing is not necessary on the lattice when one works with actions which are expressed in terms of the U_μ's, as is done in Monte Carlo calculations. The reason is that the volume of the gauge group is finite, and it can be

factored out and simplified when one normalizes the path integral with the partition function for the calculation of expectation values of operators. In perturbation theory, where one makes a saddle-point approximation of the functional integral around $U_\mu = 1$ and the A_μ's become the real variables (which moreover are decompactified), gauge fixing is instead necessary (Baaquie, 1977; Stehr and Weisz, 1983). A gauge has then to be fixed in order to avoid the zero modes in the quadratic part of the action (expressed in terms of the A_μ's). This degeneracy is due to gauge invariance.

We can see why it is necessary to fix a gauge in perturbative lattice QCD also from the following argument. Strictly speaking, perturbation theory arises as an expansion around the minimum of the action, that is the plaquette. We can see from looking at the Wilson action that the value of the plaquette $P_{\mu\nu}(x) = 1$ minimizes the pure gauge action, but this does not yet imply $U_\mu(x) = 1$. Quite on the contrary, even if one fixes $U_\mu(x) = 1$ for each link from the beginning, a gauge transformation will lead to $1 \rightarrow \Omega(x)\,\Omega^{-1}(x + a\hat\mu)$, which can assume any value. In order for perturbation theory to be a weak coupling expansion around the configuration $U_\mu(x) = 1$, one must then fix the gauge.

Gauge fixing is thus an essential step in the perturbative calculations made on the lattice, and can be implemented by using a lattice Faddeev-Popov procedure, which goes along lines similar to the continuum. The final result, however, will be rather different. In fact, as another consequence of the gauge invariance on the lattice, one obtains from the Faddeev-Popov procedure an infinite number of vertices. Although cumbersome, this procedure is consistent and gives a meaning to the lattice functional integral.

We illustrate the lattice Faddeev-Popov method for a covariant gauge, which is the most commonly used, implementing the gauge-fixing condition

$$\mathcal{F}_x^a[A, \chi] = \nabla_\mu^\star A_\mu^a(x) - \chi^a(x) = 0, \tag{5.48}$$

with χ some arbitrary fields. One chooses the backward lattice derivative $\nabla_\mu^\star$ here because only in this way the gluon propagator can be expressed in a simple form, as we will see below. The Faddeev-Popov determinant is defined by

$$1 = \Delta_{FP}[A, \chi] \cdot \int \mathcal{D}g \prod_{x,a} \delta(\mathcal{F}_x^a[A_g, \chi]), \tag{5.49}$$

where g is a gauge transformation and A_g is the result of this gauge transformation applied to A. The measure $\mathcal{D}g = \prod_x d\mu(g_x)$ is given by the product of the Haar measures over the lattice sites. From the property (5.34) of an Haar measure it is easy to see that the Faddeev-Popov determinant so constructed is gauge invariant: $\Delta_{FP}[A_g, \chi] = \Delta_{FP}[A, \chi]$.

One makes now the same steps as in the continuum, that is Eq. (5.49) is inserted into the partition function, and, as in the continuum, the gauge invariance of the Faddeev-Popov determinant is exploited so that at the end one can factorize the $\mathcal{D}g$ integration and drop it out. Finally, after adding a gauge-fixing term to the action like is done in the continuum and

integrating in χ, one obtains for the expectation value of a generic gauge-invariant operator [16]

$$\langle O \rangle = \frac{\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}A \mathcal{D}\chi \Delta_{FP}[A,\chi] \prod_{x,a} \delta(\mathcal{F}_x^a[A,\chi]) \cdot O \cdot e^{-S_{QCD}} e^{-S_{meas}} e^{-\frac{1}{2\alpha} a^4 \sum_{x,a} \chi^a(x)\chi^a(x)}}{\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}A \mathcal{D}\chi \Delta_{FP}[A,\chi] \prod_{x,a} \delta(\mathcal{F}_x^a[A,\chi]) \cdot e^{-S_{QCD}} e^{-S_{meas}} e^{-\frac{1}{2\alpha} a^4 \sum_{x,a} \chi^a(x)\chi^a(x)}}, \tag{5.50}$$

where α is the gauge parameter (particular cases are the Feynman gauge $\alpha = 1$ and the Landau gauge $\alpha = 0$). What is left at this point is the computation of the Faddeev-Popov determinant, which turns out to be independent of χ. To do this, one only needs to know A_g in the infinitesimal neighborhood of the identity transformation $g = 1$. The infinitesimal transformation with parameter $\epsilon^a(x)$ gives

$$U_\mu(x) \to e^{i\epsilon(x)} U_\mu(x) e^{-i\epsilon(x+a\hat{\mu})} = e^{iag_0 \left(A_\mu(x) + \delta_{(\epsilon)} A_\mu(x) \right)}, \tag{5.51}$$

where

$$g_0 \delta_{(\epsilon)} A_\mu^a(x) = -\sum_b \widehat{D}_\mu[A]_{ab} \epsilon^b(x), \tag{5.52}$$

with

$$\widehat{D}_\mu[A] = (M^\dagger)^{-1}(A_\mu(x)) \cdot \nabla_\mu + ig_0 A_\mu^a(x) t^a. \tag{5.53}$$

We remark that M is the same matrix of Eq. (5.37) and t^a is a matrix in the *adjoint* representation of $SU(3)$. This is similar to the continuum case. In fact $\widehat{D}_\mu[A]$ is a discretized form of the covariant derivative acting on fields in the adjoint representation. The result for the Faddeev-Popov determinant is indeed still reminiscent of the continuum: [17]

$$\Delta_{FP}[A] = \det \left(-\nabla_\mu^\star \widehat{D}_\mu[A] \right). \tag{5.54}$$

However, the important difference with the continuum case is that the lattice operator $\widehat{D}_\mu[A]$ is not linear in A, because of the expansion

$$(M^\dagger)^{-1}(A_\mu(x)) = 1 + \frac{i}{2} ag_0 A_\mu(x) - \frac{1}{12}(ag_0 A_\mu(x))^2 + \cdots, \tag{5.55}$$

and an infinite number of ghost vertices are thus generated. [18] In fact, using the well-known formula for Grassmann variables c and $\overline{c}$,

$$\int \mathcal{D}(\overline{c}c) \, e^{-a^4 \sum_{ij} \overline{c}_i Q_{ij} c_j} = \det Q, \tag{5.58}$$

---

[16] This formula is not generally valid in a nonperturbative context. It can happen that the partition function of a gauge-fixed BRS-invariant theory reduces to zero on the lattice, because of the contribution of Gribov copies (Neuberger, 1986; Neuberger, 1987). One of the possible ways to overcome this is proposed in (Testa, 1998b).

[17] For the detailed derivation, see (Rothe, 1997).

[18] The part of the lattice Faddeev-Popov determinant that gives a direct correspondence with the continuum ghosts can be obtained putting $M = 1$, as one can easily see by combining the formulae (5.52) and (5.53) to

we can write the Faddeev-Popov determinant in terms of an action involving ghosts,

$$\Delta_{FP}[A] = \int \left(\prod_{a,x} d\overline{c}^a(x) c^a(x) \right) \mathrm{e}^{a^4 \sum_x \overline{c}^a(x) \nabla^\star_\mu \widehat{D}^{ab}_\mu[A] c^b(x)}, \tag{5.59}$$

so that the expectation value of a generic gauge-invariant operator is

$$\langle O \rangle = \frac{\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}A \mathcal{D}\overline{c} \mathcal{D}c \cdot O \cdot \mathrm{e}^{-S_{QCD}} \mathrm{e}^{a^4 \sum_x \overline{c}^a(x) \nabla^\star_\mu \widehat{D}^{ab}_\mu[A] c^b(x)} \mathrm{e}^{-S_{meas}} \mathrm{e}^{-S_{gf}}}{\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}A \mathcal{D}\overline{c} \mathcal{D}c \cdot \mathrm{e}^{-S_{QCD}} \mathrm{e}^{a^4 \sum_x \overline{c}^a(x) \nabla^\star_\mu \widehat{D}^{ab}_\mu[A] c^b(x)} \mathrm{e}^{-S_{meas}} \mathrm{e}^{-S_{gf}}}, \tag{5.60}$$

where the gauge-fixing term has been written, thanks to the delta function $\delta(\mathcal{F}^a_x[A, \chi])$, as

$$S_{gf} = \frac{a^4}{2\alpha} \sum_x \left(\sum_\mu \nabla^\star_\mu A_\mu(x) \right)^2 = \frac{a^2}{2\alpha} \sum_x \left(\sum_\mu \left(A_\mu(x) - A_\mu(x - a\hat{\mu}) \right) \right)^2. \tag{5.61}$$

The ghosts are Grassmann variables which carry a color index, but they have no spin index. They transform according to the adjoint representation of $SU(3)$.

We are now at last ready to compute the gluon propagator in the covariant gauge $\partial_\mu A_\mu = 0$, which is:

$$G^{ab}_{\mu\nu}(k) = \delta^{ab} \frac{1}{\frac{4}{a^2} \sum_\lambda \sin^2 \frac{ak_\lambda}{2}} \left\{ \delta_{\mu\nu} - (1 - \alpha) \frac{\sin \frac{ak_\mu}{2} \sin \frac{ak_\nu}{2}}{\sum_\lambda \sin^2 \frac{ak_\lambda}{2}} \right\}. \tag{5.62}$$

This expression is the result of adding the free part of the gluon action,

$$\begin{aligned} S_g &= \frac{1}{4} \left(\nabla_\mu A_\nu - \nabla_\nu A_\mu \right)^2 = \frac{1}{2} \left(\nabla_\mu A_\nu \nabla_\mu A_\nu - \nabla_\mu A_\nu \nabla_\nu A_\mu \right) = -\frac{1}{2} \left(A_\nu \nabla^\star_\mu \nabla_\mu A_\nu - A_\nu \nabla^\star_\mu \nabla_\nu A_\mu \right) \\ &= -\frac{1}{2} \left(A_\nu (\Delta \delta_{\mu\nu} - \nabla^\star_\mu \nabla_\nu) A_\mu \right), \end{aligned} \tag{5.63}$$

to the gauge-fixing term,

$$S_{gf} = \frac{1}{2\alpha} (\nabla^\star_\nu A_\nu)(\nabla^\star_\mu A_\mu) = -\frac{1}{2\alpha} A_\nu \nabla_\nu \nabla^\star_\mu A_\mu, \tag{5.64}$$

where we have integrated by parts. We can see that one is forced to choose the backward lattice derivative in the gauge-fixing term because in the pure gauge action the forward derivative is

reconstruct the continuum formulae

$$g_0 \delta_{(\epsilon)} A^a_\mu(x) = -(\partial_\mu + \mathrm{i} g_0 A^c_\mu(x) t^c)^{ab} \, \epsilon^b(x) = -D_\mu[A](x) \cdot \epsilon(x) \tag{5.56}$$

and

$$\Delta_{FP}[A] = \det\left(-\partial_\mu D_\mu[A] \right). \tag{5.57}$$

present. Only in this way the longitudinal components of the lattice propagator have a simple form. In the limit $a \to 0$ the lattice gluon propagator of course reduces to the well-known expression

$$\delta^{ab} \cdot \frac{1}{k^2} \left[\delta_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2} \right]. \tag{5.65}$$

From Eqs. (5.59) and (5.53) one can derive the Feynman rules for the ghosts on the lattice. The ghost propagator is

$$\delta^{ab} \cdot \frac{1}{\dfrac{4}{a^2} \sum_\lambda \sin^2 \dfrac{ak_\lambda}{2}}, \tag{5.66}$$

and the ghost-gluon-gluon vertex (see Fig. 7) is

$$\mathrm{i} g_0 f_{abc} (\widehat{p}_2)_\mu \cos \frac{(ap_1)_\mu}{2}, \tag{5.67}$$

while the ghost-ghost-gluon-gluon vertex,

$$\frac{1}{12} g_0^2 a^2 \{t^a, t^b\}_{cd} \, \delta_{\mu\nu} \, (\widehat{p}_1)_\mu (\widehat{p}_2)_\mu, \tag{5.68}$$

is the first ghost vertex which is a lattice artifact, that is it vanishes in the continuum limit. Vertices containing three or more ghosts are at least of order g_0^3 and do not enter in 1-loop calculations.

The lattice theory so defined after gauge fixing has an exact BRS symmetry (Baaquie, 1977; Kawai, Nakayama and Seo, 1981), which is given by

$$\delta A_\mu^a(x) = \frac{1}{g_0 a} \left[M_{ab}^{-1}(A_\mu(x)) \, c^b(x) - M_{ba}^{-1}(A_\mu(x)) \, c^b(x + a\hat{\mu}) \right] \tag{5.69}$$

$$\delta c^a(x) = -\frac{1}{2} f_{abc} \, c^b(x) c^c(x) \tag{5.70}$$

$$\delta \overline{c}^a(x) = -\frac{1}{\alpha g_0 a} \sum_\nu \left(A_\nu^a(x) - A_\nu^a(x - a\hat{\nu}) \right) \tag{5.71}$$

$$\delta \psi(x) = \mathrm{i} \, c^a(x) \, T^a \, \psi(x) \tag{5.72}$$

$$\delta \overline{\psi}(x) = \mathrm{i} \, \overline{\psi}(x) \, T^a \, c^a(x). \tag{5.73}$$

The BRS variation of the gauge-fixing term in the action is the opposite of the BRS variation of the Faddeev-Popov term, and they cancel. This can be seen from the nilpotence of the BRS transformation, $\delta^2 A_\mu^a(x) = 0$. The transformations of quarks, antiquarks and ghosts are the same as in the continuum BRS. The transformation of the antighosts, which contains the gauge-fixing term, replaces the continuum derivative with the lattice backward derivative, and the difference with the continuum is then of $O(a)$. The transformation of the gauge potential is instead quite different from the continuum, and because of M is nonlinear in the gauge potential A. For $a \to 0$ it reduces to the continuum BRS transformation.

In the abelian case the Faddeev-Popov determinant reduces to a trivial factor that can be eliminated from the path integral when computing expectation values, analogously to the continuum QED theory.

The derivation of the Faddeev-Popov determinant for arbitrary linear gauge-fixing conditions and for lattices with a large class of boundaries can be found in (Lüscher and Weisz, 1986).

5.3 Fermion action

We now discuss the Feynman rules coming from the fermion part of the action. The quark propagator can be computed by inverting the lattice Dirac operator in momentum space, and is given by

$$S^{ab}(k, m_0) = \delta^{ab} \cdot a \, \frac{-\mathrm{i} \sum_\mu \gamma_\mu \sin ak_\mu + am_0 + 2r \sum_\mu \sin^2 \frac{ak_\mu}{2}}{\sum_\mu \sin^2 ak_\mu + \left(2r \sum_\mu \sin^2 \frac{ak_\mu}{2} + am_0\right)^2}. \tag{5.74}$$

In the continuum limit this is the well-known expression

$$\delta^{ab} \cdot \frac{-\mathrm{i} \sum_\mu \gamma_\mu k_\mu + m_0}{\sum_\mu k_\mu^2 + m_0^2}. \tag{5.75}$$

The vertices are obtained from the expansion of the fermionic part of the action in powers of $g_0 A$, with a Fourier transform. We get again an infinite tower of vertices, which involve two (anti-)quarks and n gluons. Fortunately, only a finite number of them is needed at any given order in g_0. Here we only give the explicit expressions for the vertices which are sufficient to do 1-loop calculations.

We have already derived the quark-quark-gluon vertex (see Eq. (5.18)), and is

$$(V_1^a)_\mu^{bc}(p_1, p_2) = -g_0 (T^a)^{bc} \left(\mathrm{i} \gamma_\mu \cos \frac{a(p_1 + p_2)_\mu}{2} + r \sin \frac{a(p_1 + p_2)_\mu}{2} \right), \tag{5.76}$$

where p_1 and p_2 are the quark momenta flowing in and out of the vertices (see Fig. 7). For $a \to 0$ this is the familiar continuum QCD vertex

$$- g_0 (T^a)^{bc} \, \mathrm{i} \gamma_\mu. \tag{5.77}$$

Doing a similar calculation, one can derive that the quark-quark-gluon-gluon vertex is given by

$$(V_2^{ab})_{\mu_1 \mu_2}^{cd}(p_1, p_2) = -\frac{1}{2} a g_0^2 \, \delta_{\mu_1 \mu_2} \left(\frac{1}{N_c} \delta^{ab} + d^{abe} T^e \right)^{cd} \left(-\mathrm{i} \gamma_\mu \sin \frac{a(p_1 + p_2)_\mu}{2} + r \cos \frac{a(p_1 + p_2)_\mu}{2} \right). \tag{5.78}$$

This vertex is zero in the continuum limit and thus has no continuum analog. However it can still give nonvanishing contributions to Feynman diagrams, like in divergent loops or in

tadpoles, which have a factor $1/a^2$ coming from the gluon propagator. Vertices of two quarks and n gluons are also irrelevant and are associated with a factor $a^{n-1}g_0^n$.

In spite of the complexities brought from the fact that the lattice gauge theory has an infinite number of vertices, it turns out that the superficial degree of divergence of a Feynman diagram depends only on the number of its external lines (Kawai, Nakayama and Seo, 1981):

$$D = 4 - E_G - E_g - \frac{3}{2}E_q. \tag{5.79}$$

The type of vertex functions which give rise to divergences is the same as in the continuum, and this is very important for the renormalizability of the theory. Thus, vertices which are of higher order in a and g_0 do not modify this continuum picture.

It is customary to define and use the vector and axial currents on the lattice as follows:

$$V_\mu(x) = \overline{\psi}(x)\,\gamma_\mu\,\psi(x) \tag{5.80}$$
$$A_\mu(x) = \overline{\psi}(x)\,\gamma_\mu\gamma_5\,\psi(x). \tag{5.81}$$

However, these currents are not conserved in the Wilson formulation (in fact, they are not equal to the currents appearing in the action), and therefore they are not protected from renormalization. One then has

$$Z_V \neq 1, \tag{5.82}$$
$$Z_A \neq 1, \tag{5.83}$$

at variance with the continuum results. The Noether current corresponding to the vector transformation that leaves the Wilson action invariant is instead given by

$$V_\mu^{cons}(x) = \frac{1}{2}\left(\overline{\psi}(x)\,(\gamma_\mu - r)\,U_\mu(x)\,\psi(x + a\hat{\mu}) + \overline{\psi}(x + a\hat{\mu})\,(\gamma_\mu + r)\,U_\mu^\dagger(x)\,\psi(x)\right), \tag{5.84}$$

and it is an extended operator, which needs for its definition two lattice sites. The renormalization constant of this conserved current is one. It is not possible to make a corresponding construction which leads to a conserved axial current because, as we will see in detail in the next Section, the Wilson action breaks chiral symmetry.

We want to conclude this Section by mentioning that, for reasons of computing power, the so-called quenched approximation is often used in Monte Carlo simulations. In order to perform a statistical sampling of the functional integral, the fermion variables (which are Grassmann numbers) are analytically integrated (using Eq. (5.58)) and one does the numerical simulations using the partition function

$$Z = \int \mathcal{D}U \ \det(\slashed{D}[U] + m_0)\,\mathrm{e}^{-S_g[U]}. \tag{5.85}$$

This is equivalent to use the effective action

$$S_{\mathrm{eff}}[U] = S_g[U] - \log\det(\slashed{D}[U] + m_0) = S_g[U] - \mathrm{Tr}\,\log(\slashed{D}[U] + m_0). \tag{5.86}$$
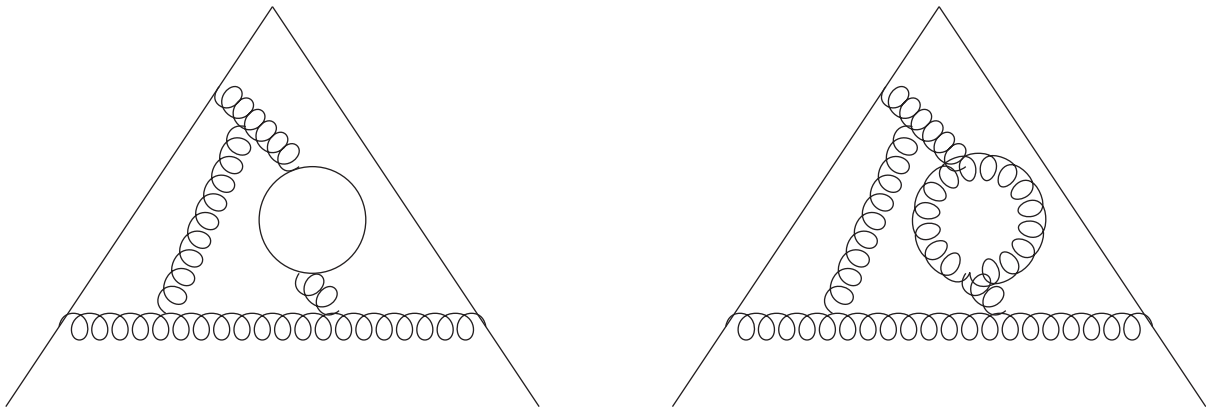
Figure 8: The diagram on the left is zero in the quenched approximation, while the diagram on the right does not contain internal quark loops and has to be included also in quenched calculations.

Simulations in full QCD have to include the full contribution of the determinant, while quenching amounts to doing simulations where one puts $\det(\slashed{D}[U] + m_0) = 1$, which saves a couple of orders of magnitudes of computer time. In physical terms, this means that there are no sea quarks in the calculations: the internal quark loops are neglected (see Fig. 8). Quenching is often summarized by saying that $N_f = 0$. Although it looks quite drastic, in many cases this does not turn out to be a bad approximation.

   This approximation has been introduced only for reasons having to do with the simulations. In perturbation theory, quenching means dropping all diagrams which contain an internal quark loop, but the inclusion of these diagrams is usually not so challenging as it is in the simulations. For consistency one can exclude the diagrams containing internal quark loops when the perturbative numbers have to be used in connection with quenched simulation results.

6   Aspects of chiral symmetry on the lattice

Due to the presence of the Wilson term (the part of the action proportional to the parameter r), the Wilson action breaks chiral symmetry, and thus Wilson fermions do not possess chiral invariance even when the bare mass of the quark is zero. This additional symmetry-spoiling term turns out however to be necessary in order to get rid of the extra fermions (also called doublers) which are unavoidable in the naive lattice discretization of the QCD action.

   Let us see what would happen putting $r = 0$ in the Wilson action (which corresponds to naive fermions). In this case, the free fermion propagator is just

$$S^{ab}(k, m_0) = \delta^{ab} \cdot a \, \frac{-\mathrm{i} \sum_\mu \gamma_\mu \sin a k_\mu + a m_0}{\sum_\mu \sin^2 a k_\mu + (a m_0)^2}.$$

(6.1)

Let us for simplicity consider the massless naive propagator, setting $m_0 = 0$ in the above equation. This propagator has a pole at $ak = (0, 0, 0, 0)$, as expected. However, there are also poles at $ak = (\pi, 0, 0, 0)$, $ak = (0, \pi, 0, 0)$, ..., $ak = (\pi, \pi, 0, 0)$, ..., $ak = (\pi, \pi, \pi, \pi)$, that is at all points at the edges of the first Brillouin zone, because $\sum_\mu \sin^2 ak_\mu$ vanishes at these 16 points, at which any component is either $k_\mu = 0$ or $k_\mu = \pi/a$. This propagator describes a fermion mode at each of these poles. All these fermions are acceptable and legitimate particles, even if they correspond to poles at the edges of the Brillouin zone. In fact, we can always think of shifting the integration in momentum space, thanks to the periodicity of the lattice; for example, we could shift it from $\int_{-\pi/a}^{\pi/a}$ to $\int_{-\pi/2a}^{3\pi/2a}$ and so no pole would then be found at the edges. At these poles we have then low-energy excitations as legitimate as the fermion at $ak = (0, 0, 0, 0)$.

For $r = 0$ we would then have to take into account all these 16 Dirac particles when doing computations with the lattice theory. Although they are a lattice artifact, they would be pair produced as soon as the interactions are switched on, and would appear in internal loops and contribute to intermediate processes.

Let us consider in more detail one of these poles, for example the one at $ak = (\pi, 0, 0, 0)$. To see things more clearly, we make the change of variables

$$k_0' = \frac{\pi}{a} - k_0, \quad k_i' = k_i, \tag{6.2}$$

and correspondingly [19]

$$\gamma_0' = -\gamma_0, \quad \gamma_i' = \gamma_i, \tag{6.3}$$

so that $\gamma_5' = -\gamma_5$. The propagator in the new variables assumes near $ak' = (0, 0, 0, 0)$ the same form as the original propagator Eq. (6.1) near $ak = (0, 0, 0, 0)$. This means that there is a fermion mode also at $ak = (\pi, 0, 0, 0)$, and moreover the chirality of this new particle is opposite to the chirality of the mode at $ak = (0, 0, 0, 0)$, because $\gamma_5' = -\gamma_5$, so that $(1+\gamma_5)/2 = (1-\gamma_5')/2$. It can be easily seen that the 16 doublers split in 8 particles of one chirality and 8 particles of the opposite chirality, so that even if the massless continuum theory had chiral symmetry and the physical particle (the pole at the origin) was a chiral mode we end up with a vector theory on the lattice. The main problem then is not that there are more species of fermions than expected, but that the doublers destroy the chirality properties of the continuum theory.

The Wilson term does precisely the work of suppressing these 15 unwanted additional fermions in the continuum limit. In fact, the Wilson action in short form can be written as

$$D_W = \frac{1}{2}\Big(\gamma_\mu(\widetilde{\nabla}_\mu^\star + \widetilde{\nabla}_\mu) - ar\widetilde{\nabla}_\mu^\star\widetilde{\nabla}_\mu\Big), \tag{6.4}$$

where the gauge covariant forward derivative is given by

$$\widetilde{\nabla}_\mu\psi(x) = \frac{1}{a}\Big(U(x, \mu)\psi(x + a\hat{\mu}) - \psi(x)\Big), \tag{6.5}$$

___
[19] The Dirac matrices that one has to use in the transformed variables are equivalent to the standard set: $\gamma_\mu' = (\gamma_0\gamma_5)\gamma_\mu(\gamma_0\gamma_5)^\dagger$.

and the Wilson term is the Laplacian part of this action,

$$-\frac{1}{2}ar\widetilde{\nabla}^{\star}_{\mu}\widetilde{\nabla}_{\mu}. \tag{6.6}$$

This term is irrelevant in the continuum limit but modifies the lattice dispersion relation for finite lattice spacing, so that this "bosonic" energy term gives the extra doublers at the edges of the Brillouin zone a mass of the order of the cutoff. This mass then becomes large in the continuum limit and decouples the doublers from the "right" physical fermion. Of course, being a generalized (momentum-dependent) mass term, it automatically breaks chiral symmetry. This connection of the doublers with chiral symmetry is a deep one, and what we have shown is a particular case of a general phenomenon, as we will shortly see.

Thus, if one uses Wilson fermions chiral symmetry can only be recovered in the continuum limit. The breaking of chiral symmetry at finite lattice spacing has serious consequences on the Wilson theory, among them the appearance of an additive renormalization to the quark mass. The nonzero value of the bare quark mass which corresponds to a vanishing renormalized quark mass is called critical mass. Its value depends on the strength of the interaction. There is then a critical line in the plane of bare parameters,

$$m_0 = m_c(g_0), \qquad m_c(0) = 0, \tag{6.7}$$

where the physical quark mass vanishes. It is only the subtracted mass $m_0 - m_c$ that is multiplicatively renormalized:

$$m_R = Z_m(m_0 - m_c). \tag{6.8}$$

Thus, the renormalized quark mass has no protection from chiral symmetry and acquires a nonzero value even when the bare quark mass is zero. The bare and the renormalized quark mass cannot vanish at the same time, and the bare mass has to be carefully tuned in order to extract physical information from the simulations. Since numerical simulations of QCD are performed with lattice spacings that are not small, the violation of chiral symmetry by these lattice effects is rather pronounced, and this mass renormalization is quite large.

The quark self-energy at one loop has a correction proportional to $1/a$, which corresponds to the mass counterterm which has to be introduced due to the breaking of chirality, and we will show in detail how to compute it in Section 15. This critical mass m_c is one of the best known quantities in lattice perturbation theory, and its high-precision determination at 1-loop, as well as its value calculated to two loop, are given in Section 19.2.2.

Another bad consequence of the loss of chirality is the appearance of more mixings under renormalization, because the mixing among operators of different chirality is not a priori forbidden for Wilson fermions. One example of this kind of mixings will be given in Section 14.2. Moreover, it is not possible to define a conserved axial current.

The impossibility of removing the doublers in the naive fermion formulation without breaking at the same time chiral symmetry (as it happens in the Wilson action) or some other important symmetry is a special case of a very important no-go theorem, established by Nielsen

and Ninomiya many years ago (Nielsen and Ninomiya, 1981a; 1981b; 1981c; Friedan, 1982). [20] In short, the theorem says that a lattice fermion formulation without species doubling and with an explicit continuous chiral symmetry is impossible, unless one is prepared to give up some other fundamental properties like locality, or unitarity. [21]

It is quite common in quantum field theory that the introduction of an ultraviolet regulator brings some unphysical features, and this chiral symmetry issue is perhaps the most serious and unpleasant drawback of the lattice regularization. The difficulties regarding chiral symmetry with lattice fermions arise already at the level of free fields, and they are present even when one does not discretize space, but only the time direction. We can see why this happens from general topological considerations on the free fermion propagator, where the close interplay between chirality and doublers follows from the continuity of the energy-momentum relation in the Brillouin zone (Karsten and Smit, 1981) (see also the lectures of (Smit, 1986)). The general form of a massless fermion propagator on the lattice which is compatible with continuous chiral invariance is

$$\frac{1}{i \sum_\mu \gamma_\mu P_\mu(k)}, \tag{6.11}$$

where the function $P_\mu(k)$ is real. This function has to cross each $P_\mu = 0$ axis only once if this propagator is meant to describe for $a \to 0$ only a single fermion.

Let us assume at first that $P_\mu(k)$ is a continuous function. If the derivative in the Dirac operator in the action is anti-hermitian (like in Wilson), the theory is unitary and the function $P_\mu(k)$ is periodic in 2π. Since it has a first order zero at $k_\mu = 0$ (that is at $k_\mu = 2\pi n_\mu$), it has to cross the $P_\mu = 0$ axis one other time somewhere else in the first Brillouin zone, and with the opposite derivative, which corresponds to the opposite chirality (see Fig. 9, top left). This crossing is then precisely the doubler, and therefore it is unavoidable to have these extra particles in the theory. This reasoning only comes from global features, and is independent of the shape of the particular function, as long as it is continuous. Another heuristic proof coming from topological arguments, as well as a more general explanation of why half of the doublers have opposite chirality, has been given in (Wilczek, 1987). In this work a particular choice of the function $P_\mu(k)$ which minimizes the numbers of doublers is also proposed, which has the drawback of the loss of the equivalence of all four directions under discrete permutations, and the need of new operators (but no $1/a$ counterterm as in Wilson).

---

[20] An alternative proof of the theorem which makes use of the Poincaré-Hopf theorem has been given in (Karsten, 1981).

[21] This statement only applies to the chiral symmetry which acts on the spinor fields like

$$\psi \quad \to \quad \psi + \epsilon \cdot \gamma_5 \, \psi, \tag{6.9}$$
$$\overline{\psi} \quad \to \quad \overline{\psi} + \epsilon \cdot \overline{\psi} \, \gamma_5. \tag{6.10}$$

As we will shortly see, one of the major theoretical advances of the last years has been the understanding that there are other kinds of transformations that can define a lattice chiral symmetry and which do not necessarily imply a fermion doubling.
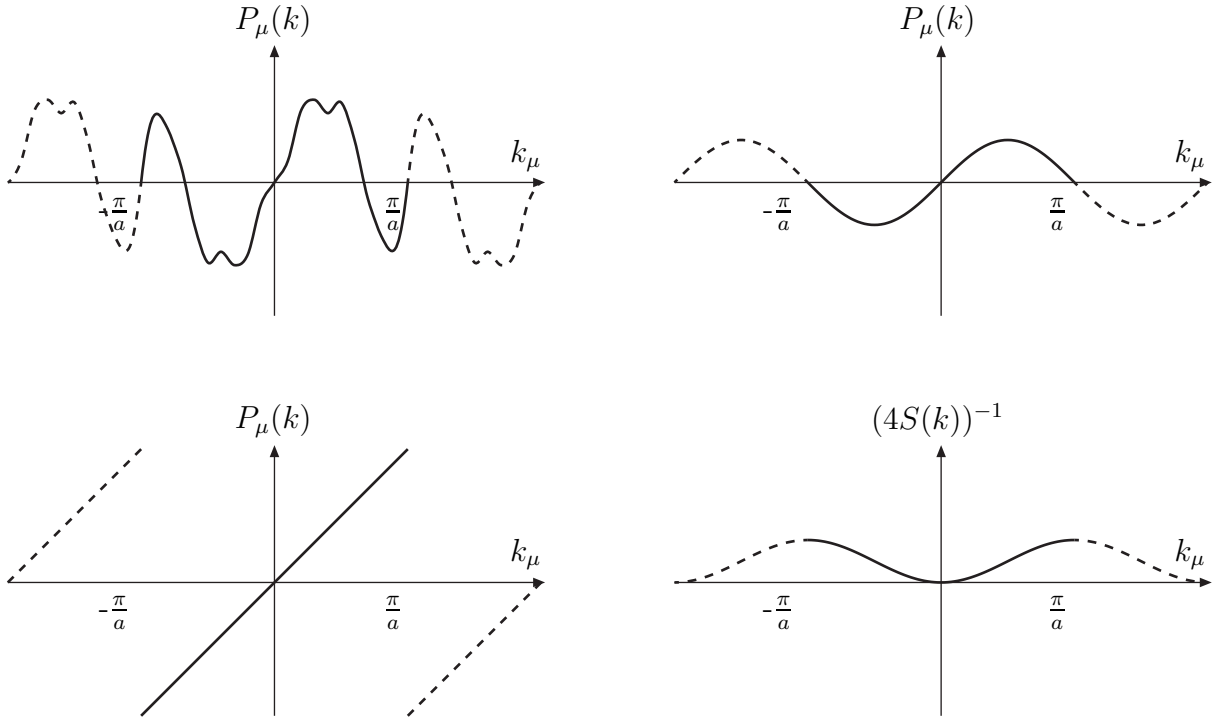
Figure 9: Examples of inverse propagator functions: a smooth function $P_\mu(k)$ which gives a particle and its three doublers (top left), $P_\mu(k)$ for the naive fermion propagator (top right) and the SLAC propagator (bottom left), and the inverse of the bosonic propagator (bottom right). The dashed parts of the curves lie outside the first Brillouin zone.

Let us come back to the naive propagator in Eq. (6.1), i.e., $P_\mu(k) = \sin ak_\mu$. One crucial feature here is that the argument of the sine function is k, which is a consequence of the fact that one uses the anti-hermitian derivative $\nabla_\mu + \nabla_\mu^\star$. This choice causes all our problems. If the argument of the sine function were $k/2$, things would be completely different, since $\sin ak/2$ is antiperiodic, that is it has a period of 4π in ak, and there would be no need to cross the $P_\mu = 0$ axis a second time *inside* the first Brillouin zone. Then no doublers would appear. However, $\sin ak/2$ can only appear in the propagator if one uses either the forward or the backward derivative, but not their sum. These derivatives are not anti-hermitian, and unfortunately in this case the theory turns out not to be unitary, which causes other kinds of serious problems. The fact is that ∇ and $\nabla^\star$ are unphysical, and they propagate the fermion only in the forward or backward direction, and so using only one of them cannot give a Lorentz invariant theory in Minkowski space. It has also been shown, in lattice QED, that if one uses only the forward or backward derivative, then the interactions generate noncovariant contributions to the quark self-energy and vertex function, and the theory is nonrenormalizable (Sadooghi and Rothe, 1997).

Note that a boson propagator does not have this problem, as it is the solution of a second

order differential equation. Therefore the function $P_\mu(k)$ is in this case quadratic (and γ_μ disappears), and crossings are replaced by second-order zeros. Therefore, even if doublers were present, there would be no problems regarding the chiral properties of the lattice theory. In any case, the Dirac operator can be chosen to be discretized using $\nabla^\star \nabla$, which is hermitian and produces a $\sin^2 k/2$ function in the propagator, as is well known. Therefore in this case the function has only a minimum at the origin without further crossings in the first Brillouin zone (see Fig. 9, bottom right), and there are no doublers.

In the fermionic case, the only other way to avoid the second crossing would be to consider a discontinuous function $P_\mu(k)$. The most famous example of this is given by the SLAC propagator (Drell, Weinstein and Yankielowicz, 1976a; 1976b), for which $P_\mu(k) = k_\mu$ throughout the whole Brillouin zone (see Fig. 9, bottom left). SLAC fermions have been studied in perturbation theory in (Karsten and Smit, 1978; Karsten and Smit, 1979). However, this implies a nonlocality in the lattice action (it corresponds to a nonlocal lattice derivative), which brings many problems in the continuum limit. The locality assumption is important in order to avoid a disaster in the straightforward weak coupling expansion, otherwise propagators and vertex functions would not be analytic in the momenta. Another chiral action which is nonlocal has been proposed in (Rebbi, 1987).

At the end of the day the origin of the fermion doubling lies in the fact that the Dirac equation is of first order. Doublers in the naive fermion action are a necessary feature, and we can understand why also by looking at quantum anomalies. We know that in continuum quantum field theory the quantum corrections, and precisely the process of regularization, break chiral symmetry. A mass scale appears in the renormalized theory, [22] which gives rise to the phenomenon of dimensional transmutation and causes the appearance of quantum anomalies. So even in theories that are chirally symmetric the axial current acquires an anomalous divergence through quantum effects.

Naive lattice fermions can be thought as a regularization that does not break chiral symmetry for any finite a. The lattice theory has no anomalies, and this implies that one needs some extra particles in order to cancel the continuum anomaly which is associated with the "right" fermion. When one tries to remove only the extra doublers from the game then the anomaly becomes again nonzero (as in the continuum), which must then correspond to a regularization which somehow has to break chiral symmetry, and in fact we end up with the lattice Wilson action. So, everything fits in the general picture.

After these general considerations we are now ready to state the Nielsen-Ninomiya theorem. In one of its formulations it says that the lattice massless Dirac operator $D = \gamma_\mu D_\mu$ in the fermionic action

$$S_F = a^4 \sum_{x,y} \overline{\psi}(x) D(x-y)\psi(y) \tag{6.12}$$

---

[22]For example, in dimensional regularization (which preserves the gauge symmetry) one needs to introduce a scale μ to define the coupling in noninteger dimensions, and in Pauli-Villars one introduces a heavy mass in the propagators.

cannot satisfy all of the following four properties at the same time:

(a) $D(x)$ is local (in the sense that is bounded by $Ce^{-\gamma|x|}$);

(b) its Fourier transform has the right continuum limit: $D(p) = i\gamma_\mu p_\mu + O(ap^2)$ for small p;

(c) $D(p)$ is invertible for $p \neq 0$ (and hence there are no massless doublers);

(d) $\gamma_5 D + D\gamma_5 = 0$ (it has chiral symmetry).

Therefore, for any given lattice action at least one of the these conditions has to fail. In particular, naive fermions have doublers and therefore do not satisfy (c), Wilson fermions break chiral symmetry and therefore do not satisfy (d), and SLAC fermions are not local and therefore do not satisfy (a). The case of staggered fermions, another widely used (and old) fermion formulation which has been useful for studying problems in which chiral symmetry is relevant, and which will be discussed in the next Section, is more complicated from this point of view: only a $U(1) \otimes U(1)$ subgroup of the $SU(N_f) \otimes SU(N_f)$ chiral group remains unbroken, and the doublers are removed only partially.

Contrary to what one would naively expect after looking at the Nielsen-Ninomiya theorem, it is still possible to construct a Dirac operator which satisfies (a), (b) and (c) and is also chiral invariant. The solution to this apparent paradox is that the corresponding chiral symmetry cannot be the one associated with a Dirac operator which anticommutes with γ_5, and the condition (d) has instead to be replaced by the Ginsparg-Wilson relation: $\gamma_5 D + D\gamma_5$ is not zero, but is proportional to $aD\gamma_5 D$. Thus, the real lattice chiral symmetry turns out not to be what one would naively expect, the Nielsen-Ninomiya theorem is still valid and one can have a nonpathological formulation of chiral fermions with no doublers. [23]

This kind of lattice chiral fermions will be discussed in detail in Section 8. Before that, we will shortly present staggered fermions. If one wants to maintain some form of chiral symmetry, but is prepared to give up flavor symmetry, then staggered fermions are the ideal fermions to work with. Otherwise, the only way to maintain chiral symmetry and flavor symmetry at the same time (and of course all other fundamental properties like locality, unitarity etc.) leads again to the Ginsparg-Wilson relation.

7   Staggered fermions

Another formulation of fermions on the lattice which is quite popular can be obtained by keeping part of the doublers of the naive fermion action and interpreting them as extra flavors. One

---

[23]When the condition that the Dirac operator anticommutes with γ_5 is released (at $a \neq 0$), the lattice quark propagator is not restricted to the form in Eq. (6.11) and the considerations about the presence of the doublers deriving from it are not anymore valid. In fact, one finds more general propagator functions which in particular do not have the simple Dirac structure of Eq. (6.11), as one can see by looking at the overlap propagator in Eqs. (8.18) and (8.19).

remains with 4 fermion flavors whose 16 components are split over a unit hypercube by assigning only a single fermion field component to each lattice site. This construction gives the staggered, or Kogut-Susskind, fermions (Kogut and Susskind, 1975; Banks, Susskind and Kogut, 1976; Susskind, 1977), and can only be carried out in an even number of spacetime dimensions. It turns out that a continuous subgroup of the original chiral transformations remains a symmetry of this lattice action even at finite lattice spacing, and thus no mass counterterms are needed for vanishing bare quark masses. All this is achieved at the expense of flavor and translational symmetry, which become in fact all mixed together.

The idea is to use only one spinor component for each Dirac spinor at each site. One has to decouple this component from the remaining three, which can then be kept out of the dynamics. This is accomplished doing a change of variables called spin-diagonalization (Kawamoto and Smit, 1981)

$$\psi(x) = \gamma(x)\chi(x), \tag{7.1}$$

$$\overline{\psi}(x) = \overline{\chi}(x)\gamma^\dagger(x), \tag{7.2}$$

where

$$\gamma(x=an) = \gamma_0^{n_0}\gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3} \tag{7.3}$$

depends only on $\mathrm{mod}_2(n_\mu)$ (because $\gamma_\mu^2 = 1$). In these new spinor variables $\chi(x)$ and $\overline{\chi}(x)$ the naive fermion action

$$S^f = a^4 \sum_x \sum_\mu \frac{1}{2a} \overline{\psi}(x)\gamma_\mu \left[U_\mu(x)\psi(x+a\hat{\mu}) - U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu}) \right] + a^4 \sum_x m_f \overline{\psi}(x)\psi(x) \tag{7.4}$$

becomes

$$S_{stagg}^f = a^4 \sum_x \sum_\mu \frac{1}{2a} \overline{\chi}(x)\eta_\mu(x) \left[U_\mu(x)\chi(x+a\hat{\mu}) - U_\mu^\dagger(x-a\hat{\mu})\chi(x-a\hat{\mu}) \right] + a^4 \sum_x m_f \overline{\chi}(x)\chi(x). \tag{7.5}$$

Up to now, what we have done is just a rewriting of the usual naive action in terms of new variables. The crucial thing now is that the Dirac matrices have disappeared, and they have been replaced by the phase factors

$$\eta_\mu(x=an) = (-1)^{\sum_{\mu<\nu} n_\nu}. \tag{7.6}$$

Thus in the action (7.5) the 4 components of the spinor $\chi(x)$ are decoupled from each other. We are then allowed to keep only one spinor component out of four, and to forget about the others. Since we started with the action of naive fermions, which presents 16 doublers, after the spin-diagonalization we end up only with 4 doublers. This theory is a theory of 4 flavors.

The phase factors $\eta_\mu(x)$ bring a minus sign in the action for every translation of one lattice spacing a, and divide the lattice in even and odd sites, so that they form two independent sublattices. This makes the action invariant only for translations of *two* lattice spacings. Translations of an even number of lattice spacings correspond to ordinary continuum translations,

49

whereas a translation of an odd number of lattice spacings interchanges the chiral components (i.e., is a chiral rotation of $\pi/2$).

It is then natural to take as fundamental objects hypercubes of linear size a (which contain 16 sites) rather than single sites. The 16 spinor components of the 4 flavors of the theory at each site x can be assigned at the vertices of such a 2^4-hypercube whose $(0,0,0,0)$ vertex is the point x. In the continuum limit each 2^4-hypercube will then be mapped to a single physical point. In this way one distributes the fermionic degrees of freedom over the lattice, leaving only one degree of freedom per lattice site. Beside the 4 components of each of the 4 continuum quarks, the components of the continuum γ matrices are also spread over these 2^4-hypercubes. With this construction there is a doubling of the effective lattice spacing, which becomes $2a$, and the staggered formulation effectively halves the size of the Brillouin zone. This is the way it (partially) solves the problem of the doublers. One works with an action in which there is only one independent Grassmann variable $\chi(x)$ per site. Only the color degrees of freedom remain at each site.

Since in d dimensions the unit hypercube has 2^d sites, and a Dirac spinor has $2^{d/2}$ components (for even d), one needs $2^{d/2}$ fermion fields to carry out this construction. In four dimensions this corresponds to 4 flavors, and everything fits together. This also shows that it is not possible to remove all the doublers in this way, because 16 is the minimal number of sites of a four-dimensional unit hypercube, and not 4.

With the construction above, the 16-fold original degeneracy has thus been reduced to a 4-fold degeneracy. Actually, strictly speaking the 4 flavors are degenerate only in the continuum limit, while at finite lattice spacing the $SU(4)$ symmetry is broken. For nonzero a the action maintains nevertheless an exact $U(1)_V$ symmetry,

$$\chi(x) \to e^{i\theta_V}\chi(x), \quad \overline{\chi}(x) \to \overline{\chi}(x)e^{-i\theta_V}, \tag{7.7}$$

corresponding to fermion number conservation, which in the case of vanishing bare masses is supplemented by a flavor nonsinglet axial $U(1)_A$ symmetry,

$$\chi(x) \to e^{i\theta_A\cdot(-1)^{n_0+n_1+n_2+n_3}}\chi(x), \quad \overline{\chi}(x) \to \overline{\chi}(x)e^{i\theta_A\cdot(-1)^{n_0+n_1+n_2+n_3}}. \tag{7.8}$$

This chiral $U(1)_L\otimes U(1)_R$ group is all what remains of the original $SU(4)_L\otimes SU(4)_R$ symmetry, but it is enough to guarantee that quark masses are not additively renormalized. Thus, no mass counterterm is required if one starts with a zero bare mass, and this is a great advantage of staggered over Wilson fermions.

The intertwining of spin and flavor is a major disadvantage of staggered fermions, and the correct spin-flavor structure is only recovered in the continuum limit. Much of the work in staggered calculations goes in the reconstruction of the continuum quark fields and operators from the 16 one-component fields (carrying flavor and spin components) spread over the hypercubes. This can be complicated, and calculations with staggered fermions can thus become rather involved. For details on the way staggered perturbation theory is set up, the reader can turn to

the works of (Sharatchandra, Thun and Weisz, 1981; van den Doel and Smit, 1983; Golterman and Smit, 1984a; Golterman and Smit, 1984b; Göckeler, 1984) and (Daniel and Sheard, 1988; Patel and Sharpe, 1993; Sharpe and Patel, 1994; Ishizuka and Shizawa, 1994). The difficult part of dealing with staggered fermions is in writing the lattice discretization of a continuum operator which has certain given symmetry properties. In fact, the construction of staggered lattice operators and the interpretation of their components in terms of spin and flavor turns out to be quite complicated. Moreover, there are various possibilities for the assignments of the various spin and flavor components. One can choose to make these assignments in momentum space as well.

This is the price to pay for the fact that staggered fermions are numerically quite cheap. The mixing of operators under renormalization when staggered fermions are used can also become quite complicated. Flavor symmetry breaking can generate mixings with operators which were not present in the original continuum theory. If for example we consider a four-fermion operator with a certain flavor structure, then using staggered fermions this operator will mix already at one loop with many other four-fermion operators which have different flavor components. This quickly renders these calculations technically involved, and complicates a lot perturbative calculations with staggered fermions. Gamma matrices are split as well, and also the presence of color matrices (if one has U fields in the operators) or derivatives contributes to entangle things more. To my knowledge no one has done the calculation of the renormalization of an operator like $\overline{\psi}\gamma_\mu D_\nu \psi$, which involves a covariant derivative, with staggered fermions. We will present the calculation of this operator with Wilson fermions in Section 15.4. On the other hand, the perturbative calculations for ϵ'/ϵ are at a rather advanced stage (for a recent work see (Lee, 2001), and other recent perturbative calculations with staggered fermions can be found in (Hein *et al.*, 2002; Nobes *et al.*, 2002) and (Lee and Sharpe, 2002a; Lee and Sharpe, 2002b)). Of course it makes much more sense to spend a lot of effort for weak operators, given the good chiral properties of staggered fermions. But recently the full understanding of the implications of the Ginsparg-Wilson relation has opened more ways for the investigation of these matrix elements on the lattice.

8   Ginsparg-Wilson fermions

8.1   The Ginsparg-Wilson relation

For many years it was believed that the Nielsen-Ninomiya theorem had said the final word with regard to the possibility of having reasonable chiral fermions on the lattice. The (now) fundamental paper by Ginsparg and Wilson (1982), in which the mildest breaking of chiral symmetry was introduced in a study of the block-spin renormalization group in lattice QCD, appeared shortly after the work of Nielsen and Ninomiya, but remained almost unnoticed. Its importance was only recognized fifteen years later.

The Ginsparg-Wilson relation lay indeed dormant for all this time, until it was rediscovered in the context of perfect actions (Hasenfratz, 1998a). Shortly after, also overlap fermions and domain wall fermions, which had been formulated some time before following ingenious ideas, and well before anyone were thinking about the Ginsparg-Wilson relation, were also recognized to be a solution of this relation. Many interesting developments have then come out after the rediscovery of the paper of Ginsparg and Wilson, and for a general overview of these developments the reader can study the excellent reviews of (Niedermayer, 1999), (Creutz, 2001), (Lüscher, 2001), and (Neuberger, 2001), which cover different aspects of them, as well as the shorter and original discussion of the main ideas given in (Hernández, Jansen and Lellouch, 2002). An up-to-date discussion of the numerical results which have been obtained using Ginsparg-Wilson fermions can be found in (Giusti, 2002).

A Dirac operator D which satisfies the Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = a\frac{1}{\rho} D\gamma_5 D \tag{8.1}$$

and the hermiticity condition $D^\dagger = \gamma_5 D\gamma_5$ defines fermions which have exact chiral symmetry, present no doublers [24] and which also possess all other fundamental principles like flavor symmetry, locality (Hernández, Jansen and Lüscher, 1999), [25] unitarity and gauge invariance.

Lüscher has shown that fermions obeying the Ginsparg-Wilson relation possess an exact chiral symmetry at finite lattice spacing, which is of the form (Lüscher, 1998)

$$\psi \;\rightarrow\; \psi + \epsilon \cdot \gamma_5 \left(1 - \frac{a}{\rho} D\right)\psi, \tag{8.2}$$

$$\overline{\psi} \;\rightarrow\; \overline{\psi} + \epsilon \cdot \overline{\psi}\,\gamma_5. \tag{8.3}$$

Note the asymmetric way in which ψ and $\overline{\psi}$ are treated. The global anomaly of the original continuum fermions is also reproduced at nonzero lattice spacing (as was already noticed in the case of domain wall fermions by (Jansen, 1992)). In terms of the quark propagator the Ginsparg-Wilson relation reads

$$S(x,y)\gamma_5 + \gamma_5 S(x,y) = a\frac{1}{\rho}\gamma_5\,\delta(x-y), \tag{8.4}$$

which implies that the propagator is chirally invariant at all nonzero distances, i.e., on the mass shell.

This surprising result is a new form of chiral symmetry that can coexist with a momentum cutoff. The fact is that chiral symmetry can be realized on the lattice in different ways other

---

[24]Although in general the Ginsparg-Wilson relation does not guarantee the absence of doublers, we are of course only interested in actions which have no doublers and the solutions that we will discuss are all of this kind.

[25]Locality in this context does not have the meaning of strict locality, but it is to be understood in the larger sense that the strength of the interaction decays exponentially with the distance in lattice units. It becomes microscopically small when one considers the continuum limit.

than the naive expectation, and this does not constitute an exception to the Nielsen-Ninomiya theorem. In this case, the condition of the theorem that is not fulfilled is the anticommutation of the Dirac operator with γ_5, which can only be recovered in the continuum limit. Chirality (in this new formulation) remains a symmetry of the lattice theory also for nonzero lattice spacing, as are flavor symmetry and the other fundamental symmetries.

The chiral symmetry associated with the Ginsparg-Wilson relation can be used to define left- and right-handed fermions. We first note that the operator

$$\widehat{\gamma}_5 = \gamma_5 \left(1 - \frac{a}{\rho} D \right) \tag{8.5}$$

satisfies

$$(\widehat{\gamma}_5)^\dagger = \widehat{\gamma}_5, \qquad (\widehat{\gamma}_5)^2 = 1, \qquad D\widehat{\gamma}_5 = -\gamma_5 D. \tag{8.6}$$

The projectors

$$\widehat{P}_\pm = \frac{1}{2}(1 \pm \widehat{\gamma}_5) \tag{8.7}$$

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

can then separate the two chiral sectors. In particular, the constraints

$$\widehat{P}_- \psi = \psi \tag{8.8}$$

$$\overline{\psi} P_+ = \overline{\psi} \tag{8.9}$$

define the left-handed fermions. Lüscher's chiral symmetry can then be rewritten in the appealing form

$$\psi \rightarrow \psi + \epsilon \cdot \widehat{\gamma}_5 \, \psi \tag{8.10}$$

$$\overline{\psi} \rightarrow \overline{\psi} + \epsilon \cdot \overline{\psi} \, \gamma_5. \tag{8.11}$$

It is to be remarked here that the definition of left-handed fields is not independent of the gauge fields U. The full implications of this fact will be seen when one wants to construct chiral gauge theories on the lattice, and we will discuss them in Section 9. One of the most remarkable developments has indeed been the discovery that Ginsparg-Wilson fermions constitute a nonperturbative regularization of gauge theories with an explicit chiral symmetry.

The known solutions of the Ginsparg-Wilson relation are given by overlap, domain wall and fixed-point ("classically perfect") fermions. We will now have a look at these three cases in some detail. A few other actions which are approximate solutions of the Ginsparg-Wilson relation have sprung up in recent years. We will not discuss them here. They are much more complicated than these three cases, especially from the point of view of perturbation theory. We will see that overlap, domain wall and fixed-point fermions bring already many complications into the perturbative features of the theory.

We mention at last that for Ginsparg-Wilson fermions it is also possible to define a conserved axial current, and of course a conserved vector current (which was possible already in the Wilson case). The form of these Noether currents is rather complicated. They extend over all lattice sites, and the kernels decay exponentially with the distance from the "physical" point (in which the corresponding continuum current is defined). These are then still local currents in the sense explained above. They are given explicitly in (Kikukawa and Yamada, 1999b) for overlap fermions.

8.2 Overlap fermions

Almost ten years ago Narayanan and Neuberger (1993a; 1993b; 1994; 1995), motivated from mathematical insights and previous theoretical developments (Callan and Harvey, 1985; Kaplan, 1992; Frolov and Slavnov, 1993), devised an ingenious construction with which it was shown how to define chiral fermions on the lattice. The chiral mode resulted from the overlap between two infinite towers of chiral fermion fields, and the infinite number of fermions for each lattice site was the crucial new feature which was understood as necessary in order to construct chiral fermions on the lattice. The Dirac operator coming from this overlap formalism was later recognized by Neuberger (1998a; 1998b; 1998c) to be a solution of the Ginsparg-Wilson relation, and its action was given a simple form.

In the massless case [26] the overlap-Dirac operator is [27]

$$D_N = \frac{1}{a} \rho \left(1 + \frac{X}{\sqrt{X^\dagger X}} \right), \qquad X = D_W - \frac{1}{a} \rho, \tag{8.14}$$

where D_W is the usual Wilson-Dirac operator:

$$D_W = \frac{1}{2} \left(\gamma_\mu (\widetilde{\nabla}_\mu^\star + \widetilde{\nabla}_\mu) - ar\widetilde{\nabla}_\mu^\star \widetilde{\nabla}_\mu \right) \tag{8.15}$$

$$\widetilde{\nabla}_\mu \psi(x) = \frac{1}{a} \left(U(x, \mu)\psi(x + a\hat{\mu}) - \psi(x) \right).$$

---

[26] When the quarks have a nonzero bare mass m_0 then the overlap-Dirac operator is given by

$$D_N^{(m_0)} = \left(1 - \frac{1}{2\rho} am_0 \right) D_N + m_0. \tag{8.12}$$

[27] We note that any X which satisfies

$$\gamma_5 X^\dagger = X\gamma_5 \tag{8.13}$$

makes D_N a solution of the Ginsparg-Wilson relation. Using the Wilson action is the standard option, although other actions have been sometimes chosen for X, which are supposed to improve things. Such generalized overlap fermions have been proposed by (Bietenholz, 1999; Bietenholz and Hip, 2000; Bietenholz, 2001; Bietenholz, 2002). They showed that when X is a truncated perfect action (see later) the convergence properties of these fermions are improved, together with their locality and other symmetries. The presence in X of fat link actions (Section 15.7) can further improve things (Bietenholz, 2001; DeGrand, 2001). Perturbation theory however becomes much more complicated in all these cases.

In the range $0 < \rho < 2r$ (at tree level) the right spectrum of massless fermions is obtained. For the pure gauge part one usually uses the Wilson plaquette action. Sometimes improved pure gauge actions are also used.

Since additive mass renormalization is forbidden by chiral symmetry, when using overlap fermions one avoids altogether a source of systematic errors that is always present with Wilson fermions.

Although the interactions in the overlap action are not limited to nearest-neighbor sites, and not even to next-to-nearest-neighbor sites, but in fact involve all sites, the strength of these interactions falls off exponentially with the distance, given in lattice units, and in this sense the theory is still local.

Let us now have a look at perturbation theory with overlap fermions. The interaction vertices and the quark propagator are much more complicated than the ones in the Wilson formulation. This causes the perturbative computations to be rather cumbersome, and one then needs computer programs, even in the simplest cases. The calculations in (Capitani, 2001a; Capitani, 2001b; Capitani and Giusti, 2000; Capitani and Giusti, 2001; Capitani, 2002a) have been carried out using an ensemble of routines written in the symbolic manipulation language FORM. These routines are an extension of the ones used to do calculations with the Wilson action in several cases.

In deriving the Feynman rules we start by noticing that in momentum space

$$
X_0(k) = \frac{1}{a}\left(i \sum_\mu \gamma_\mu \sin ak_\mu + 2r \sum_\mu \sin^2 \frac{ak_\mu}{2} - \rho \right). \tag{8.16}
$$

The massless quark propagator, which is computed inverting

$$
\frac{1}{a}\rho\left(1 + \frac{X_0(k)}{\sqrt{X_0^\dagger(k)X_0(k)}} \right), \tag{8.17}
$$

is then given by

$$
S^{ab}(k) = \delta^{ab} \cdot \left(\frac{-i \sum_\mu \gamma_\mu \sin ak_\mu}{2\rho\left(\omega(k) + b(k)\right)} + \frac{a}{2\rho} \right), \tag{8.18}
$$

where

$$
\omega(k) \;=\; \left(\sqrt{X^\dagger X}\right)_0(k) = \frac{1}{a}\sqrt{\sum_\mu \sin^2 ak_\mu + \left(2r \sum_\mu \sin^2 \frac{ak_\mu}{2} - \rho \right)^2}, \tag{8.19}
$$

$$
b(k) \;=\; \frac{1}{a}\left(2r \sum_\mu \sin^2 \frac{ak_\mu}{2} - \rho \right).
$$

To compute the interaction vertices one first expands X order by order:

$$
X(p_1, p_2) = X_0(p_1)(2\pi)^4 \delta^{(4)}(p_1 - p_2) + X_1(p_1, p_2) + X_2(p_1, p_2) + O(g_0^3), \tag{8.20}
$$

55

where X_0 is given in Eq. (8.16), and the X_i's are the same interaction vertices of the Wilson action. For one-loop calculations one needs to compute only the vertices of order g_0 and g_0^2. Let us now write the expansion of the inverse of the square root as follows:

$$\frac{1}{\sqrt{X^\dagger X}}(p_1, p_2) = \left(\frac{1}{\sqrt{X^\dagger X}}\right)_0 (p_1, p_2) + Y_1(p_1, p_2) + Y_2(p_1, p_2) + O(g_0^3). \tag{8.21}$$

Y_1 and Y_2 can be obtained imposing the identity (Kikukawa and Yamada, 1999a)

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4q}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4r}{(2\pi)^4} (X^\dagger X)(p_1, q) \left(\frac{1}{\sqrt{X^\dagger X}}\right)(q, r) \left(\frac{1}{\sqrt{X^\dagger X}}\right)(r, p_2) = (2\pi)^4 \delta^{(4)}(p_1 - p_2) \tag{8.22}$$

order by order in the coupling. For example, at first order one has

$$(X^\dagger X)_1(p_1, p_2) \left(\frac{1}{\sqrt{X^\dagger X}}\right)_0^2 (p_2) + (X^\dagger X)_0(p_1) \, Y_1(p_1, p_2) \left(\frac{1}{\sqrt{X^\dagger X}}\right)_0 (p_2)$$
$$+ (\sqrt{X^\dagger X})_0(p_1) \, Y_1(p_1, p_2) = 0, \tag{8.23}$$

which solving for Y_1 gives

$$Y_1(p_1, p_2) = -\frac{1}{(\omega(p_1) + \omega(p_2))\omega(p_1)\omega(p_2)} (X^\dagger X)_1(p_1, p_2). \tag{8.24}$$

One can easily compute $(X^\dagger X)_1$ and thus obtain an explicit expression for Y_1. With somewhat more manipulations one also arrives at an expression for Y_2. The overlap vertices can be finally computed using

$$\left(\frac{X}{\sqrt{X^\dagger X}}\right)_1 = X_0 Y_1 + X_1 \left(\frac{1}{\sqrt{X^\dagger X}}\right)_0 \tag{8.25}$$

$$\left(\frac{X}{\sqrt{X^\dagger X}}\right)_2 = X_0 Y_2 + X_1 Y_1 + X_2 \left(\frac{1}{\sqrt{X^\dagger X}}\right)_0. \tag{8.26}$$

It is clear that the interaction vertices can be entirely given in terms of the vertices of the QED Wilson action,

$$W_{1\mu}(p_1, p_2) = -g_0 \left(i\gamma_\mu \cos \frac{a(p_1 + p_2)_\mu}{2} + r \sin \frac{a(p_1 + p_2)_\mu}{2} \right) \tag{8.27}$$

$$W_{2\mu}(p_1, p_2) = -\frac{1}{2} a g_0^2 \left(-i\gamma_\mu \sin \frac{a(p_1 + p_2)_\mu}{2} + r \cos \frac{a(p_1 + p_2)_\mu}{2} \right)$$

(where p_1 and p_2 are the quark momenta flowing in and out of the vertices), and of $X_0(p)$. The quark-quark-gluon vertex in the overlap theory has then the expression

$$(V_1^a)_\mu^{bc}(p_1, p_2) = (T^a)^{bc} \cdot \rho \frac{1}{\omega(p_1) + \omega(p_2)} \left[W_{1\mu}(p_1, p_2) - \frac{1}{\omega(p_1)\omega(p_2)} X_0(p_2) W_{1\mu}^\dagger(p_1, p_2) X_0(p_1) \right], \tag{8.28}$$

which in the continuum limit is the usual QCD vertex, $-g_0(T^a)^{bc}\,\mathrm{i}\gamma_\mu$. The quark-quark-gluon-gluon vertex in the overlap theory is

$$
(V_2^{ab})_{\mu\nu}^{cd}(p_1, p_2) = \delta_{\mu\nu} \left(\frac{1}{N_c}\delta^{ab} + d^{abe}T^e \right)^{cd}
$$

$$
\times \left\{ \rho \frac{1}{\omega(p_1) + \omega(p_2)} \left[W_{2\mu}(p_1, p_2) - \frac{1}{\omega(p_1)\omega(p_2)} X_0(p_2) W_{2\mu}^\dagger(p_1, p_2) X_0(p_1) \right] \right.
$$

$$
+ \frac{1}{2}\rho \frac{1}{\omega(p_1) + \omega(p_2)} \frac{1}{\omega(p_1) + \omega(k)} \frac{1}{\omega(k) + \omega(p_2)} \tag{8.29}
$$

$$
\times \left[X_0(p_2) W_{1\mu}^\dagger(p_2, k) W_{1\nu}(k, p_1) + W_{1\mu}(p_2, k) X_0^\dagger(k) W_{1\nu}(k, p_1) + W_{1\mu}(p_2, k) W_{1\nu}^\dagger(k, p_1) X_0(p_1) \right.
$$

$$
\left. \left. - \frac{\omega(p_1) + \omega(k) + \omega(p_2)}{\omega(p_1)\omega(k)\omega(p_2)} X_0(p_2) W_{1\mu}^\dagger(p_2, k) X_0(k) W_{1\nu}^\dagger(k, p_1) X_0(p_1) \right] \right\}.
$$

This is an irrelevant operator and then, although it is not obvious by its visual appearance, it vanishes in the continuum limit.

Beyond the lowest orders the calculation of the vertices becomes increasingly complicated, but in principle it can be carried through. Perhaps in this case it is more convenient to use an alternative method to derive the Feynman rules, which involves less algebra but more integrals. It makes use of the integral representation

$$
\frac{1}{\sqrt{X^\dagger X}} = \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X^\dagger X}. \tag{8.30}
$$

The expansion of the square root to the second order in this representation is given by

$$
\frac{1}{\sqrt{X^\dagger X}} = \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0} - \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0}(X_0^\dagger X_1 + X_1^\dagger X_0)\frac{1}{t^2 + X_0^\dagger X_0} \tag{8.31}
$$

$$
- \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0}(X_0^\dagger X_2 + X_1^\dagger X_1 + X_2^\dagger X_0)\frac{1}{t^2 + X_0^\dagger X_0}
$$

$$
+ \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0}(X_0^\dagger X_1 + X_1^\dagger X_0)\frac{1}{t^2 + X_0^\dagger X_0}(X_0^\dagger X_1 + X_1^\dagger X_0)\frac{1}{t^2 + X_0^\dagger X_0},
$$

so that, using the fact that $1/(t^2 + X_0^\dagger X_0)$ commutes with X_0, we have that

$$
aD = aD_0 + \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0}(t^2 X_1 - X_0 X_1^\dagger X_0)\frac{1}{t^2 + X_0^\dagger X_0} \tag{8.32}
$$

$$
+ \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X_0^\dagger X_0}(t^2 X_2 - X_0 X_2^\dagger X_0)\frac{1}{t^2 + X_0^\dagger X_0}
$$

$$
- \int_{-\infty}^{\infty} \frac{dt}{\pi} t^2 \frac{1}{t^2 + X_0^\dagger X_0}X_1\frac{1}{t^2 + X_0^\dagger X_0}(X_0^\dagger X_1 + X_1^\dagger X_0)\frac{1}{t^2 + X_0^\dagger X_0}
$$

$$
- \int_{-\infty}^{\infty} \frac{dt}{\pi} t^2 \frac{1}{t^2 + X_0^\dagger X_0}X_0 X_1^\dagger\frac{1}{t^2 + X_0^\dagger X_0}X_1\frac{1}{t^2 + X_0^\dagger X_0}
$$

$$
+ \int_{-\infty}^{\infty} \frac{dt}{\pi} X_0\frac{1}{t^2 + X_0^\dagger X_0}X_1^\dagger X_0\frac{1}{t^2 + X_0^\dagger X_0}X_1^\dagger X_0\frac{1}{t^2 + X_0^\dagger X_0} + \cdots.
$$

The t integration can now be performed using the residue theorem, and at the end one obtain the various vertices.

For more details on one-loop perturbative calculations with overlap fermions the works of (Chiu, Wang and Zenkin, 1998; Ishibashi *et al.*, 2000) and (Yamada, 1998) are useful. One can also see (Fujikawa and Ishibashi, 2002), which contains many technical details about perturbation theory, although for generalized Ginsparg-Wilson relations.

Many 1-loop calculations with overlap fermions have been already carried out, including the relation between the Λ parameter in the lattice scheme defined by the overlap operator and in the $\overline{\text{MS}}$ scheme (Alexandrou, Panagopoulos and Vicari, 2000), and the renormalization factors of the quark bilinears $\overline{\psi}\Gamma\psi$ (Alexandrou *et al.*, 2000), of the lowest moments of all structure functions (Capitani, 2001a; Capitani, 2001b; Capitani, 2002a), and of the $\Delta S = 2$ and $\Delta S = 1$ effective weak Hamiltonians, which are important for the calculation of $\Delta I = 1/2$ amplitudes (see Section 14.2) and of the parameter ϵ'/ϵ on the lattice (Capitani and Giusti, 2000; Capitani and Giusti, 2001).

A calculation of the axial anomaly to order g_0^4 has been presented in (Chiu and Hsieh, 2002).

8.3   Domain wall fermions

Another solution of the Ginsparg-Wilson relation is given by domain wall fermions. The construction of domain wall fermions is not peculiar to the lattice, and we begin by discussing a simple continuum case. The main point, which traces back to ideas from (Callan and Harvey, 1985), is to work in a 5-dimensional spacetime where in the fifth dimension, denoted by s, there is a domain wall separating the region $s > 0$ from the region $s < 0$. This domain wall can be described by a background field $\Phi(s)$ with a behavior like in Fig. 10. An example is $\Phi(s) = M \tanh(Ms)$, but the precise form is not important. The essential thing is that it behaves like a step function with a height of order M and a width of order $1/M$.

