

Lectures on integrable Hamiltonian systems

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Abstract

We consider integrable Hamiltonian systems in a general setting of invariant submanifolds which need not be compact. For instance, this is the case a global Kepler system, non-autonomous integrable Hamiltonian systems and integrable systems with time-dependent parameters.

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Introduction

The Liouville – Arnold theorem for completely integrable systems [3, 54], the Poincaré – Lyapounov – Nekhoroshev theorem for partially integrable systems [26, 65] and the Mishchenko – Fomenko theorem for the superintegrable ones [9, 18, 61] state the existence of action-angle coordinates around a compact invariant submanifold of a Hamiltonian integrable system which is a torus T^m . However, it is well known that global extension of these action-angle coordinates meets a certain topological obstruction [4, 13, 15].

Note that superintegrable systems sometimes are called non-commutative (or non-Abelian) completely integrable systems.

In these Lectures, we consider integrable Hamiltonian systems in a general setting of invariant submanifolds which need not be compact [20, 22, 23, 24, 35, 41, 74, 86]. These invariant submanifolds are proved to be diffeomorphic to toroidal cylinders $\mathbb{R}^{m-r} \times T^r$ (Theorem 1.10). A key point is that, in accordance with Theorem 7.2, a fibred manifold whose fibres are diffeomorphic either to a compact manifold or \mathbb{R}^r is a fibre bundle, but this is not the case of toroidal cylinders.

In particular, this is the case of non-autonomous integrable Hamiltonian systems (Section 4.3) and Hamiltonian mechanics with time-dependent parameters (Section 6).

It may happen that a Hamiltonian system on a phase space Z falls into different integrable Hamiltonian systems on different open subsets of Z . For instance, this is the case of the Kepler system (Section 3). It contains two different globally superintegrable systems on different open subsets of a phase space $Z = \mathbb{R}^4$. Their integrals of motion form the Lie algebras $so(3)$ and $so(2,1)$ with compact and non-compact invariant submanifolds, respectively [41, 74].

Geometric quantization of completely integrable and superintegrable Hamiltonian systems with respect to action-angle variables is considered (Section 5.3). The reason is that, since a Hamiltonian of an integrable system depends only on action variables, it seems natural to provide the Schrödinger representation of action variables by first order differential operators on functions of angle coordinates.

Throughout the Lectures, all functions and maps are smooth, and manifolds are real smooth and paracompact. We are not concerned with the real-analytic case because a paracompact real-analytic manifold admits the partition of unity by smooth functions. As a consequence, sheaves of modules over real-analytic functions need not be acyclic that is essential for our consideration.

1 Partially integrable systems

This Section addresses partially integrable systems on Poisson and symplectic manifolds. Completely integrable systems can be regarded as the particular partially integrable ones (Remark 1.6). A key point is that a partially integrable system admits different compatible Poisson structures (Theorem 1.11).

Our goal are Theorem 1.15 on partial integrable systems on a Poisson manifold, Theorem 1.17 on partial integrable system on a symplectic manifold, and Theorem 1.18 as the global generalization of Theorem 1.17.

1.1 Geometry of symplectic and Poisson manifolds

This Section summarize some relevant material on symplectic manifolds, Poisson manifolds and symplectic foliations [1, 37, 41, 55, 84].

Let Z be a smooth manifold. Any exterior two-form Ω on Z yields a linear bundle morphism

$$\Omega^\flat : TZ \xrightarrow{Z} T^*Z, \quad \Omega^\flat : v \rightarrow -v \lrcorner \Omega(z), \quad v \in T_z Z, \quad z \in Z. \quad (1.1)$$

One says that a two-form Ω is of rank r if the morphism (1.1) has a rank r . A kernel $\text{Ker } \Omega$ of Ω is defined as the kernel of the morphism (1.1). In particular, $\text{Ker } \Omega$ contains the canonical zero section $\widehat{0}$ of $TZ \rightarrow Z$. If $\text{Ker } \Omega = \widehat{0}$, a two-form Ω is said to be non-degenerate. A closed non-degenerate two-form Ω is called symplectic. Accordingly, a manifold equipped with a symplectic form is a symplectic manifold. A symplectic manifold (Z, Ω) always is even dimensional and orientable.

A manifold morphism ζ of a symplectic manifold (Z, Ω) to a symplectic manifold (Z', Ω') is called symplectic if $\Omega = \zeta^* \Omega'$. Any symplectic morphism is an immersion. A symplectic isomorphism is called the symplectomorphism.

A vector field u on a symplectic manifold (Z, Ω) is an infinitesimal generator of a local one-parameter group of local symplectomorphism iff the Lie derivative $\mathbf{L}_u \Omega$ vanishes. It is called the canonical vector field. A canonical vector field u on a symplectic manifold (Z, Ω) is said to be Hamiltonian if a closed one-form $u \lrcorner \Omega$ is exact. Any smooth function $f \in C^\infty(Z)$ on Z defines a unique Hamiltonian vector field ϑ_f such that

$$\vartheta_f \lrcorner \Omega = -df, \quad \vartheta_f = \Omega^\sharp(df), \quad (1.2)$$

where Ω^\sharp is the inverse isomorphism to Ω^b (1.1).

Example 1.1: Given an m -dimensional manifold M coordinated by (q^i) , let

$$\pi_{*M} : T^*M \rightarrow M$$

be its cotangent bundle equipped with the holonomic coordinates $(q^i, p_i = \dot{q}_i)$. It is endowed with the canonical Liouville form

$$\Xi = p_i dq^i$$

and the canonical symplectic form

$$\Omega_T = d\Xi = dp_i \wedge dq^i. \quad (1.3)$$

Their coordinate expressions are maintained under holonomic coordinate transformations. The Hamiltonian vector field ϑ_f (1.2) with respect to the canonical symplectic form (1.3) reads

$$\vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i.$$

□

The canonical symplectic form (1.3) plays a prominent role in symplectic geometry in view of the classical Darboux theorem.

THEOREM 1.1: Each point of a symplectic manifold (Z, Ω) has an open neighborhood equipped with coordinates (q^i, p_i) , called canonical or Darboux coordinates, such that Ω takes the coordinate form (1.3). □

Let $i_N : N \rightarrow Z$ be a submanifold of a $2m$ -dimensional symplectic manifold (Z, Ω) . A subset

$$\text{Orth}_\Omega TN = \bigcup_{z \in N} \{v \in T_z Z : v \lrcorner u \lrcorner \Omega = 0, u \in T_z N\}$$

of $TZ|_N$ is called orthogonal to TN relative to a symplectic form Ω . One considers the following special types of submanifolds of a symplectic manifold such that the pull-back $\Omega_N = i_N^* \Omega$ of a symplectic form Ω onto a submanifold N is of constant rank. A submanifold N of Z is said to be:

- coisotropic if $\text{Orth}_\Omega TN \subseteq TN$, $\dim N \geq m$;
- symplectic if Ω_N is a symplectic form on N ;
- isotropic if $TN \subseteq \text{Orth}_\Omega TN$, $\dim N \leq m$.

A Poisson bracket on a ring $C^\infty(Z)$ of smooth real functions on a manifold Z (or a Poisson structure on Z) is defined as an \mathbb{R} -bilinear map

$$C^\infty(Z) \times C^\infty(Z) \ni (f, g) \rightarrow \{f, g\} \in C^\infty(Z)$$

which satisfies the following conditions:

- $\{g, f\} = -\{f, g\}$;

- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$;
- $\{h, fg\} = \{h, f\}g + f\{h, g\}$.

A Poisson bracket makes $C^\infty(Z)$ into a real Lie algebra, called the Poisson algebra. A Poisson structure is characterized by a particular bivector field as follows.

THEOREM 1.2: Every Poisson bracket on a manifold Z is uniquely defined as

$$\{f, f'\} = w(df, df') = w^{\mu\nu} \partial_\mu f \partial_\nu f' \quad (1.4)$$

by a bivector field w whose Schouten – Nijenhuis bracket $[w, w]_{\text{SN}}$ vanishes. It is called a Poisson bivector field. \square

A manifold Z endowed with a Poisson structure is called a Poisson manifold.

Example 1.2: Any manifold admits a zero Poisson structure characterized by a zero Poisson bivector field $w = 0$. \square

A function $f \in C^\infty(Z)$ is called the Casimir function of a Poisson structure on Z if its Poisson bracket with any function on Z vanishes. Casimir functions form a real ring $\mathcal{C}(Z)$.

Any bivector field w on a manifold Z yields a linear bundle morphism

$$w^\sharp : T^*Z \xrightarrow{Z} TZ, \quad w^\sharp : \alpha \rightarrow -w(z)[\alpha], \quad \alpha \in T_z^*Z. \quad (1.5)$$

One says that w is of rank r if the morphism (1.5) is of this rank. If a Poisson bivector field is of constant rank, the Poisson structure is called regular. Throughout the Lectures, only regular Poisson structures are considered. A Poisson structure determined by a Poisson bivector field w is said to be non-degenerate if w is of maximal rank.

There is one-to-one correspondence $\Omega_w \leftrightarrow w_\Omega$ between the symplectic forms and the non-degenerate Poisson bivector fields which is given by the equalities

$$\begin{aligned} w_\Omega(\phi, \sigma) &= \Omega_w(w_\Omega^\sharp(\phi), w_\Omega^\sharp(\sigma)), & \phi, \sigma \in \mathcal{O}^1(Z), \\ \Omega_w(\vartheta, \nu) &= w_\Omega(\Omega_w^\flat(\vartheta), \Omega_w^\flat(\nu)), & \vartheta, \nu \in \mathcal{T}(Z), \end{aligned}$$

where the morphisms w_Ω^\sharp (1.5) and Ω_w^\flat (1.1) are mutually inverse, i.e.,

$$w_\Omega^\sharp = \Omega_w^\flat, \quad w_\Omega^{\alpha\nu} \Omega_{w\alpha\beta} = \delta_\beta^\nu.$$

However, this correspondence is not preserved under manifold morphisms in general. Namely, let (Z_1, w_1) and (Z_2, w_2) be Poisson manifolds. A manifold morphism $\varrho : Z_1 \rightarrow Z_2$ is said to be a Poisson morphism if

$$\{f \circ \varrho, f' \circ \varrho\}_1 = \{f, f'\}_2 \circ \varrho, \quad f, f' \in C^\infty(Z_2),$$

or, equivalently, if $w_2 = T\varrho \circ w_1$, where $T\varrho$ is the tangent map to ϱ . Herewith, the rank of w_1 is superior or equal to that of w_2 . Therefore, there are no pull-back

and push-forward operations of Poisson structures in general. Nevertheless, let us mention the following construction.

THEOREM 1.3: Let (Z, w) be a Poisson manifold and $\pi : Z \rightarrow Y$ a fibration such that, for every pair of functions (f, g) on Y and for each point $y \in Y$, the restriction of a function $\{\pi^* f, \pi^* g\}$ to a fibre $\pi^{-1}(y)$ is constant, i.e., $\{\pi^* f, \pi^* g\}$ is the pull-back onto Z of some function on Y . Then there exists a coinduced Poisson structure w' on Y for which π is a Poisson morphism. \square

Example 1.3: The direct product $Z \times Z'$ of Poisson manifolds (Z, w) and (Z', w') can be endowed with the product of Poisson structures, given by a bivector field $w+w'$ such that the surjections pr_1 and pr_2 are Poisson morphisms. \square

A vector field u on a Poisson manifold (Z, w) is an infinitesimal generator of a local one-parameter group of Poisson automorphisms iff the Lie derivative

$$\mathbf{L}_u w = [u, w]_{\text{SN}} \quad (1.6)$$

vanishes. It is called the canonical vector field for a Poisson structure w . In particular, for any real smooth function f on a Poisson manifold (Z, w) , let us put

$$\vartheta_f = w^\sharp(df) = -w \lrcorner df = w^{\mu\nu} \partial_\mu f \partial_\nu. \quad (1.7)$$

It is a canonical vector field, called the Hamiltonian vector field of a function f with respect to a Poisson structure w . Hamiltonian vector fields fulfil the relations

$$\{f, g\} = \vartheta_f \lrcorner dg, \quad (1.8)$$

$$[\vartheta_f, \vartheta_g] = \vartheta_{\{f, g\}}, \quad f, g \in C^\infty(Z). \quad (1.9)$$

For instance, the Hamiltonian vector field ϑ_f (1.2) of a function f on a symplectic manifold (Z, Ω) coincides with that (1.7) with respect to the corresponding Poisson structure w_Ω . The Poisson bracket defined by a symplectic form Ω reads

$$\{f, g\} = \vartheta_g \lrcorner \vartheta_f \lrcorner \Omega.$$

Since a Poisson manifold (Z, w) is assumed to be regular, the range $\mathbf{T} = w^\sharp(T^*Z)$ of the morphism (1.5) is a subbundle of TZ called the characteristic distribution on (Z, w) . It is spanned by Hamiltonian vector fields, and it is involutive by virtue of the relation (1.9). It follows that a Poisson manifold Z admits local adapted coordinates in Theorem 7.6. Moreover, one can choose particular adapted coordinates which bring a Poisson structure into the following canonical form.

THEOREM 1.4: For any point z of a k -dimensional Poisson manifold (Z, w) , there exist coordinates

$$(z^1, \dots, z^{k-2m}, q^1, \dots, q^m, p_1, \dots, p_m) \quad (1.10)$$

on a neighborhood of z such that

$$w = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}, \quad \{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i}.$$

□

The coordinates (1.10) are called the canonical or Darboux coordinates for the Poisson structure w . The Hamiltonian vector field of a function f written in this coordinates is

$$\vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i.$$

Of course, the canonical coordinates for a symplectic form Ω in Theorem 1.1 also are canonical coordinates in Theorem 1.4 for the corresponding non-degenerate Poisson bivector field w , i.e.,

$$\Omega = dp_i \wedge dq^i, \quad w = \partial^i \wedge \partial_i.$$

With respect to these coordinates, the mutually inverse bundle isomorphisms Ω^\flat (1.1) and w^\sharp (1.5) read

$$\begin{aligned} \Omega^\flat : v^i \partial_i + v_i \partial^i &\rightarrow -v_i dq^i + v^i dp_i, \\ w^\sharp : v_i dq^i + v^i dp_i &\rightarrow v^i \partial_i - v_i \partial^i. \end{aligned}$$

Integral manifolds of the characteristic distribution \mathbf{T} of a k -dimensional Poisson manifold (Z, w) constitute a (regular) foliation \mathcal{F} of Z whose tangent bundle $T\mathcal{F}$ is \mathbf{T} . It is called the characteristic foliation of a Poisson manifold. By the very definition of the characteristic distribution $\mathbf{T} = T\mathcal{F}$, a Poisson bivector field w is subordinate to $\overset{2}{\wedge} T\mathcal{F}$. Therefore, its restriction $w|_F$ to any leaf F of \mathcal{F} is a non-degenerate Poisson bivector field on F . It provides F with a non-degenerate Poisson structure $\{, \}_F$ and, consequently, a symplectic structure. Clearly, the local Darboux coordinates for the Poisson structure w in Theorem 1.4 also are the local adapted coordinates

$$(z^1, \dots, z^{k-2m}, z^i = q^i, z^{m+i} = p_i), \quad i = 1, \dots, m,$$

(7.38) for the characteristic foliation \mathcal{F} , and the symplectic structures along its leaves read

$$\Omega_F = dp_i \wedge dq^i.$$

Since any foliation is locally simple, a local structure of an arbitrary Poisson manifold reduces to the following [84, 88].

THEOREM 1.5: Each point of a Poisson manifold has an open neighborhood which is Poisson equivalent to the product of a manifold with the zero Poisson structure and a symplectic manifold. □

Provided with this symplectic structure, the leaves of the characteristic foliation of a Poisson manifold Z are assembled into a symplectic foliation of Z as follows (see Section 7.4).

Let \mathcal{F} be an even dimensional foliation of a manifold Z . A \tilde{d} -closed non-degenerate leafwise two-form $\Omega_{\mathcal{F}}$ on a foliated manifold (Z, \mathcal{F}) is called symplectic. Its pull-back $i_F^* \Omega_{\mathcal{F}}$ onto each leaf F of \mathcal{F} is a symplectic form on F . A foliation \mathcal{F} provided with a symplectic leafwise form $\Omega_{\mathcal{F}}$ is called the symplectic foliation.

If a symplectic leafwise form $\Omega_{\mathcal{F}}$ exists, it yields a bundle isomorphism

$$\Omega_{\mathcal{F}}^b : T\mathcal{F} \xrightarrow{Z} T\mathcal{F}^*, \quad \Omega_{\mathcal{F}}^b : v \rightarrow -v \rfloor \Omega_{\mathcal{F}}(z), \quad v \in T_z \mathcal{F}.$$

The inverse isomorphism $\Omega_{\mathcal{F}}^\sharp$ determines a bivector field

$$w_{\Omega}(\alpha, \beta) = \Omega_{\mathcal{F}}(i_{\mathcal{F}}^\sharp(i_{\mathcal{F}}^* \alpha), i_{\mathcal{F}}^\sharp(i_{\mathcal{F}}^* \beta)), \quad \alpha, \beta \in T_z^* Z, \quad z \in Z, \quad (1.11)$$

on Z subordinate to $\overset{2}{\wedge} T\mathcal{F}$. It is a Poisson bivector field. The corresponding Poisson bracket reads

$$\{f, f'\}_{\mathcal{F}} = \vartheta_f \rfloor \tilde{d}f', \quad \vartheta_f \rfloor \Omega_{\mathcal{F}} = -\tilde{d}f, \quad \vartheta_f = \Omega_{\mathcal{F}}^\sharp(\tilde{d}f). \quad (1.12)$$

Its kernel is $S_{\mathcal{F}}(Z)$.

Conversely, let (Z, w) be a Poisson manifold and \mathcal{F} its characteristic foliation. Since $\text{Ann } T\mathcal{F} \subset T^*Z$ is precisely the kernel of a Poisson bivector field w , a bundle homomorphism

$$w^\sharp : T^*Z \xrightarrow{Z} TZ$$

factorizes in a unique fashion

$$w^\sharp : T^*Z \xrightarrow{\frac{i_{\mathcal{F}}^*}{Z}} T\mathcal{F}^* \xrightarrow{\frac{w_{\mathcal{F}}^\sharp}{Z}} T\mathcal{F} \xrightarrow{\frac{i_{\mathcal{F}}}{Z}} TZ \quad (1.13)$$

through a bundle isomorphism

$$w_{\mathcal{F}}^\sharp : T\mathcal{F}^* \xrightarrow{Z} T\mathcal{F}, \quad w_{\mathcal{F}}^\sharp : \alpha \rightarrow -w(z) \rfloor \alpha, \quad \alpha \in T_z \mathcal{F}^*. \quad (1.14)$$

The inverse isomorphism $w_{\mathcal{F}}^b$ yields a symplectic leafwise form

$$\Omega_{\mathcal{F}}(v, v') = w(w_{\mathcal{F}}^b(v), w_{\mathcal{F}}^b(v')), \quad v, v' \in T_z \mathcal{F}, \quad z \in Z. \quad (1.15)$$

The formulas (1.11) and (1.15) establish the equivalence between the Poisson structures on a manifold Z and its symplectic foliations.

Turn now to a group action on Poisson manifolds. By G throughout is meant a real connected Lie group, \mathfrak{g} is its right Lie algebra, and \mathfrak{g}^* is the Lie coalgebra (see Section 7.5).

We start with the symplectic case. Let a Lie group G act on a symplectic manifold (Z, Ω) on the left by symplectomorphisms. Such an action of G is called

symplectic. Since G is connected, its action on a manifold Z is symplectic iff the homomorphism $\varepsilon \rightarrow \xi_\varepsilon$, $\varepsilon \in \mathfrak{g}$, (7.45) of a Lie algebra \mathfrak{g} to a Lie algebra $\mathcal{T}_1(Z)$ of vector fields on Z is carried out by canonical vector fields for a symplectic form Ω on Z . If all these vector fields are Hamiltonian, an action of G on Z is called a Hamiltonian action. One can show that, in this case, ξ_ε , $\varepsilon \in \mathfrak{g}$, are Hamiltonian vector fields of functions on Z of the following particular type.

PROPOSITION 1.6: An action of a Lie group G on a symplectic manifold Z is Hamiltonian iff there exists a mapping

$$\widehat{J} : Z \rightarrow \mathfrak{g}^*, \quad (1.16)$$

called the momentum mapping, such that

$$\xi_\varepsilon \lrcorner \Omega = -dJ_\varepsilon, \quad J_\varepsilon(z) = \langle \widehat{J}(z), \varepsilon \rangle, \quad \varepsilon \in \mathfrak{g}. \quad (1.17)$$

□

The momentum mapping (1.16) is defined up to a constant map. Indeed, if \widehat{J} and \widehat{J}' are different momentum mappings for the same symplectic action of G on Z , then

$$d(\langle \widehat{J}(z) - \widehat{J}'(z), \varepsilon \rangle) = 0, \quad \varepsilon \in \mathfrak{g}.$$

Given $g \in G$, let us consider the difference

$$\sigma(g) = \widehat{J}(gz) - \text{Ad}^*g(\widehat{J}(z)), \quad (1.18)$$

where Ad^*g is the coadjoint representation (7.47) on \mathfrak{g}^* . One can show that the difference (1.18) is constant on a symplectic manifold Z [1]. A momentum mapping \widehat{J} is called equivariant if $\sigma(g) = 0$, $g \in G$.

Example 1.4: Let a symplectic form on Z be exact, i.e., $\Omega = d\theta$, and let θ be G -invariant, i.e.,

$$\mathbf{L}_{\xi_\varepsilon} \theta = d(\xi_\varepsilon \lrcorner \theta) + \xi_\varepsilon \lrcorner \Omega = 0, \quad \varepsilon \in \mathfrak{g}.$$

Then the momentum mapping \widehat{J} (1.16) can be given by the relation

$$\langle \widehat{J}(z), \varepsilon \rangle = (\xi_\varepsilon \lrcorner \theta)(z).$$

It is equivariant. In accordance with the relation (7.47), it suffices to show that

$$J_\varepsilon(gz) = J_{\text{Ad } g^{-1}(\varepsilon)}(z), \quad (\xi_\varepsilon \lrcorner \theta)(gz) = (\xi_{\text{Ad } g^{-1}(\varepsilon)} \lrcorner \theta)(z).$$

This holds by virtue of the relation (7.46). For instance, let T^*Q be a symplectic manifold equipped with the canonical symplectic form Ω_T (1.3). Let a left action of a Lie group G on Q have the infinitesimal generators $\tau_m = \varepsilon_m^i(q) \partial_i$. The canonical lift of this action onto T^*Q has the infinitesimal generators (7.19):

$$\xi_m = \widetilde{\tau}_m = v e_m^i \partial_i - p_j \partial_i \varepsilon_m^j \partial^i, \quad (1.19)$$

and preserves the canonical Liouville form Ξ on T^*Q . The ξ_m (1.19) are Hamiltonian vector fields of the functions $J_m = \varepsilon_m^i(q)p_i$, determined by the equivariant momentum mapping $\widehat{J} = \varepsilon_m^i(q)p_i\varepsilon^m$. \square

THEOREM 1.7: A momentum mapping \widehat{J} associated to a symplectic action of a Lie group G on a symplectic manifold Z obeys the relation

$$\{J_\varepsilon, J_{\varepsilon'}\} = J_{[\varepsilon, \varepsilon']} - \langle T_e\sigma(\varepsilon'), \varepsilon \rangle. \quad (1.20)$$

\square

In the case of an equivariant momentum mapping, the relation (1.20) leads to a homomorphism

$$\{J_\varepsilon, J_{\varepsilon'}\} = J_{[\varepsilon, \varepsilon']} \quad (1.21)$$

of a Lie algebra \mathfrak{g} to a Poisson algebra of smooth functions on a symplectic manifold Z (cf. Proposition 1.8 below).

Now let a Lie group G act on a Poisson manifold (Z, w) on the left by Poisson automorphism. This is a Poisson action. Since G is connected, its action on a manifold Z is a Poisson action iff the homomorphism $\varepsilon \rightarrow \xi_\varepsilon$, $\varepsilon \in \mathfrak{g}$, (7.45) of a Lie algebra \mathfrak{g} to a Lie algebra $\mathcal{T}_1(Z)$ of vector fields on Z is carried out by canonical vector fields for a Poisson bivector field w , i.e., the condition (1.6) holds. The equivalent conditions are

$$\begin{aligned} \xi_\varepsilon(\{f, g\}) &= \{\xi_\varepsilon(f), g\} + \{f, \xi_\varepsilon(g)\}, & f, g \in C^\infty(Z), \\ \xi_\varepsilon(\{f, g\}) &= [\xi_\varepsilon, \vartheta_f](g) - [\xi_\varepsilon, \vartheta_g](f), \\ [\xi_\varepsilon, \vartheta_f] &= \vartheta_{\xi_\varepsilon(f)}, \end{aligned}$$

where ϑ_f is the Hamiltonian vector field (1.7) of a function f .

A Hamiltonian action of G on a Poisson manifold Z is defined similarly to that on a symplectic manifold. Its infinitesimal generators are tangent to leaves of the symplectic foliation of Z , and there is a Hamiltonian action of G on every symplectic leaf. Proposition 1.6 together with the notions of a momentum mapping and an equivariant momentum mapping also are extended to a Poisson action. However, the difference σ (1.18) is constant only on leaves of the symplectic foliation of Z in general. At the same time, one can say something more on an equivariant momentum mapping (that also is valid for a symplectic action).

PROPOSITION 1.8: An equivariant momentum mapping \widehat{J} (1.16) is a Poisson morphism to the Lie coalgebra \mathfrak{g}^* , provided with the Lie – Poisson structure (7.48). \square

1.2 Poisson and symplectic Hamiltonian systems

Given a Poisson manifold (Z, w) , a Poisson Hamiltonian system (w, \mathcal{H}) on Z for a Hamiltonian $\mathcal{H} \in C^\infty(Z)$ with respect to a Poisson structure w is defined as

a set

$$S_{\mathcal{H}} = \bigcup_{z \in Z} \{v \in T_z Z : v - w^\sharp(d\mathcal{H})(z) = 0\}. \quad (1.22)$$

By a solution of this Hamiltonian system is meant a vector field ϑ on Z which takes its values into $TN \cap S_{\mathcal{H}}$. Clearly, the Poisson Hamiltonian system (1.22) has a unique solution which is the Hamiltonian vector field

$$\vartheta_{\mathcal{H}} = w^\sharp(d\mathcal{H}) \quad (1.23)$$

of \mathcal{H} . Hence, $S_{\mathcal{H}}$ (1.22) is an autonomous first order dynamic equation (see forthcoming Remark 1.5), called the Hamilton equation for a Hamiltonian \mathcal{H} with respect to a Poisson structure w .

Remark 1.5: Let u be a vector field u on Z . A closed subbundle $u(Z)$ of the tangent bundle TZ given by the coordinate relations

$$\dot{z}^\lambda = u^\lambda(z) \quad (1.24)$$

is said to be a first order autonomous dynamic equation on a manifold Z [56, 58]. By a solution of the autonomous first order dynamic equation (1.24) is meant an integral curve of a vector field u . \square

Relative to local canonical coordinates (z^λ, q^i, p_i) (1.10) for a Poisson structure w on Z and corresponding holonomic coordinates $(z^\lambda, q^i, p_i, \dot{z}^\lambda, \dot{q}^i, \dot{p}_i)$ on TZ , the Hamilton equation (1.22) and the Hamiltonian vector field (1.23) take a form

$$\dot{q}^i = \partial^i \mathcal{H}, \quad \dot{p}_i = -\partial_i \mathcal{H}, \quad \dot{z}^\lambda = 0, \quad (1.25)$$

$$\vartheta_{\mathcal{H}} = \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i. \quad (1.26)$$

Solutions of the Hamilton equation (1.25) are integral curves of the Hamiltonian vector field (1.26).

Let (Z, w, \mathcal{H}) be a Poisson Hamiltonian system. Its integral of motion is a smooth function F on Z whose Lie derivative

$$\mathbf{L}_{\vartheta_{\mathcal{H}}} F = \{\mathcal{H}, F\} \quad (1.27)$$

along the Hamiltonian vector field $\vartheta_{\mathcal{H}}$ (1.26) vanishes in accordance with the equality (4.58). The equality (1.27) is called the evolution equation.

It is readily observed that the Poisson bracket $\{F, F'\}$ of any two integrals of motion F and F' also is an integral of motion. Consequently, the integrals of motion of a Poisson Hamiltonian system constitute a real Lie algebra.

Since

$$\vartheta_{\{\mathcal{H}, F\}} = [\vartheta_{\mathcal{H}}, \vartheta_F], \quad \{\mathcal{H}, F\} = -\mathbf{L}_{\vartheta_F} \mathcal{H},$$

the Hamiltonian vector field ϑ_F of any integral of motion F of a Poisson Hamiltonian system is a symmetry both of the Hamilton equation (1.25) (Proposition 4.8) and a Hamiltonian \mathcal{H} (Definition 4.9).

Let (Z, Ω) be a symplectic manifold. The notion of a symplectic Hamiltonian system is a repetition of the Poisson one, but all expressions are rewritten in terms of a symplectic form Ω as follows.

A symplectic Hamiltonian system (Ω, \mathcal{H}) on a manifold Z for a Hamiltonian \mathcal{H} with respect to a symplectic structure Ω is a set

$$S_{\mathcal{H}} = \bigcup_{z \in Z} \{v \in T_z Z : v \lrcorner \Omega + d\mathcal{H}(z) = 0\}. \quad (1.28)$$

As in the general case of Poisson Hamiltonian systems, the symplectic one (Ω, \mathcal{H}) has a unique solution which is the Hamiltonian vector field

$$\vartheta_{\mathcal{H}} \lrcorner \Omega = -d\mathcal{H} \quad (1.29)$$

of \mathcal{H} . Hence, $S_{\mathcal{H}}$ (1.28) is an autonomous first order dynamic equation, called the Hamilton equation for a Hamiltonian \mathcal{H} with respect to a symplectic structure Ω . Relative to the local canonical coordinates (q^i, p_i) for a symplectic structure Ω , the Hamilton equation (1.28) and the Hamiltonian vector field (1.29) read

$$\dot{q}^i = \partial^i \mathcal{H}, \quad \dot{p}_i = -\partial_i \mathcal{H}, \quad (1.30)$$

$$\vartheta_{\mathcal{H}} = \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i. \quad (1.31)$$

Integrals of motion of a symplectic Hamiltonian system are defined just as those of a Poisson Hamiltonian system.

1.3 Partially integrable systems on a Poisson manifold

Completely integrable and superintegrable systems are considered with respect to a symplectic structure on a manifold which holds fixed from the beginning. As was mentioned above, partially integrable system admits different compatible Poisson structures (see Theorem 1.11 below). Treating partially integrable systems, we therefore are based on a wider notion of the dynamical algebra [35, 41].

Let we have m mutually commutative vector fields $\{\vartheta_{\lambda}\}$ on a connected smooth real manifold Z which are independent almost everywhere on Z , i.e., the set of points, where the multivector field $\overset{m}{\wedge} \vartheta_{\lambda}$ vanishes, is nowhere dense. We denote by $\mathcal{S} \subset C^{\infty}(Z)$ the \mathbb{R} -subring of smooth real functions f on Z whose derivations $\vartheta_{\lambda} \lrcorner df$ vanish for all ϑ_{λ} . Let \mathcal{A} be an m -dimensional Lie \mathcal{S} -algebra generated by the vector fields $\{\vartheta_{\lambda}\}$. One can think of one of its elements as being an autonomous first order dynamic equation on Z and of the other as being its integrals of motion in accordance with Definition 4.4. By virtue of this definition, elements of \mathcal{S} also are regarded as integrals of motion. Therefore, we agree to call \mathcal{A} a dynamical algebra.

Given a commutative dynamical algebra \mathcal{A} on a manifold Z , let G be the group of local diffeomorphisms of Z generated by the flows of these vector fields. The orbits of G are maximal invariant submanifolds of \mathcal{A} (we follow the

terminology of [81]). Tangent spaces to these submanifolds form a (non-regular) distribution $\mathcal{V} \subset TZ$ whose maximal integral manifolds coincide with orbits of G . Let $z \in Z$ be a regular point of the distribution \mathcal{V} , i.e., $\overset{m}{\wedge} \vartheta_\lambda(z) \neq 0$. Since the group G preserves $\overset{m}{\wedge} \vartheta_\lambda$, a maximal integral manifold M of \mathcal{V} through z also is regular (i.e., its points are regular). Furthermore, there exists an open neighborhood U of M such that, restricted to U , the distribution \mathcal{V} is an m -dimensional regular distribution on U . Being involutive, it yields a foliation \mathfrak{F} of U . A regular open neighborhood U of an invariant submanifold of M is called saturated if any invariant submanifold through a point of U belongs to U . For instance, any compact invariant submanifold has such an open neighborhood.

DEFINITION 1.9: Let \mathcal{A} be an m -dimensional dynamical algebra on a regular Poisson manifold (Z, w) . It is said to be a partially integrable system if:

- (a) its generators ϑ_λ are Hamiltonian vector fields of some functions $S_\lambda \in \mathcal{S}$ which are independent almost everywhere on Z , i.e., the set of points where the m -form $\overset{m}{\wedge} dS_\lambda$ vanishes is nowhere dense;
- (b) all elements of $\mathcal{S} \subset C^\infty(Z)$ are mutually in involution, i.e., their Poisson brackets equal zero. \square

It follows at once from this definition that the Poisson structure w is at least of rank $2m$, and that \mathcal{S} is a commutative Poisson algebra. We call the functions S_λ in item (a) of Definition 1.9 the generating functions of a partially integrable system, which is uniquely defined by a family (S_1, \dots, S_m) of these functions.

Remark 1.6: If $2m = \dim Z$ in Definition 1.9, we have a completely integrable system on a symplectic manifold Z (see Definition 2.2 below). \square

If $2m < \dim Z$, there exist different Poisson structures on Z which bring a dynamical algebra \mathcal{A} into a partially integrable system. Forthcoming Theorems 1.10 and 1.11 describe all these Poisson structures around a regular invariant submanifold $M \subset Z$ of \mathcal{A} [35].

THEOREM 1.10: Let \mathcal{A} be a dynamical algebra, M its regular invariant submanifold, and U a saturated regular open neighborhood of M . Let us suppose that:

- (i) the vector fields ϑ_λ on U are complete,
- (ii) the foliation \mathfrak{F} of U admits a transversal manifold Σ and its holonomy pseudogroup on Σ is trivial,
- (iii) the leaves of this foliation are mutually diffeomorphic.

Then the following hold.

- (I) The leaves of \mathcal{F} are diffeomorphic to a toroidal cylinder

$$\mathbb{R}^{m-r} \times T^r, \quad 0 \leq r \leq m. \quad (1.32)$$

- (II) There exists an open saturated neighborhood of M , say U again, which is the trivial principal bundle

$$U = N \times (\mathbb{R}^{m-r} \times T^r) \xrightarrow{\pi} N \quad (1.33)$$

over a domain $N \subset \mathbb{R}^{\dim Z - m}$ with the structure group (1.32).

(III) If $2m \leq \dim Z$, there exists a Poisson structure of rank $2m$ on U such that \mathcal{A} is a partially integrable system in accordance with Definition 1.9. \square

Proof: We follow the proof in [12, 54] generalized to the case of non-compact invariant submanifolds [35, 37, 41].

(I). Since m -dimensional leaves of the foliation \mathcal{F} admit m complete independent vector fields, they are locally affine manifolds diffeomorphic to a toroidal cylinder (1.32).

(II). By virtue of the condition (ii), the foliation \mathfrak{F} of U is a fibred manifold [62]. Then one can always choose an open fibred neighborhood of its fibre M , say U again, over a domain N such that this fibred manifold

$$\pi : U \rightarrow N \tag{1.34}$$

admits a section σ . In accordance with the well-known theorem [67, 68] complete Hamiltonian vector fields ϑ_λ define an action of a simply connected Lie group G on Z . Because vector fields ϑ_λ are mutually commutative, it is the additive group \mathbb{R}^m whose group space is coordinatized by parameters s^λ of the flows with respect to the basis $\{e_\lambda = \vartheta_\lambda\}$ for its Lie algebra. The orbits of the group \mathbb{R}^m in $U \subset Z$ coincide with the fibres of the fibred manifold (1.34). Since vector fields ϑ_λ are independent everywhere on U , the action of \mathbb{R}^m on U is locally free, i.e., isotropy groups of points of U are discrete subgroups of the group \mathbb{R}^m . Given a point $x \in N$, the action of \mathbb{R}^m on the fibre $M_x = \pi^{-1}(x)$ factorizes as

$$\mathbb{R}^m \times M_x \rightarrow G_x \times M_x \rightarrow M_x \tag{1.35}$$

through the free transitive action on M_x of the factor group $G_x = \mathbb{R}^m / K_x$, where K_x is the isotropy group of an arbitrary point of M_x . It is the same group for all points of M_x because \mathbb{R}^m is a commutative group. Clearly, M_x is diffeomorphic to the group space of G_x . Since the fibres M_x are mutually diffeomorphic, all isotropy groups K_x are isomorphic to the group \mathbb{Z}_r for some fixed $0 \leq r \leq m$. Accordingly, the groups G_x are isomorphic to the additive group (1.32). Let us bring the fibred manifold (1.34) into a principal bundle with the structure group G_0 , where we denote $\{0\} = \pi(M)$. For this purpose, let us determine isomorphisms $\rho_x : G_0 \rightarrow G_x$ of the group G_0 to the groups G_x , $x \in N$. Then a desired fibrewise action of G_0 on U is defined by the law

$$G_0 \times M_x \rightarrow \rho_x(G_0) \times M_x \rightarrow M_x. \tag{1.36}$$

Generators of each isotropy subgroup K_x of \mathbb{R}^m are given by r linearly independent vectors of the group space \mathbb{R}^m . One can show that there exist ordered collections of generators $(v_1(x), \dots, v_r(x))$ of the groups K_x such that $x \rightarrow v_i(x)$ are smooth \mathbb{R}^m -valued fields on N . Indeed, given a vector $v_i(0)$ and a section σ of the fibred manifold (1.34), each field $v_i(x) = (s_i^\alpha(x))$ is a unique smooth solution of the equation

$$g(s_i^\alpha) \sigma(x) = \sigma(x), \quad (s_i^\alpha(0)) = v_i(0),$$

on an open neighborhood of $\{0\}$. Let us consider the decomposition

$$v_i(0) = B_i^a(0)e_a + C_i^j(0)e_j, \quad a = 1, \dots, m - r, \quad j = 1, \dots, r,$$

where $C_i^j(0)$ is a non-degenerate matrix. Since the fields $v_i(x)$ are smooth, there exists an open neighborhood of $\{0\}$, say N again, where the matrices $C_i^j(x)$ are non-degenerate. Then

$$A(x) = \begin{pmatrix} \text{Id} & (B(x) - B(0))C^{-1}(0) \\ 0 & C(x)C^{-1}(0) \end{pmatrix} \quad (1.37)$$

is a unique linear endomorphism

$$(e_a, e_i) \rightarrow (e_a, e_j)A(x)$$

of the vector space \mathbb{R}^m which transforms the frame $\{v_\lambda(0)\} = \{e_a, v_i(0)\}$ into the frame $\{v_\lambda(x)\} = \{e_a, \vartheta_i(x)\}$, i.e.,

$$v_i(x) = B_i^a(x)e_a + C_i^j(x)e_j = B_i^a(0)e_a + C_i^j(0)[A_j^b(x)e_b + A_j^k(x)e_k].$$

Since $A(x)$ (1.37) also is an automorphism of the group \mathbb{R}^m sending K_0 onto K_x , we obtain a desired isomorphism ρ_x of the group G_0 to the group G_x . Let an element g of the group G_0 be the coset of an element $g(s^\lambda)$ of the group \mathbb{R}^m . Then it acts on M_x by the rule (1.36) just as the element $g((A_x^{-1})^\lambda_\beta s^\beta)$ of the group \mathbb{R}^m does. Since entries of the matrix A (1.37) are smooth functions on N , this action of the group G_0 on U is smooth. It is free, and $U/G_0 = N$. Then the fibred manifold (1.34) is a trivial principal bundle with the structure group G_0 . Given a section σ of this principal bundle, its trivialization $U = N \times G_0$ is defined by assigning the points $\rho^{-1}(g_x)$ of the group space G_0 to the points $g_x\sigma(x)$, $g_x \in G_x$, of a fibre M_x . Let us endow G_0 with the standard coordinate atlas $(r^\lambda) = (t^a, \varphi^i)$ of the group (1.32). Then U admits the trivialization (1.33) with respect to the bundle coordinates (x^A, t^a, φ^i) where x^A , $A = 1, \dots, \dim Z - m$, are coordinates on a base N . The vector fields ϑ_λ on U relative to these coordinates read

$$\vartheta_a = \partial_a, \quad \vartheta_i = -(BC^{-1})_i^a(x)\partial_a + (C^{-1})_i^k(x)\partial_k. \quad (1.38)$$

Accordingly, the subring \mathcal{S} restricted to U is the pull-back $\pi^*C^\infty(N)$ onto U of the ring of smooth functions on N .

(III). Let us split the coordinates (x^A) on N into some m coordinates (J_λ) and the rest $\dim Z - 2m$ coordinates (z^A) . Then we can provide the toroidal domain U (1.33) with the Poisson bivector field

$$w = \partial^\lambda \wedge \partial_\lambda \quad (1.39)$$

of rank $2m$. The independent complete vector fields ∂_a and ∂_i are Hamiltonian vector fields of the functions $S_a = J_a$ and $S_i = J_i$ on U which are in involution with respect to the Poisson bracket

$$\{f, f'\} = \partial^\lambda f \partial_\lambda f' - \partial_\lambda f \partial^\lambda f' \quad (1.40)$$

defined by the bivector field w (1.39). By virtue of the expression (1.38), the Hamiltonian vector fields $\{\partial_\lambda\}$ generate the \mathcal{S} -algebra \mathcal{A} . Therefore, (w, \mathcal{A}) is a partially integrable system. \square

Remark 1.7: Condition (ii) of Theorem 1.10 is equivalent to that $U \rightarrow U/G$ is a fibred manifold [62]. It should be emphasized that a fibration in invariant submanifolds is a standard property of integrable systems [3, 6, 10, 26, 30, 65]. If fibres of such a fibred manifold are assumed to be compact then this fibred

manifold is a fibre bundle (Theorem 7.2) and vertical vector fields on it (e.g., in condition (i) of Theorem 1.10) are complete (Theorem 7.5). \square

A Poisson structure in Theorem 1.10 is by no means unique. Given the toroidal domain U (1.33) provided with bundle coordinates (x^A, r^λ) , it is readily observed that, if a Poisson bivector field on U satisfies Definition 1.9, it takes the form

$$w = w_1 + w_2 = w^{A\lambda}(x^B)\partial_A \wedge \partial_\lambda + w^{\mu\nu}(x^B, r^\lambda)\partial_\mu \wedge \partial_\nu. \quad (1.41)$$

The converse also holds as follows.

THEOREM 1.11: For any Poisson bivector field w (1.41) of rank $2m$ on the toroidal domain U (1.33), there exists a toroidal domain $U' \subset U$ such that a dynamical algebra \mathcal{A} in Theorem 1.10 is a partially integrable system on U' . \square

Remark 1.8: It is readily observed that any Poisson bivector field w (1.41) fulfills condition (b) in Definition 1.9, but condition (a) imposes a restriction on the toroidal domain U . The key point is that the characteristic foliation \mathcal{F} of U yielded by the Poisson bivector fields w (1.41) is the pull-back of an m -dimensional foliation \mathcal{F}_N of the base N , which is defined by the first summand w_1 (1.41) of w . With respect to the adapted coordinates (J_λ, z^A) , $\lambda = 1, \dots, m$, on the foliated manifold (N, \mathcal{F}_N) , the Poisson bivector field w reads

$$w = w_\nu^\mu(J_\lambda, z^A)\partial^\nu \wedge \partial_\mu + w^{\mu\nu}(J_\lambda, z^A, r^\lambda)\partial_\mu \wedge \partial_\nu. \quad (1.42)$$

Then condition (a) in Definition 1.9 is satisfied if $N' \subset N$ is a domain of a coordinate chart (J_λ, z^A) of the foliation \mathcal{F}_N . In this case, the dynamical algebra \mathcal{A} on the toroidal domain $U' = \pi^{-1}(N')$ is generated by the Hamiltonian vector fields

$$\vartheta_\lambda = -w[dJ_\lambda = w_\lambda^\mu \partial_\mu] \quad (1.43)$$

of the m independent functions $S_\lambda = J_\lambda$. \square

Proof: The characteristic distribution of the Poisson bivector field w (1.41) is spanned by the Hamiltonian vector fields

$$v^A = -w[dx^A = w^{A\mu} \partial_\mu] \quad (1.44)$$

and the vector fields

$$w[dr^\lambda = w^{A\lambda} \partial_A + 2w^{\mu\lambda} \partial_\mu].$$

Since w is of rank $2m$, the vector fields ∂_μ can be expressed in the vector fields v^A (1.44). Hence, the characteristic distribution of w is spanned by the Hamiltonian vector fields v^A (1.44) and the vector fields

$$v^\lambda = w^{A\lambda} \partial_A. \quad (1.45)$$

The vector fields (1.45) are projected onto N . Moreover, one can derive from the relation $[w, w] = 0$ that they generate a Lie algebra and, consequently, span an involutive distribution \mathcal{V}_N of rank m on N . Let \mathcal{F}_N denote the corresponding foliation of N . We

consider the pull-back $\mathcal{F} = \pi^* \mathcal{F}_N$ of this foliation onto U by the trivial fibration π [62]. Its leaves are the inverse images $\pi^{-1}(F_N)$ of leaves F_N of the foliation \mathcal{F}_N , and so is its characteristic distribution

$$T\mathcal{F} = (T\pi)^{-1}(\mathcal{V}_N).$$

This distribution is spanned by the vector fields v^λ (1.45) on U and the vertical vector fields on $U \rightarrow N$, namely, the vector fields v^A (1.44) generating the algebra \mathcal{A} . Hence, $T\mathcal{F}$ is the characteristic distribution of the Poisson bivector field w . Furthermore, since $U \rightarrow N$ is a trivial bundle, each leaf $\pi^{-1}(F_N)$ of the pull-back foliation \mathcal{F} is the manifold product of a leaf F_N of N and the toroidal cylinder $\mathbb{R}^{k-m} \times T^m$. It follows that the foliated manifold (U, \mathcal{F}) can be provided with an adapted coordinate atlas

$$\{(U_\iota, J_\lambda, z^A, r^\lambda)\}, \quad \lambda = 1, \dots, k, \quad A = 1, \dots, \dim Z - 2m,$$

such that (J_λ, z^A) are adapted coordinates on the foliated manifold (N, \mathcal{F}_N) . Relative to these coordinates, the Poisson bivector field (1.41) takes the form (1.42). Let N' be the domain of this coordinate chart. Then the dynamical algebra \mathcal{A} on the toroidal domain $U' = \pi^{-1}(N')$ is generated by the Hamiltonian vector fields ϑ_λ (1.43) of functions $S_\lambda = J_\lambda$. \square

Remark 1.9: Let us note that the coefficients $w^{\mu\nu}$ in the expressions (1.41) and (1.42) are affine in coordinates r^λ because of the relation $[w, w] = 0$ and, consequently, they are constant on tori. \square

Now, let w and w' be two different Poisson structures (1.41) on the toroidal domain (1.33) which make a commutative dynamical algebra \mathcal{A} into different partially integrable systems (w, \mathcal{A}) and (w', \mathcal{A}) .

