

# LieART – A Mathematica Application for Lie Algebras and Representation Theory

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## Abstract

We present the Mathematica application “LieART” (Lie Algebras and Representation Theory) for computations frequently encountered in Lie algebras and representation theory, such as tensor product decomposition and subalgebra branching of irreducible representations. LieART can handle all classical and exceptional Lie algebras. It computes root systems of Lie algebras, weight systems and several other properties of irreducible representations. LieART’s user interface has been created with a strong focus on usability and thus allows the input of irreducible representations via their dimensional name, while the output is in the textbook style used in most particle-physics publications. The unique Dynkin labels of irreducible representations are used internally and can also be used for input and output. LieART exploits the Weyl reflection group for most of the calculations, resulting in fast computations and a low memory consumption. Extensive tables of properties, tensor products and branching rules of irreducible representations are included in the appendix.

*Keywords:* Lie algebra; Lie group; representation theory; irreducible representation; tensor product; branching rule; GUT; model building

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## Program Summary

*Authors:* Robert Feger, Thomas W. Kephart

*Program Title:* LieART

*Licensing provisions:* GNU Lesser General Public License (LGPL)

*Programming language:* Mathematica

*Computer:* x86, x86\_64, PowerPC

*Operating system:* cross-platform

*RAM:*  $\geq 1$  GB recommended. Memory usage depends strongly on the Lie algebra's rank and type, as well as the dimensionality of the representations in the computation.

*Keywords:* Lie algebra; Lie group; representation theory; irreducible representation; tensor product; branching rule; GUT; model building

*Classification:* 4.2, 11.1

*External routines/libraries:* Wolfram Mathematica 8–10

*Nature of problem:* The use of Lie algebras and their representations is widespread in physics, especially in particle physics. The description of nature in terms of gauge theories requires the assignment of fields to representations of compact Lie groups and their Lie algebras. Mass and interaction terms in the Lagrangian give rise to the need for computing tensor products of representations of Lie algebras. The mechanism of spontaneous symmetry breaking leads to the application of subalgebra decomposition. This computer code was designed for the purpose of Grand Unified Theory (GUT) Model building, where compact Lie groups beyond the U(1), SU(2) and SU(3) of the Standard Model of particle physics are needed. Tensor product decomposition and subalgebra decomposition have been implemented for all classical Lie groups SU( $N$ ), SO( $N$ ) and Sp( $2N$ ) and the exceptionals E<sub>6</sub>, E<sub>7</sub>, E<sub>8</sub>, F<sub>4</sub> and G<sub>2</sub>.

*Solution method:* LieART generates the weight system of an irreducible representation (irrep) of a Lie algebra by exploiting the Weyl reflection groups, which is inherent in all simple Lie algebras. Tensor products are computed by the application of Klimyk's formula, except for SU( $N$ )'s, where the Young-tableaux algorithm is used. Subalgebra decomposition of SU( $N$ )'s are performed by projection matrices, which are generated from an algorithm to determine maximal subalgebras as originally developed by Dynkin [1, 2].

*Restrictions:* Internally irreps are represented by their unique Dynkin label. LieART's default behavior in TraditionalForm is to print the dimensional name, which is the labeling preferred by physicist. Most Lie algebras can have more than one irrep of the same dimension and different irreps with the same dimension are usually distinguished by one or more primes (e.g. **175** and **175'** of A<sub>4</sub>). To determine the need for one or more primes of an irrep a brute-force loop over other irreps must be performed to search for irreps with the same dimensionality. Since Lie algebras have an infinite number of irreps, this loop must be cut off, which is done by limiting the maximum Dynkin digit in the loop. In rare cases for irreps of high dimensionality in high-rank algebras, if the cutoff used is too low, then the assignment of primes will be incorrect, but the problem can be avoided by raising the cutoff. However, in either case, this can only affect the display of the irrep because all computations involving this irrep are correct, since the internal unique representation of Dynkin labels is used.

*Running time:* From less than a second to hours depending on the Lie algebra's rank and type and/or the dimensionality of the representations in the computation.

## 1. Introduction

Lie groups are a key ingredient in modern physics, while smaller Lie groups like  $SU(2)$  and  $SO(3,1)$ , enter the quantum mechanics of elementary chemistry and condensed matter physics, the full spectrum of Lie groups, i.e., the classical groups  $SU(N)$ ,  $SO(N)$  and  $Sp(2N)$  and the exceptionals  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$  and  $G_2$ , have all appeared with varying degrees of frequency in particle physics. Lie groups have many other application e.g., to the theoretical physics of gravity, string theory, etc. as well as applications to engineering and elsewhere. Here we will focus on the Lie algebras of the compact forms of Lie groups that are most useful for particle physics. Most of the results are easily extended to the non-compact forms.

Shortly after the Standard Model was completed, Grand Unified Theories (GUTs) were proposed, where the Standard-Model gauge group  $SU(3)_C \times SU(2)_L \times U(1)_Y$  is embedded in a higher symmetry, typically  $SU(5)$  [3],  $SO(10)$  [4, 5] or  $E_6$  [6], although other choices have been tried. Major reviews appeared on the uses of Lie algebras [7, 8], including tables of irreducible representations (irreps) and their invariants. There are also a number of useful textbooks that cover the topic [9, 10, 11]. While extensive tables already exist for building GUT models, it has sometimes been necessary to go beyond what is tabulated in the literature. Our purpose here is to give extended tables that will satisfy most modern model-building requirements, but also provide the software that allows one to go further as the situation may require. In describing the software we will incorporate a review of most of the necessary group-theory background. This includes root and weight systems, the associated Weyl groups for all the classical and exceptional Lie algebras, orthogonal basis systems, and Weyl-group orbits, which are used in our method of calculating tensor products and irrep decompositions.

The theory of Lie algebras is in a mature state and many algorithms have been established to facilitate computations in representation theory. The correspondence of irreps to Young tableaux, especially for  $SU(N)$ 's, with algorithms for decomposing tensor products and subalgebra decomposition, even allows complex calculations involving high-dimensional irreps by hand. Lie-algebra related computations have been implemented multiple times on the computer in many different programming languages. Popular programs with a similar aim as the software presented here are [12, 13, 14]. However, at the time we started the project no such implementation existed for the computer-algebra system Mathematica. (Meanwhile a package for computations in finite-dimensional and affine Lie algebras has been published [15] that has a similar intention as our software, as well as a package for the calculation of the 2-loop renormalization-group equations of supersymmetric models based on gauge groups incorporating many Lie algebra related computations [16]). Mathematica<sup>®</sup> is a computer algebra software by Wolfram Research, Inc. which is widely used especially among particle physicists.

Originally intended as an in-house solution for a computerized grand-unified-model scan of  $SU(N)$ 's in Mathematica [17], we present here the Mathematica application LieART (Lie Algebras and Representation Theory), that makes tensor products and subalgebra branching of irreps of the classical and exceptional Lie algebras available for this platform. LieART's code exploits the Weyl reflection group, inherent in all simple Lie algebras, in many parts of the algorithms, which makes computations fast and at the same time economical on memory. We also focused on the usability of LieART with a particle physicist as user in mind: Irreps can be entered by their dimensional name, a nomenclature that physicists prefer over the more unique Dynkin label. LieART displays results in textbook style used in most particle-physics publications, e.g.,  $\overline{\mathbf{10}}$  for the conjugated 10-dimensional irrep of  $SU(5)$  instead of the corresponding Dynkin label (0010). The Dynkin label is used internally, but can also be used as input and output. LieART can also display results in terms of L<sup>A</sup>T<sub>E</sub>X commands, that are defined in a supplemental L<sup>A</sup>T<sub>E</sub>X style file for the inclusion of results in publications.

The paper is organized as follows: In Section 2 we give instructions for downloading and installing LieART, as well as locating its documentation integrated in Mathematica's help system. Section 3 comprises a quick-start tutorial for LieART, introducing the most important functions for the most common tasks in an example-based fashion. Section 4 presents a self-contained overview of the Lie algebra theory used in LieART and gives notes on its implementation. Section 5 gives benchmarks for a few tensor-product decompositions and a subalgebra decomposition of a large irrep. In Section 6 we present a L<sup>A</sup>T<sub>E</sub>X style file included in LieART for displaying weights, roots and irreps properly. In Section 7 we conclude and give an outlook on future versions. In the appendix we include an extensive collection of tables with properties of irreps, tensor

products and branching rules. These tables follow [7] in selection and presentation style, but extend most of the results. We plan to maintain and further extend our tables, which can be used directly as lookup tables without the aid of LieART.

## 2. Download and Installation

### 2.1. Download

LieART is hosted by Hepforge, IPPP Durham. The LieART project home page is

<http://lieart.hepforge.org/>

and the LieART Mathematica application can be downloaded as tar.gz archive from

<http://www.hepforge.org/downloads/lieart/>

### 2.2. Automatic Installation

Start Mathematica and in the front end select the menu entry

File → Install...

In the appearing dialog select `Application` as `Type of Item to Install` and the `tar.gz` file in the open file dialog from `Source`. (It is not necessary to decompress the `tar.gz` archive since Mathematica does this automatically.) Choose whether you want to install LieART for an individual user or system wide. For a system-wide installation you might be asked for the superuser password.

### 2.3. Manual Installation

The above procedure in Mathematica 7 only allows you to automatically install the Mathematica package file (`LieART.m`) of LieART without the documentation. We therefore suggest a manual installation of the LieART application in Mathematica 7 and in Mathematica 8 through 10 if problems with the automatic installation occur.

Extract the archive to the subdirectory `AddOns/Applications` of the directory to which `$UserBaseDirectory` is set for a user-only installation. For a system-wide installation place it in the according subdirectory of `$InstallationDirectory`. Restart Mathematica to allow it to integrate LieART's documentation in its help system.

### 2.4. Documentation

The documentation of LieART is integrated in Mathematica's help system. After restarting Mathematica the following path should lead to LieART's documentation:

Help → Documentation Center  
→ Add-Ons & Packages (at the bottom)  
→ LieART, Button labeled "Documentation"

(Alternatively, a search for "LieART" (with the correct case) in the Documentation Center leads to the same page.) The displayed page serves as the documentation home of LieART and includes links to the descriptions of its most important functions.

The documentation of LieART includes a `Quick Start Tutorial` for the impatient, which can be found near the bottom of LieART's documentation home under the section `Tutorials`.

Tables of representation properties, tensor products and branching rules generated by LieART can be found in the section `Tables` at the bottom of LieART's documentation home.

### 2.5. L<sup>A</sup>T<sub>E</sub>X Package

LieART comes with a L<sup>A</sup>T<sub>E</sub>X package that defines commands to display irreps, roots and weights properly. The style file `lieart.sty` can be found in the subdirectory `latex/` of the LieART project tree. Please copy it to a location where your L<sup>A</sup>T<sub>E</sub>X installation can find it.

### 3. Quick Start

This section provides a tutorial introducing the most important and frequently used functions of LieART for Lie-algebra and representation-theory related calculations. The functions are introduced based on simple examples that can easily be modified and extended to the user's desired application. Most examples use irreducible representations (irreps) of  $SU(5)$ , which most textbooks use in examples since it is less trivial than  $SU(3)$ , but small enough to return results almost instantly on any recent computer. Also,  $SU(5)$  frequently appears in unified model building since the Standard-Model gauge group is one of its maximal subgroups. This tutorial can also be found in the LieART documentation integrated into the Mathematica Documentation Center as "Quick Start Tutorial" under the section "Tutorials" on the LieART documentation home.

This loads the package:

```
In[1]:= « LieART`
```

#### 3.1. Entering Irreducible Representations

Irreps are internally described by their Dynkin label with a combined head of `Irrep` and the Lie algebra.

---

`Irrep[algebraClass][label]` irrep described by its *algebraClass* and Dynkin *label*.

Entering irreps by Dynkin label.

The *algebraClass* follows the Dynkin classification of simple Lie algebras and can only be A, B, C, D for the classical algebras and E6, E7, E8, F4 and G2 for the exceptional algebras. The precise classical algebra is determined by the length of the Dynkin label.

Entering the  $\overline{10}$  of  $SU(5)$  by its Dynkin label and algebra class:

```
In[2]:= Irrep[A][0,0,1,0]//FullForm
```

```
Out[2]:= Irrep[A][0,0,1,0]
```

In `StandardForm` the irrep is displayed in the textbook notation of Dynkin labels:

```
In[3]:= Irrep[A][0,0,1,0]//StandardForm
```

```
Out[3]:= (0010)
```

In `TraditionalForm` (default) the irrep is displayed by its dimensional name:

```
In[4]:= Irrep[A][0,0,1,0]
```

```
Out[4]:=  $\overline{10}$ 
```

The default output format type of LieART is `TraditionalForm`. The associated user setting is overwritten for the notebook LieART that is loaded in. For `StandardForm` as output format type please set the global variable `$DefaultOutputForm=StandardForm`.

As an example for entering an irrep of an exceptional algebra, consider the **27** of  $E_6$ :

```
In[5]:= Irrep[E6][1,0,0,0,0,0]
```

```
Out[5]:= 27
```

Irreps may also be entered by their dimensional name. The package transforms the irrep into its Dynkin label. Since the algebra of an irrep of a classical Lie algebra becomes ambiguous with only the dimensional name, it has to be specified.

---

`Irrep[algebra][dimname]` irrep entered by its *algebra* and dimensional name *dimname*.

Entering irreps by dimensional name.

Entering the  $\overline{\mathbf{10}}$  of  $SU(5)$  by its dimensional name specifying the algebra by its Dynkin classification  $A_4$ :

```
In[6]:= Irrep[A4][Bar[10]]//InputForm
Out[6]:= Irrep[A][0,0,1,0]
```

The traditional name of the algebra  $SU(5)$  may also be used:

```
In[7]:= Irrep[SU5][Bar[10]]//InputForm
Out[7]:= Irrep[A][0,0,1,0]
```

Irreps of product algebras like  $SU(3)\otimes SU(2)\otimes U(1)$  are specified by `ProductIrrep` with the individual irreps of simple Lie algebras as arguments.

---

`ProductIrrep[irreps]` head of product *irreps*, gathering irreps of simple Lie algebras.

Product irreps.

The product irrep  $(\mathbf{3}, \overline{\mathbf{3}})$  of  $SU(3)\otimes SU(3)$ :

```
In[8]:= ProductIrrep[Irrep[SU3][3], Irrep[SU3][Bar[3]]]
Out[8]:= (3, 3)
```

```
In[9]:= %//InputForm
Out[9]:= ProductIrrep[Irrep[A][1,0], Irrep[A][0,1]]
```

```
In[10]:= ProductIrrep[Irrep[A][1,0], Irrep[A][0,1]]
Out[10]:= (3, 3)
```

Take for example the left-handed quark doublet in the Standard-Model gauge group  $SU(3)\otimes SU(2)\otimes U(1)$  (The  $U(1)$  charge is not typeset in bold face):

```
In[11]:= ProductIrrep[Irrep[SU3][3], Irrep[SU2][2], Irrep[U][1/3]]
Out[11]:= (3, 2)(1/3)
```

```
In[12]:= %//InputForm
Out[12]:= ProductIrrep[Irrep[A][1,0], Irrep[A][1], Irrep[U][1/3]]
```

### 3.2. Decomposing Tensor Products

---

`DecomposeProduct[irreps]` decomposes the tensor product of several *irreps*.

Tensor product decomposition.

Decompose the tensor product  $\mathbf{3}\otimes\overline{\mathbf{3}}$  of  $SU(3)$ :

```
In[13]:= DecomposeProduct[Irrep[SU3][3], Irrep[SU3][Bar[3]]]
Out[13]:= 1 + 8
```

Decompose the tensor product  $\mathbf{27}\otimes\overline{\mathbf{27}}$  of  $E_6$ :

```
In[14]:= DecomposeProduct[Irrep[E6][27], Irrep[E6][Bar[27]]]
Out[14]:= 1 + 78 + 650
```

Decompose the tensor product  $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$  of  $SU(3)$ :

```
In[15]:= DecomposeProduct[Irrep[SU3][3], Irrep[SU3][3], Irrep[SU3][3]]
Out[15]:= 1 + 2(8) + 10
```

Decompose the tensor product  $8 \otimes 8$  of  $SU(3)$ :

```
In[16]:= DecomposeProduct[Irrep[SU3][8], Irrep[SU3][8]]
Out[16]:= 1 + 2(8) + 10 +  $\overline{10}$  + 27
```

Internally a sum of irreps is represented by `IrrepPlus` and `IrrepTimes`, an analog of the built-in functions `Plus` and `Times`:

```
In[17]:= %//InputForm
Out[17]:= IrrepPlus[Irrep[A][0,0], IrrepTimes[2, Irrep[A][1,1]],
  Irrep[A][3,0], Irrep[A][0,3], Irrep[A][2,2]]
```

Results can be transformed into a list of irreps with `IrrepList`, suitable for further processing with Mathematica built-in functions like `Select` or `Cases`:

```
In[18]:= %//IrrepList
Out[18]:= {1, 8, 8, 10,  $\overline{10}$ , 27}
```

Decompose the tensor product  $4 \otimes 4 \otimes 6 \otimes 15$  of  $SU(4)$ :

```
In[19]:= DecomposeProduct[Irrep[SU4][4], Irrep[SU4][4], Irrep[SU4][6], Irrep[SU4][15]]
Out[19]:= 2(1) + 7(15) + 4(20') + 35 + 5(45) + 3( $\overline{45}$ ) + 3(84) + 2(175) + 256
```

The Mathematica built-in command `Times` for products is replaced by `DecomposeProduct` for irreps as arguments. E.g., decompose the tensor product  $\overline{10} \otimes 24 \otimes 45$  of  $SU(5)$ :

```
In[20]:= Irrep[SU5][Bar[10]]*Irrep[SU5][24]*Irrep[SU5][45]
Out[20]:= 3(5)+6( $\overline{45}$ )+3( $\overline{50}$ )+5( $\overline{70}$ )+2( $\overline{105}$ )+ $\overline{175''}$ +6( $\overline{280}$ )+2( $\overline{280'}$ )+ $\overline{420}$ + $\overline{450'}$ +3( $\overline{480}$ )+2( $\overline{720}$ )+ $\overline{1120}$ + $\overline{2520}$ 
```

For powers of irreps the Mathematica built-in command `Power` may be used. E.g., decompose the tensor product  $27 \otimes 27 \otimes 27$  of  $E_6$ :

```
In[21]:= Irrep[E6][27]^3
Out[21]:= 1 + 2(78) + 3(650) + 2925 + 3003 + 2(5824)
```

Decompose tensor products of product irreps  $(3, \overline{3}, 1) \otimes (\overline{3}, 3, 1)$  of  $SU(3) \otimes SU(3) \otimes SU(3)$ :

```
In[22]:= DecomposeProduct[
  ProductIrrep[Irrep[SU3][3], Irrep[SU3][Bar[3]], Irrep[SU3][1]],
  ProductIrrep[Irrep[SU3][Bar[3]], Irrep[SU3][3], Irrep[SU3][1]]]
Out[22]:= (1, 1, 1) + (8, 1, 1) + (1, 8, 1) + (8, 8, 1)
```

Decompose the tensor products  $(3, 2) \otimes (\overline{3}, 1)$  of  $SU(3) \otimes SU(2)$ :

```
In[23]:= DecomposeProduct[
  ProductIrrep[Irrep[SU3][3], Irrep[SU2][2]],
  ProductIrrep[Irrep[SU3][Bar[3]], Irrep[SU2][1]]]
Out[23]:= (1, 2) + (8, 2)
```

### 3.3. Decomposition to Subalgebras

<code>DecomposeIrrep[irrep, subalgebra]</code>	decomposes <i>irrep</i> to the specified <i>subalgebra</i> .
<code>DecomposeIrrep[pirrep, subalgebra, pos]</code>	decomposes the product irrep <i>pirrep</i> at position <i>pos</i> .

Decompose irreps and product irreps.

Decompose the  $\overline{10}$  of  $SU(5)$  to  $SU(3) \otimes SU(2) \otimes U(1)$ :

```
In[24]:= DecomposeIrrep[Irrep[SU5][Bar[10]], ProductAlgebra[SU3, SU2, U1]]
Out[24]:= (1, 1)(6) + (3, 1)(-4) + ( $\overline{3}$ , 2)(1)
```



Decompose the **10** and the  $\bar{\mathbf{5}}$  of  $SU(5)$  to  $SU(3)\otimes SU(2)\otimes U(1)$  (DecomposeIrrep is Listable):

```
In[25]:= DecomposeIrrep[{Irrep[SU5][10], Irrep[SU5][Bar[5]]}, ProductAlgebra[SU3, SU2, U1]]
Out[25]:= {{(3,1)(4) + (3,2)(-1) + (1,1)(-6), (3,1)(-2) + (1,2)(3)}
```

Decompose the **16** of  $SO(10)$  to  $SU(5)\otimes U(1)$ :

```
In[26]:= DecomposeIrrep[Irrep[SO10][16], ProductAlgebra[SU5, U1]]
Out[26]:= (1)(-5) + (5)(3) + (10)(-1)
```

Decompose the **27** of  $E_6$  to  $SU(3)\otimes SU(3)\otimes SU(3)$ :

```
In[27]:= DecomposeIrrep[Irrep[E6][27], ProductAlgebra[SU3, SU3, SU3]]
Out[27]:= (3,1,3) + (1,3,3) + (3,3,1)
```

Decompose the  $SU(3)$  irrep **3** in  $(\mathbf{24}, \mathbf{3})(-3)$  of  $SU(5)\otimes SU(3)\otimes U(1)$  to  $SU(2)\otimes U'(1)$ , i.e.,  $SU(5)\otimes SU(3)\otimes U(1) \rightarrow SU(5)\otimes SU(2)\otimes U'(1)\otimes U(1)$ :

```
In[28]:= DecomposeIrrep[ProductIrrep[Irrep[SU5][24], Irrep[SU3][3], Irrep[U1][-3]],
ProductAlgebra[SU2, U1], 2]
Out[28]:= (24,1)(-2)(-3) + (24,2)(1)(-3)
```

The same decomposition as above displayed as branching rule:

```
In[29]:= IrrepRule[#, DecomposeIrrep[#, ProductAlgebra[SU2, U1], 2]] &@
ProductIrrep[Irrep[SU5][24], Irrep[SU3][3], Irrep[U1][-3]]
Out[29]:= (24,3)(-3) → (24,1)(-2)(-3) + (24,2)(1)(-3)
```

Branching rules for all totally antisymmetric irreps, so-called basic irreps, of  $SU(6)$  to  $SU(3)\otimes SU(3)\otimes U(1)$ :

```
In[30]:= IrrepRule[#, DecomposeIrrep[#, ProductAlgebra[SU3, SU3, U1]]] &/@
BasicIrreps[SU6]//TableForm
Out[30]:=
6 → (3,1)(1) + (1,3)(-1)
15 → (3,1)(2) + (1,3)(-2) + (3,3)(0)
20 → (1,1)(3) + (1,1)(-3) + (3,3)(-1) + (3,3)(1)
15 → (3,1)(-2) + (1,3)(2) + (3,3)(0)
6 → (3,1)(-1) + (1,3)(1)
```

### 3.4. Young Tableaux

The irreps of  $SU(N)$  have a correspondence to Young tableaux, which can be displayed by YoungTableau.

YoungTableau[irrep] Displays the Young tableau associated with an  $SU(N)$  irrep.

Young tableaux.

Young tableau of the **720** of  $SU(5)$ :

```
In[31]:= YoungTableau[Irrep[A][1,2,0,1]]
Out[31]:=


|  |  |  |  |  |
|--|--|--|--|--|
|  |  |  |  |  |
|  |  |  |  |  |
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```

Display Young tableaux of  $SU(4)$  irreps with a maximum of one column per box count:

```
In[32]:= Row[Row[{"#", ":", " ", YoungTableau[#]}] &/@
SortBy[Irrep[A]@@@Tuples[{0,1}, 3], Dim], Spacer[10]]
Out[32]:= 1: • 4: 

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```

## 4. Theoretical Background and Implementation

In this section we give a self-contained overview of the Lie algebra theory used and implemented in LieART. It is subdivided into parts discussing basic properties of Lie algebras, roots, weights, Weyl orbits, representations and decompositions. Every subsection begins with a list of the relevant LieART functions followed by text that introduces the necessary theory with reference to the functions and notes on their implementation. This section is not intended as a pedagogical introduction to Lie algebras and we refer the reader to the excellent literature serving this purpose [7, 9, 11].

### 4.1. Algebras

<code>Rank[<i>expr</i>]</code>	gives the rank of the algebra of <i>expr</i> , which can be an irrep, a weight a root or an algebra itself.
<code>Algebra[<i>algebraClass</i>][<i>rank</i>]</code>	represents a classical algebra of the type <i>algebraClass</i> , which can only be A, B, C or D, with rank <i>rank</i> .
<code>Algebra[<i>expr</i>]</code>	gives the algebra (classical or exceptional) of <i>expr</i> , which may be an irrep, a weight or a root in any basis.
<code>OrthogonalSimpleRoots[<i>algebra</i>]</code>	gives the simple roots of <i>algebra</i> in the orthogonal basis.
<code>CartanMatrix[<i>algebra</i>]</code>	gives the Cartan matrix of <i>algebra</i> .
<code>OmegaMatrix[<i>algebra</i>]</code>	gives the matrix of fundamental weights of <i>algebra</i> as rows.
<code>OrthogonalFundamentalWeights[<i>algebra</i>]</code>	gives the fundamental weights of <i>algebra</i> in the orthogonal basis.
<code>OrthogonalBasis[<i>expr</i>]</code>	transforms <i>expr</i> from any basis into the orthogonal basis.
<code>OmegaBasis[<i>expr</i>]</code>	transforms <i>expr</i> from any basis into the $\omega$ -basis.
<code>AlphaBasis[<i>expr</i>]</code>	transforms <i>expr</i> from any basis into the $\alpha$ -basis.
<code>DMatrix[<i>algebra</i>]</code>	gives a matrix with inverse length factors of simple roots on the main diagonal.
<code>ScalarProduct[<i>weight1</i>, <i>weight2</i>]</code>	gives the scalar product of <i>expr1</i> and <i>expr2</i> in any basis. <i>expr1</i> and <i>expr2</i> may be weights or roots.
<code>MetricTensor[<i>algebra</i>]</code>	gives the metric tensor or quadratic-form matrix of <i>algebra</i> .

Basic Algebra Properties.

#### 4.1.1. Definition

A *Lie Algebra* is a vector space  $g$  over a field  $F$  with the *Lie bracket*  $[\cdot, \cdot]$  as binary operation, which is bilinear, alternating and fulfills the Jacoby identity. The Lie bracket is often referred to as the commutator. The Lie brackets of the generators  $t_i$  of the Lie algebra are

$$[t_i, t_j] = f_{ijk} t_k \quad (1)$$

with the so-called *structure constants*  $f_{ijk}$ , that fully determine the algebra. A Lie algebra is called *simple* when it contains no non-trivial ideals. A *semi-simple* Lie algebra is a sum of simple ones.

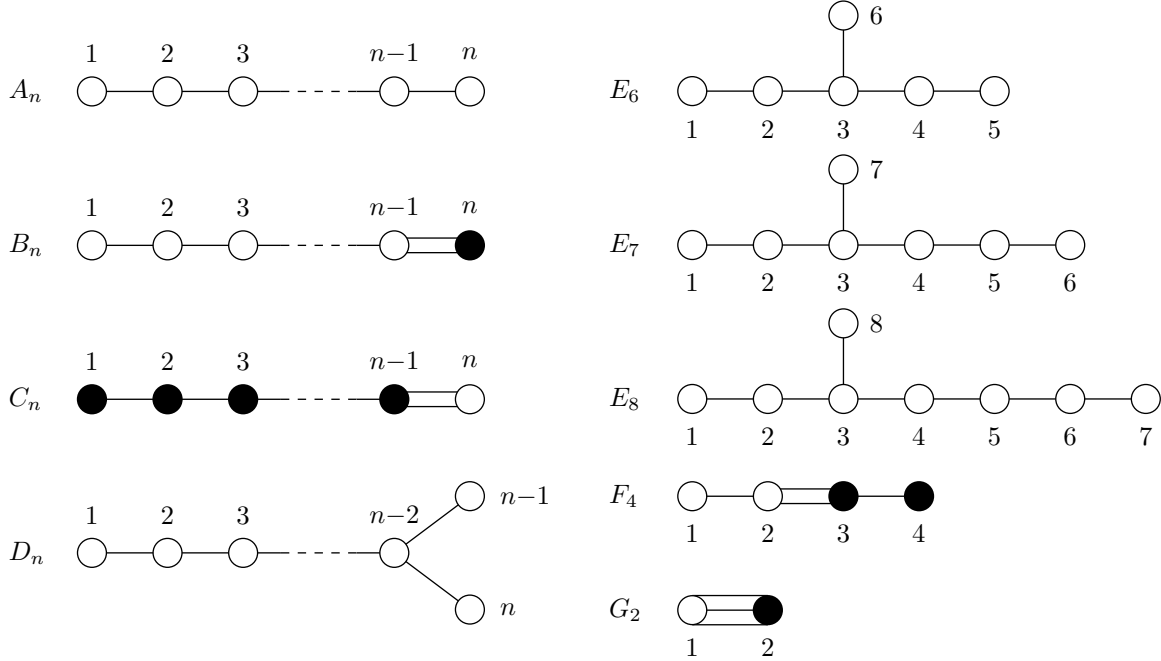


Figure 1: Dynkin Diagrams of classical and exceptional simple Lie algebras.

#### 4.1.2. Roots

The generators  $t_i$  of a simple Lie algebra in the Cartan-Weyl basis fall into two sets: The so-called *Cartan subalgebra*,  $H$ , contains all simultaneously diagonalizable generators  $h_i$ , i.e., the generators are Hermitian and mutually commute (the Cartan subalgebra is abelian):

$$h_i = h_i^\dagger, \quad [h_i, h_j] = 0, \quad i, j = 1, \dots, n. \quad (2)$$

The number of simultaneously diagonalizable generators  $n$  is called the *rank* of the algebra, and can be determined by the function `Rank[expr]` in LieART. We denote all other generators as  $e_\alpha$ . They satisfy  $n$  eigenvalue equations with the generators of the Cartan subalgebra  $h_i$ :

$$[h_i, e_\alpha] = \alpha_i e_\alpha, \quad i = 1, \dots, n, \quad (3)$$

which is a subset of (1) and thus the  $\alpha_i$  are structure constants, which are real numbers due to the hermiticity of the  $h_i$ 's. Since the  $\alpha_i$  are the solutions to the eigenvalue equation (3) the vectors  $\alpha = (\alpha_1, \dots, \alpha_n)$  are called the *root vectors*, which lie in an  $n$ -dimensional euclidian space, called the *root space*. *Roots* are functionals mapping the Cartan subalgebra  $H$  onto the real numbers (the eigenvalues), for all generators  $t_i$ , which also includes the  $h_i$  where the eigenvalues are zero. Thus, a Lie algebra has as many roots as generators. The roots are labeled by the root vectors, which we will use in its place from now on.

A zero root with an  $n$ -fold degeneracy is associated with the Cartan subalgebra. In the Cartan-Weyl basis the other generators come in conjugated pairs  $e_\alpha^\dagger = e_{-\alpha}$  and correspond to the ladder operators of  $SU(2)$ . So-called *positive roots* correspond to the raising operator  $e_\alpha$  and negative roots to the lowering operators  $e_{-\alpha}$ . If  $\alpha$  is a root so is  $-\alpha$ .

Some of the positive roots can be written as sum of others. Those for which this is not possible are called *simple roots* and a Lie algebra has as many simple roots as its rank. It is clear that specifying the simple roots fully determines a Lie algebra and thus can be used to replace (1), because all structure constants can be derived therefrom.

Type	Cartan	Name	Rank	Description
classical	$A_n$	$SU(n+1)$	$n \geq 1$	Special unitary algebras of $n+1$ complex dimension
	$B_n$	$SO(2n+1)$	$n \geq 3$	Special orthogonal algebras of odd $(2n+1)$ real dimension
	$C_n$	$Sp(2n)$	$n \geq 2$	Symplectic algebras of even $(2n)$ complex dimension
	$D_n$	$SO(2n)$	$n \geq 4$	Special orthogonal algebras of even $(2n)$ real dimension
exceptional	$E_6$	$E_6$	6	Exceptional algebra of rank 6
	$E_7$	$E_7$	7	Exceptional algebra of rank 7
	$E_8$	$E_8$	8	Exceptional algebra of rank 8
	$F_4$	$F_4$	4	Exceptional algebra of rank 4
	$G_2$	$G_2$	2	Exceptional algebra of rank 2

Table 4.1: Classification of simple Lie algebras.

#### 4.1.3. Classification of Lie Algebras

Using the commutation relations and the Jacobi identity to analyze the generators, constraints on the roots can be derived and eventually all possible root systems found, which is identical to identifying all allowed Lie algebras. It turns out that simple roots can only come in at most two lengths in one Lie algebra and at four different angles between any pair of them. The simple roots are in particular not orthogonal. The so-called *Dynkin diagrams* are an ingenious way to depict these relations: simple roots are represented by dots, which are open,  $\circ$ , for the longer roots or for all roots if they only come in one length, and filled,  $\bullet$ , for the shorter roots. Angles between two simple roots are represented by lines connecting the dots: no line for an angle of  $90^\circ$ , one line for  $120^\circ$ , two lines for  $135^\circ$  and three for  $150^\circ$ . Figure 1 shows the Dynkin diagrams for all simple Lie algebras. Semi-simple Lie algebras have disjoint parts and can thus be reduced to two or more Dynkin diagrams of simple Lie algebras.

The simple Lie algebras fall into two types: four families of infinite series algebras,  $A_n$ ,  $B_n$ ,  $C_n$  and  $D_n$ , also called the *classical Lie algebras* and five so-called *exceptional algebras*,  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$  and  $G_2$ , with their rank as subscript (see Table 4.1). The labels are according to the classification by Cartan. The classical Lie algebras are internally represented (i.e., in `FullForm`) in LieART by `Algebra[algebraClass][n]`, with *algebraClass* being either A, B, C or D and  $n$  being the rank. The exceptional algebras are defined in LieART as `E6`, `E7`, `E8`, `F4` and `G2` and short forms of the classical algebras are predefined up to rank 30 for the Cartan classification, i.e., `A1`, ..., `A30`, `B3`, ..., `B30`, `C2`, ..., `C30`, `D4`, ..., `D30`, and up to dimension 30 for the conventional name, i.e., `SU2`, ..., `SU30`, `SO7`, ..., `SO30` and `Sp4`, `Sp6`, ..., `Sp30`. In `StandardForm` the Cartan classification is explicitly displayed and in `TraditionalForm` the Lie algebra is written by its conventional name.

Note the isomorphisms for low-dimension algebras:

$$SU(2) \sim SO(3) \sim Sp(2) \qquad (A_1 \sim B_1 \sim C_1), \qquad (4a)$$

$$SO(4) \sim SU(2) \otimes SU(2) \qquad (D_2 \sim A_1 \otimes A_1), \qquad (4b)$$

$$SO(5) \sim Sp(4) \qquad (B_2 \sim C_2), \qquad (4c)$$

$$SO(6) \sim SU(4) \qquad (D_3 \sim A_3). \qquad (4d)$$

When these low-dimension Lie algebras occur in a calculation, choose the  $SU(2)$  form in (4a) and (4b), the  $Sp(4)$  form in (4c) and the  $SU(4)$  form in (4d) when using LieART.

#### 4.1.4. Bases

With respect to the Weyl reflection group, inherent in all compact Lie algebras, as we will explain later, it is convenient to express the root space in an orthogonal coordinate system, which is a subspace of Euclidian space. The specific subspace varies with the Lie algebra. For  $A_n$  it is a subspace of  $\mathbb{R}^{n+1}$ , where the coordinates sum to one. As the simple roots define the Lie algebra, they are explicitly specified in LieART

using orthogonal coordinates and can be retrieved by `OrthogonalSimpleRoots[algebra]`. E.g., the four simple roots of  $A_4$  (SU(5)) in orthogonal coordinates are:

```
In[33]:= OrthogonalSimpleRoots[A4]//Column
(1, -1, 0, 0, 0)
(0, 1, -1, 0, 0)
Out[33]:= (0, 0, 1, -1, 0)
(0, 0, 0, 1, -1)
```

The so-called *Cartan matrix* exhibits the non-orthogonality of the simple roots. It is defined as

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \quad i, j = 1, \dots, n \quad (5)$$

where the scalar product  $(\cdot, \cdot)$  is the ordinary scalar product of  $\mathbb{R}^{n+1}$  in the case of  $A_n$ . Most textbooks translate the Dynkin diagrams to the corresponding Cartan matrix as a starting point. And in fact, the rows of the Cartan matrix are the simple roots in the so-called  $\omega$ -basis, which is the bases of *fundamental weights*, also called the *Dynkin basis*. (Weights will be introduced later in the context of representations.) The Cartan matrix is implemented in LieART as the function `CartanMatrix[algebra]` following the definition of (5). The Cartan matrix for  $A_4$  reads:

```
In[34]:= CartanMatrix[A4]
Out[34]:= ( 2  -1  0  0 )
( -1  2  -1  0 )
( 0  -1  2  -1 )
( 0  0  -1  2 )
```

Besides the orthogonal basis, and the  $\omega$ -basis, the  $\alpha$ -basis is also useful. As the name indicates it is the basis of simple roots and it explicitly shows how, e.g., a root is composed out of simple roots. Neither the  $\omega$ -basis nor the  $\alpha$ -basis is orthogonal. The Cartan matrix mediates between the  $\omega$ - and  $\alpha$ -bases:

$$\alpha_i = \sum_{j=1}^n A_{ij} \omega_j, \quad \omega_i = \sum_{j=1}^n (A^{-1})_{ij} \alpha_j. \quad (6)$$

where the  $\omega_i$  are the fundamental weights, which we will define later. These bases are dual to each other in the sense that

$$\frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} \equiv (\alpha_i^\vee, \omega_j) = \delta_{ij}, \quad i, j = 1, \dots, n \quad (7)$$

where  $\alpha_i^\vee$  is the so-called *coroot* of  $\alpha_i$  defined as

$$\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}. \quad (8)$$

The transformation to the orthogonal basis can be derived from (7): Expressing  $\alpha_i$  and  $\omega_j$  in orthogonal coordinates as  $\hat{\alpha}_i$  and  $\hat{\omega}_j$  (7) reads

$$\frac{2\hat{\alpha}_i \cdot \hat{\omega}_j}{\hat{\alpha}_i \cdot \hat{\alpha}_i} \equiv \hat{\alpha}_i^\vee \cdot \hat{\omega}_j = \delta_{ij}, \quad i, j = 1, \dots, n \quad (9)$$

using the ordinary scalar product of  $\mathbb{R}^m$ , where  $m$  is the dimension of the orthogonal subspace. Using the matrices  $\hat{A}$  and  $\hat{\Omega}$  with the simple coroots  $\hat{\alpha}_i^\vee$  and the fundamental weights  $\hat{\omega}_j$  as rows, we can write (9) as the matrix equation:

$$\hat{A}\hat{\Omega}^T = I_n \quad (10)$$

where both  $\hat{A}$  and  $\hat{\Omega}$  are  $n \times m$  matrices, where  $\hat{\Omega}$  is defined below in 12. Please note that the dimension of the orthogonal space  $m$  is not necessarily the same as the rank of the algebra  $n$ . These exceptions are:  $A_n$  with  $m=n+1$ ,  $E_6$  with  $m=8$ ,  $E_7$  with  $m=8$  and  $G_2$  with  $m=3$ . For all others  $m=n$  holds. The matrix of the simple coroots in the orthogonal basis  $\hat{A}$  is easily calculated from the simple roots given in LieART, but the

matrix of fundamental weights in the orthogonal basis  $\hat{\Omega}$  must be determined by (10). In the cases where  $\hat{A}$  is not a square matrix its inverse does not exist. Because the rows of  $\hat{A}$ , which are the simple coroots in the orthogonal basis, are linear independent,  $\hat{A}\hat{A}^T$  is invertible and the so-called right-inverse  $\hat{A}^+$  can be found via

$$\hat{A}^+ = \hat{A}^T(\hat{A}\hat{A}^T)^{-1} \quad (11)$$

which satisfies:  $\hat{A}\hat{A}^+ = I_n$ , i.e., by comparing with (10) the matrix  $\hat{\Omega}^T$  can be identified with  $\hat{A}^+$ , in other words the fundamental weights as rows of  $\hat{\Omega}$  in terms of simple coroots as rows of  $\hat{A}$  are

$$\hat{\Omega} = (\hat{A}^+)^T = (\hat{A}\hat{A}^T)^{-1}\hat{A} \quad (12)$$

The Mathematica built-in function `PseudoInverse[matrix]` yields the right-inverse for our case of a *matrix* with linear independent rows, i.e., the implementation of the second equality in (12) is not needed. The matrix of the fundamental weights  $\hat{\Omega}$  is implemented as `OmegaMatrix[algebra]`, e.g., for  $A_4$ :

```
In[35]:= OmegaMatrix[A4]
Out[35]:= 
$$\begin{pmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ \frac{3}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & -\frac{4}{5} \end{pmatrix}$$

```

and the function `OrthogonalFundamentalWeights[algebra]` adds the proper heads to the rows of  $\hat{\Omega}$ , to identify them as weights in the orthogonal basis. We will discuss (fundamental) weights in Section 4.1.6 in more detail.

The matrix of the fundamental weights in the orthogonal basis  $\hat{\Omega}$  mediates between the  $\omega$ -basis and the orthogonal basis:

$$\omega_i = \sum_{j=1}^n \hat{\Omega}_{ij} e_j, \quad e_i = \sum_{j=1}^n (\hat{\Omega}^{-1})_{ij} \omega_j. \quad (13)$$

The LieART functions `AlphaBasis[weightOrRoot]`, `OmegaBasis[weightOrRoot]` and `OrthogonalBasis[weightOrRoot]` transform *weightOrRoot* from any basis into the  $\alpha$ -basis, the  $\omega$ -basis and the orthogonal basis, respectively. It is obvious how the simple roots in the  $\alpha$ -basis look:

```
In[36]:= AlphaBasis[OrthogonalSimpleRoots[A4]]//Column
Out[36]:= 
$$\begin{pmatrix} 1, 0, 0, 0 \\ 0, 1, 0, 0 \\ 0, 0, 1, 0 \\ 0, 0, 0, 1 \end{pmatrix}$$

```

and likewise the fundamental weights in the  $\omega$ -basis:

```
In[37]:= OmegaBasis[OrthogonalFundamentalWeights[A4]]//Column
Out[37]:= 
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```

(Roots and weights in the  $\omega$ -basis are displayed with framed boxes following the notation of most textbooks.) A root in LieART is represented by three different heads: `RootOrthogonal[algebraClass][label]` for a root in the orthogonal basis, `RootOmega[algebraClass][label]` in the  $\omega$ -basis and in the  $\alpha$ -basis by `RootAlpha[algebraClass][label]`. The *algebraClass* can only be A, B, C or D to indicate a classical Lie algebra or E6, E7, E8, F4 or G2 for the exceptionals. The *label* stands for the comma-separated coordinates. This form of the roots is displayed in `InputForm` and `FullForm`. E.g., the first simple root of  $A_4$  in all three bases reads:

```
In[38]:= {#, OmegaBasis[#], AlphaBasis[#]} &@First[OrthogonalSimpleRoots[A4]]//InputForm
Out[38]:= {RootOrthogonal[A][1, -1, 0, 0, 0], RootOmega[A][2, -1, 0, 0], RootAlpha[A][1, 0, 0, 0]}
```

#### 4.1.5. Scalar Product

The standard choice for the length factors  $(\alpha_j, \alpha_j)$  in (5) is 2 for the longer roots, if there are two root lengths. The factors  $2/(\alpha_j, \alpha_j)$  can only take three different values which are: 1 for all roots of  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$ ,  $E_8$  and for the long roots of  $B_n$ ,  $C_n$ ,  $F_4$  and  $G_2$ ; 2 for the short roots of  $B_n$ ,  $C_n$  and  $F_4$  and 3 for the short root of  $G_2$ . Their implementation in LieART is in the form of diagonal matrices with the inverse factors for the simple roots corresponding to the row on the main diagonal, i.e.,

$$D = \text{diag} \left( \frac{1}{2} (\alpha_1, \alpha_1), \dots, \frac{1}{2} (\alpha_n, \alpha_n) \right) \quad (14)$$

as defined in [18]. E.g., for  $F_4$ , to avoid a trivial example, we have:

```
In [39] := DMatrix[F4]
Out [39] :=  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$ 
```

In the  $\omega$ -basis the scalar product used in (5) becomes:

$$(x, y) = \sum_{i,j} x_i (A^{-1})_{ij} D_j y_j = \sum_{i,j} x_i G_{ij} y_j \quad (15)$$

where the  $x_i$  and  $y_j$  are coordinates of  $x$  and  $y$  in the  $\omega$ -basis. The matrix

$$G_{ij} = (A^{-1})_{ij} \frac{(\alpha_j, \alpha_j)}{2} = (A^{-1})_{ij} D_j \quad (16)$$

is called *quadratic-form matrix* or *metric tensor* of the Lie algebra. The scalar product is available in LieART as `ScalarProduct[weightOrRoot1, weightOrRoot2]`, where *weightOrRoot1* and *weightOrRoot2* may be roots or weights in the orthogonal basis, the  $\alpha$ -basis or the  $\omega$ -basis. The function recognizes the basis by the heads of *weightOrRoot1* and *weightOrRoot2*. The LieART function for the metric tensor  $G$  is `MetricTensor[algebra]`, e.g., for  $A_4$ :

```
In [40] := MetricTensor[A4]
Out [40] :=  $\begin{pmatrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$ 
```

#### 4.1.6. Representation

A *representation* is a linear map of the Lie algebra into the general linear group, i.e., the matrix group, that preserves the Lie bracket relations. It is a homomorphism that maps the generators  $t_i$  onto invertible matrices  $T_i$ , that satisfy the same “commutation” relations as the Lie algebra, namely

$$[T_i, T_j] = f_{ijk} T_k, \quad (17)$$

where the  $[\cdot, \cdot]$  is now the commutator.

Points in the vector space that the matrices act on can be labeled by the set of eigenvalues of the matrices representing the generators of the Cartan subalgebra. Such a set of eigenvalues is called a *weight vector*, and the associated functional *weight*, denoted by  $\lambda$ . They are defined in root space which is called *weight space* in this context. The weights and weight vectors of a representation correspond to roots and root vectors of the algebra. In fact, weights can be expressed as rational linear combinations of roots, and, as pointed out in this section, eventually by simple roots. In particular, the structure functions themselves form a representation of the algebra: the *adjoint representation*, which has the same dimension as the algebra, namely the number of roots.

## 4.2. Weyl Group Orbits

<code>Reflect[weightOrRoot, simpleroots]</code>	reflects <i>weightOrRoot</i> at the hyperplanes orthogonal to the specified <i>simpleroots</i> .
<code>Reflect[weightOrRoot]</code>	reflects <i>weightOrRoot</i> at the hyperplanes orthogonal to all simple roots of the Lie algebra of <i>weightOrRoot</i> .
<code>ReflectionMatrices[algebra]</code>	gives the reflection matrices of the Weyl group of <i>algebra</i> .
<code>Orbit[weightOrRoot, simpleroots]</code>	generates the Weyl group orbit of <i>weightOrRoot</i> using only the specified <i>simpleroots</i> .
<code>Orbit[weightOrRoot]</code>	generates the full Weyl group orbit of <i>weightOrRoot</i> using all simple roots of the Lie algebra of <i>weightOrRoot</i> .
<code>DimOrbit[weightOrRoot, simpleroots]</code>	gives the size of the orbit of <i>weightOrRoot</i> using only the <i>simpleroots</i> .
<code>DimOrbit[weightOrRoot]</code>	gives the size of the orbit of <i>weightOrRoot</i> using all simple roots of the Lie algebra of <i>weightOrRoot</i> .

### Weyl Group Orbits

The finite group  $W(L)$ , called the Weyl group of the Lie algebra  $L$ , is a reflection group inherent in the root systems of all simple Lie algebras. The Coxeter groups are an abstraction of reflection groups and the so-called *Coxeter-Dynkin diagram* describing Coxeter groups are closely related to the Dynkin diagrams presented here. In fact the Coxeter-Dynkin diagram corresponding to the Dynkin diagram describes the Weyl group of the Lie algebra.

The transformations  $r_i$  generating the Weyl group are reflections of a vector  $x$  in root space at the hyperplanes orthogonal to the simple roots  $\alpha_i$  of the Lie algebra defined by

$$r_i x = x - \frac{2(x, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i, \quad i = 1, \dots, n, \quad x \in \mathbb{R}^n. \quad (18)$$

The LieART function `Reflect[weightOrRoot, simpleroots]` implements the reflections  $r_i$  with *weightOrRoot* as  $x$  and *simpleroots* as a list of simple roots  $\alpha_i$ . The result is a list of weights, because the reflection is performed with several roots simultaneously.

If *weightOrRoots* are in the orthogonal basis and ought to be reflected using all roots, the function pattern is `Reflect[weightOrRoot]`, without the simple roots as second argument. Instead of the definition with scalar products following (18), the implementation multiplies the orthogonal coordinates with precomputed reflection matrices, which have a simple form in the orthogonal basis. The function computing the reflection matrices is `ReflectionMatrices[algebra]` and simply applies the built-in Mathematica command `ReflectionMatrix` to all simple roots and saves the result as `DownValues` of `ReflectionMatrices[algebra]`. E.g., the reflection matrices for  $A_4$  (in the 5-dimensional orthogonal basis) are:

