# Lecture Notes for Phys 621 "Electrodynamics"

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## Abstract

These lecture notes will contain some additional material related to Jackson, 3d ed. (*abbreviated* **JACK.** ), which is the main textbook. Notes for all lectures will be kept in a single file and the table of contents will be automatically updated so that each time you can print out only the updated part.

The *Mathematica* appendix is included for reference; you do not have to print it out since there will be other files which illustrate how this program works.

Please report any typos to vitaly@oak.njit.edu

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**Foreword**

These notes will provide a description of topics which are not covered in **JACK.** in order to keep the course self-contained. Topics fully explained in **JACK.** will be described more briefly and in such cases sections from **JACK.** which require an in-depth analysis will be indicated as **work-through: ...** . Occasionally you will have reading assignments - indicated as *READING: ...* . (Do not expect to understand everything in such cases, but it is always useful to see a more general picture, even if a bit faint.)

Homeworks are important part of the course; they are indicated as **HW***: ...* and include both problems from **JACK.** and unfinished proofs/verifications from notes. If done by hand the HW solutions must be clearly written in pen (black or blue), if done using *Mathematica* they must be printed out as a clear hard copy (adding neat hand-written corrections on the hard copy is ok). No electronic submissions of *Mathematica* notebooks will be accepted.

# Part I

# Mathematical introduction

## I. VECTORS AND VECTOR CALCULUS

### A. Summation convention

Components of a vector in selected coordinates are indicated by Greek indexes and summation of repeated indexes from 1 to 3 is implied, e.g.

$$a^2 = \vec{a} \cdot \vec{a} = \sum_{\alpha=1}^{3} a_\alpha a_\alpha = a_\alpha a_\alpha \tag{1}$$

(summation over other indexed will be indicated explicitly). The notation $x^\alpha$ will correspond to $x\,, y\,, z$ with $\alpha = 1\,2\,3$, respectively and $r^\alpha$ will be used in the same sense.

Scalar product:

$$\vec{a} \cdot \vec{b} = a_\alpha b_\beta$$

The Kronecker delta symbol $\delta^{\alpha\beta}$ will be used, e.g.

$$a_i^2 = a_i^\alpha a_i^\beta \delta^{\alpha\beta} \tag{2}$$

(which is a second-rank *tensor*, while $\vec{a}$ is tensor of the 1st rank). Similarly, for terms involving vector product a full antisymmetric tensor $\epsilon_{\alpha\beta\gamma}$ will be used. It is defined as $\epsilon_{1,2,3} = 1$ and so are all components which follow after an even permutation of indexes. Components which have an odd permutation, e.g. $\epsilon_{2,1,3}$ are $-1$ and all other are 0. Then

$$\left(\vec{a} \times \vec{b}\right)_\alpha = \epsilon_{\alpha\beta\gamma} a_\beta b_\gamma \tag{3}$$

**HW***: Prove the 1st three identities from the inner cover of* **JACK.**

### B. Derivatives

Operator $\hat{\nabla}$:

$$\hat{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \tag{4}$$

(Cartesian coordinates only!)

Then

$$\mathrm{grad}\Phi \equiv \hat{\nabla}\Phi \equiv \frac{\partial}{\partial \vec{r}}\Phi \tag{5}$$

or in components

$$\left(\hat{\nabla}\Phi\right)_\alpha = \frac{\partial}{\partial x^\alpha}\Phi$$

Divergence:

$$\mathrm{div}\vec{F} \equiv \hat{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x^\alpha}F_\alpha \tag{6}$$

Curl:

$$\mathrm{curl}\vec{F} \equiv \hat{\nabla}\times\vec{F} \tag{7}$$

or in components

$$\left(\mathrm{curl}\vec{F}\right)_\alpha = \epsilon_{\alpha,\beta,\gamma}\frac{\partial}{\partial x^\beta}F_\gamma$$

**HW**: *Let* $\vec{r} = (x,\, y,\, z)$ *and* $r = |\vec{r}|$. *Find* $\hat{\nabla}r$ , $\hat{\nabla}\cdot\vec{r}$ , $\hat{\nabla}\times(\vec{\omega}\times\vec{r})$ *with* $\vec{\omega} = const$

Note $\mathrm{grad}\Phi$ and $\mathrm{curl}\vec{F}$ are genuine vectors, while $\mathrm{div}\vec{F}$ is a true scalar.

Important relations:

$$\mathrm{curl}\left(\mathrm{grad}\Phi\right) = 0 \tag{8}$$

$$\mathrm{div}\left(\mathrm{curl}\vec{F}\right) = 0 \tag{9}$$

**HW**: *show that*

**HW**: *Prove formulas for* $\hat{\nabla}\times(\psi\vec{a})$ *and* $\hat{\nabla}\cdot\left(\vec{a}\times\vec{b}\right)$ *- see front inner cover of* **JACK.**

## II. MULTIDIMENSIONAL INTEGRATION

### A. Change of variables

$$\int\int_R f(x,y)\,dx\,dy = \int\int_{R^*} f\left[x(u,v)\,,y(u,v)\right]\left|\partial(x,y)/\partial(u,v)\right| \tag{10}$$

Here $|\partial(x,y)/\partial(u,v)|$ is the Jacobian. The 3D case is similar.

Cylindrical: $(r,\phi,z)$ with

$$x = \cos\phi\ ,\quad y = \sin\phi \tag{11}$$

$$J = |\partial(x,y,z)/\partial(r,\phi,z)| = |\partial(x,y)/\partial(r,\phi)| = r$$

**HW**: *show that*

Spherical: $(r,\theta,\phi)$ with

$$x = r\sin\theta\cos\phi\ ,\ y = r\sin\theta\sin\phi\ ,\ z = r\cos\theta\ , \tag{12}$$

$$J = -r^2\sin\theta \tag{13}$$

**HW**: *show that*

Solid angle:

$$d\Omega \equiv \frac{dA}{r^2} = \frac{1}{r^2}\frac{dV}{dr} = \sin\theta\,d\theta\,d\phi \tag{14}$$

$$\int d\Omega = 4\pi$$

**HW**: *show that*

### B. Divergence theorem (Gauss)

$$\int\int\int_V \left(\hat{\nabla}\cdot\vec{F}\right)\,dV = \int\int_S \vec{F}\cdot\vec{n}\,da \tag{15}$$

Proof: first prove for an infinitesimal cube oriented along $x, y, z$; then extend for the full volume **HW**: *(optional) do that*

**HW**: *verify the Divergence theorem for $\vec{F} = \vec{r}$ and spherical volume*

### C. Stokes theorem

$$\int\int_S \left(\hat{\nabla} \times \vec{F}\right) \cdot \vec{n}\, da = \oint \vec{F} \cdot d\vec{l} \tag{16}$$

Proof: first prove for a plane (”Green’s theorem”) starting from an infinitesimal square; then generalize for arbitrary, non-planar surface

**HW**: *verify Stokes theorem for $\vec{F} = \omega \times \vec{r}$ and a circular shape.*

## III. DIRAC DELTA

### A. Basic definitons

$$\begin{aligned} \delta(x) &= 0\ , \quad x \neq 0 \\ \delta(x) &= \infty\ , \quad x = 0 \\ \int_{-\epsilon}^{\epsilon} \delta(x)\, dx &= 1\ , \quad \text{for any } \epsilon > 0 \end{aligned} \tag{17}$$

Then,

$$\int_{-\epsilon}^{\epsilon} \delta(x) f(x)\, dx = f(0)\ , \quad \text{for any } \epsilon > 0 \tag{18}$$

Note: the real meaning should be given only to integrals. E.g., $\delta(x)$ can oscillate infinitely fast, which does not contradict $\delta(x) = 0$ once an integral is taken.

Sequences leading to a $\delta$-function for $n \to \infty$:

$$\delta_n(x) = n \text{ for } |x| < 1/2n\ , \quad 0 \ \text{ otherwise} \tag{19}$$

$$\delta_n(x) = \frac{n}{\pi} \frac{1}{n^2 x^2 + 1} \tag{20}$$

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} \exp\left(-n^2 x^2\right) \tag{21}$$

$$\delta_n(x) = \frac{\sin(nx)}{\pi x} \tag{22}$$

$$\delta_n(x) = \frac{n}{2} \exp\left(-n|x|\right) \tag{23}$$

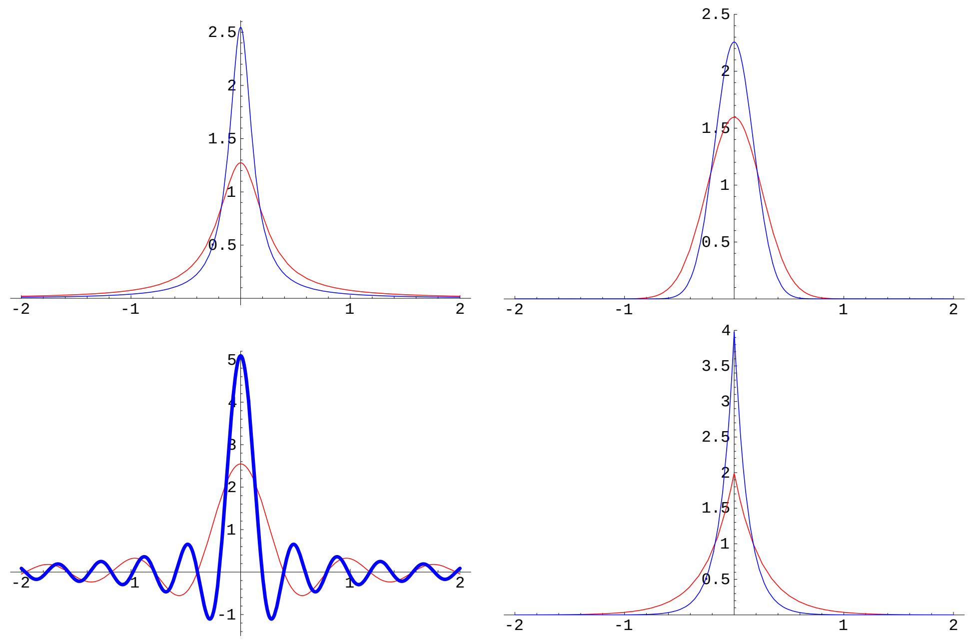


FIG. 1: Various represenations of $\delta_n$ which lead to Dirac delta-function for $n \to \infty$ - see eqs.(20-23).

**HW**: *check normalization and reproduce plots*

Derivative:

$$\int \delta'(x) f(x) = -f'(0) \tag{24}$$

**HW**: *Show that integrating by parts; verify explicitly by using eq.(19); note that for finite n derivative of eq.(19) leads to* $\pm\delta$

## B. Multidimensional

$$\delta\left(\vec{r} - \vec{r}_0\right) = \delta\left(x - x_0\right) \delta\left(y - y_0\right) \delta\left(z - z_0\right) \tag{25}$$

From Gauss theorem:

$$\hat{\nabla}^2 \left(\frac{1}{r}\right) = -4\pi\delta\left(\vec{r}\right) \tag{26}$$

the rest will be discussed in class.

## IV. SPECIAL FUNCTIONS

### A. General

Consider so-called "normal" form of a differential equation

$$u'' + I(x)u = 0 \tag{27}$$

Now if $I(x) = const \equiv I$, one has for solutions $u(x)$

$$\sin\left(\sqrt{I}x\right) ,\quad \cos\left(\sqrt{I}x\right) ,\quad I > 0$$

$$\exp\left(\pm\sqrt{-I}x\right) ,\quad I < 0$$

For $I(x) \neq const$ we have special functions. [Q. Why not always use the normal form? A. $f(x), g(x), h(x)$ are very simple, while $I(x)$ is often not -see examples below, but for qualitative analysis the normal form is better.] [Another Q. The normal form looks remarkably similar to the Schrödinger eqn. in physics (with $I(x) = E - U(x)$). Why not call it this way? A. Think. Will discuss in class.]

Now what's qualitatively new (besides the special functions being harder to compute or sometimes even imagine)? New is the fact that $I(x)$ can change sign (!!!), leading to a transition from an oscillatory to exponential-type structure of the same solution in different domains of $x$. Already the simplest, Airy function described below has this property.

The other novelty is the possibility of $I(x_0) = \infty$ at some $x_0$ called a *singular point.* Now instead of the pair of independent solutions of type sin and cos which look extremely similar to each other, one will have two solutions with very different structure, depending on their behavior near $x_0$.

Near both types of points, zero and $\infty$ for $I(x)$, very interesting physics is expected. In this sense special functions are much more exciting than the elementary, which are somewhat dull in having the same behavior for all $x$ (and an excursion into the complex plain will not make things much more exciting).

### 1. *Oscillatory behavior*

For $I(x) > 0$ in eq. 27), there will be oscillations in each of the fundamental solution, similarly to sin and cos. Moreover, there exist a theorem that zeros of $\phi_1$ and $\phi_2$ alternate with each other, making the similarity even stronger. [In this context note that *both* $\phi_1$, $\phi_2$ oscillate, or both do not].

### 2. *Singular points and classification of the solutions*

If for $x \to x_0$, one has $I(x) \to \infty$, such a point $x_0$ is called *singular.* $x_0 = \infty$ is also a possibility (and this is the only singular point for the Airy equation above). There is a more fine classification based on how fast the $\infty$ is approached, but we will not use it now. [Note: if were talking about the general form of the differential equation, eq.(31) the singular point is the one where $f(x) = 0$, not $f(x) = \infty$ (!)].

Remarkably, the solution does not always have to diverge at $x_0$. If $\phi(x_0)$ is finite, it is called the special function of the *first* kind. If it diverges, it is called of the *second* kind.

### 3. *Asymptotics, and the WKB approximation*

Let $I(x)$ in eq.(27) be large. Let us write it as $I(x) = -\rho^2 q(x)$ with $\rho \gg 1$ and make a substitution:

$$u(x) = \exp\left\{-\rho \int v(x)dx\right\}$$

which leads to

$$\rho v' + \rho^2 v^2 - \rho^2 q = 0 \tag{28}$$

[HW (optional) show this]. Now solve by iterations:

$$v_0 = \pm\sqrt{q}$$

next

$$v_1 = -\frac{1}{\rho}\frac{v_0'}{2v_0}$$

Now

$$u(x) \propto \exp\left\{\rho\left[\pm\int\sqrt{q}dx\right] - 1/2\ln|\sqrt{q}|\right\}$$

or

$$u(x) \propto \frac{1}{|I|^{1/4}} \exp\left\{\pm \int \sqrt{-I(x)} dx\right\}$$

More explicitly, for $I(x) > 0$,

$$u(x) \propto \frac{1}{I^{1/4}} \sin\left\{\int I^{1/2} dx + \alpha\right\} \tag{29}$$

($\alpha$ can have two values since 2 solutions), and for $I(x) < 0$

$$u(x) \propto \frac{1}{(-I)^{1/4}} \exp\left\{\pm \int (-I)^{1/2} dx\right\} \tag{30}$$

### B. Specific functions

Easier in the full form (simpler coefficients):

$$f(x)y'' + g(x)y' + h(x)y = 0 \tag{31}$$

| Name | $f(x)$ | $g(x)$ | $h(x)$ | Applications | *Mathematica* |
|---|---|---|---|---|---|
| sin, cos | 1 | 0 | 1 | everywhere | Sin, *Cos* |
| exp $(\pm x)$ | 1 | 0 | -1 | everywhere | Exp |
| Airy | 1 | 0 | -x | Optics (caustics), QM: transition to classically forbidden region; motion in constant field | AiryAi, AiryBi |
| Bessel (modified Bessel) | $x^2$ | $x$ | $\pm x^2 - n^2$ | (mostly) problems with cylindrical symmetry | BesselJ, BesselY $(-)$ or BesselI, BesselK $(+)$ |
| Legendre | $1 - x^2$ | $-2x$ | $n(n+1)$ | problems with asymuthal symmetry | LegendreP, LegendreQ |

One can always switch to "normal" form by changing variables in order to get rid of the 1st derivative:

$$u(x) = y(x) \exp\left\{\frac{1}{2} \int (g/f) dx\right\}$$

with

$$I = h/f - (1/4)(g/f)^2 - 1/2\,(g/f)'$$

Note: "difficulty" of a special function mostly depends on the number of free parameters.