DEFINITION 1.12: We agree to call the triple (w, w', \mathcal{A}) a bi-Hamiltonian partially integrable system if any Hamiltonian vector field $\vartheta \in \mathcal{A}$ with respect to w possesses the same Hamiltonian representation

$$\vartheta = -w \lrcorner df = -w' \lrcorner df, \quad f \in \mathcal{S}, \quad (1.46)$$

relative to w' , and *vice versa*. \square

Definition 1.12 establishes a *sui generis* equivalence between the partially integrable systems (w, \mathcal{A}) and (w', \mathcal{A}) . Theorem 1.13 below states that the triple (w, w', \mathcal{A}) is a bi-Hamiltonian partially integrable system in accordance with Definition 1.12 iff the Poisson bivector fields w and w' (1.41) differ only in the second terms w_2 and w'_2 . Moreover, these Poisson bivector fields admit a recursion operator as follows.

THEOREM 1.13: (I) The triple (w, w', \mathcal{A}) is a bi-Hamiltonian partially integrable system in accordance with Definition 1.12 iff the Poisson bivector fields w and w' (1.41) differ in the second terms w_2 and w'_2 . (II) These Poisson bivector fields admit a recursion operator. \square

Proof: (I). It is easily justified that, if Poisson bivector fields w (1.41) fulfil Definition 1.12, they are distinguished only by the second summand w_2 . Conversely, as follows

from the proof of Theorem 1.11, the characteristic distribution of a Poisson bivector field w (1.41) is spanned by the vector fields (1.44) and (1.45). Hence, all Poisson bivector fields w (1.41) distinguished only by the second summand w_2 have the same characteristic distribution, and they bring \mathcal{A} into a partially integrable system on the same toroidal domain U' . Then the condition in Definition 1.12 is easily justified. (II). The result follows from forthcoming Lemma 1.14. \square

Given a smooth real manifold X , let w and w' be Poisson bivector fields of rank $2m$ on X , and let w^\sharp and w'^\sharp be the corresponding bundle homomorphisms (1.5). A tangent-valued one-form R on X yields bundle endomorphisms

$$R : TX \rightarrow TX, \quad R^* : T^*X \rightarrow T^*X. \quad (1.47)$$

It is called a recursion operator if

$$w'^\sharp = R \circ w^\sharp = w^\sharp \circ R^*. \quad (1.48)$$

Given a Poisson bivector field w and a tangent valued one-form R such that $R \circ w^\sharp = w^\sharp \circ R^*$, the well-known sufficient condition for $R \circ w^\sharp$ to be a Poisson bivector field is that the Nijenhuis torsion (7.36) of R , seen as a tangent-valued one-form, and the Magri – Morosi concomitant of R and w vanish [11, 66]. However, as we will see later, recursion operators between Poisson bivector fields in Theorem 1.13 need not satisfy these conditions.

LEMMA 1.14: A recursion operator between Poisson structures of the same rank exists iff their characteristic distributions coincide. \square

Proof: It follows from the equalities (1.48) that a recursion operator R sends the characteristic distribution of w to that of w' , and these distributions coincide if w and w' are of the same rank. Conversely, let regular Poisson structures w and w' possess the same characteristic distribution $T\mathcal{F} \rightarrow TX$ tangent to a foliation \mathcal{F} of X . We have the exact sequences (7.39) – (7.40). The bundle homomorphisms w^\sharp and w'^\sharp (1.5) factorize in a unique fashion (1.13) through the bundle isomorphisms $w_{\mathcal{F}}^\sharp$ and $w'_{\mathcal{F}}^\sharp$ (1.13). Let us consider the inverse isomorphisms

$$w_{\mathcal{F}}^\flat : T\mathcal{F} \rightarrow T\mathcal{F}^*, \quad w'_{\mathcal{F}}^\flat : T\mathcal{F} \rightarrow T\mathcal{F}^* \quad (1.49)$$

and the compositions

$$R_{\mathcal{F}} = w'_{\mathcal{F}}^\sharp \circ w_{\mathcal{F}}^\flat : T\mathcal{F} \rightarrow T\mathcal{F}, \quad R_{\mathcal{F}}^* = w_{\mathcal{F}}^\flat \circ w'_{\mathcal{F}}^\sharp : T\mathcal{F}^* \rightarrow T\mathcal{F}^*. \quad (1.50)$$

There is the obvious relation

$$w'_{\mathcal{F}}^\sharp = R_{\mathcal{F}} \circ w_{\mathcal{F}}^\sharp = w_{\mathcal{F}}^\sharp \circ R_{\mathcal{F}}^*.$$

In order to obtain a recursion operator (1.48), it suffices to extend the morphisms $R_{\mathcal{F}}$ and $R_{\mathcal{F}}^*$ (1.50) onto TX and T^*X , respectively. For this purpose, let us consider a splitting

$$\begin{aligned} \zeta : TX &\rightarrow T\mathcal{F}, \\ TX &= T\mathcal{F} \oplus (\text{Id} - i_{\mathcal{F}} \circ \zeta)TX = T\mathcal{F} \oplus E, \end{aligned}$$

of the exact sequence (7.39) and the dual splitting

$$\begin{aligned}\zeta^* : T\mathcal{F}^* &\rightarrow T^*X, \\ T^*X &= \zeta^*(T\mathcal{F}^*) \oplus (\text{Id} - \zeta^* \circ i_{\mathcal{F}}^*)T^*X = \zeta^*(T\mathcal{F}^*) \oplus E',\end{aligned}$$

of the exact sequence (7.40). Then the desired extensions are

$$R = R_{\mathcal{F}} \times \text{Id } E, \quad R^* = (\zeta^* \circ R_{\mathcal{F}}^*) \times \text{Id } E'.$$

This recursion operator is invertible, i.e., the morphisms (1.47) are bundle isomorphisms. \square

For instance, the Poisson bivector field w (1.41) and the Poisson bivector field

$$w_0 = w^{A\lambda} \partial_A \wedge \partial_\lambda$$

admit a recursion operator $w_0^\sharp = R \circ w^\sharp$ whose entries are given by the equalities

$$R_B^A = \delta_B^A, \quad R_\nu^\mu = \delta_\nu^\mu, \quad R_\lambda^A = 0, \quad w^{\mu\lambda} = R_B^\lambda w^{B\mu}. \quad (1.51)$$

Its Nijenhuis torsion (7.36) fails to vanish, unless coefficients $w^{\mu\lambda}$ are independent of coordinates r^λ .

Given a partially integrable system (w, \mathcal{A}) in Theorem 1.11, the bivector field w (1.42) can be brought into the canonical form (1.39) with respect to partial action-angle coordinates in forthcoming Theorem 1.15. This theorem extends the Liouville – Arnold theorem to the case of a Poisson structure and a non-compact invariant submanifold [35, 41].

THEOREM 1.15: Given a partially integrable system (w, \mathcal{A}) on a Poisson manifold (U, w) , there exists a toroidal domain $U' \subset U$ equipped with partial action-angle coordinates $(I_a, I_i, z^A, \tau^a, \phi^i)$ such that, restricted to U' , a Poisson bivector field takes the canonical form

$$w = \partial^a \wedge \partial_a + \partial^i \wedge \partial_i, \quad (1.52)$$

while the dynamical algebra \mathcal{A} is generated by Hamiltonian vector fields of the action coordinate functions $S_a = I_a, S_i = I_i$. \square

Proof: First, let us employ Theorem 1.11 and restrict U to the toroidal domain, say U again, equipped with coordinates $(J_\lambda, z^A, r^\lambda)$ such that the Poisson bivector field w takes the form (1.42) and the algebra \mathcal{A} is generated by the Hamiltonian vector fields ϑ_λ (1.43) of m independent functions $S_\lambda = J_\lambda$ in involution. Let us choose these vector fields as new generators of the group G and return to Theorem 1.10. In accordance with this theorem, there exists a toroidal domain $U' \subset U$ provided with another trivialization $U' \rightarrow N' \subset N$ in toroidal cylinders $\mathbb{R}^{m-r} \times T^r$ and endowed with bundle coordinates $(J_\lambda, z^A, r^\lambda)$ such that the vector fields ϑ_λ (1.43) take the form (1.38). For the sake of simplicity, let U', N' and y^λ be denoted U, N and $r^\lambda = (t^a, \varphi^i)$ again. Herewith, the Poisson bivector field w is given by the expression (1.42) with new coefficients. Let $w^\sharp : T^*U \rightarrow TU$ be the corresponding bundle homomorphism. It factorizes in a unique fashion (1.13):

$$w^\sharp : T^*U \xrightarrow{i_{\mathcal{F}}^*} T\mathcal{F}^* \xrightarrow{w_{\mathcal{F}}^\sharp} T\mathcal{F} \xrightarrow{i_{\mathcal{F}}} TU$$

through the bundle isomorphism

$$w_{\mathcal{F}}^{\sharp} : T\mathcal{F}^* \rightarrow T\mathcal{F}, \quad w_{\mathcal{F}}^{\sharp} : \alpha \rightarrow -w(x)|_{\alpha}.$$

Then the inverse isomorphisms $w_{\mathcal{F}}^{\flat} : T\mathcal{F} \rightarrow T\mathcal{F}^*$ provides the foliated manifold (U, \mathcal{F}) with the leafwise symplectic form

$$\Omega_{\mathcal{F}} = \Omega^{\mu\nu}(J_{\lambda}, z^A, t^a) \tilde{d}J_{\mu} \wedge \tilde{d}J_{\nu} + \Omega_{\mu}^{\nu}(J_{\lambda}, z^A) \tilde{d}J_{\nu} \wedge \tilde{d}r^{\mu}, \quad (1.53)$$

$$\Omega_{\mu}^{\alpha} w_{\beta}^{\mu} = \delta_{\beta}^{\alpha}, \quad \Omega^{\alpha\beta} = -\Omega_{\mu}^{\alpha} \Omega_{\nu}^{\beta} w^{\mu\nu}. \quad (1.54)$$

Let us show that it is \tilde{d} -exact. Let F be a leaf of the foliation \mathcal{F} of U . There is a homomorphism of the de Rham cohomology $H_{\text{DR}}^*(U)$ of U to the de Rham cohomology $H_{\text{DR}}^*(F)$ of F , and it factorizes through the leafwise cohomology $H_{\mathcal{F}}^*(U)$. Since N is a domain of an adapted coordinate chart of the foliation \mathcal{F}_N , the foliation \mathcal{F}_N of N is a trivial fibre bundle

$$N = V \times W \rightarrow W.$$

Since \mathcal{F} is the pull-back onto U of the foliation \mathcal{F}_N of N , it also is a trivial fibre bundle

$$U = V \times W \times (\mathbb{R}^{k-m} \times T^m) \rightarrow W \quad (1.55)$$

over a domain $W \subset \mathbb{R}^{\dim Z - 2m}$. It follows that

$$H_{\text{DR}}^*(U) = H_{\text{DR}}^*(T^r) = H_{\mathcal{F}}^*(U).$$

Then the closed leafwise two-form $\Omega_{\mathcal{F}}$ (1.53) is exact due to the absence of the term $\Omega_{\mu\nu} dr^{\mu} \wedge dr^{\nu}$. Moreover, $\Omega_{\mathcal{F}} = \tilde{d}\Xi$ where Ξ reads

$$\Xi = \Xi^{\alpha}(J_{\lambda}, z^A, r^{\lambda}) \tilde{d}J_{\alpha} + \Xi_i(J_{\lambda}, z^A) \tilde{d}\varphi^i$$

up to a \tilde{d} -exact leafwise form. The Hamiltonian vector fields $\vartheta_{\lambda} = \vartheta_{\lambda}^{\mu} \partial_{\mu}$ (1.38) obey the relation

$$\vartheta_{\lambda} \lrcorner \Omega_{\mathcal{F}} = -\tilde{d}J_{\lambda}, \quad \Omega_{\beta}^{\alpha} \vartheta_{\lambda}^{\beta} = \delta_{\lambda}^{\alpha}, \quad (1.56)$$

which falls into the following conditions

$$\Omega_i^{\lambda} = \partial^{\lambda} \Xi_i - \partial_i \Xi^{\lambda}, \quad (1.57)$$

$$\Omega_a^{\lambda} = -\partial_a \Xi^{\lambda} = \delta_a^{\lambda}. \quad (1.58)$$

The first of the relations (1.54) shows that Ω_{β}^{α} is a non-degenerate matrix independent of coordinates r^{λ} . Then the condition (1.57) implies that $\partial_i \Xi^{\lambda}$ are independent of φ^i , and so are Ξ^{λ} since φ^i are cyclic coordinates. Hence,

$$\Omega_i^{\lambda} = \partial^{\lambda} \Xi_i, \quad (1.59)$$

$$\partial_i \lrcorner \Omega_{\mathcal{F}} = -\tilde{d}\Xi_i. \quad (1.60)$$

Let us introduce new coordinates $I_a = J_a$, $I_i = \Xi_i(J_{\lambda})$. By virtue of the equalities (1.58) and (1.59), the Jacobian of this coordinate transformation is regular. The relation (1.60) shows that ∂_i are Hamiltonian vector fields of the functions $S_i = I_i$. Consequently, we can choose vector fields ∂_{λ} as generators of the algebra \mathcal{A} . One obtains from the equality (1.58) that

$$\Xi^a = -t^a + E^a(J_{\lambda}, z^A)$$

and Ξ^i are independent of t^a . Then the leafwise Liouville form Ξ reads

$$\Xi = (-t^a + E^a(I_\lambda, z^A))\tilde{d}I_a + E^i(I_\lambda, z^A)\tilde{d}I_i + I_i\tilde{d}\phi^i.$$

The coordinate shifts

$$\tau^a = -t^a + E^a(I_\lambda, z^A), \quad \phi^i = \varphi^i - E^i(I_\lambda, z^A)$$

bring the leafwise form $\Omega_{\mathcal{F}}$ (1.53) into the canonical form

$$\Omega_{\mathcal{F}} = \tilde{d}I_a \wedge \tilde{d}\tau^a + \tilde{d}I_i \wedge \tilde{d}\phi^i$$

which ensures the canonical form (1.52) of a Poisson bivector field w . \square

1.4 Partially integrable system on a symplectic manifold

Let \mathcal{A} be a commutative dynamical algebra on a $2n$ -dimensional connected symplectic manifold (Z, Ω) . Let it obey condition (a) in Definition 1.9. However, condition (b) is not necessarily satisfied, unless $m = n$, i.e., a system is completely integrable. Therefore, we modify a definition of partially integrable systems on a symplectic manifold.

DEFINITION 1.16: A collection $\{S_1, \dots, S_m\}$ of $m \leq n$ independent smooth real functions in involution on a symplectic manifold (Z, Ω) is called a partially integrable system. \square

Remark 1.10: By analogy with Definition 1.9, one can require that functions S_λ in Definition 1.16 are independent almost everywhere on Z . However, all theorems that we have proved above are concerned with partially integrable systems restricted to some open submanifold $Z' \subset Z$ of regular points of Z . Therefore, let us restrict functions S_λ to an open submanifold $Z' \subset Z$ where they are independent, and we obtain a partially integrable system on a symplectic manifold (Z', Ω) which obeys Definition 1.16. However, it may happen that Z' is not connected. In this case, we have different partially integrable systems on different components of Z' . \square

Given a partially integrable system (S_λ) in Definition 1.16, let us consider the map

$$S : Z \rightarrow W \subset \mathbb{R}^m. \tag{1.61}$$

Since functions S_λ are everywhere independent, this map is a submersion onto a domain $W \subset \mathbb{R}^m$, i.e., S (1.61) is a fibred manifold of fibre dimension $2n - m$. Hamiltonian vector fields ϑ_λ of functions S_λ are mutually commutative and independent. Consequently, they span an m -dimensional involutive distribution on Z whose maximal integral manifolds constitute an isotropic foliation \mathcal{F} of Z . Because functions S_λ are constant on leaves of this foliation, each fibre of a fibred manifold $Z \rightarrow W$ (1.61) is foliated by the leaves of the foliation \mathcal{F} .

If $m = n$, we are in the case of a completely integrable system, and leaves of \mathcal{F} are connected components of fibres of the fibred manifold (1.61).

The Poincaré – Lyapounov – Nekhoroshev theorem [26, 65] generalizes the Liouville – Arnold one to a partially integrable system if leaves of the foliation \mathcal{F} are compact. It imposes a sufficient condition which Hamiltonian vector fields v_λ must satisfy in order that the foliation \mathcal{F} is a fibred manifold [26, 27]. Extending the Poincaré – Lyapounov – Nekhoroshev theorem to the case of non-compact invariant submanifolds, we in fact assume from the beginning that these submanifolds form a fibred manifold [35, 41].

THEOREM 1.17: Let a partially integrable system $\{S_1, \dots, S_m\}$ on a symplectic manifold (Z, Ω) satisfy the following conditions.

- (i) The Hamiltonian vector fields ϑ_λ of S_λ are complete.
- (ii) The foliation \mathcal{F} is a fibred manifold

$$\pi : Z \rightarrow N \quad (1.62)$$

whose fibres are mutually diffeomorphic.

Then the following hold.

- (I) The fibres of \mathcal{F} are diffeomorphic to the toroidal cylinder (1.32).
- (II) Given a fibre M of \mathcal{F} , there exists its open saturated neighborhood U whose fibration (1.62) is a trivial principal bundle with the structure group (1.32).
- (III) The neighborhood U is provided with the bundle (partial action-angle) coordinates

$$(I_\lambda, p_s, q^s, y^\lambda) \rightarrow (I_\lambda, p_s, q^s), \quad \lambda = 1, \dots, m, \quad s = 1, \dots, n - m,$$

such that: (i) the action coordinates (I_λ) (1.73) are expressed in the values of the functions (S_λ) , (ii) the angle coordinates (y^λ) (1.76) are coordinates on a toroidal cylinder, and (iii) the symplectic form Ω on U reads

$$\Omega = dI_\lambda \wedge dy^\lambda + dp_s \wedge dq^s. \quad (1.63)$$

□

Proof: (I) The proof of parts (I) and (II) repeats exactly that of parts (I) and (II) of Theorem 1.10. As a result, let

$$\pi : U \rightarrow \pi(U) \subset N \quad (1.64)$$

be a trivial principal bundle with the structure group $\mathbb{R}^{m-r} \times T^r$, endowed with the standard coordinate atlas $(r^\lambda) = (t^\alpha, \varphi^i)$. Then U (1.64) admits a trivialization

$$U = \pi(U) \times (\mathbb{R}^{m-r} \times T^r) \rightarrow \pi(U) \quad (1.65)$$

with respect to the fibre coordinates (t^α, φ^i) . The Hamiltonian vector fields ϑ_λ on U relative to these coordinates read (1.38):

$$\vartheta_\alpha = \partial_\alpha, \quad \vartheta_i = -(BC^{-1})_i^\alpha(x) \partial_\alpha + (C^{-1})_i^k(x) \partial_k. \quad (1.66)$$

In order to specify coordinates on the base $\pi(U)$ of the trivial bundle (1.65), let us consider the fibred manifold S (1.61). It factorizes as

$$S : U \xrightarrow{\pi} \pi(U) \xrightarrow{\pi'} S(U)$$

through the fibre bundle π . The map π' also is a fibred manifold. One can always restrict the domain $\pi(U)$ to a chart of the fibred manifold π' , say $\pi(U)$ again. Then $\pi(U) \rightarrow S(U)$ is a trivial bundle $\pi(U) = S(U) \times V$, and so is $U \rightarrow S(U)$. Thus, we have the composite bundle

$$U = S(U) \times V \times (\mathbb{R}^{m-r} \times T^r) \rightarrow S(U) \times V \rightarrow S(U). \quad (1.67)$$

Let us provide its base $S(U)$ with the coordinates (J_λ) such that

$$J_\lambda \circ S = S_\lambda. \quad (1.68)$$

Then $\pi(U)$ can be equipped with the bundle coordinates (J_λ, x^A) , $A = 1, \dots, 2(n-m)$, and $(J_\lambda, x^A, t^a, \varphi^i)$ are coordinates on U (1.33). Since fibres of $U \rightarrow \pi(U)$ are isotropic, a symplectic form Ω on U relative to the coordinates $(J_\lambda, x^A, r^\lambda)$ reads

$$\begin{aligned} \Omega = & \Omega^{\alpha\beta} dJ_\alpha \wedge dJ_\beta + \Omega_\beta^\alpha dJ_\alpha \wedge dr^\beta + \\ & \Omega_{AB} dx^A \wedge dx^B + \Omega_A^\lambda dJ_\lambda \wedge dx^A + \Omega_{A\beta} dx^A \wedge dr^\beta. \end{aligned} \quad (1.69)$$

The Hamiltonian vector fields $\vartheta_\lambda = \vartheta_\lambda^\mu \partial_\mu$ (1.66) obey the relations $\vartheta_\lambda \lrcorner \Omega = -dJ_\lambda$ which result in the coordinate conditions

$$\Omega_\beta^\alpha \vartheta_\lambda^\beta = \delta_\lambda^\alpha, \quad \Omega_{A\beta} \vartheta_\lambda^\beta = 0. \quad (1.70)$$

The first of them shows that Ω_β^α is a non-degenerate matrix independent of coordinates r^λ . Then the second one implies that $\Omega_{A\beta} = 0$. By virtue of the well-known Künneth formula for the de Rham cohomology of manifold products, the closed form Ω (1.69) is exact, i.e., $\Omega = d\Xi$ where the Liouville form Ξ is

$$\Xi = \Xi^\alpha(J_\lambda, x^B, r^\lambda) dJ_\alpha + \Xi_i(J_\lambda, x^B) d\varphi^i + \Xi_A(J_\lambda, x^B, r^\lambda) dx^A.$$

Since $\Xi_a = 0$ and Ξ_i are independent of φ^i , it follows from the relations

$$\Omega_{A\beta} = \partial_A \Xi_\beta - \partial_\beta \Xi_A = 0$$

that Ξ_A are independent of coordinates t^a and are at most affine in φ^i . Since φ^i are cyclic coordinates, Ξ_A are independent of φ^i . Hence, Ξ_i are independent of coordinates x^A , and the Liouville form reads

$$\Xi = \Xi^\alpha(J_\lambda, x^B, r^\lambda) dJ_\alpha + \Xi_i(J_\lambda) d\varphi^i + \Xi_A(J_\lambda, x^B) dx^A. \quad (1.71)$$

Because entries Ω_β^α of $d\Xi = \Omega$ are independent of r^λ , we obtain the following.

(i) $\Omega_i^\lambda = \partial^\lambda \Xi_i - \partial_i \Xi^\lambda$. Consequently, $\partial_i \Xi^\lambda$ are independent of φ^i , and so are Ξ^λ since φ^i are cyclic coordinates. Hence, $\Omega_i^\lambda = \partial^\lambda \Xi_i$ and $\partial_i \lrcorner \Omega = -d\Xi_i$. A glance at the last equality shows that ∂_i are Hamiltonian vector fields. It follows that, from the beginning, one can separate m generating functions on U , say S_i again, whose Hamiltonian vector fields are tangent to invariant tori. In this case, the matrix B in the expressions (1.37) and (1.66) vanishes, and the Hamiltonian vector fields ϑ_λ (1.66) read

$$\vartheta_a = \partial_a, \quad \vartheta_i = (C^{-1})_i^k \partial_k. \quad (1.72)$$

Moreover, the coordinates t^a are exactly the flow parameters s^a . Substituting the expressions (1.72) into the first condition (1.70), we obtain

$$\begin{aligned} \Omega = & \Omega^{\alpha\beta} dJ_\alpha \wedge dJ_\beta + dJ_a \wedge ds^a + C_k^i dJ_i \wedge d\varphi^k + \\ & \Omega_{AB} dx^A \wedge dx^B + \Omega_A^\lambda dJ_\lambda \wedge dx^A. \end{aligned}$$

It follows that Ξ_i are independent of J_a , and so are $C_i^k = \partial^k \Xi_i$.

(ii) $\Omega_a^\lambda = -\partial_a \Xi^\lambda = \delta_a^\lambda$. Hence, $\Xi^a = -s^a + E^a(J_\lambda)$ and $\Xi^i = E^i(J_\lambda, x^B)$ are independent of s^a .

In view of items (i) – (ii), the Liouville form Ξ (1.71) reads

$$\begin{aligned} \Xi &= (-s^a + E^a(J_\lambda, x^B))dJ_a + E^i(J_\lambda, x^B)dJ_i + \\ &\quad \Xi_i(J_j)d\varphi^i + \Xi_A(J_\lambda, x^B)dx^A. \end{aligned}$$

Since the matrix $\partial^k \Xi_i$ is non-degenerate, we can perform the coordinate transformations

$$\begin{aligned} I_a &= J_a, & I_i &= \Xi_i(J_j), \\ r'^a &= -s^a + E^a(J_\lambda, x^B), & r'^i &= \varphi^i - E^j(J_\lambda, x^B) \frac{\partial J_j}{\partial I_i}. \end{aligned} \quad (1.73)$$

These transformations bring Ω into the form

$$\Omega = dI_\lambda \wedge dr'^\lambda + \Omega_{AB}(I_\mu, x^C)dx^A \wedge dx^B + \Omega_A^\lambda(I_\mu, x^C)dI_\lambda \wedge dx^A. \quad (1.74)$$

Since functions I_λ are in involution and their Hamiltonian vector fields ∂_λ mutually commute, a point $z \in M$ has an open neighborhood

$$U_z = \pi(U_z) \times O_z, \quad O_z \subset \mathbb{R}^{m-r} \times T^r,$$

endowed with local Darboux coordinates $(I_\lambda, p_s, q^s, y^\lambda)$, $s = 1, \dots, n - m$, such that the symplectic form Ω (1.74) is given by the expression

$$\Omega = dI_\lambda \wedge dy^\lambda + dp_s \wedge dq^s. \quad (1.75)$$

Here, $y^\lambda(I_\lambda, x^A, r'^\alpha)$ are local functions

$$y^\lambda = r'^\lambda + f^\lambda(I_\lambda, x^A) \quad (1.76)$$

on U_z . With the above-mentioned group G of flows of Hamiltonian vector fields ϑ_λ , one can extend these functions to an open neighborhood

$$\pi(U_z) \times \mathbb{R}^{k-m} \times T^m$$

of M , say U again, by the law

$$y^\lambda(I_\lambda, x^A, G(z)^\alpha) = G(z)^\lambda + f^\lambda(I_\lambda, x^A).$$

Substituting the functions (1.76) on U into the expression (1.74), one brings the symplectic form Ω into the canonical form (1.63) on U . \square

Remark 1.11: If one supposes from the beginning that leaves of the foliation \mathcal{F} are compact, the conditions of Theorem 1.17 can be replaced with that \mathcal{F} is a fibred manifold (see Theorems 7.2 and 7.5). \square

1.5 Global partially integrable systems

As was mentioned above, there is a topological obstruction to the existence of global action-angle coordinates. Forthcoming Theorem 1.18 is a global generalization of Theorem 1.17 [23, 41, 74].

THEOREM 1.18: Let a partially integrable system $\{S_1, \dots, S_m\}$ on a symplectic manifold (Z, Ω) satisfy the following conditions.

- (i) The Hamiltonian vector fields ϑ_λ of S_λ are complete.
- (ii) The foliation \mathcal{F} is a fibre bundle

$$\pi : Z \rightarrow N. \quad (1.77)$$

(iii) Its base N is simply connected and the cohomology $H^2(N; \mathbb{Z})$ of N with coefficients in the constant sheaf \mathbb{Z} is trivial.

Then the following hold.

(I) The fibre bundle (1.77) is a trivial principal bundle with the structure group (1.32), and we have a composite fibred manifold

$$S = \zeta \circ \pi : Z \longrightarrow N \longrightarrow W, \quad (1.78)$$

where $N \rightarrow W$ however need not be a fibre bundle.

(II) The fibred manifold (1.78) is provided with the global fibred action-angle coordinates

$$(I_\lambda, x^A, y^\lambda) \rightarrow (I_\lambda, x^A) \rightarrow (I_\lambda), \quad \lambda = 1, \dots, m, \quad A = 1, \dots, 2(n - m),$$

such that: (i) the action coordinates (I_λ) (1.87) are expressed in the values of the functions (S_λ) and they possess identity transition functions, (ii) the angle coordinates (y^λ) (1.87) are coordinates on a toroidal cylinder, (iii) the symplectic form Ω on U reads

$$\Omega = dI_\lambda \wedge dy^\lambda + \Omega_A^\lambda dI_\lambda \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B. \quad (1.79)$$

□

Proof: Following part (I) of the proof of Theorems 1.10 and 1.17, one can show that a typical fibre of the fibre bundle (1.77) is the toroidal cylinder (1.32). Let us bring this fibre bundle into a principal bundle with the structure group (1.32). Generators of each isotropy subgroup K_x of \mathbb{R}^m are given by r linearly independent vectors $u_i(x)$ of a group space \mathbb{R}^m . These vectors are assembled into an r -fold covering $K \rightarrow N$. This is a subbundle of the trivial bundle

$$N \times \mathbb{R}^m \rightarrow N \quad (1.80)$$

whose local sections are local smooth sections of the fibre bundle (1.80). Such a section over an open neighborhood of a point $x \in N$ is given by a unique local solution $s^\lambda(x')e_\lambda$, $e_\lambda = \vartheta_\lambda$, of the equation

$$g(s^\lambda)\sigma(x') = \exp(s^\lambda e_\lambda)\sigma(x') = \sigma(x'), \quad s^\lambda(x)e_\lambda = u_i(x),$$

where σ is an arbitrary local section of the fibre bundle $Z \rightarrow N$ over an open neighborhood of x . Since N is simply connected, the covering $K \rightarrow N$ admits r everywhere different global sections u_i which are global smooth sections $u_i(x) = u_i^\lambda(x)e_\lambda$ of the fibre bundle (1.80). Let us fix a point of N further denoted by $\{0\}$. One can determine linear combinations of the functions S_λ , say again S_λ , such that $u_i(0) = e_i$, $i = m - r, \dots, m$, and the group G_0 is identified to the group $\mathbb{R}^{m-r} \times T^r$. Let E_x denote an r -dimensional subspace of \mathbb{R}^m passing through the points $u_1(x), \dots, u_r(x)$. The spaces E_x , $x \in N$, constitute an r -dimensional subbundle $E \rightarrow N$ of the trivial bundle (1.80). Moreover, the latter is split into the Whitney sum of vector bundles $E \oplus E'$, where $E'_x = \mathbb{R}^m/E_x$ [48]. Then there is a global smooth section γ of the trivial principal bundle $N \times GL(m, \mathbb{R}) \rightarrow N$ such that $\gamma(x)$ is a morphism of E_0 onto E_x , where

$$u_i(x) = \gamma(x)(e_i) = \gamma_i^\lambda e_\lambda.$$

This morphism also is an automorphism of the group \mathbb{R}^m sending K_0 onto K_x . Therefore, it provides a group isomorphism $\rho_x : G_0 \rightarrow G_x$. With these isomorphisms, one can define the fibrewise action of the group G_0 on Z given by the law

$$G_0 \times M_x \rightarrow \rho_x(G_0) \times M_x \rightarrow M_x. \quad (1.81)$$

Namely, let an element of the group G_0 be the coset $g(s^\lambda)/K_0$ of an element $g(s^\lambda)$ of the group \mathbb{R}^m . Then it acts on M_x by the rule (1.81) just as the coset $g((\gamma(x)^{-1})^\lambda s^\beta)/K_x$ of an element $g((\gamma(x)^{-1})^\lambda s^\beta)$ of \mathbb{R}^m does. Since entries of the matrix γ are smooth functions on N , the action (1.81) of the group G_0 on Z is smooth. It is free, and $Z/G_0 = N$. Thus, $Z \rightarrow N$ (1.77) is a principal bundle with the structure group $G_0 = \mathbb{R}^{m-r} \times T^r$.

Furthermore, this principal bundle over a paracompact smooth manifold N is trivial as follows. In accordance with the well-known theorem [48], its structure group G_0 (1.32) is reducible to the maximal compact subgroup T^r , which also is the maximal compact subgroup of the group product $\times_r GL(1, \mathbb{C})$. Therefore, the equivalence classes of T^r -principal bundles ξ are defined as

$$c(\xi) = c(\xi_1 \oplus \dots \oplus \xi_r) = (1 + c_1(\xi_1)) \cdots (1 + c_1(\xi_r))$$

by the Chern classes $c_1(\xi_i) \in H^2(N; \mathbb{Z})$ of $U(1)$ -principal bundles ξ_i over N [48]. Since the cohomology group $H^2(N; \mathbb{Z})$ of N is trivial, all Chern classes c_1 are trivial, and the principal bundle $Z \rightarrow N$ over a contractible base also is trivial. This principal bundle can be provided with the following coordinate atlas.

Let us consider the fibred manifold $S : Z \rightarrow W$ (1.61). Because functions S_λ are constant on fibres of the fibre bundle $Z \rightarrow N$ (1.77), the fibred manifold (1.61) factorizes through the fibre bundle (1.77), and we have the composite fibred manifold (1.78). Let us provide the principal bundle $Z \rightarrow N$ with a trivialization

$$Z = N \times \mathbb{R}^{m-r} \times T^r \rightarrow N, \quad (1.82)$$

whose fibres are endowed with the standard coordinates $(r^\lambda) = (t^a, \varphi^i)$ on the toroidal cylinder (1.32). Then the composite fibred manifold (1.78) is provided with the fibred coordinates

$$(J_\lambda, x^A, t^a, \varphi^i), \quad (1.83)$$

$$\lambda = 1, \dots, m, \quad A = 1, \dots, 2(n-m), \quad a = 1, \dots, m-r, \quad i = 1, \dots, r,$$

where J_λ (1.68) are coordinates on the base W induced by Cartesian coordinates on \mathbb{R}^m , and (J_λ, x^A) are fibred coordinates on the fibred manifold $\zeta : N \rightarrow W$. The coordinates J_λ on $W \subset \mathbb{R}^m$ and the coordinates (t^a, φ^i) on the trivial bundle (1.82) possess the identity transition functions, while the transition function of coordinates (x^A) depends on the coordinates (J_λ) in general.

The Hamiltonian vector fields ϑ_λ on Z relative to the coordinates (1.83) take the form

$$\vartheta_\lambda = \vartheta_\lambda^a(x) \partial_a + \vartheta_\lambda^i(x) \partial_i. \quad (1.84)$$

Since these vector fields commute (i.e., fibres of $Z \rightarrow N$ are isotropic), the symplectic form Ω on Z reads

$$\begin{aligned} \Omega = \Omega_\beta^\alpha dJ_\alpha \wedge dr^\beta + \Omega_{\alpha A} dr^\alpha \wedge dx^A + \Omega^{\alpha\beta} dJ_\alpha \wedge dJ_\beta + \\ \Omega_A^\alpha dJ_\alpha \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B. \end{aligned} \quad (1.85)$$

This form is exact (see Lemma 1.19 below). Thus, we can write

$$\begin{aligned} \Omega = d\Xi, \quad \Xi = \Xi^\lambda(J_\alpha, x^B, r^\alpha) dJ_\lambda + \Xi_\lambda(J_\alpha, x^B) dr^\lambda + \\ \Xi_A(J_\alpha, x^B, r^\alpha) dx^A. \end{aligned} \quad (1.86)$$

Up to an exact summand, the Liouville form Ξ (1.86) is brought into the form

$$\Xi = \Xi^\lambda(J_\alpha, x^B, r^\alpha) dJ_\lambda + \Xi_i(J_\alpha, x^B) d\varphi^i + \Xi_A(J_\alpha, x^B, r^\alpha) dx^A,$$

i.e., it does not contain the term $\Xi_a dt^a$.

The Hamiltonian vector fields ϑ_λ (1.84) obey the relations $\vartheta_\lambda | \Omega = -dJ_\lambda$, which result in the coordinate conditions (1.70). Then following the proof of Theorem 1.17, we can show that a symplectic form Ω on Z is given by the expression (1.79) with respect to the coordinates

$$\begin{aligned} I_a = J_a, \quad I_i = \Xi_i(J_j), \\ y^a = -\Xi^a = t^a - E^a(J_\lambda, x^B), \quad y^i = \varphi^i - \Xi^j(J_\lambda, x^B) \frac{\partial J_j}{\partial I_i}. \end{aligned} \quad (1.87)$$

□

LEMMA 1.19: The symplectic form Ω (1.85) is exact. □

Proof: In accordance with the well-known Künneth formula, the de Rham cohomology group of the product (1.82) reads

$$H_{\text{DR}}^2(Z) = H_{\text{DR}}^2(N) \oplus H_{\text{DR}}^1(N) \otimes H_{\text{DR}}^1(T^r) \oplus H_{\text{DR}}^2(T^r).$$

By the de Rham theorem [48], the de Rham cohomology $H_{\text{DR}}^2(N)$ is isomorphic to the cohomology $H^2(N; \mathbb{R})$ of N with coefficients in the constant sheaf \mathbb{R} . It is trivial since

$$H^2(N; \mathbb{R}) = H^2(N; \mathbb{Z}) \otimes \mathbb{R}$$

where $H^2(N; \mathbb{Z})$ is trivial. The first cohomology group $H_{\text{DR}}^1(N)$ of N is trivial because N is simply connected. Consequently, $H_{\text{DR}}^2(Z) = H_{\text{DR}}^2(T^r)$. Then the closed form Ω (1.85) is exact since it does not contain the term $\Omega_{ij} d\varphi^i \wedge d\varphi^j$. □

2 Superintegrable systems

In comparison with partially integrable and completely integrable systems integrals of motion of a superintegrable system need not be in involution. We consider superintegrable systems on a symplectic manifold. A key point is that invariant submanifolds of any superintegrable system are maximal integral manifolds of a certain partially integrable system (Proposition 2.4). Completely integrable systems are particular superintegrable systems (see Definition 2.2).

Our goal are Theorem 2.5 for superintegrable systems, Theorem 2.7 for completely integrable systems, Theorem 2.9 for globally superintegrable systems, and Theorem 2.10 for globally completely integrable systems.

DEFINITION 2.1: Let (Z, Ω) be a $2n$ -dimensional connected symplectic manifold, and let $(C^\infty(Z), \{, \})$ be the Poisson algebra of smooth real functions on Z . A subset

$$F = (F_1, \dots, F_k), \quad n \leq k < 2n, \quad (2.1)$$

of the Poisson algebra $C^\infty(Z)$ is called a superintegrable system if the following conditions hold.

(i) All the functions F_i (called the generating functions of a superintegrable system) are independent, i.e., the k -form $\bigwedge^k dF_i$ nowhere vanishes on Z . It follows that the map $F : Z \rightarrow \mathbb{R}^k$ is a submersion, i.e.,

$$F : Z \rightarrow N = F(Z) \quad (2.2)$$

is a fibred manifold over a domain (i.e., contractible open subset) $N \subset \mathbb{R}^k$ endowed with the coordinates (x_i) such that $x_i \circ F = F_i$.

(ii) There exist smooth real functions s_{ij} on N such that

$$\{F_i, F_j\} = s_{ij} \circ F, \quad i, j = 1, \dots, k. \quad (2.3)$$

(iii) The matrix function \mathbf{s} with the entries s_{ij} (2.3) is of constant corank $m = 2n - k$ at all points of N . \square

Remark 2.1: We restrict our consideration to the case of generating functions which are independent everywhere on a symplectic manifold Z (see Remarks 1.10 and 2.2). \square

If $k = n$, then $\mathbf{s} = 0$, and we are in the case of completely integrable systems as follows.

DEFINITION 2.2: The subset F , $k = n$, (2.1) of the Poisson algebra $C^\infty(Z)$ on a symplectic manifold (Z, Ω) is called a completely integrable system if F_i are independent functions in involution. \square

If $k > n$, the matrix \mathbf{s} is necessarily non-zero. Therefore, superintegrable systems also are called non-commutative completely integrable systems. If $k = 2n - 1$, a superintegrable system is called maximally superintegrable.

The following two assertions clarify the structure of superintegrable systems [18, 22, 41].

PROPOSITION 2.3: Given a symplectic manifold (Z, Ω) , let $F : Z \rightarrow N$ be a fibred manifold such that, for any two functions f, f' constant on fibres of F , their Poisson bracket $\{f, f'\}$ is so. By virtue of Theorem 1.3, N is provided with an unique coinduced Poisson structure $\{, \}_N$ such that F is a Poisson morphism. \square

Since any function constant on fibres of F is a pull-back of some function on N , the superintegrable system (2.1) satisfies the condition of Proposition 2.3 due to item (ii) of Definition 2.1. Thus, the base N of the fibration (2.2) is endowed with a coinduced Poisson structure of corank m . With respect to coordinates x_i in item (i) of Definition 2.1 its bivector field reads

$$w = s_{ij}(x_k) \partial^i \wedge \partial^j. \quad (2.4)$$

PROPOSITION 2.4: Given a fibred manifold $F : Z \rightarrow N$ in Proposition 2.3, the following conditions are equivalent [18, 55]:

- (i) the rank of the coinduced Poisson structure $\{, \}_N$ on N equals $2\dim N - \dim Z$,
- (ii) the fibres of F are isotropic,
- (iii) the fibres of F are maximal integral manifolds of the involutive distribution spanned by the Hamiltonian vector fields of the pull-back F^*C of Casimir functions C of the coinduced Poisson structure (2.4) on N . \square

It is readily observed that the fibred manifold F (2.2) obeys condition (i) of Proposition 2.4 due to item (iii) of Definition 2.1, namely, $k - m = 2(k - n)$.

Fibres of the fibred manifold F (2.2) are called the invariant submanifolds.

Remark 2.2: In many physical models, condition (i) of Definition 2.1 fails to hold. Just as in the case of partially integrable systems, it can be replaced with that a subset $Z_R \subset Z$ of regular points (where $\wedge^k dF_i \neq 0$) is open and dense. Let M be an invariant submanifold through a regular point $z \in Z_R \subset Z$. Then it is regular, i.e., $M \subset Z_R$. Let M admit a regular open saturated neighborhood U_M (i.e., a fibre of F through a point of U_M belongs to U_M). For instance, any compact invariant submanifold M has such a neighborhood U_M . The restriction of functions F_i to U_M defines a superintegrable system on U_M which obeys Definition 2.1. In this case, one says that a superintegrable system is considered around its invariant submanifold M . \square

Let (Z, Ω) be a $2n$ -dimensional connected symplectic manifold. Given the superintegrable system (F_i) (2.1) on (Z, Ω) , the well known Mishchenko – Fomenko theorem (Theorem 2.6) states the existence of (semi-local) generalized action-angle coordinates around its connected compact invariant submanifold [9, 18, 61]. The Mishchenko – Fomenko theorem is extended to superintegrable systems with non-compact invariant submanifolds (Theorem 2.5) [22, 24, 41, 74]. These submanifolds are diffeomorphic to a toroidal cylinder

$$\mathbb{R}^{m-r} \times T^r, \quad m = 2n - k, \quad 0 \leq r \leq m. \quad (2.5)$$

Note that the Mishchenko – Fomenko theorem is mainly applied to superintegrable systems whose integrals of motion form a compact Lie algebra. The group generated by flows of their Hamiltonian vector fields is compact. Since a fibration of a compact manifold possesses compact fibres, invariant submanifolds of such a superintegrable system are compact. With Theorem 2.5, one can describe superintegrable Hamiltonian system with an arbitrary Lie algebra of integrals of motion.

Given a superintegrable system in accordance with Definition 2.1, the above mentioned generalization of the Mishchenko – Fomenko theorem to non-compact invariant submanifolds states the following.

THEOREM 2.5: Let the Hamiltonian vector fields ϑ_i of the functions F_i be complete, and let the fibres of the fibred manifold F (2.2) be connected and mutually diffeomorphic. Then the following hold.

- (I) The fibres of F (2.2) are diffeomorphic to the toroidal cylinder (2.5).
- (II) Given a fibre M of F (2.2), there exists its open saturated neighborhood U_M which is a trivial principal bundle

$$U_M = N_M \times \mathbb{R}^{m-r} \times T^r \xrightarrow{F} N_M \quad (2.6)$$

with the structure group (2.5).

(III) The neighborhood U_M is provided with the bundle (generalized action-angle) coordinates $(I_\lambda, p_s, q^s, y^\lambda)$, $\lambda = 1, \dots, m$, $s = 1, \dots, n - m$, such that: (i) the generalized angle coordinates (y^λ) are coordinates on a toroidal cylinder, i.e., fibre coordinates on the fibre bundle (2.6), (ii) (I_λ, p_s, q^s) are coordinates on its base N_M where the action coordinates (I_λ) are values of Casimir functions of the coinduced Poisson structure $\{, \}_N$ on N_M , and (iii) the symplectic form Ω on U_M reads

$$\Omega = dI_\lambda \wedge dy^\lambda + dp_s \wedge dq^s. \quad (2.7)$$

□

Proof: It follows from item (iii) of Proposition 2.4 that every fibre M of the fibred manifold (2.2) is a maximal integral manifold of the involutive distribution spanned by the Hamiltonian vector fields v_λ of the pull-back F^*C_λ of m independent Casimir functions $\{C_1, \dots, C_m\}$ of the Poisson structure $\{, \}_N$ (2.4) on an open neighborhood N_M of a point $F(M) \in N$. Let us put $U_M = F^{-1}(N_M)$. It is an open saturated neighborhood of M . Consequently, invariant submanifolds of a superintegrable system (2.1) on U_M are maximal integral manifolds of the partially integrable system

$$C^* = (F^*C_1, \dots, F^*C_m), \quad 0 < m \leq n, \quad (2.8)$$

on a symplectic manifold (U_M, Ω) . Therefore, statements (I) – (III) of Theorem 2.5 are the corollaries of Theorem 1.17. Its condition (i) is satisfied as follows. Let M' be an arbitrary fibre of the fibred manifold $F : U_M \rightarrow N_M$ (2.2). Since

$$F^*C_\lambda(z) = (C_\lambda \circ F)(z) = C_\lambda(F_i(z)), \quad z \in M',$$

the Hamiltonian vector fields v_λ on M' are \mathbb{R} -linear combinations of Hamiltonian vector fields ϑ_i of the functions F_i . It follows that v_λ are elements of a finite-dimensional

real Lie algebra of vector fields on M' generated by the vector fields ϑ_i . Since vector fields ϑ_i are complete, the vector fields v_λ on M' also are complete (see forthcoming Remark 2.3). Consequently, these vector fields are complete on U_M because they are vertical vector fields on $U_M \rightarrow N$. The proof of Theorem 1.17 shows that the action coordinates (I_λ) are values of Casimir functions expressed in the original ones C_λ . \square

Remark 2.3: If complete vector fields on a smooth manifold constitute a basis for a finite-dimensional real Lie algebra, any element of this Lie algebra is complete [68]. \square

Remark 2.4: Since an open neighborhood U_M (2.6) in item (II) of Theorem 2.5 is not contractible, unless $r = 0$, the generalized action-angle coordinates on U sometimes are called semi-local. \square

Remark 2.5: The condition of the completeness of Hamiltonian vector fields of the generating functions F_i in Theorem 2.5 is rather restrictive (see the Kepler system in Section 3). One can replace this condition with that the Hamiltonian vector fields of the pull-back onto Z of Casimir functions on N are complete. \square

If the conditions of Theorem 2.5 are replaced with that the fibres of the fibred manifold F (2.2) are compact and connected, this theorem restarts the Mishchenko – Fomenko one as follows.