```
In[41]:= Row[MatrixForm /@ ReflectionMatrices[A4]]
Out[41]:= 
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

```

The Weyl group of  $A_n$  is particularly simple in the orthogonal basis: It is the symmetric group  $S_{n+1}$ . The reflection matrices for  $A_4$  above represent the generators of  $S_5$ , i.e., the coordinate permutations (12), (23), (34) and (45), respectively.

Acting on a vector  $x$  in root space by all elements of the Weyl group gives a set of points, of which some may coincide. The subset of distinct points is called the *orbit* of  $x$  and denoted as  $O(x)$ . The LieART function `Orbit[weightOrRoot, simpleroots]` gives the orbit of *weightOrRoot* using the *simpleroots*. If the second



argument is omitted, all simple roots of the algebra associated with *weightOrRoot* are used. The function applies `Reflect` in a nested fashion and removes duplicate points in every step. The orbit of an  $A_n$  root or weight is constructed in a special way for performance reasons: The *weightOrRoot* is transformed to the orthogonal basis and the other points of its orbit are constructed by permuting its coordinates using the built-in Mathematica function `Permutations`. For example, the orbit of the first simple root of  $A_4$  is

```
In[42]:= Orbit[First[OrthogonalSimpleRoots[A4]]]
Out[42]:= {(-1, 0, 0, 0, 1), (-1, 0, 0, 1, 0), (-1, 0, 1, 0, 0), (-1, 1, 0, 0, 0), (0, -1, 0, 0, 1),
(0, -1, 0, 1, 0), (0, -1, 1, 0, 0), (0, 0, -1, 0, 1), (0, 0, -1, 1, 0), (0, 0, 0, -1, 1),
(0, 0, 0, 1, -1), (0, 0, 1, -1, 0), (0, 0, 1, 0, -1), (0, 1, -1, 0, 0), (0, 1, 0, -1, 0),
(0, 1, 0, 0, -1), (1, -1, 0, 0, 0), (1, 0, -1, 0, 0), (1, 0, 0, -1, 0), (1, 0, 0, 0, -1)}
```

which is in fact the  $A_4$  root system without the zero roots.

With the same set of Weyl group generators, defined by the roots used, every vector is uniquely associated with only one orbit. In turn every element of an orbit allows us to generate the entire orbit by reflecting at the hyperplanes defined by the roots. The hyperplanes divide the space into so-called *Weyl chambers*. An orbit has no more than one distinct element in every chamber and the Weyl group permutes the chambers. The so-called *dominant chamber* has elements with only positive coordinates in the  $\omega$ -basis, which serves as a definite element for the orbits associated with them. The test function `DominantQ[weightOrRoot]` gives `True` if *weightOrRoot* is in the dominant chamber and `False` otherwise. The dominant root of `Out[42]` in the  $\omega$ -basis is

```
In[43]:= OmegaBasis[Select[%, DominantQ]]
Out[43]:= {1 0 0 1}
```

If an orbit is created by `LieART` it is saved as a `DownValue` of `Orbit` associated with its dominant root or weight. Whenever an orbit of a non-dominant weight or root is needed, `LieART` first seeks the `DownValues` of `Orbit` for the weight or root, to see if the orbit has already been generated. Reusing computed orbits saves CPU time especially for Lie algebras other than  $A_n$  and the described procedure avoids saving the same orbit multiple times as `DownValue` involving different roots or weights.

The size of the orbit, i.e., its numbers of elements, denoted by  $|O(x)|$ , is implemented as the function `DimOrbit[weightOrRoot, simpleroots]` or `DimOrbit[weightOrRoot]` if all simple roots of the associated Lie algebras should be used. The size of the orbit in `Out[42]` is

```
In[44]:= DimOrbit[First[OrthogonalSimpleRoots[A4]]]
Out[44]:= 20
```

### 4.3. Roots

<code>RootSystem[algebra]</code>	root system of <i>algebra</i>
<code>ZeroRoots[algebra]</code>	zero roots associated with the Cartan subalgebra of <i>algebra</i>
<code>Height[root]</code>	height of a <i>root</i> within the root system
<code>HighestRoot[algebra]</code>	highest root of the root system of <i>algebra</i>
<code>PositiveRootQ[root]</code>	gives <code>True</code> if <i>root</i> is a positive root and <code>False</code> otherwise
<code>NumberOfPositiveRoots[algebra]</code>	number of positive roots of <i>algebra</i>
<code>PositiveRoots[algebra]</code>	gives only the positive roots of <i>algebra</i>

#### Roots

The roots of a Lie algebra can be built from the simple roots. There are two traditional approaches: (a) building the roots from linear combinations of simple roots. Since not all linear combinations of simple roots are roots, the difficulty lies in filtering out combinations that are roots. (b) Starting from a *highest root* the roots can be constructed by subtracting simple roots. `LieART` uses yet another approach: It builds the orbits of the simple roots by applying the Weyl group of the Lie algebra and adds the  $n$ -fold degenerated zero roots corresponding to the Cartan subalgebra. The simple roots of the same length belong to the same

orbit, e.g., for  $A_n$  there is only one orbit besides the zero orbit (see Out [42]). Nevertheless, the orbits of all simple roots are generated and then united. The fact that non-zero roots are non-degenerate allows us to remove duplicate roots obtained by the described procedure.

The function `RootSystem[algebra]` constructs the root system by the procedure described above. As a non-trivial example we demonstrate the procedure on  $G_2$ , which has two non-trivial orbits and the zero orbit: The two simple roots of  $G_2$

```
In[45]:= OmegaBasis[OrthogonalSimpleRoots[G2]]
Out[45]:= { $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ }
```

have different lengths:

```
In[46]:= DMatrix[G2]
Out[46]:=  $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$ 
```

Generating the Weyl group orbits of each of the simple roots

```
In[47]:= Orbit /@ OmegaBasis[OrthogonalSimpleRoots[G2]]
Out[47]:=  $\left( \begin{array}{|c|c|} \hline -2 & 1 \\ \hline -3 & 1 \\ \hline \end{array} \begin{array}{|c|c|} \hline -1 & 0 \\ \hline -3 & 2 \\ \hline \end{array} \begin{array}{|c|c|} \hline -1 & 1 \\ \hline 0 & -1 \\ \hline \end{array} \begin{array}{|c|c|} \hline 1 & -1 \\ \hline 0 & 1 \\ \hline \end{array} \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 3 & -2 \\ \hline \end{array} \begin{array}{|c|c|} \hline 2 & -1 \\ \hline 3 & -1 \\ \hline \end{array} \right)$ 
```

and adding the twofold degenerated zero roots constructed by `ZeroRoots[algebra]`

```
In[48]:= ZeroRoots[G2]
Out[48]:= { $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ }
```

yields the full  $G_2$  root system, displayed in spindle shape, as defined below in section 4.4.1

```
In[49]:= RootSystem[G2, SpindleShape -> True]
Out[49]:=  $\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 3 & -1 \\ \hline 1 & 0 \\ \hline -1 & 1 \\ \hline -3 & 2 & 2 & -1 \\ \hline 0 & 0 & 0 & 0 \\ \hline -2 & 1 & 3 & -2 \\ \hline 1 & -1 \\ \hline -1 & 0 \\ \hline -3 & 1 \\ \hline 0 & -1 \\ \hline \end{array}$ 
```

where a row stands for the same height of the roots. The *height* of a root is defined as the sum of coefficients in its linear combination of simple roots, i.e., the sum of coordinates in the  $\alpha$ -basis. It is implemented by `Height[root]`. The *highest root* has the largest height, implemented in LieART as `HighestRoot[algebra]`, which simply returns the first root of the root system, since the latter is sorted by the height of the roots decreasingly. E.g. for  $G_2$ :

```
In[50]:= HighestRoot[G2]
Out[50]:=  $\begin{bmatrix} 0 & 1 \end{bmatrix}$ 
```

The *positive roots* are the roots that are only positive linear combinations of simple roots, i.e., the coordinates in the  $\alpha$ -basis are all positive, with at least one being non-zero, to exclude the zero roots. The function `PositiveRootQ[root]` tests if *root* is positive. The *root* may be in any basis and will be transformed into the  $\alpha$ -basis, where its coordinates are tested accordingly. The number of positive roots are explicitly

stated as `NumberOfPositiveRoots[algebra]` in LieART. It serves as a limiter to the nested reflections for the generation of Weyl group orbits. There is a theorem stating that the maximum number of reflections building an element of the Weyl group is equal to the number of positive roots of the corresponding Lie algebra.

Since the root system is sorted by height, the positive roots come first. `PositiveRoots[algebra]` extracts only those with the use of `NumberOfPositiveRoots[algebra]`. E.g., for  $G_2$ :

```
In[51]:= PositiveRoots[G2]
Out[51]:= { [0 1], [3 -1], [1 0], [-1 1], [-3 2], [2 -1] }
```

#### 4.4. Representations

<code>WeightOrthogonal[algebraClass][label]</code>	weight in the orthogonal basis defined by its algebra <i>algebraClass</i> and Dynkin <i>label</i>
<code>WeightAlpha[algebraClass][label]</code>	weight in the $\alpha$ -basis defined by its algebra <i>algebraClass</i> and Dynkin <i>label</i>
<code>Weight[algebraClass][label]</code>	weight in the $\omega$ -basis defined by its algebra <i>algebraClass</i> and Dynkin <i>label</i>
<code>Irrep[algebraClass][label]</code>	irrep described by its algebra <i>algebraClass</i> and Dynkin <i>label</i>
<code>WeightLevel[weight, irrep]</code>	Level of the <i>weight</i> within the <i>irrep</i>
<code>Height[irrep]</code>	height of <i>irrep</i>
<code>SingleDominantWeightSystem[irrep]</code>	dominant weights of <i>irrep</i> without their multiplicities
<code>WeightMultiplicity[weight, irrep]</code>	computes the multiplicity of <i>weight</i> within <i>irrep</i>
<code>DominantWeightSystem[irrep]</code>	dominant weights of <i>irrep</i> with their multiplicities
<code>WeightSystem[irrep]</code>	full weight system of <i>irrep</i>
<code>Irrep[algebra][dimname]</code>	irrep entered by its <i>algebra</i> and <i>dimname</i>
<code>ProductIrrep[irreps]</code>	head of product <i>irreps</i>
<code>Delta[algebra]</code>	half the sum of positive roots of <i>algebra</i> ( $\delta=(1, 1, \dots)$ )
<code>WeylDimensionFormula[algebra]</code>	explicit Weyl dimension formula for <i>algebra</i>
<code>Dim[irrep]</code>	dimension of <i>irrep</i>
<code>DimName[irrep]</code>	dimensional name of <i>irrep</i>
<code>Index[irrep]</code>	index of <i>irrep</i>
<code>CongruencyClass[irrep]</code>	congruency class number of <i>irrep</i>

##### Basic Properties of Irreps

As explained in Section 4.1.6 a *representation* is a set of matrices that satisfies the same commutation relations as the algebra. Each of the matrices can be labeled by the *weight vector* with the eigenvalues of the matrices corresponding to the generators of the Cartan subalgebra, and we will refer to the weight vector simply as *weight*. The weight vector has the dimension of the Cartan subalgebra, i.e., the rank of the algebra, and not the dimension of the space the matrices act on. The latter depends on the particular representation.

The weights  $\lambda$  can be written as linear combination of simple roots and a crucial theorem states that the so-called *Dynkin labels*  $a_i$  defined as

$$a_i = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}, \quad i = 1, \dots, n \quad (19)$$

are integers for all simple roots  $\alpha_i$ . (Please note that this is also true if  $\lambda$  is replaced by any simple root, since this constitutes an element of the Cartan matrix as defined in (5).) The Dynkin labels are in particular used to label weights (and roots). The smallest non-zero weights with  $a_i \geq 0$  are called the *fundamental*

weights  $\omega_i$ . They define the  $\omega$ -basis or Dynkin basis already introduced. They are implemented in LieART as `OrthogonalFundamentalWeights[algebra]` in the orthogonal basis and we have given an example for  $A_4$  in `Out[37]`. The Dynkin labels  $a_i$  of a weight  $\lambda$  are the coefficients of its linear combination of fundamental weights, i.e., the  $a_i$  are the coordinates in the  $\omega$ -basis, which can be displayed as a row vector with comma separated entries or as a framed box following the convention of some textbooks:

$$\lambda = \sum_{i=1}^n a_i \omega_i = (a_1, a_2, \dots, a_n) = \boxed{\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n}. \quad (20)$$

A weight in LieART is represented by three different heads, depending on its basis, in analogy with the roots: `WeightOrthogonal[algebraClass][label]` for a weight in the orthogonal basis, in the  $\alpha$ -basis `WeightAlpha[algebraClass][label]` and simply `Weight[algebraClass][label]` in the  $\omega$ -basis, where we omit the explicit “Omega” for brevity, because the  $\omega$ -basis is the natural basis for weights. (The same can be said for the  $\alpha$ -basis for roots, favoring the shorter head `Root` instead of `RootAlpha` in the  $\alpha$ -basis. Unfortunately this would clash with the built-in Mathematica function `Root[f, k]` representing the  $k$ th root of a polynomial equation defined by  $f[x] = 0$ .) The `algebraClass` can only be A, B, C or D to indicate a classical Lie algebra or E6, E7, E8, F4 or G2 for the exceptionals. The `label` stands for the comma-separated coordinates. This form of the weight is displayed in `InputForm` and `FullForm`. E.g., the first fundamental weight of  $A_4$  in all three bases reads:

```
In[52]:= {#, AlphaBasis[#], OmegaBasis[#]} &
         @First[OrthogonalFundamentalWeights[A4]]//InputForm
Out[52]:= {WeightOrthogonal[A][4/5, -1/5, -1/5, -1/5, -1/5], WeightAlpha[A][4/5, 3/5, 2/5, 1/5],
         Weight[A][1, 0, 0, 0]}
```

Since weights are linear combinations of roots, many properties of roots translate to weights. The Weyl group also applies to weights and the weight space is also divided into Weyl chambers. A weight with only positive coordinates lies in the dominant Weyl chamber and is called a *dominant weight*. In analogy with the highest root, every irreducible representation (irrep) has a non-degenerate *highest weight*, denoted as  $\Lambda$ , which is also a dominant weight, but not necessarily the only dominant weight of the irrep. The weight system of the irrep can be computed from the highest weight  $\Lambda$  by subtracting simple roots. Thus, a highest weight  $\Lambda$  uniquely defines the irrep, and since a particular Lie algebra has infinitely many irreps, it serves as a label for the irrep itself using the same denotation,  $\Lambda$ .

In LieART an irrep is represented by `Irrep[algebraClass][label]`, where `algebraClass` defines the Lie algebra class in the same manner as for weights and roots, and `label` is the comma-separated label of the highest weight of the irrep. E.g., the 10 dimensional irrep of  $A_4$  has the highest weight  $(0, 1, 0, 0)$  and thus the irrep can be entered as `Irrep[A][0, 1, 0, 0]`.

The so-called *Dynkin label* of an irrep is similar to the notation of a weight, but since the highest weight has only positive label entries the commas between them can be omitted, as long as this is unambiguous. The Dynkin label in LieART is displayed in `StandardForm`, e.g., the **10** of  $A_4$ :

```
In[53]:= Irrep[A][0, 1, 0, 0]//StandardForm
Out[53]:= (0100)
```

If at least one of the entries in the Dynkin labels has more than a single digit, all entries are separated by commas to avoid ambiguities, which is the standard textbook convention:

```
In[54]:= Irrep[A][0, 10, 3, 1] // StandardForm
Out[54]:= (0, 10, 3, 1)
```

#### 4.4.1. Weight System

The conventional approach to computing all weights of an irrep is to subtract simple roots from the highest weight  $\Lambda$  that defines the irrep. The Dynkin label of the highest weight  $\Lambda = (a_1, a_2, \dots, a_n)$  reveals how many times each simple root can be subtracted: the  $i$ th root can be subtracted  $a_i$  times. The *level* of a weight is the number of simple root that need to be subtracted from the highest weight to obtain it. A weight

may be obtained by different subtraction routes, but it always involves the same number of simple roots, thus its level is unique. As explained earlier, the  $\alpha$ -basis exhibits the coefficients of the linear combination of simple roots, which are rational numbers in general. The difference between these coefficients of the weight and the highest weight show how many times each simple roots has been subtracted from the latter. The sum over these differences, for each simple root, gives the level of the weight:

$$L(\lambda, \Lambda) = \sum_{i=1}^n (\bar{\Lambda}_i - \bar{\lambda}_i) \quad (21)$$

where  $\lambda = (\bar{\lambda}_1, \dots, \bar{\lambda}_n)$  and  $\Lambda = (\bar{\Lambda}_1, \dots, \bar{\Lambda}_n)$  is the weight and highest weight in the  $\alpha$ -basis, respectively. The LieART function `WeightLevel[weight, irrep]` implements this procedure. The highest level of an irrep is called its *height*. Please note that the highest weight has the lowest level, which is zero. The weight with the highest level has the coefficients of the highest weight in the alpha basis, with negative sign and rearranged (if the irrep is complex), i.e., the sum over them is the negative of the sum of the highest weight. Thus, the height of an irrep with highest weight  $\Lambda$  is

$$\text{ht}(\Lambda) = 2 \sum_{i=1}^n \bar{\Lambda}_i, \quad (22)$$

which is available in LieART as `Height[irrep]`.

The algorithm to compute the weight system used in LieART is an implementation of the scheme developed in [19]. It deviates from the traditional procedure described above for performance reasons. Some weights of an irrep may be degenerated and the procedure with subtracting simple roots only yields an upper limit of this degeneracy, which is the number of subtraction routes that lead to a weight. To compute the *multiplicity*  $m_\lambda$  of a weight  $\lambda$  of the irrep with highest weight  $\Lambda$ , the so-called *Freudenthal recursion formula* is usually used:

$$2 \sum_{\alpha \in \Delta^+} \sum_{k \geq 0} (\lambda + k\alpha, \alpha) m_{\lambda+k\alpha} = [(\Lambda + \delta, \Lambda + \delta) - (\lambda + \delta, \lambda + \delta)] m_\lambda \quad (23)$$

where  $\Delta^+$  denotes the positive root system,  $\delta = (1, 1, \dots, 1)$  is half the sum of all positive roots in the  $\omega$ -basis and  $m_{\lambda+k\alpha}$  is the already computed multiplicity of a weight  $\lambda + k\alpha$  that is higher than  $\lambda$ . The sum over  $k$  is finite because the weight  $\lambda + k\alpha$  must be a member of the weight system of the irrep under consideration.

The recursive nature of the Freudenthal formula makes the computation of weight multiplicities the most CPU time consuming procedure in the determination of the weight system of an irrep. The algorithm developed in [19] exploits the Weyl group in both the weight and the root system. The weight system of an irrep is a collection of Weyl group orbits, represented by their unique dominant weight. The multiplicity of the dominant weight is the same for all weights of the associated orbit. Thus, a weight system can be constructed by (a) determining the dominant weights of the irrep, (b) computing their multiplicity and (c) generating the orbits of the dominants weights with the same multiplicity by application of the Weyl group of the associated algebra.

In LieART the function `SingleDominantWeightSystem[irrep]` determines the dominant weights of *irrep* by successively subtracting positive roots starting from the highest weight and keeping only the dominant weight of the result in every step. This process terminates, because there are smallest dominant weights, i.e., the fundamental weights, constituting a lower boundary. E.g., the **40** of  $A_4$  has two distinct dominant weights:

```
In[55]:= SingleDominantWeightSystem[Irrep[A][0, 0, 1, 1]]
Out[55]:= {{0 0 1 1}, {0 1 0 0}}
```

Thus, an improved version of the Freudenthal formula considers only dominant weights  $\lambda$ . The second exploitable property is the existence of a stabilizer of the weight, a subgroup of its Weyl group  $W$  that fixes the weight:

$$\text{Stab}_W(\lambda) = W_T := \{w \in W \mid w\lambda = \lambda\}. \quad (24)$$

The stabilizer  $W_T$  reduces the number of independent scalar products and previous computed multiplicities, because for  $w \in W_T$ ,  $(\lambda + k\alpha, \alpha) = (w(\lambda + k\alpha_i), w\alpha_i) = ((\lambda + kw\alpha_i), w\alpha_i)$  and  $m_{\lambda+k\alpha_i} = m_{w(\lambda+k\alpha_i)} = m_{\lambda+kw\alpha_i}$ .

Since the elements of the Weyl group  $W$  are reflections at simple roots, the stabilizer group is defined by the reflection at simple roots that map  $\lambda$  onto itself. Because of (18) this is the case, when  $(\lambda, \alpha_i) = 0$ . If  $\lambda$  is expressed in the  $\omega$ -basis as  $\lambda = \sum n_i \omega_i$  the scalar product with the  $i$ th simple root is zero, when  $n_i = 0$ . Let  $T$  be a set of these indices, i.e.,  $T = \{i \mid n_i = 0\}$ , and let  $\Delta_T$  be the root system based on the simple roots labeled by  $T$ .

The group  $\hat{W}_T$ , which is the inclusion of  $W_T$  and the negative identity element  $w = -1$  as  $\hat{W}_T = \langle W_T, -1 \rangle$ , decomposes the root system into orbits  $o_1, \dots, o_r$ , defined by  $\hat{W}_T \alpha_i$ . Each orbit has a unique representative  $\xi_i$  in the positive roots ( $\xi_i \in \Delta^+$ ). The  $\xi_i$ 's are those positive roots, that have non-zero coefficients in the  $\omega$ -basis at the positions where  $\lambda$  has zeros, i.e.,  $\xi_i = \sum m_i \omega_i$  with  $m_i \geq 0$  for  $i \in T$ .

The computation of the multiplicity  $m_\lambda$  of the dominant weight  $\lambda$  is then accomplished by the *modified Freudenthal formula*:

$$\sum_{i=1}^n |o_i| \sum_{k=1}^{\infty} (\lambda + k\xi_i, \xi_i) m_{\lambda+k\xi_i} = [(\Lambda + \delta, \Lambda + \delta) - (\lambda + \delta, \lambda + \delta)] m_\lambda \quad (25)$$

where  $|o_i|$  are the sizes of the orbits. It is important to note that these sizes are  $|o_i| = |W_T \xi_i|$  if  $\xi_i \in \Delta_T$  and  $|o_i| = 2|W_T \xi_i|$  if  $\xi_i \notin \Delta_T$ , because in the former case the negative roots are included in  $W_T \xi_i$ , i.e., the  $-1$ , while only positive roots are in the orbit  $W_T \xi_i$  if  $\xi_i \notin \Delta_T$ , requiring a factor of 2 for the same reason as on the left-hand side of (23). It is  $\xi_i \in \Delta_T$  if  $(\lambda, \xi_i) = 0$ .

The higher weight  $\lambda + k\xi_i$  is not necessarily a dominant weight, but can always be reflected to the dominant chamber to obtain the corresponding multiplicity that is already computed.

The computation of weight multiplicities is implemented in LieART as `WeightMultiplicity[weight, irrep]` following the above algorithm, using several helper functions: `T[weight]` gives the set  $T$ , the positions of zeros of the coefficients of *weight* in the  $\omega$ -basis, `Xis[algebra, t]` determines the  $\xi$ 's based on the set  $T$  which should be supplied via the argument  $t$ , `Alphas[algebra, t]` gives  $\alpha_i$  with  $i \in T$  to construct the orbit  $W_T \xi_i$ . `XisAndMul[algebra, t]` yields a list of the  $\xi_i$ 's together with their associated orbit size  $|o_i|$ . Since the possible subsets  $T$  of zeros in the weight coefficients for a specific algebra are limited, we follow the suggestion of [19] and save this in list form as `XisAndMul` for reuse upon first evaluation in the course of a calculation. Saved values of `XisAndMul` can be retrieved by `Definition[XisAndMul]`.

Take for example the dominant weight  $\boxed{0 \ 1 \ 0 \ 0}$  of the **40** of  $A_4$  from `Out [55]`. The set of indices  $T$  is

```
In[56]:= T[Weight[A][0,1,0,0]]
```

```
Out[56]:=  $\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ 
```

or in `InputForm`: `{{1},{3},{4}}`. (The structure as “list of lists” is due to the use of the Mathematica built-in functions `Position` and `Extract`.) The  $\xi_i$ 's and the sizes of their associated orbits  $|o_i|$  are

```
In[57]:= XisAndMul[A4, T[Weight[A][0,1,0,0]]]
```

```
Out[57]:=  $\begin{pmatrix} \boxed{1 \ 0 \ 0 \ 1} & 12 \\ \boxed{0 \ -1 \ 1 \ 1} & 6 \\ \boxed{2 \ -1 \ 0 \ 0} & 2 \end{pmatrix}$ 
```

and the weight multiplicity of the dominant weight in the **40** of  $A_4$  is:

```
In[58]:= WeightMultiplicity[Weight[A][0,1,0,0], Irrep[A][0,0,1,1]]
```

```
Out[58]:= 2
```

The LieART function `DominantWeightSystem[irrep]` gives a list of the dominant weights of *irrep* along with their multiplicities:

```
In[59]:= DominantWeightSystem[Irrep[A][0,0,1,1]]
```

```
Out[59]:=  $\begin{pmatrix} \boxed{0 \ 0 \ 1 \ 1} & 1 \\ \boxed{0 \ 1 \ 0 \ 0} & 2 \end{pmatrix}$ 
```



where  $\text{ord}(L)$  is the order of the Lie algebra  $L$ , which is equivalent to the number of roots or the dimension of the adjoint irrep. The index is related to the length of the weights and has applications in renormalization group equations and elsewhere. The corresponding LieART function is `Index[irrep]`. E.g., the index of the **40** of  $A_4$  is:

```
In[64]:= Index[Irrep[A][0,0,1,1]]
```

```
Out[64]:= 22
```

The label of *irrep* does not need to be numerical, similar to `Dim`.

*Congruency Class.* The *congruency class* expands the concept of  $n$ -ality of  $SU(N)$ , which in turn is a generalization of  $SU(3)$  triality, to all other simple Lie algebras. LieART uses congruency classes to characterize irreps, especially for the distinction of irreps of the same dimension and with the same index. We follow the definitions of [8, 20], labeling the congruency class by the *congruency number*, which is a single number for  $A_n, B_n, C_n, E_6, E_7, E_8, F_4$  and  $G_2$  and a two component vector for  $D_n$ . For an irreducible representation  $(a_1 a_2 \dots a_n)$  the congruency number (vector)  $c$  is:

$$A_n : c = \sum_{k=1}^n k a_k \pmod{n+1} \quad (28a)$$

$$B_n : c = a_n \pmod{2} \quad (28b)$$

$$C_n : c = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{2k+1} \pmod{2} \quad (28c)$$

$$D_n : c = \begin{pmatrix} a_{n-1} + a_n \pmod{2} \\ 2 \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} a_{2k+1} + (n-2)a_{n-1} + n a_n \pmod{4} \end{pmatrix} \quad (28d)$$

$$E_6 : c = a_1 - a_2 + a_4 - a_5 \pmod{3} \quad (28e)$$

$$E_7 : c = a_4 + a_6 + a_7 \pmod{2} \quad (28f)$$

$$E_8, F_4, G_2 : c = 0 \quad (28g)$$

Please note that the congruency class definitions of [8, 20], which we use, differ from [7] for the  $D_n$ 's : For  $D_4$ , i.e.  $SO(8)$ , the second component of [7] is half of the definition above. The congruency class for  $D_5$  is only a single number in [7], which is the same as the second component of the  $D_5$  congruency class vector of our definition from [8, 20] (with no factor of 2).

These congruency numbers (vectors)  $c$  are implemented as `CongruencyClass[irrep]` in LieART. E.g., the congruency numbers of the three eight dimensional irreps of  $SO(8)$  are all distinguished by their congruency number vector:

```
In[65]:= CongruencyClass[{Irrep[D][1,0,0,0], Irrep[D][0,0,1,0], Irrep[D][0,0,0,1]}]
```

```
Out[65]:= {{02}, {12}, {10}}
```

The head of a congruency vector is `CongruencyVector` and it is displayed as  $(c_1 c_2)$ , i.e. without commas separating the two components similar to the Dynkin label of irreps.

#### 4.4.3. Representation Names

The Dynkin label of an irreducible representation together with its Lie algebra uniquely specifies it, e.g., (0011) of  $A_4$ . An irrep in LieART is also represented (`FullForm`) by the Dynkin label and a Mathematica head that indicates the algebra class. E.g., the irrep (0011) of  $A_4$  is represented by `Irrep[A][0,0,1,1]` in LieART.

However, it is common practice to name representations by their dimension, the *dimensional name*, which is often times shorter. The dimension of a representation is not unique, i.e., there are different irreps with the same dimensions, which might be accidental or because of a relation between them. If it is accidental, irreps



with the same dimension have primes ( $\mathbf{dim}'$ ) in their dimensional name, e.g., the  $\mathbf{175}'$  of  $A_4$  is unrelated to the  $\mathbf{175}$ . Irreps can be related by conjugation, when they are complex. One of the irreps is written with an overbar ( $\overline{\mathbf{dim}}$ ). E.g., the  $\overline{\mathbf{10}}$  of  $A_4$  is the conjugate of the  $\mathbf{10}$ . Due to the high symmetry of  $SO(8)$  irrep, more than two related irreps of the same dimension exist. In the case of  $SO(8)$  subscripts specify the irreps completely.

The introduced properties of representations, the dimension, the index and the congruency class serve us well to discriminate between irreps with the same dimension. LieART has an algorithm implemented that determines the dimensional name of an irrep, following the naming conventions of [7]:

1. To determine the dimensional name of a specific irrep, LieART collects other irreps of the same dimensionality by brute-force scanning through a generated set of irreps.
2. Irreps that are related by conjugation or the symmetry of  $SO(8)$  not only have the same dimension, but also the same index. *Unrelated* irreps of equal dimension have different indices and can be organized and labeled by their indices. They are sorted by ascending index and labeled with primes ( $\mathbf{dim}'$ ) accordingly, starting with no prime. E.g., the names of the two unrelated 70 dimensional irreps of  $A_4$  are (the congruency class of  $A_4$  is called “Quintality”):

Dynkin	Dimension	Index	Quintality	Name
(2001)	70	49	1	$\mathbf{70}$
(0004)	70	84	1	$\mathbf{70}'$

3. Related irreps of the same dimensionality have the same index, but mostly (see below) different congruency class numbers. For Lie algebras other than  $SO(8)$ , only the conjugate of complex irreps are related. The convention here is that the irrep with *higher* congruency class number of the conjugated pair is labeled with an overbar ( $\overline{\mathbf{dim}}$ ). Since e.g. the  $\mathbf{70}'$  is a complex irrep it has a related conjugated irrep, the  $\overline{\mathbf{70}'}$ , i.e., overbars and primes may both appear in the labeling of an irrep. The above table for the determination of the primes involves only the lower congruency class number irreps of same dimensional and same index irreps. Consider the  $\mathbf{70}'$  and its conjugate, the  $\overline{\mathbf{70}'}$ :

Dynkin	Dimension	Index	Quintality	Name
(0004)	70	84	1	$\mathbf{70}'$
(4000)	70	84	4	$\overline{\mathbf{70}'}$

If the congruency class number of a complex irrep is zero, its conjugate also has a congruency class number of zero. In this case, where all three, the dimension, the index and the congruency class number are the same, the structure of the Dynkin labels are consulted: With the Dynkin label interpreted as digits of an integer number, the *smaller* “number” is labeled with the overbar. E.g., the 126 dimensional irreps of  $A_4$  are

Dynkin	Dimension	Index	Quintality	Name
(2010)	126	105	0	$\mathbf{126}$
(0102)	126	105	0	$\overline{\mathbf{126}}$
(5000)	126	210	0	$\mathbf{126}'$
(0005)	126	210	0	$\overline{\mathbf{126}'}$

Observe that this rule only applies for zero congruency class number: The  $\overline{\mathbf{70}'}$  has a “larger” number (4000) than the  $\mathbf{70}'$  with (0004).

4. For  $SO(8)$  irreps the convention for the labeling with primes are the same as for all other Lie algebras. Due to the three-fold symmetry most irreps come in sets of three with the same dimension and index. If only one digit of the Dynkin label is non-zero it is called the spinor, vector or conjugate irrep, depending on the dot in the Dynkin diagram it corresponds to. Usually they can be distinguished by the congruency class number, which is a two component vector for  $SO(8)$ : (01) for a vector irrep, (10) for a spinor and (11) for the conjugate. The irrep is then labeled by the subscripts “v”, “s” and “c”, resp. E.g., the three 8 dimensional irreps of  $SO(8)$  are

Dynkin	Dimension	Index	Congruency vector	Name
(1000)	8	1	(01)	$\mathbf{8}_v$
(0001)	8	1	(10)	$\mathbf{8}_s$
(0010)	8	1	(11)	$\mathbf{8}_c$

Some irreps with more than one non-zero digit of the Dynkin label with the same congruency vectors as above are labeled the same way if they are unique. However, if there is more than one irrep with the same dimension, index and also congruency vector, there is more than one digit of the Dynkin label non-zero. The subscript label is then a mixture like “sv”, and the ordering is determined by the Dynkin digit beginning with the largest. E.g. the 224 dimensional irreps of SO(8):

Dynkin	Dimension	Index	Congruency vector	Name
(2010)	224	100	(12)	$\mathbf{224}_{vc}$
(2001)	224	100	(10)	$\mathbf{224}_{vs}$
(1020)	224	100	(02)	$\mathbf{224}_{cv}$
(1002)	224	100	(02)	$\mathbf{224}_{sv}$
(0021)	224	100	(10)	$\mathbf{224}_{cs}$
(0012)	224	100	(12)	$\mathbf{224}_{sc}$

There are also cases where the congruency vector is zero in both components for all irreps of the same dimension and index. In this case subtracting the same integer from every Dynkin digit to obtain irreps with non-zero congruency class vector has proven to be a reliable way to label the irreps. E.g., the 35 dimensional irreps can be related to the 8 dimensional ones and, e.g., the primed 840 dimensional irreps to the 56 dimensional ones:

Dynkin	Dimension	Index	Congruency vector	Name
(1000)	8	1	(02)	$\mathbf{8}_v$
(0010)	8	1	(12)	$\mathbf{8}_c$
(0001)	8	1	(10)	$\mathbf{8}_s$
(2000)	35	10	(00)	$\mathbf{35}_v$
(0020)	35	10	(00)	$\mathbf{35}_c$
(0002)	35	10	(00)	$\mathbf{35}_s$
(1010)	56	15	(10)	$\mathbf{56}_s$
(1001)	56	15	(12)	$\mathbf{56}_c$
(0011)	56	15	(02)	$\mathbf{56}_v$
(2020)	840	540	(00)	$\mathbf{840}'_s$
(2002)	840	540	(00)	$\mathbf{840}'_c$
(0022)	840	540	(00)	$\mathbf{840}'_v$

In LieART the function `DimName[irrep]` determines the dimensional name according to the algorithm described above, which is automatically displayed if an irrep is displayed in `TraditionalForm`. Several internal helper functions are called by `DimName`. The function `GetIrrepByDim[algebra, dim, maxDynkinDigit]` provides irreps with the same dimension, which are then gathered into sublists by `DimName`. The function `SortSameDimAndIndex` sorts the irreps of same dimension and index by their congruency class, and automatically by the Dynkin label viewed as “numbers”, if the congruency class numbers are the same. The positions of the lists of same-index irreps determines the number of primes to apply and the position of the irrep within the same-index list, whether it should be labeled by an overbar. In case of an SO(8) irrep `DimName` branches to the function `SO8Label[irrep]`, which uses `SimpleSO8Label` to give a subscript of “v”, “s” and “c” in the case where the congruency vector is unique. If the congruency vector is not unique, but non-zero, `ConcatSO8Label` concatenates the mixed subscripts like “sv” in the correct ordering. If the congruency vector is zero in both components the irrep is related to irreps with non-zero congruency vector by `ReducedDynkinLabel`.

*Limitations.* The determination of the primes has one limitation, which requires explanation: The function `GetIrrepByDim[algebra, dim, maxDynkinDigit]` determines irreps of the same dimension. In a brute-force fashion it generates “all” irreps and extracts only those that have the dimension *dim*. Since there are infinite many irreps of any Lie algebra, it must be constrained. This is done by imposing a maximum Dynkin digit to use for the generation of Dynkin labels. Since the numbers of possible Dynkin labels grow rapidly with the maximum Dynkin digit allowed, the limit should be very low. To compare the irrep in question with others it should be at least its maximum Dynkin digit, e.g., for (2031) it is “3”. The related irreps only have a permutation of the Dynkin label, thus they are included in the generated list of irreps up to a Dynkin digit of “3” in the example. However, for the determination of the primes for the unrelated irreps it may not suffice to generate irreps only up to the maximum Dynkin digit of the irrep in question: The number of primes are determined by the position in a list of same-dimensional irreps sorted by the index. If there is an irrep with a higher maximal Dynkin digit, e.g., “4” in our example, but at the same time has a lower index than the irrep in question, this procedure would give the irrep in question too few primes. This situation rarely happens, especially in  $A_n$ ’s, but e.g. for  $G_2$  it happens as early as the 77 dimensional irrep:

Dynkin	Dimension	Index	Congruency number	Name
(30)	77	44	0	<b>77</b>
(02)	77	55	0	<b>77'</b>

When determining the name of (02) of  $G_2$  the Dynkin labels would only be generated up to a maximum Dynkin digit of “2”, the (30) would not appear and thus the (02) would be labeled without any prime. The determination of the name of (30) would “see” the (02), but would determine no prime for (30) because of its lower index compared to (02). For these two irreps the problem can be solved by generating irreps up to the maximum Dynkin digit *plus one* for the irrep in question, i.e., up to “3” for (02). Because the Dynkin label of a  $G_2$  irrep is small, this is easily manageable. In fact, we have implemented the addition of three to the maximum Dynkin digit for  $G_2$ , because for some higher dimensions the problem will reappear. However, for Lie algebras with long Dynkin labels, the number of generated Dynkin labels becomes large and its construction slows LieART down and consumes a large amount of memory. We have found a balance between accuracy and efficiency, which pushes this problem to very high dimensional irreps, by defining the following number to add to the maximum Dynkin digit of the irrep in question: 1 for  $A_n, B_n, C_n$  and  $D_n$  with  $n \leq 4$ , 0 for  $A_n, B_n, C_n$  and  $D_n$  with  $n \geq 5$ , 1 for  $E_6$  and  $F_4$ , 0 for  $E_7$  and  $E_8$  and 3 for  $G_2$ . Please note that this limitation is only connected to the labeling of irreps with primes. Computations in LieART are always performed using the Dynkin label as in the `FullForm`. If in doubt one can always use the Dynkin label displayed in `StandardForm`, `InputForm` and `FullForm` which serves as the unique identifier of an irrep.

Besides its Dynkin label, the alternative definition of `Irrep` as `Irrep[algebra][dimname]` can be used to specify an irrep by its dimensional name as *dimname* and its algebra as *algebra*. The *algebra* must be fully specified, such as `A4`, `SU5`, `E6`, not only the algebra class such as `A`. The *dimname* is an integer for the dimension with a `Bar[dim]` wrapped around it for a conjugated irrep or an `IrrepPrime[dim, n]` for an irrep with *n* primes. If only one prime is needed the second argument *n* may be omitted. The `Bar` and `IrrepPrime` can be combined in any sequence. E.g., the  $\overline{175'}$  can be entered by

```
In[66]:= Irrep[A4][IrrepPrime[Bar[175]]]//InputForm
Out[66]:= Irrep[A][0, 0, 2, 1]
```

Alternatives are `Irrep[SU5][IrrepPrime[Bar[175]]]`, `Irrep[SU5][Bar[IrrepPrime[175]]]` and `Irrep[A4][IrrepPrime[Bar[175], 1]]`. Internally the function `GetIrrepByDimName[algebra, dimname]` determines the corresponding Dynkin label. It uses the function `GetIrrepByDim` mentioned above to find all irreps with the same dimension and then extract the irrep with the identical dimensional name. If the user specifies an irrep that does not exist, e.g. an  $\overline{11}$  of  $A_4$ , the comparison must stop at some point. It has been chosen that `GetIrrepByDim` generates only irreps with a maximum Dynkin digit as set by the global variable `$MaxDynkinDigit`. The default is `$MaxDynkinDigit=3`. The consequence is that the determination of the correct Dynkin label of the entered irrep may abort, because the irrep does not exist or that it involves a Dynkin digit higher than `$MaxDynkinDigit=3`. The latter is the case for the **70'** with a Dynkin label of (0004). LieART prints an error message indicating the two possible scenarios:

```

In[67]:= Irrep[A4][IrrepPrime[70]]//InputForm
      Irrep::noirrep: Either an irrep with the dimension name 70' does not exist
Out[67]:= in SU(5) or it has at least one Dynkin digit higher than 3. Try with
      $MaxDynkinDigit set to a higher value than 3. »
Increasing $MaxDynkinDigit to 4 resolves the issue:
In[68]:= $MaxDynkinDigit=4;
      Irrep[A4][IrrepPrime[70]]//InputForm
Out[68]:= Irrep[A][0,0,0,4]

```

#### 4.5. Tensor Product Decomposition

DecomposeProduct[ <i>irreps</i> ]	decomposes the tensor product of <i>irreps</i>
DominantWeightsAndMul[ <i>weights</i> ]	filters and tallies dominant weights of <i>weights</i> by multiplicities
SortOutIrrep[ <i>dominantWeightsAndMul</i> ]	sorts out the irrep of largest height from the collection of dominant weights <i>dominantWeightsAndMul</i>
WeightSystemWithMul[ <i>irrep</i> ]	full weight system of <i>irrep</i> with multiplicities
TrivialStabilizerWeights[ <i>weights</i> ]	drops weights that lie on a chamber wall
ReflectToDominantWeightWithMul[ <i>weightAndMul</i> ]	reflects <i>weightAndMul</i> to the dominant chamber and multiplies the parity of the reflection to the multiplicity

Tensor product decomposition.

Tensor products of irreps can be decomposed into a direct sum of irreps. The product of two irreps  $R_1$  and  $R_2$  can be decomposed as

$$R_1 \otimes R_2 = \sum_i m_i R_i \quad (29)$$

with the following dimension and index sum rules:

$$\dim(R_1 \otimes R_2) = \dim(R_1) \cdot \dim(R_2) = \sum_i m_i \dim(R_i) \quad (30)$$

$$l(R_1 \otimes R_2) = l(R_1) \dim(R_2) + \dim(R_1) l(R_2) = \sum_i m_i l(R_i). \quad (31)$$

##### 4.5.1. Generic Algorithm

A straight-forward method to compute the right-side of (29) is the following: Add all weights of  $R_2$  to each weight of  $R_1$ . The resulting  $\dim(R_1) \cdot \dim(R_2)$  weights belong to the different irreps  $R_i$ , which must be sorted out. Instead of all weights, one can consider just the dominant weights in the product, as each of the dominant weights represents an orbit in the irreps  $R_i$ . As an irrep is a collection of orbits, some of the dominant weights in the product represent the highest weight of an irrep in the decomposition. There is a unique dominant weight that represents the irrep of largest height in the decomposition. The sorting procedure should start with this dominant weight viewed as the highest weight of an irrep and then construct the dominant weight system of the corresponding irrep. The dominant weight system of the irrep with largest height should then be subtracted from the combined dominant weights of the product to filter it out. The same procedure is applied recursively to the remaining set of dominant weights until it is empty, i.e., all irreps have been filtered out.

LieART provides the function `DecomposeProduct[irreps]` for the decomposition of the tensor product of arbitrary many *irreps* of any classical or exceptional Lie algebra as argument. As a demonstration of generic algorithm we consider the decomposition of the SU(3) tensor product  $\mathbf{8} \otimes \mathbf{8}$ , which is

```
In[69]:= DecomposeProduct[Irrep[SU3][8], Irrep[SU3][8]]
```

```
Out[69]:= 1 + 2(8) + 10 +  $\overline{10}$  + 27
```

In the straight forward approach one adds all weights of the second  $\mathbf{8}$  to each weight of the first  $\mathbf{8}$ , using the built-in Mathematica function `Outer`. One filters out only the dominant weights and tallies multiple occurrences thereof using the LieART function `DominantWeightsAndMul[weights]`, which also sorts the weights according to their height, when viewed as a highest weight of an irrep. For the  $SU(3)$  tensor product  $\mathbf{8} \otimes \mathbf{8}$  the dominant weights along with their multiplicities are

```
In[70]:= DominantWeightsAndMul[Flatten[Outer[Plus, WeightSystem[Irrep[SU3][8]],
WeightSystem[Irrep[SU3][8]]]]]
```

```
Out[70]:=  $\left( \begin{array}{cc|c} \boxed{2} & \boxed{2} & 1 \\ \boxed{0} & \boxed{3} & 2 \\ \boxed{3} & \boxed{0} & 2 \\ \boxed{1} & \boxed{1} & 6 \\ \boxed{0} & \boxed{0} & 10 \end{array} \right)$ 
```

The dominant weight with the largest height  $\boxed{2 \ 2}$  must be the highest weight of an irrep. The dominant weight system of the (22) (the  $\mathbf{27}$ ) of  $SU(3)$  is

```
In[71]:= DominantWeightSystem[Irrep[A][2, 2]]
```

```
Out[71]:=  $\left( \begin{array}{cc|c} \boxed{2} & \boxed{2} & 1 \\ \boxed{0} & \boxed{3} & 1 \\ \boxed{3} & \boxed{0} & 1 \\ \boxed{1} & \boxed{1} & 2 \\ \boxed{0} & \boxed{0} & 3 \end{array} \right)$ 
```

It contains all dominant weights appearing in the tensor product, but with mostly smaller multiplicities. The irrep (22) can be filtered out by subtracting the multiplicities in `Out[71]` from `Out[70]`. The LieART function `SortOutIrrep[dominantWeightsAndMul]` performs the task of computing the dominant weight system of the irrep corresponding to the largest height weight and subtracting it from the tensor product. It returns the dominant weights of the tensor product with the ones of the irrep removed and passes the latter using the `Sow` and `Reap` mechanism of Mathematica:

```
In[72]:= Reap[SortOutIrrep[%%]]
```

```
Out[72]:= {  $\left( \begin{array}{cc|c} \boxed{0} & \boxed{3} & 1 \\ \boxed{3} & \boxed{0} & 1 \\ \boxed{1} & \boxed{1} & 4 \\ \boxed{0} & \boxed{0} & 7 \end{array} \right), (\mathbf{27})$  }
```

The function `SortOutIrrep` is applied recursively until the list of dominant weights with multiplicities of the tensor product is empty. E.g., applying `SortOutIrrep` to the dominants weights of `Out[72]` filters out the (03) (the  $\overline{\mathbf{10}}$ ) of  $SU(3)$ :

```
In[73]:= Reap[SortOutIrrep[First[%]]]
```

```
Out[73]:= {  $\left( \begin{array}{cc|c} \boxed{3} & \boxed{0} & 1 \\ \boxed{1} & \boxed{1} & 3 \\ \boxed{0} & \boxed{0} & 6 \end{array} \right), (\overline{\mathbf{10}})$  }
```

The irreps filtered out by `SortOutIrrep` are collected by the LieART function `GetIrreps` and can be displayed as the result of the decomposition. However, LieART computes the tensor product decomposition by an implementation of Klimyk's formula, which is far more efficient than the straight-forward procedure described above, but the sorting-out algorithm is still used for subalgebra decomposition in Section 4.6.

#### 4.5.2. Algorithm Based on Klimyk's Formula

Adding all weights of  $R_2$  to each weight of  $R_1$  is costly for large irreps. LieART's algorithm to decompose tensor-product implements Klimyk's formula [21, 22], which improves the runtime of tensor product decompositions considerably: Let  $\lambda_1$  and  $\lambda_2$  be weights and  $\Lambda_1$  and  $\Lambda_2$  the highest weights of  $R_1$  and  $R_2$ , respectively. Instead of adding all weights  $\lambda_1$  to each weight  $\lambda_2$ , the weights  $\lambda_1$  are added only to the highest weight  $\Lambda_2$  of  $R_2$  together with half the sum of positive simple roots,  $\delta=(1, \dots, 1)$ , building the set of weights

$$\mu = \lambda_1 + \Lambda_2 + \delta. \quad (32)$$

Each  $\mu$  is reflected to the dominant chamber, yielding a highest weight denoted as  $\{\mu\}$ , with  $\text{sgn}(\mu)$  as the parity of the reflection. Of these dominant weights all that lie on a chamber wall are dropped. The irreps in the decomposition are  $R(\{\mu\} - \delta)$ . Klimyk's formula reads

$$R_1(\Lambda_1) \otimes R_2(\Lambda_2) = \sum_{\lambda_1} m_{1\lambda_1} t(\lambda_1 + \Lambda_2 + \delta) R(\{\lambda_1 + \Lambda_2 + \delta\} - \delta), \quad (33)$$

where  $m_{1\lambda_1}$  denotes the multiplicity of  $\lambda_1$  in  $R_1$ . We define  $t(\mu)$  to be  $\text{sgn}(\mu)$  if  $\mu$  has a trivial-stabilizer subgroup  $\text{Stab}(\mu) = \{1\}$  (see (24)), and zero if the stabilizer subgroup is non-trivial, i.e. the weight lies on a chamber wall:

$$t(\mu) = \begin{cases} \text{sgn}(\mu) & : \text{Stab}(\mu) = \{1\} \\ 0 & : \text{else} \end{cases}. \quad (34)$$

We will demonstrate this algorithm with LieART in the following paragraphs.

As a demonstration of the algorithm implemented we consider the decomposition of the SU(3) tensor product  $\mathbf{6} \otimes \mathbf{3}$ , which is

```
In[74]:= DecomposeProduct[Irrep[SU3][6], Irrep[SU3][3]]
Out[74]:= 8 + 10
```

LieART normally reorders the arguments of `DecomposeProduct` by their dimension, which usually simplifies the application of Klimyk's formula. But for didactical reasons we assume in the following that the irreps are not reordered. LieART generates the weight system with weight multiplicities of the  $\mathbf{6}$ :