The 5-dimensional free Dirac operator in this theory is formed adding to the usual term a derivative term in the fifth dimension which is proportional to γ_5, and the background field:

$$D_5 = \gamma_\mu \partial_\mu + \gamma_5 \partial_s - \Phi(s). \tag{8.33}$$

The eigenvectors of this operator can be written in the form

$$\chi(x, s) = e^{ipx} u(s), \tag{8.34}$$

where the 4-dimensional plane waves are solution of the usual 4-dimensional Dirac equation, while $u(s)$ satisfies

$$\left(\gamma_5 \partial_s - \Phi(s)\right) u(s) = -i\gamma_\mu p_\mu u(s). \tag{8.35}$$

All solutions $\chi(x, s)$ have a definite chirality and describe fermions which have a mass of order M (the only available scale), except a mode which is massless and which obeys the equations

$$(\gamma_5 \partial_s - \Phi(s)) u(s) = 0, \qquad \gamma_\mu p_\mu u(s) = 0. \tag{8.36}$$
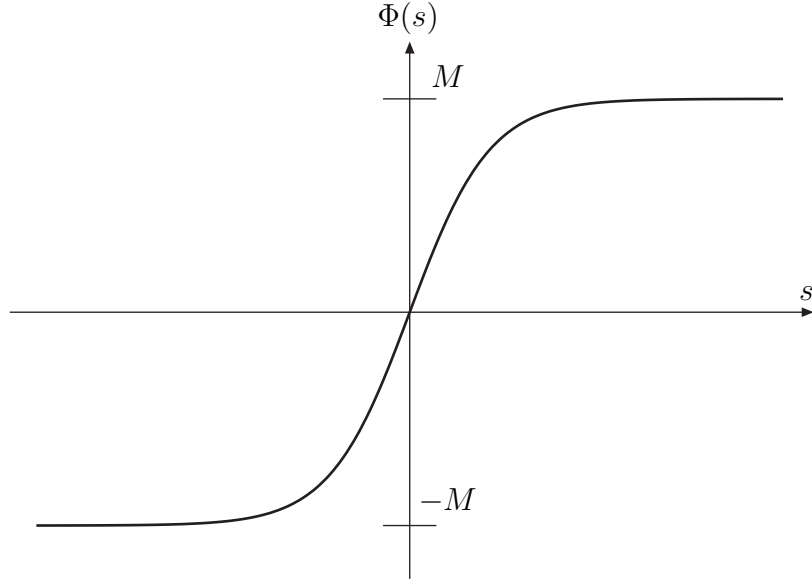
58

Figure 10: A background field for the domain wall.

The massless chiral solutions are then given by

$$u(s) = \exp\left(\pm \int_0^s dt\, \Phi(t)\right)v, \qquad \gamma_\mu p_\mu v = 0, \qquad P_\pm v = v, \tag{8.37}$$

and we see that the solution with positive chirality is nonnormalizable for $|s| \to \infty$, while the interesting result is the solution with negative chirality. This is the massless chiral mode, the aim of the domain wall construction, which from the above equation can be seen to fall off exponentially in the fifth dimension. This mode is thus confined near the domain wall $s = 0$, and at energies which are far below M this is the only particle left in the theory, which has then effectively become, after this dimensional reduction, a chiral theory in 4 dimensions. [28]

This chiral mode is present also when the theory is formulated on a lattice (Kaplan 1992; 1993). The scale M is now meant to be of order $1/a$, so that for any finite lattice spacing only the massless chiral mode can propagate, while the massive particles decouple from the theory.

As already noted, the precise form of the background field is not important. Let us consider the half space $s \geq 0$, with $P_+\chi(x,s)|_{s=0} = 0$, in which we can take $\Phi(s) = M$. The Dirac operator is now just

$$D_5 = D_4 + \gamma_5\partial_s - M, \tag{8.38}$$

with $D_4 = \gamma_\mu\partial_\mu$. It can be shown that the 5-dimensional propagator on the domain wall is then given by

$$G(x,s;y,t)\Big|_{s=t=0} = 2M \cdot P_- S(x,y)P_+, \tag{8.39}$$

[28]For more details on this derivation see also (Lüscher, 2001).

where the propagator $S(x, y)$ is the inverse of the 4-dimensional operator

$$D = M + \frac{D_4 - M}{\sqrt{1 - \left(\dfrac{D_4}{M}\right)^2}},$$ (8.40)

which satisfies the relation

$$\gamma_5 D + D\gamma_5 = \frac{1}{M} D\gamma_5 D.$$ (8.41)

If we make at this point the identification $M = \rho/a$, the above equations, which we have obtained after doing the dimensional reduction, are just the overlap solution (when $D_4 = D_W$) and the Ginsparg-Wilson relation. In this sense, overlap and domain wall fermions are equivalent, and one can consider the overlap case like a domain wall seen from the 4-dimensional world.

The lattice formulation of domain wall fermions that is commonly used in simulations has been proposed in (Shamir, 1993; Furman and Shamir, 1995). For a review see also (Jansen, 1996). The 5-dimensional action is constructed starting from the 4-dimensional Wilson action (otherwise some doublers mode would still be present), and is given by [29]

$$
\begin{aligned}
S_{DW} \;=\; & \sum_x \sum_{s=1}^{N_s} \left[\frac{1}{2} \sum_\mu \left(\overline{\psi}_s(x)(-r + \gamma_\mu) U_\mu(x) \psi_s(x + \hat{\mu}) + \overline{\psi}_s(x)(-r - \gamma_\mu) U_\mu^\dagger(x - \hat{\mu}) \psi_s(x - \hat{\mu}) \right) \right. \\
& \left. + \frac{1}{2} \left(\overline{\psi}_s(x)(1 + \gamma_5)\psi_{s+1}(x) + \overline{\psi}_s(x)(1 - \gamma_5)\psi_{s-1}(x) \right) + (\rho - 1 + 4r)\overline{\psi}_s(x)\psi_s(x) \right] \\
& + m \sum_x \left(\overline{\psi}_{N_s}(x) P_+ \psi_1(x) + \overline{\psi}_1(x) P_+ \psi_{N_s}(x) \right).
\end{aligned}
$$ (8.42)

There is a parameter ρ, which corresponds to the one already seen in the overlap case, and has to take values between zero and two (at tree level). The Wilson parameter is set to $r = -1$, and the sign of the Wilson term is different. This domain wall action can be thought as a Wilson action with an infinite number of flavors (labeled by the index s) and a special mass matrix (Eqs. (8.47) and (8.48) below). The infinite number of flavors corresponds to the infinite tower of fermions for each lattice site of the original overlap formulation.

The pure gauge part is the usual Wilson plaquette action. The gauge fields are 4-dimensional, and therefore there is no gauge interaction along the fifth dimension. The measure term, the gauge-fixing term and the Faddeev-Popov term are also the same as in Wilson. This means that the gluon propagator and the quark-gluon vertices are the same as in the Wilson action. The quark propagator is instead different, and much more complicated. In practical terms this lattice action can be looked at as describing a theory of N_s fermion flavors which have a complicated propagator.

---

[29] In the following we put $a = 1$.

As the reader will have noticed, the range of s in Eq. (8.42) is finite, $1 \leq s \leq N_s$, because this is how the simulations have to be made. All modes have a mass of order ρ/a except two modes which are nearly massless (that is, their masses are exponentially small in N_s) and which are localized near the boundaries $s = 1$ and $s = N_s$. These two modes have opposite chirality. As long as N_s is finite there is a small residual interaction (exponentially small in N_s) between these two chiral modes. [30] It is only in the limit in which the number of points in the fifth dimension goes to infinity that the chiral mode at $s = 1$ becomes massless and decouples entirely from the other massless mode, giving an exact chiral theory. In the perturbative calculations made so far (Aoki and Hirose, 1996; Aoki and Taniguchi, 1999a; Aoki *et al.*, 1999a; Aoki and Taniguchi, 1999b; Aoki *et al.*, 1999b; Aoki and Kuramashi, 2001; Aoki *et al.*, 2002) it has been assumed that one can take the limit $N_s \to \infty$ before doing the Feynman integrals. The quark propagator which one uses in perturbation theory is then the one for $N_s = \infty$.

The domain wall Dirac operator in momentum space is

$$D_{st}(p) = \delta_{s,t} \sum_\mu \mathrm{i}\gamma_\mu \sin p_\mu + (W_{st}^+(p) + mM_{st}^+)P_+ + (W_{st}^-(p) + mM_{st}^-)P_-, \tag{8.43}$$

with

$$\begin{align}
W_{st}^+(p) &= -W(p)\,\delta_{s,t} + \delta_{s+1,t}, \tag{8.44} \\
W_{st}^-(p) &= -W(p)\,\delta_{s,t} + \delta_{s-1,t}, \tag{8.45} \\
W(p) &= 1 - \rho - 2r\sum_\mu \sin^2 \frac{p_\mu}{2}, \tag{8.46}
\end{align}$$

and the mass matrix is

$$\begin{align}
M_{st}^+ &= \delta_{s,N_s}\,\delta_{t,1}, \tag{8.47} \\
M_{st}^- &= \delta_{s,1}\,\delta_{t,N_s}, . \tag{8.48}
\end{align}$$

By inverting it one gets the quark propagator

$$\begin{align}
S_{st}(p) = \langle \psi_s(-p)\overline{\psi}_t(p)\rangle &= (-\mathrm{i}\gamma_\mu \sin p_\mu\, \delta_{s,u} + W_{su}^-(p) + mM_{su}^-(p))\, G_{ut}^R(p) P_+ \tag{8.49} \\
&\quad + (-\mathrm{i}\gamma_\mu \sin p_\mu\, \delta_{s,u} + W_{su}^+(p) + mM_{su}^+(p))\, G_{ut}^L(p) P_-,
\end{align}$$

where

$$\begin{align}
G_{st}^R(p) &= \frac{A}{F}\Big[-(1-m^2)(1-W\mathrm{e}^{-\alpha})\mathrm{e}^{\alpha(-2N_s+s+t)} - (1-m^2)(1-W\mathrm{e}^{\alpha})\mathrm{e}^{-\alpha(s+t)} \\
&\quad -2Wm(\mathrm{e}^{\alpha(-N+s-t)} + \mathrm{e}^{\alpha(-N_s-s+t)})\sinh\alpha \Big] + A\mathrm{e}^{-\alpha|s-t|}, \tag{8.50} \\
G_{st}^L(p) &= \frac{A}{F}\Big[-(1-m^2)(1-W\mathrm{e}^{\alpha})\mathrm{e}^{\alpha(-2N_s+s+t-2)} - (1-m^2)(1-W\mathrm{e}^{-\alpha})\mathrm{e}^{-\alpha(s+t-2)}
\end{align}$$

[30]The overlap between the chiral modes on the two walls depends also on the strength of the gauge coupling, and for strong couplings these chiral modes tend to acquire some nonnegligible overlap.

$$-2Wm(\mathrm{e}^{\alpha(-N+s-t)} + \mathrm{e}^{\alpha(-N_s-s+t)})\sinh\alpha\Big] + A\mathrm{e}^{-\alpha|s-t|}, \tag{8.51}$$

$$\cosh(\alpha) = \frac{1 + W^2 + \sum_\mu \sin^2 p_\mu}{2|W|}, \tag{8.52}$$

$$A = \frac{1}{2W\sinh\alpha}, \tag{8.53}$$

$$F = 1 - \mathrm{e}^\alpha W - m^2(1 - W\mathrm{e}^{-\alpha}). \tag{8.54}$$

These formulae are valid only for positive W. For $1 < \rho \le 2$ and small momenta W can be negative, and in this case the propagator is given by the above equations with the replacements

$$W \to -|W|, \tag{8.55}$$

$$\mathrm{e}^{\pm\alpha} \to -\mathrm{e}^{\pm\alpha}, \tag{8.56}$$

which also imply $\sinh\alpha \to -\sinh\alpha$.

The massless fermion is given at tree level by

$$\chi_0 = \sqrt{1 - w_0^2}\,(P_+ w_0^{s-1}\psi_s + P_- w_0^{N_s-s}\psi_s), \tag{8.57}$$

with $w_0 = 1 - \rho$. We can see that this damping factor confines the two chiralities on the two different domain walls. Since this factor is renormalized by the interactions due to the additive renormalization of ρ (this is a mass term not protected by chiral symmetry), [31] it is more convenient to work instead with the "physical" field

$$q(x) = P_+\psi_1(x) + P_-\psi_{N_s}(x) \tag{8.58}$$

$$\overline{q}(x) = \overline{\psi}_{N_s}(x)P_+ + \overline{\psi}_1(x)P_-, \tag{8.59}$$

whose renormalization is simple. The corresponding propagator is given by

$$S_q(p) = \langle q(-p)\overline{q}(p)\rangle = \frac{-\mathrm{i}\gamma_\mu \sin p_\mu + (1 - W\mathrm{e}^{-\alpha})m}{-(1 - \mathrm{e}^\alpha W) + m^2(1 - W\mathrm{e}^{-\alpha})}. \tag{8.60}$$

While the 1-loop propagator $\langle\psi(-p)\overline{\psi}(p)\rangle$ gets an additive mass correction, this 1-loop physical propagator is protected from such terms, as it should be thanks to chiral symmetry. Composite operators in the theory are then constructed using the physical field. The bilinears are for example given by

$$O(x) = \overline{q}(x)\Gamma q(x). \tag{8.61}$$

One sees that, when computing Feynman diagrams in which the physical field is present as an external state, the propagators of the external lines are also needed,

$$\langle q(-p)\overline{\psi}_s(p)\rangle = \frac{1}{F}(\mathrm{i}\gamma_\mu \sin p_\mu - m(1 - W\mathrm{e}^{-\alpha}))(\mathrm{e}^{-\alpha(N_s-s)}P_+ + \mathrm{e}^{-\alpha(s-1)}P_-) \tag{8.62}$$

[31]The renormalizations of w_0 and of the wave function turn out to be quite large. The former in particular is of order $O(10^2)$. This is claimed to be cured by tadpole improvement. The wave function at one loop has been also computed in (Shamir, 2000).
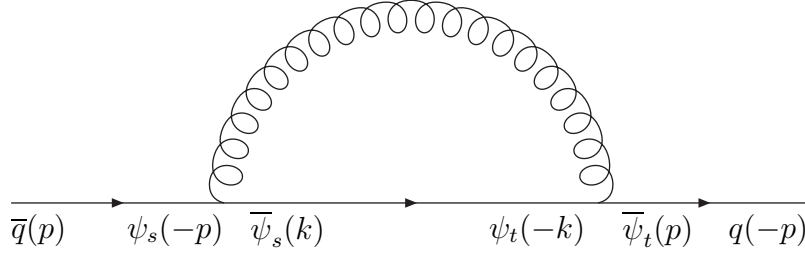
Figure 11: A typical correction to the physical quark propagator showing the various fermionic fields which form the various propagators.

$$
\langle \psi_s(-p)\overline{q}(p) \rangle =
\begin{aligned}
&+ \frac{1}{F}[m(\mathrm{i}\gamma_\mu \sin p_\mu - m(1 - W\mathrm{e}^{-\alpha})) - F]\mathrm{e}^{-\alpha}(\mathrm{e}^{-\alpha(s-1)}P_+ + \mathrm{e}^{-\alpha(N_s-s)}P_-), \\
&\quad \frac{1}{F}(\mathrm{e}^{-\alpha(N_s-s)}P_- + \mathrm{e}^{-\alpha(s-1)}P_+)(\mathrm{i}\gamma_\mu \sin p_\mu - m(1 - W\mathrm{e}^{-\alpha})) \\
&+ \frac{1}{F}(\mathrm{e}^{-\alpha(s-1)}P_- + \mathrm{e}^{-\alpha(N_s-s)}P_+)\mathrm{e}^{-\alpha}[m(\mathrm{i}\gamma_\mu \sin p_\mu - m(1 - W\mathrm{e}^{-\alpha})) - F].
\end{aligned}
\tag{8.63}
$$

A typical situation in which these propagators are required is depicted in Fig. 11, a self-energy diagram. It is interesting to note that logarithmic divergences, for example in the self-energy, are localized at the boundaries $s = 1$ and $s = N_s$, because they arise when the masses are going to zero.

Given the expressions of the quark propagators, perturbative calculations with domain wall fermions tend to be rather cumbersome, even in the limit $N_s = \infty$. Many quantities have been calculated so far, using gluon propagators in Feynman gauge, but no operators with covariant derivatives have been considered, like for instance the ones measuring moments of structure functions.

8.4 Fixed-point fermions

Historically the fixed-point action, developed in the context of perfect actions (Hasenfratz and Niedermayer, 1994; Wiese, 1993; Bietenholz and Wiese, 1994; DeGrand *et al.*, 1995; DeGrand *et al.*, 1996; DeGrand *et al.*, 1997), was the actual chiral battleground where the Ginsparg-Wilson relation was rediscovered (Hasenfratz, 1998a). The key idea here goes under the name of classically perfect actions, that is actions for which their classical predictions (and in particular the properties of the chiral modes) agree with those of the continuum. This is true even when

the lattice spacing is not small. [32]

The construction of perfect actions employs ideas which strongly resemble Wilson's renormalization group in statistical mechanics (Wilson and Kogut, 1974; Wilson, 1975). A renormalization group step in this context, also called a block transformation, consists in doubling the lattice spacing, which means that in lattice units the correlation length is halved after each step. In this way the short-scale fluctuations in the functional integral can be eliminated step by step. The fixed points of these renormalization group transformations, which is reached after they are iterated an infinite number of times,

$$\mathcal{A} \longrightarrow \mathcal{A}' \longrightarrow \mathcal{A}'' \longrightarrow \mathcal{A}''' \longrightarrow \cdots \longrightarrow \mathcal{A}^{FP}, \tag{8.64}$$

are the fixed-point actions. Their properties can be investigated using classical equations.

A block variable of the lattice with double lattice spacing can be constructed from the fine variable as follows:

$$\chi(x_B) = b \sum_x \omega(2x_B - x)\phi(x), \qquad \left(\sum_x \omega(2x_B - x) = 1\right). \tag{8.65}$$

One integrates out the original variables keeping the new block averages fixed. Blocking is then equivalent to going to a coarse lattice. The action that one obtains after the blocking transformations will in general contain all kinds of interactions, even when one starts with a very simple action. The fixed-point action to which one at the end arrives contains an infinite number of interactions, with an infinite number of corresponding couplings.

A fixed point is by definition an action that is invariant under further block transformations. One is then on a critical surface and the correlation length is infinite. This means that in asymptotically free theories, and in particular in QCD, the fixed-point action can only be reached for $g_0 \to 0$, that is $\beta \to \infty$. Strictly speaking a fixed-point action $\mathcal{A}^{FP}$ could then only be simulated at $\beta = \infty$. However, if one considers an action with the same form as the fixed-point action but for finite β, this action will be quite close, for large β, to the true renormalization group trajectory (see Fig. 12). The lattice artifacts will then be quite small. In this way one has linearized the fixed-point transformation and in the only relevant direction this action $\beta\mathcal{A}^{FP}$ is classically perfect. [33]

Doing a good averaging using appropriate block transformations is a kind of an art. A bad choice can lead to a fixed-point action with bad locality properties or even to no fixed point at all. A good choice can instead lead to a good parameterization of the fixed-point action which is convenient for simulations. The task is then to find a good blocking transformation which leads to a fixed-point action that can be well approximated by taking a small number of terms which are not difficult to simulate.

---

[32]For a nice pedagogical review of these ideas and techniques, see (Hasenfratz, 1998b); a short review can be found in (Hasenfratz, 1998a).

[33]However, it is not quantum perfect, as 1-loop calculations have shown (Hasenfratz and Niedermayer, 1997).
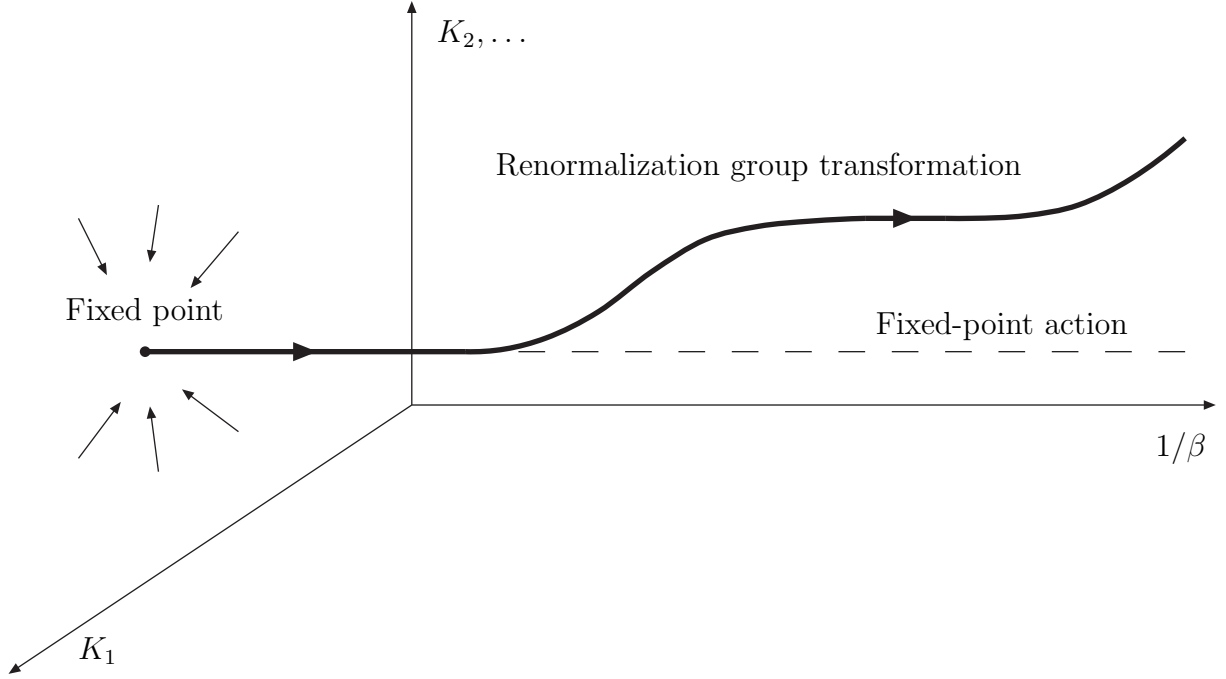
Figure 12: The renormalization group flow and the classically perfect fixed-point trajectory. The directions $K_1, K_2, \ldots$ represent the parameter space.

Let us now consider the Wilson action

$$
\begin{aligned}
\beta \mathcal{A}_g(U) + \mathcal{A}_f(\overline{\psi}, \psi, U) \;=\; & \beta \sum_P \left(1 - \frac{1}{N_c} \mathrm{Re}\, \mathrm{Tr} U_P\right) \\
& + \frac{1}{2} \sum_{x,\mu} \left(\overline{\psi}(x)(\gamma_\mu - 1)U_\mu(x)\psi(x + \hat{\mu}) - \overline{\psi}(x + \hat{\mu})(\gamma_\mu + 1)U_\mu^\dagger(x)\psi(x)\right) \\
& + \sum_x \overline{\psi}(x)(m + 4)\psi(x).
\end{aligned}
\tag{8.66}
$$

The renormalization group transformed action is given, after one step, by

$$
e^{-\left(\beta' \mathcal{A}_g'(V) + \mathcal{A}_f'(\overline{\chi},\chi,V)\right)} = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}U \, e^{-\left(\beta(\mathcal{A}_g(U) + T_g(V,U)) + \mathcal{A}_f(\overline{\psi},\psi,U) + T_f(\overline{\chi},\chi,\overline{\psi},\psi,U)\right)},
\tag{8.67}
$$

where the blocking kernels are

$$
T_g(V,U) = \sum_{x_B,\mu} \left[-\frac{\kappa_g}{N_c} \mathrm{Re}\, \mathrm{Tr}(V_\mu(x_B)Q_\mu^\dagger(x_B)) + \mathcal{N}(Q_\mu(x_B)) \right]
\tag{8.68}
$$

for the pure gauge part (where $\mathcal{N}$ is needed to normalize correctly the block variables), and

$$
T_f(\overline{\chi}, \chi, \overline{\psi}, \psi, U) = \kappa_f \sum_{x_B} \left(\overline{\chi}(x_B) - b_f \sum_x \overline{\psi}(x)\omega_{x,x_B}^\dagger(U)\right)\left(\chi(x_B) - b_f \sum_x \omega_{x_B,x}(U)\psi(x)\right)
\tag{8.69}
$$

65

for the fermion part. $Q_\mu(x_B)$ is an average of the fine links near x_B, while ω is an averaging function for the quark fields. The details of these function are complicated and will not interest us here. A simple case is given by $\omega(2x_B - x) = 2^{-d}$ for the points x in the hypercube containing x_B.

Note that for finite κ_g, κ_f the average of the fine variables in a block is allowed to fluctuate around the block variables. When instead $\kappa_g, \kappa_f \to \infty$ the blocking kernels become delta functions. Thus, the parameters κ_g and κ_f determine the stiffness of the averaging.

In QCD the fixed point lies at $\beta = \infty$. In this limit one can perform the computations using a saddle-point approximation. Hence for the pure gauge part we have

$$\mathcal{A}'_g(V) = \min_{\{U\}} \left[\mathcal{A}_g(U) + T_g^\infty(V, U) \right], \tag{8.70}$$

where T_g^∞ is the $\beta = \infty$ limit of the blocking kernel, and the fixed-point action satisfies

$$\mathcal{A}_g^{FP}(V) = \min_{\{U\}} \left[\mathcal{A}_g^{FP}(U) + T_g^\infty(V, U) \right]. \tag{8.71}$$

Substituting the $U_{\min}$ that minimizes Eq. (8.70) in the fermion part one obtains a recursion relation and finally arrives at the equation that determines the fixed-point fermion action,

$$h_{x_B, x'_B}^{FP}(V)^{-1} = \frac{1}{\kappa_f} \delta_{x_B, x'_B} + b_f^2 \sum_{x, x'} \omega_{x_B, x}(U_{min}) h_{x, x'}^{FP}(U_{\min})^{-1} \omega_{x', x'_B}^\dagger(U_{\min}), \tag{8.72}$$

where

$$\mathcal{A}_f(\overline{\psi}, \psi, U) = \sum_{x, x'} \overline{\psi}(x) h_{x, x'}(U) \psi(x'). \tag{8.73}$$

So far no approximations have been made. The blocking kernels are quadratic, and if the actions are also quadratic the renormalization group transformations can be computed using Gaussian integrals. The above is then an exact formula, and Gaussian integrals are equivalent to minimization. The solutions are in general obtained doing some approximations and truncations, but in the case of the free theory they can be still computed exactly.

The fixed-point propagator which is obtained starting from massless Wilson fermions and using $\omega(2x_B - x) = 2^{-d}$ and $U_{\min} = 1$ is given by

$$h^{-1}(q) = \sum_{l \in Z^d} \frac{\gamma_\mu(q_\mu + 2\pi l_\mu)}{(q + 2\pi l)^2} \prod_\nu \frac{\sin^2 \dfrac{q_\nu}{2}}{\left(\dfrac{q_\nu}{2} + \pi l_\nu \right)^2} + \frac{2}{\kappa_f}. \tag{8.74}$$

This is an analytic function and corresponds to a local action which has no doublers. It is then not surprising to learn that this action breaks chiral symmetry. This breaking is due to the term $2/\kappa_f$, which comes entirely from the fermionic blocking kernel T_f. The remarkable point is that this fixed-point propagator satisfies

$$\{h^{-1}, \gamma_5\} = \frac{4}{\kappa_f} \gamma_5, \tag{8.75}$$

66

or equivalently the fixed-point action obeys

$$\{h, \gamma_5\} = \frac{4}{\kappa_f}\, h\gamma_5 h, \tag{8.76}$$

which we can recognize to be a form of the Ginsparg-Wilson relation. Thus this action, although it breaks the naive chiral symmetry, has a remnant of chiral symmetry (from the point of view of the continuum) which is given by Lüscher's symmetry, and which is a good chiral symmetry for any finite value of the lattice spacing. It follows that the pion mass is zero when the bare quark mass in the action is zero, that the index theorem is satisfied, that the theory reproduces the correct global anomalies, and so on.

We would like to point out that in the limit $\kappa_f = \infty$ this action becomes chirally symmetric in the naive sense, $\{h, \gamma_5\} = 0$, but then it becomes also nonlocal, and so it does not contradict the Nielsen-Ninomiya theorem.

We have thus learned that the fixed-point action satisfies the Ginsparg-Wilson relation. As we observed at the beginning, it is actually when studying the solution in Eq. (8.74) that the Ginsparg-Wilson was rediscovered in all its strength after fifteen years of neglect.

In Monte Carlo computations a truncation of the fixed-point action is necessary. It is not possible to simulate more than the first few terms. It is however hoped that the use of the truncated action still brings small errors with respect to the results that would come out using the true fixed-point action.

Perturbative calculations, even for a truncated action, are more cumbersome than average. Even when the action contains just a few terms, their complexity can increase swiftly, because they contain higher powers of the fundamental fields and of their derivatives. The propagator and vertices are obtained from summing the contributions of all these terms, which in general will give some complicated expressions. The coefficients of these various terms can only be determined numerically, and this constitutes another limitation on the accuracy of fixed-point results. When one wants to make simulations using a truncated action that has a small residual symmetry breaking and is not too far from the true fixed-point action, a relatively large number of terms is needed, and in this case perturbation theory looks quite demanding. In fact not many perturbative calculations with fixed-point actions have appeared in the literature. Apart from (Hasenfratz and Niedermayer, 1997), examples of this kind of perturbative calculations can be found in (Bietenholz and Struckmann, 1999), where a perfect action for the anharmonic oscillator was perturbatively constructed, and (Farchioni *et al.*, 1995), where the mass gap of the nonlinear σ-model was computed in perturbation theory. However, no renormalization of operators has been attempted.

Finally, we would like to point out that, although perturbation theory is more complicated here compared to overlap and domain wall fermions, the fixed-point action possesses some better physical properties than them. The fixed-point action is in fact classically perfect, while the overlap and domain wall actions are only aimed to recover the classical predictions for chiral symmetry, and do not care to what happens to other physical quantities. Moreover,

the strength of the interaction decays slower for overlap and domain wall fermions, that is the effective range is greater, and in this sense the fixed-point action is more local. Of course, the overlap action can be given explicitly in a simple form, while to obtain the fixed-point action, even in an approximate form, one needs to solve complicated equations.

At the end of the day domain wall and overlap fermions are equivalent formulations, while fixed-point fermions can be considered to be something different because they possess better "continuum" properties than the other two cases.

For recent developments on the subject of fixed-point actions, see (Hasenfratz *et al.*, 2002). Many technical details can also be found in (Jörg, 2002).

8.5 Concluding remarks

Among the various formulations of Ginsparg-Wilson fermions, we have discussed in some detail the perturbation theory of the overlap solution and also given an elementary introduction to perturbative calculations for the domain wall and, for what is possible without being too technical, the perfect action solutions.

The practical aspects of the implementation of these various fermions, and of the computations made with them, are quite different. In particular, the control over the numerical chiral symmetry breaking which necessarily occurs in actual simulations seems to be easier for overlap fermions (Hernández, Jansen and Lüscher, 2000).

In the domain wall formulation, where chirality appears in the reduction of a five-dimensional theory to our four-dimensional world, one has to remember that the exact chiral symmetry is attained only when there is an infinite number of sites in the fifth dimension, which is never the case in Monte Carlo simulations. Reducing the amount of chiral breaking in this case means doing new simulations using larger lattices.

In the fixed point action it is the necessary truncation which breaks chiral symmetry, and one has to add more and more terms in order to decrease the amount of breaking.

For overlap fermions, on the other hand, the numerical chiral symmetry breaking can be reduced by using more and more refined methods to compute the square root in the action. Techniques which use polynomial approximations of the action and an exact evaluation of the lowest eigenvalues of the Dirac operator are widely employed. One does not need to repeat the simulations on new and larger lattices, or to add more terms to the action. Here the road to smaller chiral symmetry breaking goes via the extraction of more eigenvalues and the refinement of the polynomials, which can be easier to accomplish. It looks instead much more expensive and complicated to make simulations on lattices which are longer and longer in the domain wall fifth dimension, or to include into the perfect action calculations more and more terms of the expansion, with all that this implies in terms of costs and complexities.

We mention in passing that also perturbation theory seems to be more difficult to carry out in the case of domain wall and fixed-point actions.

9 Perturbation theory of lattice regularized chiral gauge theories

The recent developments in the understanding of chiral symmetry on the lattice have also led to very interesting insights into the subject of chiral gauge theories, in which left- and right-handed fermions do not couple to the gauge fields in the same way. This is the case of neutrino interactions, in which as is well known no right-handed neutrino components couple to the electroweak gauge field, and in general of the electroweak theory.

We have learned in the previous Section that a Dirac operator satisfying the Ginsparg-Wilson relation describes fermions that have an exact chiral symmetry at finite lattice spacing. In this Section we discuss the fact that when this symmetry is gauged, that is when these fermions become chiral gauge fermions, the lattice regularization can at the same time preserve the gauge invariance at all orders in perturbation theory, and at any finite lattice spacing. It has been shown by Lüscher (2000c) that using Ginsparg-Wilson fermions it is indeed possible to regularize chiral gauge theories in such a way that the regularization does not break gauge invariance and at the same time maintains at all orders chiral symmetry, locality and all other fundamental principles. We think that this has been one the major theoretical advances in the theory of quantum fields on the lattice in recent times, and we would like to give a short account of it.

Ginsparg-Wilson lattice fermions provide in this way the only known regularization of chiral gauge theories which is consistent at the nonperturbative level and which does not violate the gauge symmetry or other fundamental principles. For all other widely used regularization methods (and also the BPHZ finite-part prescription) this is impossible, because chirality and gauge invariance cannot be maintained together as symmetries beyond the classical level. [34] If one wants to keep chiral invariance unbroken, one is compelled to introduce new counter-terms order by order in perturbation theory to maintain the gauge invariance, making these regularizations much less appealing. Even on the lattice itself new counterterms need to be introduced when one just uses Wilson fermions, and a delicate fine tuning of the corresponding coefficients is required in order to recover chiral invariance. This is what was done in the Rome approach (Borrelli *et al.*, 1989; 1990; Testa, 1998a). The problem here is not only that one has to introduce counterterms that are not gauge invariant, but that the fermion modes couple to the longitudinal modes of the gauge field (because of the lack of gauge invariance), and the effect of these unwanted gauge modes on the fermions, which can lead to modifications of the spectrum and in particular to the appearance of doublers, has to be controlled and removed in some way. While in the rest of this Section we are only interested in chiral gauge theories which

---

[34] A well-known 1-loop example in the electroweak theory is given by the divergent triangle diagrams containing γ_5, which give rise to the chiral anomaly (Adler, 1969; Bell and Jackiw, 1969; Bardeen, 1969). The recent lectures of (Zinn-Justin, 2002) contain much material about chiral anomalies in various regularizations and their connection with topology.

employ Ginsparg-Wilson fermions and which maintain an exact gauge invariance, there is another nonperturbative approach which is worth mentioning where the coupling of the fermions to the unphysical degrees of freedom of the gauge field is controlled by means of a suitable gauge fixing (Bock, Golterman and Shamir, 1998a; Bock, Golterman and Shamir, 1998b; Bock *et al.*, 2000). This gauge fixing approach uses Wilson fermions and the naive notion of chirality on the lattice, and skirts the consequences of the Nielsen-Ninomiya theorem because it has no explicit gauge invariance, which can in fact only be recovered in the continuum limit. It thus needs counterterms which are not gauge covariant (and which have to be appropriately tuned), but it still ensures renormalizability and at the end it achieves the decoupling of the longitudinal degrees of freedom while keeping the fermion spectrum intact, so that one can define a lattice chiral gauge theory where the fermions have no doublers. Some nonperturbative considerations are also needed in order to define a valid perturbation theory around $A_\mu = 0$ which is still a reliable approximation of the full lattice theory at weak coupling (Bock, Golterman and Shamir, 1998c). A naive gauge fixing does not do the job, and an appropriate classical potential containing higher powers of the gauge potential has to be introduced. A gauge fixing action which leads to a chiral gauge theory has been so constructed in the abelian case (Golterman and Shamir, 1997; Shamir, 1998), and it needs only one counterterm to be tuned nonperturbatively. For a recent presentation of this method see (Golterman and Shamir, 2002). Reviews have also been given in (Shamir, 1996) and (Golterman, 2001). [35]

It thus seems that the lattice, when Ginsparg-Wilson fermions are used, is an exception to this vicious pattern which is common to all other regularizations. The lattice acquires in this sense a predominance over other regularizations, as it is the only nonperturbative technique which makes possible to regularize theories which are chiral and gauge invariant, and it does a much better job than to introduce counterterms. No noninvariant counterterms in the action are needed. One can thus regularize chiral gauge theories without breaking the gauge invariance and using a cutoff, which was thought impossible.

Such a powerful chiral regularization, which can only be realized using Ginsparg-Wilson fermions on the lattice, works because the gauge anomaly cancels when radiative corrections are included. Simple schemes which do not take this fact into account and try to construct chiral fermions only at the tree level cannot work out. Many years have been indeed necessary to understand the structure of chiral fermions on the lattice and go beyond the apparent impossibility given by the Nielsen-Ninomiya theorem. Of course at the end one does not contradict the Nielsen-Ninomiya theorem at all, because it is the condition (d) (see Section 6) which is not fulfilled. This was a nontrivial conceptual advance.

We know from general results in algebraic renormalization theory that if chiral gauge anomalies can be shown to be canceled at one loop, then they are absent at all orders because no further anomalies can be generated at higher orders (Adler and Bardeen, 1969). A feature of Ginsparg-Wilson fermions is that the gauge anomaly descends from a topological field in 4+2

---

[35]I thank M. Golterman and Y. Shamir for correspondence.

dimensions, and that the anomaly cancellation at one loop can be formulated in terms of a local cohomology problem in $4 + 2$ dimensions, which has been studied and solved in (Lüscher, 1999a; 1999b; 2000a; 2000b; Suzuki, 1999), for abelian gauge theories. In general one needs a classification of the topological fields in 6 dimensions, and the local cohomology problem is then solved.

Let us now explain some of the details of the construction of chiral gauge theories using Ginsparg-Wilson fermions. In chiral theories the fermion integral in the path-integral formulation is restricted to left-handed fields, and the propagator involves a chiral projector (defined in Eq. (8.7)):

$$\{\psi(x)\overline{\psi}(y)\}_F = \langle 1 \rangle_F \times \widehat{P}_- S(x, y) P_+, \tag{9.1}$$

where $S(x, y)$ is the inverse of the Dirac operator: $\sum_z D(x, z) S(z, y) = a^{-4} \delta_{xy}$. Thus, only the left-handed components of the fermion field propagate.

The fundamental point is that the definition of the measure for left-handed fermions, which is needed to construct the quantum theory, is highly nontrivial, because the projectors $\widehat{P}$ contain a $\widehat{\gamma}_5 = \gamma_5(1 - a/\rho D)$ matrix, and thus depend on the gauge fields. This produces an ambiguity in the measure which can be expressed in terms of a pure phase factor, and can be seen in the following way. The measure can be defined by giving a complete basis of complex-valued fermionic fields v_j which are eigenstates of the projection operators. The quantum field can then be expanded like

$$\psi(x) = \sum_j c_j v_j(x), \tag{9.2}$$

and changing the basis means that the measure gets multiplied by the determinant of a unitary transformation matrix, which is a phase factor. Since the projectors depend on the gauge fields, the basis and the corresponding phase factor also depend on the gauge fields and cannot be fixed independently of them (see Fig. 13). This is the source of the phase ambiguity.

Thus, in chiral gauge theories the fermion integration has a nontrivial phase ambiguity and the fermion measure is not a simple product of local measures. Since this phase ambiguity depends on the gauge fields, it does not cancel in ratios of expectation values when one normalizes the path integral with the partition function. The right-handed projectors on the other hand give a constant phase factor, which can be factored out and does cancel in such ratios.

The phase problem, that is the definition of the measure, is the key issue in the construction of chiral gauge theories. The phase ambiguity for the left-handed components can be consistently removed only if the fermion multiplet is nonanomalous, that is when

$$d_R^{abc} = 2\mathrm{i}\,\mathrm{Tr}\left\{ R(T^a) \left[R(T^b) R(T^c) + R(T^c) R(T^b) \right] \right\} \tag{9.3}$$

is zero, where $R(T^a)$ are the anti-hermitian generators of the fermion representation of the gauge group. [36] For $U(1)$ gauge theories coupled to N left-handed Weyl fermions of charges e_α this condition becomes $\sum_{\alpha=1}^{N} e_\alpha^3 = 0$.

---

[36]In this Section we use, as in the original papers, anti-hermitian color matrices, that is they satisfy
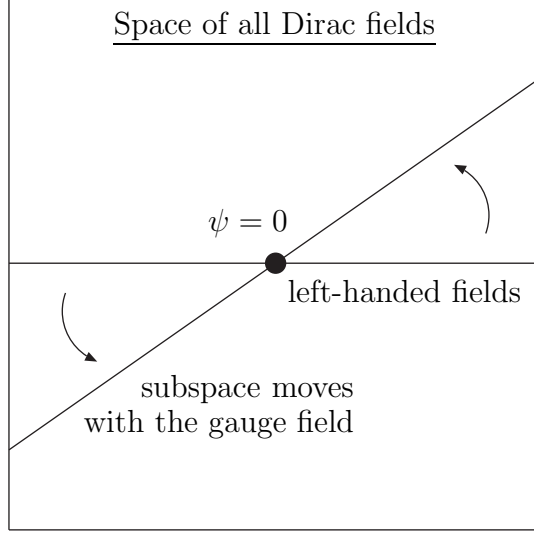
Figure 13: Dependence of the phase of the fermion measure on the gauge fields.

The key step forward is that the phase problem can be equivalently formulated in terms of a local current which is gauge covariant. To make this step let us consider the effective action when the fermionic degrees of freedom are integrated out,

$$\mathrm{e}^{-S_{\mathrm{eff}}[U]} = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathrm{e}^{-S_F[U,\overline{\psi},\psi]}. \tag{9.4}$$

An infinitesimal deformation of the gauge field

$$\delta_\eta U_\mu(x) = a\eta_\mu(x)U_\mu(x) \tag{9.5}$$

(where $\eta_\mu(x) = \eta_\mu^a(x)T^a$, and μ is not summed) induces a variation of the effective action

$$\delta_\eta S_{\mathrm{eff}} = -\mathrm{Tr}\left\{\delta_\eta D\widehat{P}_- D^{-1}P_+\right\} + \mathrm{i}\mathcal{L}_\eta. \tag{9.6}$$

The first term in the above formula is the naive expression which one would normally obtain, while the second term

$$\mathcal{L}_\eta = a^4 \sum_x \eta_\mu^a(x)j_\mu^a(x), \tag{9.7}$$

which is linear in the field variation, is the new term proper of Ginsparg-Wilson fermions which arises because the fermion measure depends on the gauge fields, and in fact this term can also be written as $\mathcal{L}_\eta = \mathrm{i}\sum_j(v_j, \delta_\eta v_j)$. The axial current $j_\mu(x)$ contains all the information about

$\mathrm{Tr}\{T^aT^b\} = -1/2\,\delta^{ab}$ and $[T^a, T^b] = f^{abc}T^c$. This means that $U_\mu = \exp\{ag_0A_\mu^aT^a\}$. The interested reader can then turn for more details to these papers without problems.

the phase of the fermion measure, provided it is given by a gauge-covariant local field which satisfies the *integrability condition*, which in its differential form reads

$$\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta + a \mathcal{L}_{[\eta,\zeta]} = \mathrm{i}\,\mathrm{Tr}\Big\{\widehat{P}_-\Big[\delta_\eta \widehat{P}_-, \delta_\zeta \widehat{P}_-\Big]\Big\}, \tag{9.8}$$

for all field variations $\eta_\mu(x)$ and $\zeta_\mu(x)$ that do not depend on the gauge field. The reconstruction theorem then says that a given current with these requirements fixes the phase of the measure, and knowing the current is then equivalent to knowing the fermion measure. Once the measure is fixed, the functional integral is well defined and the effective action is gauge invariant. We have then constructed a chiral gauge theory which is not spoiled by radiative corrections.

There is thus a one-to-one correspondence between the current and the measure, up to a constant phase factor in each topological sector which is not relevant. The current $j_\mu(x)$ defines the chiral gauge theory, and the problem of constructing the fermion measure is reduced to the problem of constructing this current. The measure does not need to be explicitly specified.

Let us now make an expansion in the coupling constant of the various quantities introduced above. The exact cancellation of the gauge anomaly can then be proven recursively, order by order in g_0. We think it is interesting to sketch how this construction is carried out, skipping the points that are technically more involved, for which the reader is referred to the original papers. A more detailed treatment is beyond the scope of this review.

To write an expansion in the coupling, which we also need in order to be able to derive the additional interaction vertices coming from the measure term $\mathcal{L}_\eta$, we use the variation with respect to the gauge potential A rather than to the link U, which is

$$\overline{\delta}_\eta A_\mu(x) = \eta_\mu(x). \tag{9.9}$$

Because

$$\delta_\eta = g_0^{-1}\overline{\delta}_\eta + O(1), \tag{9.10}$$

the corresponding variation of the effective action contains an explicit dependence on the coupling,

$$\overline{\delta}_\eta S_{\mathrm{eff}} = -\mathrm{Tr}\Big\{\overline{\delta}_\eta D\widehat{P}_- D^{-1} P_+\Big\} + \mathrm{i}g_0 \mathcal{L}_{\overline{\eta}}, \tag{9.11}$$

with

$$\overline{\eta}_\mu(x) = \left\{1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)!}\Big(g_0 a\, \widetilde{A}_\mu(x)\Big)^k\right\}\eta_\mu(x), \tag{9.12}$$

where the gauge potential entering in $\overline{\eta}$ is in the adjoint representation. To construct the current, one starts from the curvature term in the right-hand side of the integrability condition Eq. (9.8),

$$\mathcal{F}_{\eta\zeta} = \mathrm{i}\,\mathrm{Tr}\Big\{\widehat{P}_-\Big[\delta_\eta \widehat{P}_-, \delta_\zeta \widehat{P}_-\Big]\Big\}, \tag{9.13}$$

which has an expansion in the coupling constant whose leading term is of order g_0^3. The reason why the terms of order g_0^2 and lower are zero is the anomaly cancellation condition $d_R^{abc} = 0$,

and the fact that these lowest-order contributions are zero will be crucial in the following. The lowest-order part of the curvature is then

$$\widetilde{\mathcal{F}}_{\eta\zeta} = \frac{1}{3!}\frac{\partial^3}{\partial g_0^3}\mathcal{F}_{\eta\zeta}\bigg|_{g_0=0}, \tag{9.14}$$

and the lowest-order part of the measure term is

$$\widetilde{\mathcal{L}}_{\eta} = \frac{1}{4!}\frac{\partial^4}{\partial g_0^4}\mathcal{L}_{\eta}\bigg|_{g_0=0}, \tag{9.15}$$

where $\widetilde{\mathcal{L}}$ has to be invariant under the linearized gauge transformations [37]

$$A_\mu(x) \to A_\mu(x) + \nabla_\mu\omega(x). \tag{9.18}$$

In terms of these quantities the lowest-order form of the integrability condition reads

$$\bar{\delta}_\eta\widetilde{\mathcal{L}}_\zeta - \bar{\delta}_\zeta\widetilde{\mathcal{L}}_\eta = \widetilde{\mathcal{F}}_{\eta\zeta}. \tag{9.19}$$

From the lowest-order term of the curvature $\widetilde{\mathcal{F}}$ we now define the functional

$$\mathcal{H}_\eta = -\frac{1}{5}\widetilde{\mathcal{F}}_{\eta\lambda}\bigg|_{\lambda_\mu=A_\mu}, \tag{9.20}$$

which satisfies the lowest-oder form of the integrability condition [38] and is linear in η, which allows to define the current $h_\mu(x)$:

$$\mathcal{H}_\eta = a^4\sum_x \eta^a(x)h_\mu^a(x). \tag{9.23}$$

This current in general is not gauge invariant, but from it we can construct

$$q(x) = \nabla_\mu^\star h_\mu(x), \tag{9.24}$$

---

[37]One has

$$\eta_\mu(x) = -\widetilde{D}_\mu\omega(x), \tag{9.16}$$

with

$$\widetilde{D}_\mu\omega(x) = \frac{1}{a}\Big[U_\mu(x)\omega(x+a\hat{\mu})U_\mu^\dagger(x) - \omega(x)\Big]. \tag{9.17}$$

[38]To prove that

$$\bar{\delta}_\eta\mathcal{H}_\zeta - \bar{\delta}_\zeta\mathcal{H}_\eta = \widetilde{\mathcal{F}}_{\eta\zeta} \tag{9.21}$$

one has to make use of the Bianchi identity

$$\bar{\delta}_\eta\widetilde{\mathcal{F}}_{\zeta\lambda} + \bar{\delta}_\zeta\widetilde{\mathcal{F}}_{\lambda\eta} + \bar{\delta}_\lambda\widetilde{\mathcal{F}}_{\eta\zeta} = 0. \tag{9.22}$$

which is invariant under the linearized gauge transformations (9.18). It can be shown that $q(x)$ is a topological field, that is it satisfies

$$a^4 \sum_x \overline{\delta}_\lambda q(x) = 0 \qquad (9.25)$$

for all variations $\lambda_\mu(x)$ of the gauge potential. Not surprisingly, this topological field turns out to correspond to the anomaly $-1/2 \operatorname{Tr}\{\gamma_5 T D(x, x)\}$. We stress that $q(x)$ has been derived in a unique way from the current $j_\mu(x)$ (which also determines the fermion measure), via the curvature term of the integrability condition, $\widetilde{\mathcal{F}}$.

Topological fields like $q(x)$ have been classified in the U(1) lattice gauge theory, and in absence of matter they are equal to a sum of Chern polynomials plus a divergence term which is topologically trivial. The field $q(x)$ that we have just obtained is homogeneous of degree 4 in the gauge potential, and since Chern polynomials in four dimensions have degree 2, they cannot contribute in this case. All that is left is thus the topologically trivial term, so that we have

$$q(x) = \partial_\mu^\star k_\mu(x), \qquad (9.26)$$

where $k_\mu(x)$ turns out to be a local current invariant under linearized gauge transformations. The lowest-order part of the measure term is now given by

$$\widetilde{\mathcal{L}}_\eta = \mathcal{H}_\eta + \overline{\delta}_\eta \cdot \frac{1}{4} a^4 \sum_x A_\mu^a(x) k_\mu^a(x), \qquad (9.27)$$

which, since the last term has zero curvature, satisfies the lowest-oder form of the integrability condition.

This completes the construction at leading order. The higher-order terms of $\mathcal{L}$ can be then computed recursively according to the following procedure. If one has already calculated the $O(g_0^n)$ term in the expansion of $\mathcal{L}$, then he subtracts to $\mathcal{L}$ another function, $\mathcal{L}^{(n)}$, whose first n terms in the coupling expansion are the same as $\mathcal{L}$. Applying then the above construction (with some slight changes) to the difference $\mathcal{L} - \mathcal{L}^{(n)}$, one can compute its leading-order term, which determines $\mathcal{L}$ at $O(g_0^{n+1})$, and repeating this procedure one can obtain $\mathcal{L}$ to the desired order.

This construction is unique up to terms that are of higher orders in the lattice spacing and that therefore are irrelevant in the continuum limit (apart for a finite renormalization). As we have already mentioned, it can be carried out only if the fermion multiplet is anomaly-free. If this is not instead the case, i.e., if $d_R^{abc} \neq 0$, the lowest term of the curvature turns out to be of order g_0 instead of g_0^3, and the lowest term of the measure is of order g_0^2 instead of g_0^4. The topological field obtained along the lines explained above is now homogeneous of degree 2 in the gauge potential, and then this time the Chern polynomials do contribute, and $q(x)$ is topologically nontrivial. It can be shown that this leads to the presence of a lattice field corresponding to $F_{\mu\nu}\widetilde{F}_{\mu\nu}$:

$$q(x) = -\frac{1}{192\pi^2} d_R^{abc} \epsilon_{\mu\nu\rho\sigma} T^a F_{\mu\nu}^b(x) F_{\rho\sigma}^c(x + a\hat{\mu} + a\hat{\nu}) + \nabla_\mu^\star k_\mu(x), \qquad (9.28)$$

where $F_{\mu\nu} = \nabla_\mu A_\nu(x) - \nabla_\nu A_\mu(x)$ is the linearized gauge field-strength tensor. This is the well-known covariant anomaly. Thus, for $d_R^{abc} \neq 0$, the theory is not chiral invariant after quantum corrections, which corresponds to the fact that the construction of a $\widetilde{\mathcal{L}}$ which satisfies the integrability condition, and hence of the measure term, cannot be accomplished.

For nonabelian gauge groups the cohomology problem has not yet been solved, that is a classification of the topological fields in $4 + 2$ dimensions has not yet been done. Thus, the general structure of the nonabelian anomaly on the lattice is currently not known. However, although we cannot yet prove that $q(x)$ is topologically trivial for nonabelian gauge groups, there is no reason to suspect that the theorem could not be valid after all in that case. The cancellation of the anomalies can in fact be proven for topological fields in the continuum limit, and if the topological structure at finite a matches with the one in the continuum limit (which would not constitute a surprising result) then the theorem can be proven also at finite lattice spacing. [39] Thus, the remaining open issues are mostly technical and not of principle and one should expect to overcome these difficulties at some point. There is little doubt that the construction is valid also in this case. [40]

We conclude this Section discussing the implications of this remarkable construction for perturbative calculations. The measure term

$$\mathcal{L}_\eta = \sum_{k=4}^{\infty} \frac{g_0^k}{k!} \, a^{4k+4} \sum_{x,\ldots,z_k} L^{(k)}(x, z_1, \ldots, z_k)^{aa_1\ldots a_k}_{\mu\mu_1\ldots\mu_k} \, \eta_\mu^a(x) A_{\mu_1}^{a_1}(z_1) \cdots A_{\mu_k}^{a_k}(z_k) \qquad (9.29)$$

can be considered as a local counterterm to the action, and the implicit dependence of the measure on the gauge fields generates additional gauge-field vertices in the effective action. Operating with $\overline{\delta}_\eta A_\mu(x)$ on $\overline{\mathcal{L}}_\eta$ an appropriate number of times one can obtain all these additional vertices which derive from the measure term. These vertices only appear at the one-loop level, and the k-th order vertex is given by

$$\mathrm{i} g_0 \, \overline{\delta}_\eta \cdots \overline{\delta}_\eta \, \mathcal{L}_{\overline{\eta}}\Big|_{A_\mu=0} = g_0^k \, a^{4k} \sum_{z_1,\ldots,z_k} V_M^{(k)}(z_1, \ldots, z_k)^{a_1\ldots a_k}_{\mu_1\ldots\mu_k} \, \eta_{\mu_1}^{a_1}(z_1) \cdots \eta_{\mu_k}^{a_k}(z_k). \qquad (9.30)$$

In this formula the operator $\overline{\delta}_\eta$ has been applied $k - 1$ times. The expansion of $\mathcal{L}_\eta$ as a power series in the coupling only begins with the g_0^4 term, and these vertices are only of fifth and

---

[39]Although in this case one should also show that there are no global topological obstructions at the nonperturbative level.

[40]The only nonabelian chiral gauge theory for which the solution of the cohomology problem is known is the $SU(2)_L \otimes U(1)_Y$ electroweak theory, which is easier to deal with because the representation of $SU(2)$ are pseudo-real (Kikukawa and Nakayama, 2001). Theoretical advances have also been reported in (Suzuki, 2000; Igarashi, Okuyama and Suzuki, 2000), where the cancellation of the lattice gauge anomaly to all orders in powers of the gauge potential for any compact group was established, and in (Kikukawa, 2002), where an explicit construction of the chiral measure which employs domain wall fermions was given. In the abelian case a nonperturbative construction has been so far carried out for weak fields satisfying $|F_{\mu\nu}| < \epsilon$ (with $\epsilon < 1/30$), however one can make them the only statistically relevant fields in the functional integral by using a modified version of the plaquette action which lies in the same universality class of the standard plaquette action (Lüscher, 1999c).

higher orders in the gauge coupling. They can be determined, using a recursive procedure, once a local gauge-covariant current is given that satisfies the integrability condition.

These interaction vertices are not explicitly known at present. Luckily, they are not needed in most cases of interest. In fact, since they are only of order g_0^5 and higher, and moreover they are proportional to positive powers of the lattice spacing (as can be seen by naive dimensional counting), they do not contribute to the continuum limit of one-loop diagrams. At the two-loop level the vertices coming from the measure term can come into play only in some very special circumstances. In fact, the lowest-order vertex, which is a five-point vertex, is totally symmetric in the gauge group indices, so that in two-loop calculations this vertex can contribute only if it gives rise to more than three external lines.

All propagators and vertices of these chiral gauge theories satisfy the conditions for the validity of the Reisz power counting theorem (see Section 15). Although their renormalizability has not yet been proven, it seems unlikely that these theories are nonrenormalizable.

In this Section we have discussed only the theory of left-handed fermions. The introduction of Higgs fields does not affect the measure term, because the chiral projectors do not refer to the Higgs sector. Higgs field or other fields that couple vectorially can then be easily incorporated in this theory.

We have seen that lattice Feynman diagrams for Ginsparg-Wilson fermions are more complicated to calculate than continuum ones, and the vertices coming from the fermionic measure are also quite involved. On the other hand, having a nonperturbative regularization that is both chiral and gauge invariant after quantum corrections are included can be a big advantage, especially when one makes calculations in the electroweak theory. It can then be worth to pay this price.

We have thus shown that, although certainly not simple, there is now a construction of chiral gauge theories, at least in the abelian case, which is suitable for quantum calculations, and which makes uses of the lattice. A consistent formulation of the standard model now exists beyond perturbation theory.

10   The approach to the continuum limit

The range of couplings for which perturbation theory aspires to be a reasonable expansion, that is when g_0 is small, is closely related to the approach to the continuum limit of lattice QCD. This approach can be described using Callan-Symanzik renormalization group equations which are similar to the continuum ones. Since the first two coefficients of the β function, which determines the perturbative running of the coupling, are independent of the scheme (Caswell and Wilczek, 1974; Espriu and Tarrach, 1982), they have the same values also for lattice QCD, which is therefore asymptotically free. This implies that the lattice QCD coupling goes to zero in the limit in which the momenta go to infinity, which when translated in the lattice language means the limit of vanishing lattice spacing, or in other words the continuum limit.

Let us consider, in a massless theory, a physical quantity P of mass dimension n which is computed on a lattice of spacing a. The product $a^n P$ is a dimensionless quantity, which can only depend on the bare coupling, which in turn depends on the scale, that on the lattice is determined by the spacing a. We thus have $a^n P = f(g_0(a))$. Since the physical quantity P has some well-defined value in the continuum limit, for $a \to 0$ it must happen that

$$\lim_{a \to 0} \frac{f(g_0(a))}{a^n} \tag{10.1}$$

is a constant. Near the continuum limit g_0 is a smooth function of a which satisfies

$$\lim_{a \to 0} g_0(a) = g_c. \tag{10.2}$$

This critical point for QCD turns out to be $g_c = 0$. This is a stable ultraviolet fixed point of the theory.

In the continuum limit the correlation length, the rate of the exponential falloff of the two-point correlation functions in position space, goes to infinity. This can be understood by thinking that the correlation length is fixed in physical units, and thus when a decreases it just becomes larger and larger when instead is measured in units of the lattice spacing. For $a \to 0$ the correlation length must diverge in lattice units, and the discretization effects disappear. The continuum limit is thus a critical point of the theory.

What shown above is a completely nonperturbative argument. This is just a particular case of the well-known fact that the bare parameters depend in general on the cutoff, which in our case is the lattice spacing a. Therefore there is always a well-defined relation between g_0 and a. For couplings small enough for the perturbative β function to be a reasonable approximation to the real running of the coupling, the approach to the continuum limit can be studied in perturbation theory, where g_0 and a tend together to zero along a trajectory determined by renormalization group equations. The β function appearing in the renormalization group equation for lattice QCD is defined by

$$a \frac{dg_0}{da} = -\beta(g_0). \tag{10.3}$$

This β function has for small couplings the expansion

$$\beta(g_0) = -g_0^3 \left[b_0 + b_1 g_0^2 + b_2 g_0^4 + \dots \right]. \tag{10.4}$$

We see that things are similar to the continuum, with μ replaced by $1/a$. The first two coefficients of the β function are universal, and thus b_0 and b_1 in Eq. (10.4) are the same as in the continuum, were they were computed by (Gross and Wilczek, 1973; Politzer, 1973) and by (Jones, 1974; Caswell, 1974):

$$b_0 = \frac{1}{(4\pi)^2} \left(11 - \frac{2}{3} N_f \right), \tag{10.5}$$

$$b_1 = \frac{1}{(4\pi)^4} \left(102 - \frac{38}{3} N_f \right). \tag{10.6}$$

The coefficient b_2 depends on the scheme. The first calculations of this coefficient for the Wilson action were attempted in (Ellis and Martinelli, 1984a; Ellis and Martinelli, 1984b; Ellis, 1984). The coefficient b_2 was then fully computed in (Lüscher and Weisz, 1995a; Lüscher and Weisz, 1995d; Allés, Feo and Panagopoulos, 1997; Christou et al., 1998), with the result

$$b_2 = -0.00159983232(13) + 0.0000799(4)\, N_f - 0.00000605(2)\, N_f^2 \quad (c_{sw} = 0), \quad (10.7)$$

$$b_2 = -0.00159983232(13) - 0.0009449(4)\, N_f + 0.00006251(2)\, N_f^2, \quad (c_{sw} = 1), \quad (10.8)$$

were in the last line we have also given its value in the tree-level improved theory (which we will introduce in the next Section), which has been computed by (Bode and Panagopoulos, 2002) for general c_{sw}. The value of b_2 in the Schrödinger functional has been computed in (Bode, 1998; Bode, Wolff and Weisz, 1999; Bode, Weisz and Wolff, 2000a; Bode, Weisz and Wolff, 2000b), and for $T = L$, $\theta = \pi/5$ and $N_f = 0$ has the value

$$b_2 = \frac{1}{(4\pi)^3}\, 0.482(7), \tag{10.9}$$

while for $N_f = 2$ is

$$b_2 = \frac{1}{(4\pi)^3}\, 0.064(10). \tag{10.10}$$

For comparison, b_2 is given in the continuum $\overline{\text{MS}}$ scheme by (Tarasov, Vladimirov and Zharkov, 1980)

$$b_2 = \frac{1}{(4\pi)^6}\left(\frac{2857}{2} - \frac{5033}{18}\, N_f + \frac{325}{54}\, N_f^2\right). \tag{10.11}$$

In this scheme some higher-order coefficients are also known (van Ritbergen, Vermaseren and Larin, 1997).

Solving Eqs. (10.3) and (10.4) at lowest order gives the solution

$$g_0^2 \sim -\frac{1}{b_0 \log a^2 \Lambda_{lat}^2} + O(1/\log^2 a^2). \tag{10.12}$$

The evolution of the bare lattice coupling with the scale a defines, like in the continuum, a Λ parameter. The value of the Λ parameter cannot be determined using the lowest-order solution, since a rescaling of Λ is of the same order as terms which have been dropped. A reasonable definition of the Λ parameter has to take into account higher orders:

$$\Lambda = a^{-1} \cdot (b_0 g_0^2)^{-\frac{b_1}{2b_0^2}} \cdot e^{-\frac{1}{2b_0 g_0^2}} \cdot \exp\left\{-\int_0^g dt\left(\frac{1}{\beta(t)} + \frac{1}{b_0 t^3} - \frac{b_1}{b_0^2 t}\right)\right\}. \tag{10.13}$$

This definition has been used in the calculations of (Capitani et al., 1999c) reported in Section 2, which use the Schrödinger functional. A Λ parameter in a given scheme specifies the value of the coupling constant in that scheme for any given scale μ, and all dimensionful quantities are

proportional to Λ. Since the Λ parameter depends on the scheme, its values on the lattice will not be equal to Λ_{QCD} as is known from the continuum. It happens indeed that the values of Λ on the lattice are in general quite different from the continuum. Their ratio can be computed using lattice (and continuum) perturbation theory (Hasenfratz and Hasenfratz, 1980; Dashen and Gross, 1981; Weisz, 1981; Hasenfratz and Hasenfratz, 1981). For example, for the pure gauge Wilson action one has

$$\frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_{lat}} = 28.80934(1). \tag{10.14}$$

The huge change in scales between the lattice theories and the continuum is a common phenomenon, and is more pronounced for some lattice actions than for others.

The two scales Λ_{lat} and Λ_{QCD} can thus be related. Once the relation between the lattice and continuum coupling constants is known, combining the results of Monte Carlo simulations with the knowledge of Λ_{lat} and the quark masses allows in principle to predict all physical quantities. In this way one can predict for example the value of $\alpha_S(M_Z)$ using only nonperturbative lattice calculations of the spectrum.

We have seen that the bare coupling g_0 automatically determines the size of the lattice spacing a, and computing lattice quantities near the continuum limit means taking both of them to zero in such a way that Eq. (10.12) (or even better Eq. (10.13)) is satisfied. The renormalized physical quantities should remain constant along this trajectory, and this defines the "scaling region" near the continuum limit. If when doing some lattice simulations we can show that g_0 and a follow this relation, then we now that we are extracting continuum physics from the Monte Carlo results.

For sufficiently large scales one can also expand the running couplings in different schemes directly in powers of each other. We can then relate the couplings by doing a matching at a finite scale p (Celmaster and Gonsalves, 1979a; 1979b). On the lattice the relation between the bare and the renormalized coupling (defined as the three-point function at a certain momentum p) is

$$g_R(p) = g_0\Big[1 + g_0^2\Big(-b_0 \log ap + C^L + O(a^2 p^2 \log ap)\Big) + O(g_0^4)\Big], \tag{10.15}$$

while for the continuum coupling one has

$$g_R(p) = g_{\overline{\text{MS}}}\Big[1 + g_{\overline{\text{MS}}}^2\Big(-b_0 \log \frac{p}{\mu} + C^{\overline{\text{MS}}}\Big) + O(g_{\overline{\text{MS}}}^4)\Big]. \tag{10.16}$$

Combining these two equations we have then that

$$g_0 = g_{\overline{\text{MS}}}\Big[1 + g_0^2\Big(C^{\overline{\text{MS}}} - C^L + b_0 \log a\mu\Big) + O(g_0^4) + O(a^2)\Big]. \tag{10.17}$$

The most effective way of computing the matching between different couplings is provided by background field techniques (De Wit, 1967a; De Wit, 1967a; Abbott, 1981). For most of the calculations cited above a background field method has been indeed employed. In the calculations of (Lüscher and Weisz, 1995a; Lüscher and Weisz, 1995d) this was supplemented

by evaluations of the Feynman diagrams using the coordinate space method. We will discuss this method in detail in Section 19, and we will show there some integrals that occur in the calculation of b_2 using the background field.

A gauge theory on the lattice with a background gauge field is renormalizable to all orders of perturbation theory. A nice thing is that no new counterterms are needed, that is the ones already required in the lattice theory without background fields are sufficient (Lüscher and Weisz, 1995c), as it happens in the continuum case (Kluberg-Stern and Zuber, 1975a; Kluberg-Stern and Zuber, 1975b). Thus, introducing a background field does not affect the renormalization of the lattice theory. The renormalization of gauge theories on the lattice without background fields has been proven in (Reisz, 1989).

The propagator of the background field is proportional to $1/g^2$, and for the renormalization of the coupling and the perturbative determination of the lowest coefficients of the β function the computation of the self-energy of the background field is then sufficient. This is the main advantage of using a background field method. It is much easier to compute diagrams with two legs instead of three, as would be the case without background field, and the number of diagrams to be computed is also smaller. It is true that to control the renormalization of the gauge-fixing parameter the corrections to the gauge field propagator must also be calculated, but this is only needed to lower orders.

In the continuum the fields A_μ are decomposed as follows:

$$A_\mu(x) = B_\mu(x) + g_0 q_\mu(x), \tag{10.18}$$

where the background field B_μ is a smooth external source field (which is not required to satisfy the Yang-Mills equations), while q_μ is the quantum field. To the gauge action reexpressed in terms of B_μ and q_μ one has to add the gauge-fixing term

$$\frac{1}{\alpha} \int dx \, \mathrm{Tr}\Big(D_\mu q_\mu(x) D_\nu q_\nu(x)\Big) \tag{10.19}$$

and the ghost action (coming from the Faddeev-Popov procedure)

$$2 \int dx \, \mathrm{Tr}\Big(D_\mu \overline{c}(x) \left(D_\mu + \mathrm{i} g_0 \widetilde{q}_\mu(x)\right) c(x)\Big), \tag{10.20}$$

where the covariant derivative is

$$D_\mu = \partial_\mu + \mathrm{i}\widetilde{B}_\mu, \tag{10.21}$$

and $\widetilde{q}$ and $\widetilde{B}$ denote the quantum field and the background field in the adjoint representation: $\widetilde{B}_\mu = B_\mu^a t^a$ (like in Section 5.2.1).

On the lattice, as expected, there is more than one choice for extending the theory with the introduction of a nonzero background field. A convenient decomposition of the gauge links is

$$U_\mu(x) = \mathrm{e}^{\mathrm{i}aB_\mu(x)} \mathrm{e}^{\mathrm{i}a g_0 q_\mu(x)}. \tag{10.22}$$

In this case the background gauge transformations take a simple form,

$$B_\mu^\Omega = -\frac{i}{a} \log\left(\Omega(x) \, e^{iaB_\mu(x)} \, \Omega^{-1}(x + a\hat{\mu})\right) \tag{10.23}$$

$$q_\mu^\Omega = \Omega(x) \, q_\mu(x) \, \Omega^{-1}(x), \tag{10.24}$$

with Ω a classical field. The gauge-fixing term is

$$\frac{1}{\alpha} \cdot a^4 \sum_x \, \text{Tr}\left(D_\mu^\star q_\mu(x) D_\nu^\star q_\nu(x)\right) \tag{10.25}$$

and the ghost action is

$$2a^4 \sum_x \, \text{Tr}\left(D_\mu \bar{c}(x) \left((M^\dagger)^{-1}(q_\mu(x)) \cdot D_\mu + ig_0 \tilde{q}_\mu(x)\right) c(x)\right), \tag{10.26}$$

where the forward and backward lattice covariant derivatives in this case act as follows:

$$D_\mu f(x) = \frac{1}{a}\left(e^{iaB_\mu(x)} \, f(x + a\hat{\mu}) \, e^{-iaB_\mu(x)} - f(x)\right), \tag{10.27}$$

$$D_\mu^\star f(x) = \frac{1}{a}\left(f(x) - e^{-iaB_\mu(x-a\hat{\mu})} \, f(x - a\hat{\mu}) \, e^{iaB_\mu(x-a\hat{\mu})}\right), \tag{10.28}$$

and M is the same matrix defined in Eq. (5.37).

The theory so defined is gauge invariant, and has a BRS symmetry and a shift transformation symmetry (Lüscher and Weisz, 1995c). The Feynman rules of pure QCD in the presence of a background field are given in (Lüscher and Weisz, 1995d). The usual lattice gauge theory with the standard covariant gauge-fixing term can be recovered in the limit of zero background field.