THEOREM 2.6: Let the fibres of the fibred manifold F (2.2) be connected and compact. Then they are diffeomorphic to a torus T^m , and statements (II) – (III) of Theorem 2.5 hold. \square

Remark 2.6: In Theorem 2.6, the Hamiltonian vector fields v_λ are complete because fibres of the fibred manifold F (2.2) are compact. As well known, any vector field on a compact manifold is complete. \square

If F (2.1) is a completely integrable system, the coinduced Poisson structure on N equals zero, and the generating functions F_i are the pull-back of n independent functions on N . Then Theorems 2.6 and 2.5 come to the Liouville – Arnold theorem [3, 54] and its generalization (Theorem 2.7) to the case of non-compact invariant submanifolds [20, 37], respectively. In this case, the partially integrable system C^* (2.8) is exactly the original completely integrable system F .

THEOREM 2.7: Given a completely integrable system, F in accordance with Definition 2.2, let the Hamiltonian vector fields ϑ_i of the functions F_i be complete, and let the fibres of the fibred manifold F (2.2) be connected and mutually diffeomorphic. Then items (I) and (II) of Theorem 2.5 hold, and its item (III) is replaced with the following one.

(III') The neighborhood U_M (2.6) where $m = n$ is provided with the bundle (generalized action-angle) coordinates (I_λ, y^λ) , $\lambda = 1, \dots, n$, such that the angle coordinates (y^λ) are coordinates on a toroidal cylinder, and the symplectic form Ω on U_M reads

$$\Omega = dI_\lambda \wedge dy^\lambda. \quad (2.9)$$

□

To study a superintegrable system, one conventionally considers it with respect to generalized action-angle coordinates. A problem is that, restricted to an action-angle coordinate chart on an open subbundle U of the fibred manifold $Z \rightarrow N$ (2.2), a superintegrable system becomes different from the original one since there is no morphism of the Poisson algebra $C^\infty(U)$ on (U, Ω) to that $C^\infty(Z)$ on (Z, Ω) . Moreover, a superintegrable system on U need not satisfy the conditions of Theorem 2.5 because it may happen that the Hamiltonian vector fields of the generating functions on U are not complete. To describe superintegrable systems in terms of generalized action-angle coordinates, we therefore follow the notion of a globally superintegrable system [41, 74].

DEFINITION 2.8: A superintegrable system F (2.1) on a symplectic manifold (Z, Ω) in Definition 2.1 is called globally superintegrable if there exist global generalized action-angle coordinates

$$(I_\lambda, x^A, y^\lambda), \quad \lambda = 1, \dots, m, \quad A = 1, \dots, 2(n - m), \quad (2.10)$$

such that: (i) the action coordinates (I_λ) are expressed in the values of some Casimir functions C_λ on the Poisson manifold $(N, \{, \}_N)$, (ii) the angle coordinates (y^λ) are coordinates on the toroidal cylinder (1.32), and (iii) the symplectic form Ω on Z reads

$$\Omega = dI_\lambda \wedge dy^\lambda + \Omega_{AB}(I_\mu, x^C) dx^A \wedge dx^B. \quad (2.11)$$

□

It is readily observed that the semi-local generalized action-angle coordinates on U in Theorem 2.5 are global on U in accordance with Definition 2.8.

Forthcoming Theorem 2.9 provides the sufficient conditions of the existence of global generalized action-angle coordinates of a superintegrable system on a symplectic manifold (Z, Ω) [23, 41, 74]. It generalizes the well-known result for the case of compact invariant submanifolds [13, 18].

THEOREM 2.9: A superintegrable system F on a symplectic manifold (Z, Ω) is globally superintegrable if the following conditions hold.

- (i) Hamiltonian vector fields ϑ_i of the generating functions F_i are complete.
- (ii) The fibred manifold F (2.2) is a fibre bundle with connected fibres.
- (iii) Its base N is simply connected and the cohomology $H^2(V; \mathbb{Z})$ is trivial
- (iv) The coinduced Poisson structure $\{, \}_N$ on a base N admits m independent Casimir functions C_λ . □

Proof: Theorem 2.9 is a corollary of Theorem 1.18. In accordance with Theorem 1.18, we have a composite fibred manifold

$$Z \xrightarrow{F} N \xrightarrow{C} W, \quad (2.12)$$

where $C : N \rightarrow W$ is a fibred manifold of level surfaces of the Casimir functions C_λ (which coincides with the symplectic foliation of a Poisson manifold N). The composite

fibred manifold (2.12) is provided with the adapted fibred coordinates $(J_\lambda, x^A, r^\lambda)$ (1.83), where J_λ are values of independent Casimir functions and $(r^\lambda) = (t^a, \varphi^i)$ are coordinates on a toroidal cylinder. Since $C_\lambda = J_\lambda$ are Casimir functions on N , the symplectic form Ω (1.85) on Z reads

$$\Omega = \Omega_\beta^\alpha dJ_\alpha \wedge r^\beta + \Omega_{\alpha A} dy^\alpha \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B. \quad (2.13)$$

In particular, it follows that transition functions of coordinates x^A on N are independent of coordinates J_λ , i.e., $C : V \rightarrow W$ is a trivial bundle. By virtue of Lemma 1.19, the symplectic form (2.13) is exact, i.e., $\Omega = d\Xi$, where the Liouville form Ξ (1.86) is

$$\Xi = \Xi^\lambda(J_\alpha, y^\mu) dJ_\lambda + \Xi_i(J_\alpha) d\varphi^i + \Xi_A(x^B) dx^A.$$

Then the coordinate transformations (1.87):

$$\begin{aligned} I_a &= J_a, & I_i &= \Xi_i(J_j), \\ y^a &= -\Xi^a = t^a - E^a(J_\lambda), & y^i &= \varphi^i - \Xi^j(J_\lambda) \frac{\partial J_j}{\partial I_i}, \end{aligned} \quad (2.14)$$

bring Ω (2.13) into the form (2.11). In comparison with the general case (1.87), the coordinate transformations (2.14) are independent of coordinates x^A . Therefore, the angle coordinates y^i possess identity transition functions on N . \square

Theorem 2.9 restarts Theorem 2.5 if one considers an open subset V of N admitting the Darboux coordinates x^A on the symplectic leaves of U .

Note that, if invariant submanifolds of a superintegrable system are assumed to be connected and compact, condition (i) of Theorem 2.9 is unnecessary since vector fields ϑ_λ on compact fibres of F are complete. Condition (ii) also holds by virtue of Theorem 7.2. In this case, Theorem 2.9 reproduces the well known result in [13].

If F in Theorem 2.9 is a completely integrable system, the coinduced Poisson structure on N equals zero, the generating functions F_i are the pull-back of n independent functions on N , and Theorem 2.9 takes the following form [23, 41].

THEOREM 2.10: Let a completely integrable system $\{F_1, \dots, F_n\}$ on a symplectic manifold (Z, Ω) satisfy the following conditions.

- (i) The Hamiltonian vector fields ϑ_i of F_i are complete.
- (ii) The fibred manifold F (2.2) is a fibre bundle with connected fibres over a simply connected base N whose cohomology $H^2(N, \mathbb{Z})$ is trivial.

Then the following hold.

(I) The fibre bundle F (2.2) is a trivial principal bundle with the structure group $\mathbb{R}^{2n-r} \times T^r$.

(II) The symplectic manifold Z is provided with the global Darboux coordinates (I_λ, y^λ) such that $\Omega = dI_\lambda \wedge dy^\lambda$. \square

It follows from the proof of Theorem 1.18 that its condition (iii) and, accordingly, condition (iii) of Theorem 2.9 guarantee that fibre bundles F in conditions (ii) of these theorems are trivial. Therefore, Theorem 2.9 can be reformulated as follows.

THEOREM 2.11: A superintegrable system F on a symplectic manifold (Z, Ω) is globally superintegrable iff the following conditions hold.

- (i) The fibred manifold F (2.2) is a trivial fibre bundle.
- (ii) The coinduced Poisson structure $\{, \}_N$ on a base N admits m independent Casimir functions C_λ such that Hamiltonian vector fields of their pull-back F^*C_λ are complete. \square

Remark 2.7: It follows from Remark 2.3 and condition (ii) of Theorem 2.11 that a Hamiltonian vector field of the the pull-back F^*C of any Casimir function C on a Poisson manifold N is complete. \square

In autonomous Hamiltonian mechanics, one considers superintegrable systems whose generating functions are integrals of motion, i.e., they are in involution with a Hamiltonian \mathcal{H} , and the functions $(\mathcal{H}, F_1, \dots, F_k)$ are nowhere independent, i.e.,

$$\{\mathcal{H}, F_i\} = 0, \quad (2.15)$$

$$d\mathcal{H} \wedge (\wedge^k dF_i) = 0. \quad (2.16)$$

In order that an evolution of a Hamiltonian system can be defined at any instant $t \in \mathbb{R}$, one supposes that the Hamiltonian vector field of its Hamiltonian is complete. By virtue of Remark 2.7 and forthcoming Proposition 2.12, a Hamiltonian of a superintegrable system always satisfies this condition.

PROPOSITION 2.12: It follows from the equality (2.16) that a Hamiltonian \mathcal{H} is constant on the invariant submanifolds. Therefore, it is the pull-back of a function on N which is a Casimir function of the Poisson structure (2.4) because of the conditions (2.15). \square

Proposition 2.12 leads to the following.

PROPOSITION 2.13: Let \mathcal{H} be a Hamiltonian of a globally superintegrable system provided with the generalized action-angle coordinates $(I_\lambda, x^A, y^\lambda)$ (2.10). Then a Hamiltonian \mathcal{H} depends only on the action coordinates I_λ . Consequently, the Hamilton equation of a globally superintegrable system take the form

$$\dot{y}^\lambda = \frac{\partial \mathcal{H}}{\partial I_\lambda}, \quad I_\lambda = \text{const.}, \quad x^A = \text{const.}$$

\square

Following the original Mishchenko – Fomenko theorem, let us mention superintegrable systems whose generating functions $\{F_1, \dots, F_k\}$ form a k -dimensional real Lie algebra \mathfrak{g} of corank m with the commutation relations

$$\{F_i, F_j\} = c_{ij}^h F_h, \quad c_{ij}^h = \text{const.} \quad (2.17)$$

Then F (2.2) is a momentum mapping of Z to the Lie coalgebra \mathfrak{g}^* provided with the coordinates x_i in item (i) of Definition 2.1 [41, 45]. In this case, the

coinduced Poisson structure $\{\cdot, \cdot\}_N$ coincides with the canonical Lie – Poisson structure on \mathfrak{g}^* given by the Poisson bivector field

$$w = \frac{1}{2} c_{ij}^h x_h \partial^i \wedge \partial^j.$$

Let V be an open subset of \mathfrak{g}^* such that conditions (i) and (ii) of Theorem 2.11 are satisfied. Then an open subset $F^{-1}(V) \subset Z$ is provided with the generalized action-angle coordinates.

Remark 2.8: Let Hamiltonian vector fields ϑ_i of the generating functions F_i which form a Lie algebra \mathfrak{g} be complete. Then they define a locally free Hamiltonian action on Z of some simply connected Lie group G whose Lie algebra is isomorphic to \mathfrak{g} [67, 68]. Orbits of G coincide with k -dimensional maximal integral manifolds of the regular distribution \mathcal{V} on Z spanned by Hamiltonian vector fields ϑ_i [81]. Furthermore, Casimir functions of the Lie – Poisson structure on \mathfrak{g}^* are exactly the coadjoint invariant functions on \mathfrak{g}^* . They are constant on orbits of the coadjoint action of G on \mathfrak{g}^* which coincide with leaves of the symplectic foliation of \mathfrak{g}^* . \square

THEOREM 2.14: Let a globally superintegrable Hamiltonian system on a symplectic manifold Z obey the following conditions.

- (i) It is maximally superintegrable.
- (ii) Its Hamiltonian \mathcal{H} is regular, i.e, $d\mathcal{H}$ nowhere vanishes.
- (iii) Its generating functions F_i constitute a finite dimensional real Lie algebra and their Hamiltonian vector fields are complete.

Then any integral of motion of this Hamiltonian system is the pull-back of a function on a base N of the fibration F (2.2). In other words, it is expressed in the integrals of motion F_i . \square

Proof: The proof is based on the following. A Hamiltonian vector field of a function f on Z lives in the one-codimensional regular distribution \mathcal{V} on Z spanned by Hamiltonian vector fields ϑ_i iff f is the pull-back of a function on a base N of the fibration F (2.2). A Hamiltonian \mathcal{H} brings Z into a fibred manifold of its level surfaces whose vertical tangent bundle coincide with \mathcal{V} . Therefore, a Hamiltonian vector field of any integral of motion of \mathcal{H} lives in \mathcal{V} . \square

It may happen that, given a Hamiltonian \mathcal{H} of a Hamiltonian system on a symplectic manifold Z , we have different superintegrable Hamiltonian systems on different open subsets of Z . For instance, this is the case of the Kepler system.

3 Global Kepler system

We consider the Kepler system on a plane \mathbb{R}^2 [41, 74].

Its phase space is $T^*\mathbb{R}^2 = \mathbb{R}^4$ provided with the Cartesian coordinates (q_i, p_i) , $i = 1, 2$, and the canonical symplectic form

$$\Omega_T = \sum_i dp_i \wedge dq_i. \tag{3.1}$$

Let us denote

$$p = \left(\sum_i (p_i)^2 \right)^{1/2}, \quad r = \left(\sum_i (q_i)^2 \right)^{1/2}, \quad (p, q) = \sum_i p_i q_i.$$

An autonomous Hamiltonian of the Kepler system reads

$$\mathcal{H} = \frac{1}{2} p^2 - \frac{1}{r}. \quad (3.2)$$

The Kepler system is a Hamiltonian system on a symplectic manifold

$$Z = \mathbb{R}^4 \setminus \{0\} \quad (3.3)$$

endowed with the symplectic form Ω_T (3.1).

Let us consider the functions

$$M_{12} = -M_{21} = q_1 p_2 - q_2 p_1, \quad (3.4)$$

$$A_i = \sum_j M_{ij} p_j - \frac{q_i}{r} = q_i p^2 - p_i (p, q) - \frac{q_i}{r}, \quad i = 1, 2, \quad (3.5)$$

on the symplectic manifold Z (3.3). It is readily observed that they are integrals of motion of the Hamiltonian \mathcal{H} (3.2) where M_{12} is an angular momentum and (A_i) is a Rung – Lenz vector. Let us denote

$$M^2 = (M_{12})^2, \quad A^2 = (A_1)^2 + (A_2)^2 = 2M^2 \mathcal{H} + 1. \quad (3.6)$$

Let $Z_0 \subset Z$ be a closed subset of points where $M_{12} = 0$. A direct computation shows that the functions (M_{12}, A_i) (3.4) – (3.5) are independent of an open submanifold

$$U = Z \setminus Z_0 \quad (3.7)$$

of Z . At the same time, the functions $(\mathcal{H}, M_{12}, A_i)$ are independent nowhere on U because it follows from the expression (3.6) that

$$\mathcal{H} = \frac{A^2 - 1}{2M^2} \quad (3.8)$$

on U (3.7). The well known dynamics of the Kepler system shows that the Hamiltonian vector field of its Hamiltonian is complete on U (but not on Z).

The Poisson bracket of integrals of motion M_{12} (3.4) and A_i (3.5) obeys the relations

$$\{M_{12}, A_i\} = \eta_{2i} A_1 - \eta_{1i} A_2, \quad (3.9)$$

$$\{A_1, A_2\} = 2\mathcal{H} M_{12} = \frac{A^2 - 1}{M_{12}}, \quad (3.10)$$

where η_{ij} is an Euclidean metric on \mathbb{R}^2 . It is readily observed that these relations take the form (2.3). However, the matrix function \mathbf{s} of the relations (3.9) –

(3.10) fails to be of constant rank at points where $\mathcal{H} = 0$. Therefore, let us consider the open submanifolds $U_- \subset U$ where $\mathcal{H} < 0$ and U_+ where $\mathcal{H} > 0$. Then we observe that the Kepler system with the Hamiltonian \mathcal{H} (3.2) and the integrals of motion (M_{ij}, A_i) (3.4) – (3.5) on U_- and the Kepler system with the Hamiltonian \mathcal{H} (3.2) and the integrals of motion (M_{ij}, A_i) (3.4) – (3.5) on U_+ are superintegrable Hamiltonian systems. Moreover, these superintegrable systems can be brought into the form (2.17) as follows.

Let us replace the integrals of motions A_i with the integrals of motion

$$L_i = \frac{A_i}{\sqrt{-2\mathcal{H}}} \quad (3.11)$$

on U_- , and with the integrals of motion

$$K_i = \frac{A_i}{\sqrt{2\mathcal{H}}} \quad (3.12)$$

on U_+ .

The superintegrable system (M_{12}, L_i) on U_- obeys the relations

$$\{M_{12}, L_i\} = \eta_{2i}L_1 - \eta_{1i}L_2, \quad (3.13)$$

$$\{L_1, L_2\} = -M_{12}. \quad (3.14)$$

Let us denote $M_{i3} = -L_i$ and put the indexes $\mu, \nu, \alpha, \beta = 1, 2, 3$. Then the relations (3.13) – (3.14) are brought into the form

$$\{M_{\mu\nu}, M_{\alpha\beta}\} = \eta_{\mu\beta}M_{\nu\alpha} + \eta_{\nu\alpha}M_{\mu\beta} - \eta_{\mu\alpha}M_{\nu\beta} - \eta_{\nu\beta}M_{\mu\alpha} \quad (3.15)$$

where $\eta_{\mu\nu}$ is an Euclidean metric on \mathbb{R}^3 . A glance at the expression (3.15) shows that the integrals of motion M_{12} (3.4) and L_i (3.11) constitute the Lie algebra $\mathfrak{g} = so(3)$. Its corank equals 1. Therefore the superintegrable system (M_{12}, L_i) on U_- is maximally superintegrable. The equality (3.8) takes the form

$$M^2 + L^2 = -\frac{1}{2\mathcal{H}}. \quad (3.16)$$

The superintegrable system (M_{12}, K_i) on U_+ obeys the relations

$$\{M_{12}, K_i\} = \eta_{2i}K_1 - \eta_{1i}K_2, \quad (3.17)$$

$$\{K_1, K_2\} = M_{12}. \quad (3.18)$$

Let us denote $M_{i3} = -K_i$ and put the indexes $\mu, \nu, \alpha, \beta = 1, 2, 3$. Then the relations (3.17) – (3.18) are brought into the form

$$\{M_{\mu\nu}, M_{\alpha\beta}\} = \rho_{\mu\beta}M_{\nu\alpha} + \rho_{\nu\alpha}M_{\mu\beta} - \rho_{\mu\alpha}M_{\nu\beta} - \rho_{\nu\beta}M_{\mu\alpha} \quad (3.19)$$

where $\rho_{\mu\nu}$ is a pseudo-Euclidean metric of signature $(+, +, -)$ on \mathbb{R}^3 . A glance at the expression (3.19) shows that the integrals of motion M_{12} (3.4) and K_i (3.12) constitute the Lie algebra $so(2, 1)$. Its corank equals 1. Therefore the

superintegrable system (M_{12}, K_i) on U_+ is maximally superintegrable. The equality (3.8) takes the form

$$K^2 - M^2 = \frac{1}{2\mathcal{H}}. \quad (3.20)$$

Thus, the Kepler system on a phase space \mathbb{R}^4 falls into two different maximally superintegrable systems on open submanifolds U_- and U_+ of \mathbb{R}^4 . We agree to call them the Kepler superintegrable systems on U_- and U_+ , respectively.

Let us study the first one and put

$$\begin{aligned} F_1 &= -L_1, & F_2 &= -L_2, & F_3 &= -M_{12}, \\ \{F_1, F_2\} &= F_3, & \{F_2, F_3\} &= F_1, & \{F_3, F_1\} &= F_2. \end{aligned} \quad (3.21)$$

We have a fibred manifold

$$F : U_- \rightarrow N \subset \mathfrak{g}^*, \quad (3.22)$$

which is the momentum mapping to the Lie coalgebra $\mathfrak{g}^* = so(3)^*$, endowed with the coordinates (x_i) such that integrals of motion F_i on \mathfrak{g}^* read $F_i = x_i$. A base N of the fibred manifold (3.22) is an open submanifold of \mathfrak{g}^* given by the coordinate condition $x_3 \neq 0$. It is a union of two contractible components defined by the conditions $x_3 > 0$ and $x_3 < 0$. The coinduced Lie – Poisson structure on N takes the form

$$w = x_2 \partial^3 \wedge \partial^1 + x_3 \partial^1 \wedge \partial^2 + x_1 \partial^2 \wedge \partial^3. \quad (3.23)$$

The coadjoint action of $so(3)$ on N reads

$$\varepsilon_1 = x_3 \partial^2 - x_2 \partial^3, \quad \varepsilon_2 = x_1 \partial^3 - x_3 \partial^1, \quad \varepsilon_3 = x_2 \partial^1 - x_1 \partial^2. \quad (3.24)$$

The orbits of this coadjoint action are given by the equation

$$x_1^2 + x_2^2 + x_3^2 = \text{const.} \quad (3.25)$$

They are the level surfaces of the Casimir function

$$C = x_1^2 + x_2^2 + x_3^2$$

and, consequently, the Casimir function

$$h = -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)^{-1}. \quad (3.26)$$

A glance at the expression (3.16) shows that the pull-back F^*h of this Casimir function (3.26) onto U_- is the Hamiltonian \mathcal{H} (3.2) of the Kepler system on U_- .

As was mentioned above, the Hamiltonian vector field of F^*h is complete. Furthermore, it is known that invariant submanifolds of the superintegrable Kepler system on U_- are compact. Therefore, the fibred manifold F (3.22) is a fibre bundle in accordance with Theorem 7.2. Moreover, this fibre bundle

is trivial because N is a disjoint union of two contractible manifolds. Consequently, it follows from Theorem 2.11 that the Kepler superintegrable system on U_- is globally superintegrable, i.e., it admits global generalized action-angle coordinates as follows.

The Poisson manifold N (3.22) can be endowed with the coordinates

$$(I, x_1, \gamma), \quad I < 0, \quad \gamma \neq \frac{\pi}{2}, \frac{3\pi}{2}, \quad (3.27)$$

defined by the equalities

$$\begin{aligned} I &= -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)^{-1}, \\ x_2 &= \left(-\frac{1}{2I} - x_1^2\right)^{1/2} \sin \gamma, \quad x_3 = \left(-\frac{1}{2I} - x_1^2\right)^{1/2} \cos \gamma. \end{aligned} \quad (3.28)$$

It is readily observed that the coordinates (3.27) are Darboux coordinates of the Lie – Poisson structure (3.23) on U_- , namely,

$$w = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial \gamma}. \quad (3.29)$$

Let ϑ_I be the Hamiltonian vector field of the Casimir function I (3.28). By virtue of Proposition 2.4, its flows are invariant submanifolds of the Kepler superintegrable system on U_- . Let α be a parameter along the flow of this vector field, i.e.,

$$\vartheta_I = \frac{\partial}{\partial \alpha}. \quad (3.30)$$

Then U_- is provided with the generalized action-angle coordinates (I, x_1, γ, α) such that the Poisson bivector associated to the symplectic form Ω_T on U_- reads

$$W = \frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial \gamma}. \quad (3.31)$$

Accordingly, Hamiltonian vector fields of integrals of motion F_i (3.21) take the form

$$\begin{aligned} \vartheta_1 &= \frac{\partial}{\partial \gamma}, \\ \vartheta_2 &= \frac{1}{4I^2} \left(-\frac{1}{2I} - x_1^2\right)^{-1/2} \sin \gamma \frac{\partial}{\partial \alpha} - x_1 \left(-\frac{1}{2I} - x_1^2\right)^{-1/2} \sin \gamma \frac{\partial}{\partial \gamma} - \\ &\quad \left(-\frac{1}{2I} - x_1^2\right)^{1/2} \cos \gamma \frac{\partial}{\partial x_1}, \\ \vartheta_3 &= \frac{1}{4I^2} \left(-\frac{1}{2I} - x_1^2\right)^{-1/2} \cos \gamma \frac{\partial}{\partial \alpha} - x_1 \left(-\frac{1}{2I} - x_1^2\right)^{-1/2} \cos \gamma \frac{\partial}{\partial \gamma} + \\ &\quad \left(-\frac{1}{2I} - x_1^2\right)^{1/2} \sin \gamma \frac{\partial}{\partial x_1}. \end{aligned}$$

A glance at these expressions shows that the vector fields ϑ_1 and ϑ_2 fail to be complete on U_- (see Remark 2.5).

One can say something more about the angle coordinate α . The vector field ϑ_I (3.30) reads

$$\frac{\partial}{\partial \alpha} = \sum_i \left(\frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

This equality leads to the relations

$$\frac{\partial q_i}{\partial \alpha} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{\partial p_i}{\partial \alpha} = -\frac{\partial \mathcal{H}}{\partial q_i},$$

which take the form of the Hamilton equation. Therefore, the coordinate α is a cyclic time $\alpha = t \bmod 2\pi$ given by the well-known expression

$$\begin{aligned} \alpha &= \phi - a^{3/2} e \sin(a^{-3/2} \phi), & r &= a(1 - e \cos(a^{-3/2} \phi)), \\ a &= -\frac{1}{2I}, & e &= (1 + 2IM^2)^{1/2}. \end{aligned}$$

Now let us turn to the Kepler superintegrable system on U_+ . It is a globally superintegrable system with non-compact invariant submanifolds as follows.

Let us put

$$\begin{aligned} S_1 &= -K_1, & S_2 &= -K_2, & S_3 &= -M_{12}, \\ \{S_1, S_2\} &= -S_3, & \{S_2, S_3\} &= S_1, & \{S_3, S_1\} &= S_2. \end{aligned} \quad (3.32)$$

We have a fibred manifold

$$S : U_+ \rightarrow N \subset \mathfrak{g}^*, \quad (3.33)$$

which is the momentum mapping to the Lie coalgebra $\mathfrak{g}^* = so(2, 1)^*$, endowed with the coordinates (x_i) such that integrals of motion S_i on \mathfrak{g}^* read $S_i = x_i$. A base N of the fibred manifold (3.33) is an open submanifold of \mathfrak{g}^* given by the coordinate condition $x_3 \neq 0$. It is a union of two contractible components defined by the conditions $x_3 > 0$ and $x_3 < 0$. The coinduced Lie – Poisson structure on N takes the form

$$w = x_2 \partial^3 \wedge \partial^1 - x_3 \partial^1 \wedge \partial^2 + x_1 \partial^2 \wedge \partial^3. \quad (3.34)$$

The coadjoint action of $so(2, 1)$ on N reads

$$\varepsilon_1 = -x_3 \partial^2 - x_2 \partial^3, \quad \varepsilon_2 = x_1 \partial^3 + x_3 \partial^1, \quad \varepsilon_3 = x_2 \partial^1 - x_1 \partial^2.$$

The orbits of this coadjoint action are given by the equation

$$x_1^2 + x_2^2 - x_3^2 = \text{const.}$$

They are the level surfaces of the Casimir function

$$C = x_1^2 + x_2^2 - x_3^2$$

and, consequently, the Casimir function

$$h = \frac{1}{2}(x_1^2 + x_2^2 - x_3^2)^{-1}. \quad (3.35)$$

A glance at the expression (3.20) shows that the pull-back S^*h of this Casimir function (3.35) onto U_+ is the Hamiltonian \mathcal{H} (3.2) of the Kepler system on U_+ .

As was mentioned above, the Hamiltonian vector field of S^*h is complete. Furthermore, it is known that invariant submanifolds of the superintegrable Kepler system on U_+ are diffeomorphic to \mathbb{R} . Therefore, the fibred manifold S (3.33) is a fibre bundle in accordance with Theorem 7.2. Moreover, this fibre bundle is trivial because N is a disjoint union of two contractible manifolds. Consequently, it follows from Theorem 2.11 that the Kepler superintegrable system on U_+ is globally superintegrable, i.e., it admits global generalized action-angle coordinates as follows.

The Poisson manifold N (3.33) can be endowed with the coordinates

$$(I, x_1, \lambda), \quad I > 0, \quad \lambda \neq 0,$$

defined by the equalities

$$I = \frac{1}{2}(x_1^2 + x_2^2 - x_3^2)^{-1},$$

$$x_2 = \left(\frac{1}{2I} - x_1^2\right)^{1/2} \cosh \lambda, \quad x_3 = \left(\frac{1}{2I} - x_1^2\right)^{1/2} \sinh \lambda.$$

These coordinates are Darboux coordinates of the Lie – Poisson structure (3.34) on N , namely,

$$w = \frac{\partial}{\partial \lambda} \wedge \frac{\partial}{\partial x_1}. \quad (3.36)$$

Let ϑ_I be the Hamiltonian vector field of the Casimir function I (3.28). By virtue of Proposition 2.4, its flows are invariant submanifolds of the Kepler superintegrable system on U_+ . Let τ be a parameter along the flows of this vector field, i.e.,

$$\vartheta_I = \frac{\partial}{\partial \tau}. \quad (3.37)$$

Then U_+ (3.33) is provided with the generalized action-angle coordinates (I, x_1, λ, τ) such that the Poisson bivector associated to the symplectic form Ω_T on U_+ reads

$$W = \frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \lambda} \wedge \frac{\partial}{\partial x_1}. \quad (3.38)$$

Accordingly, Hamiltonian vector fields of integrals of motion S_i (3.32) take the

form

$$\begin{aligned}
\vartheta_1 &= -\frac{\partial}{\partial \lambda}, \\
\vartheta_2 &= \frac{1}{4I^2} \left(\frac{1}{2I} - x_1^2 \right)^{-1/2} \cosh \lambda \frac{\partial}{\partial \tau} + x_1 \left(\frac{1}{2I} - x_1^2 \right)^{-1/2} \cosh \lambda \frac{\partial}{\partial \lambda} + \\
&\quad \left(\frac{1}{2I} - x_1^2 \right)^{1/2} \sinh \lambda \frac{\partial}{\partial x_1}, \\
\vartheta_3 &= \frac{1}{4I^2} \left(\frac{1}{2I} - x_1^2 \right)^{-1/2} \sinh \lambda \frac{\partial}{\partial \tau} + x_1 \left(\frac{1}{2I} - x_1^2 \right)^{-1/2} \sinh \lambda \frac{\partial}{\partial \lambda} + \\
&\quad \left(\frac{1}{2I} - x_1^2 \right)^{1/2} \cosh \lambda \frac{\partial}{\partial x_1}.
\end{aligned}$$

Similarly to the angle coordinate α (3.30), the generalized angle coordinate τ (3.37) obeys the Hamilton equation

$$\frac{\partial q_i}{\partial \tau} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{\partial p_i}{\partial \tau} = -\frac{\partial \mathcal{H}}{\partial q_i}.$$

Therefore, it is the time $\tau = t$ given by the well-known expression

$$\begin{aligned}
\tau &= s - a^{3/2} e \sinh(a^{-3/2} s), & r &= a(e \cosh(a^{-3/2} s) - 1), \\
a &= \frac{1}{2I}, & e &= (1 + 2IM^2)^{1/2}.
\end{aligned}$$

4 Non-autonomous integrable systems

The generalization of Liouville – Arnold and Mishchenko – Fomenko theorems to the case of non-compact invariant submanifolds (Theorems 2.5 and 2.7) enables one to analyze completely integrable and superintegrable non-autonomous Hamiltonian systems whose invariant submanifolds are necessarily non-compact [30, 41, 76].

A non-autonomous classical mechanics is described on a configuration space Q which is a fibre bundle $Q \rightarrow \mathbb{R}$ over the time axis \mathbb{R} . Its phase space is the vertical cotangent bundle V^*Q of $Q \rightarrow \mathbb{R}$ provided with the canonical Poisson structure (4.24). However, non-autonomous mechanics fails to be a familiar Poisson Hamiltonian system on V^*Q . At the same time, it is equivalent to an autonomous Hamiltonian system on the cotangent bundle T^*Q provided with the canonical symplectic form (4.18).

This formulation of non-relativistic mechanics is similar to that of classical field theory on fibre bundles over a base of dimension > 1 [28, 40, 73]. A difference between mechanics and field theory however lies in the fact that connections on bundles over \mathbb{R} are flat, and they fail to be dynamic variables, but describe reference frames.

4.1 Geometry of fibre bundle over \mathbb{R}

This Section summarizes some peculiarities of fibre bundles over \mathbb{R} .

Let

$$\pi : Q \rightarrow \mathbb{R} \quad (4.1)$$

be a fibred manifold whose base is treated as a time axis. Throughout the Lectures, the time axis \mathbb{R} is parameterized by the Cartesian coordinate t with the transition functions $t' = t + \text{const}$. Relative to the Cartesian coordinate t , the time axis \mathbb{R} is provided with the standard vector field ∂_t and the standard one-form dt which also is the volume element on \mathbb{R} . The symbol dt also stands for any pull-back of the standard one-form dt onto a fibre bundle over \mathbb{R} .

Remark 4.1: Point out one-to-one correspondence between the vector fields $f\partial_t$, the densities $f dt$ and the real functions f on \mathbb{R} . Roughly speaking, we can neglect the contribution of $T\mathbb{R}$ and $T^*\mathbb{R}$ to some expressions. \square

In order that the dynamics of a mechanical system can be defined at any instant $t \in \mathbb{R}$, we further assume that a fibred manifold $Q \rightarrow \mathbb{R}$ is a fibre bundle with a typical fibre M .

Remark 4.2: In accordance with Remark 7.5, a fibred manifold $Q \rightarrow \mathbb{R}$ is a fibre bundle iff it admits an Ehresmann connection Γ , i.e., the horizontal lift $\Gamma\partial_t$ onto Q of the standard vector field ∂_t on \mathbb{R} is complete. By virtue of Theorem 7.3, any fibre bundle $Q \rightarrow \mathbb{R}$ is trivial. Its different trivializations

$$\psi : Q = \mathbb{R} \times M \quad (4.2)$$

differ from each other in fibrations $Q \rightarrow M$. \square

Given bundle coordinates (t, q^i) on the fibre bundle $Q \rightarrow \mathbb{R}$ (4.1), the first order jet manifold J^1Q of $Q \rightarrow \mathbb{R}$ is provided with the adapted coordinates (t, q^i, \dot{q}^i) possessing transition functions (7.49) which read

$$q_t^i = (\partial_t + q_t^j \partial_j) q^i.$$

Note that, if $Q = \mathbb{R} \times M$ coordinated by (t, \bar{q}^i) , there is the canonical isomorphism

$$J^1(\mathbb{R} \times M) = \mathbb{R} \times TM, \quad \bar{q}_t^i = \dot{\bar{q}}^i, \quad (4.3)$$

that one can justify by inspection of the transition functions of the coordinates \bar{q}_t^i and $\dot{\bar{q}}^i$ when transition functions of q^i are time-independent. Due to the isomorphism (4.3), every trivialization (4.2) yields the corresponding trivialization of the jet manifold

$$J^1Q = \mathbb{R} \times TM. \quad (4.4)$$

The canonical imbedding (7.53) of J^1Q takes the form

$$\begin{aligned} \lambda_{(1)} : J^1Q \ni (t, q^i, \dot{q}^i) &\rightarrow (t, q^i, \dot{t} = 1, \dot{q}^i = q_t^i) \in TQ, \\ \lambda_{(1)} &= d_t = \partial_t + q_t^i \partial_i, \end{aligned} \quad (4.5)$$

where by d_t is meant the total derivative. From now on, a jet manifold J^1Q is identified with its image in TQ .

In view of the morphism $\lambda_{(1)}$ (4.5), any connection

$$\Gamma = dt \otimes (\partial_t + \Gamma^i \partial_i) \quad (4.6)$$

on a fibre bundle $Q \rightarrow \mathbb{R}$ can be identified with a nowhere vanishing horizontal vector field

$$\Gamma = \partial_t + \Gamma^i \partial_i \quad (4.7)$$

on Q which is the horizontal lift $\Gamma \partial_t$ (7.57) of the standard vector field ∂_t on \mathbb{R} by means of the connection (4.6). Conversely, any vector field Γ on Q such that $dt \lrcorner \Gamma = 1$ defines a connection on $Q \rightarrow \mathbb{R}$. Therefore, the connections (4.6) further are identified with the vector fields (4.7). The integral curves of the vector field (4.7) coincide with the integral sections for the connection (4.6).

Connections on a fibre bundle $Q \rightarrow \mathbb{R}$ constitute an affine space modelled over the vector space of vertical vector fields on $Q \rightarrow \mathbb{R}$. Accordingly, the covariant differential (7.61), associated with a connection Γ on $Q \rightarrow \mathbb{R}$, takes its values into the vertical tangent bundle VQ of $Q \rightarrow \mathbb{R}$:

$$D^\Gamma : J^1Q \xrightarrow{Q} VQ, \quad \dot{q}^i \circ D^\Gamma = \dot{q}_t^i - \Gamma^i. \quad (4.8)$$

Its kernel, given by the coordinate equation

$$q_t^i = \Gamma^i(t, q^i), \quad (4.9)$$

is a closed subbundle of the jet bundle $J^1Q \rightarrow \mathbb{R}$. This is a first order dynamic differential equation on a fibre bundle $Q \rightarrow \mathbb{R}$ [41, 71].

A connection Γ on a fibre bundle $Q \rightarrow \mathbb{R}$ is obviously flat. It yields a horizontal distribution on Q . The integral manifolds of this distribution are integral curves of the vector field (4.7) which are transversal to fibres of a fibre bundle $Q \rightarrow \mathbb{R}$.

THEOREM 4.1: By virtue of Theorem 7.9, every connection Γ on a fibre bundle $Q \rightarrow \mathbb{R}$ defines an atlas of local constant trivializations of $Q \rightarrow \mathbb{R}$ such that the associated bundle coordinates (t, q^i) on Q possess the transition functions $q^i \rightarrow q'^i(q^j)$ independent of t , and

$$\Gamma = \partial_t \quad (4.10)$$

with respect to these coordinates. Conversely, every atlas of local constant trivializations of the fibre bundle $Q \rightarrow \mathbb{R}$ determines a connection on $Q \rightarrow \mathbb{R}$ which is equal to (4.10) relative to this atlas. \square

A connection Γ on a fibre bundle $Q \rightarrow \mathbb{R}$ is said to be complete if the horizontal vector field (4.7) is complete. In accordance with Remark 7.5, a connection on a fibre bundle $Q \rightarrow \mathbb{R}$ is complete iff it is an Ehresmann connection. The following holds [56].

THEOREM 4.2: Every trivialization of a fibre bundle $Q \rightarrow \mathbb{R}$ yields a complete connection on this fibre bundle. Conversely, every complete connection Γ on

$Q \rightarrow \mathbb{R}$ defines its trivialization (4.2) such that the horizontal vector field (4.7) equals ∂_t relative to the bundle coordinates associated with this trivialization. \square

4.2 Non-autonomous Hamiltonian systems

In non-autonomous mechanics on a configuration space $Q \rightarrow \mathbb{R}$, the jet manifold J^1Q plays a role of the velocity space. To describe non-autonomous mechanics, let us restrict our consideration to first order Lagrangian theory on a fibre bundle $Q \rightarrow \mathbb{R}$ [39, 41]. A first order Lagrangian is defined as a density

$$L = \mathcal{L}dt, \quad \mathcal{L} : J^1Q \rightarrow \mathbb{R}, \quad (4.11)$$

on a velocity space J^1Q . The corresponding second-order Lagrange operator reads

$$\delta L = (\partial_i \mathcal{L} - d_t \partial_i^t \mathcal{L}) \theta^i \wedge dt. \quad (4.12)$$

Let us further use the notation

$$\pi_i = \partial_i^t \mathcal{L}, \quad \pi_{ji} = \partial_j^t \partial_i^t \mathcal{L}. \quad (4.13)$$

The kernel $\text{Ker } \delta L \subset J^2Q$ of the Lagrange operator defines the second order Lagrange equation

$$(\partial_i - d_t \partial_i^t) \mathcal{L} = 0. \quad (4.14)$$

Its solutions are (local) sections c of the fibre bundle $Q \rightarrow \mathbb{R}$ whose second order jet prolongations \check{c} live in (4.14). They obey the equations

$$\partial_i \mathcal{L} \circ \check{c} - \frac{d}{dt} (\pi_i \circ \check{c}) = 0. \quad (4.15)$$

As was mentioned above, a phase space of non-relativistic mechanics on a configuration space $Q \rightarrow \mathbb{R}$ is the vertical cotangent bundle

$$V^*Q \xrightarrow{\pi_{\Pi}} Q \xrightarrow{\pi} \mathbb{R},$$

of $Q \rightarrow \mathbb{R}$ equipped with the holonomic coordinates $(t, q^i, p_i = \dot{q}_i)$ with respect to the fibre bases $\{\bar{d}q^i\}$ for the bundle $V^*Q \rightarrow Q$.

The cotangent bundle T^*Q of the configuration space Q is endowed with the holonomic coordinates (t, q^i, p_0, p_i) , possessing the transition functions

$$p'_i = \frac{\partial q^j}{\partial q'^i} p_j, \quad p'_0 = \left(p_0 + \frac{\partial q^j}{\partial t} p_j \right). \quad (4.16)$$

It admits the Liouville form

$$\Xi = p_0 dt + p_i dq^i, \quad (4.17)$$

the symplectic form

$$\Omega_T = d\Xi = dp_0 \wedge dt + dp_i \wedge dq^i, \quad (4.18)$$

and the corresponding Poisson bracket

$$\{f, g\}_T = \partial^0 f \partial_t g - \partial^0 g \partial_t f + \partial^i f \partial_i g - \partial^i g \partial_i f, \quad f, g \in C^\infty(T^*Q). \quad (4.19)$$

Provided with the structures (4.18) – (4.19), the cotangent bundle T^*Q of Q plays a role of the homogeneous phase space of Hamiltonian non-relativistic mechanics.

There is the canonical one-dimensional affine bundle

$$\zeta : T^*Q \rightarrow V^*Q. \quad (4.20)$$

A glance at the transformation law (4.16) shows that it is a trivial affine bundle. Indeed, given a global section h of ζ , one can equip T^*Q with the global fibre coordinate

$$I_0 = p_0 - h, \quad I_0 \circ h = 0, \quad (4.21)$$

possessing the identity transition functions. With respect to the coordinates

$$(t, q^i, I_0, p_i), \quad i = 1, \dots, m, \quad (4.22)$$

the fibration (4.20) reads

$$\zeta : \mathbb{R} \times V^*Q \ni (t, q^i, I_0, p_i) \rightarrow (t, q^i, p_i) \in V^*Q. \quad (4.23)$$

Let us consider the subring of $C^\infty(T^*Q)$ which comprises the pull-back ζ^*f onto T^*Q of functions f on the vertical cotangent bundle V^*Q by the fibration ζ (4.20). This subring is closed under the Poisson bracket (4.19). Then by virtue of Theorem 1.3, there exists the degenerate coinduced Poisson structure

$$\{f, g\}_V = \partial^i f \partial_i g - \partial^i g \partial_i f, \quad f, g \in C^\infty(V^*Q), \quad (4.24)$$

on a phase space V^*Q such that

$$\zeta^*\{f, g\}_V = \{\zeta^*f, \zeta^*g\}_T. \quad (4.25)$$

The holonomic coordinates on V^*Q are canonical for the Poisson structure (4.24).

With respect to the Poisson bracket (4.24), the Hamiltonian vector fields of functions on V^*Q read

$$\vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i, \quad f \in C^\infty(V^*Q), \quad (4.26)$$

$$[\vartheta_f, \vartheta_{f'}] = \vartheta_{\{f, f'\}_V}. \quad (4.27)$$

They are vertical vector fields on $V^*Q \rightarrow \mathbb{R}$. Accordingly, the characteristic distribution of the Poisson structure (4.24) is the vertical tangent bundle $VV^*Q \subset TV^*Q$ of a fibre bundle $V^*Q \rightarrow \mathbb{R}$. The corresponding symplectic foliation on the phase space V^*Q coincides with the fibration $V^*Q \rightarrow \mathbb{R}$.

It is readily observed that the ring $\mathcal{C}(V^*Q)$ of Casimir functions on a Poisson manifold V^*Q consists of the pull-back onto V^*Q of functions on \mathbb{R} . Therefore, the Poisson algebra $C^\infty(V^*Q)$ is a Lie $C^\infty(\mathbb{R})$ -algebra.

Remark 4.3: The Poisson structure (4.24) can be introduced in a different way [56, 71]. Given any section h of the fibre bundle (4.20), let us consider the pull-back forms

$$\begin{aligned}\Theta &= h^*(\Xi \wedge dt) = p_i dq^i \wedge dt, \\ \Omega &= h^*(d\Xi \wedge dt) = dp_i \wedge dq^i \wedge dt\end{aligned}\quad (4.28)$$

on V^*Q . They are independent of the choice of h . With Ω (4.28), the Hamiltonian vector field ϑ_f (4.26) for a function f on V^*Q is given by the relation

$$\vartheta_f \lrcorner \Omega = -df \wedge dt,$$

while the Poisson bracket (4.24) is written as

$$\{f, g\}_V dt = \vartheta_g \lrcorner \vartheta_f \lrcorner \Omega.$$

Moreover, one can show that a projectable vector field ϑ on V^*Q such that $\vartheta \lrcorner dt = \text{const.}$ is a canonical vector field for the Poisson structure (4.24) iff

$$\mathbf{L}_\vartheta \Omega = d(\vartheta \lrcorner \Omega) = 0. \quad (4.29)$$

□

In contrast with autonomous Hamiltonian mechanics, the Poisson structure (4.24) fails to provide any dynamic equation on a fibre bundle $V^*Q \rightarrow \mathbb{R}$ because Hamiltonian vector fields (4.26) of functions on V^*Q are vertical vector fields, but not connections on $V^*Q \rightarrow \mathbb{R}$. Hamiltonian dynamics on V^*Q is described as a particular Hamiltonian dynamics on fibre bundles [28, 40, 73].

A Hamiltonian on a phase space $V^*Q \rightarrow \mathbb{R}$ of non-relativistic mechanics is defined as a global section

$$h : V^*Q \rightarrow T^*Q, \quad p_0 \circ h = \mathcal{H}(t, q^j, p_j), \quad (4.30)$$

of the affine bundle ζ (4.20). Given the Liouville form Ξ (4.17) on T^*Q , this section yields the pull-back Hamiltonian form

$$H = (-h)^* \Xi = p_k dq^k - \mathcal{H} dt \quad (4.31)$$

on V^*Q . This is the well-known invariant of Poincaré – Cartan [3].