```
In[75]:= WeightSystemWithMul[Irrep[SU3][6]]
Out[75]:=
```

$$\begin{pmatrix} \begin{matrix} 2 & 0 \\ -2 & 2 \\ 0 & -2 \\ 0 & 1 \\ 1 & -1 \\ -1 & 0 \end{matrix} & \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix} \end{pmatrix}$$

We add all weights of the  $\mathbf{6}$  to the highest weight of the  $\mathbf{3}$ , i.e.  $\text{HighestWeight}[\text{Irrep}[\text{SU3}][3]] = \begin{bmatrix} 1 & 0 \end{bmatrix}$ , and  $\text{Delta}[\text{SU3}] = \begin{bmatrix} 1 & 1 \end{bmatrix}$ , according to (32). The highest weight  $\Lambda_2$  and  $\delta$  can be added directly, but to add the sum to all weights in the weight system with multiplicities LieART provides the command `Add`:

```
In[76]:= mu = Add[WeightSystemWithMul[Irrep[SU3][6]], HighestWeight[Irrep[SU3][3]] + Delta[SU3]]
Out[76]:=
```

$$\begin{pmatrix} \begin{matrix} 4 & 1 \\ 0 & 3 \\ 2 & -1 \\ 2 & 2 \\ 3 & 0 \\ 1 & 1 \end{matrix} & \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix} \end{pmatrix}$$

LieART reflects these weights to the dominant chamber yielding the corresponding dominant weights and

the parity of the reflection, i.e.  $\text{sgn}(\mu) = (-1)^l$ , where  $l$  is the number of reflections needed to reach the dominant chamber:

```
In[77]:= ReflectToDominantWeightWithMul /@ mu
```

```
Out[77]:= 
$$\begin{pmatrix} \boxed{4\ 1} & 1 \\ \boxed{0\ 3} & 1 \\ \boxed{1\ 1} & -1 \\ \boxed{2\ 2} & 1 \\ \boxed{3\ 0} & 1 \\ \boxed{1\ 1} & 1 \end{pmatrix}$$

```

The weights  $\boxed{4\ 1}$ ,  $\boxed{0\ 3}$ ,  $\boxed{2\ 2}$ ,  $\boxed{3\ 0}$  and  $\boxed{1\ 1}$  were already in the dominant chamber, while  $\boxed{2\ -1}$  needed one reflection to become the dominant weight  $\boxed{1\ 1}$  with a parity of  $-1$ .

Weights on walls of the dominant chamber do not contribute in Klimyk's formula and must be dropped. Weights not lying on a chamber wall have a trivial-stabilizer subgroup. To keep only these weights LieART applies the command `TrivialStabilizerWeights`, which drops all weights containing at least one zero anywhere in their Dynkin label, which corresponds to lying on a chamber wall:

```
In[78]:= TrivialStabilizerWeights[ReflectToDominantWeightWithMul /@ mu]
```

```
Out[78]:= 
$$\begin{pmatrix} \boxed{4\ 1} & 1 \\ \boxed{1\ 1} & -1 \\ \boxed{2\ 2} & 1 \\ \boxed{1\ 1} & 1 \end{pmatrix}$$

```

LieART subtracts  $\delta$  yielding highest weights and constructs the irreps in the decomposition  $R(\{\lambda_1 + \Lambda_2 + \delta\} - \delta)$ :

```
In[79]:= ToIrrep/@Add[TrivialStabilizerWeights[ReflectToDominantWeightWithMul/@mu],  
-delta]
```

```
Out[79]:= 
$$\begin{pmatrix} \mathbf{10} & 1 \\ \mathbf{1} & -1 \\ \mathbf{8} & 1 \\ \mathbf{1} & 1 \end{pmatrix}$$

```

While the irreps in the left column correspond to  $R(\{\lambda_1 + \Lambda_2 + \delta\} - \delta)$  in Klimy's formula, the multiplicities in the right column correspond to  $m_{1\lambda_1} \text{sgn}(\lambda_1 + \Lambda_2 + \delta)$ . Summing up the decomposition accordingly we see that the  $\mathbf{1}$  drops out and we obtain the result  $\mathbf{8} + \mathbf{10}$  as already stated in Out [74].

#### 4.5.3. $SU(N)$ Decomposition via Young Tableaux

A correspondence of  $SU(N)$  irreps and Young tableaux is very useful for the calculation of tensor products and subalgebra decomposition by hand. We have found that the algorithm for the  $SU(N)$  tensor product decomposition via Young tableaux also performs better on the computer, with respect to CPU time and memory consumption, than the procedure described in the previous section. Thus, LieART uses the Young tableaux algorithm for the tensor-product decomposition of  $SU(N)$  irreps and the procedure of adding weights and filtering out irreps for all other classical and exceptional Lie algebras.

A *Young tableau* is a left-aligned set of boxes, with successive rows having an equal or smaller number of boxes. Young tableaux correspond to the symmetry of the tensors of  $SU(N)$  irreps, by first writing each index of the tensor into one box of a Young tableau and the prescription that they ought to be first symmetrized in the rows and then antisymmetrized in the columns. Please see Out [31] in Section 3 for a non-trivial example for a Young tableau displayed by LieART.

To demonstrate the algorithm for the tensor product decomposition via Young tableaux we use the same  $SU(3)$  tensor product as in Section 4.5.1,  $\mathbf{8} \otimes \mathbf{8}$ .

The construction principle (by hand) is to put the Young tableau with the most boxes to the left and bump all boxes of the right Young tableau row by row to the left one following certain rules. To understand these





#### 4.6. Subalgebra Decomposition

<code>DecomposeIrrep[irrep, subalgebra]</code>	decomposes <i>irrep</i> to the specified <i>subalgebra</i> .
<code>DecomposeIrrep[productIrrep, subalgebra, pos]</code>	decomposes <i>productIrrep</i> at position <i>pos</i> .
<code>ProjectionMatrix[origin, target]</code>	defines the projection matrix for the algebra-subalgebra pair specified by <i>origin</i> and <i>target</i>
<code>Project[projectionMatrix, weights]</code>	applies the <i>projectionMatrix</i> to the <i>weights</i>
<code>GroupProjectedWeights[projectedWeights, target]</code>	groups the projected weights according to the subalgebra specified in <i>target</i>
<code>NonSemiSimpleSubalgebra[origin, simpleRootToDrop]</code>	computes the projection matrix of a maximal non-semi-simple subalgebra by dropping one dot of the Dynkin diagram <i>simpleRootToDrop</i> and turning it into a U(1) charge
<code>SemiSimpleSubalgebra[origin, simpleRootToDrop]</code>	computes the projection matrix of a maximal semi-simple subalgebra by dropping one dot from the extended Dynkin diagram.
<code>ExtendedWeightScheme[algebra, simpleRootToDrop]</code>	adds the Dynkin label associated with the extended simple root $-\gamma$ to each weight of the lowest orbit of <i>algebra</i> and drops the simple root <i>simpleRootToDrop</i>
<code>SpecialSubalgebra[origin, targetirreps]</code>	computes the projection matrix of a maximal special subalgebra by specifying the branching rule of the generating irrep.

Subalgebra decomposition of irreps and product algebra irreps.

The LieART function `DecomposeIrrep[irrep, subalgebra]` decomposes an irrep of a simple Lie algebra into a maximal subalgebra specified by *subalgebra*, which can be simple, semi-simple or non-semi-simple. To decompose an irrep of a semi-simple or non-semi-simple irrep, a third argument *pos* allows one to specify which part of *productIrrep* should be decomposed into the *subalgebra*.

The implementation of `DecomposeIrrep` in LieART uses so-called projection matrices. These matrices project the weights of an irrep into the specified subalgebra. The resulting weights are further processed in the same manner as in Section 4.6: Only the dominant weights of the decomposed weights are kept, because they uniquely define the orbits of the subalgebra and thus its irreps. In the next step the irreps comprised in the collection of dominant weights are sorted out using the same LieART functions as for the generic tensor product decomposition, discussed in section 4.5.1. It is clear that the major task is the determination of the projection matrices. They are different for each algebra-maximal-subalgebra pair and are not unique. I.e., a projection is not unique and thus the matrices are not unique in general. Our matrices are correct and consistent, but one may find different projection matrices in the literature, which are also correct. (An extensive collection of projection matrices can be found in [23] for the Lie algebra  $A_n$  and in [24] for the Lie algebras  $B_n$ ,  $C_n$  and  $D_n$ .) Once a projection matrix is known it can be used for the decomposition of all irreps of the algebra-maximal-subalgebra pair. E.g., the projection matrix for the branching  $SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$  is

```
In[80]:= ProjectionMatrix[SU5, ProductAlgebra[SU3, SU2, U1]]
```

$$\text{Out [80]} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 6 & 3 \end{pmatrix}$$

The determination of the projection matrices is closely connected to the problem of finding maximal subalgebras and we defer the description of its implementation in LieART to the next section. Taking the projection matrix `Out [80]` as given we demonstrate the algorithm of `DecomposeIrrep` to find the branching rule for the **10** of  $SU(5)$  to  $SU(3) \otimes SU(2) \otimes U(1)$ , which is

```
In[81]:= DecomposeIrrep[Irrep[SU5][10], ProductAlgebra[SU3, SU2, U1]]
Out[81]:= (1, 1)(-6) + (3-bar, 1)(4) + (3, 2)(-1)
```

The LieART function `Project[projectionMatrix, weights]` applies the *projectionMatrix* to each of the *weights* and a subsequent `GroupProjectedWeights[projectedWeights, target]` groups the Dynkin label of each of the *projectedWeights* according to the subalgebra specified by *target*. In the case of our example each weight of the **10** of  $SU(5)$  decomposes to  $SU(3) \otimes SU(2) \otimes U(1)$  as:

```
In[82]:= IrrepRule @@@ Transpose[{WeightSystem[Irrep[SU5][10]],
  Row/@GroupProjectedWeights[Project[ProjectionMatrix[SU5,
  ProductAlgebra[SU3, SU2, U1]], WeightSystem[Irrep[SU5][10]],
  ProductAlgebra[SU3, SU2, U1]]}]
```

$$\begin{array}{l} \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 4 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline 1 & -1 & 1 & 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 1 & -1 & 0 & 4 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline -1 & 0 & 1 & 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline -1 & 0 & 0 & 4 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline 1 & 0 & -1 & 1 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 1 & 0 & 1 & -1 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline -1 & 1 & -1 & 1 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline -1 & 1 & 1 & -1 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline 1 & 0 & 0 & -1 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 1 & 0 & -1 & -1 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline -1 & 1 & 0 & -1 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline -1 & 1 & -1 & -1 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline 0 & -1 & 0 & 1 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 0 & -1 & 1 & -1 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline 0 & -1 & 1 & -1 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 0 & -1 & -1 & -1 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline 0 & 0 & -1 & 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & -6 \\ \hline \end{array} \end{array}$$

The algorithm of `DecomposeIrrep` differs slightly and keeps only the dominant weights after projection and groups only them yielding

$$\left( \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 4 \\ \hline 1 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & -6 \\ \hline \end{array} \right) \quad (40)$$

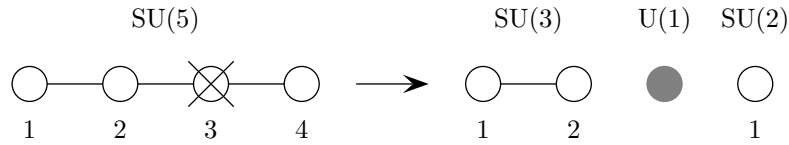
for our example. A combination of the functions `GetAllProductIrrep` and `GetProductIrrep` filter out the product irreps,  $(\bar{\mathbf{3}}, \mathbf{1})(4)$ ,  $(\mathbf{1}, \mathbf{1})(-6)$  and  $(\mathbf{3}, \mathbf{2})(-1)$  in our case, by applying the function `GetIrrep` known from section 4.5.1 to the weights.

#### 4.6.1. Branching Rules and Maximal Subalgebras

To determine the projection matrices we start with the algorithm to find maximal subalgebras. Subalgebras fall into two classes: *regular* and *special* subalgebras, with the first one being further categorized into non-semisimple and semisimple subalgebras. In the following we describe the derivation of the three types of maximal subalgebras: regular non-semisimple, regular semisimple and special subalgebras, originally developed by Dynkin [1, 2] and demonstrate how it is utilized by LieART to determine the projection matrices.

*Non-Semisimple Subalgebras.* A non-semisimple subalgebra is a semisimple subalgebra times a  $U(1)$  factor, e.g.  $SU(3) \otimes SU(2) \otimes U(1)$ . A subalgebra of this type is obtained by removing a dot from the Dynkin diagram. The resulting two or more disconnected Dynkin diagrams symbolize the semisimple subalgebra

and the removed dot, i.e., simple root, becomes the U(1) generator. E.g., the non-semisimple subalgebra  $SU(3)\otimes SU(2)\otimes U(1)$  can be obtained from  $SU(5)$  by removing the third dot from its Dynkin diagram:



Since the Dynkin label of a weight represents its composition of simple roots (explicitly in the  $\alpha$ -basis), dropping a simple root (dot) from the Dynkin diagram corresponds to dropping the associated digit from the Dynkin label. The U(1) charge is the coefficient of the dropped simple root in the weight's linear combination of simple roots, i.e., the associated digit of the Dynkin label in the  $\alpha$ -basis, which is often normalized to give integer values. Accordingly, in Out [82] the third Dynkin digit of the weight of the **10** has been removed after the projection and by expressing the weight system in the  $\alpha$ -basis

```
In [83] := AlphaBasis[WeightSystem[Irrep[SU5][10]]]//Column
(3/5, 6/5, 4/5, 2/5)
(3/5, 1/5, 4/5, 2/5)
(-2/5, 1/5, 4/5, 2/5)
(3/5, 1/5, -1/5, 2/5)
(-2/5, 1/5, -1/5, 2/5)
Out [83] := (3/5, 1/5, -1/5, -3/5)
(-2/5, 1/5, -1/5, -3/5)
(-2/5, -4/5, -1/5, 2/5)
(-2/5, -4/5, -1/5, -3/5)
(-2/5, -4/5, -6/5, -3/5)
```

we see that the U(1) charge at the end are the third coordinate of the weight in the  $\alpha$ -basis multiplied by 5 to give integer values. To summarize, the third Dynkin digit has been moved to the end (i.e., third and fourth digits have been interchanged) and only it has been projected into the  $\alpha$  basis and rescaled.

Writing the weights of the **10** as *columns* of a matrix  $\hat{W}$  and the weights with the third digit expressed in non-normalized  $\alpha$ -basis coordinates moved to the end as rows of a matrix  $\hat{W}'$ , the projection matrix  $\hat{P}$  can be determined from

$$\hat{P}\hat{W} = \hat{W}' \quad (41)$$

with the right-inverse  $\hat{W}^+$  of  $\hat{W}$  (see section 4.1.4), since  $\hat{W}$  is in general not a rectangular matrix:

$$\hat{P} = \hat{W}'\hat{W}^+. \quad (42)$$

As mentioned above the projection matrix found by this procedure can now be used to decompose any  $SU(5)$  irrep into  $SU(3)\otimes SU(2)\otimes U(1)$ . The **10** is actually not the smallest irrep needed for the determination of the projection matrix. The **10** as well as all other irreps can be built from tensor products of the **5**, which we call the *generating irrep* of  $SU(5)$ . In the orthogonal algebras only tensor products of the so-called *spinor representations* can construct all other irreps of the algebra. Thus, they must be used for the determination of the projection matrices. The generating irreps of representative Lie algebras are listed in Table 4.2.

In fact LieART excludes the zero-weights from the generating irreps, if any, i.e., only the lowest non-trivial orbit is needed for the determination of the projection matrices.

The calculation of a projection requires the knowledge of the simple root to drop from the Dynkin diagram for a specified algebra-subalgebra pair. LieART provides an extra package file called `BranchingRules.m`, listing this information for the implemented branching rules along with special embeddings to be discussed later. The file will be extended to encompass more branching rules in future versions of LieART, but may also be extended by the user. The definition for the more general branching rule  $SU(n) \rightarrow SU(N-k)\otimes SU(k)\otimes U(1)$ , including the demonstrated case  $SU(5) \rightarrow SU(3)\otimes SU(2)\otimes U(1)$ , reads:

```
ProjectionMatrix[origin:Algebra[A][n_],
ProductAlgebra[Algebra[A][m_],Algebra[A][k_],Algebra[U][1]]] :=
NonSemiSimpleSubalgebra[origin,-k-1] /; m==(n-k-1)
```

Algebra	Irrep (Dynkin)	Irrep (Name)
A <sub>4</sub> (SU(5))	(1000)	<b>5</b>
B <sub>4</sub> (SO(9))	(0001)	<b>16</b>
C <sub>4</sub> (Sp(8))	(1000)	<b>8</b>
D <sub>4</sub> (SO(8))	(0001)	<b>8<sub>s</sub></b>
E <sub>6</sub>	(100000)	<b>27</b>
E <sub>7</sub>	(0000010)	<b>56</b>
E <sub>8</sub>	(00000010)	<b>248</b>
F <sub>4</sub>	(0001)	<b>26</b>
G <sub>2</sub>	(10)	<b>7</b>

Table 4.2: Generating Irreps of representative Lie algebras

*Semisimple Subalgebras.* To obtain a semisimple subalgebra without a U(1) generator, a root from the so-called *extended Dynkin diagram* is removed. The extended Dynkin diagram is constructed by adding the most negative root to the set of simple roots. (The negative of the highest root  $\gamma$  gives the most negative root  $-\gamma$  to form the extended Dynkin diagram.) The resulting set of roots is linearly dependent, but removing one root restores the linear independence yielding a valid system of simple root of a subalgebra, which in general is semisimple. The highest roots  $\gamma$  and the according extended root  $-\gamma$  for representative Lie algebras are listed in Table 4.3. The non-zero entries in the Dynkin label of  $-\gamma$  prescribe to which existing dot in

Algebra	Highest Root ( $\gamma$ )	Extended Root ( $-\gamma$ )
A <sub>4</sub> (SU(5))	1 0 0 1	-1 0 0 -1
B <sub>4</sub> (SO(9))	0 1 0 0	0 -1 0 0
C <sub>4</sub> (Sp(8))	2 0 0 0	-2 0 0 0
D <sub>4</sub> (SO(8))	0 1 0 0	0 -1 0 0
E <sub>6</sub>	0 0 0 0 0 1	0 0 0 0 0 -1
E <sub>7</sub>	1 0 0 0 0 0 0	-1 0 0 0 0 0 0
E <sub>8</sub>	0 0 0 0 0 0 1 0	0 0 0 0 0 0 -1 0
F <sub>4</sub>	1 0 0 0	-1 0 0 0
G <sub>2</sub>	0 1	0 -1

Table 4.3: Highest roots  $\gamma$  and most negative roots  $-\gamma$  of representative Lie algebras.

the Dynkin diagram it should connect, since the Dynkin label in the  $\omega$ -basis encode the angle between two simple roots. A “1” is an angle of 120°, symbolized by a single connected line in the Dynkin diagram. A “2” is an angle of 135°, expressed by a double line in the Dynkin diagram. The minus sign gives negative angles or reverses the order of roots. The extended Dynkin diagrams for all classical and exceptional Lie Algebras are shown in Figure 2. Please note that the double line connecting the extended root  $-\gamma$  for  $C_n$  is according to the “-2” in its Dynkin label.

To demonstrate the determination of the projection matrix from using the generating irrep, we cannot use an irrep of  $SU(N)$ , because dropping a root from the extended Dynkin diagram of  $SU(N)$  returns  $SU(N)$ . Thus,  $SU(N)$  has no *regular maximal* semisimple subalgebra. (Please note that some  $SU(N)$ ’s have *special* maximal semisimple subalgebras, e.g.,  $SU(4) \rightarrow SU(2) \otimes SU(2)$ .) Instead we consider the subalgebra branching of  $SO(7) \rightarrow SU(2) \otimes SU(2) \otimes SU(2)$  ( $B_3 \rightarrow A_1 \otimes A_1 \otimes A_1$ ). The maximal subalgebra  $SU(2) \otimes SU(2) \otimes SU(2)$  is obtained from the extended Dynkin diagram of  $SO(7)$  ( $B_3$ ) by removing the second dot:

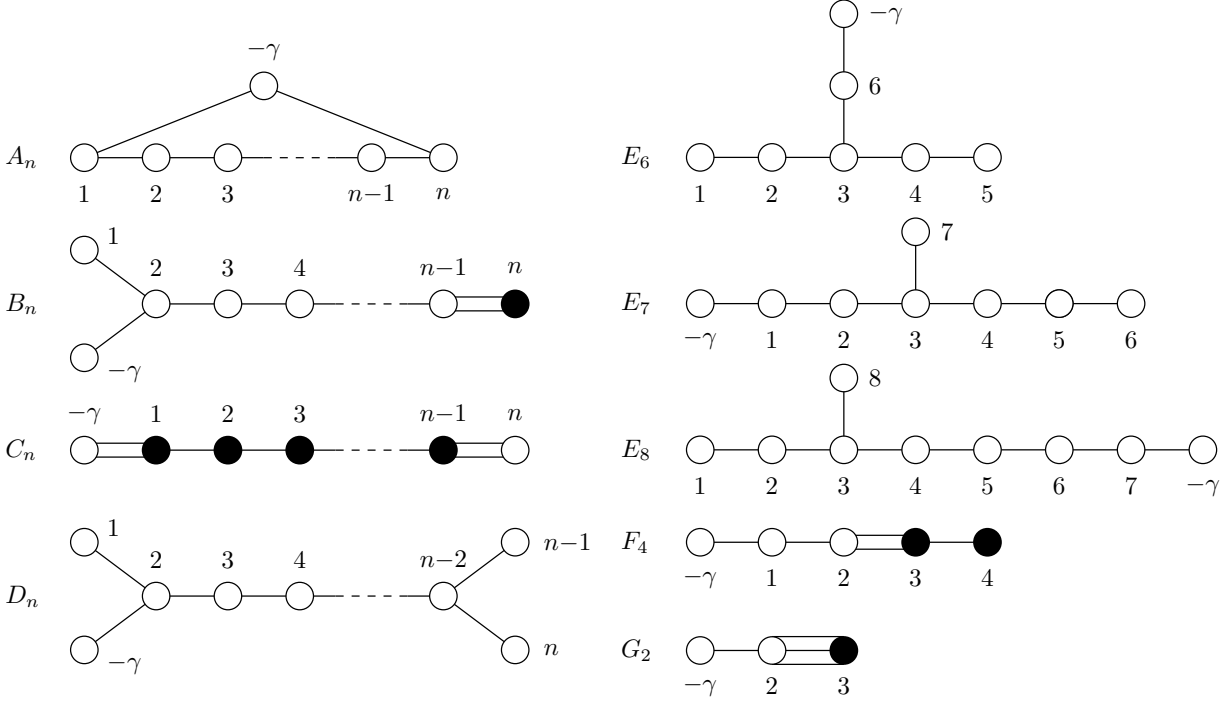
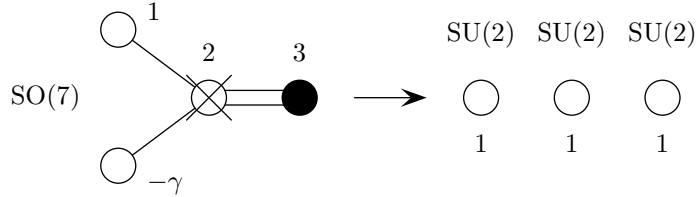


Figure 2: Extended Dynkin Diagrams of classical and exceptional simple Lie algebras.



To derive the projection matrix, we investigate the decomposition of the  $SO(7)$  generating irrep (the **8**) into three  $SU(2)$ s. Extending the Dynkin diagram with  $-\gamma$  has the effect that each weight  $w$  gets extended by one entry with the coefficient of the weight relative to  $-\gamma$ , obtained by their scalar product:  $(w, -\gamma)$ . The so-called *extended weight scheme* of the lowest non-trivial orbit of a generating irrep is determined by the LieART function `ExtendedWeightScheme[algebra, simpleRootToDrop]`, which directly removes the Dynkin digits associated to the simple root to drop, specified by `simpleRootToDrop`. For the lowest non-trivial orbit of the generating irrep of  $SO(7)$  these two steps are:

$$\begin{array}{ccc}
 \begin{array}{|c|} \hline 0 \ 0 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ -1 \ 0 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ -1 \ 1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 0 \ 1 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ -1 \ 1 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ -1 \ -1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 1 \ -1 \ 1 \\ \hline \end{array} & \xrightarrow{\text{insert } -\gamma} & \begin{array}{|c|} \hline 1 \ 0 \ -1 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 1 \ 0 \ 1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline -1 \ 0 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline -1 \ 0 \ 0 \ 1 \\ \hline \end{array} & \xrightarrow{\text{drop } 2} & \begin{array}{|c|} \hline -1 \ 0 \ 1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 1 \ 0 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 1 \ 0 \ 0 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 1 \ 0 \ -1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline -1 \ 1 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline -1 \ 0 \ 1 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline -1 \ 0 \ -1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 0 \ -1 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ 1 \ -1 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ 1 \ 1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 0 \ 0 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ 1 \ 0 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ 1 \ -1 \\ \hline \end{array}
 \end{array} \tag{43}$$

With the weight of the  $SO(7)$  generating irrep as columns of the matrix  $\hat{W}$  and the weights in the  $3SU(2)$  decomposition (right-hand side of (43)) as columns of  $\hat{W}'$  the projection matrix  $\hat{P}$  is computed as described for non-semisimple regular subalgebras as  $\hat{P} = \hat{W}' \hat{W}^+$  with the right-inverse  $\hat{W}^+$  of  $\hat{W}$ .

The definition for the branching rule  $SO(7) \rightarrow SU(2) \otimes SU(2) \otimes SU(2)$  in the file `BranchingRules.m` reads:

```
ProjectionMatrix[origin:Algebra[B][3],
  ProductAlgebra[Algebra[A][1],Algebra[A][1],Algebra[A][1]]]:=
  SemiSimpleSubalgebra[origin,2]
```

*Special Subalgebras.* Special maximal subalgebras cannot be derived from the root system. The embedding of a special subalgebra does not follow a general pattern and must be derived for every algebra-subalgebra pair individually. Generating irreps are also used to derive the subalgebra embedding, which may be simple or semisimple and can involve more than one irrep of the subalgebra. LieART is not equipped with an algorithm to determine the maximal special subalgebras, but provides an interface to declare the embeddings (`BranchingRules.m`), which can be taken from the literature [7, 8].

As an example we consider  $SO(7)$  ( $B_3$ ) again, which has  $G_2$  as special maximal subalgebra. The generating spinor irrep of  $SO(7)$ , the **8**, decomposes to the  $G_2$  singlet plus the **7**. The weights of the **8** of  $SO(7)$  and the weights of both the **1** and **7** of  $G_2$  are brought into lexicographical order to define the projection matrix:

$$\begin{array}{ccc}
 \boxed{1 \ 0 \ -1} & \rightarrow & \boxed{2 \ -1} \\
 \boxed{1 \ -1 \ 1} & \rightarrow & \boxed{1 \ 0} \\
 \boxed{0 \ 1 \ -1} & \rightarrow & \boxed{1 \ -1} \\
 \boxed{0 \ 0 \ 1} & \rightarrow & \boxed{0 \ 0} \\
 \boxed{0 \ 0 \ -1} & \rightarrow & \boxed{0 \ 0} \\
 \boxed{0 \ -1 \ 1} & \rightarrow & \boxed{-1 \ 1} \\
 \boxed{-1 \ 1 \ -1} & \rightarrow & \boxed{-1 \ 0} \\
 \boxed{-1 \ 0 \ 1} & \rightarrow & \boxed{-2 \ 1}
 \end{array} \tag{44}$$

Arranging the weights on the left-hand side of (44) as columns of  $\hat{W}$  and the weights of the right-hand side as columns of  $\hat{W}'$ , the projection matrix  $\hat{P}$  is again computed via  $\hat{P} = \hat{W}' \hat{W}^+$  with the right-inverse  $\hat{W}^+$  of  $\hat{W}$ .

These procedures are performed by the LieART function `SpecialSubalgebra[origin, targetirreps]`. The definition for the branching rule  $SO(7) \rightarrow G_2$  in the file `BranchingRules.m` reads:

```
ProjectionMatrix[origin:Algebra[B][3],ProductAlgebra[G2]]:=
  SpecialSubalgebra[origin,
  {ProductIrrep[Irrep[G2][0,0],ProductIrrep[Irrep[G2][1,0]]}]
```

Please note that irreps of the subalgebra must be gathered in a list (`{...}`), even if it is a single irrep. The projection matrix for  $SO(7) \rightarrow G_2$  is

```
In[84]:= ProjectionMatrix[B3,ProductAlgebra[G2]]
Out[84]:=  $\begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$ 
```

## 5. Benchmarks

In this section we give runtime benchmarks for the tensor decomposition and subalgebra decomposition. Because exceptional algebras have complicated Weyl reflection groups of high orders, computations involving them are much more CPU and memory demanding than with classical algebras of equal rank. In the following we give runtime benchmarks for the tensor-product decomposition and subalgebra decomposition of irreps of exceptional algebras.

We use the Mathematica command `Timing[]`, which gives the CPU time spent in the Mathematica kernel in seconds. It does not include the time needed for the display of results in the front end. As pointed out in Section 4.4.3, in `TraditionalForm` irreps are displayed by their dimensional names, while computations are performed by their Dynkin labels. The determination of the dimensional name can be very time consuming depending on the algebra and the irrep. This time is allotted to the display of computation results and thus not measured with `Timing[]`. For most of the examples below you will find significant differences between the displayed time and the wall-clock time due to this effect, but this is intended as we want to give an estimate of the actual time needed for decompositions and not the display of results. Please note that each of the following timings are taken with a newly launched Kernel to avoid speedup due to caching of intermediate results from previous computations.

The following timings were taken with Mathematica 8.0.1. on an Apple<sup>®</sup> iMac<sup>®</sup> with an Intel<sup>®</sup> Core<sup>™</sup> i5 750 (2.66 GHz) processor and 4 GB RAM.

We compute  $27^n$  for  $n = 2, \dots, 8$  in  $E_6$ :

```
In[85]:= Timing[Irrep[E6][27]^2]
Out[85]:= {0.136494,  $\overline{27} + \overline{351} + \overline{351}'$ }
```

```
In[86]:= Timing[Irrep[E6][27]^3]
Out[86]:= {0.143171,  $1 + 2(\overline{78}) + 3(\overline{650}) + 2925 + 3003 + 2(\overline{5824})$ }
```

```
In[87]:= Timing[Irrep[E6][27]^4]
Out[87]:= {0.43606,  $6(\overline{27}) + 6(\overline{351}) + 3(\overline{351}') + 8(\overline{1728}) + 6(\overline{7371}) + 6(\overline{7722}) + 17550 + 19305' + 2(\overline{34398}) + 3(\overline{51975}) + 3(\overline{54054})$ }
```

```
In[88]:= Timing[Irrep[E6][27]^5]
Out[88]:= {0.715819,  $15(\overline{27}) + 26(\overline{351}) + 20(\overline{351}') + 24(\overline{1728}) + 30(\overline{7371}) + 15(\overline{7722}) + 20(\overline{17550}) + 20(\overline{19305}) + 34398 + 46332 + 10(\overline{51975}) + 10(\overline{61425}) + 100386 + 20(\overline{112320}) + 4(\overline{314496}) + 4(\overline{359424}) + 5(\overline{386100}) + 5(\overline{412776}) + 6(\overline{494208})$ }
```

```
In[89]:= Timing[Irrep[E6][27]^6]
Out[89]:= {1.53349,  $15(1) + 65(\overline{78}) + 130(\overline{650}) + 45(\overline{2430}) + 110(\overline{2925}) + 50(\overline{3003}) + 15(\overline{3003}) + 136(\overline{5824}) + 80(\overline{5824}) + 144(\overline{34749}) + 43758 + 90(\overline{70070}) + 90(\overline{78975}) + 45(\overline{78975}) + 45(\overline{85293}) + 40(\overline{105600}) + 40(\overline{146432}) + 80(\overline{252252}) + 16(\overline{252252}) + 15(\overline{371800}) + 442442 + 5(\overline{600600}) + 30(\overline{600600}) + 5(\overline{812175}) + 45(\overline{852930}) + 45(\overline{972972}) + 5(\overline{1337050}) + 5(\overline{1559376}) + 5(\overline{1896180}) + 9(\overline{2453814}) + 10(\overline{2977975}) + 9(\overline{3007368}) + 10(\overline{3309696}) + 16(\overline{4752384})$ }
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In[90]:= Timing[Irrep[E6][27]^7]
Out[90]:= {12.9962,  $210(\overline{27}) + 385(\overline{351}) + 225(\overline{351}') + 630(\overline{1728}) + 735(\overline{7371}) + 595(\overline{7722}) + 525(\overline{17550}) + 300(\overline{19305}) + 105(\overline{19305}') + 336(\overline{34398}) + 315(\overline{46332}) + 735(\overline{51975}) + 441(\overline{54054}) + 105(\overline{61425}) + 560(\overline{112320}) + 504(\overline{314496}) + 504(\overline{359424}) + 210(\overline{386100}) + 71(\overline{393822}) + 21(\overline{412776}) + 21(\overline{459459}) + 106(\overline{494208}) + 210(\overline{579150}) + 105(\overline{638820}) + 6(\overline{741312}) + 70(\overline{853281}) + 420(\overline{967680}) + 210(\overline{1123200}) + 140(\overline{1253070}) + 210(\overline{1640925}) + 1706562 + 21(\overline{1837836}) + 210(\overline{2088450}) + 90(\overline{4200768}) + 14(\overline{4582656}) + 15(\overline{5553900}) + 105(\overline{5776056}) + 84(\overline{6110208}) + 14(\overline{6243237}) + 105(\overline{6747300}) + 15(\overline{7528950}) + 126(\overline{7601958}) + 6(\overline{8401536}) + 14(\overline{10378368}) + 14(\overline{14805504}) + 14(\overline{16540524}) + 15(\overline{17453475}) + 21(\overline{17918901}) + 21(\overline{19297278}) + 20(\overline{19768320}) + 35(\overline{30115800}) + 35(\overline{34906950})$ }
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In[91]:= Timing[Irrep[E6][27]^8]
{93.8079, 820(27) + 1960(351) + 1435(351') + 2800(1728) + 4165(7371) + 2520(7722) + 3780(17550) +
3220(19305) + 105(19305') + 1316(34398) + 1736(46332) + 3675(51975) + 1092(54054) +
1960(61425) + 196(100386) + 5040(112320) + 4144(314496) + 2400(359424) + 1120(359424) +
2660(386100) + 1260(393822) + 1652(412776) + 1260(459459) + 2856(494208) + 421(579150) +
420(638820) + 112(741312) + 2240(967680) + 2240(1123200) + 420(1253070) + 1345(1640925) +
238(2088450) + 1344(2559843) + 420(3281850) + 210(3675672) + 112(4088448) + 2688(4200768) +
840(4582656) + 20(5501925) + 420(5553900) + 1260(5776056) + 5895396 + 455(6243237) +
Out[91]:= 21(6675669) + 140(6747300) + 140(7528950) + 637(7601958) + 1260(7757100) + 28(7757100) +
630(9189180) + 560(10378368) + 448(12648636) + 560(13478400) + 672(14017536) +
140(17918901) + 28(19297278) + 14(22007700) + 140(23629320) + 28(26702676) +
272(30115800) + 140(30718116) + 7(32424678) + 35(36100350) + 301(37459422) + 14(41442192) +
252(46542600) + 280(48243195) + 35(49017150) + 64(54991872) + 448(66830400) +
20(74826180) + 64(75119616) + 21(77026950) + 28(89791416) + 42(93459366) + 35(103169430) +
56(123803316) + 56(136547775) + 70(138881925) + 70(184864680) + 64(192067200) +
90(219490128)}

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and  $78^n$  for  $n = 2, \dots, 7$  in  $E_6$ :

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In[92]:= Timing[Irrep[E6][78]^2]
Out[92]:= {0.182129, 1 + 78 + 650 + 2430 + 2925}

In[93]:= Timing[Irrep[E6][78]^3]
Out[93]:= {0.20391, 1 + 5(78) + 4(650) + 3(2430) + 4(2925) + 2(5824) + 2(5824) + 3(34749) + 43758 + 70070 +
2(105600)}

In[94]:= Timing[Irrep[E6][78]^4]
Out[94]:= {2.33734, 5(1) + 17(78) + 24(650) + 18(2430) + 26(2925) + 2(3003) + 2(3003) + 16(5824) + 16(5824) +
27(34749) + 6(43758) + 15(70070) + 6(78975) + 6(78975) + 3(85293) + 16(105600) + 8(252252) +
8(252252) + 537966 + 600600 + 600600 + 6(812175) + 6(852930) + 2(1337050) + 3(1911195) +
3(2453814)}

In[95]:= Timing[Irrep[E6][78]^5]
Out[95]:= {5.25617, 17(1) + 90(78) + 150(650) + 110(2430) + 175(2925) + 24(3003) + 24(3003) + 140(5824) +
140(5824) + 255(34749) + 50(43758) + 170(70070) + 90(78975) + 90(78975) + 51(85293) +
160(105600) + 16(146432) + 16(146432) + 120(252252) + 120(252252) + 10(537966) + 40(600600) +
40(600600) + 95(812175) + 120(852930) + 24(972972) + 24(972972) + 30(1337050) + 1559376 +
Out[95]:= 1559376 + 40(1911195) + 75(2453814) + 30(2977975) + 30(2977975) + 15(3490695) + 20(4752384) +
20(4752384) + 4969107 + 20(5054400) + 20(5054400) + 10(11655930) + 11(12514788) +
4(19160064) + 4(19160064) + 4(22843392) + 20(23795200) + 5(29422393) + 5(34906950) +
6(42134742)}

In[96]:= Timing[Irrep[E6][78]^6]
Out[96]:= {14.811, 90(1) + 542(78) + 1100(650) + 840(2430) + 1390(2925) + 270(3003) + 270(3003) + 1264(5824) +
1264(5824) + 2496(34749) + 465(43758) + 1905(70070) + 1170(78975) + 1170(78975) + 720(85293) +
1680(105600) + 320(146432) + 320(146432) + 1615(252252) + 1615(252252) + 40(371800) +
40(371800) + 115(537966) + 745(600600) + 745(600600) + 1370(812175) + 1890(852930) +
585(972972) + 585(972972) + 495(1337050) + 65(1559376) + 65(1559376) + 585(1911195) +
1350(2453814) + 760(2977975) + 760(2977975) + 45(3007368) + 45(3007368) + 66(3162159) +
66(3162159) + 90(3309696) + 90(3309696) + 450(3490695) + 15(4548180) + 560(4752384) +
560(4752384) + 15(4969107) + 480(5054400) + 480(5054400) + 80(7779200) + 80(7779200) +
245(11655930) + 515(12514788) + 240(19160064) + 240(19160064) + 80(22843392) +
640(23795200) + 115(29422393) + 144(32752512) + 144(32752512) + 210(34906950) +
36685506 + 225(42134742) + 5(44767800) + 5(44767800) + 41(47783736) + 41(47783736) +
90(47849373) + 90(47849373) + 90(53557504) + 90(53557504) + 15(59073300) + 15(59073300) +
46(64205141) + 40(64414350) + 40(64414350) + 45(66023100) + 80(115287744) + 80(115287744) +
15(119189070) + 5(200449886) + 5(203365305) + 5(221077350) + 30(226459233) + 9(252808452) +
9(252808452) + 10(303388800) + 10(303388800) + 50(348985350) + 45(350895402) +
9(366235506) + 10(476952476) + 16(734557824)}

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In[97]:= Timing[Irrep[E6][78]^7]
{334.518, 542(1) + 3962(78) + 9156(650) + 7413(2430) + 12481(2925) + 3024(3003) + 3024(3003) +
12474(5824) + 12474(5824) + 26481(34749) + 5055(43758) + 22204(70070) + 14910(78975) +
14910(78975) + 9786(85293) + 19376(105600) + 5236(146432) + 5236(146432) + 21420(252252) +
21420(252252) + 1155(371800) + 1155(371800) + 1505(537966) + 12145(600600) + 12145(600600) +
19621(812175) + 28161(852930) + 10899(972972) + 10899(972972) + 8085(1337050) +
1701(1559376) + 1701(1559376) + 175(1896180) + 175(1896180) + 8910(1911195) +
22365(2453814) + 14770(2977975) + 14770(2977975) + 1575(3007368) + 1575(3007368) +
2016(3162159) + 2016(3162159) + 2870(3309696) + 2870(3309696) + 9729(3490695) +
686(4548180) + 11760(4752384) + 11760(4752384) + 231(4969107) + 9450(5054400) +
9450(5054400) + 2990(7779200) + 2990(7779200) + 5075(11655930) + 13216(12514788) +
420(14152320) + 420(14152320) + 6804(19160064) + 6804(19160064) + 1624(22843392) +
15114(23795200) + 2730(29422393) + 5544(32752512) + 5544(32752512) + 6111(34906950) +
21(36685506) + 5901(42134742) + 196(42398720) + 196(42398720) + 455(44767800) +
455(44767800) + 490(45741696) + 490(45741696) + 2156(47783736) + 2156(47783736) +
3870(47849373) + 3870(47849373) + 3430(53557504) + 3430(53557504) + 300(54991872) +
Out[97]:= 300(54991872) + 1260(59073300) + 1260(59073300) + 2541(64205141) + 1400(64414350) +
1400(64414350) + 2079(66023100) + 560(85974525) + 560(85974525) + 315(89791416) +
315(89791416) + 3360(115287744) + 3360(115287744) + 525(119189070) + 106(152423700) +
175(200449886) + 140(203365305) + 70(212838912) + 70(212838912) + 386(221077350) +
225961450 + 1575(226459233) + 560(236487680) + 560(236487680) + 819(252808452) +
819(252808452) + 840(303388800) + 840(303388800) + 3465(348985350) + 2100(350895402) +
294(366235506) + 210(392837445) + 210(392837445) + 505(466237200) + 505(466237200) +
525(476952476) + 14(532097280) + 14(532097280) + 504(537567030) + 504(537567030) +
70(598998400) + 70(598998400) + 210(625532544) + 210(625532544) + 15(649806300) +
15(649806300) + 126(688740975) + 966(734557824) + 105(797489550) + 140(929510400) +
140(929510400) + 21(944929700) + 420(1051315200) + 420(1051315200) + 216(1177830720) +
216(1177830720) + 6(1445558400) + 216(1478062080) + 210(1525620096) + 210(1525620096) +
84(1544524800) + 84(1544524800) + 14(1643241600) + 14(1643241600) + 14(3116305920) +
20(3203785728) + 20(3203785728) + 14(3256917300) + 84(3548188800) + 119(3863940795) +
15(4035297123) + 105(4129204716) + 21(4790483775) + 35(4942962024) + 35(4942962024) +
21(4973434830) + 141(4991693850) + 35(8713554850)}

```

and  $248^n$  for  $n = 2, \dots, 7$  in  $E_8$ :

```

In[98]:= Timing[Irrep[E8][248]^2]
Out[98]:= {0.850115, 1 + 248 + 3875 + 27000 + 30380}

In[99]:= Timing[Irrep[E8][248]^3]
Out[99]:= {0.883473, 1 + 5(248) + 3(3875) + 3(27000) + 4(30380) + 2(147250) + 3(779247) + 1763125 + 2450240 +
2(4096000)}

In[100]:= Timing[Irrep[E8][248]^4]
Out[100]:= {31.9855, 5(1) + 16(248) + 17(3875) + 18(27000) + 23(30380) + 13(147250) + 21(779247) +
6(1763125) + 12(2450240) + 16(4096000) + 3(4881384) + 6(6696000) + 8(26411008) +
6(70680000) + 6(76271625) + 79143000 + 146325270 + 2(203205000) + 3(281545875) +
3(344452500)}

In[101]:= Timing[Irrep[E8][248]^5]
Out[101]:= {49.9295, 16(1) + 79(248) + 90(3875) + 100(27000) + 136(30380) + 100(147250) + 170(779247) +
50(1763125) + 109(2450240) + 140(4096000) + 36(4881384) + 70(6696000) + 100(26411008) +
75(70680000) + 90(76271625) + 10(79143000) + 36(146325270) + 30(203205000) +
40(281545875) + 24(301694976) + 60(344452500) + 15(820260000) + 30(1094951000) +
20(2172667860) + 20(2275896000) + 2642777280 + 10(3929713760) + 10(4825673125) +
6899079264 + 20(8634368000) + 4(12692520960) + 5(17535336000) + 4(20288765952) +
5(21039669000) + 6(23592339045)}

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In[102]:= **Timing[Irrep[E8][248]^6]**

{90.112,79(1) + 421(248) + 575(3875) + 675(27000) + 924(30380) + 775(147250) + 1386(779247) + 415(1763125) + 1011(2450240) + 1240(4096000) + 405(4881384) + 765(6696000) + 1144(26411008) + 895(70680000) + 1125(76271625) + 115(79143000) + 554(146325270) + 410(203205000) + 510(281545875) + 456(301694976) + 855(344452500) + 315(820260000) + 605(1094951000) + 405(2172667860) + 470(2275896000) + 15(2642777280) + 15(2903770000) + 195(3929713760) + 45(4076399250) + 325(4825673125) + 125(6899079264) + 480(8634368000) + 80(8634368000') + 80(12692520960) + 115(17535336000) + 216(20288765952) + 165(21039669000) + 180(23592339045) + 144(45329752170) + 45(63513702720) + 45(66393847000) + 69176971200 + 90(83080364250) + 90(85424220000) + 40(110977024000) + 40(124436480000) + 15(152883490500) + 15(220778105625) + 80(234550030000) + 267413986840 + 30(355647996000) + 5(417933862500) + 5(492957660000) + 45(508731738750) + 45(574197082368) + 5(627099023250) + 9(841900509450) + 5(919045960000) + 10(1041872676000) + 9(1283242632840) + 10(1349926375875) + 16(1813461073920)}

In[103]:= **Timing[Irrep[E8][248]^7]**

{7636.29,421(1) + 2674(248) + 4081(3875) + 5061(27000) + 7007(30380) + 6580(147250) + 12306(779247) + 3850(1763125) + 9779(2450240) + 11830(4096000) + 4452(4881384) + 8226(6696000) + 12830(26411008) + 10465(70680000) + 13566(76271625) + 1330(79143000) + 7651(146325270) + 5370(203205000) + 6405(281545875) + 7014(301694976) + 11445(344452500) + 5250(820260000) + 9744(1094951000) + 6714(2172667860) + 8204(2275896000) + 231(2642777280) + 455(2903770000) + 3290(3929713760) + 1281(4076399250) + 6475(4825673125) + 3171(6899079264) + 8840(8634368000) + 2296(8634368000') + 1414(12692520960) + 2240(17535336000) + 5124(20288765952) + 3696(21039669000) + 3717(23592339045) + 4410(45329752170) + 1449(63513702720) + 1674(66393847000) + 21(69176971200) + 2765(83080364250) + 3150(85424220000) + 1190(110977024000) + 1820(124436480000) + 420(152883490500) + 1155(220778105625) + 300(223850628000) + 2835(234550030000) + 546(267413986840) + 1155(355647996000) + 105(417532087000) + 140(417933862500) + 175(492957660000) + 1575(508731738750) + 315(560213725500) + 2170(574197082368) + 280(627099023250) + 294(841900509450) + 875(919045960000) + 420(1041872676000) + 560(1198018560000) + 735(1283242632840) + 756(1349926375875) + 1473701482500 + 756(1813461073920) + 105(3067797300750) + 504(3191795712000) + 504(3233052753920) + 105(3431612952000) + 3754721200320 + 210(3950782290000) + 70(4007202600000) + 210(4189713446646) + 21(4490627295000) + 70(4779643627500) + 5006235840320 + 210(6458110083072) + 21(7723951192125) + 420(8145764352000) + 140(8715491428800) + 6(10701806469120) + 210(12737135385000) + 210(13532264250750) + 84(19994148864000) + 84(26125438976000) + 105(26461348084080) + 14(28123973939490) + 105(29369472656250) + 14(30014459904000) + 15(33372802062000) + 6(33699815424000) + 126(33943999320000) + 14(46678711926784) + 21(53540697687750) + 14(56860936405000) + 21(57306919524192) + 20(57591234560000) + 15(58549130859375) + 35(85471274280000) + 35(107701303073000)}