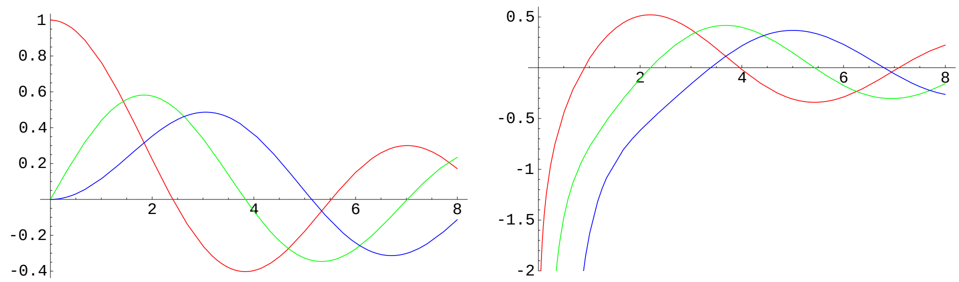


FIG. 2: Bessel functions $J_n(x)$ (1st kind) and $Y_n(x)$ (2d kind) for $n = 0$ (red), $n = 1$ (green) and $n = 2$ (blue).

### C. Bessel functions

*READING:* **JACK.** *sec. 3.7* . Note: **JACK.** uses $N$ instead of $Y$.

The singular point is $x = 0$. So, $y_1(x) = J_n(x)$ -first kind, and $y_2(x) = Y_n(x)$ - second kind, see Fig. 2. Let us transform to normal form:

$$u(x) = y(x) \exp\left\{\frac{1}{2}\int (x/x^2)dx\right\}$$

or

$$u(x) = y(x)\sqrt{x}$$

and

$$I(x) = 1 - n^2/x^2 + 1/4x^2$$

Note that for large $x$ the function $I(x)$ approaches 1. Thus, according to WKB, $u(x)$ for $x \gg 1$ is oscillatory, with period $2\pi$ and constant amplitude, while $y(x)$ oscillates with amplitude decaying as $1/\sqrt{x}$. Indeed, one has:

### D. Asymptotes

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{2}n\pi - \frac{\pi}{4}\right) \tag{32}$$

and

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{1}{2}n\pi - \frac{\pi}{4}\right) \tag{33}$$

[HW: plot a few Bessel functions together with their asymptotes].

For large $n$ the Bessel function will be non-oscillatory for a large interval of $x$ (since $I(x)$ will be negative). For large $n$ and *finite* $x$ one can use the small-$x$ expressions described below. More interesting is the case when both $x$ and $n$ are large, when transition to oscillations is expected for $x \sim n$. Here one recovers a relation between the Bessel functions and the Airy functions - see Abramowitz, 9.3.4, etc.

#### 1. *Small $x$*

Near the singular point and for $n \neq 0$ one can neglect $x^2$ in the free term. This leads to what is known as a *homogeneous* equation which can be solved by a power-law. We look for a solution $y(x) \propto x^\mu$, and get

$$\mu(\mu - 1) + \mu - n^2 = 0$$

or

$$\mu = \pm n$$

Indeed, one has for $x \to 0$

$$J_n \sim \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)} \ , \quad n \geq 0 \tag{34}$$

and

$$Y_n \sim -\frac{1}{\pi} \left(\frac{x}{2}\right)^{-n} \Gamma(n) \ , \quad n > 0 \tag{35}$$

A special case is

$$Y_0(x) \sim \frac{2}{\pi} \left[\ln \frac{z}{2} + \gamma\right] \ , \quad \gamma = 0.5772... \tag{36}$$

### E. Integer vs. non-integer $n$, generating function, recurrence relations and integral representation

For non-integer $n = \nu$ the solution $J_{-\nu}$ is linearly independent from $J_\nu$. In particular, there exists a formula expressing $Y_\nu$ through $J_\nu$ and $J_{-\nu}$ (see, e.g., Abramowitz, 9.1.2). For integer $n$ such solutions are not independent:

$$J_{-n}(x) = (-1)^n J_n(x) \tag{37}$$

This follows from the so-called *generating function*

$$e^{x/2(t-1/t)} = \sum_{n=-\infty}^{n=\infty} J_n(x) t^n \tag{38}$$

(note: here $n$ are strictly integer!). Many recurrence relations which can be obtained directly from the above equation, e.g.

$$J_{n-1} - J_{n+1} = 2J_n' \tag{39}$$

From here, and from the symmetry relation, one gets

$$J_0' = -J_1$$

. Another relation is

$$J_n(x) = x^{-n} \int^x z^n J_{n-1}(z) dz \tag{40}$$

There are several integral representations, the "best" is

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin\theta} d\theta \tag{41}$$

which is famous for optics applications. (A similar expression for any $n$ is available - see Abramowitz or Arfken).

### 1. *Roots of $J_\nu$*

$$J_\nu \left( x_{\nu\, n} \right) = 0$$

$$x_{\nu\, n} \simeq \pi \left( n + \nu/2 - 1/4 \right) \;, \quad n \gg 1 \tag{42}$$

For arb. $n$ - see *Mathematica* appendix.

### 2. *Orthogonality*

The 2-d order operator in the Bessel equation can be made in self-adjoined form if divided by $x$. Then orthogonality of $\sqrt{x} J_\nu \left( x_{\nu\, n}\, x \right)$ is achieved on $[0, 1]$ for different $n$ (Note: $\nu$ must be the same - different $\nu$ correspond to different operators and are not related).

$$\int_0^1 x J_\nu \left( x_{\nu\, n}\, x \right) J_\nu \left( x_{\nu\, m}\, x \right)\, dx = 0 \;, \quad n \neq m \tag{43}$$

and

$$\int_0^1 x J_\nu \left(x_{\nu\, n}\, x\right)^2 \, dx = \frac{1}{2} J_{\nu+1} \left(x_{\nu\, n}\right) \tag{44}$$

(Note: rhs is non-zero since $x_{\nu\, n}$ is a root of another function.)

### *3. Relation to Laplacian*

In cylindrical coordinates

$$\hat{\Delta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \tag{45}$$

Separating variables and using $Z(z) \propto \exp\left(\pm kz\right)$ and $Q(\phi) \propto \exp\left(\pm i\nu\phi\right)$ and switching to $x = kr$ one gets the standard Bessel equation.

**HW***: show that*

### *4. Modified Bessel functions*

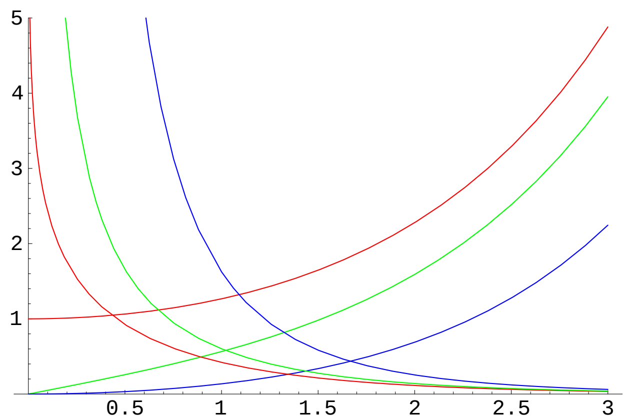


FIG. 3: Modified Bessel functions of the 1st kind , $I_n(x)$ (ascending lines) and 2d kind, $K_n(x)$ (descending lines) for $n = 0$ (red), $n = 1$ (green) and $n = 2$ (blue).

Note that going from $x^2$ to $-x^2$ in the Bessel equation can be achieved by going from $x$ to $ix$. Strictly speaking, great care must be shown since we (i.e., us) do not know well enough analytical properties in the complex plain. Nevertheless,

$$I_n(x) = (-i)^n J_n(ix) \tag{46}$$

will be a real-value function, with the same asymptotes as $J_n$ for $x \to 0$, but with an exponential growth as $x \to \infty$

$$I_n(x) \sim e^x/\sqrt{2\pi x} \tag{47}$$

One can expect that a similar construction will be used starting from $Y_n(x)$. Unfortunately, there is an extra factor $\pi/2$. The power-law asymptotes of $K_n(x)$ $(n \neq 0)$ as $x \to 0$ remain similar to those of $Y_n(x)$, but with this extra factor. For $n = 0$ one has

$$K_0(x) \sim -\ln(x/2) - \gamma\ , \quad x \to 0 \tag{48}$$

Alternatively, for $x \to \infty$ and any $n$ one has

$$K_n(x) \sim e^{-x}/\sqrt{2\pi x} \cdot \pi/2 \tag{49}$$

The general structure of the functions is shown in Fig. 3. Note that all functions are now positive. Unlike the regular Bessel case, there are now no "good" functions which remain finite for both $x = 0$ and $x \to \infty$.

**HW**: *Plot a few modified Bessel functions together with their asymptotes* .

## F. Legendre

### 1. *Generating function for Legendre polynomials (LP)*

Consider a point charge $q$ placed on the $z$-axes at a distance $a$ from the origin (figure will be shown in class). We are interested in the electrostatic potential produced by this charge at an observation point at a distance $r$ from the the origin (not from this charge!), which is seen at an angle $\theta$. If the distance from the charge is $r_1$, then the potential is

$$\phi = kq/r_1\ , \quad k \approx 9 \cdot 10^9 Nm^2/C^2 \equiv 1/(4\pi\epsilon_0)$$

From geometry,

$$r_1^2 = r^2 + a^2 - 2ar\cos\theta$$

Consider now $r \gg a$ and perform a series expansion of the potential in powers of $a/r$. The coefficients of this expansion are functions of $\cos\theta$ and *by definition* they correspond to Legendre polynomials. More precisely:

$$\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

Here $P_n$ are known are the "Legendre polynomials" (LP), and the function on the left is the "generating function". (Of course we can replace $\cos\theta$ by $x$, but it is useful to remember the connection).

[HW: use the Series command in Mathematica to get the first 5 LP]

### *2. Explicit expressions and recurrence relations*

For small $n$ LP are simple:

$$P_0(x) = 1\ ,\ \ P_1(x) = x\ ,\ \ P_2(x) = \left(3x^2 - 1\right)/2\ldots$$

there are many recurrence relations which allow to generate LP fast, without the actual series expansion, e.g.

$$(n+1)P_{n+1}(x) = x(2n+1)P_n(x) - nP_{n-1}(x)$$

[HW: Use the above recurrence relation, and start with $P_0$, $P_1$ to obtain other LP. For some reasonably large $n$ compare with standard LegendreP in Mathematica, and compare the evaluation time using the Timing command. Do not try to view the polynomial on the screen for large $n$ !].

Note that

$$P_n(1) = 1\ ,\ \ P_n(-1) = (-1)^n$$

this follows, e.g. from the recurrence relation [HW (optional) show this]. Also

$$P_{-n}(x) = P_{n-1}(x) \tag{50}$$

[HW: check this using FullSimlify]

### *3. Relation to Laplacian*

In spherical coordinates one has

$$\hat{\Delta} = \hat{\Delta}_R - \frac{1}{r^2}\hat{l}^2 \tag{51}$$

with $l^2$ given by

$$\hat{l}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{52}$$

[HW: (optional) check this using Calculus‘VectorAnalysis‘) Making a substitution

$$x = \cos\theta \ , \ \ \sin^2\theta = 1 - x^2$$

one gets an equation

$$\hat{l}^2 = -\frac{\partial}{\partial x}\left[\left(1 - x^2\right)\frac{\partial}{\partial x}\right] - \frac{1}{1 - x^2}\frac{\partial^2}{\partial \phi^2} \tag{53}$$

which leads for LP in $x$ (or in $\cos\theta$) if no $\phi$-dependence.

Another relation

$$\hat{\Delta} r^n P_n(\cos\theta) = 0 \text{ for any } n \tag{54}$$

[HW (optional): check this using Calculus‘VectorAnalysis‘] I.e. in problems with no dependence on $\phi$ an expansion in $r^n P_n(\cos\theta)$ with $-\infty < n < \infty$ is possible. We will use it in the problem of a conducting sphere in a uniform field.

### 4. *The other solution*

Consider now the differential equation for the LP - see Table. There are singular points at $x = \pm 1$. The LP are solutions of the 1st kind, which are regular. The other solutions are the Legendre function of the 2d kind, $Q_n(x)$. They are *not* polynomials (why?), and are given by LegendreQ in Mathematica. Both types of solutions are shown in Fig. 4. The functions $Q_n(x)$ have logarithmic singularities, e.g.

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x} \ , \ \ Q_1(x) = \frac{1}{2}x\ln\frac{1+x}{1-x} - 1 \ , \ldots$$

[HW (optional): using Mathematica check for some $n$, or in general, that $Q_n$ satisfy the same recurrence relation as $P_n$.]

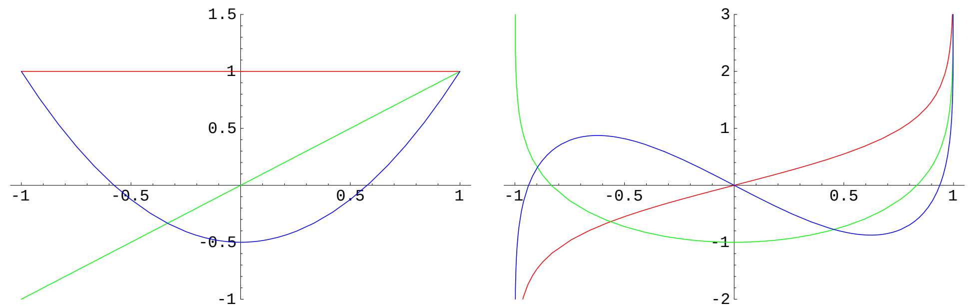


FIG. 4: Legendre polynomials $P_n(x)$ (left) and Legendre functions of the 2d kind, $Q_n(x)$ (right) for $n = 0, 1, 2$

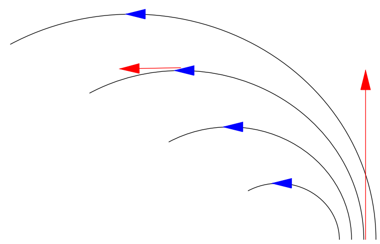

FIG. 5: Example of vector field lines. At each point the direction of vector field is tangent to the line. The magnitude of the vector field at a given point is proportional to the density of lines. Such pictures are useful since the field lines, as a rule, do not start or end, except for very special points, electric charges (and magnetic field lines of $\vec{B}$ *never* start or end).

*Dr. Vitaly A. Shneidman, Phys 621, 4th Lecture*

# Part II

# Overview of fields

*READING: Jack. Ch.1,2*

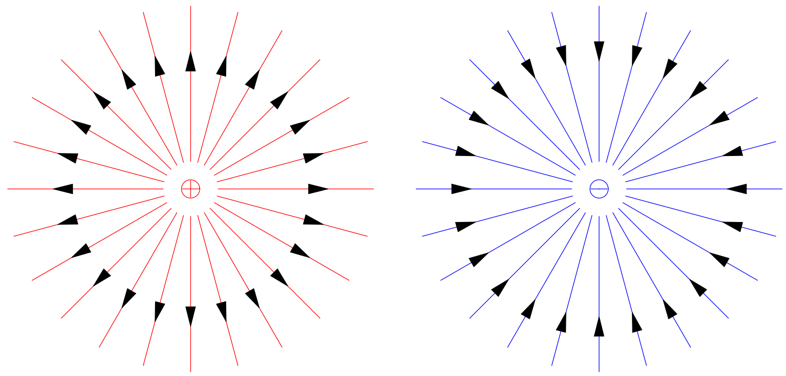

FIG. 6: Electric field lines due to single point charges. Lines point to a singularity when charge is approached.