It should be emphasized that, in contrast with a Hamiltonian in autonomous mechanics, the Hamiltonian \mathcal{H} (4.30) is not a function on V^*Q , but it obeys the transformation law

$$\mathcal{H}'(t, q'^i, p'_i) = \mathcal{H}(t, q^i, p_i) + p'_i \partial_i q'^i. \quad (4.32)$$

Remark 4.4: Any connection Γ (4.7) on a configuration bundle $Q \rightarrow \mathbb{R}$ defines the global section $h_\Gamma = p_i \Gamma^i$ (4.30) of the affine bundle ζ (4.20) and the corresponding Hamiltonian form

$$H_\Gamma = p_k dq^k - \mathcal{H}_\Gamma dt = p_k dq^k - p_i \Gamma^i dt. \quad (4.33)$$

Furthermore, given a connection Γ , any Hamiltonian form (4.31) admits the splitting

$$H = H_\Gamma - \mathcal{E}_\Gamma dt, \quad (4.34)$$

where

$$\mathcal{E}_\Gamma = \mathcal{H} - \mathcal{H}_\Gamma = \mathcal{H} - p_i \Gamma^i \quad (4.35)$$

is a function on V^*Q . It is called the Hamiltonian function relative to a reference frame Γ . With respect to the coordinates adapted to a reference frame Γ , we have $\mathcal{E}_\Gamma = \mathcal{H}$. Given different reference frames Γ and Γ' , the decomposition (4.34) leads at once to the relation

$$\mathcal{E}_{\Gamma'} = \mathcal{E}_\Gamma + \mathcal{H}_\Gamma - \mathcal{H}_{\Gamma'} = \mathcal{E}_\Gamma + (\Gamma^i - \Gamma'^i) p_i \quad (4.36)$$

between the Hamiltonian functions with respect to different reference frames. \square

Given a Hamiltonian form H (4.31), there exists a unique horizontal vector field (4.7):

$$\gamma_H = \partial_t - \gamma^i \partial_i - \gamma_i \partial^i,$$

on V^*Q (i.e., a connection on $V^*Q \rightarrow \mathbb{R}$) such that

$$\gamma_H \rfloor dH = 0. \quad (4.37)$$

This vector field, called the Hamilton vector field, reads

$$\gamma_H = \partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k. \quad (4.38)$$

In a different way (Remark 4.3), the Hamilton vector field γ_H is defined by the relation

$$\gamma_H \rfloor \Omega = dH.$$

Consequently, it is canonical for the Poisson structure $\{, \}_V$ (4.24). This vector field yields the first order dynamic Hamilton equation

$$q_t^k = \partial^k \mathcal{H}, \quad (4.39)$$

$$p_{tk} = -\partial_k \mathcal{H} \quad (4.40)$$

on $V^*Q \rightarrow \mathbb{R}$, where $(t, q^k, p_k, q_t^k, \dot{p}_{tk})$ are the adapted coordinates on the first order jet manifold $J^1 V^*Q$ of $V^*Q \rightarrow \mathbb{R}$.

Due to the canonical imbedding $J^1V^*Q \rightarrow TV^*Q$ (4.5), the Hamilton equation (4.39) – (4.40) is equivalent to the autonomous first order dynamic equation

$$\dot{t} = 1, \quad \dot{q}^i = \partial^i \mathcal{H}, \quad \dot{p}_i = -\partial_i \mathcal{H} \quad (4.41)$$

on a manifold V^*Q (Remark 1.5).

A solution of the Hamilton equation (4.39) – (4.40) is an integral section r for the connection γ_H .

We agree to call (V^*Q, H) the Hamiltonian system of $k = \dim Q - 1$ degrees of freedom.

In order to describe evolution of a Hamiltonian system at any instant, the Hamilton vector field γ_H (4.38) is assumed to be complete, i.e., it is an Ehresmann connection (Remark 4.2). In this case, the Hamilton equation (4.39) – (4.40) admits a unique global solution through each point of the phase space V^*Q . By virtue of Theorem 4.2, there exists a trivialization of a fibre bundle $V^*Q \rightarrow \mathbb{R}$ (not necessarily compatible with its fibration $V^*Q \rightarrow Q$) such that

$$\gamma_H = \partial_t, \quad H = \bar{p}_i d\bar{q}^i \quad (4.42)$$

with respect to the associated coordinates $(t, \bar{q}^i, \bar{p}_i)$. A direct computation shows that the Hamilton vector field γ_H (4.38) satisfies the relation (4.29) and, consequently, it is an infinitesimal generator of a one-parameter group of automorphisms of the Poisson manifold $(V^*Q, \{, \}_V)$. Then one can show that $(t, \bar{q}^i, \bar{p}_i)$ are canonical coordinates for the Poisson manifold $(V^*Q, \{, \}_V)$ [56], i.e.,

$$w = \frac{\partial}{\partial \bar{p}_i} \wedge \frac{\partial}{\partial \bar{q}^i}.$$

Since $\mathcal{H} = 0$, the Hamilton equation (4.39) – (4.40) in these coordinates takes the form

$$\bar{q}_t^i = 0, \quad \bar{p}_{ti} = 0,$$

i.e., $(t, \bar{q}^i, \bar{p}_i)$ are the initial data coordinates.

As was mentioned above, one can associate to any Hamiltonian system on a phase space V^*Q an equivalent autonomous symplectic Hamiltonian system on the cotangent bundle T^*Q (Theorem 4.3).

Given a Hamiltonian system (V^*Q, H) , its Hamiltonian \mathcal{H} (4.30) defines the function

$$\mathcal{H}^* = \partial_t \lrcorner (\Xi - \zeta^*(-h)^*\Xi) = p_0 + h = p_0 + \mathcal{H} \quad (4.43)$$

on T^*Q . Let us regard \mathcal{H}^* (4.43) as a Hamiltonian of an autonomous Hamiltonian system on the symplectic manifold (T^*Q, Ω_T) . The corresponding autonomous Hamilton equation on T^*Q takes the form

$$\dot{t} = 1, \quad \dot{p}_0 = -\partial_t \mathcal{H}, \quad \dot{q}^i = \partial^i \mathcal{H}, \quad \dot{p}_i = -\partial_i \mathcal{H}. \quad (4.44)$$

Remark 4.5: Let us note that the splitting $\mathcal{H}^* = p_0 + \mathcal{H}$ (4.43) is ill defined. At the same time, any reference frame Γ yields the decomposition

$$\mathcal{H}^* = (p_0 + \mathcal{H}_\Gamma) + (\mathcal{H} - \mathcal{H}_\Gamma) = \mathcal{H}_\Gamma^* + \mathcal{E}_\Gamma, \quad (4.45)$$

where \mathcal{H}_Γ is the Hamiltonian (4.33) and \mathcal{E}_Γ (4.35) is the Hamiltonian function relative to a reference frame Γ . \square

The Hamiltonian vector field $\vartheta_{\mathcal{H}^*}$ of \mathcal{H}^* (4.43) on T^*Q is

$$\vartheta_{\mathcal{H}^*} = \partial_t - \partial_t \mathcal{H} \partial^0 + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i. \quad (4.46)$$

Written relative to the coordinates (4.22), this vector field reads

$$\vartheta_{\mathcal{H}^*} = \partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i. \quad (4.47)$$

It is identically projected onto the Hamilton vector field γ_H (4.38) on V^*Q such that

$$\zeta^*(\mathbf{L}_{\gamma_H} f) = \{\mathcal{H}^*, \zeta^* f\}_T, \quad f \in C^\infty(V^*Q). \quad (4.48)$$

Therefore, the Hamilton equation (4.39) – (4.40) is equivalent to the autonomous Hamilton equation (4.44).

Obviously, the Hamiltonian vector field $\vartheta_{\mathcal{H}^*}$ (4.47) is complete if the Hamilton vector field γ_H (4.38) is complete.

Thus, the following has been proved [14, 41, 57].

THEOREM 4.3: A Hamiltonian system (V^*Q, H) of k degrees of freedom is equivalent to an autonomous Hamiltonian system (T^*Q, \mathcal{H}^*) of $k+1$ degrees of freedom on a symplectic manifold (T^*Q, Ω_T) whose Hamiltonian is the function \mathcal{H}^* (4.43). \square

We agree to call (T^*Q, \mathcal{H}^*) the homogeneous Hamiltonian system and \mathcal{H}^* (4.43) the homogeneous Hamiltonian.

It is readily observed that the Hamiltonian form H (4.31) is the Poincaré – Cartan form of the Lagrangian

$$L_H = h_0(H) = (p_i q_t^i - \mathcal{H}) dt \quad (4.49)$$

on the jet manifold $J^1 V^*Q$ of $V^*Q \rightarrow \mathbb{R}$ [40, 59, 73].

The Lagrange operator (4.12) associated to the Lagrangian L_H reads

$$\mathcal{E}_H = \delta L_H = [(q_t^i - \partial^i \mathcal{H}) dp_i - (p_{ti} + \partial_i \mathcal{H}) dq^i] \wedge dt. \quad (4.50)$$

The corresponding Lagrange equation (4.14) is of first order, and it coincides with the Hamilton equation (4.39) – (4.40) on $J^1 V^*Q$.

Due to this fact, the Lagrangian L_H (4.49) plays a prominent role in Hamiltonian non-relativistic mechanics.

In particular, let

$$u = u^t \partial_t + u^i \partial_i, \quad u^t = 0, 1,$$

be a projectable vector field on a configuration space Q . Its functorial lift (7.22) onto the cotangent bundle T^*Q is

$$\tilde{u} = u^t \partial_t + u^i \partial_i - p_j \partial_i u^j \partial^i. \quad (4.51)$$

This vector field is identically projected onto a vector field, also given by the expression (4.51), on the phase space V^*Q as a base of the trivial fibre bundle (4.20). Then we have the equality

$$\mathbf{L}_{\tilde{u}}H = \mathbf{L}_{J^1\tilde{u}}L_H = (-u^t\partial_t\mathcal{H} + p_i\partial_tu^i - u^i\partial_i\mathcal{H} + p_i\partial_ju^i\partial^j\mathcal{H})dt. \quad (4.52)$$

This equality enables us to study conservation laws in Hamiltonian mechanics similarly to those in Lagrangian mechanics.

Let an equation of motion of a mechanical system on a fibre bundle $Y \rightarrow \mathbb{R}$ be described by an r -order differential equation \mathfrak{E} given by a closed subbundle of the jet bundle $J^rY \rightarrow \mathbb{R}$ [40, 73].

DEFINITION 4.4: An integral of motion of this mechanical system is defined as a ($k < r$)-order differential operator Φ on Y such that \mathfrak{E} belongs to the kernel of an r -order jet prolongation of the differential operator $d_t\Phi$, i.e.,

$$J^{r-k-1}(d_t\Phi)|_{\mathfrak{E}} = J^{r-k}\Phi|_{\mathfrak{E}} = 0. \quad (4.53)$$

□

It follows that an integral of motion Φ is constant on solutions s of a differential equation \mathfrak{E} , i.e., there is the differential conservation law

$$(J^k s)^*\Phi = \text{const.}, \quad (J^{k+1} s)^*d_t\Phi = 0. \quad (4.54)$$

We agree to write the condition (4.53) as the weak equality

$$J^{r-k-1}(d_t\Phi) \approx 0, \quad (4.55)$$

which holds on-shell, i.e., on solutions of a differential equation \mathfrak{E} by the formula (4.54).

In non-relativistic mechanics, we can restrict our consideration to integrals of motion Φ which are functions on J^kY . As was mentioned above, equations of motion of non-relativistic mechanics mainly are of first or second order. Accordingly, their integrals of motion are functions on Y or J^kY . In this case, the corresponding weak equality (4.53) takes the form

$$d_t\Phi \approx 0 \quad (4.56)$$

of a weak conservation law or, simply, a conservation law.

Different integrals of motion need not be independent. Let integrals of motion Φ_1, \dots, Φ_m of a mechanical system on Y be functions on J^kY . They are called independent if

$$d\Phi_1 \wedge \dots \wedge d\Phi_m \neq 0 \quad (4.57)$$

everywhere on J^kY . In this case, any motion $J^k s$ of this mechanical system lies in the common level surfaces of functions Φ_1, \dots, Φ_m which bring J^kY into a fibred manifold.

Integrals of motion can come from symmetries. This is the case of Lagrangian and Hamiltonian mechanics.

DEFINITION 4.5: Let an equation of motion of a mechanical system be an r -order differential equation $\mathfrak{E} \subset J^r Y$. Its infinitesimal symmetry (or, simply, a symmetry) is defined as a vector field on $J^r Y$ whose restriction to \mathfrak{E} is tangent to \mathfrak{E} . \square

For instance, let us consider first order dynamic equations.

PROPOSITION 4.6: Let \mathfrak{E} be the autonomous first order dynamic equation (1.24) given by a vector field u on a manifold Z . A vector field ϑ on Z is its symmetry iff $[u, \vartheta] \approx 0$. \square

One can show that a smooth real function F on a manifold Z is an integral of motion of the autonomous first order dynamic equation (1.24) (i.e., it is constant on solutions of this equation) iff its Lie derivative along u vanishes:

$$\mathbf{L}_u F = u^\lambda \partial_\lambda \Phi = 0. \quad (4.58)$$

PROPOSITION 4.7: Let \mathfrak{E} be the first order dynamic equation (4.9) given by a connection Γ (4.7) on a fibre bundle $Y \rightarrow \mathbb{R}$. Then a vector field ϑ on Y is its symmetry iff $[\Gamma, \vartheta] \approx 0$. \square

A smooth real function Φ on Y is an integral of motion of the first order dynamic equation (4.9) in accordance with the equality (4.56) iff

$$\mathbf{L}_\Gamma \Phi = (\partial_t + \Gamma^i \partial_i) \Phi = 0. \quad (4.59)$$

Following Definition 4.5, let us introduce the notion of a symmetry of differential operators in the following relevant case. Let us consider an r -order differential operator on a fibre bundle $Y \rightarrow \mathbb{R}$ which is represented by an exterior form \mathcal{E} on $J^r Y$. Let its kernel $\text{Ker } \mathcal{E}$ be an r -order differential equation on $Y \rightarrow \mathbb{R}$.

PROPOSITION 4.8: It is readily justified that a vector field ϑ on $J^r Y$ is a symmetry of the equation $\text{Ker } \mathcal{E}$ in accordance with Definition 4.5 iff

$$\mathbf{L}_\vartheta \mathcal{E} \approx 0. \quad (4.60)$$

\square

Motivated by Proposition 4.8, we come to the following.

DEFINITION 4.9: Let \mathcal{E} be the above mentioned differential operator. A vector field ϑ on $J^r Y$ is called a symmetry of a differential operator \mathcal{E} if the Lie derivative $\mathbf{L}_\vartheta \mathcal{E}$ vanishes. \square

By virtue of Proposition 4.8, a symmetry of a differential operator \mathcal{E} also is a symmetry of the differential equation $\text{Ker } \mathcal{E}$.

4.3 Non-autonomous integrable systems

Let us consider a non-autonomous mechanical system on a configuration space $Q \rightarrow \mathbb{R}$ in Section 4.2. Its phase space is the vertical cotangent bundle $V^*Q \rightarrow Q$ of $Q \rightarrow \mathbb{R}$ endowed with the Poisson structure $\{\cdot, \cdot\}_V$ (4.24). A Hamiltonian of a non-autonomous mechanical system is a section h (4.30) of the one-dimensional fibre bundle (4.20) – (4.23):

$$\zeta : T^*Q \rightarrow V^*Q, \quad (4.61)$$

where T^*Q is the cotangent bundle of Q endowed with the canonical symplectic form Ω_T (4.18). The Hamiltonian h (4.30) yields the pull-back Hamiltonian form H (4.31) on V^*Q and defines the Hamilton vector field γ_H (4.38) on V^*Q . A smooth real function F on V^*Q is an integral of motion of a Hamiltonian system (V^*Q, H) if its Lie derivative $\mathbf{L}_{\gamma_H} F$ vanishes.

DEFINITION 4.10: A non-autonomous Hamiltonian system (V^*Q, H) of $n = \dim Q - 1$ degrees of freedom is called superintegrable if it admits $n \leq k < 2n$ integrals of motion Φ_1, \dots, Φ_k , obeying the following conditions.

(i) All the functions Φ_α are independent, i.e., the k -form $d\Phi_1 \wedge \dots \wedge d\Phi_k$ nowhere vanishes on V^*Q . It follows that the map

$$\Phi : V^*Q \rightarrow N = (\Phi_1(V^*Q), \dots, \Phi_k(V^*Q)) \subset \mathbb{R}^k \quad (4.62)$$

is a fibred manifold over a connected open subset $N \subset \mathbb{R}^k$.

(ii) There exist smooth real functions $s_{\alpha\beta}$ on N such that

$$\{\Phi_\alpha, \Phi_\beta\}_V = s_{\alpha\beta} \circ \Phi, \quad \alpha, \beta = 1, \dots, k. \quad (4.63)$$

(iii) The matrix function with the entries $s_{\alpha\beta}$ (4.63) is of constant corank $m = 2n - k$ at all points of N . \square

In order to describe this non-autonomous superintegrable Hamiltonian system, we use the fact that there exists an equivalent autonomous Hamiltonian system (T^*Q, \mathcal{H}^*) of $n + 1$ degrees of freedom on a symplectic manifold (T^*Q, Ω_T) whose Hamiltonian is the function \mathcal{H}^* (4.43) (Theorem 4.3), and that this Hamiltonian system is superintegrable (Theorem 4.15). Our goal is the following.

THEOREM 4.11: Let Hamiltonian vector fields of the functions Φ_α be complete, and let fibres of the fibred manifold Φ (4.62) be connected and mutually diffeomorphic. Then there exists an open neighborhood U_M of a fibre M of Φ (4.62) which is a trivial principal bundle with the structure group

$$\mathbb{R}^{1+m-r} \times T^r \quad (4.64)$$

whose bundle coordinates are the generalized action-angle coordinates

$$(p_A, q^A, I_\lambda, t, y^\lambda), \quad A = 1, \dots, k - n, \quad \lambda = 1, \dots, m, \quad (4.65)$$

such that:

- (i) (t, y^λ) are coordinates on the toroidal cylinder (4.64),
- (ii) the Poisson bracket $\{, \}_V$ on U_M reads

$$\{f, g\}_V = \partial^A f \partial_A g - \partial^A g \partial_A f + \partial^\lambda f \partial_\lambda g - \partial^\lambda g \partial_\lambda f,$$

- (iii) a Hamiltonian \mathcal{H} depends only on the action coordinates I_λ ,
- (iv) the integrals of motion Φ_1, \dots, Φ_k are independent of coordinates (t, y^λ) .

□

Let us start with the case $k = n$ of a completely integrable non-autonomous Hamiltonian system (Theorem 4.14).

DEFINITION 4.12: A non-autonomous Hamiltonian system (V^*Q, H) of n degrees of freedom is said to be completely integrable if it admits n independent integrals of motion F_1, \dots, F_n which are in involution with respect to the Poisson bracket $\{, \}_V$ (4.24). □

By virtue of the relations (4.27), the vector fields

$$(\gamma_H, \vartheta_{F_1}, \dots, \vartheta_{F_n}), \quad \vartheta_{F_\alpha} = \partial^i F_\alpha \partial_i - \partial_i F_\alpha \partial^i, \quad (4.66)$$

mutually commute and, therefore, they span an $(n + 1)$ -dimensional involutive distribution \mathcal{V} on V^*Q . Let G be the group of local diffeomorphisms of V^*Q generated by the flows of vector fields (4.66). Maximal integral manifolds of \mathcal{V} are the orbits of G and invariant submanifolds of vector fields (4.66). They yield a foliation \mathcal{F} of V^*Q .

Let (V^*Q, H) be a non-autonomous Hamiltonian system and (T^*Q, \mathcal{H}^*) an equivalent autonomous Hamiltonian system on T^*Q . An immediate consequence of the relations (4.25) and (4.48) is the following.

THEOREM 4.13: Given a non-autonomous completely integrable Hamiltonian system

$$(\gamma_H, F_1, \dots, F_n) \quad (4.67)$$

of n degrees of freedom on V^*Q , the associated autonomous Hamiltonian system

$$(\mathcal{H}^*, \zeta^* F_1, \dots, \zeta^* F_n) \quad (4.68)$$

of $n + 1$ degrees of freedom on T^*Q is completely integrable. □

The Hamiltonian vector fields

$$(u_{\mathcal{H}^*}, u_{\zeta^* F_1}, \dots, u_{\zeta^* F_n}), \quad u_{\zeta^* F_\alpha} = \partial^i F_\alpha \partial_i - \partial_i F_\alpha \partial^i, \quad (4.69)$$

of the autonomous integrals of motion (4.68) span an $(n + 1)$ -dimensional involutive distribution \mathcal{V}_T on T^*Q such that

$$T\zeta(\mathcal{V}_T) = \mathcal{V}, \quad Th(\mathcal{V}) = \mathcal{V}_T|_{h(V^*Q)=I_0=0}, \quad (4.70)$$

where

$$\begin{aligned} Th : TV^*Q \ni (t, q^i, p_i, \dot{t}, \dot{q}^i, \dot{p}_i) \rightarrow \\ (t, q^i, p_i, I_0 = 0, \dot{t}, \dot{q}^i, \dot{p}_i, \dot{I}_0 = 0) \in TT^*Q. \end{aligned}$$

It follows that, if M is an invariant submanifold of the non-autonomous completely integrable Hamiltonian system (4.67), then $h(M)$ is an invariant submanifold of the autonomous completely integrable Hamiltonian system (4.68).

In order to introduce generalized action-angle coordinates around an invariant submanifold M of the non-autonomous completely integrable Hamiltonian system (4.67), let us suppose that the vector fields (4.66) on M are complete. It follows that M is a locally affine manifold diffeomorphic to a toroidal cylinder

$$\mathbb{R}^{1+n-r} \times T^r. \quad (4.71)$$

Moreover, let us assume that there exists an open neighborhood U_M of M such that the foliation \mathcal{F} of U_M is a fibred manifold $\phi : U_M \rightarrow N$ over a domain $N \subset \mathbb{R}^n$ whose fibres are mutually diffeomorphic.

Because the morphism Th (4.70) is a bundle isomorphism, the Hamiltonian vector fields (4.69) on the invariant submanifold $h(M)$ of the autonomous completely integrable Hamiltonian system are complete. Since the affine bundle ζ (4.61) is trivial, the open neighborhood $\zeta^{-1}(U_M)$ of the invariant submanifold $h(M)$ is a fibred manifold

$$\tilde{\phi} : \zeta^{-1}(U_M) = \mathbb{R} \times U_M \xrightarrow{(\text{Id}_{\mathbb{R}}, \phi)} \mathbb{R} \times N = N'$$

over a domain $N' \subset \mathbb{R}^{n+1}$ whose fibres are diffeomorphic to the toroidal cylinder (4.71). In accordance with Theorem 2.7, the open neighborhood $\zeta^{-1}(U_M)$ of $h(M)$ is a trivial principal bundle

$$\zeta^{-1}(U_M) = N' \times (\mathbb{R}^{1+n-r} \times T^r) \rightarrow N' \quad (4.72)$$

with the structure group (4.71) whose bundle coordinates are the generalized action-angle coordinates

$$(I_0, I_1, \dots, I_n, t, z^1, \dots, z^n) \quad (4.73)$$

such that:

- (i) (t, z^a) are coordinates on the toroidal cylinder (4.71),
- (ii) the symplectic form Ω_T on $\zeta^{-1}(U)$ reads

$$\Omega_T = dI_0 \wedge dt + dI_a \wedge dz^a,$$

- (iii) $\mathcal{H}^* = I_0$,
- (iv) the integrals of motion $\zeta^*F_1, \dots, \zeta^*F_n$ depend only on the action coordinates I_1, \dots, I_n .

Provided with the coordinates (4.73),

$$\zeta^{-1}(U_M) = U_M \times \mathbb{R}$$

is a trivial bundle possessing the fibre coordinate I_0 (4.21). Consequently, the non-autonomous open neighborhood U_M of an invariant submanifold M of the completely integrable Hamiltonian system (4.66) is diffeomorphic to the Poisson annulus

$$U_M = N \times (\mathbb{R}^{1+n-r} \times T^r) \quad (4.74)$$

endowed with the generalized action-angle coordinates

$$(I_1, \dots, I_n, t, z^1, \dots, z^n) \quad (4.75)$$

such that:

(i) the Poisson structure (4.24) on U_M takes the form

$$\{f, g\}_V = \partial^a f \partial_a g - \partial^a g \partial_a f,$$

(ii) the Hamiltonian (4.30) reads $\mathcal{H} = 0$,

(iii) the integrals of motion F_1, \dots, F_n depend only on the action coordinates I_1, \dots, I_n .

The Hamilton equation (4.39) – (4.40) relative to the generalized action-angle coordinates (4.75) takes the form

$$z_t^a = 0, \quad I_{ta} = 0.$$

It follows that the generalized action-angle coordinates (4.75) are the initial date coordinates.

Note that the generalized action-angle coordinates (4.75) by no means are unique. Given a smooth function \mathcal{H}' on \mathbb{R}^n , one can provide $\zeta^{-1}(U_M)$ with the generalized action-angle coordinates

$$t, \quad z'^a = z^a - t \partial^a \mathcal{H}', \quad I'_0 = I_0 + \mathcal{H}'(I_b), \quad I'_a = I_a. \quad (4.76)$$

With respect to these coordinates, a Hamiltonian of the autonomous Hamiltonian system on $\zeta^{-1}(U_M)$ reads $\mathcal{H}'^* = I'_0 - \mathcal{H}'$. A Hamiltonian of the non-autonomous Hamiltonian system on U endowed with the generalized action-angle coordinates (I_a, t, z'^a) is \mathcal{H}' .

Thus, the following has been proved.

THEOREM 4.14: Let $(\gamma_H, F_1, \dots, F_n)$ be a non-autonomous completely integrable Hamiltonian system. Let M be its invariant submanifold such that the vector fields (4.66) on M are complete and that there exists an open neighborhood U_M of M which is a fibred manifold in mutually diffeomorphic invariant submanifolds. Then U_M is diffeomorphic to the Poisson annulus (4.74), and it can be provided with the generalized action-angle coordinates (4.75) such that the integrals of motion (F_1, \dots, F_n) and the Hamiltonian \mathcal{H} depend only on the action coordinates I_1, \dots, I_n . \square

Let now $(\gamma_H, \Phi_1, \dots, \Phi_k)$ be a non-autonomous superintegrable Hamiltonian system in accordance with Definition 4.10. The associated autonomous Hamiltonian system on T^*Q possesses $k + 1$ integrals of motion

$$(\mathcal{H}^*, \zeta^* \Phi_1, \dots, \zeta^* \Phi_k) \quad (4.77)$$

with the following properties.

(i) The functions (4.77) are mutually independent, and the map

$$\begin{aligned} \tilde{\Phi} : T^*Q \rightarrow (\mathcal{H}^*(T^*Q), \zeta^*\Phi_1(T^*Q), \dots, \zeta^*\Phi_k(T^*Q)) = \\ (I_0, \Phi_1(V^*Q), \dots, \Phi_k(V^*Q)) = \mathbb{R} \times N = N' \end{aligned} \quad (4.78)$$

is a fibred manifold.

(ii) The functions (4.77) obey the relations

$$\{\zeta^*\Phi_\alpha, \zeta^*\Phi_\beta\} = s_{\alpha\beta} \circ \zeta^*\Phi, \quad \{\mathcal{H}^*, \zeta^*\Phi_\alpha\} = s_{0\alpha} = 0$$

so that the matrix function with the entries $(s_{0\alpha}, s_{\alpha\beta})$ on N' is of constant corank $2n + 1 - k$.

Refereing to Definition 2.1 of an autonomous superintegrable system, we come to the following.

THEOREM 4.15: Given a non-autonomous superintegrable Hamiltonian system (γ_H, Φ_α) on V^*Q , the associated autonomous Hamiltonian system (4.77) on T^*Q is superintegrable. \square

There is the commutative diagram

$$\begin{array}{ccc} T^*Q & \xrightarrow{\zeta} & V^*Q \\ \tilde{\Phi} \downarrow & & \downarrow \Phi \\ N' & \xrightarrow{\xi} & N \end{array}$$

where ζ (4.61) and

$$\xi : N' = \mathbb{R} \times N \rightarrow N$$

are trivial bundles. It follows that the fibred manifold (4.78) is the pull-back $\tilde{\Phi} = \xi^*\Phi$ of the fibred manifold Φ (4.62) onto N' .

Let the conditions of Theorem 2.5 hold. If the Hamiltonian vector fields

$$(\gamma_H, \vartheta_{\Phi_1}, \dots, \vartheta_{\Phi_k}), \quad \vartheta_{\Phi_\alpha} = \partial^i \Phi_\alpha \partial_i - \partial_i \Phi_\alpha \partial^i,$$

of integrals of motion Φ_α on V^*Q are complete, the Hamiltonian vector fields

$$(u_{\mathcal{H}^*}, u_{\zeta^*\Phi_1}, \dots, u_{\zeta^*\Phi_k}), \quad u_{\zeta^*\Phi_\alpha} = \partial^i \Phi_\alpha \partial_i - \partial_i \Phi_\alpha \partial^i,$$

on T^*Q are complete. If fibres of the fibred manifold Φ (4.62) are connected and mutually diffeomorphic, the fibres of the fibred manifold $\tilde{\Phi}$ (4.78) also are well.

Let M be a fibre of Φ (4.62) and $h(M)$ the corresponding fibre of $\tilde{\Phi}$ (4.78). In accordance Theorem 2.5, there exists an open neighborhood U' of $h(M)$ which is a trivial principal bundle with the structure group (4.64) whose bundle coordinates are the generalized action-angle coordinates

$$(I_0, I_\lambda, t, y^\lambda, p_A, q^A), \quad A = 1, \dots, n - m, \quad \lambda = 1, \dots, k, \quad (4.79)$$

such that:

- (i) (t, y^λ) are coordinates on the toroidal cylinder (4.64),
- (ii) the symplectic form Ω_T on U' reads

$$\Omega_T = dI_0 \wedge dt + dI_\alpha \wedge dy^\alpha + dp_A \wedge dq^A,$$

- (iii) the action coordinates (I_0, I_α) are expressed in the values of the Casimir functions $C_0 = I_0, C_\alpha$ of the coinduced Poisson structure

$$w = \partial^A \wedge \partial_A$$

on N' ,

- (iv) a homogeneous Hamiltonian \mathcal{H}^* depends on the action coordinates, namely, $\mathcal{H}^* = I_0$,

- (iv) the integrals of motion $\zeta^* \Phi_1, \dots, \zeta^* \Phi_k$ are independent of the coordinates (t, y^λ) .

Provided with the generalized action-angle coordinates (4.79), the above mentioned neighborhood U' is a trivial bundle $U' = \mathbb{R} \times U_M$ where $U_M = \zeta(U')$ is an open neighborhood of the fibre M of the fibre bundle Φ (4.62). As a result, we come to Theorem 4.11.

5 Quantum superintegrable systems

To quantize classical Hamiltonian systems, one usually follows canonical quantization which replaces the Poisson bracket $\{f, f'\}$ of smooth functions with the bracket $[\widehat{f}, \widehat{f}']$ of Hermitian operators in a Hilbert space such that Dirac's condition

$$[\widehat{f}, \widehat{f}'] = -i\{\widehat{f}, \widehat{f}'\} \quad (5.1)$$

holds. Canonical quantization of Hamiltonian non-relativistic mechanics on a configuration space $Q \rightarrow \mathbb{R}$ is geometric quantization [31, 37, 41]. In the case of integrable Hamiltonian systems, there is a reason that, since a Hamiltonian of an integrable system depends only on action variables (Proposition 2.13), it seems natural to provide the Schrödinger representation of action variables by first order differential operators on functions of angle coordinates

For the sake of simplicity, symplectic and Poisson manifolds throughout this Section are assumed to be simple connected (see Remark 5.1). Geometric quantization of toroidal cylinders possessing a non-trivial first homotopy group is considered in Section 6.4.

5.1 Geometric quantization of symplectic manifolds

We start with the basic geometric quantization of symplectic manifolds [16, 37, 41, 79]. It falls into the following three steps: prequantization, polarization and metaplectic correction.

Let (Z, Ω) be a $2m$ -dimensional simply connected symplectic manifold. Let $C \rightarrow Z$ be a complex line bundle whose typical fibre is \mathbb{C} . It is coordinated by (z^λ, c) where c is a complex coordinate.

PROPOSITION 5.1: By virtue of the well-known theorems [48, 58], the structure group of a complex line bundle $C \rightarrow Z$ is reducible to $U(1)$ such that:

- given a bundle atlas of $C \rightarrow Z$ with $U(1)$ -valued transition functions and associated bundle coordinates (z^λ, c) , there exists a Hermitian fibre metric

$$g(c, c) = c\bar{c} \quad (5.2)$$

in C ;

- for any Hermitian fibre metric g in $C \rightarrow Z$, there exists a bundle atlas of $C \rightarrow Z$ with $U(1)$ -valued transition functions such that g takes the form (5.2) with respect to the associated bundle coordinates. \square

Let \mathcal{K} be a linear connection on a fibre bundles $C \rightarrow Z$. It reads

$$\mathcal{K} = dz^\lambda \otimes (\partial_\lambda + \mathcal{K}_\lambda c \partial_c), \quad (5.3)$$

where \mathcal{K}_λ are local complex functions on Z . The corresponding covariant differential $D^\mathcal{K}$ (7.61) takes the form

$$D^\mathcal{K} = (c_\lambda - \mathcal{K}_\lambda c) dz^\lambda \otimes \partial_c. \quad (5.4)$$

The curvature two-form (7.66) of the connection \mathcal{K} (5.3) reads

$$R = \frac{1}{2} (\partial_\nu \mathcal{K}_\mu - \partial_\mu \mathcal{K}_\nu) c dz^\nu \wedge dz^\mu \otimes \partial_c. \quad (5.5)$$

PROPOSITION 5.2: A connection A on a complex line bundle $C \rightarrow Z$ is a $U(1)$ -principal connection iff there exists an A -invariant Hermitian fibre metric g in C , i.e.,

$$d_H(g(c, c)) = g(D^A c, c) + g(c, D^A c).$$

With respect to the bundle coordinates (z^λ, c) in Proposition 5.1, this connection reads

$$A = dz^\lambda \otimes (\partial_\lambda + i A_\lambda c \partial_c), \quad (5.6)$$

where A_λ are local real functions on Z . \square

The curvature R (5.5) of the $U(1)$ -principal connection A (5.6) defines the first Chern characteristic form

$$c_1(A) = -\frac{1}{4\pi} (\partial_\nu A_\mu - \partial_\mu A_\nu) c dz^\nu \wedge dz^\mu, \quad (5.7)$$

$$R = -2\pi i c_1 \otimes u_C, \quad (5.8)$$

where

$$u_C = c \partial_c \quad (5.9)$$

is the Liouville vector field (7.23) on C . The Chern form (5.7) is closed, but it need not be exact because $A_\mu dz^\mu$ is not a one-form on Z in general.

DEFINITION 5.3: A complex line bundle $C \rightarrow Z$ over a symplectic manifold (Z, Ω) is called a prequantization bundle if a form $(2\pi)^{-1}\Omega$ on Z belongs to the first Chern characteristic class of C . \square

A prequantization bundle, by definition, admits a $U(1)$ -principal connection A , called an admissible connection, whose curvature R (5.5) obeys the relation

$$R = -i\Omega \otimes u_C, \quad (5.10)$$

called the admissible condition.

Remark 5.1: Let A be the admissible connection (5.6) and $B = B_\mu dz^\mu$ a closed one-form on Z . Then

$$A' = A + icB \otimes \partial_c \quad (5.11)$$

also is an admissible connection. Since a manifold Z is assumed to be simply connected, a closed one-form B is exact. In this case, connections A and A' (5.11) are gauge conjugate [58]. \square

Given an admissible connection A , one can assign to each function $f \in C^\infty(Z)$ the C -valued first order differential operator \hat{f} on a fibre bundle $C \rightarrow Z$ in accordance with Kostant – Souriau formula

$$\hat{f} = -i\vartheta_f]D^A - fu_C = -[i\vartheta_f^\lambda(c_\lambda - iA_\lambda c) + fc]\partial_c, \quad (5.12)$$

where D^A is the covariant differential (5.4) and ϑ_f is the Hamiltonian vector field of f . It is easily justified that the operators (5.12) obey Dirac's condition (5.1) for all elements f of the Poisson algebra $C^\infty(Z)$.

The Kostant – Souriau formula (5.12) is called the prequantization because, in order to obtain Hermitian operators \hat{f} (5.12) acting on a Hilbert space, one should restrict both a class of functions $f \in C^\infty(Z)$ and a class of sections of $C \rightarrow Z$ in consideration as follows.

Given a symplectic manifold (Z, Ω) , by its polarization is meant a maximal involutive distribution $\mathbf{T} \subset TZ$ such that

$$\Omega(\vartheta, v) = 0, \quad \vartheta, v \in \mathbf{T}_z, \quad z \in Z.$$

This term also stands for the algebra \mathcal{T}_Ω of sections of the distribution \mathbf{T} . We denote by $\mathcal{A}_\mathcal{T}$ the subalgebra of the Poisson algebra $C^\infty(Z)$ which consists of the functions f such that

$$[\vartheta_f, \mathcal{T}_\Omega] \subset \mathcal{T}_\Omega.$$

It is called the quantum algebra of a symplectic manifold (Z, Ω) . Elements of this algebra only are quantized.

In order to obtain the carrier space of the algebra $\mathcal{A}_{\mathcal{T}}$, let us assume that Z is oriented and that its cohomology $H^2(Z; \mathbb{Z}_2)$ with coefficients in the constant sheaf \mathbb{Z}_2 vanishes. In this case, one can consider the metilinear complex line bundle $\mathcal{D}_{1/2}[Z] \rightarrow Z$ characterized by a bundle atlas $\{(U; z^\lambda, r)\}$ with the transition functions

$$r' = Jr, \quad J\bar{J} = \left| \det \left(\frac{\partial z^\mu}{\partial z'^\nu} \right) \right|. \quad (5.13)$$

Global sections ρ of this bundle are called the half-densities on Z [16, 41]. Note that the metilinear bundle $\mathcal{D}_{1/2}[Z] \rightarrow Z$ admits the canonical lift of any vector field u on Z such that the corresponding Lie derivative of its sections reads

$$\mathbf{L}_u = u^\lambda \partial_\lambda + \frac{1}{2} \partial_\lambda u^\lambda. \quad (5.14)$$

Given an admissible connection A , the prequantization formula (5.12) is extended to sections $s \otimes \rho$ of the fibre bundle

$$C \otimes_Z \mathcal{D}_{1/2}[Z] \rightarrow Z \quad (5.15)$$

as follows:

$$\begin{aligned} \widehat{f}(s \otimes \rho) &= (-i\nabla_{\vartheta_f} - f)(s \otimes \rho) = (\widehat{f}s) \otimes \rho + s \otimes \mathbf{L}_{\vartheta_f} \rho, \\ \nabla_{\vartheta_f}(s \otimes \rho) &= (\nabla_{\vartheta_f}^A s) \otimes \rho + s \otimes \mathbf{L}_{\vartheta_f} \rho, \end{aligned} \quad (5.16)$$

where $\mathbf{L}_{\vartheta_f} \rho$ is the Lie derivative (5.14) acting on half-densities. This extension is said to be the metaplectic correction, and the tensor product (5.15) is called the quantization bundle. One can think of its sections ϱ as being C -valued half-forms. It is readily observed that the operators (5.16) obey Dirac's condition (5.1). Let us denote by \mathfrak{E}_Z a complex vector space of sections ϱ of the fibre bundle $C \otimes \mathcal{D}_{1/2}[Z] \rightarrow Z$ of compact support such that

$$\begin{aligned} \nabla_v \varrho &= 0, \quad v \in \mathcal{T}_\Omega, \\ \nabla_v \varrho &= \nabla_v(s \otimes \rho) = (\nabla_v^A s) \otimes \rho + s \otimes \mathbf{L}_v \rho. \end{aligned} \quad (5.17)$$

LEMMA 5.4: For any function $f \in \mathcal{A}_{\mathcal{T}}$ and an arbitrary section $\varrho \in \mathfrak{E}_Z$, the relation $\widehat{f}\varrho \in \mathfrak{E}_Z$ holds. \square

Thus, we have a representation of the quantum algebra $\mathcal{A}_{\mathcal{T}}$ in the space \mathfrak{E}_Z . Therefore, by quantization of a function $f \in \mathcal{A}_{\mathcal{T}}$ is meant the restriction of the operator \widehat{f} (5.16) to \mathfrak{E}_Z .

Let g be an A -invariant Hermitian fibre metric in $C \rightarrow Z$ in accordance with Proposition 5.2. If $\mathfrak{E}_Z \neq 0$, the Hermitian form

$$\langle s_1 \otimes \rho_1 | s_2 \otimes \rho_2 \rangle = \int_Z g(s_1, s_2) \rho_1 \bar{\rho}_2 \quad (5.18)$$

brings \mathfrak{E}_Z into a pre-Hilbert space. Its completion $\overline{\mathfrak{E}}_Z$ is called a quantum Hilbert space, and the operators \widehat{f} (5.16) in this Hilbert space are Hermitian.

In particular, let us consider the standard geometric quantization of a cotangent bundle [16, 37, 41, 79].

Let M be an m -dimensional simply connected smooth manifold coordinated by (q^i) . Its cotangent bundle T^*M is simply connected. It is provided with the canonical symplectic form Ω_T (1.3) written with respect to holonomic coordinates $(q^i, p_i = \dot{q}_i)$ on T^*M . Let us consider the trivial complex line bundle

$$C = T^*M \times \mathbb{C} \rightarrow T^*M. \quad (5.19)$$

The canonical symplectic form (1.3) on T^*M is exact, i.e., it has the same zero de Rham cohomology class as the first Chern class of the trivial $U(1)$ -bundle C (5.19). Therefore, C is a prequantization bundle in accordance with Definition 5.3.

Coordinated by (q^i, p_i, c) , this bundle is provided with the admissible connection (5.6):

$$A = dp_j \otimes \partial^j + dq^j \otimes (\partial_j - ip_j c \partial_c) \quad (5.20)$$

such that the condition (5.10) is satisfied. The corresponding A -invariant fibre metric in C is given by the expression (5.2). The covariant derivative of sections s of the prequantization bundle C (5.19) relative to the connection A (5.20) along the vector field $u = u^j \partial_j + u_j \partial^j$ on T^*M reads

$$\nabla_u s = u^j (\partial_j + ip_j) s + u_j \partial^j s. \quad (5.21)$$

Given a function $f \in C^\infty(T^*M)$ and its Hamiltonian vector field

$$\vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i,$$

the covariant derivative (5.21) along ϑ_f is

$$\nabla_{\vartheta_f} s = \partial^i f (\partial_i + ip_i) s - \partial_i f \partial^i s.$$

With the connection A (5.20), the prequantization (5.12) of elements f of the Poisson algebra $C^\infty(T^*M)$ takes the form

$$\widehat{f} = -i \partial^j f (\partial_j + ip_j) + i \partial_j f \partial^j - f. \quad (5.22)$$

Let us note that, since the complex line bundle (5.19) is trivial, its sections are simply smooth complex functions on T^*M . Then the prequantum operators (5.22) can be written in the form

$$\widehat{f} = -i \mathbf{L}_{\vartheta_f} + (\mathbf{L}_v f - f), \quad (5.23)$$

where $v = p_j \partial^j$ is the Liouville vector field (7.23) on $T^*M \rightarrow M$.

It is readily observed that the vertical tangent bundle VT^*M of the cotangent bundle $T^*M \rightarrow M$ provides a polarization of T^*M . Certainly, it is not a unique polarization of T^*M . We call VT^*M the vertical polarization. The

corresponding quantum algebra $\mathcal{A}_T \subset C^\infty(T^*M)$ consists of affine functions of momenta

$$f = a^i(q^j)p_i + b(q^j) \quad (5.24)$$

on T^*M . Their Hamiltonian vector fields read

$$\vartheta_f = a^i \partial_i - (p_j \partial_i a^j + \partial_i b) \partial^i. \quad (5.25)$$

We call \mathcal{A}_T the quantum algebra of a cotangent bundle.

Since the Jacobain of holonomic coordinate transformations of the cotangent bundle T^*M equals 1, the geometric quantization of T^*M need no metaplectic correction. Consequently, the quantum algebra \mathcal{A}_T of the affine functions (5.24) acts on the subspace $\mathfrak{E}_{T^*M} \subset C^\infty(T^*M)$ of complex functions of compact support on T^*M which obey the condition (5.17):

$$\nabla_v s = v_i \partial^i s = 0, \quad \mathcal{T}_\Omega \ni v = v_i \partial^i.$$

A glance at this equality shows that elements of \mathfrak{E}_{T^*M} are independent of momenta p_i , i.e., they are the pull-back of complex functions on M with respect to the fibration $T^*M \rightarrow M$. These functions fail to be of compact support, unless $s = 0$. Consequently, the carrier space \mathfrak{E}_{T^*M} of the quantum algebra \mathcal{A}_T is reduced to zero. One can overcome this difficulty as follows.

Given the canonical zero section $\widehat{0}(M)$ of the cotangent bundle $T^*M \rightarrow M$, let

$$C_M = \widehat{0}(M)^* C \quad (5.26)$$

be the pull-back of the complex line bundle C (5.19) over M . It is a trivial complex line bundle $C_M = M \times \mathbb{C}$ provided with the pull-back Hermitian fibre metric $g(c, c') = c\bar{c}'$ and the pull-back (7.60):

$$A_M = \widehat{0}(M)^* A = dq^j \otimes (\partial_j - ip_j c \partial_c)$$

of the connection A (5.20) on C . Sections of C_M are smooth complex functions on M . One can consider a representation of the quantum algebra \mathcal{A}_T of the affine functions (5.24) in the space of complex functions on M by the prequantum operators (5.22):

$$\widehat{f} = -ia^j \partial_j - b.$$

However, this representation need a metaplectic correction.

Let us assume that M is oriented and that its cohomology $H^2(M; \mathbb{Z}_2)$ with coefficients in the constant sheaf \mathbb{Z}_2 vanishes. Let $\mathcal{D}_{1/2}[M]$ be the metalinear complex line over M . Since the complex line bundle C_M (5.26) is trivial, the quantization bundle (5.15):

$$C_M \otimes_M \mathcal{D}_{1/2}[M] \rightarrow M \quad (5.27)$$

is isomorphic to $\mathcal{D}_{1/2}[M]$.

Because the Hamiltonian vector fields (5.25) of functions f (5.24) project onto vector fields $a^j \partial_j$ on M and $\mathbf{L}_v f - f = -b$ in the formula (5.23) is a function on M , one can assign to each element f of the quantum algebra \mathcal{A}_T the following first order differential operator in the space $\mathcal{D}_{1/2}(M)$ of complex half-densities ρ on M :

$$\widehat{f}\rho = (-i\mathbf{L}_{a^j \partial_j} - b)\rho = (-ia^j \partial_j - \frac{i}{2}\partial_j a^j - b)\rho, \quad (5.28)$$

where $\mathbf{L}_{a^j \partial_j}$ is the Lie derivative (5.14) of half-densities. A glance at the expression (5.28) shows that it is the Schrödinger representation of the quantum algebra A_T of the affine functions (5.24). We call \widehat{f} (5.28) the Schrödinger operators.

Let $\mathfrak{E}_M \subset \mathcal{D}_{1/2}(M)$ be a space of complex half-densities ρ of compact support on M and $\overline{\mathfrak{E}}_M$ the completion of \mathfrak{E}_M with respect to the non-degenerate Hermitian form

$$\langle \rho | \rho' \rangle = \int_Q \rho \overline{\rho'}. \quad (5.29)$$

The (unbounded) Schrödinger operators (5.28) in the domain \mathfrak{E}_M of the Hilbert space $\overline{\mathfrak{E}}_M$ are Hermitian.