As an example for subalgebra decomposition of a large irrep we decompose the **600600** of  $E_6$  to  $SU(3) \times SU(3) \times SU(3)$  :

```
In[104]:= Timing[DecomposeIrrep[Irrep[E6][600600], ProductAlgebra[SU3, SU3, SU3]]]
{124.17, (1, 1, 1) + 10(3, 3, 3) + 10(3, 3, 3) + 5(8, 1, 1) + 5(1, 8, 1) + 5(1, 1, 8) + 10(6, 3, 3) + 10(3, 3, 6) +
10(3, 6, 3) + 10(3, 6, 3) + 10(3, 3, 6) + 10(6, 3, 3) + (10, 1, 1) + (10, 1, 1) + (1, 10, 1) + (1, 10, 1) +
(1, 1, 10) + (1, 1, 10) + 8(6, 3, 6) + 8(6, 6, 3) + 8(3, 6, 6) + 8(6, 6, 3) + 8(3, 6, 6) + 8(6, 6, 3) + 8(6, 3, 6) +
11(8, 8, 1) + 11(8, 1, 8) + 11(1, 8, 8) + 7(6, 6, 6) + 7(6, 6, 6) + 5(10, 8, 1) + 4(10, 1, 8) + 5(8, 10, 1) +
4(8, 10, 1) + 4(8, 1, 10) + 5(8, 1, 10) + 4(10, 8, 1) + 5(10, 1, 8) + 4(1, 10, 8) + 5(1, 8, 10) + 4(1, 8, 10) +
5(1, 10, 8) + 10(15, 3, 3) + 10(3, 15, 3) + 10(3, 3, 15) + 10(15, 3, 3) + 10(3, 15, 3) + 10(3, 3, 15) +
(10, 10, 1) + 2(10, 10, 1) + 2(10, 1, 10) + 2(10, 1, 10) + 2(10, 10, 1) + 2(10, 10, 1) + 2(10, 1, 10) +
(10, 1, 10) + 2(1, 10, 10) + 2(1, 10, 10) + (1, 10, 10) + 2(1, 10, 10) + 2(15', 3, 3) + 2(3, 15', 3) +
2(3, 3, 15') + 2(15', 3, 3) + 2(3, 15', 3) + 2(3, 3, 15') + 25(8, 8, 8) + 9(6, 15, 3) + 9(6, 3, 15) + 9(15, 3, 6) +
9(15, 6, 3) + 9(3, 15, 6) + 9(3, 6, 15) + 9(15, 6, 3) + 9(15, 3, 6) + 9(3, 6, 15) + 9(3, 15, 6) + 9(6, 15, 3) +
9(6, 3, 15) + (6, 15', 3) + (6, 3, 15') + (15', 3, 6) + (15', 6, 3) + (3, 15', 6) + (3, 6, 15') + (15', 6, 3) +
(15', 3, 6) + (3, 6, 15') + (3, 15', 6) + (6, 15', 3) + (6, 3, 15') + 8(10, 8, 8) + 8(8, 10, 8) + 8(8, 8, 10) +
8(8, 8, 10) + 8(8, 10, 8) + 8(10, 8, 8) + 6(6, 15, 6) + 6(6, 6, 15) + 6(15, 6, 6) + 6(15, 6, 6) + 6(6, 6, 15) +
6(6, 15, 6) + (6, 15', 6) + (6, 6, 15') + (15', 6, 6) + (15', 6, 6) + (6, 6, 15') + (6, 15', 6) + 2(10, 10, 8) +
2(10, 8, 10) + 2(10, 8, 10) + 2(10, 10, 8) + 2(8, 10, 10) + 2(8, 10, 10) + 2(8, 10, 10) + 2(8, 10, 10) +
2(10, 10, 8) + 2(10, 8, 10) + 2(10, 8, 10) + 2(10, 10, 8) + 2(27, 1, 1) + 2(1, 27, 1) + 2(1, 1, 27) +
4(24, 3, 3) + 4(3, 24, 3) + 4(3, 3, 24) + 4(24, 3, 3) + 4(3, 24, 3) + 4(3, 3, 24) + (21, 6, 3) + (6, 3, 21) +
(3, 21, 6) + (3, 6, 21) + (21, 3, 6) + (6, 21, 3) + 8(15, 15, 3) + 8(15, 3, 15) + 8(3, 15, 15) + 8(15, 15, 3) +
8(15, 3, 15) + 8(3, 15, 15) + 3(24, 3, 6) + 4(24, 6, 3) + 3(6, 24, 3) + 4(6, 3, 24) + 2(15, 15', 3) +
(15, 3, 15') + (15', 15, 3) + 2(15', 3, 15) + (3, 15', 15) + 2(3, 15, 15') + 4(3, 24, 6) + 3(3, 6, 24) +
2(15', 15, 3) + (15', 3, 15) + (15, 15', 3) + 2(15, 3, 15') + 3(24, 6, 3) + 4(24, 3, 6) + 3(3, 24, 6) +
4(3, 6, 24) + (3, 15, 15') + 2(3, 15', 15) + 4(6, 24, 3) + 3(6, 3, 24) + 6(6, 15, 15) + 6(15, 15, 6) +
6(15, 6, 15) + 6(15, 6, 15) + 6(15, 15, 6) + 6(6, 15, 15) + 4(27, 8, 1) + 4(27, 1, 8) + 4(8, 27, 1) +
4(8, 1, 27) + 4(1, 27, 8) + 4(1, 8, 27) + 2(24, 6, 6) + (6, 15, 15') + 2(6, 24, 6) + 2(6, 6, 24) + (15, 15', 6) +
(15', 6, 15) + (15', 15, 6) + (15, 6, 15') + 2(24, 6, 6) + 2(6, 24, 6) + 2(6, 6, 24) + (6, 15', 15) +
(10, 27, 1) + (10, 1, 27) + (27, 10, 1) + (27, 10, 1) + (27, 1, 10) + (27, 1, 10) + (10, 27, 1) + (10, 1, 27) +
(1, 10, 27) + (1, 27, 10) + (1, 27, 10) + (1, 10, 27) + 3(24, 15, 3) + 2(24, 3, 15) + 2(15, 24, 3) +
3(15, 3, 24) + 2(3, 15, 24) + 3(3, 24, 15) + 3(15, 24, 3) + 2(15, 3, 24) + 2(24, 15, 3) + 3(24, 3, 15) +
2(3, 24, 15) + 3(3, 15, 24) + 7(27, 8, 8) + 7(8, 27, 8) + 7(8, 8, 27) + (35, 8, 1) + (35, 1, 8) + (8, 35, 1) +
(8, 1, 35) + (1, 35, 8) + (1, 8, 35) + 4(15, 15, 15) + 4(15, 15, 15) + 2(10, 27, 8) + (10, 8, 27) +
2(27, 10, 8) + (27, 8, 10) + 2(27, 8, 10) + (27, 10, 8) + (8, 10, 27) + 2(8, 27, 10) + (8, 27, 10) +
2(8, 10, 27) + (10, 27, 8) + 2(10, 8, 27) + 2(24, 15, 6) + (24, 6, 15) + (6, 15, 24) + 2(6, 24, 15) +
(15, 24, 6) + 2(15, 6, 24) + 2(15, 24, 6) + (15, 6, 24) + 2(24, 6, 15) + (24, 15, 6) + (6, 24, 15) +
2(6, 15, 24) + (35, 10, 1) + (10, 1, 35) + (35, 1, 10) + (10, 35, 1) + (1, 10, 35) + (1, 35, 10) +
(10, 27, 10) + (27, 10, 10) + (10, 10, 27) + (42, 3, 3) + (3, 42, 3) + (3, 3, 42) + (42, 3, 3) + (3, 42, 3) +
(3, 3, 42) + (24, 24, 3) + (24, 3, 24) + (3, 24, 24) + (24, 24, 3) + (24, 3, 24) + (3, 24, 24) + (35, 8, 8) +
(35, 8, 8) + (8, 35, 8) + (8, 35, 8) + (8, 8, 35) + (8, 8, 35) + (42, 3, 6) + (42, 6, 3) + (6, 42, 3) +
(6, 3, 42) + (3, 42, 6) + (3, 6, 42) + (42, 6, 3) + (42, 3, 6) + (3, 42, 6) + (3, 6, 42) + (6, 42, 3) + (6, 3, 42) +
(24, 15, 15) + (15, 15, 24) + (15, 24, 15) + (15, 24, 15) + (15, 15, 24) + (24, 15, 15) + (27, 27, 1) +
(27, 1, 27) + (1, 27, 27) + (42, 3, 15) + (15, 42, 3) + (3, 15, 42) + (42, 15, 3) + (15, 3, 42) + (3, 42, 15) +
(27, 27, 8) + (27, 8, 27) + (8, 27, 27) + (42, 15, 6) + (6, 42, 15) + (15, 6, 42) + (42, 6, 15) + (15, 42, 6) +
(6, 15, 42)}
```

Out [104] :=

## 6. L<sup>A</sup>T<sub>E</sub>X Package

LieART comes with a L<sup>A</sup>T<sub>E</sub>X package (`lieart.sty` in the subdirectory `latex/`) that defines commands to display irreps, roots and weights properly (see Table 6.1), which are displayed by LieART using the LaTeXForm on an appropriate expression, e.g.:

```
In[105]:= DecomposeProduct[Irrep[SU3][8], Irrep[SU3][8]]//LaTeXForm
Out[105]:=  $\mathbb{10} + 2\overline{\mathbb{10}} + \mathbb{10} + \overline{\mathbb{10}} + \mathbb{10}$ 
```

Command Example	Output	Description
<code>\irrep{10}</code>	<b>10</b>	dimensional name of irrep
<code>\irrepbar{10}</code>	$\overline{\mathbb{10}}$	dimensional name of conjugated irrep
<code>\irrep[2]{175}</code>	<b>175''</b>	number of primes as optional parameter
<code>\irrepsub{8}{s}</code>	$\mathbf{8}_s$	irrep with subscript, e.g., irreps of SO(8)
<code>\irrepbarsub{10}{a}</code>	$\overline{\mathbb{10}}_a$	conjugated irrep with subscript, e.g., for labeling anti-symmetric product
<code>\dynkin{0,1,0,0}</code>	(0100)	Dynkin label of irrep
<code>\dynkincomma{0,10,0,0}</code>	(0,10,0,0)	for Dynkin labels with at least one digit larger than 9 (also available as <code>\root</code> , <code>\rootorthogonal</code> , <code>\weightalpha</code> and <code>\weightorthogonal</code> for negative integers)
<code>\weight{0,1,0,-1}</code>	$\begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}$	weight in $\omega$ -basis
<code>\rootomega{-1,2,-1,0}</code>	$\begin{bmatrix} -1 & 2 & -1 & 0 \end{bmatrix}$	root in $\omega$ -basis

Table 6.1: L<sup>A</sup>T<sub>E</sub>X commands defined in supplemental style file `lieart.sty`

## 7. Conclusions and Outlook

We have programmed the Mathematica application LieART, which brings Lie-algebra and representation-theory related computations to Mathematica. It provides functions for the decomposition of tensor products and branching rules of irreducible representations, which are of high interest in particle physics, especially unified model building. LieART exploits the Weyl reflection group in most of its applications, making it fast and memory efficient. The user interface focuses on usability, allowing one to enter irreducible representations by their dimensional name and giving results in textbook style. We have reproduced and extended existing tabulated data on irreducible representations, their tensor products and branching rules.

In future versions we plan to add more branching rules to LieART. Currently, only a selection of common branching rules used in the tables are implemented. We consider the tables given in the appendix as dynamical: They are included in LieART as Mathematica notebooks and can easily be modified and extended by the user. Tables for algebras of high rank and/or higher dimensional irreducible representations have large CPU time and high memory consumption. Nevertheless, we plan to extend the tables even further and make them available online in a standard format (pdf and/or html).

## 8. Acknowledgments

We thank Carl Albright for many useful discussions that led to the development of LieART. We also thank Tanja Feger for checking the tables against those found in [7]. We thank Savdeep Sethi and Bruce Westbury for runtime comparisons with LiE [12], which led to the implementation of Klymuk's formula for tensor products. We thank Florian Hartmann, Constantin Sluka and Giulia Ferlito for reporting bugs in the initial version.

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## A. Tables

We present here tables of properties of irreps, such as Dynkin labels, dimensional names, indices, congruency classes and the number of singlets in various subalgebra branchings in Section A.1, as well as tables of tensor products in Section A.2 and subalgebra branching rules in Section A.3 for many classical and all exceptional Lie algebras. In presentation style, selection of irreps and subalgebra branching we closely follow [7], which has been the definitive reference for unified model building since its publication. The tables were created by the supplemental package `Tables.m`, which uses LieART for the computation. The tables can also be found as Mathematica notebooks in the LieART documentation integrated into the Mathematica documentation center as “Representation Properties”, “Tensor Products” and “Branching Rules” under the section “Tables” on the LieART documentation home. Since LieART comes with the functions that generate the tables, the user may extend them to the limit of his or her computer power.

Algebra	Irrep Properties		Tensor Products		Branching Rules	
	Number	Page	Number	Page	Number	Page
SU(2)	A.2	46	A.35	84	A.68	118
SU(3)	A.3	47	A.36	85	A.69	118
SU(4)	A.4	49	A.37	87	A.70	119
SU(5)	A.5	51	A.38	89	A.71	120
SU(6)	A.6	53	A.39	91	A.72	121
SU(7)	A.7	55	A.40	93	A.73	123
SU(8)	A.8	56	A.41	94	A.74	125
SU(9)	A.9	57	A.42	95	A.75	127
SU(10)	A.10	57	A.43	95	A.76	129
SU(11)	A.11	58	A.44	96	A.77	131
SU(12)	A.12	58	A.45	96	A.78	133
SO(7)	A.13	59	A.46	97	A.79	135
SO(8)	A.14	61	A.47	99	A.80	136
SO(9)	A.15	63	A.48	101	A.81	138
SO(10)	A.16	65	A.49	103	A.82	140
SO(11)	A.17	67	A.50	105	–	–
SO(12)	A.18	68	A.51	106	–	–
SO(13)	A.19	69	A.52	107	–	–
SO(14)	A.20	69	A.53	107	A.83	143
SO(18)	A.21	70	A.54	108	A.84	144
SO(22)	A.22	70	A.55	108	A.85	144
SO(26)	A.23	70	A.56	108	A.86	144
Sp(4)	A.24	71	A.57	109	–	–
Sp(6)	A.25	73	A.58	110	–	–
Sp(8)	A.26	75	A.59	111	–	–
Sp(10)	A.27	76	A.60	112	–	–
Sp(12)	A.28	77	A.61	113	–	–
E <sub>6</sub>	A.30	78	A.63	114	A.87	145
E <sub>7</sub>	A.31	79	A.64	115	A.88	147
E <sub>8</sub>	A.32	79	A.65	115	A.89	148
F <sub>4</sub>	A.33	80	A.66	115	–	–
G <sub>2</sub>	A.34	82	A.67	117	–	–

Table A.1: Table of tables

A.1. Properties of Irreducible Representations

A.1.1.  $SU(N)$

Table A.2:  $SU(2)$  Irreps

Dynkin label	Dimension (name)	l (index)	Duality
(1)	<b>2</b>	1	1
(2)	<b>3</b>	4	0
(3)	<b>4</b>	10	1
(4)	<b>5</b>	20	0
(5)	<b>6</b>	35	1
(6)	<b>7</b>	56	0
(7)	<b>8</b>	84	1
(8)	<b>9</b>	120	0
(9)	<b>10</b>	165	1
(10)	<b>11</b>	220	0
(11)	<b>12</b>	286	1
(12)	<b>13</b>	364	0
(13)	<b>14</b>	455	1
(14)	<b>15</b>	560	0
(15)	<b>16</b>	680	1
(16)	<b>17</b>	816	0
(17)	<b>18</b>	969	1
(18)	<b>19</b>	1140	0
(19)	<b>20</b>	1330	1
(20)	<b>21</b>	1540	0
(21)	<b>22</b>	1771	1
(22)	<b>23</b>	2024	0
(23)	<b>24</b>	2300	1
(24)	<b>25</b>	2600	0
(25)	<b>26</b>	2925	1
(26)	<b>27</b>	3276	0
(27)	<b>28</b>	3654	1
(28)	<b>29</b>	4060	0
(29)	<b>30</b>	4495	1
(30)	<b>31</b>	4960	0
(31)	<b>32</b>	5456	1
(32)	<b>33</b>	5984	0
(33)	<b>34</b>	6545	1
(34)	<b>35</b>	7140	0
(35)	<b>36</b>	7770	1
(36)	<b>37</b>	8436	0
(37)	<b>38</b>	9139	1
(38)	<b>39</b>	9880	0
(39)	<b>40</b>	10660	1
(40)	<b>41</b>	11480	0
(41)	<b>42</b>	12341	1
(42)	<b>43</b>	13244	0
(43)	<b>44</b>	14190	1
(44)	<b>45</b>	15180	0
(45)	<b>46</b>	16215	1
(46)	<b>47</b>	17296	0

Table A.3: SU(3) Irreps

Dynkin label	Dimension (name)	1 (index)	Triality	SU(2) singlets
(10)	<b>3</b>	1	1	1
(20)	<b>6</b>	5	2	1
(11)	<b>8</b>	6	0	1*
(30)	<b>10</b>	15	0	1
(21)	<b>15</b>	20	1	1
(40)	<b>15'</b>	35	1	1
(05)	<b>21</b>	70	1	1
(13)	<b>24</b>	50	1	1
(22)	<b>27</b>	54	0	1*
(60)	<b>28</b>	126	0	1
(41)	<b>35</b>	105	0	1
(70)	<b>36</b>	210	1	1
(32)	<b>42</b>	119	1	1
(08)	<b>45</b>	330	1	1
(51)	<b>48</b>	196	1	1
(90)	<b>55</b>	495	0	1
(24)	<b>60</b>	230	1	1
(16)	<b>63</b>	336	1	1
(33)	<b>64</b>	240	0	1*
(10, 0)	<b>66</b>	715	1	1
(0, 11)	<b>78</b>	1001	1	1
(71)	<b>80</b>	540	0	1
(52)	<b>81</b>	405	0	1
(43)	<b>90</b>	435	1	1
(12, 0)	<b>91</b>	1365	0	1
(81)	<b>99</b>	825	1	1
(62)	<b>105</b>	665	1	1
(13, 0)	<b>105'</b>	1820	1	1
(35)	<b>120</b>	730	1	1
(19)	<b>120'</b>	1210	1	1
(0, 14)	<b>120''</b>	2380	1	1
(44)	<b>125</b>	750	0	1*
(27)	<b>132</b>	1034	1	1
(15, 0)	<b>136</b>	3060	0	1
(10, 1)	<b>143</b>	1716	0	1
(16, 0)	<b>153</b>	3876	1	1
(63)	<b>154</b>	1155	0	1
(82)	<b>162</b>	1539	0	1
(54)	<b>165</b>	1210	1	1
(11, 1)	<b>168</b>	2366	1	1
(0, 17)	<b>171</b>	4845	1	1
(18, 0)	<b>190</b>	5985	0	1
(73)	<b>192</b>	1744	1	1
(92)	<b>195</b>	2210	1	1
(1, 12)	<b>195'</b>	3185	1	1
(46)	<b>210</b>	1855	1	1
(19, 0)	<b>210'</b>	7315	1	1
(55)	<b>216</b>	1890	0	1*

\*SU(2) $\times$ U(1) singlet.

Table A.3: SU(3) Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Triality	SU(2) singlets
(13, 1)	<b>224</b>	4200	0	1
(2, 10)	<b>231</b>	3080	1	1
(0, 20)	<b>231'</b>	8855	1	1
(38)	<b>234</b>	2535	1	1
(21, 0)	<b>253</b>	10626	0	1
(14, 1)	<b>255</b>	5440	1	1
(74)	<b>260</b>	2730	0	1
(11, 2)	<b>270</b>	4185	0	1
(65)	<b>273</b>	2821	1	1
(22, 0)	<b>276</b>	12650	1	1
(93)	<b>280</b>	3570	0	1
(1, 15)	<b>288</b>	6936	1	1
(0, 23)	<b>300</b>	14950	1	1
(12, 2)	<b>312</b>	5564	1	1
(84)	<b>315</b>	3885	1	1
(16, 1)	<b>323</b>	8721	0	1
(24, 0)	<b>325</b>	17550	0	1
(10, 3)	<b>330</b>	4895	1	1
(57)	<b>336</b>	4060	1	1
(66)	<b>343</b>	4116	0	1*
(25, 0)	<b>351</b>	20475	1	1
(2, 13)	<b>357</b>	7259	1	1
(17, 1)	<b>360</b>	10830	1	1
(49)	<b>375</b>	5375	1	1
(0, 26)	<b>378</b>	23751	1	1
(3, 11)	<b>384</b>	6560	1	1
(1, 18)	<b>399</b>	13300	1	1
(85)	<b>405</b>	5670	0	1
(14, 2)	<b>405'</b>	9315	0	1
(27, 0)	<b>406</b>	27405	0	1
(76)	<b>420</b>	5810	1	1
(28, 0)	<b>435</b>	31465	1	1
(10, 4)	<b>440</b>	7260	0	1
(19, 1)	<b>440'</b>	16170	0	1
(12, 3)	<b>442</b>	8619	0	1
(15, 2)	<b>456</b>	11780	1	1
(0, 29)	<b>465</b>	35960	1	1
(95)	<b>480</b>	7720	1	1
(20, 1)	<b>483</b>	19481	1	1
(30, 0)	<b>496</b>	40920	0	1
(68)	<b>504</b>	7980	1	1
(13, 3)	<b>504'</b>	11130	1	1
(11, 4)	<b>510</b>	9605	1	1
(2, 16)	<b>510'</b>	14705	1	1
(77)	<b>512</b>	8064	0	1*

\*SU(2) $\times$ U(1) singlet.



Table A.4: SU(4) Irreps

Dynkin label	Dimension (name)	l (index)	Quadrality	SU(3) singlets
(100)	<b>4</b>	1	1	1
(010)	<b>6</b>	2	2	0
(200)	<b>10</b>	6	2	1
(101)	<b>15</b>	8	0	1*
(011)	<b>20</b>	13	1	0
(020)	<b>20'</b>	16	0	0
(003)	<b>20''</b>	21	1	1
(400)	<b>35</b>	56	0	1
(201)	<b>36</b>	33	1	1
(210)	<b>45</b>	48	0	0
(030)	<b>50</b>	70	2	0
(500)	<b>56</b>	126	1	1
(120)	<b>60</b>	71	1	0
(111)	<b>64</b>	64	2	0
(301)	<b>70</b>	98	2	1
(202)	<b>84</b>	112	0	1*
(310)	<b>84'</b>	133	1	0
(600)	<b>84''</b>	252	2	1
(040)	<b>105</b>	224	0	0
(104)	<b>120</b>	238	1	1
(007)	<b>120'</b>	462	1	1
(220)	<b>126</b>	210	2	0
(112)	<b>140</b>	203	1	0
(031)	<b>140'</b>	259	1	0
(410)	<b>140''</b>	308	2	0
(302)	<b>160</b>	296	1	1
(800)	<b>165</b>	792	0	1
(121)	<b>175</b>	280	0	0
(501)	<b>189</b>	504	0	1
(050)	<b>196</b>	588	2	0
(015)	<b>216</b>	630	1	0
(900)	<b>220</b>	1287	1	1
(023)	<b>224</b>	504	1	0
(311)	<b>256</b>	512	0	0
(402)	<b>270</b>	666	2	1
(230)	<b>280</b>	672	0	0
(140)	<b>280'</b>	742	1	0
(601)	<b>280''</b>	966	1	1
(10, 0, 0)	<b>286</b>	2002	2	1
(212)	<b>300</b>	580	2	0
(303)	<b>300'</b>	720	0	1*
(610)	<b>315</b>	1176	0	0
(060)	<b>336</b>	1344	0	0
(221)	<b>360</b>	762	1	0
(420)	<b>360'</b>	1056	0	0
(0, 0, 11)	<b>364</b>	3003	1	1
(131)	<b>384</b>	896	2	0
(701)	<b>396</b>	1716	2	1

\*SU(3) $\times$ U(1) singlet.

Table A.4: SU(4) Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Quadrality	SU(3) singlets
(411)	<b>420</b>	1113	1	0
(205)	<b>420'</b>	1337	1	1
(710)	<b>440</b>	2046	1	0
(12, 0, 0)	<b>455</b>	4368	0	1
(330)	<b>480</b>	1464	1	0
(403)	<b>500</b>	1525	1	1
(051)	<b>504</b>	1806	1	0
(213)	<b>540</b>	1359	1	0
(240)	<b>540'</b>	1764	2	0
(520)	<b>540''</b>	2007	1	0
(070)	<b>540'''</b>	2772	2	0
(108)	<b>540''''</b>	2871	1	1
(13, 0, 0)	<b>560</b>	6188	1	1
(810)	<b>594</b>	3366	2	0
(602)	<b>616</b>	2464	0	1
(321)	<b>630</b>	1722	2	0
(511)	<b>640</b>	2176	2	0
(14, 0, 0)	<b>680</b>	8568	2	1
(901)	<b>715</b>	4576	0	1
(222)	<b>729</b>	1944	0	0
(141)	<b>735</b>	2352	0	0
(430)	<b>750</b>	2850	2	0
(132)	<b>756</b>	2205	1	0
(503)	<b>770</b>	2926	2	1
(620)	<b>770'</b>	3542	2	0
(019)	<b>780</b>	5291	1	0
(0, 0, 15)	<b>816</b>	11628	1	1
(404)	<b>825</b>	3080	0	1*
(080)	<b>825'</b>	5280	0	0
(160)	<b>840</b>	3906	1	0
(702)	<b>864</b>	4248	1	1
(412)	<b>875</b>	2800	0	0
(043)	<b>900</b>	3585	1	0
(116)	<b>924</b>	3927	1	0
(10, 0, 1)	<b>924'</b>	7007	1	1
(250)	<b>945</b>	4032	0	0
(313)	<b>960</b>	3008	2	0
(16, 0, 0)	<b>969</b>	15504	0	1
(124)	<b>1000</b>	3450	1	0
(10, 1, 0)	<b>1001</b>	8008	0	0
(027)	<b>1056</b>	5896	1	0
(035)	<b>1100</b>	5115	1	0
(306)	<b>1120</b>	5208	1	1
(17, 0, 0)	<b>1140</b>	20349	1	1
(802)	<b>1170</b>	6942	2	1
(11, 0, 1)	<b>1170'</b>	10374	2	1

\*SU(3) $\times$ U(1) singlet.

Table A.5: SU(5) Irreps

Dynkin label	Dimension (name)	1 (index)	Quintality	SU(4) singlets	SU(3) $\times$ SU(2) singlets
(1000)	<b>5</b>	1	1	1	0
(0100)	<b>10</b>	3	2	0	1
(2000)	<b>15</b>	7	2	1	0
(1001)	<b>24</b>	10	0	1*	1*
(0003)	<b>35</b>	28	2	1	0
(0011)	<b>40</b>	22	2	0	0
(0101)	<b>45</b>	24	1	0	0
(0020)	<b>50</b>	35	1	0	1
(2001)	<b>70</b>	49	1	1	0
(0004)	<b>70'</b>	84	1	1	0
(0110)	<b>75</b>	50	0	0	1*
(0012)	<b>105</b>	91	1	0	0
(2010)	<b>126</b>	105	0	0	0
(5000)	<b>126'</b>	210	0	1	0
(3001)	<b>160</b>	168	2	1	0
(1101)	<b>175</b>	140	2	0	1
(1200)	<b>175'</b>	175	0	0	0
(0300)	<b>175''</b>	210	1	0	1
(2002)	<b>200</b>	200	0	1*	1*
(1020)	<b>210</b>	203	2	0	0
(6000)	<b>210'</b>	462	1	1	0
(3100)	<b>224</b>	280	0	0	0
(1110)	<b>280</b>	266	1	0	0
(3010)	<b>280'</b>	336	1	0	0
(0210)	<b>315</b>	357	2	0	1
(1004)	<b>315'</b>	462	2	1	0
(7000)	<b>330</b>	924	2	1	0
(2200)	<b>420</b>	574	1	0	0
(4100)	<b>420'</b>	714	1	0	0
(1012)	<b>450</b>	510	2	0	0
(3002)	<b>450'</b>	615	1	1	0
(1102)	<b>480</b>	536	1	0	0
(0040)	<b>490</b>	882	2	0	1
(0008)	<b>495</b>	1716	2	1	0
(4010)	<b>540</b>	882	2	0	0
(0202)	<b>560</b>	728	2	0	0
(1300)	<b>560'</b>	868	2	0	0
(1005)	<b>560''</b>	1092	1	1	0
(2110)	<b>700</b>	910	2	0	0
(1030)	<b>700'</b>	1050	0	0	0
(0009)	<b>715</b>	3003	1	1	0
(1021)	<b>720</b>	924	1	0	1
(5100)	<b>720'</b>	1596	2	0	0
(3200)	<b>840</b>	1512	2	0	0
(4002)	<b>875</b>	1575	2	1	0
(6001)	<b>924</b>	2310	0	1	0
(1013)	<b>945</b>	1449	1	0	0
(0105)	<b>945'</b>	2016	2	0	0

\*SU(4) $\times$ U(1) and SU(3) $\times$ SU(2) $\times$ U(1) singlets resp.

Table A.5: SU(5) Irreps (continued)

Dynkin label	Dimension (name)	1 (index)	Quintality	SU(4) singlets	SU(3) $\times$ SU(2) singlets
(0130)	<b>980</b>	1666	1	0	1
(3003)	<b>1000</b>	1750	0	1*	1*
(10, 0, 0, 0)	<b>1001</b>	5005	0	1	0
(1111)	<b>1024</b>	1280	0	0	1*
(0121)	<b>1050</b>	1540	2	0	0
(3011)	<b>1050'</b>	1575	0	0	0
(0211)	<b>1120</b>	1624	1	0	0
(0016)	<b>1155</b>	3234	2	0	0
(0220)	<b>1176</b>	1960	0	0	1*
(0500)	<b>1176'</b>	2940	0	0	1
(0203)	<b>1200</b>	2040	1	0	0
(2102)	<b>1215</b>	1782	2	0	1
(0032)	<b>1260</b>	2478	2	0	0
(11, 0, 0, 0)	<b>1365</b>	8008	1	1	0
(0113)	<b>1440</b>	2472	2	0	0
(7001)	<b>1440'</b>	4488	1	1	0
(0041)	<b>1470</b>	3234	1	0	0
(0024)	<b>1500</b>	3450	2	0	0
(2005)	<b>1540</b>	3542	2	1	0
(0106)	<b>1540'</b>	4158	1	0	0
(2201)	<b>1701</b>	2835	0	0	0
(4101)	<b>1750</b>	3500	0	0	0
(0017)	<b>1760</b>	6072	1	0	0
(2030)	<b>1800</b>	3360	1	0	0
(12, 0, 0, 0)	<b>1820</b>	12376	2	1	0
(2021)	<b>1890</b>	3087	2	0	0
(0401)	<b>1890'</b>	4032	2	0	0
(4003)	<b>1925</b>	4235	1	1	0
(4011)	<b>2000</b>	3900	1	0	0
(8001)	<b>2145</b>	8151	2	1	0
(1301)	<b>2205</b>	4116	1	0	1
(4020)	<b>2250</b>	4875	0	0	0
(7010)	<b>2376</b>	7920	0	0	0
(0, 0, 0, 13)	<b>2380</b>	18564	2	1	0
(0033)	<b>2400</b>	5880	1	0	0
(0122)	<b>2430</b>	4536	1	0	0
(0025)	<b>2475</b>	7095	1	0	0
(2111)	<b>2520</b>	4074	1	0	0
(2013)	<b>2520'</b>	4746	2	0	0
(0410)	<b>2520''</b>	5964	1	0	1
(2006)	<b>2520'''</b>	7224	1	1	0
(0600)	<b>2520''''</b>	8316	2	0	1
(8100)	<b>2574</b>	10725	0	0	0
(2103)	<b>2625</b>	4900	1	0	0
(0114)	<b>2625'</b>	5775	1	0	0
(2120)	<b>2700</b>	4950	0	0	0
(5101)	<b>2970</b>	7524	1	0	0

\*SU(4) $\times$ U(1) and SU(3) $\times$ SU(2) $\times$ U(1) singlets resp.

Table A.6: SU(6) Irreps

Dynkin label	Dimension (name)	l (index)	Sextality	SU(5) singlets	SU(4)×SU(2) singlets	SU(3)×SU(3) singlets
(10000)	<b>6</b>	1	1	1	0	0
(01000)	<b>15</b>	4	2	0	1	0
(00100)	<b>20</b>	6	3	0	0	2
(20000)	<b>21</b>	8	2	1	0	0
(10001)	<b>35</b>	12	0	1*	1*	1*
(30000)	<b>56</b>	36	3	1	0	0
(11000)	<b>70</b>	33	3	0	0	0
(01001)	<b>84</b>	38	1	0	0	0
(00101)	<b>105</b>	52	2	0	0	0
(00020)	<b>105'</b>	64	2	0	1	0
(20001)	<b>120</b>	68	1	1	0	0
(00004)	<b>126</b>	120	2	1	0	0
(00200)	<b>175</b>	120	0	0	0	2+1*
(01010)	<b>189</b>	108	0	0	1*	1*
(00110)	<b>210</b>	131	1	0	0	0
(00012)	<b>210'</b>	152	2	0	0	0
(00005)	<b>252</b>	330	1	1	0	0
(20010)	<b>280</b>	192	0	0	0	0
(30001)	<b>315</b>	264	2	1	0	0
(00102)	<b>336</b>	248	1	0	0	0
(11001)	<b>384</b>	256	2	0	1	0
(20002)	<b>405</b>	324	0	1*	1*	1*
(00021)	<b>420</b>	358	1	0	0	0
(60000)	<b>462</b>	792	0	1	0	0
(03000)	<b>490</b>	504	0	0	1	0
(00013)	<b>504</b>	516	1	0	0	0
(10101)	<b>540</b>	378	3	0	0	2
(10020)	<b>560</b>	456	3	0	0	0
(40001)	<b>700</b>	810	3	1	0	0
(30010)	<b>720</b>	696	1	0	0	0
(70000)	<b>792</b>	1716	1	1	0	0
(11010)	<b>840</b>	668	1	0	0	0
(10200)	<b>840'</b>	764	1	0	0	0
(30100)	<b>840''</b>	864	0	0	0	0
(11100)	<b>896</b>	768	0	0	0	0
(00300)	<b>980</b>	1134	3	0	0	4
(10110)	<b>1050</b>	880	2	0	0	0
(30002)	<b>1050'</b>	1135	1	1	0	0
(41000)	<b>1050''</b>	1440	0	0	0	0
(21001)	<b>1134</b>	1053	3	0	0	0
(22000)	<b>1134'</b>	1296	0	0	0	0
(02010)	<b>1176</b>	1120	2	0	1	0
(02100)	<b>1176'</b>	1204	1	0	0	0
(11002)	<b>1260</b>	1146	1	0	0	0
(80000)	<b>1287</b>	3432	2	1	0	0
(10005)	<b>1386</b>	2112	2	1	0	0
(01200)	<b>1470</b>	1568	2	0	0	0
(40010)	<b>1575</b>	2040	2	0	0	0

\*SU(5)×U(1) and SU(4)×SU(2)×U(1) and SU(3)×SU(3)×U(1) singlets resp.

Table A.6: SU(6) Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Sextality	SU(5) singlets	SU(4)×SU(2) singlets	SU(3)×SU(3) singlets
(10102)	<b>1701</b>	1620	2	0	0	0
(13000)	<b>1764</b>	2310	1	0	0	0
(04000)	<b>1764'</b>	2688	2	0	1	0
(02002)	<b>1800</b>	1920	2	0	0	0
(40100)	<b>1800'</b>	2460	1	0	0	0
(01110)	<b>1960</b>	1932	3	0	0	2
(51000)	<b>1980</b>	3498	1	0	0	0
(90000)	<b>2002</b>	6435	3	1	0	0
(10021)	<b>2205</b>	2352	2	0	1	0
(40002)	<b>2310</b>	3256	2	1	0	0
(21010)	<b>2430</b>	2592	2	0	0	0
(21100)	<b>2520</b>	2868	1	0	0	0
(20200)	<b>2520'</b>	2976	2	0	0	0
(10030)	<b>2520''</b>	3156	1	0	0	0
(32000)	<b>2520'''</b>	3732	1	0	0	0
(10006)	<b>2520''''</b>	4884	1	1	0	0
(10013)	<b>2688</b>	3328	2	0	0	0
(30003)	<b>2695</b>	3696	0	1*	1*	1*
(0, 0, 0, 0, 10)	<b>3003</b>	11440	2	1	0	0
(50010)	<b>3080</b>	5148	3	0	0	0
(30011)	<b>3200</b>	3840	0	0	0	0
(20110)	<b>3240</b>	3564	3	0	0	0
(50100)	<b>3465</b>	6072	2	0	0	0
(61000)	<b>3465'</b>	7656	2	0	0	0
(11011)	<b>3675</b>	3780	0	0	1*	1*
(21002)	<b>3675'</b>	4340	2	0	1	0
(10201)	<b>3969</b>	4536	0	0	0	2+1*
(00400)	<b>4116</b>	7056	0	0	0	4+1*
(10103)	<b>4200</b>	5260	1	0	0	0
(70001)	<b>4290</b>	10296	0	1	0	0
(0, 0, 0, 0, 11)	<b>4368</b>	19448	1	1	0	0
(10111)	<b>4410</b>	4767	1	0	0	0
(12100)	<b>4410'</b>	5712	2	0	0	0
(00301)	<b>4410''</b>	6216	2	0	0	0
(23000)	<b>4410'''</b>	7224	2	0	0	0
(02003)	<b>4500</b>	6150	1	0	0	0
(12010)	<b>4536</b>	5508	3	0	0	0
(50002)	<b>4536'</b>	8100	3	1	0	0
(03100)	<b>4704</b>	7056	3	0	0	0
(42000)	<b>4950</b>	9240	2	0	0	0
(02011)	<b>5040</b>	6024	1	0	0	0
(01030)	<b>5040'</b>	7104	2	0	1	0
(20102)	<b>5292</b>	6426	3	0	0	2
(00050)	<b>5292'</b>	11088	2	0	1	0
(10014)	<b>5544</b>	8844	1	0	0	0
(01006)	<b>5544'</b>	11616	2	0	0	0
(10120)	<b>5670</b>	7128	0	0	0	0
(31010)	<b>5670'</b>	7857	3	0	0	0

\*SU(5)×U(1) and SU(4)×SU(2)×U(1) and SU(3)×SU(3)×U(1) singlets resp.

Table A.7: SU(7) Irreps

Dynkin label	Dimension (name)	1 (index)	Septality	SU(6) singlets	SU(5) $\times$ SU(2) singlets	SU(4) $\times$ SU(3) singlets
(10000)	<b>7</b>	1	1	1	0	0
(01000)	<b>21</b>	5	2	0	1	0
(20000)	<b>28</b>	9	2	1	0	0
(00100)	<b>35</b>	10	3	0	0	1
(10001)	<b>48</b>	14	0	1*	1*	1*
(30000)	<b>84</b>	45	3	1	0	0
(11000)	<b>112</b>	46	3	0	0	0
(01001)	<b>140</b>	55	1	0	0	0
(20001)	<b>189</b>	90	1	1	0	0
(00020)	<b>196</b>	105	3	0	1	0
(000101)	<b>210</b>	95	3	0	0	0
(00004)	<b>210'</b>	165	3	1	0	0
(001001)	<b>224</b>	100	2	0	0	0
(000012)	<b>378</b>	234	3	0	0	0
(010010)	<b>392</b>	196	0	0	1*	1*
(000005)	<b>462</b>	495	2	1	0	0
(000110)	<b>490</b>	280	2	0	0	0
(000200)	<b>490'</b>	315	1	0	0	1
(200010)	<b>540</b>	315	0	0	0	0
(300001)	<b>560</b>	390	2	1	0	0
(001010)	<b>588</b>	329	1	0	0	0
(110001)	<b>735</b>	420	2	0	1	0
(200002)	<b>735'</b>	490	0	1*	1*	1*
(000102)	<b>756</b>	495	2	0	0	0
(001100)	<b>784</b>	490	0	0	0	1*
(001002)	<b>840</b>	540	1	0	0	0
(000021)	<b>882</b>	651	2	0	0	0
(000006)	<b>924</b>	1287	1	1	0	0
(000013)	<b>1008</b>	870	2	0	0	0
(000030)	<b>1176</b>	1050	1	0	1	0
(020001)	<b>1260</b>	885	3	0	0	0
(101001)	<b>1323</b>	819	3	0	0	1
(400001)	<b>1386</b>	1320	3	1	0	0
(300010)	<b>1575</b>	1275	1	0	0	0
(700000)	<b>1716</b>	3003	0	1	0	0
(110010)	<b>2016</b>	1380	1	0	0	0
(000103)	<b>2100</b>	1875	1	0	0	0
(300002)	<b>2156</b>	1925	1	1	0	0
(000014)	<b>2310</b>	2640	1	0	0	0
(000111)	<b>2352</b>	1806	1	0	0	0
(300100)	<b>2400</b>	2100	0	0	0	0
(210001)	<b>2450</b>	1925	3	0	0	0
(102000)	<b>2646</b>	2205	0	0	0	0
(000022)	<b>2646'</b>	2583	1	0	0	0
(110002)	<b>2800</b>	2150	1	0	0	0
(110100)	<b>2940</b>	2205	0	0	0	0
(100200)	<b>2940'</b>	2415	2	0	0	0

\*SU(6) $\times$ U(1) and SU(5) $\times$ SU(2) $\times$ U(1) and SU(4) $\times$ SU(3) $\times$ U(1) singlets resp.

Table A.8: SU(8) Irreps

Dynkin label	Dimension (name)	l (index)	Octality	SU(7) singlets	SU(6) $\times$ SU(2) singlets	SU(5) $\times$ SU(3) singlets	SU(4) $\times$ SU(4) singlets
(100000)	<b>8</b>	1	1	1	0	0	0
(010000)	<b>28</b>	6	2	0	1	0	0
(200000)	<b>36</b>	10	2	1	0	0	0
(001000)	<b>56</b>	15	3	0	0	1	0
(100001)	<b>63</b>	16	0	1*	1*	1*	1*
(000100)	<b>70</b>	20	4	0	0	0	2
(300000)	<b>120</b>	55	3	1	0	0	0
(110000)	<b>168</b>	61	3	0	0	0	0
(010001)	<b>216</b>	75	1	0	0	0	0
(200001)	<b>280</b>	115	1	1	0	0	0
(400000)	<b>330</b>	220	4	1	0	0	0
(020000)	<b>336</b>	160	4	0	1	0	0
(101000)	<b>378</b>	156	4	0	0	0	0
(001001)	<b>420</b>	170	2	0	0	0	0
(000101)	<b>504</b>	215	3	0	0	0	0
(210000)	<b>630</b>	340	4	0	0	0	0
(010010)	<b>720</b>	320	0	0	1*	1*	1*
(000005)	<b>792</b>	715	3	1	0	0	0
(300001)	<b>924</b>	550	2	1	0	0	0
(200010)	<b>945</b>	480	0	0	0	0	0
(0000110)	<b>1008</b>	526	3	0	0	0	0
(0000200)	<b>1176</b>	700	2	0	0	1	0
(200002)	<b>1232</b>	704	0	1*	1*	1*	1*
(110001)	<b>1280</b>	640	2	0	1	0	0
(0010010)	<b>1344</b>	680	1	0	0	0	0
(0001010)	<b>1512</b>	804	2	0	0	0	0
(0000102)	<b>1512'</b>	885	3	0	0	0	0
(0000021)	<b>1680</b>	1090	3	0	0	0	0
(0000006)	<b>1716</b>	2002	2	1	0	0	0
(0002000)	<b>1764</b>	1120	0	0	0	0	2+1*
(0010002)	<b>1800</b>	1025	1	0	0	0	0
(0000013)	<b>1848</b>	1375	3	0	0	0	0
(0001002)	<b>2100</b>	1250	2	0	0	0	0
(0010100)	<b>2352</b>	1344	0	0	0	1*	1*
(0001100)	<b>2352'</b>	1414	1	0	0	0	0
(0200001)	<b>2520</b>	1555	3	0	0	0	0
(0000030)	<b>2520'</b>	1980	2	0	1	0	0
(4000001)	<b>2520''</b>	2035	3	1	0	0	0
(1010001)	<b>2800</b>	1550	3	0	0	1	0
(3000010)	<b>3080</b>	2145	1	0	0	0	0
(0000007)	<b>3432</b>	5005	1	1	0	0	0
(1001001)	<b>3584</b>	2048	4	0	0	0	2
(3000002)	<b>4032</b>	3064	1	1	0	0	0
(1100010)	<b>4200</b>	2525	1	0	0	0	0
(0000103)	<b>4620</b>	3630	2	0	0	0	0
(0000014)	<b>4620'</b>	4510	2	0	0	0	0
(2100001)	<b>4752</b>	3234	3	0	0	0	0
(0000111)	<b>5376</b>	3712	2	0	0	0	0

\*SU(7) $\times$ U(1) and SU(6) $\times$ SU(2) $\times$ U(1) and SU(5) $\times$ SU(3) $\times$ U(1) and SU(4) $\times$ SU(4) $\times$ U(1) singlets resp.



Table A.9: SU(9) Irreps

Dynkin label	Dimension (name)	l (index)	1 Nonality	SU(8) singlets	SU(7)×SU(2) singlets	SU(6)×SU(3) singlets	SU(5)×SU(4) singlets
(1000000)	<b>9</b>	1	1	1	0	0	0
(0100000)	<b>36</b>	7	2	0	1	0	0
(2000000)	<b>45</b>	11	2	1	0	0	0
(1000001)	<b>80</b>	18	0	1*	1*	1*	1*
(0010000)	<b>84</b>	21	3	0	0	1	0
(0001000)	<b>126</b>	35	4	0	0	0	1
(3000000)	<b>165</b>	66	3	1	0	0	0
(1100000)	<b>240</b>	78	3	0	0	0	0
(0100001)	<b>315</b>	98	1	0	0	0	0
(2000001)	<b>396</b>	143	1	1	0	0	0
(4000000)	<b>495</b>	286	4	1	0	0	0
(0200000)	<b>540</b>	231	4	0	1	0	0
(1010000)	<b>630</b>	238	4	0	0	0	0
(0010001)	<b>720</b>	266	2	0	0	0	0
(2100000)	<b>990</b>	473	4	0	0	0	0
(0001001)	<b>1008</b>	406	4	0	0	0	0
(00010001)	<b>1050</b>	420	3	0	0	0	0
(0100010)	<b>1215</b>	486	0	0	1*	1*	1*
(0000005)	<b>1287</b>	1001	4	1	0	0	0
(3000001)	<b>1440</b>	748	2	1	0	0	0
(2000010)	<b>1540</b>	693	0	0	0	0	0
(0000110)	<b>1890</b>	903	4	0	0	0	0
(2000002)	<b>1944</b>	972	0	1*	1*	1*	1*

\*SU(8)×U(1) and SU(7)×SU(2)×U(1) and SU(6)×SU(3)×U(1) and SU(5)×SU(4)×U(1) singlets resp.

Table A.10: SU(10) Irreps

Dynkin label	Dimension (name)	l (index)	1 Decality	SU(9) singlets	SU(8)×SU(2) singlets	SU(7)×SU(3) singlets	SU(6)×SU(4) singlets	SU(5)×SU(5) singlets
(10000000)	<b>10</b>	1	1	1	0	0	0	0
(01000000)	<b>45</b>	8	2	0	1	0	0	0
(20000000)	<b>55</b>	12	2	1	0	0	0	0
(10000001)	<b>99</b>	20	0	1*	1*	1*	1*	1*
(00100000)	<b>120</b>	28	3	0	0	1	0	0
(00010000)	<b>210</b>	56	4	0	0	0	1	0
(30000000)	<b>220</b>	78	3	1	0	0	0	0
(00001000)	<b>252</b>	70	5	0	0	0	0	2
(11000000)	<b>330</b>	97	3	0	0	0	0	0
(01000001)	<b>440</b>	124	1	0	0	0	0	0
(20000001)	<b>540</b>	174	1	1	0	0	0	0
(40000000)	<b>715</b>	364	4	1	0	0	0	0
(02000000)	<b>825</b>	320	4	0	1	0	0	0
(10100000)	<b>990</b>	344	4	0	0	0	0	0
(00100001)	<b>1155</b>	392	2	0	0	0	0	0
(21000000)	<b>1485</b>	636	4	0	0	0	0	0
(10010000)	<b>1848</b>	700	5	0	0	0	0	0
(01000010)	<b>1925</b>	700	0	0	1*	1*	1*	1*
(00010001)	<b>1980</b>	742	3	0	0	0	0	0

\*SU(9)×U(1) and SU(8)×SU(2)×U(1) and SU(7)×SU(3)×U(1) and SU(6)×SU(4)×U(1) and SU(5)×SU(5)×U(1) singlets resp.

Table A.11: SU(11) Irreps

Dynkin label	Dimension (name)	1 (index)	Decal	SU(10) singlets	SU(9)×SU(2) singlets	SU(8)×SU(3) singlets	SU(7)×SU(4) singlets	SU(6)×SU(5) singlets
(1000000000)	<b>11</b>	1	1	1	0	0	0	0
(0100000000)	<b>55</b>	9	2	0	1	0	0	0
(2000000000)	<b>66</b>	13	2	1	0	0	0	0
(1000000001)	<b>120</b>	22	0	1*	1*	1*	1*	1*
(0010000000)	<b>165</b>	36	3	0	0	1	0	0
(3000000000)	<b>286</b>	91	3	1	0	0	0	0
(0001000000)	<b>330</b>	84	4	0	0	0	1	0
(1100000000)	<b>440</b>	118	3	0	0	0	0	0
(0000100000)	<b>462</b>	126	5	0	0	0	0	1
(0100000001)	<b>594</b>	153	1	0	0	0	0	0

\*SU(10)×U(1) and SU(9)×SU(2)×U(1) and SU(8)×SU(3)×U(1) and SU(7)×SU(4)×U(1) and SU(6)×SU(5)×U(1) singlets resp.

Table A.12: SU(12) Irreps

Dynkin label	Dimension (name)	1 (index)	Decal	SU(11) singlets	SU(10)×SU(2) singlets	SU(9)×SU(3) singlets	SU(7)×SU(5) singlets	SU(6)×SU(6) singlets
(10000000000)	<b>12</b>	1	1	1	0	0	0	0
(01000000000)	<b>66</b>	10	2	0	1	0	0	0
(20000000000)	<b>78</b>	14	2	1	0	0	0	0
(10000000001)	<b>143</b>	24	0	1*	1*	1*	1*	1*
(00100000000)	<b>220</b>	45	3	0	0	1	0	0
(30000000000)	<b>364</b>	105	3	1	0	0	0	0
(00010000000)	<b>495</b>	120	4	0	0	0	0	0
(11000000000)	<b>572</b>	141	3	0	0	0	0	0
(01000000001)	<b>780</b>	185	1	0	0	0	0	0
(00001000000)	<b>792</b>	210	5	0	0	0	1	0

\*SU(11)×U(1) and SU(10)×SU(2)×U(1) and SU(9)×SU(3)×U(1) and SU(7)×SU(5)×U(1) and SU(6)×SU(6)×U(1) singlets resp.