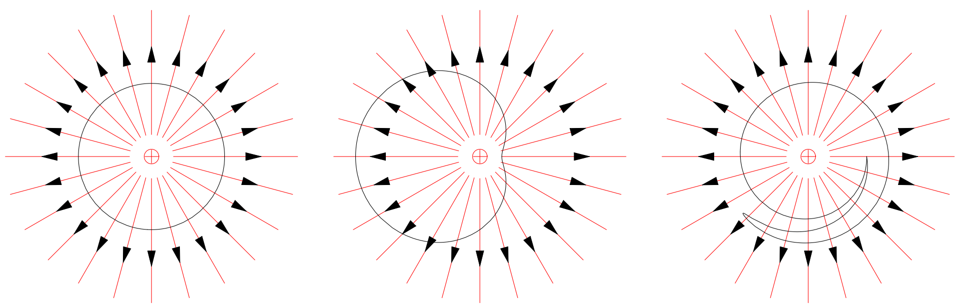

FIG. 7: Geometric meaning of Gauss theorem. The total number of lines which cross the surface (with account for sign) does not change as long as the charge remains inside.

## V. FIELD LINES; SOURCES AND CURLES; CHARGE

### A. Field lines

#### *1. Geometric meaning of the Gauss theorem*

### B. Intensity of sources and curles

- Intensity of a source - flux
- Intensity of a curl - circulation

Note that for localized source (curl) the flux (circulation) does not depend on location of the surrounding surface (contour).

Examples: charge - source of $\vec{E}$; current - curl of $\vec{B}$ (and $\partial\vec{B}/\partial t$ is a curl of $\vec{E}$.

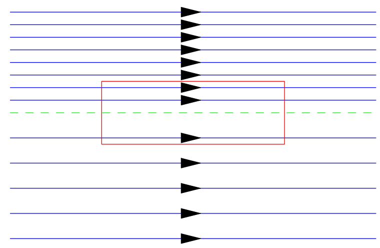

FIG. 8: A surface curl. A field (e.g $\vec{B}$ - not $\vec{E}$) with such structure is possible, but requires a surface current which goes into the page along the green (dashed) line.

### C. Force on a charge

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{55}$$

and

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{56}$$

determines dynamics (Why not just $m\vec{a}$?)

This is "experimental" definition of $\vec{E}$ and $\vec{B}$.

Electron:

$$e = -1.6 \cdot 10^{-19}\,C \tag{57}$$

(not that small!)

Difficulties with point charges (will be discussed in class).

## VI. MAXWELL EQUATIONS IN FREE SPACE

$$\oint_{surface} \vec{E} \cdot d\,\vec{a} = q_{enc}/\epsilon_0 \tag{58}$$

$$\oint_{surface} \vec{B} \cdot d\,\vec{a} = 0 \tag{59}$$

$$\oint_{line} \vec{E} \cdot d\,\vec{s} = -\frac{d\,\Phi_B}{d\,t} \tag{60}$$

$$\oint_{line} \vec{B} \cdot d\,\vec{s} = \mu_0 i_{enc} + \frac{1}{c^2}\frac{d\,\Phi_E}{d\,t} \tag{61}$$

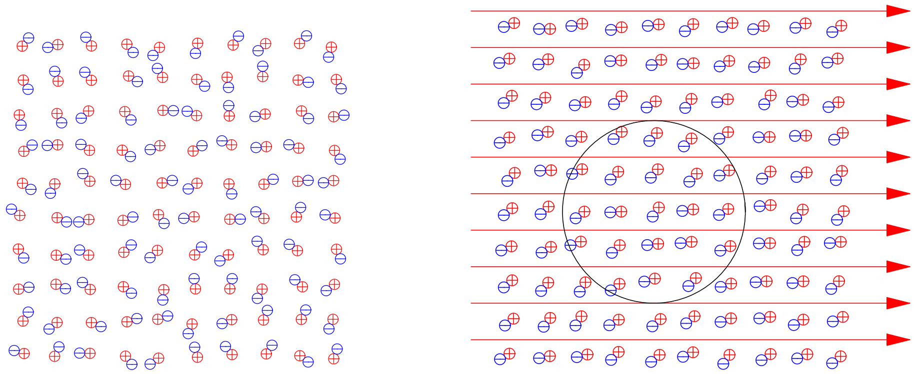

FIG. 9: Electric properties of a dielectric. The left figure shows (schematically) a dielectric of polar molecules. Molecules are randomly oriented, as in a liquid or gas. When an external field is applied (right figure), molecules tend to orient along the field. Any macroscopic inner volume (as the sphere shown in the picture) will still contain, on average, zero charge. Nevertheless, the left-hand side of the dielectric gets a negative, and the right-hand side a positive surface charges. Those create additional, *polarization* field (not shown in the picture) which points left, effectively reducing the field $\vec{E}$ inside the dielectric. Field $\vec{D}$, on the o.h., is not affected by polarization charges and will remain the same inside and outside of the dielectric, as long as it is normal to the surface.

with $c = 1/\sqrt{\mu_0\epsilon_0} \simeq 3 \cdot 10^8\ m/s$, the speed of light.

**HW**: *Make sure this is equivalent to differential form in* **JACK.** *, p.2 with* $\vec{D} = \epsilon_0\vec{E}$ *and* $\vec{H} = \vec{B}/\mu_0$.

## VII. VECTORS $\vec{D}$ AND $\vec{H}$

Why need additional fields? $\vec{E}$ and $\vec{B}$ are good, but they are sensitive to *all* charges and currents, including microscopic.

### A. Electric displacement, $\vec{D}$

The physical reality is described by the field $\vec{E}$ which determines the force on any charge, $\vec{F} = q\vec{E}$. But this field depends on all charges, including the charges which make up the matterial or appear on its surface. (such charges are often called "bound", and will be distinguished by an index "b"). Thus one attempts to introduce $\vec{D}$ which depens only on *free* charges and which in some sense is simpler. There is an issue of course how $\vec{D}$ and $\vec{E}$ are related; this will be a property of material. We consider the simplest case of an isotropic dielectric with a dielectric constant $\kappa$ and with linear relation

$$\vec{D} = \kappa\epsilon_0\vec{E} \tag{62}$$

(note that in SI units dimensions of $D$ and $E$ are different since $\epsilon_0$ is included). The "Gauss theorem" will now read

$$\oint \vec{D} \cdot d\,\vec{A} = q_f \tag{63}$$

where only free charges are counted.

The simplest example of the power of $\vec{D}$: Consider a parallel-plate capacitor with charge density $\pm\sigma$, which is disconnected from a battery and filled with several dilectric plates with $\kappa_1\,, \kappa_2\,, \ldots$. Find $\vec{E}$ in each plate.
*Solution.* First find

$$D = \epsilon_0\frac{\sigma}{\epsilon_0} = \sigma \tag{64}$$

(dielectrics absolutely do not matter!). Then, inside each dielectric

$$E_i = D/\kappa_i\epsilon_0 \tag{65}$$

(In a general case when $\vec{D}$ is not normal to the surface the dielectric will matter- see below, but the above was a good illusration).

### B. refraction of $\vec{D}$ near the surface of a dielectric

There will be polarization, bound charges on the surface of a dielectric. Nevertheless, due to Gauss theorem for $\vec{D}$,

$$D_n^I = D_n^{II} \tag{66}$$

i.e. the normal component does not change when one goes from dielectric $I$ to dielectric $II$. (Note: **JACK.** also adds possible $\sigma$, the surface density of free charges, when $D_n$ will change. We do not need it here). The tangential component, however, does change since $E^I_{||} = E^{II}_{||}$ (absence of circulation) and

$$D^I_{||} = \kappa_1 \epsilon_0 E^I_{||} \ , \quad D^{II}_{||} = \kappa_2 \epsilon_0 E^{II}_{||} \ , \tag{67}$$

Since (unlike $E$) the field lines of $D$ do not vanish (or appear) on the surface of a dielectric) the picture will look like a refraction.

### C. Comparing $\vec{D}$ and $\vec{E}$

Here we consider electrostatics only. Then

- Sources:
  $\vec{E}$ all charges
  $\vec{D}$ free charges

- Curles:
  $\vec{E}$ none (i.e. curl-free field)
  $\vec{D}$ - surfaces of a dielectric (discontinuity of the tangential component will lead to non-zero circulation)

- Potential:
  $\vec{E} = -\hat{\nabla}\Phi$ (Yes!)
  $\vec{D}$ - no potential (Why?)

#### 1. *"Measuring" $\vec{D}$ and $\vec{E}$ inside a dielectric*

Can measure only in vacuum. Thus need a cavity with field approximately equal to field in the dielectric.
For $\vec{E}$ - long, needle-like cavity in the direction of field. (Why?)
For $\vec{D}$ - short, coin-like cavity (Why?)

### D. $\vec{H}$

Nature of surface current. (in class)

Importance of $\vec{H}$ for the Lab (in class).

Connection to $\vec{B}$:

$$\vec{B} = \mu_0 \hat{\mu} \cdot \vec{H} \tag{68}$$

(usually written for Fourier components)

Boundary conditions at interfaces:

$$\vec{n} \times \left( H^{II} - H^{I} \right) = \vec{K}_{macro} \tag{69}$$

Here $\vec{K}_{macro}$ is the macroscopic surface current, and if there is none tangential components of $\vec{H}$ are continuos.

On the o.h., $B_n^I = B_n^{II}$ and $H_n$ will have a discontinuity.

### E. Maxwell equations in media

In eq. (58)

$$\vec{E} \, \rightarrow \, \vec{D}/\epsilon_0 \;, \quad q \, \rightarrow \, \int \rho_{free} \, dV$$

in eq. (61)

$$\vec{B} \, \rightarrow \, \mu_0 \vec{H} \;, \quad i \, \rightarrow \, \int \vec{J}_{macro} \cdot d\vec{a}$$

with $\vec{J}$ being current density. (In practice, differential form is more convenient). Note that coninuity of $\vec{J}$ is satisfied automatically:

$$\frac{\partial}{\partial t} \rho + \hat{\nabla} \cdot \vec{J} = 0 \tag{70}$$

To close equations need $\vec{D}\left(\vec{E}\right)$ (and possibly of $\vec{B}$) and $\vec{H}\left(\vec{E}\,,\,\vec{B}\right)$ - depends on material. Also, need $\vec{J}\left(\vec{E}\,,\,\vec{B}\right)$ (Ohm's law").

### F. Summary: sources and curles of $\vec{E}$, $\vec{D}$, $\vec{B}$, $\vec{H}$

| Field | Sources | Curls | Normal comp. at interface | Tang. comp. at interface |
|---|---|---|---|---|
| $\vec{E}$ | all charges | $\frac{\partial}{\partial t}\vec{B}(t)$ only (i.e. NONE in statics) | discontinuos | continuous |
| $\vec{D}$ | free charges | interfaces of dielectrics (non-normal approach) + time-dependent magnetic field | continuos for dielectrics (unless free charge deposited); discontinuous for charged conductors | discontinuos |
| $\vec{B}$ | NONE | all currents + time-dependent electric field | continuous | discontinuous |
| $\vec{H}$ | interfaces of magnetics (non-tangential approach) | macrocurrents + $\frac{\partial}{\partial t}\vec{D}(t)$ ("displacement current") | discontinuos | continuos (unless macro current on surface) |

### G. Beyond continuos medium

X-rays (in class).

## VIII. THE WAVE

Consider Maxwell equations in free space (no charges or currents). The have wave-like solutions. E.g. a plane wave:

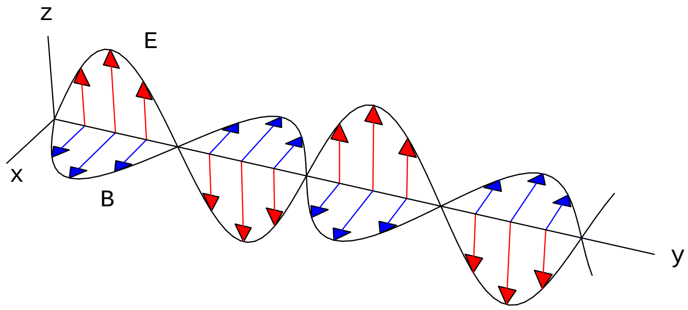


FIG. 10:

*Dr. Vitaly A. Shneidman, Phys 621, 5th Lecture*

# Part III

# Electrostatics

## IX. FREE CHARGES

**work-through: JACK. . pp.24-41; p.103**

### A. Field

Coulomb's law:

$$\vec{F}_{12} = k\frac{q_1 q_2}{r^3}\vec{r}_{12} \ , \ \ k = \frac{1}{4\pi\epsilon_0} \simeq 9\,10^9 \ n\,m^2/C^2 \tag{71}$$

with $r = |\vec{r}_{12}|$.

Field:

$$\vec{E} = k\frac{q}{r^3}\vec{r}_{12} \tag{72}$$

Several charges:

$$\vec{E}\,(\vec{r}) = k\sum_i \frac{q_i}{R_i^3}\vec{R}_i \ , \ \ \vec{R}_i \equiv \vec{r} - \vec{r}_i \tag{73}$$

Continuous distribution:

$$\vec{E}(\vec{r}) = k \int d\vec{r}' \rho(\vec{r}') \frac{\vec{R}}{R^3}, \quad \vec{R} \equiv \vec{r} - \vec{r}' \tag{74}$$

**HW**: *Consider a rectangular plate $a \times b$ in the $x, y$ plane (with 0 at the center) charged with surface density $\sigma$. Find the field on the $z$-axes. Explore all three (!) limits.*

## B. Potential

Potential $\Phi$ (we will not use "$\Phi$" for flux anymore to comply with **JACK.** ):

$$\Phi(\vec{r}) = k \int d\vec{r}' \rho(\vec{r}') \frac{1}{R} \tag{75}$$

**HW**: *Find explicitly (a) the electric field and (b) the potential in the plane of a thin uniformly charged ring both inside and outside of the ring. (Note, you might have to use special functions, such as elliptic integrals).*

### 1. *Surface charge and dipole layer*

Surface charge:

$$d\vec{r}'\rho \rightarrow da\,\sigma \tag{76}$$

Dipole layer: consider close surface layers $\pm\sigma$ separated by $d$ with

$$\sigma(\vec{r})\, d(\vec{r}) = D(\vec{r})$$

Then,

$$\vec{R}'' = \vec{R}' + \vec{n}d$$

and

$$\vec{R}'' = \frac{1}{\vec{R}'} + \vec{n} \cdot \hat{\nabla}' \frac{1}{\vec{R}'}$$

**HW**: *show that*

Note that $-\hat{\nabla}' \frac{1}{\vec{R}'}$ is $\propto$ to the field of point charge and when multiplied by $\vec{n}$ leads to a spherical angle (compare to Gauss). Thus,

$$\Phi(\vec{r}) = k \int da' D(\vec{r}')\, \vec{n} \cdot \frac{\vec{R}'}{R^3} = -k \int da' D(\vec{r}')\, d\Omega' \tag{77}$$

### C. Poisson and Laplace equations

Poisson:

$$\hat{\nabla}^2\Phi = -\rho/\epsilon_0 \tag{78}$$

Laplace ($\rho = 0$)

$$\hat{\nabla}^2\Phi = 0 \tag{79}$$

**HW**: *show that eq. (75) indeed complies with the Poisson equation. Use*

$$\hat{\nabla}^2\frac{1}{R} = -4\pi\delta\left(\vec{R}\right) \ , \ \ \vec{R} \equiv \vec{r} - \vec{r}' \tag{80}$$

*and note that $\vec{r}\prime$ is unaffected by Laplacian. See also* **JACK.** *p. 35.*

### D. Green's theorem

Start with Gauss:

$$\int d\vec{r}\,\hat{\nabla}\cdot\vec{A} = \oint da\,\vec{n}\cdot\vec{A}$$

use

$$\vec{A} = \phi\hat{\nabla}\psi - \psi\hat{\nabla}\phi$$

Note:

$$\vec{n}\cdot\hat{\nabla}\phi = \partial\phi/\partial n$$

Thus,

$$\int d\vec{r}\left(\phi\hat{\nabla}^2\psi - \psi\hat{\nabla}^2\phi\right) = \oint da\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) \tag{81}$$

**HW**: *Restore all the steps. See also* **JACK.** *p. 36*

Next: select

$$\psi = \frac{1}{R}\ , \ \ \phi = \Phi$$

and switch to $\vec{r}'$ in integration. Using eq. (80) one obtains for $\vec{r}$ inside the volume:

$$\Phi\left(\vec{r}\right) = k\int d\vec{r}'\frac{\rho\left(r'\right)}{R} + \frac{1}{4\pi}\oint da'\left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] \tag{82}$$

### E. Uniqueness of the solution

Assume 2 different solutions exist and construct

$$U = \phi_2 - \Phi_1$$

Note

$$\hat{\nabla}^2 U = 0$$

even if thre are charges. Then, in Gauss theorem (GT) select

$$\vec{A} = U \hat{\nabla} U$$

with

$$\hat{\nabla} \vec{A} = U \hat{\nabla}^2 U + \left(\hat{\nabla} U\right)^2$$

Thus, from GT

$$\int dV \left(\hat{\nabla} U\right)^2 = \oint da\, U \frac{\partial U}{\partial n}$$

which is 0 for "good" boundary conditions.