The β function can also be defined for describing the scale evolution of the renormalized coupling. In this form it has been used to determine the perturbative running of the strong coupling in the Schrödinger functional scheme of which we have discussed in Section 2. There we showed also the scale evolution of the renormalized masses of the quarks, which is described in lattice QCD by the τ function:

$$a\frac{dm}{da} = -\tau(g)m. \tag{10.29}$$

The expansion of the τ function for small g is given by

$$\tau(g) = -g^2\left[d_0 + d_1 g^2 + d_2 g^4 + \dots\right]. \tag{10.30}$$

The leading-order coefficient coefficient d_0 does not depend on the scheme, and has been computed in (Nanopoulos and Ross, 1975)

$$d_0 = \frac{8}{(4\pi)^2}. \tag{10.31}$$

The coefficient d_1 is different in the various schemes. In the Schrödinger functional d_1 has been computed by (Sint and Weisz, 1999), and is given (for $T = L$ and $\theta = 0.5$) by

$$d_1 = \frac{1}{(4\pi)^2}\Big(0.217(1) + 0.084(1)\,N_f\Big). \tag{10.32}$$

In continuum $\overline{\mathrm{MS}}$ is instead given by (Nanopoulos and Ross, 1979; Tarrach, 1981; Espriu and Tarrach, 1982)

$$d_1 = \frac{1}{(4\pi)^4}\Big(\frac{404}{3} - \frac{40}{9}\,N_f\Big). \tag{10.33}$$

In the $\overline{\mathrm{MS}}$ scheme some higher-order coefficients are also known (Chetyrkin, 1997; Vermaseren, Larin and van Ritbergen, 1997).

A very useful quantity is the renormalization group invariant mass, defined by (Gasser and Leutwyler, 1982; Gasser and Leutwyler, 1984; Gasser and Leutwyler, 1985)

$$M = m \cdot (2b_0 g^2)^{-\frac{d_0}{2b_0}} \cdot \exp\left(-\int_0^g dt\Big[\frac{\tau(t)}{\beta(t)} - \frac{d_0}{b_0 t}\Big]\right). \tag{10.34}$$

This quantity, at variance with the Λ parameter, is even scheme independent. Moreover, renormalization group masses present the convenience that they are nonperturbatively defined, whereas the masses renormalized at a certain scale in the $\overline{\mathrm{MS}}$ scheme are not.

11   Improvement

The results of lattice simulations are affected by statistical errors (which arise because only a finite number of configurations can be generated) and systematic errors.

The systematic errors are of various nature. Rather important are those coming from the finiteness of the lattice spacing, and often also finite volume effects, quenching effects (i.e., when internal quark loops are neglected) and extrapolations to the chiral limit constitute significant systematic errors.

The techniques which go under the name of "improvement" aim at removing the systematic error due to the finiteness of the lattice spacing, which are generally of order a with respect to the continuum limit, as in the formal expansion

$$\left\langle p \left|\hat{\mathcal{O}}_L\right| p'\right\rangle_{\text{Monte Carlo}} = a^d\left[\left\langle p\left|\hat{\mathcal{O}}\right|p'\right\rangle_{\text{phys}} + O(a)\right]. \tag{11.1}$$

We know that at the values of the coupling which are presently simulated these discretization errors can often have a significant effect on the results.

It is very expensive to reduce these cutoff effects by "brute force", that is simply decreasing the lattice spacing a, since the computing time required for this would grow with the fifth

power of the inverse lattice spacing in the quenched approximation, and even faster in full QCD. This means for example that halving the discretization errors by brute force would require a calculational effort at least thirty times bigger, all other things being equal. A better way of reducing these errors can be achieved by improving the theory. These unphysical terms are then systematically removed by adding irrelevant terms to actions and operators (that is, they vanish in the continuum limit and do not change the continuum theory). We will now discuss what this implies in terms of perturbation theory. For a pedagogical introduction on these topics, which we will not here discuss much in detail, the reader can turn to the reviews of (Lüscher, 1986) and (Lüscher, 1999a).

11.1   Improved quarks

A systematic improvement program to reduce the cutoff errors order by order in the lattice spacing a for Wilson fermions was first proposed by Symanzik (1979; 1981; 1983a; 1983b) and then further developed, and applied to on-shell matrix elements, in (Lüscher and Weisz, 1985a; Lüscher and Weisz, 1985b; Sheikholeslami and Wohlert, 1985; Heatlie *et al.*, 1991).

In this formulation, an irrelevant operator is added to the Wilson action in order to cancel, in on-shell matrix elements, all terms that in the continuum limit are effectively of order a. [41] In this way one can reduce the cutoff errors coming from the discretization of the action from $O(a)$ to $O(a^2)$:

$$\left\langle p \left| \widehat{\mathcal{O}}_L \right| p' \right\rangle_{Monte\ Carlo} = a^d \left[\left\langle p \left| \widehat{\mathcal{O}} \right| p' \right\rangle_{phys} + O(a^2) \right]. \tag{11.2}$$

This is a remarkable decrease of the systematic error due to the finiteness of the lattice spacing. The continuum limit is reached much faster, with a rate proportional to a^2. [42]

For this purpose one has to introduce the improved "clover" fermion action, first proposed in (Sheikholeslami and Wohlert, 1985):

$$\Delta S_I^f = c_{sw} \cdot i g_0 a^4 \sum_{x,\mu\nu} \frac{r}{4a} \, \overline{\psi}(x) \sigma_{\mu\nu} F_{\mu\nu}^{clover}(x) \psi(x), \tag{11.3}$$

which vanishes in the continuum limit and has the same symmetries of the original unimproved Wilson action. This counterterm is the only one required after exploiting all the symmetries

---

[41]This means that at n loops these terms, because of Eq. (10.12), have the form $g_0^{2n} a \log^n a$.

[42]A similar thing can be found for example in numerical integration methods, where the trapezoidal rule reduces the discretization error to the cubic power of the integration step, and Simpson's rule further reduces it to the fifth power.
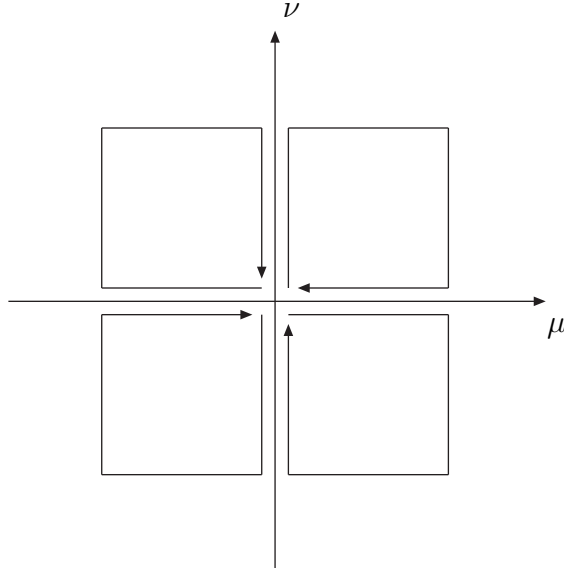
Figure 14: The combination of the four plaquettes which builds the clover lattice approximation of the $F_{\mu\nu}$ tensor at the point x.

and the equations of motion. [43] The "clover" gluon field is here defined as

$$F_{\mu\nu}^{clover}(x) = \frac{1}{4}\frac{1}{2ig_0 a^2} \sum_{\mu\nu=\pm} (P_{\mu\nu}(x) - P_{\mu\nu}^\dagger(x)),$$ (11.8)

that is the average of the four plaquettes lying in the plane $\mu\nu$ stemming from the point x (see Fig. 14). This definition maximizes the symmetry of the field-strength tensor on the lattice (Mandula, Zweig and Govaerts, 1983b).

Improved actions like this allow one to extract continuum physics already for lattice couplings which are not too small. One just exploits the fact that more than one lattice action corresponds to a given continuum action, and looks for lattice actions with smaller discretization errors. This of course implies a change in the Feynman rules (and usually not for the best).

----

[43]The other dimension-five terms that are gauge invariant and compatible with the symmetries of the Wilson action are

$$\overline{\psi}\,\overrightarrow{D}_\mu \overrightarrow{D}_\mu\,\psi + \overline{\psi}\,\overleftarrow{D}_\mu \overleftarrow{D}_\mu\,\psi,$$ (11.4)

$$m\left(\overline{\psi}\gamma_\mu \overrightarrow{D}_\mu\,\psi - \overline{\psi}\,\overleftarrow{D}_\mu\,\gamma_\mu\psi\right),$$ (11.5)

which can be eliminated in on-shell matrix elements using two equations of motion, and

$$m\,\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right),$$ (11.6)

$$m^2\,\overline{\psi}\psi,$$ (11.7)

which can be reabsorbed into a rescaling of the coupling and mass.

Adding the Sheikholeslami-Wohlert term to the Wilson Lagrangian means that we have to add to the Wilson quark-quark-gluon interaction vertex

$$(V^a)^{bc}_\rho(p_1, p_2) = -g_0(T^a)^{bc}\left[i\gamma_\rho \cos\frac{a(p_1+p_2)_\rho}{2} + r\sin\frac{a(p_1+p_2)_\rho}{2}\right] \tag{11.9}$$

the improved quark-quark-gluon interaction vertex

$$(V^a_{IMP})^{bc}_\rho(p_1, p_2) = -c_{sw}\cdot g_0\frac{r}{2}(T^a)^{bc}\cos\frac{a(p_1-p_2)_\rho}{2}\sum_\lambda \sigma_{\rho\lambda}\sin a(p_1-p_2)_\lambda. \tag{11.10}$$

The fermion propagator and the vertices with an even number of gluons are instead not modified by the improved action.

Neither is the gluon propagator. In fact, as we have seen in Section 5, the first corrections to the pure gauge term in the Wilson action are already of order a^2, and thus there is no need of improvement to this order, although sometimes it is useful to improve the gluon action reducing the discretization errors from $O(a^2)$ to $O(a^3)$ (in fact $O(a^4)$), as we will see shortly.

Only with the appropriate value of the improvement coefficient c_{sw} (for a given value of the coupling) the cancellation of $O(a)$ effects can be achieved. At lowest order in perturbation theory one just needs the tree-level value $c_{sw} = 1$, and in the work of (Heatlie *et al.*, 1991) it has been explicitly shown that all terms that are effectively of order a are absent in the 1-loop matrix elements of the quark currents for $c_{sw} = 1$. Perturbative determinations of c_{sw} at order g_0^2 have been made in (Wohlert, 1987; Naik, 1993; Lüscher and Weisz, 1996). Nonperturbative determinations of c_{sw} have been carried out by the ALPHA Collaboration, using the Schrödinger functional and imposing the cancellation of all $O(a^2)$ effects in the PCAC relation. The formula that summarizes the ALPHA Collaboration quenched results when $0 \le g_0 \le 1$ is (Lüscher *et al.*, 1997):

$$c_{sw} = \frac{1 - 0.656\,g_0^2 - 0.152\,g_0^4 - 0.054\,g_0^6}{1 - 0.922\,g_0^2}, \tag{11.11}$$

while in the $N_f = 2$ case one has (Jansen and Sommer, 1998):

$$c_{sw} = \frac{1 - 0.454\,g_0^2 - 0.175\,g_0^4 + 0.012\,g_0^6 + 0.045\,g_0^8}{1 - 0.720\,g_0^2}. \tag{11.12}$$

The $O(g_0^2)$ terms in these formulae correspond to the 1-loop perturbative result.

All the above machinery is not enough to improve completely a lattice theory. In addition to improving the action, one also has to improve each operator that happens to be studied. This means that one must add a basis of higher-dimensional irrelevant operators with the same symmetry properties as the original unimproved operator. Bases of improved operators have been constructed for quark currents (Sint and Sommer, 1996; Sint and Weisz, 1997) and operators measuring unpolarized structure functions (Capitani *et al.*, 2001b), and improved renormalization factors have been computed in (Gabrielli *et al.*, 1991; Frezzotti *et al.*, 1992;

Borrelli *et al.*, 1993) and (Capitani *et al.*, 1998b; Capitani *et al.*, 2001b). We will not deal with these calculations here.

Of course one can also attempt to improve the theory to the next order, canceling all contributions which are effectively of order a^2. For the fermion part this is quite complicated. In this case four-quark operators are also necessary, besides a certain number of two-quark operators of dimension six.

We would like briefly to mention what improving the theory looks like when overlap fermions are used (Capitani *et al.*, 1999b; Capitani *et al.*, 2000c; Capitani *et al.*, 2001b). One of the many qualities of overlap fermions is that the overlap action is already improved to $O(a)$, and thus one needs only to improve operators. An operator $O = \overline{\psi}\widetilde{O}\psi$ is improved, to all orders in perturbation theory, by taking the expression

$$O^{imp} = \overline{\psi}\Big(1 - \frac{1}{2\rho}aD_N\Big)\widetilde{O}\Big(1 - \frac{1}{2\rho}aD_N\Big)\psi. \tag{11.13}$$

Moreover, O^{imp} and $\widetilde{O}$ have the same renormalization constants.

We remind, for comparison, that for Wilson fermions an additional interaction vertex has instead to be introduced, the Sheikholeslami-Wohlert clover term (proportional to c_{sw}), and furthermore for each operator one needs to construct a complete basis of operators which are one dimension higher and whose coefficients have then to be tuned to cancel the residual $O(a)$ contributions not coming from the action.

For example, for the first moment of the quark momentum distribution the relevant operator is

$$O_{\{\mu\nu\}} = \overline{\psi}\gamma_{\{\mu}D_{\nu\}}\psi, \tag{11.14}$$

symmetrized in μ and ν, and one basis for the improvement is given by

$$\overline{\psi}\gamma_{\{\mu}D_{\nu\}}\psi - \frac{1}{4}aic_1\sum_\lambda\overline{\psi}\sigma_{\lambda\{\mu}\big[D_{\nu\}}, D_\lambda\big]\psi - \frac{1}{4}ac_2\overline{\psi}\big\{D_\mu, D_\nu\big\}\psi. \tag{11.15}$$

Only for some particular values of the improvement coefficients

$$\begin{aligned} c_1(g_0^2) &= 1 + g_0^2 c_1^{(1)} + O(g_0^4) \\ c_2(g_0^2) &= 1 + g_0^2 c_2^{(1)} + O(g_0^4) \end{aligned} \tag{11.16}$$

the operator if effectively improved, i.e., all $O(a)$ corrections are exactly canceled. Even for this simple case, these two coefficients are not known, because at present one knows only a relation between them. One of the coefficients then remains unknown (Capitani *et al.*, 2001b).

One could determine all remaining coefficients using some Ward Identities or physical conditions, but this involves a lot of effort. Moreover, for higher moments, which contain more covariant derivatives, the number of operator counterterms becomes larger and larger. This means that many improvement coefficients have to be determined, and they need as many

conditions to be set. Moreover, one has also to compute the contribution of each one of these operator counterterms to the total renormalization constant. This looks a formidable task.

Thus, improving the theory is much simpler for overlap fermions.

11.2 Improved gluons

The plaquette is not the only possibility for the construction of the discretized version of the gauge field strength. One can also consider larger closed loops. As we noted in Section 5, the corrections to the pure gauge part of the Wilson action with respect to the continuum are of order a^2. The gluon part of the action is then already $O(a)$ improved. The next step consists in implementing the improvement to $O(a^2)$, that is adding to the Wilson action some counterterms of dimension 6 that (with the appropriate values of their coefficients) can cancel all $O(a^2)$ effects, so that the first corrections that are left are then at least of order a^3. [44] There are 3 terms of dimension 6 with the right quantum numbers, and the improved gauge action can be written as

$$S_g = \frac{6}{g_0^2}\left[c_0(g_0^2)\,\mathcal{L}^{(4)} + a^2 \sum_{i=1}^{3} c_i(g_0^2)\,\mathcal{L}_i^{(6)}\right], \qquad (11.17)$$

where $\mathcal{L}^{(4)}$ is the usual Wilson plaquette action, and the dimension-6 terms are six-link closed loops which are called planar, twisted and L-shaped respectively (see Fig. 15). Each of these terms contains the $\sum_{\mu\nu}\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right)$ operator plus a linear combination of the following operators:

$$\sum_{\mu\nu}\mathrm{Tr}\left(D_\mu F_{\mu\nu}D_\mu F_{\mu\nu}\right), \qquad \sum_{\mu\nu\rho}\mathrm{Tr}\left(D_\mu F_{\nu\rho}D_\mu F_{\nu\rho}\right), \qquad \sum_{\mu\nu\rho}\mathrm{Tr}\left(D_\mu F_{\mu\rho}D_\nu F_{\nu\rho}\right). \qquad (11.18)$$

Lüscher and Weisz have computed the coefficients of these linear combinations and determined, imposing improvement conditions on the large Wilson loops, the values of the improvement coefficients that accomplish the cancellation of the $O(a^2)$ corrections (Lüscher and Weisz, 1985a; Lüscher and Weisz, 1985b). At tree level one has

$$c_0 = \frac{5}{3}, \qquad c_1 = -\frac{1}{12}, \qquad c_2 = 0, \qquad c_3 = 0, \qquad (11.19)$$

so that only the $\mathcal{L}_1^{(6)}$ counterterm (the planar loop) is needed. We have then

$$S_g = \frac{6}{g_0^2}\left[\frac{5}{3}\mathcal{L}^{(4)} - \frac{1}{12}a^2\,\mathcal{L}_1^{(6)}\right] = \frac{1}{2}\int \mathrm{Tr}\, F_{\mu\nu}^2(x) + O(a^4). \qquad (11.20)$$

This action is a discretization of the continuum QCD pure gauge action in which the discretization errors have been reduced to order a^4. Although both $\mathcal{L}^{(4)}$ and $\mathcal{L}_1^{(6)}$ have discretization errors

---

[44]Actually, due to symmetry considerations they are of order a^4. The fact that the corrections to the pure gauge action are of order a^2 or a^4 essentially comes from the fact that one can construct gauge-invariant terms of dimension 6 and 8, but not of dimension 5 and 7.

of order a^2, the above combination cancels these contributions and the first correction to the continuum becomes of order a^4. Of course this is only a tree-level cancellation, and the coefficients get corrected by quantum effects. The values that improve the pure gauge action to 1-loop are

$$
\begin{aligned}
c_0(g_0^2) &= \frac{5}{3} + 0.2370 g_0^2, \\
c_1(g_0^2) &= -\frac{1}{12} - 0.02521 g_0^2, \\
c_2(g_0^2) &= -0.00441 g_0^2, \\
c_3(g_0^2) &= 0.
\end{aligned}
\tag{11.21}
$$

They define the Lüscher-Weisz action. These coefficients satisfy the normalization condition

$$
c_0 + 8c_1 + 8c_2 + 16c_3 = 1,
\tag{11.22}
$$

which is valid to all orders of perturbation theory. Since at one loop c_2 is small one usually drops $\mathcal{L}_2^{(6)}$, and $\mathcal{L}_1^{(6)}$ remains the only counterterm (as it was at tree level). In this case the normalization condition is used to write the action in the form

$$
\begin{aligned}
S_g &= \frac{6}{g_0^2}\left[(1 - 8c_1)\, \mathcal{L}^{(4)} + a^2 c_1\, \mathcal{L}_1^{(6)} \right] \\
&= \frac{6}{g_0^2}\left[(1 - 8c_1) \sum_x \sum_{\mu<\nu} P_{\mu\nu}^{1\times1} + a^2 c_1 \sum_x \sum_{\mu<\nu} \left(P_{\mu\mu,\nu}^{1\times2} + P_{\nu\nu,\mu}^{1\times2} \right) \right],
\end{aligned}
\tag{11.23}
$$

where $P_{\mu\mu,\nu}^{1\times2}$ denotes the rectangle which is two lattice spacings long in the μ direction. These actions has been extensively investigated also in (Weisz, 1983; Weisz and Wohlert, 1984; Wohlert, Weisz and Wetzel, 1985; Curci, Menotti and Paffuti, 1983; Bernreuther, Wetzel and Wohlert, 1984).

There are also other actions which go under the name of improved gauge actions where improvement is not done à la Symanzik, but instead following renormalization group arguments (and then they still have $O(a^4)$ correction terms). In this case one looks for actions which are close to what one obtains after doing blocking transformations, that is renormalization group transformations in which the lattice spacing is doubled at each step. A perturbative calculation gives the action proposed by Iwasaki (1983a; 1983b), which is Eq. (11.23) with

$$
c_1 = -0.331,
\tag{11.24}
$$

while nonperturbative calculations which use Schwinger-Dyson equations lead to the DBW2 action (Takaishi, 1996; de Forcrand *et al.*, 2000), which is Eq. (11.23) with [45]

$$
c_1 \simeq -1.40686.
\tag{11.25}
$$

---

[45]The acronym DBW2 stands for doubly-blocked Wilson 1×2 plaquette. We remark that in this case the relation between the coefficients is not linear, and these are only rather approximate values.

$$\mathcal{L}_1^{(6)} \qquad\qquad \mathcal{L}_2^{(6)} \qquad\qquad \mathcal{L}_3^{(6)}$$
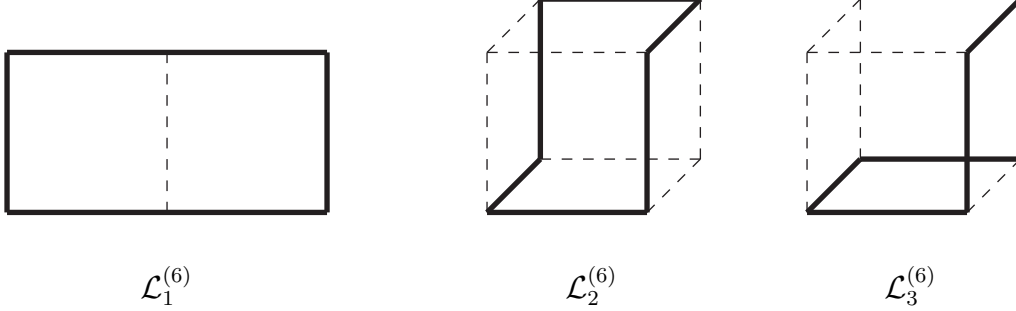
Figure 15: The planar, twisted and L-shaped six-link loops.

There are other proposals in which c_2 and c_3 are nonzero, which we will not consider here.

All these improved gauge actions have the aim to achieve a better convergence to the continuum limit. Their main drawback is that they have no reflection positivity (Lüscher and Weisz, 1984), and thus there is no corresponding theory in the continuum, that is it is not possible to construct a Hilbert space in the usual way. This also causes problems with numerical simulations. The violation of physical positivity in fact leads to unphysical poles in the propagators, which correspond to unphysical states that create a sizeable disturb in the extraction of physical observables (Necco, 2002b).

All these actions are quite complicated to use in perturbation theory. The gluon propagator in a covariant gauge for generic c_1 is given by

$$G_{\mu\nu}(k) = \frac{1}{(\widehat{k}^2)^2}\left(\alpha\widehat{k}_\mu\widehat{k}_\nu + \sum_\sigma(\widehat{k}_\sigma\delta_{\mu\nu} - \widehat{k}_\nu\delta_{\mu\sigma})\widehat{k}_\sigma A_{\sigma\nu}(k)\right), \qquad (11.26)$$

with

$$
\begin{aligned}
A_{\mu\nu}(k) \;\; = \;\; A_{\nu\mu}(k) = (1 - \delta_{\mu\nu})\,\Delta(k)^{-1}&\left[(\widehat{k}^2)^2 - c_1\widehat{k}^2\left(2\sum_\rho\widehat{k}_\rho^4 + \widehat{k}^2\sum_{\rho\neq\mu,\nu}\widehat{k}_\rho^2\right)\right.\\
&\left.+c_1^2\left(\left(\sum_\rho\widehat{k}_\rho^4\right)^2 + \widehat{k}^2\sum_\rho\widehat{k}_\rho^4\sum_{\tau\neq\mu,\nu}\widehat{k}_\tau^2 + (\widehat{k}^2)^2\prod_{\rho\neq\mu,\nu}\widehat{k}_\rho^2\right)\right], \qquad (11.27)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta(k) \;\; = \;\; &\left(\widehat{k}^2 - c_1\sum_\rho\widehat{k}_\rho^4\right)\left[\widehat{k}^2 - c_1\left((\widehat{k}^2)^2 + \sum_\tau\widehat{k}_\tau^4\right) + \frac{1}{2}c_1^2\left((\widehat{k}^2)^3 + 2\sum_\tau\widehat{k}_\tau^6 - \widehat{k}^2\sum_\tau\widehat{k}_\tau^4\right)\right]\\
&-4c_1^3\sum_\rho\widehat{k}_\rho^4\sum_{\tau\neq\rho}\widehat{k}_\tau^2. \qquad (11.28)
\end{aligned}
$$

The gluon vertices are quite complicated, and we will not report them here. They can be found in (Weisz and Wohlert, 1984). The quark-gluon vertices are of course not modified.

90

Perturbative calculations using improved gauge actions have been recently presented in (Aoki *et al.*, 2000) for three-quark operators and in (DeGrand, Hasenfratz and Kovács, 2002) for two- and four-quark operators, and they have even been employed in domain wall calculations (Aoki *et al.*, 2002), where the use of improved gauge actions is thought to diminish the residual chiral symmetry breaking which one has when working at finite N_s.

12 The Schrödinger functional

We present here a short introduction to a powerful framework for lattice calculations that goes under the name of Schrödinger functional. This is a field of research that has grown very much in recent years and would need a separate review in itself, given its peculiarities and technical complexities as well as the number and importance of the results that has produced. For an introductory review the lectures of (Lüscher, 1999a) are recommended.

The Schrödinger functional was extensively investigated on the lattice in (Symanzik, 1981; Lüscher, 1985; Lüscher *et al.*, 1992; Sint, 1994) and used in various physical situations. [46] It has been essential for the calculation of c_{sw} perturbatively and nonperturbatively (Jansen *et al.*, 1996; Lüscher *et al.*, 1996; Lüscher and Weisz, 1996; Lüscher *et al.*, 1997; Jansen and Sommer, 1998) and for the nonperturbative computation of the running coupling in QCD (which has allowed a quite precise determination of the Λ parameter) and of the masses of the quarks and their scale evolution (Lüscher, Weisz and Wolff, 1991; Lüscher *et al.*, 1993; Lüscher *et al.*, 1994; Capitani *et al.*, 1998c; Capitani *et al.*, 1999c; Bode *et al.*, 2001; Garden *et al.*, 2000; Knechtli *et al.*, 2002).

The Schrödinger functional constitutes a finite volume renormalization scheme. It is a standard functional integral in which fixed boundary conditions are imposed, and where the time direction assumes a special meaning. In fact, on the space directions there are generalized periodic conditions, while on the time direction one puts Dirichlet boundary conditions (see Fig. 16).

This means that at $x_0 = 0$ and $x_0 = T$ one fixes the spatial components of the links U to some particular values (usually constant abelian fields), while the temporal components, $U_0(x)$, remain unconstrained, and they are defined only for $0 \leq x_0 \leq T - 1$. The pure gauge action is given by the sum of the Wilson plaquettes which are fully contained between the timeslices at $x_0 = 0$ and $x_0 = T$. The spatial plaquettes at the boundaries $x_0 = 0$ and $x_0 = T$ contribute to the action only with a weight $1/2$ to avoid double counting. The gauge group is local in the bulk but global on the boundaries.

The fermion fields are dynamical only for $1 \leq x_0 \leq T - 1$, while at the two temporal

---

[46]The Schrödinger functional had also been studied in (Rossi and Testa, 1980a; Rossi and Testa, 1980b; Rossi and Testa, 1984; Rossi and Yoshida, 1989; Leroy *et al.*, 1990), using the temporal gauge and boundary conditions different from the ones that we are going to introduce in the following.

boundaries only half their components are defined, and they are fixed to some particular values:

$$P_+\psi(x)\Big|_{x_0=0} = \rho(\vec{x}), \qquad P_-\psi(x)\Big|_{x_0=T} = \rho'(\vec{x}), \tag{12.1}$$
$$\overline{\psi}(x)P_-\Big|_{x_0=0} = \overline{\rho}(\vec{x}), \qquad \overline{\psi}(x)P_+\Big|_{x_0=T} = \overline{\rho}'(\vec{x}),$$

where the projectors are

$$P_\pm = \frac{1 \pm \gamma_0}{2}. \tag{12.2}$$

The complementary components $(P_-\psi(x)\Big|_{x_0=0}$ etc.) vanish for consistency. In the spatial directions the quark fields are periodic up to a phase, and the generalized periodic conditions can be written as

$$\psi(x + L\hat{k}) = e^{i\theta_k}\psi(x), \qquad \overline{\psi}(x + L\hat{k}) = \overline{\psi}(x)e^{-i\theta_k}, \qquad k = 1,2,3. \tag{12.3}$$

It turns out to be more convenient to work with the equivalent setting in which the fermion fields have strictly periodic boundary conditions in the spatial directions and this phase θ is moved inside the definition of the covariant derivative,

$$D_\mu\psi(x) = \frac{1}{a}\Big[\lambda_\mu U_\mu(x)\psi(x + a\hat{\mu}) - \psi(x)\Big] \tag{12.4}$$
$$D_\mu^\star\psi(x) = \frac{1}{a}\Big[\psi(x) - \lambda_\mu^{-1}U_\mu^{-1}(x - a\hat{\mu})\psi(x - a\hat{\mu})\Big],$$

with

$$\lambda_\mu = e^{ia\theta_\mu/L}, \quad \theta_0 = 0, \quad -\pi < \theta_k \leq \pi. \tag{12.5}$$

This phase θ is called a finite-size momentum, and it is not quantized (although we are by definition in a finite volume). It can then be chosen to be smaller than the minimal quantized momentum, $p_{\min} = 2\pi/L$, thereby reducing this kind of lattice artifacts. θ is a free parameter, and can be tuned in such a way that one obtains the best numerical signals or the best perturbative expansions (or both, if one is lucky).

The boundaries do not influence the integration measure of the functional integral. Correlation functions in the path integral formulation can be computed in the usual way, the only difference being that the action has a more complicate form at the boundaries. The correlation functions can then involve functional derivatives of the Boltzmann factor acting also on the boundary values of the quark fields,

$$\zeta(\vec{x}) = \frac{\delta}{\delta\overline{\rho}(\vec{x})}, \qquad \overline{\zeta}(\vec{x}) = -\frac{\delta}{\delta\rho(\vec{x})}, \tag{12.6}$$
$$\zeta'(\vec{x}) = \frac{\delta}{\delta\overline{\rho}'(\vec{x})}, \qquad \overline{\zeta}'(\vec{x}) = -\frac{\delta}{\delta\rho'(\vec{x})},$$

if the operators under consideration contain fermions fields close to the boundaries. After taking the appropriate number of derivatives to build the desired operator, one puts $\rho = \rho' = \overline{\rho} = \overline{\rho}' = 0$ as usual.
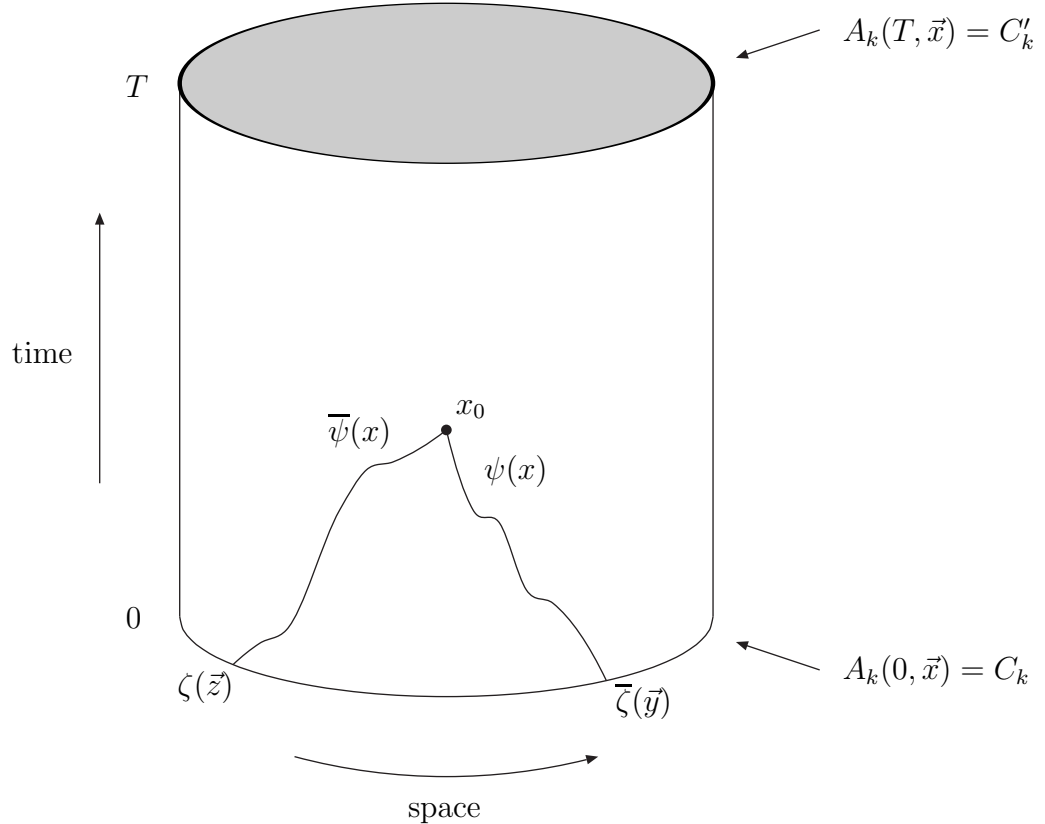
Figure 16: The Schrödinger functional, with its time boundaries. Shown is also a correlation function involving the boundary fields.

The Schrödinger functional is not linked to a particular regularization. In fact, it is originally a continuum construction, and corresponds to the amplitude for the quantum evolution from a field configuration at the time zero to a field configuration at the time T. A lattice regularization is of course useful for studying the nonperturbative aspects of QCD, from first principles. When one chooses to set the Schrödinger functional on the lattice, [47] the fact that it is defined in a finite volume brings interesting features.

The choice of temporal boundary conditions specified above implies that zero modes are absent at the lowest oder of perturbation theory. For zero quark masses the lowest eigenvalue of the Dirac operator is then of order $1/L$. The frequency gap on the quark and gluon fields remains of order $1/L$ even in the interacting theory, and this means that one can perform

---

[47]Usually one takes as lattice action in the interior the Wilson action, which is probably the simplest setting for a lattice Schrödinger functional.

simulations with quark masses close to zero without encountering singularities, because an infrared cutoff is provided by the lattice size, L.

A lattice of finite volume usually gives rise to systematic errors, but here the situation is completely different. The Schrödinger functional is used as a finite volume renormalization scheme, where the renormalized quantities are specified at the scale $\mu = 1/L$ and for vanishing quark masses. [48] This is something quite different from the usual lattice calculations, where the renormalization scale is independent of the lattice size, and is instead determined by the lattice spacing. Thus, there cannot be by definition any finite volume effects in the Schrödinger functional. The finite volume is instead used to probe the theory and specify the renormalization prescriptions.

We stress again that the Schrödinger functional is a continuum scheme, because the scale at which the theory is renormalized is not proportional to the inverse of the lattice spacing, and so it can also be defined for a theory with zero lattice spacing. In practice however everything, from Monte Carlo simulations to perturbation theory, is carried out using lattice techniques. For instance, the perturbative calculations used for the running coupling in the Schrödinger functional that we mentioned in Section 2 were all done on the lattice. [49] At the end one can extrapolate the results obtained at different lattice spacings to the continuum limit, where the Schrödinger functional is well defined. Then it helps that the Λ parameter is much closer to the continuum that usual Λ parameters defined on the lattice. For the theory with zero flavors one has

$$\frac{\Lambda_{\mathrm{SF}}}{\Lambda_{\overline{\mathrm{MS}}}} = 0.48811(1). \tag{12.7}$$

This is related to the fact that this scheme seems "closer" to the $\overline{\mathrm{MS}}$ scheme, as we have also discussed at the end of Section 2.

The Schrödinger functional scheme can also be supplemented with powerful finite-size recursive techniques, which allow to perform renormalization calculations which span a wide range of energies. Wilson suggested to introduce a renormalization group transformation so that one can cover large scale differences in a recursive manner (Wilson, 1980), and these ideas were then developed in (Lüscher, 1983), (Lüscher *et al.*, 1992) and (Jansen *et al.*, 1996; Lüscher *et al.*, 1996). Nice summaries of these techniques are also given in (Lüscher, 1997), (Sommer, 1997) and (Lüscher, 2002) The evolution of renormalized coupling and masses can then be studied from rather low to rather high energies. This allows nonperturbative studies to be carried out with a good control over the systematic errors.

We would like to sketch how this nonperturbative renormalization over many scales is accomplished. One takes a sequence of pairs of lattices, where in each pair a lattice has size L and

---

[48]Usually one chooses $T = 2L$ or $T = L$, and everything in the theory is then referred to the scale L. It seems that some systematic errors are more pronounced in the case $T = 2L$, and $T = L$ is then usually preferred (Sint and Weisz, 1999).

[49]A one-loop calculation of the Schrödinger functional in dimensional regularization can be found in (Sint, 1995).

the other one has size $L' = 2L$, and computes the renormalized parameters at the new scale $\mu' = \mu/2$ knowing them at the scale $\mu = 1/L$, maintaining in the process the bare parameters fixed. [50] This defines step scaling functions σ, which for the renormalized coupling and masses are given by [51]

$$g^2(2L) = \sigma(g^2(L)) \tag{12.10}$$

$$Z_P(2L) = \sigma_P(Z_P(L)) \cdot Z_P(L). \tag{12.11}$$

These are a kind of integrated form of the β and τ functions (see Eqs. (10.3) and (10.29)). The errors on these step scaling functions can be rendered rather small when for each L one repeats the lattice computations for a few different values of a and then extrapolates them to the continuum limit in the $O(a)$ improved theory. One does not need large lattices at all, and in fact lattices as small as $L/a = 5$ have been used. For the coupling and mass calculations it has never been necessary to consider lattices larger than $L'/a = 32$.

The step scaling functions σ and σ_P are used recursively to go from very high to very low scales. Once the renormalized quantities are computed at the new scale $L' = 2L$, this is taken as the starting scale for the computation of the renormalized quantities at $L'' = 2L' = 4L$, and this process is repeated until after n steps one reaches the scale $L^{(n)} = 2^n L$. One has then completed the evolution of the renormalized parameters from the energy μ to the energy $\mu/2^n$. The low-energy end of this evolution should correspond to a scale at which one can safely match the renormalized quantities to some low-energy hadronic quantities. In this way one fixes the relations between the bare coupling and masses and the renormalized coupling and masses at that scale. This matching can again be done without the use of large lattices. At the high-energy end one uses perturbation theory, which is completely safe there (see Figs. 1 and 2), to compute the Λ parameter and the renormalization group invariant masses (Eqs. (10.13) and (10.34) respectively). Once these numbers are known, one can easily carry out the matching of the Λ parameter to the $\overline{\text{MS}}$ scheme or to other continuum schemes, via continuum calculations only. We remind that the renormalization group invariant masses do not depend on the scheme, and thus is only the Λ parameter which needs an additional conversion to $\overline{\text{MS}}$.

---

[50]The renormalized strong coupling is defined as the response of the functional to the variation of the gauge fields at the boundaries.

[51]In practice the PCAC relation

$$\partial_\mu A_\mu(x) = 2m \cdot P(x) \tag{12.8}$$

is used to define the masses, so that the renormalization of the mass is proportional to Z_A/Z_P, and since Z_A does not evolve with the scale the scale evolution of the renormalized mass is proportional to the inverse of the renormalization of the pseudoscalar density:

$$m(\mu) = \frac{Z_A}{Z_P(L)} m_0, \qquad \mu = \frac{1}{L}. \tag{12.9}$$

We remind that the local currents are not conserved on the lattice, and they are renormalized by the strong forces, so that their Z's are different from one.

We have thus been able to connect the nonperturbative infrared sector of the theory with the high-energy perturbative regime. This is the only method which allows such a remarkable thing. With the Schrödinger functional coupled with the recursive finite-size scaling techniques one can cover large scale differences, which can be of more than two orders of magnitude. One can arrive at energies of more than 100 GeV (as in Figs. 1 and 2). In order to achieve this with conventional methods it would be necessary to contain all relevant scales in a single lattice, which is impossible. The Schrödinger functional acts in this process only as an intermediate renormalization scheme. Everything can be computed nonperturbatively and when improvement is also implemented the systematic errors are then completely controlled.

The methods that we have just described have allowed to establish the best lattice result that we currently have for the Λ parameter of QCD,

$$\Lambda_{\overline{\text{MS}}} = 238 \pm 19 \text{ M}eV, \tag{12.12}$$

in the theory with zero flavors (Capitani *et al.*, 1999c). In this work also the nonperturbative relation between the bare masses and the renormalization group invariant masses has been determined with a rather small error. A spinoff of the latter calculation has been the computation of the nonperturbative renormalization of the scalar quark condensate using overlap fermions (Hernández *et al.*, 2001; Hernández *et al.*, 2002). This is possible for one thing because in a regularization which respects chiral symmetry one has

$$Z_S = Z_P = \frac{1}{Z_M}, \tag{12.13}$$

and therefore the knowledge of the renormalization of the mass translates immediately in the determination of the renormalization of the scalar current, and furthermore because the renormalization group invariant masses are independent of the scheme, and thus they are the same whichever lattice action is used. This means that one can then compute the renormalization of the mass in the overlap formulation by comparing

$$M_{RGI} = Z_M^{\text{ov}}(g_0)\, m_{\text{ov}}(g_0) \tag{12.14}$$
$$M_{RGI} = Z_M^{\text{W}}(g_0')\, m_{\text{W}}(g_0'), \tag{12.15}$$

where Z_M^{W}, the relation between the bare mass in the Wilson formulation and the renormalization group invariant mass, has been nonperturbatively determined using the finite-size scaling techniques and the Schrödinger functional as an intermediate scheme (Capitani *et al.*, 1999c). Thus, Z_M^{ov} can be computed by evaluating the ratio of the bare masses in the two schemes, which can be done imposing that some renormalized quantity at a certain scale assumes the same value in both schemes. Finally, the knowledge of Z_M^{ov}, because of Eq. (12.13), allows the determination of the renormalization factor connecting the bare condensate to the renormalization group invariant condensate.

This kind of reasoning can be applied to other physical situations and other lattice actions. It shows how important the calculations made using the Schrödinger functional can be in practice, and how the results obtained using the finite-size scaling techniques can have implications beyond the Schrödinger functional and the Wilson action.

Let us now turn our attention to perturbation theory, which in the Schrödinger functional has its own peculiarities, and is more complicated than average. For details on the perturbative setting the reader can consult the article of (Lüscher and Weisz, 1996). There is an asymmetry between spatial indices and the temporal index, and the perturbative calculations are not visually 4-dimensional covariant. This is due to the fact that the Schrödinger functional has no periodic boundary conditions in the time direction, and so translational invariance is lost and Fourier transforms can be done only in the spatial directions. Thus one works in the time-momentum representation, and uses quantities like

$$q(x_0, \vec{p}), \tag{12.16}$$

the three-dimensional Fourier transform of a function $q(x)$ of coordinate space. The presence of the boundaries also renders the form of the propagators quite complicated. Moreover, usually the improved theory is considered and this causes the calculations to be even more involved. In fact, to improve the Schrödinger functional to order a, in addition to the usual Sheikholeslami-Wohlert term one also needs some $O(a)$ boundary counterterms, both for the gluon part and for the quark part of the action.

Let us then consider the $O(a)$ improved theory. The quark propagator obeys the equation

$$\left(D + \delta D_v + \delta D_b + m_0\right)S(x, y) = a^{-4}\delta_{xy}, \qquad 0 < x_0 < T, \tag{12.17}$$

with the boundary conditions

$$P_+ S(x, y)\big|_{x_0=0} = P_- S(x, y)\big|_{x_0=T} = 0, \tag{12.18}$$

where the improvement counterterm to the Dirac operator in the interior is the same as for the Wilson action, namely the Sheikholeslami-Wohlert term

$$\delta D_v \psi(x) = c_{sw} \frac{\mathrm{i}}{4} a \sigma_{\mu\nu} F_{\mu\nu}^{clover}(x)\psi(x), \tag{12.19}$$

but there are also additional counterterms at the boundaries,

$$\delta D_b \psi(x) = (\tilde{c}_t - 1)\frac{1}{a}\left[\delta_{x_0,a}\left[\psi(x) - U_0^\dagger(x - a\hat{0})P_+\psi(x - a\hat{0})\right] + \delta_{x_0,T-a}\left[\psi(x) - U_0(x)P_-\psi(x + a\hat{0})\right]\right], \tag{12.20}$$

where $\tilde{c}_t$ is an improvement coefficient corresponding to these fermion boundary counterterms. In perturbation theory it is convenient to make the decomposition

$$\psi(x) = \psi_{cl}(x) + \chi(x), \qquad \overline{\psi}(x) = \overline{\psi}_{cl}(x) + \overline{\chi}(x), \tag{12.21}$$

where the classical field satisfies the Dirac equation

$$\left(D + \delta D_v + \delta D_b + m_0\right)\psi_{cl}(x) = 0, \qquad 0 < x_0 < T, \tag{12.22}$$

and has boundary values

$$P_+\psi_{cl}(x)|_{x=0} = \rho(\vec{x}), \qquad P_-\psi_{cl}(x)|_{x=T} = \rho'(\vec{x}) \tag{12.23}$$

(see Eq. (12.1)). A similar decomposition holds for $\overline{\psi}_{cl}$. In terms of the boundaries the classical field has the expression

$$\psi_{cl}(x) = a^3 \sum_{\vec{y}} \widetilde{c}_t \left[S(x, y)U_0^\dagger(y - a\hat{0})\rho(\vec{y})\Big|_{y_0=a} + S(x, y)U_0(y)\rho'(\vec{y})\Big|_{y_0=T-a} \right]. \tag{12.24}$$

A useful property of this decomposition is that the quantum components $\chi(x)$ have vanishing boundary values. The fermionic action splits as

$$S_f^{imp}[U, \overline{\psi}, \psi] = S_f^{imp}[U, \overline{\psi}_{cl}, \psi_{cl}] + S_f^{imp}[U, \overline{\chi}, \chi], \tag{12.25}$$

and the quantum and classical components are then completely independent. The generating functional with fermionic sources $\eta, \overline{\eta}$ is given by

$$\begin{aligned}
\log Z_f &= \log Z_f\Big|_{\overline{\rho}'=\dots=\eta=0} \\
&\quad -a^3 \sum_{\vec{x}} \left[\frac{1}{2}a\widetilde{c}_s \left[\overline{\rho}(\vec{x})\gamma_k(\nabla_k^\star + \nabla_k)\rho(\vec{x}) + \overline{\rho}'(\vec{x})\gamma_k(\nabla_k^\star + \nabla_k)\rho'(\vec{x})\right] \right. \\
&\quad \left. -\widetilde{c}_t\left[\overline{\rho}(\vec{x})U_0(x - a\hat{0})\psi_{cl}(x)\Big|_{x_0=a} + \overline{\rho}'(\vec{x})U_0^\dagger(x)\psi_{cl}(x)\Big|_{x_0=T-a}\right] \right] \\
&\quad +a^8 \sum_{x,y} \overline{\eta}(x)S(x, y)\eta(y) + a^4 \sum_x \left[\overline{\eta}(x)\psi_{cl}(x) + \overline{\psi}_{cl}(x)\eta(x)\right], \tag{12.26}
\end{aligned}$$

where $\widetilde{c}_s$ is an improvement coefficient corresponding to this other kind of fermion counterterms at the boundaries (entirely contained in the timeslices at $x_0 = 0$ and $x_0 = T$). This now allows upon differentiation to derive the basic contractions, among which we find [52]

$$\begin{aligned}
\left[\psi(x)\overline{\psi}(y)\right] &= S(x, y), \tag{12.27} \\
\left[\psi(x)\overline{\zeta}(\vec{y})\right] &= \frac{\delta\psi_{cl}(x)}{\delta\rho(\vec{y})} = \widetilde{c}_t S(x, y)U_0^\dagger(y - a\hat{0})P_+\Big|_{y_0=a}, \tag{12.28} \\
\left[\zeta(\vec{z})\overline{\psi}(x)\right] &= \frac{\delta\overline{\psi}_{cl}(x)}{\delta\overline{\rho}(\vec{z})} = \widetilde{c}_t P_- U_0(z - a\hat{0})S(z, x)\Big|_{z_0=a}. \tag{12.29}
\end{aligned}$$

---

[52]The square brackets denote fermion correlations in a given gauge field.

Using these contractions one can construct for example the correlation function of the axial current, inserted at the time x_0, which is (see Fig. 16)

$$f_A(x_0) = a^6 \sum_{\vec{y},\vec{z}} \frac{1}{2} \left\langle \mathrm{Tr}\left\{ \left[\zeta(\vec{z})\overline{\psi}(x) \right] \gamma_0\gamma_5 \left[\psi(x)\overline{\zeta}(\vec{y}) \right] \gamma_5 \right\} \right\rangle, \qquad (12.30)$$

where $\langle \cdots \rangle$ denotes gluon averages. The complete list of contractions is given in (Lüscher and Weisz, 1996).

We give here the explicit expression of the quark propagator, which has a quite complicated form, even in the unimproved theory ($\widetilde{c}_t = \widetilde{c}_s = 1$). One has

$$S(x,y) = \left(D^\dagger + m_0 \right) G(x,y), \qquad 0 < x_0, y_0 < T, \qquad (12.31)$$

where

$$G(x,y) = \frac{1}{L^3} \sum_{\vec{p}} \frac{e^{i\vec{p}(\vec{x}-\vec{y})}}{-2ia^{-1}\sin ap_0^+ A(\vec{p}^+)R(p^+)} \qquad (12.32)$$

$$\times \left\{ (M(p^+) - ia^{-1}\sin ap_0^+)e^{-\omega(\vec{p}^+)|x_0-y_0|} + (M(p^+) + ia^{-1}\sin ap_0^+)e^{-\omega(\vec{p}^+)(2T-|x_0-y_0|)} \right.$$

$$\left. - (M(p^+) + i\gamma_0 a^{-1}\sin ap_0^+)e^{-\omega(\vec{p}^+)(x_0+y_0)} - (M(p^+) - i\gamma_0 a^{-1}\sin ap_0^+)e^{-\omega(\vec{p}^+)(2T-x_0-y_0)} \right\},$$

for $0 \le x_0 \le T$, with

$$p_\mu^+ = p_\mu + \frac{\theta_\mu}{L} \qquad (12.33)$$

$$A(\vec{q}) = 1 + a\left(m_0 + \frac{2}{a}\sum_{k=1}^3 \sin^2 \frac{aq_k}{2} \right) \qquad (12.34)$$

$$R(q) = M(q)\left[1 - e^{-2\omega(\vec{q})T} \right] - \frac{i}{a}\sin aq_0\left[1 + e^{-2\omega(\vec{q})T} \right] \qquad (12.35)$$

$$M(q) = m_0 + \frac{2}{a}\sum_\mu \sin^2 \frac{aq_\mu}{2} \qquad (12.36)$$

and

$$p_0 = p_0^+ = i\omega(\vec{p}^+) \,\mathrm{mod}\,\frac{2\pi}{a}, \qquad (12.37)$$

where $\omega(\vec{q})$ is given by

$$\sinh\left[\frac{a}{2}\omega(\vec{q}) \right] = \frac{a}{2}\sqrt{\frac{\frac{1}{a^2}\sum_{k=1}^3 \sin^2 aq_k + \frac{1}{a^2}(A(\vec{q})-1)^2}{A(\vec{q})}}. \qquad (12.38)$$

The angle θ in the formulae above is the finite-size momentum which comes from the boundary conditions in the spatial directions (or rather from the modified covariant derivative, Eq. (12.4)).

We can see that things easily become rather technical. We want here to report also the expression of the gluon propagator, which in the Feynman gauge is given by

$$D_{\mu\nu}(x,y) = \frac{1}{L^3} \sum_{\vec{p}} e^{i\vec{p}(\vec{x}-\vec{y})} d_{\mu\nu}(x_0,y_0;\vec{p}), \tag{12.39}$$

with

$$d_{00}(x_0,y_0;\vec{p}) = \frac{a}{\sinh(\varepsilon a)\sinh(\varepsilon T)} \cosh\left[\varepsilon\left(T-x_0-\frac{1}{2}a\right)\right] \cosh\left[\varepsilon\left(y_0+\frac{1}{2}a\right)\right] \tag{12.40}$$

$$d_{kj}(x_0,y_0;\vec{p}) = \delta_{kj} \frac{a}{\sinh(\varepsilon a)\sinh(\varepsilon T)} \sinh\left[\varepsilon(T-x_0)\right] \sinh\left(\varepsilon y_0\right) \tag{12.41}$$

for nonzero $\vec{p}$, and

$$d_{00}(x_0,y_0;\vec{0}) = y_0 + a \tag{12.42}$$

$$d_{kj}(x_0,y_0;\vec{0}) = \delta_{kj}(T-x_0)\frac{y_0}{T} \tag{12.43}$$

for $\vec{p}=0$. The "energy" ε is here given by

$$\cosh(a\varepsilon) = 1 + 2\sum_{k=1}^{3} \sin^2\frac{ap_k}{2}. \tag{12.44}$$

The mixed components d_{0k} and d_{k0} vanish. The above expression is valid for $x_0 \geq y_0$, and for $y_0 > x_0$ one uses the symmetry

$$d_{\mu\nu}(y_0,x_0;\vec{p}) = d_{\nu\mu}(x_0,y_0;\vec{p}). \tag{12.45}$$

For more technical details, and for the remaining parts of the perturbative setting like the gauge fixing (which is quite complicated), see (Lüscher *et al.*, 1992; Lüscher and Weisz, 1996) and also (Kurth, 2002), where the renormalization of the quark mass has been computed at one loop. The Feynman rules for the gluon vertices are given in (Palombi, Petronzio and Shindler, 2002).

The Schrödinger functional has been very useful also for the computation of many improvement coefficients. In particular the $O(a)$ corrections to the PCAC relation were used to fix c_{sw}, and because a mass term is present in this relation the Schrödinger functional is much more appropriate for this determination than, say, usual Wilson fermions. Also calculations of moments of structure functions have been done using the Schrödinger functional, coupled to the recursive finite-size scaling technique with which to define appropriate step scaling functions for the relevant operators. These calculations are reported in (Bucarelli *et al.*, 1999), in (Guagnelli, Jansen and Petronzio, 1999a; 1999b; 1999c; 2000) and in (Jansen, 2000; Palombi, Petronzio and Shindler, 2002). Perturbation theory is in this case quite cumbersome. The covariant derivatives make everything more complicated, because of the phase factors and the boundary fields.

We conclude mentioning that also a few two-loop calculations have been completed using the Schrödinger functional (Wolff, 1995; Narayanan and Wolff, 1995; Bode, Wolff and Weisz, 1999; Bode, Weisz and Wolff, 2000a; Bode, Weisz and Wolff, 2000b).

13 The hypercubic group

With this Section we begin to explain in a more detailed way how lattice perturbative calculations are actually done. As a first thing, it is useful to discuss the symmetry group of the lattice and see what are the consequences of the breaking of Lorentz invariance.

On the lattice one inevitably ends up with a discrete group. The symmetry group of the discrete rotations of a four-dimensional hypercubic lattice onto itself is a crystallographic group, denoted by W_4 and called the hypercubic group. This is the group of $\pi/2$ rotations on the six lattice planes with the addition of the reflections (so that parity transformations are also included). It consists of 384 elements and 20 irreducible representations (Baake, Gemünden and Oedingen, 1982; Baake, Gemünden and Oedingen, 1983; Mandula, Zweig and Govaerts, 1983a).

The hypercubic group is a subgroup of the orthogonal group $O(4)$, which is the Lorentz group analytically continued to euclidean space. One of the main difficulties in doing perturbative calculations on the lattice arises from the fact that the (euclidean) Lorentz symmetry breaks down to the hypercubic W_4 symmetry. The lattice has then a reduced symmetry with respect to the continuum, and many more mixings arise, as we will see in the next Section.

Let us first consider the special hypercubic group, SW_4, consisting of proper rotations without reflections, which has 192 elements and 13 irreducible representations. Five of these representations (of dimensions 1, 1, 2, 3 and 3) are connected to the 4-dimensional group of permutations, S_4. There are then four representations of $O(4)$ which remain irreducible under the special hypercubic group:

$$(\mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{1}), (\tfrac{\mathbf{1}}{\mathbf{2}}, \tfrac{\mathbf{1}}{\mathbf{2}}), (\tfrac{\mathbf{3}}{\mathbf{2}}, \tfrac{\mathbf{1}}{\mathbf{2}}). \tag{13.1}$$

The product of the first three of these representations with the completely antisymmetric representation of the permutation group S_4 gives three other representations of SW_4 (which have the same dimensionality), while $(\tfrac{3}{2}, \tfrac{1}{2})$ is invariant under this operation and the hypercubic representations $(\tfrac{1}{2}, \tfrac{3}{2})$ and $(\tfrac{3}{2}, \tfrac{1}{2})$ turn out to be the same.

So far we have then been able to identify 12 representations. These is still another representation, which has dimension 6. The complete list of the representations of the special hypercubic group SW_4 is thus given by:

$$\mathbf{1_1}, \mathbf{1_2}, \mathbf{2}, \mathbf{3_1}, \mathbf{3_2}, \mathbf{3_3}, \mathbf{3_4}, \mathbf{3_5}, \mathbf{3_6}, \mathbf{4_1}, \mathbf{4_2}, \mathbf{6}, \mathbf{8}, \tag{13.2}$$

where subscripts label different representations with the same dimensionality. This group is a subgroup of $SO(4)$, the special orthogonal group.

We now discuss the representations of the group W_4. Including the reflections doubles the number of group elements, but not the number of the representations. This comes out from the fact that, contrary to what happens in three dimensions for the case of the cubic group, the hypercubic group is not the direct product of the rotation group and the reflection group.

The reason is that the reflection of all four axes is still a rotation, which is not true for the reflection of three axes in three dimensions. Therefore, the representations do not double, and from 13 they only become 20. What happens is that 9 of these 13 just double (generating the representations with opposite parity), while the remaining 4, all of dimension 3, merge into two 6-dimensional representations which are reflection invariant. In particular, the $\mathbf{3_3}$ and $\mathbf{3_4}$ of SW_4 merge into the $\mathbf{6_1}$ of W_4, and the $\mathbf{3_5}$ and $\mathbf{3_6}$ of SW_4 merge into the $\mathbf{6_2}$ of W_4. The other representations of SW_4 just double. We can then give the complete list of the representations of W_4: [53]

$$\mathbf{1_1}, \mathbf{1_2}, \mathbf{1_3}, \mathbf{1_4}, \mathbf{2_1}, \mathbf{2_2}, \mathbf{3_1}, \mathbf{3_2}, \mathbf{3_3}, \mathbf{3_4}, \mathbf{4_1}, \mathbf{4_2}, \mathbf{4_3}, \mathbf{4_4}, \mathbf{6_1}, \mathbf{6_2}, \mathbf{6_3}, \mathbf{6_4}, \mathbf{8_1}, \mathbf{8_2}. \tag{13.3}$$

The representation $\mathbf{4_1}$ is the canonical one, which corresponds to an object with a Lorentz index, like $(\frac{1}{2}, \frac{1}{2})$ is in the continuum. When we are interested in the behavior of lattice operators which have more than one Lorentz index, we have to look at which representations of the hypercubic group are contained in the tensor products of the $\mathbf{4_1}$ with itself, and compare the result with what happens in the continuum, where one has to consider the tensor products of the $(\frac{1}{2}, \frac{1}{2})$ with itself. The relation between these two expansions determines what kind of mixings arise when one computes radiative corrections of lattice matrix elements (apart from additional mixings due to the breaking of chiral symmetry or of other symmetries).

14 Operator mixing on the lattice

Since W_4 is a subgroup of $O(4)$, a continuum operator which belongs to a given irreducible representation the (euclidean) Lorentz group becomes in general a sum of irreducible representations of the part of that group that still remains a symmetry on the lattice, the hypercubic group. The continuum operator can then belong to various distinct lattice representations, according to the way in which its indices are chosen. This implies than on the lattice the possibilities for mixings under renormalization are larger than in the continuum, and mixings can arise which are pure lattice artifacts and which have to be carefully treated. The number of independent renormalization factors in a lattice calculation is then in general larger than in the continuum. In particular, operators which are multiplicatively renormalizable in the continuum may lose this property on the lattice. [54]

Thus, the mixing of lattice operators under renormalization presents features which are often significantly different from the corresponding continuum theory. In general the mixing patterns

---

[53]It could be useful to note that our notation $\mathbf{N_m}$ corresponds to the representation $\tau_m^{(N)}$ (or $^s\tau_m^{(N)}$ for SW_4) in (Baake, Gemünden and Oedingen, 1982).

[54]The fact that additional operators appear in the mixings is a feature that occurs not only using the lattice regularization. For example, in continuum calculations using dimensional regularization in the version known as DRED "evanescent" operators, coming from the additional -2ϵ dimensions, are generated in the intermediate stages of the calculations.

are more complicated. Moreover, for Wilson fermions additional mixings (beside those due to the breaking of Lorentz invariance) can arise because of the breaking of chiral symmetry. For staggered fermions, the loss of flavor invariance also opens the door for more mixings, although of a different kind. All these additional mixings are not physical, are just lattice artifacts which have to be subtracted in order to get physical results from the lattice. In practical terms the worst situation occurs in the case of mixings with operators of lower dimensions. This implies that the corresponding lattice renormalization factors contain a power divergent coefficient, proportional to $1/a^n$. These are a kind of lattice artifact which is not possible to subtract in perturbation theory.

In short, the Lorentz breaking, as well as the breaking of chiral, flavor or other symmetries that occur in specific lattice actions, means in general that the multiplicative renormalizability of continuum operators becomes a mixing when the theory is put on a lattice, in some cases even with power divergences. The necessary condition for not having any mixing at all is that the operator belongs to an irreducible representation of W_4, but this is sometimes not sufficient, as we will see shortly.

14.1   Unpolarized structure functions

Let us discuss some examples involving operators which measure moments of unpolarized structure functions. [55] These operators appear in the operator product expansion of two electromagnetic or weak hadronic currents, and have the form

$$O_{\{\mu\mu_1\cdots\mu_n\}}(x) = \overline{\psi}(x)\, \gamma_{\{\mu}\, D_{\mu_1} \cdots D_{\mu_n\}}\, \psi(x). \tag{14.1}$$

They are symmetric in all their indices and traceless. The operator $O_{\{\mu\mu_1\cdots\mu_n\}}$ measures the n-th moment, $\langle x^n \rangle$, of the unpolarized structure functions, that is of the distribution of the momentum of the quarks inside the hadrons. The renormalization of these operators, which presents particular computational difficulties due to the presence of the covariant derivatives, has been first studied with Wilson fermions in (Kronfeld and Photiadis, 1985; Corbò, Franco and Rossi, 1989; Corbò, Franco and Rossi, 1990; Caracciolo, Curci, Menotti and Pelissetto, 1990; Caracciolo, Curci, Menotti and Pelissetto, 1989) and then in (Capitani and Rossi, 1995a; Beccarini *et*

---

[55]It is not possible to compute a complete structure function directly on the lattice. The reason is that the structure functions describe the physics close to the light cone, and this region of Minkowski space shrinks to a point when one goes to Euclidean space, where Monte Carlo simulations are performed. However, on a Euclidean lattice it is possible to compute the moments of the structure functions, using an operator product expansion:

$$\int_0^1 x^n \mathcal{F}^{(i)}(x, Q^2) \sim C^{(n,i)}\left(\frac{Q^2}{\mu^2}\right) \cdot \langle h|O^{(n,i)}(\mu)|h\rangle.$$

The Wilson coefficients contain the short-distance physics, and can be perturbatively computed in the continuum. The matrix elements contain the long-distance physics, and can computed using numerical simulations, supplemented by a lattice renormalization of the relevant operators.

al., 1995; Göckeler *et al.*, 1996b; Brower *et al.*, 1997) and (Capitani, 2001a; Capitani, 2001b). Recent numerical works in full QCD which have made use of these perturbative renormalization factors are (Dolgov *et al.*, 2001; Dolgov *et al.*, 2002; Dreher *et al.*, 2002) and (Göckeler *et al.*, 2002), and recent short reviews of perturbative and nonperturbative methods and results can be found in (Capitani, 2002b; Capitani, 2002c; Negele, 2002).

Perturbative renormalization factors for all these operators have also been computed using overlap fermions, and are reported in (Capitani, 2001a; Capitani, 2001b; Capitani, 2002a).

We point out that all mixings discussed below, which are artifacts of the lattice, are only due to the breaking of Lorentz invariance. They have nothing to do with the breaking of chiral symmetry for Wilson fermions, and therefore they are still present, in exactly the same form, even when one uses Ginsparg-Wilson fermions.

In the continuum each of these structure function operators belongs to an irreducible representation of the Lorentz group. On the lattice they are instead in general reducible, they become linear combinations of irreducible representations of the hypercubic group, and this is the reason of the mixings which appear when radiative corrections are computed. More detailed analyses of these mixings can be found in (Beccarini *et al.*, 1995) and (Göckeler *et al.*, 1996a).

14.1.1 First moment

The operator is $O_{\mu\nu} = \overline{\psi}\,\gamma_\mu\,D_\nu\,\psi$, symmetric and traceless.

An object with a single Lorentz index belongs in the continuum to the $(\frac{1}{2}, \frac{1}{2})$ representation of the euclidean Lorentz group $O(4)$, while on the lattice it belongs to the $\mathbf{4_1}$ representation of the hypercubic group W_4.

The general decomposition of the 16 (nonsymmetrized) tensor components is in the continuum:

$$(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \otimes (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) = (\mathbf{0}, \mathbf{0}) \oplus (\mathbf{1}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}), \tag{14.2}$$

while on the lattice is:

$$\mathbf{4_1} \otimes \mathbf{4_1} = \mathbf{1_1} \oplus \mathbf{3_1} \oplus \mathbf{6_1} \oplus \mathbf{6_3}. \tag{14.3}$$

We have essentially two choices here for the symmetrized operators, that is the two indices can be different or can be equal. In the latter case, one has also to subtract the trace component.

The first case can be exemplified by considering the operator $O_{\{01\}}$, which belongs to the $\mathbf{6_1}$ and is multiplicatively renormalizable. We will compute its renormalization constant in detail in Section 15.4. A representative of the second case is $O_{\{00\}} - \frac{1}{3}(O_{\{11\}} + O_{\{22\}} + O_{\{33\}})$, which belongs to the $\mathbf{3_1}$ and is also multiplicatively renormalizable. The subtracted trace part belongs to the $\mathbf{1_1}$. Finally, the antisymmetric components (which do not enter in the operator product expansion for the moments), for example the operator $O_{[01]}$, belong to the remaining representation in the expansion, the $\mathbf{6_3}$.

Since they belong to different representations of W_4, the lattice renormalization factors of the operators $O_{\{01\}}$ and $O_{\{00\}} - \frac{1}{3}(O_{\{11\}} + O_{\{22\}} + O_{\{33\}})$ are different, as has been verified by

explicit calculations; in the continuum however they are the same, as both operators belong to the $(\mathbf{1}, \mathbf{1})$.

We mention here that from the point of view of Monte Carlo simulations the choice of two different indices is worse, because in this case one has to choose one component of the hadron momentum to be different from zero, and this leads to larger systematic effects due to the granularity of the lattice.

14.1.2 Second moment

The operator is $O_{\mu\nu\sigma} = \overline{\psi}\,\gamma_\mu\,D_\nu D_\sigma\,\psi$, symmetric and traceless.

The general decomposition of the 64 (nonsymmetrized) tensor components of this rank-three operator is in the continuum:

$$(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \otimes (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \otimes (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) = 4 \cdot (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \oplus 2 \cdot (\frac{\mathbf{3}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \oplus 2 \cdot (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{3}}{\mathbf{2}}) \oplus (\frac{\mathbf{3}}{\mathbf{2}}, \frac{\mathbf{3}}{\mathbf{2}}), \qquad (14.4)$$

while on the lattice is:

$$\mathbf{4_1} \otimes \mathbf{4_1} \otimes \mathbf{4_1} = 4 \cdot \mathbf{4_1} \oplus \mathbf{4_2} \oplus \mathbf{4_4} \oplus 3 \cdot \mathbf{8_1} \oplus 2 \cdot \mathbf{8_2}. \qquad (14.5)$$

We have essentially three choices here for the symmetrized components. One is represented by the operator $O_{\{123\}}$, which belongs to the $\mathbf{4_2}$ and is multiplicatively renormalizable. This choice however is quite unsatisfactory from the point of view of simulations, because two components of the hadron momentum have to be different from zero, leading to rather large systematic errors. One should minimize these systematic errors by including as few nonzero components of the hadron momentum as possible. From this point of view, the optimal choice is the operator $O_{\{111\}}$, which belongs to the $\mathbf{4_1}$. Unfortunately this operator mixes with $\overline{\psi}\,\gamma_1\,\psi$, which is a $\mathbf{4_1}$ as well. Moreover, the coefficient of this mixing can be seen from dimensional arguments to be power divergent, $1/a^2$, and thus this mixing cannot be studied in perturbation theory.

There is an intermediate choice between having the indices all different or all equal, and is given by the operator $O_{\{011\}} - \frac{1}{2}(O_{\{022\}} + O_{\{033\}})$, which has no power divergences due to the particular combination chosen. This operator belongs to an irreducible representation of W_4, but nonetheless is not multiplicatively renormalizable, and mixes with other operators. The way in which this happens is not trivial, and was first understood in (Beccarini *et al.*, 1995). The point is that this operator belongs to the $\mathbf{8_1}$, but there are three "copies" of the $\mathbf{8_1}$ in the decomposition of $O_{\mu\nu\sigma}$. It turns out that two of these copies mix in our case, at least at the 1-loop level. What happens is that $O_A = O_{011} - \frac{1}{2}(O_{022} + O_{033})$ and $O_B = O_{101} + O_{110} - \frac{1}{2}(O_{202} + O_{220} + O_{303} + O_{330})$, which have different tree levels ($\gamma_0 p_1^2 - \frac{1}{2}(\gamma_0 p_2^2 + \gamma_0 p_3^2)$ and $2\gamma_1 p_0 p_1 - (\gamma_2 p_0 p_2 + \gamma_3 p_0 p_3)$ respectively), have different 1-loop corrections and renormalize with different numerical factors. In other words, $O_{\{011\}} - \frac{1}{2}(O_{\{022\}} + O_{\{033\}}) = \frac{1}{3}(O_A + O_B)$ mixes with an operator of mixed symmetry (nonsymmetrized).

105

We have thus seen that the choice of indices for this rank-three operator is very important, and has practical consequences for the Monte Carlo simulations as well as for the calculation of renormalization factors.

In the continuum, all the cases discussed above for $O_{\{\mu\nu\sigma\}}$, including $O_{\{111\}}$, belong to the $(\frac{3}{2}, \frac{3}{2})$, and therefore they have the same renormalization constant, and of course no mixing problem.

14.1.3 Third moment

The operator is $O_{\mu\nu\sigma\rho} = \overline{\psi}\,\gamma_\mu\,D_\nu D_\sigma D_\rho\,\psi$, symmetric and traceless.