5.2 Leafwise geometric quantization

Developed for symplectic manifolds [16, 79], the geometric quantization technique has been generalized to Poisson manifolds in terms of contravariant connections [83, 84]. Though there is one-to-one correspondence between the Poisson structures on a smooth manifold and its symplectic foliations, geometric quantization of a Poisson manifold need not imply quantization of its symplectic leaves [41, 85].

- Firstly, contravariant connections fail to admit the pull-back operation. Therefore, prequantization of a Poisson manifold does not determine straightforwardly prequantization of its symplectic leaves.

- Secondly, polarization of a Poisson manifold is defined in terms of sheaves of functions, and it need not be associated to any distribution. As a consequence, its pull-back onto a leaf is not polarization of a symplectic manifold in general.

- Thirdly, a quantum algebra of a Poisson manifold contains the center of a Poisson algebra. However, there are models where quantization of this center has no physical meaning. For instance, a center of the Poisson algebra of a mechanical system with classical parameters consists of functions of these parameters.

Geometric quantization of symplectic foliations disposes of these problems. A quantum algebra $\mathcal{A}_{\mathcal{F}}$ of a symplectic foliation \mathcal{F} also is a quantum algebra of the associated Poisson manifold such that its restriction to each symplectic leaf F is a quantum algebra of F . Thus, geometric quantization of a symplectic foliation provides leafwise quantization of a Poisson manifold [32, 37, 41].

Geometric quantization of a symplectic foliation is phrased in terms of leafwise connections on a foliated manifold (see Definition 5.5 below). Any leafwise connection on a complex line bundle over a Poisson manifold is proved to come from a connection on this bundle (Theorem 5.7). Using this fact, one can state the equivalence of prequantization of a Poisson manifold to prequantization of its symplectic foliation [41], which also yields prequantization of each symplectic leaf (Proposition 5.10). We show that polarization of a symplectic foliation is associated to particular polarization of a Poisson manifold (Proposition 5.11), and its restriction to any symplectic leaf is polarization of this leaf (Proposition 5.12). Therefore, a quantum algebra of a symplectic foliation is both a quantum algebra of a Poisson manifold and, restricted to each symplectic leaf, a quantum algebra of this leaf.

We define metaplectic correction of a symplectic foliation so that its quantum algebra is represented by Hermitian operators in the pre-Hilbert module of leafwise half-forms, integrable over the leaves of this foliation.

Let $(Z, \{, \})$ be a Poisson manifold and $(\mathcal{F}, \Omega_{\mathcal{F}})$ its symplectic foliation such that $\{, \} = \{, \}_{\mathcal{F}}$ (see (1.12)). Let leaves of \mathcal{F} be simply connected.

Prequantization of a symplectic foliation $(\mathcal{F}, \Omega_{\mathcal{F}})$ provides a representation

$$f \rightarrow \widehat{f}, \quad [\widehat{f}, \widehat{f'}] = -i\widehat{\{f, f'\}_{\mathcal{F}}}, \quad (5.30)$$

of the Poisson algebra $(C^\infty(Z), \{, \}_{\mathcal{F}})$ by first order differential operators on sections s of some complex line bundle $C \rightarrow Z$, called the prequantization bundle. These operators are given by the Kostant – Souriau prequantization formula

$$\widehat{f} = -i\nabla_{\vartheta_f}^{\mathcal{F}} s + \varepsilon f s, \quad \vartheta_f = \Omega_{\mathcal{F}}^\sharp(\widetilde{d}f), \quad \varepsilon \neq 0, \quad (5.31)$$

where $\nabla^{\mathcal{F}}$ is an admissible leafwise connection on $C \rightarrow Z$ such that its curvature form \widetilde{R} (5.39) obeys the admissible condition

$$\widetilde{R} = i\varepsilon\Omega_{\mathcal{F}} \otimes u_C, \quad (5.32)$$

where u_C is the Liouville vector field (5.9) on C .

Using the above mentioned fact that any leafwise connection comes from a connection, we show that prequantization of a symplectic foliation yields prequantization of its symplectic leaves.

Remark 5.2: If Z is a symplectic manifold whose symplectic foliation is reduced to Z itself, the formulas (5.31) – (5.32), $\varepsilon = -1$, of leafwise prequantization restart the formulas (5.12) and (5.10) of geometric quantization of a symplectic manifold Z . \square

Let $S_{\mathcal{F}}(Z) \subset C^\infty(Z)$ be a subring of functions constant on leaves of a foliation \mathcal{F} , and let $\mathcal{T}_1(\mathcal{F})$ be the real Lie algebra of global sections of the tangent bundle $T\mathcal{F} \rightarrow Z$ to \mathcal{F} . It is the Lie $S_{\mathcal{F}}(Z)$ -algebra of derivations of $C^\infty(Z)$, regarded as a $S_{\mathcal{F}}(Z)$ -ring.

DEFINITION 5.5: In the framework of the leafwise differential calculus $\mathfrak{F}^*(Z)$ (7.41), a (linear) leafwise connection on a complex line bundle $C \rightarrow Z$ is defined

as a connection $\nabla^{\mathcal{F}}$ on the $C^\infty(Z)$ -module $C(Z)$ of global sections of this bundle, where $C^\infty(Z)$ is regarded as an $S_{\mathcal{F}}(Z)$ -ring (see Definition 7.11). It associates to each element $\tau \in \mathcal{T}_1(\mathcal{F})$ an $S_{\mathcal{F}}(Z)$ -linear endomorphism $\nabla_\tau^{\mathcal{F}}$ of $C(Z)$ which obeys the Leibniz rule

$$\nabla_\tau^{\mathcal{F}}(fs) = (\tau \lrcorner \tilde{d}f)s + f\nabla_\tau^{\mathcal{F}}(s), \quad f \in C^\infty(Z), \quad s \in C(Z). \quad (5.33)$$

□

A linear connection on $C \rightarrow Z$ can be equivalently defined as a connection on the module $C(Z)$ which assigns to each vector field $\tau \in \mathcal{T}_1(Z)$ on Z an \mathbb{R} -linear endomorphism of $C(Z)$ obeying the Leibniz rule (5.33). Restricted to $\mathcal{T}_1(\mathcal{F})$, it obviously yields a leafwise connection. In order to show that any leafwise connection is of this form, we appeal to an alternative definition of a leafwise connection in terms of leafwise forms.

The inverse images $\pi^{-1}(F)$ of leaves F of the foliation \mathcal{F} of Z provide a (regular) foliation $C_{\mathcal{F}}$ of the line bundle C . Given the (holomorphic) tangent bundle $TC_{\mathcal{F}}$ of this foliation, we have the exact sequence of vector bundles

$$0 \rightarrow VC \xrightarrow{C} TC_{\mathcal{F}} \xrightarrow{C} C \times_Z T\mathcal{F} \rightarrow 0, \quad (5.34)$$

where VC is the (holomorphic) vertical tangent bundle of $C \rightarrow Z$.

DEFINITION 5.6: A (linear) leafwise connection on the complex line bundle $C \rightarrow Z$ is a splitting of the exact sequence (5.34) which is linear over C . □

One can choose an adapted coordinate atlas $\{(U_\xi; z^\lambda, z^i)\}$ (7.38) of a foliated manifold (Z, \mathcal{F}) such that U_ξ are trivialization domains of the complex line bundle $C \rightarrow Z$. Let (z^λ, z^i, c) , $c \in \mathbb{C}$, be the corresponding bundle coordinates on $C \rightarrow Z$. They also are adapted coordinates on the foliated manifold $(C, C_{\mathcal{F}})$. With respect to these coordinates, a (linear) leafwise connection is represented by a $TC_{\mathcal{F}}$ -valued leafwise one-form

$$A_{\mathcal{F}} = \tilde{d}z^i \otimes (\partial_i + A_i c \partial_c), \quad (5.35)$$

where A_i are local complex functions on C .

The exact sequence (5.34) is obviously a subsequence of the exact sequence

$$0 \rightarrow VC \xrightarrow{C} TC \xrightarrow{C} C \times_Z TZ \rightarrow 0,$$

where TC is the holomorphic tangent bundle of C . Consequently, any connection

$$\mathcal{K} = dz^\lambda \otimes (\partial_\lambda + \mathcal{K}_\lambda c \partial_c) + dz^i \otimes (\partial_i + \mathcal{K}_i c \partial_c) \quad (5.36)$$

on the complex line bundle $C \rightarrow Z$ yields a leafwise connection

$$\mathcal{K}_{\mathcal{F}} = \tilde{d}z^i \otimes (\partial_i + \mathcal{K}_i c \partial_c). \quad (5.37)$$

THEOREM 5.7: Any leafwise connection on the complex line bundle $C \rightarrow Z$ comes from a connection on it [41]. □

In particular, it follows that Definitions 5.5 and 5.6 of a leafwise connection are equivalent, namely,

$$\nabla^{\mathcal{F}} s = \tilde{d}s - A_i s \tilde{d}z^i, \quad s \in C(Z).$$

The curvature of a leafwise connection $\nabla^{\mathcal{F}}$ is defined as a $C^\infty(Z)$ -linear endomorphism

$$\tilde{R}(\tau, \tau') = \nabla_{[\tau, \tau']}^{\mathcal{F}} - [\nabla_\tau^{\mathcal{F}}, \nabla_{\tau'}^{\mathcal{F}}] = \tau^i \tau'^j R_{ij}, \quad R_{ij} = \partial_i A_j - \partial_j A_i, \quad (5.38)$$

of $C(Z)$ for any vector fields $\tau, \tau' \in \mathcal{T}_1(\mathcal{F})$. It is represented by the vertical-valued leafwise two-form

$$\tilde{R} = \frac{1}{2} R_{ij} \tilde{d}z^i \wedge \tilde{d}z^j \otimes u_C. \quad (5.39)$$

If a leafwise connection $\nabla^{\mathcal{F}}$ comes from a connection ∇ , its curvature leafwise form \tilde{R} (5.39) is an image $\tilde{R} = i_{\mathcal{F}}^* R$ of the curvature form R (7.62) of the connection ∇ with respect to the morphism $i_{\mathcal{F}}^*$ (7.42).

Now let us turn to the admissible condition (5.32).

LEMMA 5.8: Let us assume that there exists a leafwise connection $\mathcal{K}_{\mathcal{F}}$ on the complex line bundle $C \rightarrow Z$ which fulfils the admissible condition (5.32). Then, for any Hermitian fibre metric g in $C \rightarrow Z$, there exists a leafwise connection $A_{\mathcal{F}}^g$ on $C \rightarrow Z$ which:

- (i) satisfies the admissible condition (5.32),
- (ii) preserves g ,
- (iii) comes from a $U(1)$ -principal connection on $C \rightarrow Z$.

This leafwise connection $A_{\mathcal{F}}^g$ is called admissible. \square

Proof: Given a Hermitian fibre metric g in $C \rightarrow Z$, let $\Psi^g = \{(z^\lambda, z^i, c)\}$ an associated bundle atlas of C with $U(1)$ -valued transition functions such that $g(c, c') = c\bar{c}'$ (Proposition 5.1). Let the above mentioned leafwise connection $\mathcal{K}_{\mathcal{F}}$ come from a linear connection \mathcal{K} (5.36) on $C \rightarrow Z$ written with respect to the atlas Ψ^g . The connection \mathcal{K} is split into the sum $A^g + \gamma$ where

$$A^g = dz^\lambda \otimes (\partial_\lambda + \text{Im}(\mathcal{K}_\lambda) c \partial_c) + dz^i \otimes (\partial_i + \text{Im}(\mathcal{K}_i) c \partial_c) \quad (5.40)$$

is a $U(1)$ -principal connection, preserving the Hermitian fibre metric g . The curvature forms R of \mathcal{K} and R^g of A^g obey the relation $R^g = \text{Im}(R)$. The connection A^g (5.40) defines the leafwise connection

$$A_{\mathcal{F}}^g = i_{\mathcal{F}}^* A = \tilde{d}z^i \otimes (\partial_i + i A_i^g c \partial_c), \quad i A_i^g = \text{Im}(\mathcal{K}_i), \quad (5.41)$$

preserving the Hermitian fibre metric g . Its curvature fulfils a desired relation

$$\tilde{R}^g = i_{\mathcal{F}}^* R^g = \text{Im}(i_{\mathcal{F}}^* R) = i\varepsilon \Omega_{\mathcal{F}} \otimes u_C. \quad (5.42)$$

\square

Since A^g (5.40) is a $U(1)$ -principal connection, its curvature form R^g is related to the first Chern form of integral de Rham cohomology class by the

formula (5.8). If the admissible condition (5.32) holds, the relation (5.42) shows that the leafwise cohomology class of the leafwise form $-(2\pi)^{-1}\varepsilon\Omega_{\mathcal{F}}$ is an image of an integral de Rham cohomology class with respect to the cohomology morphism $[i_{\mathcal{F}}^*]$. Conversely, if a leafwise symplectic form $\Omega_{\mathcal{F}}$ on a foliated manifold (Z, \mathcal{F}) is of this type, there exist a prequantization bundle $C \rightarrow Z$ and a $U(1)$ -principal connection A on $C \rightarrow Z$ such that the leafwise connection $i_{\mathcal{F}}^*A$ fulfils the relation (5.32). Thus, we have stated the following.

PROPOSITION 5.9: A symplectic foliation $(\mathcal{F}, \Omega_{\mathcal{F}})$ of a manifold Z admits the prequantization (5.31) iff the leafwise cohomology class of $-(2\pi)^{-1}\varepsilon\Omega_{\mathcal{F}}$ is an image of an integral de Rham cohomology class of Z . \square

Let F be a leaf of a symplectic foliation $(\mathcal{F}, \Omega_{\mathcal{F}})$ provided with the symplectic form

$$\Omega_F = i_F^*\Omega_{\mathcal{F}}.$$

In accordance with Proposition 7.8 and the commutative diagram

$$\begin{array}{ccc} H^*(Z; \mathbb{Z}) & \longrightarrow & H_{\text{DR}}^*(Z) \\ \downarrow & & \downarrow \\ H^*(F; \mathbb{Z}) & \longrightarrow & H_{\text{DR}}^*(F) \end{array}$$

of groups of the de Rham cohomology $H_{\text{DR}}^*(*)$ and the cohomology $H^*(*; \mathbb{Z})$ with coefficients in the constant sheaf \mathbb{Z} , the symplectic form $-(2\pi)^{-1}\varepsilon\Omega_F$ belongs to an integral de Rham cohomology class if a leafwise symplectic form $\Omega_{\mathcal{F}}$ fulfils the condition of Proposition 5.9. This states the following.

PROPOSITION 5.10: If a symplectic foliation admits prequantization, each its symplectic leaf does prequantization too. \square

The corresponding prequantization bundle for F is the pull-back complex line bundle i_F^*C , coordinated by (z^i, c) . Furthermore, let $A_{\mathcal{F}}^g$ (5.41) be a leafwise connection on the prequantization bundle $C \rightarrow Z$ which obeys Lemma 5.8, i.e., comes from a $U(1)$ -principal connection A^g on $C \rightarrow Z$. Then the pull-back

$$A_F = i_F^*A^g = dz^i \otimes (\partial_i + ii_F^*(A_i^g)c\partial_c) \quad (5.43)$$

of the connection A^g onto $i_F^*C \rightarrow F$ satisfies the admissible condition

$$R_F = i_F^*R = i\varepsilon\Omega_F,$$

and preserves the pull-back Hermitian fibre metric i_F^*g in $i_F^*C \rightarrow F$.

Let us define polarization of a symplectic foliation $(\mathcal{F}, \Omega_{\mathcal{F}})$ of a manifold Z as a maximal (regular) involutive distribution $\mathbf{T} \subset T\mathcal{F}$ on Z such that

$$\Omega_{\mathcal{F}}(u, v) = 0, \quad u, v \in \mathbf{T}_z, \quad z \in Z. \quad (5.44)$$

Given the Lie algebra $\mathbf{T}(Z)$ of \mathbf{T} -subordinate vector fields on Z , let $\mathcal{A}_{\mathcal{F}} \subset C^\infty(Z)$ be the complexified subalgebra of functions f whose leafwise Hamiltonian vector fields ϑ_f (1.12) fulfil the condition

$$[\vartheta_f, \mathbf{T}(Z)] \subset \mathbf{T}(Z).$$

It is called the quantum algebra of a symplectic foliation $(\mathcal{F}, \Omega_{\mathcal{F}})$ with respect to the polarization \mathbf{T} . This algebra obviously contains the center $S_{\mathcal{F}}(Z)$ of the Poisson algebra $(C^\infty(Z), \{, \}_{\mathcal{F}})$, and it is a Lie $S_{\mathcal{F}}(Z)$ -algebra.

PROPOSITION 5.11: Every polarization \mathbf{T} of a symplectic foliation $(\mathcal{F}, \Omega_{\mathcal{F}})$ yields polarization of the associated Poisson manifold (Z, w_Ω) . \square

Proof: Let us consider the presheaf of local smooth functions f on Z whose leafwise Hamiltonian vector fields ϑ_f (1.12) are subordinate to \mathbf{T} . The sheaf Φ of germs of these functions is polarization of the Poisson manifold (Z, w_Ω) (see Remark 5.3 below). Equivalently, Φ is the sheaf of germs of functions on Z whose leafwise differentials are subordinate to the codistribution $\Omega_{\mathcal{F}}^b \mathbf{T}$. \square

Remark 5.3: Let us recall that polarization of a Poisson manifold $(Z, \{, \})$ is defined as a sheaf \mathbf{T}^* of germs of complex functions on Z whose stalks \mathbf{T}_z^* , $z \in Z$, are Abelian algebras with respect to the Poisson bracket $\{, \}$ [85]. Let $\mathbf{T}^*(Z)$ be the structure algebra of global sections of the sheaf \mathbf{T}^* ; it also is called the Poisson polarization [83, 84]. A quantum algebra \mathcal{A} associated to the Poisson polarization \mathbf{T}^* is defined as a subalgebra of the Poisson algebra $C^\infty(Z)$ which consists of functions f such that

$$\{f, \mathbf{T}^*(Z)\} \subset \mathbf{T}^*(Z).$$

Polarization of a symplectic manifold yields its Poisson one. \square

Let us note that the polarization Φ in the proof of Proposition 5.11) need not be maximal, unless \mathbf{T} is of maximal dimension $\dim \mathcal{F}/2$. It belongs to the following particular type of polarizations of a Poisson manifold. Since the cochain morphism $i_{\mathcal{F}}^*$ (7.42) is an epimorphism, the leafwise differential calculus \mathfrak{F}^* is universal, i.e., the leafwise differentials $\tilde{d}f$ of functions $f \in C^\infty(Z)$ on Z make up a basis for the $C^\infty(Z)$ -module $\mathfrak{F}^1(Z)$. Let $\Phi(Z)$ denote the structure \mathbb{R} -module of global sections of the sheaf Φ . Then the leafwise differentials of elements of $\Phi(Z)$ make up a basis for the $C^\infty(Z)$ -module of global sections of the codistribution $\Omega_{\mathcal{F}}^b \mathbf{T}$. Equivalently, the leafwise Hamiltonian vector fields of elements of $\Phi(Z)$ constitute a basis for the $C^\infty(Z)$ -module $\mathbf{T}(Z)$. Then one can easily show that polarization \mathbf{T} of a symplectic foliation $(\mathcal{F}, \Omega_{\mathcal{F}})$ and the corresponding polarization Φ of the Poisson manifold (Z, w_Ω) in Proposition 5.11 define the same quantum algebra $\mathcal{A}_{\mathcal{F}}$.

Let (F, Ω_F) be a symplectic leaf of a symplectic foliation $(\mathcal{F}, \Omega_{\mathcal{F}})$. Given polarization $\mathbf{T} \rightarrow Z$ of $(\mathcal{F}, \Omega_{\mathcal{F}})$, its restriction

$$\mathbf{T}_F = i_F^* \mathbf{T} \subset i_F^* T\mathcal{F} = TF$$

to F is an involutive distribution on F . It obeys the condition

$$i_F^* \Omega_{\mathcal{F}}(u, v) = 0, \quad u, v \in \mathbf{T}_{Fz}, \quad z \in F,$$

i.e., it is polarization of the symplectic manifold (F, Ω_F) . Thus, we have stated the following.

PROPOSITION 5.12: Polarization of a symplectic foliation defines polarization of each symplectic leaf. \square

Clearly, the quantum algebra \mathcal{A}_F of a symplectic leaf F with respect to the polarization \mathbf{T}_F contains all elements $i_F^* f$ of the quantum algebra $\mathcal{A}_{\mathcal{F}}$ restricted to F .

Since $\mathcal{A}_{\mathcal{F}}$ is the quantum algebra both of a symplectic foliation $(\mathcal{F}, \Omega_{\mathcal{F}})$ and the associated Poisson manifold (Z, w_{Ω}) , let us follow the standard metaplectic correction technique [16, 41].

Assuming that Z is oriented and that $H^2(Z; \mathbb{Z}_2) = 0$, let us consider the metaplectic complex line bundle $\mathcal{D}_{1/2}[Z] \rightarrow Z$ characterized by an atlas

$$\Psi_Z = \{(U; z^\lambda, z^i, r)\}$$

with the transition functions (5.13). Global sections ρ of this bundle are half-densities on Z . Their Lie derivative (5.14) along a vector field u on Z reads

$$\mathbf{L}_u \rho = u^\lambda \partial_\lambda \rho + u^i \partial_i \rho + \frac{1}{2}(\partial_\lambda u^\lambda + \partial_i u^i) \rho. \quad (5.45)$$

Given an admissible connection $A_{\mathcal{F}}^g$, the prequantization formula (5.31) is extended to sections $\varrho = s \otimes \rho$ of the fibre bundle

$$C \otimes_Z \mathcal{D}_{1/2}[Z] \quad (5.46)$$

as follows

$$\begin{aligned} \widehat{f} &= -i[(\nabla_{\vartheta_f}^{\mathcal{F}} + i\varepsilon f) \otimes \text{Id} + \text{Id} \otimes \mathbf{L}_{\vartheta_f}] = \\ &= -i[\nabla_{\vartheta_f}^{\mathcal{F}} + i\varepsilon f + \frac{1}{2}\partial_i \vartheta_f^i], \quad f \in \mathcal{A}_{\mathcal{F}}. \end{aligned} \quad (5.47)$$

This extension is the metaplectic correction of leafwise quantization. It is readily observed that the operators (5.47) obey Dirac's condition (5.30). Let us denote by \mathfrak{E}_Z the complex space of sections ϱ of the fibre bundle (5.46) of compact support such that

$$(\nabla_{\vartheta}^{\mathcal{F}} \otimes \text{Id} + \text{Id} \otimes \mathbf{L}_{\vartheta})\varrho = (\nabla_{\vartheta}^{\mathcal{F}} + \frac{1}{2}\partial_i \vartheta^i)\varrho = 0$$

for all \mathbf{T} -subordinate leafwise Hamiltonian vector fields ϑ .

LEMMA 5.13: For any function $f \in \mathcal{A}_{\mathcal{F}}$ and an arbitrary section $\varrho \in \mathfrak{E}_Z$, the relation $\widehat{f}\varrho \in \mathfrak{E}_Z$ holds. \square

Thus, we have a representation of the quantum algebra $\mathcal{A}_{\mathcal{F}}$ in the space \mathfrak{E}_Z . Therefore, by quantization of a function $f \in \mathcal{A}_{\mathcal{F}}$ is meant the restriction of the operator \widehat{f} (5.47) to \mathfrak{E}_Z .

The space \mathfrak{E}_Z is provided with the non-degenerate Hermitian form

$$\langle \rho | \rho' \rangle = \int_Z \rho \rho', \quad (5.48)$$

which brings \mathfrak{E}_Z into a pre-Hilbert space. Its completion carries a representation of the quantum algebra $\mathcal{A}_{\mathcal{F}}$ by (unbounded) Hermitian operators.

However, it may happen that the above quantization has no physical meaning because the Hermitian form (5.48) on the carrier space \mathfrak{E}_Z and, consequently, the mean values of operators (5.47) are defined by integration over the whole manifold Z . For instance, it implies integration over time and classical parameters. Therefore, we suggest a different scheme of quantization of symplectic foliations.

Let us consider the exterior bundle $\wedge^{2m} T\mathcal{F}^*$, $2m = \dim \mathcal{F}$. Its structure group $GL(2m, \mathbb{R})$ is reducible to the group $GL^+(2m, \mathbb{R})$ since a symplectic foliation is oriented. One can regard this fibre bundle as being associated to a $GL(2m, \mathbb{C})$ -principal bundle $P \rightarrow Z$. As earlier, let us assume that $H^2(Z; \mathbb{Z}_2) = 0$. Then the principal bundle P admits a two-fold covering principal bundle with the structure metilinear group $ML(2m, \mathbb{C})$ [16]. As a consequence, there exists a complex line bundle $\mathcal{D}_{\mathcal{F}} \rightarrow Z$ characterized by an atlas

$$\Psi_{\mathcal{F}} = \{(U_{\xi}; z^{\lambda}, z^i, r)\}$$

with the transition functions $r' = J_{\mathcal{F}} r$ such that

$$J_{\mathcal{F}} \bar{J}_{\mathcal{F}} = \det \left(\frac{\partial z'^i}{\partial z^j} \right). \quad (5.49)$$

One can think of its sections as being complex leafwise half-densities on Z . The metilinear bundle $\mathcal{D}_{1/2}[\mathcal{F}] \rightarrow Z$ admits the canonical lift of any \mathbf{T} -subordinate vector field u on Z . The corresponding Lie derivative of its sections reads

$$\mathbf{L}_u^{\mathcal{F}} = u^i \partial_i + \frac{1}{2} \partial_i u^i. \quad (5.50)$$

We define the quantization bundle as the tensor product

$$Y_{\mathcal{F}} = C \otimes_Z \mathcal{D}_{1/2}[\mathcal{F}] \rightarrow Z. \quad (5.51)$$

Its sections are C -valued leafwise half-forms. Given an admissible leafwise connection $A_{\mathcal{F}}^g$ and the Lie derivative $\mathbf{L}_u^{\mathcal{F}}$ (5.50), let us associate the first order differential operator

$$\begin{aligned} \hat{f} &= -i[(\nabla_{\vartheta_f}^{\mathcal{F}} + i\varepsilon f) \otimes \text{Id} + \text{Id} \otimes \mathbf{L}_{\vartheta_f}^{\mathcal{F}}] = \\ &= -i[\nabla_{\vartheta_f}^{\mathcal{F}} + i\varepsilon f + \frac{1}{2} \partial_i \vartheta_f^i], \quad f \in \mathcal{A}_{\mathcal{F}}, \end{aligned} \quad (5.52)$$

on sections $\varrho_{\mathcal{F}}$ of $Y_{\mathcal{F}}$ to each element of the quantum algebra $\mathcal{A}_{\mathcal{F}}$. A direct computation with respect to the local Darboux coordinates on Z proves the following.

LEMMA 5.14: The operators (5.52) obey Dirac's condition (5.30). \square

LEMMA 5.15: If a section $\varrho_{\mathcal{F}}$ fulfils the condition

$$(\nabla_{\vartheta}^{\mathcal{F}} \otimes \text{Id} + \text{Id} \otimes \mathbf{L}_{\vartheta}^{\mathcal{F}})\varrho_{\mathcal{F}} = (\nabla_{\vartheta}^{\mathcal{F}} + \frac{1}{2}\partial_i\vartheta^i)\varrho_{\mathcal{F}} = 0 \quad (5.53)$$

for all \mathbf{T} -subordinate leafwise Hamiltonian vector field ϑ , then $\widehat{f}\varrho_{\mathcal{F}}$ for any $f \in \mathcal{A}_{\mathcal{F}}$ possesses the same property. \square

Let us restrict the representation of the quantum algebra $\mathcal{A}_{\mathcal{F}}$ by the operators (5.52) to the subspace $\mathfrak{E}_{\mathcal{F}}$ of sections $\varrho_{\mathcal{F}}$ of the quantization bundle (5.51) which obey the condition (5.53) and whose restriction to any leaf of \mathcal{F} is of compact support. The last condition is motivated by the following.

Since $i_F^*T\mathcal{F}^* = T^*F$, the pull-back $i_F^*\mathcal{D}_{1/2}[\mathcal{F}]$ of $\mathcal{D}_{1/2}[\mathcal{F}]$ onto a leaf F is a metilinear bundle of half-densities on F . By virtue of Propositions 5.10 and 5.12, the pull-back $i_F^*Y_{\mathcal{F}}$ of the quantization bundle $Y_{\mathcal{F}} \rightarrow Z$ onto F is a quantization bundle for the symplectic manifold $(F, i_F^*\Omega_{\mathcal{F}})$. Given the pull-back connection A_F (5.43) and the polarization $\mathbf{T}_F = i_F^*\mathbf{T}$, this symplectic manifold is subject to the standard geometric quantization by the first order differential operators

$$\widehat{f} = -i(i_F^*\nabla_{\vartheta_f}^{\mathcal{F}} + i\varepsilon f + \frac{1}{2}\partial_i\vartheta_f^i), \quad f \in \mathcal{A}_F, \quad (5.54)$$

on sections ϱ_F of $i_F^*Y_{\mathcal{F}} \rightarrow F$ of compact support which obey the condition

$$(i_F^*\nabla_{\vartheta}^{\mathcal{F}} + \frac{1}{2}\partial_i\vartheta^i)\varrho_F = 0 \quad (5.55)$$

for all \mathbf{T}_F -subordinate Hamiltonian vector fields ϑ on F . These sections constitute a pre-Hilbert space \mathfrak{E}_F with respect to the Hermitian form

$$\langle \rho_F | \rho'_F \rangle = \int_F \varrho_F \varrho'_F.$$

The key point is the following.

PROPOSITION 5.16: We have $i_F^*\mathfrak{E}_{\mathcal{F}} \subset \mathfrak{E}_F$, and the relation

$$i_F^*(\widehat{f}\varrho_{\mathcal{F}}) = \widehat{(i_F^*f)}(i_F^*\varrho_{\mathcal{F}}) \quad (5.56)$$

holds for all elements $f \in \mathcal{A}_{\mathcal{F}}$ and $\varrho_{\mathcal{F}} \in \mathfrak{E}_{\mathcal{F}}$. \square

Proof: One can use the fact that the expressions (5.54) and (5.55) have the same coordinate form as the expressions (5.52) and (5.53) where $z^\lambda = \text{const}$. \square

The relation (5.56) enables one to think of the operators \widehat{f} (5.52) as being the leafwise quantization of the $S_{\mathcal{F}}(Z)$ -algebra $\mathcal{A}_{\mathcal{F}}$ in the pre-Hilbert $S_{\mathcal{F}}(Z)$ -module $\mathfrak{E}_{\mathcal{F}}$ of leafwise half-forms.

5.3 Quantization of integrable systems in action-angle variables

In accordance with Theorem 2.5, any superintegrable Hamiltonian system (2.3) on a symplectic manifold (Z, Ω) restricted to some open neighborhood U_M (2.6) of its invariant submanifold M is characterized by generalized action-angle coordinates $(I_\lambda, p_A, q^A, y^\lambda)$, $\lambda = 1, \dots, m$, $A = 1, \dots, n - m$. They are canonical for the symplectic form Ω (2.7) on U_M . Then one can treat the coordinates (I_λ, p_A) as n independent functions in involution on a symplectic annulus (U_M, Ω) which constitute a completely integrable system in accordance with Definition 2.2. Strictly speaking, its quantization fails to be a quantization of the original superintegrable system (2.3) because $F_i(I_\lambda, q^A, p_A)$ are not linear functions and, consequently, the algebra (2.3) and the algebra

$$\{I_\lambda, p_A\} = \{I_\lambda, q^A\} = 0, \quad \{p_A, q^B\} = \delta_A^B \quad (5.57)$$

are not isomorphic in general. However, one can obtain the Hamilton operator $\widehat{\mathcal{H}}$ and the Casimir operators \widehat{C}_λ of an original superintegrable system and their spectra.

There are different approaches to quantization of completely integrable and superintegrable systems [19, 33, 42, 46, 64]. It should be emphasized that action-angle coordinates need not be globally defined on a phase space, but form an algebra of the Poisson canonical commutation relations (5.57) on an open neighborhood U_M of an invariant submanifold M . Therefore, quantization of an integrable system with respect to the action-angle variables is a quantization of the Poisson algebra $C^\infty(U_M)$ of real smooth functions on U_M . Since there is no morphism $C^\infty(U_M) \rightarrow C^\infty(Z)$, this quantization is not equivalent to quantization of an original integrable system on Z and, from on a physical level, is interpreted as quantization around an invariant submanifold M . A key point is that, since U_M is not a contractible manifold, the geometric quantization technique should be called into play in order to quantize an integrable system around its invariant submanifold. A peculiarity of the geometric quantization procedure is that it remains equivalent under symplectic isomorphisms, but essentially depends on the choice of a polarization [5, 69].

Geometric quantization of completely integrable systems has been studied at first with respect to the polarization spanned by Hamiltonian vector fields of integrals of motion [64]. For example, the well-known Simms quantization of a harmonic oscillator is of this type [16]. However, one meets a problem that the associated quantum algebra contains affine functions of angle coordinates on a torus which are ill defined. As a consequence, elements of the carrier space of this quantization fail to be smooth, but are tempered distributions. We have developed a different variant of geometric quantization of completely integrable systems [19, 33, 37]. Since a Hamiltonian of a completely integrable system depends only on action variables, it seems natural to provide the Schrödinger representation of action variables by first order differential operators on functions of angle coordinates. For this purpose, one should choose the angle polarization

of a symplectic manifold spanned by almost-Hamiltonian vector fields of angle variables.

Given an open neighborhood U_M (2.6) in Theorem 2.5, let us consider its fibration

$$U_M = N_M \times \mathbb{R}^{m-r} \times T^r \rightarrow V \times \mathbb{R}^{m-r} \times T^r = \mathcal{M}, \quad (5.58)$$

$$(I_\lambda, p_A, q^A, y^\lambda) \rightarrow (q^A, y^\lambda). \quad (5.59)$$

Then one can think of a symplectic annulus (U_M, Ω) as being an open subbundle of the cotangent bundle $T^*\mathcal{M}$ endowed with the canonical symplectic form $\Omega_T = \Omega$ (2.7). This fact enables us to provide quantization of any superintegrable system on a neighborhood of its invariant submanifold as geometric quantization of the cotangent bundle $T^*\mathcal{M}$ over the toroidal cylinder \mathcal{M} (5.58) [38]. Note that this quantization however differs from that in Section 5.1 because \mathcal{M} (5.58) is not simply connected in general.

Let (q^A, r^a, α^i) be coordinates on the toroidal cylinder \mathcal{M} (5.58), where $(\alpha^1, \dots, \alpha^r)$ are angle coordinates on a torus T^r , and let (p_A, I_a, I_i) be the corresponding action coordinates (i.e., the holonomic fibre coordinates on $T^*\mathcal{M}$). Since the symplectic form Ω (2.7) is exact, the quantum bundle is defined as a trivial complex line bundle C over $T^*\mathcal{M}$. Let its trivialization hold fixed. Any other trivialization leads to an equivalent quantization of $T^*\mathcal{M}$. Given the associated fibre coordinate $c \in \mathbb{C}$ on $C \rightarrow T^*\mathcal{M}$, one can treat its sections as smooth complex functions on $T^*\mathcal{M}$.

The Kostant – Souriau prequantization formula (5.12) associates to every smooth real function f on $T^*\mathcal{M}$ the first order differential operator

$$\widehat{f} = -i\vartheta_f \rfloor D^A - fc\partial_c$$

on sections of $C \rightarrow T^*\mathcal{M}$, where ϑ_f is the Hamiltonian vector field of f and D^A is the covariant differential (5.4) with respect to an admissible $U(1)$ -principal connection A on C . This connection preserves the Hermitian fibre metric $g(c, c') = c\bar{c}'$ in \mathcal{C} , and its curvature obeys the prequantization condition (5.10). Such a connection reads

$$A = A_0 - ic(p_A dq^A + I_a dr^a + I_i d\alpha^i) \otimes \partial_c, \quad (5.60)$$

where A_0 is a flat $U(1)$ -principal connection on $C \rightarrow T^*\mathcal{M}$.

The classes of gauge non-conjugate flat principal connections on C are indexed by the set $\mathbb{R}^r/\mathbb{Z}^r$ of homomorphisms of the de Rham cohomology group

$$H_{\text{DR}}^1(T^*\mathcal{M}) = H_{\text{DR}}^1(\mathcal{M}) = H_{\text{DR}}^1(T^r) = \mathbb{R}^r$$

of $T^*\mathcal{M}$ to $U(1)$. We choose their representatives of the form

$$A_0[(\lambda_i)] = dp_A \otimes \partial^A + dI_a \otimes \partial^a + dI_j \otimes \partial^j + dq^A \otimes \partial_A + dr^a \otimes \partial_a + d\alpha^j \otimes (\partial_j - i\lambda_j c \partial_c), \quad \lambda_i \in [0, 1).$$

Accordingly, the relevant connection (5.60) on C reads

$$\begin{aligned} A[(\lambda_i)] = & dp_A \otimes \partial^A + dI_a \otimes \partial^a + dI_j \otimes \partial^j + \\ & dq^A \otimes (\partial_A - ip_A c \partial_c) + dr^a \otimes (\partial_a - iI_a c \partial_c) + \\ & d\alpha^j \otimes (\partial_j - i(I_j + \lambda_j) c \partial_c). \end{aligned} \quad (5.61)$$

For the sake of simplicity, we further assume that the numbers λ_i in the expression (5.61) belong to \mathbb{R} , but bear in mind that connections $A[(\lambda_i)]$ and $A[(\lambda'_i)]$ with $\lambda_i - \lambda'_i \in \mathbb{Z}$ are gauge conjugate.

Let us choose the above mentioned angle polarization coinciding with the vertical polarization $VT^*\mathcal{M}$. Then the corresponding quantum algebra \mathcal{A} of $T^*\mathcal{M}$ consists of affine functions

$$f = a^A(q^B, r^b, \alpha^j) p_A + a^b(q^B, r^a, \alpha^j) I_b + a^i(q^B, r^a, \alpha^j) I_i + b(q^B, r^a, \alpha^j)$$

in action coordinates (p_A, I_a, I_i) . Given a connection (5.61), the corresponding Schrödinger operators (5.28) read

$$\begin{aligned} \widehat{f} = & \left(-ia^A \partial_A - \frac{i}{2} \partial_A a^A \right) + \left(-ia^b \partial_i - \frac{i}{2} \partial_b a^b \right) + \\ & \left(-ia^i \partial_i - \frac{i}{2} \partial_i a^i + a^i \lambda_i \right) - b. \end{aligned} \quad (5.62)$$

They are Hermitian operators in the pre-Hilbert space $\mathfrak{E}_{\mathcal{M}}$ of complex half-densities ψ of compact support on \mathcal{M} endowed with the Hermitian form

$$\langle \psi | \psi' \rangle = \int_{\mathcal{M}} \psi \overline{\psi'} d^{n-m} q d^{m-r} r d^r \alpha.$$

Note that, being a complex function on a toroidal cylinder $\mathbb{R}^{m-r} \times T^r$, any half-density $\psi \in \mathfrak{E}_{\mathcal{M}}$ is expanded into the series

$$\psi = \sum_{(n_\mu)} \phi(q^B, r^a)_{(n_j)} \exp[in_j \alpha^j], \quad (n_j) = (n_1, \dots, n_r) \in \mathbb{Z}^r, \quad (5.63)$$

where $\phi(q^B, r^a)_{(n_\mu)}$ are half-densities of compact support on \mathbb{R}^{n-r} . In particular, the action operators (5.62) read

$$\widehat{p}_A = -i\partial_A, \quad \widehat{I}_a = -i\partial_a, \quad \widehat{I}_j = -i\partial_j + \lambda_j. \quad (5.64)$$

It should be emphasized that

$$\widehat{a\widehat{p}}_A \neq \widehat{a p}_A, \quad \widehat{a\widehat{I}}_b \neq \widehat{a I}_b, \quad \widehat{a\widehat{I}}_j \neq \widehat{a I}_j, \quad a \in C^\infty(\mathcal{M}). \quad (5.65)$$

The operators (5.62) provide a desired quantization of a superintegrable Hamiltonian system written with respect to the action-angle coordinates. They satisfy Dirac's condition (5.1). However, both a Hamiltonian \mathcal{H} and original

integrals of motion F_i do not belong to the quantum algebra \mathcal{A} , unless they are affine functions in the action coordinates (p_A, I_a, I_i) . In some particular cases, integrals of motion F_i can be represented by differential operators, but this representation fails to be unique because of inequalities (5.65), and Dirac's condition need not be satisfied. At the same time, both the Casimir functions C_λ and a Hamiltonian \mathcal{H} (Proposition 2.13) depend only on action variables I_a, I_i . If they are polynomial in I_a , one can associate to them the operators $\widehat{C}_\lambda = C_\lambda(\widehat{I}_a, \widehat{I}_j)$, $\widehat{\mathcal{H}} = \mathcal{H}(\widehat{I}_a, \widehat{I}_j)$, acting in the space \mathfrak{E}_M by the law

$$\begin{aligned}\widehat{\mathcal{H}}\psi &= \sum_{(n_j)} \mathcal{H}(\widehat{I}_a, n_j + \lambda_j) \phi(q^A, r^a)_{(n_j)} \exp[in_j \alpha^j], \\ \widehat{C}_\lambda \psi &= \sum_{(n_j)} C_\lambda(\widehat{I}_a, n_j + \lambda_j) \phi(q^A, r^a)_{(n_j)} \exp[in_j \alpha^j].\end{aligned}$$

Example 5.4: Let us consider a superintegrable system with the Lie algebra $\mathfrak{g} = so(3)$ of integrals of motion $\{F_1, F_2, F_3\}$ on a four-dimensional symplectic manifold (Z, Ω) , namely,

$$\{F_1, F_2\} = F_3, \quad \{F_2, F_3\} = F_1, \quad \{F_3, F_1\} = F_2.$$

Since it is compact, an invariant submanifold of a superintegrable system in question is a circle $M = S^1$. We have a fibred manifold $F : Z \rightarrow N$ (3.22) onto an open subset $N \subset \mathfrak{g}^*$ of the Lie coalgebra \mathfrak{g}^* . This fibred manifold is a fibre bundle since its fibres are compact (Theorem 7.2). Its base N is endowed with the coordinates (x_1, x_2, x_3) such that integrals of motion $\{F_1, F_2, F_3\}$ on Z read

$$F_1 = x_1, \quad F_2 = x_2, \quad F_3 = x_3.$$

The coinduced Poisson structure on N is the Lie – Poisson structure (3.23). The coadjoint action of $so(3)$ is given by the expression (3.24). An orbit of the coadjoint action of dimension 2 is given by the equality (3.25). Let M be an invariant submanifold such that the point $F(M) \in \mathfrak{g}^*$ belongs to the orbit (3.25). Let us consider an open fibred neighborhood $U_M = N_M \times S^1$ of M which is a trivial bundle over an open contractible neighborhood N_M of $F(M)$ endowed with the coordinates (I, x_1, γ) defined by the equalities (3.27). Here, I is the Casimir function (3.28) on \mathfrak{g}^* . These coordinates are the Darboux coordinates of the Lie – Poisson structure (3.29) on N_M . Let ϑ_I be the Hamiltonian vector field of the Casimir function I (3.28). Its flows are invariant submanifolds. Let α be a parameter (3.30) along the flows of this vector field. Then U_M is provided with the action-angle coordinates (I, x_1, γ, α) such that the Poisson bivector on U_M takes the form (3.31). The action-angle variables $\{I, H_1 = x_1, \gamma\}$ constitute a superintegrable system

$$\{I, F_1\} = 0, \quad \{I, \gamma\} = 0, \quad \{F_1, \gamma\} = 1, \quad (5.66)$$

on U_M . It is related to the original one by the transformations

$$I = -\frac{1}{2}(F_1^2 + F_2^2 + F_3^2)^{1/2},$$

$$F_2 = \left(-\frac{1}{2I} - F_1^2\right)^{1/2} \sin \gamma, \quad F_3 = \left(-\frac{1}{2I} - H_1^2\right)^{1/2} \cos \gamma.$$

Its Hamiltonian is expressed only in the action variable I . Let us quantize the superintegrable system (5.66). We obtain the algebra of operators

$$\hat{f} = a \left(-i\frac{\partial}{\partial\alpha} - \lambda\right) - ib\frac{\partial}{\partial\gamma} - \frac{i}{2} \left(\frac{\partial a}{\partial\alpha} + \frac{\partial b}{\partial\gamma}\right) - c,$$

where a, b, c are smooth functions of angle coordinates (γ, α) on the cylinder $\mathbb{R} \times S^1$. In particular, the action operators read

$$\hat{I} = -i\frac{\partial}{\partial\alpha} - \lambda, \quad \hat{F}_1 = -i\frac{\partial}{\partial\gamma}.$$

These operators act in the space of smooth complex functions

$$\psi(\gamma, \alpha) = \sum_k \phi(\gamma)_k \exp[ik\alpha]$$

on T^2 . A Hamiltonian $\mathcal{H}(I)$ of a classical superintegrable system also can be represented by the operator

$$\hat{\mathcal{H}}(I)\psi = \sum_k \mathcal{H}(I - \lambda)\phi(\gamma)_k \exp[ik\alpha]$$

on this space. \square

6 Mechanics with time-dependent parameters

At present, quantum systems with classical parameters attract special attention in connection with holonomic quantum computation.

This Section addresses mechanical systems with time-dependent parameters. These parameters can be seen as sections of some smooth fibre bundle $\Sigma \rightarrow \mathbb{R}$ called the parameter bundle. Then a configuration space of a mechanical system with time-dependent parameters is a composite fibre bundle

$$Q \xrightarrow{\pi_{Q\Sigma}} \Sigma \longrightarrow \mathbb{R} \tag{6.1}$$

[37, 56, 72]. Indeed, given a section $\varsigma(t)$ of a parameters bundle $\Sigma \rightarrow \mathbb{R}$, the pull-back bundle

$$Q_\varsigma = \varsigma^*Q \rightarrow \mathbb{R} \tag{6.2}$$

is a subbundle $i_\varsigma : Q_\varsigma \rightarrow Q$ of a fibre bundle $Q \rightarrow \mathbb{R}$ which is a configuration space of a mechanical system with a fixed parameter function $\varsigma(t)$.

In order to obtain the Lagrange and Hamilton equations, we treat parameters on the same level as dynamic variables. The corresponding total velocity and phase spaces are the first order jet manifold J^1Q and the vertical cotangent bundle V^*Q of the configuration bundle $Q \rightarrow \mathbb{R}$, respectively.

Section 6.2 addresses quantization of mechanical systems with time-dependent parameters. Since parameters remain classical, a phase space, that we quantize, is the vertical cotangent bundle V_Σ^*Q of a fibre bundle $Q \rightarrow \Sigma$. We apply to $V_\Sigma^*Q \rightarrow \Sigma$ the technique of leafwise geometric quantization (Section 5.2).

Berry's phase factor is a phenomenon peculiar to quantum systems depending on classical time-dependent parameters [2, 8, 50, 63]. It is described by driving a carrier Hilbert space of a Hamilton operator over a parameter manifold. Berry's phase factor depending only on the geometry of a path in a parameter manifold is called geometric (Section 6.3). It is characterized by a holonomy operator. A problem lies in separation of a geometric phase factor from the total evolution operator without using an adiabatic assumption.