### F. Formal properties of the Green's function

$$\hat{\nabla}^2 G\left(\vec{r}, \vec{r}\,'\right) = -4\pi \delta \vec{r} - \vec{r}\,' \tag{83}$$

$$G = \frac{1}{R} + F\left(\vec{r}, \vec{r}\,'\right) \tag{84}$$

Start with Green's theorem, eq. (81) and define

$$\phi = \Phi\ , \quad \psi = G$$

(rest in class)

### G. Energy

(in class - no conductors and capacitors yet)

## X. MULTIPOLE EXPANSION

### A. Notations

We consider $N$ charges $q_i$ with coordinates $\vec{a}_i$. Components of a vector are indicated by upper Greek indexes and summation of repeated indexes from 1 to 3 is implied, e.g.

$$a_i^2 = \vec{a}_i \cdot \vec{a}_i = a_i^\alpha a_i^\alpha \tag{85}$$

(summation over $i$ from 1 till $N$ will be indicated explicitly).

$\vec{r}$ denotes the position of the obsevation point and $\hat{r}$ is a unit vector with components $\hat{r}^\alpha$. The Kronecker delta symbol $\delta^{\alpha\beta}$ will be used, e.g.

$$a_i^2 = a_i^\alpha a_i^\beta \delta^{\alpha\beta} \tag{86}$$

For a continuous charge distribution with density $\rho$ one would use

$$\sum_i q_i \left(\ldots\right) \;\rightarrow\; \int d^3\vec{r}' \rho\left(\vec{r}'\right)\left(\ldots\right) \tag{87}$$

Derivatives: let us derive the familiar $\nabla r = \hat{r}$

$$\frac{\partial}{\partial r^\beta}\left(r^\alpha r^\alpha\right) = \frac{1}{2\left(r^\alpha r^\alpha\right)^{1/2}} \frac{\partial}{\partial r^\beta} r^\gamma r^\gamma = \frac{r^\beta}{r} = \hat{r}^\beta \tag{88}$$

### B. Expansion

$$V\left(\vec{r}\right) = \frac{k}{r}\sum_i \frac{q_i}{\sqrt{1 + a_i^2/r^2 - 2\hat{r}\cdot\vec{a}_i/r}} = \tag{89}$$

$$\frac{k}{r}\sum_i q_i \left\{1 + \frac{\hat{r}\cdot\vec{a}_i}{r} + \left(-a_i^2/2 + \frac{3}{2}(\hat{r}\cdot\vec{a}_i)^2\right)/r^2 + \ldots\right\} =$$

$$= \frac{k}{r}\left\{Q + \hat{r}\cdot\vec{p}/r + Q^{\alpha\beta}\hat{r}^\alpha\hat{r}^\beta/r^2 + \ldots\right\}$$

with

$$Q = \sum_i q_i \;, \quad \vec{p} = \sum_i q_i \vec{a}_i \;, \quad Q^{\alpha\beta} = \sum_i q_i \left[\frac{3}{2}a_i^\alpha a_i^\beta - \frac{1}{2}a_i^2\delta^{\alpha\beta}\right] \;, \quad \ldots \tag{90}$$

Here we used

$$(\hat{r}\cdot\vec{a}_i)^2 = \hat{r}^\alpha a_i^\alpha \hat{r}^\beta a_i^\beta \tag{91}$$

and that $\vec{r}$ and its components do not depend on $i$.

### C. Comments

- The leading term is unique; the other terms are not and depend on the selection of origin of coordinates.

- the objects which appear in the expansion, $Q^{\alpha\beta\cdots}$ are known as *tensors* and they give scalars once convoluted (i.e. summation is performed over repeated indices) with components $\hat{r}^\alpha \hat{r}^\beta \ldots$. The number of indices is it rank of a tensor. Rank 1 corresponds to a vector and rank 0 to a scalar.

- Each term in the expansion has a physical meaning: the $1/r$ corresponds to a point charge; the $1/r^2$ to potential at a large distance from a dipole. The next term is field of a "quadrupole" - 4 identical alternating charges in the corners of a square, also far away. Next comes the "octopole" - alternating charges at the verteces of a cube, etc.

- in view of above, each term satisfies the Laplace equation. Thus, often one can "guess" the potential by using superposition of fields due to several multipoles if those satisfy the boundary conditions. This is a "physicists way" to solve the Laplace equation, useful if one has a good intuition (and quite rigorous due to the uniquiness theorem). Favorite examples - sphere (conducting or dielectric) in a uniform field. [There can be of course other terms with similar structure which do not decay as $r \to \infty$ and which do not appear in the multipole expansion; they are also part of the "guessing game". Best known example - the potential $-\vec{E}_0 \cdot \vec{r}$ which corresponds to a uniform field $\vec{E} = \vec{E}_0$]

- Gravitational analogy: as usual $q_i \to m_i$ and $k \to -G$, the gravitational constant. Since masses are positive, there will always be the $1/r$ term (Newton potential). The dipole term will vanish if the origin is selected at the center of mass. Higher order terms are small, and below quadrupole are known only for Earth.

### D. Dipole

It is more convenient (for future diferentiation) to use $\vec{r}$ rather than $\hat{r}$ and use summation over repeated indices:

$$V_{dip} = \frac{k}{r^3} p^\alpha r^\alpha \tag{92}$$

Note that this is field of an "infinitesimal" dipole with a finite dipole moment: consider a real dipole with charge $\pm q$ separated by a distanse $d$ and think of a limit $d \to 0\,,\, q \to \infty$ with $qd = p = const.$

Field of a real dipole is different, of course. If one uses the miltipole expansion for a real diple, the leading term will be $V_{dip}$. If the origin of coordinates is selected at the midpoint, the quadrupole term will be zero (show this!). There will be an octopole term, however.

Now let us calculate the field which corresponds to $V_{dip}$:

$$E^\beta = -k\frac{\partial}{\partial r^\beta}\left[\frac{p^\alpha r^\alpha}{(r^\gamma r^\gamma)^{3/2}}\right] = -k\frac{p^\beta}{r^3} + k\frac{3}{2}\frac{p^\alpha r^\alpha}{r^{5/2}} 2r^\beta = \frac{k}{r^3}\left(-p^\beta + 3(p^\alpha \hat{r}^\alpha)\hat{r}^\beta\right) \tag{93}$$

or in vector form

$$\vec{E} = \frac{k}{r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right) \tag{94}$$

In the direction perpendicular to $\vec{p}$ this gives

$$\vec{E} = -\frac{k}{r^3}\vec{p} \tag{95}$$

This is consistent with the elementary formula for a *real* dipole

$$E_{dip} = \frac{kqd}{L^3} \tag{96}$$

since $L$, the distance from both charges, tends to $r$ as $d \to 0$.

<u>Force on a dipole in non-uniform field</u>

Let $\vec{d}$ go from $-q$ to $+q$ (i.e $\vec{p} = q\vec{d}$) and $+q$ is at $\vec{r}$. Then

$$\vec{F} = q\vec{E}\left(\vec{r}\right) - q\vec{E}\left(\vec{r} - \vec{d}\right) = q\left(\vec{d}\cdot\nabla\right)\vec{E} \tag{97}$$

or

$$\vec{F} = \left(\vec{p}\cdot\nabla\right)\vec{E} \tag{98}$$

With indices one has:

$$q\vec{E}\left(\vec{r}-\vec{d}\right) \simeq q\vec{E}\left(\vec{r}\right) - qd^{\alpha}\frac{\partial}{\partial r^{\alpha}}\vec{E}\left(\vec{r}\right) = \tag{99}$$

(note summation over $\alpha$), which gives the same expression.

### E. Multipole expansions for fields with asimuthal symmetry

Any such field can be expanded in Legendre polynomials

$$\Phi\left(r,\theta\right) = \frac{k}{r}\sum_{n=0}^{\infty} B_n \left(\frac{1}{r}\right)^n P_n\left(\cos\theta\right) \tag{100}$$

(assuming the sum converges for sufficiently large $r$).

Where to get $B_n$? Note on the axes $(z = r)$

$$\Phi\left(z=r\right) = \frac{k}{r}\sum_{n=0}^{\infty} B_n \left(\frac{1}{z}\right)^n \tag{101}$$

Thus, if know $\Phi\left(z=r\right)$, which is often simple - see below-, exapnd it for large $z$ and get $B_n$ (!).

Example: ring of radius $a$

$$\Phi\left(z=r\right) = kQ/\sqrt{z^2+a^2}$$

$$\Phi\left(z=r\right) = \frac{kQ}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{2n} P_{2n}\left(0\right)$$

(note these Legendre polynomials appear "by chance" ). Now,

$$\Phi\left(r,\theta\right) = \frac{kQ}{r}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{2n} P_{2n}\left(0\right) P_{2n}\left(\cos\theta\right) \tag{102}$$

see Fig. 11 and LegExpand.nb (also p. 103 in **JACK.** ).

A similar expansion can be done for small $z$ (positive powers) - see LegExpand.nb

Example: physical dipole with $kq = \pm 1$ and $d = \pm 1/2$.

$$\Phi(z) = \frac{1}{z^2 - 1/4} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n (2z)^{-2n}$$

Thus,

$$\Phi\left(r,\theta\right) = \frac{1}{r}\sum_{n=0}^{\infty}(-1)^n (2r)^{-2n} P_n\left(\cos\theta\right) \tag{103}$$

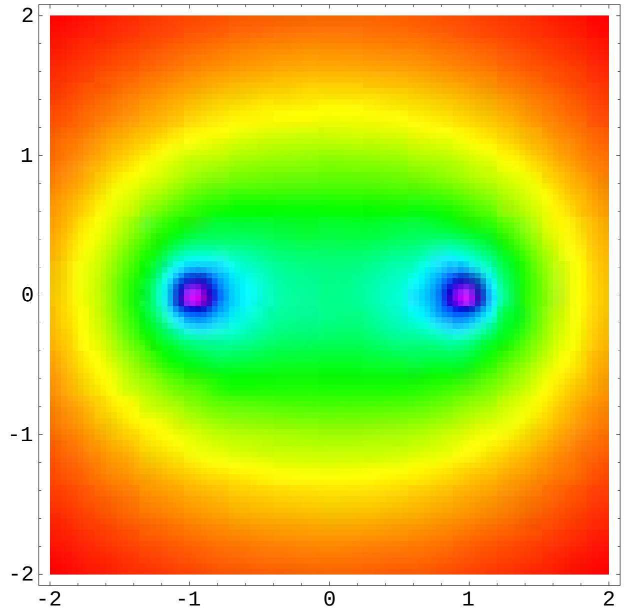


FIG. 11: Potential of a charged ring in the $z, y$ plane ($z$ is the axis of the ring) obtained from the large- and small-$r$ expansions with Legendre polynomials.

*Dr. Vitaly A. Shneidman, Phys 621, 6th Lecture*

## XI. CONDUCTORS

### A. Elementary properties of a conductor in electrostatics

Field:

- inside: electric field is zero (otherwise - current!)
- outside: field lines approach the surface at $90^o$ (otherwise - surface current).

Charge:

- inside: no free charge (follows from $\vec{E} = 0$ and the GT)
- surface: there can be charges only on *external surface* (from GT)

$$E_n = \sigma/\epsilon_0 \tag{104}$$

Potential:

- inside + surface: constant
- (by convention) zero at infinity or at a grounded conductor

Force on a unit surface (note 1/2 !):

$$F_n = \frac{1}{2} E_n \sigma = \frac{\sigma^2}{2\epsilon_0} \tag{105}$$

Capacitance for 1 or 2 conductors:

$$Q = CV \tag{106}$$

(see **JACK.** p. 43 for more than 2 - will not need).

Plotting field lines - see Appendix B.

## B. Charge near a conducting sphere

### *1. Method of images*

**work-through: JACK. Sect. 2.2**

$$q' = -\frac{a}{r} q \ , \ r' = \frac{a^2}{r} \tag{107}$$

Can prove from geometry - see **JACK.** or from Legendre polynomials (both charge and image are on the $z$-axes) (will be discussed in class)

### *2. Surface chage density*

Introduce

$$\vec{R} = \vec{r} - \vec{a} \ , \quad \vec{R}' = \vec{r}' - \vec{a} = \frac{a^2}{r^2} \vec{r} - \vec{a}$$

with

$$\frac{R'}{R} = -\frac{q}{q'} = \frac{r}{a}$$

Then, the field near the surface

$$\vec{E} = kq \frac{\vec{R}}{R^3} + kq' \frac{\vec{R}'}{R'^3} = \tag{108}$$

$$kq\frac{1}{R^3}\left[\vec{R}-\frac{r^2}{a^2}\vec{R}'\right]=$$
$$kq\frac{1}{R^3}\left[\vec{a}\left(\frac{r^2}{a^2}-1\right)\right]$$

**HW**: *identify with eq.(2.5) in* **JACK.** Note that the field is in direction of $\vec{a}$, i.e. automatically normal to the surface.

*3. Non-zero potential or charge on the sphere*

in class

*READING:* **JACK.** *Sects. 2.3, 2.4*

*4. Inversion properties of Laplace equation*

briefly - not in **JACK.** .

$$\hat{\nabla}^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{1}{r^2}\hat{l}^2(\theta,\,\phi) \tag{109}$$

$$r\;\rightarrow\;a/r'\;,\;\;\Phi\;\rightarrow\;r'/a\,\Phi' \tag{110}$$

**HW**: *(optional) try to show that*

*5. The Green's function*

**work-through: JACK. , Sec. 2.6**

*READING:* **JACK.** *, Sec. 2.7*

$$q\;\rightarrow\;4\pi\epsilon_0=1/k$$

Introduce

$$\vec{R}=\vec{\rho}-\vec{a}\;,\;\;\vec{R}'=\vec{\rho}'-\vec{a}=\frac{a^2}{\rho^2}\vec{\rho}-\vec{a}$$

with

$$R^2=\rho^2+\rho'^2-2\rho\rho'\cos\gamma$$

and

$$R'^2 = \frac{a^4}{\rho^2} + \rho'^2 - 2\frac{a^2}{\rho}\rho' \cos\gamma$$

Then:

$$G(\vec{\rho},\ \vec{\rho}') = \frac{1}{R} - \frac{a}{\rho}\frac{1}{R'} \tag{111}$$

same as eq.(2.17) in **JACK.** ( **HW**: *identify that* ).