The general decomposition of the 256 (nonsymmetrized) tensor components is in the continuum:

$$
\begin{aligned}
(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \otimes (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \otimes (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \otimes (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \;=\; & 4\cdot(\mathbf{0},\mathbf{0}) \oplus 6\cdot(\mathbf{1},\mathbf{0}) \oplus 6\cdot(\mathbf{0},\mathbf{1}) \oplus 2\cdot(\mathbf{2},\mathbf{0}) \oplus 2\cdot(\mathbf{0},\mathbf{2}) \\
& \oplus 9\cdot(\mathbf{1},\mathbf{1}) \oplus 3\cdot(\mathbf{2},\mathbf{1}) \oplus 3\cdot(\mathbf{1},\mathbf{2}) \oplus 2\cdot(\mathbf{2},\mathbf{2}), \quad (14.6)
\end{aligned}
$$

while on the lattice is:

$$
\begin{aligned}
\mathbf{4_1} \otimes \mathbf{4_1} \otimes \mathbf{4_1} \otimes \mathbf{4_1} \;=\; & 4\cdot\mathbf{1_1} \oplus \mathbf{1_2} \oplus \mathbf{1_4} \oplus 3\cdot\mathbf{2_1} \oplus 2\cdot\mathbf{2_2} \oplus 7\cdot\mathbf{3_1} \oplus 3\cdot\mathbf{3_2} \oplus 3\cdot\mathbf{3_3} \oplus 3\cdot\mathbf{3_4} \\
& \oplus 10\cdot\mathbf{6_1} \oplus 6\cdot\mathbf{6_2} \oplus 10\cdot\mathbf{6_3} \oplus 6\cdot\mathbf{6_4}. \quad (14.7)
\end{aligned}
$$

Without entering a detailed discussion, here we only say that it turns out that among the symmetrized operators only $O_{\{0123\}}$, which belongs to the $\mathbf{1_2}$ representation, is multiplicatively renormalizable. Any other choice leads to some mixings, in some cases with power-divergent coefficients.

A special case is given by the operator $O_{\{0011\}} + O_{\{3322\}} - O_{\{0022\}} - O_{\{3311\}}$, which belongs to the $\mathbf{2_1}$ and in principle mixes with two other operators of mixed symmetry which have very complicated expressions, as shown in (Göckeler *et al.*, 1996a). However, at one loop this mixing is not seen and we can consider this operator to be multiplicatively renormalizable. The tadpole coming from this operator (as well as the one corresponding to $O_{\{0123\}}$) will be computed in detail in Section 18.3.

14.1.4 Higher moments

The operator for the fourth moment is $O_{\mu\nu\sigma\rho\tau} = \overline{\psi}\,\gamma_\mu\,D_\nu D_\sigma D_\rho D_\tau\,\psi$, symmetric and traceless.

We have seen that going from the first to the second and then to the third moment the mixing structure becomes more and more complicated. For the fourth moment and higher, that is for operators at least of rank five, a new feature occur: mixings which imply power-divergent coefficients becomes unavoidable, because at least two of the indices are bound to be equal. One can always find lower-dimensional operators with the same transformation properties, and it is not possible to avoid a power-divergent renormalization factor.

Furthermore, even the finite mixings become much more complicated and quite entangled, since there happen to be a lot of "copies" of the same representations around. For example, the rank-five operator has the decomposition

$$\mathbf{4_1} \otimes \mathbf{4_1} \otimes \mathbf{4_1} \otimes \mathbf{4_1} \otimes \mathbf{4_1} = 31 \cdot \mathbf{4_1} \oplus 20 \cdot \mathbf{4_2} \oplus 15 \cdot \mathbf{4_3} \oplus 20 \cdot \mathbf{4_4} \oplus 45 \cdot \mathbf{8_1} \oplus 40 \cdot \mathbf{8_2}. \tag{14.8}$$

For the higher moments it seems then rather unlikely to find an operator which does not mix and does not have power divergences.

14.2 A mixing due to breaking of chiral symmetry: $\Delta I = 1/2$ operators

We now discuss a case in which some of the operator mixings that take place are entirely due to the breaking of chiral symmetry by Wilson fermions. We show then that the calculations on the lattice become much simpler and more manageable when overlap fermions are instead used. The physics is the one of strangeness-changing weak decays, and the operators that we consider appear in the part of the $\Delta S = 1$ effective weak nonleptonic Hamiltonian which is relevant for $\Delta I = 1/2$ transitions (Gaillard and Lee, 1974; Altarelli and Maiani, 1974; Altarelli *et al.*, 1981; Buras and Weisz, 1990; Buras *et al.*, 1992). The $\Delta I = 1/2$ amplitudes, as is well known, are experimentally much greater than the $\Delta I = 3/2$ amplitudes. This phenomenon is called "octet enhancement" or "$\Delta I = 1/2$ rule", and to this day has not been theoretically understood.

The $\Delta I = 1/2$ bare operators that we consider are, for scales above the charm mass and below the bottom mass,

$$O_{\pm} = (O_1 - O_1^c) \pm (O_2 - O_2^c), \tag{14.9}$$

with [56]

$$O_1 = (\overline{s}^a \gamma_L^\mu u^b)(\overline{u}^b \gamma_L^\mu d^a) \tag{14.10}$$
$$O_2 = (\overline{s}\gamma_L^\mu u)(\overline{u}\gamma_L^\mu d) \tag{14.11}$$
$$O_1^c = (\overline{s}^a \gamma_L^\mu c^b)(\overline{c}^b \gamma_L^\mu d^a) \tag{14.12}$$
$$O_2^c = (\overline{s}\gamma_L^\mu c)(\overline{c}\gamma_L^\mu d). \tag{14.13}$$

Without entering into many details, we sketch the structure of their mixing on the lattice.

When Wilson fermions are used, these operators need in order to be renormalized the subtraction of several other operators of the same and of lower dimensionality. The pattern of mixing is as follows:

$$\widehat{O}_{\pm} = Z_{\pm}^{\mathrm{W}}\left[O_{\pm} + \sum_i C_{\pm}^{6,i} \cdot O_{\pm}^{6,i}\right. \tag{14.14}$$

[56]When color indices are not shown they are trivially contracted, i.e., the operators are color singlets.

$$+(m_c - m_u)\, C_{\pm}^5 \cdot \mathrm{i}\overline{s}\sigma_{\mu\nu}F^{\mu\nu}d + a(m_c - m_u)(m_d - m_s)\, \widetilde{C}_{\pm}^5 \cdot \mathrm{i}\overline{s}\sigma_{\mu\nu}\widetilde{F}^{\mu\nu}d$$

$$+ \frac{1}{a^2}(m_c - m_u)\, C_{\pm}^3 \cdot \overline{s}d + \frac{1}{a}(m_c - m_u)(m_d - m_s)\, \widetilde{C}_{\pm}^3 \cdot \overline{s}\gamma_5 d \Bigg] + O(a^2).$$

All dimension-6 operators $O_{\pm}^{6,i}$ have opposite chirality with respect to $O_{\pm}$. The precise form of them, which is given in (Martinelli, 1984), will not interest us here. [57]

A remarkable thing is that the coefficients of the mixings with the dimension-5 operators, which could in principle diverge like $1/a$, are instead, thanks to the Glashow-Iliopoulos-Maiani (GIM) mechanism, finite. This happens because the GIM mechanism says that this mixing should be zero for $m_c = m_u$, and then a mass factor $m_c - m_u$ takes the place in which a factor $1/a$ would otherwise be.

The coefficients of the mixings with the dimension-3 operators could in principle diverge like $1/a^3$. The GIM mechanism and, for the parity-violating operator, also a factor $m_d - m_s$ due to the CPS symmetry (which combines C, P and the exchange of the s and d quarks (Bernard *et al.*, 1985)), renders these power divergences less severe. Still, the fact that these mixings remain power divergent makes impossible to calculate the full renormalization of the $\Delta I = 1/2$ operators using perturbation theory. What can be computed in perturbation theory is only the overall renormalization Z^W, which is logarithmically divergent, the coefficients $C_{\pm}^{6,i}$, and the coefficients $C_{\pm}^5$ and $\widetilde{C}_{\pm}^5$, which however require a two-loop calculation, which has been done in (Curci *et al.*, 1988) for $C_{\pm}^5$.

Let us now see what happens when fermions which respect chiral symmetry are used. The renormalization is in this case given by: [58]

$$\widehat{O}_{\pm} = Z_{\pm}^{ov}\Bigg[O_{\pm} + (m_c^2 - m_u^2)\, C_{\pm}^m \cdot \Big((m_d + m_s)\,\overline{s}d + (m_d - m_s)\,\overline{s}\gamma_5 d\Big)\Bigg] + O(a^2). \qquad (14.15)$$

We can see that chiral symmetry has brought a big change in the pattern of subtractions.

First of all, chiral symmetry forbids in a direct way any mixings with the other dimension-6 operators, $O_{\pm}^{6,i}$, which are of opposite chirality. Furthermore, the GIM mechanism from linear becomes now, when combined with chiral symmetry, quadratic, and thus it gives coefficients proportional to $m_c^2 - m_u^2$, like in the continuum. This mass factor counts for *two* powers of $1/a$. Finally, the mixing coefficients with parity-conserving and parity-violating operators are

---

[57]The study of weak operators on the lattice has a long history, and several perturbative calculations have been done using Wilson fermions (Cabibbo, Martinelli and Petronzio, 1984; Martinelli, 1984; Maiani *et al.*, 1987; Bernard, Soni and Draper, 1987). Recent results for four-fermion operators can be found in (Gupta, Bhattacharya and Sharpe, 1997). Four-fermion operators are also useful for other problems, like the renormalization of higher-twist operators in deep inelastic scattering, and in this case they have a different color, spin and flavor structure (Capitani *et al.*, 1999a; Capitani *et al.*, 2000a; Capitani *et al.*, 2000b; Capitani *et al.*, 2001a).

[58]We should mention that to construct overlap operators which have the right chiral properties γ_L^{μ} has to be replaced by $\gamma_L^{\mu}(1 - \frac{a}{2\rho}D)$, and similarly for the other bilinears, so that for example $\overline{s}d$ becomes $\overline{s}(1 - \frac{a}{2\rho}D)d$.

now the same. The parity-conserving operators then acquire an additional factor $(m_d + m_s)$ which mirrors the factor $(m_d - m_s)$ coming from the CPS symmetry for the parity-violating operators. This eats away another factor of a.

As a result, the mixings with the dimension-5 magnetic operators, which were finite in Wilson, become now of order a^2, and hence one does not have to take into account these mixings at all, even in the improved theory. What is remarkable is that after all these new mass factors are inserted the mixings with the dimension-3 operators $\overline{s}d$ and $\overline{s}\gamma_5 d$ (which were power divergent in Wilson) become finite. Thus, the renormalization of the $\Delta I = 1/2$ matrix elements can now be carried out entirely with perturbative methods, because there are no power-divergent coefficients when one uses overlap fermions (Capitani and Giusti, 2001).

15   Analytic computations

Analytic computations of Feynman diagrams in lattice QCD present quite a few new and interesting features with respect to the continuum. Of course standard rules like a minus sign for each fermionic loop, which derive from the general properties of the path integral and the Wick theorem, continue to be valid on the lattice. The combinatorial rules are also similar to the continuum. But there are a few technicalities, many of them connected to the breaking of Lorentz invariance, which the reader should be aware of. We will discuss many of them in this Section. We will first introduce the power counting theorem on the lattice and see how divergent integrals can be treated. We will then show in detail the calculation of a matrix element at one loop, using Wilson fermions. A few comments about calculations with overlap fermions and with fat links will be also made.

15.1   The power counting theorem of Reisz

On the lattice the functions to be integrated are periodic (with period $2\pi/a$), and a power counting theorem which is appropriate for this kind of integrals, and which accounts for their properties in the the continuum limit, has been established by Reisz (1988a; 1988b; 1988c; 1988d). This power counting theorem, like the one in the continuum (Hahn and Zimmermann, 1968), is very useful for the treatment of divergent integrals, and is fundamental for proving the renormalizability of lattice gauge theories.

We will follow the presentation of (Lüscher, 1990), where somewhat milder conditions are required than in the original papers. Let us then consider a generic lattice integral at L loops, which will have the general form

$$I = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k_1}{(2\pi)^4} \cdots \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k_L}{(2\pi)^4} \frac{V(k, q; m, a)}{C(k, q; m, a)}, \tag{15.1}$$

where q_i ($i = 1, \ldots, E$) are the external momenta and m stands for the masses of the theory. The numerator V contains all vertices and the numerators of the various propagators, while

109

the denominator C is the product of the denominators of these propagators. This overall denominator is assumed to have the structure

$$C(k, q; m, a) = \prod_{i=1}^{I} C_i(l_i; m, a), \tag{15.2}$$

where I is the number of internal lines of the diagram, and the line momenta $l_i(k, q)$ carried by them are linear combinations of the integration variables k_j and the external momenta q_j. For the power counting theorem to be valid, a few conditions have to be satisfied by the numerator V, the denominators C_i and the line momenta l_i. These conditions can be stated as follows.

(V1) There exists an integer ω and a smooth function F such that

$$V(k, q; m, a) = a^{-\omega} F(ak, aq; am), \tag{15.3}$$

and F is periodic in ak_i and a polynomial in am.

(V2) The continuum limit of the numerator,

$$P(k, q; m) = \lim_{a \to 0} V(k, q; m, a), \tag{15.4}$$

exists.

(C1) There exist smooth functions G_i satisfying

$$C_i(l_i; m, a) = a^{-2} G_i(al_i; am), \tag{15.5}$$

and the G_i's are periodic in al_i and polynomials in am.

(C2) The continuum limit of all C_i's exists, and is given by

$$\lim_{a \to 0} C_i(l_i; m, a) = l_i^2 + m_i^2, \tag{15.6}$$

where the positive masses m_i are combinations of the original masses m.

(C3) There exist positive constants a_0 and A such that

$$|C_i(l_i; m, a)| \geq A(\hat{l}_i^2 + m_i^2) \tag{15.7}$$

for all $a \leq a_0$ and all l_i's.

(L1) All line momenta satisfy

$$l_i(k, q) = \sum_{j=1}^{L} a_{ij} k_j + \sum_{l=1}^{E} b_{il} q_l, \tag{15.8}$$

for a_{ij} integer and b_{il} real.

(L2) Given the linear combinations

$$p_i(k) = \sum_{j=1}^{L} a_{ij} k_j \tag{15.9}$$

and the associated set

$$\mathcal{L} = \{k_1, \ldots, k_L, p_1, \ldots, p_I\}, \tag{15.10}$$

and considering $u_1, \ldots, u_L$ linearly independent elements of $\mathcal{L}$, then

$$k_i = \sum_{j=1}^{L} c_{ij} u_j \tag{15.11}$$

holds, with c_{ij} integer.

The conditions L1 and L2 define a "natural" choice of line momenta. It is important that the coefficients a_{ij} in L1 and c_{ij} in L2 are integers. These conditions guarantee that shifting integration variables by $2\pi/a$ and choosing some of the line momenta as new integration variables still gives a periodic integrand and does not change the domain of integration.

The condition C3 is one of the most significant. While the other conditions are rather weak, and are fulfilled by any reasonable theory, this one is strongly discriminating against certain type of integrands. Condition C3 is in fact satisfied by scalars as well as by Wilson and overlap fermions, but not by naive fermions and staggered fermions. Essentially this condition asks that the denominators C_i diverge like $1/a^2$ when the momenta l_i are at the edges of the Brillouin zone, which is sufficient to forbid any doublers in that region.

We need now a definition of the degree of divergence of an integrand. The degree of divergence of the numerator is defined from its asymptotic behavior,

$$V(\lambda k, q; m, a\lambda) \stackrel{\lambda \to \infty}{=} K\lambda^{\deg V} + O(\lambda^{\deg V - 1}), \tag{15.12}$$

where $K \neq 0$, and similarly for the degree of divergence of the denominator, $\deg C$. The lattice degree of divergence takes into account the behavior of the integrand functions not only for small lattice spacing, but also for large loop momenta $k \sim 1/a$. The degree of divergence of the integral I is then given (in four dimensions) by

$$\deg I = 4 + \deg V - \deg C. \tag{15.13}$$

Finally, for integrals beyond one loop we need to introduce the notion of Zimmermann subspaces, which are linear subspaces of the momenta. Let us consider L linear independent elements of the set of momenta $\mathcal{L}$ defined in condition L2,

$$u_1, \ldots, u_d, v_1, \ldots, v_{L-d}, \qquad (d \geq 1), \tag{15.14}$$

and take them as new integration variables. If we now fix $v_1, \ldots, v_{L-d}$ to some value, we obtain a 4d-dimensional Zimmermann subspace, spanned by $u_1, \ldots, u_d$. [59] The degree of divergence for V in this Zimmermann subspace is then defined as

$$V(k(\lambda u, v), q; m, a\lambda) \overset{\lambda \to \infty}{=} K\lambda^{\deg_Z V} + O(\lambda^{\deg_Z V - 1}), \tag{15.15}$$

where $K \neq 0$, and similarly for C_i. This allows to study the behavior of the integrand when only some of the momenta are large.

The theorem of Reisz says that the continuum limit of the integral I in Eq. (15.1) exists if $\deg_Z I < 0$ for all its possible Zimmermann subspaces Z, and in this case is given by integrating the naive continuum limit of the integrand (as in conditions V2 and C2):

$$\lim_{a \to 0} I = \int_{-\infty}^{\infty} \frac{d^4 k_1}{(2\pi)^4} \cdots \int_{-\infty}^{\infty} \frac{d^4 k_L}{(2\pi)^4} \frac{P(k, q; m)}{\prod_{i=1}^{I}(l_i^2 + m_i^2)}. \tag{15.16}$$

Thus, in this case we have reduced the initial problem to the computation of a simpler continuum integral, which is absolutely convergent. The theorem can also be formulated in the case in which massless propagators are present, but then one has also to introduce infrared degrees of divergence (Reisz, 1988b; 1988d).

The proof of the power counting theorem is rather complicated and goes beyond the scope of this review. We refer the interested reader to the original papers (Reisz, 1988a; 1988b; 1988c; 1988d).

We now discuss a couple of examples which illustrate the meaning of the theorem of Reisz. Let us consider the one-dimensional integral

$$I(N) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk}{2\pi} \frac{1}{\frac{N^2}{a^2} \sin^2 \frac{ak}{N} + m^2}, \tag{15.17}$$

which when $N = 1$ describes naive fermions and when $N = 2$ describes scalar particles. Doing *naively* the limit $a \to 0$ of the integrand and of the integration region gives

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{k^2 + m^2} = \frac{1}{2m}, \tag{15.18}$$

which is independent of N. The *true* value of the integral at finite a can be computed using the Schwinger representation

$$\frac{1}{x^2 + m^2} = \int_0^{\infty} d\alpha\, e^{-\alpha(x^2 + m^2)}, \tag{15.19}$$

so that it becomes

$$I(N) = \frac{2a}{N^2} \int_0^{\infty} dy\, e^{-\left(1 + \frac{2a^2 m^2}{N^2}\right)y} I_0(y), \tag{15.20}$$

[59]One does not distinguish between subspaces corresponding to different values of the fixed momenta $v_1, \ldots, v_{L-d}$.

where I_0 is a modified Bessel function, which for large y behaves as

$$I_0(y) \xrightarrow{y \to \infty} \frac{1}{\sqrt{2\pi y}} \, e^y. \tag{15.21}$$

We can now compute the limit $a \to 0$ of $I(N)$ by replacing in the integrand the modified Bessel function with its asymptotic expression, and using $\int_0^\infty e^{-bx} dx/\sqrt{x} = \sqrt{\pi/a}$ one obtains

$$I(N) \xrightarrow{a \to 0} \frac{1}{Nm}, \tag{15.22}$$

which is the correct continuum limit of $I(N)$. We can see that it depends on N. For $N = 2$ this result is the same as the naive continuum limit (15.18), whereas for $N = 1$ the naive continuum limit gives only half of the true value (15.22). This mismatch corresponds to a case in which the Reisz theorem cannot be applied, because the propagator of naive fermions ($N = 1$) does not satisfy condition C3. In fact, this propagator has a doubler, and the true value of the integral in the continuum limit is then precisely twice the result which one would obtain just doing the naive continuum limit.

In the above example all integrals have a negative degree of divergence. The integral

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \frac{1 - \cos ak_\mu}{\frac{4}{a^2} \sum_\lambda \sin^2 \frac{ak_\lambda}{2} + m^2}, \tag{15.23}$$

instead, is divergent like $1/a^2$ in the continuum limit, as can be seen by dimensional counting using the rescaled variable $k' = ka$. Therefore, the theorem of Reisz cannot be used, and in fact a naive continuum limit of the integrand gives a zero result, which is incorrect.

The Reisz power counting theorem is quite useful for the calculation of lattice integrals, especially when they are divergent, as we will see in the next Section. It is also of great help for the proof of the renormalizability of lattice theories at all orders of perturbation theory, which has been given for pure nonabelian gauge theories by (Reisz, 1989).

15.2 Divergent integrals

For the treatment of divergent integrals on the lattice it is convenient to use a method which was introduced in (Kawai, Nakayama and Seo, 1981). It consists in making an expansion of these integrals in powers of the external momenta, and computing on the lattice only the integrals with vanishing momentum, which are technically much simpler. As an example we consider the case of a quadratically divergent integral depending on two external momenta p and q,

$$I = \int dk \, \mathcal{I}(k, p, q). \tag{15.24}$$

This integral can be split as

$$I = J + (I - J), \tag{15.25}$$

where

$$
\begin{aligned}
J \quad &= \int \mathrm{d}k \; \mathcal{I}(k,0,0) + \sum_{\rho,\sigma} \left[p_\rho q_\sigma \int \mathrm{d}k \; \frac{\partial^2 \mathcal{I}(k,p,q)}{\partial p_\rho \partial q_\sigma} \bigg|_{p=q=0} \right. \\
&\left. + \frac{p_\rho p_\sigma}{2} \int \mathrm{d}k \; \frac{\partial^2 \mathcal{I}(k,p,0)}{\partial p_\rho \partial p_\sigma} \bigg|_{p=0} + \frac{q_\rho q_\sigma}{2} \int \mathrm{d}k \; \frac{\partial^2 \mathcal{I}(k,0,q)}{\partial q_\rho \partial q_\sigma} \bigg|_{q=0} \right]
\end{aligned}
\tag{15.26}
$$

is the Taylor expansion of the original integral to second order. The integrals appearing in J do not depend on the external momenta and are thus much easier to calculate on the lattice. The whole dependence on the external momenta remains in $I - J$ which, because of the subtraction, is ultraviolet-finite for $a \to 0$ and can thus be computed, according to the theorem of Reisz, just by taking the naive continuum limit. Thanks to this fact, only zero-momentum integrals have to be evaluated on the lattice.

Notice that for $p, q \neq 0$ and finite lattice spacing I is well defined, but J and $I - J$ are infrared divergent. To compute J and $I - J$ separately, one must then introduce an intermediate regularization. The associated divergences will at the end cancel out in the sum $J + (I - J)$. This intermediate regularization is completely independent from the main regularization used in the lattice theory, and in particular can be different from it. It just comes out because the splitting is somewhat unnatural.

To give an explicit illustration of this method, let us take the logarithmically divergent integral

$$
I = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \frac{1}{\left(\frac{4}{a^2} \sum_\mu \sin^2 \frac{a(k+p)_\mu}{2} \right)^2}.
\tag{15.27}
$$

The splitting is then made as follows:

$$
J \;=\; I(p=0) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \frac{1}{\left(\frac{4}{a^2} \sum_\mu \sin^2 \frac{ak_\mu}{2} \right)^2},
\tag{15.28}
$$

$$
\begin{aligned}
I - J \;&=\; \lim_{a \to 0} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \left\{ \frac{1}{\left(\frac{4}{a^2} \sum_\mu \sin^2 \frac{a(k+p)_\mu}{2} \right)^2} - \frac{1}{\left(\frac{4}{a^2} \sum_\mu \sin^2 \frac{ak_\mu}{2} \right)^2} \right\} \\
&=\; \int_{-\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} \left\{ \frac{1}{((k+p)^2)^2} - \frac{1}{(k^2)^2} \right\}.
\end{aligned}
\tag{15.29}
$$

Taking common denominators, it is easy to see that the degree of divergence of the above integral is negative, and therefore it can be safely computed in the continuum.

If we use dimensional regularization we have the result [60]

$$
J \;=\; \frac{1}{16\pi^2} \left(\frac{2}{d-4} - \log 4\pi + F_0 \right),
\tag{15.30}
$$

---

[60]See Eq. (18.30) later. Notice that the integral of the second term in $I - J$ is zero in this regularization.

$$I - J = \frac{1}{16\pi^2}\Big(-\frac{2}{d-4} - \log a^2 p^2 + \log 4\pi - \gamma_E\Big), \tag{15.31}$$

where $\gamma_E = 0.57721566490153286\ldots$ and the lattice constant is $F_0 = 4.369225233874758\ldots$ (see Eqs. (18.25) and (18.26) and Table 2 later), while if we regularize adding a small mass term m^2 to k^2 in the denominators we obtain [61]

$$J_m = \frac{1}{16\pi^2}\Big(-\log a^2 m^2 - \gamma_E + F_0\Big), \tag{15.32}$$

$$(I - J)_m = \frac{1}{16\pi^2}\Big(\log a^2 m^2 - \log a^2 p^2\Big). \tag{15.33}$$

In both cases, adding up J and $I - J$ we obtain for the original integral the result

$$I = -\log a^2 p^2 - \gamma_E + F_0. \tag{15.34}$$

To summarize, for the computation of any divergent integral which depends on external momenta it is sufficient to compute some lattice integrals at zero momenta and some continuum integrals.

In computer programs, a convenient way to deal with a generic divergent integral (which has to be processed in an automated way) is to subtract to it a simple integral with the same divergent behavior for which the numerical value is exactly known. The difference is then finite and can be computed with reasonable precision using simple integration routines. This is extremely convenient in the case of actions which give rise to complicated denominators, like for example overlap fermions. In this case, Wilson integrals with the same divergence are subtracted to the original overlap integral, and then overlap denominators, which are much more complicated, appear only in the numerical calculation of finite integrals. The calculation of divergent integrals is made only using Wilson fermions.

Of course, this is not the only available method for computing divergent integrals. In Section 19.2.1 we will show another technique based on the coordinate space method.

15.3   General aspects of the calculations

We have seen that the Feynman rules on the lattice are rather different from the continuum ones. The structure of lattice integrals is also completely different. The integrands are periodic in the momenta, and the basic objects are trigonometric functions and not simple polynomials of the momenta. Many standard methods which are very useful in continuum perturbation theory, like Feynman parameterization and partial integration, are then not of much relevance for perturbative lattice calculations. [62]

We have thought of doing a useful thing by showing a complete lattice calculation which illustrates the peculiar aspects of lattice perturbation theory.

---

[61]See Eq. (18.29) later.

[62]We mention that recently a computational method has been presented (Becher and Melnikov, 2002) that through a change of variables ($t = \tan k/2$) transforms lattice integrals in continuum-like integrals and employs known techniques of continuum calculations which use asymptotic expansions.

15.4 Example (Wilson): the first moment of the quark momentum distribution

We describe in this Section, as a pedagogical example, a typical lattice perturbative calculation. We explain in detail the main steps of the computation, with the Wilson action, of the renormalization constant of the forward matrix element $\langle q|O_{\{\mu\nu\}}|q\rangle$ on one-quark states of the operator

$$O_{\{\mu\nu\}} = \overline{\psi}\gamma_{\{\mu}D_{\nu\}}\frac{\lambda^f}{2}\psi, \qquad \mu \neq \nu. \tag{15.35}$$

This operator, which is symmetrized in the indices μ and ν, measures the first moment of the fraction of the momentum of the proton carried by the quarks. The λ's are flavor matrices, which means that we are considering a flavor nonsinglet operator. The corresponding singlet operator (proportional to the identity flavor matrix) mixes, when radiative corrections are included, with the gluon operator $\sum_\rho \mathrm{Tr}\,(F_{\mu\rho}F_{\rho\nu})$, which measures the first moment of the momentum distribution of the gluon, and to make things not too complicated we will not consider it here.

This example is rather simple (compared to other operators) and contains all the main interesting features one can think of: a logarithmic divergence, a covariant derivative, symmetrized indices and of course the special use of Kronecker deltas in lattice perturbative calculations. Moreover, it is an example of a calculation which needs an expansion of the various propagators and vertices in the lattice spacing a (in this case, at first order). [63]

For what concerns the value of the renormalization constant, the flavor matrices are not important (that is, except for the fact that they forbid the mixing with gluonic operators), and we will then carry out the explicit computations using the operator

$$\frac{1}{2}\big(\overline{\psi}\gamma_\mu D_\nu\psi + \overline{\psi}\gamma_\nu D_\mu\psi\big), \tag{15.36}$$

where $\mu \neq \nu$ is chosen, so that this operator belongs to the representation $\mathbf{6_1}$ of the hypercubic group (see Section 13). The choice $\mu = \nu$ would compel us to consider the operator $O_{\{00\}} - \frac{1}{3}(O_{\{11\}} + O_{\{22\}} + O_{\{33\}})$, which belongs to the representation $\mathbf{3_1}$ and has a different lattice renormalization constant. This would render the calculations a bit more cumbersome, without teaching us much new.

The covariant derivative is defined as follows:

$$D = \overset{\leftrightarrow}{D} = \frac{1}{2}\big(\vec{D} - \overset{\leftarrow}{D}\big), \tag{15.37}$$

and the discretization of it that we use for this calculation is

$$\vec{D}_\mu \psi(x) = \frac{1}{2a}\Big[U_\mu(x)\psi(x + a\hat{\mu}) - U_\mu^\dagger(x - a\hat{\mu})\psi(x - a\hat{\mu})\Big], \tag{15.38}$$

---

[63] A simpler calculation with no covariant derivatives and no need of expansions in a, that the reader could try as a warm-up, is given by the renormalization of quark currents. This was first computed in (Martinelli and Zhang, 1983a), and for the case of extended currents, which are defined on more than one lattice site like the conserved vector current in Eq. (5.84), in (Martinelli and Zhang, 1983b).
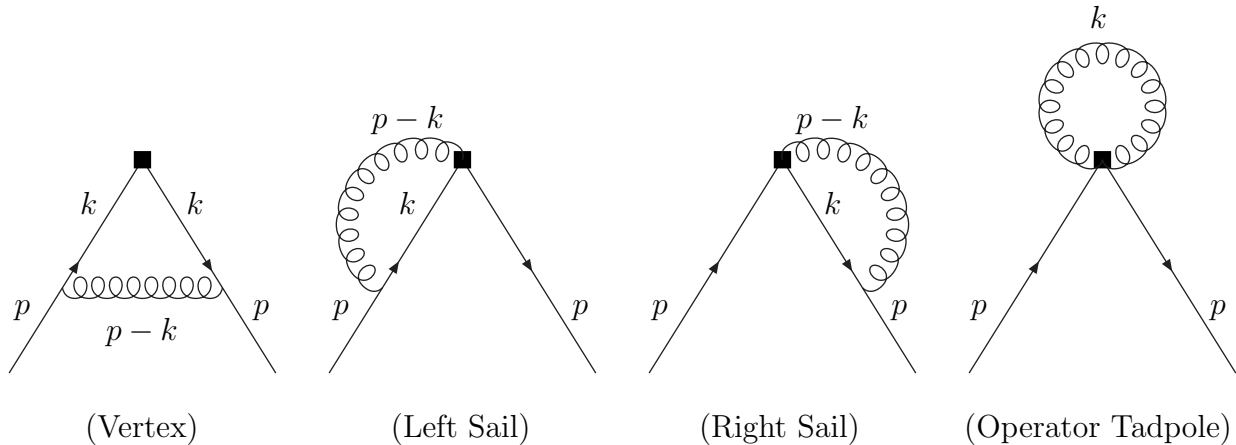
(Vertex)   (Left Sail)   (Right Sail)   (Operator Tadpole)

Figure 17: "Proper" diagrams for the 1-loop correction of the matrix element $\langle q|\overline{\psi}\gamma_{\{\mu}D_{\nu\}}\psi|q\rangle$. The black squares indicates the insertion of the operator. Shown is also the choice of momenta used in the calculations.

$$\overline{\psi}(x)\overleftarrow{D}_\mu \;\; = \;\; \frac{1}{2a}\Big[\overline{\psi}(x+a\hat{\mu})U_\mu^\dagger(x) - \overline{\psi}(x-a\hat{\mu})U_\mu(x-a\hat{\mu})\Big].$$

We consider amputated Green's functions, that is the external propagators are removed. The tree level of the amputated forward quark matrix element of the operator above is easily seen to be

$$\langle q|O_{\{\mu\nu\}}|q\rangle\Big|_{tree} = \frac{1}{2}i(\gamma_\mu p_\nu + \gamma_\nu p_\mu), \tag{15.39}$$

and the 1-loop QCD result has, as we will see from the calculation, the form

$$\langle q|O_{\{\mu\nu\}}|q\rangle\Big|_{1\ loop} = \frac{1}{2}i(\gamma_\mu p_\nu + \gamma_\nu p_\mu)\cdot\frac{g_0^2}{16\pi^2}C_F\Big(c_1\log a^2p^2 + c_2\Big), \tag{15.40}$$

i.e., it is proportional to the tree level and this operator is thus multiplicatively renormalized. The renormalization constant for the matching to the $\overline{\text{MS}}$ scheme can then be read off from the above 1-loop result plus the corresponding continuum calculations made in the $\overline{\text{MS}}$ scheme (see Eq. (3.3) and Section 3). For the computation of the lattice part it is necessary to evaluate six Feynman diagrams, which are given in Figs. 17 and 18. The two diagrams in Fig. 18 compute the quark self-energy, and give the renormalization of the wave function. The remaining four diagrams, in Fig. 17, are specific to the operator considered, and we will call them "proper" diagrams.

15.4.1 Preliminaries

We work in Feynman gauge ($\alpha = 1$), where the form of the gluon propagator is simpler, and we set $r = 1$. We perform the calculations using massless fermions. [64] This is the simplest

---

[64]Calculations in which the quark propagator is massive are more complicated. A few examples of these calculations, which use simpler operators, can be found in (Kronfeld and Mertens, 1984; El-Khadra, Kronfeld
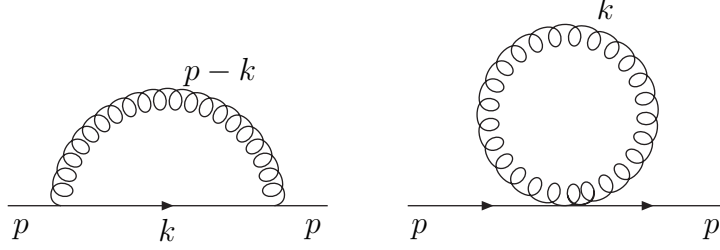
Figure 18: Diagrams for the quark self-energy. On the left the sunset diagram, on the right the tadpole diagram.

situation one can think of, although it already leads to complicated manipulations, as we will shortly see. We carry out these manipulations starting from the operator

$$O_{\mu\nu} = \overline{\psi}\gamma_\mu D_\nu \psi, \tag{15.41}$$

and implement the symmetrization in μ and ν at a later stage. Due to the presence of the link variable U in the covariant derivative, this operator has an expansion in the coupling,

$$O_{\mu\nu} = O_{\mu\nu}^{(0)} + g_0 O_{\mu\nu}^{(1)} + g_0^2 O_{\mu\nu}^{(2)} + O(g_0^3). \tag{15.42}$$

To evaluate the one-loop Feynman diagrams in momentum space one has to compute the Fourier transforms of the operators in this expansion including the term of $O(g_0^2)$. It turns out that to work out these momentum-space insertions for our forward matrix element we can use the operator defined with the right derivative only, instead of the one involving the difference between the right and the left derivative (which would lead to more complicated manipulations). We have then that the expansion of $a^4 \sum_x \left(\overline{\psi}\gamma_\mu \vec{D}_\nu \psi \right)(x)$ is

$$a^4 \frac{1}{2a} \sum_x \left(\overline{\psi}(x)\gamma_\mu U_\nu(x)\psi(x+a\hat{\nu}) - \overline{\psi}(x)\gamma_\mu U_\nu^\dagger(x-a\hat{\nu})\psi(x-a\hat{\nu}) \right)$$

$$= a^4 \left\{ \frac{1}{2a} \sum_x \left(\overline{\psi}(x)\gamma_\mu\psi(x+a\hat{\nu}) - \overline{\psi}(x)\gamma_\mu\psi(x-a\hat{\nu}) \right) \right. \tag{15.43}$$

$$+ \frac{1}{2}ig_0 T^a \sum_x \left(\overline{\psi}(x)\gamma_\mu A_\nu^a(x)\psi(x+a\hat{\nu}) + \overline{\psi}(x)\gamma_\mu A_\nu^a(x-a\hat{\nu})\psi(x-a\hat{\nu}) \right)$$

$$- \frac{1}{4}ag_0^2 T^a T^b \sum_x \left(\overline{\psi}(x)\gamma_\mu A_\nu^a(x)A_\nu^b(x)\psi(x+a\hat{\nu}) - \overline{\psi}(x)\gamma_\mu A_\nu^a(x-a\hat{\nu})A_\nu^b(x-a\hat{\nu})\psi(x-a\hat{\nu}) \right)$$

$$\left. + O(a^2 g_0^3) \right\}.$$

and Mackenzie, 1997; Mertens, Kronfeld and El-Khadra, 1998) and (Kuramashi, 1998).

The Fourier transform of the lowest order is

$$a^4 \frac{1}{2a} \sum_x \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k'}{(2\pi)^4} \overline{\psi}(k')\gamma_\mu \psi(k) e^{-ik'x}\left(e^{ik(x+a\hat{\nu})} - e^{ik(x-a\hat{\nu})}\right) \quad (15.44)$$

$$= \frac{1}{2a} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \overline{\psi}(k)\gamma_\mu \psi(k)\left(e^{iak_\nu} - e^{-iak_\nu}\right) \quad (15.45)$$

$$= \frac{i}{a} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \overline{\psi}(k)\gamma_\mu \psi(k) \sin ak_\nu, \quad (15.46)$$

where we have used $a^4 \sum_x e^{-ik'x}e^{ikx} = (2\pi)^4 \delta^{(4)}(k - k')$. Similar delta functions arise at each order of the expansion, expressing the conservation of momentum at the various vertices. The first order term in g_0 is

$$a^4 \frac{1}{2} i g_0 T^a \sum_x \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4q}{(2\pi)^4} \overline{\psi}(p)\gamma_\mu \psi(k)A_\nu^a(q)$$
$$\times e^{-ipx}e^{iqx}e^{ikx}\left(e^{iq_\nu a/2}e^{ik_\nu a} + e^{-iq_\nu a/2}e^{-ik_\nu a}\right)$$

$$= \frac{1}{2} i g_0 T^a \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \overline{\psi}(p)\gamma_\mu \psi(k)A_\nu^a(p-k)\left(e^{i(p-k)_\nu a/2}e^{ik_\nu a} + e^{-i(p-k)_\nu a/2}e^{-ik_\nu a}\right)$$

$$= i g_0 T^a \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \overline{\psi}(p)\gamma_\mu \psi(k)A_\nu^a(p-k) \cos \frac{a(k+p)_\nu}{2}. \quad (15.47)$$

With our convention for the Fourier transform of the gauge fields the gluons are always entering the vertices. The calculation above corresponds to the left sail in Fig. 17, and the reader can check that the calculation for the right sail gives the same function:

$$i g_0 T^a \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \overline{\psi}(k)\gamma_\mu \psi(p)A_\nu^a(k-p) \cos \frac{a(k+p)_\nu}{2}. \quad (15.48)$$

Finally, since the only second-order contribution in which we are interested is the operator tadpole, we exploit the simplifications which this situation brings. In particular, since the gluon is emitted and reabsorbed at the same vertex, there is a Kronecker delta in color space coming from the gluon propagator and the color factor becomes $\sum_a (T^a)_{bb}^2 = (N_c^2 - 1)/(2N_c) = C_F$, the quadratic Casimir invariant of $SU(N_c)$. We have then that the insertion of the operator tadpole is

$$-\frac{1}{4} a g_0^2 C_F \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \overline{\psi}(p)\gamma_\mu \psi(p)A_\nu^a(k)A_\nu^a(k)\left(e^{ip_\nu a} - e^{-ip_\nu a}\right)$$

$$= -\frac{1}{2} i a g_0^2 C_F \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \overline{\psi}(p)\gamma_\mu \psi(p)A_\nu^a(k)A_\nu^a(k) \sin ap_\nu, \quad (15.49)$$

where now the color index a is not summed. Note that the factors $\exp(\pm iak_\nu/2)$ coming from the gluons have canceled, again because it is the same gluon that is emitted and absorbed at the vertex. This insertion does not depend on the momentum of the gluon.
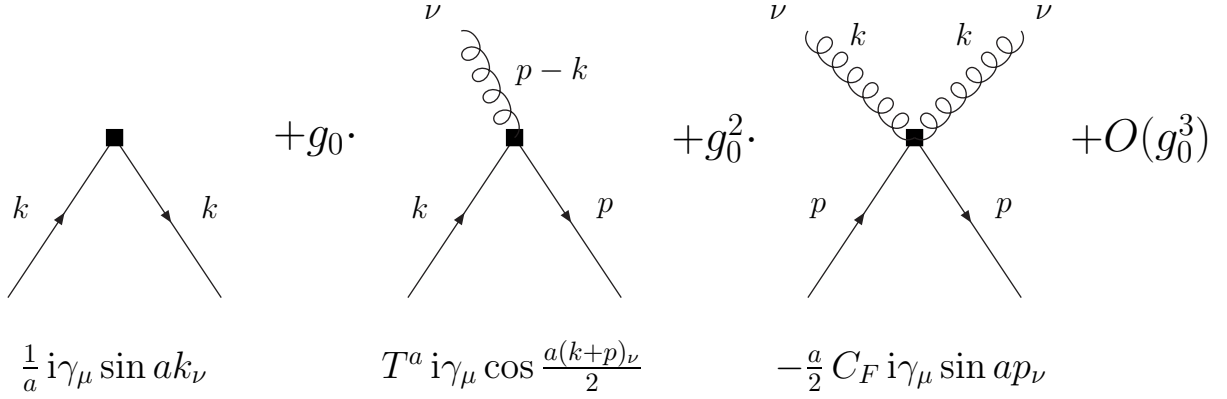
$$+g_0 \cdot$$

$$+g_0^2 \cdot$$

$$+O(g_0^3)$$

$$\frac{1}{a}\,\mathrm{i}\gamma_\mu \sin ak_\nu$$

$$T^a\,\mathrm{i}\gamma_\mu \cos \frac{a(k+p)_\nu}{2}$$

$$-\frac{a}{2}\,C_F\,\mathrm{i}\gamma_\mu \sin ap_\nu$$

Figure 19: Operator insertions for $\overline{\psi}\gamma_{\{\mu}D_{\nu\}}\psi$. Please note that in the second-order term the color factors have been already worked out as they occur in the tadpole.

We have thus obtained, in the momentum choice of Fig. 17, the operator insertions

$$O^{(0)}_{\mu\nu}(k) \;=\; \frac{1}{a}\,\mathrm{i}\gamma_\mu \sin ak_\nu, \tag{15.50}$$

$$O^{(1)}_{\mu\nu}(k,p) \;=\; T^a\,\mathrm{i}\gamma_\mu \cos \frac{a(k+p)_\nu}{2}, \tag{15.51}$$

$$O^{(2)}_{\mu\nu}(p) \;=\; -\frac{a}{2}\,C_F\,\mathrm{i}\gamma_\mu \sin ap_\nu. \tag{15.52}$$

These operator insertions are shown in Fig. 19. They generate the vertex, sails and operator tadpole respectively (see Fig. 17). The gluons have the same index as the trigonometric functions.

In the intermediate stages of the calculations many terms are present. A lot of them originate from the Taylor expansions of the denominators of the propagators. With our choice of momenta [65] only the gluon propagator has to be expanded in the lattice spacing, whereas the quark propagator does not contain any external momenta, and does not need to be expanded. We have found that this leads to less complicated manipulations than the other choice in which the quark propagators have momentum $p - k$ and the gluon propagator has momentum k.

15.4.2 Vertex

The first diagram that we consider, and for which we will give a rather detailed explanation of how is computed, is the vertex function in Fig. 17. Since this Feynman diagram is divergent, it is convenient to split its computation into two parts, as explained in Section 15.2: the integral at zero momentum, J, and the rest of the integral, $I - J$, which can be computed in the

---

[65]We always have that the external momentum is p, the quark propagator has momentum k and the gluon propagator has momentum $p - k$ (except in the tadpoles).

continuum thanks to the theorem of Reisz. We use dimensional regularization as regulator of the intermediate divergences, with $d = 4 - 2\epsilon$.

We have then [66]

$$J = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d k}{(2\pi)^d} \sum_{\rho, \lambda} G_{\rho\lambda}(p - k) \cdot V_\rho(k, p) \cdot S(k) \cdot O^{(0)}_{\mu\nu}(k) \cdot S(k) \cdot V_\lambda(p, k) \bigg|_{ap=0}, \quad (15.53)$$

where (in Feynman gauge)

$$S^{ab}(k) = a\,\delta^{ab} \cdot \frac{-i \sum_\lambda \gamma_\lambda \sin ak_\lambda + 2 \sum_\lambda \sin^2 \frac{ak_\lambda}{2}}{\sum_\lambda \sin^2 ak_\lambda + 4\left(\sum_\lambda \sin^2 \frac{ak_\lambda}{2}\right)^2}, \quad (15.54)$$

$$(V^a)^{bc}_\rho(k, p) = (V^a)^{bc}_\rho(p, k) = -g_0 (T^a)^{bc} \left(\sin \frac{a(k+p)_\rho}{2} + i\gamma_\rho \cos \frac{a(k+p)_\rho}{2} \right) \quad (15.55)$$

$$= -g_0 (T^a)^{bc} \left(\sin \frac{ak_\rho}{2} + i\gamma_\rho \cos \frac{ak_\rho}{2} + \frac{ap_\rho}{2} \left(\cos \frac{ak_\rho}{2} - i\gamma_\rho \sin \frac{ak_\rho}{2} \right) \right)$$

$$+ O(a^2),$$

$$G^{ab}_{\rho\lambda}(p - k) = \delta^{ab} \delta_{\rho\lambda} \cdot \frac{1}{\frac{4}{a^2} \sum_\lambda \sin^2 \frac{a(p-k)_\lambda}{2}} \quad (15.56)$$

$$= \delta^{ab} \delta_{\rho\lambda} \cdot a^2 \left\{ \frac{1}{4 \sum_\lambda \sin^2 \frac{ak_\lambda}{2}} + a \frac{2 \sum_\lambda p_\lambda \sin ak_\lambda}{\left(4 \sum_\lambda \sin^2 \frac{ak_\lambda}{2}\right)^2} + O(a^2) \right\}.$$

Putting everything together we have

$$J = a^4 \cdot \frac{g_0^2}{a} C_F \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d k}{(2\pi)^d} \cdot \sum_\rho \left\{ \frac{1}{4 \sum_\lambda \sin^2 \frac{ak_\lambda}{2}} + \frac{2a \sum_\lambda p_\lambda \sin ak_\lambda}{\left(4 \sum_\lambda \sin^2 \frac{ak_\lambda}{2}\right)^2} \right\} \quad (15.57)$$

$$\times \left(\sin \frac{ak_\rho}{2} + i\gamma_\rho \cos \frac{ak_\rho}{2} + \frac{ap_\rho}{2} \left(\cos \frac{ak_\rho}{2} - i\gamma_\rho \sin \frac{ak_\rho}{2} \right) \right)$$

$$\times \left\{ \frac{-i \sum_\lambda \gamma_\lambda \sin ak_\lambda + 2 \sum_\lambda \sin^2 \frac{ak_\lambda}{2}}{\sum_\lambda \sin^2 ak_\lambda + 4\left(\sum_\lambda \sin^2 \frac{ak_\lambda}{2}\right)^2} \right\} \cdot \gamma_\mu \sin ak_\nu \cdot \left\{ \frac{-i \sum_\lambda \gamma_\lambda \sin ak_\lambda + 2 \sum_\lambda \sin^2 \frac{ak_\lambda}{2}}{\sum_\lambda \sin^2 ak_\lambda + 4\left(\sum_\lambda \sin^2 \frac{ak_\lambda}{2}\right)^2} \right\}$$

---

[66]We will assume the implicit summation convention for the indices ρ and λ in the continuum, while on the lattice even repeated indices, unless explicitly stated, are not summed.

$$\times \left(\sin \frac{ak_\rho}{2} + i\gamma_\rho \cos \frac{ak_\rho}{2} + \frac{ap_\rho}{2} \left(\cos \frac{ak_\rho}{2} - i\gamma_\rho \sin \frac{ak_\rho}{2} \right) \right) + O(a^5).$$

The color factors are, for this diagram, the same as in the continuum,

$$\sum_a \sum_b (T^a)_{cb} (T^a)_{bc} = \sum_a (T^a)^2_{cc} = \frac{N_c^2 - 1}{2N_c} = C_F, \tag{15.58}$$

and C_F becomes an overall factor in front of the expressions. We now rescale the integration variable:

$$k \rightarrow k' = ak, \tag{15.59}$$

so that [67]

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d k}{(2\pi)^d} f(ak, ap) = \int_{-\pi}^{\pi} \frac{d^d k'}{(2\pi)^d} \frac{1}{a^4} f(k', ap). \tag{15.60}$$

Note that the domain of integration after the rescaling becomes independent of a. The factor a^4 in front of Eq. (15.57) cancels the factor $1/a^4$ coming from the rescaling in Eq. (15.60), and thus we have an overall factor $1/a$ left. This means that in order to take the continuum limit of this lattice integral we have to expand the whole integrand including factors of order ap. This is not surprising, as it goes in the right direction of recovering the tree level $\gamma_\mu p_\nu$. We continue to call k the integration variable, with the understanding that is the rescaled one. We use from now on the shorthand notation

$$\Gamma_\lambda = \sin k_\lambda, \tag{15.61}$$

$$W = 2 \sum_\lambda \sin^2 \frac{k_\lambda}{2}, \tag{15.62}$$

$$N_\rho = \sin \frac{k_\rho}{2}, \tag{15.63}$$

$$M_\rho = \cos \frac{k_\rho}{2}, \tag{15.64}$$

so that the calculations will appear somewhat easier to follow visually. Of course we also have

$$\slashed{\Gamma} = \sum_\lambda \gamma_\lambda \sin k_\lambda \tag{15.65}$$

and

$$\Gamma^2 = \sum_\lambda \sin^2 k_\lambda. \tag{15.66}$$

---

[67]We have introduced noninteger dimensions because we are using dimensional regularization to compute the finite part of the divergent integrals. While in principle there should be a factor $1/a^d$ in the right-hand side of this equation, the difference between d and 4 is not relevant for the expansion in a and we can set from the beginning this factor to be equal to $1/a^4$.

It is worthwhile to keep in mind during the calculations that Γ and N are odd in k, while M and W are even. Expanding everything to order a (which compensates the remaining $1/a$ factor in Eq. (15.57)) we have:

$$
\begin{aligned}
J \;=\;& \mathrm{i}g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_{\rho} \Bigg\{ \frac{2\sum_{\lambda} p_\lambda \Gamma_\lambda}{(2W)^2}(N_\rho + \mathrm{i}\gamma_\rho M_\rho)\frac{-\mathrm{i}\slashed{\Gamma}+W}{\Gamma^2+W^2}\cdot\gamma_\mu\Gamma_\nu\cdot\frac{-\mathrm{i}\slashed{\Gamma}+W}{\Gamma^2+W^2}(N_\rho+\mathrm{i}\gamma_\rho M_\rho) \\
&+\frac{1}{2W}\frac{p_\rho}{2}(M_\rho-\mathrm{i}\gamma_\rho N_\rho)\frac{-\mathrm{i}\slashed{\Gamma}+W}{\Gamma^2+W^2}\cdot\gamma_\mu\Gamma_\nu\cdot\frac{-\mathrm{i}\slashed{\Gamma}+W}{\Gamma^2+W^2}(N_\rho+\mathrm{i}\gamma_\rho M_\rho) \\
&+\frac{1}{2W}\frac{p_\rho}{2}(N_\rho+\mathrm{i}\gamma_\rho M_\rho)\frac{-\mathrm{i}\slashed{\Gamma}+W}{\Gamma^2+W^2}\cdot\gamma_\mu\Gamma_\nu\cdot\frac{-\mathrm{i}\slashed{\Gamma}+W}{\Gamma^2+W^2}(M_\rho-\mathrm{i}\gamma_\rho N_\rho)\Bigg\}.
\end{aligned}
\tag{15.67}
$$

We emphasize at this point that it has been necessary to perform the Taylor expansion before doing the gamma algebra, because the Dirac structure is "hidden" inside the unexpanded propagators and vertices. After a few manipulations we get

$$
\begin{aligned}
J \;=\;& \mathrm{i}g_0^2 C_F \int_{-\pi}^{\pi}\frac{d^d k}{(2\pi)^d}\sum_{\rho}\Bigg\{\frac{2\sum_\lambda p_\lambda\Gamma_\lambda}{(2W)^2(\Gamma^2+W^2)^2}\Gamma_\nu\Big[\gamma_\mu N_\rho^2 W^2 - \gamma_\rho\gamma_\mu\gamma_\rho M_\rho^2 W^2 - \slashed{\Gamma}\gamma_\mu\slashed{\Gamma}N_\rho^2 \\
&+\gamma_\rho\slashed{\Gamma}\gamma_\mu\slashed{\Gamma}\gamma_\rho M_\rho^2 + (\gamma_\rho\slashed{\Gamma}\gamma_\mu+\gamma_\mu\slashed{\Gamma}\gamma_\rho+\gamma_\rho\gamma_\mu\slashed{\Gamma}+\slashed{\Gamma}\gamma_\mu\gamma_\rho)N_\rho M_\rho W + \ldots\Big] \\
&+\frac{1}{2W(\Gamma^2+W^2)^2}\frac{p_\rho}{2}\Gamma_\nu\Big[2\gamma_\mu N_\rho M_\rho W^2 - 2\slashed{\Gamma}\gamma_\mu\slashed{\Gamma}N_\rho M_\rho + 2\gamma_\rho\gamma_\mu\gamma_\rho N_\rho M_\rho W^2 \\
&-2\gamma_\rho\slashed{\Gamma}\gamma_\mu\slashed{\Gamma}\gamma_\rho N_\rho M_\rho + (\gamma_\rho\slashed{\Gamma}\gamma_\mu+\gamma_\mu\slashed{\Gamma}\gamma_\rho+\gamma_\rho\gamma_\mu\slashed{\Gamma}+\slashed{\Gamma}\gamma_\mu\gamma_\rho)(M_\rho^2-N_\rho^2)W + \ldots\Big]\Bigg\},
\end{aligned}
\tag{15.68}
$$

where the dots denote terms which are odd in k and therefore do not contribute to the final result because their integral is zero by parity. Notice that the terms which are even in k possess an odd number of Dirac matrices, and this again goes in the right direction of recovering the tree-level expression.

At this stage we can finally perform the gamma algebra. However, this is one of the most delicate points of the whole calculation, and one must be careful here. What happens is that the expressions are not in general tensors in the usual sense. Summed indices can appear more than twice in each monomial, due to the breaking of Lorentz invariance. Thus, the well-known formulae for doing the gamma algebra in the continuum cannot be used straightforwardly on the lattice.

The lattice reduction formulae which we need in order to proceed with the calculation of the vertex are:

$$
\slashed{\Gamma}\gamma_\mu\slashed{\Gamma} = -\gamma_\mu\Gamma^2 + 2\slashed{\Gamma}\Gamma_\mu,
\tag{15.69}
$$

which is just as it would be in the continuum,

$$
\sum_\rho \gamma_\rho\gamma_\mu\gamma_\rho = \sum_\rho \gamma_\rho(-\gamma_\rho\gamma_\mu + 2\delta_{\rho\mu}) = \gamma_\mu\sum_\rho(-\gamma_\rho^2 + 2\delta_{\rho\mu}),
\tag{15.70}
$$

which is instead different from the continuum result, and

$$\sum_\rho \gamma_\rho \slashed{\Gamma} \gamma_\mu \slashed{\Gamma} \gamma_\rho \;=\; \sum_\rho \gamma_\rho (-\gamma_\mu \Gamma^2 + 2\slashed{\Gamma}\Gamma_\mu)\gamma_\rho \tag{15.71}$$

$$=\; \sum_\rho (\gamma_\mu \gamma_\rho^2 \Gamma^2 - 2\delta_{\rho\mu}\gamma_\mu \Gamma^2 - 2\slashed{\Gamma}\gamma_\rho^2 \Gamma_\mu + 4\gamma_\rho \Gamma_\rho \Gamma_\mu).$$

The Kronecker deltas in the formulae above, as well as the various γ_ρ^2 factors, are very important and for the moment they have to be kept, because their outcome depends on what is present in the rest of each monomial. This can be seen in the following examples:

$$\sum_\rho \gamma_\rho \gamma_\mu \gamma_\rho \cos k_\mu \cos k_\rho \;=\; \sum_\rho (-\gamma_\mu \gamma_\rho^2 + 2\delta_{\rho\mu}\gamma_\mu)\cos k_\mu \cos k_\rho \tag{15.72}$$

$$=\; -\gamma_\mu \cos k_\mu \sum_\rho \cos k_\rho + 2\gamma_\mu \cos^2 k_\mu$$

$$\sum_\rho \gamma_\rho \gamma_\mu \gamma_\rho \cos^2 k_\mu \;=\; \sum_\rho (-\gamma_\mu \gamma_\rho^2 + 2\delta_{\rho\mu}\gamma_\mu)\cos^2 k_\mu \tag{15.73}$$

$$=\; -(4 - 2\epsilon)\gamma_\mu \cos^2 k_\mu + 2\gamma_\mu \cos^2 k_\mu = -(2 - 2\epsilon)\gamma_\mu \cos^2 k_\mu.$$

Similarly, a term like $\gamma_\rho \Gamma_\rho$ in the J expression above can be contracted only if in the rest of the monomial there are no other functions with index ρ. Another gamma algebra relation that we need is

$$\gamma_\alpha \gamma_\beta \gamma_\gamma + \gamma_\gamma \gamma_\beta \gamma_\alpha = 2(\delta_{\alpha\beta}\gamma_\gamma + \delta_{\beta\gamma}\gamma_\alpha - \delta_{\alpha\gamma}\gamma_\beta), \tag{15.74}$$

so that

$$\gamma_\rho \slashed{\Gamma} \gamma_\mu + \gamma_\mu \slashed{\Gamma} \gamma_\rho + \gamma_\rho \gamma_\mu \slashed{\Gamma} + \slashed{\Gamma} \gamma_\mu \gamma_\rho = 4\gamma_\rho \Gamma_\mu. \tag{15.75}$$

Using these formulae for the reduction of γ matrices we then obtain:

$$J \;=\; \mathrm{i}g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \frac{2\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2(\Gamma^2 + W^2)^2} \Gamma_\nu \left[\sum_\rho \left(\gamma_\mu N_\rho^2 W^2 + \gamma_\mu \Gamma^2 (N_\rho^2 + M_\rho^2) \right. \right. \right.$$

$$\left. -2\slashed{\Gamma}\Gamma_\mu(N_\rho^2 + M_\rho^2) + \gamma_\mu M_\rho^2 W^2 + 4\gamma_\rho \Gamma_\rho \Gamma_\mu M_\rho^2 + 4\gamma_\rho \Gamma_\mu N_\rho M_\rho W \right)$$

$$\left. -2\gamma_\mu \Gamma^2 M_\mu^2 - 2\gamma_\mu M_\mu^2 W^2 \right]$$

$$+\frac{1}{2W(\Gamma^2 + W^2)^2} \sum_\rho \frac{p_\rho}{2} \Gamma_\nu \left[\left(4\gamma_\mu \Gamma^2 N_\mu M_\mu - 8\gamma_\rho \Gamma_\rho \Gamma_\mu N_\rho M_\rho \right. \right. \tag{15.76}$$

$$\left. \left. +4\gamma_\rho \Gamma_\mu (M_\rho^2 - N_\rho^2)W \right) + 4\gamma_\mu N_\mu M_\mu W^2 \right] \right\}.$$

We now exploit the symmetry $k \to -k$ of the integration region one more time: odd powers of sine functions of the various Lorentz components of the momentum must be combined together

in such a way that only even powers, which have a nonzero final integral, are left:

$$\sum_{\lambda\rho} \sin k_\rho \sin k_\lambda = \sum_{\lambda\rho} \sin^2 k_\rho \delta_{\lambda\rho} \tag{15.77}$$

$$\sum_{\lambda\rho} \sin k_\rho \sin k_\lambda \sin_\mu \sin_\nu = \sum_{\lambda\rho} \sin^2 k_\rho \sin^2 k_\lambda (\delta_{\rho\mu}\delta_{\lambda\nu} + \delta_{\rho\nu}\delta_{\lambda\mu}) \qquad (\mu \neq \nu). \tag{15.78}$$

Of course, since in our case we have $\mu \neq \nu$, terms of the kind $\sin k_\mu \sin k_\nu$ are zero when integrated and can be safely dropped (unless they appear together with sines of other indices like in $\sin k_\mu \sin k_\nu \sin k_\alpha \sin k_\beta$, as in the formula above). Examples which are relevant for the J expression that we are computing are:

$$\sum_{\lambda\rho} p_\lambda \Gamma_\lambda \Gamma_\nu \gamma_\mu N_\rho^2 W = \sum_\rho \gamma_\mu p_\nu \Gamma_\nu^2 N_\rho^2 W, \tag{15.79}$$

$$\sum_\rho \gamma_\mu p_\rho \Gamma_\nu N_\mu M_\mu W^2 = 0, \tag{15.80}$$

$$\sum_{\lambda\rho} p_\lambda \Gamma_\lambda \Gamma_\nu \slashed{\Gamma} \Gamma_\mu (N_\rho^2 + M_\rho^2) = \sum_\rho (p_\nu \Gamma_\nu^2 \gamma_\mu \Gamma_\mu^2 + p_\mu \Gamma_\mu^2 \gamma_\nu \Gamma_\nu^2)(N_\rho^2 + M_\rho^2), \tag{15.81}$$

and another interesting example is

$$\sum_\rho p_\rho \slashed{\Gamma} \Gamma_\mu \Gamma_\rho N_\rho M_\rho = \gamma_\mu p_\mu \Gamma_\mu^2 \Gamma_\nu N_\nu M_\nu + \gamma_\nu p_\nu \Gamma_\nu^2 \Gamma_\mu N_\mu M_\mu. \tag{15.82}$$

It is easy to see that after this step all that is left in J is multiplied for either $\gamma_\mu p_\nu$ or $\gamma_\nu p_\mu$, while p_ρ has disappeared altogether:

$$\begin{aligned}
J = & \; ig_0^2 C_F \int_{-\pi}^\pi \frac{d^d k}{(2\pi)^d} \left\{ \frac{2\gamma_\mu p_\nu \Gamma_\nu^2}{(2W)^2(\Gamma^2 + W^2)^2} \left[\sum_\rho \left((N_\rho^2 + M_\rho^2)W^2 + \Gamma^2(N_\rho^2 + M_\rho^2) \right) \right. \right. \\
& \left. - 2\Gamma_\mu^2(N_\rho^2 + M_\rho^2) \right) - 2\Gamma^2 M_\mu^2 - 2M_\mu^2 W^2 + 4\Gamma_\mu^2 M_\mu^2 + 4\Gamma_\mu N_\mu M_\mu W \right] \\
& \left. + \frac{2\gamma_\nu p_\mu \Gamma_\mu^2}{(2W)^2(\Gamma^2 + W^2)^2} \left[-2\Gamma_\nu^2 \sum_\rho (N_\rho^2 + M_\rho^2) + 4\Gamma_\nu^2 M_\nu^2 + 4\Gamma_\nu N_\nu M_\nu W \right] \right\}.
\end{aligned} \tag{15.83}$$

Notice also that the whole part proportional to $1/(2W(\Gamma^2 + W^2)^2)$ has disappeared, because all terms were of the kind $\sin k_\mu \sin k_\nu$.

At this point we remark that we started to do our calculation only using the operator $\gamma_\mu D_\nu$, and what we have just obtained is a 1-loop expression of the form

$$\gamma_\mu D_\nu \xrightarrow{\text{1 loop}} \frac{g_0^2}{16\pi^2} i \left(\gamma_\mu p_\nu A + \gamma_\nu p_\mu B \right). \tag{15.84}$$

It seems that we have not obtained a 1-loop expression which is proportional to the tree level, which is the necessary condition in order for the operator to be multiplicatively renormalized.

However, if we now take the symmetrization in μ and ν into account everything is fine, because the operator $O_{\nu\mu}$ gives

$$\gamma_\nu D_\mu \xrightarrow{1 \text{ loop}} \frac{g_0^2}{16\pi^2} \,\mathrm{i} \left(\gamma_\mu p_\nu B + \gamma_\nu p_\mu A\right), \tag{15.85}$$

with the same A and B of $\gamma_\mu D_\nu$, and this means that

$$O_{\{\mu\nu\}} = \frac{1}{2}\left(\gamma_\mu D_\nu + \gamma_\nu D_\mu\right) \xrightarrow{1 \text{ loop}} \frac{g_0^2}{16\pi^2} \frac{1}{2} \,\mathrm{i} \left(\gamma_\mu p_\nu + \gamma_\nu p_\mu\right)(A + B). \tag{15.86}$$

The symmetrized operator is thus multiplicatively renormalized (at least, for now, its vertex contribution), and looking at Eq. (3.1) we see that the lattice part of the 1-loop renormalization constant that matches to the continuum is given by

$$-\gamma_{vert}^{(0)} \cdot \log a^2 p^2 + R_{vert}^{lat} = A + B. \tag{15.87}$$

In practical terms, in order to get the 1-loop expression for the symmetrized operator, we need to exchange, in the expression in Eq. (15.83) for J, the indices μ and ν in the terms proportional to $\gamma_\nu p_\mu$. We have then the result

$$
\begin{aligned}
J &= \mathrm{i}g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \Bigg\{ \frac{\gamma_\mu p_\nu \Gamma_\nu^2}{2(\Gamma^2 + W^2)^2}\left[\sum_\rho (N_\rho^2 + M_\rho^2) - 2M_\mu^2 \right] + \frac{2\gamma_\mu p_\nu \Gamma_\mu^2 \Gamma_\nu^2}{W(\Gamma^2 + W^2)^2} \\
&\quad + \frac{2\gamma_\mu p_\nu \Gamma_\nu^2}{(2W)^2(\Gamma^2 + W^2)^2}\left(\sum_\rho \left(\Gamma^2(N_\rho^2 + M_\rho^2) - 4\Gamma_\mu^2(N_\rho^2 + M_\rho^2) \right) - 2\Gamma^2 M_\mu^2 + 8\Gamma_\mu^2 M_\mu^2 \right) \Bigg\},
\end{aligned}
$$

in which we have also made some simplifications and used the trigonometric identity

$$N_\rho M_\rho = \frac{1}{2}\Gamma_\rho. \tag{15.88}$$

We can now write the final result in a more explicit way, using also

$$N_\rho^2 + M_\rho^2 = 1, \tag{15.89}$$

and we obtain our final expression:

$$
\begin{aligned}
J &= \mathrm{i}g_0^2 C_F \gamma_\mu p_\nu \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \Bigg\{ \frac{\sin^2 k_\nu \left(4 - 2\epsilon - 2\cos^2 \frac{k_\mu}{2}\right)}{2\left(\sum_\lambda \sin^2 k_\lambda + \left(2\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2 \right)^2} \\
&\quad + \frac{\sin^2 k_\mu \sin^2 k_\nu}{\left(\sum_\lambda \sin^2 \frac{k_\lambda}{2} \right)\left(\sum_\lambda \sin^2 k_\lambda + \left(2\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2 \right)^2} \\
&\quad + \frac{\sin^2 k_\nu \left(\left(4 - 2\epsilon - 2\cos^2 \frac{k_\mu}{2}\right)\left(\sum_\lambda \sin^2 k_\lambda - 4\sin^2 k_\mu\right)\right)}{8\left(\sum_\lambda \sin^2 \frac{k_\lambda}{2} \right)^2 \left(\sum_\lambda \sin^2 k_\lambda + \left(2\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2 \right)^2} \Bigg\}.
\end{aligned}
\tag{15.90}
$$

It should be noticed that the function $\cos(k_\alpha/2)$ is present in the final expression of the Feynman diagrams only with an even power (even though in the Feynman rules, for example in the quark-gluon vertex, Eq. (5.76), it appears also at the first power), while $\cos k_\alpha$ can be present also with an odd power. The same will be true for the sails, and in general for every Feynman diagram in this kind of theories. This is due to the fact that $\cos(k_\alpha/2)$ has not a period of 2π, and hence is not an admissible function, while $\cos^2(k_\alpha/2)$ has a period of 2π indeed. Of course the sine functions, due to the symmetry of the integration region, appear in any case only with an even power.

The last term in the final result for J is logarithmically divergent. The coefficient of the divergence is easily extracted, and the finite rest can be computed with great precision using the algebraic method that we will introduce in Section 18. We can however proceed here in a much faster (although less precise) way by subtracting a simple lattice integral which the same divergence, for example

$$-\frac{1}{3}\int_{-\pi}^{\pi}\frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}\frac{1}{\left(4\sum_\lambda \sin^2\dfrac{k_\lambda}{2}\right)^2} = \frac{1}{16\pi^2}\left[-\frac{1}{3}\left(-\frac{1}{\epsilon}-\log 4\pi + F_0\right)\right], \qquad (15.91)$$

which we have taken from Section 18. The finite difference can then be computed numerically with a simple integration routine. In this way the value of the zero-momentum integral can be obtained, and the result is

$$J = \mathrm{i}\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2}C_F\left[-\frac{1}{3}\left(\frac{1}{\epsilon}-F_0+\log 4\pi\right)+0.473493\right] = \mathrm{i}\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2}C_F\left(-\frac{1}{3\epsilon}+1.086227\right).$$
$$(15.92)$$

The continuum part, $I - J$, is just

$$
\begin{aligned}
I - J &= -g_0^2 C_F \frac{1}{(p-k)^2}\gamma_\rho\frac{-\mathrm{i}k\!\!\!/}{k^2}\mathrm{i}\gamma_\mu k_\nu\frac{-\mathrm{i}k\!\!\!/}{k^2}\gamma_\rho &(15.93)\\
&= \mathrm{i}\gamma_\mu p_\nu\frac{g_0^2}{16\pi^2}C_F\left[-\frac{1}{3}\left(-\frac{1}{\epsilon}+\log p^2+\gamma_E-\log 4\pi\right)+\frac{5}{9}\right]\\
&= \mathrm{i}\gamma_\mu p_\nu\frac{g_0^2}{16\pi^2}C_F\left(\frac{1}{3\epsilon}-\frac{1}{3}\log p^2+1.206825\right).
\end{aligned}
$$

In principle in $I - J$ also terms of the form $p_\mu p_\nu p\!\!\!/ /p^2$ appear, but they belong to the matrix element and not to the renormalization constant, and in any case they cancel when the difference between continuum and lattice renormalization factors is taken, because they have the same coefficient in both cases.