A.1.2.  $SO(N)$

Table A.13:  $SO(7)$  Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SU(4) singlets
(100)	<b>7</b>	1	0	1
(001)	<b>8</b>	1	1	0
(010)	<b>21</b>	5	0	0
(200)	<b>27</b>	9	0	1
(002)	<b>35</b>	10	0	0
(101)	<b>48</b>	14	1	0
(300)	<b>77</b>	44	0	1
(110)	<b>105</b>	45	0	0
(011)	<b>112</b>	46	1	0
(003)	<b>112'</b>	54	1	0
(201)	<b>168</b>	85	1	0
(020)	<b>168'</b>	96	0	0
(400)	<b>182</b>	156	0	1
(102)	<b>189</b>	90	0	0
(004)	<b>294</b>	210	0	0
(210)	<b>330</b>	220	0	0
(012)	<b>378</b>	234	0	0
(500)	<b>378'</b>	450	0	1
(301)	<b>448</b>	344	1	0
(111)	<b>512</b>	320	1	0
(103)	<b>560</b>	390	1	0
(202)	<b>616</b>	440	0	0
(005)	<b>672</b>	660	1	0
(120)	<b>693</b>	561	0	0
(600)	<b>714</b>	1122	0	1
(021)	<b>720</b>	570	1	0
(310)	<b>819</b>	780	0	0
(030)	<b>825</b>	825	0	0
(013)	<b>1008</b>	870	1	0
(401)	<b>1008'</b>	1086	1	0
(700)	<b>1254</b>	2508	0	1
(104)	<b>1386</b>	1320	0	0
(006)	<b>1386'</b>	1782	0	0
(211)	<b>1512</b>	1341	1	0
(302)	<b>1560</b>	1560	0	0
(112)	<b>1617</b>	1386	0	0
(203)	<b>1728</b>	1656	1	0
(410)	<b>1750</b>	2250	0	0
(220)	<b>1911</b>	2093	0	0
(501)	<b>2016</b>	2892	1	0
(022)	<b>2079</b>	2178	0	0
(800)	<b>2079'</b>	5148	0	1
(014)	<b>2310</b>	2640	0	0
(007)	<b>2640</b>	4290	1	0
(121)	<b>2800</b>	2950	1	0
(130)	<b>3003</b>	3861	0	0
(040)	<b>3003'</b>	4576	0	0
(105)	<b>3024</b>	3762	1	0

Table A.13: SO(7) Irreps (continued)

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SU(4) singlets
(031)	<b>3080</b>	3905	1	0
(900)	<b>3289</b>	9867	0	1
(510)	<b>3366</b>	5610	0	0
(402)	<b>3375</b>	4500	0	0
(311)	<b>3584</b>	4288	1	0
(601)	<b>3696</b>	6798	1	0
(204)	<b>4095</b>	5070	0	0
(113)	<b>4096</b>	4608	1	0
(303)	<b>4200</b>	5325	1	0
(320)	<b>4312</b>	6160	0	0
(212)	<b>4550</b>	5200	0	0
(008)	<b>4719</b>	9438	0	0
(015)	<b>4752</b>	6930	1	0
(023)	<b>4928</b>	6600	1	0
(10, 0, 0)	<b>5005</b>	17875	0	1
(610)	<b>5985</b>	12540	0	0
(106)	<b>6006</b>	9438	0	0
(701)	<b>6336</b>	14520	1	0
(502)	<b>6545</b>	11220	0	0
(11, 0, 0)	<b>7371</b>	30888	0	1
(221)	<b>7392</b>	10076	1	0
(411)	<b>7392'</b>	11484	1	0
(230)	<b>7560</b>	12240	0	0
(122)	<b>7722</b>	10296	0	0
(032)	<b>8008</b>	12584	0	0
(009)	<b>8008'</b>	19305	1	0
(420)	<b>8568</b>	15504	0	0
(205)	<b>8624</b>	13398	1	0
(403)	<b>8800</b>	14300	1	0
(050)	<b>8918</b>	19110	0	0
(114)	<b>9009</b>	12870	0	0
(016)	<b>9009'</b>	16302	0	0
(304)	<b>9625</b>	15125	0	0
(140)	<b>10010</b>	18590	0	0
(710)	<b>10010'</b>	25740	0	0
(041)	<b>10192</b>	18746	1	0
(024)	<b>10296</b>	17160	0	0
(801)	<b>10296'</b>	28743	1	0
(312)	<b>10395</b>	15345	0	0
(12, 0, 0)	<b>10556</b>	51272	0	1
(131)	<b>10752</b>	16960	1	0
(213)	<b>11088</b>	15906	1	0
(107)	<b>11088'</b>	21450	1	0
(602)	<b>11704</b>	25080	0	0
(0, 0, 10)	<b>13013</b>	37180	0	0
(511)	<b>13824</b>	27072	1	0
(13, 0, 0)	<b>14756</b>	82212	0	1
(520)	<b>15561</b>	34827	0	0
(330)	<b>15912</b>	31824	0	0

Table A.14: SO(8) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(7) singlets
(0001)	$\mathbf{8}_s$	1	(10)	1
(1000)	$\mathbf{8}_v$	1	(02)	0
(0010)	$\mathbf{8}_c$	1	(12)	0
(0100)	$\mathbf{28}$	6	(00)	0
(2000)	$\mathbf{35}_v$	10	(00)	0
(0020)	$\mathbf{35}_c$	10	(00)	0
(0002)	$\mathbf{35}_s$	10	(00)	1
(1010)	$\mathbf{56}_s$	15	(10)	0
(0011)	$\mathbf{56}_v$	15	(02)	0
(1001)	$\mathbf{56}_c$	15	(12)	0
(0003)	$\mathbf{112}_s$	54	(10)	1
(3000)	$\mathbf{112}_v$	54	(02)	0
(0030)	$\mathbf{112}_c$	54	(12)	0
(0101)	$\mathbf{160}_s$	60	(10)	0
(1100)	$\mathbf{160}_v$	60	(02)	0
(0110)	$\mathbf{160}_c$	60	(12)	0
(2001)	$\mathbf{224}_{vs}$	100	(10)	0
(0021)	$\mathbf{224}_{cs}$	100	(10)	0
(1020)	$\mathbf{224}_{cv}$	100	(02)	0
(1002)	$\mathbf{224}_{sv}$	100	(02)	0
(2010)	$\mathbf{224}_{vc}$	100	(12)	0
(0012)	$\mathbf{224}_{sc}$	100	(12)	0
(4000)	$\mathbf{294}_v$	210	(00)	0
(0040)	$\mathbf{294}_c$	210	(00)	0
(0004)	$\mathbf{294}_s$	210	(00)	1
(0200)	$\mathbf{300}$	150	(00)	0
(1011)	$\mathbf{350}$	150	(00)	0
(2100)	$\mathbf{567}_v$	324	(00)	0
(0120)	$\mathbf{567}_c$	324	(00)	0
(0102)	$\mathbf{567}_s$	324	(00)	0
(3010)	$\mathbf{672}_{vc}$	444	(10)	0
(1030)	$\mathbf{672}_{cv}$	444	(10)	0
(0031)	$\mathbf{672}_{cs}$	444	(02)	0
(0013)	$\mathbf{672}_{sc}$	444	(02)	0
(3001)	$\mathbf{672}_{vs}$	444	(12)	0
(1003)	$\mathbf{672}_{sv}$	444	(12)	0
(0005)	$\mathbf{672}'_s$	660	(10)	1
(5000)	$\mathbf{672}'_v$	660	(02)	0
(0050)	$\mathbf{672}'_c$	660	(12)	0
(1110)	$\mathbf{840}_s$	465	(10)	0
(0111)	$\mathbf{840}_v$	465	(02)	0
(1101)	$\mathbf{840}_c$	465	(12)	0
(2020)	$\mathbf{840}'_s$	540	(00)	0
(2002)	$\mathbf{840}'_c$	540	(00)	0
(0022)	$\mathbf{840}'_v$	540	(00)	0
(1012)	$\mathbf{1296}_s$	810	(10)	0
(2011)	$\mathbf{1296}_v$	810	(02)	0
(1021)	$\mathbf{1296}_c$	810	(12)	0
(6000)	$\mathbf{1386}_v$	1782	(00)	0

Table A.14: SO(8) Irreps (continued)

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(7) singlets
(0060)	<b>1386<sub>c</sub></b>	1782	(00)	0
(0006)	<b>1386<sub>s</sub></b>	1782	(00)	1
(0201)	<b>1400<sub>s</sub></b>	975	(10)	0
(1200)	<b>1400<sub>v</sub></b>	975	(02)	0
(0210)	<b>1400<sub>c</sub></b>	975	(12)	0
(0103)	<b>1568<sub>s</sub></b>	1260	(10)	0
(3100)	<b>1568<sub>v</sub></b>	1260	(02)	0
(0130)	<b>1568<sub>c</sub></b>	1260	(12)	0
(4001)	<b>1680<sub>vs</sub></b>	1530	(10)	0
(0041)	<b>1680<sub>cs</sub></b>	1530	(10)	0
(1040)	<b>1680<sub>cv</sub></b>	1530	(02)	0
(1004)	<b>1680<sub>sv</sub></b>	1530	(02)	0
(4010)	<b>1680<sub>vc</sub></b>	1530	(12)	0
(0014)	<b>1680<sub>sc</sub></b>	1530	(12)	0
(0300)	<b>1925</b>	1650	(00)	0
(2003)	<b>2400<sub>sv</sub></b>	2100	(10)	0
(0023)	<b>2400<sub>sc</sub></b>	2100	(10)	0
(3020)	<b>2400<sub>vc</sub></b>	2100	(02)	0
(3002)	<b>2400<sub>vs</sub></b>	2100	(02)	0
(2030)	<b>2400<sub>cv</sub></b>	2100	(12)	0
(0032)	<b>2400<sub>cs</sub></b>	2100	(12)	0
(0007)	<b>2640<sub>s</sub></b>	4290	(10)	1
(7000)	<b>2640<sub>v</sub></b>	4290	(02)	0
(0070)	<b>2640<sub>c</sub></b>	4290	(12)	0
(2101)	<b>2800<sub>vs</sub></b>	2150	(10)	0
(0121)	<b>2800<sub>cs</sub></b>	2150	(10)	0
(1120)	<b>2800<sub>cv</sub></b>	2150	(02)	0
(1102)	<b>2800<sub>sv</sub></b>	2150	(02)	0
(2110)	<b>2800<sub>vc</sub></b>	2150	(12)	0
(0112)	<b>2800<sub>sc</sub></b>	2150	(12)	0
(3011)	<b>3675<sub>v</sub></b>	3150	(00)	0
(1031)	<b>3675<sub>c</sub></b>	3150	(00)	0
(1013)	<b>3675<sub>s</sub></b>	3150	(00)	0
(4100)	<b>3696<sub>v</sub></b>	3960	(00)	0
(0140)	<b>3696<sub>c</sub></b>	3960	(00)	0
(0104)	<b>3696<sub>s</sub></b>	3960	(00)	0
(5010)	<b>3696'<sub>vc</sub></b>	4422	(10)	0
(1050)	<b>3696'<sub>cv</sub></b>	4422	(10)	0
(0051)	<b>3696'<sub>cs</sub></b>	4422	(02)	0
(0015)	<b>3696'<sub>sc</sub></b>	4422	(02)	0
(5001)	<b>3696'<sub>vs</sub></b>	4422	(12)	0
(1005)	<b>3696'<sub>sv</sub></b>	4422	(12)	0
(1111)	<b>4096</b>	3072	(00)	0
(2200)	<b>4312<sub>v</sub></b>	4004	(00)	0
(0220)	<b>4312<sub>c</sub></b>	4004	(00)	0
(0202)	<b>4312<sub>s</sub></b>	4004	(00)	0
(2021)	<b>4536<sub>s</sub></b>	3807	(10)	0
(1022)	<b>4536<sub>v</sub></b>	3807	(02)	0
(2012)	<b>4536<sub>c</sub></b>	3807	(12)	0

Table A.15: SO(9) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(8) singlets	SU(4)×SU(2) singlets
(1000)	<b>9</b>	1	0	1	0
(0001)	<b>16</b>	2	1	0	0
(0100)	<b>36</b>	7	0	0	0
(2000)	<b>44</b>	11	0	1	1
(0010)	<b>84</b>	21	0	0	1
(0002)	<b>126</b>	35	0	0	0
(1001)	<b>128</b>	32	1	0	0
(3000)	<b>156</b>	65	0	1	0
(1100)	<b>231</b>	77	0	0	0
(0101)	<b>432</b>	150	1	0	0
(4000)	<b>450</b>	275	0	1	1
(0200)	<b>495</b>	220	0	0	1
(2001)	<b>576</b>	232	1	0	0
(1010)	<b>594</b>	231	0	0	0
(0003)	<b>672</b>	308	1	0	0
(0011)	<b>768</b>	320	1	0	0
(2100)	<b>910</b>	455	0	0	0
(1002)	<b>924</b>	385	0	0	0
(5000)	<b>1122</b>	935	0	1	0
(0110)	<b>1650</b>	825	0	0	0
(3001)	<b>1920</b>	1120	1	0	0
(0020)	<b>1980</b>	1155	0	0	1
(2010)	<b>2457</b>	1365	0	0	1
(6000)	<b>2508</b>	2717	0	1	1
(1101)	<b>2560</b>	1280	1	0	0
(1200)	<b>2574</b>	1573	0	0	0
(0102)	<b>2772</b>	1463	0	0	0
(0004)	<b>2772'</b>	1848	0	0	0
(3100)	<b>2772''</b>	1925	0	0	0
(2002)	<b>3900</b>	2275	0	0	0
(0300)	<b>4004</b>	3003	0	0	0
(0012)	<b>4158</b>	2541	0	0	0
(1003)	<b>4608</b>	2816	1	0	0
(0201)	<b>4928</b>	3080	1	0	0
(1011)	<b>5040</b>	2870	1	0	0
(7000)	<b>5148</b>	7007	0	1	0
(4001)	<b>5280</b>	4180	1	0	0
(4100)	<b>7140</b>	6545	0	0	0
(3010)	<b>7700</b>	5775	0	0	0
(2200)	<b>8748</b>	7047	0	0	1
(1110)	<b>9009</b>	6006	0	0	0
(2101)	<b>9504</b>	6468	1	0	0
(0005)	<b>9504'</b>	8580	1	0	0
(8000)	<b>9867</b>	16445	0	1	1
(1020)	<b>12012</b>	9009	0	0	0
(3002)	<b>12375</b>	9625	0	0	0
(0111)	<b>12672</b>	8800	1	0	0
(0103)	<b>12672'</b>	9328	1	0	0
(5001)	<b>12672''</b>	13024	1	0	0

Table A.15: SO(9) Irreps (continued)

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(8) singlets	SU(4)×SU(2) singlets
(0021)	<b>13200</b>	10450	1	0	0
(1102)	<b>15444</b>	10725	0	0	0
(0210)	<b>15444'</b>	12441	0	0	1
(5100)	<b>16302</b>	19019	0	0	0
(0013)	<b>16896</b>	14080	1	0	0
(9000)	<b>17875</b>	35750	0	1	0
(1004)	<b>18018</b>	15015	0	0	0
(1300)	<b>18018'</b>	17017	0	0	0
(2003)	<b>18480</b>	14630	1	0	0
(2011)	<b>19712</b>	14784	1	0	0
(4010)	<b>20196</b>	19635	0	0	1
(0400)	<b>22932</b>	25480	0	0	1
(0030)	<b>23595</b>	23595	0	0	1
(3200)	<b>23868</b>	24531	0	0	0
(1201)	<b>24192</b>	19488	1	0	0
(1012)	<b>25740</b>	20020	0	0	0
(0120)	<b>27027</b>	24024	0	0	0
(0202)	<b>27456</b>	22880	0	0	0
(6001)	<b>27456'</b>	35464	1	0	0
(3101)	<b>27648</b>	24576	1	0	0
(0006)	<b>28314</b>	33033	0	0	0
(2110)	<b>31500</b>	27125	0	0	0
(4002)	<b>32725</b>	32725	0	0	0
(6100)	<b>33957</b>	49049	0	0	0
(0301)	<b>34944</b>	33488	1	0	0
(2020)	<b>44352</b>	41888	0	0	1
(0104)	<b>46332</b>	45045	0	0	0
(5010)	<b>46683</b>	57057	0	0	0
(2102)	<b>54675</b>	48600	0	0	0
(7001)	<b>54912</b>	86944	1	0	0
(2300)	<b>54978</b>	64141	0	0	0
(3003)	<b>56320</b>	56320	1	0	0
(4200)	<b>56430</b>	72105	0	0	1
(0022)	<b>56628</b>	58201	0	0	0
(0014)	<b>56628'</b>	61347	0	0	0
(3011)	<b>59136</b>	56672	1	0	0
(1005)	<b>59136'</b>	64064	1	0	0
(0112)	<b>60060</b>	55055	0	0	0
(1111)	<b>65536</b>	57344	1	0	0
(7100)	<b>65780</b>	115115	0	0	0
(1103)	<b>67200</b>	61600	1	0	0
(4101)	<b>68640</b>	77220	1	0	0
(2004)	<b>69300</b>	71225	0	0	0
(1210)	<b>71500</b>	71500	0	0	0
(0007)	<b>75504</b>	110110	1	0	0
(1021)	<b>76032</b>	73920	1	0	0
(5002)	<b>76076</b>	95095	0	0	0
(2201)	<b>78624</b>	79716	1	0	0
(3110)	<b>87516</b>	94809	0	0	0



Table A.16: SO(10) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SU(5) singlets	SU(2) $\times$ SU(2) $\times$ SU(4) singlets	SO(9) singlets	SU(2) $\times$ SO(7) singlets
(10000)	<b>10</b>	1	(02)	0	0	1	0
(00001)	<b>16</b>	2	(11)	1	0	0	0
(01000)	<b>45</b>	8	(00)	1*	0	0	0
(20000)	<b>54</b>	12	(00)	0	1	1	1
(00100)	<b>120</b>	28	(02)	0	0	0	1
(00020)	<b>126</b>	35	(02)	1	0	0	0
(10010)	<b>144</b>	34	(11)	0	0	0	0
(00011)	<b>210</b>	56	(00)	1*	1	0	0
(30000)	<b>210'</b>	77	(02)	0	0	1	0
(11000)	<b>320</b>	96	(02)	0	0	0	0
(01001)	<b>560</b>	182	(11)	1	0	0	0
(40000)	<b>660</b>	352	(00)	0	1	1	1
(00030)	<b>672</b>	308	(11)	1	0	0	0
(20001)	<b>720</b>	266	(11)	0	0	0	0
(02000)	<b>770</b>	308	(00)	1*	1	0	1
(10100)	<b>945</b>	336	(00)	0	0	0	0
(10020)	<b>1050</b>	420	(00)	0	0	0	0
(00110)	<b>1200</b>	470	(11)	0	0	0	0
(21000)	<b>1386</b>	616	(00)	0	0	0	0
(00012)	<b>1440</b>	628	(11)	1	0	0	0
(10011)	<b>1728</b>	672	(02)	0	0	0	0
(50000)	<b>1782</b>	1287	(02)	0	0	1	0
(30010)	<b>2640</b>	1386	(11)	0	0	0	0
(00040)	<b>2772</b>	1848	(00)	1	0	0	0
(01100)	<b>2970</b>	1353	(02)	0	0	0	0
(11010)	<b>3696</b>	1694	(11)	0	0	0	0
(01020)	<b>3696'</b>	1848	(02)	1	0	0	0
(00200)	<b>4125</b>	2200	(00)	0	1	0	1
(60000)	<b>4290</b>	4004	(00)	0	1	1	1
(20100)	<b>4312</b>	2156	(02)	0	0	0	1
(12000)	<b>4410</b>	2401	(02)	0	0	0	0
(31000)	<b>4608</b>	2816	(02)	0	0	0	0
(20020)	<b>4950</b>	2695	(02)	0	0	0	0
(10003)	<b>5280</b>	3124	(11)	0	0	0	0
(01011)	<b>5940</b>	2904	(00)	1*	0	0	0
(00120)	<b>6930</b>	4004	(00)	0	0	0	0
(00031)	<b>6930'</b>	4389	(02)	1	0	0	0
(03000)	<b>7644</b>	5096	(00)	1*	0	0	0
(40001)	<b>7920</b>	5566	(11)	0	0	0	0
(02001)	<b>8064</b>	4592	(11)	1	0	0	0
(20011)	<b>8085</b>	4312	(00)	0	1	0	0
(10101)	<b>8800</b>	4620	(11)	0	0	0	0
(00022)	<b>8910</b>	5544	(00)	1*	1	0	0
(70000)	<b>9438</b>	11011	(02)	0	0	1	0
(00005)	<b>9504</b>	8580	(11)	1	0	0	0
(00111)	<b>10560</b>	5984	(02)	0	0	0	0
(10021)	<b>11088</b>	6314	(11)	0	0	0	0
(41000)	<b>12870</b>	10296	(00)	0	0	0	0

\*SU(5) $\times$ U(1) singlets resp.

Table A.16: SO(10) Irreps (continued)

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SU(5) singlets	SU(2) $\times$ SU(2) $\times$ SU(4) singlets	SO(9) singlets	SU(2) $\times$ SO(7) singlets
(30100)	<b>14784</b>	9856	(00)	0	0	0	0
(21001)	<b>15120</b>	9282	(11)	0	0	0	0
(22000)	<b>16380</b>	11648	(00)	0	1	0	1
(01030)	<b>17280</b>	12144	(11)	1	0	0	0
(30020)	<b>17325</b>	12320	(00)	0	0	0	0
(11100)	<b>17920</b>	10752	(00)	0	0	0	0
(80000)	<b>19305</b>	27456	(00)	0	1	1	1
(50010)	<b>20592</b>	18590	(11)	0	0	0	0
(10040)	<b>20790</b>	16863	(02)	0	0	0	0
(11020)	<b>23040</b>	14848	(00)	0	0	0	0
(20030)	<b>23760</b>	17754	(11)	0	0	0	0
(01110)	<b>25200</b>	16030	(11)	0	0	0	0
(00041)	<b>26400</b>	22660	(11)	1	0	0	0
(10200)	<b>27720</b>	18788	(02)	0	0	0	0
(30011)	<b>28160</b>	19712	(02)	0	0	0	0
(00060)	<b>28314</b>	33033	(02)	1	0	0	0
(00103)	<b>29568</b>	23408	(11)	0	0	0	0
(00201)	<b>30800</b>	22330	(11)	0	0	0	0
(51000)	<b>31680</b>	32032	(02)	0	0	0	0
(02100)	<b>34398</b>	24843	(02)	0	0	0	1
(01012)	<b>34992</b>	23814	(11)	1	0	0	0
(11011)	<b>36750</b>	23275	(02)	0	0	0	0
(90000)	<b>37180</b>	63206	(02)	0	0	1	0
(13000)	<b>37632</b>	31360	(02)	0	0	0	0
(20110)	<b>38016</b>	25872	(11)	0	0	0	0
(00023)	<b>39600</b>	33110	(11)	1	0	0	0
(40100)	<b>42120</b>	36036	(02)	0	0	0	1
(12010)	<b>43680</b>	31668	(11)	0	0	0	0
(02020)	<b>46800</b>	35880	(02)	1	0	0	0
(31010)	<b>48048</b>	38038	(11)	0	0	0	0
(60001)	<b>48048'</b>	54054	(11)	0	0	0	0
(10120)	<b>48114</b>	34749	(02)	0	0	0	0
(32000)	<b>48510</b>	43659	(02)	0	0	0	0
(20012)	<b>49280</b>	35728	(11)	0	0	0	0
(40020)	<b>50050</b>	45045	(02)	0	0	0	0
(10031)	<b>50688</b>	39424	(00)	0	0	0	0
(04000)	<b>52920</b>	51744	(00)	1*	1	0	1
(00121)	<b>55440</b>	42658	(11)	0	0	0	0
(01040)	<b>64350</b>	60060	(00)	1	0	0	0
(10022)	<b>64680</b>	49588	(02)	0	0	0	0
(10, 0, 0, 0, 0)	<b>68068</b>	136136	(00)	0	1	1	1
(21100)	<b>68640</b>	52624	(02)	0	0	0	0
(10050)	<b>68640'</b>	72644	(11)	0	0	0	0
(01200)	<b>70070</b>	56056	(00)	0	0	0	0
(00300)	<b>70070'</b>	63063	(02)	0	0	0	1
(03001)	<b>70560</b>	60564	(11)	1	0	0	0
(61000)	<b>70785</b>	88088	(00)	0	0	0	0
(10111)	<b>72765</b>	51744	(00)	0	0	0	0

\*SU(5) $\times$ U(1) singlets resp.

Table A.17: SO(11) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(10000)	<b>11</b>	1	0
(00001)	<b>32</b>	4	1
(01000)	<b>55</b>	9	0
(20000)	<b>65</b>	13	0
(00100)	<b>165</b>	36	0
(30000)	<b>275</b>	90	0
(10001)	<b>320</b>	72	1
(00010)	<b>330</b>	84	0
(11000)	<b>429</b>	117	0
(00002)	<b>462</b>	126	0
(40000)	<b>935</b>	442	0
(02000)	<b>1144</b>	416	0
(01001)	<b>1408</b>	432	1
(10100)	<b>1430</b>	468	0
(20001)	<b>1760</b>	604	1
(21000)	<b>2025</b>	810	0
(50000)	<b>2717</b>	1729	0
(10010)	<b>3003</b>	1092	0
(00101)	<b>3520</b>	1304	1
(00003)	<b>4224</b>	1872	1
(10002)	<b>4290</b>	1638	0
(01100)	<b>5005</b>	2093	0
(00011)	<b>5280</b>	2196	1
(60000)	<b>7007</b>	5733	0
(30001)	<b>7040</b>	3376	1
(20100)	<b>7128</b>	3240	0
(12000)	<b>7150</b>	3510	0
(31000)	<b>7293</b>	3978	0
(00200)	<b>7865</b>	3861	0
(11001)	<b>10240</b>	4352	1
(01010)	<b>11583</b>	5265	0
(03000)	<b>13650</b>	8190	0
(20010)	<b>15400</b>	7560	0
(70000)	<b>16445</b>	16744	0
(01002)	<b>17160</b>	8112	0
(41000)	<b>21945</b>	15561	0
(20002)	<b>22275</b>	11340	0
(40001)	<b>22880</b>	14508	1
(00110)	<b>23595</b>	12441	0
(00020)	<b>23595'</b>	13728	0
(02001)	<b>24960</b>	13104	1
(30100)	<b>26520</b>	15912	0
(00004)	<b>28314</b>	18018	0
(10101)	<b>28512</b>	13932	1
(22000)	<b>28798</b>	18326	0
(11100)	<b>33033</b>	18018	0
(80000)	<b>35750</b>	44200	0
(10003)	<b>36960</b>	20748	1
(00102)	<b>37752</b>	20592	0

Table A.18: SO(12) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(10000)	<b>12</b>	1	(02)
(00010)	<b>32</b>	4	(10)
(01000)	<b>66</b>	10	(00)
(20000)	<b>77</b>	14	(00)
(00100)	<b>220</b>	45	(02)
(10001)	<b>352</b>	76	(10)
(30000)	<b>352'</b>	104	(02)
(00020)	<b>462</b>	126	(00)
(00010)	<b>495</b>	120	(00)
(11000)	<b>560</b>	140	(02)
(00011)	<b>792</b>	210	(02)
(40000)	<b>1287</b>	546	(00)
(02000)	<b>1638</b>	546	(00)
(01010)	<b>1728</b>	504	(10)
(10100)	<b>2079</b>	630	(00)
(20010)	<b>2112</b>	680	(10)
(21000)	<b>2860</b>	1040	(00)
(50000)	<b>4004</b>	2275	(02)
(00030)	<b>4224</b>	1872	(10)
(10020)	<b>4752</b>	1764	(02)
(10010)	<b>4928</b>	1680	(02)
(00101)	<b>4928'</b>	1736	(10)
(01100)	<b>8008</b>	3094	(02)
(10011)	<b>8085</b>	2940	(00)
(00011)	<b>8800</b>	3500	(10)
(30001)	<b>9152</b>	4056	(10)
(00012)	<b>9504</b>	4068	(10)
(60000)	<b>11011</b>	8008	(00)
(12000)	<b>11088</b>	4956	(02)
(31000)	<b>11088'</b>	5460	(02)
(20100)	<b>11232</b>	4680	(02)
(11001)	<b>13728</b>	5460	(10)
(00200)	<b>14014</b>	6370	(00)
(01010)	<b>21021</b>	8918	(00)
(01020)	<b>21450</b>	9750	(00)
(03000)	<b>23100</b>	12600	(00)
(20020)	<b>27027</b>	13104	(00)
(20010)	<b>27456</b>	12480	(00)
(70000)	<b>27456'</b>	24752	(02)
(00040)	<b>28314</b>	18018	(00)
(40010)	<b>32032</b>	18564	(10)
(41000)	<b>35750</b>	22750	(00)
(01011)	<b>36036</b>	16107	(02)
(02001)	<b>36960</b>	18060	(10)
(10003)	<b>41184</b>	22620	(10)
(10101)	<b>43680</b>	20020	(10)
(30100)	<b>45045</b>	24570	(00)
(20011)	<b>45760</b>	21840	(02)
(00110)	<b>48048</b>	23660	(02)

Table A.19: SO(13) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(100000)	<b>13</b>	1	0
(000001)	<b>64</b>	8	1
(010000)	<b>78</b>	11	0
(200000)	<b>90</b>	15	0
(001000)	<b>286</b>	55	0
(300000)	<b>442</b>	119	0
(110000)	<b>715</b>	165	0
(100001)	<b>768</b>	160	1
(000010)	<b>1287</b>	330	0
(000002)	<b>1716</b>	462	0
(400000)	<b>1729</b>	665	0
(020000)	<b>2275</b>	700	0
(101000)	<b>2925</b>	825	0
(210000)	<b>3927</b>	1309	0
(010001)	<b>4160</b>	1160	1
(200001)	<b>4992</b>	1520	1
(500000)	<b>5733</b>	2940	0
(100100)	<b>7722</b>	2475	0
(011000)	<b>12285</b>	4410	0
(001001)	<b>13312</b>	4480	1
(100010)	<b>14300</b>	4950	0
(310000)	<b>16302</b>	7315	0
(120000)	<b>16575</b>	6800	0
(600000)	<b>16744</b>	10948	0
(201000)	<b>17017</b>	6545	0

Table A.20: SO(14) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SU(2)×SU(2)×SO(10) singlets
(1000000)	<b>14</b>	1	(02)	0
(0000010)	<b>64</b>	8	(11)	0
(0100000)	<b>91</b>	12	(00)	0
(2000000)	<b>104</b>	16	(00)	1
(0010000)	<b>364</b>	66	(02)	0
(3000000)	<b>546</b>	135	(02)	0
(1000001)	<b>832</b>	168	(11)	0
(1100000)	<b>896</b>	192	(02)	0
(0001000)	<b>1001</b>	220	(00)	1
(0000020)	<b>1716</b>	462	(02)	0
(0000100)	<b>2002</b>	495	(02)	0
(4000000)	<b>2275</b>	800	(00)	1
(0000011)	<b>3003</b>	792	(00)	0
(0200000)	<b>3080</b>	880	(00)	1
(1010000)	<b>4004</b>	1056	(00)	0
(0100010)	<b>4928</b>	1320	(11)	0
(2100000)	<b>5265</b>	1620	(00)	0
(2000010)	<b>5824</b>	1688	(11)	0

Table A.21: SO(18) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(8)×SO(10) singlets
(100000000)	<b>18</b>	1	(02)	0
(010000000)	<b>153</b>	16	(00)	0
(200000000)	<b>170</b>	20	(00)	1
(000000001)	<b>256</b>	32	(11)	0
(001000000)	<b>816</b>	120	(02)	0
(300000000)	<b>1122</b>	209	(02)	0
(110000000)	<b>1920</b>	320	(02)	0
(000100000)	<b>3060</b>	560	(00)	0
(100000010)	<b>4352</b>	800	(11)	0

Table A.22: SO(22) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(12)×SO(10) singlets
(10000000000)	<b>22</b>	1	(02)	0
(01000000000)	<b>231</b>	20	(00)	0
(20000000000)	<b>252</b>	24	(00)	1
(00000000010)	<b>1024</b>	128	(11)	0
(00100000000)	<b>1540</b>	190	(02)	0
(11000000000)	<b>3520</b>	480	(02)	0
(00010000000)	<b>7315</b>	1140	(00)	0

Table A.23: SO(26) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(16)×SO(10) singlets
(1000000000000)	<b>26</b>	1	(02)	0
(0100000000000)	<b>325</b>	24	(00)	0
(0010000000000)	<b>2600</b>	276	(02)	0
(0000000000001)	<b>4096</b>	512	(11)	0
(1100000000000)	<b>5824</b>	672	(02)	0
(0001000000000)	<b>14950</b>	2024	(00)	0
(1010000000000)	<b>52325</b>	7728	(00)	0
(0000100000000)	<b>65780</b>	10626	(02)	0
(1000000000010)	<b>102400</b>	16896	(11)	0
(0000010000000)	<b>230230</b>	42504	(00)	0
(1001000000000)	<b>320320</b>	56672	(02)	0
(0110000000000)	<b>450450</b>	83853	(02)	0
(0000001000000)	<b>657800</b>	134596	(02)	0

A.1.3.  $Sp(N)$

Table A.24:  $Sp(4)$  Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(10)	<b>4</b>	1	1
(01)	<b>5</b>	2	0
(20)	<b>10</b>	6	0
(02)	<b>14</b>	14	0
(11)	<b>16</b>	12	1
(30)	<b>20</b>	21	1
(03)	<b>30</b>	54	0
(21)	<b>35</b>	42	0
(40)	<b>35'</b>	56	0
(12)	<b>40</b>	58	1
(04)	<b>55</b>	154	0
(50)	<b>56</b>	126	1
(31)	<b>64</b>	112	1
(13)	<b>80</b>	188	1
(22)	<b>81</b>	162	0
(60)	<b>84</b>	252	0
(05)	<b>91</b>	364	0
(41)	<b>105</b>	252	0
(70)	<b>120</b>	462	1
(32)	<b>140</b>	371	1
(14)	<b>140'</b>	483	1
(06)	<b>140''</b>	756	0
(23)	<b>154</b>	462	0
(51)	<b>160</b>	504	1
(80)	<b>165</b>	792	0
(07)	<b>204</b>	1428	0
(42)	<b>220</b>	748	0
(90)	<b>220'</b>	1287	1
(15)	<b>224</b>	1064	1
(61)	<b>231</b>	924	0
(33)	<b>256</b>	960	1
(24)	<b>260</b>	1092	0
(08)	<b>285</b>	2508	0
(10,0)	<b>286</b>	2002	0
(71)	<b>320</b>	1584	1
(52)	<b>324</b>	1377	1
(16)	<b>336</b>	2100	1
(11,0)	<b>364</b>	3003	1
(09)	<b>385</b>	4158	0
(43)	<b>390</b>	1794	0
(25)	<b>405</b>	2268	0
(34)	<b>420</b>	2121	1
(81)	<b>429</b>	2574	0
(62)	<b>455</b>	2366	0
(12,0)	<b>455'</b>	4368	0
(17)	<b>480</b>	3816	1
(0,10)	<b>506</b>	6578	0
(53)	<b>560</b>	3108	1

Table A.24: Sp(4) Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Congruency class
(91)	<b>560'</b>	4004	1
(13, 0)	<b>560''</b>	6188	1
(26)	<b>595</b>	4284	0
(72)	<b>616</b>	3850	1
(44)	<b>625</b>	3750	0
(35)	<b>640</b>	4192	1
(0, 11)	<b>650</b>	10010	0
(18)	<b>660</b>	6501	1
(14, 0)	<b>680</b>	8568	0
(10, 1)	<b>715</b>	6006	0
(63)	<b>770</b>	5082	0
(82)	<b>810</b>	5994	0
(15, 0)	<b>816</b>	11628	1
(0, 12)	<b>819</b>	14742	0
(27)	<b>836</b>	7524	0
(54)	<b>880</b>	6204	1
(19)	<b>880'</b>	10516	1
(11, 1)	<b>896</b>	8736	1
(36)	<b>924</b>	7623	1
(45)	<b>935</b>	7106	0
(16, 0)	<b>969</b>	15504	0
(0, 13)	<b>1015</b>	21112	0
(73)	<b>1024</b>	7936	1
(92)	<b>1040</b>	8996	1
(12, 1)	<b>1105</b>	12376	0
(28)	<b>1134</b>	12474	0
(17, 0)	<b>1140</b>	20349	1
(1, 10)	<b>1144</b>	16302	1
(64)	<b>1190</b>	9758	0
(0, 14)	<b>1240</b>	29512	0
(37)	<b>1280</b>	12992	1
(55)	<b>1296</b>	11340	1
(10, 2)	<b>1309</b>	13090	0
(83)	<b>1326</b>	11934	0
(46)	<b>1330</b>	12502	0
(18, 0)	<b>1330'</b>	26334	0
(13, 1)	<b>1344</b>	17136	1
(1, 11)	<b>1456</b>	24388	1
(29)	<b>1495</b>	19734	0
(0, 15)	<b>1496</b>	40392	0
(19, 0)	<b>1540</b>	33649	1
(74)	<b>1560</b>	14742	1
(14, 1)	<b>1615</b>	23256	0
(11, 2)	<b>1620</b>	18549	1
(93)	<b>1680</b>	17388	1
(38)	<b>1716</b>	21021	1
(65)	<b>1729</b>	17290	0
(20, 0)	<b>1771</b>	42504	0
(0, 16)	<b>1785</b>	54264	0



Table A.25: Sp(6) Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(100)	<b>6</b>	1	1
(010)	<b>14</b>	4	0
(001)	<b>14'</b>	5	1
(200)	<b>21</b>	8	0
(300)	<b>56</b>	36	1
(110)	<b>64</b>	32	1
(101)	<b>70</b>	40	0
(002)	<b>84</b>	72	0
(020)	<b>90</b>	60	0
(011)	<b>126</b>	93	1
(400)	<b>126'</b>	120	0
(210)	<b>189</b>	144	0
(201)	<b>216</b>	180	1
(500)	<b>252</b>	330	1
(003)	<b>330</b>	495	1
(120)	<b>350</b>	325	1
(102)	<b>378</b>	423	1
(030)	<b>385</b>	440	0
(310)	<b>448</b>	480	1
(600)	<b>462</b>	792	0
(111)	<b>512</b>	512	0
(301)	<b>525</b>	600	0
(012)	<b>594</b>	792	0
(021)	<b>616</b>	748	1
(700)	<b>792</b>	1716	1
(220)	<b>924</b>	1144	0
(410)	<b>924'</b>	1320	0
(004)	<b>1001</b>	2288	0
(202)	<b>1078</b>	1540	0
(401)	<b>1100</b>	1650	1
(040)	<b>1274</b>	2184	0
(800)	<b>1287</b>	3432	0
(130)	<b>1344</b>	1952	1
(211)	<b>1386</b>	1815	1
(103)	<b>1386'</b>	2508	0
(510)	<b>1728</b>	3168	1
(013)	<b>2002</b>	4147	1
(900)	<b>2002'</b>	6435	1
(320)	<b>2016</b>	3216	1
(501)	<b>2079</b>	3960	0
(031)	<b>2184</b>	3900	1
(121)	<b>2205</b>	3360	0
(112)	<b>2240</b>	3680	1
(022)	<b>2457</b>	4680	0
(302)	<b>2464</b>	4400	1
(005)	<b>2548</b>	8190	1
(610)	<b>3003</b>	6864	0
(10, 0, 0)	<b>3003'</b>	11440	0
(311)	<b>3072</b>	5120	0

Table A.25: Sp(6) Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Congruency class
(230)	<b>3276</b>	5928	0
(050)	<b>3528</b>	8400	0
(601)	<b>3640</b>	8580	1
(203)	<b>3744</b>	8112	1
(420)	<b>3900</b>	7800	0
(104)	<b>4004</b>	10582	1
(140)	<b>4116</b>	8526	1
(11, 0, 0)	<b>4368</b>	19448	1
(402)	<b>4914</b>	10764	0
(710)	<b>4928</b>	13728	1
(221)	<b>5460</b>	10270	1
(014)	<b>5460'</b>	16120	0
(006)	<b>5712</b>	24480	0
(212)	<b>5720</b>	11440	0
(411)	<b>6006</b>	12441	1
(701)	<b>6006'</b>	17160	0
(12, 0, 0)	<b>6188</b>	31824	0
(041)	<b>6300</b>	15450	1
(330)	<b>6720</b>	14880	1
(520)	<b>6930</b>	16995	1
(131)	<b>7168</b>	15360	0
(113)	<b>7168'</b>	17408	0
(023)	<b>7392</b>	20240	1
(032)	<b>7700</b>	19800	0
(810)	<b>7722</b>	25740	0
(303)	<b>8190</b>	21060	0
(122)	<b>8316</b>	18810	1
(060)	<b>8568</b>	26928	0
(13, 0, 0)	<b>8568'</b>	50388	1
(502)	<b>8918</b>	23569	1
(240)	<b>9450</b>	23400	0
(801)	<b>9450'</b>	32175	1
(105)	<b>9828</b>	35568	0
(204)	<b>10395</b>	31680	0
(511)	<b>10752</b>	27136	0
(150)	<b>10752'</b>	29952	1
(321)	<b>11319</b>	25872	0
(620)	<b>11550</b>	34100	0
(007)	<b>11628</b>	63954	1
(14, 0, 0)	<b>11628'</b>	77520	0
(910)	<b>11648</b>	45760	1
(312)	<b>12096</b>	29088	1
(430)	<b>12375</b>	33000	0
(015)	<b>12852</b>	51102	1
(901)	<b>14300</b>	57200	0
(602)	<b>15092</b>	47432	0
(15, 0, 0)	<b>15504</b>	116280	1
(051)	<b>15708</b>	50490	1
(403)	<b>15750</b>	47625	1

Table A.26: Sp(8) Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(1000)	<b>8</b>	1	1
(0100)	<b>27</b>	6	0
(2000)	<b>36</b>	10	0
(0001)	<b>42</b>	14	0
(0010)	<b>48</b>	14	1
(3000)	<b>120</b>	55	1
(1100)	<b>160</b>	60	1
(1001)	<b>288</b>	140	1
(0200)	<b>308</b>	154	0
(1010)	<b>315</b>	140	0
(4000)	<b>330</b>	220	0
(2100)	<b>594</b>	330	0
(0002)	<b>594'</b>	462	0
(0110)	<b>792</b>	451	1
(0101)	<b>792'</b>	484	0
(5000)	<b>792''</b>	715	1
(0020)	<b>825</b>	550	0
(0011)	<b>1056</b>	748	1
(2001)	<b>1155</b>	770	0
(2010)	<b>1232</b>	770	1
(1200)	<b>1512</b>	1029	1
(6000)	<b>1716</b>	2002	0
(3100)	<b>1728</b>	1320	1
(0300)	<b>2184</b>	1820	0
(7000)	<b>3432</b>	5005	1
(3001)	<b>3520</b>	3080	1
(3010)	<b>3696</b>	3080	0
(1002)	<b>3696'</b>	3542	1
(1110)	<b>4096</b>	3072	0
(1101)	<b>4200</b>	3325	1
(4100)	<b>4290</b>	4290	0
(0003)	<b>4719</b>	6292	0
(1020)	<b>4752</b>	4026	1
(2200)	<b>4914</b>	4368	0
(1011)	<b>6237</b>	5544	0
(8000)	<b>6435</b>	11440	0
(0210)	<b>6552</b>	5915	1
(0201)	<b>7020</b>	6630	0
(0030)	<b>8008</b>	9009	1
(4001)	<b>9009</b>	10010	0
(0102)	<b>9009</b>	10010	0
(4010)	<b>9360</b>	10010	1
(1300)	<b>9408</b>	9800	1
(5100)	<b>9504</b>	12012	1
(0120)	<b>10010</b>	10010	0
(0012)	<b>10296</b>	12727	1
(0400)	<b>11340</b>	13860	0
(9000)	<b>11440</b>	24310	1
(0021)	<b>12012</b>	14014	0

Table A.27: Sp(10) Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(10000)	<b>10</b>	1	1
(01000)	<b>44</b>	8	0
(20000)	<b>55</b>	12	0
(00100)	<b>110</b>	27	1
(00001)	<b>132</b>	42	1
(00010)	<b>165</b>	48	0
(30000)	<b>220</b>	78	1
(11000)	<b>320</b>	96	1
(40000)	<b>715</b>	364	0
(02000)	<b>780</b>	312	0
(10100)	<b>891</b>	324	0
(10001)	<b>1155</b>	504	0
(10010)	<b>1408</b>	576	1
(21000)	<b>1430</b>	624	0
(50000)	<b>2002</b>	1365	1
(01100)	<b>2860</b>	1326	1
(00200)	<b>4004</b>	2184	0
(20100)	<b>4212</b>	2106	1
(01001)	<b>4290</b>	2301	1
(12000)	<b>4620</b>	2478	1
(00002)	<b>4719</b>	3432	0
(31000)	<b>4928</b>	2912	1
(01010)	<b>5005</b>	2548	0
(60000)	<b>5005'</b>	4368	0
(20001)	<b>5720</b>	3276	1
(20010)	<b>6864</b>	3744	0
(00020)	<b>7865</b>	5148	0
(03000)	<b>8250</b>	5400	0
(00101)	<b>8580</b>	5304	0
(00110)	<b>9152</b>	5408	1
(00011)	<b>9438</b>	6435	1
(70000)	<b>11440</b>	12376	1
(41000)	<b>14300</b>	10920	0
(30100)	<b>15015</b>	9828	0
(22000)	<b>17820</b>	12312	0
(11100)	<b>17920</b>	10752	0
(30001)	<b>21021</b>	15288	0
(80000)	<b>24310</b>	31824	0
(30010)	<b>24960</b>	17472	1
(10200)	<b>28028</b>	19110	1
(11001)	<b>28160</b>	18944	0
(11010)	<b>32340</b>	20874	1
(02100)	<b>35640</b>	25596	1
(51000)	<b>36608</b>	34944	1
(10002)	<b>37752</b>	32604	1
(13000)	<b>42240</b>	34176	1
(40100)	<b>44550</b>	36855	1
(90000)	<b>48620</b>	75582	1

Table A.28: Sp(12) Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(100000)	<b>12</b>	1	1
(010000)	<b>65</b>	10	0
(200000)	<b>78</b>	14	0
(001000)	<b>208</b>	44	1
(300000)	<b>364</b>	105	1
(000100)	<b>429</b>	110	0
(000001)	<b>429'</b>	132	0
(110000)	<b>560</b>	140	1
(000010)	<b>572</b>	165	1
(400000)	<b>1365</b>	560	0
(020000)	<b>1650</b>	550	0
(101000)	<b>2002</b>	616	0
(210000)	<b>2925</b>	1050	0
(100100)	<b>4368</b>	1540	1
(500000)	<b>4368'</b>	2380	1
(100001)	<b>4576</b>	1848	1
(100010)	<b>6006</b>	2310	0
(011000)	<b>7800</b>	3050	1
(201000)	<b>11088</b>	4620	1
(120000)	<b>11440</b>	5060	1
(310000)	<b>11648</b>	5600	1
(600000)	<b>12376</b>	8568	0
(002000)	<b>13650</b>	6300	0
(010100)	<b>18954</b>	8262	0
(010001)	<b>21450</b>	10450	0
(030000)	<b>24310</b>	13090	0

Table A.29: Sp(14) Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(1000000)	<b>14</b>	1	1
(0100000)	<b>90</b>	12	0
(2000000)	<b>105</b>	16	0
(0010000)	<b>350</b>	65	1
(3000000)	<b>560</b>	136	1
(1100000)	<b>896</b>	192	1
(0001000)	<b>910</b>	208	0
(0000001)	<b>1430</b>	429	1
(0000100)	<b>1638</b>	429	1
(0000010)	<b>2002</b>	572	0
(4000000)	<b>2380</b>	816	0
(0200000)	<b>3094</b>	884	0
(1010000)	<b>3900</b>	1040	0
(2100000)	<b>5355</b>	1632	0
(5000000)	<b>8568</b>	3876	1

A.1.4. Exceptional Algebras

Table A.30:  $E_6$  Irreps

Dynkin label	Dimension (name)	1/6 (index)	Triality	SO(10) singlets	SU(6) $\times$ SU(2) singlets	SU(3) $\times$ SU(3) $\times$ SU(3) singlets
(10000)	<b>27</b>	1	1	1	0	0
(00001)	<b>78</b>	4	0	1*	0	0
(00010)	<b>351</b>	25	1	0	0	0
(00020)	<b>351'</b>	28	1	1	0	0
(10010)	<b>650</b>	50	0	1*	1	2
(10001)	<b>1728</b>	160	1	1	0	0
(00002)	<b>2430</b>	270	0	1*	1	1
(00100)	<b>2925</b>	300	0	0	0	1
(30000)	<b>3003</b>	385	0	1	0	1
(11000)	<b>5824</b>	672	0	0	0	0
(01010)	<b>7371</b>	840	1	0	0	0
(20010)	<b>7722</b>	946	1	1	0	0
(000101)	<b>17550</b>	2300	1	0	0	0
(00021)	<b>19305</b>	2695	1	1	0	0
(40000)	<b>19305'</b>	3520	1	1	0	0
(02000)	<b>34398</b>	5390	1	0	0	0
(10011)	<b>34749</b>	4752	0	1*	0	2
(00003)	<b>43758</b>	7854	0	1*	0	1
(10002)	<b>46332</b>	7260	1	1	0	0
(10100)	<b>51975</b>	7700	1	0	0	0
(21000)	<b>54054</b>	8932	1	0	0	0
(10030)	<b>61425</b>	10675	1	1	0	0
(01010)	<b>70070</b>	10780	0	0	1	3
(20010)	<b>78975</b>	12825	0	0	0	1
(20020)	<b>85293</b>	14580	0	1*	1	3
(00050)	<b>100386</b>	24310	1	1	0	0
(001001)	<b>105600</b>	17600	0	0	0	0
(100110)	<b>112320</b>	18080	1	0	0	0
(300001)	<b>146432</b>	28160	0	1	0	0
(110001)	<b>252252</b>	45276	0	0	0	2
(010011)	<b>314496</b>	56000	1	0	0	0
(200011)	<b>359424</b>	67072	1	1	0	0
(000130)	<b>359424'</b>	79360	1	0	0	0
(400010)	<b>371800</b>	85800	0	1	0	2
(001100)	<b>386100</b>	73700	1	0	0	0
(000102)	<b>393822</b>	78540	1	0	0	0
(000210)	<b>412776</b>	85848	1	0	0	0
(600000)	<b>442442</b>	136136	0	1	0	1
(000022)	<b>459459</b>	95557	1	1	0	0
(001020)	<b>494208</b>	98560	1	0	0	0
(000004)	<b>537966</b>	137940	0	1*	1	1
(300100)	<b>579150</b>	125400	1	0	0	0
(100200)	<b>600600</b>	123200	0	0	0	1
(300020)	<b>638820</b>	143780	1	1	0	0
(100003)	<b>741312</b>	170016	1	1	0	0
(100012)	<b>812175</b>	166600	0	1*	1	3
(101010)	<b>852930</b>	167670	0	0	0	3

\*SO(10) $\times$ U(1) singlets resp.