In Section 6.4, we address the Berry phase phenomena in completely integrable systems. The reason is that, being constant under an internal dynamic evolution, action variables of a completely integrable system are driven only by a perturbation holonomy operator without any adiabatic approximation [36, 37, 41].

6.1 Lagrangian and Hamiltonian mechanics with parameters

Let the composite bundle (6.1), treated as a configuration space of a mechanical system with parameters, be equipped with bundle coordinates (t, σ^m, q^i) where (t, σ^m) are coordinates on a fibre bundle $\Sigma \rightarrow \mathbb{R}$.

Remark 6.1: Though $Q \rightarrow \mathbb{R}$ is a trivial bundle, a fibre bundle $Q \rightarrow \Sigma$ need not be trivial. \square

For a time, it is convenient to regard parameters as dynamic variables. Then a total velocity space of a mechanical system with parameters is the first order jet manifold J^1Q of the fibre bundle $Q \rightarrow \mathbb{R}$. It is equipped with the adapted coordinates $(t, \sigma^m, q^i, \sigma_t^m, q_t^i)$.

Let a fibre bundle $Q \rightarrow \Sigma$ be provided with a connection

$$A_\Sigma = dt \otimes (\partial_t + A_t^i \partial_i) + d\sigma^m \otimes (\partial_m + A_m^i \partial_i). \quad (6.3)$$

Then the corresponding vertical covariant differential (7.78):

$$\tilde{D} : J^1Q \rightarrow V_\Sigma Q, \quad \tilde{D} = (q_t^i - A_t^i - A_m^i \sigma_t^m) \partial_i, \quad (6.4)$$

is defined on a configuration bundle $Q \rightarrow \mathbb{R}$.

Given a section ς of a parameter bundle $\Sigma \rightarrow \mathbb{R}$, the restriction of \tilde{D} to $J^1 i_\varsigma(J^1 Q_\varsigma) \subset J^1 Q$ is the familiar covariant differential on a fibre bundle Q_ς (6.2) corresponding to the pull-back (7.79):

$$A_\varsigma = \partial_t + [(A_m^i \circ \varsigma) \partial_t \varsigma^m + (A \circ \varsigma)_t^i] \partial_i, \quad (6.5)$$

of the connection A_Σ (6.3) onto $Q_\zeta \rightarrow \mathbb{R}$. Therefore, one can use the vertical covariant differential \tilde{D} (6.4) in order to construct a Lagrangian for a mechanical system with parameters on the configuration space Q (6.1).

We suppose that such a Lagrangian L depends on derivatives of parameters σ_t^m only via the vertical covariant differential \tilde{D} (6.4), i.e.,

$$L = \mathcal{L}(t, \sigma^m, q^i, \tilde{D}^i = q_t^i - A_t^i - A_m^i \sigma_t^m) dt. \quad (6.6)$$

Obviously, this Lagrangian is non-regular because of the Lagrangian constraint

$$\partial_m^t \mathcal{L} + A_m^i \partial_i^t \mathcal{L} = 0.$$

As a consequence, the corresponding Lagrange equation

$$(\partial_i - d_t \partial_i^t) \mathcal{L} = 0, \quad (6.7)$$

$$(\partial_m - d_t \partial_m^t) \mathcal{L} = 0 \quad (6.8)$$

is overdefined, and it admits a solution only if a rather particular relation

$$(\partial_m + A_m^i \partial_i) \mathcal{L} + \partial_i^t \mathcal{L} d_t A_m^i = 0$$

is satisfied.

However, if a parameter function ζ holds fixed, the equation (6.8) is replaced with the condition

$$\sigma^m = \zeta^m(t), \quad (6.9)$$

and the Lagrange equation (6.7) only should be considered. One can think of this equation under the condition (6.9) as being the Lagrange equation for the Lagrangian

$$L_\zeta = J^1 \zeta^* L = \mathcal{L}(t, \zeta^m, q^i, \tilde{D}^i = q_t^i - A_t^i - A_m^i \partial_t \zeta^m) dt \quad (6.10)$$

on a velocity space $J^1 Q_\zeta$.

A total phase space of a mechanical system with time-dependent parameters on the composite bundle (6.1) is the vertical cotangent bundle V^*Q of $Q \rightarrow \mathbb{R}$. It is coordinated by $(t, \sigma^m, q^i, p_m, p_i)$.

Let us consider Hamiltonian forms on a phase space V^*Q which are associated with the Lagrangian L (6.6). The Lagrangian constraint space $N_L \subset V^*Q$ defined by this Lagrangian is given by the equalities

$$p_i = \partial_i^t \mathcal{L}, \quad p_m + A_m^i p_i = 0, \quad (6.11)$$

where A_Σ is the connection (6.3) on a fibre bundle $Q \rightarrow \Sigma$.

Let

$$\Gamma = \partial_t + \Gamma^m(t, \sigma^r) \partial_m \quad (6.12)$$

be some connection on a parameter bundle $\Sigma \rightarrow \mathbb{R}$, and let

$$\gamma = \partial_t + \Gamma^m \partial_m + (A_t^i + A_m^i \Gamma^m) \partial_i \quad (6.13)$$

be the composite connection (7.71) on a fibre bundle $Q \rightarrow \mathbb{R}$ which is defined by the connection A_Σ (6.3) on $Q \rightarrow \Sigma$ and the connection Γ (6.12) on $\Sigma \rightarrow \mathbb{R}$. Then a desired L -associated Hamiltonian form reads

$$H = (p_m d\sigma^m + p_i dq^i) - [p_m \Gamma^m + p_i (A_t^i + A_m^i \Gamma^m) + \mathcal{E}_\gamma(t, \sigma^m, q^i, p_i)] dt, \quad (6.14)$$

where a Hamiltonian function \mathcal{E}_γ satisfies the relations

$$\partial_i^t \mathcal{L}(t, \sigma^m, q^i, \tilde{D}^i) = \partial^i \mathcal{E}_\gamma(t, \sigma^m, q^i, \partial_i^t \mathcal{L}) = \partial_i^t \mathcal{L}, \quad (6.15)$$

$$p_i \partial^i \mathcal{E}_\gamma - \mathcal{E}_\gamma = \mathcal{L}(t, \sigma^m, q^i, \tilde{D}^i) = \partial^i \mathcal{E}_\gamma. \quad (6.16)$$

A key point is that the Hamiltonian form (6.14) is affine in momenta p_m and that the relations (6.15) – (6.16) are independent of the connection Γ (6.12).

The Hamilton equation (4.39) – (4.40) for the Hamiltonian form H (6.14) reads

$$q_t^i = A_t^i + A_m^i \Gamma^m + \partial^i \mathcal{E}_\gamma, \quad (6.17)$$

$$p_{ti} = -p_j (\partial_i A_t^j + \partial_i A_m^j \Gamma^m) - \partial_i \mathcal{E}_\gamma, \quad (6.18)$$

$$\sigma_t^m = \Gamma^m, \quad (6.19)$$

$$p_{tm} = -p_i (\partial_m A_t^i + \Gamma^n \partial_m A_n^i) - \partial_m \mathcal{E}_\gamma, \quad (6.20)$$

whereas the Lagrangian constraint (6.11) takes the form

$$p_i = \partial_i^t \mathcal{L}(t, q^i, \sigma^m, \partial^i \mathcal{E}_\gamma(t, \sigma^m, q^i, p_i)), \quad (6.21)$$

$$p_m + A_m^i p_i = 0. \quad (6.22)$$

If a parameter function $\varsigma(t)$ holds fixed, we ignore the equation (6.20) and treat the rest ones as follows.

Given $\varsigma(t)$, the equations (6.9) and (6.22) define a subbundle

$$P_\varsigma \rightarrow Q_\varsigma \rightarrow \mathbb{R} \quad (6.23)$$

over \mathbb{R} of a total phase space $V^*Q \rightarrow \mathbb{R}$. With the connection (6.3), we have the splitting (7.77) of V^*Q which reads

$$V^*Q = A_\Sigma(V_\Sigma^*Q) \oplus_Q (Q \times V^*\Sigma),$$

$$p_i \bar{d}q^i + p_m \bar{d}\sigma^m = p_i (\bar{d}q^i - A_m^i \bar{d}\sigma^m) + (p_m + A_m^i p_i) \bar{d}\sigma^m,$$

where V_Σ^*Q is the vertical cotangent bundle of $Q \rightarrow \Sigma$. Then $V^*Q \rightarrow Q$ can be provided with the bundle coordinates

$$\bar{p}_i = p_i, \quad \bar{p}_m = p_m + A_m^i p_i$$

compatible with this splitting. Relative to these coordinates, the equation (6.22) takes the form $\bar{p}_m = 0$. It follows that the subbundle

$$i_P : P_\varsigma = i_\varsigma^*(A_\Sigma(V_\Sigma^*Q)) \rightarrow V^*Q, \quad (6.24)$$

coordinated by (t, q^i, p_i) , is isomorphic to the vertical cotangent bundle

$$V^*Q_\varsigma = i_\varsigma^* V_\Sigma^* Q$$

of the configuration space $Q_\varsigma \rightarrow \mathbb{R}$ (6.2) of a mechanical system with a parameter function $\varsigma(t)$. Consequently, the fibre bundle P_ς (6.23) is a phase space of this system.

Given a parameter function ς , there exists a connection Γ on a parameter bundle $\Sigma \rightarrow \mathbb{R}$ such that $\varsigma(t)$ is its integral section, i.e., the equation (6.19) takes the form

$$\partial_t \varsigma^m(t) = \Gamma^m(t, \varsigma(t)). \quad (6.25)$$

Then a system of equations (6.17), (6.18) and (6.21) under the conditions (6.9) and (6.25) describes a mechanical system with a given parameter function $\varsigma(t)$ on a phase space P_ς . Moreover, this system is the Hamilton equation for the pull-back Hamiltonian form

$$H_\varsigma = i_P^* H = p_i dq^i - [p_i (A_t^i + A_m^i \partial_t \varsigma^m) + \varsigma^* \mathcal{E}_\gamma] dt \quad (6.26)$$

on P_ς where

$$A_t^i + A_m^i \partial_t \varsigma^m = (i_\varsigma^* \gamma)_t^i$$

is the pull-back connection (7.79) on $Q_\varsigma \rightarrow \mathbb{R}$.

It is readily observed that the Hamiltonian form H_ς (6.26) is associated with the Lagrangian L_ς (6.10) on $J^1 Q_\varsigma$, and the equations (6.17), (6.18) and (6.21) are corresponded to the Lagrange equation (6.7).

6.2 Quantum mechanics with classical parameters

This Section is devoted to quantization of mechanical systems with time-dependent parameters on the composite bundle Q (6.1). Since parameters remain classical, a phase space that we quantize is the vertical cotangent bundle $V_\Sigma^* Q$ of a fibre bundle $Q \rightarrow \Sigma$. This phase space is equipped with holonomic coordinates (t, σ^m, q^i, p_i) . It is provided with the following canonical Poisson structure. Let T^*Q be the cotangent bundle of Q equipped with the holonomic coordinates $(t, \sigma^m, q^i, p_0, p_m, p_i)$. It is endowed with the canonical Poisson structure $\{, \}_T$ (4.19). There is the canonical fibration

$$\zeta_\Sigma : T^*Q \xrightarrow{\zeta} V^*Q \longrightarrow V_\Sigma^*Q \quad (6.27)$$

(see the exact sequence (7.73)). Then the Poisson bracket $\{, \}_\Sigma$ on the space $C^\infty(V_\Sigma^*Q)$ of smooth real functions on V_Σ^*Q is defined by the relation

$$\zeta_\Sigma^* \{f, f'\}_\Sigma = \{\zeta_\Sigma^* f, \zeta_\Sigma^* f'\}_T, \quad (6.28)$$

$$\{f, f'\}_\Sigma = \partial^k f \partial_k f' - \partial_k f \partial^k f', \quad f, f' \in C^\infty(V_\Sigma^*Q). \quad (6.29)$$

The corresponding characteristic symplectic foliation \mathcal{F} coincides with the fibration $V_\Sigma^*Q \rightarrow \Sigma$. Therefore, we can apply to a phase space $V_\Sigma^*Q \rightarrow \Sigma$ the technique of leafwise geometric quantization [32, 41].

Let us assume that a manifold Q is oriented, that fibres of $V_\Sigma^*Q \rightarrow \Sigma$ are simply connected, and that

$$H^2(Q; \mathbb{Z}_2) = H^2(V_\Sigma^*Q; \mathbb{Z}_2) = 0.$$

Being the characteristic symplectic foliation of the Poisson structure (6.29), the fibration $V_\Sigma^*Q \rightarrow \Sigma$ is endowed with the symplectic leafwise form (1.15):

$$\Omega_{\mathcal{F}} = \tilde{d}p_i \wedge \tilde{d}q^i.$$

Since this form is \tilde{d} -exact, its leafwise de Rham cohomology class equals zero and, consequently, it is the image of the zero de Rham cohomology class. Then, in accordance with Proposition 5.9, the symplectic foliation $(V_\Sigma^*Q \rightarrow \Sigma, \Omega_{\mathcal{F}})$ admits prequantization.

Since the leafwise form $\Omega_{\mathcal{F}}$ is \tilde{d} -exact, the prequantization bundle $C \rightarrow V_\Sigma^*Q$ is trivial. Let its trivialization

$$C = V_\Sigma^*Q \times \mathbb{C} \tag{6.30}$$

hold fixed, and let $(t, \sigma^m, q^k, p_k, c)$ be the corresponding bundle coordinates. Then $C \rightarrow V_\Sigma^*Q$ admits a leafwise connection

$$A_{\mathcal{F}} = \tilde{d}p_k \otimes \partial^k + \tilde{d}q^k \otimes (\partial_k - ip_k c \partial_c).$$

This connection preserves the Hermitian fibre metric g (4.51) in C , and its curvature fulfils the prequantization condition (5.32):

$$\tilde{R} = -i\Omega_{\mathcal{F}} \otimes u_C.$$

The corresponding prequantization operators (5.31) read

$$\begin{aligned} \hat{f} &= -i\vartheta_f + (p_k \partial^k f - f), & f &\in C^\infty(V_\Sigma^*Q), \\ \vartheta_f &= \partial^k f \partial_k - \partial_k f \partial^k. \end{aligned}$$

Let us choose the canonical vertical polarization of the symplectic foliation $(V_\Sigma^*Q \rightarrow \Sigma, \Omega_{\mathcal{F}})$ which is the vertical tangent bundle $\mathbf{T} = VV_\Sigma^*Q$ of a fibre bundle

$$\pi_{VQ} : V_\Sigma^*Q \rightarrow Q.$$

It is readily observed that the corresponding quantum algebra $\mathcal{A}_{\mathcal{F}}$ consists of functions

$$f = a^i(t, \sigma^m, q^k) p_i + b(t, \sigma^m, q^k) \tag{6.31}$$

on V_Σ^*Q which are affine in momenta p_k .

Following the quantization procedure in Section 5.2, one should consider the quantization bundle (5.51) which is isomorphic to the prequantization bundle C (6.30) because the metilinear bundle $\mathcal{D}_{1/2}[\mathcal{F}]$ of complex fibrewise half-densities on $V_\Sigma^*Q \rightarrow \Sigma$ is trivial owing to the identity transition functions $J_{\mathcal{F}} = 1$ (5.49).

Then we define the representation (5.52) of the quantum algebra $\mathcal{A}_{\mathcal{F}}$ of functions f (6.31) in the space $\mathfrak{E}_{\mathcal{F}}$ of sections ρ of the prequantization bundle $C \rightarrow V_{\Sigma}^*Q$ which obey the condition (5.53) and whose restriction to each fibre of $V_{\Sigma}^*Q \rightarrow \Sigma$ is of compact support. Since the trivialization (6.30) of C holds fixed, its sections are complex functions on V_{Σ}^*Q , and the above mentioned condition (5.53) reads

$$\partial_k f \partial^k \rho = 0, \quad f \in C^{\infty}(Q),$$

i.e., elements of $\mathfrak{E}_{\mathcal{F}}$ are constant on fibres of $V_{\Sigma}^*Q \rightarrow Q$. Consequently, $\mathfrak{E}_{\mathcal{F}}$ reduces to zero $\rho = 0$.

Therefore, we modify the leafwise quantization procedure as follows. Given a fibration

$$\pi_{Q\Sigma} : Q \rightarrow \Sigma,$$

let us consider the corresponding metlinear bundle $\mathcal{D}_{1/2}[\pi_{Q\Sigma}] \rightarrow Q$ of leafwise half-densities on $Q \rightarrow \Sigma$ and the tensor product

$$Y_Q = C_Q \otimes \mathcal{D}_{1/2}[\pi_{Q\Sigma}] = \mathcal{D}_{1/2}[\pi_{Q\Sigma}] \rightarrow Q,$$

where $C_Q = \mathbb{C} \times Q$ is the trivial complex line bundle over Q . It is readily observed that the Hamiltonian vector fields

$$\vartheta_f = a^k \partial_k - (p_j \partial_k a^j + \partial_k b) \partial^k$$

of elements $f \in \mathcal{A}_{\mathcal{F}}$ (6.31) are projectable onto Q . Then one can associate to each element f of the quantum algebra $\mathcal{A}_{\mathcal{F}}$ the first order differential operator

$$\begin{aligned} \widehat{f} &= (-i\nabla_{\pi_{VQ}(\vartheta_f)} + f) \otimes \text{Id} + \text{Id} \otimes \mathbf{L}_{\pi_{VQ}(\vartheta_f)} = \\ & -ia^k \partial_k - \frac{i}{2} \partial_k a^k - b \end{aligned} \quad (6.32)$$

in the space \mathfrak{E}_Q of sections of the fibre bundle $Y_Q \rightarrow Q$ whose restriction to each fibre of $Q \rightarrow \Sigma$ is of compact support. Since the pull-back of $\mathcal{D}_{1/2}[\pi_{Q\Sigma}]$ onto each fibre Q_{σ} of $Q \rightarrow \Sigma$ is the metlinear bundle of half-densities on Q_{σ} , the restrictions ρ_{σ} of elements of $\rho \in \mathfrak{E}_Q$ to Q_{σ} constitute a pre-Hilbert space with respect to the non-degenerate Hermitian form

$$\langle \rho_{\sigma} | \rho'_{\sigma} \rangle_{\sigma} = \int_{Q_{\sigma}} \rho_{\sigma} \overline{\rho'_{\sigma}}.$$

Then the Schrödinger operators (6.32) are Hermitian operators in the pre-Hilbert $C^{\infty}(\Sigma)$ -module \mathfrak{E}_Q , and provide the desired geometric quantization of the symplectic foliation $(V_{\Sigma}^*Q \rightarrow \Sigma, \Omega_{\mathcal{F}})$.

In order to quantize the evolution equation of a mechanical system on a phase space V_{Σ}^*Q , one should bear in mind that this equation is not reduced to the Poisson bracket $\{, \}_{\Sigma}$ on V_{Σ}^*Q , but is expressed in the Poisson bracket $\{, \}_{\mathcal{T}}$

on the cotangent bundle T^*Q [32, 41]. Therefore, let us start with the classical evolution equation.

Given the Hamiltonian form H (6.14) on a total phase space V^*Q , let (T^*Q, \mathcal{H}^*) be an equivalent homogeneous Hamiltonian system with the homogeneous Hamiltonian \mathcal{H}^* (4.43):

$$\mathcal{H}^* = p_0 + [p_m \Gamma^m + p_i (A_t^i + A_m^i \Gamma^m) + \mathcal{E}_\gamma(t, \sigma^m, q^i, p_i)]. \quad (6.33)$$

Let us consider the homogeneous evolution equation (4.48) where F are functions on a phase space V_Σ^*Q . It reads

$$\begin{aligned} \{\mathcal{H}^*, \zeta_\Sigma^* F\}_T &= 0, & F &\in C^\infty(V_\Sigma^*Q), \\ \partial_t F + \Gamma^m \partial_m F + (A_t^i + A_m^i \Gamma^m + \partial^i \mathcal{E}_\gamma) \partial_i F - \\ & [p_j (\partial_i A_t^j + \partial_i A_m^j \Gamma^m) + \partial_i \mathcal{E}_\gamma] \partial^i F &= 0. \end{aligned} \quad (6.34)$$

It is readily observed that a function $F \in C^\infty(V_\Sigma^*Q)$ obeys the equality (6.34) iff it is constant on solutions of the Hamilton equation (6.17) – (6.19). Therefore, one can think of the relation (6.34) as being a classical evolution equation on $C^\infty(V_\Sigma^*Q)$.

In order to quantize the evolution equation (6.34), one should quantize a symplectic manifold $(T^*Q, \{\cdot, \cdot\}_T)$ so that its quantum algebra \mathcal{A}_T contains the pull-back $\zeta_\Sigma^* \mathcal{A}_\mathcal{F}$ of the quantum algebra $\mathcal{A}_\mathcal{F}$ of the functions (6.31). For this purpose, we choose the vertical polarization VT^*Q on the cotangent bundle T^*Q . The corresponding quantum algebra \mathcal{A}_T consists of functions on T^*Q which are affine in momenta (p_0, p_m, p_i) (see Section 5.2). Clearly, $\zeta_\Sigma^* \mathcal{A}_\mathcal{F}$ is a subalgebra of the quantum algebra \mathcal{A}_T of T^*Q .

Let us restrict our consideration to the subalgebra $\mathcal{A}'_T \subset \mathcal{A}_T$ of functions

$$f = a(t, \sigma^r) p_0 + a^m(t, \sigma^r) p_m + a^i(t, \sigma^m, q^j) p_i + b(t, \sigma^m, q^j),$$

where a and a^λ are the pull-back onto T^*Q of functions on a parameter space Σ . Of course, $\zeta_\Sigma^* \mathcal{A}_\mathcal{F} \subset \mathcal{A}'_T$. Moreover, \mathcal{A}'_T admits a representation by the Hermitian operators

$$\widehat{f} = -i(a \partial_t + a^m \partial_m + a^i \partial_i) - \frac{i}{2} \partial_k a^k - b \quad (6.35)$$

in the carrier space \mathfrak{E}_Q of the representation (6.32) of $\mathcal{A}_\mathcal{F}$. Then, if $\mathcal{H}^* \in \mathcal{A}'_T$, the evolution equation (6.34) is quantized as the Heisenberg equation

$$i[\widehat{\mathcal{H}}^*, \widehat{f}] = 0, \quad f \in \mathcal{A}_\mathcal{F}. \quad (6.36)$$

A problem is that the function \mathcal{H}^* (6.33) fails to belong to the algebra \mathcal{A}'_T , unless the Hamiltonian function \mathcal{E}_γ (6.14) is affine in momenta p_i . Let us assume that \mathcal{E}_γ is polynomial in momenta. This is the case of almost all physically relevant models.

LEMMA 6.1: Any smooth function f on V_Σ^*Q which is a polynomial of momenta p_k is decomposed in a finite sum of products of elements of the algebra $\mathcal{A}_\mathcal{F}$. \square

By virtue of Lemma 6.1, one can associate to a polynomial Hamiltonian function \mathcal{E}_γ an element of the enveloping algebra $\overline{\mathcal{A}}_\mathcal{F}$ of the Lie algebra $\mathcal{A}_\mathcal{F}$ (though it by no means is unique). Accordingly, the homogeneous Hamiltonian \mathcal{H}^* (6.33) is represented by an element of the enveloping algebra $\overline{\mathcal{A}}'_T$ of the Lie algebra \mathcal{A}'_T . Then the Schrödinger representation (6.32) and (6.35) of the Lie algebras $\mathcal{A}_\mathcal{F}$ and \mathcal{A}'_T is naturally extended to their enveloping algebras $\overline{\mathcal{A}}_\mathcal{F}$ and $\overline{\mathcal{A}}'_T$ that provides quantization

$$\widehat{\mathcal{H}}^* = -i[\partial_t + \Gamma^m \partial_m + (A_t^k + A_m^k \Gamma^m) \partial_k] - \frac{i}{2} \partial_k (A_t^k + A_m^k \Gamma^m) + \widehat{\mathcal{E}}_\gamma \quad (6.37)$$

of the homogeneous Hamiltonian \mathcal{H}^* (6.33).

It is readily observed that the operator $i\widehat{\mathcal{H}}^*$ (6.37) obeys the Leibniz rule

$$i\widehat{\mathcal{H}}^*(r\rho) = \partial_t r \rho + r(i\widehat{\mathcal{H}}^* \rho), \quad r \in C^\infty(\mathbb{R}), \quad \rho \in \mathfrak{E}_Q. \quad (6.38)$$

Therefore, it is a connection on pre-Hilbert $C^\infty(\mathbb{R})$ -module \mathfrak{E}_Q . The corresponding Schrödinger equation reads

$$i\widehat{\mathcal{H}}^* \rho = 0, \quad \rho \in \mathfrak{E}_Q.$$

Given a trivialization

$$Q = \mathbb{R} \times M, \quad (6.39)$$

there is the corresponding global decomposition

$$\widehat{\mathcal{H}}^* = -i\partial_t + \widehat{\mathcal{H}},$$

where $\widehat{\mathcal{H}}$ plays a role of the Hamilton operator. Then we can introduce the evolution operator U which obeys the equation

$$\partial_t U(t) = -i\widehat{\mathcal{H}}^* \circ U(t), \quad U(0) = \mathbf{1}.$$

It can be written as the formal time-ordered exponent

$$U = T \exp \left[-i \int_0^t \widehat{\mathcal{H}} dt' \right].$$

Given the quantum operator $\widehat{\mathcal{H}}^*$ (6.37), the bracket

$$\nabla \widehat{f} = i[\widehat{\mathcal{H}}^*, \widehat{f}] \quad (6.40)$$

defines a derivation of the quantum algebra $\overline{\mathcal{A}}_\mathcal{F}$. Since $\widehat{p}_0 = -i\partial_t$, the derivation (6.40) obeys the Leibniz rule

$$\nabla(r\widehat{f}) = \partial_t r \widehat{f} + r \nabla \widehat{f}, \quad r \in C^\infty(\mathbb{R}).$$

Therefore, it is a connection on the $C^\infty(\mathbb{R})$ -algebra $\overline{\mathcal{A}}_\mathcal{F}$, which enables one to treat quantum evolution of $\overline{\mathcal{A}}_\mathcal{F}$ as a parallel displacement along time. In

particular, \widehat{f} is parallel with respect to the connection (6.40) if it obeys the Heisenberg equation (6.36).

Now let us consider a mechanical system depending on a given parameter function $\varsigma : \mathbb{R} \rightarrow \Sigma$. Its configuration space is the pull-back bundle Q_ς (6.2). The corresponding phase space is the fibre bundle P_ς (6.24). The pull-back H_ς of the Hamiltonian form H (6.14) onto P_ς takes the form (6.26).

The homogeneous phase space of a mechanical system with a parameter function ς is the pull-back

$$\overline{P}_\varsigma = i_P^* T^* Q \quad (6.41)$$

onto P_ς of the fibre bundle $T^* Q \rightarrow V^* Q$ (4.20). The homogeneous phase space \overline{P}_ς (6.41) is coordinated by (t, q^i, p_0, p_i) , and it isomorphic to the cotangent bundle $T^* Q_\varsigma$. The associated homogeneous Hamiltonian on \overline{P}_ς reads

$$\mathcal{H}_\varsigma^* = p_0 + [p_i(A_t^i + A_m^i \partial_t \varsigma^m) + \varsigma^* \mathcal{E}_\gamma]. \quad (6.42)$$

It characterizes the dynamics of a mechanical system with a given parameter function ς .

In order to quantize this system, let us consider the pull-back bundle

$$\mathcal{D}_{1/2}[Q_\varsigma] = i_\varsigma^* \mathcal{D}_{1/2}[\pi_{Q\Sigma}]$$

over Q_ς and its pull-back sections $\rho_\varsigma = i_\varsigma^* \rho$, $\rho \in \mathfrak{E}_Q$. It is easily justified that these are fibrewise half-densities on a fibre bundle $Q_\varsigma \rightarrow \mathbb{R}$ whose restrictions to each fibre $i_t : Q_t \rightarrow Q_\varsigma$ are of compact support. These sections constitute a pre-Hilbert $C^\infty(\mathbb{R})$ -module \mathfrak{E}_ς with respect to the Hermitian forms

$$\langle i_t^* \rho_\varsigma | i_t^* \rho'_\varsigma \rangle_t = \int_{Q_t} i_t^* \rho_\varsigma \overline{i_t^* \rho'_\varsigma}.$$

Then the pull-back operators

$$\begin{aligned} (\varsigma^* \widehat{f}) \rho_\varsigma &= (\widehat{f} \rho)_\varsigma, \\ \varsigma^* \widehat{f} &= -i a^k(t, \varsigma^m(t), q^j) \partial_k - \frac{i}{2} \partial_k a^k(t, \varsigma^m(t), q^j) - b(t, \varsigma^m(t), q^j), \end{aligned}$$

in \mathfrak{E}_ς provide the representation of the pull-back functions

$$i_\varsigma^* f = a^k(t, \varsigma^m(t), q^j) p_k + b(t, \varsigma^m(t), q^j), \quad f \in \mathcal{A}_\mathcal{F},$$

on $V^* Q_\varsigma$. Accordingly, the quantum operator

$$\widehat{\mathcal{H}}_\varsigma^* = -i \partial_t - i(A_t^i + A_m^i \partial_t \varsigma^m) \partial_i - \frac{i}{2} \partial_i (A_t^i + A_m^i \partial_t \varsigma^m) - \widehat{\varsigma^* \mathcal{E}_\gamma} \quad (6.43)$$

coincides with the pull-back operator $\varsigma^* \widehat{\mathcal{H}}^*$, and it yields the Heisenberg equation

$$i[\widehat{\mathcal{H}}_\varsigma^*, \varsigma^* \widehat{f}] = 0$$

of a quantum system with a parameter function ς .

The operator $\widehat{\mathcal{H}}_\varsigma^*$ (6.43) acting in the pre-Hilbert $C^\infty(\mathbb{R})$ -module \mathfrak{E}_ς obeys the Leibniz rule

$$i\widehat{\mathcal{H}}_\varsigma^*(r\rho_\varsigma) = \partial_t r\rho_\varsigma + r(i\widehat{\mathcal{H}}_\varsigma^*\rho_\varsigma), \quad r \in C^\infty(\mathbb{R}), \quad \rho_\varsigma \in \mathfrak{E}_Q, \quad (6.44)$$

and, therefore, it is a connection on \mathfrak{E}_ς . The corresponding Schrödinger equation reads

$$\begin{aligned} i\widehat{\mathcal{H}}_\varsigma^*\rho_\varsigma &= 0, \quad \rho_\varsigma \in \mathfrak{E}_\varsigma, \\ \left[\partial_t + (A_t^i + A_m^i \partial_t \varsigma^m) \partial_i + \frac{1}{2} \partial_i (A_t^i + A_m^i \partial_t \varsigma^m) - i\widehat{\varsigma^* \mathcal{E}_\gamma} \right] \rho_\varsigma &= 0. \end{aligned} \quad (6.45)$$

With the trivialization (6.39) of Q , we have a trivialization of $Q_\varsigma \rightarrow \mathbb{R}$ and the corresponding global decomposition

$$\widehat{\mathcal{H}}_\varsigma^* = -i\partial_t + \widehat{\mathcal{H}}_\varsigma,$$

where

$$\widehat{\mathcal{H}}_\varsigma = -i(A_t^i + A_m^i \partial_t \varsigma^m) \partial_i - \frac{i}{2} \partial_i (A_t^i + A_m^i \partial_t \varsigma^m) + \widehat{\varsigma^* \mathcal{E}_\gamma} \quad (6.46)$$

is a Hamilton operator. Then we can introduce an evolution operator U_ς which obeys the equation

$$\partial_t U_\varsigma(t) = -i\widehat{\mathcal{H}}_\varsigma^* \circ U_\varsigma(t), \quad U_\varsigma(0) = \mathbf{1}.$$

It can be written as the formal time-ordered exponent

$$U_\varsigma(t) = T \exp \left[-i \int_0^t \widehat{\mathcal{H}}_\varsigma dt' \right]. \quad (6.47)$$

6.3 Berry geometric factor

As was mentioned above, the Berry phase factor is a standard attribute of quantum mechanical systems with time-dependent classical parameters [8, 58]. The quantum Berry phase factor is described by driving a carrier Hilbert space of a Hamilton operator over cycles in a parameter manifold. The Berry geometric factor depends only on the geometry of a path in a parameter manifold and, therefore, provides a possibility to perform quantum gate operations in an intrinsically fault-tolerant way. A problem lies in separation of the Berry geometric factor from the total evolution operator without using an adiabatic assumption. Firstly, holonomy quantum computation implies exact cyclic evolution, but exact adiabatic cyclic evolution almost never exists. Secondly, an adiabatic condition requires that the evolution time must be long enough.

In a general setting, let us consider a linear (not necessarily finite-dimensional) dynamical system

$$\partial_t \psi = \widehat{S} \psi$$

whose linear (time-dependent) dynamic operator \widehat{S} falls into the sum

$$\widehat{S} = \widehat{S}_0 + \Delta = \widehat{S}_0 + \partial_t \zeta^m \Delta_m, \quad (6.48)$$

where $\zeta(t)$ is a parameter function given by a section of some smooth fibre bundle $\Sigma \rightarrow \mathbb{R}$ coordinated by (t, σ^m) . Let assume the following:

- (i) the operators $\widehat{S}_0(t)$ and $\Delta(t')$ commute for all instants t and t' ,
 - (ii) the operator Δ depends on time only through a parameter function $\zeta(t)$.
- Then the corresponding evolution operator $U(t)$ can be represented by the product of time-ordered exponentials

$$U(t) = U_0(t) \circ U_1(t) = T \exp \left[\int_0^t \Delta dt' \right] \circ T \exp \left[\int_0^t \widehat{S}_0 dt' \right], \quad (6.49)$$

where the first one is brought into the ordered exponential

$$U_1(t) = T \exp \left[\int_0^t \Delta_m(\zeta(t')) \partial_t \zeta^m(t') dt' \right] = \quad (6.50)$$

$$T \exp \left[\int_{\zeta[0,t]} \Delta_m(\sigma) d\sigma^m \right]$$

along the curve $\zeta[0, t]$ in a parameter bundle Σ . It is the Berry geometric factor depending only on a trajectory of a parameter function ζ . Therefore, one can think of this factor as being a displacement operator along a curve $\zeta[0, t] \subset \Sigma$. Accordingly,

$$\Delta = \Delta_m \partial_t \zeta^m \quad (6.51)$$

is called the holonomy operator.

However, a problem is that the above mentioned commutativity condition (i) is rather restrictive.

Turn now to the quantum Hamiltonian system with classical parameters in Section 6.2. The Hamilton operator $\widehat{\mathcal{H}}_\zeta$ (6.46) in the evolution operator U (6.47) takes the form (6.48):

$$\widehat{\mathcal{H}}_\zeta = -i \left[A_m^k \partial_k + \frac{1}{2} \partial_k A_m^k \right] \partial_t \zeta^m + \widehat{\mathcal{H}}'(\zeta). \quad (6.52)$$

Its second term $\widehat{\mathcal{H}}'$ can be regarded as a dynamic Hamilton operator of a quantum system, while the first one is responsible for the Berry geometric factor as follows.

Bearing in mind possible applications to holonomic quantum computations, let us simplify the quantum system in question. The above mentioned trivialization (6.39) of Q implies a trivialization of a parameter bundle $\Sigma = \mathbb{R} \times W$ such that a fibration $Q \rightarrow \Sigma$ reads

$$\mathbb{R} \times M \xrightarrow{\text{Id} \times \pi_M} \mathbb{R} \times W,$$

where $\pi_M : M \rightarrow W$ is a fibre bundle. Let us suppose that components A_m^k of the connection A_Σ (6.3) are independent of time. Then one can regard the second term in this connection as a connection on a fibre bundle $M \rightarrow W$. It also follows that the first term in the Hamilton operator (6.52) depends on time only through parameter functions $\zeta^m(t)$. Furthermore, let the two terms in the Hamilton operator (6.52) mutually commute on $[0, t]$. Then the evolution operator U (6.47) takes the form

$$U = T \exp \left[- \int_{\zeta([0,t])} \left(A_m^k \partial_k + \frac{1}{2} \partial_k A_m^k \right) d\sigma^m \right] \circ \quad (6.53)$$

$$T \exp \left[-i \int_0^t \widehat{\mathcal{H}}' dt' \right].$$

One can think of its first factor as being the parallel displacement operator along the curve $\zeta([0, t]) \subset W$ with respect to the connection

$$\nabla_m \rho = \left(\partial_m + A_m^k \partial_k + \frac{1}{2} \partial_k A_m^k \right) \rho, \quad \rho \in \mathfrak{E}_Q, \quad (6.54)$$

called the Berry connection on a $C^\infty(W)$ -module \mathfrak{E}_Q . A peculiarity of this factor in comparison with the second one lies in the fact that integration over time through a parameter function $\zeta(t)$ depends only on a trajectory of this function in a parameter space, but not on parametrization of this trajectory by time. Therefore, the first term of the evolution operator U (6.53) is the Berry geometric factor. The corresponding holonomy operator (6.51) reads

$$\Delta = \left(A_m^k \partial_k + \frac{1}{2} \partial_k A_m^k \right) \partial_t \zeta^m.$$

6.4 Non-adiabatic holonomy operator

We address the Berry phase phenomena in a completely integrable system of m degrees of freedom around its invariant torus T^m . The reason is that, being constant under an internal evolution, its action variables are driven only by a perturbation holonomy operator Δ . We construct such an operator for an arbitrary connection on a fibre bundle

$$W \times T^m \rightarrow W, \quad (6.55)$$

without any adiabatic approximation [36, 37, 41]. In order that a holonomy operator and a dynamic Hamiltonian mutually commute, we first define a holonomy operator with respect to initial data action-angle coordinates and, afterwards, return to the original ones. A key point is that both classical evolution of action variables and mean values of quantum action operators relative to

original action-angle coordinates are determined by the dynamics of initial data action and angle variables.

A generic phase space of a Hamiltonian system with time-dependent parameters is a composite fibre bundle

$$P \rightarrow \Sigma \rightarrow \mathbb{R},$$

where $\Pi \rightarrow \Sigma$ is a symplectic bundle (i.e., a symplectic foliation whose leaves are fibres of $\Pi \rightarrow \Sigma$), and

$$\Sigma = \mathbb{R} \times W \rightarrow \mathbb{R}$$

is a parameter bundle whose sections are parameter functions. In the case of a completely integrable system with time-dependent parameters, we have the product

$$P = \Sigma \times U = \Sigma \times (V \times T^m) \rightarrow \Sigma \rightarrow \mathbb{R},$$

equipped with the coordinates $(t, \sigma^\alpha, I_k, \varphi^k)$. Let us suppose for a time that parameters also are dynamic variables. The total phase space of such a system is the product

$$\Pi = V^*\Sigma \times U$$

coordinated by $(t, \sigma^\alpha, p_\alpha = \dot{\sigma}_\alpha, I_k, \varphi^k)$. Its dynamics is characterized by the Hamiltonian form (6.14):

$$\begin{aligned} H_\Sigma &= p_\alpha d\sigma^\alpha + I_k d\varphi^k - \mathcal{H}_\Sigma(t, \sigma^\beta, p_\beta, I_j, \varphi^j) dt, \\ \mathcal{H}_\Sigma &= p_\alpha \Gamma^\alpha + I_k (\Lambda_t^k + \Lambda_\alpha^k \Gamma_t^\alpha) + \tilde{\mathcal{H}}, \end{aligned} \quad (6.56)$$

where $\tilde{\mathcal{H}}$ is a function, $\partial_t + \Gamma^\alpha \partial_\alpha$ is the connection (6.12) on the parameter bundle $\Sigma \rightarrow \mathbb{R}$, and

$$\Lambda = dt \otimes (\partial_t + \Lambda_t^k \partial_k) + d\sigma^\alpha \otimes (\partial_\alpha + \Lambda_\alpha^k \partial_k) \quad (6.57)$$

is the connection (6.3) on the fibre bundle

$$\Sigma \times T^m \rightarrow \Sigma.$$

Bearing in mind that σ^α are parameters, one should choose the Hamiltonian \mathcal{H}_Σ (6.56) to be affine in their momenta p_α . Then a Hamiltonian system with a fixed parameter function $\sigma^\alpha = \zeta^\alpha(t)$ is described by the pull-back Hamiltonian form (6.26):

$$\begin{aligned} H_\zeta &= I_k d\varphi^k - \{I_k [\Lambda_t^k(t, \varphi^j) + \\ &\quad \Lambda_\alpha^k(t, \zeta^\beta, \varphi^j) \partial_t \zeta^\alpha] + \tilde{\mathcal{H}}(t, \zeta^\beta, I_j, \varphi^j)\} dt \end{aligned} \quad (6.58)$$

on a Poisson manifold

$$\mathbb{R} \times U = \mathbb{R} \times (V \times T^m). \quad (6.59)$$

Let $\tilde{\mathcal{H}} = \mathcal{H}(I_i)$ be a Hamiltonian of an original autonomous completely integrable system on the toroidal domain U (6.59) equipped with the action-angle coordinates (I_k, ϕ^k) . We introduce a desired holonomy operator by the appropriate choice of the connection Λ (6.57).

For this purpose, let us choose the initial data action-angle coordinates (I_k, ϕ^k) by the converse to the canonical transformation (4.76):

$$\varphi^k = \phi^k - t\partial^k \mathcal{H}. \quad (6.60)$$

With respect to these coordinates, the Hamiltonian of an original completely integrable system vanishes and the Hamiltonian form (6.58) reads

$$H_\zeta = I_k d\phi^k - I_k [\Lambda_t^k(t, \phi^j) + \Lambda_\alpha^k(t, \zeta^\beta, \phi^j) \partial_t \zeta^\alpha] dt. \quad (6.61)$$

Let us put $\Lambda_t^k = 0$ by the choice of a reference frame associated to the initial data coordinates ϕ^k , and let us assume that coefficients Λ_α^k are independent of time, i.e., the part

$$\Lambda_W = d\sigma^\alpha \otimes (\partial_\alpha + \Lambda_\alpha^k \partial_k) \quad (6.62)$$

of the connection Λ (6.57) is a connection on the fibre bundle (6.55). Then the Hamiltonian form (6.61) reads

$$H_\zeta = I_k d\phi^k - I_k \Lambda_\alpha^k(\zeta^\beta, \phi^j) \partial_t \zeta^\alpha dt. \quad (6.63)$$

Its Hamilton vector field (4.38) is

$$\gamma_H = \partial_t + \Lambda_\alpha^i \partial_t \zeta^\alpha \partial_i - I_k \partial_i \Lambda_\alpha^k \partial_t \zeta^\alpha \partial^i, \quad (6.64)$$

and it leads to the Hamilton equation

$$d_t \phi^i = \Lambda_\alpha^i(\zeta(t), \phi^l) \partial_t \zeta^\alpha, \quad (6.65)$$

$$d_t I_i = -I_k \partial_i \Lambda_\alpha^k(\zeta(t), \phi^l) \partial_t \zeta^\alpha. \quad (6.66)$$

Let us consider the lift

$$V^* \Lambda_W = d\sigma^\alpha \otimes (\partial_\alpha + \Lambda_\alpha^i \partial_i - I_k \partial_i \Lambda_\alpha^k \partial^i) \quad (6.67)$$

of the connection Λ_W (6.62) onto the fibre bundle

$$W \times (V \times T^m) \rightarrow W,$$

seen as a subbundle of the vertical cotangent bundle

$$V^*(W \times T^m) = W \times T^* T^m$$

of the fibre bundle (6.55). It follows that any solution $I_i(t), \phi^i(t)$ of the Hamilton equation (6.65) – (6.66) (i.e., an integral curve of the Hamilton vector field (6.64)) is a horizontal lift of the curve $\zeta(t) \subset W$ with respect to the connection $V^* \Lambda_W$ (6.67), i.e.,

$$I_i(t) = I_i(\zeta(t)), \quad \phi^i(t) = \phi^i(\zeta(t)).$$

Thus, the right-hand side of the Hamilton equation (6.65) – (6.66) is the holonomy operator

$$\Delta = (\Lambda_\alpha^i \partial_t \zeta^\alpha, -I_k \partial_i \Lambda_\alpha^k \partial_t \zeta^\alpha). \quad (6.68)$$

It is not a linear operator, but the substitution of a solution $\phi(\zeta(t))$ of the equation (6.65) into the Hamilton equation (6.66) results in a linear holonomy operator on the action variables I_i .

Let us show that the holonomy operator (6.68) is well defined. Since any vector field ϑ on $\mathbb{R} \times T^m$ such that $\vartheta \lrcorner dt = 1$ is complete, the Hamilton equation (6.65) has solutions for any parameter function $\zeta(t)$. It follows that any connection Λ_W (6.62) on the fibre bundle (6.55) is an Ehresmann connection, and so is its lift (6.67). Because $V^* \Lambda_W$ (6.67) is an Ehresmann connection, any curve $\zeta([0, 1]) \subset W$ can play a role of the parameter function in the holonomy operator Δ (6.68).

Now, let us return to the original action-angle coordinates (I_k, φ^k) by means of the canonical transformation (6.60). The perturbed Hamiltonian reads

$$\mathcal{H}' = I_k \Lambda_\alpha^k(\zeta(t), \varphi^i - t \partial^i \mathcal{H}(I_j)) \partial_t \zeta^\alpha(t) + \mathcal{H}(I_j),$$

while the Hamilton equation (6.65) – (6.66) takes the form

$$\begin{aligned} \partial_t \varphi^i &= \partial^i \mathcal{H}(I_j) + \Lambda_\alpha^i(\zeta(t), \varphi^l - t \partial^l \mathcal{H}(I_j)) \partial_t \zeta^\alpha(t) \\ &\quad - t I_k \partial^i \partial^s \mathcal{H}(I_j) \partial_s \Lambda_\alpha^k(\zeta(t), \varphi^l - t \partial^l \mathcal{H}(I_j)) \partial_t \zeta^\alpha(t), \\ \partial_t I_i &= -I_k \partial_i \Lambda_\alpha^k(\zeta(t), \varphi^l - t \partial^l \mathcal{H}(I_j)) \partial_t \zeta^\alpha(t). \end{aligned}$$

Its solution is

$$I_i(\zeta(t)), \quad \varphi^i(t) = \phi^i(\zeta(t)) + t \partial^i \mathcal{H}(I_j(\zeta(t))),$$

where $I_i(\zeta(t))$, $\phi^i(\zeta(t))$ is a solution of the Hamilton equation (6.65) – (6.66). We observe that the action variables I_k are driven only by the holonomy operator, while the angle variables φ^i have a non-geometric summand.

Let us emphasize that, in the construction of the holonomy operator (6.68), we do not impose any restriction on the connection Λ_W (6.62). Therefore, any connection on the fibre bundle (6.55) yields a holonomy operator of a completely integrable system. However, a glance at the expression (6.68) shows that this operator becomes zero on action variables if all coefficients Λ_α^k of the connection Λ_W (6.62) are constant, i.e., Λ_W is a principal connection on the fibre bundle (6.55) seen as a principal bundle with the structure group T^m .