Normal derivative at the surface (same as when calculating electric field):

$$\frac{\partial}{\partial n'} G(\vec{\rho},\ \vec{\rho}')\bigg|_{\rho'=a} = \frac{\rho^2 - a^2}{a R_s^3} \tag{112}$$

with $R_s$ the distance from the surface:

$$R_s^2 = \rho^2 + a^2 - 2a\rho\cos\gamma$$

Thus, if potential is known at the surface (and there are no charges)

$$\Phi(\vec{\rho}) = -\frac{1}{4\pi}\oint \Phi_s \frac{\partial G}{\partial n}\, d\Omega\, a^2 \tag{113}$$

with

$$\cos\gamma = cos\theta\cos\theta' + sin\theta\sin\theta'\cos(\phi - \phi') \tag{114}$$

**HW**: *prove that*

Actual calculations, however, can be very cumbersome.

## C. Conducting sphere in uniform field $\vec{E}_0$

### *1. Method of images*

*READING:* **JACK.** *, sect. 2.5*

**HW**: *get the result a la* **JACK.** *, but with only one charge at infinity*

### *2. With Legendre polynomials*

$$r^n P_n(\cos\theta)$$

is the solution of the Laplce equation for any $n$. From the recurrence relations

$$P_{-n}(x) = P_{n-1}(x)$$

Thus

$$V(r,\theta) = \sum_{n=0}^{\infty} \left\{ a_n r^n P_n(\cos\theta) + b_n \frac{P_n(\cos\theta)}{r^{n+1}} \right\} \tag{115}$$

Boundary condition (1st one):

$$V(r \to \pm\infty) = \mp E_0 r \cos\theta$$

Thus,

$$a_1 = -E_0 \ , \quad a_n = 0 \text{ for } n \neq 1$$

2d BC:

$$V(r = R) = 0$$

(uncharged sphere). Thus

$$b_0/R + \left(\frac{b_1}{R^2} + a_1 R\right) P_1(\cos\theta) + \sum_{n=2}^{\infty} b_n \frac{(\cos\theta)}{r^{n+1}} = 0$$

Thus,

$$b_1 = -a_1 R^3 = E_0 R^3 \ , \quad b_n = 0 \text{ for } n \neq 1$$

or

$$V(r,\theta) = -E_0 r P_1(\cos\theta)\left(1 - (R/r)^3\right)$$

Charged sphere: $V(R) = Q/R$ (CGS). Thus, $b_0 = Q$ and $Q/r$ is added to previous solution at $r \geq R$ (superposition!)

Plotting of lines is in condSphere.m. The result is in Fig. 12

(this is approximately the end of class 6)

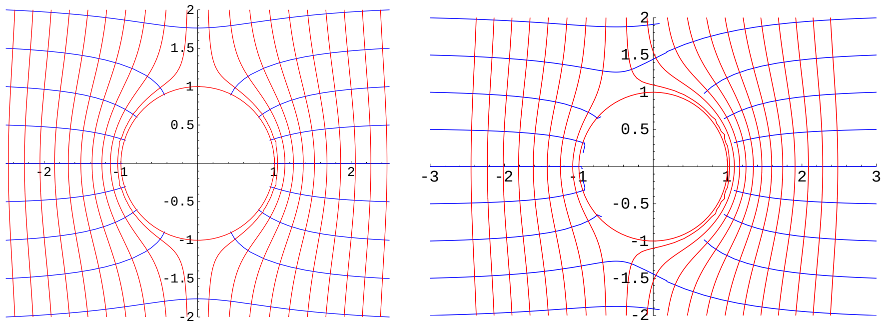


FIG. 12: Left: Electric field lines (blue) and equipotential surfaces (red) for a conducting sphere in a uniform field. Right: same, when the sphere is charged wit a positive charge

*Dr. Vitaly A. Shneidman, Phys 621, 7th Lecture*

## XII. LAPLACE EQUATION IN A SEMI-INFINITE STRIPE

Consider the following 2D problem:
two parallel conducting planes are separated by a distance $a = 1$ in the $x$-direction and are at zero potential. Between the planes there is a semi-infinite conducting slab with thickness $a - 0$ which down extends from $y = 0$ and which is at potential $V$. Find the distribution of potential in the stripe $0 < x < a$ and $0 < y < \infty$.

This is a rich problem which will allow us to

- consider the solution via separation of variables
- examin singularities near the corners and discuss the dimensional analysis
- briefly consider relation to the theory of analytic functions (complex variables) where the problem does have a closed solution (though, lets forget about it for the moment
- introduce numerical methods of solving the Laplace equation

### A. Separation of variables

Since we have rectangular geometry, natural to consider Cartesian coordinates. Look for a solution

$$V(x, y) = \sum_k a_k X_k(x) Y_k(y) \tag{116}$$

(can do this because the Laplace equation is linear). Consider now a term in the above sum (drop the $k$-index for the moment)

$$\hat{\Delta}\left(X(x)Y(y)\right) = Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

or

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \tag{117}$$

Now

$$Y(y) \propto e^{-\lambda y}$$

thus $\lambda > 0$. Next,

$$X(x) \propto \sin\left(\lambda x\right)$$

and from $X(1) = 0$

$$\lambda_n = n\pi\ , \quad n = 1, 2, \ldots$$

To get coefficients in eq. (116) consider $y = 0$:

$$V = \sum_n a_n \sin\left(\lambda_n x\right)$$

multiply by $\sin\left(\lambda_m x\right)$ and integrate. From orthogonality (HW - check!)

$$a_n = 2V\int_0^1 \sin(n\pi x)dx$$

which is $4V/(\pi n)$ for $n$ odd and 0 for $n$ even (HW - check). Thus,

$$V(x,y) = \frac{4V}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left\{-(2k+1)\pi y\right\}\sin\left\{(2k+1)\pi x\right\} \tag{118}$$

This is a formally exact solution. In practice need to know, how many terms contribute? The extereme is $k = 0$ with

$$V(x,y) \simeq \frac{4V}{\pi}\exp\left\{-\pi y\right\}\sin\left\{\pi x\right\} \quad y \to \infty$$

More general, suppose we fix the desired relative accuracy, $\epsilon$. Then,

$$n \sim \frac{\ln(1/\epsilon)}{\pi y}$$

this is weakly sensitive to $\epsilon$, and because of $\pi$ gives a small $n$ for finite $y$. Thus, for $y \gtrsim 1$ the leading asymptote is an excellent approximation. But what about $y \to 0$? Need an *infinite* number of terms! Any program, including *Mathematica*, will fail. What to do? Rearrange!

$$V(x,y) = V + \frac{4V}{\pi}\sum_k \frac{1}{2k+1}\left[\exp\{-(2k+1)\pi y\} - 1\right]\sin\{(2k+1)\pi x\} \tag{119}$$

(HW - check that the part of the sum with $-1$ indeed cancels $V$). With just a few $k$ this should give a good approximation for small $y$.

**HW**: *write a Fourier series if the stripe is replaced by a finite rectangle with the same zero potential on the upper side as on the vertical sides. Explore the limits of a "tall" and "short" rectangle.*

### B. Edge

Expand eq. (118) to obtain

$$V(x,y) \simeq V - 4Vy\sum_k \sin\{(2k+1)\pi x\} \tag{120}$$

After summation:

$$V(x,y) \simeq V\left[1 - \frac{2y}{\sin(\pi x)}\right], \quad y \to 0 \tag{121}$$

**HW**: *check this (Mathematica is fine)*

### C. Corner

Hard to get from a series; better start again:

$$\Delta V = 0$$

Look for $V = V(r,\theta)$, but no length scale!!!. Thus,

$$V = V(\theta)$$

Now use

$$\Delta V(\theta) = \frac{1}{r^2}\frac{d^2V}{d\theta^2} = 0$$

or (with BC)

$$V(\theta) = \frac{2V}{\pi}\left(\frac{\pi}{2} - \theta\right)$$

In cartesian coordinates (since we use them elsewhere)

$$V(x,y) \sim \frac{2V}{\pi}\left(\frac{\pi}{2} - \arctan\frac{y}{x}\right) = \frac{2V}{\pi}\arctan\frac{x}{y} \tag{122}$$

For small $x$ this is consistent with the edge expression, which is valid however only for $x \gg y$.

## D. Relation to complex variables - 2D only!

This is not in **JACK.** , but allows to solve his problem analytically

### *1. Cauchy-Riemann conditions*

If

$$f(z) = u(x, y) + i\, v(x, y) \tag{123}$$

then for the limit

$$\lim_{\Delta\, z \to 0} \frac{f\,(z + \Delta\, z) - f(z)}{\Delta\, z}$$

to be independent of the direction of $\Delta\, z$:

$$\frac{\partial}{\partial x} u = \frac{\partial}{\partial y} v\ , \quad \frac{\partial}{\partial y} u = -\frac{\partial}{\partial x} v \tag{124}$$

Thus,

$$\Delta u = \Delta v = 0 \tag{125}$$

### *2. Complex potential*

We used

$$\hat{\nabla} \times \vec{E} = 0$$

to introduce $\vec{V} = -\hat{\nabla}\Phi$. Another condition (with no charges)

$$\hat{\nabla} \cdot \vec{E} = 0 \tag{126}$$

Thus, one can introduce $\vec{A}$ with

$$\vec{E} = \hat{\nabla} \times A \tag{127}$$

(Note: In ch. 6 **JACK.** does something similar with magnetic field $\vec{B}$ -for which it is much more meaningful since $\hat{\nabla} \cdot B$ is *always* zero).

For a $3D$ space this $\vec{A}$ is not a big help, but if one has a $2D$ field $\vec{E}(x, y)$ one can select $\vec{A}(x, y)$ in the $z$-directions so that

$$E_x = \frac{\partial}{\partial y} A(x, y) = -\frac{\partial}{\partial x}\Phi(x, y)\ , \quad E_y = -\frac{\partial}{\partial x} A(x, y) = -\frac{\partial}{\partial y}\Phi(x, y)$$

Comapare this with Cauchy-Riemann! In other words,

$$\phi = \Phi - iA \tag{128}$$

is an *analytic* function of $z = x_i y$.

**HW**: *Construct $\vec{A}$ for a uninform field*

**HW**: *Show that $d\phi/dz = -E_x + iE_y$. Note that you can take the derivative in any convenient direction, e,g, $x$ or $y$ since $w$ is analytic.*

Equation for a field line now reads

$$dx/E_x = dy/E_y$$

or

$$dx\frac{\partial}{\partial x}A(x,y) + dy\frac{\partial}{\partial y}A(x,y) \equiv dA = 0 \tag{129}$$

In other words elctric field lines correspond to $-Im[\phi] = const$ and are just as easy to plot as equipotential surfaces $Re[\phi] = const$.

**HW**: *Plot lines of $Re[\phi] = const$ and $-Im[\phi] = const$ for (a) $\phi = z$ and (b) $\phi = -\ln z$. To which electric fields they correspond?)*

*READING: (optional) more general features are in Landau-Lifshits, vol.8, Ch. 1.3 and many practical examples of conformal mapping, both with and without electrostatic context, are in Kreyszig, Advanced Ehngineering Mathematics*

### 3. Conformal mapping and solution of the problem

Returning to our problem we need the following:

- find a function $w(z)$ which mapps the stripe to a simpler shape with "good" boundary conditions (in our case it will be a right angle with potenatial $V$ on one side and 0 on the other) - see Fig. 13.

- solve the problem, obtainig $\phi(w)$ (note: any analytical function $\phi(w)$ will satisfy the Laplace equation, but the proper one will also satisfy boundary conditions. (in our case we already solved the problem for the angle, so expect $\phi \propto \theta = -i\ln(w)$

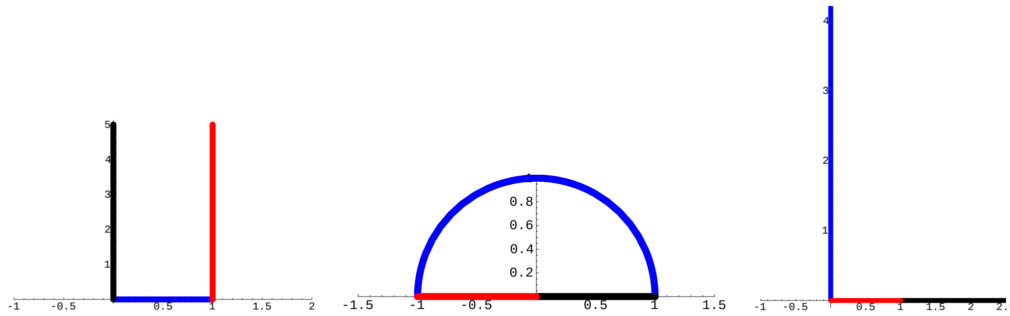

FIG. 13: Transformation from a semi-infite stripe in the $z$-plane to an angle in the $w$-plane via an intermediate semi-circle in the $Z$-plane. Colors track each side of the stripe. The functions are: $Z(z) = e^{i\pi z}$ and $w(Z) = (1+Z)/(1-Z)$. Direct conversion from stripe to corner is achieved by $w\left(Z(z)\right) = \left(1+\exp(i\pi z)\right)/\left(1-\exp(i\pi z)\right) = i\cot(\pi z/2)$ - see 621_stripe.nb

- the function

$$\phi\left(w(z)\right)$$

will be the solution of the problem.

For the corner we have

$$\phi = -\frac{2V}{\pi} i \ln(w) \tag{130}$$

Thus, in original variables

$$\phi = -\frac{2V}{\pi} i \ln\left(i\cot\frac{\pi z}{2}\right) \tag{131}$$

which gives Fig. 14.

(THE REST WILL BE DISCUSSED IN CLASS)

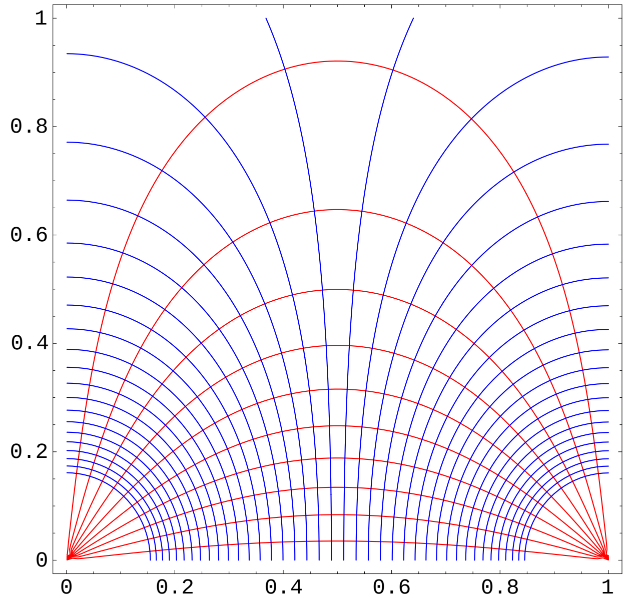


FIG. 14: Equipotential surfaces (red) and field lines (blue) for a stripe (semi-infinite in $y$-direction)

*Dr. Vitaly A. Shneidman, Phys 621, 8th Lecture*

### E. Mixed boundary conditions - thin disc

*READING: (optional)* **JACK.** *. Sect. 3.13. The problem below is technically similar to the one in* **JACK.** *, although somewhat simpler and of more historical interest.*

Use cylindrcial coordiantes with asimuthal symmatry.

$$\hat{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} = 0 \tag{132}$$

look for

$$\Phi(r,z) = R(r)Z(z)$$

Then

$$Z''_{zz}/Z = -\frac{1}{rR}\left(rR'_r\right)'_r \equiv \lambda^2$$

Thus, ($0 \le z \le \infty$):

$$Z \propto e^{-\lambda z}\ , \quad R \propto J_0(\lambda r) \tag{133}$$

and

$$\Phi(r,z) = \int_0^\infty d\lambda\, A(\lambda) J_0(\lambda r) e^{-\lambda z} \tag{134}$$

with boundary conditions

$$\Phi(r,0) = \int_0^\infty d\lambda\, A(\lambda) J_0(\lambda r) \tag{135}$$

(which is $\Phi_0$ for $r < a$).