Table A.31:  $E_7$  Irreps

Dynkin label	Dimension (name)	1/12 (index)	Congruency class
(0000010)	<b>56</b>	1	1
(1000000)	<b>133</b>	3	0
(0000001)	<b>912</b>	30	1
(0000020)	<b>1463</b>	55	0
(0000100)	<b>1539</b>	54	0
(1000010)	<b>6480</b>	270	1
(2000000)	<b>7371</b>	351	0
(0100000)	<b>8645</b>	390	0
(0000030)	<b>24320</b>	1440	1
(0001000)	<b>27664</b>	1430	1
(0000011)	<b>40755</b>	2145	0
(0000110)	<b>51072</b>	2832	1
(1000001)	<b>86184</b>	4995	1
(1000020)	<b>150822</b>	9450	0
(1000100)	<b>152152</b>	9152	0
(3000000)	<b>238602</b>	17940	0
(0000002)	<b>253935</b>	17820	0
(0000040)	<b>293930</b>	24310	0
(2000010)	<b>320112</b>	21762	1
(0100010)	<b>362880</b>	23760	1
(0010000)	<b>365750</b>	24750	0
(1100000)	<b>573440</b>	40960	0
(0000200)	<b>617253</b>	46410	0
(0000101)	<b>861840</b>	61830	1
(0000021)	<b>885248</b>	65728	1
(0000120)	<b>915705</b>	71145	0
(0001010)	<b>980343</b>	71253	0
(1000030)	<b>2273920</b>	194480	1
(1001000)	<b>2282280</b>	178035	1

Table A.32:  $E_8$  Irreps

Dynkin label	Dimension (name)	1/60 (index)	Congruency class
(00000010)	<b>248</b>	1	0
(10000000)	<b>3875</b>	25	0
(00000020)	<b>27000</b>	225	0
(00000100)	<b>30380</b>	245	0
(00000001)	<b>147250</b>	1425	0
(10000010)	<b>779247</b>	8379	0
(00000030)	<b>1763125</b>	22750	0
(00001000)	<b>2450240</b>	29640	0
(00000110)	<b>4096000</b>	51200	0
(20000000)	<b>4881384</b>	65610	0
(01000000)	<b>6696000</b>	88200	0
(00000011)	<b>26411008</b>	372736	0
(10000020)	<b>70680000</b>	1083000	0
(10000100)	<b>76271625</b>	1148175	0

Table A.33:  $F_4$  Irreps

Dynkin label	Dimension (name)	1/6 (index)	Congruency class
(0001)	<b>26</b>	1	0
(1000)	<b>52</b>	3	0
(0010)	<b>273</b>	21	0
(0002)	<b>324</b>	27	0
(1001)	<b>1053</b>	108	0
(2000)	<b>1053'</b>	135	0
(0100)	<b>1274</b>	147	0
(0003)	<b>2652</b>	357	0
(0011)	<b>4096</b>	512	0
(1010)	<b>8424</b>	1242	0
(1002)	<b>10829</b>	1666	0
(3000)	<b>12376</b>	2618	0
(0004)	<b>16302</b>	3135	0
(2001)	<b>17901</b>	3213	0
(0101)	<b>19278</b>	3213	0
(0020)	<b>19448</b>	3366	0
(1100)	<b>29172</b>	5610	0
(0012)	<b>34749</b>	6237	0
(1003)	<b>76076</b>	16093	0
(0005)	<b>81081</b>	20790	0
(4000)	<b>100776</b>	31008	0
(1011)	<b>106496</b>	21504	0
(0110)	<b>107406</b>	23409	0
(2010)	<b>119119</b>	27489	0
(0102)	<b>160056</b>	35910	0
(2002)	<b>160056'</b>	37962	0
(3001)	<b>184756</b>	49742	0
(0021)	<b>205751</b>	47481	0
(0013)	<b>212992</b>	51200	0
(0200)	<b>226746</b>	61047	0
(2100)	<b>340119</b>	95931	0
(0006)	<b>342056</b>	111826	0
(1101)	<b>379848</b>	94962	0
(1004)	<b>412776</b>	113778	0
(1020)	<b>420147</b>	107730	0
(5000)	<b>627912</b>	261630	0
(0030)	<b>629356</b>	181545	0
(1012)	<b>787644</b>	207009	0
(0103)	<b>952952</b>	274890	0
(2003)	<b>1002456</b>	302022	0
(0014)	<b>1042899</b>	320892	0
(3010)	<b>1074944</b>	351424	0
(0111)	<b>1118208</b>	311808	0
(0007)	<b>1264120</b>	510510	0
(2011)	<b>1327104</b>	387072	0
(0022)	<b>1341522</b>	395577	0
(4001)	<b>1360476</b>	505818	0
(3002)	<b>1484406</b>	494802	0
(1110)	<b>1801371</b>	554268	0



Table A.33:  $F_4$  Irreps (continued)

Dynkin label	Dimension (name)	1/6 (index)	Congruency class
(1005)	<b>1850212</b>	640458	0
(0201)	<b>2488563</b>	829521	0
(3100)	<b>2674763</b>	1028755	0
(1102)	<b>2792556</b>	877149	0
(6000)	<b>3187041</b>	1716099	0
(1200)	<b>3195192</b>	1167474	0
(0120)	<b>3508596</b>	1192023	0
(1021)	<b>3921372</b>	1256850	0
(2101)	<b>3955952</b>	1369368	0
(0008)	<b>4188834</b>	2040714	0
(1013)	<b>4313088</b>	1423872	0
(0015)	<b>4313088'</b>	1645056	0
(0104)	<b>4528953</b>	1625778	0
(2020)	<b>4582656</b>	1615680	0
(2004)	<b>4940676</b>	1836918	0
(0031)	<b>5218304</b>	1856512	0
(0023)	<b>6680856</b>	2441082	0
(0112)	<b>7113106</b>	2462229	0
(4010)	<b>7142499</b>	3113397	0
(1006)	<b>7147140</b>	3023790	0
(5001)	<b>7822737</b>	3811077	0
(3003)	<b>8498776</b>	3432198	0
(2012)	<b>8843094</b>	3174444	0
(0210)	<b>9683388</b>	3848526	0
(4002)	<b>10044008</b>	4442542	0
(1030)	<b>10482472</b>	4031720	0
(3011)	<b>10862592</b>	4282368	0
(0040)	<b>11955216</b>	5057976	0
(0009)	<b>12664184</b>	7306260	0
(0300)	<b>13530946</b>	6245052	0
(7000)	<b>13748020</b>	9253475	0
(1103)	<b>15031926</b>	5781510	0
(0016)	<b>15611882</b>	7205484	0
(0202)	<b>15997696</b>	6460608	0
(4100)	<b>16016924</b>	8008462	0
(2110)	<b>16665831</b>	6837264	0
(1111)	<b>16777216</b>	6291456	0
(0105)	<b>18206370</b>	7936110	0
(1014)	<b>19214624</b>	7759752	0
(2005)	<b>20407140</b>	9157050	0
(1022)	<b>23056488</b>	9015678	0
(1007)	<b>24488568</b>	12401262	0
(2200)	<b>26108082</b>	12384603	0
(2102)	<b>26476956</b>	11032065	0
(0024)	<b>27625000</b>	12218750	0
(0032)	<b>28068768</b>	12055176	0
(0121)	<b>28481544</b>	11684736	0
(3101)	<b>28481544'</b>	12962754	0
(1201)	<b>31702671</b>	13819113	0

Table A.34:  $G_2$  Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(10)	<b>7</b>	1	0
(01)	<b>14</b>	4	0
(20)	<b>27</b>	9	0
(11)	<b>64</b>	32	0
(30)	<b>77</b>	44	0
(02)	<b>77'</b>	55	0
(40)	<b>182</b>	156	0
(21)	<b>189</b>	144	0
(03)	<b>273</b>	351	0
(12)	<b>286</b>	286	0
(50)	<b>378</b>	450	0
(31)	<b>448</b>	480	0
(60)	<b>714</b>	1122	0
(22)	<b>729</b>	972	0
(04)	<b>748</b>	1496	0
(13)	<b>896</b>	1472	0
(41)	<b>924</b>	1320	0
(70)	<b>1254</b>	2508	0
(32)	<b>1547</b>	2652	0
(51)	<b>1728</b>	3168	0
(05)	<b>1729</b>	4940	0
(23)	<b>2079</b>	4257	0
(80)	<b>2079'</b>	5148	0
(14)	<b>2261</b>	5491	0
(42)	<b>2926</b>	6270	0
(61)	<b>3003</b>	6864	0
(90)	<b>3289</b>	9867	0
(06)	<b>3542</b>	13662	0
(33)	<b>4096</b>	10240	0
(24)	<b>4914</b>	14274	0
(71)	<b>4928</b>	13728	0
(15)	<b>4928'</b>	16544	0
(10, 0)	<b>5005</b>	17875	0
(52)	<b>5103</b>	13365	0
(07)	<b>6630</b>	33150	0
(43)	<b>7293</b>	21879	0
(11, 0)	<b>7371</b>	30888	0
(81)	<b>7722</b>	25740	0
(62)	<b>8372</b>	26312	0
(34)	<b>9177</b>	31464	0
(16)	<b>9660</b>	42780	0
(25)	<b>10206</b>	39852	0
(12, 0)	<b>10556</b>	51272	0
(08)	<b>11571</b>	72732	0
(91)	<b>11648</b>	45760	0
(53)	<b>12096</b>	42912	0
(72)	<b>13090</b>	48620	0
(13, 0)	<b>14756</b>	82212	0
(44)	<b>15625</b>	62500	0

Table A.34:  $G_2$  Irreps (continued)

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(10, 1)	<b>17017</b>	77792	0
(17)	<b>17472</b>	98592	0
(35)	<b>18304</b>	82368	0
(63)	<b>19019</b>	78793	0
(09)	<b>19096</b>	147312	0
(26)	<b>19278</b>	97308	0
(82)	<b>19683</b>	85293	0
(14, 0)	<b>20196</b>	127908	0
(11, 1)	<b>24192</b>	127296	0
(54)	<b>24948</b>	115236	0
(15, 0)	<b>27132</b>	193800	0
(92)	<b>28652</b>	143260	0
(73)	<b>28672</b>	137216	0
(18)	<b>29667</b>	207669	0
(45)	<b>30107</b>	154836	0
(0, 10)	<b>30107'</b>	279565	0
(36)	<b>33495</b>	191400	0
(12, 1)	<b>33592</b>	201552	0
(27)	<b>33858</b>	214434	0
(16, 0)	<b>35853</b>	286824	0
(64)	<b>37961</b>	200651	0
(10, 2)	<b>40579</b>	231880	0
(83)	<b>41769</b>	228735	0
(0, 11)	<b>45695</b>	502645	0
(13, 1)	<b>45696</b>	310080	0
(55)	<b>46656</b>	272160	0
(17, 0)	<b>46683</b>	415701	0
(19)	<b>47872</b>	406912	0
(46)	<b>53599</b>	344565	0
(74)	<b>55614</b>	333684	0
(11, 2)	<b>56133</b>	363528	0
(28)	<b>56133'</b>	435699	0
(37)	<b>57344</b>	405504	0
(93)	<b>59136</b>	367488	0
(18, 0)	<b>59983</b>	591261	0
(14, 1)	<b>61047</b>	465120	0
(0, 12)	<b>67158</b>	863460	0
(65)	<b>69160</b>	454480	0
(1, 10)	<b>74074</b>	751322	0
(12, 2)	<b>76076</b>	554268	0
(19, 0)	<b>76153</b>	826804	0
(84)	<b>79002</b>	534204	0
(15, 1)	<b>80256</b>	682176	0
(56)	<b>81081</b>	583011	0
(10, 3)	<b>81719</b>	572033	0
(29)	<b>88803</b>	828828	0
(47)	<b>89726</b>	704990	0
(38)	<b>93093</b>	797940	0
(20, 0)	<b>95634</b>	1138500	0

A.2. Tensor Products

A.2.1.  $SU(N)$

Table A.35:  $SU(2)$  Tensor Products

$2 \times 2 = 1 + 3$
$3 \times 2 = 2 + 4$
$3 \times 3 = 1 + 3 + 5$
$4 \times 2 = 3 + 5$
$4 \times 3 = 2 + 4 + 6$
$4 \times 4 = 1 + 3 + 5 + 7$
$5 \times 2 = 4 + 6$
$5 \times 3 = 3 + 5 + 7$
$5 \times 4 = 2 + 4 + 6 + 8$
$5 \times 5 = 1 + 3 + 5 + 7 + 9$
$6 \times 2 = 5 + 7$
$6 \times 3 = 4 + 6 + 8$
$6 \times 4 = 3 + 5 + 7 + 9$
$6 \times 5 = 2 + 4 + 6 + 8 + 10$
$6 \times 6 = 1 + 3 + 5 + 7 + 9 + 11$
$7 \times 2 = 6 + 8$
$7 \times 3 = 5 + 7 + 9$
$7 \times 4 = 4 + 6 + 8 + 10$
$7 \times 5 = 3 + 5 + 7 + 9 + 11$
$7 \times 6 = 2 + 4 + 6 + 8 + 10 + 12$
$7 \times 7 = 1 + 3 + 5 + 7 + 9 + 11 + 13$
$8 \times 2 = 7 + 9$
$8 \times 3 = 6 + 8 + 10$
$8 \times 4 = 5 + 7 + 9 + 11$
$8 \times 5 = 4 + 6 + 8 + 10 + 12$
$8 \times 6 = 3 + 5 + 7 + 9 + 11 + 13$
$8 \times 7 = 2 + 4 + 6 + 8 + 10 + 12 + 14$
$8 \times 8 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$
$9 \times 2 = 8 + 10$
$9 \times 3 = 7 + 9 + 11$
$9 \times 4 = 6 + 8 + 10 + 12$
$9 \times 5 = 5 + 7 + 9 + 11 + 13$
$9 \times 6 = 4 + 6 + 8 + 10 + 12 + 14$
$9 \times 7 = 3 + 5 + 7 + 9 + 11 + 13 + 15$
$9 \times 8 = 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16$
$9 \times 9 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$
$10 \times 2 = 9 + 11$
$10 \times 3 = 8 + 10 + 12$
$10 \times 4 = 7 + 9 + 11 + 13$
$10 \times 5 = 6 + 8 + 10 + 12 + 14$
$10 \times 6 = 5 + 7 + 9 + 11 + 13 + 15$
$10 \times 7 = 4 + 6 + 8 + 10 + 12 + 14 + 16$
$10 \times 8 = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$
$10 \times 9 = 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18$
$10 \times 10 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$

Table A.36: SU(3) Tensor Products

$\bar{3} \times 3 = 1 + 8$
$3 \times 3 = \bar{3} + 6$
$\bar{6} \times 3 = \bar{3} + \bar{15}$
$6 \times 3 = 8 + 10$
$\bar{6} \times \bar{6} = 6 + \bar{15} + \bar{15}'$
$6 \times \bar{6} = 1 + 8 + 27$
$8 \times 3 = 3 + \bar{6} + 15$
$8 \times \bar{6} = 3 + \bar{6} + 15 + 24$
$8 \times 8 = 1 + 2(8) + 10 + \bar{10} + 27$
$\bar{10} \times 3 = \bar{6} + 24$
$10 \times 3 = 15 + 15'$
$\bar{10} \times \bar{6} = 15 + 21 + 24$
$10 \times \bar{6} = 3 + 15 + 42$
$10 \times 8 = 8 + 10 + 27 + 35$
$\bar{10} \times 10 = 1 + 8 + 27 + 64$
$10 \times 10 = \bar{10} + 27 + 28 + 35$
$\bar{15}' \times 3 = \bar{10} + \bar{35}$
$\bar{15} \times 3 = 8 + \bar{10} + 27$
$15 \times 3 = 6 + \bar{15} + \bar{24}$
$15' \times 3 = \bar{21} + \bar{24}$
$\bar{15}' \times \bar{6} = 27 + \bar{28} + \bar{35}$
$\bar{15} \times \bar{6} = 8 + 10 + \bar{10} + 27 + \bar{35}$
$15 \times \bar{6} = \bar{3} + 6 + \bar{15} + \bar{24} + \bar{42}$
$15' \times \bar{6} = 6 + \bar{24} + \bar{60}$
$15 \times 8 = 3 + \bar{6} + 2(15) + 15' + 24 + 42$
$15' \times 8 = 15 + 15' + 42 + 48$
$\bar{15}' \times 10 = \bar{3} + \bar{15} + \bar{42} + \bar{90}$
$\bar{15} \times 10 = \bar{3} + 6 + \bar{15} + \bar{24} + \bar{42} + \bar{60}$
$15 \times 10 = \bar{6} + 15 + 15' + 24 + 42 + 48$
$15' \times 10 = 24 + 36 + 42 + 48$
$\bar{15}' \times 15 = 8 + \bar{10} + 27 + \bar{35} + 64 + \bar{81}$
$\bar{15}' \times 15' = 1 + 8 + 27 + 64 + 125$
$\bar{15} \times 15 = 1 + 2(8) + 10 + \bar{10} + 2(27) + 35 + \bar{35} + 64$
$15 \times 15 = \bar{3} + 6 + 2(\bar{15}) + \bar{15}' + \bar{21} + 2(\bar{24}) + \bar{42} + \bar{60}$
$15' \times 15 = \bar{15} + \bar{21} + \bar{24} + \bar{42} + \bar{60} + \bar{63}$
$15' \times 15' = \bar{15}' + \bar{42} + \bar{45} + \bar{60} + \bar{63}$
$21 \times 3 = \bar{15}' + \bar{48}$
$21 \times 3 = 28 + 35$
$21 \times \bar{6} = \bar{36} + \bar{42} + \bar{48}$
$21 \times \bar{6} = 10 + 35 + 81$
$21 \times 8 = 21 + 24 + 60 + 63$
$21 \times 10 = \bar{6} + 24 + 60 + 120$
$21 \times 10 = \bar{42} + \bar{45} + \bar{60} + \bar{63}$
$21 \times 15 = \bar{15} + \bar{15}' + \bar{42} + \bar{48} + \bar{90} + \bar{105}$
$21 \times 15' = \bar{3} + \bar{15} + \bar{42} + \bar{90} + \bar{165}$
$21 \times 15 = 27 + 28 + 35 + 64 + 80 + 81$

Table A.36: SU(3) Tensor Products (continued)

$$\begin{aligned}
\overline{21} \times 15' &= \overline{35} + \overline{55} + \overline{64} + \overline{80} + \overline{81} \\
21 \times 21 &= \overline{21} + \overline{60} + \overline{66} + \overline{90} + \overline{99} + \overline{105} \\
\overline{21} \times 21 &= 1 + 8 + 27 + 64 + 125 + 216 \\
24 \times 3 &= \overline{15} + \overline{15'} + \overline{42} \\
\overline{24} \times 3 &= 10 + 27 + 35 \\
24 \times \overline{6} &= \overline{15} + \overline{15'} + \overline{24} + \overline{42} + \overline{48} \\
\overline{24} \times \overline{6} &= 8 + 10 + 27 + 35 + 64 \\
24 \times 8 &= \overline{6} + 15 + 21 + 2(24) + 42 + 60 \\
24 \times 10 &= 3 + \overline{6} + 15 + 24 + 42 + 60 + 90 \\
\overline{24} \times 10 &= \overline{15} + \overline{15'} + \overline{21} + \overline{24} + \overline{42} + \overline{60} + \overline{63} \\
24 \times 15 &= \overline{3} + 6 + 2(\overline{15}) + \overline{15'} + \overline{24} + 2(\overline{42}) + \overline{48} + \overline{60} + \overline{90} \\
24 \times 15' &= \overline{3} + 6 + \overline{15} + \overline{24} + \overline{42} + \overline{60} + \overline{90} + \overline{120} \\
\overline{24} \times 15 &= 8 + 10 + \overline{10} + 2(27) + 28 + 2(35) + \overline{35} + 64 + 81 \\
\overline{24} \times 15' &= \overline{10} + 27 + 28 + 35 + \overline{35} + 64 + 80 + 81 \\
24 \times 21 &= \overline{24} + \overline{36} + \overline{42} + \overline{48} + \overline{60} + \overline{90} + \overline{99} + \overline{105} \\
\overline{24} \times 21 &= 8 + \overline{10} + 27 + \overline{35} + 64 + \overline{81} + 125 + \overline{154} \\
24 \times 24 &= 6 + \overline{15} + \overline{15'} + \overline{21} + 2(\overline{24}) + \overline{36} + 2(\overline{42}) + 2(\overline{48}) + \overline{60} + \overline{90} + \overline{105} \\
\overline{24} \times 24 &= 1 + 2(8) + 10 + \overline{10} + 2(27) + 35 + \overline{35} + 2(64) + 81 + \overline{81} + 125 \\
27 \times 3 &= 15 + 24 + 42 \\
27 \times \overline{6} &= \overline{6} + 15 + 15' + 24 + 42 + 60 \\
27 \times 8 &= 8 + 10 + \overline{10} + 2(27) + 35 + \overline{35} + 64 \\
27 \times 10 &= 8 + 10 + \overline{10} + 27 + 35 + \overline{35} + 64 + 81 \\
27 \times 15 &= 3 + \overline{6} + 2(15) + 15' + 21 + 2(24) + 2(42) + 48 + 60 + 90 \\
27 \times 15' &= \overline{6} + 15 + 15' + 24 + 42 + 48 + 60 + 90 + 105 \\
27 \times 21 &= 15 + 21 + 24 + 42 + 60 + 63 + 90 + 120 + 132 \\
27 \times 24 &= 3 + \overline{6} + 2(15) + 15' + 21 + 2(24) + 2(42) + 48 + 2(60) + 63 + 90 + 120 \\
27 \times 27 &= 1 + 2(8) + 10 + \overline{10} + 3(27) + 28 + 28 + 2(35) + 2(\overline{35}) + 2(64) + 81 + \overline{81} + 125 \\
\overline{28} \times 3 &= 21 + 63 \\
28 \times 3 &= 36 + 48 \\
\overline{28} \times \overline{6} &= 45 + 60 + 63 \\
28 \times \overline{6} &= 15' + 48 + 105 \\
28 \times 8 &= 28 + 35 + 80 + 81 \\
\overline{28} \times 10 &= \overline{10} + \overline{35} + \overline{81} + \overline{154} \\
28 \times 10 &= 55 + 64 + 80 + 81 \\
\overline{28} \times 15 &= 21 + 24 + 60 + 63 + 120 + 132 \\
\overline{28} \times 15' &= \overline{6} + 24 + 60 + 120 + 210 \\
28 \times 15 &= 36 + 42 + 48 + 90 + 99 + 105 \\
28 \times 15' &= 60 + 66 + 90 + 99 + 105 \\
\overline{28} \times 21 &= 48 + 78 + 90 + 120 + 120' + 132 \\
28 \times 21 &= 3 + 15 + 42 + 90 + 165 + 273 \\
\overline{28} \times 24 &= 42 + 45 + 60 + 63 + 90 + 120 + 120' + 132 \\
28 \times 24 &= 15 + 15' + 42 + 48 + 90 + 105 + 165 + 192 \\
28 \times 27 &= 27 + 28 + 35 + 64 + 80 + 81 + 125 + 154 + 162 \\
\overline{28} \times 28 &= 1 + 8 + 27 + 64 + 125 + 216 + 343 \\
28 \times 28 &= \overline{28} + \overline{81} + 91 + 125 + 143 + 154 + 162
\end{aligned}$$

Table A.37: SU(4) Tensor Products

$\bar{4} \times 4 = 1 + 15$
$4 \times 4 = 6 + 10$
$6 \times 4 = \bar{4} + \bar{20}$
$6 \times 6 = 1 + 15 + 20'$
$\bar{10} \times 4 = \bar{4} + \bar{36}$
$10 \times 4 = \bar{20} + \bar{20}''$
$10 \times 6 = 15 + 45$
$\bar{10} \times 10 = 1 + 15 + 84$
$10 \times 10 = 20' + 35 + 45$
$15 \times 4 = 4 + 20 + 36$
$15 \times 6 = 6 + 10 + \bar{10} + 64$
$15 \times 10 = 6 + 10 + 64 + 70$
$15 \times 15 = 1 + 2(15) + 20' + 45 + \bar{45} + 84$
$20'' \times 4 = \bar{10} + \bar{70}$
$20 \times 4 = 6 + \bar{10} + 64$
$20' \times 4 = 20 + 60$
$\bar{20} \times 4 = 15 + 20' + 45$
$\bar{20}'' \times 4 = 35 + 45$
$20'' \times 6 = \bar{36} + \bar{84}'$
$20 \times 6 = \bar{4} + \bar{20} + \bar{36} + \bar{60}$
$20' \times 6 = 6 + 50 + 64$
$20'' \times 10 = \bar{4} + \bar{36} + \bar{160}$
$20 \times 10 = \bar{4} + \bar{20} + \bar{36} + \bar{140}$
$20' \times 10 = \bar{10} + 64 + 126$
$\bar{20} \times 10 = 20 + 36 + 60 + 84'$
$\bar{20}'' \times 10 = 56 + 60 + 84'$
$20'' \times 15 = 20 + 20'' + 120 + 140$
$20 \times 15 = 4 + 2(20) + 20'' + 36 + 60 + 140$
$20' \times 15 = 15 + 20' + 45 + \bar{45} + 175$
$20'' \times 20'' = 50 + \bar{84}'' + \bar{126} + \bar{140}''$
$20'' \times 20 = 64 + \bar{70} + \bar{126} + \bar{140}''$
$20'' \times 20' = 36 + 140 + 224$
$20 \times 20 = 6 + 10 + \bar{10} + 50 + 2(64) + \bar{70} + \bar{126}$
$20' \times 20 = 4 + 20 + 36 + 60 + 140 + 140'$
$20' \times 20' = 1 + 15 + 20' + 84 + 105 + 175$
$\bar{20} \times 20 = 1 + 2(15) + 20' + 45 + \bar{45} + 84 + 175$
$\bar{20}'' \times 20'' = 1 + 15 + 84 + 300'$
$\bar{20}'' \times 20 = 15 + 45 + 84 + 256$
$\bar{35} \times 4 = 20'' + 120$
$35 \times 4 = 56 + 84'$
$35 \times 6 = 70 + 140''$
$\bar{35} \times 10 = \bar{10} + \bar{70} + \bar{270}$
$35 \times 10 = 84'' + 126 + 140''$
$35 \times 15 = 35 + 45 + 189 + 256$
$\bar{35} \times 20'' = 120' + 140' + 216 + 224$
$\bar{35} \times 20 = 120 + 140 + 216 + 224$
$35 \times 20'' = 4 + 36 + 160 + 500$

Table A.37: SU(4) Tensor Products (continued)

$35 \times 20 = 36 + 84' + 160 + 420$
$35 \times 20' = 84 + 256 + 360'$
$\overline{35} \times 35 = 1 + 15 + 84 + 300' + 825$
$35 \times 35 = 105 + 165 + 280 + 315 + 360'$
$\overline{36} \times 4 = 15 + \overline{45} + 84$
$36 \times 4 = 10 + 64 + 70$
$36 \times 6 = \overline{20} + \overline{20}'' + \overline{36} + \overline{140}$
$\overline{36} \times 10 = 4 + 20 + 36 + 140 + 160$
$36 \times 10 = \overline{20} + \overline{20}'' + \overline{60} + \overline{120} + \overline{140}$
$36 \times 15 = 4 + 20 + 2(36) + 60 + 84' + 140 + 160$
$\overline{36} \times 20'' = 20' + \overline{35} + \overline{45} + 175 + \overline{189} + \overline{256}$
$\overline{36} \times 20 = 15 + 20' + \overline{35} + 45 + 2(\overline{45}) + 84 + 175 + \overline{256}$
$36 \times 20'' = 6 + \overline{10} + 64 + \overline{70} + \overline{270} + 300$
$36 \times 20 = 6 + 10 + \overline{10} + 2(64) + 70 + \overline{70} + 126 + 300$
$36 \times 20' = 20 + 20'' + 36 + 60 + 84' + 140 + 360$
$\overline{36} \times 35 = \overline{20} + \overline{20}'' + \overline{120} + \overline{140} + \overline{420}' + \overline{540}$
$36 \times 35 = 56 + 60 + 84' + 280'' + 360 + 420$
$\overline{36} \times 36 = 1 + 2(15) + 20' + 45 + \overline{45} + 2(84) + 175 + 256 + \overline{256} + 300'$
$36 \times 36 = 6 + 10 + 50 + 2(64) + 2(70) + 126 + \overline{126} + 140'' + 270 + 300$
$\overline{45} \times 4 = 20 + 20'' + 140$
$45 \times 4 = 36 + 60 + 84'$
$45 \times 6 = 10 + 64 + 70 + 126$
$\overline{45} \times 10 = 6 + \overline{10} + 64 + \overline{70} + 300$
$45 \times 10 = 50 + 64 + 70 + 126 + 140''$
$45 \times 15 = 15 + 20' + 35 + 2(45) + 84 + 175 + 256$
$\overline{45} \times 20'' = 60 + 120 + 140 + 140' + 216 + 224$
$\overline{45} \times 20 = 20 + 20'' + 36 + 60 + 120 + 2(140) + 140' + 224$
$45 \times 20'' = 4 + 20 + 36 + 140 + 160 + 540$
$45 \times 20 = 4 + 20 + 2(36) + 60 + 84' + 140 + 160 + 360$
$45 \times 20' = 15 + 45 + \overline{45} + 84 + 175 + 256 + 280$
$\overline{45} \times 35 = 15 + 45 + 84 + 256 + 300' + 875$
$45 \times 35 = 175 + 189 + 256 + 280 + 315 + 360'$
$\overline{45} \times 36 = 4 + 2(20) + 20'' + 36 + 60 + 120 + 2(140) + 160 + 360 + 540$
$45 \times 36 = 20 + 36 + 56 + 2(60) + 2(84') + 140 + 140' + 160 + 360 + 420$
$\overline{45} \times 45 = 1 + 2(15) + 20' + 45 + \overline{45} + 2(84) + 175 + 256 + \overline{256} + 300' + 729$
$45 \times 45 = 20' + 35 + 45 + \overline{45} + 84 + 105 + 2(175) + 189 + 2(256) + 280 + 360'$
$50 \times 4 = \overline{60} + \overline{140}'$
$50 \times 6 = 20' + 105 + 175$
$50 \times 10 = \overline{45} + 175 + 280$
$50 \times 15 = 50 + 64 + 126 + \overline{126} + 384$
$50 \times 20'' = \overline{20}'' + \overline{140} + \overline{360} + \overline{480}$
$50 \times 20 = \overline{20} + \overline{60} + \overline{140} + \overline{140}' + \overline{280}' + \overline{360}$
$50 \times 20' = 6 + 50 + 64 + 196 + 300 + 384$
$50 \times 35 = \overline{70} + 300 + 630 + 750$
$50 \times 36 = \overline{36} + \overline{60} + \overline{84}' + \overline{140} + \overline{140}' + \overline{224} + \overline{360} + \overline{756}$
$50 \times 45 = \overline{10} + 64 + \overline{70} + 126 + \overline{126} + 300 + 384 + 540' + 630$
$50 \times 50 = 1 + 15 + 20' + 84 + 105 + 175 + 300' + 336 + 729 + 735$



Table A.38: SU(5) Tensor Products

$\bar{5} \times 5 = 1 + 24$
$5 \times 5 = 10 + 15$
$\bar{10} \times 5 = \bar{5} + \bar{45}$
$10 \times 5 = \bar{10} + \bar{40}$
$\bar{10} \times 10 = 1 + 24 + \bar{75}$
$10 \times 10 = \bar{5} + \bar{45} + \bar{50}$
$\bar{15} \times 5 = \bar{5} + \bar{70}$
$15 \times 5 = \bar{35} + \bar{40}$
$\bar{15} \times 10 = 24 + \bar{126}$
$15 \times 10 = \bar{45} + \bar{105}$
$\bar{15} \times 15 = 1 + 24 + 200$
$15 \times 15 = \bar{50} + \bar{70}' + \bar{105}$
$24 \times 5 = 5 + 45 + 70$
$24 \times 10 = 10 + 15 + 40 + 175$
$24 \times 15 = 10 + 15 + 160 + 175$
$24 \times 24 = 1 + 2(24) + 75 + 126 + \bar{126} + 200$
$35 \times 5 = \bar{15} + \bar{160}$
$\bar{35} \times 5 = \bar{70}' + \bar{105}$
$35 \times 10 = \bar{70} + \bar{280}'$
$\bar{35} \times 10 = 126 + 224$
$35 \times 15 = \bar{5} + \bar{70} + \bar{450}'$
$\bar{35} \times 15 = 126' + 175' + 224$
$35 \times 24 = 35 + 40 + 315' + 450$
$35 \times 35 = \bar{175}'' + \bar{210}' + \bar{420} + \bar{420}'$
$\bar{35} \times 35 = 1 + 24 + 200 + 1000$
$40 \times 5 = \bar{10} + \bar{15} + \bar{175}$
$\bar{40} \times 5 = \bar{45} + \bar{50} + \bar{105}$
$40 \times 10 = \bar{5} + \bar{45} + \bar{70} + \bar{280}$
$\bar{40} \times 10 = 24 + 75 + 126 + 175'$
$40 \times 15 = \bar{5} + \bar{45} + \bar{70} + \bar{480}$
$\bar{40} \times 15 = 75 + 126 + 175' + 224$
$40 \times 24 = 10 + 35 + 2(40) + 175 + 210 + 450$
$40 \times 35 = \bar{280} + \bar{280}' + \bar{420} + \bar{420}'$
$\bar{40} \times 35 = 24 + \bar{126} + 200 + \bar{1050}'$
$40 \times 40 = \bar{45} + \bar{50} + \bar{70} + \bar{175}'' + 2(\bar{280}) + \bar{280}' + \bar{420}$
$\bar{40} \times 40 = 1 + 2(24) + 75 + 126 + \bar{126} + 200 + 1024$
$45 \times 5 = 10 + 40 + 175$
$\bar{45} \times 5 = 24 + 75 + 126$
$45 \times 10 = \bar{10} + \bar{15} + \bar{40} + \bar{175} + \bar{210}$
$\bar{45} \times 10 = 5 + 45 + 50 + 70 + 280$
$45 \times 15 = \bar{10} + \bar{40} + \bar{175} + \bar{450}$
$\bar{45} \times 15 = 45 + 70 + 280 + 280'$
$45 \times 24 = 5 + 2(45) + 50 + 70 + 105 + 280 + 480$
$45 \times 35 = \bar{160} + \bar{175} + \bar{540} + \bar{700}$
$\bar{45} \times 35 = 45 + 105 + 480 + 945$
$45 \times 40 = \bar{10} + \bar{15} + \bar{40} + \bar{160} + 2(\bar{175}) + \bar{210} + \bar{315} + \bar{700}$

Table A.38: SU(5) Tensor Products (continued)

$\overline{45} \times 40 = 5 + 2(45) + 50 + 70 + 105 + 280 + 480 + 720$
$45 \times 45 = 10 + 15 + 35 + 2(40) + 2(175) + 210 + 315 + 450 + 560$
$\overline{45} \times 45 = 1 + 2(24) + 2(75) + 126 + \overline{126} + 175' + \overline{175}' + 200 + 1024$
$50 \times 5 = 40 + 210$
$\overline{50} \times 5 = 75 + 175'$
$50 \times 10 = \overline{10} + \overline{175} + \overline{315}$
$\overline{50} \times 10 = 45 + 175'' + 280$
$50 \times 15 = \overline{15} + \overline{175} + \overline{560}$
$\overline{50} \times 15 = 50 + 280 + 420$
$50 \times 24 = 45 + 50 + 105 + 280 + 720$
$50 \times 35 = \overline{210} + \overline{700} + \overline{840}$
$\overline{50} \times 35 = 70 + 480 + 1200$
$50 \times 40 = \overline{40} + \overline{175} + \overline{210} + \overline{315} + \overline{560}' + \overline{700}$
$\overline{50} \times 40 = 5 + 45 + 70 + 280 + 480 + 1120$
$50 \times 45 = 10 + 40 + 175 + 210 + 315 + 450 + 1050$
$\overline{50} \times 45 = 24 + 75 + 126 + \overline{126} + 175' + \overline{700}' + 1024$
$50 \times 50 = 15 + 175 + 210 + 490 + 560 + 1050$
$\overline{50} \times 50 = 1 + 24 + 75 + 200 + 1024 + 1176$
$\overline{70} \times 5 = 24 + \overline{126} + 200$
$70 \times 5 = 15 + 160 + 175$
$\overline{70} \times 10 = 45 + 70 + 105 + 480$
$70 \times 10 = \overline{35} + \overline{40} + \overline{175} + \overline{450}$
$\overline{70} \times 15 = 5 + 45 + 70 + 450' + 480$
$70 \times 15 = \overline{35} + \overline{40} + \overline{210} + \overline{315}' + \overline{450}$
$70 \times 24 = 5 + 45 + 2(70) + 280 + 280' + 450' + 480$
$\overline{70} \times 35 = 50 + 70' + 105 + 560'' + 720 + 945$
$70 \times 35 = \overline{10} + \overline{15} + \overline{160} + \overline{175} + \overline{875} + \overline{1215}$
$\overline{70} \times 40 = 45 + 50 + 70' + 2(105) + 280 + 480 + 720 + 945$
$70 \times 40 = \overline{10} + \overline{15} + \overline{40} + \overline{160} + 2(\overline{175}) + \overline{450} + \overline{560} + \overline{1215}$
$\overline{70} \times 45 = 24 + 75 + 126 + 2(\overline{126}) + \overline{175}' + 200 + \overline{224} + 1024 + \overline{1050}'$
$70 \times 45 = 10 + 15 + 40 + 160 + 2(175) + 210 + 450 + 700 + 1215$
$\overline{70} \times 50 = 75 + \overline{126} + 175' + \overline{175}' + \overline{224} + 1024 + \overline{1701}$
$70 \times 50 = 35 + 40 + 175 + 210 + 450 + 700 + 1890$
$\overline{70} \times 70 = 1 + 2(24) + 75 + 126 + \overline{126} + 2(200) + 1000 + 1024 + 1050' + \overline{1050}'$
$70 \times 70 = 10 + 15 + 2(160) + 2(175) + 315 + 540 + 560 + 700 + 875 + 1215$
$75 \times 5 = 45 + 50 + 280$
$75 \times 10 = 10 + 40 + 175 + 210 + 315$
$75 \times 15 = 40 + 175 + 210 + 700$
$75 \times 24 = 24 + 2(75) + 126 + \overline{126} + 175' + \overline{175}' + 1024$
$75 \times 35 = 175 + 450 + 560 + 1440$
$75 \times 40 = 10 + 15 + 40 + 2(175) + 210 + 315 + 450 + 560 + 1050$
$75 \times 45 = 5 + 2(45) + 50 + 70 + 105 + 175'' + 2(280) + 480 + 720 + 1120$
$75 \times 50 = 5 + 45 + 50 + 70 + 280 + 480 + 720 + 980 + 1120$
$75 \times 70 = 45 + 50 + 70 + 105 + 2(280) + 280' + 420 + 480 + 720 + 2520$
$75 \times 75 = 1 + 2(24) + 2(75) + 126 + \overline{126} + 175' + \overline{175}' + 200 + 700' + \overline{700}' + 2(1024) + 1176$

Table A.39: SU(6) Tensor Products

$\bar{6} \times 6 = 1 + 35$
$6 \times 6 = 15 + 21$
$\bar{15} \times 6 = \bar{6} + \bar{84}$
$15 \times 6 = 20 + 70$
$\bar{15} \times 15 = 1 + 35 + 189$
$15 \times 15 = \bar{15} + \bar{105} + \bar{105}'$
$20 \times 6 = \bar{15} + \bar{105}$
$20 \times 15 = \bar{6} + \bar{84} + \bar{210}$
$20 \times 20 = 1 + 35 + 175 + 189$
$\bar{21} \times 6 = \bar{6} + \bar{120}$
$21 \times 6 = 56 + 70$
$\bar{21} \times 15 = 35 + \bar{280}$
$21 \times 15 = \bar{105} + \bar{210}'$
$21 \times 20 = \bar{84} + \bar{336}$
$\bar{21} \times 21 = 1 + 35 + 405$
$21 \times 21 = \bar{105}' + \bar{126} + \bar{210}'$
$35 \times 6 = 6 + 84 + 120$
$35 \times 15 = 15 + 21 + 105 + 384$
$35 \times 20 = 20 + 70 + \bar{70} + 540$
$35 \times 21 = 15 + 21 + 315 + 384$
$35 \times 35 = 1 + 2(35) + 189 + 280 + \bar{280} + 405$
$\bar{56} \times 6 = \bar{21} + \bar{315}$
$56 \times 6 = \bar{126} + \bar{210}'$
$\bar{56} \times 15 = \bar{120} + \bar{720}$
$56 \times 15 = \bar{336} + \bar{504}$
$56 \times 20 = 280 + 840''$
$\bar{56} \times 21 = \bar{6} + \bar{120} + \bar{1050}'$
$56 \times 21 = \bar{252} + \bar{420} + \bar{504}$
$56 \times 35 = 56 + 70 + 700 + 1134$
$\bar{56} \times 56 = 1 + 35 + 405 + 2695$
$56 \times 56 = 462 + 490 + 1050'' + 1134'$
$\bar{70} \times 6 = \bar{15} + \bar{21} + \bar{384}$
$70 \times 6 = \bar{105} + \bar{105}' + \bar{210}'$
$\bar{70} \times 15 = \bar{6} + \bar{84} + \bar{120} + \bar{840}$
$70 \times 15 = \bar{84} + \bar{210} + \bar{336} + \bar{420}$
$70 \times 20 = 35 + 189 + 280 + 896$
$\bar{70} \times 21 = \bar{6} + \bar{84} + \bar{120} + \bar{1260}$
$70 \times 21 = \bar{210} + \bar{336} + \bar{420} + \bar{504}$
$70 \times 35 = 20 + 56 + 2(70) + 540 + \bar{560} + 1134$
$\bar{70} \times 56 = 35 + 280 + 405 + 3200$
$70 \times 56 = 840'' + 896 + 1050'' + 1134'$
$\bar{70} \times 70 = 1 + 2(35) + 189 + 280 + \bar{280} + 405 + 3675$
$70 \times 70 = 175 + 189 + 280 + 490 + 840'' + 2(896) + 1134'$
$84 \times 6 = 15 + 105 + 384$
$\bar{84} \times 6 = 35 + 189 + 280$
$84 \times 15 = 20 + 70 + \bar{70} + 540 + \bar{560}$

Table A.39: SU(6) Tensor Products (continued)

$\overline{84} \times 15 = 6 + 84 + 120 + 210 + 840$
$84 \times 20 = \overline{15} + \overline{21} + \overline{105} + \overline{105}' + \overline{384} + \overline{1050}$
$84 \times 21 = 20 + 70 + 540 + 1134$
$\overline{84} \times 21 = 84 + 120 + 720 + 840$
$84 \times 35 = 6 + 2(84) + 120 + 210 + 336 + 840 + 1260$
$84 \times 56 = \overline{105} + \overline{210}' + \overline{1701} + \overline{2688}$
$\overline{84} \times 56 = 315 + 384 + 1575 + 2430$
$84 \times 70 = \overline{15} + 2(\overline{105}) + \overline{105}' + \overline{210}' + \overline{384} + \overline{1050} + \overline{1701} + \overline{2205}$
$\overline{84} \times 70 = 15 + 21 + 105 + 315 + 2(384) + 1050 + 1176 + 2430$
$84 \times 84 = 15 + 21 + 2(105) + 105' + 210' + 2(384) + 1050 + 1176 + 1701 + 1800$
$\overline{84} \times 84 = 1 + 2(35) + 175 + 2(189) + 280 + \overline{280} + 405 + 896 + \overline{896} + 3675$
$105' \times 6 = \overline{70} + 560$
$105 \times 6 = 20 + \overline{70} + 540$
$\overline{105}' \times 6 = \overline{210} + \overline{420}$
$\overline{105} \times 6 = \overline{84} + \overline{210} + \overline{336}$
$105' \times 15 = \overline{15} + \overline{384} + \overline{1176}$
$105 \times 15 = \overline{15} + \overline{21} + \overline{105} + \overline{384} + \overline{1050}$
$\overline{105}' \times 15 = 189 + 490 + 896$
$\overline{105} \times 15 = 35 + 175 + 189 + 280 + 896$
$105' \times 20 = \overline{84} + \overline{840} + \overline{1176}'$
$105 \times 20 = \overline{6} + \overline{84} + \overline{120} + \overline{210} + \overline{840} + \overline{840}'$
$105' \times 21 = \overline{21} + \overline{384} + \overline{1800}$
$105 \times 21 = \overline{15} + \overline{105} + \overline{384} + \overline{1701}$
$\overline{105}' \times 21 = 175 + 896 + 1134'$
$\overline{105} \times 21 = 189 + 280 + 840'' + 896$
$105' \times 35 = 105 + 105' + 210' + 1050 + 2205$
$105 \times 35 = 15 + 2(105) + 105' + 210' + 384 + 1050 + 1701$
$105' \times 56 = \overline{120} + \overline{1260} + \overline{4500}$
$105 \times 56 = \overline{84} + \overline{336} + \overline{1260} + \overline{4200}$
$\overline{105}' \times 56 = 840' + 2520 + 2520'''$
$\overline{105} \times 56 = 720 + 840 + 1800' + 2520$
$105' \times 70 = \overline{6} + \overline{84} + \overline{120} + \overline{840} + \overline{1260} + \overline{5040}$
$105 \times 70 = \overline{6} + 2(\overline{84}) + \overline{120} + \overline{210} + \overline{336} + \overline{840} + \overline{1260} + \overline{4410}$
$\overline{105}' \times 70 = 210 + 840 + 840' + 1176' + 1764 + 2520$
$\overline{105} \times 70 = 84 + 120 + 210 + 720 + 2(840) + 840' + 1176' + 2520$
$105' \times 84 = 20 + \overline{70} + 540 + 560 + \overline{1134} + 1960 + \overline{4536}$
$105 \times 84 = 20 + \overline{56} + 70 + 2(\overline{70}) + 2(540) + 560 + \overline{560} + \overline{1134} + 1960 + \overline{3240}$
$\overline{105}' \times 84 = \overline{84} + \overline{210} + \overline{336} + \overline{420} + \overline{840} + \overline{2520}'' + \overline{4410}$
$\overline{105} \times 84 = \overline{6} + 2(\overline{84}) + \overline{120} + 2(\overline{210}) + \overline{336} + \overline{420} + \overline{840} + \overline{840}' + \overline{1260} + \overline{4410}$
$105' \times 105' = \overline{105}' + \overline{1050} + \overline{1176} + \overline{1764}' + \overline{2520}' + \overline{4410}'$
$105' \times 105 = \overline{105} + \overline{384} + \overline{1050} + \overline{1176} + \overline{1470} + \overline{2430} + \overline{4410}'$
$105 \times 105 = \overline{15} + \overline{21} + \overline{105} + \overline{105}' + 315 + 2(\overline{384}) + 2(\overline{1050}) + \overline{1176} + \overline{1470} + \overline{2430} + \overline{2520}'$
$\overline{105}' \times 105 = 35 + 189 + 280 + \overline{280} + 896 + 3675 + \overline{5670}$
$\overline{105} \times 105 = 1 + 2(35) + 175 + 2(189) + 280 + \overline{280} + 405 + 896 + \overline{896} + 3675 + 3969$