The final result for the Feynman diagram is (Capitani, 2001a):

$$I = \mathrm{i}\gamma_\mu p_\nu\frac{g_0^2}{16\pi^2}C_F\left(-\frac{1}{3}\log p^2+2.293052\right). \qquad (15.94)$$

We could have chosen a mass regularization instead of dimensional regularization (adding a mass m^2 to the gluon propagator). In this case we would have obtained

$$J_m = \mathrm{i}\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2}C_F\left[-\frac{1}{3}\left(\log m^2 + \gamma_E - F_0\right) - 0.193173\right] = \mathrm{i}\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2}C_F\left(-\frac{1}{3}\log m^2 + 1.070830\right) \tag{15.95}$$

and

$$(I - J)_m = \mathrm{i}\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2}C_F\left[-\frac{1}{3}\left(\log p^2 - \log m^2\right) + \frac{11}{9}\right]. \tag{15.96}$$

Of course the sum of J_m and $(I - J)_m$ is still given by Eq. (15.94).

Looking at the expression in Eq. (15.57), it can be seen that after the a^4 rescaling it contains also $1/a$ terms. One could ask where have the $1/a$ terms gone. The fact is that they are zero for this diagram. They are zero also for the sails, while in the case of the self-energy the $1/a$ terms are finite and give an important contribution, which is linked to the breaking of chiral symmetry.

15.4.3 Sails

We now turn to the sails in Fig. 17. In these diagrams there is only one interaction vertex coming from QCD, and the gluon is the contraction of the A_ν coming from the covariant derivative in the operator and of the A_ρ in the QCD vertex V_ρ, which therefore (in Feynman gauge) becomes V_ν.

The first order expansion of the covariant derivative in the operator gives, as we have seen,

$$O_{\mu\nu}^{(1)}(k, p) = \mathrm{i}\gamma_\mu \cos\frac{a(k + p)_\nu}{2} = \mathrm{i}\gamma_\mu\left(\cos\frac{ak_\nu}{2} - \frac{ap_\nu}{2}\sin\frac{ak_\nu}{2}\right), \tag{15.97}$$

and this expression is valid for both sails. We evaluate the two sails together, since they are similar and some simplifications will take place at intermediate stages of the calculations when taking their sum. Again, we first compute the J part at zero-momentum,

$$
\begin{aligned}
J &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d k}{(2\pi)^d} G_{\nu\nu}(p - k) \cdot g_0\left[V_\nu(k, p) \cdot S(k) \cdot O_{\mu\nu}^{(1)}(k, p) + O_{\mu\nu}^{(1)}(k, p) \cdot S(k) \cdot V_\nu(p, k)\right]\Bigg|_{ap=0} \\
&= -\frac{\mathrm{i}g_0^2}{a}C_F\int_{-\pi}^{\pi}\frac{d^d k}{(2\pi)^d}\left(\frac{1}{2W} + \frac{2a\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2}\right)\left[\left(N_\nu + \mathrm{i}\gamma_\nu M_\nu\right)\frac{-\mathrm{i}\slashed{\Gamma} + W}{\Gamma^2 + W^2}\gamma_\mu\left(M_\nu - \frac{ap_\nu}{2}N_\nu\right)\right.\\
&\qquad\left. + \gamma_\mu\left(M_\nu - \frac{ap_\nu}{2}N_\nu\right)\frac{-\mathrm{i}\slashed{\Gamma} + W}{\Gamma^2 + W^2}\left(N_\nu + \mathrm{i}\gamma_\nu M_\nu\right)\right],
\end{aligned}
\tag{15.98}
$$

where we have already rescaled the integration variable as we did in the vertex. It is easy to see that the integral in the first line contains an overall factor a^3 (a from $S(k)$ and a^2 from $G_{\nu\nu}(p - k)$), and after rescaling (which gives a factor $1/a^4$) one is left with an overall $1/a$ factor.

We have then, in the limit $a \to 0$:

$$
\begin{aligned}
J &= -ig_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \\
&\quad \times \Bigg\{ \frac{2\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2} \Bigg((N_\nu + i\gamma_\nu M_\nu) \frac{-i\slashed{\Gamma} + W}{\Gamma^2 + W^2} \cdot \gamma_\mu M_\nu + \gamma_\mu M_\nu \cdot \frac{-i\slashed{\Gamma} + W}{\Gamma^2 + W^2}(N_\nu + i\gamma_\nu M_\nu) \Bigg) \\
&\quad + \frac{1}{2W} \frac{p_\nu}{2} \Bigg((M_\nu - i\gamma_\nu N_\nu) \frac{-i\slashed{\Gamma} + W}{\Gamma^2 + W^2} \cdot \gamma_\mu M_\nu + \gamma_\mu M_\nu \cdot \frac{-i\slashed{\Gamma} + W}{\Gamma^2 + W^2}(M_\nu - i\gamma_\nu N_\nu) \quad (15.99) \\
&\qquad + (N_\nu + i\gamma_\nu M_\nu) \frac{-i\slashed{\Gamma} + W}{\Gamma^2 + W^2} \cdot \gamma_\mu(-N_\nu) + \gamma_\mu(-N_\nu) \cdot \frac{-i\slashed{\Gamma} + W}{\Gamma^2 + W^2}(N_\nu + i\gamma_\nu M_\nu) \Bigg) \Bigg\} \\
&= -ig_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \Bigg\{ \frac{2\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2(\Gamma^2 + W^2)} \Big(2\gamma_\mu N_\nu M_\nu W + (\gamma_\nu \slashed{\Gamma} \gamma_\mu + \gamma_\mu \slashed{\Gamma} \gamma_\nu) M_\nu^2 + \dots \Big) \\
&\quad + \frac{1}{2W(\Gamma^2 + W^2)} \frac{p_\nu}{2} \Big(2\gamma_\mu (M_\nu^2 - N_\nu^2)W - 2(\gamma_\nu \slashed{\Gamma} \gamma_\mu + \gamma_\mu \slashed{\Gamma} \gamma_\nu) N_\nu M_\nu + \dots \Big) \Bigg\}.
\end{aligned}
$$

Again we drop terms odd in k and perform the gamma algebra, using

$$
\gamma_\nu \slashed{\Gamma} \gamma_\mu + \gamma_\mu \slashed{\Gamma} \gamma_\nu = 2(\gamma_\mu \Gamma_\nu + \gamma_\nu \Gamma_\mu), \tag{15.100}
$$

which is valid for $\mu \neq \nu$. After combining the sine functions and exchanging the indices μ and ν in the $\gamma_\nu p_\mu$ terms we arrive at

$$
\begin{aligned}
J &= -ig_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \Bigg\{ \frac{1}{2W^2(\Gamma^2 + W^2)} \gamma_\mu p_\nu \Big(\Gamma_\nu^2 W + 2\Gamma_\nu^2 (M_\nu^2 + M_\mu^2) \Big) \\
&\quad + \frac{1}{2W(\Gamma^2 + W^2)} \Big(\gamma_\mu p_\nu (M_\nu^2 - N_\nu^2)W - \Gamma_\nu^2 \Big) \Bigg\}, \tag{15.101}
\end{aligned}
$$

where we have also replaced the $N_\nu M_\nu$ factor with $\Gamma_\nu/2$. At the end, after some further simplifications (which include the replacement of $M_\rho^2 - N_\rho^2$ with $\cos k_\rho$), we obtain the explicit form

$$
\begin{aligned}
J &= -ig_0^2 C_F \gamma_\mu p_\nu \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \Bigg\{ \frac{\cos k_\nu}{2\Big(\sum_\lambda \sin^2 k_\lambda + \big(2\sum_\lambda \sin^2 \frac{k_\lambda}{2} \big)^2 \Big)^2} \\
&\quad + \frac{\sin^2 k_\nu \Big(\cos^2 \frac{k_\mu}{2} + \cos^2 \frac{k_\nu}{2} \Big)}{4\big(\sum_\lambda \sin^2 \frac{k_\lambda}{2} \big)^2 \Big(\sum_\lambda \sin^2 k_\lambda + \big(2\sum_\lambda \sin^2 \frac{k_\lambda}{2} \big)^2 \Big)^2} \Bigg\}. \tag{15.102}
\end{aligned}
$$

This final expression for the sails gives, when integrated,

$$
J = i\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2} C_F \left[2\left(\frac{1}{\epsilon} - F_0 + \log 4\pi \right) + 6.506752 \right] = i\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2} C_F \left(\frac{2}{\epsilon} + 2.830350 \right), \tag{15.103}
$$

and

$$I - J = g_0^2 C_F \frac{1}{(p-k)^2} \left[\gamma_\nu \frac{-i\slashed{k}}{k^2} \gamma_\mu + \gamma_\nu \frac{-i\slashed{k}}{k^2} \gamma_\mu \right] \tag{15.104}$$

$$= i\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2} C_F \left[2\left(-\frac{1}{\epsilon} + \log p^2 + \gamma_E - \log 4\pi \right) - 4 \right]$$

$$= i\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2} C_F \left(-\frac{2}{\epsilon} + 2\log p^2 - 7.907617 \right).$$

The final result for the Feynman diagram is (Capitani, 2001a):

$$I = i\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2} C_F \left(2\log p^2 - 5.077267 \right). \tag{15.105}$$

With a mass regularization the intermediate results would be:

$$J_m = i\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2} C_F \left[2\left(\log m^2 + \gamma_E - F_0 \right) + 6.506752 \right] = i\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2} C_F \left(2\log m^2 - 1.077267 \right), \tag{15.106}$$

and

$$(I - J)_m = i\gamma_\mu p_\nu \frac{g_0^2}{16\pi^2} C_F \left[2\left(\log p^2 - \log m^2 \right) - 4 \right]. \tag{15.107}$$

Also in this diagram the $1/a$ terms are zero.

15.4.4 Operator tadpole

We now consider the diagram arising from the second order expansion of the covariant derivative in the operator, whose Fourier transform is, in the special case in which the two gluons are contracted to make a tadpole,

$$O_{\mu\nu}^{(2)}(p) = -\frac{a}{2} C_F i\gamma_\mu \sin ap_\nu. \tag{15.108}$$

The operator that enters into the tadpole depends only on the external quark momentum, but not on the integration variable, which is carried by the gluon, and so it has a simple continuum limit. The integral is rather easy to compute, and the result is

$$I = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d k}{(2\pi)^d} G_{\nu\nu}(k) \cdot g_0^2 O_{\mu\nu}^{(2)}(p)$$

$$= g_0^2 C_F \left(-\frac{a}{2} i\gamma_\mu \sin ap_\nu \right) \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d k}{(2\pi)^d} a^2 \frac{1}{4 \sum_\lambda \sin^2 \frac{ak_\lambda}{2}}$$

$$= g_0^2 C_F \left(-\frac{1}{2} i\gamma_\mu \frac{\sin ap_\nu}{a} \right) \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{4 \sum_\lambda \sin^2 \frac{k_\lambda}{2}}$$

130

$$= -\frac{1}{2}g_0^2 C_F\, i\gamma_\mu p_\nu \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d}\, \frac{1}{4\sum_\lambda \sin^2 \dfrac{k_\lambda}{2}}. \tag{15.109}$$

Although this diagram comes from $O(a^2 A^2)$ terms in the action which are naively zero in the continuum limit, the gluon loop gives an extra factor of $1/a^2$ and so these contributions do not disappear in the limit $a = 0$ of the diagram.

So, this diagram is not zero, and we can see that it is also not divergent. This finite integral can be encountered very frequently in lattice calculations, is in fact one of the most basic quantities of perturbation theory and is taken as a fundamental constant, called Z_0, in the algebraic method (see Section 18). This constant can in principle can be computed with arbitrary precision. It is now known with a very great precision, about 400 significant decimal places, as we will see in Section 19. For our calculation is enough to know that

$$\int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d}\, \frac{1}{4\sum_\lambda \sin^2 \dfrac{k_\lambda}{2}} = Z_0 = 0.15493339, \tag{15.110}$$

so that the value of the operator tadpole is

$$I = -\frac{g_0^2}{16\pi^2} C_F i\gamma_\mu p_\nu \cdot 12.233050. \tag{15.111}$$

Of course the symmetrization of this result is trivial.

The computation of operator tadpoles for more complicated operators, which can be done in an exact way (that is, expressing the results in terms of only two integrals known with arbitrary precision) using the algebraic method, is discussed in Section 18.

15.4.5 Quark self-energy (sunset diagram)

The zero-momentum part for the sunset diagram of the quark self-energy (that is, without considering the tadpole) is:

$$
\begin{aligned}
J &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d k}{(2\pi)^d} \sum_\rho G_{\rho\rho}(p-k) \cdot \Big[V_\rho(k,p) \cdot S(k) \cdot V_\rho(p,k) \Big] \Big|_{ap=0} \\
&= \frac{g_0^2}{a} C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \left(\frac{1}{2W} + \frac{2a\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2} \right)\left(N_\rho + i\gamma_\rho M_\rho + \frac{ap_\rho}{2}(M_\rho - i\gamma_\rho N_\rho) \right) \\
&\quad \times \frac{-i\Gamma + W}{\Gamma^2 + W^2}\left(N_\rho + i\gamma_\rho M_\rho + \frac{ap_\rho}{2}(M_\rho - i\gamma_\rho N_\rho) \right),
\end{aligned} \tag{15.112}
$$

where we have already rescaled the integration variable. After combining the various factors a coming from the propagator and the vertices as well as from the rescaling of k, what remains

in front of the expression is an overall factor $1/a$. This means that we have to keep all terms of order ap. We have then

$$
\begin{aligned}
J &= \frac{g_0^2}{a} C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \frac{1}{2W} (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \frac{-\mathrm{i}\slashed{\Gamma} + W}{\Gamma^2 + W^2} (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \qquad (15.113) \\
&+ g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \left\{ \frac{2\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2} (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \frac{-\mathrm{i}\slashed{\Gamma} + W}{\Gamma^2 + W^2} (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \right. \\
&\quad + \frac{1}{2W} \frac{p_\rho}{2} \left[(M_\rho - \mathrm{i}\gamma_\rho N_\rho) \frac{-\mathrm{i}\slashed{\Gamma} + W}{\Gamma^2 + W^2} (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \right. \\
&\quad \left. \left. + (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \frac{-\mathrm{i}\slashed{\Gamma} + W}{\Gamma^2 + W^2} (M_\rho - \mathrm{i}\gamma_\rho N_\rho) \right] \right\},
\end{aligned}
$$

which gives

$$
\begin{aligned}
J &= \frac{g_0^2}{a} C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \frac{1}{2W(\Gamma^2 + W^2)} \left(N_\rho^2 W - \gamma_\rho^2 M_\rho^2 W + (\gamma_\rho \slashed{\Gamma} + \slashed{\Gamma}\gamma_\rho) N_\rho M_\rho \right) \\
&+ g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \left\{ \frac{2\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2 (\Gamma^2 + W^2)} \left(-\mathrm{i}\slashed{\Gamma} N_\rho^2 + \mathrm{i}\gamma_\rho \slashed{\Gamma}\gamma_\rho M_\rho^2 + 2\mathrm{i}\gamma_\rho N_\rho M_\rho W \right) \right. \\
&\quad \left. + \frac{1}{2W(\Gamma^2 + W^2)} \frac{p_\rho}{2} \left(2\mathrm{i}\gamma_\rho (M_\rho^2 - N_\rho^2) W - 2\mathrm{i}\slashed{\Gamma} N_\rho M_\rho - 2\mathrm{i}\gamma_\rho \slashed{\Gamma}\gamma_\rho N_\rho M_\rho \right) \right\}, \qquad (15.114)
\end{aligned}
$$

where again we have not considered terms which are odd in k. Here we have also kept the contribution proportional to $1/a$ because, contrary to vertex and sails, for this diagram it turns out not to be zero. In fact, this is a very important quantity for Wilson fermions. It contributes to the linearly divergent Σ_0/a term in the 1-loop self-energy [68]

$$
\frac{g_0^2}{16\pi^2} \left(\frac{\Sigma_0}{a} + \mathrm{i}\slashed{p}\, \Sigma_1 + m_0\, \Sigma_2 \right), \qquad (15.115)
$$

which is due to the breaking of chiral symmetry for Wilson fermions (in fact it is proportional to r, and vanishes for naive fermions), and gives the critical mass to one loop:

$$
m_c = \Sigma_0. \qquad (15.116)
$$

We can see that after reduction there are no Dirac matrices in the contribution to m_c coming from this diagram. The corresponding integral is finite and given by

$$
m_c^{(a)} = g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \frac{1}{2W(\Gamma^2 + W^2)} \left((N_\rho^2 - M_\rho^2) W + \Gamma_\rho^2 \right) \qquad (15.117)
$$

---

[68] Of course, since in our example we are doing the computations using a massless fermion propagator, we do not obtain the Σ_2 factor.

$$= g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \frac{\sum_\rho \cos k_\rho}{2\left(\sum_\lambda \sin^2 k_\lambda + \left(2\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2\right)^2} \right.$$

$$\left. + \frac{\sum_\rho \sin^2 k_\rho}{4\left(\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)\left(\sum_\lambda \sin^2 k_\lambda + \left(2\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2\right)^2} \right\}$$

$$= -\frac{g_0^2}{16\pi^2} C_F \cdot 2.502511.$$

This is the contribution to the critical mass coming from the sunset diagram of the self-energy. Another contribution to it, $m_c^{(b)}$, comes from the tadpole and we will compute it soon. Let us now turn to the rest of the expression (15.114), which contributes to the renormalization of the operator that we are studying. From now on J will only denote the $a = 0$ part of the zero-momentum expression, in which after reduction there is one Dirac matrix, and which gives Σ_1. We have then

$$J = g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \left\{ \frac{2\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2(\Gamma^2 + W^2)} \left(-i\slashed{\Gamma}(N_\rho^2 + M_\rho^2) + 2i\gamma_\rho \Gamma_\rho M_\rho^2 + i\gamma_\rho \Gamma_\rho W \right) \right.$$

$$\left. + \frac{1}{2W(\Gamma^2 + W^2)} p_\rho \left(i\gamma_\rho (M_\rho^2 - N_\rho^2)W - i\gamma_\rho \Gamma_\rho^2 \right) \right\} \tag{15.118}$$

$$= g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \frac{i\slashed{p}}{(2W)^2(\Gamma^2 + W^2)} \left(-2\Gamma_\nu^2 \sum_\rho (N_\rho^2 + M_\rho^2) + 4\sum_\rho \Gamma_\rho^2 M_\rho^2 + 2\Gamma_\nu^2 W \right) \right.$$

$$\left. + \frac{i\slashed{p}}{2W(\Gamma^2 + W^2)} \left((M_\nu^2 - N_\nu^2)W - \Gamma_\nu^2 \right) \right\}. \tag{15.119}$$

In the last passage we have used the substitution

$$\sum_\lambda \gamma_\lambda p_\lambda \int f_\lambda(k) = \slashed{p} \int f_\mu(k), \tag{15.120}$$

since this kind of integrals does not depend on the direction, with the understanding that the index μ is fixed and must not appear in the rest of the monomial. This reconstructs the $i\slashed{p}$ factor and allows the extraction of the value of Σ_1. We thus have:

$$J = g_0^2 C_F i\slashed{p} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \frac{\cos k_\nu}{2\left(\sum_\lambda \sin^2 k_\lambda + \left(2\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2\right)^2} \right.$$

$$\left. + \frac{-(4 - 2\epsilon)\sin^2 k_\nu + 2\sum_\rho \sin^2 k_\rho \cos^2 \frac{k_\rho}{2}}{8\left(\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2 \left(\sum_\lambda \sin^2 k_\lambda + \left(2\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2\right)^2} \right\}. \tag{15.121}$$

The lattice result at zero momentum is then

$$J = i\slashed{p}\frac{g_0^2}{16\pi^2} C_F \left(\frac{1}{\epsilon} - F_0 + \log 4\pi + 4.411364 \right) = i\slashed{p}\frac{g_0^2}{16\pi^2} C_F \left(\frac{1}{\epsilon} + 2.573163 \right). \tag{15.122}$$

We have still to consider the continuum $(I - J)$:

$$
\begin{aligned}
I - J &= -g_0^2 C_F \frac{1}{(p-k)^2} \gamma_\rho \frac{-i\slashed{k}}{k^2} \gamma_\rho \quad &(15.123)\\
&= i\slashed{p} \frac{g_0^2}{16\pi^2} C_F \left(-\frac{1}{\epsilon} + \log p^2 + \gamma_E - \log 4\pi - 1 \right)\\
&= i\slashed{p} \frac{g_0^2}{16\pi^2} C_F \left(-\frac{1}{\epsilon} + \log p^2 - 2.953809 \right).
\end{aligned}
$$

The final result for the $i\slashed{p}$ part of the sunset diagram of the lattice self-energy is then:

$$
\Sigma_1^{(a)} = \frac{g_0^2}{16\pi^2} C_F \left(\log p^2 - 0.380646 \right). \quad (15.124)
$$

This term, together with the analog term coming from the self-energy tadpole which we will compute soon, needs to be added to the results of the proper diagrams (vertex, sails and operator tadpole). In fact this is the wave function renormalization, and inserted into the tree-level diagram for the operator gives a leg correction:

$$
i\slashed{p}\, \Sigma_1 \cdot S(p) \cdot i\gamma_\mu p_\nu = i\slashed{p}\, \Sigma_1 \cdot \frac{-i\slashed{p}}{p^2} \cdot i\gamma_\mu p_\nu = i\gamma_\mu p_\nu\, \Sigma_1 \quad (15.125)
$$

If a mass regularization is used one gets

$$
J_m = i\slashed{p} \frac{g_0^2}{16\pi^2} C_F \left(\log m^2 + \gamma_E - F_0 + 5.411364 \right) = i\slashed{p} \frac{g_0^2}{16\pi^2} C_F \left(\log m^2 + 1.619354 \right), \quad (15.126)
$$

and

$$
(I - J)_m = i\slashed{p} \frac{g_0^2}{16\pi^2} C_F \left(\log p^2 - \log m^2 - 2 \right). \quad (15.127)
$$

15.4.6   Quark self-energy (tadpole diagram)

The last diagram, which completes the lattice self-energy and has no analog in the continuum, is the self-energy tadpole, which originates from the irrelevant vertex in Eq. (5.78). This vertex, when put in the tadpole diagram, depends only on the external momentum. The diagram is then [69]

$$
\begin{aligned}
I &= \frac{1}{2} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d k}{(2\pi)^d} \sum_\rho G_{\rho\rho}(k) \cdot (V_2^{aa})_{\rho\rho}(p, p)\\
&= \frac{1}{2} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d k}{(2\pi)^d} a^2 \frac{1}{4 \sum_\lambda \sin^2 \frac{ak_\lambda}{2}} \left(-\frac{1}{2} a g_0^2 \sum_a \{T^a, T^a\}_{cc} \right) \sum_\rho \left(-i\gamma_\rho \sin a p_\rho + \cos a p_\rho \right)
\end{aligned}
$$

---

[69] Note that we have used $\lim_{a \to 0} \sum_\rho \cos a p_\rho = 4$.

$$= -\frac{1}{2}g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{4\sum_{\lambda}\sin^2\frac{k_\lambda}{2}} \left(-i\not{p} + \frac{4}{a}\right)$$

$$= -\frac{1}{2}g_0^2 C_F Z_0 \left(-i\not{p} + \frac{4}{a}\right). \tag{15.128}$$

The last term diverges like $1/a$, and therefore is part of Σ_0. In fact it gives a substantial contribution to the critical mass:

$$m_c^{(b)} = -g_0^2 C_F \cdot 2Z_0 = -\frac{g_0^2}{16\pi^2}C_F \cdot 48.932201. \tag{15.129}$$

The 1-loop critical mass for Wilson fermions is then: [70]

$$m_c = m_c^{(a)} + m_c^{(b)} = -\frac{g_0^2}{16\pi^2}C_F \cdot 51.434712 = -g_0^2 C_F \cdot 0.325714. \tag{15.130}$$

The term proportional to $i\not{p}$ gives the contribution of the self-energy tadpole to the renormalization of the first moment operator:

$$\Sigma_1^{(b)} = \frac{g_0^2}{16\pi^2}C_F \cdot 12.233050. \tag{15.131}$$

The total result for the 1-loop self-energy on the lattice in the massless case is then

$$\Sigma_1 = \Sigma_1^{(a)} + \Sigma_1^{(b)} = \frac{g_0^2}{16\pi^2}C_F \left(\log p^2 + 11.852404 + (1-\alpha)\left(-\log p^2 + 4.792010\right)\right), \tag{15.132}$$

where we have also included the part which one would obtain if the calculation were carried out in a general covariant gauge.

15.4.7 Concluding remarks

We have thus computed all diagrams which are necessary for the 1-loop renormalization of the operator $O_{\{\mu\nu\}} = \bar{\psi}\gamma_{\{\mu}D_{\nu\}}\frac{\lambda^\alpha}{2}\psi$, with different indices. Collecting the results of the various diagrams together, we have the final result (Capitani, 2001a)

$$\langle q|O_{\{01\}}|q\rangle\Big|_{1\ loop} = \frac{1}{2}i(\gamma_0 p_1 + \gamma_1 p_0) \cdot \frac{g_0^2}{16\pi^2}C_F \left(\frac{8}{3}\log a^2 p^2 - 3.16486\right). \tag{15.133}$$

This allows us to specify Eq. (3.1) for this operator, with $R_{ij}^{lat} = -3.16486 \cdot C_F$. For the 1-loop matching to the $\overline{\text{MS}}$ scheme one needs also to know the $R_{ij}^{\overline{\text{MS}}}$ factor in Eq. (3.2). Its value is $-40/9 \cdot C_F$, and can be inferred from the calculations of the various $I - J$ integrals that we

---

[70] We will later see that the 1-loop critical mass can be expressed in terms of only two constants (Eq. (18.117)), which are calculable with very high precision. Its 2-loop value is given at the end of Section 19.

have just done, by summing the contributions of vertex, sails and the sunset diagram of the self-energy (which are 5/9, −4 and −1 respectively). We then obtain for this operator

$$\langle q|O^{lat}_{\{01\}}|q\rangle \;=\; \left(1 + \frac{g_0^2}{16\pi^2}C_F\Big(\frac{8}{3}\log a^2p^2 - 3.16486\Big)\right)\cdot\langle q|O^{tree}_{\{01\}}|q\rangle \qquad (15.134)$$

$$\langle q|O^{\overline{MS}}_{\{01\}}|q\rangle \;=\; \left(1 + \frac{g_{\overline{MS}}^2}{16\pi^2}C_F\Big(\frac{8}{3}\log\frac{p^2}{\mu^2} - \frac{40}{9}\Big)\right)\cdot\langle q|O^{tree}_{\{01\}}|q\rangle, \qquad (15.135)$$

so that the renormalization factor that converts the raw lattice results (in the Wilson formulation) to the $\overline{MS}$ scheme is (for $\mu = 1/a$)

$$\langle q|O^{\overline{MS}}_{\{01\}}|q\rangle = \Big(1 - \frac{g_0^2}{16\pi^2}C_F \cdot 1.27958\Big)\cdot\langle q|O^{lat}_{\{01\}}|q\rangle. \qquad (15.136)$$

For the typical value $g_0 = 1$ (which corresponds to a scale $1/a$ of about 2 GeV in the quenched approximation) we then obtain

$$\langle q|O^{\overline{MS}}_{\{01\}}|q\rangle = 0.98920 \cdot \langle q|O^{lat}_{\{01\}}|q\rangle. \qquad (15.137)$$

We would now like to make some comments on these lattice calculations. First we notice that the magnitude of the self-energy tadpole is much larger than the result for the vertex and the sails, and this is a general feature of lattice calculations, which is called "tadpole dominance" of perturbation theory. It has been in some cases used to estimate the value of the radiative corrections of matrix elements, by neglecting all diagrams other than the tadpole.

However, even if the tadpole of the self-energy is large, sometimes this is not the end of the story. An uncritical application of tadpole dominance can be at times misleading. For example, in the calculations that we have just done we have encountered a situation in which the operator tadpole, which also gives a large number, exactly cancels the self-energy tadpole, so that the final result for the matrix element is given only from the contributions of the vertex and the sails (plus the sunset diagram of the self-energy). This cancellation only happens for the first moment. For higher moments, that is the case in which we have an operator with n covariant derivatives with indices all different from each other, the result of the operator tadpole is $nZ_0/2$ (see also Section 18.3), [71] so the final result has even a different sign from the one that would be inferred from computing the self-energy tadpole alone.

As discussed in Section 14, there exists also another class of operators which measure the first moment of the quark momentum distribution and which belong to another representation of the hypercubic group:

$$O_{00} - \frac{1}{3}(O_{11} + O_{22} + O_{33}). \qquad (15.138)$$

---

[71] This result is only valid for n between 2 and 4, as for higher n is impossible to have all indices distinct from each other, and the contractions of the A_μ's become more complicated and do not give just $nZ_0/2$. The numerical result for $n \geq 5$ is however not far from this number, and one has again a final positive result when all diagrams are summed, with dominance of the operator tadpole and not of the self-energy tadpole.

Due to the breaking of the Lorentz invariance, on the lattice this operator has a renormalization constant different from $O_{\{01\}}$. We leave as an exercise for the interested reader to reproduce the final 1-loop result [72]

$$\langle q|O_{00} - \frac{1}{3}\sum_{i=1}^{3} O_{ii}|q\rangle\Big|_{1\ loop} = \frac{1}{2}\mathrm{i}\Big(\gamma_0 p_0 - \frac{1}{3}\sum_{i=1}^{3}\gamma_i p_i\Big)\cdot\frac{g_0^2}{16\pi^2}C_F\Big(\frac{8}{3}\log a^2 p^2 - 1.88259\Big), \quad (15.139)$$

which gives the 1-loop matching factor to the $\overline{\mathrm{MS}}$ scheme

$$\langle q|(O_{00} - 1/3\sum_{i=1}^{3} O_{ii})^{\overline{\mathrm{MS}}}|q\rangle = 0.97837\cdot\langle q|(O_{00} - 1/3\sum_{i=1}^{3} O_{ii})^{lat}|q\rangle. \quad (15.140)$$

We want to conclude this Section by mentioning that in the case of a calculation with the improved action we have to add the Sheikholeslami-Wohlert improved vertex of Eq. (11.10), and also the operators have to be improved. This renders the calculations much more cumbersome. Since the Sheikholeslami-Wohlert vertex includes a σ matrix, more Dirac matrices appear in the manipulations. In the case in which both vertices in the vertex function are taken to be the improved ones, the chains of Dirac matrices can become quite long.

The improvement of the operators, in the case that we have just calculated, means that we have to consider also the contribution to the renormalization constants coming from the operators in Eq. (11.15). These calculations are quite complicated, and the results are given in (Capitani *et al.*, 2001b).

15.5 Example of overlap results

To follow an overlap calculation step by step in the same way as we did for the Wilson case would be quite cumbersome. We give here, as an example of results, the analytic expressions for the tadpole of the self-energy of the quark, $\Sigma_1^{(b)}$, and for the vertex of the scalar current $\overline{\psi}\psi$ (in the Feynman gauge, for $r = 1$). In order to be able to write them in a compact form it is convenient to introduce further abbreviations:

$$B = b(k) = 2\sum_{\lambda}\sin^2\frac{k_\lambda}{2} - \rho, \quad (15.141)$$

$$D = 2\rho\Big(\omega(k) + b(k)\Big), \quad (15.142)$$

$$A = \frac{\omega^2(k)}{\rho^2} = 1 - \frac{4}{\rho}\sum_{\lambda}\sin^2\frac{k_\lambda}{2} + \frac{1}{\rho^2}\Big(\sum_{\lambda}\sin^2 k_\lambda + \Big(2\sum_{\lambda}\sin^2\frac{k_\lambda}{2}\Big)^2\Big), \quad (15.143)$$

---

[72]It might turn useful to know that the numerical results for all individual diagrams are the same as in the calculation with different indices, except for the vertex diagram, in which the number for the finite part is now 3.575320.

with $\omega(k)$ given in Eq. (8.19). The result for the 1-loop tadpole of the quark self-energy for overlap fermions is then given by

$$\frac{1}{2}g_0^2 \int \frac{d^4k}{2\pi^4} G_{\mu\mu}(k)\left(1 - \frac{4}{\rho}\right) + g_0^2 \int \frac{d^4k}{2\pi^4} G_{\mu\mu}(k)\frac{1}{\rho^2(1 + \sqrt{A})^2} \tag{15.144}$$

$$\times \sum_\lambda \left[M_\lambda^2 + N_\lambda^2 + \left(1 + \frac{1}{\sqrt{A}}\right)\left(-\Gamma_\mu^2 + B(M_\mu^2 - N_\mu^2)\right) + \frac{2 + \sqrt{A}}{\rho\sqrt{A}}\left(-B(M_\lambda^2 - N_\lambda^2) + \Gamma^2\right)\right]$$

$$+ g_0^2 \int \frac{d^4k}{2\pi^4} G_{\mu\mu}^2(k)\frac{1}{\rho^2(1 + \sqrt{A})^2} \sum_\lambda \left[-2\Gamma_\mu^2 N_\lambda^2 + \left(1 + \frac{1}{\sqrt{A}}\right)\left(2\Gamma_\mu^2(B + 2M_\mu^2 - M_\lambda^2)\right)\right].$$

The first term comes from the part of the overlap vertex V_2 in Eq. (8.29) containing W_2 and $W_2^\dagger$, and its value for $\rho = 1$ is quite large,

$$\frac{1}{2}g_0^2 \cdot Z_0\left(1 - \frac{4}{\rho}\right)\bigg|_{\rho=1} = -\frac{g_0^2}{16\pi^2} 36.69915, \tag{15.145}$$

while the 1-loop result for the whole self-energy tadpole in Eq. (15.144) is slightly smaller, but still large:

$$-\frac{g_0^2}{16\pi^2} 23.35975. \tag{15.146}$$

Adding now the value $-14.27088\, g_0^2/(16\pi^2)$ for the sunset diagram of the overlap self-energy of the quark, which is much harder to compute by hand and would produce a very lengthy analytic expression, gives the result $-37.63063\, g_0^2/(16\pi^2)$ for the complete self-energy in the Feynman gauge, for $\rho = 1$ (Alexandrou *et al.*, 2000; Capitani, 2001a). The values of the overlap self-energy for various choices of ρ and in a general covariant gauge are given in Table 1. One advantage of using overlap fermions is that Σ_0, the power divergent part of the self-energy that for Wilson fermions gives a nonzero additive mass renormalization, vanishes here.

The results for the 1-loop vertex diagram of the scalar operator [73] can be written in the form

$$g_0^2 \int \frac{d^4k}{2\pi^4} G(p - k)\frac{1}{(1 + \sqrt{A})^2}\left[\left(-\frac{\Gamma^2}{D^2} + \frac{1}{4\rho^2}\right) \cdot X + \frac{1}{\rho D} \cdot Y\right] \tag{15.147}$$

for small p, where

$$X = \sum_\lambda \left[-(M_\lambda^2 - N_\lambda^2) + \frac{1}{\rho\sqrt{A}}2(M_\lambda^2 + N_\lambda^2) + \frac{1}{\rho^2 A}\left((\Gamma^2 - B^2)(M_\lambda^2 - N_\lambda^2) + 2B\Gamma^2\right)\right]$$

$$Y = \sum_\lambda \left[\Gamma^2 + \frac{1}{\rho\sqrt{A}}2\Gamma^2(M_\lambda^2 + N_\lambda^2) + \frac{1}{\rho^2 A}\Gamma^2\left(-2B(M_\lambda^2 - N_\lambda^2) + \Gamma^2 - B^2\right)\right]. \tag{15.148}$$

These are the only diagrams for which the author has had enough patience to perform their calculation by hand using the overlap action (Capitani, 2001a). Other overlap calculations of

---

[73]We note that the sails are not present in this case, as there are no covariant derivatives.

ρ	self-energy (sunset)	self-energy (tadpole)	total self-energy
0.2	-27.511695 +11.911596 ξ	-213.087934 -7.119586 ξ	-240.599629 +4.792010 ξ
0.3	-23.687573 +11.098129 ξ	-131.723110 -6.306119 ξ	-155.410693 +4.792010 ξ
0.4	-21.172454 +10.520210 ξ	-91.817537 -5.728200 ξ	-112.989991 +4.792010 ξ
0.5	-19.337313 +10.071356 ξ	-68.315503 -5.279346 ξ	-87.652816 +4.792010 ξ
0.6	-17.912921 +9.704142 ξ	-52.931363 -4.912132 ξ	-70.844284 +4.792010 ξ
0.7	-16.760616 +9.393275 ξ	-42.140608 -4.601265 ξ	-58.901224 +4.792010 ξ
0.8	-15.800204 +9.123666 ξ	-34.193597 -4.331656 ξ	-49.993801 +4.792010 ξ
0.9	-14.981431 +8.885590 ξ	-28.125054 -4.093580 ξ	-43.106485 +4.792010 ξ
1.0	-14.270881 +8.672419 ξ	-23.359746 -3.880409 ξ	-37.630627 +4.792010 ξ
1.1	-13.645294 +8.479438 ξ	-19.534056 -3.687428 ξ	-33.179350 +4.792010 ξ
1.2	-13.087876 +8.303183 ξ	-16.407174 -3.511173 ξ	-29.495050 +4.792010 ξ
1.3	-12.586126 +8.141044 ξ	-13.813486 -3.349034 ξ	-26.399612 +4.792010 ξ
1.4	-12.130497 +7.991018 ξ	-11.635482 -3.199008 ξ	-23.765979 +4.792010 ξ
1.5	-11.713524 +7.851554 ξ	-9.787582 -3.059544 ξ	-21.501106 +4.792010 ξ
1.6	-11.329238 +7.721442 ξ	-8.206069 -2.929432 ξ	-19.535307 +4.792010 ξ
1.7	-10.972744 +7.599750 ξ	-6.842630 -2.807740 ξ	-17.815374 +4.792010 ξ
1.8	-10.639905 +7.485778 ξ	-5.660084 -2.693768 ξ	-16.299989 +4.792010 ξ
1.9	-10.327042 +7.379023 ξ	-4.629539 -2.587013 ξ	-14.956581 +4.792010 ξ

Table 1: Results for the finite constant of the quark self-energy with overlap fermions in a general covariant gauge, from (Capitani, 2001a). We have used the abbreviation $\xi = 1 - \alpha$. The first column (sunset) refers to the diagram on the left in Fig. 18, and the second column (tadpole) to the diagram on the right.

renormalization factors made using FORM codes are reported there and in (Capitani, 2001b; Capitani and Giusti, 2000; Capitani and Giusti, 2001).

The renormalization constants of operators computed using overlap fermions are sometimes large, and in general larger than the corresponding Wilson results (Capitani, 2001a; Capitani, 2001b). For example for the first moment of the unpolarized quark distribution (operator $O_{\{01\}}$) the constant is -53.25571, while (as we have just seen) in the Wilson case is -3.16486. However, for the proper diagrams the overlap results show only relatively small differences from the Wilson numbers. The biggest contribution to the renormalization constant comes from the operator tadpole, and it is exactly the same for overlap and Wilson fermions. The difference between overlap and Wilson results is then almost entirely due to the quark self-energy. In the Feynman gauge, the constant of the finite part of the self-energy in the overlap (for $\rho = 1$) is -37.63063, while for Wilson fermions is $+11.85240$; their difference is a large number, -49.48303, and quite close to the difference of the total renormalization constants for

the two kinds of fermions, -50.09085. We can notice from Table 1 that the value of the overlap self-energy decreases when ρ increases, and if one would consider the overlap fermions for $\rho = 1.9$, the difference between the self-energies would go from -49.48303 down to -26.80898. However, the quark propagator becomes singular for $\rho = 2$, so simulations would likely be more expensive when approaching this value of ρ.

We note that, contrary to the Wilson case, where the tadpole gives a much larger contribution than the sunset, in the overlap things are more entangled, and for certain values of ρ it is the sunset that gives a larger result than the tadpole.

15.6   Tadpole improvement

We have seen that gluon tadpoles give large numerical results compared to other diagrams (although sometimes they happen to cancel with each other). Quite often the large corrections which occur in lattice perturbation theory are caused by these tadpole diagrams. Since they are an artifact of the lattice (the corresponding interaction vertex is zero in the naive continuum limit), in order to attempt to make the lattice perturbative expansions closer to the continuum ones a tadpole resummation method has been proposed in (Parisi, 1980) and (Lepage and Mackenzie, 1993). The lectures of (Mackenzie, 1995) contain a pedagogical introduction to these ideas.

This resummation of tadpoles amounts to a mean-field improvement, in which one makes a redefinition of the link, separating the contributions of their infrared modes:

$$U_\mu(x) = \mathrm{e}^{ig_0 a A_\mu(x)} = u_0 \, \mathrm{e}^{ig_0 a A_\mu^{IR}(x)} = u_0 \, \widetilde{U}_\mu(x). \tag{15.149}$$

The rescaling factor u_0 (which is a number between 0 and 1) contains the high-energy part of the link variables. One makes in the gluon action the substitution

$$U_\mu(x) = u_0 \, \widetilde{U}_\mu(x), \tag{15.150}$$

and takes $\widetilde{U}_\mu(x)$ as the new link variable. This implies $\widetilde{\beta} = u_0^4 \beta$. The effective coupling becomes then $\widetilde{g}_0^2 = g_0^2/u_0^4$, and is claimed to be closer to the coupling defined in the $\overline{\mathrm{MS}}$ scheme than the original lattice coupling. In fact, the perturbative expansions in terms of $\widetilde{g}_0$ have often smaller coefficients and are better behaved than the standard perturbative expansions.

The value of the rescaling factor u_0 is taken from the mean value of the link in the Landau gauge,

$$u_0 = \left\langle \frac{1}{N_c} \, \mathrm{Re} \, \mathrm{Tr} U_\mu(x) \right\rangle, \tag{15.151}$$

because in this gauge its value is higher (in other gauges u_0 can assume very small values). However, the fourth root of the mean value of the plaquette,

$$u_P = \left\langle \frac{1}{N_c} \, \mathrm{Re} \, \mathrm{Tr} P_{\mu\nu}(x) \right\rangle^{1/4}, \tag{15.152}$$

is easier to compute and is then taken in place of u_0 in most applications, although one should always keep in mind that this is a good approximation only for small lattice spacings. When $a \sim 0.4$ fm the two definitions differ already by about 10%. In this case, if one sticks to the plaquette definition, every perturbative quantity should be computed at least to 2 loop. There is then a small ambiguity in this choice.

In the resummed theory one works with tadpole-improved actions and operators. Of course all quantities that contain a field U have to be rescaled accordingly, and will be multiplied by some positive or negative power of u_0. This then applies also to the covariant derivatives appearing in operators measuring moments of structure functions. When all variables are properly rescaled the large tadpole contributions can be reabsorbed, and after this tadpole improvement has been implemented the coefficients in the results of perturbative lattice calculations get in general smaller.

The above construction can be seen as a different choice of the strong coupling (and of other parameters), and is equivalent to a reorganization of lattice perturbation theory. At the end of the day, this is a theory in which a redefinition of the coupling is taking place, and then it has a different value of the Λ parameter. [74] So, one should be careful and do things consistently, because one is effectively changing scheme. This would not matter if one knew the complete series, but at the lowest orders it makes a difference. It is however difficult to estimate the error that results from this. Moreover, not always this different definition of the coupling gives a better perturbative expansions. Not in all cases the contributions get smaller.

It is possible to set up tadpole improvement also on Symanzik-improved theories, rescaling the affected quantities with appropriate powers of u_0. For example, the improvement coefficient c_{sw} undergoes the rescaling $\widetilde{c}_{sw} = u_0^3 c_{sw}$. The coefficients of the Lüscher-Weisz improved gauge action are modified as follows:

$$c_0 = \frac{5}{3u_0^4}, \qquad c_1 = -\frac{1}{12u_0^6}. \tag{15.153}$$

A different kind of improvement of lattice perturbation theory has been proposed and used in (Panagopoulos and Vicari, 1998; Panagopoulos and Vicari, 1999). It works through the resummation of "cactus" diagrams, i.e., tadpoles which become disconnected if any one of their vertices is removed. These diagrams are gauge invariant and it seems that in some cases, for example the renormalization of the topological charge, this kind of resummation achieves better results than standard tadpole improvement.

We conclude mentioning that for overlap fermions simple tadpole improvements like the ones discussed here seem not to be of much help, because also the sunset diagram of the self-energy gives large results, and this is a diagram that is present also in the continuum. Probably

---

[74]This is similar to what happens in continuum QCD renormalized in the minimal subtraction (MS) scheme, where some second-order perturbative corrections can sometimes be large. If however one systematically drops the factors γ_E and $\log 4\pi$ these corrections become much smaller, and this defines the $\overline{\text{MS}}$ scheme, where one has a different renormalized coupling and a different value of the Λ parameter.
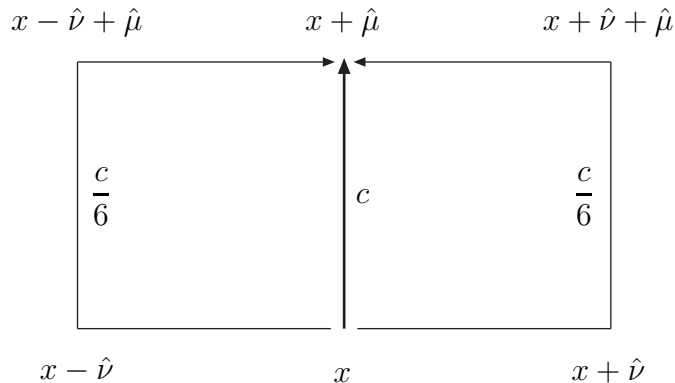
Figure 20: The APE smearing of the thin link $U_\mu(x)$ which produces the fat link.

a different kind of diagram resummation has to be devised in this case.

15.7   Perturbation theory for fat links

Another way to reduce the magnitude of the large renormalization factors seems to be given, at least in some cases, by fat link actions (DeGrand, Hasenfratz and Kovács, 1999). Under the name of fat links are meant actions in which the quarks couple to gauge links which are smeared (see Fig. 20). They present some advantages, like the facts that exceptional configurations are suppressed and the additive mass renormalization is small. Furthermore, these actions exhibit better chiral properties. From the point of view of perturbation theory the interesting feature is that renormalization factors often have values quite close to unity, and hence this could be worth of some consideration in the cases in which the 1-loop corrections of matrix elements on the lattice are large compared to their tree-level values. Damping the large perturbative corrections can be particularly useful in the case of overlap fermions. Simulations using overlap-like fermions with fat links have been reported in (DeGrand, 2001), and the improvement of the locality and topological properties using overlap and overlap-like fermions with fat links has been studied in (DeGrand, Hasenfratz and Kovács, 2002) and (Kovács, 2002).

We follow (Bernard and DeGrand, 2000), and we consider the particular construction known as APE blocking (Albanese *et al.*, 1987), where the smearing of a link is done as follows. Starting with the original link, also known as thin link,

$$V_\mu^{(0)}(x) = U_\mu(x), \tag{15.154}$$

the first smearing step is done like in Fig. 20. The general smearing step constructs the fat link recursively as

$$V_\mu^{(m+1)}(x) = \mathcal{P}\left((1-c)V_\mu^{(m)}(x) + \frac{c}{6}\sum_{\nu\neq\mu}\left[V_\nu^{(m)}(x)V_\mu^{(m)}(x+\hat{\nu})V_\nu^{\dagger(m)}(x+\hat{\mu})\right.\right. \tag{15.155}$$

$$+V_\nu^{\dagger(m)}(x - \hat\nu)V_\mu^{(m)}(x - \hat\nu)V_\nu^{(m)}(x - \hat\nu + \hat\mu)\Big]\Big),$$

where $\mathcal{P}$ projects back into $SU(3)$ matrices. A typical value that is used is $c = 0.45$, and a typical value for the total number of iterations is $N = 10$ (see for example (Bernard *et al.*, 2000)).

In perturbation theory the basic variables are the A_μ's, and for 1-loop computations of quark operators only the linear part of the relation between them,

$$A_\mu^{(1)}(x) = \sum_y \sum_\nu h_{\mu\nu}(y)A_\nu^{(0)}(x + y), \tag{15.156}$$

is relevant, because the quadratic part, being antisymmetric, gives no contributions to the tadpoles (which at one loop are the only diagrams that can be constructed from two gluons stemming from the same point). In momentum space this convolution becomes a form factor,

$$A_\mu^{(1)}(q) = \sum_\nu \widetilde{h}_{\mu\nu}(q)A_\nu^{(0)}(q), \tag{15.157}$$

where

$$\widetilde{h}_{\mu\nu}(q) = \Big(1 - \frac{c}{6}\widehat{q}^2\Big)\Big(\delta_{\mu\nu} - \frac{\widehat{q}_\mu\widehat{q}_\nu}{\widehat{q}^2}\Big) + \frac{\widehat{q}_\mu\widehat{q}_\nu}{\widehat{q}^2}. \tag{15.158}$$

We notice that the longitudinal part $\widetilde{h}$ is not affected by the smearing, because this part is unphysical and does not depend on which path of links between two fixed sites is chosen. After N smearings one gets

$$A_\mu^{(N)}(q) = \sum_\nu \widetilde{h}_{\mu\nu}^{(N)}(q)A_\nu^{(0)}(q), \tag{15.159}$$

with

$$\widetilde{h}_{\mu\nu}^{(N)}(q) = \Big(1 - \frac{c}{6}\widehat{q}^2\Big)^N\Big(\delta_{\mu\nu} - \frac{\widehat{q}_\mu\widehat{q}_\nu}{\widehat{q}^2}\Big) + \frac{\widehat{q}_\mu\widehat{q}_\nu}{\widehat{q}^2}. \tag{15.160}$$

Since this is a correction to the gauge interaction of the quarks, the effect is that each quark-gluon vertex is multiplied by a form factor $\widetilde{h}_{\mu\nu}^{(N)}(q)$, where q is the gluon momentum. Now we consider the case, which often happens in practice, in which all gluon lines of a diagram start and end on quark lines. If this is true one can imagine these form factors as attached to the gluon propagator instead than to the vertices. The effect of smearing can then be summarized in a simple modification of the gluon propagator [75]

$$G_{\mu\nu}(q) \longrightarrow \widetilde{h}_{\mu\lambda}^{(N)}(q)G_{\lambda\rho}(q)\widetilde{h}_{\rho\nu}^{(N)}(q). \tag{15.161}$$

---

[75]Here $G_{\mu\nu}(q)$ is the standard Wilson gluon propagator, because the fat link action changes the quark-gluon interaction but leaves the plaquette action unaltered, and in particular the gluon propagator. This is in some sense complementary to the case of improved gluons (Section 11.2), where the pure gauge action is modified but not the quark-gluon interaction. Recently actions which contain both fat links and an improvement of the pure gauge part have also been considered in perturbation theory (DeGrand, Hasenfratz and Kovács, 2002).

It is now clear that the Landau gauge is the most natural setup for fat link calculations. The Landau gauge propagator kills the longitudinal components of $\widetilde{h}$, and it is then quite easy to convert a thin link calculation made in Landau gauge to a fat link result, provided the loop integration momentum was chosen to be the same as the gluon momentum q (which we have *not* done in Section 15.4):

$$\int \frac{d^4q}{(2\pi)^4}\, \mathcal{I}(q) \to \int \frac{d^4q}{(2\pi)^4} \left(1 - \frac{c}{6}\widehat{q}^2\right)^{2N} \mathcal{I}(q). \tag{15.162}$$

Thus, no matter how many iterations N one makes, the final result is simply given by the multiplication of the old thin link integrand with a form factor. In the case in which the thin link calculation was instead done in Feynman gauge, one can observe that doing the same calculation using fat links in Landau gauge will generate new terms, coming from the $\left(1 - \frac{c}{6}\widehat{q}^2\right)^N \frac{\widehat{q}_\mu \widehat{q}_\nu}{\widehat{q}^2}$ part of $\widetilde{h}^{(N)}_{\mu\nu}(q)$. However, when $\left(1 - \frac{c}{6}\widehat{q}^2\right) = 1$, the total contribution of these terms must vanish by gauge invariance. If this cancellation already occurs at the level of the integrands, the further multiplication by $\left(1 - \frac{c}{6}\widehat{q}^2\right)^N$ will not spoil it, and then one can use again Eq. (15.162) for passing from thin to fat links, even if the thin link calculation was done in Feynman gauge. This has been verified for the additive and multiplicative renormalization of the mass (Bernard and DeGrand, 2000). However, if the cancellation is more subtle, for example involving integration by parts, then Eq. (15.162) is not valid anymore and these new fat link terms have to be computed from scratch.

As we mentioned at the beginning, one of the attractive features of fat links is that renormalization factors seem to be much closer to their tree level values. The Wilson tadpole diagram $12.233050\, g_0^2/(16\pi^2)C_F$ (see Eq. (15.131)), which is responsible for many of the large corrections in lattice perturbation theory, becomes in the fat link case $0.346274\, g_0^2/(16\pi^2)C_F$ (when $c = 0.45$ and $N = 10$ are used). Thus tadpole improvement is not necessary in this kind of calculations. Other quantities, like the 1-loop correction to the renormalization of the various currents, have been verified to be smaller of nearly two orders of magnitudes with respect to the Wilson results.

The reason of these small factors is that if c is not too large, $c < 0.75$, the absolute value of $\left(1 - \frac{c}{6}\widehat{q}^2\right)$ is less than one, and so $\left(1 - \frac{c}{6}\widehat{q}^2\right)^N$ is a very small factor, except perhaps for the region of very small momenta. Thus, the region of large momenta is completely suppressed, and this is precisely the dominant contribution to the tadpole diagrams. We can then understand why these fat link integrals are strongly suppressed. [76] This damping however does not necessarily take place when divergent diagrams are considered. In this case one has to perform appropriate subtractions, and not all terms in the original integral remain proportional to $\left(1 - \frac{c}{6}\widehat{q}^2\right)^{2N}$. The finite part of a divergent fat link integral is then not necessarily small, and more investigations are needed to understand the general situation of divergent integrals.

[76]We should however point out that a too strong suppression could render the integrals infrared sensitive.

Recently a new bunch of perturbative calculations with fat link actions has been reported (De-Grand, 2002; DeGrand, Hasenfratz and Kovács, 2002). They mainly use another smearing choice known as hypercubic blocking (Hasenfratz and Knechtli, 2001; Hasenfratz, Hoffmann and Knechtli, 2002; Hasenfratz and Knechtli, 2002), which is more complicated and involves three free parameters which need to be fixed. Using this version of fat links the renormalization constants of the quark currents as well as of weak four-fermion operators have been computed, with the fattening of the links applied to the Wilson, the improved (for the quark as well as the gluon part) and the overlap actions. The case of fat links applied to staggered fermions has been extensively studied in (Lee and Sharpe, 2002a; Lee, 2002; Lee and Sharpe, 2002b).

16   Computer codes

The calculation of the 1-loop renormalization constant of the unimproved matrix element $\langle q|O_{\{\mu\nu\}}|q\rangle$, that we have described in detail in the previous Section, can be done entirely by hand. However, even in this relatively simple case a computer program (written in the FORM language) has also been used to have some cross-checks on the correctness of all results. It is in general useful to have this kind of mutual checks between calculations made by hand and calculations made using a computer.

If one wants to compute the matrix element $\langle q|O_{\{\mu\nu\}}|q\rangle$ in the improved theory, adding the vertex in Eq. (11.10) to the usual Wilson vertex, or even including the improvement of the operator as well, which means computing the renormalization of the operators in Eq. (11.15), or even worse if one considers more complicated operators, containing perhaps more covariant derivatives, computer programs become necessary, as the huge number of manipulations and the size of the typical monomials makes the completion of these calculations entirely by hand almost impossible. Of course these codes turn out at the end to be necessary also because they can provide an output file of the result of the analytic manipulations which is already formatted (for example in Fortran) as an input file for the numerical integration.

One of the main reasons for the increasing difficulty in the manipulations for the moments of unpolarized structure functions is that the covariant derivative is proportional to the inverse of the lattice spacing, $D \sim 1/a$, so that one has

$$\langle x^n \rangle \quad \sim \quad \langle \overline{\psi}\, \gamma_\mu\, D_{\mu_1} \cdots D_{\mu_n}\, \psi \rangle \quad \sim \quad \frac{1}{a^n}. \tag{16.1}$$

This means that to compute the n-th moment one needs to perform a Taylor expansion in a to order n, and then every single object (propagators, vertices, operator insertions) has to be expanded to this order. One does not need too much imagination to see what happens. It is sufficient to have a look to the Wilson quark-quark-gluon vertex to order a^2,

$$(V^a)_\mu^{bc}(k, ap) \;\; = \;\; -g_0\, (T^a)^{bc} \cdot \left\{ \mathrm{i}\gamma_\mu \Big[\cos\frac{k_\mu}{2} - \frac{1}{2}ap_\mu\, \sin\frac{k_\mu}{2} - \frac{1}{8}a^2 p_\mu^2\, \cos\frac{k_\mu}{2} \Big] \right. \tag{16.2}$$

$$
+r\Big[\sin\frac{k_\mu}{2} + \frac{1}{2}ap_\mu\cos\frac{k_\mu}{2} - \frac{1}{8}a^2p_\mu^2\sin\frac{k_\mu}{2}\Big]\Big\},
$$

or to the expansion of the Wilson quark propagator even only to order a,

$$
S^{ab}(k+aq, am_0) = \delta^{ab}\cdot\Bigg\{\frac{-\mathrm{i}\sum_\mu\gamma_\mu\sin k_\mu + 2r\sum_\mu\sin^2\frac{k_\mu}{2}}{\sum_\mu\sin^2 k_\mu + \big[2r\sum_\mu\sin^2\frac{k_\mu}{2}\big]^2} \tag{16.3}
$$

$$
+a\cdot\Bigg[\frac{-\mathrm{i}\sum_\mu\gamma_\mu q_\mu\cos k_\mu + r\sum_\mu q_\mu\sin k_\mu + m_0}{\sum_\mu\sin^2 k_\mu + \big[2r\sum_\mu\sin^2\frac{k_\mu}{2}\big]^2}
$$

$$
-\Big(-\mathrm{i}\sum_\rho\gamma_\rho\sin k_\rho + 2r\sum_\rho\sin^2\frac{k_\rho}{2}\Big)\frac{\sum_\mu q_\mu\sin 2k_\mu + 4r\sum_\mu\sin^2\frac{k_\mu}{2}\Big(r\sum_\nu q_\nu\sin k_\nu + m_0\Big)}{\Big\{\sum_\mu\sin^2 k_\mu + \big[2r\sum_\mu\sin^2\frac{k_\mu}{2}\big]^2\Big\}^2}\Bigg]\Bigg\}.
$$

The algebraic manipulations become thus quite complex. The main consequence of all this is the generation of a huge number of terms, at least in the initial stages of the manipulations, even in the case of matrix elements where all Lorentz indices are contracted. The multiplication of two vertices and two quark propagators which are expanded to order a can be seen from the formulae above to give rise to about $4^2\cdot 11^2 \sim 2000$ monomial terms. Initial expansions of Feynman diagrams containing operators which measure the second and third moment of structure functions can easily reach the order of 10^6 terms. This slows down the execution of the codes considerably. Most of these terms become zero after doing the Dirac algebra, or do not contribute to the sought Dirac structure, or are zero after integration. The terms which do not contribute to the final expression have to be killed as early as possible to speed up the computations. Of course this is not easy; for example terms like the ones proportional to $1/(2W(\Gamma^2+W^2)^2)$ which we have mentioned after Eq. (15.83) are zero because $\mu\neq\nu$, but this can be seen only after the Dirac algebra has been solved.

Thus, the fact that an operator with n covariant derivatives requires Taylor expansions in a to order n also implies a limitation on the number of moments of structure functions that one can practically compute on the lattice. This is something different from the limitation coming from operator mixings, seen in Section 14, and the combination of these two computational challenges renders in practice the computation of the renormalization of the fourth moment or higher very difficult.

Also the gamma algebra becomes cumbersome to do by hand. This is even worse when one adds the improvement, because, as noted in the previous Section, adding a σ matrix for each improved vertex at the end builds up long chains of Dirac matrices.

At the end of the day, perturbation theory, even only at 1-loop level, is quite cumbersome on the lattice, and due to the complexity of the calculations, to the great number of diagrams, and to the huge amount of terms for each diagram, computer codes have to be used. To evaluate

the Feynman diagrams and obtain the algebraic expressions for the renormalization factors, the author has developed sets of computer codes written in the symbolic manipulation language FORM (for recent developments see (Vermaseren, 2000)). These codes are able to take as input the Feynman rules for the particular combination of operators, propagators (Wilson or overlap) and vertices (Wilson, and Sheikholeslami-Wohlert-improved, or overlap) appearing in each diagram, to expand them in the lattice spacing a at the appropriate order, to evaluate the gamma algebra on the lattice, and then to work out everything until the final sought-for expressions are obtained. Due to the enormous number of terms in the initial stages of the manipulations one needs in many cases a large working memory.

To properly deal with the γ_5 matrices additional computer routines have also been written which are able to perform computations in the 't Hooft-Veltman scheme, the only scheme proven to have consistent trace properties when γ_5 matrices are involved. [77] This has been important especially for doing calculations involving weak interactions and four-fermion operators (Capitani *et al.*, 1999a; Capitani *et al.*, 2000a; Capitani *et al.*, 2000b; Capitani *et al.*, 2001a). There is however an increase of about one order of magnitude in computing times when one uses the 't Hooft-Veltman scheme, due to the sum splittings, and also a careful memory management is required. These FORM codes are able to use Dimensional Regularization (NDR or 't Hooft-Veltman), and a mass regularization, and some independent checks are thus possible. As a further check on the codes, the author has in many cases performed the calculations also by hand.

Computer codes are of course often employed nowadays also for continuum perturbation theory, where however in general there are more external legs and one can reach higher loops, because the building blocks (the various Feynman rules, the typical monomials, etc.) are much simpler. But there are differences also in the codes themselves.

We have already mentioned several times that the Lorentz symmetry is broken on the lattice. This gives rise to a whole new series of problems, regarding for example the validity of the Einstein summation convention. One of the biggest challenges of computer codes for lattice perturbation theory is to deal with the fact that the summation convention on repeated indices is suspended. FORM, and similar programs, have been developed having in mind the usual continuum calculations. [78] There are therefore many useful built-in features that are in principle somewhat of an hindrance in doing lattice perturbative calculations. These built-in functions cannot be used straightforwardly on the lattice. This is for example what FORM would normally do, because two equal indices are assumed to be contracted:

$$\sum_\lambda \gamma_\lambda p_\lambda \quad \longrightarrow \quad \not{p} \tag{16.4}$$

[77] A definition of the Dirac matrices in this scheme is given in (Veltman, 1989). The work of (Jegerlehner, 2001) contains an interesting discussion about dimensional regularization, the use of chiral fields and their relation with the properties of γ_5 in noninteger dimensions.

[78] A recent description of this kind of programs can be found in (Weinzierl, 2002).

$$\sum_{\lambda} \gamma_{\lambda} p_{\lambda} \sin k_{\lambda} \quad \longrightarrow \quad \not{p} \sin k_{\lambda} \tag{16.5}$$

$$\sum_{\lambda} \gamma_{\lambda} \sin k_{\lambda} \cos^2 k_{\lambda} \quad \longrightarrow \quad (\gamma \cdot \sin k) \cos^2 k_{\lambda} \tag{16.6}$$

$$\sum_{\lambda,\rho} \gamma_{\rho} \gamma_{\lambda} \gamma_{\rho} \sin k_{\lambda} \cos^2 k_{\rho} \quad \longrightarrow \quad -2 \sum_{\lambda} \gamma_{\lambda} \sin k_{\lambda} \cos^2 k_{\rho}. \tag{16.7}$$

Here however the typical terms are monomials which contain more than twice the same index. Only the first case is then correctly handled by FORM. For example, in the last case the right answer is instead

$$-\sum_{\lambda,\rho} \gamma_{\lambda} \sin k_{\lambda} \cos^2 k_{\rho} + 2 \sum_{\rho} \gamma_{\rho} \sin k_{\rho} \cos^2 k_{\rho}. \tag{16.8}$$

For this reason one needs the development of special routines to deal with the gamma algebra on the lattice (for more details see also (Capitani and Rossi, 1995b; Capitani and Rossi, 1995a)). One solution is to introduce generalized Kronecker delta symbols (Lüscher and Weisz, 1995d)

$$\delta_{\mu_1 \mu_2 \dots \mu_n} \tag{16.9}$$

which are equal to one only if all indices are equal, $\mu_1 = \mu_2 = \dots = \mu_n$, and are zero otherwise. In general one needs special routines, with appropriate modifications to the usual commands, to properly treat Dirac matrices and handle terms like in Eq. (15.71).

In Feynman diagrams that involve a few gluons the color structure can become quite involved, especially if the 4-gluon vertex is present. A program for the automatic generation of gluon vertices and the reduction of the color structure has been described in (Lüscher and Weisz, 1986). Expression containing color tensors can be computed by repeatedly using the identities (Lüscher and Weisz, 1995d)

$$\mathrm{Tr}(T^a X T^a Y) = \frac{1}{2N_c} \mathrm{Tr}(XY) - \frac{1}{2} \mathrm{Tr}(X) \mathrm{Tr}(Y) \tag{16.10}$$

$$\mathrm{Tr}(T^a X) \mathrm{Tr}(T^a Y) = \frac{1}{2N_c} \mathrm{Tr}(X) \mathrm{Tr}(Y) - \frac{1}{2} \mathrm{Tr}(XY), \tag{16.11}$$

where X and Y stand for general complex $N_c \times N_c$ matrices.

Of course when the computations become so complicated that it is not possible to carry them out by hand, a number of additional checks on the codes is desirable. One can use different regularizations, like a mass regularization and dimensional regulation (in its various forms), and one can develop various routines which use different methods.

The case of overlap fermions is one in which manual checks are much more difficult, because only a few computations can be performed by hand. When computing renormalization factors it is useful then to exploit the cancellation of the gauge-dependent part between the continuum and the lattice results (as we noted in Section 3). In particular, in the calculations made in covariant gauge the contributions proportional to $(1-\alpha)$ must be independent of the parameter ρ, and this can only be seen after the numerical integration, since in the thousands of terms

which have to be integrated the dependence on ρ cannot be factored out. The dependence on ρ is in fact highly nontrivial, and a look at the quark propagator shows that the monomials in the integrand are not even rational functions of ρ. Thus, this is really a nontrivial check. Of course calculations made in a general covariant gauge are much more expensive than the restriction to the Feynman gauge, due to the more complicated form of the gluon propagator, but it is worthwhile to do them. They can give a strong check of the behavior of the FORM codes in the case of overlap fermions, as well as of the integration routines.

The contributions proportional to $(1 - \alpha)$, besides being independent of ρ, seem to a certain extent also to be independent of the fermion action used. In particular, for overlap fermions they have the same value as for the Wilson case. [79] These terms are in general equal to their Wilson counterparts at the level of the single diagrams. An exception is given by the self-energy, where only the sum of the sunset and the tadpole diagrams is independent of ρ, as can be seen in Table 1, and has the same value as in the Wilson action, as can be seen looking at Eq. (15.132). [80] The component of the total self-energy proportional to $(1 - \alpha)$ is proportional only to the combination $F_0 - \gamma_E + 1 = 4.792009568973\cdots$ (see Eqs. (18.25) and (18.26) and Table 2 later), and therefore is the same for all plaquette actions, irrespective of the fermionic part. Another case in which we have found that individual overlap diagrams do not correspond to their Wilson result is given by operators measuring the moment of the structure function g_2 (Capitani, 2001b). These operators are of the form

$$\overline{\psi}\gamma_{[\mu}\gamma_5 D_{\{\nu]}D_\rho \cdots D_{\lambda\}}\psi, \tag{16.12}$$

that is they involve both a symmetrization and an antisymmetrization of indices, and in the Wilson case they mix with other operators of lower dimension, with a $1/a$ divergent coefficient, while in the overlap case they are multiplicatively renormalizable, because these mixings are forbidden by chiral symmetry. For this operator only the sum of vertex and sails is independent of the fermion action used.

Lattice perturbative calculations generally involve the manipulation of a huge number of terms, but often a large number of terms remain also in the final analytic expressions which have to be numerically integrated. The integrations then require a lot of computer time, which in some cases can be of the order of hundreds of hours.

---

[79] In the case of overlap fermions however, due do the greater number of terms and the more complicated functional forms, one gets less precise numbers for these contributions (for a given computing time).

[80] This is probably connected to the fact that the 1-loop self-energy for Wilson fermions has a nonvanishing Σ_0 contribution, which gives the additive quark mass renormalization due to the breaking of chiral symmetry, while for overlap fermions this term is zero.

17 Lattice integrals

After the analytical manipulations in Feynman diagrams have been carried out, it still remains to integrate the resulting expressions. Lattice integrals are quite complicated rational functions of trigonometric expressions, and so far it has not been possible to compute them analytically. The only way to obtain a number from these expressions is to use numerical integration methods. The results of the analytic calculations obtained with computer programs can be stored as a sum of terms suitable to be integrated, which can be formatted in an appropriate way and passed on to become the input of a, say, Fortran program.

Integrals can be approximated by discrete sums. At the basic level, one computes the function to be integrated in a number of points, and uses the sum over these points (sometimes with appropriate weights) as an approximation to the true value of the integral. Since we are working in four dimensions (usually in momentum space), even choosing only 100 points in each direction means that the function has to be evaluated in 10^8 points, and the computational requirements grow then quite rapidly. This becomes much worse for 2-loop integrals, where one has to evaluate sums in eight dimensions. It is therefore invaluable, in order to compute 2-loop integrals, to use the coordinate space methods that we will introduce in Section 19, with which these 8-dimensional integrals can be expressed in terms of only 4-dimensional sums.

Using simple integration routines it is possible to evaluate numerically lattice 1-loop integrals with reasonable precision, and one can without much effort obtain results with five or six significant decimal places. There are also some more refined methods which have been devised and used to obtain faster and cheaper evaluations of these integrals and to improve on the precision of the integrations. We are going to explain some of them in the rest of this Section.