Use the identity

$$\int_0^\infty d\lambda\, J_0(\lambda r)\frac{\sin(a\lambda)}{\lambda} = \frac{\pi}{2}\ , \quad r \leq a \tag{136}$$

$$\int_0^\infty d\lambda\, J_0(\lambda r)\frac{\sin(a\lambda)}{\lambda} = \arcsin\left(a/r\right)\ , \quad r > a \tag{137}$$

**HW**: *Check this - Mathematica is fine.*

Thus,

$$\Phi(r,z) = \frac{2\Phi_0}{\pi}\int_0^\infty d\lambda\, J_0(\lambda r)\frac{\sin(a\lambda)}{\lambda}e^{-\lambda|z|} \tag{138}$$

(we now include $z < 0$, from symmetry).

### *1. Charge density*

$$\frac{1}{\epsilon_0}\sigma(r) = -\frac{\partial}{\partial z}\Phi(r,z)\bigg|_{z=0} = \frac{2\Phi_0}{\pi}\int_0^\infty d\lambda\, J_0(\lambda r)\sin(a\lambda) = \frac{2\Phi_0}{\pi}\frac{1}{\sqrt{a^2-r^2}}\ , \quad r < a \tag{139}$$

**HW**: *check this and that the above expression is* $0$ *at* $r > a$

### *2. Capacitance*

Charge: because of two sides

$$Q = 2\int_0^a \sigma(r) 2\pi r\, dr = 8a\Phi_0\epsilon_0$$

**HW**: *check this*

Thus,

$$C = Q/\Phi_0 = 8a\epsilon_0 \tag{140}$$

(Cavendish).

For $r = 0$ one has (we consider $a = 1$)

$$\Phi(0, z) = \frac{2\Phi_0}{\pi} \tan^{-1} \frac{1}{z} \tag{141}$$

**HW**: *Check this*

Expanding for large $z$ gives

$$\Phi(0, z) \sim \frac{2\Phi_0}{\pi z}, \quad z \to \infty$$

Note that the leading term in the multipole expansion should correspond to the charge $Q$ (which gives an alternative way to find $C$, without integrating $\sigma(r)$).

To express solution in terms of Legendre polynomials, we use the full expansion of *arctan*:

$$\Phi(0, z) = \frac{2\Phi_0}{\pi} \sum_{k=0}^{\infty} (-1)^k (1/z)^{2k+1} \frac{1}{2k+1} \tag{142}$$

and following the standard routine replace every $(1/z)^{2k+1}$ term by $(1/\rho)^{2k+1} P_{2k} (\cos\theta)$ with $\rho$ and $\theta$ corresponding to spherical coordinates:

$$\Phi(\rho, \theta) = \frac{2\Phi_0}{\pi} \sum_{k=0}^{\infty} (-1)^k (1/\rho)^{2k+1} P_{2k} (\cos\theta) \frac{1}{2k+1} \tag{143}$$

The expansion is accurate and evaluates fast (compared to the Bessel integral) - see 621_disc.nb

## XIII. DIELECTRICS

### A. Polarization: Elementary treatment

Define

$$\vec{P}$$

as dipole moment of a unit volume.

Consider the simplest case of a ||-plate capacitor with area $A$ distance between plates $d$ and with a dielectric plate $\kappa$ (thickness almost $d$) inside. Capacitor is charged to "free" charge $\pm Q_f = \pm\sigma_f A$.

Treat external capacitor as a dipole with

$$p_{ext} = Q_f d = \sigma_f \,\text{volume} \tag{144}$$

Similarly, the dielectric plate also creates an opposite "dipole" due to bound charges:

$$p_b = q_b d = \sigma_b \,\text{volume} \tag{145}$$

Since $p_b = P \times$ volume the latter gives

$$\sigma_b = P \tag{146}$$

This is the first important relation, more generally the surface charge density is defined by the normal component of $\vec{P}$

$$\sigma_b = P_n \tag{147}$$

Now bound charges create a field $E_b = \sigma_b/\epsilon_0$ which is subtracted from the "free" (external) field $E_f = \sigma_f/\epsilon_0 = D/\epsilon_0$ to give the physical field $E$:

$$E = D/\epsilon_0 - \sigma_b/\epsilon_0 = D/\epsilon_0 - P/\epsilon_0 \tag{148}$$

or

$$D = \epsilon_0 E + P \tag{149}$$

Note that $\kappa$ does not enter here, so that this relation is more general than the linear case we discussed. Similarly, it is valid in a more general form.

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \tag{150}$$

#### *1. Linear dielectric*

If $\vec{D} = \kappa\epsilon_0 \vec{E}$, then

$$\vec{E} = \frac{1}{\epsilon_0 (\kappa - 1)} \vec{P} \tag{151}$$

### B. Charge near a flat dielectric

**work-through: JACK. Sec. 4.4, pp. 155-57** In class

**HW**: *Consider a 2D version of the problem: an infinite charged line parallel to a flat surface. Note: use for potential $-\lambda/(2\pi\epsilon_0)\ln r$; otherwise repeat the same steps as in 3D.*

## C. Dielectric sphere

**work-through: JACK. Sec. 4.4, pp. 157-59**

We consider a dielectric sphere with radius $R$ placed in external field $\vec{E}_0$ which points from "south pole" to "north pole" (no physical meaning, just orientation). Strictly speaking, we need to solve the Laplace equation bouth outside and inside the sphere, and match on the boundary. In fact, we will do that but guess a bit. If our guesses are wrong, we will be unable to match solutions on the boundary; if we can match, the solution is uniquely correct.

### 1. *a "physicists solution"*

The solution is determined by a single vector $\vec{E}_0$ and the only way to make a scalar out of it (which is required for the potential and which should be linear in $\vec{E}_0$) is $\vec{r} \cdot \vec{E}_0$.

The only two possibilities which satisfy the Laplace eqaution are $\vec{E}_0 \cdot \vec{r}$ and $E_0 \cdot \vec{r}/r^3$. So the solution must be constructed out of such expressions, which are then matched at the boundaries.

Inside only $\vec{E}_0 \cdot \vec{r}$ can be used (the other is singular at $r = 0$). That means, inside field is uniform and parallel to $\vec{E}_0$

$$\vec{E}_{in} = \frac{1}{\epsilon_0 (\kappa - 1)} \vec{P} \tag{152}$$

The potential is

$$V_{in} = -\vec{E}_{in} \cdot \vec{r} \tag{153}$$

Outside: Both terms contribute (do not contradict conditions at $\infty$). The potential is

$$V_{out} = -\vec{E}_0 \cdot \vec{r} + \frac{1}{4\pi\epsilon_0} \vec{p} \cdot \vec{r}/r^3 \tag{154}$$

with

$$\vec{p} = \frac{4\pi}{3} R^3 \vec{P} \tag{155}$$

the dipole moment of the sphere.

From the continuity of the potential we have (polarization is expressed through $\vec{E}$)

$$-\vec{E}_{in} \cdot \vec{R} = \frac{\kappa - 1}{3} \vec{E}_{in} \cdot \vec{R} - \vec{E}_0 \cdot \vec{R} \tag{156}$$

from here (since $\vec{R}$ can be in arbitrary direction)

$$\vec{E}_1 = \vec{E}_0 \frac{1}{1 + (\kappa - 1)/3} \tag{157}$$

Due to a few guesses, we did not have to use the other boundary condition, but it is useful to check if it is ok. The outside field - uniform plus that of a dipole is

$$\vec{E}_{out} = \vec{E}_0 + \frac{1}{4\pi\epsilon_0 r^3} \left(3(\vec{p} \cdot \vec{r})\vec{r}/r^2 - \vec{p}\right) \tag{158}$$

The normal component of this field should have a discontinuity at the surface due to polarization charges. Or, equivalently we can test the *continuity* of the normal of $\vec{D}$. One has

$$\vec{D}_{out} = \epsilon_0 \vec{E}_{out} \ , \quad \vec{D}_{in} = \kappa\epsilon_0 \vec{E}_{in} \tag{159}$$

With the normal component

$$D^n = \vec{D} \cdot \vec{R}/R \tag{160}$$

one has for $\vec{r} \to \vec{R}$:

$$D^n_{out} = \epsilon_0 \vec{E}_0 \cdot \vec{R}/R + \epsilon_0 \frac{1}{4\pi\epsilon_0 R^3} \left(3(\vec{p} \cdot \vec{R}) - \vec{p} \cdot \vec{R}\right)/R \tag{161}$$

and

$$D^n_{in} = \kappa\epsilon_0 \vec{E}_{in} \cdot \vec{R}/R \tag{162}$$

The two latter expressions should be the same for the solution (157)
**HW***: check if this is correct*

*2. Solution with Legendre polynomials*

General solution can be written as

$$V(r, \theta) = \sum_{n=0}^{\infty} \left\{ A_n r^n P_n(\cos\theta) + B_n \frac{P_n(\cos\theta)}{r^{n+1}} \right\} \tag{163}$$

The expansion is the same as for a conducting sphere, but now *out* and *in*.

Out, the same BC at $r \to \infty$. Thus,

$$A_1^{out} = -E_0 \ , \quad A_n^{out} = 0 \text{ for } n \neq 1$$

Inside - no singularity at $r = 0$:

$$B_n^{in} = 0$$

Boundary conditions on the surface: $\vec{E}_{||}$ continuous (otherwise circulation $\neq 0$)
$D_{normal}$ continuous; $\vec{D}^{in,out} = \kappa_{1,2}\vec{E}^{in,out}$.
Thus

$$-\frac{1}{R}\frac{\partial V^{in}(R,\theta)}{\partial \theta} = -\frac{1}{R}\frac{\partial V^{out}(R,\theta)}{\partial \theta}$$

and

$$-\kappa_1 \frac{\partial V^{in}(r=R,\theta)}{\partial r} = -\kappa_2 \frac{\partial V^{out}(r=R,\theta)}{\partial r}$$

From these BC:

$$A_1^{in} = A_1^{out} + B_1^{out}/R^3$$

and

$$\kappa_1 A_1^{in} = \kappa_2 \left( A_1^{out} - 2B_1^{out}/R^3 \right)$$

(linear system of equations). Thus,

$$A_1^{in} = -\frac{3\kappa_2}{2\kappa_2 + \kappa_1} E_0 \ , \quad B_1^{out} = \frac{\kappa_1 - \kappa_2}{\kappa_1 + 2\kappa_2}$$

and

$$V^{in} = -\frac{3\kappa_2}{2\kappa_2 + \kappa_1} E_0 r \cos\theta$$

$$V^{out} = -E_0 r \cos\theta + \frac{\kappa_1 - \kappa_2}{\kappa_1 + 2\kappa_2} E_0 \frac{R^3}{r^2} \cos\theta$$

Checkpoint1: $\kappa_1 = \kappa_2$:

$$V^{in} = V^{out} = -E_0 r \cos\theta$$

Checkpoint2: $\kappa_1/\kappa_2 \to \infty$:

$$V^{in} = 0 \ , \quad V^{out} = -E_0 r \cos\theta \left( 1 - \frac{R^3}{r^3} \right)$$

(conductor).

General:

$$E^{in} = \frac{3\kappa_2}{2\kappa_2 + \kappa_1} E_0$$

Uniform field(!)

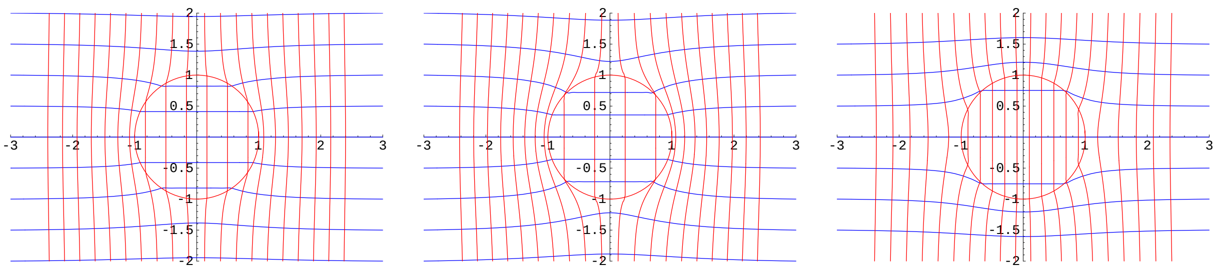

FIG. 15: Electric field lines (blue) and equipotential surfaces (red) for a dielectric sphere in a uniform field. $\epsilon \equiv \kappa^{in}/\kappa^{out}$; from top left to right: $\epsilon = 2$, $\epsilon = 4$ and $\epsilon = 0.33$ (cavity in a dielectric). Note that the field lines better represent the vector $\vec{D}$, rather than $\vec{E}$. (Think why?)

*Dr. Vitaly A. Shneidman, Phys 621, 9th Lecture*

# Part IV

# Magnetostatics

(WILL BE DISCUSSED IN CLASS)

## XIV. INTRODUCTION

### A. Math

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\gamma} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} \qquad (164)$$

**HW**: *use the above identity to prove 2d and 6th formulas from inner cover of* **JACK.**

#### *1. Vector operations*

#### *2. Elliptic integrals*

### B. Motion of charges and constant currents

### C. Macroscopic ME

#### *1. Amper's law*

**HW**: *Consider a planar surface current with density $\lambda$ (in A/m). Find $\vec{B}$ above and below the plane*

## XV. VECTOR POTENTIAL

Basic relations, gauge invariance.

**HW**: *Find $\vec{A}$ for a constant $\vec{B}$ (solution is not unique) One possibility is $\vec{A} = 1/2\,\vec{B}\times\vec{r}$, check it; find* $\mathrm{div}\vec{A}$

**HW**: *Find $\vec{A}$ for $\vec{B}$ of a straight wire (solution is not unique)*

**HW**: *Show that for*

$$\vec{A} = \frac{1}{2}\left(u\hat{\nabla}v - v\hat{\nabla}u\right)$$

*(any scalar $u, v$) one has*

$$\vec{B} = \hat{\nabla}u \times \hat{\nabla}v$$

**HW**: *Show that*

$$\oint \vec{A}\cdot d\vec{r}$$

*is invariant under gauge transformation $\vec{A} \to \vec{A} + \hat{\nabla}\phi$*

### A. Biot-Savarat law

### B. Force between wire loops

**HW**: *restore the elementary formula for interaction between || wires*

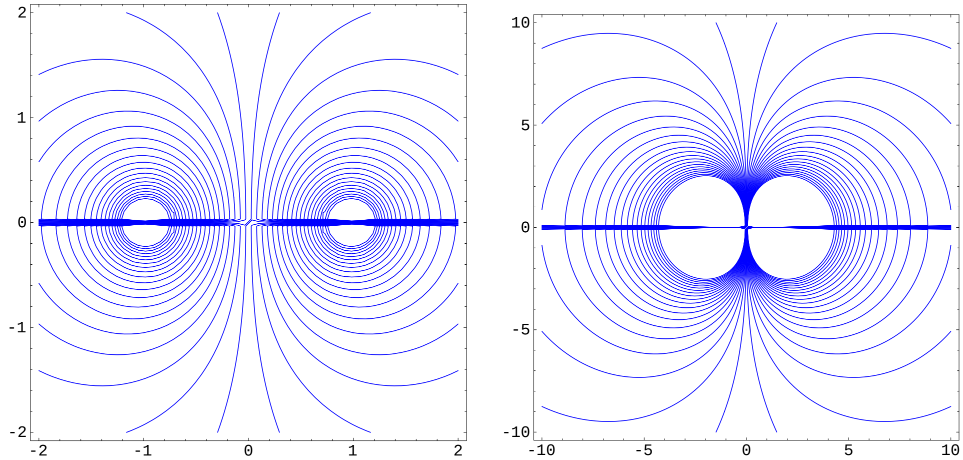


FIG. 16: Lines of constant values of the vector potential $\vec{A}$ for a ring current close to the ring (left) and far away (tight). The ring is perpendicular to the plane of the picture and has radius 1.