Table A.40: SU(7) Tensor Products

$\bar{7} \times 7 = 1 + 48$
$7 \times 7 = 21 + 28$
$\bar{21} \times 7 = \bar{7} + \bar{140}$
$21 \times 7 = 35 + 112$
$\bar{21} \times 21 = 1 + 48 + 392$
$21 \times 21 = \bar{35} + \bar{196} + \bar{210}$
$\bar{28} \times 7 = \bar{7} + \bar{189}$
$28 \times 7 = 84 + 112$
$\bar{28} \times 21 = 48 + \bar{540}$
$28 \times 21 = \bar{210} + \bar{378}$
$\bar{28} \times 28 = 1 + 48 + 735'$
$28 \times 28 = \bar{196} + \bar{210}' + \bar{378}$
$\bar{35} \times 7 = \bar{21} + \bar{224}$
$35 \times 7 = \bar{35} + \bar{210}$
$\bar{35} \times 21 = \bar{7} + \bar{140} + \bar{588}$
$35 \times 21 = \bar{21} + \bar{224} + \bar{490}$
$\bar{35} \times 28 = \bar{140} + \bar{840}$
$35 \times 28 = \bar{224} + \bar{756}$
$\bar{35} \times 35 = 1 + 48 + 392 + 784$
$35 \times 35 = \bar{7} + \bar{140} + \bar{490}' + \bar{588}$
$48 \times 7 = 7 + 140 + 189$
$48 \times 21 = 21 + 28 + 224 + 735$
$48 \times 28 = 21 + 28 + 560 + 735$
$48 \times 35 = 35 + 112 + 210 + 1323$
$48 \times 48 = 1 + 2(48) + 392 + 540 + \bar{540} + 735'$
$\bar{84} \times 7 = \bar{28} + \bar{560}$
$84 \times 7 = \bar{210}' + \bar{378}$
$\bar{84} \times 21 = \bar{189} + \bar{1575}$
$84 \times 21 = \bar{756} + \bar{1008}$
$\bar{84} \times 28 = \bar{7} + \bar{189} + \bar{2156}$
$84 \times 28 = \bar{462} + \bar{882} + \bar{1008}$
$\bar{84} \times 35 = \bar{540} + \bar{2400}$
$84 \times 35 = \bar{840} + \bar{2100}$
$84 \times 48 = 84 + 112 + 1386 + 2450$
$84 \times 84 = \bar{924} + \bar{1176} + \bar{2310} + \bar{2646}'$
$\bar{112} \times 7 = \bar{21} + \bar{28} + \bar{735}$
$112 \times 7 = \bar{196} + \bar{210} + \bar{378}$
$\bar{112} \times 21 = \bar{7} + \bar{140} + \bar{189} + \bar{2016}$
$112 \times 21 = \bar{224} + \bar{490} + \bar{756} + \bar{882}$
$\bar{112} \times 28 = \bar{7} + \bar{140} + \bar{189} + \bar{2800}$
$112 \times 28 = \bar{490} + \bar{756} + \bar{882} + \bar{1008}$
$\bar{112} \times 35 = 48 + 392 + \bar{540} + \bar{2940}$
$112 \times 35 = \bar{140} + \bar{588} + \bar{840} + \bar{2352}$
$112 \times 48 = 35 + 84 + 2(112) + 1260 + 1323 + 2450$
$112 \times 84 = \bar{2100} + \bar{2310} + \bar{2352} + \bar{2646}'$
$112 \times 112 = \bar{490}' + \bar{588} + \bar{840} + \bar{1176} + \bar{2100} + 2(\bar{2352}) + \bar{2646}'$

Table A.41: SU(8) Tensor Products

$\overline{8} \times 8 = 1 + 63$
$8 \times 8 = 28 + 36$
$\overline{28} \times 8 = \overline{8} + \overline{216}$
$28 \times 8 = 56 + 168$
$\overline{28} \times 28 = 1 + 63 + 720$
$28 \times 28 = 70 + 336 + 378$
$\overline{36} \times 8 = \overline{8} + \overline{280}$
$36 \times 8 = 120 + 168$
$\overline{36} \times 28 = 63 + \overline{945}$
$36 \times 28 = 378 + 630$
$\overline{36} \times 36 = 1 + 63 + 1232$
$36 \times 36 = 330 + 336 + 630$
$\overline{56} \times 8 = \overline{28} + \overline{420}$
$56 \times 8 = 70 + 378$
$\overline{56} \times 28 = \overline{8} + \overline{216} + \overline{1344}$
$56 \times 28 = \overline{56} + 504 + 1008$
$\overline{56} \times 36 = \overline{216} + \overline{1800}$
$56 \times 36 = \overline{504} + \overline{1512}'$
$\overline{56} \times 56 = 1 + 63 + 720 + 2352$
$56 \times 56 = \overline{28} + \overline{420} + \overline{1176} + \overline{1512}$
$63 \times 8 = 8 + 216 + 280$
$63 \times 28 = 28 + 36 + 420 + 1280$
$63 \times 36 = 28 + 36 + 924 + 1280$
$63 \times 56 = 56 + 168 + 504 + 2800$
$63 \times 63 = 1 + 2(63) + 720 + 945 + \overline{945} + 1232$
$70 \times 8 = \overline{56} + 504$
$70 \times 28 = \overline{28} + \overline{420} + \overline{1512}$
$70 \times 36 = \overline{420} + \overline{2100}$
$70 \times 56 = \overline{8} + \overline{216} + \overline{1344} + \overline{2352}'$
$70 \times 63 = 70 + 378 + 378 + 3584$
$70 \times 70 = 1 + 63 + 720 + 1764 + 2352$
$\overline{120} \times 8 = \overline{36} + \overline{924}$
$120 \times 8 = 330 + 630$
$\overline{120} \times 28 = \overline{280} + \overline{3080}$
$120 \times 28 = \overline{1512}' + \overline{1848}$
$\overline{120} \times 36 = \overline{8} + \overline{280} + \overline{4032}$
$120 \times 36 = \overline{792} + \overline{1680} + \overline{1848}$
$120 \times 56 = \overline{2100} + \overline{4620}$
$120 \times 63 = 120 + 168 + 2520'' + 4752$
$\overline{168} \times 8 = \overline{28} + \overline{36} + \overline{1280}$
$168 \times 8 = 336 + 378 + 630$
$\overline{168} \times 28 = \overline{8} + \overline{216} + \overline{280} + \overline{4200}$
$168 \times 28 = \overline{504} + \overline{1008} + \overline{1512}' + \overline{1680}$
$168 \times 36 = \overline{1008} + \overline{1512}' + \overline{1680} + \overline{1848}$
$168 \times 56 = \overline{420} + \overline{1512} + \overline{2100} + \overline{5376}$
$168 \times 63 = 56 + 120 + 2(168) + 2520 + 2800 + 4752$

Table A.42: SU(9) Tensor Products

$\overline{9} \times 9 = 1 + 80$
$9 \times 9 = 36 + 45$
$\overline{36} \times 9 = \overline{9} + \overline{315}$
$36 \times 9 = 84 + 240$
$\overline{36} \times 36 = 1 + 80 + 1215$
$36 \times 36 = 126 + 540 + 630$
$\overline{45} \times 9 = \overline{9} + \overline{396}$
$45 \times 9 = 165 + 240$
$\overline{45} \times 36 = 80 + \overline{1540}$
$45 \times 36 = 630 + 990$
$\overline{45} \times 45 = 1 + 80 + 1944$
$45 \times 45 = 495 + 540 + 990$
$80 \times 9 = 9 + 315 + 396$
$80 \times 80 = 1 + 2(80) + 1215 + 1540 + \overline{1540} + 1944$
$\overline{84} \times 9 = \overline{36} + \overline{720}$
$84 \times 9 = 126 + 630$
$84 \times 36 = \overline{126} + \overline{1008} + \overline{1890}$
$\overline{126} \times 9 = \overline{84} + \overline{1050}$
$126 \times 9 = \overline{126} + \overline{1008}$
$\overline{165} \times 9 = \overline{45} + \overline{1440}$
$165 \times 9 = 495 + 990$
$240 \times 9 = 540 + 630 + 990$

Table A.43: SU(10) Tensor Products

$\overline{10} \times 10 = 1 + 99$
$10 \times 10 = 45 + 55$
$\overline{45} \times 10 = \overline{10} + \overline{440}$
$45 \times 10 = 120 + 330$
$\overline{45} \times 45 = 1 + 99 + 1925$
$45 \times 45 = 210 + 825 + 990$
$\overline{55} \times 10 = \overline{10} + \overline{540}$
$55 \times 10 = 220 + 330$
$55 \times 45 = 990 + 1485$
$55 \times 55 = 715 + 825 + 1485$
$99 \times 10 = 10 + 440 + 540$
$\overline{120} \times 10 = \overline{45} + \overline{1155}$
$120 \times 10 = 210 + 990$
$\overline{210} \times 10 = \overline{120} + \overline{1980}$
$210 \times 10 = 252 + 1848$

Table A.44: SU(11) Tensor Products

$\overline{11} \times 11 = 1 + 120$
$11 \times 11 = 55 + 66$
$\overline{55} \times 11 = \overline{11} + \overline{594}$
$55 \times 11 = 165 + 440$
$\overline{66} \times 11 = \overline{11} + \overline{715}$
$66 \times 11 = 286 + 440$
$120 \times 11 = 11 + 594 + 715$

Table A.45: SU(12) Tensor Products

$\overline{12} \times 12 = 1 + 143$
$12 \times 12 = 66 + 78$
$\overline{66} \times 12 = \overline{12} + \overline{780}$
$66 \times 12 = 220 + 572$
$\overline{78} \times 12 = \overline{12} + \overline{924}$
$78 \times 12 = 364 + 572$
$143 \times 12 = 12 + 780 + 924$



Table A.46:  $SO(7)$  Tensor Products

$7 \times 7 = 1 + 21 + 27$
$8 \times 7 = 8 + 48$
$8 \times 8 = 1 + 7 + 21 + 35$
$21 \times 7 = 7 + 35 + 105$
$21 \times 8 = 8 + 48 + 112$
$21 \times 21 = 1 + 21 + 27 + 35 + 168' + 189$
$27 \times 7 = 7 + 77 + 105$
$27 \times 8 = 48 + 168$
$27 \times 21 = 21 + 27 + 189 + 330$
$27 \times 27 = 1 + 21 + 27 + 168' + 182 + 330$
$35 \times 7 = 21 + 35 + 189$
$35 \times 8 = 8 + 48 + 112 + 112'$
$35 \times 21 = 7 + 21 + 35 + 105 + 189 + 378$
$35 \times 27 = 35 + 105 + 189 + 616$
$35 \times 35 = 1 + 7 + 21 + 27 + 35 + 105 + 168' + 189 + 294 + 378$
$48 \times 7 = 8 + 48 + 112 + 168$
$48 \times 8 = 7 + 21 + 27 + 35 + 105 + 189$
$48 \times 21 = 8 + 2(48) + 112 + 112' + 168 + 512$
$48 \times 27 = 8 + 48 + 112 + 168 + 448 + 512$
$48 \times 35 = 8 + 2(48) + 2(112) + 112' + 168 + 512 + 560$
$48 \times 48 = 1 + 7 + 2(21) + 27 + 2(35) + 77 + 2(105) + 168' + 2(189) + 330 + 378 + 616$
$77 \times 7 = 27 + 182 + 330$
$77 \times 8 = 168 + 448$
$77 \times 21 = 77 + 105 + 616 + 819$
$77 \times 27 = 7 + 77 + 105 + 378' + 693 + 819$
$77 \times 35 = 189 + 330 + 616 + 1560$
$77 \times 48 = 48 + 168 + 448 + 512 + 1008' + 1512$
$77 \times 77 = 1 + 21 + 27 + 168' + 182 + 330 + 714 + 825 + 1750 + 1911$
$105 \times 7 = 21 + 27 + 168' + 189 + 330$
$105 \times 8 = 48 + 112 + 168 + 512$
$105 \times 21 = 7 + 35 + 77 + 2(105) + 189 + 378 + 616 + 693$
$105 \times 27 = 7 + 35 + 77 + 2(105) + 378 + 616 + 693 + 819$
$105 \times 35 = 21 + 27 + 35 + 105 + 168' + 2(189) + 330 + 378 + 616 + 1617$
$105 \times 48 = 8 + 2(48) + 2(112) + 112' + 2(168) + 448 + 2(512) + 560 + 720 + 1512$
$105 \times 77 = 21 + 27 + 168' + 182 + 189 + 2(330) + 1560 + 1617 + 1750 + 1911$
$105 \times 105 = 1 + 2(21) + 2(27) + 35 + 2(168') + 182 + 3(189) + 294 + 3(330) + 378 + 616 + 825 + 1560 + 2(1617) + 1911$
$112' \times 7 = 112 + 112' + 560$
$112 \times 7 = 48 + 112 + 112' + 512$
$112' \times 8 = 35 + 189 + 294 + 378$
$112 \times 8 = 21 + 35 + 105 + 168' + 189 + 378$
$112' \times 21 = 48 + 112 + 112' + 512 + 560 + 1008$
$112 \times 21 = 8 + 48 + 2(112) + 112' + 168 + 512 + 560 + 720$
$112' \times 27 = 112 + 112' + 512 + 560 + 1728$
$112 \times 27 = 48 + 112 + 112' + 168 + 512 + 560 + 1512$

Table A.46: SO(7) Tensor Products (continued)

$112' \times 35 = 8 + 48 + 112 + 112' + 168 + 512 + 560 + 672 + 720 + 1008$
$112 \times 35 = 8 + 2(48) + 2(112) + 112' + 168 + 2(512) + 560 + 720 + 1008$
$112' \times 48 = 21 + 35 + 105 + 168' + 2(189) + 294 + 2(378) + 616 + 1386 + 1617$
$112 \times 48 = 7 + 21 + 27 + 2(35) + 2(105) + 168' + 3(189) + 294 + 330 + 2(378) + 616 + 693 + 1617$
$112' \times 77 = 112' + 512 + 560 + 1512 + 1728 + 4200$
$112 \times 77 = 112 + 168 + 448 + 512 + 560 + 1512 + 1728 + 3584$
$112' \times 105 = 48 + 2(112) + 112' + 168 + 2(512) + 2(560) + 720 + 1008 + 1512 + 1728 + 4096$
$112 \times 105 = 8 + 2(48) + 2(112) + 2(112') + 2(168) + 448 + 3(512) + 2(560) + 720 + 1008 + 1512 + 1728 + 2800$
$112' \times 112' = 1 + 7 + 21 + 27 + 35 + 77 + 105 + 168' + 189 + 294 + 330 + 378 + 616 + 693 + 825 + 1386 + 1386' + 1617 + 2079 + 2310$
$112' \times 112 = 7 + 21 + 27 + 35 + 2(105) + 168' + 2(189) + 294 + 330 + 2(378) + 616 + 693 + 1386 + 2(1617) + 2079 + 2310$
$112 \times 112 = 1 + 7 + 2(21) + 27 + 2(35) + 77 + 2(105) + 2(168') + 3(189) + 294 + 330 + 3(378) + 2(616) + 693 + 825 + 1386 + 2(1617) + 2079$
$168' \times 7 = 105 + 378 + 693$
$168 \times 7 = 48 + 168 + 448 + 512$
$168' \times 8 = 112 + 512 + 720$
$168 \times 8 = 27 + 77 + 105 + 189 + 330 + 616$
$168' \times 21 = 21 + 168' + 189 + 330 + 378 + 825 + 1617$
$168 \times 21 = 48 + 112 + 2(168) + 448 + 512 + 560 + 1512$
$168' \times 27 = 27 + 168' + 189 + 294 + 330 + 1617 + 1911$
$168 \times 27 = 8 + 48 + 112 + 168 + 448 + 512 + 720 + 1008' + 1512$
$168' \times 35 = 35 + 105 + 168' + 189 + 378 + 616 + 693 + 1617 + 2079$
$168 \times 35 = 48 + 112 + 112' + 2(168) + 448 + 2(512) + 560 + 1512 + 1728$
$168' \times 48 = 48 + 112 + 112' + 168 + 2(512) + 560 + 720 + 1008 + 1512 + 2800$
$168 \times 48 = 7 + 21 + 27 + 35 + 77 + 2(105) + 168' + 182 + 2(189) + 2(330) + 378 + 2(616) + 693 + 819 + 1560 + 1617$
$168' \times 77 = 77 + 105 + 378 + 616 + 693 + 819 + 1386 + 4312 + 4550$
$168 \times 77 = 8 + 48 + 112 + 168 + 448 + 512 + 720 + 1008' + 1512 + 2016 + 2800 + 3584$
$168' \times 105 = 7 + 35 + 77 + 2(105) + 189 + 294 + 2(378) + 2(616) + 2(693) + 819 + 1386 + 1617 + 2079 + 3003 + 4550$
$168 \times 105 = 8 + 2(48) + 2(112) + 112' + 2(168) + 2(448) + 3(512) + 560 + 720 + 1008 + 1008' + 2(1512) + 1728 + 2800 + 3584$
$168' \times 112' = 48 + 112 + 112' + 168 + 2(512) + 560 + 720 + 1008 + 1512 + 1728 + 2800 + 4096 + 4928$
$168' \times 112 = 8 + 48 + 2(112) + 112' + 168 + 448 + 2(512) + 2(560) + 2(720) + 1008 + 1512 + 1728 + 2800 + 3080 + 4096$
$168 \times 112' = 35 + 105 + 168' + 2(189) + 294 + 330 + 2(378) + 2(616) + 693 + 1386 + 1560 + 2(1617) + 4095 + 4550$
$168 \times 112 = 21 + 27 + 35 + 77 + 2(105) + 168' + 3(189) + 294 + 2(330) + 2(378) + 3(616) + 693 + 819 + 1386 + 1560 + 2(1617) + 1911 + 4550$
$168 \times 168 = 1 + 7 + 2(21) + 27 + 2(35) + 77 + 2(105) + 2(168') + 182 + 2(189) + 2(330) + 2(378) + 378' + 2(616) + 2(693) + 2(819) + 825 + 2(1560) + 2(1617) + 1750 + 1911 + 2079 + 3375 + 4550$

Table A.47: SO(8) Tensor Products

$8_s \times 8_s = 1 + 28 + 35_s$
$8_c \times 8_s = 8_v + 56_v$
$8_c \times 8_v = 8_s + 56_s$
$8_v \times 8_s = 8_c + 56_c$
$8_v \times 8_v = 1 + 28 + 35_v$
$28 \times 8_s = 8_s + 56_s + 160_s$
$28 \times 8_v = 8_v + 56_v + 160_v$
$28 \times 28 = 1 + 28 + 35_v + 35_c + 35_s + 300 + 350$
$35_s \times 8_s = 8_s + 112_s + 160_s$
$35_s \times 8_v = 56_v + 224_{sv}$
$35_c \times 8_s = 56_s + 224_{cs}$
$35_c \times 8_v = 56_v + 224_{cv}$
$35_v \times 8_s = 56_s + 224_{vs}$
$35_v \times 8_c = 56_c + 224_{vc}$
$35_v \times 8_v = 8_v + 112_v + 160_v$
$35_c \times 28 = 28 + 35_c + 350 + 567_c$
$35_v \times 28 = 28 + 35_v + 350 + 567_v$
$35_s \times 35_c = 35_v + 350 + 840'_v$
$35_s \times 35_v = 35_c + 350 + 840'_c$
$35_c \times 35_c = 1 + 28 + 35_c + 294_c + 300 + 567_c$
$35_c \times 35_v = 35_s + 350 + 840'_s$
$35_v \times 35_v = 1 + 28 + 35_v + 294_v + 300 + 567_v$
$56_v \times 8_s = 8_c + 56_c + 160_c + 224_{sc}$
$56_v \times 8_v = 28 + 35_c + 35_s + 350$
$56_c \times 8_s = 8_v + 56_v + 160_v + 224_{sv}$
$56_c \times 8_v = 8_s + 56_s + 160_s + 224_{vs}$
$56_s \times 8_s = 28 + 35_v + 35_c + 350$
$56_s \times 8_c = 8_v + 56_v + 160_v + 224_{cv}$
$56_s \times 8_v = 8_c + 56_c + 160_c + 224_{vc}$
$56_c \times 28 = 8_c + 2(56_c) + 160_c + 224_{vc} + 224_{sc} + 840_c$
$56_s \times 28 = 8_s + 2(56_s) + 160_s + 224_{vs} + 224_{cs} + 840_s$
$56_v \times 35_c = 8_v + 56_v + 160_v + 224_{cv} + 672_{cs} + 840_v$
$56_v \times 35_v = 56_v + 160_v + 224_{cv} + 224_{sv} + 1296_v$
$56_c \times 35_c = 56_c + 160_c + 224_{vc} + 224_{sc} + 1296_c$
$56_c \times 35_v = 8_c + 56_c + 160_c + 224_{vc} + 672_{vs} + 840_c$
$56_s \times 35_s = 56_s + 160_s + 224_{vs} + 224_{cs} + 1296_s$
$56_s \times 35_c = 8_s + 56_s + 160_s + 224_{cs} + 672_{cv} + 840_s$
$56_s \times 35_v = 8_s + 56_s + 160_s + 224_{vs} + 672_{vc} + 840_s$
$56_v \times 56_s = 8_c + 2(56_c) + 112_c + 2(160_c) + 224_{vc} + 224_{sc} + 840_c + 1296_c$
$56_c \times 56_c = 1 + 2(28) + 35_v + 35_c + 35_s + 300 + 2(350) + 567_v + 567_s + 840'_c$
$56_c \times 56_s = 8_v + 2(56_v) + 112_v + 2(160_v) + 224_{cv} + 224_{sv} + 840_v + 1296_v$
$56_s \times 56_s = 1 + 2(28) + 35_v + 35_c + 35_s + 300 + 2(350) + 567_v + 567_c + 840'_s$
$112_s \times 8_s = 35_s + 294_s + 567_s$
$112_s \times 8_v = 224_{sc} + 672_{sv}$
$112_c \times 8_s = 224_{cv} + 672_{cs}$
$112_c \times 8_v = 224_{cs} + 672_{cv}$
$112_v \times 8_s = 224_{vc} + 672_{vs}$

Table A.47: SO(8) Tensor Products (continued)

$$\begin{aligned}
112_v \times 8_c &= 224_{vs} + 672_{vc} \\
112_v \times 8_v &= 35_v + 294_v + 567_v \\
112_s \times 28 &= 112_s + 160_s + 1296_s + 1568_s \\
112_v \times 28 &= 112_v + 160_v + 1296_v + 1568_v \\
112_s \times 35_c &= 224_{vs} + 1296_s + 2400_{sc} \\
112_s \times 35_v &= 224_{cs} + 1296_s + 2400_{sv} \\
112_c \times 35_c &= 8_c + 112_c + 160_c + 672'_c + 1400_c + 1568_c \\
112_c \times 35_v &= 224_{sc} + 1296_c + 2400_{cv} \\
112_v \times 35_s &= 224_{cv} + 1296_v + 2400_{vs} \\
112_v \times 35_c &= 224_{sv} + 1296_v + 2400_{vc} \\
112_v \times 35_v &= 8_v + 112_v + 160_v + 672'_v + 1400_v + 1568_v \\
112_s \times 56_s &= 350 + 567_s + 840'_c + 840'_v + 3675_s \\
112_c \times 56_c &= 350 + 567_c + 840'_s + 840'_v + 3675_c \\
112_c \times 56_s &= 56_v + 224_{cv} + 672_{cs} + 840_v + 1680_{cv} + 2800_{cv} \\
112_v \times 56_v &= 350 + 567_v + 840'_s + 840'_c + 3675_v \\
112_v \times 56_c &= 56_s + 224_{vs} + 672_{vc} + 840_s + 1680_{vs} + 2800_{vs} \\
112_v \times 56_s &= 56_c + 224_{vc} + 672_{vs} + 840_c + 1680_{vc} + 2800_{vc} \\
112_s \times 112_s &= 1 + 28 + 35_s + 294_s + 300 + 567_s + 1386_s + 1925 + 3696_s + 4312_s \\
112_v \times 112_v &= 1 + 28 + 35_v + 294_v + 300 + 567_v + 1386_v + 1925 + 3696_v + 4312_v \\
160_s \times 8_s &= 28 + 35_s + 300 + 350 + 567_s \\
160_s \times 8_v &= 56_c + 160_c + 224_{sc} + 840_c \\
160_c \times 8_s &= 56_v + 160_v + 224_{cv} + 840_v \\
160_c \times 8_v &= 56_s + 160_s + 224_{cs} + 840_s \\
160_v \times 8_s &= 56_c + 160_c + 224_{vc} + 840_c \\
160_v \times 8_c &= 56_s + 160_s + 224_{vs} + 840_s \\
160_v \times 8_v &= 28 + 35_v + 300 + 350 + 567_v \\
160_s \times 28 &= 8_s + 56_s + 112_s + 2(160_s) + 224_{vs} + 224_{cs} + 840_s + 1296_s + 1400_s \\
160_v \times 28 &= 8_v + 56_v + 112_v + 2(160_v) + 224_{cv} + 224_{sv} + 840_v + 1296_v + 1400_v \\
160_s \times 35_c &= 56_s + 160_s + 224_{vs} + 224_{cs} + 840_s + 1296_s + 2800_{cs} \\
160_s \times 35_v &= 56_s + 160_s + 224_{vs} + 224_{cs} + 840_s + 1296_s + 2800_{vs} \\
160_c \times 35_c &= 8_c + 56_c + 112_c + 2(160_c) + 840_c + 1296_c + 1400_c + 1568_c \\
160_c \times 35_v &= 56_c + 160_c + 224_{vc} + 224_{sc} + 840_c + 1296_c + 2800_{vc} \\
160_v \times 35_s &= 56_v + 160_v + 224_{cv} + 224_{sv} + 840_v + 1296_v + 2800_{sv} \\
160_v \times 35_c &= 56_v + 160_v + 224_{cv} + 224_{sv} + 840_v + 1296_v + 2800_{cv} \\
160_v \times 35_v &= 8_v + 56_v + 112_v + 2(160_v) + 840_v + 1296_v + 1400_v + 1568_v \\
160_s \times 56_s &= 28 + 35_v + 35_c + 35_s + 300 + 3(350) + 567_v + 567_c + 567_s + 840'_c + 840'_v + 4096 \\
160_c \times 56_c &= 28 + 35_v + 35_c + 35_s + 300 + 3(350) + 567_v + 567_c + 567_s + 840'_s + 840'_v + 4096 \\
160_c \times 56_s &= 8_v + 2(56_v) + 2(160_v) + 2(224_{cv}) + 224_{sv} + 672_{cs} + 2(840_v) + 1296_v + 1400_v + 2800_{cv} \\
160_v \times 56_v &= 28 + 35_v + 35_c + 35_s + 300 + 3(350) + 567_v + 567_c + 567_s + 840'_s + 840'_c + 4096 \\
160_v \times 56_c &= 8_s + 2(56_s) + 2(160_s) + 2(224_{vs}) + 224_{cs} + 672_{vc} + 2(840_s) + 1296_s + 1400_s + 2800_{vs} \\
160_v \times 56_s &= 8_c + 2(56_c) + 2(160_c) + 2(224_{vc}) + 224_{sc} + 672_{vs} + 2(840_c) + 1296_c + 1400_c + 2800_{vc} \\
160_s \times 112_s &= 28 + 35_s + 294_s + 300 + 350 + 2(567_s) + 3675_s + 3696_s + 4096 + 4312_s \\
160_v \times 112_v &= 28 + 35_v + 294_v + 300 + 350 + 2(567_v) + 3675_v + 3696_v + 4096 + 4312_v \\
160_s \times 160_s &= 1 + 2(28) + 35_v + 35_c + 2(35_s) + 294_s + 2(300) + 3(350) + 567_v + 567_c + 3(567_s) + 840'_s + 840'_c + 840'_v + 1925 + 3675_s + 2(4096) + 4312_s \\
160_v \times 160_v &= 1 + 2(28) + 2(35_v) + 35_c + 35_s + 294_v + 2(300) + 3(350) + 3(567_v) + 567_c + 567_s + 840'_s + 840'_c + 840'_v + 1925 + 3675_v + 2(4096) + 4312_v
\end{aligned}$$

Table A.48: SO(9) Tensor Products

$9 \times 9 = 1 + 36 + 44$
$16 \times 9 = 16 + 128$
$16 \times 16 = 1 + 9 + 36 + 84 + 126$
$36 \times 9 = 9 + 84 + 231$
$36 \times 16 = 16 + 128 + 432$
$36 \times 36 = 1 + 36 + 44 + 126 + 495 + 594$
$44 \times 9 = 9 + 156 + 231$
$44 \times 16 = 128 + 576$
$44 \times 36 = 36 + 44 + 594 + 910$
$44 \times 44 = 1 + 36 + 44 + 450 + 495 + 910$
$84 \times 9 = 36 + 126 + 594$
$84 \times 16 = 16 + 128 + 432 + 768$
$84 \times 36 = 9 + 84 + 126 + 231 + 924 + 1650$
$84 \times 44 = 84 + 231 + 924 + 2457$
$84 \times 84 = 1 + 36 + 44 + 84 + 126 + 495 + 594 + 924 + 1980 + 2772$
$126 \times 9 = 84 + 126 + 924$
$126 \times 16 = 16 + 128 + 432 + 672 + 768$
$126 \times 36 = 36 + 84 + 126 + 594 + 924 + 2772$
$126 \times 44 = 126 + 594 + 924 + 3900$
$126 \times 84 = 9 + 36 + 84 + 126 + 231 + 594 + 924 + 1650 + 2772 + 4158$
$126 \times 126 = 1 + 9 + 36 + 44 + 84 + 126 + 231 + 495 + 594 + 924 + 1650 + 1980 + 2772 + 2772' + 4158$
$128 \times 9 = 16 + 128 + 432 + 576$
$128 \times 16 = 9 + 36 + 44 + 84 + 126 + 231 + 594 + 924$
$128 \times 36 = 16 + 2(128) + 432 + 576 + 768 + 2560$
$128 \times 44 = 16 + 128 + 432 + 576 + 1920 + 2560$
$128 \times 84 = 16 + 2(128) + 2(432) + 576 + 672 + 768 + 2560 + 5040$
$128 \times 126 = 16 + 2(128) + 2(432) + 576 + 672 + 2(768) + 2560 + 4608 + 5040$
$128 \times 128 = 1 + 9 + 2(36) + 44 + 2(84) + 2(126) + 156 + 2(231) + 495 + 2(594) + 910 + 2(924) + 1650 + 2457 + 2772 + 3900$
$156 \times 9 = 44 + 450 + 910$
$156 \times 16 = 576 + 1920$
$156 \times 36 = 156 + 231 + 2457 + 2772''$
$156 \times 44 = 9 + 156 + 231 + 1122 + 2574 + 2772''$
$156 \times 84 = 594 + 910 + 3900 + 7700$
$156 \times 126 = 924 + 2457 + 3900 + 12375$
$156 \times 128 = 128 + 576 + 1920 + 2560 + 5280 + 9504$
$156 \times 156 = 1 + 36 + 44 + 450 + 495 + 910 + 2508 + 4004 + 7140 + 8748$
$231 \times 9 = 36 + 44 + 495 + 594 + 910$
$231 \times 16 = 128 + 432 + 576 + 2560$
$231 \times 36 = 9 + 84 + 156 + 2(231) + 924 + 1650 + 2457 + 2574$
$231 \times 44 = 9 + 84 + 156 + 2(231) + 1650 + 2457 + 2574 + 2772''$
$231 \times 84 = 36 + 44 + 126 + 495 + 2(594) + 910 + 924 + 2772 + 3900 + 9009$
$231 \times 126 = 84 + 126 + 231 + 594 + 2(924) + 1650 + 2457 + 2772 + 3900 + 15444$
$231 \times 128 = 16 + 2(128) + 2(432) + 2(576) + 768 + 1920 + 2(2560) + 4928 + 5040 + 9504$
$231 \times 156 = 36 + 44 + 450 + 495 + 594 + 2(910) + 7140 + 7700 + 8748 + 9009$
$231 \times 231 = 1 + 2(36) + 2(44) + 126 + 450 + 2(495) + 3(594) + 3(910) + 1980 + 2772 + 3900 + 4004 + 7700 + 8748 + 2(9009)$

Table A.48: SO(9) Tensor Products (continued)

$$432 \times 9 = 128 + 432 + 768 + 2560$$

$$432 \times 16 = 36 + 84 + 126 + 231 + 495 + 594 + 924 + 1650 + 2772$$

$$432 \times 36 = 16 + 128 + 2(432) + 576 + 672 + 768 + 2560 + 4928 + 5040$$

$$432 \times 44 = 128 + 432 + 576 + 768 + 2560 + 5040 + 9504$$

$$432 \times 84 = 16 + 2(128) + 2(432) + 576 + 672 + 2(768) + 2(2560) + 4608 + 4928 + 5040 + 12672$$

$$432 \times 126 = 16 + 2(128) + 3(432) + 576 + 672 + 2(768) + 2(2560) + 4608 + 4928 + 2(5040) + 12672 + 12672'$$

$$432 \times 128 = 9 + 36 + 44 + 2(84) + 2(126) + 2(231) + 495 + 3(594) + 910 + 3(924) + 2(1650) + 1980 + 2457 + 2574 + 2(2772) + 3900 + 4158 + 9009 + 15444$$

$$432 \times 156 = 432 + 576 + 1920 + 2560 + 5040 + 9504 + 19712 + 27648$$

$$432 \times 231 = 16 + 2(128) + 2(432) + 2(576) + 672 + 2(768) + 1920 + 3(2560) + 4608 + 4928 + 2(5040) + 9504 + 12672 + 19712 + 24192$$

$$432 \times 432 = 1 + 9 + 2(36) + 44 + 2(84) + 3(126) + 156 + 2(231) + 2(495) + 3(594) + 910 + 4(924) + 3(1650) + 1980 + 2(2457) + 2574 + 4(2772) + 2772' + 2(3900) + 4004 + 2(4158) + 2(9009) + 12012 + 2(15444) + 15444' + 25740 + 27456$$

$$450 \times 9 = 156 + 1122 + 2772''$$

$$450 \times 16 = 1920 + 5280$$

$$450 \times 36 = 450 + 910 + 7140 + 7700$$

$$450 \times 44 = 44 + 450 + 910 + 2508 + 7140 + 8748$$

$$450 \times 84 = 2457 + 2772'' + 12375 + 20196$$

$$450 \times 126 = 3900 + 7700 + 12375 + 32725$$

$$450 \times 128 = 576 + 1920 + 5280 + 9504 + 12672'' + 27648$$

$$450 \times 156 = 9 + 156 + 231 + 1122 + 2574 + 2772'' + 5148 + 16302 + 18018' + 23868$$

$$450 \times 231 = 156 + 231 + 1122 + 2457 + 2574 + 2(2772'') + 16302 + 20196 + 23868 + 31500$$

$$450 \times 432 = 1920 + 2560 + 5280 + 9504 + 19712 + 27648 + 59136 + 68640$$

$$450 \times 450 = 1 + 36 + 44 + 450 + 495 + 910 + 2508 + 4004 + 7140 + 8748 + 9867 + 22932 + 33957 + 54978 + 56430$$

$$495 \times 9 = 231 + 1650 + 2574$$

$$495 \times 16 = 432 + 2560 + 4928$$

$$495 \times 36 = 36 + 495 + 594 + 910 + 2772 + 4004 + 9009$$

$$495 \times 44 = 44 + 495 + 594 + 910 + 1980 + 8748 + 9009$$

$$495 \times 84 = 84 + 231 + 924 + 1650 + 2457 + 2574 + 2772 + 15444 + 15444'$$

$$495 \times 126 = 126 + 495 + 594 + 924 + 1650 + 2772 + 3900 + 9009 + 15444 + 27456$$

$$495 \times 128 = 128 + 432 + 576 + 768 + 2(2560) + 4928 + 5040 + 9504 + 12672 + 24192$$

$$495 \times 156 = 156 + 231 + 1650 + 2457 + 2574 + 2772'' + 12012 + 23868 + 31500$$

$$495 \times 231 = 9 + 84 + 156 + 2(231) + 924 + 2(1650) + 2(2457) + 2(2574) + 2772'' + 4158 + 12012 + 15444 + 15444' + 18018' + 31500$$

$$495 \times 432 = 16 + 128 + 2(432) + 576 + 672 + 768 + 1920 + 2(2560) + 4608 + 2(4928) + 2(5040) + 9504 + 12672 + 12672' + 19712 + 24192 + 34944 + 65536$$

$$495 \times 450 = 450 + 495 + 910 + 7140 + 7700 + 8748 + 9009 + 44352 + 56430 + 87516$$

$$495 \times 495 = 1 + 36 + 44 + 126 + 450 + 2(495) + 594 + 910 + 1980 + 2772 + 2772' + 3900 + 4004 + 7700 + 8748 + 2(9009) + 22932 + 25740 + 27456 + 44352 + 71500$$


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Table A.49: SO(10) Tensor Products

$10 \times 10 = 1 + 45 + 54$
$16 \times 10 = \overline{16} + \overline{144}$
$16 \times 16 = 10 + 120 + \overline{126}$
$\overline{16} \times 16 = 1 + 45 + 210$
$45 \times 10 = 10 + 120 + 320$
$45 \times 16 = 16 + 144 + 560$
$45 \times 45 = 1 + 45 + 54 + 210 + 770 + 945$
$54 \times 10 = 10 + 210' + 320$
$54 \times 16 = 144 + 720$
$54 \times 45 = 45 + 54 + 945 + 1386$
$54 \times 54 = 1 + 45 + 54 + 660 + 770 + 1386$
$120 \times 10 = 45 + 210 + 945$
$120 \times 16 = \overline{16} + \overline{144} + \overline{560} + \overline{1200}$
$120 \times 45 = 10 + 120 + 126 + \overline{126} + 320 + 1728 + 2970$
$120 \times 54 = 120 + 320 + 1728 + 4312$
$120 \times 120 = 1 + 45 + 54 + 2(210) + 770 + 945 + 1050 + \overline{1050} + 4125 + 5940$
$126 \times 10 = 210 + 1050$
$\overline{126} \times 16 = \overline{144} + \overline{672} + \overline{1200}$
$126 \times 16 = \overline{16} + \overline{560} + \overline{1440}$
$126 \times 45 = 120 + 126 + 1728 + 3696'$
$126 \times 54 = \overline{126} + 1728 + 4950$
$126 \times 120 = 45 + 210 + 945 + 1050 + 5940 + 6930$
$\overline{126} \times 126 = 1 + 45 + 210 + 770 + 5940 + 8910$
$126 \times 126 = 54 + 945 + 1050 + 2772 + 4125 + 6930$
$144 \times 10 = \overline{16} + \overline{144} + \overline{560} + \overline{720}$
$\overline{144} \times 16 = 45 + 54 + 210 + 945 + \overline{1050}$
$144 \times 16 = 10 + 120 + 126 + 320 + 1728$
$144 \times 45 = 16 + 2(144) + 560 + 720 + 1200 + 3696$
$144 \times 54 = 16 + 144 + 560 + 720 + 2640 + 3696$
$144 \times 120 = \overline{16} + 2(\overline{144}) + 2(\overline{560}) + \overline{720} + \overline{1200} + \overline{1440} + \overline{3696} + \overline{8800}$
$\overline{144} \times 126 = 16 + 144 + 560 + 1200 + 1440 + 3696 + 11088$
$144 \times 126 = \overline{144} + \overline{560} + \overline{720} + \overline{1200} + \overline{1440} + \overline{5280} + \overline{8800}$
$\overline{144} \times 144 = 1 + 2(45) + 54 + 2(210) + 770 + 2(945) + 1050 + \overline{1050} + 1386 + 5940 + 8085$
$144 \times 144 = 10 + 2(120) + 126 + \overline{126} + 210' + 2(320) + 2(1728) + 2970 + 3696' + 4312 + 4950$
$210 \times 10 = 120 + 126 + \overline{126} + 1728$
$210' \times 10 = 54 + 660 + 1386$
$210 \times 16 = 16 + 144 + 560 + 1200 + 1440$
$210' \times 16 = \overline{720} + \overline{2640}$
$210 \times 45 = 45 + 2(210) + 945 + 1050 + \overline{1050} + 5940$
$210' \times 45 = 210' + 320 + 4312 + 4608$
$210 \times 54 = 210 + 945 + 1050 + \overline{1050} + 8085$
$210' \times 54 = 10 + 210' + 320 + 1782 + 4410 + 4608$
$210 \times 120 = 10 + 2(120) + 126 + \overline{126} + 320 + 2(1728) + 2970 + 3696' + \overline{3696}' + 10560$
$210' \times 120 = 945 + 1386 + 8085 + 14784$
$210 \times 126 = 10 + 120 + 126 + 320 + 1728 + 2970 + 3696' + 6930' + 10560$
$210' \times 126 = \overline{1050} + 8085 + 17325$
$210 \times 144 = 16 + 2(144) + 2(560) + 672 + 720 + 2(1200) + 1440 + 3696 + 8800 + 11088$
$210' \times 144 = \overline{144} + \overline{720} + \overline{2640} + \overline{3696} + \overline{7920} + \overline{15120}$

Table A.49: SO(10) Tensor Products (continued)

$210 \times 210$	$= 1 + 2(45) + 54 + 2(210) + 770 + 2(945) + 1050 + \overline{1050} + 4125 + 2(5940) + 6930 + \overline{6930} + 8910$
$210' \times 210$	$= 1728 + 4312 + 4950 + \overline{4950} + 28160$
$210' \times 210'$	$= 1 + 45 + 54 + 660 + 770 + 1386 + 4290 + 7644 + 12870 + 16380$
$320 \times 10$	$= 45 + 54 + 770 + 945 + 1386$
$320 \times 16$	$= \overline{144} + \overline{560} + \overline{720} + \overline{3696}$
$320 \times 45$	$= 10 + 120 + 210' + 2(320) + 1728 + 2970 + 4312 + 4410$
$320 \times 54$	$= 10 + 120 + 210' + 2(320) + 2970 + 4312 + 4410 + 4608$
$320 \times 120$	$= 45 + 54 + 210 + 770 + 2(945) + 1050 + \overline{1050} + 1386 + 5940 + 8085 + 17920$
$320 \times 126$	$= 210 + 945 + 1050 + \overline{1050} + 5940 + 8085 + 23040$
$320 \times 144$	$= \overline{16} + 2(\overline{144}) + 2(\overline{560}) + 2(\overline{720}) + \overline{1200} + \overline{2640} + 2(\overline{3696}) + \overline{8064} + \overline{8800} + \overline{15120}$
$320 \times 210$	$= 120 + 126 + \overline{126} + 320 + 3(1728) + 2970 + 3696' + \overline{3696}' + 4312 + 4950 + \overline{4950} + 36750$
$320 \times 210'$	$= 45 + 54 + 660 + 770 + 945 + 2(1386) + 12870 + 14784 + 16380 + 17920$
$320 \times 320$	$= 1 + 2(45) + 2(54) + 210 + 660 + 2(770) + 3(945) + 3(1386) + 4125 + 5940 + 7644 + 8085 + 14784 + 16380 + 2(17920)$
$560 \times 10$	$= \overline{144} + \overline{560} + \overline{1200} + \overline{3696}$
$560 \times 16$	$= 120 + \overline{126} + 320 + 1728 + 2970 + \overline{3696}'$
$\overline{560} \times 16$	$= 45 + 210 + 770 + 945 + 1050 + 5940$
$560 \times 45$	$= 16 + 144 + 2(560) + 720 + 1200 + 1440 + 3696 + 8064 + 8800$
$560 \times 54$	$= 144 + 560 + 720 + 1200 + 3696 + 8800 + 15120$
$560 \times 120$	$= \overline{16} + 2(\overline{144}) + 2(\overline{560}) + \overline{672} + \overline{720} + 2(\overline{1200}) + \overline{1440} + 2(\overline{3696}) + \overline{8064} + \overline{8800} + \overline{11088} + \overline{25200}$
$560 \times 126$	$= \overline{16} + \overline{144} + 2(\overline{560}) + \overline{1200} + \overline{1440} + \overline{3696} + \overline{8064} + \overline{8800} + \overline{11088} + \overline{34992}$
$\overline{560} \times 126$	$= 144 + 560 + 672 + 720 + 2(1200) + 3696 + 8800 + 11088 + 17280 + 25200$
$560 \times 144$	$= 10 + 2(120) + 126 + \overline{126} + 2(320) + 3(1728) + 2(2970) + 3696' + \overline{3696}' + 4312 + 4410 + \overline{4950} + 10560 + 36750$
$\overline{560} \times 144$	$= 45 + 54 + 2(210) + 770 + 3(945) + 2(1050) + \overline{1050} + 1386 + 4125 + 2(5940) + 6930 + 8085 + 17920 + 23040$
$560 \times 210$	$= 16 + 2(144) + 3(560) + 720 + 2(1200) + 2(1440) + 2(3696) + 5280 + 8064 + 2(8800) + 11088 + 25200 + 34992$
$560 \times 210'$	$= \overline{560} + \overline{720} + \overline{2640} + \overline{3696} + \overline{8800} + \overline{15120} + \overline{38016} + \overline{48048}$
$560 \times 320$	$= \overline{16} + 2(\overline{144}) + 2(\overline{560}) + 2(\overline{720}) + 2(\overline{1200}) + \overline{1440} + \overline{2640} + 3(\overline{3696}) + \overline{8064} + 2(\overline{8800}) + \overline{11088} + \overline{15120} + \overline{25200} + \overline{38016} + \overline{43680}$
$560 \times 560$	$= 10 + 2(120) + 126 + 2(\overline{126}) + 210' + 2(320) + 4(1728) + 3(2970) + 3696' + 3(\overline{3696}') + 2(4312) + 4410 + 4950 + \overline{4950} + \overline{6930}' + 2(10560) + 27720 + 34398 + 2(36750) + 46800 + 48114$
$\overline{560} \times 560$	$= 1 + 2(45) + 54 + 3(210) + 2(770) + 3(945) + 2(1050) + 2(\overline{1050}) + 1386 + 4125 + 4(5940) + 6930 + \overline{6930} + 7644 + 2(8085) + 8910 + 2(17920) + 23040 + \overline{23040} + 72765 + 73710$
$660 \times 10$	$= 210' + 1782 + 4608$
$660 \times 16$	$= 2640 + 7920$
$660 \times 45$	$= 660 + 1386 + 12870 + 14784$
$660 \times 54$	$= 54 + 660 + 1386 + 4290 + 12870 + 16380$
$660 \times 120$	$= 4312 + 4608 + 28160 + 42120$
$660 \times 126$	$= \overline{4950} + 28160 + 50050$
$660 \times 144$	$= 720 + 2640 + 7920 + 15120 + 20592 + 48048$
$660 \times 210'$	$= 10 + 210' + 320 + 1782 + 4410 + 4608 + 9438 + 31680 + 37632 + 48510$
$660 \times 320$	$= 210' + 320 + 1782 + 4312 + 4410 + 2(4608) + 31680 + 42120 + 48510 + 68640$



Table A.50: SO(11) Tensor Products

$11 \times 11 = 1 + 55 + 65$
$32 \times 11 = 32 + 320$
$32 \times 32 = 1 + 11 + 55 + 165 + 330 + 462$
$55 \times 11 = 11 + 165 + 429$
$55 \times 32 = 32 + 320 + 1408$
$55 \times 55 = 1 + 55 + 65 + 330 + 1144 + 1430$
$65 \times 11 = 11 + 275 + 429$
$65 \times 32 = 320 + 1760$
$65 \times 55 = 55 + 65 + 1430 + 2025$
$65 \times 65 = 1 + 55 + 65 + 935 + 1144 + 2025$
$165 \times 11 = 55 + 330 + 1430$
$165 \times 32 = 32 + 320 + 1408 + 3520$
$165 \times 55 = 11 + 165 + 429 + 462 + 3003 + 5005$
$165 \times 65 = 165 + 429 + 3003 + 7128$
$165 \times 165 = 1 + 55 + 65 + 330 + 462 + 1144 + 1430 + 4290 + 7865 + 11583$
$275 \times 11 = 65 + 935 + 2025$
$275 \times 32 = 1760 + 7040$
$275 \times 55 = 275 + 429 + 7128 + 7293$
$275 \times 65 = 11 + 275 + 429 + 2717 + 7150 + 7293$
$275 \times 165 = 1430 + 2025 + 15400 + 26520$
$275 \times 275 = 1 + 55 + 65 + 935 + 1144 + 2025 + 7007 + 13650 + 21945 + 28798$
$320 \times 11 = 32 + 320 + 1408 + 1760$
$320 \times 32 = 11 + 55 + 65 + 165 + 330 + 429 + 462 + 1430 + 3003 + 4290$
$320 \times 55 = 32 + 2(320) + 1408 + 1760 + 3520 + 10240$
$320 \times 65 = 32 + 320 + 1408 + 1760 + 7040 + 10240$
$320 \times 165 = 32 + 2(320) + 2(1408) + 1760 + 3520 + 5280 + 10240 + 28512$
$320 \times 320 = 1 + 11 + 2(55) + 65 + 2(165) + 275 + 2(330) + 2(429) + 2(462) + 1144 + 2(1430) + 2025 + 2(3003) + 2(4290) + 5005 + 7128 + 11583 + 15400 + 17160 + 22275$
$330 \times 11 = 165 + 462 + 3003$
$330 \times 32 = 32 + 320 + 1408 + 3520 + 5280$
$330 \times 55 = 55 + 330 + 462 + 1430 + 4290 + 11583$
$330 \times 65 = 330 + 1430 + 4290 + 15400$
$330 \times 165 = 11 + 165 + 330 + 429 + 462 + 3003 + 4290 + 5005 + 17160 + 23595$
$330 \times 330 = 1 + 55 + 65 + 165 + 330 + 462 + 1144 + 1430 + 3003 + 4290 + 7865 + 11583 + 17160 + 23595' + 37752$
$429 \times 11 = 55 + 65 + 1144 + 1430 + 2025$
$429 \times 32 = 320 + 1408 + 1760 + 10240$
$429 \times 55 = 11 + 165 + 275 + 2(429) + 3003 + 5005 + 7128 + 7150$
$429 \times 65 = 11 + 165 + 275 + 2(429) + 5005 + 7128 + 7150 + 7293$
$429 \times 165 = 55 + 65 + 330 + 1144 + 2(1430) + 2025 + 4290 + 11583 + 15400 + 33033$
$429 \times 275 = 55 + 65 + 935 + 1144 + 1430 + 2(2025) + 21945 + 26520 + 28798 + 33033$
$429 \times 429 = 1 + 2(55) + 2(65) + 330 + 935 + 2(1144) + 3(1430) + 3(2025) + 7865 + 11583 + 13650 + 15400 + 26520 + 28798 + 2(33033)$
$462 \times 11 = 330 + 462 + 4290$
$462 \times 32 = 32 + 320 + 1408 + 3520 + 4224 + 5280$
$462 \times 55 = 165 + 330 + 462 + 3003 + 4290 + 17160$
$462 \times 65 = 462 + 3003 + 4290 + 22275$
$462 \times 165 = 55 + 165 + 330 + 462 + 1430 + 3003 + 4290 + 11583 + 17160 + 37752$