One expects (Symanzik, 1983b) that a lattice Feynman diagram D with l loops will have near the continuum limit the asymptotic behavior

$$D(a) \overset{a\to 0}{\sim} a^{-\omega} \sum_{n=0}^{\infty} \sum_{m=0}^{l} c_{nm} a^n (\log a)^m, \tag{17.1}$$

where the nonnegative exponent ω is related to the convergence properties of D and its subdiagrams. Lüscher and Weisz have devised a recursive blocking method for the calculation of the first coefficients of the expansion in a of a 1-loop Feynman diagram:

$$D(a) \overset{a\to 0}{\sim} a^{-\omega} \sum_{n=0}^{\infty} a^n \left(c_{n0} + c_{n1} \log a \right), \tag{17.2}$$

so that one does not need to evaluate the diagram for very small lattice spacings, which would be numerically quite demanding. To use this method (Lüscher and Weisz, 1986), the diagram has to be known numerically (with a good precision) for a sequence of different lattice spacings, which for simplicity will be taken to be $a_k = 1/\mu k$, with k integer and μ a constant. The continuum limit $a \to 0$ means $k \to \infty$. We consider the case in which only even powers of a

appear in the expansion, which includes diagrams which are $O(a)$ improved (if one is interested in the coefficients with $n \leq 2$). It is also assumed that one knows the coefficients of the logarithm terms exactly, so that the task is reduced to compute numerically only the coefficient c_{00} (which is the finite part of the limit $a = 0$ of the Feynman diagram), and c_{20}. Let us see how one can compute c_{00}.

One starts by defining an auxiliary function

$$f_0(k) = \{a^\omega D(a) + (c_{01} + a^2 c_{21}) \log k\}|_{a=a_k}, \tag{17.3}$$

which can be seen to have the asymptotic expansion

$$f_0(k) \stackrel{k \to \infty}{\sim} A_0 + \frac{A_1}{(\mu k)^2} + \sum_{n=2}^{\infty} \frac{(A_n + B_n \log k)}{(\mu k)^{2n}}, \tag{17.4}$$

where

$$
\begin{aligned}
A_0 &= c_{00} - c_{01} \log \mu, \\
A_1 &= c_{20} - c_{21} \log \mu,
\end{aligned}
\tag{17.5}
$$

and so on. We can then consider taking

$$A_0 \simeq f_0(k_{max}) \tag{17.6}$$

as a first approximation to compute c_{00} (remember that c_{01} is exactly known). It is however easy to do better. In fact, the function

$$f_1(k) = \frac{(k + \delta_0)^2}{4\delta_0 k} f_0(k + \delta_0) - \frac{(k - \delta_0)^2}{4\delta_0 k} f_0(k - \delta_0) \tag{17.7}$$

has an expansion similar to Eq. (17.4) with the same initial term A_0 but no $1/k^2$ term, so that the first correction becomes of order $1/k^4$ only. Therefore the approach to the limiting value for $k \to \infty$ is faster, and

$$A_0 \simeq f_1(k_{max} - \delta_0) \tag{17.8}$$

gives a better approximation for the computation of c_{00}. Notice that f_1 is not defined at k_{max}, because it is a discrete difference involving the nearest point, and thus one has to use its value at $k_{max} - \delta_0$. Usually one chooses $\delta_0 = 1$ or $\delta_0 = 2$.

The above blocking transformation can be iterated i times to give closer and closer approximations to the right value of A_0:

$$f_i(k) = A_0 + O(1/k^{2i+2}). \tag{17.9}$$

However, because of the logarithms present in Eq. (17.4) for $n \geq 2$, the transformations become now slightly more complicated:

$$
\begin{align}
f_{i+1}(k) &= w_1 f_i(k + \delta_i) + w_2 f_i(k) + w_3 f_i(k - \delta_i), \tag{17.10} \\
w_j &= v_j/(v_1 + v_2 + v_3), \qquad (j = 1, 2, 3), \tag{17.11} \\
v_1 &= (k + \delta_i)^{2i+2} \log(1 - \delta_i/k), \tag{17.12} \\
v_2 &= k^{2i+2}[\log(1 + \delta_i/k) - \log(1 - \delta_i/k)], \tag{17.13} \\
v_3 &= -(k - \delta_i)^{2i+2} \log(1 + \delta_i/k). \tag{17.14}
\end{align}
$$

While each new iteration gives in principle a better approximation to A_0, it is to be noticed that the domain of the function f_i becomes smaller and smaller with every new blocking step, and therefore after a while (depending on the total number of different lattice measurements available, k) there is a natural halt to the blocking steps. Moreover, the numerical precision is lost a little after each iteration; Lüscher and Weisz estimated that one loses one or two decimal places with each blocking step.

There is an optimal choice for the number of iterations, and in (Lüscher and Weisz, 1986) a stopping criterion and an estimate for the total error are given. At the end there is an optimal estimate of c_{00} given by

$$
A_0 \simeq f_{i^*}(k^*) \tag{17.15}
$$

where k^* and i^* minimize the error.

Another method which is useful for the extraction of the leading behavior of lattice integrals has been used in (Bode, Weisz and Wolff, 2000a; Bode, Weisz and Wolff, 2000b). One assumes as before to know the integrals $F(k)$ for a sequence of different lattice spacings k_i, $(i = 1, \ldots, n)$, with the necessary computer precision. One wants then to determine the first n_f coefficients of the expansion

$$
F(k) = \sum_{i=1}^{n_f} \alpha_i f_i(k) + R(k) \tag{17.16}
$$

in terms of general functions $f_i(k) = \log^\mu k/k^\nu$ (which also include $f_0(k) = 1$).

Of course one has always $n_f \leq n$, because the parameters to be determined cannot exceed the data set. In matrix form the above equation reads

$$
F = f\alpha + R, \tag{17.17}
$$

where F is the n-dimensional vector of the data $F(k_i)$, α is the n_f-dimensional vector of the leading coefficients that we want to determine, and f is an $n \times n_f$ matrix.

The method calls for α to be determined by minimizing the quadratic form in the residues

$$
\chi^2 = (F - f\alpha)^T W^2 (F - f\alpha), \tag{17.18}
$$

where W^2 is a matrix of positive weights. [81] This leads to

$$f^T W^2 f \alpha = f^T W^2 F, \tag{17.19}$$

and if the columns of the matrix Wf are linearly independent, and P is a projector on the corresponding n_f-dimensional subspace, then the task is reduced to finding a solution of

$$Wf\alpha = PWF. \tag{17.20}$$

This can be done using the singular value decomposition for Wf,

$$Wf = USV^T, \tag{17.21}$$

where S and V are diagonal and orthonormal $n_f \times n_f$ matrices respectively, and U is a column-orthonormal $n_f \times n_f$ matrix, i.e., $U^T U = 1$ and $UU^T = P$. The solution for the n_f leading coefficients in Eq. (17.16) is then

$$\alpha = V S^{-1} U^T W F. \tag{17.22}$$

The blocking procedure of Lüscher and Weisz discussed previously can be considered as a particular case of this method. In fact, if one repeats these improved fits modifying the component proportional to one of the functions f_j, then only the corresponding α_j changes. Then, if $n = n_f$ (i.e., one has the minimal data set which is able to determine the coefficients), this method is equivalent to performing blocking transformations which cancel the $n_f - 1$ components not related to α_j.

We want to conclude this Section by discussing a simple method which is useful to accelerate the convergence of the numerical evaluation of the integral of a periodic analytic function over a compact domain. This kind of integrals arises in theories with twisted boundary conditions (Lüscher and Weisz, 1986),

$$U_\mu(x + L\hat\nu) = \Omega_\nu U_\mu(x)\Omega_\nu^{-1}, \tag{17.23}$$

where the gauge fields cross through the lattice boundaries. At least two directions must be twisted, otherwise the twisting is equivalent to a field redefinition of a theory with standard boundary conditions. The twist matrices Ω are constant and gauge-field independent, and are $SU(N_c)$ matrices which satisfy the algebra

$$\Omega_\mu\Omega_\sigma = e^{\frac{2\pi i}{N_c}} \Omega_\sigma\Omega_\mu. \tag{17.24}$$

Explicit representations of these matrices are not needed. These boundary conditions remove the zero modes and the theory acquires a mass gap, so that the gluon propagator is not singular.

---

[81]These weights could be useful in order to give more importance to particular data points, as for larger a there is less roundoff error, while for smaller a the asymptotic expansions are better satisfied. In (Bode, Weisz and Wolff, 2000a; Bode, Weisz and Wolff, 2000b) it was however reported that uniform weights $W = 1$ work just fine as well.

Beside inducing an infrared cutoff in Feynman diagrams, they also cause the spectrum of momenta to be continuous, because the momentum components are not quantized by the boundary conditions. Similarly, the ghosts are also twisted periodic fields, and the Faddeev-Popov determinant has no zero modes. One then ends up with integrals of periodic analytic functions. More technicalities and the Feynman rules can be found in (Lüscher and Weisz, 1986).

For simplicity we consider one-dimensional integrals. We want to compute

$$I = \int_{-\pi}^{\pi} \frac{dk}{2\pi} f(k), \tag{17.25}$$

where $f(k)$ is analytic and periodic, $f(k + 2\pi) = f(k)$. A first approximation of this integral is given by the sum

$$I(N) = \frac{1}{N} \sum_{j=1}^{N} f(\frac{2\pi}{N} j) \tag{17.26}$$

for N large enough. While in general this is not a very efficient approximation, for periodic functions the convergence turns out to be exponential,

$$I(N) - I = O(e^{-\epsilon N}), \tag{17.27}$$

where ϵ is the absolute value of the imaginary part of the singularity of the integrand which is closest to the real axis.

One can see that problems can arise when ϵ happens to be very small, because in this case the convergence of $I(N)$ to I becomes quite slow. However, things can be improved by making a change of variable (Lüscher and Weisz, 1986),

$$k = k' - \alpha \sin k', \qquad 0 \le \alpha(\epsilon) < 1, \tag{17.28}$$

where α is chosen to be near to one, so that the singularity in the new variables is pushed away from the real axis: $\hat{\epsilon} = O(1)$. The sums

$$\hat{I}(N) = \frac{1}{N} \sum_{j=1}^{N} \hat{f}(\frac{2\pi}{N} j), \tag{17.29}$$

calculated using the transformed function

$$\hat{f}(k') = (1 - \alpha \cos k') f(k(k')), \tag{17.30}$$

become then better convergent to the integral I:

$$\hat{I}(N) - I = O(e^{-\hat{\epsilon} N}). \tag{17.31}$$

Thus, much lower values of N, and less computing power, are sufficient to get the same precision in the numerical evaluation of the integral I.

18 Algebraic method for 1-loop integrals

If one is happy with computing 1-loop lattice integrals with a precision of only five or six significant decimal places, their values can be easily estimated by using a simple rectangle integration. In order to have some more precise results one can implement the methods discussed in the previous Section.

The algebraic method for Wilson fermions, that we are going to explain in this Section, is an enormous improvement on all that. It allows every integral coming from 1-loop Feynman diagrams to be computed in a completely symbolic way and to be reduced to a linear combination of a few basic constants. This means that the generic integral can then be calculated numerically with a very large precision with a very small effort. In fact, once determined these few basic constants with the desired precision once for all, the original integral is just an appropriate linear combination of them. It is then possible to compute every integral with a very high precision, for example with seventy significant decimal places (Capitani *et al.*, 1998a), and in some cases with even nearly 400 significant decimal places (see Appendix B). Computing 1-loop integrals with sixty or seventy significant decimal places turns out to be absolutely necessary if one wants to evaluate 2-loop integrals with a precision of at least ten significant decimal places, which can be accomplished using the coordinate space method (Section 19).

It is sufficient to apply the algebraic method only to integrals with zero external momenta, since the momentum-dependent part can be evaluated in the continuum, as we have seen in Section 15.2. Furthermore, the general zero-momentum integral on the lattice can always be written as a linear combination of terms of the form

$$\mathcal{F}(p, q; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{\hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t}}{D_F(k, m_f)^p D_B(k, m_b)^q}. \tag{18.1}$$

The algebraic method allows then to express these terms in terms of a certain number of basic integrals. The complete reduction of a generic $\mathcal{F}(p, q; n_x, n_y, n_z, n_t)$ is achieved using an iteration procedure which makes use of appropriate recursion relations in noninteger dimensions.

At the end of this procedure, using the algebraic method every purely bosonic integral can be expressed in terms of 3 basic constants, every purely fermionic integral in terms of 9 basic constants, and every general fermionic-bosonic integral in terms of 15 basic constants.

18.1 The bosonic case

We are now going to explain in detail the recursive algorithm in the bosonic case, when the gluon action is given by the Wilson plaquette (Caracciolo, Menotti and Pelissetto, 1991; Caracciolo, Menotti and Pelissetto, 1992).

We note first of all that it is enough to consider lattice integrals at zero external momenta.

We are in fact interested in the continuum limit of

$$G(p) = \int \frac{d^4 k}{(2\pi)^4} F(k; p) \tag{18.2}$$

where p is some external momentum. As explained in Section 15.2, we can split this integral in two parts: a subtracted (ultraviolet-finite) integral, which can be evaluated in the continuum, and a certain number of lattice integrals with zero external momenta. We have then

$$
\begin{aligned}
G(p) &= \int \frac{d^4 k}{(2\pi)^4} \left[F(k; p) - (T^{n_F} F)(k; p) \right] \\
&\quad + \int \frac{d^4 k}{(2\pi)^4} (T^{n_F} F)(k; p),
\end{aligned}
\tag{18.3}
$$

where n_F is the degree of the divergence of the integral, and

$$(T^{n_F} F)(k; p) = \sum_{n=0}^{n_F} \frac{1}{n!} p_{\mu_1} \dots p_{\mu_n} \left[\frac{\partial}{\partial p_{\mu_1}} \dots \frac{\partial}{\partial p_{\mu_n}} F(k; p) \right]_{p=0}. \tag{18.4}$$

If the propagators are massless, we need to at this point to introduce an intermediate regularization for $k = 0$. It is convenient to choose an infrared mass cutoff m or to use dimensional regularization. The singularities will cancel when the contribution of the momentum-dependent part is added at the end.

Any zero-momentum integral coming from the calculation of lattice Feynman diagrams in the pure gauge Wilson theory can be expressed as a linear combination of terms of the form [82]

$$\mathcal{B}(p; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{\hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t}}{D_B(k, m)^p}, \tag{18.5}$$

where p and n_i are positive integers, and the inverse bosonic propagator, which we take in general to be massive in order to regularize the divergences coming from the separation in J and $I - J$, is

$$D_B(k, m) = \hat{k}^2 + m^2. \tag{18.6}$$

Actually, due to the appearance of other kind of singularities at some intermediate stages of the reductions, we have to consider the more general integrals

$$\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{\hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t}}{D_B(k, m)^{p+\delta}},$$

where p is an arbitrary integer (not necessarily positive) and δ is a real number which will be set to zero at the end of the calculations.

---

[82]It is always possible to reduce any numerator to contain only factors of $\sin^2 k_\mu/2$, using $\sin^2 k_\mu = 4\sin^2 k_\mu/2 - 4\sin^4 k_\mu/2$ and similar formulae for the cosine functions.

To begin with, each integral $\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t)$ can be reduced through purely algebraic manipulations to a sum of integrals of the same type with $n_x = n_y = n_z = n_t = 0$ (i.e., pure denominators). This is done by using the recursion relations [83]

$$\mathcal{B}_\delta(p; 1) = \frac{1}{4}[\mathcal{B}_\delta(p-1) - m^2\mathcal{B}_\delta(p)] \tag{18.7}$$

$$\mathcal{B}_\delta(p; x, 1) = \frac{1}{3}[\mathcal{B}_\delta(p-1; x) - \mathcal{B}_\delta(p; x+1) - m^2\mathcal{B}_\delta(p; x)] \tag{18.8}$$

$$\mathcal{B}_\delta(p; x, y, 1) = \frac{1}{2}[\mathcal{B}_\delta(p-1; x, y) - \mathcal{B}_\delta(p; x+1, y) - \mathcal{B}_\delta(p; x, y+1)$$
$$-m^2\mathcal{B}_\delta(p; x, y)] \tag{18.9}$$

$$\mathcal{B}_\delta(p; x, y, z, 1) = \mathcal{B}_\delta(p-1; x, y, z) - \mathcal{B}_\delta(p; x+1, y, z)$$
$$-\mathcal{B}_\delta(p; x, y+1, z) - \mathcal{B}_\delta(p; x, y, z+1) - m^2\mathcal{B}_\delta(p; x, y, z), \tag{18.10}$$

which can be obtained from the trivial identity

$$D_B(k, m) = \sum_{i=1}^{4} \hat{k}_i^2 + m^2. \tag{18.11}$$

With these recursion relations one can eliminate each numerator argument, n_i, of the $\mathcal{B}_\delta$ function, provided that it has the value 1. When it is greater than 1, one has to lower its value until it reaches 1, so that it is then possible to use the above set of recursion relations. The lowering of n_i is done by using another recursion relation,

$$\mathcal{B}_\delta(p; \ldots, r) = \frac{r-1}{p+\delta-1}\mathcal{B}_\delta(p-1; \ldots, r-1) - \frac{4r-6}{p+\delta-1}\mathcal{B}_\delta(p-1; \ldots, r-2) + 4\mathcal{B}_\delta(p; \ldots, r-1),$$
$$\tag{18.12}$$

which is obtained integrating by parts the equation (for $r > 1$)

$$\frac{(\hat{k}_w^2)^r}{D_B(k, m)^{p+\delta}} = 4\frac{(\hat{k}_w^2)^{r-1}}{D_B(k, m)^{p+\delta}} + 2\frac{(\hat{k}_w^2)^{r-2}}{p+\delta-1}\sin k_w \frac{\partial}{\partial k_w} \frac{1}{D_B(k, m)^{p+\delta-1}}. \tag{18.13}$$

Notice that for $p = 1$ some coefficients in this recursion relation diverge as $1/\delta$, and therefore in order to compute $\mathcal{B}_\delta(1; \ldots)$ for $\delta = 0$ we need to compute $\mathcal{B}_\delta(0; \ldots)$ including terms of order δ. In general one needs to compute the intermediate expressions for the integrals $\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t)$ with $p \leq 0$ keeping all terms of order δ.

Using the recursion relations introduced so far, every integral $\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t)$ can thus be reduced to a sum of the form

$$\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t) = \sum_{r=p-n_x-n_y-n_z-n_t}^{p} a_r(m, \delta)\mathcal{B}_\delta(r), \tag{18.14}$$

where $a_r(m, \delta)$ are polynomials in m^2, which may diverge as $1/\delta$ for $p > 0$ and $r \leq 0$.

---

[83]In the following when one of the arguments n_i is zero it will be omitted.

At this point, it only remains to reexpress all $\mathcal{B}_\delta(p)$'s appearing in the above formula in terms of a small finite number of them. To accomplish this we need some other recursion relations, which can be obtained considering the trivial identity

$$\mathcal{B}_\delta(p; 1, 1, 1, 1) - 4\mathcal{B}_\delta(p + 1; 2, 1, 1, 1) - m^2 \mathcal{B}_\delta(p + 1; 1, 1, 1, 1) = 0, \tag{18.15}$$

and applying to it the previous procedure until it is reduced to a relation between the $\mathcal{B}_\delta(r)$'s only. One then arrives to a nontrivial relation of the form

$$\sum_{r=p-4}^{p} b_r(p; \delta)\mathcal{B}_\delta(r) + \mathcal{S}(p; m, \delta) = 0, \tag{18.16}$$

where $\mathcal{S}(p; m, \delta) = O(m^2)$ for $p \leq 2$, while for $p > 2$ is a polynomial in $1/m^2$ (which is finite for $\delta \to 0$).

We can now use the last relation to express all $\mathcal{B}_\delta(p)$'s in terms of $\mathcal{B}_\delta(r)$'s which are only in the range $0 \leq r \leq 3$. To do this, when $p \geq 4$ we just write $\mathcal{B}_\delta(p)$ in terms of $\mathcal{B}_\delta(p-1), \ldots, \mathcal{B}_\delta(p-4)$ and iterate until needed. When $p \leq -1$, we solve the relation in terms of $\mathcal{B}_\delta(p-4)$, make the shift $p \to p + 4$, and then use it to write $\mathcal{B}_\delta(p)$ in terms of $\mathcal{B}_\delta(p+1), \ldots, \mathcal{B}_\delta(p+4)$. Again we iterate until needed. Applying recursively these two relations we get then, for $p \neq 0, 1, 2, 3$:

$$\mathcal{B}_\delta(p) = \sum_{r=0}^{3} c_r(p; \delta)\mathcal{B}_\delta(r) + \mathcal{T}(p; m, \delta), \tag{18.17}$$

where $\mathcal{T}(p; m, \delta)$ is a polynomial in $1/m^2$.

The above procedure allows the general bosonic integral to be written, after a finite number of steps, as

$$\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t) = A(\delta)\mathcal{B}_\delta(0) + B(\delta)\mathcal{B}_\delta(1) + C(\delta)\mathcal{B}_\delta(2) + D(\delta)\mathcal{B}_\delta(3) + E(m, \delta), \tag{18.18}$$

where $E(m, \delta)$ is a polynomial in $1/m^2$. It can be shown that the limit $\delta \to 0$ is safe at this stage, and one finally obtains

$$\mathcal{B}(p; n_x, n_y, n_z, n_t) = A(0) + B(0)\mathcal{B}(1) + C(0)\mathcal{B}(2) + D(0)\mathcal{B}(3) + E(m, 0), \tag{18.19}$$

in terms of three basic constants, $\mathcal{B}(1)$, $\mathcal{B}(2)$ and $\mathcal{B}(3)$. This is a minimal set, i.e, no further reduction can be done.

It is common practice to write the bosonic results in terms of the three constants Z_0, Z_1 and F_0, which are defined by

$$Z_0 = \mathcal{B}(1)|_{m=0} \tag{18.20}$$

$$Z_1 = \frac{1}{4}\mathcal{B}(1; 1, 1)|_{m=0} \tag{18.21}$$

$$F_0 = \lim_{m \to 0}\left(16\pi^2\mathcal{B}(2) + \log m^2 + \gamma_E\right). \tag{18.22}$$

Z_0	0.1549333902310602140848372O8
Z_1	0.10778131353987400134339155O
F_0	4.369225233874758

<div align="center">Table 2: Numerical values of the basic bosonic constants.</div>

Explicitly,

$$Z_0 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{4\sum_\lambda \sin^2 \frac{k_\lambda}{2}}, \tag{18.23}$$

$$Z_1 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin^2 \frac{k_1}{2} \sin^2 \frac{k_2}{2}}{\sum_\lambda \sin^2 \frac{k_\lambda}{2}}, \tag{18.24}$$

and [84]

$$\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\left(4\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2 + m^2} = \frac{1}{16\pi^2}\left(-\log m^2 - \gamma_E + F_0\right). \tag{18.26}$$

Their values are given in Table 2. We remind that $\gamma_E = 0.57721566490153286\dots$ is the well-known Euler's constant appearing in continuum integrals.

Rewriting $\mathcal{B}(1)$ and $\mathcal{B}(2)$ in terms of F_0 and Z_0 is rather trivial. For $\mathcal{B}(3)$ one has

$$\mathcal{B}(3) = \frac{1}{32\pi^2 m^2} - \frac{1}{128\pi^2}\left(\log m^2 + \gamma_E - F_0\right) - \frac{1}{1024} - \frac{13}{1536\pi^2} + \frac{Z_1}{256}, \tag{18.27}$$

which is a special case of Eq. (18.29).

In general $d-1$ basic constants are enough for all bosonic integrals in d dimensions, and $d-2$ if one only considers finite integrals. This means that in two spacetime dimensions any finite bosonic integral can be written in terms of rational numbers and $1/\pi^2$ factors only, and one constant has to be introduced for divergent integrals.

This concludes the illustration of the algebraic method. However, if one wants to use it, it comes handy to know some formulae regarding divergent integrals, that is the $\mathcal{B}_\delta(p)$'s for $p \geq 2$. These formulae can be derived using their expression in terms of the modified Bessel function $I_0(x)$,

$$\mathcal{B}_\delta(p) = \frac{1}{2^{p+\delta}\Gamma(p+\delta)} \int_0^\infty dx\, x^{p+\delta-1} e^{-m^2 x/(2-4x)} I_0^4(x), \tag{18.28}$$

---

[84]In case dimensional regularization is used, F_0 is defined by

$$\int_{-\pi}^{\pi} \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \frac{1}{\left(4\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2} = \frac{1}{16\pi^2}\left(-\frac{1}{\epsilon} - \log 4\pi + F_0\right). \tag{18.25}$$

obtained using the Schwinger representation. The basic formula for the divergent integrals is then

$$\mathcal{B}(r) = \frac{1}{\Gamma(r)} \sum_{i=2}^{r-1} \frac{b_{i-2}\Gamma(r-i)}{2^i(m^2)^{r-i}} + \frac{b_{r-2}}{2^r\Gamma(r)}\Big(-\log m^2 - \gamma_E + F_{r-2}\Big) + H_{r-2}. \tag{18.29}$$

If dimensional regularization is instead used one has

$$\mathcal{B}_d(r) = \frac{b_{r-2}}{2^r\Gamma(r)}\Big(\frac{2}{d-4} - \log 4\pi + F_{r-2}\Big) + G_{r-2}. \tag{18.30}$$

In the latter case, the recursion relations have to be extended to noninteger dimensions, which can be done without too much effort. These formulae can be found in (Caracciolo, Menotti and Pelissetto, 1992).

The constants b_i appearing in the formulae for divergent integrals are defined by the asymptotic expansion of the modified Bessel function I_0,

$$I_0^d(x) = \frac{\mathrm{e}^{dx}}{(2\pi x)^{d/2-2}} \sum_{i=0}^{\infty} \frac{b_i - (d-4)c_i + O((d-4)^2)}{x^{i+2}} + O(\mathrm{e}^{-x}), \tag{18.31}$$

and the constants F_i are also related to this modified Bessel function: [85]

$$F_p = \frac{1}{b_p}\left\{ \int_0^2 dx\, x^{p+1}\mathrm{e}^{-4x}I_0^4(x) + \int_2^\infty dx\, x^{p+1}\Big[\mathrm{e}^{-4x}I_0^4(x) - \sum_{i=0}^{p} \frac{b_i}{x^{i+2}}\Big]\right\}. \tag{18.33}$$

The first F_i's are given explicitly by [86]

$$F_1 = F_0 - \frac{1}{8}\pi^2 + \frac{35}{12} + \frac{1}{2}\pi^2\, Z_1 \tag{18.34}$$

$$F_2 = F_0 - \frac{31}{144}\pi^2 + \frac{1349}{216} + \frac{1}{9}\pi^2\, Z_0 + \frac{31}{36}\pi^2\, Z_1 \tag{18.35}$$

$$F_3 = F_0 - \frac{523}{1872}\pi^2 + \frac{24257}{2808} + \frac{25}{117}\pi^2\, Z_0 + \frac{523}{468}\pi^2\, Z_1 \tag{18.36}$$

$$F_4 = F_0 - \frac{7145}{22176}\pi^2 + \frac{294919}{33264} + \frac{401}{1386}\pi^2\, Z_0 + \frac{7145}{5544}\pi^2\, Z_1 \tag{18.37}$$

$$F_5 = F_0 - \frac{27971}{80190}\pi^2 + \frac{3347101}{481140} + \frac{13582}{40095}\pi^2\, Z_0 + \frac{55942}{40095}\pi^2\, Z_1 \tag{18.38}$$

$$F_6 = F_0 - \frac{27039607}{74221920}\pi^2 + \frac{448657133}{111332880} + \frac{1708783}{4638870}\pi^2\, Z_0 + \frac{27039607}{18555480}\pi^2\, Z_1 \tag{18.39}$$

$$F_7 = F_0 - \frac{751956319}{2016614880}\pi^2 + \frac{3823946741}{3024922320} + \frac{48529351}{126038430}\pi^2\, Z_0 + \frac{751956319}{504153720}\pi^2\, Z_1, \tag{18.40}$$

---

[85] These and other constants to them related were introduced in (González-Arroyo and Korthals-Altes, 1982). Note that F_0 was called F_{0000} there. Many properties of these functions are discussed there and in (Ellis and Martinelli, 1984a). We also note that the finite constant Z_0 can also be expressed in terms of modified Bessel functions:

$$Z_0 = \frac{1}{2}\int_0^\infty dx\, \mathrm{e}^{-4x}I_0^4(x). \tag{18.32}$$

[86] These relations can be obtained by applying the recursion relations to an appropriate identity like Eq. (18.15).

r	b_r	c_r	G_r	H_r
0	$\dfrac{1}{4\pi^2}$	0	0	0
1	$\dfrac{1}{8\pi^2}$	$-\dfrac{1}{32\pi^2}$	$-\dfrac{7}{256\pi^2}$	$-\dfrac{1}{32\pi^2}$
2	$\dfrac{3}{32\pi^2}$	$-\dfrac{1}{32\pi^2}$	$-\dfrac{11}{1536\pi^2}$	$-\dfrac{1}{128\pi^2}$
3	$\dfrac{13}{128\pi^2}$	$-\dfrac{55}{1536\pi^2}$	$-\dfrac{793}{589824\pi^2}$	$-\dfrac{53}{36864\pi^2}$
4	$\dfrac{77}{512\pi^2}$	$-\dfrac{5}{96\pi^2}$	$-\dfrac{311}{1474560\pi^2}$	$-\dfrac{331}{1474560\pi^2}$
5	$\dfrac{297}{1024\pi^2}$	$-\dfrac{1973}{20480\pi^2}$	$-\dfrac{27251}{943718400\pi^2}$	$-\dfrac{3653}{11796480\pi^2}$
6	$\dfrac{5727}{8192\pi^2}$	$-\dfrac{54583}{245760\pi^2}$	$-\dfrac{559001}{158544691200\pi^2}$	$-\dfrac{4261}{110100480\pi^2}$
7	$\dfrac{66687}{32768\pi^2}$	$-\dfrac{8558131}{13762560\pi^2}$	$-\dfrac{7910171}{20293720473600\pi^2}$	$-\dfrac{1331861}{295950090240\pi^2}$

Table 3: Values of b_r, c_r, G_r and H_r.

while the first b_i's, together with the first H_i's and G_i's, which are defined by

$$H_p = \frac{1}{\Gamma(p+2)} \sum_{i=0}^{p-1} \frac{b_i}{2^{i+2}(i-p)}, \tag{18.41}$$

$$G_p = H_p - \frac{1}{\Gamma(p+2)} \frac{c_p}{2^{p+1}}, \tag{18.42}$$

are given in Table 3.

If the integral to be computed is finite from the start, all $\log m^2$ terms must cancel, and this means that the F_0 constant is not present. In fact F_0 appears only in divergent integrals $(\mathcal{B}(2), \mathcal{B}(3), \ldots)$, and always in the combination

$$\log m^2 + \gamma_E - F_0. \tag{18.43}$$

The absence of $\log m^2$ factors implies then the cancellation of the γ_E and F_0 terms, and thus all finite integrals turn out to be functions of Z_0 and Z_1 only.

These two basic constants are now known with an incredible high precision, about 400 significant decimal places, as it will be shown in Section 19.1 and Appendix B, where they will

be computed using coordinate space methods. Thus, every finite bosonic integral can now be calculated with about 400 significant decimal places.

18.2 Examples of bosonic integrals

As an illustration of the algebraic method, we show here in detail how to compute a few bosonic integrals which will afterwards be used for the calculation of the operator tadpoles necessary for the renormalization of the operators corresponding to the third moment of the quark momentum distribution. We will start from the simpler integrals, which in some cases will be needed as intermediate results for the calculation of the more complicated ones.

Since these integrals all are finite, in general it is sufficient to use the recursion relations of the previous subsection with $m = 0$. Exceptions will be duly noted. We will not write the subscript δ explicitly, although the integrals are supposed to be computed at $\delta \neq 0$ when necessary. We also remind that $\mathcal{B}(1) = Z_0$, from its definition.

1. $\mathcal{B}(1; 1)$

   This very simple integral is given, using the first recursion relation, by

   $$\mathcal{B}(1; 1) = \frac{1}{4}\mathcal{B}(0) = \frac{1}{4}. \tag{18.44}$$

2. $\mathcal{B}(1; 1, 1)$

   We have

   $$\mathcal{B}(1; 1, 1) = 4Z_1, \tag{18.45}$$

   from its definition.

3. $\mathcal{B}(2; 1, 1)$

   By applying the various recursion relations we get

   $$
   \begin{aligned}
   \mathcal{B}(2; 1, 1) &= \frac{1}{3}\left(\mathcal{B}(1; 1) - \mathcal{B}(2; 2)\right) \\
   &= \frac{1}{3}\left(\frac{1}{4} - \left(\frac{1}{1+\delta}\mathcal{B}(1; 1) - \frac{2}{1+\delta}\mathcal{B}(1) + 4\mathcal{B}(2; 1)\right)\right) \\
   &= \frac{1}{3}\left(\frac{1}{4} - \left(\frac{1}{4} - 2Z_0 + 4 \cdot \frac{1}{4}\mathcal{B}(1)\right)\right) \\
   &= \frac{1}{3}Z_0. \tag{18.46}
   \end{aligned}
   $$

   We have taken the limit $\delta = 0$, because it is safe to do so here.

4. $\mathcal{B}(2; 2, 1)$

   We have to compute this integral because it is necessary for $\mathcal{B}(2; 1, 1, 1)$ (the last example). The manipulations are as follows:

$$
\begin{aligned}
\mathcal{B}(2; 2, 1) &= \frac{1}{3}\left(\mathcal{B}(1; 2) - \mathcal{B}(2; 3)\right) \\
&= \frac{1}{3}\left(\left(\frac{1}{\delta}\mathcal{B}(0; 1) - \frac{2}{\delta}\mathcal{B}(0) + 4\mathcal{B}(1; 1)\right) - \left(2\mathcal{B}(1; 2) - 6\mathcal{B}(1; 1) + 4\mathcal{B}(2; 1)\right)\right) \\
&= \frac{1}{3}\left(\frac{1}{\delta}\mathcal{B}(0; 1) - \frac{2}{\delta}\mathcal{B}(0) + 4\mathcal{B}(1; 1)\right. \\
&\qquad \left. -\frac{2}{\delta}\mathcal{B}(0; 1) + \frac{4}{\delta}\mathcal{B}(0) - 8\mathcal{B}(1; 1) + 6\mathcal{B}(1; 1) - 4\mathcal{B}(2; 1)\right) \\
&= \frac{1}{3}\left(-\frac{1}{\delta}\mathcal{B}(0; 1) + \frac{2}{\delta}\mathcal{B}(0) + 2\mathcal{B}(1; 1) - 4\mathcal{B}(2; 1)\right),
\end{aligned}
\tag{18.47}
$$

   where we have taken the limit $\delta = 0$ when safe. Now

$$
\mathcal{B}(0; 1) = \frac{1}{4}\mathcal{B}(-1), \tag{18.48}
$$

   and we need the expression of $\mathcal{B}(-1)$ including terms of order δ. Applying the recursion relations to the identity (18.15), [87]

$$
\mathcal{B}_\delta(3; 1, 1, 1, 1) - 4\mathcal{B}_\delta(4; 2, 1, 1, 1) - m^2\,\mathcal{B}_\delta(4; 1, 1, 1, 1) = 0, \tag{18.49}
$$

   we obtain

$$
\mathcal{B}(-1) = 8 + \delta \cdot \left(-20Z_0 - 48Z_1 + 8\right) + O(\delta^2). \tag{18.50}
$$

   We have then

$$
-\frac{1}{\delta}\mathcal{B}(0; 1) + \frac{2}{\delta}\mathcal{B}(0) = -\frac{1}{4\delta}\left(8 + \delta\left(-20Z_0 - 48Z_1 + 8\right)\right) + \frac{2}{\delta} = 5Z_0 + 12Z_1 - 2, \tag{18.51}
$$

   which is finite, as it should be. The final result is

$$
\mathcal{B}(2; 2, 1) = \frac{4}{3}Z_0 + 4Z_1 - \frac{1}{2}. \tag{18.52}
$$

5. $\mathcal{B}(2; 1, 2)$

   Also this integral appears in the computation of $\mathcal{B}(2; 1, 1, 1)$. Of course it has to be equal to $\mathcal{B}(2; 2, 1)$ by symmetry, but the manipulations are slightly different. Actually, they are

---

[87]Now we have to keep the m^2 term in this identity, because some intermediate integrals will be divergent.

much simpler than the previous one:

$$
\begin{aligned}
\mathcal{B}(2;1,2) &= \mathcal{B}(1;1,1) - 2\mathcal{B}(1;1) + 4\mathcal{B}(2;1,1) \\
&= 4Z_1 - 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{3}Z_0 \\
&= \frac{4}{3}Z_0 + 4Z_1 - \frac{1}{2}.
\end{aligned}
\tag{18.53}
$$

This example shows that by a judicious choice of the order of the indices it is possible to obtain the same result with less effort.

6. $\mathcal{B}(2;1,1,1)$

The first decomposition of the integral gives

$$
\mathcal{B}(2;1,1,1) = \frac{1}{2}\Big(\mathcal{B}(1;1,1) - \mathcal{B}(2;2,1) - \mathcal{B}(2;1,2) \Big).
\tag{18.54}
$$

Taking the results of the two integrals that we have just computed, we have then

$$
\begin{aligned}
\mathcal{B}(2;1,1,1) &= \frac{1}{2}\left(4Z_1 - 2 \cdot \left(\frac{4}{3}Z_0 + 4Z_1 - \frac{1}{2} \right) \right) \\
&= -\frac{4}{3}Z_0 - 2Z_1 + \frac{1}{2}.
\end{aligned}
\tag{18.55}
$$

These are all the integrals which will be needed in the next Section. In (Panagopoulos and Vicari, 1990) the exact expressions of various other bosonic integrals (useful for the renormalization of the trilinear gluon condensate) in terms of the constants Z_0 and Z_1 can also be found.

18.3   Operator tadpoles

A very important class of diagrams on the lattice is given by the operator tadpoles, which are generated when an operator in a matrix element contains in its definition U fields. In the case in which the pure gauge action is given by the plaquette, and this includes Wilson and Ginsparg-Wilson fermions, these tadpoles can be computed exactly, using the algebraic methods that we have just discussed.

We give here the results of the operator tadpoles in a general covariant gauge for operators which contain one, two and three covariant derivatives. These operators are useful for example for the calculations of the renormalization constants of operators measuring the lowest moments of the structure functions. Dirac matrices can then be added at will, since they do not influence the calculation of the tadpoles. What is important instead is the choice of the indices of the derivatives, and this is not surprising because these operators fall in different representations

of the hypercubic group depending on this choice, and the results will differ accordingly. The operator tadpoles for the various operators in terms of Z_0 and Z_1 are

$$\mathcal{T}_{\overline{\psi}D_\mu\psi} = \frac{1}{2}Z_0 + (1-\alpha)\frac{1}{8}Z_0, \tag{18.56}$$

$$\mathcal{T}_{\overline{\psi}D_\mu D_\nu\psi} = -Z_0 + (1-\alpha)\frac{1}{6}Z_0, \tag{18.57}$$

$$\mathcal{T}_{\overline{\psi}D_\mu D_\mu\psi} = -2Z_0 + \frac{1}{8} + (1-\alpha)\left(Z_0 - \frac{1}{8}\right), \tag{18.58}$$

$$\mathcal{T}_{\overline{\psi}D_\mu D_\nu D_\sigma\psi} = -\frac{3}{2}Z_0 + (1-\alpha)\left(-\frac{1}{24}Z_0 - \frac{1}{4}Z_1 + \frac{1}{16}\right), \tag{18.59}$$

$$\mathcal{T}_{\overline{\psi}D_\mu D_\mu D_\nu\psi} = -\frac{5}{2}Z_0 + \frac{1}{8} + (1-\alpha)\left(\frac{9}{8}Z_0 + Z_1 - \frac{1}{4}\right), \tag{18.60}$$

$$\mathcal{T}_{\overline{\psi}D_\nu D_\mu D_\mu\psi} = -\frac{5}{2}Z_0 + \frac{1}{8} + (1-\alpha)\left(\frac{9}{8}Z_0 + Z_1 - \frac{1}{4}\right), \tag{18.61}$$

$$\mathcal{T}_{\overline{\psi}D_\mu D_\nu D_\mu\psi} = -\frac{5}{2}Z_0 - Z_1 + \frac{1}{4} + (1-\alpha)\left(\frac{9}{8}Z_0 + Z_1 - \frac{1}{4}\right), \tag{18.62}$$

where $\mu \neq \nu \neq \sigma$, and repeated indices are not summed.

These results, as we mentioned, are valid also for Wilson and overlap fermions (among others), because they depend only on the structure of the gluon propagator, whereas this is not true for the self-energy tadpole, which for overlap fermions is different from the Wilson result (as we have seen in Section 15.5), because there the interaction vertex is different.

We now show explicitly how these computations are done, deriving the results of Eqs. (18.59), (18.60), (18.61) and (18.62), and taking as final task the calculation of the operator tadpoles of the operators

$$O_{\{0123\}} = \overline{\psi}\gamma_{\{0}D_1 D_2 D_{3\}}\psi \tag{18.63}$$

and

$$O_{\{0011\}} + O_{\{3322\}} - O_{\{0022\}} - O_{\{3311\}} \tag{18.64}$$
$$= \overline{\psi}\gamma_{\{0}D_0 D_1 D_{1\}}\psi + \overline{\psi}\gamma_{\{3}D_3 D_2 D_{2\}}\psi - \overline{\psi}\gamma_{\{0}D_0 D_2 D_{2\}}\psi - \overline{\psi}\gamma_{\{3}D_3 D_1 D_{1\}}\psi,$$

which was carried out in (Capitani, 2001b). These operators are multiplicatively renormalizable at one loop, and measure the third moment of the unpolarized quark distribution.

Each of the four terms in Eq. (18.64) gives the same value for the tadpole. Furthermore

$$O_{\{0011\}} = \frac{1}{6}\left(O_{0011} + O_{0101} + O_{0110} + O_{1001} + O_{1010} + O_{1100}\right), \tag{18.65}$$

and since for the tadpole what is important is the position of the covariant derivatives, and not of the Dirac matrices, we have

$$\mathcal{T}_{O_{\{0011\}}} = \frac{1}{3}\left(\mathcal{T}_{O_{0011}} + \mathcal{T}_{O_{0101}} + \mathcal{T}_{O_{0110}}\right) \tag{18.66}$$

(the remaining terms have the indices 0 and 1 exchanged, and therefore they give the same result). The results corresponding to these three terms are given in Eqs. (18.61), (18.62) and (18.60) respectively.

Let us begin by computing the tadpoles for operators with three covariant derivatives which have general indices μ, ν and ρ (that is, for the moment they are left open to be equal or different). We have again that one can consider, for the sake of the computation of the tadpole, $\overline{\psi} \, \vec{D}\vec{D}\vec{D} \, \psi$ instead of $1/8 \, \overline{\psi} \, \overleftrightarrow{D}\overleftrightarrow{D}\overleftrightarrow{D} \, \psi$, which would be a lot more tedious to calculate. Applying the three covariant derivatives in cascade gives

$$
\begin{aligned}
\vec{D}_\rho\vec{D}_\nu\vec{D}_\mu \, \psi(x) \;=\; & \frac{1}{8a^3}\Bigg[U_\rho(x)U_\nu(x+a\hat{\rho})U_\mu(x+a\hat{\nu}+a\hat{\rho})\psi(x+a\hat{\mu}+a\hat{\nu}+a\hat{\rho}) \qquad (18.67)\\
& -U_\rho(x)U_\nu(x+a\hat{\rho})U_\mu^\dagger(x-a\hat{\mu}+a\hat{\nu}+a\hat{\rho})\psi(x-a\hat{\mu}+a\hat{\nu}+a\hat{\rho})\\
& -U_\rho(x)U_\nu^\dagger(x-a\hat{\nu}+a\hat{\rho})U_\mu(x-a\hat{\nu}+a\hat{\rho})\psi(x+a\hat{\mu}-a\hat{\nu}+a\hat{\rho})\\
& +U_\rho(x)U_\nu^\dagger(x-a\hat{\nu}+a\hat{\rho})U_\mu^\dagger(x-a\hat{\mu}-a\hat{\nu}+a\hat{\rho})\psi(x-a\hat{\mu}-a\hat{\nu}+a\hat{\rho})\\
& -U_\rho^\dagger(x-a\hat{\rho})U_\nu(x-a\hat{\rho})U_\mu(x+a\hat{\nu}-a\hat{\rho})\psi(x+a\hat{\mu}+a\hat{\nu}-a\hat{\rho})\\
& +U_\rho^\dagger(x-a\hat{\rho})U_\nu(x-a\hat{\rho})U_\mu^\dagger(x-a\hat{\mu}+a\hat{\nu}-a\hat{\rho})\psi(x-a\hat{\mu}+a\hat{\nu}-a\hat{\rho})\\
& +U_\rho^\dagger(x-a\hat{\rho})U_\nu^\dagger(x-a\hat{\nu}-a\hat{\rho})U_\mu(x-a\hat{\nu}-a\hat{\rho})\psi(x+a\hat{\mu}-a\hat{\nu}-a\hat{\rho})\\
& -U_\rho^\dagger(x-a\hat{\rho})U_\nu^\dagger(x-a\hat{\nu}-a\hat{\rho})U_\mu^\dagger(x-a\hat{\mu}-a\hat{\nu}-a\hat{\rho})\psi(x-a\hat{\mu}-a\hat{\nu}-a\hat{\rho})\Bigg].
\end{aligned}
$$

For the calculation of the tadpoles we have now to expand the U's in the above expression to second order, and keep the A^2 terms coming from the same U as well as the products $A \cdot A$ coming from two different U's. Regarding the latter, if we were doing the calculations in Feynman gauge only the pairs of A with the same index would give nonzero contributions. Here however we work in general covariant gauge, so we must consider all terms. In addition, we have to expand the ψ terms to order a^3p^3, which reconstructs the tree level of the operator,

$$
- \mathrm{i}p_\mu p_\nu p_\rho. \qquad (18.68)
$$

For example, one of such terms gives

$$
\begin{aligned}
\frac{1}{a^3}\psi(x+a\hat{\mu}+a\hat{\nu}+a\hat{\rho}) \;\longrightarrow\; & \frac{1}{a^3}\mathrm{e}^{\mathrm{i}ap_\mu}\mathrm{e}^{\mathrm{i}ap_\nu}\mathrm{e}^{\mathrm{i}ap_\rho} \qquad (18.69)\\
=\; & -\mathrm{i}p_\mu p_\nu p_\rho - \delta_{\mu\nu}\mathrm{i}p_\mu^2 p_\rho - \delta_{\mu\rho}\mathrm{i}p_\mu^2 p_\nu - \delta_{\nu\rho}\mathrm{i}p_\nu^2 p_\mu - \frac{\mathrm{i}}{2}\delta_{\mu\nu}\delta_{\nu\rho}p_\mu^3.
\end{aligned}
$$

This factor a^3 compensates the factor $1/a^3$ coming from the covariant derivatives, while the factor a^2 coming from the U expansions multiplied with the factor a^2 present in the gluon propagator cancels the rescaling factor of the integration variable, a^4. The tadpoles are then finite in the limit $a \to 0$.

Let us first consider the terms where both A's come from the expansion of the same U. This is a simple calculation which gives

$$C_F \frac{1}{8a^3}\left(-\frac{g_0^2 a^2}{2}\right)\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4}\left(A_\mu^a(k)A_\mu^a(-k) + A_\nu^a(k)A_\nu^a(-k) + A_\rho^a(k)A_\rho^a(-k)\right)\cdot 8a^3 \mathrm{i}p_\mu \mathrm{i}p_\nu \mathrm{i}p_\rho$$

$$= -\frac{3g_0^2 a^2}{2}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^4 k}{(2\pi)^4}G_{\mu\mu}^{aa}(k)\cdot(-\mathrm{i}p_\mu p_\nu p_\rho), \tag{18.70}$$

where we have contracted the A's to form the propagator (note that now the color index a is not summed). Inserting the covariant gauge propagator (Eq. (5.62)) and dividing for the tree level we have that the above expression is reduced to $g_0^2 C_F$ multiplied by [88]

$$\mathcal{T}_1 = -\frac{3}{2}\Big(\mathcal{B}(1) - (1-\alpha)\mathcal{B}(2;1)\Big) = -\frac{3}{2}Z_0 + (1-\alpha)\frac{3}{8}Z_0. \tag{18.71}$$

The remaining part of the calculation of the tadpoles involves the terms in which the A's that have to be contracted come from the expansion of different U's, and is much more complicated. In the case in which all indices all different, which corresponds to the operator $O_{\{0123\}}$, the relevant part of the expansion of Eq. (18.67) is

$$\frac{1}{8a^3}(-g_0^2 a^2)\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^4 k}{(2\pi)^4} \tag{18.72}$$

$$\times\Bigg[\Big(A_\rho(-k)A_\nu(k)\mathrm{e}^{\mathrm{i}ak_\rho/2}\mathrm{e}^{\mathrm{i}ak_\nu/2} + A_\nu(-k)A_\mu(k)\mathrm{e}^{\mathrm{i}ak_\nu/2}\mathrm{e}^{\mathrm{i}ak_\mu/2}$$

$$+A_\rho(-k)A_\mu(k)\mathrm{e}^{\mathrm{i}ak_\rho/2}\mathrm{e}^{\mathrm{i}ak_\mu/2}\mathrm{e}^{\mathrm{i}ak_\nu}\Big)\Big(-\mathrm{i}p_\mu p_\nu p_\rho - \mathrm{i}\delta_{\nu\rho}p_\nu^2 p_\mu - \mathrm{i}\delta_{\mu\nu}p_\mu^2 p_\rho - \mathrm{i}\delta_{\mu\rho}p_\mu^2 p_\nu\Big)$$

$$+\Big(A_\rho(-k)A_\nu(k)\mathrm{e}^{\mathrm{i}ak_\rho/2}\mathrm{e}^{\mathrm{i}ak_\nu/2} - A_\nu(-k)A_\mu(k)\mathrm{e}^{\mathrm{i}ak_\nu/2}\mathrm{e}^{-\mathrm{i}ak_\mu/2}$$

$$-A_\rho(-k)A_\mu(k)\mathrm{e}^{\mathrm{i}ak_\rho/2}\mathrm{e}^{-\mathrm{i}ak_\mu/2}\mathrm{e}^{\mathrm{i}ak_\nu}\Big)\Big(-\mathrm{i}p_\mu p_\nu p_\rho + \mathrm{i}\delta_{\nu\rho}p_\nu^2 p_\mu - \mathrm{i}\delta_{\mu\nu}p_\mu^2 p_\rho + \mathrm{i}\delta_{\mu\rho}p_\mu^2 p_\nu\Big)$$

$$+\Big(-A_\rho(-k)A_\nu(k)\mathrm{e}^{\mathrm{i}ak_\rho/2}\mathrm{e}^{-\mathrm{i}ak_\nu/2} - A_\nu(-k)A_\mu(k)\mathrm{e}^{-\mathrm{i}ak_\nu/2}\mathrm{e}^{\mathrm{i}ak_\mu/2}$$

$$+A_\rho(-k)A_\mu(k)\mathrm{e}^{\mathrm{i}ak_\rho/2}\mathrm{e}^{\mathrm{i}ak_\mu/2}\mathrm{e}^{-\mathrm{i}ak_\nu}\Big)\Big(-\mathrm{i}p_\mu p_\nu p_\rho - \mathrm{i}\delta_{\nu\rho}p_\nu^2 p_\mu + \mathrm{i}\delta_{\mu\nu}p_\mu^2 p_\rho + \mathrm{i}\delta_{\mu\rho}p_\mu^2 p_\nu\Big)$$

$$+\Big(-A_\rho(-k)A_\nu(k)\mathrm{e}^{\mathrm{i}ak_\rho/2}\mathrm{e}^{-\mathrm{i}ak_\nu/2} + A_\nu(-k)A_\mu(k)\mathrm{e}^{-\mathrm{i}ak_\nu/2}\mathrm{e}^{-\mathrm{i}ak_\mu/2}$$

$$-A_\rho(-k)A_\mu(k)\mathrm{e}^{\mathrm{i}ak_\rho/2}\mathrm{e}^{-\mathrm{i}ak_\mu/2}\mathrm{e}^{-\mathrm{i}ak_\nu}\Big)\Big(-\mathrm{i}p_\mu p_\nu p_\rho + \mathrm{i}\delta_{\nu\rho}p_\nu^2 p_\mu + \mathrm{i}\delta_{\mu\nu}p_\mu^2 p_\rho - \mathrm{i}\delta_{\mu\rho}p_\mu^2 p_\nu\Big)$$

$$+\Big(-A_\rho(-k)A_\nu(k)\mathrm{e}^{-\mathrm{i}ak_\rho/2}\mathrm{e}^{\mathrm{i}ak_\nu/2} + A_\nu(-k)A_\mu(k)\mathrm{e}^{\mathrm{i}ak_\nu/2}\mathrm{e}^{\mathrm{i}ak_\mu/2}$$

$$-A_\rho(-k)A_\mu(k)\mathrm{e}^{-\mathrm{i}ak_\rho/2}\mathrm{e}^{\mathrm{i}ak_\mu/2}\mathrm{e}^{\mathrm{i}ak_\nu}\Big)\Big(-\mathrm{i}p_\mu p_\nu p_\rho + \mathrm{i}\delta_{\nu\rho}p_\nu^2 p_\mu + \mathrm{i}\delta_{\mu\nu}p_\mu^2 p_\rho - \mathrm{i}\delta_{\mu\rho}p_\mu^2 p_\nu\Big)$$

$$+\Big(-A_\rho(-k)A_\nu(k)\mathrm{e}^{-\mathrm{i}ak_\rho/2}\mathrm{e}^{\mathrm{i}ak_\nu/2} - A_\nu(-k)A_\mu(k)\mathrm{e}^{\mathrm{i}ak_\nu/2}\mathrm{e}^{-\mathrm{i}ak_\mu/2}$$

$$+A_\rho(-k)A_\mu(k)\mathrm{e}^{-\mathrm{i}ak_\rho/2}\mathrm{e}^{-\mathrm{i}ak_\mu/2}\mathrm{e}^{\mathrm{i}ak_\nu}\Big)\Big(-\mathrm{i}p_\mu p_\nu p_\rho - \mathrm{i}\delta_{\nu\rho}p_\nu^2 p_\mu + \mathrm{i}\delta_{\mu\nu}p_\mu^2 p_\rho + \mathrm{i}\delta_{\mu\rho}p_\mu^2 p_\nu\Big)$$

$$+\Big(A_\rho(-k)A_\nu(k)\mathrm{e}^{-\mathrm{i}ak_\rho/2}\mathrm{e}^{-\mathrm{i}ak_\nu/2} - A_\nu(-k)A_\mu(k)\mathrm{e}^{-\mathrm{i}ak_\nu/2}\mathrm{e}^{\mathrm{i}ak_\mu/2}$$

---

[88] It is easy to see that $\mathcal{B}(2;1) = 1/4\cdot\mathcal{B}(1) = 1/4\cdot Z_0$.

$$-A_\rho(-k)A_\mu(k)\mathrm{e}^{-\mathrm{i}ak_\rho/2}\mathrm{e}^{\mathrm{i}ak_\mu/2}\mathrm{e}^{-\mathrm{i}ak_\nu}\Big)\Big(-\mathrm{i}p_\mu p_\nu p_\rho+\mathrm{i}\delta_{\nu\rho}p_\nu^2 p_\mu-\mathrm{i}\delta_{\mu\nu}p_\mu^2 p_\rho+\mathrm{i}\delta_{\mu\rho}p_\mu^2 p_\nu\Big)$$

$$+\Big(A_\rho(-k)A_\nu(k)\mathrm{e}^{-\mathrm{i}ak_\rho/2}\mathrm{e}^{-\mathrm{i}ak_\nu/2}+A_\nu(-k)A_\mu(k)\mathrm{e}^{-\mathrm{i}ak_\nu/2}\mathrm{e}^{-\mathrm{i}ak_\mu/2}$$

$$+A_\rho(-k)A_\mu(k)\mathrm{e}^{-\mathrm{i}ak_\rho/2}\mathrm{e}^{-\mathrm{i}ak_\mu/2}\mathrm{e}^{-\mathrm{i}ak_\nu}\Big)\Big(-\mathrm{i}p_\mu p_\nu p_\rho-\mathrm{i}\delta_{\nu\rho}p_\nu^2 p_\mu-\mathrm{i}\delta_{\mu\nu}p_\mu^2 p_\rho-\mathrm{i}\delta_{\mu\rho}p_\mu^2 p_\nu\Big)\bigg],$$

which gives

$$(-\mathrm{i}p_\mu p_\nu p_\rho)g_0^2 a^2\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^4k}{(2\pi)^4}\left(G_{\rho\nu}(k)\sin\frac{ak_\rho}{2}\sin\frac{ak_\nu}{2}\right.\tag{18.73}$$

$$\left.+G_{\nu\mu}(k)\sin\frac{ak_\nu}{2}\sin\frac{ak_\mu}{2}+G_{\rho\mu}(k)\cos ak_\nu\sin\frac{ak_\rho}{2}\sin\frac{ak_\mu}{2}\right).$$

The result is

$$\mathcal{T}_2^{(\mu\nu\sigma)}=-(1-\alpha)\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\left(\frac{4\sin^2\frac{k_\rho}{2}\sin^2\frac{k_\nu}{2}+4\sin^2\frac{k_\nu}{2}\sin^2\frac{k_\mu}{2}+\left(1-2\sin^2\frac{k_\nu}{2}\right)\sin^2\frac{k_\rho}{2}\sin^2\frac{k_\mu}{2}}{\left(4\sum_\lambda\sin^2\frac{k_\lambda}{2}\right)^2}\right)$$

$$=(1-\alpha)\left(-\frac{3}{4}\mathcal{B}(2;1,1)+\frac{1}{8}\mathcal{B}(2;1,1,1)\right)$$

$$=(1-\alpha)\left(-\frac{5}{12}Z_0-\frac{1}{4}Z_1+\frac{1}{16}\right).\tag{18.74}$$

In the last line we have used the results which we have obtained in the previous subsection applying the recursion relations of the algebraic method. This result, added to $\mathcal{T}_1$, gives the operator tadpole for $O_{\{0123\}}$, Eq. (18.59):

$$\mathcal{T}^{(\mu\nu\sigma)}=-\frac{3}{2}Z_0+(1-\alpha)\left(-\frac{1}{24}Z_0-\frac{1}{4}Z_1+\frac{1}{16}\right)=36.69915049+(1-\alpha)\,4.59514785.\tag{18.75}$$

We now consider the cases in which two of the indices are equal, which are necessary for the computation of the operator tadpole of $O_{\{0011\}}$. We have

$$\mathcal{T}_2^{(\nu\nu\mu)}=a^2\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^4k}{(2\pi)^4}\left(G_{\nu\nu}(k)\left(\sin^2\frac{ak_\nu}{2}-\cos^2\frac{ak_\nu}{2}\right)\right.$$

$$\left.+G_{\mu\nu}(k)\left(\sin\frac{ak_\mu}{2}\sin\frac{ak_\nu}{2}+\cos ak_\nu\sin\frac{ak_\mu}{2}\sin\frac{ak_\nu}{2}+\sin ak_\nu\sin\frac{ak_\mu}{2}\sin\frac{ak_\nu}{2}\right)\right)$$

$$=\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\left(\frac{\sin^2\frac{k_\nu}{2}-\cos^2\frac{k_\nu}{2}}{\left(4\sum_\lambda\sin^2\frac{k_\lambda}{2}\right)}-(1-\alpha)\times\right.$$

$$\left.\times\frac{\sin^4\frac{k_\nu}{2}-\sin^2\frac{k_\nu}{2}\cos^2\frac{k_\nu}{2}+\sin^2\frac{k_\mu}{2}\sin^2\frac{k_\nu}{2}\left(1+\cos k_\nu\right)+\sin^2\frac{k_\mu}{2}\sin\frac{k_\nu}{2}\cos\frac{k_\nu}{2}\sin k_\nu}{\left(4\sum_\lambda\sin^2\frac{k_\lambda}{2}\right)^2}\right)$$

$$=\frac{1}{2}\mathcal{B}(1;1)-\mathcal{B}(1)-(1-\alpha)\left(\frac{1}{2}\mathcal{B}(2;2)-\mathcal{B}(2;1)+\mathcal{B}(2;1,1)-\frac{1}{4}\mathcal{B}(2;2,1)\right)$$

168

$$= -Z_0 + \frac{1}{8} - (1-\alpha)\left(\frac{1}{2}\left(\frac{1}{4} - Z_0\right) - \frac{1}{4}Z_0 + \frac{1}{3}Z_0 - \frac{1}{4}\left(\frac{4}{3}Z_0 + 4Z_1 - \frac{1}{2}\right)\right)$$

$$= -Z_0 + \frac{1}{8} + (1-\alpha)\left(\frac{3}{4}Z_0 + Z_1 - \frac{1}{4}\right). \tag{18.76}$$

Again, in the last line we have substituted the results obtained in the previous subsection using the algebraic method. The willing reader can check that

$$\mathcal{T}_2^{(\rho\mu\mu)} = \mathcal{T}_2^{(\nu\nu\mu)}, \tag{18.77}$$

but the remaining combination gives a different result:

$$\mathcal{T}_2^{(\mu\nu\mu)} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4}\left(\frac{\cos k_\nu\left(\sin^2\frac{k_\nu}{2} - \cos^2\frac{k_\nu}{2}\right)}{\left(4\sum_\lambda \sin^2\frac{k_\lambda}{2}\right)}\right.$$

$$\left. -(1-\alpha)\frac{\cos k_\nu\left(\sin^4\frac{k_\nu}{2} - \sin^2\frac{k_\nu}{2}\cos^2\frac{k_\nu}{2}\right) + 2\sin^2\frac{k_\mu}{2}\sin^2\frac{k_\nu}{2}}{\left(4\sum_\lambda \sin^2\frac{k_\lambda}{2}\right)^2}\right)$$

$$= -\frac{1}{4}\mathcal{B}(1;1,1) + \mathcal{B}(1;1) - \mathcal{B}(1) - (1-\alpha)\left(\frac{1}{2}\mathcal{B}(2;2) - \mathcal{B}(2;1) + \mathcal{B}(2;1,1) - \frac{1}{4}\mathcal{B}(2;2,1)\right)$$

$$= -Z_0 - Z_1 + \frac{1}{4} + (1-\alpha)\left(\frac{3}{4}Z_0 + Z_1 - \frac{1}{4}\right). \tag{18.78}$$

Adding the term $\mathcal{T}_1$ to each of the above cases we have

$$\mathcal{T}^{(\rho\mu\mu)} = \mathcal{T}^{(\nu\nu\mu)} = -\frac{5}{2}Z_0 + \frac{1}{8} + (1-\alpha)\left(\frac{9}{8}Z_0 + Z_1 - \frac{1}{4}\right) \tag{18.79}$$

$$\mathcal{T}^{(\mu\nu\mu)} = -\frac{5}{2}Z_0 - Z_1 + \frac{1}{4} + (1-\alpha)\left(\frac{9}{8}Z_0 + Z_1 - \frac{1}{4}\right), \tag{18.80}$$

so that the operator tadpole for the operator $O_{\{0011\}}$, and hence for the operator $O_{\{0011\}} + O_{\{3322\}} - O_{\{0022\}} - O_{\{3311\}}$, is finally

$$\mathcal{T}_{O_{\{0011\}}} = \frac{1}{3}\left(\mathcal{T}^{(\mu\nu\mu)} + 2\mathcal{T}^{(\nu\nu\mu)}\right) \tag{18.81}$$

$$= -\frac{5}{2}Z_0 - \frac{1}{3}Z_1 + \frac{1}{6} + (1-\alpha)\left(\frac{9}{8}Z_0 + Z_1 - \frac{1}{4}\right) \tag{18.82}$$

$$= -40.5196866756 + (1-\alpha)\,5.0660880895. \tag{18.83}$$

All these results can be given with much more precision using the numbers reported in Appendix B.

The tadpoles of operators with one or two covariant derivatives, which are given at the beginning of this Section, are much simpler to calculate and are left as an exercise for the reader.

18.4 The first moment of the gluon momentum distribution

We want now to show the 1-loop result of a gluonic matrix element, which is probably the most complicated case in which the resulting integrals have been reduced to an expression containing only the two bosonic constants Z_0 and Z_1. The calculation of the operator measuring the first moment of the gluon momentum distribution, which also corresponds to the gluonic contribution to the energy-momentum tensor, has been done analytically and then the integrals reduced using the algebraic method (Caracciolo, Menotti and Pelissetto, 1992; Capitani and Rossi, 1995a). The renormalization of this operator can be obtained by computing the radiative corrections to the gluonic matrix element

$$\langle g| \sum_\rho \text{Tr}\,(F_{\mu\rho}F_{\rho\nu})|g\rangle. \tag{18.84}$$

The relevant diagrams are shown in Fig. 21.

The vertex function gives

$$N_c \cdot \left(-\frac{13}{192} + \frac{23}{48\pi^2} - \frac{53}{144}Z_0 + \frac{1}{3}Z_1 - \frac{3}{16\pi^2}\Big(\log p^2 + \gamma_E - F_0\Big)\right), \tag{18.85}$$

the result for the sails is

$$N_c \cdot \left(\frac{1}{192} - \frac{7}{9\pi^2} + \frac{31}{24}Z_0 - \frac{19}{48}Z_1 + \frac{7}{24\pi^2}\Big(\log p^2 + \gamma_E - F_0\Big)\right), \tag{18.86}$$

and the operator tadpole gives

$$N_c \cdot \left(-\frac{3}{64} - \frac{4}{3}Z_0\right) + \frac{1}{4N_c}. \tag{18.87}$$

The diagram containing the 4-gluon vertex (the rightmost in Fig. 21) is zero for this operator.

One has still to add the gluon self-energy at one loop (Caracciolo, Menotti and Pelissetto, 1992; Capitani and Rossi, 1995a),

$$N_c \cdot \left(\frac{1}{16} + \frac{7}{36\pi^2} + \frac{7}{72}Z_0 - \frac{5}{48\pi^2}\Big(\log p^2 + \gamma_E - F_0\Big)\right) - \frac{1}{8N_c}. \tag{18.88}$$

This is the case $N_f = 0$, that is the quenched approximation. The diagrams are as in Fig. 6, without the quark loops. The numerical result for the gluon self-energy is

$$\frac{1}{16\pi^2}\left(21.679380\,N_c - 19.739209\,\frac{1}{N_c}\right). \tag{18.89}$$

Summing everything together we have that the complete renormalization of the operator $\sum_\rho \text{Tr}\,(F_{\mu\rho}F_{\rho\nu})$ at 1-loop is given by

$$N_c \cdot \left(-\frac{3}{64} - \frac{5}{48\pi^2} - \frac{5}{16}Z_0 - \frac{1}{16}Z_1\right) + \frac{1}{8N_c}. \tag{18.90}$$
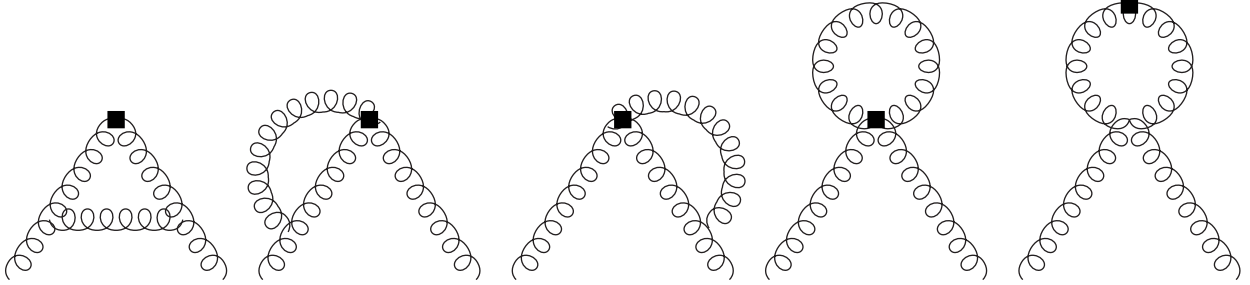
Figure 21: "Proper" diagrams for the 1-loop correction to the matrix element $\langle g| \sum_\rho \mathrm{Tr}\,(F_{\mu\rho}F_{\rho\nu})|g\rangle$. The black squares indicate the insertion of the operator. Notice that the last two diagrams are quite different: the last diagram contains a 4-gluon vertex (and vanishes for this matrix element), while the previous one is the tadpole coming from the second-order expansion of the operator.

The divergences of the individual diagrams have canceled, and so the energy-momentum tensor has zero anomalous dimensions, as it should be. Numerically one has

$$\frac{1}{16\pi^2}\left(-17.778285\,N_c + 19.739209\,\frac{1}{N_c}\right). \tag{18.91}$$

Using the values of Z_0 and Z_1 reported in Appendix B this result (as well as the gluon self-energy Eq. (18.89) can be stated with almost 400 significant decimal places.

We want to point out that working out the color structure for the diagrams considered here is more complicated than the case of the quark matrix elements measuring the moments of the unpolarized quark distributions. The reduction of the color factors, the f_{abc} tensors etc. can become quite cumbersome. On the other hand, the momentum integrals are much simpler in these gluonic matrix elements than in the quark case, because the fermion propagator is missing. Numerically evaluating integrals that have only gluon propagators poses much less of a computational challenge.

18.5   The general fermionic case

Similarly to the bosonic case, one only needs to compute lattice integrals with vanishing external momenta. Any lattice zero-momentum integral coming from the calculation of lattice Feynman diagrams in the general Wilson case can be written as a linear combination of terms of the form

$$\mathcal{F}(p, q; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\hat{k}_x^{2n_x}\hat{k}_y^{2n_y}\hat{k}_z^{2n_z}\hat{k}_t^{2n_t}}{D_F(k, m_f)^p D_B(k, m_b)^q}, \tag{18.92}$$

where p, q and n_i are positive integers, the inverse bosonic propagator is

$$D_B(k, m_b) = \hat{k}^2 + m_b^2, \tag{18.93}$$

and the denominator appearing in the propagator of Wilson fermions is taken to be

$$D_F(k, m_f) = \sum_i \sin^2 k_i + \frac{r^2}{4}(\hat{k}^2)^2 + m_f^2. \tag{18.94}$$

Actually, the correct Wilson denominator would be

$$\hat{D}_F(k, m_f) = \sum_i \sin^2 k_i + \left(\frac{r}{2}\hat{k}^2 + m_f\right)^2, \tag{18.95}$$

however in this algorithm m_f only play the role of an infrared regulator and thus it does not need to be the true fermion mass. This form of the fermion propagator turns out to be much easier to handle than the true one. Moreover, for integrals which are only logarithmically divergent, the two forms give exactly the same result in the limit of small quark masses.

As we did in the fermionic case, we first generalize the integrals to noninteger dimensions introducing

$$\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t}}{D_F(k, m_f)^{p+\delta} D_B(k, m_b)^q}, \tag{18.96}$$

where p and q are arbitrary integers (not necessarily positive) and δ is used in the intermediate steps of the calculations and then safely set to zero at the end of them.