### C. Vector potential for a ring current

(see 621_ring.nb)

### D. Expansion at large distances

in class

$$\vec{A} = \frac{mu_0}{4\pi}\frac{\vec{m}\times\vec{R}}{R^3} \tag{165}$$

$$\vec{m} = \frac{1}{2}\int d\vec{r}\,\vec{r}\times\vec{J} \tag{166}$$

### E. Magnetic moment and gyromagnetic ratio

**work-through: JACK. , p.187**

Micro-current density:

$$\vec{j} = \sum_a q_a \vec{v}_a(t)\delta\left(\vec{r} - \vec{r}_a(t)\right)$$

From eq.(166)

$$\vec{m}(t) = \frac{1}{2}\sum_a q_a \vec{r}_a(t) \times \vec{v}_a(t) \tag{167}$$

Compare this to

$$\vec{L}(t) = \sum_a M_a \vec{r}_a(t) \times \vec{v}_a(t) \tag{168}$$

if all $q_a/M_a = q/M$

$$\vec{m} = \frac{q}{2M}\vec{L} \tag{169}$$

**HW**: *It is known that values of $L$ are quantized in units of $\hbar$; find quantization of $|\vec{m}|$ for electrons (give numbers)*

**HW**: *(hard) two different charges of opposite sign revolve around center of mass. Relate $m$ to $L$. Check the limits $q_1 = -q_2\,, m_1 = m_2$ and $q_1 = -q_2\,, m_1 \gg m_2$ ("proton and electron")*

Briefly discuss: relation to Burnett effect (magnetization of a rotating body) and Einstein-de Haas effect (rotation of a freely suspended body upon magnetization) - not in **JACK.**

### F. Torque and Larmor precession

### G. Magnetization $\vec{M}$

in class **work-through: JACK. 5.9**

#### *1. Scalar magnetic potential $\Phi_M$*

in class

### 2. Vector potential $\vec{A}$

in class

### 3. Sphere in a uniform field

in class

**work-through: JACK. 5.11**

**HW**: *(optional) try to solve this problem with vector potential*

# XVI. ENERGY IN MAGNETIC FIELD

## A. Interaction of linear currents and mutual inductance

start with eq. (5.10) in **JACK.**

$$U_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{12}} \equiv M_{12} I_1 I_2 \tag{170}$$

Transition to distributed current

$$I\,d\vec{l} \;\rightarrow\; \vec{J} d\vec{r}$$

## B. Self induction

in class

$$\text{flux} = LI \tag{171}$$

$$U = \frac{1}{2} L I^2 \tag{172}$$

### 1. Energy density

in class

$$u = \frac{1}{2} \frac{B^2}{\mu_0} = \frac{1}{2} BH \tag{173}$$

*2. Thin wire loop*

in class

$$L = \frac{\mu_0}{2\pi} l \ln \frac{l}{a} \tag{174}$$

**HW**: *Consider a coaxial cable with hollow inner cable of radius a and outside radius b. (a) calculate the inductance per unit length,* $L/\Delta l$*; (b) the same for capacitance* $C/\Delta l$*; (c) calculate* $\Delta l/\sqrt{LC}$

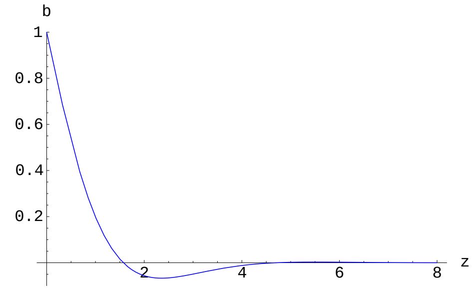


FIG. 17: Skin-depth profile. Note that oscillations quickly become "invisible" upon departure from the surface.

*Dr. Vitaly A. Shneidman, Phys 621, 11th Lecture*

## XVII. QUASISTATIC MAGNETIC FIELD

in class

Diffusion approximation:

$$\hat{\nabla}^2 \vec{B} = \mu\sigma \frac{\partial}{\partial t} \vec{B} \tag{175}$$

How to solve diffusion equation - see Appendix C

**HW**: *(a) derive the Green's function by switching to* $X = x/\sqrt{Dt}$*; (b) verify normalization of the Green's function; (c) reproduce the erfc solution*

### A. Diffusion of magnetic field

in class - see Fig. 22.

### B. Skin depth

in class - see Fig. 17.

### C. Skin effect

not in **JACK.** .

### 1. The problem

Consider a cylindrical wire of radius $a$; the wire is oriented in the $z$-direction. There is a high-frequency current in the $z$-direction, due to a field $E_z(r)$ accompanied by a circular magnetic field $B_\phi(r)$.

### 2. math: Vector Laplacian

(see Arfken, p. 116) In cylindrical coordinates $r, \phi, z$ applications of $\hat{\nabla}^2$ to a *vector* fied $\vec{A}(\vec{r}) = \vec{e}_r A_r + \vec{e}_\phi A_\phi + \vec{e}_z A_z$ gives

$$\hat{\nabla}^2 \vec{A}\Big|_\phi = \hat{\nabla}^2 A_\phi - \frac{1}{r^2} A_\phi + \frac{2}{r^2} \frac{\partial}{\partial \phi} A_r \tag{176}$$

$$\hat{\nabla}^2 \vec{A}\Big|_z = \hat{\nabla}^2 A_z$$

(and we will not need the $r$-component).

### 3. Electric field

$$\mu\sigma \frac{\partial}{\partial t} \vec{E} = \hat{\nabla}^2 \vec{E} \tag{177}$$

or from the 2d eq.(177) and laplacian in cylindrical coordinates (see **JACK.** - inner back cover)

$$\mu\sigma \frac{\partial}{\partial t} E_z = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} E_z \right) \tag{178}$$

With

$$E_z(r,t) = E_z^0(r) e^{-i\omega t}$$

one gets:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} E_z^0 \right) + k^2 E_z^0 = 0 \ , \quad k^2 \equiv i\omega\mu\sigma \tag{179}$$

and

$$E_z \sim J_0(kr)\, e^{-i\omega t} \tag{180}$$

See Fig. 18 (red).

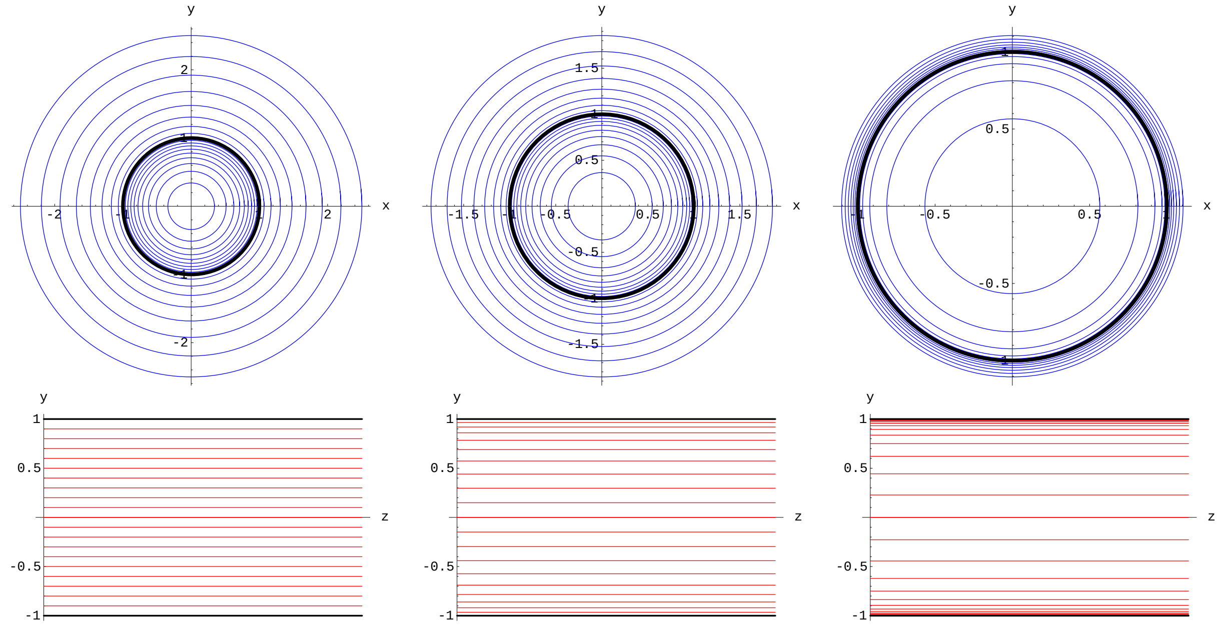

FIG. 18: Magnetic and electric fields in a long cylindrical conductor. (magnetic field -upper row- is shown both inside and outside, the electric only inside the conductor). Left column - static field, middle - intermediate frequency, right column - high frequency. Note concentration of the inner field near the surface -"skin effect"; electric current has the same structure as electric field.

### *4. Magnetic field*

$$\mu\sigma \frac{\partial}{\partial t}\vec{B} = \hat{\nabla}^2 \vec{B} \tag{181}$$

or from the 1st eq.(177) (without last term, no $B_r$) and laplacian in cylindrical coordinates (see **JACK.** - inner back cover)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}B^0_\phi\right) - \frac{B^0_\phi}{r^2} = -i\omega\mu\sigma B^0_\phi \tag{182}$$

and

$$B_\phi \sim J_1(kr)\, e^{-i\omega t} \tag{183}$$

See Fig. 18 (blue). Magnetic field outside is given by the static relation

$$B_\phi(r,t) = \mu_0 I/(2\pi r)\ , \quad I = \sigma \int_0^a 2\pi r\, dr\, E_z(r)$$

**HW**: *get eq.(183) from*

$$\vec{B}^0 = \frac{1}{i\omega}\hat{\nabla} \times \vec{E}$$

*and the Bessel function identity* $J_0'(x) = -J_1(x)$

# XVIII. MAXWELL EQUATIONS

## A. Vector and scalar potential

in class

## B. Green function of the wave equation

in class

## C. Field of a moving charge

### *1. Lienard-Wiechert potential*

see Ch. 14 in **JACK.** although he uses relativity. We will follow here a more historical path.

Start with

$$G(\vec{r}, t, \vec{r}', t') = \frac{1}{R}\delta\left(t - t' - R/c\right) \tag{184}$$

with $\vec{R} = \vec{r} - \vec{r}'$.

Then from

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -4\pi k \rho \tag{185}$$

one has

$$\Phi(\vec{r}) = k\int d\vec{r}'\frac{1}{R}\rho\left(\vec{r}', t_r\right) \tag{186}$$

with

$$t_r = t - R/c$$

Similarly, from

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J} \tag{187}$$

one has

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\vec{r}'\frac{1}{R}\vec{J}\left(\vec{r}', t_r\right) \tag{188}$$

Now, for a point charge

$$\rho(\vec{r}, t) = q\delta\left(\vec{r} - \vec{r}_0(t)\right)\ , \quad \vec{J} = \rho\vec{v}$$

For integration need to replace $\vec{r}$ by $\vec{r}'$ and $t$ by $t_r$. Note the appearance of the Jacobian in denominator:

$$\partial\left(\vec{r}' - \vec{r}_0(t_r)\right)/\,\partial\left(\vec{r}'\right) = 1 - \frac{\vec{R}\cdot\vec{v}}{Rc}$$

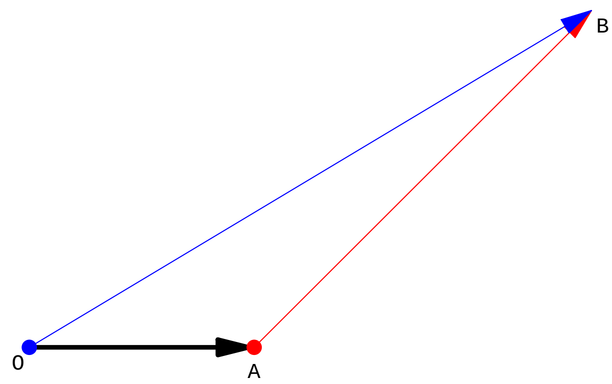


FIG. 19: Evaluation of the potential of a charge moving with constant velocity $\vec{v}$ in the direction $OA$. Point $O$ is the retraded position, point $A$ is the current position and $B$ is the observation point. $\vec{R}' \equiv \vec{OB}$, $\vec{R} \equiv \vec{AB}$. Note that $\left|\vec{R}'\right|/c = \left|\vec{R}' - \vec{R}\right|/v$

Thus,

$$\Phi = kq\frac{1}{R_r - \vec{v}\cdot\vec{R}_r/c}\ , \quad \vec{A} = \vec{v}\,\Phi/c^2 \tag{189}$$

(where we indicate retarded position).

### *2. Charge moving with constant velocity*

see Fig. 19.

We use

$$\vec{r}_0(t) = \vec{v}t$$

and

$$\vec{R} = \vec{r} - \vec{r}_0(t)\ , \quad \vec{R}' = \vec{r} - \vec{r}'$$

where $\vec{r}'$ is the retarded position.

From the diagram note

$$OA = OB \cdot v/c$$

or

$$\vec{R}' - \vec{R} = \vec{v}\,R'/c$$

and

$$R^2 = R'^2 + \frac{v^2}{c^2}R'^2 - 2\frac{\vec{v}\cdot\vec{R}}{c}R' \tag{190}$$

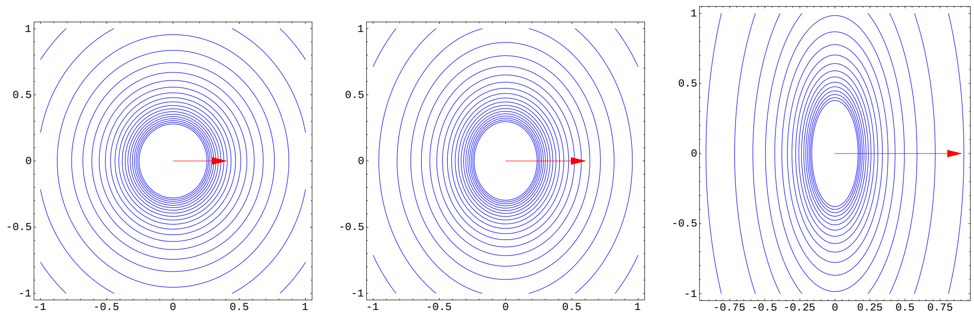

FIG. 20: Potential of a charge moving with a constant velocity, $v/c = 0.4$, 0.6 and 0.9, respectively.