In order to quantize a non-autonomous completely integrable system on the Poisson toroidal domain $(U, \{, \}_V)$ (6.59) equipped with action-angle coordinates (I_i, φ^i) , one may follow the instantwise geometric quantization of non-autonomous mechanics. As a result, we can simply replace functions on T^m with those on $\mathbb{R} \times T^m$ [19]. Namely, the corresponding quantum algebra $\mathcal{A} \subset C^\infty(U)$ consists of affine functions

$$f = a^k(t, \varphi^j) I_k + b(t, \varphi^j) \quad (6.69)$$

of action coordinates I_k represented by the operators (6.32) in the space

$$\mathfrak{E} = \mathbb{C}^\infty(\mathbb{R} \times T^m) \quad (6.70)$$

of smooth complex functions $\psi(t, \varphi)$ on $\mathbb{R} \times T^m$. This space is provided with the structure of the pre-Hilbert $\mathbb{C}^\infty(\mathbb{R})$ -module endowed with the non-degenerate $\mathbb{C}^\infty(\mathbb{R})$ -bilinear form

$$\langle \psi | \psi' \rangle = \left(\frac{1}{2\pi} \right)^m \int_{T^m} \psi \overline{\psi'} d^m \varphi, \quad \psi, \psi' \in \mathfrak{E}.$$

Its basis consists of the pull-back onto $\mathbb{R} \times T^m$ of the functions

$$\psi_{(n_r)} = \exp[i(n_r \phi^r)], \quad (n_r) = (n_1, \dots, n_m) \in \mathbb{Z}^m. \quad (6.71)$$

Furthermore, this quantization of a non-autonomous completely integrable system on the Poisson manifold $(U, \{, \}_V)$ is extended to the associated homogeneous completely integrable system on the symplectic annulus (4.72):

$$U' = \zeta^{-1}(U) = N' \times T^m \rightarrow N'$$

by means of the operator $\widehat{I}_0 = -i\partial_t$ in the pre-Hilbert module \mathfrak{E} (6.70). Accordingly, the homogeneous Hamiltonian $\widehat{\mathcal{H}}^*$ is quantized as

$$\widehat{\mathcal{H}}^* = -i\partial_t + \widehat{\mathcal{H}}.$$

It is a Hamiltonian of a quantum non-autonomous completely integrable system. The corresponding Schrödinger equation is

$$\widehat{\mathcal{H}}^* \psi = -i\partial_t \psi + \widehat{\mathcal{H}} \psi = 0, \quad \psi \in \mathfrak{E}. \quad (6.72)$$

For instance, a quantum Hamiltonian of an original autonomous completely integrable system seen as the non-autonomous one is

$$\widehat{\mathcal{H}}^* = -i\partial_t + \mathcal{H}(\widehat{I}_j).$$

Its spectrum

$$\widehat{\mathcal{H}}^* \psi_{(n_r)} = E_{(n_r)} \psi_{(n_r)}$$

on the basis $\{\psi_{(n_r)}\}$ (6.71) for \mathfrak{E} (6.71) coincides with that of the autonomous Hamiltonian $\widehat{\mathcal{H}}(I_k) = \mathcal{H}(\widehat{I}_k)$. The Schrödinger equation (6.72) reads

$$\widehat{\mathcal{H}}^* \psi = -i\partial_t \psi + \mathcal{H}(-i\partial_k + \lambda_k) \psi = 0, \quad \psi \in E.$$

Its solutions are the Fourier series

$$\psi = \sum_{(n_r)} B_{(n_r)} \exp[-itE_{(n_r)}] \psi_{(n_r)}, \quad B_{(n_r)} \in \mathbb{C}.$$

Now, let us quantize this completely integrable system with respect to the initial data action-angle coordinates (I_i, ϕ^i) . As was mentioned above, it is given on a toroidal domain U (6.59) provided with another fibration over \mathbb{R} . Its quantum algebra $\mathcal{A}_0 \subset C^\infty(U)$ consists of affine functions

$$f = a^k(t, \phi^j)I_k + b(t, \phi^j). \quad (6.73)$$

The canonical transformation (4.76) ensures an isomorphism of Poisson algebras \mathcal{A} and \mathcal{A}_0 . Functions f (6.73) are represented by the operators \widehat{f} (6.32) in the pre-Hilbert module \mathfrak{E}_0 of smooth complex functions $\Psi(t, \phi)$ on $\mathbb{R} \times T^m$. Given its basis

$$\Psi_{(n_r)}(\phi) = [in_r\phi^r],$$

the operators \widehat{I}_k and $\widehat{\psi}_{(n_r)}$ take the form

$$\begin{aligned} \widehat{I}_k \psi_{(n_r)} &= (n_k + \lambda_k) \psi_{(n_r)}, \\ \psi_{(n_r)} \psi_{(n'_r)} &= \psi_{(n_r)} \psi_{(n'_r)} = \psi_{(n_r+n'_r)}. \end{aligned} \quad (6.74)$$

The Hamiltonian of a quantum completely integrable system with respect to the initial data variables is $\widehat{\mathcal{H}}_0^* = -i\partial_t$. Then one easily obtains the isometric isomorphism

$$R(\psi_{(n_r)}) = \exp[itE_{(n_r)}]\Psi_{(n_r)}, \quad \langle R(\psi)|R(\psi') \rangle = \langle \psi|\psi' \rangle, \quad (6.75)$$

of the pre-Hilbert modules \mathfrak{E} and \mathfrak{E}_0 which provides the equivalence

$$\widehat{I}_i = R^{-1}\widehat{I}_i R, \quad \widehat{\psi}_{(n_r)} = R^{-1}\widehat{\Psi}_{(n_r)} R, \quad \widehat{\mathcal{H}}^* = R^{-1}\widehat{\mathcal{H}}_0^* R \quad (6.76)$$

of the quantizations of a completely integrable system with respect to the original and initial data action-angle variables.

In view of the isomorphism (6.76), let us first construct a holonomy operator of a quantum completely integrable system $(\mathcal{A}_0, \widehat{\mathcal{H}}_0^*)$ with respect to the initial data action-angle coordinates. Let us consider the perturbed homogeneous Hamiltonian

$$\mathbf{H}_\zeta = \mathcal{H}_0^* + \mathbf{H}_1 = I_0 + \partial_t \zeta^\alpha(t) \Lambda_\alpha^k(\zeta(t), \phi^j) I_k$$

of the classical perturbed completely integrable system (6.63). Its perturbation term \mathbf{H}_1 is of the form (6.69) and, therefore, is quantized by the operator

$$\widehat{\mathbf{H}}_1 = -i\partial_t \zeta^\alpha \widehat{\Delta}_\alpha = -i\partial_t \zeta^\alpha \left[\Lambda_\alpha^k \partial_k + \frac{1}{2} \partial_k (\Lambda_\alpha^k) + i\lambda_k \Lambda_\alpha^k \right].$$

The quantum Hamiltonian $\widehat{\mathbf{H}}_\zeta = \widehat{\mathcal{H}}_0^* + \widehat{\mathbf{H}}_1$ defines the Schrödinger equation

$$\partial_t \Psi + \partial_t \zeta^\alpha \left[\Lambda_\alpha^k \partial_k + \frac{1}{2} \partial_k (\Lambda_\alpha^k) + i\lambda_k \Lambda_\alpha^k \right] \Psi = 0. \quad (6.77)$$

If its solution exists, it can be written by means of the evolution operator $U(t)$ which is reduced to the geometric factor

$$U_1(t) = T \exp \left[i \int_0^t \partial_{t'} \varsigma^\alpha(t') \widehat{\Delta}_\alpha(t') dt' \right].$$

The latter can be viewed as a displacement operator along the curve $\varsigma[0, 1] \subset W$ with respect to the connection

$$\widehat{\Lambda}_W = d\sigma^\alpha(\partial_\alpha + \widehat{\Delta}_\alpha) \quad (6.78)$$

on the $\mathbb{C}^\infty(W)$ -module $\mathbb{C}^\infty(W \times T^m)$ of smooth complex functions on $W \times T^m$. Let us study whether this displacement operator exists.

Given a connection Λ_W (6.62), let $\Phi^i(t, \phi)$ denote the flow of the complete vector field

$$\partial_t + \Lambda_\alpha(\varsigma, \phi) \partial_t \varsigma^\alpha$$

on $\mathbb{R} \times T^m$. It is a solution of the Hamilton equation (6.65) with the initial data ϕ . We need the inverse flow $(\Phi^{-1})^i(t, \phi)$ which obeys the equation

$$\begin{aligned} \partial_t(\Phi^{-1})^i(t, \phi) &= -\partial_t \varsigma^\alpha \Lambda_\alpha^i(\varsigma, (\Phi^{-1})^i(t, \phi)) = \\ &= -\partial_t \varsigma^\alpha \Lambda_\alpha^k(\varsigma, \phi) \partial_k(\Phi^{-1})^i(t, \phi). \end{aligned}$$

Let Ψ_0 be an arbitrary complex half-form Ψ_0 on T^m possessing identical transition functions, and let the same symbol stand for its pull-back onto $\mathbb{R} \times T^m$. Given its pull-back

$$(\Phi^{-1})^* \Psi_0 = \det \left(\frac{\partial(\Phi^{-1})^i}{\partial \phi^k} \right)^{1/2} \Psi_0(\Phi^{-1}(t, \phi)), \quad (6.79)$$

it is readily observed that

$$\Psi = (\Phi^{-1})^* \Psi_0 \exp[i\lambda_k \phi^k] \quad (6.80)$$

obeys the Schrödinger equation (6.77) with the initial data Ψ_0 . Because of the multiplier $\exp[i\lambda_k \phi^k]$, the function Ψ (6.80) however is ill defined, unless all numbers λ_k equal 0 or $\pm 1/2$. Let us note that, if some numbers λ_k are equal to $\pm 1/2$, then $\Psi_0 \exp[i\lambda_k \phi^k]$ is a half-density on T^m whose transition functions equal ± 1 , i.e., it is a section of a non-trivial metlinear bundle over T^m .

Thus, we observe that, if λ_k equal 0 or $\pm 1/2$, then the displacement operator always exists and $\Delta = i\mathbf{H}_1$ is a holonomy operator. Because of the action law (6.74), it is essentially infinite-dimensional.

For instance, let Λ_W (6.62) be the above mentioned principal connection, i.e., $\Lambda_\alpha^k = \text{const}$. Then the Schrödinger equation (6.77) where $\lambda_k = 0$ takes the form

$$\partial_t \Psi(t, \phi^j) + \partial_t \varsigma^\alpha(t) \Lambda_\alpha^k \partial_k \Psi(t, \phi^j) = 0,$$

and its solution (6.79) is

$$\Psi(t, \phi^j) = \Psi_0(\phi^j - (\zeta^\alpha(t) - \zeta^\alpha(0))\Lambda_\alpha^j).$$

The corresponding evolution operator $U(t)$ reduces to Berry's phase multiplier

$$U_1 \Psi_{(n_r)} = \exp[-in_j(\zeta^\alpha(t) - \zeta^\alpha(0))\Lambda_\alpha^j] \Psi_{(n_r)}, \quad n_j \in (n_r).$$

It keeps the eigenvectors of the action operators \widehat{I}_i .

In order to return to the original action-angle variables, one can employ the morphism R (6.75). The corresponding Hamiltonian reads

$$\mathbf{H} = R^{-1} \mathbf{H}_\zeta R.$$

The key point is that, due to the relation (6.76), the action operators \widehat{I}_i have the same mean values

$$\langle I_k \psi | \psi \rangle = \langle I_k \Psi | \Psi \rangle, \quad \Psi = R(\psi),$$

with respect both to the original and the initial data action-angle variables. Therefore, these mean values are defined only by the holonomy operator.

In conclusion, let us note that, since action variables are driven only by a holonomy operator, one can use this operator in order to perform a dynamic transition between classical solutions or quantum states of an unperturbed completely integrable system by an appropriate choice of a parameter function ζ . A key point is that this transition can take an arbitrary short time because we are entirely free with time parametrization of ζ and can choose it quickly changing, in contrast with slowly varying parameter functions in adiabatic models. This fact makes non-adiabatic holonomy operators in completely integrable systems promising for several applications, e.g., quantum control and quantum computation.

7 Appendix

For the sake of convenience of the reader, this Section summarizes the relevant material on differential geometry of fibre bundles [44, 58, 73].

7.1 Geometry of fibre bundles

Throughout this work, all morphisms are smooth, and manifolds are smooth real and finite-dimensional. A smooth manifold is customarily assumed to be Hausdorff, second-countable and, consequently, paracompact. Being paracompact, a smooth manifold admits a partition of unity by smooth real functions. Unless otherwise stated, manifolds are assumed to be connected. The symbol $C^\infty(Z)$ stands for a ring of smooth real functions on a manifold Z .

Given a smooth manifold Z , by $\pi_Z : TZ \rightarrow Z$ is denoted its tangent bundle. Given manifold coordinates (z^α) on Z , the tangent bundle TZ is equipped with the holonomic coordinates

$$(z^\lambda, \dot{z}^\lambda), \quad \dot{z}^\lambda = \frac{\partial z'^\lambda}{\partial z^\mu} \dot{z}^\mu,$$

with respect to the holonomic frames $\{\partial_\lambda\}$ in the tangent spaces to Z . Any manifold morphism $f : Z \rightarrow Z'$ yields the tangent morphism

$$Tf : TZ \rightarrow TZ', \quad \dot{z}'^\lambda \circ Tf = \frac{\partial f^\lambda}{\partial z^\mu} \dot{z}^\mu,$$

of their tangent bundles. A morphism f is said to be an immersion if $T_z f$, $z \in Z$, is injective, and a submersion if $T_z f$, $z \in Z$, is surjective. Note that a submersion is an open map (i.e., an image of any open set is open).

If $f : Z \rightarrow Z'$ is an injective immersion, its range is called a submanifold of Z' . A submanifold is said to be imbedded if it also is a topological subspace. In this case, f is called an imbedding. If $Z \subset Z'$, its natural injection is denoted by $i_Z : Z \rightarrow Z'$.

If a manifold morphism

$$\pi : Y \rightarrow X, \quad \dim X = n > 0, \quad (7.1)$$

is a surjective submersion, one says that: (i) its domain Y is a fibred manifold, (ii) X is its base, (iii) π is a fibration, and (iv) $Y_x = \pi^{-1}(x)$ is a fibre over $x \in X$. A fibred manifold admits an atlas of fibred coordinate charts $(U_Y; x^\lambda, y^i)$ such that (x^λ) are coordinates on $\pi(U_Y) \subset X$ and coordinate transition functions read

$$x'^\lambda = f^\lambda(x^\mu), \quad y'^i = f^i(x^\mu, y^j).$$

For each point $y \in Y$ of a fibred manifold, there exists a local section s of $Y \rightarrow X$ passing through y . By a local section of the fibration (7.1) is meant an injection $s : U \rightarrow Y$ of an open subset $U \subset X$ such that $\pi \circ s = \text{Id} U$, i.e., a section sends any point $x \in X$ into the fibre Y_x over this point. A local section also is defined over any subset $N \subset X$ as the restriction to N of a local section over an open set containing N . If $U = X$, one calls s the global section. A range $s(U)$ of a local section $s : U \rightarrow Y$ of a fibred manifold $Y \rightarrow X$ is an imbedded submanifold of Y . A local section is a closed map, sending closed subsets of U onto closed subsets of Y . If s is a global section, then $s(X)$ is a closed imbedded submanifold of Y . Global sections of a fibred manifold need not exist.

THEOREM 7.1: Let $Y \rightarrow X$ be a fibred manifold whose fibres are diffeomorphic to \mathbb{R}^m . Any its section over a closed imbedded submanifold (e.g., a point) of X is extended to a global section [80]. In particular, such a fibred manifold always has a global section. \square

Given fibred coordinates $(U_Y; x^\lambda, y^i)$, a section s of a fibred manifold $Y \rightarrow X$ is represented by collections of local functions $\{s^i = y^i \circ s\}$ on $\pi(U_Y)$.

Morphisms of fibred manifolds, by definition, are fibrewise morphisms, sending a fibre to a fibre. Namely, a fibred morphism of a fibred manifold $\pi : Y \rightarrow X$ to a fibred manifold $\pi' : Y' \rightarrow X'$ is defined as a pair (Φ, f) of manifold morphisms which form a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & Y' \\ \pi \downarrow & & \downarrow \pi' \\ X & \xrightarrow{f} & X' \end{array}, \quad \pi' \circ \Phi = f \circ \pi.$$

Fibred injections and surjections are called monomorphisms and epimorphisms, respectively. A fibred diffeomorphism is called an isomorphism or an automorphism if it is an isomorphism to itself. For the sake of brevity, a fibred morphism over $f = \text{Id } X$ usually is said to be a fibred morphism over X , and is denoted by $Y \xrightarrow[X]{} Y'$. In particular, a fibred automorphism over X is called a vertical automorphism.

A fibred manifold $Y \rightarrow X$ is said to be trivial if Y is isomorphic to the product $X \times V$. Different trivializations of $Y \rightarrow X$ differ from each other in surjections $Y \rightarrow V$.

A fibred manifold $Y \rightarrow X$ is called a fibre bundle if it is locally trivial, i.e., if it admits a fibred coordinate atlas $\{(\pi^{-1}(U_\xi); x^\lambda, y^i)\}$ over a cover $\{\pi^{-1}(U_\xi)\}$ of Y which is the inverse image of a cover $\mathfrak{U} = \{U_\xi\}$ of X . In this case, there exists a manifold V , called a typical fibre, such that Y is locally diffeomorphic to the splittings

$$\psi_\xi : \pi^{-1}(U_\xi) \rightarrow U_\xi \times V, \quad (7.2)$$

glued together by means of transition functions

$$\varrho_{\xi\zeta} = \psi_\xi \circ \psi_\zeta^{-1} : U_\xi \cap U_\zeta \times V \rightarrow U_\xi \cap U_\zeta \times V \quad (7.3)$$

on overlaps $U_\xi \cap U_\zeta$. Restricted to a point $x \in X$, trivialization morphisms ψ_ξ (7.2) and transition functions $\varrho_{\xi\zeta}$ (7.3) define diffeomorphisms of fibres

$$\psi_\xi(x) : Y_x \rightarrow V, \quad x \in U_\xi, \quad (7.4)$$

$$\varrho_{\xi\zeta}(x) : V \rightarrow V, \quad x \in U_\xi \cap U_\zeta. \quad (7.5)$$

Trivialization charts (U_ξ, ψ_ξ) together with transition functions $\varrho_{\xi\zeta}$ (7.3) constitute a bundle atlas

$$\Psi = \{(U_\xi, \psi_\xi), \varrho_{\xi\zeta}\} \quad (7.6)$$

of a fibre bundle $Y \rightarrow X$. Two bundle atlases are said to be equivalent if their union also is a bundle atlas, i.e., there exist transition functions between trivialization charts of different atlases. All atlases of a fibre bundle are equivalent, and a fibre bundle $Y \rightarrow X$ is uniquely defined by a bundle atlas.

Given a bundle atlas Ψ (7.6), a fibre bundle Y is provided with the fibred coordinates

$$x^\lambda(y) = (x^\lambda \circ \pi)(y), \quad y^i(y) = (y^i \circ \psi_\xi)(y), \quad y \in \pi^{-1}(U_\xi),$$

called the bundle coordinates, where y^i are coordinates on a typical fibre V .

There is the following useful criterion for a fibred manifold to be a fibre bundle.

THEOREM 7.2: A fibred manifold whose fibres are diffeomorphic either to a compact manifold or \mathbb{R}^r is a fibre bundle [60]. \square

In particular, a compact fibred manifold is a fibre bundle.

THEOREM 7.3: Any fibre bundle over a contractible base is trivial [44]. \square

Note that a fibred manifold over a contractible base need not be trivial. It follows from Theorem 7.3 that any cover of a base X by domains (i.e., contractible open subsets) is a bundle cover.

A fibred morphism of fibre bundles is called a bundle morphism. A bundle monomorphism $\Phi : Y \rightarrow Y'$ over X onto a submanifold $\Phi(Y)$ of Y' is called a subbundle of a fibre bundle $Y' \rightarrow X$.

The following are the standard constructions of new fibre bundles from old ones.

- Given a fibre bundle $\pi : Y \rightarrow X$ and a manifold morphism $f : X' \rightarrow X$, the pull-back of Y by f is called the manifold

$$f^*Y = \{(x', y) \in X' \times Y : \pi(y) = f(x')\}$$

together with the natural projection $(x', y) \rightarrow x'$. It is a fibre bundle over X' such that the fibre of f^*Y over a point $x' \in X'$ is that of Y over the point $f(x') \in X$. Any section s of a fibre bundle $Y \rightarrow X$ yields the pull-back section $f^*s(x') = (x', s(f(x')))$ of $f^*Y \rightarrow X'$.

- If $X' \subset X$ is a submanifold of X and $i_{X'}$ is the corresponding natural injection, then the pull-back bundle

$$i_{X'}^*Y = Y|_{X'}$$

is called the restriction of a fibre bundle Y to the submanifold $X' \subset X$. If X' is an imbedded submanifold, any section of the pull-back bundle $Y|_{X'} \rightarrow X'$ is the restriction to X' of some section of $Y \rightarrow X$.

- Let $\pi : Y \rightarrow X$ and $\pi' : Y' \rightarrow X$ be fibre bundles over the same base X . Their bundle product $Y \times_X Y'$ over X is defined as the pull-back

$$Y \times_X Y' = \pi^*Y' \quad \text{or} \quad Y \times_X Y' = \pi'^*Y$$

together with its natural surjection onto X . Fibres of the bundle product $Y \times_X Y'$ are the Cartesian products $Y_x \times Y'_x$ of fibres of fibre bundles Y and Y' .

- Let us consider the composite fibre bundle

$$Y \rightarrow \Sigma \rightarrow X. \tag{7.7}$$

It is provided with bundle coordinates $(x^\lambda, \sigma^m, y^i)$, where (x^λ, σ^m) are bundle coordinates on a fibre bundle $\Sigma \rightarrow X$, i.e., transition functions of coordinates

σ^m are independent of coordinates y^i . Let h be a global section of a fibre bundle $\Sigma \rightarrow X$. Then the restriction $Y_h = h^*Y$ of a fibre bundle $Y \rightarrow \Sigma$ to $h(X) \subset \Sigma$ is a subbundle of a fibre bundle $Y \rightarrow X$.

A fibre bundle $\pi : Y \rightarrow X$ is called a vector bundle if both its typical fibre and fibres are finite-dimensional real vector spaces, and if it admits a bundle atlas whose trivialization morphisms and transition functions are linear isomorphisms. Then the corresponding bundle coordinates on Y are linear bundle coordinates (y^i) possessing linear transition functions $y'^i = A_j^i(x)y^j$. We have

$$y = y^i e_i(\pi(y)) = y^i \psi_\xi(\pi(y))^{-1}(e_i), \quad \pi(y) \in U_\xi,$$

where $\{e_i\}$ is a fixed basis for a typical fibre V of Y and $\{e_i(x)\}$ are the fibre bases (or the frames) for the fibres Y_x of Y associated to a bundle atlas Ψ .

By virtue of Theorem 7.1, any vector bundle has a global section, e.g., the canonical global zero-valued section $\widehat{0}(x) = 0$.

Global sections of a vector bundle $Y \rightarrow X$ constitute a projective $C^\infty(X)$ -module $Y(X)$ of finite rank. It is called the structure module of a vector bundle.

There are the following particular constructions of new vector bundles from the old ones.

- Let $Y \rightarrow X$ be a vector bundle with a typical fibre V . By $Y^* \rightarrow X$ is denoted the dual vector bundle with the typical fibre V^* , dual of V . The interior product of Y and Y^* is defined as a fibred morphism

$$\lrcorner : Y \otimes Y^* \xrightarrow{X} X \times \mathbb{R}.$$

- Let $Y \rightarrow X$ and $Y' \rightarrow X$ be vector bundles with typical fibres V and V' , respectively. Their Whitney sum $Y \oplus_X Y'$ is a vector bundle over X with the typical fibre $V \oplus V'$.

- Let $Y \rightarrow X$ and $Y' \rightarrow X$ be vector bundles with typical fibres V and V' , respectively. Their tensor product $Y \otimes_X Y'$ is a vector bundle over X with the typical fibre $V \otimes V'$. Similarly, the exterior product of vector bundles $Y \wedge_X Y'$ is defined. The exterior product

$$\wedge Y = X \times \mathbb{R} \oplus_X Y \oplus_X \wedge^2 Y \oplus_X \cdots \oplus_X \wedge^k Y, \quad k = \dim Y - \dim X, \quad (7.8)$$

is called the exterior bundle.

- If Y' is a subbundle of a vector bundle $Y \rightarrow X$, the factor bundle Y/Y' over X is defined as a vector bundle whose fibres are the quotients Y_x/Y'_x , $x \in X$.

By a morphism of vector bundles is meant a linear bundle morphism, which is a linear fibrewise map whose restriction to each fibre is a linear map.

Given a linear bundle morphism $\Phi : Y' \rightarrow Y$ of vector bundles over X , its kernel $\text{Ker } \Phi$ is defined as the inverse image $\Phi^{-1}(\widehat{0}(X))$ of the canonical zero-valued section $\widehat{0}(X)$ of Y . If Φ is of constant rank, its kernel and its range are vector subbundles of the vector bundles Y' and Y , respectively. For instance, monomorphisms and epimorphisms of vector bundles fulfil this condition.

Remark 7.1: Given vector bundles Y and Y' over the same base X , every

linear bundle morphism

$$\Phi : Y_x \ni \{e_i(x)\} \rightarrow \{\Phi_i^k(x)e'_k(x)\} \in Y'_x$$

over X defines a global section

$$\Phi : x \rightarrow \Phi_i^k(x)e^i(x) \otimes e'_k(x)$$

of the tensor product $Y \otimes Y'^*$, and *vice versa*. \square

A sequence of vector bundles

$$0 \rightarrow Y' \xrightarrow{i} Y \xrightarrow{j} Y'' \rightarrow 0 \quad (7.9)$$

over X is said to be a short exact sequence if it is exact at all terms Y' , Y , and Y'' . This means that i is a bundle monomorphism, j is a bundle epimorphism, and $\text{Ker } j = \text{Im } i$. Then Y'' is isomorphic to a factor bundle Y/Y' .

One says that the exact sequence (7.9) is split if there exists a bundle monomorphism $s : Y'' \rightarrow Y$ such that $j \circ s = \text{Id } Y''$ or, equivalently,

$$Y = i(Y') \oplus s(Y'') = Y' \oplus Y''.$$

THEOREM 7.4: Every exact sequence of vector bundles (7.9) is split. \square

The tangent bundle TZ and the cotangent bundle T^*Z of a manifold Z exemplify vector bundles. The cotangent bundle of a manifold Z is the dual $T^*Z \rightarrow Z$ of the tangent bundle $TZ \rightarrow Z$. It is equipped with the holonomic coordinates

$$(z^\lambda, \dot{z}_\lambda). \quad \dot{z}'_\lambda = \frac{\partial z^\mu}{\partial z'^\lambda} \dot{z}_\mu,$$

with respect to the coframes $\{dz^\lambda\}$ for T^*Z which are the duals of $\{\partial_\lambda\}$.

The tensor product of tangent and cotangent bundles

$$T = (\otimes^m TZ) \otimes (\otimes^k T^*Z), \quad m, k \in \mathbb{N}, \quad (7.10)$$

is called a tensor bundle, provided with holonomic bundle coordinates $\dot{z}_{\beta_1 \dots \beta_k}^{\alpha_1 \dots \alpha_m}$ possessing transition functions

$$\dot{z}_{\beta_1 \dots \beta_k}^{\alpha_1 \dots \alpha_m} = \frac{\partial z'^{\alpha_1}}{\partial z^{\mu_1}} \dots \frac{\partial z'^{\alpha_m}}{\partial z^{\mu_m}} \frac{\partial z^{\nu_1}}{\partial z'^{\beta_1}} \dots \frac{\partial z^{\nu_k}}{\partial z'^{\beta_k}} \dot{z}_{\nu_1 \dots \nu_k}^{\mu_1 \dots \mu_m}.$$

Let $\pi_Y : TY \rightarrow Y$ be the tangent bundle of a fibred manifold $\pi : Y \rightarrow X$. Given fibred coordinates (x^λ, y^i) on Y , it is equipped with the holonomic coordinates $(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i)$. The tangent bundle $TY \rightarrow Y$ has the subbundle $VY = \text{Ker}(T\pi)$, which consists of the vectors tangent to fibres of Y . It is called the vertical tangent bundle of Y , and it is provided with the holonomic coordinates $(x^\lambda, y^i, \dot{y}^i)$ with respect to the vertical frames $\{\partial_i\}$. Every fibred

morphism $\Phi : Y \rightarrow Y'$ yields the linear bundle morphism over Φ of the vertical tangent bundles

$$V\Phi : VY \rightarrow VY', \quad \dot{y}'^i \circ V\Phi = \frac{\partial \Phi^i}{\partial y^j} \dot{y}^j. \quad (7.11)$$

It is called the vertical tangent morphism.

In many important cases, the vertical tangent bundle $VY \rightarrow Y$ of a fibre bundle $Y \rightarrow X$ is trivial, and it is isomorphic to the bundle product

$$VY = Y \times_X \bar{Y}, \quad (7.12)$$

where $\bar{Y} \rightarrow X$ is some vector bundle. One calls (7.12) the vertical splitting. For instance, every vector bundle $Y \rightarrow X$ admits the canonical vertical splitting

$$VY = Y \oplus_X Y. \quad (7.13)$$

The vertical cotangent bundle $V^*Y \rightarrow Y$ of a fibred manifold $Y \rightarrow X$ is defined as the dual of the vertical tangent bundle $VY \rightarrow Y$. It is not a subbundle of the cotangent bundle T^*Y , but there is the canonical surjection

$$\zeta : T^*Y \ni \dot{x}_\lambda dx^\lambda + \dot{y}_i dy^i \rightarrow \dot{y}_i \bar{d}y^i \in V^*Y, \quad (7.14)$$

where the bases $\{\bar{d}y^i\}$, possessing transition functions

$$\bar{d}y'^i = \frac{\partial y'^i}{\partial y^j} \bar{d}y^j,$$

are the duals of the vertical frames $\{\partial_i\}$ of the vertical tangent bundle VY .

For any fibred manifold Y , there exist the exact sequences of vector bundles

$$0 \rightarrow VY \rightarrow TY \xrightarrow{\pi_X} Y \times_X TX \rightarrow 0, \quad (7.15)$$

$$0 \rightarrow Y \times_X T^*X \rightarrow T^*Y \rightarrow V^*Y \rightarrow 0. \quad (7.16)$$

Their splitting, by definition, is a connection on $Y \rightarrow X$.

Let $\bar{\pi} : \bar{Y} \rightarrow X$ be a vector bundle with a typical fibre \bar{V} . An affine bundle modelled over the vector bundle $\bar{Y} \rightarrow X$ is a fibre bundle $\pi : Y \rightarrow X$ whose typical fibre V is an affine space modelled over \bar{V} , all the fibres Y_x of Y are affine spaces modelled over the corresponding fibres \bar{Y}_x of the vector bundle \bar{Y} , and there is an affine bundle atlas

$$\Psi = \{(U_\alpha, \psi_\alpha), \varrho_{\alpha\zeta}\}$$

of $Y \rightarrow X$ whose local trivializations morphisms ψ_α (7.4) and transition functions $\varrho_{\alpha\zeta}$ (7.5) are affine isomorphisms. Dealing with affine bundles, we use only affine bundle coordinates (y^i) associated to an affine bundle atlas Ψ .

By virtue of Theorem 7.1, affine bundles have global sections, but in contrast with vector bundles, there is no canonical global section of an affine bundle.

By a morphism of affine bundles is meant a bundle morphism $\Phi : Y \rightarrow Y'$ whose restriction to each fibre of Y is an affine map. It is called an affine bundle morphism. Every affine bundle morphism $\Phi : Y \rightarrow Y'$ of an affine bundle Y modelled over a vector bundle \bar{Y} to an affine bundle Y' modelled over a vector bundle \bar{Y}' yields a unique linear bundle morphism

$$\bar{\Phi} : \bar{Y} \rightarrow \bar{Y}', \quad \bar{y}'^i \circ \bar{\Phi} = \frac{\partial \Phi^i}{\partial y^j} \bar{y}^j,$$

called the linear derivative of Φ .

Every affine bundle $Y \rightarrow X$ modelled over a vector bundle $\bar{Y} \rightarrow X$ admits the canonical vertical splitting

$$VY = Y \times_X \bar{Y}. \quad (7.17)$$

7.2 Vector and multivector fields

Vector fields on a manifold Z are global sections of the tangent bundle $TZ \rightarrow Z$.

The set $\mathcal{T}_1(Z)$ of vector fields on Z is both a $C^\infty(Z)$ -module and a real Lie algebra with respect to the Lie bracket

$$\begin{aligned} u &= u^\lambda \partial_\lambda, & v &= v^\lambda \partial_\lambda, \\ [v, u] &= (v^\lambda \partial_\lambda u^\mu - u^\lambda \partial_\lambda v^\mu) \partial_\mu. \end{aligned}$$

Given a vector field u on X , a curve $c : \mathbb{R} \supset (\cdot) \rightarrow Z$ in Z is said to be an integral curve of u if $Tc = u(c)$. Every vector field u on a manifold Z can be seen as an infinitesimal generator of a local one-parameter group of local diffeomorphisms (a flow), and *vice versa* [51]. One-dimensional orbits of this group are integral curves of u .

Remark 7.2: Let $U \subset Z$ be an open subset and $\epsilon > 0$. Recall that by a local one-parameter group of local diffeomorphisms of Z defined on $(-\epsilon, \epsilon) \times U$ is meant a map

$$G : (-\epsilon, \epsilon) \times U \ni (t, z) \rightarrow G_t(z) \in Z$$

which possesses the following properties:

- for each $t \in (-\epsilon, \epsilon)$, the mapping G_t is a diffeomorphism of U onto the open subset $G_t(U) \subset Z$;
- $G_{t+t'}(z) = (G_t \circ G_{t'})(z)$ if $t + t' \in (-\epsilon, \epsilon)$.

If such a map G is defined on $\mathbb{R} \times Z$, it is called the one-parameter group of diffeomorphisms of Z . If a local one-parameter group of local diffeomorphisms of Z is defined on $(-\epsilon, \epsilon) \times Z$, it is uniquely prolonged onto $\mathbb{R} \times Z$ to a one-parameter group of diffeomorphisms of Z [51]. As was mentioned above, a local one-parameter group of local diffeomorphisms G on $U \subset Z$ defines a local vector field u on U by setting $u(z)$ to be the tangent vector to the curve $s(t) = G_t(z)$ at $t = 0$. Conversely, let u be a vector field on a manifold Z . For each $z \in Z$, there

exist a number $\epsilon > 0$, a neighborhood U of z and a unique local one-parameter group of local diffeomorphisms on $(-\epsilon, \epsilon) \times U$, which determines u . \square

A vector field is called complete if its flow is a one-parameter group of diffeomorphisms of Z .

THEOREM 7.5: Any vector field on a compact manifold is complete. \square

A vector field u on a fibred manifold $Y \rightarrow X$ is called projectable if it is projected onto a vector field on X , i.e., there exists a vector field τ on X such that

$$\tau \circ \pi = T\pi \circ u.$$

A projectable vector field takes the coordinate form

$$u = u^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i, \quad \tau = u^\lambda\partial_\lambda. \quad (7.18)$$

A projectable vector field is called vertical if its projection onto X vanishes, i.e., if it lives in the vertical tangent bundle VY .

A vector field $\tau = \tau^\lambda\partial_\lambda$ on a base X of a fibred manifold $Y \rightarrow X$ gives rise to a vector field on Y by means of a connection on this fibre bundle (see the formula (7.57) below). Nevertheless, every tensor bundle (7.10) admits the functorial lift of vector fields

$$\tilde{\tau} = \tau^\mu\partial_\mu + [\partial_\nu\tau^{\alpha_1}\dot{x}_{\beta_1\cdots\beta_k}^{\nu\alpha_2\cdots\alpha_m} + \cdots - \partial_{\beta_1}\tau^\nu\dot{x}_{\nu\beta_2\cdots\beta_k}^{\alpha_1\cdots\alpha_m} - \cdots]\dot{\partial}_{\alpha_1\cdots\alpha_m}^{\beta_1\cdots\beta_k}, \quad (7.19)$$

where we employ the compact notation

$$\dot{\partial}_\lambda = \frac{\partial}{\partial \dot{x}^\lambda}. \quad (7.20)$$

This lift is an \mathbb{R} -linear monomorphism of the Lie algebra $\mathcal{T}_1(X)$ of vector fields on X to the Lie algebra $\mathcal{T}_1(Y)$ of vector fields on Y . In particular, we have the functorial lift

$$\tilde{\tau} = \tau^\mu\partial_\mu + \partial_\nu\tau^\alpha\dot{x}^\nu\frac{\partial}{\partial \dot{x}^\alpha} \quad (7.21)$$

of vector fields on X onto the tangent bundle TX and their functorial lift

$$\tilde{\tau} = \tau^\mu\partial_\mu - \partial_\beta\tau^\nu\dot{x}_\nu\frac{\partial}{\partial \dot{x}_\beta} \quad (7.22)$$

onto the cotangent bundle T^*X .

Let $Y \rightarrow X$ be a vector bundle. Using the canonical vertical splitting (7.13), we obtain the canonical vertical vector field

$$u_Y = y^i\partial_i \quad (7.23)$$

on Y , called the Liouville vector field.

A multivector field ϑ of degree $|\vartheta| = r$ (or, simply, an r -vector field) on a manifold Z is a section

$$\vartheta = \frac{1}{r!} \vartheta^{\lambda_1 \dots \lambda_r} \partial_{\lambda_1} \wedge \dots \wedge \partial_{\lambda_r} \quad (7.24)$$

of the exterior product $\overset{r}{\wedge} TZ \rightarrow Z$. Let $\mathcal{T}_r(Z)$ denote the $C^\infty(Z)$ -module space of r -vector fields on Z . All multivector fields on a manifold Z make up the graded commutative algebra $\mathcal{T}_*(Z)$ of global sections of the exterior bundle $\wedge TZ$ (7.8) with respect to the exterior product \wedge .

The graded commutative algebra $\mathcal{T}_*(Z)$ is endowed with the Schouten – Nijenhuis bracket

$$\begin{aligned} [\cdot, \cdot]_{\text{SN}} : \mathcal{T}_r(Z) \times \mathcal{T}_s(Z) &\rightarrow \mathcal{T}_{r+s-1}(Z), \\ [\vartheta, v]_{\text{SN}} &= \vartheta \bullet v + (-1)^{rs} v \bullet \vartheta, \\ \vartheta \bullet v &= \frac{r}{r!s!} (\vartheta^{\mu\lambda_2 \dots \lambda_r} \partial_\mu v^{\alpha_1 \dots \alpha_s} \partial_{\lambda_2} \wedge \dots \wedge \partial_{\lambda_r} \wedge \partial_{\alpha_1} \wedge \dots \wedge \partial_{\alpha_s}). \end{aligned} \quad (7.25)$$

This generalizes the Lie bracket of vector fields. It obeys the relations

$$\begin{aligned} [\vartheta, v]_{\text{SN}} &= (-1)^{|\vartheta||v|} [v, \vartheta]_{\text{SN}}, \\ [\nu, \vartheta \wedge v]_{\text{SN}} &= [\nu, \vartheta]_{\text{SN}} \wedge v + (-1)^{(|\nu|-1)|\vartheta|} \vartheta \wedge [\nu, v]_{\text{SN}}. \end{aligned}$$

The Lie derivative of a multivector field ϑ along a vector field u is defined as

$$\mathbf{L}_u v = [u, \vartheta]_{\text{SN}}, \quad \mathbf{L}_u(\vartheta \wedge v) = \mathbf{L}_u \vartheta \wedge v + \vartheta \wedge \mathbf{L}_u v.$$

7.3 Differential forms

An exterior r -form on a manifold Z is a section

$$\phi = \frac{1}{r!} \phi_{\lambda_1 \dots \lambda_r} dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r}$$

of the exterior product $\overset{r}{\wedge} T^*Z \rightarrow Z$, where

$$\begin{aligned} dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r} &= \frac{1}{r!} \epsilon^{\lambda_1 \dots \lambda_r}_{\mu_1 \dots \mu_r} dz^{\mu_1} \otimes \dots \otimes dz^{\mu_r}, \\ \epsilon^{\dots \lambda_i \dots \lambda_j \dots}_{\dots \mu_p \dots \mu_k \dots} &= -\epsilon^{\dots \lambda_j \dots \lambda_i \dots}_{\dots \mu_p \dots \mu_k \dots} = -\epsilon^{\dots \lambda_i \dots \lambda_j \dots}_{\dots \mu_k \dots \mu_p \dots}, \\ \epsilon^{\lambda_1 \dots \lambda_r}_{\lambda_1 \dots \lambda_r} &= 1. \end{aligned}$$

Sometimes, it is convenient to write

$$\phi = \phi'_{\lambda_1 \dots \lambda_r} dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r}$$

without the coefficient $1/r!$.

Let $\mathcal{O}^r(Z)$ denote the $C^\infty(Z)$ -module of exterior r -forms on a manifold Z . By definition, $\mathcal{O}^0(Z) = C^\infty(Z)$ is the ring of smooth real functions on Z . All

exterior forms on Z constitute the graded algebra $\mathcal{O}^*(Z)$ of global sections of the exterior bundle $\wedge T^*Z$ (7.8) endowed with the exterior product

$$\begin{aligned}\phi &= \frac{1}{r!} \phi_{\lambda_1 \dots \lambda_r} dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r}, & \sigma &= \frac{1}{s!} \sigma_{\mu_1 \dots \mu_s} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_s}, \\ \phi \wedge \sigma &= \frac{1}{r!s!} \phi_{\nu_1 \dots \nu_r} \sigma_{\nu_{r+1} \dots \nu_{r+s}} dz^{\nu_1} \wedge \dots \wedge dz^{\nu_{r+s}} = \\ &= \frac{1}{r!s!(r+s)!} \epsilon^{\nu_1 \dots \nu_{r+s}} \alpha_{\alpha_1 \dots \alpha_{r+s}} \phi_{\nu_1 \dots \nu_r} \sigma_{\nu_{r+1} \dots \nu_{r+s}} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_{r+s}},\end{aligned}$$

such that

$$\phi \wedge \sigma = (-1)^{|\phi||\sigma|} \sigma \wedge \phi,$$

where the symbol $|\phi|$ stands for the form degree. The algebra $\mathcal{O}^*(Z)$ also is provided with the exterior differential

$$d\phi = dz^\mu \wedge \partial_\mu \phi = \frac{1}{r!} \partial_\mu \phi_{\lambda_1 \dots \lambda_r} dz^\mu \wedge dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r}$$

which obeys the relations

$$d \circ d = 0, \quad d(\phi \wedge \sigma) = d(\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge d(\sigma).$$

The exterior differential d makes $\mathcal{O}^*(Z)$ into a differential graded algebra, called the exterior algebra.

Given a manifold morphism $f : Z \rightarrow Z'$, any exterior k -form ϕ on Z' yields the pull-back exterior form $f^*\phi$ on Z given by the condition

$$f^*\phi(v^1, \dots, v^k)(z) = \phi(Tf(v^1), \dots, Tf(v^k))(f(z))$$

for an arbitrary collection of tangent vectors $v^1, \dots, v^k \in T_z Z$. We have the relations

$$f^*(\phi \wedge \sigma) = f^*\phi \wedge f^*\sigma, \quad df^*\phi = f^*(d\phi).$$

In particular, given a fibred manifold $\pi : Y \rightarrow X$, the pull-back onto Y of exterior forms on X by π provides the monomorphism of graded commutative algebras $\mathcal{O}^*(X) \rightarrow \mathcal{O}^*(Y)$. Elements of its range $\pi^*\mathcal{O}^*(X)$ are called basic forms. Exterior forms

$$\phi : Y \rightarrow \wedge^r T^*X, \quad \phi = \frac{1}{r!} \phi_{\lambda_1 \dots \lambda_r} dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r},$$

on Y such that $u \lrcorner \phi = 0$ for an arbitrary vertical vector field u on Y are said to be horizontal forms. Horizontal forms of degree $n = \dim X$ are called densities.

In the case of the tangent bundle $TX \rightarrow X$, there is a different way to lift exterior forms on X onto TX [43]. Let f be a function on X . Its tangent lift onto TX is defined as the function

$$\tilde{f} = \dot{x}^\lambda \partial_\lambda f. \quad (7.26)$$

Let σ be an r -form on X . Its tangent lift onto TX is said to be the r -form $\tilde{\sigma}$ given by the relation

$$\tilde{\sigma}(\tilde{\tau}_1, \dots, \tilde{\tau}_r) = \sigma(\widetilde{\tau_1, \dots, \tau_r}), \quad (7.27)$$

where τ_i are arbitrary vector fields on X and $\tilde{\tau}_i$ are their functorial lifts (7.21) onto TX . We have the coordinate expression

$$\begin{aligned} \sigma &= \frac{1}{r!} \sigma_{\lambda_1 \dots \lambda_r} dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r}, \\ \tilde{\sigma} &= \frac{1}{r!} [\dot{x}^\mu \partial_\mu \sigma_{\lambda_1 \dots \lambda_r} dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r} + \\ &\quad \sum_{i=1}^r \sigma_{\lambda_1 \dots \lambda_r} dx^{\lambda_1} \wedge \dots \wedge d\dot{x}^{\lambda_i} \wedge \dots \wedge dx^{\lambda_r}]. \end{aligned} \quad (7.28)$$

The following equality holds:

$$d\tilde{\sigma} = \widetilde{d\sigma}. \quad (7.29)$$

The interior product (or contraction) of a vector field u and an exterior r -form ϕ on a manifold Z is given by the coordinate expression

$$\begin{aligned} u \rfloor \phi &= \sum_{k=1}^r \frac{(-1)^{k-1}}{r!} u^{\lambda_k} \phi_{\lambda_1 \dots \lambda_k \dots \lambda_r} dz^{\lambda_1} \wedge \dots \wedge \widehat{dz}^{\lambda_k} \wedge \dots \wedge dz^{\lambda_r} = \\ &\quad \frac{1}{(r-1)!} u^\mu \phi_{\mu \alpha_2 \dots \alpha_r} dz^{\alpha_2} \wedge \dots \wedge dz^{\alpha_r}, \end{aligned}$$

where the caret $\widehat{}$ denotes omission. It obeys the relations

$$\begin{aligned} \phi(u_1, \dots, u_r) &= u_r \rfloor \dots u_1 \rfloor \phi, \\ u \rfloor (\phi \wedge \sigma) &= u \rfloor \phi \wedge \sigma + (-1)^{|\phi|} \phi \wedge u \rfloor \sigma. \end{aligned} \quad (7.30)$$

The Lie derivative of an exterior form ϕ along a vector field u is

$$\mathbf{L}_u \phi = u \rfloor d\phi + d(u \rfloor \phi), \quad (7.31)$$

$$\mathbf{L}_u (\phi \wedge \sigma) = \mathbf{L}_u \phi \wedge \sigma + \phi \wedge \mathbf{L}_u \sigma. \quad (7.32)$$

In particular, if f is a function, then

$$\mathbf{L}_u f = u(f) = u \rfloor df.$$

An exterior form ϕ is invariant under a local one-parameter group of diffeomorphisms G_t of Z (i.e., $G_t^* \phi = \phi$) iff its Lie derivative along the infinitesimal generator u of this group vanishes, i.e.,

$$\mathbf{L}_u \phi = 0.$$

Following physical terminology (Definition 4.9), we say that a vector field u is a symmetry of an exterior form ϕ .