It turns out that every $\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t)$ with $q \leq 0$ (i.e., a purely fermionic integral) can be expressed iteratively in terms of nine purely fermionic integrals, in our case $\mathcal{F}(1, 0)$, $\mathcal{F}(1, -1)$, $\mathcal{F}(1, -2)$, $\mathcal{F}(2, 0)$, $\mathcal{F}(2, -1)$, $\mathcal{F}(2, -2)$, $\mathcal{F}(3, -2)$, $\mathcal{F}(3, -3)$ and $\mathcal{F}(3, -4)$. Purely fermionic integrals can always be expressed in terms of integrals of the same type. This is a general property of all recursion relations. The integral $\mathcal{F}(2, 0)$ appears only in the case of divergent integrals. Only eight constants are then needed if the original purely fermionic integral is finite (i.e., $q \leq 0$ and $p \leq 1$).

In the general case in which q can be positive one needs three additional constants, called Y_1, Y_2, and Y_3, to describe the mixed fermionic-bosonic integrals, plus the constants Z_0, Z_1 and F_0 which already appeared in the purely bosonic case.

While for the bosonic case we could give a complete treatment of the reduction steps, for fermions, due to the complexity of the procedure, we can only sketch them. For the readers interested to apply the method, in (Burgio, Caracciolo and Pelissetto, 1996) all details can be found.

There are four steps in this general fermionic method.

The first step consists in expressing each integral $\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t)$ in terms of $\mathcal{F}_\delta(p, q)$ only (that is, pure denominators). For this we need three sets of recursion relations. From the

trivial identity $D_B(k, m_b) = \sum_i \hat{k}_i^2 + m_b^2$ we obtain the first set of recursion relations:

$$\mathcal{F}_\delta(p, q; 1) = \frac{1}{4}\left[\mathcal{F}_\delta(p, q-1) - m_b^2 \mathcal{F}_\delta(p, q)\right], \tag{18.97}$$

$$\mathcal{F}_\delta(p, q; x, 1) = \frac{1}{3}\left[\mathcal{F}_\delta(p, q-1; x) - m_b^2 \mathcal{F}_\delta(p, q; x)\right. \tag{18.98}$$
$$\left. - \mathcal{F}_\delta(p, q; x+1)\right],$$

$$\mathcal{F}_\delta(p, q; x, y, 1) = \frac{1}{2}\left[\mathcal{F}_\delta(p, q-1; x, y) - m_b^2 \mathcal{F}_\delta(p, q; x, y)\right. \tag{18.99}$$
$$\left. -\mathcal{F}_\delta(p, q; x+1, y) - \mathcal{F}_\delta(p, q; x, y+1)\right],$$

$$\mathcal{F}_\delta(p, q; x, y, z, 1) = \mathcal{F}_\delta(p, q-1; x, y, z) - m_b^2 \mathcal{F}_\delta(p, q; x, y, z) \tag{18.100}$$
$$-\mathcal{F}_\delta(p, q; x+1, y, z) - \mathcal{F}_\delta(p, q; x, y+1, z) - \mathcal{F}_\delta(p, q; x, y, z+1).$$

From the identity

$$\sum_i \hat{k}_i^4 = 4(D_B(k, m_b) - D_F(k, m_f) - m_b^2 + m_f^2) + r^2(D_B(k, m_b) - m_b^2)^2 \tag{18.101}$$

we get a second set of recursion relations, of which we give here only some examples:

$$\mathcal{F}_\delta(p, q; x, y, 2) = 2\Big[\mathcal{F}_\delta(p, q-1; x, y) - \mathcal{F}_\delta(p-1, q; x, y) \tag{18.102}$$
$$+(m_f^2 - m_b^2)\mathcal{F}_\delta(p, q; x, y) - \frac{1}{4}\mathcal{F}_\delta(p, q; x+2, y) - \frac{1}{4}\mathcal{F}_\delta(p, q; x, y+2)\Big]$$
$$+\frac{r^2}{2}\Big[\mathcal{F}_\delta(p, q-2; x, y) - 2m_b^2 \mathcal{F}_\delta(p, q-1; x, y) + m_b^4 \mathcal{F}_\delta(p, q; x, y)\Big],$$

$$\mathcal{F}_\delta(p, q; x, y, z, 2) = 4\Big[\mathcal{F}_\delta(p, q-1; x, y, z) - \mathcal{F}_\delta(p-1, q; x, y, z) \tag{18.103}$$
$$+(m_f^2 - m_b^2)\mathcal{F}_\delta(p, q; x, y, z) - \frac{1}{4}\mathcal{F}_\delta(p, q; x+2, y, z)$$
$$-\frac{1}{4}\mathcal{F}_\delta(p, q; x, y+2, z) - \frac{1}{4}\mathcal{F}_\delta(p, q; x, y; z+2)\Big]$$
$$+r^2\Big[\mathcal{F}_\delta(p, q-2; x, y, z) - 2m_b^2 \mathcal{F}_\delta(p, q-1; x, y, z) + m_b^4 \mathcal{F}_\delta(p, q; x, y, z)\Big].$$

Integrating by parts, for $s \geq 3$, the equation

$$\frac{(\hat{k}_w^2)^s}{D_F(k, m_f)^{p+\delta}} = \frac{4(\hat{k}_w^2)^{s-1} - 4(2 + r^2\hat{k}^2)(\hat{k}_w^2)^{s-3}\sin^2 k_w}{D_F(k, m_f)^{p+\delta}}$$
$$-\frac{4(\hat{k}_w^2)^{s-3}}{p+\delta-1}\sin k_w \frac{\partial}{\partial k_w}\frac{1}{D_F(k, m_f)^{p-1+\delta}}, \tag{18.104}$$

we obtain the third set of recursion relations:

$$\mathcal{F}_\delta(p, q; \ldots, s) = 6\mathcal{F}_\delta(p, q; \ldots, s-1) - 8\mathcal{F}_\delta(p, q; \ldots, s-2)$$

$$-4r^2\mathcal{F}_\delta(p, q-1; \ldots, s-2) + 4r^2 m_b^2 \mathcal{F}_\delta(p, q; \ldots, s-2)$$
$$+r^2 \mathcal{F}_\delta(p, q-1; \ldots, s-1) - r^2 m_b^2 \mathcal{F}_\delta(p, q; \ldots, s-1)$$
$$+\frac{4}{p+\delta-1}\Big[-2q\mathcal{F}_\delta(p-1, q+1, \ldots, s-2)$$
$$+\frac{q}{2}\mathcal{F}_\delta(p-1, q+1; \ldots, s-1) + (2s-5)\mathcal{F}_\delta(p-1, q; \ldots, s-3)$$
$$-\frac{1}{2}(s-2)\mathcal{F}_\delta(p-1, q; \ldots, s-2)\Big]. \tag{18.105}$$

As before, by looking at this relation for $p = 1$ we see that in general we have to keep contributions of order δ in the intermediate stages of the algebraic reductions.

Using the recursion relations introduced up to now each integral $\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t)$ can be reduced at the end of the first step to a sum of pure denominator integrals of the form

$$\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) = \sum_{r=p-k+1}^{p} \sum_{s=q-k}^{q+k} a_{rs}(m, \delta)\mathcal{F}_\delta(r, s), \tag{18.106}$$

where $k = (n_x + n_y + n_z + n_t)$, and $m = m_b = m_f$. In the limit $m \to 0$ this becomes

$$\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) = \sum_{r=p-k+1}^{p} \sum_{s=q-k}^{q+k} a_{rs}(0, \delta)\mathcal{F}_\delta(r, s) + \mathcal{R}(m, \delta) + O(m^2), \tag{18.107}$$

where $\mathcal{R}(m, \delta)$ depends on the values of p and q as follows:

1. $p > 0$, $q \leq 0$: $\mathcal{R}(m, \delta)$ is a polynomial in $1/m^2$, finite for $\delta \to 0$;

2. $p > 0$, $q > 0$: $\mathcal{R}(m, \delta) = \frac{1}{\delta}\left(1 - \delta \log m^2\right)\mathcal{R}^{(1)}(m) + \mathcal{R}^{(2)}(m) + O(\delta)$, where $\mathcal{R}^{(i)}(m)$ are polynomials in $1/m^2$;

3. $p \leq 0, q \leq 0$: $\mathcal{R}(m, \delta) = 0$;

4. $p \leq 0$, $q > 0$: $\mathcal{R}(m, \delta) = \left(1 - \delta \log m^2\right)\mathcal{R}^{(1)}(m) + \delta\,\mathcal{R}^{(2)}(m) + O(\delta^2)$.

In the second step, with the systematic use of the identity

$$\mathcal{F}_\delta(p, q; 1, 1, 1, 1) - 4\mathcal{F}_\delta(p, q+1; 2, 1, 1, 1) - m^2 \mathcal{F}_\delta(p, q+1; 1, 1, 1, 1) = 0, \tag{18.108}$$

one obtains a nontrivial relation of the form

$$\sum_{r,s} f_{rs}(p, q; \delta)\mathcal{F}_\delta(r, s) + \mathcal{R}_\delta(p, q; m, \delta) = 0, \tag{18.109}$$

where $p - 4 \leq r \leq p$. As in the bosonic case, it can be used to obtain new recursion relations; in this case, one can exploit it in three different ways.

At the end of the second step one is then able to reduce every $\mathcal{F}_\delta(p, q)$ in terms of only the $\mathcal{F}_\delta(r, s)$'s with $0 \leq r \leq 3$ and arbitrary s, or $r \leq -1$ and $s = 1, 2, 3$, or $r \geq 4$ and $s = 0, -1, -2$.

In the third step we systematically use the identity

$$
\mathcal{F}_\delta(p, q; 1, 1, 1, 1) - \mathcal{F}_\delta(p + 1, q - 1; 1, 1, 1, 1) + \mathcal{F}_\delta(p + 1, q; 3, 1, 1, 1)
$$

$$
- \frac{1}{4}\Big[\mathcal{F}_\delta(p + 1, q - 2; 1, 1, 1, 1) - 2m^2 \mathcal{F}_\delta(p + 1, q - 1; 1, 1, 1, 1)
$$

$$
+ m^4 \mathcal{F}_\delta(p + 1, q; 1, 1, 1, 1)\Big] = 0, \tag{18.110}
$$

which is applied in four different ways according to the particular properties of the four regions $q \leq 0$ and $0 \leq p \leq 3$, $q > 0$ and $0 \leq p \leq 3$, $q = 1, 2, 3$ and $p \leq -1$, $q = -2, -1, 0$ and $p \geq 4$.

We can at the end of the third step express the remaining $\mathcal{F}_\delta(p, q)$ in terms of only the $\mathcal{F}_\delta(r, s)$'s with $r = 3$ and $-4 \leq s \leq 0$, or $r = 2$ and $-4 \leq s \leq 2$, or $r = 1$ and $-4 \leq s \leq 4$, or $r = 0$ and $-4 \leq s \leq 6$, or $r = -1$ and $s = 2$.

In the fourth step the identities in Eqs. (18.108) and (18.110), which so far have not been used for all possible values of p and q, are further exploited to provide additional relations between the remaining integrals. One has to look systematically for further values of p and q for which these two identities were not trivially satisfied. At the end of this process one can then achieve a further decrease of the number of independent constants. We give an example of these additional relations:

$$
\begin{aligned}
\mathcal{F}_\delta(0, 4) &= (1 - \delta \log m^2)\left[\frac{1}{96\pi^2 m^4} - \frac{19}{4608\pi^2 m^2} + \frac{1}{9216\pi^2}\right] \tag{18.111} \\
&\quad + \frac{31}{144}\mathcal{F}_\delta(0, 3) - \frac{13}{1152}\mathcal{F}_\delta(0, 2) + \frac{1}{9216}\mathcal{F}_\delta(0, 1) \\
&\quad + \delta\Big[-\frac{5}{576\pi^2 m^4} + \frac{61}{18432\pi^2 m^2} + \frac{80989}{88473600\pi^2} + \frac{347}{1440}\mathcal{F}_\delta(0, 3) \\
&\quad - \frac{83}{2560}\mathcal{F}_\delta(0, 2) + \frac{137}{184320}\mathcal{F}_\delta(0, 1) - \frac{689}{2880}\mathcal{F}_\delta(1, 2) + \frac{1139}{23040}\mathcal{F}_\delta(1, 1) \\
&\quad - \frac{415}{147456}\mathcal{F}_\delta(1, 0) + \frac{23}{4423680}\mathcal{F}_\delta(1, -1) - \frac{329}{11520}\mathcal{F}_\delta(2, 0) + \frac{13283}{1105920}\mathcal{F}_\delta(2, -1) \\
&\quad - \frac{391}{221184}\mathcal{F}_\delta(2, -2) - \frac{30479}{2211840}\mathcal{F}_\delta(3, -2) + \frac{437}{245760}\mathcal{F}_\delta(3, -3) + \frac{161}{589824}\mathcal{F}_\delta(3, -4)\Big].
\end{aligned}
$$

At the end of these complicated reductions, one is able to write a generic purely fermionic integral ($p > 0$ and $q \leq 0$) in terms of eight infrared-finite integrals: $\mathcal{F}(1, 0)$, $\mathcal{F}(1, -1)$, $\mathcal{F}(1, -2)$, $\mathcal{F}(2, -1)$, $\mathcal{F}(2, -2)$, $\mathcal{F}(3, -2)$, $\mathcal{F}(3, -3)$ and $\mathcal{F}(3, -4)$, plus a constant, Y_0, which appears in the logarithmic divergent integral

$$
\mathcal{F}(2, 0) = -\frac{1}{16\pi^2}\left(\log m^2 + \gamma_E - F_0\right) + Y_0. \tag{18.112}
$$

For the mixed fermionic-bosonic integrals ($p > 0$ and $q > 0$) one must introduce three additional constants, which have been chosen as

$$
Y_1 = \frac{1}{8}\mathcal{F}(1, 1; 1, 1, 1), \tag{18.113}
$$

$\mathcal{F}(1,0)$	0.08539036359532067914
$\mathcal{F}(1,-1)$	0.46936331002699614475
$\mathcal{F}(1,-2)$	3.39456907367713000586
$\mathcal{F}(2,-1)$	0.05188019503901136636
$\mathcal{F}(2,-2)$	0.23874773756341478520
$\mathcal{F}(3,-2)$	0.03447644143803223145
$\mathcal{F}(3,-3)$	0.13202727122781293085
$\mathcal{F}(3,-4)$	0.75167199030295682254
Y_0	$-$ 0.01849765846791657356
Y_1	0.00376636333661866811
Y_2	0.00265395729487879354
Y_3	0.00022751540615147107

Table 4: New constants appearing in the general fermionic case.

$$Y_2 \;=\; \frac{1}{16}\,\mathcal{F}(1,1;1,1,1,1), \tag{18.114}$$

$$Y_3 \;=\; \frac{1}{16}\,\mathcal{F}(1,2;1,1,1). \tag{18.115}$$

In Table 4 we report the values of the new basic constants introduced in the general case.

Most of the basic recursion relations for fermions presented above can be easily generalized to integrals which are dimensionally regularized, however some of them are instead intrinsically four-dimensional identities. For this reason it is more convenient to apply the above reductions which use a mass as a regulator. We don't give here the complicated relation between the two schemes, but only say that fortunately the case of logarithmically divergent integrals can still be dealt with rather easily, because it is enough to make the substitution $2/(d-4) + \log 4\pi$ for $\log m^2 + \gamma_E$. The relation between integrals computed using the true fermion propagator and integrals computed using the propagator (18.94) is also rather complicated. However, if the integrals are only logarithmically divergent, for $m \to 0$ the two results coincide. For power-divergent expressions instead, if one uses the true fermion propagator the divergent part is a polynomial in $1/m$ instead of $1/m^2$, and the expressions are in general more cumbersome than those involving the propagator in Eq. (18.94).

The method discussed here depends on the form of the quark propagator, but not on the vertices. It can thus be applied to $O(a)$ improved fermions as well. A version for, say, overlap fermions, which involves more complicated denominators, has not yet been developed. In this case one has to find a generalization of the method, but it can be expected that the corresponding recursion relations will turn out to be much more complicated, since already Wilson fermions bring rather cumbersome recursion relations. In these cases, a convenient method to reduce all integrals could be the one of (Laporta, 2000), which has been used

in (Becher and Melnikov, 2002). It uses a brute force approach which reduces integrals to simpler ones integrals (with lower values of the indices) by means of a classification which uses a lexicographic order, without the need of finding a complicated system of recurrence relations like the one exposed above.

18.6 The quark self-energy

Using the algebraic methods explained above, it is possible to give a purely algebraic result for the 1-loop quark self-energy (for $r = 1$, in the Feynman gauge). This was first computed in (González-Arroyo, Ynduráin and Martinelli, 1982; Hamber and Wu, 1983; Groot, Hoek and Smit, 1984). The result for the 1-loop quark self-energy

$$\Sigma(p^2, m^2) = g_0^2 C_F \left(\widetilde{m}_c + i\not{p}\, \widetilde{\Sigma}_1(p^2, m^2) + m\, \widetilde{\Sigma}_2(p^2, m^2) \right), \tag{18.116}$$

in terms of the basic constants is (Burgio, Caracciolo and Pelissetto, 1996)

$$\begin{aligned}
\widetilde{m}_c &= -Z_0 - 2\mathcal{F}(1,0) \approx -0.32571411742170157236, \tag{18.117}\\
\widetilde{\Sigma}_1(p^2, m^2) &= \frac{1}{16\pi^2}(2G(p^2 a^2, m^2 a^2) + \gamma_E - F_0) + \frac{1}{8}Z_0 + \frac{1}{192}\\
&\quad -\frac{1}{32\pi^2} - Y_0 + \frac{1}{4}Y_1 - \frac{1}{16}Y_2 + 12\,Y_3 - \frac{1}{768}\mathcal{F}(1,-2)\\
&\quad -\frac{1}{192}\mathcal{F}(1,-1) + \frac{109}{192}\mathcal{F}(1,0) - \frac{1}{768}\mathcal{F}(2,-2) + \frac{25}{48}\mathcal{F}(2,-1)\\
&\approx \frac{1}{8\pi^2}G(p^2 a^2, m^2 a^2) + 0.0877213749, \tag{18.118}\\
\widetilde{\Sigma}_2(p^2, m^2) &= \frac{1}{4\pi^2}(F(p^2 a^2, m^2 a^2) + \gamma_E - F_0) + \frac{1}{48} - \frac{1}{4\pi^2}\\
&\quad -4\,Y_0 + Y_1 - \frac{1}{4}Y_2 - \frac{1}{192}\mathcal{F}(1,-2) - \frac{1}{48}\mathcal{F}(1,-1)\\
&\quad -\frac{83}{48}\mathcal{F}(1,0) - \frac{1}{192}\mathcal{F}(2,-2) + \frac{49}{12}\mathcal{F}(2,-1)\\
&\approx \frac{1}{4\pi^2}F(p^2 a^2, m^2 a^2) + 0.0120318529, \tag{18.119}
\end{aligned}$$

where

$$\begin{aligned}
F(p^2 a^2, m^2 a^2) &= \int_0^1 dx \log[(1-x)(p^2 x + m^2)a^2], \tag{18.120}\\
G(p^2 a^2, m^2 a^2) &= \int_0^1 dx\, x \log[(1-x)(p^2 x + m^2)a^2]. \tag{18.121}
\end{aligned}$$

The importance of having explicit expressions like these cannot be underestimated. Thanks to them, the 1-loop self-energy $\Sigma(p^2, m^2)$ can now be computed with many significant decimal places, provided the basic constants are determined with sufficient accuracy.

19 Coordinate space methods

It is very useful to consider propagators in coordinate space. In this Section we are going to illustrate the coordinate space method, developed by (Lüscher and Weisz, 1995b), which is very powerful for a variety of reasons. In particular it turns out to be very important for the calculation of 1-loop integrals with very high precision. Having 1-loop integrals determined with such precision is necessary for the implementation of the only known method (which will be also presented here) with which the computation of 2-loop integrals with good precision can be carried out.

A fundamental object for the coordinate space method is the mixed fermionic-bosonic propagator in position space, which will be denoted by

$$\mathcal{G}_F(p, q; x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{D_F(k, m)^p D_B(k, m)^q}. \tag{19.1}$$

Any of these propagators $\mathcal{G}_F(p, q; x)$ can be expressed as a linear combination of the integrals $\mathcal{F}$ introduced in Section 18.5, and consequently as a linear combination of the fifteen basic constants $Z_0, Z_1, F_0, \ldots, Y_0, Y_1, Y_2, Y_3$. Of course the general position space propagator $\mathcal{G}_F(p, q; x)$ can also be expressed in terms of a different set of fifteen basic constants, and general recursion relations between the $\mathcal{G}_F$'s can be derived. We can always consider the $\mathcal{G}_F$'s instead of the $\mathcal{F}$'s as an intermediate representation of the general integral coming from the calculations of Feynman diagrams, and then express every $\mathcal{G}_F$ in terms of a chosen set of fifteen constants.

To begin with we consider the finite bosonic case, in which, as we know, only two basic constants are needed. In the bosonic case a simple reduction algorithm was developed by (Lüscher and Weisz, 1995b), following ideas by Vohwinkel. For $(p, q) = (0, 1)$ (i.e., the standard boson propagator), a recursion relation which involves only terms with the same $(p, q) = (0, 1)$ was obtained. This simple recursion relation avoids the introduction of noninteger dimensions and nonpositive values of q, as is instead the case for algorithms for general p and q. The free lattice gluon propagator in position space can then be evaluated recursively, and is a linear function of its values near the origin. Lüscher and Weisz chose as basic constants two values of the propagator close to the origin:

$$\mathcal{G}_F(0, 1; (0, 0, 0, 0)) = Z_0, \tag{19.2}$$
$$\mathcal{G}_F(0, 1; (1, 1, 0, 0)) = Z_0 + Z_1 - \frac{1}{4},$$

where the relation between these two constants and the constants of Section 18 is also shown.

We denote the gluon propagator by $G(x) = \mathcal{G}_F(0, 1; x)$. The key observation in the Lüscher-Weisz algorithm is that

$$\left(\nabla_\mu^\star + \nabla_\mu\right) G(x) = x_\mu H(x), \tag{19.3}$$

where the function

$$H(x) = \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} e^{ipx} \log \hat{p}^2 \tag{19.4}$$

is independent of μ. Summing now this key formula over μ and using $-\Delta G(x) = \delta_{x,0}$ (where $\Delta = \sum_{\mu=0}^{3} \nabla_\mu^\star \nabla_\mu$) we get

$$H(x) = \frac{2}{\rho} \sum_{\mu=0}^{3} [G(x) - G(x - \hat{\mu})], \qquad \rho = \sum_{\mu=0}^{3} x_\mu, \tag{19.5}$$

which can be used back to eliminate $H(x)$ from the key formula. We then obtain the fundamental recursion relation for the gluon propagator in position space:

$$G(x + \hat{\mu}) = G(x - \hat{\mu}) + \frac{2x_\mu}{\rho} \sum_{\nu=0}^{3} [G(x) - G(x - \hat{\nu})]. \tag{19.6}$$

This formula is to be used for $\rho \neq 0$. Since the propagator is independent of the sign and order of the four coordinates, we can restrict ourselves to $x_0 \geq x_1 \geq x_2 \geq x_3 \geq 0$. In this sector, the recursion relation allows to express $G(x)$ in terms of its values in five points: $G(0,0,0,0)$, $G(1,0,0,0)$, $G(1,1,0,0)$, $G(1,1,1,0)$, $G(1,1,1,1)$. Now, using the properties of the propagator three more relations between these five constants can be found,

$$G(0,0,0,0) - G(1,0,0,0) = 1/8 \tag{19.7}$$
$$G(0,0,0,0) - 3\,G(1,1,0,0) - 2\,G(1,1,1,0) = 1/\pi^2$$
$$G(0,0,0,0) - 6\,G(1,1,0,0) - 8\,G(1,1,1,0) - 3\,G(1,1,1,1) = 0,$$

and thus we can finally write the generic bosonic propagator $G(x)$ in terms of only two basis constants:

$$G(x) = r_1(x)\,G(0,0,0,0) + r_2(x)\,G(1,1,0,0) + \frac{r_3(x)}{\pi^2} + r_4(x). \tag{19.8}$$

The coefficients $r_k(x)$ are rational numbers which can be computed in a recursive manner. We have thus expressed the free gluon propagator in terms of two basic constants, which can be reinterpreted as the values of the propagator near the origin.

The generalization of the coordinate method by Lüscher and Weisz to fermions is more complicated, and we will not discuss it here. For fermions a recursion relation analog to Eq. (19.6) which only involves the fermion propagator itself (i.e., without excursions to other values of p and q) has not yet been found, and the reduction to the basic constants has to be carried out along more complicated procedures. The free fermion propagator can be written at the end in terms of eight basic constants, which can be reinterpreted as some values of the quark propagator near the origin.

19.1 High-precision integrals

The reduction of the coordinate space propagators to a small set of basic constants, which we have just described, has interesting properties that turn out to be very useful for the

high-precision computation of these basic constants. One can show that the coefficients r_k in Eq. (19.8) increase exponentially with the distance x, while the propagator $G(x)$ remains of course bounded. There are therefore huge cancellations and significance losses when x is large.

One can look at this apparent nuisance with other eyes, and exploit this numerical instability the other way around to compute the basic constants with very high precision (Lüscher and Weisz, 1995b). Let us consider the boson propagator at the points

$$
\begin{aligned}
x_1 &= (n, 0, 0, 0) \\
x_2 &= (n, 1, 0, 0)
\end{aligned}
\tag{19.9}
$$

for large n. The associated sets of coefficients $r_k(x_1)$ and $r_k(x_2)$ in Eq. (19.8),

$$
\begin{aligned}
G(x_1) &= r_1(x_1)\, G(0,0,0,0) + r_2(x_1)\, G(1,1,0,0) + \frac{r_3(x_1)}{\pi^2} + r_4(x_1) \\
G(x_2) &= r_1(x_2)\, G(0,0,0,0) + r_2(x_2)\, G(1,1,0,0) + \frac{r_3(x_2)}{\pi^2} + r_4(x_2),
\end{aligned}
\tag{19.10}
$$

are of order 10^n. If this 2×2 algebraic system is now inverted in terms of the unknowns $G(0,0,0,0)$ and $G(1,1,0,0)$, the coefficients multiplying $G(x_1)$ and $G(x_2)$ will be of order 10^{-n}. Therefore the constants $G(0,0,0,0)$ and $G(1,1,0,0)$ can be determined to this level of precision, if one just neglects $G(x_1)$ and $G(x_2)$ in the 2×2 inverse system. $G(0,0,0,0)$ and $G(1,1,0,0)$ then remain functions only of the coefficients r_3 and r_4.

It costs very little to go to very high n and obtain the values of these two constants with very high precision. One can then systematically improve on the accuracy until the desired level of precision is reached. The method is exponentially convergent, and the error can also be estimated with good accuracy. On the contrary, a direct evaluation of the integrals defining the basic constants can never provide such accurate results.

Using this method we have calculated Z_0 and Z_1, the two constants which are the basis of finite bosonic integrals, with almost 400 significant decimal places. These new results are given in Appendix B.

In the general mixed fermionic case, the expressions for a generic $\mathcal{G}_F(p, q; x)$ in terms of fifteen values of it near the origin are also numerically unstable for $|x| \to \infty$. If we consider for example the fermion propagator, we can choose eight points with $|x| \approx n$ (say $y_1, \ldots, y_8$) and then we can express the propagator for $|x| < n$ in terms of $\mathcal{G}_F(1, 0; y_i)$. These expressions are numerically stable: considering for example the set of eight points $X^{(n)} \equiv \{(n, [0-3], 0, 0), (n+1, [0-3], 0, 0)\}$ for high n one can obtain the values of the propagator near the origin with great precision. It is also possible to apply the same procedure to $\mathcal{G}_F(1, q; x)$ with $q < 0$. The main advantage is that, using larger negative values of q, one can obtain more precise estimates of $\mathcal{G}_F(1, q; x)$ at this set of points.

Once one has extracted the fifteen values for the mixed propagators near the origin, it is straightforward to change basis and compute the fifteen basic constants for 1-loop integrals,

$Z_0, Z_1, \ldots, Y_2, Y_3$. All constants (except F_0) are now known with a precision of sixty decimal places (Caracciolo, Pelissetto and Rago, 2001).

19.2 Coordinate space methods for 2-loop computations

The determination of 1-loop integrals with high precision is an essential component of refined 2-loop calculations which are carried out using coordinate space methods. They allow to reach a precision unmatched by more conventional methods.

A few 2-loop calculations have also been performed by means of sophisticated extrapolations to infinite L of momentum sums, using fitting functions and blocking transformations of the kind that we have discussed in Section 17. [89] The precision reached with these conventional methods can be good in some cases, but the precision which can be achieved using the coordinate method is already higher at present, and can be easily increased (by computing the 1-loop "building blocks" with higher precision). Sometimes the absolute values of the integrals to be computed are rather small, less than 10^{-6}, [90] and in order to get a good relative error the coordinate method seems to be the most suitable.

The 2-loop calculations completed so far, using either conventional or coordinate space methods, only concern quantities that are finite, like the coefficient b_2 of the β function, of which we have spoken in Section 10. No divergent matrix elements have yet been computed. Although some of the individual diagrams for finite matrix elements can contain subdivergences, the situation in which the matrix element are themselves divergent is certainly more challenging.

19.2.1 Bosonic case

We will now illustrate the calculation of two loop integrals using the coordinate space method by (Lüscher and Weisz, 1995b). At first we discuss the case of integrals with zero external momenta, which is simpler. Let us then consider the 2-loop integral in momentum space

$$I_1 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \frac{1}{\widehat{k}^2 \widehat{q}^2 \widehat{(k+q)}^2}. \tag{19.11}$$

This integral can be reexpressed as follows:

$$\begin{aligned}
I_1 &= \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4r}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k+q+r) \frac{1}{\widehat{k}^2 \widehat{q}^2 \widehat{r}^2} \\
&= \sum_x \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4r}{(2\pi)^4} \, \mathrm{e}^{ikx} \frac{1}{\widehat{k}^2} \, \mathrm{e}^{iqx} \frac{1}{\widehat{q}^2} \, \mathrm{e}^{irx} \frac{1}{\widehat{r}^2}
\end{aligned}$$

---

[89]In a couple of cases even three-loop calculations have been carried out (Allés *et al.*, 1994a; Allés *et al.*, 1994b; Allés, Feo and Panagopoulos, 1998).

[90]An example of this is given by the calculation of the coefficients $C_\pm^5$ of Eq. (14.14), which has been carried out using conventional 2-loop integration methods (Curci *et al.*, 1988).

$$= \sum_x \left(\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{ikx} \frac{1}{\widehat{k}^2} \right)^3$$

$$= \sum_x G^3(x). \tag{19.12}$$

The 2-loop integral in momentum space has thus been written in terms of a sum of a simple function of the position-space propagator $G(x)$ over the lattice sites. We have changed from an eight-dimensional sum in momentum space to a four-dimensional sum in position space. Since $G(x) \sim 1/x^2$, this sum is absolutely convergent. An evaluation of the sum over a finite domain containing the origin, say the region $|x| < 20$, can then be taken as a first approximation of the integral I_1. Of course one must also know the propagator $G(x)$ with very good accuracy, because some precision will be lost when computing the sums. It is here that the high-precision technique which we have just discussed turns out to be vary handy. In particular, the recursion relations in Eq. (19.6) allow to compute $G(x)$ in terms of two constants (which can be chosen to be Z_0 and Z_1) which can be computed with arbitrary precision, and thus $G(x)$ can be computed with that precision.

If $G(x)$ can be determined (for any x in the domain) with arbitrary precision, the only remaining challenge is to evaluate the sums over a reasonable domain of sites in the most effective way. Of course the domain cannot be too large because of computational limitations. The sums can then be better evaluated by exploiting the knowledge of the asymptotic expansion of the propagator for large x, which is given by [91]

$$G(x) \xrightarrow{x \to \infty} \frac{1}{4\pi^2 x^2} \left\{ 1 - \frac{1}{x^2} + 2\frac{x^4}{(x^2)^3} - 4\frac{1}{(x^2)^2} + 16\frac{x^4}{(x^2)^4} - 48\frac{x^6}{(x^2)^5} + 40\frac{(x^4)^2}{(x^2)^6} + \dots \right\}. \tag{19.13}$$

Here and in the following the notation

$$x^n = \sum_{\mu=0}^{3} (x_\mu)^n \tag{19.14}$$

is used. We have then

$$G^3(x) \xrightarrow{x \to \infty} \frac{1}{(4\pi^2)^3} \frac{1}{(x^2)^3} \left\{ 1 - \frac{3}{x^2} + 6\frac{x^4}{(x^2)^3} - 9\frac{1}{(x^2)^2} + 36\frac{x^4}{(x^2)^4} \right.$$

$$\left. - 144\frac{x^6}{(x^2)^5} + 132\frac{(x^4)^2}{(x^2)^6} \right\} + O(|x|^{-12}) \tag{19.15}$$

$$= \frac{1}{(4\pi^2)^3} \left\{ \left[\frac{1}{(x^2)^3} + \frac{3}{10(x^2)^5} \right] h_0(x) + \left[\frac{3}{(x^2)^6} + \frac{24}{7(x^2)^7} \right] h_1(x) \right.$$

$$\left. - \frac{3}{4(x^2)^8} h_2(x) + \frac{33}{140(x^2)^9} h_3(x) \right\} + O(|x|^{-12}), \tag{19.16}$$

[91]For the derivation of this and similar asymptotic expansions, see (Lüscher and Weisz, 1995b).

where in the last line we have rewritten $G^3(x)$ in terms of the homogeneous harmonic polynomials

$$
\begin{align}
h_0(x) &= 1, & (19.17)\\
h_1(x) &= 2x^4 - (x^2)^2, & (19.18)\\
h_2(x) &= 16x^6 - 20x^2x^4 + 5(x^2)^3, & (19.19)\\
h_3(x) &= 560(x^4)^2 - 560x^2x^6 + 60(x^2)^2x^4 - 9(x^2)^4. & (19.20)
\end{align}
$$

It is useful at this point to define a generalized zeta function

$$
Z(s,h) = \sum_{x:\,x\neq 0} h(x)(x^2)^{-s}, \tag{19.21}
$$

where the site $x = 0$ is not to be included in the sum. These zeta functions can be calculated with the necessary accuracy. To this end one introduces the heat kernel

$$
k(t,h) = \sum_{x:\,x\neq 0} h(x)\mathrm{e}^{-\pi t x^2}, \tag{19.22}
$$

so that the generalized zeta function can be reexpressed as

$$
\begin{align}
Z(s,h) &= \frac{\pi^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}[k(t,h)-h(0)] \tag{19.23}\\
&= \frac{\pi^s}{\Gamma(s)}\left\{\frac{2h(0)}{s(s-2)} + \int_1^\infty dt\,[t^{s-1}+(-1)^{d/2}t^{d-s+1}][k(t,h)-h(0)]\right\}, \tag{19.24}
\end{align}
$$

where in the last line the integration has been split into two parts, and the formula

$$
k(t,h) = (-1)^{d/2}\, t^{-d-2}\, k(1/t,h) \tag{19.25}
$$

has been used. [92] We have then

$$
\begin{align}
\sum_x G^3(x) &= \sum_{x:\,x\neq 0}\left[G^3(x) - G^3_{\mathrm{as}}(x)\right]\\
&\quad + \frac{1}{(4\pi^2)^3}\left\{Z(3,h_0) + \frac{3}{10}Z(5,h_0) + 3Z(6,h_1) + \frac{24}{7}Z(7,h_1)\right.\\
&\qquad\qquad\left. -\frac{3}{4}Z(8,h_2) + \frac{33}{140}Z(9,h_3)\right\}\\
&\quad + G^3(0). \tag{19.27}
\end{align}
$$

---

[92]This formula derives from the fact that $h(x)$ is harmonic and from the Poisson summation formula

$$
\sum_x \mathrm{e}^{-iqx}\mathrm{e}^{-\pi t x^2} = t^{-2}\sum_x \mathrm{e}^{-(q+2\pi x)^2/(4\pi t)}. \tag{19.26}
$$

We can see that once the values of the zeta function are known, it only remains to compute $\sum_{x:\,x\neq 0}\left[G^3(x) - G^3_{\mathrm{as}}(x)\right]$, which is rapidly convergent since each term goes at least like $|x|^{-12}$. This sum can then be evaluated with a reasonable approximation using domains which are not too large, and in this way one can obtain

$$I_1 = 0.0040430548122(3). \tag{19.28}$$

The coordinate space method can be used to compute more complicated integrals in which nontrivial numerators are present, like

$$
\begin{aligned}
I_2 &= \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \sum_{\mu=0}^{3} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^2}{\widehat{k}^2 \widehat{q}^2 (\widehat{k+q})^2} \\
&= \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4r}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k+q+r) \sum_{\mu=0}^{3} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^2}{\widehat{k}^2 \widehat{q}^2 \widehat{r}^2} \\
&= \sum_{x} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4r}{(2\pi)^4} \mathrm{e}^{ikx} \mathrm{e}^{iqx} \mathrm{e}^{irx} \sum_{\mu=0}^{3} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^2}{\widehat{k}^2 \widehat{q}^2 \widehat{r}^2} \\
&= \sum_{x} \sum_{\mu=0}^{3} \left(\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \mathrm{e}^{ikx} \frac{\widehat{k}_\mu^2}{\widehat{k}^2} \right)^2 \left(\int_{-\pi}^{\pi} \frac{d^4r}{(2\pi)^4} \mathrm{e}^{irx} \frac{1}{\widehat{r}^2} \right) \\
&= \sum_{x} \sum_{\mu=0}^{3} \left(-\nabla_\mu^* \nabla_\mu G(x) \right)^2 G(x) \\
&= 0.0423063684(1). \tag{19.29}
\end{aligned}
$$

Here the derivatives of the propagator in coordinate space generate the $\widehat{k}_\mu^2$ and $\widehat{q}_\mu^2$ factors in the numerators of the momentum-space integrals. Similarly,

$$
\begin{aligned}
I_3 &= \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \sum_{\mu=0}^{3} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^2 (\widehat{k+q})_\mu^2}{\widehat{k}^2 \widehat{q}^2 (\widehat{k+q})^2} \\
&= \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4r}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k+q+r) \sum_{\mu=0}^{3} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^2 \widehat{r}_\mu^2}{\widehat{k}^2 \widehat{q}^2 \widehat{r}^2} \\
&= \sum_{x} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4r}{(2\pi)^4} \mathrm{e}^{ikx} \mathrm{e}^{iqx} \mathrm{e}^{irx} \sum_{\mu=0}^{3} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^2 \widehat{r}_\mu^2}{\widehat{k}^2 \widehat{q}^2 \widehat{r}^2} \\
&= \sum_{x} \sum_{\mu=0}^{3} \left(\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \mathrm{e}^{ikx} \frac{\widehat{k}_\mu^2}{\widehat{k}^2} \right)^3 \\
&= \sum_{x} \sum_{\mu=0}^{3} \left(-\nabla_\mu^* \nabla_\mu G(x) \right)^3 \\
&= 0.054623978180(1). \tag{19.30}
\end{aligned}
$$

In general, integrals where the numerator is a polynomial in sines and cosines can be computed using these techniques. However, in more complicated cases, where for example the denominator

has a higher power than the integrals discussed so far, it is necessary to introduce auxiliary functions.

One of such cases is given by the calculation of

$$I_4 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \sum_{\mu=0}^{3} \frac{\widehat{k}_\mu^2 \widehat{q}_\mu^2}{(\widehat{k}^2)^2 \widehat{q}^2 (\widehat{k+q})^2}. \tag{19.31}$$

The problem here is that the factor $1/(\widehat{k}^2)^2$ cannot be related in a simple way to $G(x)$. In this case, the auxiliary function

$$K(x) = \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \frac{(e^{ipx} - 1)}{(\widehat{p}^2)^2} \tag{19.32}$$

is just what we need. In fact,

$$\nabla_\mu^* \nabla_\mu K(x) = -\int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \widehat{p}_\mu^2 \frac{e^{ipx}}{(\widehat{p}^2)^2}, \tag{19.33}$$

so that

$$\begin{aligned}
I_4 &= \sum_{x} \sum_{\mu=0}^{3} \left(-\nabla_\mu^* \nabla_\mu K(x) \right) \left(-\nabla_\mu^* \nabla_\mu G(x) \right) G(x) \\
&= 0.006603075727(1). \tag{19.34}
\end{aligned}$$

The function $K(x)$ is related to $G(x)$ by

$$\begin{aligned}
-\Delta K(x) &= G(x), \tag{19.35} \\
(\nabla_\mu^* + \nabla_\mu) K(x) &= -x_\mu G(x), \tag{19.36}
\end{aligned}$$

so that it can be recursively computed in terms of $G(x)$ and of the values of $K(x)$ at the corner of the unit hypercube.

Finally, the integral

$$I_5 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \sum_{\mu=0}^{3} \frac{\widehat{k}_\mu^4}{(\widehat{k}^2)^3 \widehat{q}^2 (\widehat{k+q})^2} \tag{19.37}$$

can be computed introducing the auxiliary function

$$L(x) = \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \frac{\left(e^{ipx} - 1 - i \sum_\mu x \sin p_\mu + \frac{1}{2}(\sum_\mu x \sin p_\mu)^2 \right)}{(\widehat{p}^2)^3}, \tag{19.38}$$

which has the properties

$$\begin{aligned}
-\Delta L(x) &= K(x) + \frac{1}{8} G(0) - \frac{1}{32\pi^2}, \tag{19.39} \\
(\nabla_\mu^* + \nabla_\mu) L(x) &= -\frac{1}{2} x_\mu \left[K(x) + \frac{1}{8} G(0) \right], \tag{19.40}
\end{aligned}$$

185

and can be computed recursively. The integral is then given by

$$
\begin{aligned}
I_5 &= \sum_x \sum_{\mu=0}^{3} \left(\nabla_\mu^* \nabla_\mu \nabla_\mu^* \nabla_\mu L(x) \right) G^2(x) \\
&= 0.00173459425(1).
\end{aligned}
\tag{19.41}
$$

We now turn to 2-loop integrals which also depend on an external momentum. Of course these integrals can also be computed using the decomposition explained in Section 15.2 and the theorem of Reisz, thanks to which only zero-momentum integrals have at the end to be really computed on the lattice. However, it is also possible to compute these 2-loop integrals using the coordinate space methods directly, as we are going to show.

To understand how external momenta are incorporated into the method, let us first discuss the simpler case of a 1-loop integral depending on an external momentum p. The logarithmically divergent integral

$$
I_6 = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{1}{\widehat{k}^2 (\widehat{k-p})^2}
\tag{19.42}
$$

can be rewritten in position space as follows:

$$
\begin{aligned}
I_6 &= \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q + k - p) \frac{1}{\widehat{k}^2 \widehat{q}^2} \\
&= \sum_x e^{-ipx} \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} e^{ikx} \frac{1}{\widehat{k}^2} e^{iqx} \frac{1}{\widehat{q}^2} \\
&= \sum_x e^{-ipx} G^2(x).
\end{aligned}
\tag{19.43}
$$

The integral can then be computed by evaluating the sums

$$
I_6 = \lim_{\epsilon \to 0} \sum_x e^{-\epsilon x^2} e^{-ipx} G^2(x).
\tag{19.44}
$$

It is now useful to employ the function $H(x)$ which we have introduced in Eq. (19.4) when deriving the recursion relations for $G(x)$ in coordinate space:

$$
H(x) = \int_{-\pi}^{\pi} \frac{d^4 p}{(2\pi)^4} e^{ipx} \log \widehat{p}^2.
\tag{19.45}
$$

Its asymptotic expansion for large x is

$$
\begin{aligned}
H(x) \overset{x \to \infty}{\longrightarrow} -\frac{1}{\pi^2} \frac{1}{(x^2)^2} \Big\{ & 1 - \frac{4}{x^2} + 8\frac{x^4}{(x^2)^3} - 7\frac{1}{(x^2)^2} + 40\frac{x^4}{(x^2)^4} \\
& - 288\frac{x^6}{(x^2)^5} + 280\frac{(x^4)^2}{(x^2)^6} + \dots \Big\},
\end{aligned}
\tag{19.46}
$$

186

and thus its leading order term has (up to a constant) the same asymptotic behavior of $G^2(x)$. We now subtract and add to the original integral an appropriate expression containing $H(x)$:

$$I_6 = -\frac{1}{16\pi^2} \lim_{\epsilon \to 0} \sum_x e^{-\epsilon x^2} e^{-ipx} H(x) + \sum_x e^{-ipx} \left[G^2(x) + \frac{1}{16\pi^2} H(x) \right]. \tag{19.47}$$

The first part is just the Fourier transform of $H(x)$, which can be read off from its definition above, and is thus given by

$$-\frac{1}{16\pi^2} \lim_{\epsilon \to 0} \sum_x e^{-\epsilon x^2} e^{-ipx} H(x) = -\frac{1}{16\pi^2} \log \hat{p}^2 = -\frac{1}{16\pi^2} \log p^2 + O(p^2). \tag{19.48}$$

The finite constant is obtained from the second part, in which the leading $1/|x|^4$ terms of $G^2(x)$ and $H(x)$ cancel in the subtraction, so that this part goes like the Fourier transform of $1/|x|^6$ (and is therefore finite). In the limit $p \to 0$ we have then

$$I_6 = -\frac{1}{16\pi^2} \log p^2 + \sum_x \left[G^2(x) + \frac{1}{16\pi^2} H(x) \right]. \tag{19.49}$$

The sum can be computed using generalized zeta functions as explained before. The result is

$$I_6 = -\frac{1}{16\pi^2} \log p^2 + \frac{1}{8\pi^2} + 0.02401318111946489(1). \tag{19.50}$$

Let us now illustrate this powerful method for a 2-loop integral with an external momentum p. We consider

$$I_7 = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \frac{1}{\widehat{k}^2 \widehat{q}^2 (k + q - p)^2}, \tag{19.51}$$

which by dimensional arguments will give a result of the form

$$I_7 = c_1 + c_2 \, p^2 \log p^2 + c_3 \, p^2 + O(p^4). \tag{19.52}$$

We would like to determine the coefficients c_2 and c_3. We already know the constant c_1, which can be computed setting $p = 0$ and is given by I_1, Eq. (19.11). This integral can be written in position space as

$$\begin{aligned} I_7 &= \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 s}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(s + q + k - p) \frac{1}{\widehat{k}^2 \widehat{q}^2 \widehat{s}^2} \\ &= \sum_x e^{-ipx} G^3(x). \end{aligned} \tag{19.53}$$

A function that has the same leading asymptotic behavior (up to a constant) of $G^3(x)$, and for which the Fourier transform is easily calculable in an exact way, is the 4-dimensional Laplacian of $H(x)$. We thus make the decomposition

$$I_7 = -\frac{1}{2(4\pi)^4} \sum_x e^{-ipx} \Delta H(x) + \sum_x e^{-ipx} \left[G^3(x) + \frac{1}{2(4\pi)^4} \Delta H(x) \right]. \tag{19.54}$$

The first part is the Fourier transform of $\Delta H(x)$, which can be easily computed and gives

$$-\frac{1}{2(4\pi)^4}\sum_x \mathrm{e}^{-ipx}\Delta H(x) = -\frac{1}{2(4\pi)^4}\,\hat{p}^2\log\hat{p}^2 = -\frac{1}{2(4\pi)^4}\,p^2\log p^2 + O(p^4). \qquad (19.55)$$

We have thus computed the coefficient c_2. The coefficient c_3 can be obtained from the second part, in which the leading $1/|x|^6$ terms of $G^3(x)$ and $H(x)$ cancel in the subtraction, which then goes like $1/|x|^8$. This fact is important, because to compute c_3 we have to expand the exponential to order p^2, and then the second part becomes

$$-\frac{1}{8}\sum_x p^2 x^2 \Big[G^3(x) + \frac{1}{2(4\pi)^4}\Delta H(x)\Big], \qquad (19.56)$$

and only in this way the function to be summed goes like $1/|x|^6$ again and is then finite and can be computed using generalized zeta functions. Putting everything together we have the result

$$\begin{aligned} I_7 &=& I_1 - \frac{1}{2(4\pi)^4}\,p^2\log p^2 - \frac{1}{8}\,p^2\sum_x x^2\Big[G^3(x) + \frac{1}{2(4\pi)^4}\Delta H(x)\Big] + O(p^4) \qquad (19.57) \\ &=& 0.0040430548122(3) - \frac{1}{2(4\pi)^4}\,p^2\log p^2 - p^2\cdot 0.00007447695(1) + O(p^4). \end{aligned}$$

A more complicated example can be found in (Lüscher and Weisz, 1995b), to which we also refer for further details on the method. These and other integrals have been used for the calculation of the coefficient b_2 of the β function in the pure Wilson gauge theory (Lüscher and Weisz, 1995a; Lüscher and Weisz, 1995d). The complete 2-loop calculation of this coefficient requires a combination of momentum space and coordinate space methods .

To summarize, coordinate space is of great help for the calculation of important 2-loop momentum space integrals. One can make use of the important advantages that only four-dimensional lattice sums must be performed, instead of eight-dimensional ones, and one can also exploit the asymptotic expansion of the gluon propagator $G(x)$ for large values of x to improve on the convergence.

19.2.2   Fermionic case

We now briefly discuss the computations of 2-loop lattice diagrams with Wilson fermions based on the coordinate space method by Lüscher and Weisz, which have been presented in (Capitani *et al.*, 1998a; Caracciolo, Pelissetto and Rago, 2001).

An essential ingredient of these 2-loop calculations is the high-precision determination of 1-loop mixed fermionic-bosonic propagators (Section 18.5). The algebraic method for general Wilson fermions, thanks to which any 1-loop lattice integral with Wilson fermions can be written as a combination of fifteen basic constants, which can be computed once for all with arbitrarily high precision, allows then the implementation of the coordinate space method to

two-loop integrals with Wilson fermions. We remind that the algebraic method depends only on the structure of the Wilson propagators, and not on the vertices, and thus it can be applied in calculations with the Wilson action as well as with the improved clover action.

Let us consider a very simple integral

$$I = \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 r}{(2\pi)^4} \frac{1}{D_F(l) D_F(r) D_F(l+r)}, \tag{19.58}$$

where D_F is the denominator of the quark propagator. The standard alternative (in momentum space) would consists in replacing each integration with a discrete sum over L points and then extrapolate to infinite L:

$$I = \frac{1}{L^8} \sum_{l,r,l+r \neq 0} \frac{1}{D_F(l) D_F(r) D_F(l+r)}. \tag{19.59}$$

Here l and r run over the set $(n+1/2) \cdot 2\pi/L$, $n = 0, \ldots, L-1$, and one excludes from the sum the points $l + r = 0$ mod 2π (where the third propagator would diverge). Increasing the values of L one gets the following approximations for the integral I:

$$L = 10 \qquad 0.000799652 \tag{19.60}$$
$$L = 18 \qquad 0.000848862 \tag{19.61}$$
$$L = 20 \qquad 0.000853822 \tag{19.62}$$
$$L = 26 \qquad 0.000863064. \tag{19.63}$$

Then, using an extrapolation function of the form

$$a_0 + \frac{a_1 \log L + a_2}{L^2} + \frac{a_3 \log L + a_4}{L^4} \tag{19.64}$$

and fitting with it the results of the sum for $6 \leq L \leq 26$, one gets the estimate $I \approx 0.000879776$. Note that the computation at $L = 26$ requires the evaluation of the function on an integration grid of $26^8 \sim 2 \cdot 10^{11}$ points. With the 2-loop techniques based on the coordinate space method one can instead without much effort obtain $I \approx 0.0008797779181(12)$.

In order to compute 2-loop momentum-space integrals with high precision, they are first rewritten in coordinate space along the lines shown by Lüscher and Weisz, trading an additional x integration for the $\delta^{(4)}(l+r+s)$ which expresses the vanishing of external momenta (Capitani *et al.*, 1998a). For example one can write

$$I = \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 r}{(2\pi)^4} \frac{1}{D_F^{p_1}(l) D_B^{q_1}(l)} \frac{1}{D_F^{p_2}(r) D_B^{q_2}(r)} \frac{1}{D_F^{p_3}(l+r) D_B^{q_3}(l+r)}$$

$$= \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 r}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{d^4 s}{(2\pi)^4} \frac{1}{D_F^{p_1}(l) D_B^{q_1}(l)} \frac{1}{D_F^{p_2}(r) D_B^{q_2}(r)} \frac{1}{D_F^{p_3}(s) D_B^{q_3}(s)} \delta^{(4)}(l+r+s)$$

$$= \sum_x \mathcal{G}_F(p_1, q_1; x) \, \mathcal{G}_F(p_2, q_2; x) \, \mathcal{G}_F(p_3, q_3; x). \tag{19.65}$$

To compute this simple integral we have to evaluate in coordinate space the lattice sums in the last line. The price to be paid is to compute all necessary $\mathcal{G}_F(p_j, q_j; x)$ integrals with huge precision. This does not constitute a particular challenge. These $\mathcal{G}_F(p_j, q_j; x)$ integrals can be determined with the desired precision (for a sufficiently large domain of values of x) by using the 1-loop algebraic algorithm, and one can exploit their asymptotic expansions for large values of x. After the subtraction of the asymptotic behavior, the sums are much better convergent.

The task is then to evaluate with enough accuracy sums of the kind

$$\Sigma = \sum_\Lambda f(x) \tag{19.66}$$

where Λ should in principle be the whole lattice, using only a not too big finite lattice domain, and be able to estimate the error. The sums are restricted over domains defined as

$$D_p = \{x \in \Lambda : |x|_1 \le p\} \tag{19.67}$$

where $|x|_1 = \sum_\mu |x_\mu|$. In (Capitani *et al.*, 1998a) it was found that the domain D_{21} is a reasonable choice. For the computation of the generic Feynman diagram one then chooses some convenient representation of 2-loop integrals, and assembles a database of them. To do this, the 1-loop integrals necessary for their computations can be recursively decomposed in terms of the fifteen basic 1-loop constants and then evaluated with very high precision.

Recently with this method the 2-loop critical mass m_c, defined by the vanishing of the inverse renormalized propagator $S^{-1}(p, m_0)$ at $p = 0$,

$$S^{-1}(0, m_c) = 0, \tag{19.68}$$

has been computed in the Wilson case. Using one-dimensional integrals calculated with high accuracy (about sixty significant decimal places), the 2-loop diagrams relevant for the critical mass have been obtained with a precision of about ten significant decimal places. The result for $SU(N_c)$ with N_f fermion flavors can be put in the form

$$m_c = g_0^2 \frac{N_c^2 - 1}{N_c} c_1 + g_0^4 (N_c^2 - 1)\left(c_{2,1} + \frac{1}{N_c^2} c_{2,2} + \frac{N_f}{N_c} c_{2,3}\right) \tag{19.69}$$

where

$$c_1 = -0.16285705871085078618 \tag{19.70}$$

is the 1-loop result, and

$$c_{2,1} = -0.0175360218(2) \tag{19.71}$$
$$c_{2,2} = 0.0165663304(2) \tag{19.72}$$
$$c_{2,3} = 0.001186203(6) \tag{19.73}$$

is the result of the 2-loop computations made with the coordinate space method. These 2-loop numbers are in agreement with the less precise results given in (Follana and Panagopoulos, 2001), obtained using conventional momentum-space methods, which have also produced the value of the critical mass for improved fermions (Panagopoulos and Proestos, 2002).

20 Numerical perturbation theory

We want to conclude this review mentioning some numerical methods that are completely different from all that we have presented so far, but represent interesting alternative ideas for the computation of perturbative expansions, and in principle can be able to perform high-loop calculations, at least in some cases. In the last decade a numerical approach to perturbation theory, in which one extracts perturbative coefficients using Monte Carlo simulations instead of calculating Feynman diagrams analytically, has emerged and has produced a few interesting results.

One of these techniques is given by the so-called numerical stochastic perturbation theory (Di Renzo *et al.*, 1994; Di Renzo, Onofri and Marchesini, 1995), which is based on the stochastic quantization by Parisi and Wu (1981). It uses numerical simulations of the Langevin equation. An additional parameter, the stochastic time τ, is introduced in the theory, and the gauge field is taken to be a random variable, $U_\mu(x; \tau)$, which evolves according to the Langevin equation:

$$\frac{d}{d\tau} U_\mu(x; \tau) = -\frac{\delta S[U]}{\delta U_\mu} + \eta(x; \tau). \tag{20.1}$$

The last term, $\eta(x; \tau)$, is a Gaussian noise matrix, i.e., its expectation value is

$$\langle \eta(x; \tau) \eta(x'; \tau') \rangle = 2\delta_{x,x'} \delta_{\tau,\tau'}, \tag{20.2}$$

and is the stochastic part of the equation. This approach lends itself quite naturally to lattice investigations. One has to discretize the stochastic time τ, introducing a nonzero Langevin time step ϵ. This causes an $O(\epsilon)$ systematic error, but for $\epsilon \to 0$ and $\tau \to \infty$ the time average is expected to reach asymptotically the expectation values corresponding to a path integral with action $S[U]$. At the end of the lattice calculations one has then to make extrapolations to the limit $\epsilon = 0$.

The numerical solution to the Langevin equation consists in updating the gauge field according to (Batrouni *et al.*, 1985),

$$U_\mu(x; \tau + \epsilon) = e^{-F[U(\tau), \eta]} U_\mu(x; \tau), \tag{20.3}$$

where the driving function is

$$F[U, \eta] = \sum_i T^i (\epsilon \nabla^i_{x,\mu} S[U] + \sqrt{\epsilon} \eta^i), \tag{20.4}$$

∇ being the Lie derivative on the group. For the Wilson plaquette action one has

$$\sum_i T^i \nabla^i_{x,\mu} S_G[U] = \frac{\beta}{4N_c} \sum_{U_P \ni U_\mu(x)} (U_P - U_P^\dagger)_{\mathrm{Tr}}, \tag{20.5}$$

where Tr stands for the traceless part.

Using stochastic perturbation theory it has been possible for simulations involving only gluons to reach much higher orders in g_0^2 than in conventional perturbative calculations, where only in a few cases one has reached to two-loop stage. Simulations done using stochastic perturbation theory have instead reached something like the tenth loop order in the case of the plaquette (Di Renzo, Onofri and Marchesini, 1995). Where results are available for stochastic and conventional perturbation theory, they agree within errors.

The inclusion of fermions in stochastic perturbation theory has been accomplished only recently (Di Renzo and Scorzato, 2001; Di Renzo, Miccio and Scorzato, 2002). Although passing from quenched to unquenched calculations requires little computational overhead, one needs to use a fast Fourier transform.

To perform the field updating in the unquenched theory one has to include a more complicated derivative term in the driving function,

$$\nabla S_G \to \nabla S_G - \nabla(\mathrm{Tr}\log M) = \nabla S_G - \mathrm{Tr}((\nabla M)M^{-1}), \tag{20.6}$$

where M is the fermionic matrix. We see that the Lie derivative generates an inverse fermionic matrix M^{-1}, which is nonlocal and quite expensive to compute numerically. An ingenious way to simulate this term has been proposed long ago (Batrouni $et\ al.$, 1985), and consists in taking instead

$$\nabla S_G - \mathrm{Re}(\xi^\dagger(\nabla M)M^{-1}\xi), \tag{20.7}$$

where ξ is a Gaussian random variable: $\langle \xi_i \xi_j \rangle = \delta_{ij}$, so that after averaging over this new variable one recovers the fermionic term

$$\langle \xi(\nabla M)M^{-1}\xi \rangle = \mathrm{Tr}((\nabla M)M^{-1}). \tag{20.8}$$

This random fermionic term can be computed recursively, expanding the relevant quantities order by order in the coupling. In practical terms one generates the random variable ξ_0 (which does not depend on the coupling and is then of order zero) and then recursively computes the variable ψ in

$$M\psi = \xi. \tag{20.9}$$

This is a lot easier than doing the inversion of the Dirac operator, and gives a local evolution, because Eq. (20.7) becomes

$$\nabla S_G - \mathrm{Re}(\xi^\dagger(\nabla M)\psi). \tag{20.10}$$

To compute ψ order by order one needs the expansions

$$M = M^{(0)} + \sum_{k>0} \beta^{-k/2} M^{(k)} \tag{20.11}$$

and its inverse

$$M^{-1} = M^{(0)^{-1}} + \sum_{k>0} \beta^{-k/2} M^{-1(k)}. \tag{20.12}$$

Note that inverting the zeroth order of the fermionic matrix is trivial: $M^{-1(0)} = M^{(0)-1}$. The non-trivial orders of M^{-1} can be obtained recursively as follows:

$$
\begin{aligned}
M^{-1(1)} &= -M^{(0)-1}M^{(1)}M^{(0)-1} \\
M^{-1(2)} &= -M^{(0)-1}M^{(2)}M^{(0)-1} - M^{(0)-1}M^{(1)}M^{-1(1)} \\
M^{-1(3)} &= -M^{(0)-1}M^{(3)}M^{(0)-1} - M^{(0)-1}M^{(2)}M^{-1(1)} - M^{(0)-1}M^{(1)}M^{-1(2)},
\end{aligned}
\tag{20.13}
$$

and so forth. Since $M^{(0)-1}$ is diagonal in momentum space, one can perform its computation going to momentum space, back and forth, and for this reason a fast Fourier transform code is needed. At the end one has

$$
\begin{aligned}
\psi^{(0)} &= M^{(0)-1}\xi_0 \\
\psi^{(1)} &= M^{-1(1)}\xi_0 = -M^{(0)-1}M^{(1)}\psi^{(0)} \\
\psi^{(2)} &= M^{-1(2)}\xi_0 = -M^{(0)-1}M^{(2)}\psi^{(0)} - M^{(0)-1}M^{(1)}\psi^{(1)},
\end{aligned}
\tag{20.14}
$$

and so forth. These are the lowest order terms in the expansion

$$
\psi^{(i)} = M^{-1(i)}\xi_0,
\tag{20.15}
$$

with which one can compute recursively Eq. (20.7), and hence simulate the fermion system using the Langevin equation.

Every quantity that one wants to compute with stochastic perturbation theory has to be expanded in powers of

$$
\beta^{-1/2} = \frac{g_0}{\sqrt{6}}.
\tag{20.16}
$$

For example,

$$
U_\mu(x;\tau) = 1 + \sum_{k>0}\beta^{-k/2}U_\mu^{(k)}(x;\tau), \qquad A_\mu(x;\tau) = \sum_{k>0}\beta^{-k/2}A_\mu^{(k)}(x;\tau),
\tag{20.17}
$$

where

$$
U_\mu(x;\tau) = \exp(A_\mu(x;\tau)/\sqrt{\beta}).
\tag{20.18}
$$

Observables are composite operators, and every observable depending on U can be expanded in the coupling. For example the first order of the plaquette (Eq. (5.6)) is given by

$$
\begin{aligned}
P_{\mu\nu}^{(1)}(x;\tau) = \quad & U_\mu^{(1)}(x;\tau)U_\nu^{(0)}(x+a\hat{\mu};\tau)(U_\mu^\dagger)^{(0)}(x+a\hat{\nu};\tau)(U_\nu^\dagger)^{(0)}(x;\tau) \\
& + U_\mu^{(0)}(x;\tau)U_\nu^{(1)}(x+a\hat{\mu};\tau)(U_\mu^\dagger)^{(0)}(x+a\hat{\nu};\tau)(U_\nu^\dagger)^{(0)}(x;\tau) \\
& + U_\mu^{(0)}(x;\tau)U_\nu^{(0)}(x+a\hat{\mu};\tau)(U_\mu^\dagger)^{(1)}(x+a\hat{\nu};\tau)(U_\nu^\dagger)^{(0)}(x;\tau) \\
& + U_\mu^{(0)}(x;\tau)U_\nu^{(0)}(x+a\hat{\mu};\tau)(U_\mu^\dagger)^{(0)}(x+a\hat{\nu};\tau)(U_\nu^\dagger)^{(1)}(x;\tau).
\end{aligned}
\tag{20.19}
$$

So far the calculations which use stochastic perturbation theory have been limited to finite quantities. The computation of the quark currents, of operators measuring structure functions

and also weak operators, which in general have nonvanishing anomalous dimensions, still seems a long way to go.

There is another way to compute the coefficients of perturbative expansions numerically. It makes use of Monte Carlo simulations at very weak couplings to measure short-distance quantities (Dimm, Lepage and Mackenzie, 1995; Trottier *et al.*, 2002). The perturbative coefficients are then extracted by making fits to the results of these numerical simulations. In this method some input from conventional perturbation theory is still required, to define a physical coupling constant.

Monte Carlo results are thus fitted to truncated polynomial expansions in the coupling. One must be careful in doing this, because if the truncation is too short a poor fit to the Monte Carlo data will come out, while if it is too long then the lowest coefficients, which are the dominant contributions, become poorly constrained. These fits can be improved by constraining some of the parameters by means of the techniques known as "constrained curve fitting" (Lepage *et al.*, 2002). They are especially useful in order to constrain parameters which are poorly determined statistically.

Couplings and volumes are chosen in such a way that the lattice momenta are perturbative. The computations are done at very small lattice spacings and couplings. For example in (Trottier *et al.*, 2002) Wilson fermions have been used with couplings ranging from $\alpha_{lat} = 0.008$ to $\alpha_{lat} = 0.053$. The lattice spacing and the volume must satisfy

$$q^\star \ll \frac{aL}{2\pi} \ll \frac{1}{\Lambda_{QCD}} \tag{20.20}$$

where $q^\star$ is a typical gluonic momentum scale. This condition also insures that the density of the discrete momenta is not small. In recent works the lattice spacing was able to cover a wide range of values,

$$10^{-29} < a\Lambda_{QCD} < 10^{-3}. \tag{20.21}$$

Unfortunately when doing these simulations one has to take into account the problem of the appearance of zero modes, and these infrared effects are potentially dangerous. Twisted boundary conditions, like the ones in Eq. (17.23), can however eliminate these zero modes, and they are also useful to suppress nonperturbative finite-volume effects.

With this method one has studied problems like the mass renormalization, small Wilson loops and the static-quark self-energy. In the case of Wilson loops, their three-loop coefficients have been extracted and very good agreement has been found with existing conventional perturbative results at two loops (Hattori and Kawai, 1981; Di Giacomo and Paffuti, 1982; Curci and Petronzio, 1983; Curci, Paffuti and Tripiccione, 1984; Hasenfratz *et al.*, 1984; Heller and Karsch, 1985; Bali and Boyle, 2002).

Numerical perturbation theory is still in its infancy. Studies have been mostly limited to gluonic quantities, and moreover finite ones, although some progress has been seen recently in the fermionic case. It seems that much work still needs to be done before one can think of

reproducing for example the results of Section 15.4 for the renormalization of the first moment of the unpolarized quark distribution, which involves a fermionic operator with a nonzero anomalous dimension.

21   Conclusions

We have discussed in this review many different aspects of the perturbative calculations made with gauge fields and fermions defined on a hypercubic lattice. Much progress has been made in the last decade. Perturbative calculations have been carried out using a variety of actions and in a variety of physical situations, and recently they have been of great help in the study of chiral fermions on the lattice. This long-standing issue has been solved, and the construction of chiral gauge theories on the lattice presents features that are theoretically relevant also for general quantum field theories.

We have seen the consequences of the loss of exact Lorentz invariance, and discussed the mixings that derive from this, as well as mixings caused by the breaking of chiral symmetry. In this respect Ginsparg-Wilson fermions represent a great step forward in lattice calculations, and they promise to solve long-standing problems like the calculation from first principles of $\Delta I = 1/2$ weak amplitudes and of the CP-violating parameter ϵ'/ϵ.

From the more technical side, new methods for the calculation of one- and two-loop integrals, with at times incredible precision, have been invented. It is now possible to compute one-loop bosonic integrals with very high precision, and also fermionic integrals with rather good precision. More challenging have been the calculations of two-loop Feynman diagrams. Here new methods based on the calculation of propagators in position space have been very useful.

These coordinate space methods present many advantages for the computation of two-loop integrals, and is likely that more development on this side will make two-loop calculations much easier to compute and more precise.

Acknowledgments

A Notation and conventions

Generally we use Greek letters $\mu, \nu, \lambda, \ldots$ for the 4-dimensional Lorentz (or Euclidean) indices, which run from 0 to 3, and Latin letters $i, j, k, \ldots$ for the 3-dimensional indices. Latin letters $a, b, c, \ldots$ are instead used for color indices. Matrices in the fundamental representation of $SU(N_c)$ are denoted by T, while in the adjoint representation they are denoted by t. The normalization is

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \qquad [T^a, T^b] = \mathrm{i} f^{abc} T^c. \tag{A.1}$$

The symbol ∂_μ is used for continuum derivatives, while the forward and backward lattice derivatives are

$$\nabla_\mu f(x) = \frac{f(x + \hat{\mu}) - f(x)}{a}, \qquad \nabla_\mu^\star f(x) = \frac{f(x) - f(x - \hat{\mu})}{a}. \tag{A.2}$$

The gauge covariant lattice derivatives are denoted by $\widetilde{\nabla}_\mu$, $\widetilde{\nabla}_\mu^\star$ and are given in the main text.

The shorthand notation

$$\widehat{ak}_\mu = \frac{2}{a} \sin \frac{ak_\mu}{2} \tag{A.3}$$

is also often used.

The euclidean Dirac matrices in the chiral representation are

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.4}$$

where σ_i are the Pauli matrices, and 1 is the 2×2 identity matrix. The chiral projectors are

$$P_\pm = \frac{1 \pm \gamma_5}{2} = \frac{1}{2}\begin{pmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{pmatrix}. \tag{A.5}$$

Also,

$$\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]. \tag{A.6}$$

B High-precision values of Z_0 and Z_1

We give here the new results of a high-precision calculation of the fundamental bosonic constants Z_0 and Z_1, defined in Eqs. (18.20) and (18.21). They have been obtained using the recursion relation Eq. (19.6) starting with the gluon propagator at $x = 396$. [93] The resulting values are:

$$
\begin{aligned}
Z_0 \quad = \quad & 0.15493339023106021408483720810737508876916113364521\% \quad \text{(B.1)} \\
& 98321191752313395351673319454163790491630919236741\% \\
& 07489754149497376290387736082594941817577598499678\% \\
& 92951387264251940296570608026229566408322643387967\% \\
& 84914774223913881583813529174816118783903355821052\% \\
& 96552782448948240231078335735055832848473775143559\% \\
& 80401738187671539786446652153505144942596811258480\% \\
& 80432512804639834740681281583412121641451856 69(1),
\end{aligned}
$$

$$
\begin{aligned}
Z_1 \quad = \quad & 0.10778131353987400134339155028381651483289553031166\% \quad \text{(B.2)} \\
& 39233465607465024738935201734450177503973077057462\% \\
& 05621844437484365688328635749227594895147284883092\% \\
& 15513746596880936011669949517608632177321337226921\% \\
& 37793898141534158628242723648006344189984806003623\% \\
& 35938862675314675890326849096822755901010023056671\% \\
& 38098768018557508588203302625651590185630674198459\% \\
& 55091053345931137243147409155042038820050 31989(1).
\end{aligned}
$$

These numbers have 396 significant decimal places. Thus, every 1-loop Feynman diagram in the pure gauge theory can always be given with such numerical accuracy (if the plaquette action is used), since it can be expressed as a linear combination of Z_0 and Z_1 only.