Compare this with the square of denominator in Lienard-Wiechert:

$$\left(R' - \frac{\vec{R}\,' \cdot \vec{v}}{c}\right)^2 = R^2 - \frac{v^2}{c^2} R'^2 + \frac{1}{c^2}\left(\vec{R}\,' \cdot \vec{v}\right)^2 = R^2 - \frac{1}{c^2}\left(\vec{R}\,' \times \vec{v}\right)^2 = R^2 - \frac{1}{c^2}\left(\vec{R} \times \vec{v}\right)^2 \tag{191}$$

Potential is given by

$$\Phi\left(\vec{r}, t\right) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{R^2 - \left|\vec{R} \times \vec{v}\right|^2 / c^2}} \tag{192}$$

$$\vec{E}\left(\vec{r}, t\right) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(R^2 - \left|\vec{R} \times \vec{v}\right|^2 / c^2\right)^{3/2}} \frac{\vec{R}}{R^3} \tag{193}$$

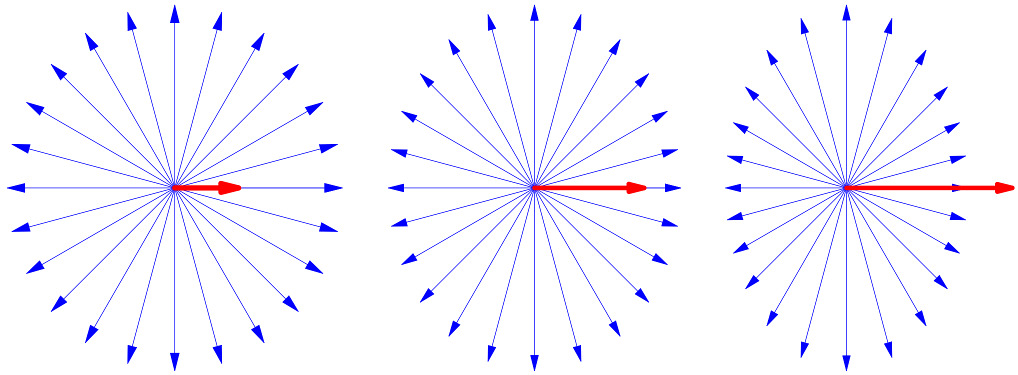

FIG. 21: Electric field for $v/c\ = 0.4$, 0.6 and 0.75

## APPENDIX A: SOME *MATHEMATICA* COMMANDS

### 1. Basic elementary commands

```
HELP:
 1) if you know the exact command , but want to refresh what
argument it requires, use ?.  E.g.
?Sin
2) if you approximately know the spelling, use ? with * for the
unknown part, e.g.
 ?*Plot*
gives all commands which have Plot in them
3) if you have no idea - use Help button (will not work in "math"
mode).


Frequent type- and space-saving commands:
1) % uses the last output as input.
       Similarly, %% uses the one before last output, etc. Or, %12
2) space - can be used instead of * for multiplication:
3) ; will not produce an output on the screen (but can
work with it further!)
(main typesaving - defining your own functions, etc. - will study
```

later).

Saving your work:

There are two ways:

1) Save["filename", symbol] appends definitions associated with the specified symbol to a file.

if symbol includes previous definitions, will save everything which is required! "filename" usually includes .m at the end (for convenience), but you can be creative. Graphics cannot be saved this way, but you can save the last command used to generate it, and then recreate the picture upon restarting Mathematica. Files are in plain text and relatively small.

Example:

In[1]:= fig:=Plot[Sin[x]/x, {x,-8,8}] (we defined a plot function, fig)

In[2]:= Save["figSinc.m", fig] (saved this function in a file figSinc.m)

In[3]:= !!figSinc.m (this shows the contents of the file)

fig := Plot[Sin[x]/x, {x, -8, 8}]

Now, if you start a new Mathematica session, you can type

<< figSinc.m

and you will have all saved definitions. Command fig will plot your picture.

2) you save as a notebook, with all graphics you created (and all the junk). Saved files are BIG, and can quickly overflow your directory if caution is not used. Use sparingly, and only for work you feel you really need and which you cannot save using the Save command.

## 2. Numbers

```
1) Integer

2) Exact - 1/2, 10^-10, Pi, E, Sqrt[2], EulerGamma, etc.

3) approximate - 2. , 10.^-10,  pi= N[Pi,15], e=N[E,7], etc.

4) Complex numbers:

I represents the imaginary unit Sqrt[-1], e.g.

z = 2+3I and then Abs[z]=..., Arg[z]=..., etc.

5) Random numbers, e.g.

Random[]
```

## 3. Symbolic math

```
Sum, e.g.:

In[74]:= Sum[i^2, {i,1,n}]

or

In[74]:= Sum[i^-2, {i, 1, Infinity}]


Derivatives and integration:

In[75]:= D[x^n,x]

            -1 + n
Out[75]= n x

In[76]:= D[%,x]

                     -2 + n
Out[76]= (-1 + n) n x

In[77]:= Integrate[%,x]

            -1 + n
Out[77]= n x


 Algebraic operations:

Expand, Factor, Collect, Simplify, etc.
```

```
Trigonometry:

TrigExpand and TrigReduce


Connection with exponential notations:

In[8]:=  ExpToTrig[Exp[I x]]

Out[8]= Cos[x] + I Sin[x]

or

In[9]:= TrigToExp[Cos[x]+I Sin[x]]

           I x
Out[9]= E


Power serieses:

In[118]:= Series[Exp[a x], {x, 0, 5}]

To make a polynomial by truncating a series:

In[119]:= Normal[%]


Limit: example

In[123]:= Limit[(1+x/n)^n, n->Infinity]

           x
Out[123]= E
```

## 4. Defining your own functions

```
In[1]:= f[x_]:=Sin[x]

In[3]:=Plot[f[x]/x, {x,-6,6}] (*will give a plot*)

Can define and save a plotting function:

In[23]:= plotf:=Plot[f[x]/x, {x,-6,6}]


Difference between := and =

In[19]:= r=Random[];
```

```
In[20]:= Table[r, {i,5}]

Out[20]= {0.307826, 0.307826, 0.307826, 0.307826, 0.307826}
```

Gives identical numbers since r was assigned a fixed value

but

```
In[21]:= Clear[r]; r:=Random[]

In[22]:=  Table[r, {i,5}]

Out[22]= {0.0592439, 0.981402, 0.944823, 0.0902293, 0.598816}
```

gives different values each time r is evaluated

A third assignment (dangerous!):

```
Clear[x]; f[x_]=Sin[x]
```

Must use Clear (!)

## 5. Graphics (2D)

Main functions: Plot, Show, ListPlot Options: PlotStyle, etc. Text, arrows, etc.

Plot[f, {x, xmin, xmax}] generates a plot of f as a function of x from xmin to xmax. Plot[{f1, f2, ... }, {x, xmin, xmax}] plots several functions.

PlotRange is an option for graphics functions that specifies what points to include in a plot.

Show[graphics, options] displays two- and three-dimensional graphics, using the options specified. Show[g1, g2, ... ] shows several plots combined.

Examples:

```
In[12]:= Clear[plo]

In[13]:= plo[n_]:=Plot[Sin[n x]/x, {x,-2,2}, PlotRange -> {-1,2},
PlotStyle -> Dashing[{0.01*n, 0.02}]]

In[15]:= sho:=Show[Table[plo[n], {n,1,3}]]

In[16]:= sho (*will give graphics*)
```

Plotting discrete data points:

ListPlot[{y1, y2, ... }] plots a list of values. The x coordinates for each point are taken to be 1, 2, ... .

Example:

```
In[21]:= list=Table[Sin[i/100.]+.1*Random[], {i,100}];
In[22]:= ListPlot[list]
Out[22]= -Graphics-
```

Main extra options: e.g., PlotStyle -> PointSize[0.02], or PlotJoined->True

Parametric plot:

ParametricPlot[{fx, fy}, {t, tmin, tmax}] produces a parametric plot with x and y coordinates fx and fy generated as a function of t. Example:

```
In[2]:= x[phi_]:=Cos[phi];
In[3]:= y[phi_]:=Sin[phi];
In[4]:= ParametricPlot[{x[phi],y[phi]}, {phi, 0, 2Pi},
, AspectRatio -> Automatic]
 -Graphics-
```

Arrow: need to use a package.

```
<<Graphics‘Arrow‘ (*note direction of quotes!*)
```

e.g.,

```
arr = Graphics[Arrow[{0,0}, {1.1}, HeadLength ->0.03]];
(*HeadLength is optional*)
Show[Plot[-x, {x,-1.5,1.5}], arr]
```

```
(*suppose we like what we see from command fig and want to create
a postscript file, e.g. t.ps *)
disp:=Display["t.ps", fig, "EPS"]
```

(*now disp will create t.ps, as a GOOD postscript
which is outside of Mathematica and can be further used
independently*)

Hiding graphics:

use DisplayFunction->Identity

Then will give output Graphics on the screen (if no errors), but no picture. Used mostly to avoid intermediate plots. Restore picture in Show using DisplayFunction->$DisplayFunction. Example:

```
In[13]:= plo[n_]:=Plot[Sin[n x]/x, {x,-2,2}, PlotRange -> {-1,2},
PlotStyle ->Dashing[{0.01*n, 0.02}], DisplayFunction->Identity ]
In[14]:= plo[1]
Out[14]= -Graphics-
In[15]:= Show[Table[plo[n], {n,1,5}],
DisplayFunction->$DisplayFunction]
```

Graphics Primitives:

Line[{pt1, pt2, ... }] is a graphics primitive which represents a line joining a sequence of points.

Point[coords] is a graphics primitive that represents a point.

Circle[{x, y}, r] is a two-dimensional graphics primitive that represents a circle of radius r centered at the point x, y.

Color:

RGBColor[red, green, blue] is a graphics directive which specifies that graphical objects which follow are to be displayed, if possible, in the color given.

Example: plotting two different Bessel functions:

```
In[2]:= red=RGBColor[1, 0, 0]; green=RGBColor[0, 1, 0];
In[3]:= Plot[{BesselI[1, x], BesselI[2, x]}, {x, 0, 5},
```

```
PlotStyle -> {red, green}]
```

### 6. Main package

```
In[1]:= <<Calculus`VectorAnalysis`
In[2]:= ?Grad
Grad[f] gives the gradient of the scalar function f in the default
coordinate system. Grad[f, coordsys] gives the gradient of f in
the coordinate system coordsys.
Also, Div[...], Curl[...], Laplacian[...], for different systems
of coordinates, JacobianDeterminant, etc.
```

### 7. Bessel zeros

```
In[2]:= << NumericalMath`BesselZeros`
In[5]:= BesselJZeros[0,3]
Out[5]= {2.40483, 5.52008, 8.65373}
In[6]:= BesselJZeros[1,3]
Out[6]= {3.83171, 7.01559, 10.1735}
```

compare with **JACK.** , p.114.

## APPENDIX B: PLOTTING ELECTRIC FIELD (EF) FOR A KNOWN POTENTIAL

### 1. Parametrization of the line with potential

Consider the following problem. Suppose we already know the potential $V(x, y)$ (we use $V$ instead of $\phi$, just to avoid Greek letters in Mathematica). How to plot the EF lines?

Let $s$ be the "length" along the line and $\alpha(x, y)$ an arbitrary scaling factor. Then, with $\vec{r} = \{x, y\}$, along the line

$$\frac{d\vec{r}}{ds} = \vec{E}\alpha$$

Let us multiply the above by $\vec{E}$. One has $\vec{E}\cdot d\vec{r} = -dV$, the change in the potential. Then,

$$-dV/ds = E^2\alpha$$

Now if we choose $\alpha = -1/E^2$, the length will coincide with the potential (!), which is a very physical parametrization of the curve. Now one has

$$\frac{d\vec{r}}{dV} = -\frac{\vec{E}}{E^2} \tag{B1}$$

If we can integrate this equation, we will get the line.

### 2. *Mathematica* realization

Here is the field of a dipole. We define potential $V(x,y)$, our own gradient *grad* (not to use package) and field $e$.

*rep* is a replacement of arbitrary $x, y$ by a point $x[v]\,, y[v]$ belonging to a given line (to be found from a differential equation).

Then *sol* finds a single line starting from $x_0\, y_0$ by solving a differential equation. (*Derivative* is the same as $x'[v]$). Then *plot* plots just one line.

For a good picture we select many initial points - $xx[n], yy[n]$ around the left charge and $xX[n], yY[n]$ around the right one. (if charges were different we would select a different number of points on left and on right to comply with Gauss theorem). Complex numbers are convenient here, e.g. $z = -1 + e^{2\pi i n/24}$ and $xx = Re[z]\,, yy = Im[z]$, with $n = 1, 2, \ldots, 24$.

*DisplayFunction– >Identity* is not to see each intermediate graphics. Full final graphics will be restored when *Show* is applied, using the *DisplayFunction– >$DisplayFunction*

*ploV* is plot of eqipotential "surfaces" (lines). This is easy compared to field lines.

*disp* is a Pure function designed to create a good postscript output, file "t.ps" using the Display command (note, "t.ps" will exist after *Mathematica* is closed and can be used independently).

Here is the routine:

```
V[x_, y_] := 1/Sqrt[(x - 1)^2 + y^2] - 1/Sqrt[(x + 1)^2 + y^2]
grad := {D[#1, x], D[#1, y]} &
e := -grad[#1] &
rep = {x -> x[v], y -> y[v]};
```

```
sol[x0_, y0_] :=
  Block[{ef = e[V[x, y]] /. rep, e2 = ef . ef, v0 = V[x0, y0]},
   NDSolve[{Derivative[1][x][v] == ef[[1]]/e2,
     Derivative[1][y][v] == ef[[2]]/e2, x[0] == x0, y[0] == y0},
{x, y},
    {v, v0, -v0}]]
plot[x0_, y0_] :=
  Block[{v0 = V[x0, y0]}, ParametricPlot[Evaluate[{x[v], y[v]} /.
      sol[x0, y0]], {v, v0, -v0},
DisplayFunction->Identity]]


xx[n_]:=Re[-1+0.1*Exp[I n 2Pi/24]]
yy[n_]:=Im[0.1*Exp[I n 2Pi/24]]
xX[n_]:=Re[1+0.1*Exp[I n 2Pi/24]]
yY[n_]:=Im[0.1*Exp[I n 2Pi/24]]
ploV:=ContourPlot[V[x,y], {x,-3,3}, {y,-2,2},
ContourShading->False, ContourStyle->Hue[0],
DisplayFunction->Identity]
disp:=Display["t.ps", #, "EPS"]&
(*final output:*)
disp[Show[Table[plot[xx[n],yy[n]],{n,24}],
Table[plot[xX[n],yY[n]],{n,24}], ploV,
DisplayFunction->$DisplayFunction,PlotRange->{-2,2}]]
```

## APPENDIX C: DIFFUSION EQUATION

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \tag{C1}$$

The solution for $c(x,0) = \delta(x)$ ("Greens function") is given by

$$G(x,t) = \frac{1}{2\sqrt{\pi D t}} \exp\left\{-\frac{x^2}{4Dt}\right\} \tag{C2}$$

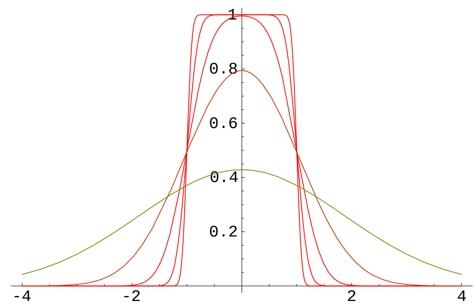


FIG. 22: Diffusion of an initially constant distribution in free space

General solution (1d):

$$c(x,t) = \int_{-\infty}^{\infty} dyG[x-y,t]c_0(y)$$

the above is general for initial distribution $c_0$; below is an example of a localized intial distribution between $x = 0$ and $-\infty$:

$$c_1[x,t] = \frac{1}{2}\text{erfc}\left[\frac{x}{2\sqrt{Dt}}\right] \tag{C3}$$

Another example: $c_0 = 1$ for $-1 < x < 1$. From superposition principle:

$$c_{box}[x,t] = (c_1[x-1,t] - c_1[x+1,t])/2 \tag{C4}$$