A tangent-valued r -form on a manifold Z is a section

$$\phi = \frac{1}{r!} \phi_{\lambda_1 \dots \lambda_r}^\mu dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r} \otimes \partial_\mu$$

of the tensor bundle

$$\overset{r}{\wedge} T^*Z \otimes TZ \rightarrow Z.$$

Remark 7.3: There is one-to-one correspondence between the tangent-valued one-forms ϕ on a manifold Z and the linear bundle endomorphisms

$$\widehat{\phi} : TZ \rightarrow TZ, \quad \widehat{\phi} : T_z Z \ni v \rightarrow v] \phi(z) \in T_z Z, \quad (7.33)$$

$$\widehat{\phi}^* : T^*Z \rightarrow T^*Z, \quad \widehat{\phi}^* : T_z^* Z \ni v^* \rightarrow \phi(z)] v^* \in T_z^* Z, \quad (7.34)$$

over Z (Remark 7.1). For instance, the canonical tangent-valued one-form

$$\theta_Z = dz^\lambda \otimes \partial_\lambda \quad (7.35)$$

on Z corresponds to the identity morphisms (7.33) and (7.34). \square

The space $\mathcal{O}^*(Z) \otimes \mathcal{T}_1(Z)$ of tangent-valued forms is provided with the Frölicher – Nijenhuis bracket

$$\begin{aligned} [\cdot, \cdot]_{\text{FN}} : \mathcal{O}^r(Z) \otimes \mathcal{T}_1(Z) \times \mathcal{O}^s(Z) \otimes \mathcal{T}_1(Z) &\rightarrow \mathcal{O}^{r+s}(Z) \otimes \mathcal{T}_1(Z), \\ [\alpha \otimes u, \beta \otimes v]_{\text{FN}} &= (\alpha \wedge \beta) \otimes [u, v] + (\alpha \wedge \mathbf{L}_u \beta) \otimes v - \\ &\quad (\mathbf{L}_v \alpha \wedge \beta) \otimes u + (-1)^r (d\alpha \wedge u] \beta) \otimes v + (-1)^r (v] \alpha \wedge d\beta) \otimes u, \\ \alpha \in \mathcal{O}^r(Z), \quad \beta \in \mathcal{O}^s(Z), \quad u, v \in \mathcal{T}_1(Z). \end{aligned}$$

Its coordinate expression is

$$\begin{aligned} [\phi, \sigma]_{\text{FN}} &= \frac{1}{r!s!} (\phi_{\lambda_1 \dots \lambda_r}^\nu \partial_\nu \sigma_{\lambda_{r+1} \dots \lambda_{r+s}}^\mu - \sigma_{\lambda_{r+1} \dots \lambda_{r+s}}^\nu \partial_\nu \phi_{\lambda_1 \dots \lambda_r}^\mu - \\ &\quad r \phi_{\lambda_1 \dots \lambda_{r-1} \nu}^\mu \partial_{\lambda_r} \sigma_{\lambda_{r+1} \dots \lambda_{r+s}}^\nu + s \sigma_{\nu \lambda_{r+2} \dots \lambda_{r+s}}^\mu \partial_{\lambda_{r+1}} \phi_{\lambda_1 \dots \lambda_r}^\nu) \\ &\quad dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_{r+s}} \otimes \partial_\mu, \\ \phi \in \mathcal{O}^r(Z) \otimes \mathcal{T}_1(Z), \quad \sigma \in \mathcal{O}^s(Z) \otimes \mathcal{T}_1(Z). \end{aligned}$$

There are the relations

$$\begin{aligned} [\phi, \sigma]_{\text{FN}} &= (-1)^{|\phi||\sigma|+1} [\sigma, \phi]_{\text{FN}}, \\ [\phi, [\sigma, \theta]_{\text{FN}}]_{\text{FN}} &= [[\phi, \sigma]_{\text{FN}}, \theta]_{\text{FN}} + (-1)^{|\phi||\sigma|} [\sigma, [\phi, \theta]_{\text{FN}}]_{\text{FN}}, \\ \phi, \sigma, \theta &\in \mathcal{O}^*(Z) \otimes \mathcal{T}_1(Z). \end{aligned}$$

Given a tangent-valued form θ , the Nijenhuis differential on $\mathcal{O}^*(Z) \otimes \mathcal{T}_1(Z)$ is defined as the morphism

$$d_\theta : \psi \rightarrow d_\theta \psi = [\theta, \psi]_{\text{FN}}, \quad \psi \in \mathcal{O}^*(Z) \otimes \mathcal{T}_1(Z).$$

In particular, if ϕ is a tangent-valued one-form, the Nijenhuis differential

$$d_\phi \phi = [\phi, \phi]_{\text{FN}} = (\phi_\nu^\mu \partial_\mu \phi_\beta^\alpha - \phi_\beta^\mu \partial_\mu \phi_\nu^\alpha - \phi_\mu^\alpha \partial_\nu \phi_\beta^\mu + \phi_\mu^\alpha \partial_\beta \phi_\nu^\mu) dz^\nu \wedge dz^\beta \otimes \partial_\alpha \quad (7.36)$$

is called the Nijenhuis torsion.

Let $Y \rightarrow X$ be a fibred manifold. We consider the following subspaces of the space $\mathcal{O}^*(Y) \otimes \mathcal{T}_1(Y)$ of tangent-valued forms on Y :

- horizontal tangent-valued forms

$$\begin{aligned} \phi : Y &\rightarrow \bigwedge^r T^*X \otimes_Y TY, \\ \phi &= dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes \frac{1}{r!} [\phi_{\lambda_1 \dots \lambda_r}^\mu(y) \partial_\mu + \phi_{\lambda_1 \dots \lambda_r}^i(y) \partial_i], \end{aligned}$$

- vertical-valued form

$$\phi : Y \rightarrow \bigwedge^r T^*X \otimes_Y VY, \quad \phi = \frac{1}{r!} \phi_{\lambda_1 \dots \lambda_r}^i(y) dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes \partial_i,$$

- vertical-valued one-forms, called soldering forms,

$$\sigma = \sigma_\lambda^i(y) dx^\lambda \otimes \partial_i. \quad (7.37)$$

7.4 Distributions and foliations

A subbundle \mathbf{T} of the tangent bundle TZ of a manifold Z is called a regular distribution (or, simply, a distribution). A vector field u on Z is said to be subordinate to a distribution \mathbf{T} if it lives in \mathbf{T} . A distribution \mathbf{T} is called involutive if the Lie bracket of \mathbf{T} -subordinate vector fields also is subordinate to \mathbf{T} .

A subbundle of the cotangent bundle T^*Z of Z is called a codistribution \mathbf{T}^* on a manifold Z . For instance, the annihilator $\text{Ann } \mathbf{T}$ of a distribution \mathbf{T} is a codistribution whose fibre over $z \in Z$ consists of covectors $w \in T_z^*$ such that $v \lrcorner w = 0$ for all $v \in \mathbf{T}_z$.

The following local coordinates can be associated to an involutive distribution [87].

THEOREM 7.6: Let \mathbf{T} be an involutive r -dimensional distribution on a manifold Z , $\dim Z = k$. Every point $z \in Z$ has an open neighborhood U which is a domain of an adapted coordinate chart (z^1, \dots, z^k) such that, restricted to U , the distribution \mathbf{T} and its annihilator $\text{Ann } \mathbf{T}$ are spanned by the local vector fields $\partial/\partial z^1, \dots, \partial/\partial z^r$ and the local one-forms dz^{r+1}, \dots, dz^k , respectively. \square

A connected submanifold N of a manifold Z is called an integral manifold of a distribution \mathbf{T} on Z if $TN \subset \mathbf{T}$. Unless otherwise stated, by an integral manifold is meant an integral manifold of dimension of \mathbf{T} . An integral manifold

is called maximal if no other integral manifold contains it. The following is the classical theorem of Frobenius [51, 87].

THEOREM 7.7: Let \mathbf{T} be an involutive distribution on a manifold Z . For any $z \in Z$, there exists a unique maximal integral manifold of \mathbf{T} through z , and any integral manifold through z is its open subset. \square

Maximal integral manifolds of an involutive distribution on a manifold Z are assembled into a regular foliation \mathcal{F} of Z . A regular r -dimensional foliation (or, simply, a foliation) \mathcal{F} of a k -dimensional manifold Z is defined as a partition of Z into connected r -dimensional submanifolds (the leaves of a foliation) F_ι , $\iota \in I$, which possesses the following properties [70, 82].

A manifold Z admits an adapted coordinate atlas

$$\{(U_\xi; z^\lambda, z^i)\}, \quad \lambda = 1, \dots, k-r, \quad i = 1, \dots, r, \quad (7.38)$$

such that transition functions of coordinates z^λ are independent of the remaining coordinates z^i . For each leaf F of a foliation \mathcal{F} , the connected components of $F \cap U_\xi$ are given by the equations $z^\lambda = \text{const}$. These connected components and coordinates (z^i) on them make up a coordinate atlas of a leaf F . It follows that tangent spaces to leaves of a foliation \mathcal{F} constitute an involutive distribution $T\mathcal{F}$ on Z , called the tangent bundle to the foliation \mathcal{F} . The factor bundle $V\mathcal{F} = TZ/T\mathcal{F}$, called the normal bundle to \mathcal{F} , has transition functions independent of coordinates z^i . Let $T\mathcal{F}^* \rightarrow Z$ denote the dual of $T\mathcal{F} \rightarrow Z$. There are the exact sequences

$$0 \rightarrow T\mathcal{F} \xrightarrow{i_{\mathcal{F}}} TX \rightarrow V\mathcal{F} \rightarrow 0, \quad (7.39)$$

$$0 \rightarrow \text{Ann } T\mathcal{F} \rightarrow T^*X \xrightarrow{i_{\mathcal{F}}^*} T\mathcal{F}^* \rightarrow 0 \quad (7.40)$$

of vector bundles over Z .

A pair (Z, \mathcal{F}) , where \mathcal{F} is a foliation of Z , is called a foliated manifold. It should be emphasized that leaves of a foliation need not be closed or imbedded submanifolds. Every leaf has an open saturated neighborhood U , i.e., if $z \in U$, then a leaf through z also belongs to U .

Any submersion $\zeta : Z \rightarrow M$ yields a foliation

$$\mathcal{F} = \{F_p = \zeta^{-1}(p)\}_{p \in \zeta(Z)}$$

of Z indexed by elements of $\zeta(Z)$, which is an open submanifold of M , i.e., $Z \rightarrow \zeta(Z)$ is a fibred manifold. Leaves of this foliation are closed imbedded submanifolds. Such a foliation is called simple. Any (regular) foliation is locally simple.

Example 7.4: Every smooth real function f on a manifold Z with nowhere vanishing differential df is a submersion $Z \rightarrow \mathbb{R}$. It defines a one-codimensional foliation whose leaves are given by the equations

$$f(z) = c, \quad c \in f(Z) \subset \mathbb{R}.$$

This is the foliation of level surfaces of the function f , called a generating function. Every one-codimensional foliation is locally a foliation of level surfaces of some function on Z . The level surfaces of an arbitrary smooth real function f on a manifold Z define a singular foliation \mathcal{F} on Z [49]. Its leaves are not submanifolds in general. Nevertheless if $df(z) \neq 0$, the restriction of \mathcal{F} to some open neighborhood U of z is a foliation with the generating function $f|_U$. \square

Let \mathcal{F} be a (regular) foliation of a k -dimensional manifold Z provided with the adapted coordinate atlas (7.38). The real Lie algebra $\mathcal{T}_1(\mathcal{F})$ of global sections of the tangent bundle $T\mathcal{F} \rightarrow Z$ to \mathcal{F} is a $C^\infty(Z)$ -submodule of the derivation module of the \mathbb{R} -ring $C^\infty(Z)$ of smooth real functions on Z . Its kernel $S_{\mathcal{F}}(Z) \subset C^\infty(Z)$ consists of functions constant on leaves of \mathcal{F} . Therefore, $\mathcal{T}_1(\mathcal{F})$ is the Lie $S_{\mathcal{F}}(Z)$ -algebra of derivations of $C^\infty(Z)$, regarded as a $S_{\mathcal{F}}(Z)$ -ring. Then one can introduce the leafwise differential calculus [32, 37] as the Chevalley – Eilenberg differential calculus over the $S_{\mathcal{F}}(Z)$ -ring $C^\infty(Z)$. It is defined as a subcomplex

$$0 \rightarrow S_{\mathcal{F}}(Z) \rightarrow C^\infty(Z) \xrightarrow{\tilde{d}} \mathfrak{F}^1(Z) \cdots \xrightarrow{\tilde{d}} \mathfrak{F}^{\dim \mathcal{F}}(Z) \rightarrow 0 \quad (7.41)$$

of the Chevalley – Eilenberg complex of the Lie $S_{\mathcal{F}}(Z)$ -algebra $\mathcal{T}_1(\mathcal{F})$ with coefficients in $C^\infty(Z)$ which consists of $C^\infty(Z)$ -multilinear skew-symmetric maps

$$\times^r \mathcal{T}_1(\mathcal{F}) \rightarrow C^\infty(Z), \quad r = 1, \dots, \dim \mathcal{F}.$$

These maps are global sections of exterior products $\wedge^r T\mathcal{F}^*$ of the dual $T\mathcal{F}^* \rightarrow Z$ of $T\mathcal{F} \rightarrow Z$. They are called the leafwise forms on a foliated manifold (Z, \mathcal{F}) , and are given by the coordinate expression

$$\phi = \frac{1}{r!} \phi_{i_1 \dots i_r} \tilde{d}z^{i_1} \wedge \cdots \wedge \tilde{d}z^{i_r},$$

where $\{\tilde{d}z^i\}$ are the duals of the holonomic fibre bases $\{\partial_i\}$ for $T\mathcal{F}$. Then one can think of the Chevalley – Eilenberg coboundary operator

$$\tilde{d}\phi = \tilde{d}z^k \wedge \partial_k \phi = \frac{1}{r!} \partial_k \phi_{i_1 \dots i_r} \tilde{d}z^k \wedge \tilde{d}z^{i_1} \wedge \cdots \wedge \tilde{d}z^{i_r}$$

as being the leafwise exterior differential. Accordingly, the complex (7.41) is called the leafwise de Rham complex (or the tangential de Rham complex).

Let us consider the exact sequence (7.40) of vector bundles over Z . Since it admits a splitting, the epimorphism $i_{\mathcal{F}}^*$ yields that of the algebra $\mathcal{O}^*(Z)$ of exterior forms on Z to the algebra $\mathfrak{F}^*(Z)$ of leafwise forms. It obeys the condition $i_{\mathcal{F}}^* \circ d = \tilde{d} \circ i_{\mathcal{F}}^*$, and provides the cochain morphism

$$\begin{aligned} i_{\mathcal{F}}^* : (\mathbb{R}, \mathcal{O}^*(Z), d) &\rightarrow (S_{\mathcal{F}}(Z), \mathcal{F}^*(Z), \tilde{d}), \\ dz^\lambda &\rightarrow 0, \quad dz^i \rightarrow \tilde{d}z^i, \end{aligned} \quad (7.42)$$

of the de Rham complex of Z to the leafwise de Rham complex (7.41).

Given a leaf $i_F : F \rightarrow Z$ of \mathcal{F} , we have the pull-back homomorphism

$$(\mathbb{R}, \mathcal{O}^*(Z), d) \rightarrow (\mathbb{R}, \mathcal{O}^*(F), d) \quad (7.43)$$

of the de Rham complex of Z to that of F .

PROPOSITION 7.8: The homomorphism (7.43) factorize through the homomorphism [37]. \square

7.5 Differential geometry of Lie groups

Let G be a real Lie group of $\dim G > 0$, and let $L_g : G \rightarrow gG$ and $R_g : G \rightarrow Gg$ denote the action of G on itself by left and right multiplications, respectively. Clearly, L_g and $R_{g'}$ for all $g, g' \in G$ mutually commute, and so do the tangent maps TL_g and $TR_{g'}$.

A vector field ξ_l (resp. ξ_r) on a group G is said to be left-invariant (resp. right-invariant) if $\xi_l \circ L_g = TL_g \circ \xi_l$ (resp. $\xi_r \circ R_g = TR_g \circ \xi_r$). Left-invariant (resp. right-invariant) vector fields make up the left Lie algebra \mathfrak{g}_l (resp. the right Lie algebra \mathfrak{g}_r) of G .

There is one-to-one correspondence between the left-invariant vector field ξ_l (resp. right-invariant vector fields ξ_r) on G and the vectors $\xi_l(e) = TL_{g^{-1}}\xi_l(g)$ (resp. $\xi_r(e) = TR_{g^{-1}}\xi_r(g)$) of the tangent space T_eG to G at the unit element e of G . This correspondence provides T_eG with the left and the right Lie algebra structures. Accordingly, the left action L_g of a Lie group G on itself defines its adjoint representation

$$\xi_r \rightarrow \text{Ad } g(\xi_r) = TL_g \circ \xi_r \circ L_{g^{-1}} \quad (7.44)$$

in the right Lie algebra \mathfrak{g}_r .

Let $\{\epsilon_m\}$ (resp. $\{\varepsilon_m\}$) denote the basis for the left (resp. right) Lie algebra, and let c_{mn}^k be the right structure constants

$$[\varepsilon_m, \varepsilon_n] = c_{mn}^k \varepsilon_k.$$

There is the morphism

$$\rho : \mathfrak{g}_l \ni \epsilon_m \rightarrow \varepsilon_m = -\epsilon_m \in \mathfrak{g}_r$$

between left and right Lie algebras such that

$$[\epsilon_m, \epsilon_n] = -c_{mn}^k \epsilon_k.$$

The tangent bundle $\pi_G : TG \rightarrow G$ of a Lie group G is trivial. There are the following two canonical isomorphisms

$$\begin{aligned} \varrho_l : TG \ni q &\rightarrow (g = \pi_G(q), TL_g^{-1}(q)) \in G \times \mathfrak{g}_l, \\ \varrho_r : TG \ni q &\rightarrow (g = \pi_G(q), TR_g^{-1}(q)) \in G \times \mathfrak{g}_r. \end{aligned}$$

Therefore, any action

$$G \times Z \ni (g, z) \rightarrow gz \in Z$$

of a Lie group G on a manifold Z on the left yields the homomorphism

$$\mathfrak{g}_r \ni \varepsilon \rightarrow \xi_\varepsilon \in \mathcal{T}_1(Z) \quad (7.45)$$

of the right Lie algebra \mathfrak{g}_r of G into the Lie algebra of vector fields on Z such that

$$\xi_{\text{Ad } g(\varepsilon)} = Tg \circ \xi_\varepsilon \circ g^{-1}. \quad (7.46)$$

Vector fields ξ_ε are said to be the infinitesimal generators of a representation of the Lie group G in Z .

In particular, the adjoint representation (7.44) of a Lie group in its right Lie algebra yields the *adjoint representation*

$$\varepsilon' : \varepsilon \rightarrow \text{ad } \varepsilon'(\varepsilon) = [\varepsilon', \varepsilon], \quad \text{ad } \varepsilon_m(\varepsilon_n) = c_{mn}^k \varepsilon_k,$$

of the right Lie algebra \mathfrak{g}_r in itself.

The dual $\mathfrak{g}^* = T_e^*G$ of the tangent space T_eG is called the Lie coalgebra). It is provided with the basis $\{\varepsilon^m\}$ which is the dual of the basis $\{\varepsilon_m\}$ for T_eG . The group G and the right Lie algebra \mathfrak{g}_r act on \mathfrak{g}^* by the coadjoint representation

$$\begin{aligned} \langle \text{Ad}^*g(\varepsilon^*), \varepsilon \rangle &= \langle \varepsilon^*, \text{Ad } g^{-1}(\varepsilon) \rangle, \quad \varepsilon^* \in \mathfrak{g}^*, \quad \varepsilon \in \mathfrak{g}_r, \\ \langle \text{ad}^*\varepsilon'(\varepsilon^*), \varepsilon \rangle &= -\langle \varepsilon^*, [\varepsilon', \varepsilon] \rangle, \quad \varepsilon' \in \mathfrak{g}_r, \\ \text{ad}^*\varepsilon_m(\varepsilon^n) &= -c_{mk}^n \varepsilon^k. \end{aligned} \quad (7.47)$$

The Lie coalgebra \mathfrak{g}^* of a Lie group G is provided with the canonical Poisson structure, called the Lie – Poisson structure [1, 55]. It is given by the bracket

$$\{f, g\}_{\text{LP}} = \langle \varepsilon^*, [df(\varepsilon^*), dg(\varepsilon^*)] \rangle, \quad f, g \in C^\infty(\mathfrak{g}^*), \quad (7.48)$$

where $df(\varepsilon^*), dg(\varepsilon^*) \in \mathfrak{g}_r$ are seen as linear mappings from $T_{\varepsilon^*}\mathfrak{g}^* = \mathfrak{g}^*$ to \mathbb{R} . Given coordinates z_k on \mathfrak{g}^* with respect to the basis $\{\varepsilon^k\}$, the Lie – Poisson bracket (7.48) and the corresponding Poisson bivector field w read

$$\{f, g\}_{\text{LP}} = c_{mn}^k z_k \partial^m f \partial^n g, \quad w_{mn} = c_{mn}^k z_k.$$

One can show that symplectic leaves of the Lie – Poisson structure on the coalgebra \mathfrak{g}^* of a connected Lie group G are orbits of the coadjoint representation (7.47) of G on \mathfrak{g}^* [88].

7.6 Jet manifolds

This Section collects the relevant material on jet manifolds of sections of fibre bundles [40, 52, 58, 73, 78].

Given a fibre bundle $Y \rightarrow X$ with bundle coordinates (x^λ, y^i) , let us consider the equivalence classes $j_x^1 s$ of its sections s , which are identified by their values

$s^i(x)$ and the values of their partial derivatives $\partial_\mu s^i(x)$ at a point $x \in X$. They are called the first order jets of sections at x . One can justify that the definition of jets is coordinate-independent. A key point is that the set J^1Y of first order jets $j_x^1 s$, $x \in X$, is a smooth manifold with respect to the adapted coordinates $(x^\lambda, y^i, y_\lambda^i)$ such that

$$y_\lambda^i(j_x^1 s) = \partial_\lambda s^i(x), \quad y_\lambda^i = \frac{\partial x^\mu}{\partial x'^\lambda} (\partial_\mu + y_\mu^j \partial_j) y^i. \quad (7.49)$$

It is called the first order jet manifold of a fibre bundle $Y \rightarrow X$.

A jet manifold J^1Y admits the natural fibrations

$$\pi^1 : J^1Y \ni j_x^1 s \rightarrow x \in X, \quad (7.50)$$

$$\pi_0^1 : J^1Y \ni j_x^1 s \rightarrow s(x) \in Y. \quad (7.51)$$

A glance at the transformation law (7.49) shows that π_0^1 is an affine bundle modelled over the vector bundle

$$T^*X \otimes_Y VY \rightarrow Y. \quad (7.52)$$

It is convenient to call π^1 (7.50) the jet bundle, while π_0^1 (7.51) is said to be the affine jet bundle.

Let us note that, if $Y \rightarrow X$ is a vector or an affine bundle, the jet bundle π_1 (7.50) is so.

Jets can be expressed in terms of familiar tangent-valued forms as follows. There are the canonical imbeddings

$$\begin{aligned} \lambda_{(1)} : J^1Y &\xrightarrow{Y} T^*X \otimes_Y TY, \\ \lambda_{(1)} &= dx^\lambda \otimes (\partial_\lambda + y_\lambda^i \partial_i) = dx^\lambda \otimes d_\lambda, \end{aligned} \quad (7.53)$$

$$\begin{aligned} \theta_{(1)} : J^1Y &\xrightarrow{Y} T^*Y \otimes_Y VY, \\ \theta_{(1)} &= (dy^i - y_\lambda^i dx^\lambda) \otimes \partial_i = \theta^i \otimes \partial_i, \end{aligned} \quad (7.54)$$

where d_λ are said to be total derivatives, and θ^i are called contact forms.

We further identify the jet manifold J^1Y with its images under the canonical morphisms (7.53) and (7.54), and represent the jets $j_x^1 s = (x^\lambda, y^i, y_\mu^i)$ by the tangent-valued forms $\lambda_{(1)}$ (7.53) and $\theta_{(1)}$ (7.54).

Sections and morphisms of fibre bundles admit prolongations to jet manifolds as follows.

Any section s of a fibre bundle $Y \rightarrow X$ has the jet prolongation to the section

$$(J^1 s)(x) = j_x^1 s, \quad y_\lambda^i \circ J^1 s = \partial_\lambda s^i(x),$$

of the jet bundle $J^1Y \rightarrow X$. A section of the jet bundle $J^1Y \rightarrow X$ is called integrable if it is the jet prolongation of some section of a fibre bundle $Y \rightarrow X$.

Any bundle morphism $\Phi : Y \rightarrow Y'$ over a diffeomorphism f admits a jet prolongation to a bundle morphism of affine jet bundles

$$J^1\Phi : J^1Y \xrightarrow[\Phi]{} J^1Y', \quad y'^i_\lambda \circ J^1\Phi = \frac{\partial(f^{-1})^\mu}{\partial x'^\lambda} d_\mu \Phi^i.$$

Any projectable vector field u (7.18) on a fibre bundle $Y \rightarrow X$ has a jet prolongation to the projectable vector field

$$J^1u = u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y^i_\mu \partial_\lambda u^\mu) \partial_i^\lambda,$$

on the jet manifold J^1Y .

7.7 Connections on fibre bundles

There are different equivalent definitions of a connection on a fibre bundle $Y \rightarrow X$. We define it both as a splitting of the exact sequence (7.15) and a global section of the affine jet bundle $J^1Y \rightarrow Y$ [40, 58, 73].

A connection on a fibred manifold $Y \rightarrow X$ is defined as a splitting (called the horizontal splitting)

$$\begin{aligned} \Gamma : Y \times_{X} TX &\xrightarrow{Y} TY, & \Gamma : \dot{x}^\lambda \partial_\lambda &\rightarrow \dot{x}^\lambda (\partial_\lambda + \Gamma_\lambda^i(y) \partial_i), & (7.55) \\ \dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i &= \dot{x}^\lambda (\partial_\lambda + \Gamma_\lambda^i \partial_i) + (\dot{y}^i - \dot{x}^\lambda \Gamma_\lambda^i) \partial_i, \end{aligned}$$

of the exact sequence (7.15). Its range is a subbundle of $TY \rightarrow Y$ called the horizontal distribution. By virtue of Theorem 7.4, a connection on a fibred manifold always exists. A connection Γ (7.55) is represented by the horizontal tangent-valued one-form

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^i \partial_i) \quad (7.56)$$

on Y which is projected onto the canonical tangent-valued form θ_X (7.35) on X .

Given a connection Γ on a fibred manifold $Y \rightarrow X$, any vector field τ on a base X gives rise to the projectable vector field

$$\Gamma\tau = \tau \rfloor \Gamma = \tau^\lambda (\partial_\lambda + \Gamma_\lambda^i \partial_i) \quad (7.57)$$

on Y which lives in the horizontal distribution determined by Γ . It is called the horizontal lift of τ by means of a connection Γ .

The splitting (7.55) also is given by the vertical-valued form

$$\Gamma = (dy^i - \Gamma_\lambda^i dx^\lambda) \otimes \partial_i, \quad (7.58)$$

which yields an epimorphism $TY \rightarrow VY$.

In an equivalent way, connections on a fibred manifold $Y \rightarrow X$ are introduced as global sections of the affine jet bundle $J^1Y \rightarrow Y$. Indeed, any global section Γ of $J^1Y \rightarrow Y$ defines the tangent-valued form $\lambda_1 \circ \Gamma$ (7.56). It follows from this

definition that connections on a fibred manifold $Y \rightarrow X$ constitute an affine space modelled over the vector space of soldering forms σ (7.37). One also deduces from (7.49) the coordinate transformation law of connections

$$\Gamma_\lambda'^i = \frac{\partial x^\mu}{\partial x'^\lambda} (\partial_\mu + \Gamma_\mu^j \partial_j) y'^i.$$

Remark 7.5: Any connection Γ on a fibred manifold $Y \rightarrow X$ yields a horizontal lift of a vector field on X onto Y , but need not defines the similar lift of a path in X into Y . Let

$$\mathbb{R} \supset [\cdot] \ni t \rightarrow x(t) \in X, \quad \mathbb{R} \ni t \rightarrow y(t) \in Y,$$

be smooth paths in X and Y , respectively. Then $t \rightarrow y(t)$ is called a horizontal lift of $x(t)$ if

$$\pi(y(t)) = x(t), \quad \dot{y}(t) \in H_{y(t)}Y, \quad t \in \mathbb{R},$$

where $HY \subset TY$ is the horizontal subbundle associated to the connection Γ . If, for each path $x(t)$ ($t_0 \leq t \leq t_1$) and for any $y_0 \in \pi^{-1}(x(t_0))$, there exists a horizontal lift $y(t)$ ($t_0 \leq t \leq t_1$) such that $y(t_0) = y_0$, then Γ is called the Ehresmann connection. A fibred manifold is a fibre bundle iff it admits an Ehresmann connection [44]. \square

Hereafter, we restrict our consideration to connections on fibre bundles. The following are two standard constructions of new connections from old ones.

- Let Y and Y' be fibre bundles over the same base X . Given connections Γ on Y and Γ' on Y' , the bundle product $Y \times_X Y'$ is provided with the product connection

$$\Gamma \times \Gamma' = dx^\lambda \otimes \left(\partial_\lambda + \Gamma_\lambda^i \frac{\partial}{\partial y^i} + \Gamma'^j_\lambda \frac{\partial}{\partial y'^j} \right). \quad (7.59)$$

- Given a fibre bundle $Y \rightarrow X$, let $f : X' \rightarrow X$ be a manifold morphism and f^*Y the pull-back of Y over X' . Any connection Γ (7.58) on $Y \rightarrow X$ yields the pull-back connection

$$f^*\Gamma = \left(dy^i - \Gamma_\lambda^i (f^\mu(x'^\nu), y^j) \frac{\partial f^\lambda}{\partial x'^\mu} dx'^\mu \right) \otimes \partial_i \quad (7.60)$$

on the pull-back bundle $f^*Y \rightarrow X'$.

Every connection Γ on a fibre bundle $Y \rightarrow X$ defines the first order differential operator

$$D^\Gamma : J^1Y \xrightarrow{Y} T^*X \otimes_Y VY, \quad (7.61)$$

$$D^\Gamma = \lambda_1 - \Gamma \circ \pi_0^1 = (y_\lambda^i - \Gamma_\lambda^i) dx^\lambda \otimes \partial_i,$$

on Y called the covariant differential. If $s : X \rightarrow Y$ is a section, its covariant differential

$$\nabla^\Gamma s = D^\Gamma \circ J^1 s = (\partial_\lambda s^i - \Gamma_\lambda^i \circ s) dx^\lambda \otimes \partial_i$$

and its covariant derivative $\nabla_\tau^\Gamma s = \tau \rfloor \nabla^\Gamma s$ along a vector field τ on X are introduced. In particular, a (local) section s of $Y \rightarrow X$ is called an integral section for a connection Γ (or parallel with respect to Γ) if s obeys the equivalent conditions

$$\nabla^\Gamma s = 0 \quad \text{or} \quad J^1 s = \Gamma \circ s.$$

Let Γ be a connection on a fibre bundle $Y \rightarrow X$. Given vector fields τ, τ' on X and their horizontal lifts $\Gamma\tau$ and $\Gamma\tau'$ (7.57) on Y , let us consider the vertical vector field

$$\begin{aligned} R(\tau, \tau') &= \Gamma[\tau, \tau'] - [\Gamma\tau, \Gamma\tau'] = \tau^\lambda \tau'^\mu R_{\lambda\mu}^i \partial_i, \\ R_{\lambda\mu}^i &= \partial_\lambda \Gamma_\mu^i - \partial_\mu \Gamma_\lambda^i + \Gamma_\lambda^j \partial_j \Gamma_\mu^i - \Gamma_\mu^j \partial_j \Gamma_\lambda^i. \end{aligned}$$

It can be seen as the contraction of vector fields τ and τ' with the vertical-valued horizontal two-form

$$R = \frac{1}{2}[\Gamma, \Gamma]_{\text{FN}} = \frac{1}{2}R_{\lambda\mu}^i dx^\lambda \wedge dx^\mu \otimes \partial_i \quad (7.62)$$

on Y called the curvature form of a connection Γ .

A flat (or curvature-free) connection is a connection Γ on a fibre bundle $Y \rightarrow X$ which satisfies the following equivalent conditions:

- its curvature vanishes everywhere on Y ;
- its horizontal distribution is involutive;
- there exists a local integral section for the connection Γ through any point $y \in Y$.

By virtue of Theorem 7.7, a flat connection Γ yields a foliation of Y which is transversal to the fibration $Y \rightarrow X$. It called a horizontal foliation. Its leaf through a point $y \in Y$ is locally defined by an integral section s_y for the connection Γ through y . Conversely, let a fibre bundle $Y \rightarrow X$ admit a horizontal foliation such that, for each point $y \in Y$, the leaf of this foliation through y is locally defined by a section s_y of $Y \rightarrow X$ through y . Then the map

$$\Gamma : Y \ni y \rightarrow j_{\pi(y)}^1 s_y \in J^1 Y$$

sets a flat connection on $Y \rightarrow X$. Hence, there is one-to-one correspondence between the flat connections and the horizontal foliations of a fibre bundle $Y \rightarrow X$.

Given a horizontal foliation of a fibre bundle $Y \rightarrow X$, there exists the associated atlas of bundle coordinates (x^λ, y^i) on Y such that every leaf of this foliation is locally given by the equations $y^i = \text{const.}$, and the transition functions $y^i \rightarrow y'^i(y^j)$ are independent of the base coordinates x^λ [40]. It is called the atlas of constant local trivializations. Two such atlases are said to be equivalent if their union also is an atlas of the same type. They are associated to the same horizontal foliation. Thus, the following is proved.

THEOREM 7.9: There is one-to-one correspondence between the flat connections Γ on a fibre bundle $Y \rightarrow X$ and the equivalence classes of atlases of constant

local trivializations of Y such that $\Gamma = dx^\lambda \otimes \partial_\lambda$ relative to the corresponding atlas. \square

Example 7.6: Any trivial bundle has flat connections corresponding to its trivializations. Fibre bundles over a one-dimensional base have only flat connections. \square

Let $Y \rightarrow X$ be a vector bundle equipped with linear bundle coordinates (x^λ, y^i) . It admits a linear connection

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^i{}_j(x) y^j \partial_i). \quad (7.63)$$

There are the following standard constructions of new linear connections from old ones.

• Any linear connection Γ (7.63) on a vector bundle $Y \rightarrow X$ defines the dual linear connection

$$\Gamma^* = dx^\lambda \otimes (\partial_\lambda - \Gamma_\lambda^j{}_i(x) y_j \partial^i) \quad (7.64)$$

on the dual bundle $Y^* \rightarrow X$.

• Let Γ and Γ' be linear connections on vector bundles $Y \rightarrow X$ and $Y' \rightarrow X$, respectively. The direct sum connection $\Gamma \oplus \Gamma'$ on the Whitney sum $Y \oplus Y'$ of these vector bundles is defined as the product connection (7.59).

• Similarly, the tensor product $Y \otimes Y'$ of vector bundles possesses the tensor product connection

$$\Gamma \otimes \Gamma' = dx^\lambda \otimes \left[\partial_\lambda + (\Gamma_\lambda^i{}_j y^{ja} + \Gamma'_\lambda{}^a{}_b y^{ib}) \frac{\partial}{\partial y^{ia}} \right]. \quad (7.65)$$

The curvature of a linear connection Γ (7.63) on a vector bundle $Y \rightarrow X$ is usually written as a Y -valued two-form

$$R = \frac{1}{2} R_{\lambda\mu}{}^i{}_j(x) y^j dx^\lambda \wedge dx^\mu \otimes e_i, \quad (7.66)$$

$$R_{\lambda\mu}{}^i{}_j = \partial_\lambda \Gamma_\mu{}^i{}_j - \partial_\mu \Gamma_\lambda{}^i{}_j + \Gamma_\lambda{}^h{}_j \Gamma_\mu{}^i{}_h - \Gamma_\mu{}^h{}_j \Gamma_\lambda{}^i{}_h,$$

due to the canonical vertical splitting $VY = Y \times Y$, where $\{\partial_i\} = \{e_i\}$. For any two vector fields τ and τ' on X , this curvature yields the zero order differential operator

$$R(\tau, \tau')s = ([\nabla_\tau^\Gamma, \nabla_{\tau'}^\Gamma] - \nabla_{[\tau, \tau']}^\Gamma)s \quad (7.67)$$

on section s of a vector bundle $Y \rightarrow X$.

Let us consider the composite bundle $Y \rightarrow \Sigma \rightarrow X$ (7.7), coordinated by $(x^\lambda, \sigma^m, y^i)$. Let us consider the jet manifolds $J^1\Sigma$, $J_\Sigma^1 Y$, and $J^1 Y$ of the fibre bundles $\Sigma \rightarrow X$, $Y \rightarrow \Sigma$ and $Y \rightarrow X$, respectively. They are parameterized respectively by the coordinates

$$(x^\lambda, \sigma^m, \sigma_\lambda^m), \quad (x^\lambda, \sigma^m, y^i, \tilde{y}_\lambda^i, y_m^i), \quad (x^\lambda, \sigma^m, y^i, \sigma_\lambda^m, y_\lambda^i).$$

There is the canonical map

$$\varrho : J^1\Sigma \times_{\Sigma} J_\Sigma^1 Y \longrightarrow J^1 Y, \quad y_\lambda^i \circ \varrho = y_m^i \sigma_\lambda^m + \tilde{y}_\lambda^i. \quad (7.68)$$

Using the canonical map (7.68), we can consider the relations between connections on fibre bundles $Y \rightarrow X$, $Y \rightarrow \Sigma$ and $\Sigma \rightarrow X$ [58, 78].

Connections on fibre bundles $Y \rightarrow X$, $Y \rightarrow \Sigma$ and $\Sigma \rightarrow X$ read

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma_\lambda^m \partial_m + \gamma_\lambda^i \partial_i), \quad (7.69)$$

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + A_\lambda^i \partial_i) + d\sigma^m \otimes (\partial_m + A_m^i \partial_i), \quad (7.70)$$

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^m \partial_m).$$

The canonical map ϱ (7.68) enables us to obtain a connection γ on $Y \rightarrow X$ in accordance with the diagram

$$\begin{array}{ccc}
 J^1\Sigma \times_{\Sigma} J^1_Y Y & \xrightarrow{\varrho} & J^1 Y \\
 (\Gamma, A) \quad \uparrow & & \uparrow \gamma \\
 \Sigma \times_X Y & \longleftarrow & Y
 \end{array}$$

This connection, called the composite connection, reads

$$\gamma = dx^\lambda \otimes [\partial_\lambda + \Gamma_\lambda^m \partial_m + (A_\lambda^i + A_m^i \Gamma_\lambda^m) \partial_i]. \quad (7.71)$$

It is a unique connection such that the horizontal lift $\gamma\tau$ on Y of a vector field τ on X by means of the connection γ (7.71) coincides with the composition $A_\Sigma(\Gamma\tau)$ of horizontal lifts of τ onto Σ by means of the connection Γ and then onto Y by means of the connection A_Σ . For the sake of brevity, let us write $\gamma = A_\Sigma \circ \Gamma$.

Given the composite bundle Y (7.7), there are the exact sequences

$$0 \rightarrow V_\Sigma Y \rightarrow VY \rightarrow Y \times_{\Sigma} V\Sigma \rightarrow 0, \quad (7.72)$$

$$0 \rightarrow Y \times_{\Sigma} V^*\Sigma \rightarrow V^*Y \rightarrow V^*_\Sigma Y \rightarrow 0, \quad (7.73)$$

where $V_\Sigma Y$ denotes the vertical tangent bundle of a fibre bundle $Y \rightarrow \Sigma$ coordinated by $(x^\lambda, \sigma^m, y^i, \dot{y}^i)$. Let us consider the splitting

$$\begin{aligned}
 B : VY \ni v = \dot{y}^i \partial_i + \dot{\sigma}^m \partial_m \rightarrow v] B = & \quad (7.74) \\
 (\dot{y}^i - \dot{\sigma}^m B_m^i) \partial_i \in V_\Sigma Y, \\
 B = (\bar{d}y^i - B_m^i \bar{d}\sigma^m) \otimes \partial_i \in V^*Y \otimes_Y V_\Sigma Y,
 \end{aligned}$$

of the exact sequence (7.72). Then the connection γ (7.69) on $Y \rightarrow X$ and the splitting B (7.74) define a connection

$$\begin{aligned}
 A_\Sigma = B \circ \gamma : TY \rightarrow VY \rightarrow V_\Sigma Y, \\
 A_\Sigma = dx^\lambda \otimes (\partial_\lambda + (\gamma_\lambda^i - B_m^i \gamma_\lambda^m) \partial_i) + & \quad (7.75) \\
 d\sigma^m \otimes (\partial_m + B_m^i \partial_i),
 \end{aligned}$$

on the fibre bundle $Y \rightarrow \Sigma$.

Conversely, every connection A_Σ (7.70) on a fibre bundle $Y \rightarrow \Sigma$ provides the splittings

$$VY = V_\Sigma Y \oplus_Y A_\Sigma(Y \times V\Sigma), \quad (7.76)$$

$$\begin{aligned} \dot{y}^i \partial_i + \dot{\sigma}^m \partial_m &= (\dot{y}^i - A_m^i \dot{\sigma}^m) \partial_i + \dot{\sigma}^m (\partial_m + A_m^i \partial_i), \\ V^*Y &= (Y \times_\Sigma V^*\Sigma) \oplus_Y A_\Sigma(V_\Sigma^*Y), \end{aligned} \quad (7.77)$$

$$\dot{y}_i \bar{d}y^i + \dot{\sigma}_m \bar{d}\sigma^m = \dot{y}_i (\bar{d}y^i - A_m^i \bar{d}\sigma^m) + (\dot{\sigma}_m + A_m^i \dot{y}_i) \bar{d}\sigma^m,$$

of the exact sequences (7.72) – (7.73). Using the splitting (7.76), one can construct the first order differential operator

$$\tilde{D} : J^1Y \rightarrow T^*X \otimes_Y V_\Sigma Y, \quad \tilde{D} = dx^\lambda \otimes (y_\lambda^i - A_\lambda^i - A_m^i \sigma_\lambda^m) \partial_i, \quad (7.78)$$

called the vertical covariant differential, on the composite fibre bundle $Y \rightarrow X$.

The vertical covariant differential (7.78) possesses the following important property. Let h be a section of a fibre bundle $\Sigma \rightarrow X$, and let $Y_h \rightarrow X$ be the restriction of a fibre bundle $Y \rightarrow \Sigma$ to $h(X) \subset \Sigma$. This is a subbundle $i_h : Y_h \rightarrow Y$ of a fibre bundle $Y \rightarrow X$. Every connection A_Σ (7.70) induces the pull-back connection (7.60):

$$A_h = i_h^* A_\Sigma = dx^\lambda \otimes [\partial_\lambda + ((A_m^i \circ h) \partial_\lambda h^m + (A \circ h)_\lambda^i) \partial_i] \quad (7.79)$$

on $Y_h \rightarrow X$. Then the restriction of the vertical covariant differential \tilde{D} (7.78) to $J^1 i_h(J^1 Y_h) \subset J^1 Y$ coincides with the familiar covariant differential D^{A_h} (7.61) on Y_h relative to the pull-back connection A_h (7.79).

7.8 Differential operators and connections on modules

This Section addresses the notion of a linear differential operator on a module over an arbitrary commutative ring [53, 58, 77, 75].

Let \mathcal{K} be a commutative ring and \mathcal{A} a commutative \mathcal{K} -ring. Let P and Q be \mathcal{A} -modules. The \mathcal{K} -module $\text{Hom}_{\mathcal{K}}(P, Q)$ of \mathcal{K} -module homomorphisms $\Phi : P \rightarrow Q$ can be endowed with the two different \mathcal{A} -module structures

$$(a\Phi)(p) = a\Phi(p), \quad (\Phi \bullet a)(p) = \Phi(ap), \quad a \in \mathcal{A}, \quad p \in P. \quad (7.80)$$

For the sake of convenience, we refer to the second one as an \mathcal{A}^\bullet -module structure. Let us put

$$\delta_a \Phi = a\Phi - \Phi \bullet a, \quad a \in \mathcal{A}.$$

DEFINITION 7.10: An element $\Delta \in \text{Hom}_{\mathcal{K}}(P, Q)$ is called a Q -valued differential operator of order s on P if

$$\delta_{a_0} \circ \dots \circ \delta_{a_s} \Delta = 0$$

for any tuple of $s + 1$ elements a_0, \dots, a_s of \mathcal{A} . The set $\text{Diff}_s(P, Q)$ of these operators inherits the \mathcal{A} - and \mathcal{A}^\bullet -module structures (7.80). \square

For instance, zero order differential operators obey the condition

$$\delta_a \Delta(p) = a\Delta(p) - \Delta(ap) = 0, \quad a \in \mathcal{A}, \quad p \in P,$$

and, consequently, they coincide with \mathcal{A} -module morphisms $P \rightarrow Q$. A first order differential operator Δ satisfies the condition

$$\delta_b \circ \delta_a \Delta(p) = ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0, \quad a, b \in \mathcal{A}.$$

DEFINITION 7.11: A connection on an \mathcal{A} -module P is an \mathcal{A} -module morphism

$$\mathfrak{d}\mathcal{A} \ni u \rightarrow \nabla_u \in \text{Diff}_1(P, P) \quad (7.81)$$

such that the first order differential operators ∇_u obey the Leibniz rule

$$\nabla_u(ap) = u(a)p + a\nabla_u(p), \quad a \in \mathcal{A}, \quad p \in P.$$

\square

Though ∇_u (7.81) is called a connection, it in fact is a covariant differential on a module P .

For instance, let $Y \rightarrow X$ be a smooth vector bundle. Its global sections form a $C^\infty(X)$ -module $Y(X)$. The well-known Serre – Swan theorem [40] states the categorial equivalence between the vector bundles over a smooth manifold X and projective modules of finite rank over the ring $C^\infty(X)$ of smooth real functions on X . A corollary of this equivalence is that the derivation module of the real ring $C^\infty(X)$ coincides with the $C^\infty(X)$ -module $\mathcal{T}(X)$ of vector fields on X . If P is a $C^\infty(X)$ -module, one can reformulate Definition 7.11 of a connection on P as follows.

DEFINITION 7.12: A connection on a $C^\infty(X)$ -module P is a $C^\infty(X)$ -module morphism

$$\nabla : P \rightarrow \mathcal{O}^1(X) \otimes P,$$

which satisfies the Leibniz rule

$$\nabla(fp) = df \otimes p + f\nabla(p), \quad f \in C^\infty(X), \quad p \in P.$$

It associates to any vector field $\tau \in \mathcal{T}(X)$ on X a first order differential operator ∇_τ on P which obeys the Leibniz rule

$$\nabla_\tau(fp) = (\tau]df)p + f\nabla_\tau p.$$

\square

In particular, let $Y \rightarrow X$ be a vector bundle and $Y(X)$ its structure module. The notion of a connection on the structure module $Y(X)$ is equivalent to the standard geometric notion of a connection on a vector bundle $Y \rightarrow X$ [58].

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