---

[93]This was the maximum value compatible with the computational resources at my disposal.

References

Abbott L.F. (1981). *The Background Field Method Beyond One Loop*, Nucl. Phys. **B185**, 189.

Adler S.L. (1969). *Axial Vector Vertex In Spinor Electrodynamics*, Phys. Rev. **177**, 2426.

Adler S.L. and W.A. Bardeen (1969). *Absence Of Higher Order Corrections In The Anomalous Axial Vector Divergence Equation*, Phys. Rev. **182**, 1517.

Albanese M. *et al.* (1987). *Glueball Masses And String Tension In Lattice QCD*, Phys. Lett. **B192**, 163.

Alexandrou C., E. Follana, H. Panagopoulos and E. Vicari (2000). *One-loop renormalization of fermionic currents with the overlap-Dirac operator*, Nucl. Phys. **B580**, 394.

Alexandrou C., H. Panagopoulos and E. Vicari (2000). Λ-*parameter of lattice QCD with the overlap-Dirac operator*, Nucl. Phys. **B571**, 257.

Allés B., M. Campostrini, A. Feo and H. Panagopoulos (1994a). *Lattice perturbation theory by computer algebra: a three loop result for the topological susceptibility*, Nucl. Phys. **B413**, 553.

Allés B., M. Campostrini, A. Feo and H. Panagopoulos (1994b). *The three loop lattice free energy*, Phys. Lett. **B324**, 433.

Allés B., A. Feo and H. Panagopoulos (1997). *The three-loop beta function in SU(N) lattice gauge theories*, Nucl. Phys. **B491**, 498.

Allés B., A. Feo and H. Panagopoulos (1998). *Asymptotic scaling corrections in QCD with Wilson fermions from the 3-loop average plaquette*, Phys. Lett. **B426**, 361.

Altarelli G., G. Curci, G. Martinelli and S. Petrarca (1981). *QCD Nonleading Corrections To Weak Decays As An Application Of Regularization By Dimensional Reduction*, Nucl. Phys. **B187**, 461.

Altarelli G. and L. Maiani (1974). *Octet Enhancement Of Nonleptonic Weak Interactions In Asymptotically Free Gauge Theories*, Phys. Lett. **B52**, 351.

Aoki S. and H. Hirose (1996). *Perturbative study for domain-wall fermions in 4+1 dimensions*, Phys. Rev. **D54**, 3471.

Aoki S., T. Izubuchi, Y. Kuramashi and Y. Taniguchi (1999a). *Perturbative renormalization factors of quark bilinear operators for domain-wall QCD*, Phys. Rev. **D59**, 094505.

Aoki S., T. Izubuchi, Y. Kuramashi and Y. Taniguchi (1999b). *Perturbative renormalization factors of three- and four-quark operators for domain-wall QCD*, Phys. Rev. **D60**, 114504.

Aoki S., T. Izubuchi, Y. Kuramashi and Y. Taniguchi (2002). *Perturbative renormalization factors in domain-wall QCD with improved gauge actions*, arXiv:hep-lat/0206013.

Aoki S. and Y. Kuramashi (2001). *Perturbative renormalization factors of $\Delta S = 1$ four-quark operators for domain-wall QCD*, Phys. Rev. **D63**, 054504.

Aoki S., Y. Kuramashi, T. Onogi and N. Tsutsui (2000). *Perturbative renormalization factors of baryon number violating operators for improved quark and gauge actions in lattice QCD*, Int. J. Mod. Phys. **A15**, 3521.

Aoki S. and Y. Taniguchi (1999a). *One loop calculation in lattice QCD with domain-wall quarks*, Phys. Rev. **D59**, 054510.

Aoki S. and Y. Taniguchi (1999b). *One loop renormalization for the axial Ward-Takahashi identity in domain-wall QCD*, Phys. Rev. **D59**, 094506.

Baake M., B. Gemünden and R. Oedingen (1982). *Structure And Representations Of The Symmetry Group Of The Four-Dimensional Cube*, J. Math. Phys. **23**, 944. [Erratum-ibid. **23**, 2595 (1982)].

Baake M., B. Gemünden and R. Oedingen (1983). *On The Relations Between Irreducible Representations Of The Hyperoctahedral Group And O(4) And SO(4)*, J. Math. Phys. **24**, 1021.

Baaquie B.E. (1977). *Gauge Fixing And Mass Renormalization In The Lattice Gauge Theory*, Phys. Rev. **D16**, 2612.

Bali G.S. and P. Boyle (2002). *Perturbative Wilson loops with massive sea quarks on the lattice*, arXiv:hep-lat/0210033.

Banks T., L. Susskind and J.B. Kogut (1976). *Strong Coupling Calculations Of Lattice Gauge Theories: (1+1)-Dimensional Exercises*, Phys. Rev. **D13**, 1043.

Bardeen W.A. (1969). *Anomalous Ward Identities In Spinor Field Theories*, Phys. Rev. **184**, 1848.

Batrouni G.G., G.R. Katz, A.S. Kronfeld, G.P. Lepage, B. Svetitsky and K.G. Wilson (1985). *Langevin Simulations Of Lattice Field Theories*, Phys. Rev. **D32**, 2736.

Beccarini G., M. Bianchi, S. Capitani and G. Rossi (1995). *Deep inelastic scattering in improved lattice QCD. 2. The second moment of structure functions*, Nucl. Phys. **B456**, 271.

Becher T. and K. Melnikov (2002). *The asymptotic expansion of lattice loop integrals around the continuum limit*, Phys. Rev. **D66**, 074508.

Bell J.S. and R. Jackiw (1969). *A PCAC Puzzle: $\pi^0 \to \gamma\gamma$ in the σ-Model*, Nuovo Cim. **A60**, 47.

Bernard C.W. and T. DeGrand (2000). *Perturbation theory for fat-link fermion actions*, Nucl. Phys. Proc. Suppl. **B83**, 845.

Bernard C.W., T.A. DeGrand, C.E. DeTar, S. Gottlieb, U.M. Heller, J.E. Hetrick, C. McNeile, K. Orginos, R.L. Sugar and D. Toussaint (2000). *Semileptonic decays of heavy mesons with the fat clover action*, Nucl. Phys. Proc. Suppl. **B83**, 274.

Bernard C.W., T. Draper, A. Soni, H.D. Politzer and M.B. Wise (1985). *Application Of Chiral Perturbation Theory To $K \to 2\pi$ Decays*, Phys. Rev. **D32**, 2343.

Bernard C.W., A. Soni and T. Draper (1987). *Perturbative corrections to four-fermion operators on the lattice*, Phys. Rev. **D36**, 3224.

Bernreuther W., W. Wetzel and R. Wohlert (1984). Λ-*Parameters For Lattice Yang-Mills Actions Containing Plaquettes With Six Links*, Phys. Lett. **B142**, 407.

Bietenholz W. (1999). *Solutions of the Ginsparg-Wilson relation and improved domain wall fermions*, Eur. Phys. J. **C6**, 537.

Bietenholz W. (2001). *Approximate Ginsparg-Wilson fermions for QCD*, arXiv:hep-lat/0007017. Proceedings of the International Workshop on Non-Perturbative Methods and Lattice QCD, Guangzhou, China, 15-21 May 2000, X.-Q. Luo and E.B. Gregory eds. (World Scientific, Singapore).

Bietenholz W. (2002). *Convergence rate and locality of improved overlap fermions*, Nucl. Phys. **B644**, 223.

Bietenholz W. and I. Hip (2000). *The scaling of exact and approximate Ginsparg-Wilson fermions*, Nucl. Phys. **B570**, 423.

Bietenholz W. and T. Struckmann (1999). *Perfect lattice perturbation theory: A study of the anharmonic oscillator*, Int. J. Mod. Phys. **C10**, 531.

Bietenholz W. and U.J. Wiese (1994). *Fixed Point Actions For Lattice Fermions*, Nucl. Phys. Proc. Suppl. **B34**, 516.

Bock W., M.F. Golterman and Y. Shamir (1998a). *On the phase diagram of a lattice U(1) gauge theory with gauge fixing*, Phys. Rev. **D58**, 054506.

Bock W., M.F. Golterman and Y. Shamir (1998b). *Lattice chiral fermions through gauge fixing*, Phys. Rev. Lett. **80**, 3444.

Bock W., M.F. Golterman and Y. Shamir (1998c). *Chiral fermions on the lattice through gauge fixing: Perturbation theory*, Phys. Rev. **D58**, 034501.

Bock W., K.C. Leung, M.F. Golterman and Y. Shamir (2000). *The phase diagram and spectrum of gauge-fixed Abelian lattice gauge theory*, Phys. Rev. **D62**, 034507.

Bode A. (1998). *Two loop expansion of the Schrödinger functional coupling α_{SF} in SU(3) lattice gauge theory*, Nucl. Phys. Proc. Suppl. **B63**, 796.

Bode A., R. Frezzotti, B. Gehrmann, M. Hasenbusch, J. Heitger, K. Jansen, S. Kurth, J. Rolf, H. Simma, S. Sint, R. Sommer, P. Weisz, H. Wittig and U. Wolff (2001). *First results on the running coupling in QCD with two massless flavors*, Phys. Lett. **B515**, 49.

Bode A. and H. Panagopoulos (2002). *The three-loop β-function of QCD with the clover action*, Nucl. Phys. **B625**, 198.

Bode A., P. Weisz and U. Wolff (2000a). *Two loop lattice expansion of the Schrödinger functional coupling in improved QCD*, Nucl. Phys. Proc. Suppl. **B83**, 920.

Bode A., P. Weisz and U. Wolff (2000b). *Two loop computation of the Schrödinger functional in lattice QCD*, Nucl. Phys. **B576**, 517. [Erratum-ibid. **B600**, 453 (2001); **B608**, 481 (2001)].

Bode A., U. Wolff and P. Weisz (1999). *Two-loop computation of the Schrödinger functional in pure SU(3) lattice gauge theory*, Nucl. Phys. **B540**, 491.

Borrelli A., L. Maiani, R. Sisto, G.C. Rossi and M. Testa (1989). *Yukawa Lattice Theory And Non-perturbative Upper Bounds To The Fermion Mass*, Phys. Lett. **B221**, 360.

Borrelli A., L. Maiani, R. Sisto, G.C. Rossi and M. Testa (1990). *Neutrinos On The Lattice: The Regularization Of A Chiral Gauge Theory*, Nucl. Phys. **B333**, 335.

Borrelli A., C. Pittori, R. Frezzotti and E. Gabrielli (1993). *New improved operators: a convenient redefinition*, Nucl. Phys. **B409**, 382.

Boulware D.G. (1970). *Renormalizeability of Massive Non-Abelian Gauge Fields: A Functional Integral Approach*, Annals Phys. **56**, 140.

Brower R.C., S. Huang, J.W. Negele, A. Pochinsky and B. Schreiber (1997). *Calculation of moments of nucleon structure functions*, Nucl. Phys. Proc. Suppl. **B53**, 318.

Bucarelli A., F. Palombi, R. Petronzio and A. Shindler (1999). *Moments of parton evolution probabilities on the lattice within the Schrödinger functional scheme*, Nucl. Phys. **B552**, 379.

Buras A.J., M. Jamin, M.E. Lautenbacher and P.H. Weisz (1992). *Effective hamiltonians for $\Delta S = 1$ and $\Delta B = 1$ non-leptonic decays beyond the leading logarithmic approximation*, Nucl. Phys. **B370**, 69.

Buras A.J. and P.H. Weisz (1990). *QCD Nonleading Corrections To Weak Decays In Dimensional Regularization And 'T Hooft-Veltman Schemes*, Nucl. Phys. **B333**, 66.

Burgio G., S. Caracciolo and A. Pelissetto (1996). *Algebraic algorithm for the computation of one-loop Feynman diagrams in lattice QCD with Wilson fermions*, Nucl. Phys. **B478**, 687.

Cabibbo N., G. Martinelli and R. Petronzio (1984). *Weak Interactions On The Lattice*, Nucl. Phys. **B244**, 381.

Callan C.G. and J.A. Harvey (1985). *Anomalies And Fermion Zero Modes On Strings And Domain Walls*, Nucl. Phys. **B250**, 427.

Capitani S. (2001a). *Perturbative renormalization of the first two moments of non-singlet quark distributions with overlap fermions*, Nucl. Phys. **B592**, 183.

Capitani S. (2001b). *Perturbative renormalization of moments of quark momentum, helicity and transversity distributions with overlap and Wilson fermions*, Nucl. Phys. **B597**, 313.

Capitani S. (2002a). *Perturbative renormalization for overlap fermions*, Nucl. Phys. Proc. Suppl. **B106**, 826.

Capitani S. (2002b). *Status of lattice structure function calculations*, Acta Phys. Polon. **B33**, 3025.

Capitani S. (2002c). *Perturbative and non-perturbative lattice calculations for the study of parton distributions*, arXiv:hep-ph/0210076. Presented at the 6th International Symposium on Radiative Corrections "Application of Quantum Field Theory Phenomenology" (RADCOR 2002) and 6th Zeuthen Workshop on Elementary Particle Theory "Loops and Legs in Quantum Field Theory", Kloster Banz, Germany, 8-13 Sep 2002.

Capitani S., S. Caracciolo, A. Pelissetto and G. Rossi (1998a). *High-precision computation of two-loop Feynman diagrams with Wilson fermions*, Nucl. Phys. Proc. Suppl. **B63**, 802.

Capitani S. and L. Giusti (2000). *Perturbative renormalization of weak Hamiltonian four-fermion operators with overlap fermions*, Phys. Rev. **D62**, 114506.

Capitani S. and L. Giusti (2001). *Analysis of the $\Delta I = 1/2$ rule and ϵ'/ϵ with overlap fermions*, Phys. Rev. **D64**, 014506.

Capitani S., M. Göckeler, R. Horsley, B. Klaus, W. Kürzinger, D. Petters, D. Pleiter, P.E.L. Rakow, S. Schaefer, A. Schäfer and G. Schierholz (2001a). *Four-quark operators in hadrons*, Nucl. Phys. Proc. Suppl. **B94**, 299.

Capitani S., M. Göckeler, R. Horsley, B. Klaus, V. Linke, P.E.L. Rakow, A. Schäfer and G. Schierholz (2000a). *Higher-twist contributions to the structure functions coming from 4-fermion operators*, Nucl. Phys. Proc. Suppl. **B83**, 232.

Capitani S., M. Göckeler, R. Horsley, B. Klaus, V. Linke, P.E.L. Rakow, A. Schäfer and G .Schierholz (2000b). *Higher-twist contribution to pion structure function: 4-Fermi operators*, Nucl. Phys. **B570**, 393.

Capitani S., M. Göckeler, R. Horsley, H. Perlt, P.E.L. Rakow G. Schierholz and A. Schiller (1998b). *Perturbative renormalization of improved lattice operators*, Nucl. Phys. Proc. Suppl. **B63**, 874.

Capitani S., M. Göckeler, R. Horsley, H. Perlt, P.E.L. Rakow, G. Schierholz and A. Schiller (1999a). *Renormalization of four-fermion operators for higher twist calculations*, Nucl. Phys. Proc. Suppl. **B73**, 285.

Capitani S., M. Göckeler, R. Horsley, H. Perlt, P.E.L. Rakow, G. Schierholz and A. Schiller (2001b). *Renormalisation and off-shell improvement in lattice perturbation theory*, Nucl. Phys. **B593**, 183.

Capitani S., M. Göckeler, R. Horsley, P.E.L. Rakow and G. Schierholz (1999b). *Operator improvement for Ginsparg-Wilson fermions*, Phys. Lett. **B468**, 150.

Capitani S., M. Göckeler, R. Horsley, P.E.L. Rakow and G. Schierholz (2000c). *On-shell and off-shell improvement for Ginsparg-Wilson fermions*, Nucl. Phys. Proc. Suppl. **B83**, 893.

Capitani S., M. Guagnelli, M. Lüscher, S. Sint, R. Sommer, P. Weisz and H. Wittig (1998c). *Non-perturbative quark mass renormalization*, Nucl. Phys. Proc. Suppl. **B63**, 153.

Capitani S., M. Lüscher, R. Sommer and H. Wittig (1999c). *Non-perturbative quark mass renormalization in quenched lattice QCD*, Nucl. Phys. **B544**, 669.

Capitani S. and G. Rossi (1995a). *Deep inelastic scattering in improved lattice QCD. 1. The first moment of structure functions*, Nucl. Phys. **B433**, 351.

Capitani S. and G. Rossi (1995b). *The use of SCHOONSCHIP and FORM in perturbative lattice calculations*, arXiv:hep-lat/9504014. In: "New Computing Techniques in Physics Research IV", Proceedings of the 4th International Workshop on Software Engineering and Artificial Intelligence for High-energy and Nuclear Physics (AIHENP95), Pisa, Italy, 3-8 April 1995, B. Denby and D. Perret-Gallix eds. (World Scientific, Singapore).

Caracciolo S., G. Curci, P. Menotti and A. Pelissetto (1989). *Renormalization Of The Energy Momentum Tensor And The Trace Anomaly In Lattice QED*, Phys. Lett. **B228**, 375.

Caracciolo S., G. Curci, P. Menotti and A. Pelissetto (1990). *The Energy Momentum Tensor For Lattice Gauge Theories*, Annals Phys. **197**, 119.

Caracciolo S., P. Menotti and A. Pelissetto (1991). *Analytic determination at one loop of the energy momentum tensor for lattice QCD*, Phys. Lett. **B260**, 401.

Caracciolo S., P. Menotti and A. Pelissetto (1992). *One loop analytic computation of the energy momentum tensor for lattice gauge theories*, Nucl. Phys. **B375**, 195.

Caracciolo S., A. Pelissetto and A. Rago (2001). *Two-loop critical mass for Wilson fermions*, Phys. Rev. **D64**, 094506.

Caswell W.E. (1974). *Asymptotic Behavior Of Nonabelian Gauge Theories To Two Loop Order*, Phys. Rev. Lett. **33**, 244.

Caswell W.E. and F. Wilczek (1974). *On The Gauge Dependence Of Renormalization Group Parameters*, Phys. Lett. **B49**, 291.

Celmaster W. and R.J. Gonsalves (1979a). *QCD Perturbation Expansions In A Coupling Constant Renormalized By Momentum Space Subtraction*, Phys. Rev. Lett. **42**, 1435.

Celmaster W. and R.J. Gonsalves (1979b). *The Renormalization Prescription Dependence Of The QCD Coupling Constant*, Phys. Rev. **D20**, 1420.

Chetyrkin K.G. (1997). *Quark mass anomalous dimension to $O(\alpha_s^4)$*, Phys. Lett. **B404**, 161.

Chiu T.W. and T.H. Hsieh (2002). *A perturbative calculation of the axial anomaly of a Ginsparg-Wilson Dirac operator*, Phys. Rev. **D65**, 054508.

Chiu T.W., C.W. Wang and S.V. Zenkin (1998). *Chiral structure of the solutions of the Ginsparg-Wilson relation*, Phys. Lett. **B438**, 321.

Christou C., A. Feo, H. Panagopoulos and E. Vicari (1998). *The three-loop beta-function of SU(N) lattice gauge theories with Wilson fermions*, Nucl. Phys. **B525**, 387. [Erratum-ibid. **B608**, 479 (2001)].

Corbò G., E. Franco and G.C. Rossi (1989). *Perturbative Renormalization Of The Lowest Moment Operators Of DIS In Lattice QCD*, Phys. Lett. **B221**, 367. [Erratum-ibid. **B225**, 463 (1989)].

Corbò G., E. Franco and G.C. Rossi (1990). *Mixing Of DIS Operators In Lattice QCD*, Phys. Lett. **B236**, 196.

Creutz M. (1983). *Quarks, gluons and lattices*, Cambridge Monographs On Mathematical Physics (Cambridge University Press).

Creutz M. (1987). *Species Doubling And Transfer Matrices For Fermionic Fields*, Phys. Rev. **D35**, 1460.

Creutz M. (2001). *Aspects of chiral symmetry and the lattice*, Rev. Mod. Phys. **73**, 119.

Curci G., E. Franco, L. Maiani and G. Martinelli (1988). *Mixing Coefficients Of The Lattice Weak Hamiltonian With Dimension Five Operators*, Phys. Lett. **B202**, 363.

Curci G., P. Menotti and G. Paffuti (1983). *Symanzik's Improved Lagrangian For Lattice Gauge Theory*, Phys. Lett. **B130**, 205. [Erratum-ibid. **B135**, 516 (1984)].

Curci G., G. Paffuti and R. Tripiccione (1984). *Perturbative Background To Monte Carlo Calculations In Lattice Gauge Theories*, Nucl. Phys. **B240**, 91.

Curci G. and R. Petronzio (1983). *On The Perturbative Contributions To The String Tension Estimates*, Phys. Lett. **B132**, 133.

Daniel D. and S.N. Sheard (1988). *Perturbative Corrections To Staggered Fermion Lattice Operators*, Nucl. Phys. **B302**, 471.

Dashen R.F. and D.J. Gross (1981). *The Relationship Between Lattice And Continuum Definitions Of The Gauge Theory Coupling*, Phys. Rev. **D23**, 2340.

Davies C.T.H. (2002). *Lattice QCD*, arXiv:hep-ph/0205181. Lectures given at the 55th Scottish Universities Summer School in Physics on "Heavy Flavor Physics", St. Andrews, Scotland, 7-23 Aug 2001. Scottish Graduate Textbook Series, Institute of Physics 2002, C.T.H. Davies and S.M. Playfer eds.

Davies C.T.H., K. Hornbostel, G.P. Lepage, P. McCallum, J. Shigemitsu and J.H. Sloan (1997). *Further precise determinations of α_s from lattice QCD*, Phys. Rev. **D56**, 2755.

de Forcrand P., M. García Pérez, T. Hashimoto, S. Hioki, H. Matsufuru, O. Miyamura, A. Nakamura, I.O. Stamatescu, T. Takaishi and T. Umeda (2000). *Renormalization group flow of SU(3) lattice gauge theory: Numerical studies in a two coupling space*, Nucl. Phys. **B577**, 263.

DeGrand T. (1996). *Nonperturbative quantum field theory on the lattice*, arXiv:hep-th/9610132. Talk given at the 1996 Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): "Fields, Strings, and Duality", Boulder, CO, 2-28 Jun 1996.

DeGrand T. (1997). *Lattice gauge theory for QCD*, arXiv:hep-ph/9610391. Proceedings of the 24th SLAC Summer Institute on Particle Physics: "The Strong Interaction, From Hadrons to Partons", Stanford, CA, 19-30 Aug 1996, J. Chan, L. DePorcel and L. Dixon eds.

DeGrand T. (2001). *A variant approach to the overlap action*, Phys. Rev. **D63**, 034503.

DeGrand T. (2002). *One loop matching coefficients for a variant overlap action–and some of its simpler relatives*, arXiv:hep-lat/0210028.

DeGrand T., A. Hasenfratz, P. Hasenfratz, P. Kunszt and F. Niedermayer (1997). *Fixed-point action for fermions in QCD*, Nucl. Phys. Proc. Suppl. **B53**, 942.

DeGrand T., A. Hasenfratz, P. Hasenfratz and F. Niedermayer (1995). *The classically perfect fixed point action for SU(3) gauge theory*, Nucl. Phys. **B454**, 587.

DeGrand T., A. Hasenfratz, P. Hasenfratz and F. Niedermayer (1996). *Fixed point actions for SU(3) gauge theory*, Phys. Lett. **B365**, 233.

DeGrand T., A. Hasenfratz and T.G. Kovács (1999). *Instantons and exceptional configurations with the clover action*, Nucl. Phys. **B547**, 259.

DeGrand T., A. Hasenfratz and T.G. Kovács (2002). *Improving the Chiral Properties of Lattice Fermions*, arXiv:hep-lat/0211006.

De Wit B. (1967a). *Quantum Theory of Gravity. 2. The Manifestly Covariant Theory*, Phys. Rev. **162**, 1195.

De Wit B. (1967b). *Quantum Theory of Gravity. 3. Applications of the Covariant Theory*, Phys. Rev. **162**, 1239.

Di Giacomo A. and G. Paffuti (1982). *Some Results Related To The Continuum Limit Of Lattice Gauge Theories*, Nucl. Phys. **B205**, 313.

Dimm W., G.P. Lepage and P.B. Mackenzie (1995). *Nonperturbative "lattice perturbation theory"*, Nucl. Phys. Proc. Suppl. **B42**, 403.

Di Renzo F., V. Miccio and L. Scorzato (2002). *Unquenched Numerical Stochastic Perturbation Theory*, arXiv:hep-lat/0209018.

Di Renzo F., E. Onofri and G. Marchesini (1995). *Renormalons from eight loop expansion of the gluon condensate in lattice gauge theory*, Nucl. Phys. **B457**, 202.

Di Renzo F., E. Onofri, G. Marchesini and P. Marenzoni (1994). *Four loop result in SU(3) lattice gauge theory by a stochastic method: Lattice correction to the condensate*, Nucl. Phys. **B426**, 675.

Di Renzo F. and L. Scorzato (2001). *Fermionic loops in numerical stochastic perturbation theory*, Nucl. Phys. Proc. Suppl. **B94**, 567.

Dolgov D., R. Brower, S. Capitani, P. Dreher, J.W. Negele, A. Pochinsky, D.B. Renner, N. Eicker, T. Lippert, K. Schilling, R.G. Edwards and U.M. Heller (2002). *Moments of nucleon light cone quark distributions calculated in full lattice QCD*, Phys. Rev. **D66**, 034506.

Dolgov D., R. Brower, S. Capitani, J.W. Negele, A. Pochinsky, D.B. Renner, N. Eicker, T. Lippert, K. Schilling, R.G. Edwards and U.M. Heller (2001). *Moments of structure functions in full QCD*, Nucl. Phys. Proc. Suppl. **B94**, 303.

Dreher P., R. Brower, S. Capitani, D. Dolgov, R.G. Edwards, N. Eicker, U.M. Heller, T. Lippert, J.W. Negele, A. Pochinsky, D.B. Renner and K. Schilling (2002). *Continuum Extrapolation of Moments of Nucleon Quark Distributions in Full QCD*, arXiv:hep-lat/0211021.

Drell S.D., M. Weinstein and S. Yankielowicz (1976a). *Strong-coupling field theory. 1. Variational approach to ϕ^4 theory*, Phys. Rev. **D14**, 487.

Drell S.D., M. Weinstein and S. Yankielowicz (1976b). *Strong-coupling field theories. 2. Fermions and gauge fields on a lattice*, Phys. Rev. **D14**, 1627.

El-Khadra A.X., A.S. Kronfeld and P.B. Mackenzie (1997). *Massive Fermions in Lattice Gauge Theory*, Phys. Rev. **D55**, 3933.

Ellis R.K. (1984). *Perturbative Corrections To Universality And Renormalization Group Behavior*, Preprint FERMILAB-CONF-84/41. Presented at the ANL Workshop on Gauge Theory on a Lattice, Argonne, IL, Apr 5-7, 1984.

Ellis R.K. and G. Martinelli (1984a). *Two Loop Corrections To The Λ Parameters Of One-Plaquette Actions*, Nucl. Phys. **B235**, 93. [Erratum-ibid. **B249**, 750 (1985)].

Ellis R.K. and G. Martinelli (1984b). *Perturbative Corrections To Renormalization Group Behavior In Lattice QCD*, Phys. Lett. **B141**, 111.

Espriu D. and R. Tarrach (1982). *On Prescription Dependence Of Renormalization Group Functions*, Phys. Rev. **D25**, 1073.

Farchioni F., P. Hasenfratz, F. Niedermayer and A. Papa (1995). *The absence of cutoff effects for the fixed point action in one loop perturbation theory*, Nucl. Phys. **B454**, 638.

Follana E. and H. Panagopoulos (2001). *The critical mass of Wilson fermions: A comparison of perturbative and Monte Carlo results*, Phys. Rev. **D63**, 017501.

Frezzotti R., E. Gabrielli, C. Pittori and G.C. Rossi (1992). *Four fermion operators with improved nearest neighbor action*, Nucl. Phys. **B373**, 781.

Friedan D. (1982). *A Proof Of The Nielsen-Ninomiya Theorem*, Commun. Math. Phys. **85**, 481.

Frolov S.A. and A.A. Slavnov (1993). *An Invariant Regularization Of The Standard Model*, Phys. Lett. **B309**, 344.

Fujikawa K. and M. Ishibashi (2002). *A perturbative study of a general class of lattice Dirac operators*, Phys. Rev. **D65**, 114504.

Furman V. and Y. Shamir (1995). *Axial Symmetries In Lattice QCD With Kaplan Fermions*, Nucl. Phys. **B439**, 54.

Gabrielli E., G. Martinelli, C. Pittori, G. Heatlie and C.T. Sachrajda (1991). *Renormalization Of Lattice Two Fermion Operators With Improved Nearest Neighbor Action*, Nucl. Phys. **B362**, 475.

Gaillard M.K. and B.W. Lee (1974). $\Delta I = 1/2$ *Rule For Nonleptonic Decays In Asymptotically Free Field Theories*, Phys. Rev. Lett. **33**, 108.

Garden J., J. Heitger, R. Sommer and H. Wittig (2000). *Precision computation of the strange quark's mass in quenched QCD*, Nucl. Phys. **B571**, 237.

Gasser J. and H. Leutwyler (1982). *Quark Masses*, Phys. Rept. **87**, 77.

Gasser J. and H. Leutwyler (1984). *Chiral Perturbation Theory To One Loop*, Annals Phys. **158**, 142.

Gasser J. and H. Leutwyler (1985). *Chiral Perturbation Theory: Expansions In The Mass Of The Strange Quark*, Nucl. Phys. **B250**, 465.

Ginsparg P.H. and K.G. Wilson (1982). *A Remnant Of Chiral Symmetry On The Lattice*, Phys. Rev. **D25**, 2649.

Giusti L. (2002). *Exact Chiral Symmetry on the Lattice: QCD Applications*, arXiv:hep-lat/0211009.

Golterman M.F. (2001). *Lattice chiral gauge theories*, Nucl. Phys. Proc. Suppl. **B94**, 189.

Golterman M.F. and Y. Shamir (1997). *A gauge-fixing action for lattice gauge theories*, Phys. Lett. **B399**, 148.

Golterman M.F. and Y. Shamir (2002). *Lattice chiral gauge theories through gauge fixing*, arXiv:hep-lat/0205001. Talk given at the NATO Advanced Research Workshop on Confinement, Topology, and other Nonperturbative Aspects of QCD, Stara Lesna, Slovakia, 21-27 Jan 2002.

Golterman M.F. and J. Smit (1984a). *Relation Between QCD Parameters On The Lattice And In The Continuum*, Phys. Lett. **B140**, 392.

Golterman M.F. and J. Smit (1984b). *Selfenergy And Flavor Interpretation Of Staggered Fermions*, Nucl. Phys. **B245**, 61.

González-Arroyo A. and C.P. Korthals-Altes (1982). *Asymptotic Freedom Scales For Any Lattice Action*, Nucl. Phys. **B205**, 46.

González-Arroyo A., F.J. Ynduráin and G. Martinelli (1982). *Computation Of The Relation Between The Quark Masses In Lattice Gauge Theories And On The Continuum*, Phys. Lett. **B117**, 437. [Erratum-ibid. **B122**, 486 (1983)].

Göckeler M. (1984). *Mass Terms And Mass Renormalization For Susskind Fermions*, Phys. Lett. **B142**, 197.

Göckeler M., R. Horsley, E.M. Ilgenfritz, H. Perlt, P. Rakow, G. Schierholz and A. Schiller (1996a). *Lattice Operators for Moments of the Structure Functions and their Transformation under the Hypercubic Group*, Phys. Rev. **D54**, 5705.

Göckeler M., R. Horsley, E.M. Ilgenfritz, H. Perlt, P. Rakow, G. Schierholz and A. Schiller (1996b). *Perturbative Renormalization of Lattice Bilinear Quark Operators*, Nucl. Phys. **B472**, 309.

Göckeler M., R. Horsley, D. Pleiter, P.E. Rakow, A. Schäfer and G. Schierholz (2002). *Calculation of moments of structure functions*, arXiv:hep-lat/0209160.

Groot R., J. Hoek and J. Smit (1984). *Normalization Of Currents In Lattice QCD*, Nucl. Phys. **B237**, 111.

Gross D.J. and F. Wilczek (1973). *Ultraviolet Behavior Of Non-Abelian Gauge Theories*, Phys. Rev. Lett. **30**, 1343.

Guagnelli M., K. Jansen and R. Petronzio (1999a). *Non-perturbative running of the average momentum of non-singlet parton densities*, Nucl. Phys. **B542**, 395.

Guagnelli M., K. Jansen and R. Petronzio (1999b). *Universal continuum limit of non-perturbative lattice non-singlet moment evolution*, Phys. Lett. **B457**, 153.

Guagnelli M., K. Jansen and R. Petronzio (1999c). *Renormalization group invariant average momentum of non-singlet parton densities*, Phys. Lett. **B459**, 594.

Guagnelli M., K. Jansen and R. Petronzio (2000). *Lattice hadron matrix elements with the Schrödinger functional: The case of the first moment of non-singlet quark density*, Phys. Lett. **B493**, 77.

Gupta R. (1999). *Lattice QCD*, arXiv:hep-lat/9807028. Lectures given at the Les Houches Summer School "Probing the Standard Model of Particle Interactions", Session LXVIII, 1997, R. Gupta, A. Morel, E. de Rafael and F. David eds. (Elsevier, Amsterdam).

Gupta R., T. Bhattacharya and S.R. Sharpe (1997). *Matrix elements of four-fermion operators with quenched Wilson fermions*, Phys. Rev. **D55**, 4036.

Hahn Y. and W. Zimmermann (1968). *An Elementary Proof of Dyson's Power Counting Theorem*, Commun. Math. Phys. **10**, 330.

Hamber H.W. and C.M. Wu (1983). *Some Predictions For An Improved Fermion Action On The Lattice*, Phys. Lett. **B133**, 351.

Hasenfratz A. and P. Hasenfratz (1980). *The Connection Between The Λ Parameters Of Lattice And Continuum QCD*, Phys. Lett. **B93**, 165.

Hasenfratz A. and P. Hasenfratz (1981). *The Scales Of Euclidean And Hamiltonian Lattice QCD*, Nucl. Phys. **B193**, 210.

Hasenfratz A., P. Hasenfratz, U.M. Heller and F. Karsch (1984). *The β-function of the SU(3) Wilson action*, Phys. Lett. **B143**, 193.

Hasenfratz A., R. Hoffmann and F. Knechtli (2002). *The static potential with hypercubic blocking*, Nucl. Phys. Proc. Suppl. **B106**, 418.

Hasenfratz A. and F. Knechtli (2001). *Flavor symmetry and the static potential with hypercubic blocking*, Phys. Rev. **D64**, 034504.

Hasenfratz A. and F. Knechtli (2002). *Simulation of dynamical fermions with smeared links*, Comput. Phys. Commun. **148**, 81.

Hasenfratz P. (1998a). *Prospects for perfect actions*, Nucl. Phys. Proc. Suppl. **B63**, 53.

Hasenfratz P. (1998b). *The theoretical background and properties of perfect actions*, arXiv:hep-lat/9803027. Prepared for the Advanced Summer School on "Nonperturbative Quantum Field Physics", Peñiscola, Spain, 2-6 Jun 1997.

Hasenfratz P., S. Hauswirth, T. Jörg, F. Niedermayer and K. Holland (2002). *Testing the fixed-point QCD action and the construction of chiral currents*, Nucl. Phys. **B643**, 280.

Hasenfratz P. and F. Niedermayer (1994). *Perfect lattice action for asymptotically free theories*, Nucl. Phys. **B414**, 785.

Hasenfratz P. and F. Niedermayer (1997). *Fixed-point actions in 1-loop perturbation theory*, Nucl. Phys. **B507**, 399.

Hattori T. and H. Kawai (1981). *Weak Coupling Perturbative Calculations Of Wilson Loop In Lattice Gauge Theory*, Phys. Lett. **B105**, 43.

Heatlie G., G. Martinelli, C. Pittori, G.C. Rossi and C.T. Sachrajda (1991). *The Improvement Of Hadronic Matrix Elements In Lattice QCD*, Nucl. Phys. **B352**, 266.

Hein J., Q. Mason, G.P. Lepage and H. Trottier (2002). *Mass renormalisation for improved staggered quarks*, Nucl. Phys. Proc. Suppl. **B106**, 236.

Heller U.M. and F. Karsch (1985). *One Loop Perturbative Calculation Of Wilson Loops On Finite Lattices*, Nucl. Phys. **B251**, 254.

Hernández P., K. Jansen and L. Lellouch (2002). *From enemies to friends: Chiral symmetry on the lattice*, arXiv:hep-lat/0203029. Contributed to the NIC Symposium 2001, Jülich, Germany, 5-6 Dec 2001.

Hernández P., K. Jansen, L. Lellouch and H. Wittig (2001). *Non-perturbative renormalization of the quark condensate in Ginsparg-Wilson regularizations*, JHEP **0107**, 018.

Hernández P., K. Jansen, L. Lellouch and H. Wittig (2002). *Scalar condensate and light quark masses from overlap fermions*, Nucl. Phys. Proc. Suppl. **B106**, 766.

Hernández P., K. Jansen and M. Lüscher (1999). *Locality properties of Neuberger's lattice Dirac operator*, Nucl. Phys. **B552**, 363.

Hernández P., K. Jansen and M. Lüscher (2000). *A note on the practical feasibility of domain-wall fermions*, arXiv:hep-lat/0007015. Talk given at the Workshop on Current Theoretical Problems in Lattice Field Theory, Ringberg, Germany, 2-8 Apr 2000.

Igarashi H., K. Okuyama and H. Suzuki (2000). *Errata and addenda to 'Anomaly cancellation condition in lattice gauge theory'*, arXiv:hep-lat/0012018.

Ishibashi M., Y. Kikukawa, T. Noguchi and A. Yamada (2000). *One-loop analyses of lattice QCD with the overlap Dirac operator*, Nucl. Phys. **B576**, 501.

Ishizuka M. and Y. Shizawa (1994). *Perturbative renormalization factors for bilinear and four quark operators for Kogut-Susskind fermions on the lattice*, Phys. Rev. **D49**, 3519.

Iwasaki Y. (1983a). *Renormalization Group Analysis Of Lattice Theories And Improved Lattice Action. 1. Two-Dimensional Nonlinear O(N) Sigma Model*, University of Tsukuba Report UTHEP-117.

Iwasaki Y. (1983b). *Renormalization Group Analysis Of Lattice Theories And Improved Lattice Action. 2. Four-Dimensional Nonabelian SU(N) Gauge Model*, University of Tsukuba Report UTHEP-118.

Jansen K. (1992). *Chiral fermions and anomalies on a finite lattice*, Phys. Lett. **B288**, 348.

Jansen K. (1996). *Domain wall fermions and chiral gauge theories*, Phys. Rept. **273**, 1.

Jansen K. (2000). *Structure functions on the lattice*, arXiv:hep-lat/0010038. Proceedings of the 30th International Conference on High-Energy Physics (ICHEP 2000), Osaka, Japan, 27 July - 2 August 2000, C.S. Lim and T. Yamanaka eds. (World Scientific, Singapore).

Jansen K., C. Liu, M. Lüscher, H. Simma, S. Sint, R. Sommer, P. Weisz and U. Wolff (1996). *Non-perturbative renormalization of lattice QCD at all scales*, Phys. Lett. **B372**, 275.

Jansen K. and R. Sommer (1998). *O(a) improvement of lattice QCD with two flavors of Wilson quarks*, Nucl. Phys. **B530**, 185.

Jegerlehner F. (2001). *Facts of life with γ_5*, Eur. Phys. J. **C18**, 673.

Ji X.D. (1995). *Exact matching condition for matrix elements in lattice and $\overline{\mathrm{MS}}$ schemes*, arXiv:hep-lat/9506034. Preprint MIT-CTP-2447, unpublished.

Jones D.R. (1974). *Two Loop Diagrams In Yang-Mills Theory*, Nucl. Phys. **B75**, 531.

Jörg T. (2002). *Chiral measurements in quenched lattice QCD with fixed point fermions*, arXiv:hep-lat/0206025. Ph.D. Thesis (Bern), Preprint BUTP-2002-9.

Kaplan D.B. (1992). *A method for simulating chiral fermions on the lattice*, Phys. Lett. **B288**, 342.

Kaplan D.B. (1993). *Chiral Fermions On The Lattice*, Nucl. Phys. Proc. Suppl. **B30**, 597.

Karsten L.H. (1981). *Lattice Fermions In Euclidean Space-Time*, Phys. Lett. **B104**, 315.

Karsten L.H. and J. Smit (1978). *Axial Symmetry In Lattice Theories*, Nucl. Phys. **B144**, 536.

Karsten L.H. and J. Smit (1979). *The Vacuum Polarization With Slac Lattice Fermions*, Phys. Lett. **B85**, 100.

Karsten L.H. and J. Smit (1981). *Lattice Fermions: Species Doubling, Chiral Invariance, And The Triangle Anomaly*, Nucl. Phys. **B183**, 103.

Kawai H., R. Nakayama and K. Seo (1981). *Comparison Of The Lattice Λ Parameter With The Continuum Λ Parameter In Massless QCD*, Nucl. Phys. **B189**, 40.

Kawamoto N. and J. Smit (1981). *Effective Lagrangian And Dynamical Symmetry Breaking In Strongly Coupled Lattice QCD*, Nucl. Phys. **B192**, 100.

Kikukawa Y. (2002). *Domain wall fermion and chiral gauge theories on the lattice with exact gauge invariance*, Phys. Rev. **D65**, 074504.

Kikukawa Y. and Y. Nakayama (2001). *Gauge anomaly cancellations in $SU(2)_L \times U(1)_Y$ electroweak theory on the lattice*, Nucl. Phys. **B597**, 519.

Kikukawa Y. and A. Yamada (1999a). *Weak coupling expansion of massless QCD with a Ginsparg-Wilson fermion and axial U(1) anomaly*, Phys. Lett. **B448**, 265.

Kikukawa Y. and A. Yamada (1999b). *Axial vector current of exact chiral symmetry on the lattice*, Nucl. Phys. **B547**, 413.

Kluberg-Stern H. and J.B. Zuber (1975a). *Renormalization of non-Abelian gauge theories in a background-field gauge. 1. Green's functions*, Phys. Rev. **D12**, 482.

Kluberg-Stern H. and J.B. Zuber (1975b). *Renormalization of non-Abelian gauge theories in a background-field gauge. 2. Gauge-invariant operators*, Phys. Rev. **D12**, 3159.

Knechtli F., M. Della Morte, J. Rolf, R. Sommer, I. Wetzorke and U. Wolff (2002). *Running quark mass in two flavor QCD*, arXiv:hep-lat/0209025.

Kogut J.B. (1983). *The Lattice Gauge Theory Approach To Quantum Chromodynamics*, Rev. Mod. Phys. **55**, 775.

Kogut J.B. and L. Susskind (1975). *Hamiltonian Formulation Of Wilson's Lattice Gauge Theories*, Phys. Rev. **D11**, 395.

Kovács T.G. (2002). *Locality and topology with fat link overlap actions*, arXiv:hep-lat/0209125.

Kronfeld A.S. (2002). *Progress in lattice QCD*, arXiv:hep-ph/0209231. Presented at the 22nd Physics in Collision Conference (PIC 2002), Stanford, California, 20-22 Jun 2002.

Kronfeld A.S. and B.P. Mertens (1984). *Renormalization Of Massive Lattice Fermions*, Nucl. Phys. Proc. Suppl. **B34**, 495.

Kronfeld A.S. and D.M. Photiadis (1985). *Phenomenology On The Lattice: Composite Operators In Lattice Gauge Theory*, Phys. Rev. **D31**, 2939.

Kuramashi Y. (1998). *Perturbative renormalization factors of bilinear operators for massive Wilson quarks on the lattice*, Phys. Rev. **D58**, 034507.

Kurth S. (2002). *The renormalised quark mass in the Schrödinger functional of lattice QCD - a one-loop calculation with a non-vanishing background field*, arXiv:hep-lat/0211011. Ph.D. Thesis (Humboldt Universität, Berlin).

Laporta S. (2000). *High-precision calculation of multi-loop Feynman integrals by difference equations*, Int. J. Mod. Phys. **A15**, 5087.

Lee W.j. (2001). *Perturbative matching of the staggered four-fermion operators for ϵ'/ϵ*, Phys. Rev. **D64**, 054505.

Lee W.j. (2002). *Perturbative improvement of staggered fermions using fat links*, arXiv:hep-lat/0208032.

Lee W.j. and S.R. Sharpe (2002a). *One-loop matching coefficients for improved staggered bilinears*, arXiv:hep-lat/0208018.

Lee W.j. and S.R. Sharpe (2002b). *Matching coefficients for improved staggered bilinears*, arXiv:hep-lat/0208036.

Lepage G.P., B. Clark, C.T. Davies, K. Hornbostel, P.B. Mackenzie, C. Morningstar and H. Trottier (2002). *Constrained curve fitting*, Nucl. Phys. Proc. Suppl. **B106**, 12.

Lepage G.P. and P.B. Mackenzie (1993). *On the viability of lattice perturbation theory*, Phys. Rev. **D48**, 2250.

Leroy J.P, J. Micheli, G.C. Rossi and K. Yoshida (1990). *QCD Perturbation Theory In The Temporal Gauge*, Z. Phys. **C48**, 653.

Lüscher M. (1977). *Construction Of A Selfadjoint, Strictly Positive Transfer Matrix For Euclidean Lattice Gauge Theories*, Commun. Math. Phys. **54**, 283.

Lüscher M. (1983). *Project proposal for the EMC^2 collaboration*, unpublished notes.

Lüscher M. (1985). *Schrödinger Representation In Quantum Field Theory*, Nucl. Phys. **B254**, 52.

Lüscher M. (1986). *Improved Lattice Gauge Theories*, Lectures given at the Les Houches Summer School "Critical Phenomena, Random Systems, Gauge Theories", Session XLIII, 1984, K. Osterwalder and R. Stora eds. (Elsevier, Amsterdam).

Lüscher M. (1990). *Selected Topics In Lattice Field Theory*, Lectures given at the Les Houches Summer School "Fields, Strings and Critical Phenomena", Session XLIX, 1988, E. Brézin and J. Zinn-Justin eds. (Elsevier, Amsterdam).

Lüscher M. (1997). *Theoretical advances in lattice QCD*, arXiv:hep-ph/9711205. Talk given at the 18th International Symposium on Lepton-Photon Interactions, Hamburg, 28 July – 1 August 1997.

Lüscher M. (1998). *Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation*, Phys. Lett. **B428**, 342.

Lüscher M. (1999a). *Advanced lattice QCD*, arXiv:hep-lat/9802029. Lectures given at the Les Houches Summer School "Probing the Standard Model of Particle Interactions", Session LXVIII, 1997, R. Gupta, A. Morel, E. de Rafael and F. David eds. (Elsevier, Amsterdam).

Lüscher M. (1999b). *Topology and the axial anomaly in abelian lattice gauge theories*, Nucl. Phys. **B538**, 515.

Lüscher M. (1999c). *Abelian chiral gauge theories on the lattice with exact gauge invariance*, Nucl. Phys. **B549**, 295.

Lüscher M. (2000a). *Weyl fermions on the lattice and the non-abelian gauge anomaly*, Nucl. Phys. **B568**, 162.

Lüscher M. (2000b). *Chiral gauge theories on the lattice with exact gauge invariance*, Nucl. Phys. Proc. Suppl. **B83**, 34.

Lüscher M. (2000c). *Lattice regularization of chiral gauge theories to all orders of perturbation theory*, JHEP **0006**, 028.

Lüscher M. (2001). *Chiral gauge theories revisited*, arXiv:hep-th/0102028. Lectures given at the International School of Subnuclear Physics, 38th Course: "Theory and Experiment Heading for New Physics", Erice, Italy, 27 Aug - 5 Sep 2000.

Lüscher M. (2002). *Lattice QCD - from quark confinement to asymptotic freedom*, arXiv:hep-ph/0211220. Plenary talk at the International Conference on Theoretical Physics (TH 2002), Paris, UNESCO, 22–27 July 2002.

Lüscher M., R. Narayanan, P. Weisz and U. Wolff (1992). *The Schrödinger Functional – a Renormalizable Probe for Non-Abelian Gauge Theories*, Nucl. Phys. **B384**, 168.

Lüscher M., S. Sint, R. Sommer and P. Weisz (1996). *Chiral symmetry and O(a) improvement in lattice QCD*, Nucl. Phys. **B478**, 365.

Lüscher M., S. Sint, R. Sommer, P. Weisz and U. Wolff (1997). *Non-perturbative O(a) improvement of lattice QCD*, Nucl. Phys. **B491**, 323.

Lüscher M., R. Sommer, P. Weisz and U. Wolff (1994). *A precise determination of the running coupling in the SU(3) Yang-Mills theory*, Nucl. Phys. **B413**, 481.

Lüscher M., R. Sommer, U. Wolff and P. Weisz (1993). *Computation of the running coupling in the SU(2) Yang-Mills theory*, Nucl. Phys. **B389**, 247.

Lüscher M. and P. Weisz (1984). *Definition And General Properties Of The Transfer Matrix In Continuum Limit Improved Lattice Gauge Theories*, Nucl. Phys. **B240**, 349.

Lüscher M. and P. Weisz (1985a). *On-Shell Improved Lattice Gauge Theories*, Commun. Math. Phys. **97**, 59. [Erratum-ibid. **98**, 433 (1985)].

Lüscher M. and P. Weisz (1985b). *Computation Of The Action For On-Shell Improved Lattice Gauge Theories At Weak Coupling*, Phys. Lett. **B158**, 250.

Lüscher M. and P. Weisz (1986). *Efficient Numerical Techniques For Perturbative Lattice Gauge Theory Computations*, Nucl. Phys. **B266**, 309.

Lüscher M. and P. Weisz (1995a). *Two loop relation between the bare lattice coupling and the $\overline{\text{MS}}$ coupling in pure SU(N) gauge theories*, Phys. Lett. **B349**, 165.

Lüscher M. and P. Weisz (1995b). *Coordinate space methods for the evaluation of Feynman diagrams in lattice field theories*, Nucl. Phys. **B445**, 429.

Lüscher M. and P. Weisz (1995c). *Background field technique and renormalization in lattice gauge theory*, Nucl. Phys. **B452**, 213.

Lüscher M. and P. Weisz (1995d). *Computation of the relation between the bare lattice coupling and the $\overline{\text{MS}}$ coupling in SU(N) gauge theories to two loops*, Nucl. Phys. **B452**, 234.

Lüscher M. and P. Weisz (1996). *O(a) improvement of the axial current in lattice QCD to one-loop order of perturbation theory*, Nucl. Phys. **B479**, 429.

Lüscher M., P. Weisz and U. Wolff (1991). *A numerical method to compute the running coupling in asymptotically free theories*, Nucl. Phys. **B359**, 221.

Mackenzie P.B. (1995). *Standard Model Phenomenology Using Lattice QCD*. In: "Phenomenology and Lattice QCD" - Proceedings of the 1993 Uehling Summer School, Seattle, WA, 21 Jun - 2 Jul 1993, G. Kilcup and S.R. Sharpe eds. (World Scientific, Singapore).

Maiani L., G. Martinelli, G.C. Rossi and M. Testa (1987). *The Octet Nonleptonic Hamiltonian And Current Algebra On The Lattice With Wilson Fermions*, Nucl. Phys. **B289**, 505.

Mandula J.E., G. Zweig and J. Govaerts (1983a). *Representations Of The Rotation Reflection Symmetry Group Of The Four-Dimensional Cubic Lattice*, Nucl. Phys. **B228**, 91.

Mandula J.E., G. Zweig and J. Govaerts (1983b). *Covariant Lattice Glueball Fields*, Nucl. Phys. **B228**, 109 .

Martinelli G. (1984). *The Four Fermion Operators Of The Weak Hamiltonian On The Lattice And In The Continuum*, Phys. Lett. **B141**, 395.

Martinelli G., C. Pittori, C.T. Sachrajda, M. Testa and A. Vladikas (1995). *A general method for nonperturbative renormalization of lattice operators*, Nucl. Phys. **B445**, 81.

Martinelli G. and Y.C. Zhang (1983a). *The Connection Between Local Operators On The Lattice And In The Continuum And Its Relation To Meson Decay Constants*, Phys. Lett. **B123**, 433.

Martinelli G. and Y.C. Zhang (1983b). *One Loop Corrections To Extended Operators On The Lattice*, Phys. Lett. **B125**, 77.

Mertens B.P., A.S. Kronfeld and A.X. El-Khadra (1998). *The self energy of massive lattice fermions*, Phys. Rev. **D58**, 034505.

Montvay I. and G. Münster (1994). *Quantum Fields On A Lattice*, Cambridge Monographs On Mathematical Physics (Cambridge University Press).

Morningstar C.J. (1996). *Lattice Perturbation Theory*, Nucl. Phys. Proc. Suppl. **B47**, 92.

Münster G. and M. Walzl (2000). *Lattice gauge theory: A short primer*, arXiv:hep-lat/0012005. Lectures given at the Zuoz Summer School on "Phenomenology of Gauge Interactions", Zuoz, Engadin, Switzerland, 13-19 Aug 2000.

Naik S. (1993). *O(a) perturbative improvement for Wilson fermions*, Phys. Lett. **B311**, 230.

Nanopoulos D.V. and D.A. Ross (1979). *Limits On The Number Of Flavors In Grand Unified Theories From Higher Order Corrections To Fermion Masses*, Nucl. Phys. **B157**, 273.

Nanopoulos D.V. and G.G. Ross (1975). *Rare Decay Modes Of The K Mesons And $K_L - K_S$ Mass Difference In An Asymptotically Free Gauge Theory*, Phys. Lett. **B56**, 279.

Narayanan R. and H. Neuberger (1993a). *Infinitely Many Regulator Fields For Chiral Fermions*, Phys. Lett. **B302**, 62.

Narayanan R. and H. Neuberger (1993b). *Chiral Fermions On The Lattice*, Phys. Rev. Lett. **71**, 3251.

Narayanan R. and H. Neuberger (1994). *Chiral Determinant As An Overlap Of Two Vacua*, Nucl. Phys. **B412**, 574.

Narayanan R. and H. Neuberger (1995). *A construction of lattice chiral gauge theories*, Nucl. Phys. **B443**, 305.

Narayanan R. and U. Wolff (1995). *Two loop computation of a running coupling lattice Yang-Mills theory*, Nucl. Phys. **B444**, 425.

Necco S. (2002a). *The $N_f = 0$ heavy quark potential and perturbation theory*, Nucl. Phys. Proc. Suppl. **B106**, 862.

Necco S. (2002b). *Universality and RG-improved gauge actions*, arXiv:hep-lat/0208052.

Necco S. and R. Sommer (2001). *Testing perturbation theory on the $N_f = 0$ static quark potential*, Phys. Lett. **B523**, 135.

Negele J.W. (2002). *Understanding Parton Distributions from Lattice QCD: Present Limitations and Future Promise*, arXiv:hep-lat/0211022. To appear in the proceedings of the European Workshop on the QCD Structure of the Nucleon (QCD-N'02), Ferrara, Italy, 3-6 Apr 2002.

Neuberger H. (1986). *Nonperturbative BRS Invariance*, Phys. Lett. **B175**, 69.

Neuberger H. (1987). *Nonperturbative BRS Invariance And The Gribov Problem*, Phys. Lett. **B183**, 337.

Neuberger H. (1998a). *Exactly massless quarks on the lattice*, Phys. Lett. **B417**, 141.

Neuberger H. (1998b). *Vector like gauge theories with almost massless fermions on the lattice*, Phys. Rev. **D57**, 5417.

Neuberger H. (1998c). *More about exactly massless quarks on the lattice*, Phys. Lett. **B427**, 353.

Neuberger H. (2001). *Exact chiral symmetry on the lattice*, Ann. Rev. Nucl. Part. Sci. **51**, 23.

Niedermayer F. (1999). *Exact chiral symmetry, topological charge and related topics*, Nucl. Phys. Proc. Suppl. **B73**, 105.

Nielsen H.B. and M. Ninomiya (1981a). *Absence Of Neutrinos On A Lattice. 1. Proof By Homotopy Theory*, Nucl. Phys. **B185**, 20. [Erratum-ibid. **B195**, 541 (1981)].

Nielsen H.B. and M. Ninomiya (1981b). *Absence Of Neutrinos On A Lattice. 2. Intuitive Topological Proof*, Nucl. Phys. **B193**, 173.

Nielsen H.B. and M. Ninomiya (1981c). *No Go Theorem For Regularizing Chiral Fermions*, Phys. Lett. **B105**, 219.

Nobes M.A., H.D. Trottier, G.P. Lepage and Q. Mason (2002). *Second order perturbation theory for improved gluon and staggered quark actions*, Nucl. Phys. Proc. Suppl. **B106**, 838.

Osterwalder K. and R. Schrader (1973). *Axioms For Euclidean Green's Functions*, Commun. Math. Phys. **31**, 83.

Osterwalder K. and R. Schrader (1975). *Axioms For Euclidean Green's Functions. 2*, Commun. Math. Phys. **42**, 281.

Palombi F., R. Petronzio and A. Shindler (2002). *Moments of singlet parton densities on the lattice in the Schrödinger functional scheme*, Nucl. Phys. **B637**, 243.

Panagopoulos H. and Y. Proestos (2002). *The critical hopping parameter in O(a) improved lattice QCD*, Phys. Rev. **D65**, 014511.

Panagopoulos H. and E. Vicari (1990). *The Trilinear Gluon Condensate On The Lattice*, Nucl. Phys. **B332**, 261.

Panagopoulos H. and E. Vicari (1998). *Resummation of cactus diagrams in lattice QCD*, Phys. Rev. **D58**, 114501.

Panagopoulos H. and E. Vicari (1999). *Resummation of cactus diagrams in the clover improved lattice formulation of QCD*, Phys. Rev. **D59**, 057503.

Parisi G. (1980). *Recent Progresses In Gauge Theories*, Preprint LNF-80/52-P. Presented at the 20th International Conference on High Energy Physics, Madison, Wis., Jul 17-23, 1980. In C80-07-17.140.

Parisi G. and Y.s. Wu (1981). *Perturbation Theory Without Gauge Fixing*, Sci. Sin. **24**, 483.

Patel A. and S.R. Sharpe (1993). *Perturbative corrections for staggered fermion bilinears*, Nucl. Phys. **B395**, 701.

Politzer H.D. (1973). *Reliable Perturbative Results For Strong Interactions?*, Phys. Rev. Lett. **30**, 1346.

Rebbi C. (1987). *Chiral Invariant Regularization Of Fermions On The Lattice*, Phys. Lett. **B186**, 200.

Reisz T. (1988a). *A Power Counting Theorem For Feynman Integrals On The Lattice*, Commun. Math. Phys. **116**, 81.

Reisz T. (1988b). *A Convergence Theorem For Lattice Feynman Integrals With Massless Propagators*, Commun. Math. Phys. **116**, 573.

Reisz T. (1988c). *Renormalization Of Feynman Integrals On The Lattice*, Commun. Math. Phys. **117**, 79.

Reisz T. (1988d). *Renormalization Of Lattice Feynman Integrals With Massless Propagators*, Commun. Math. Phys. **117**, 639.

Reisz T. (1989). *Lattice Gauge Theory: Renormalization To All Orders In The Loop Expansion*, Nucl. Phys. **B318**, 417.

Rossi G.C. and M. Testa (1980a). *The Structure Of Yang-Mills Theories In The Temporal Gauge. 1. General Formulation*, Nucl. Phys. **B163**, 109.

Rossi G.C. and M. Testa (1980b). *The Structure Of Yang-Mills Theories In The Temporal Gauge. 2. Perturbation Theory*, Nucl. Phys. **B176**, 477.

Rossi G.C. and M. Testa (1984). *The Structure Of Yang-Mills Theories In The Temporal Gauge. 3. The Instanton Sector*, Nucl. Phys. **B237**, 442.

Rossi G.C. and K. Yoshida (1989). *Fermionic Functional Integral And The Temporal Gauge Of QCD*, Nuovo Cim. **11D**, 101.

Rothe H.J. (1997). *Lattice Gauge Theories: An Introduction (2nd Edition)*, Lecture Notes in Physics **59**, 1 (World Scientific, Singapore).

Sachrajda C.T. (1990). *Lattice Perturbation Theory*, Preprint SHEP 89/90-4. In: "From Actions to Answers", Proceedings of the 1989 Theoretical Advanced Study Institute (TASI 1989), Boulder, CO, June 4-30, 1989, T. DeGrand and D. Toussaint eds. (World Scientific, Singapore).

Sadooghi N. and H.J. Rothe (1997). *Continuum behaviour of lattice QED, discretized with one-sided lattice differences, in one-loop order*, Phys. Rev. **D55**, 6749.

Shamir Y. (1993). *Chiral fermions from lattice boundaries*, Nucl. Phys. **B406**, 90.

Shamir Y. (1996). *Lattice Chiral Fermions*, Nucl. Phys. Proc. Suppl. **B47**, 212.

Shamir Y. (1998). *The Standard Model from a New Phase Transition on the Lattice*, Phys. Rev. **D57**, 132.

Shamir Y. (2000). *New domain-wall fermion actions*, Phys. Rev. **D62**, 054513.

Sharatchandra H.S. (1978). *The Continuum Limit Of Lattice Gauge Theories In The Context Of Renormalized Perturbation Theory*, Phys. Rev. **D18**, 2042.

Sharatchandra H.S., H.J. Thun and P. Weisz (1981). *Susskind Fermions On A Euclidean Lattice*, Nucl. Phys. **B192**, 205.

Sharpe S.R. (1994). *Phenomenology from the lattice*, arXiv:hep-ph/9412243. In: "CP Violation and the limits of the Standard Model", Proceedings of the 1994 Theoretical Advanced Study Institute (TASI 1994), Boulder, CO, May 29 - June 24, 1994, J. Donoghue ed. (World Scientific, Singapore).

Sharpe S.R. (1995). *Introduction To Lattice Field Theory*. In: "Phenomenology and Lattice QCD" - Proceedings of the 1993 Uehling Summer School, Seattle, WA, 21 Jun - 2 Jul 1993, G. Kilcup and S.R. Sharpe eds. (World Scientific, Singapore).

Sharpe S.R. (1999). *Progress in lattice gauge theory*, arXiv:hep-lat/9811006. Proceedings of the 29th International Conference on High-Energy Physics (ICHEP 1998), Vancouver, Canada, 23-29 Jul 1998, A. Astbury, D. Axen and J. Robinson eds. (World Scientific, Singapore).

Sharpe S.R. and A. Patel (1994). *Perturbative corrections for staggered four fermion operators*, Nucl. Phys. **B417**, 307.

Sheikholeslami B. and R. Wohlert (1985). *Improved Continuum Limit Lattice Action For QCD With Wilson Fermions*, Nucl. Phys. **B259**, 572.

Sint S. (1994). *On the Schrödinger functional in QCD*, Nucl. Phys. **B421**, 135.

Sint S. (1995). *One loop renormalization of the QCD Schrödinger functional*, Nucl. Phys. **B451**, 416.

Sint S. and R. Sommer (1996). *The running coupling from the QCD Schrödinger functional - a one loop analysis*, Nucl. Phys. **B465**, 71.

Sint S. and P. Weisz (1997). *Further results on O(a) improved lattice QCD to one-loop order of perturbation theory*, Nucl. Phys. **B502**, 251.

Sint S. and P. Weisz (1999). *The running quark mass in the SF scheme and its two-loop anomalous dimension*, Nucl. Phys. **B545**, 529.

Smit J. (1986). *Fermions On A Lattice*, Acta Phys. Polon. **B17**, 531.

Smit J. (2002). *Introduction To Quantum Fields On A Lattice: A Robust Mate*, Cambridge Lecture Notes in Physics, 15 (Cambridge University Press).

Sommer R. (1997). *Non-perturbative renormalization of QCD*, arXiv:hep-ph/9711243. Talk given at the 36th Internationale Universitätswochen Für Kernphysik und Teilchenphysik "Computing particle properties", Schladming, Austria, 1-8 Mar 1997.

Sommer R. (2002). *Non-perturbative renormalization of HQET and QCD*, arXiv:hep-lat/0209162.

Stehr J. and P.H. Weisz (1983). *Note On Gauge Fixing In Lattice QCD*, Lett. Nuovo Cim. **37**, 173.

Susskind L. (1977). *Lattice Fermions*, Phys. Rev. **D16**, 3031.

Suzuki H. (1999). *Gauge invariant effective action in Abelian chiral gauge theory on the lattice*, Prog. Theor. Phys. **101**, 1147.

Suzuki H. (2000). *Anomaly cancellation condition in lattice gauge theory*, Nucl. Phys. **B585**, 471.

Symanzik K. (1980). *Cutoff dependence in lattice ϕ_4^4 theory*. In: "Recent developments in gauge theories" (Cargèse 1979), G. 't Hooft *et al.* eds. (Plenum, New York).

Symanzik K. (1981). *Schrödinger Representation And Casimir Effect In Renormalizable Quantum Field Theory*, Nucl. Phys. **B190**, 1.

Symanzik K. (1982). *Some topics in quantum field theory*. Presented at the 6th International Conference on Mathematical Physics, Berlin, West Germany, Aug 11-21, 1981. In: "Mathematical problems in theoretical physics", Lecture Notes in Physics, Vol. 153, R. Schrader *et al.* eds. (Springer, New York).

Symanzik K. (1983a). *Continuum limit and improved action in lattice theories. 1. Principles and ϕ^4 theory*, Nucl. Phys. **B226**, 187.

Symanzik K. (1983b). *Continuum limit and improved action in lattice theories. 2. O(N) non-linear sigma model in perturbation theory*, Nucl. Phys. **B226**, 205.

Takaishi Y. (1996). *Heavy quark potential and effective actions on blocked configurations*, Phys. Rev. **D54**, 1050.

Tarasov O.V., A.A. Vladimirov and A.Y. Zharkov (1980). *The Gell-Mann-Low Function Of QCD In The Three Loop Approximation*, Phys. Lett. **B93**, 429.

Tarrach R. (1981). *The Pole Mass In Perturbative QCD*, Nucl. Phys. **B183**, 384.

Testa M. (1998a). *The Rome approach to chirality*, arXiv:hep-lat/9707007. In: "Recent Developments in Nonperturbative Quantum Field Theory", Proceedings of the APCTP-ICTP Joint International Conference, Seoul, Korea, 26 - 30 May 1997, Y.M. Cho and M. Virasoro eds. (World Scientific, Singapore).

Testa M. (1998b). *Lattice gauge fixing, Gribov copies and BRST symmetry*, Phys. Lett. **B429**, 349.

Trottier H.D., N.H. Shakespeare, G.P. Lepage and P.B. Mackenzie (2002). *Perturbative expansions from Monte Carlo simulations at weak coupling: Wilson loops and the static-quark self-energy*, Phys. Rev. **D65**, 094502.

van den Doel C. and J. Smit (1983). *Dynamical Symmetry Breaking In Two Flavor SU(N) And SO(N) Lattice Gauge Theories*, Nucl. Phys. **B228**, 122.

van Ritbergen T., J.A. Vermaseren and S.A. Larin (1997). *The four-loop beta function in quantum chromodynamics*, Phys. Lett. **B400**, 379.

Veltman M.J. (1989). *Gammatrica*, Nucl. Phys. **B319**, 253.

Vermaseren J.A.M. (2000). *New features of FORM*, arXiv:math-ph/0010025. Preprint NIKHEF-00-032

Vermaseren J.A.M., S.A. Larin and T. van Ritbergen (1997). *The 4-loop quark mass anomalous dimension and the invariant quark mass*, Phys. Lett. **B405**, 327.

Weinzierl S. (2002). *Computer algebra in particle physics*, arXiv:hep-ph/0209234.

Weisz P. (1981). *On The Connection Between The Λ-Parameters Of Euclidean Lattice And Continuum QCD*, Phys. Lett. **B100**, 331.

Weisz P. (1983). *Continuum Limit Improved Lattice Action For Pure Yang-Mills Theory 1*, Nucl. Phys. **B212**, 1.

Weisz P. (1996). *Lattice Investigations of the Running Coupling*, Nucl. Phys. Proc. Suppl. **B47**, 71.

Weisz P. and R. Wohlert (1984). *Continuum Limit Improved Lattice Action For Pure Yang-Mills Theory 2*, Nucl. Phys. **B236**, 397. [Erratum-ibid. **B247**, 544 (1984)].

Wiese U.J. (1993). *Fixed point actions for Wilson fermions*, Phys. Lett. **B315**, 417.

Wilczek F. (1987). *Lattice Fermions*, Phys. Rev. Lett. **59**, 2397.

Wilson K.G. (1974). *Confinement Of Quarks*, Phys. Rev. **D10**, 2445.

Wilson K.G. (1975). *The Renormalization Group: Critical Phenomena And The Kondo Problem*, Rev. Mod. Phys. **47**, 773.

Wilson K.G. (1977). *Quarks And Strings On A Lattice.* In: "New Phenomena In Subnuclear Physics", Part A, Proceedings of the First Half of the 1975 International School of Subnuclear Physics, Erice, Sicily, July 11 - August 1, 1975, A. Zichichi ed. (Plenum Press, New York). CLNS-321; C75-07-11.10.

Wilson K.G. (1980). *Monte Carlo Calculations For The Lattice Gauge Theory*, In: "Recent developments in gauge theories" (Cargèse 1979), G. 't Hooft *et al.* eds. (Plenum, New York).

Wilson K.G. and J.B. Kogut (1974). *The renormalization group and the ϵ expansion*, Phys. Rept. **12**, 75.

Wittig H. (1999). *Lattice gauge theory*, arXiv:hep-ph/9911400. Presented at the International Europhysics Conference on High-Energy Physics (EPS-HEP 99), Tampere, Finland, 15-21 Jul 1999.

Wohlert R. (1987). *Improved continuum limit lattice action for quarks*, Ph.D. Thesis (Hamburg), Preprint DESY 87-069, unpublished.

Wohlert R., P. Weisz and W. Wetzel (1985). *Weak Coupling Perturbative Calculations Of The Wilson Loop For The Standard Action*, Nucl. Phys. **B259**, 85.

Wolff U. (1995). *Two loop computation of a finite volume running coupling on the lattice*, Nucl. Phys. Proc. Suppl. **B42**, 291.

Yamada A. (1998). *Lattice perturbation theory in the overlap formulation for the Yukawa and gauge interactions*, Nucl. Phys. **B529**, 483.

Zinn-Justin J. (2002). *Chiral anomalies and topology*, arXiv:hep-th/0201220. Contributed to the Autumn School 2001 "Topology and Geometry in Physics", Rot an der Rot, Germany, 24-28 Sep 2001.