Table A.51: SO(12) Tensor Products

$12 \times 12 = 1 + 66 + 77$
$32 \times 12 = \overline{32} + \overline{352}$
$\overline{32} \times 32 = 12 + 220 + 792$
$32 \times 32 = 1 + 66 + 462 + 495$
$66 \times 12 = 12 + 220 + 560$
$66 \times 32 = 32 + 352 + 1728$
$66 \times 66 = 1 + 66 + 77 + 495 + 1638 + 2079$
$77 \times 12 = 12 + 352' + 560$
$77 \times 32 = 352 + 2112$
$77 \times 66 = 66 + 77 + 2079 + 2860$
$77 \times 77 = 1 + 66 + 77 + 1287 + 1638 + 2860$
$220 \times 12 = 66 + 495 + 2079$
$220 \times 32 = \overline{32} + \overline{352} + \overline{1728} + \overline{4928}'$
$220 \times 66 = 12 + 220 + 560 + 792 + 4928 + 8008$
$220 \times 77 = 220 + 560 + 4928 + 11232$
$220 \times 220 = 1 + 66 + 77 + 462 + \overline{462} + 495 + 1638 + 2079 + 8085 + 14014 + 21021$
$352 \times 12 = \overline{32} + \overline{352} + \overline{1728} + \overline{2112}$
$352' \times 12 = 77 + 1287 + 2860$
$352 \times 32 = 66 + 77 + \overline{462} + 495 + 2079 + 8085$
$\overline{352} \times 32 = 12 + 220 + 560 + 792 + 4752 + 4928$
$352' \times 32 = \overline{2112} + \overline{9152}$
$352 \times 66 = 32 + 2(352) + 1728 + 2112 + 4928' + 13728$
$352' \times 66 = 352' + 560 + 11088' + 11232$
$352 \times 77 = 32 + 352 + 1728 + 2112 + 9152 + 13728$
$352' \times 77 = 12 + 352' + 560 + 4004 + 11088 + 11088'$
$352 \times 220 = \overline{32} + 2(\overline{352}) + 2(\overline{1728}) + \overline{2112} + \overline{4928}' + \overline{8800} + \overline{13728} + \overline{43680}$
$352' \times 220 = 2079 + 2860 + 27456 + 45045$
$352 \times 352 = 1 + 2(66) + 77 + 462 + \overline{462} + 2(495) + 1638 + 2(2079) + 2860 + 2(8085) + 21021 + \overline{21450} + \overline{27027} + 27456$
$\overline{352} \times 352 = 12 + 2(220) + 352' + 2(560) + 2(792) + 4752 + \overline{4752} + 2(4928) + 8008 + 11232 + 36036 + 45760$
$462 \times 12 = 792 + 4752$
$\overline{462} \times 32 = 352 + 4928' + 9504$
$462 \times 32 = 32 + 1728 + 4224 + 8800$
$462 \times 66 = 462 + 495 + 8085 + 21450$
$462 \times 77 = \overline{462} + 8085 + 27027$
$495 \times 12 = 220 + 792 + 4928$
$495 \times 32 = 32 + 352 + \overline{1728} + 4928' + 8800$
$495 \times 66 = 66 + 462 + \overline{462} + 495 + 2079 + 8085 + 21021$
$495 \times 77 = 495 + 2079 + 8085 + 27456$
$495 \times 220 = 12 + 220 + 560 + 2(792) + 4752 + \overline{4752} + 4928 + 8008 + 36036 + 48048$
$560 \times 12 = 66 + 77 + 1638 + 2079 + 2860$
$560 \times 32 = \overline{352} + \overline{1728} + \overline{2112} + \overline{13728}$
$560 \times 66 = 12 + 220 + 352' + 2(560) + 4928 + 8008 + 11088 + 11232$
$560 \times 77 = 12 + 220 + 352' + 2(560) + 8008 + 11088 + 11088' + 11232$

Table A.52: SO(13) Tensor Products

$13 \times 13 = 1 + 78 + 90$
$64 \times 13 = 64 + 768$
$64 \times 64 = 1 + 13 + 78 + 286 + \overline{715} + 1287 + 1716$
$78 \times 13 = 13 + 286 + 715$
$78 \times 64 = 64 + 768 + 4160$
$78 \times 78 = 1 + 78 + 90 + \overline{715} + 2275 + 2925$
$90 \times 13 = 13 + 442 + 715$
$90 \times 64 = 768 + 4992$
$90 \times 78 = 78 + 90 + 2925 + 3927$
$90 \times 90 = 1 + 78 + 90 + 1729 + 2275 + 3927$
$286 \times 13 = 78 + \overline{715} + 2925$
$286 \times 64 = 64 + 768 + 4160 + 13312$
$286 \times 78 = 13 + 286 + 715 + 1287 + 7722 + 12285$
$286 \times 90 = 286 + 715 + 7722 + 17017$
$442 \times 13 = 90 + 1729 + 3927$
$442 \times 78 = 442 + 715 + 16302 + 17017$
$442 \times 90 = 13 + 442 + 715 + 5733 + 16302 + 16575$
$715 \times 13 = 78 + 90 + 2275 + 2925 + 3927$
$715 \times 78 = 13 + 286 + 442 + 2(715) + 7722 + 12285 + 16575 + 17017$
$715 \times 90 = 13 + 286 + 442 + 2(715) + 12285 + 16302 + 16575 + 17017$
$768 \times 13 = 64 + 768 + 4160 + 4992$
$1287 \times 13 = \overline{715} + 1716 + 14300$
$1729 \times 13 = 442 + 5733 + 16302$
$2275 \times 13 = 715 + 12285 + 16575$
$2925 \times 13 = 286 + 715 + 7722 + 12285 + 17017$

Table A.53: SO(14) Tensor Products

$14 \times 14 = 1 + 91 + 104$
$64 \times 14 = \overline{64} + \overline{832}$
$\overline{64} \times 64 = 1 + 91 + 1001 + 3003$
$64 \times 64 = 14 + 364 + 1716 + 2002$
$91 \times 14 = 14 + 364 + 896$
$91 \times 64 = 64 + 832 + 4928$
$91 \times 91 = 1 + 91 + 104 + 1001 + 3080 + 4004$
$104 \times 14 = 14 + 546 + 896$
$104 \times 64 = 832 + 5824$
$104 \times 91 = 91 + 104 + 4004 + 5265$
$104 \times 104 = 1 + 91 + 104 + 2275 + 3080 + 5265$
$364 \times 14 = 91 + 1001 + 4004$
$546 \times 14 = 104 + 2275 + 5265$
$832 \times 14 = \overline{64} + \overline{832} + 4928 + \overline{5824}$

Table A.54: SO(18) Tensor Products

$$\begin{aligned}
 \mathbf{18} \times \mathbf{18} &= \mathbf{1} + \mathbf{153} + \mathbf{170} \\
 \mathbf{153} \times \mathbf{18} &= \mathbf{18} + \mathbf{816} + \mathbf{1920} \\
 \mathbf{170} \times \mathbf{18} &= \mathbf{18} + \mathbf{1122} + \mathbf{1920} \\
 \mathbf{256} \times \mathbf{18} &= \overline{\mathbf{256}} + \overline{\mathbf{4352}}
 \end{aligned}$$

Table A.55: SO(22) Tensor Products

$$\begin{aligned}
 \mathbf{22} \times \mathbf{22} &= \mathbf{1} + \mathbf{231} + \mathbf{252} \\
 \mathbf{231} \times \mathbf{22} &= \mathbf{22} + \mathbf{1540} + \mathbf{3520} \\
 \mathbf{252} \times \mathbf{22} &= \mathbf{22} + \mathbf{2002} + \mathbf{3520}
 \end{aligned}$$

Table A.56: SO(26) Tensor Products

$$\begin{aligned}
 \mathbf{26} \times \mathbf{26} &= \mathbf{1} + \mathbf{325} + \mathbf{350} \\
 \mathbf{325} \times \mathbf{26} &= \mathbf{26} + \mathbf{2600} + \mathbf{5824} \\
 \mathbf{325} \times \mathbf{325} &= \mathbf{1} + \mathbf{325} + \mathbf{350} + \mathbf{14950} + \mathbf{37674} + \mathbf{52325} \\
 \mathbf{2600} \times \mathbf{26} &= \mathbf{325} + \mathbf{14950} + \mathbf{52325} \\
 \mathbf{2600} \times \mathbf{325} &= \mathbf{26} + \mathbf{2600} + \mathbf{5824} + \mathbf{65780} + \mathbf{320320} + \mathbf{450450}
 \end{aligned}$$

A.2.3.  $Sp(N)$

Table A.57:  $Sp(4)$  Tensor Products

$4 \times 4 = 1 + 5 + 10$
$5 \times 4 = 4 + 16$
$5 \times 5 = 1 + 10 + 14$
$10 \times 4 = 4 + 16 + 20$
$10 \times 5 = 5 + 10 + 35$
$10 \times 10 = 1 + 5 + 10 + 14 + 35 + 35'$
$14 \times 4 = 16 + 40$
$14 \times 5 = 5 + 30 + 35$
$14 \times 10 = 10 + 14 + 35 + 81$
$14 \times 14 = 1 + 10 + 14 + 35' + 55 + 81$
$16 \times 4 = 5 + 10 + 14 + 35$
$16 \times 5 = 4 + 16 + 20 + 40$
$16 \times 10 = 4 + 2(16) + 20 + 40 + 64$
$16 \times 14 = 4 + 16 + 20 + 40 + 64 + 80$
$16 \times 16 = 1 + 5 + 2(10) + 14 + 30 + 2(35) + 35' + 81$
$20 \times 4 = 10 + 35 + 35'$
$20 \times 5 = 16 + 20 + 64$
$20 \times 10 = 4 + 16 + 20 + 40 + 56 + 64$
$20 \times 14 = 16 + 20 + 40 + 64 + 140$
$20 \times 16 = 5 + 10 + 14 + 2(35) + 35' + 81 + 105$
$20 \times 20 = 1 + 5 + 10 + 14 + 30 + 35 + 35' + 81 + 84 + 105$
$30 \times 4 = 40 + 80$
$30 \times 5 = 14 + 55 + 81$
$30 \times 10 = 30 + 35 + 81 + 154$
$30 \times 14 = 5 + 30 + 35 + 91 + 105 + 154$
$30 \times 16 = 16 + 40 + 64 + 80 + 140 + 140'$
$30 \times 20 = 20 + 40 + 64 + 80 + 140 + 256$
$30 \times 30 = 1 + 10 + 14 + 35' + 55 + 81 + 84 + 140'' + 220 + 260$
$35 \times 4 = 16 + 20 + 40 + 64$
$35' \times 4 = 20 + 56 + 64$
$35 \times 5 = 10 + 14 + 35 + 35' + 81$
$35' \times 5 = 35 + 35' + 105$
$35 \times 10 = 5 + 10 + 14 + 30 + 2(35) + 35' + 81 + 105$
$35' \times 10 = 10 + 35 + 35' + 81 + 84 + 105$
$35 \times 14 = 5 + 10 + 30 + 2(35) + 35' + 81 + 105 + 154$
$35' \times 14 = 14 + 35 + 35' + 81 + 105 + 220$
$35 \times 16 = 4 + 2(16) + 2(20) + 2(40) + 56 + 2(64) + 80 + 140$
$35' \times 16 = 16 + 20 + 40 + 56 + 2(64) + 140 + 160$
$35 \times 20 = 4 + 2(16) + 20 + 2(40) + 56 + 2(64) + 80 + 140 + 160$
$35' \times 20 = 4 + 16 + 20 + 40 + 56 + 64 + 80 + 120 + 140 + 160$
$35 \times 30 = 10 + 14 + 35 + 35' + 55 + 2(81) + 105 + 154 + 220 + 260$
$35' \times 30 = 30 + 35 + 35' + 81 + 105 + 154 + 220 + 390$
$35 \times 35 = 1 + 5 + 2(10) + 2(14) + 30 + 3(35) + 2(35') + 55 + 3(81) + 84 + 2(105) + 154 + 220$
$35' \times 35 = 5 + 10 + 14 + 30 + 2(35) + 35' + 2(81) + 84 + 2(105) + 154 + 220 + 231$
$35' \times 35' = 1 + 5 + 10 + 14 + 30 + 35 + 35' + 55 + 81 + 84 + 105 + 154 + 165 + 220 + 231$

Table A.58: Sp(6) Tensor Products

$6 \times 6 = 1 + 14 + 21$
$14' \times 6 = 14 + 70$
$14 \times 6 = 6 + 14' + 64$
$14' \times 14' = 1 + 21 + 84 + 90$
$14' \times 14 = 6 + 64 + 126$
$14 \times 14 = 1 + 14 + 21 + 70 + 90$
$21 \times 6 = 6 + 56 + 64$
$21 \times 14' = 14' + 64 + 216$
$21 \times 14 = 14 + 21 + 70 + 189$
$21 \times 21 = 1 + 14 + 21 + 90 + 126' + 189$
$56 \times 6 = 21 + 126' + 189$
$56 \times 14' = 70 + 189 + 525$
$56 \times 14 = 56 + 64 + 216 + 448$
$56 \times 21 = 6 + 56 + 64 + 252 + 350 + 448$
$56 \times 56 = 1 + 14 + 21 + 90 + 126' + 189 + 385 + 462 + 924 + 924'$
$64 \times 6 = 14 + 21 + 70 + 90 + 189$
$64 \times 14' = 14 + 21 + 70 + 90 + 189 + 512$
$64 \times 14 = 6 + 14' + 56 + 2(64) + 126 + 216 + 350$
$64 \times 21 = 6 + 14' + 56 + 2(64) + 126 + 216 + 350 + 448$
$64 \times 56 = 14 + 21 + 70 + 90 + 126' + 2(189) + 512 + 525 + 924 + 924'$
$64 \times 64 = 1 + 2(14) + 2(21) + 3(70) + 84 + 2(90) + 126' + 3(189) + 385 + 2(512) + 525 + 924$
$70 \times 6 = 14' + 64 + 126 + 216$
$70 \times 14' = 6 + 56 + 64 + 126 + 350 + 378$
$70 \times 14 = 14 + 21 + 70 + 84 + 90 + 189 + 512$
$70 \times 21 = 14 + 2(70) + 90 + 189 + 512 + 525$
$70 \times 56 = 14' + 64 + 126 + 2(216) + 350 + 448 + 1100 + 1386$
$70 \times 64 = 6 + 14' + 56 + 3(64) + 2(126) + 2(216) + 2(350) + 378 + 448 + 616 + 1386$
$70 \times 70 = 1 + 14 + 2(21) + 70 + 84 + 2(90) + 126' + 2(189) + 385 + 2(512) + 594 + 924 + 1078$
$84 \times 6 = 126 + 378$
$84 \times 14' = 14' + 216 + 330 + 616$
$84 \times 14 = 70 + 512 + 594$
$84 \times 21 = 84 + 90 + 512 + 1078$
$84 \times 56 = 126 + 350 + 378 + 1386 + 2464$
$84 \times 64 = 64 + 126 + 216 + 350 + 378 + 616 + 1386 + 2240$
$84 \times 70 = 14 + 70 + 189 + 385 + 512 + 525 + 594 + 1386' + 2205$
$84 \times 84 = 1 + 21 + 84 + 90 + 126' + 924 + 1001 + 1078 + 1274 + 2457$
$90 \times 6 = 64 + 126 + 350$
$90 \times 14' = 14' + 64 + 216 + 350 + 616$
$90 \times 14 = 14 + 70 + 90 + 189 + 385 + 512$
$90 \times 21 = 21 + 70 + 84 + 90 + 189 + 512 + 924$
$90 \times 56 = 56 + 64 + 126 + 216 + 350 + 378 + 448 + 1386 + 2016$
$90 \times 64 = 6 + 14' + 56 + 2(64) + 2(126) + 2(216) + 2(350) + 378 + 448 + 616 + 1344 + 1386$
$90 \times 70 = 14 + 21 + 2(70) + 90 + 2(189) + 385 + 2(512) + 525 + 594 + 924 + 2205$
$90 \times 84 = 21 + 84 + 90 + 189 + 512 + 924 + 1078 + 2205 + 2457$
$90 \times 90 = 1 + 14 + 21 + 70 + 84 + 2(90) + 126' + 189 + 385 + 2(512) + 525 + 924 + 1078 + 1274 + 2205$

Table A.59: Sp(8) Tensor Products

$8 \times 8 = 1 + 27 + 36$
$27 \times 8 = 8 + 48 + 160$
$27 \times 27 = 1 + 27 + 36 + 42 + 308 + 315$
$36 \times 8 = 8 + 120 + 160$
$36 \times 27 = 27 + 36 + 315 + 594$
$36 \times 36 = 1 + 27 + 36 + 308 + 330 + 594$
$42 \times 8 = 48 + 288$
$42 \times 27 = 27 + 315 + 792'$
$42 \times 36 = 42 + 315 + 1155$
$42 \times 42 = 1 + 36 + 308 + 594' + 825$
$48 \times 8 = 27 + 42 + 315$
$48 \times 27 = 8 + 48 + 160 + 288 + 792$
$48 \times 36 = 48 + 160 + 288 + 1232$
$48 \times 42 = 8 + 160 + 792 + 1056$
$48 \times 48 = 1 + 27 + 36 + 308 + 315 + 792' + 825$
$120 \times 8 = 36 + 330 + 594$
$120 \times 27 = 120 + 160 + 1232 + 1728$
$120 \times 36 = 8 + 120 + 160 + 792'' + 1512 + 1728$
$120 \times 42 = 288 + 1232 + 3520$
$120 \times 48 = 315 + 594 + 1155 + 3696$
$120 \times 120 = 1 + 27 + 36 + 308 + 330 + 594 + 1716 + 2184 + 4290 + 4914$
$160 \times 8 = 27 + 36 + 308 + 315 + 594$
$160 \times 27 = 8 + 48 + 120 + 2(160) + 288 + 792 + 1232 + 1512$
$160 \times 36 = 8 + 48 + 120 + 2(160) + 792 + 1232 + 1512 + 1728$
$160 \times 42 = 48 + 160 + 288 + 792 + 1232 + 4200$
$160 \times 48 = 27 + 36 + 42 + 308 + 2(315) + 594 + 792' + 1155 + 4096$
$160 \times 120 = 27 + 36 + 308 + 315 + 330 + 2(594) + 3696 + 4096 + 4290 + 4914$
$160 \times 160 = 1 + 2(27) + 2(36) + 42 + 2(308) + 3(315) + 330 + 3(594) + 792' + 825 + 1155 + 2184 + 3696 + 2(4096) + 4914$
$288 \times 8 = 42 + 315 + 792' + 1155$
$288 \times 27 = 48 + 160 + 288 + 792 + 1056 + 1232 + 4200$
$288 \times 36 = 48 + 2(288) + 792 + 1232 + 3520 + 4200$
$288 \times 42 = 8 + 120 + 160 + 792 + 1056 + 1512 + 3696' + 4752$
$288 \times 48 = 27 + 36 + 308 + 315 + 594 + 594' + 792' + 825 + 4096 + 6237$
$308 \times 8 = 160 + 792 + 1512$
$308 \times 27 = 27 + 308 + 315 + 594 + 792' + 2184 + 4096$
$308 \times 36 = 36 + 308 + 315 + 594 + 825 + 4096 + 4914$
$308 \times 42 = 42 + 308 + 315 + 1155 + 4096 + 7020$
$308 \times 48 = 48 + 160 + 288 + 792 + 1232 + 1512 + 4200 + 6552$

Table A.60: Sp(10) Tensor Products

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<b><math>10 \times 10 = 1 + 44 + 55</math></b>
<b><math>44 \times 10 = 10 + 110 + 320</math></b>
<b><math>44 \times 44 = 1 + 44 + 55 + 165 + 780 + 891</math></b>
<b><math>55 \times 10 = 10 + 220 + 320</math></b>
<b><math>55 \times 44 = 44 + 55 + 891 + 1430</math></b>
<b><math>55 \times 55 = 1 + 44 + 55 + 715 + 780 + 1430</math></b>
<b><math>110 \times 10 = 44 + 165 + 891</math></b>
<b><math>110 \times 44 = 10 + 110 + 132 + 320 + 1408 + 2860</math></b>
<b><math>110 \times 55 = 110 + 320 + 1408 + 4212</math></b>
<b><math>110 \times 110 = 1 + 44 + 55 + 165 + 780 + 891 + 1155 + 4004 + 5005</math></b>
<b><math>132 \times 10 = 165 + 1155</math></b>
<b><math>132 \times 44 = 110 + 1408 + 4290</math></b>
<b><math>132 \times 55 = 132 + 1408 + 5720</math></b>
<b><math>132 \times 110 = 44 + 891 + 5005 + 8580</math></b>
<b><math>132 \times 132 = 1 + 55 + 780 + 4004 + 4719 + 7865</math></b>
<b><math>165 \times 10 = 110 + 132 + 1408</math></b>
<b><math>165 \times 44 = 44 + 165 + 891 + 1155 + 5005</math></b>
<b><math>165 \times 55 = 165 + 891 + 1155 + 6864</math></b>
<b><math>165 \times 110 = 10 + 110 + 320 + 1408 + 2860 + 4290 + 9152</math></b>
<b><math>165 \times 132 = 10 + 320 + 2860 + 9152 + 9438</math></b>
<b><math>165 \times 165 = 1 + 44 + 55 + 780 + 891 + 4004 + 5005 + 7865 + 8580</math></b>
<b><math>220 \times 10 = 55 + 715 + 1430</math></b>
<b><math>220 \times 44 = 220 + 320 + 4212 + 4928</math></b>
<b><math>220 \times 55 = 10 + 220 + 320 + 2002 + 4620 + 4928</math></b>
<b><math>220 \times 110 = 891 + 1430 + 6864 + 15015</math></b>
<b><math>220 \times 132 = 1155 + 6864 + 21021</math></b>
<b><math>220 \times 165 = 1408 + 4212 + 5720 + 24960</math></b>
<b><math>220 \times 220 = 1 + 44 + 55 + 715 + 780 + 1430 + 5005' + 8250 + 14300 + 17820</math></b>
<b><math>320 \times 10 = 44 + 55 + 780 + 891 + 1430</math></b>
<b><math>320 \times 44 = 10 + 110 + 220 + 2(320) + 1408 + 2860 + 4212 + 4620</math></b>
<b><math>320 \times 55 = 10 + 110 + 220 + 2(320) + 2860 + 4212 + 4620 + 4928</math></b>
<b><math>320 \times 110 = 44 + 55 + 165 + 780 + 2(891) + 1155 + 1430 + 5005 + 6864 + 17920</math></b>
<b><math>320 \times 132 = 165 + 891 + 1155 + 5005 + 6864 + 28160</math></b>
<b><math>320 \times 165 = 110 + 132 + 320 + 2(1408) + 2860 + 4212 + 4290 + 5720 + 32340</math></b>
<b><math>320 \times 220 = 44 + 55 + 715 + 780 + 891 + 2(1430) + 14300 + 15015 + 17820 + 17920</math></b>
<b><math>320 \times 320 = 1 + 2(44) + 2(55) + 165 + 715 + 2(780) + 3(891) + 3(1430) + 4004 + 5005 + 6864 + 8250 + 15015 + 17820 + 2(17920)</math></b>

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Table A.61: Sp(12) Tensor Products

$12 \times 12 = 1 + 65 + 78$
$65 \times 12 = 12 + 208 + 560$
$65 \times 65 = 1 + 65 + 78 + 429 + 1650 + 2002$
$78 \times 12 = 12 + 364 + 560$
$78 \times 65 = 65 + 78 + 2002 + 2925$
$78 \times 78 = 1 + 65 + 78 + 1365 + 1650 + 2925$
$208 \times 12 = 65 + 429 + 2002$
$208 \times 65 = 12 + 208 + 560 + 572 + 4368 + 7800$
$208 \times 78 = 208 + 560 + 4368 + 11088$
$208 \times 208 = 1 + 65 + 78 + 429 + 429' + 1650 + 2002 + 6006 + 13650 + 18954$
$364 \times 12 = 78 + 1365 + 2925$
$364 \times 65 = 364 + 560 + 11088 + 11648$
$364 \times 78 = 12 + 364 + 560 + 4368' + 11440 + 11648$
$429' \times 12 = 572 + 4576$
$429 \times 12 = 208 + 572 + 4368$
$429' \times 65 = 429 + 6006 + 21450$
$429 \times 65 = 65 + 429 + 429' + 2002 + 6006 + 18954$

Table A.62: Sp(14) Tensor Products

$14 \times 14 = 1 + 90 + 105$
$90 \times 14 = 14 + 350 + 896$
$90 \times 90 = 1 + 90 + 105 + 910 + 3094 + 3900$
$105 \times 14 = 14 + 560 + 896$
$105 \times 90 = 90 + 105 + 3900 + 5355$
$105 \times 105 = 1 + 90 + 105 + 2380 + 3094 + 5355$
$350 \times 14 = 90 + 910 + 3900$
$560 \times 14 = 105 + 2380 + 5355$
$896 \times 14 = 90 + 105 + 3094 + 3900 + 5355$

Table A.63:  $E_6$  Tensor Products

$\overline{27} \times 27 = 1 + 78 + 650$
$27 \times 27 = \overline{27} + \overline{351} + \overline{351}'$
$78 \times 27 = 27 + 351 + 1728$
$78 \times 78 = 1 + 78 + 650 + 2430 + 2925$
$351' \times 27 = \overline{27} + \overline{1728} + \overline{7722}$
$351 \times 27 = \overline{27} + \overline{351} + \overline{1728} + \overline{7371}$
$\overline{351} \times 27 = 78 + 650 + 2925 + 5824$
$\overline{351}' \times 27 = 650 + 3003 + 5824$
$351' \times 78 = 351 + 351' + 7371 + 19305$
$351 \times 78 = 27 + 351 + 351' + 1728 + 7371 + 17550$
$351' \times 351' = \overline{351}' + \overline{7371} + \overline{7722} + \overline{19305}' + \overline{34398} + \overline{54054}$
$351' \times 351 = \overline{351} + \overline{1728} + \overline{7371} + \overline{7722} + \overline{51975} + \overline{54054}$
$351 \times 351 = \overline{27} + \overline{351} + \overline{351}' + 2(\overline{1728}) + \overline{7371} + \overline{7722} + \overline{17550} + \overline{34398} + \overline{51975}$
$\overline{351} \times 351 = 1 + 78 + 2(650) + 2430 + 2925 + 5824 + \overline{5824} + 34749 + 70070$
$\overline{351}' \times 351' = 1 + 78 + 650 + 2430 + 34749 + 85293$
$\overline{351}' \times 351 = 78 + 650 + 2925 + 5824 + 34749 + 78975$
$650 \times 27 = 27 + 351 + 351' + 1728 + 7371 + 7722$
$650 \times 78 = 78 + 2(650) + 2925 + 5824 + \overline{5824} + 34749$
$650 \times 351' = 27 + 351 + 351' + 1728 + 7371 + 7722 + 17550 + 19305 + 61425 + 112320$
$650 \times 351 = 27 + 2(351) + 351' + 2(1728) + 2(7371) + 7722 + 17550 + 19305 + 51975 + 112320$
$650 \times 650 = 1 + 2(78) + 3(650) + 2430 + 2(2925) + 3003 + \overline{3003} + 2(5824) + 2(\overline{5824}) + 2(34749) + 70070 + 78975 + \overline{78975} + 85293$
$\overline{1728} \times 27 = 78 + 650 + 2430 + 2925 + \overline{5824} + 34749$
$1728 \times 27 = \overline{351} + \overline{351}' + \overline{1728} + \overline{7371} + \overline{17550} + \overline{19305}$
$1728 \times 78 = 27 + 351 + 2(1728) + 7371 + 7722 + 17550 + 46332 + 51975$
$\overline{1728} \times 351' = 650 + 2925 + \overline{3003} + 5824 + 2(\overline{5824}) + 34749 + 70070 + \overline{78975} + \overline{146432} + \overline{252252}$
$\overline{1728} \times 351 = 78 + 2(650) + 2430 + 2(2925) + \overline{3003} + 5824 + 2(\overline{5824}) + 2(34749) + 70070 + \overline{78975} + 105600 + \overline{252252}$
$1728 \times 351' = \overline{27} + \overline{351} + 2(\overline{1728}) + \overline{7371} + \overline{7722} + \overline{17550} + \overline{46332} + \overline{51975} + \overline{112320} + \overline{359424}$
$1728 \times 351 = \overline{27} + 2(\overline{351}) + \overline{351}' + 2(\overline{1728}) + 2(\overline{7371}) + \overline{7722} + 2(\overline{17550}) + \overline{19305} + \overline{46332} + \overline{51975} + \overline{112320} + \overline{314496}$
$1728 \times 650 = 27 + 2(351) + 351' + 3(1728) + 3(7371) + 2(7722) + 2(17550) + 19305 + 34398 + 46332 + 2(51975) + 54054 + 112320 + 314496 + 359424$
$\overline{1728} \times 1728 = 1 + 2(78) + 3(650) + 2(2430) + 3(2925) + 2(5824) + 2(\overline{5824}) + 4(34749) + 43758 + 2(70070) + 78975 + \overline{78975} + 85293 + 2(105600) + 252252 + \overline{252252} + 812175 + 852930$
$1728 \times 1728 = \overline{27} + 2(\overline{351}) + 2(\overline{351}') + 2(\overline{1728}) + 4(\overline{7371}) + \overline{7722} + 3(\overline{17550}) + 3(\overline{19305}) + \overline{34398} + \overline{46332} + 2(\overline{51975}) + \overline{61425} + 2(\overline{112320}) + 2(\overline{314496}) + \overline{386100} + \overline{393822} + \overline{459459} + \overline{494208}$
$2430 \times 27 = 1728 + 17550 + 46332$
$2430 \times 78 = 78 + 2430 + 2925 + 34749 + 43758 + 105600$
$2430 \times 351' = 351' + 7371 + 17550 + 19305 + 34398 + 314496 + 459459$
$2430 \times 351 = 351 + 1728 + 7371 + 17550 + 19305 + 46332 + 51975 + 314496 + 393822$
$2430 \times 650 = 650 + 2430 + 2925 + 5824 + \overline{5824} + 2(34749) + 70070 + 105600 + 252252 + \overline{252252} + 812175$

Table A.64:  $E_7$  Tensor Products

$56 \times 56 = 1 + 133 + 1463 + 1539$
$133 \times 56 = 56 + 912 + 6480$
$133 \times 133 = 1 + 133 + 1539 + 7371 + 8645$
$912 \times 56 = 133 + 1539 + 8645 + 40755$
$912 \times 133 = 56 + 912 + 6480 + 27664 + 86184$
$912 \times 912 = 1 + 133 + 1463 + 1539 + 7371 + 8645 + 40755 + 152152 + 253935 + 365750$
$1463 \times 56 = 56 + 6480 + 24320 + 51072$
$1463 \times 133 = 1463 + 1539 + 40755 + 150822$
$1463 \times 912 = 912 + 6480 + 27664 + 51072 + 362880 + 885248$
$1463 \times 1463 = 1 + 133 + 1463 + 1539 + 7371 + 150822 + 152152 + 293930 + 617253 + 915705$
$1539 \times 56 = 56 + 912 + 6480 + 27664 + 51072$
$1539 \times 133 = 133 + 1463 + 1539 + 8645 + 40755 + 152152$
$1539 \times 912 = 56 + 912 + 2(6480) + 27664 + 51072 + 86184 + 362880 + 861840$
$1539 \times 1463 = 133 + 1463 + 1539 + 8645 + 40755 + 150822 + 152152 + 915705 + 980343$
$1539 \times 1539 = 1 + 133 + 1463 + 2(1539) + 7371 + 8645 + 2(40755) + 150822 + 152152 + 365750 + 617253 + 980343$

Table A.65:  $E_8$  Tensor Products

$248 \times 248 = 1 + 248 + 3875 + 27000 + 30380$
$3875 \times 248 = 248 + 3875 + 30380 + 147250 + 779247$
$3875 \times 3875 = 1 + 248 + 3875 + 27000 + 30380 + 147250 + 779247 + 2450240 + 4881384 + 6696000$

Table A.66:  $F_4$  Tensor Products

$26 \times 26 = 1 + 26 + 52 + 273 + 324$
$52 \times 26 = 26 + 273 + 1053$
$52 \times 52 = 1 + 52 + 324 + 1053' + 1274$
$273 \times 26 = 26 + 52 + 273 + 324 + 1053 + 1274 + 4096$
$273 \times 52 = 26 + 273 + 324 + 1053 + 4096 + 8424$
$273 \times 273 = 1 + 26 + 52 + 2(273) + 2(324) + 2(1053) + 1053' + 1274 + 2652 + 2(4096) + 8424 + 10829 + 19278 + 19448$
$324 \times 26 = 26 + 273 + 324 + 1053 + 2652 + 4096$
$324 \times 52 = 52 + 273 + 324 + 1274 + 4096 + 10829$
$324 \times 273 = 26 + 52 + 2(273) + 324 + 2(1053) + 1274 + 2652 + 2(4096) + 8424 + 10829 + 19278 + 34749$
$324 \times 324 = 1 + 26 + 52 + 273 + 2(324) + 1053 + 1053' + 1274 + 2652 + 2(4096) + 8424 + 10829 + 16302 + 19448 + 34749$
$1053 \times 26 = 52 + 273 + 324 + 1053 + 1053' + 1274 + 4096 + 8424 + 10829$
$1053' \times 26 = 1053 + 8424 + 17901$
$1053 \times 52 = 26 + 273 + 2(1053) + 2652 + 4096 + 8424 + 17901 + 19278$
$1053' \times 52 = 52 + 1053' + 1274 + 10829 + 12376 + 29172$
$1053 \times 273 = 26 + 52 + 2(273) + 2(324) + 2(1053) + 1053' + 2(1274) + 2652 + 3(4096) + 2(8424) + 2(10829) + 17901 + 19278 + 19448 + 29172 + 34749 + 106496$
$1053' \times 273 = 273 + 1053 + 4096 + 8424 + 10829 + 17901 + 19278 + 106496 + 119119$

Table A.66:  $F_4$  Tensor Products (continued)
$$\begin{aligned}
1053 \times 324 &= 26 + 2(273) + 324 + 3(1053) + 1274 + 2652 + 3(4096) + 2(8424) + 10829 + 17901 + \\
&\quad 2(19278) + 19448 + 34749 + 76076 + 106496 \\
1053' \times 324 &= 324 + 1053' + 1274 + 4096 + 8424 + 10829 + 19448 + 29172 + 106496 + 160056' \\
1053 \times 1053 &= 1 + 26 + 2(52) + 2(273) + 3(324) + 2(1053) + 2(1053') + 3(1274) + 2652 + 4(4096) + \\
&\quad 3(8424) + 4(10829) + 12376 + 16302 + 17901 + 2(19278) + 2(19448) + 2(29172) + \\
&\quad 2(34749) + 2(106496) + 107406 + 119119 + 160056 + 160056' \\
1053' \times 1053 &= 26 + 273 + 2(1053) + 2652 + 4096 + 2(8424) + 2(17901) + 2(19278) + 34749 + \\
&\quad 76076 + 106496 + 107406 + 119119 + 184756 + 379848 \\
1053' \times 1053' &= 1 + 52 + 324 + 2(1053') + 1274 + 10829 + 12376 + 16302 + 19448 + 2(29172) + \\
&\quad 100776 + 160056 + 160056' + 226746 + 340119 \\
1274 \times 26 &= 273 + 1053 + 4096 + 8424 + 19278 \\
1274 \times 52 &= 52 + 324 + 1053' + 1274 + 4096 + 10829 + 19448 + 29172 \\
1274 \times 273 &= 26 + 273 + 324 + 2(1053) + 1274 + 2652 + 2(4096) + 2(8424) + 10829 + 17901 + \\
&\quad 19278 + 19448 + 34749 + 106496 + 107406 \\
1274 \times 324 &= 52 + 273 + 324 + 1053 + 1053' + 2(1274) + 2(4096) + 8424 + 2(10829) + 19278 + \\
&\quad 19448 + 29172 + 34749 + 106496 + 160056 \\
1274 \times 1053 &= 26 + 2(273) + 324 + 3(1053) + 2(2652) + 3(4096) + 3(8424) + 10829 + 2(17901) + \\
&\quad 3(19278) + 19448 + 2(34749) + 76076 + 2(106496) + 107406 + 119119 + 205751 + \\
&\quad 379848 \\
1274 \times 1053' &= 52 + 324 + 1053' + 2(1274) + 4096 + 2(10829) + 12376 + 19448 + 2(29172) + \\
&\quad 34749 + 106496 + 160056 + 160056' + 340119 + 420147 \\
1274 \times 1274 &= 1 + 52 + 273 + 2(324) + 2(1053') + 2(1274) + 2652 + 2(4096) + 8424 + 3(10829) + \\
&\quad 12376 + 16302 + 19278 + 2(19448) + 2(29172) + 34749 + 2(106496) + 160056 + \\
&\quad 160056' + 205751 + 226746 + 420147 \\
2652 \times 26 &= 324 + 2652 + 4096 + 10829 + 16302 + 34749 \\
2652 \times 52 &= 1053 + 2652 + 4096 + 19278 + 34749 + 76076 \\
2652 \times 273 &= 273 + 324 + 1053 + 1274 + 2652 + 2(4096) + 8424 + 2(10829) + 16302 + 19278 + \\
&\quad 19448 + 2(34749) + 76076 + 106496 + 160056 + 212992 \\
2652 \times 324 &= 26 + 273 + 324 + 1053 + 2(2652) + 2(4096) + 8424 + 10829 + 16302 + 17901 + \\
&\quad 19278 + 19448 + 2(34749) + 76076 + 81081 + 106496 + 205751 + 212992 \\
2652 \times 1053 &= 52 + 273 + 324 + 1053 + 1053' + 2(1274) + 2652 + 3(4096) + 2(8424) + 3(10829) + \\
&\quad 16302 + 2(19278) + 2(19448) + 29172 + 3(34749) + 76076 + 2(106496) + 107406 + \\
&\quad 2(160056) + 160056' + 205751 + 212992 + 412776 + 787644 \\
2652 \times 1053' &= 1053 + 2652 + 4096 + 8424 + 17901 + 2(19278) + 19448 + 34749 + 76076 + 106496 + \\
&\quad 107406 + 205751 + 379848 + 787644 + 1002456 \\
2652 \times 1274 &= 273 + 2(1053) + 1274 + 2652 + 2(4096) + 2(8424) + 10829 + 17901 + 3(19278) + \\
&\quad 19448 + 2(34749) + 2(76076) + 2(106496) + 107406 + 160056 + 205751 + 212992 + \\
&\quad 379848 + 787644 + 952952 \\
2652 \times 2652 &= 1 + 26 + 52 + 273 + 2(324) + 1053 + 1053' + 1274 + 2(2652) + 2(4096) + 8424 + \\
&\quad 2(10829) + 12376 + 2(16302) + 17901 + 19278 + 2(19448) + 29172 + 3(34749) + \\
&\quad 76076 + 81081 + 2(106496) + 107406 + 119119 + 160056 + 160056' + 2(205751) + \\
&\quad 2(212992) + 342056 + 412776 + 420147 + 629356 + 787644 + 1042899 + 1341522
\end{aligned}$$

Table A.67:  $G_2$  Tensor Products

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$7 \times 7 = 1 + 7 + 14 + 27$
$14 \times 7 = 7 + 27 + 64$
$14 \times 14 = 1 + 14 + 27 + 77 + 77'$
$27 \times 7 = 7 + 14 + 27 + 64 + 77$
$27 \times 14 = 7 + 14 + 27 + 64 + 77 + 189$
$27 \times 27 = 1 + 7 + 14 + 2(27) + 2(64) + 77 + 77' + 182 + 189$
$64 \times 7 = 14 + 27 + 64 + 77 + 77' + 189$
$64 \times 14 = 7 + 27 + 2(64) + 77 + 182 + 189 + 286$
$64 \times 27 = 7 + 14 + 2(27) + 2(64) + 2(77) + 77' + 182 + 2(189) + 286 + 448$
$64 \times 64 = 1 + 7 + 2(14) + 2(27) + 2(64) + 3(77) + 2(77') + 2(182) + 3(189) + 273 + 286 + 378 + 2(448) + 729$
$77' \times 7 = 64 + 189 + 286$
$77 \times 7 = 27 + 64 + 77 + 182 + 189$
$77' \times 14 = 14 + 77 + 77' + 189 + 273 + 448$
$77 \times 14 = 14 + 27 + 64 + 77 + 77' + 182 + 189 + 448$
$77' \times 27 = 27 + 64 + 77 + 77' + 182 + 189 + 286 + 448 + 729$
$77 \times 27 = 7 + 14 + 27 + 2(64) + 2(77) + 77' + 182 + 2(189) + 286 + 378 + 448$
$77' \times 64 = 7 + 27 + 2(64) + 77 + 2(182) + 2(189) + 2(286) + 378 + 448 + 729 + 896 + 924$
$77 \times 64 = 7 + 14 + 2(27) + 3(64) + 2(77) + 77' + 2(182) + 3(189) + 2(286) + 378 + 2(448) + 729 + 924$
$77' \times 77' = 1 + 14 + 27 + 77 + 2(77') + 182 + 189 + 273 + 378 + 2(448) + 714 + 729 + 748 + 1547$
$77' \times 77 = 14 + 27 + 64 + 2(77) + 77' + 182 + 2(189) + 273 + 286 + 378 + 2(448) + 729 + 924 + 1547$
$77 \times 77 = 1 + 7 + 14 + 2(27) + 2(64) + 2(77) + 2(77') + 2(182) + 3(189) + 273 + 2(286) + 378 + 2(448) + 714 + 729 + 924$
$182 \times 7 = 77 + 182 + 189 + 378 + 448$
$182 \times 14 = 64 + 77 + 182 + 189 + 286 + 378 + 448 + 924$
$182 \times 27 = 27 + 64 + 77 + 77' + 2(182) + 2(189) + 286 + 378 + 2(448) + 714 + 729 + 924$
$182 \times 64 = 14 + 27 + 2(64) + 2(77) + 2(77') + 2(182) + 3(189) + 273 + 2(286) + 2(378) + 3(448) + 714 + 2(729) + 2(924) + 1547 + 1728$
$182 \times 77' = 27 + 2(64) + 77 + 77' + 2(182) + 2(189) + 2(286) + 378 + 2(448) + 714 + 2(729) + 896 + 2(924) + 1547 + 1728 + 2926$
$182 \times 77 = 7 + 14 + 27 + 2(64) + 2(77) + 77' + 2(182) + 3(189) + 273 + 3(286) + 2(378) + 3(448) + 714 + 2(729) + 896 + 2(924) + 1254 + 1547 + 1728$
$182 \times 182 = 1 + 7 + 14 + 2(27) + 2(64) + 2(77) + 2(77') + 3(182) + 3(189) + 2(273) + 3(286) + 2(378) + 4(448) + 2(714) + 4(729) + 748 + 2(896) + 3(924) + 1254 + 2(1547) + 2(1728) + 2079 + 2079' + 2926 + 3003$
$189 \times 7 = 64 + 77 + 77' + 182 + 189 + 286 + 448$
$189 \times 14 = 27 + 64 + 77 + 77' + 182 + 2(189) + 286 + 378 + 448 + 729$
$189 \times 27 = 14 + 27 + 2(64) + 2(77) + 77' + 2(182) + 3(189) + 273 + 2(286) + 378 + 2(448) + 729 + 924$
$189 \times 64 = 7 + 14 + 2(27) + 3(64) + 3(77) + 2(77') + 3(182) + 4(189) + 273 + 3(286) + 2(378) + 4(448) + 714 + 2(729) + 896 + 2(924) + 1547$
$189 \times 77' = 7 + 14 + 27 + 2(64) + 2(77) + 77' + 2(182) + 3(189) + 273 + 2(286) + 2(378) + 3(448) + 714 + 2(729) + 896 + 2(924) + 1547 + 1728 + 2079$
$189 \times 77 = 7 + 14 + 2(27) + 3(64) + 3(77) + 2(77') + 3(182) + 4(189) + 273 + 3(286) + 2(378) + 4(448) + 714 + 3(729) + 896 + 2(924) + 1547 + 1728$
$189 \times 182 = 7 + 14 + 2(27) + 3(64) + 3(77) + 2(77') + 3(182) + 5(189) + 2(273) + 4(286) + 3(378) + 5(448) + 2(714) + 4(729) + 2(896) + 4(924) + 1254 + 3(1547) + 2(1728) + 2079 + 2926 + 3003$
$189 \times 189 = 1 + 7 + 2(14) + 3(27) + 4(64) + 4(77) + 3(77') + 5(182) + 6(189) + 2(273) + 5(286) + 4(378) + 6(448) + 2(714) + 5(729) + 748 + 2(896) + 5(924) + 1254 + 3(1547) + 2(1728) + 2079 + 2926$

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A.3. Branching Rules

A.3.1.  $SU(N)$

Table A.68:  $SU(2)$  Branching Rules

$SU(2) \rightarrow U(1)$
$\mathbf{2} = (1) + (-1)$
$\mathbf{3} = (2) + (0) + (-2)$
$\mathbf{4} = (3) + (1) + (-1) + (-3)$
$\mathbf{5} = (4) + (2) + (0) + (-2) + (-4)$
$\mathbf{6} = (5) + (3) + (1) + (-1) + (-3) + (-5)$
$\mathbf{7} = (6) + (4) + (2) + (0) + (-2) + (-4) + (-6)$
$\mathbf{8} = (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7)$
$\mathbf{9} = (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8)$
$\mathbf{10} = (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9)$
$\mathbf{11} = (10) + (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8) + (-10)$
$\mathbf{12} = (11) + (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9) + (-11)$
$\mathbf{13} = (12) + (10) + (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8) + (-10) + (-12)$
$\mathbf{14} = (13) + (11) + (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9) + (-11) + (-13)$
$\mathbf{15} = (14) + (12) + (10) + (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8) + (-10) + (-12) + (-14)$
$\mathbf{16} = (15) + (13) + (11) + (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9) + (-11) + (-13) + (-15)$
$\mathbf{17} = (16) + (14) + (12) + (10) + (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8) + (-10) + (-12) + (-14) + (-16)$
$\mathbf{18} = (17) + (15) + (13) + (11) + (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9) + (-11) + (-13) + (-15) + (-17)$

Table A.69:  $SU(3)$  Branching Rules

$SU(3) \rightarrow SU(2) \times U(1)$
$\mathbf{3} = (1)(-2) + (2)(1)$
$\mathbf{6} = (1)(-4) + (2)(-1) + (3)(2)$
$\mathbf{8} = (1)(0) + (2)(3) + (2)(-3) + (3)(0)$
$\mathbf{10} = (1)(-6) + (2)(-3) + (3)(0) + (4)(3)$
$\mathbf{15} = (1)(-2) + (2)(1) + (2)(-5) + (3)(4) + (3)(-2) + (4)(1)$
$\mathbf{15}' = (1)(-8) + (2)(-5) + (3)(-2) + (4)(1) + (5)(4)$
$\mathbf{21} = (1)(10) + (2)(7) + (3)(4) + (4)(1) + (5)(-2) + (6)(-5)$
$\mathbf{24} = (1)(4) + (2)(7) + (2)(1) + (3)(4) + (3)(-2) + (4)(1) + (4)(-5) + (5)(-2)$
$\mathbf{27} = (1)(0) + (2)(3) + (2)(-3) + (3)(6) + (3)(0) + (3)(-6) + (4)(3) + (4)(-3) + (5)(0)$
$\mathbf{28} = (1)(-12) + (2)(-9) + (3)(-6) + (4)(-3) + (5)(0) + (6)(3) + (7)(6)$
$\mathbf{35} = (1)(-6) + (2)(-3) + (2)(-9) + (3)(0) + (3)(-6) + (4)(3) + (4)(-3) + (5)(6) + (5)(0) + (6)(3)$
$\mathbf{36} = (1)(-14) + (2)(-11) + (3)(-8) + (4)(-5) + (5)(-2) + (6)(1) + (7)(4) + (8)(7)$
$\mathbf{42} = (1)(-2) + (2)(1) + (2)(-5) + (3)(4) + (3)(-2) + (3)(-8) + (4)(7) + (4)(1) + (4)(-5) + (5)(4) + (5)(-2) + (6)(1)$
$\mathbf{45} = (1)(16) + (2)(13) + (3)(10) + (4)(7) + (5)(4) + (6)(1) + (7)(-2) + (8)(-5) + (9)(-8)$
$\mathbf{48} = (1)(-8) + (2)(-5) + (2)(-11) + (3)(-2) + (3)(-8) + (4)(1) + (4)(-5) + (5)(4) + (5)(-2) + (6)(7) + (6)(1) + (7)(4)$
$\mathbf{55} = (1)(-18) + (2)(-15) + (3)(-12) + (4)(-9) + (5)(-6) + (6)(-3) + (7)(0) + (8)(3) + (9)(6) + (10)(9)$

Table A.70: SU(4) Branching Rules

SU(4) $\rightarrow$ SU(3) $\times$ U(1)	
$4 = (1)(-3) + (3)(1)$	
$6 = (3)(-2) + (\bar{3})(2)$	
$10 = (1)(-6) + (3)(-2) + (6)(2)$	
$15 = (1)(0) + (3)(4) + (\bar{3})(-4) + (8)(0)$	
$20 = (3)(1) + (\bar{3})(5) + (\bar{6})(1) + (8)(-3)$	
$20' = (\bar{6})(4) + (6)(-4) + (8)(0)$	
$20'' = (1)(9) + (\bar{3})(5) + (\bar{6})(1) + (\bar{10})(-3)$	
$35 = (1)(-12) + (3)(-8) + (6)(-4) + (10)(0) + (15')(4)$	
$36 = (1)(-3) + (3)(1) + (\bar{3})(-7) + (6)(5) + (8)(-3) + (15)(1)$	
$45 = (3)(-8) + (\bar{3})(-4) + (6)(-4) + (8)(0) + (10)(0) + (15)(4)$	
$50 = (10)(-6) + (\bar{10})(6) + (15)(-2) + (\bar{15})(2)$	
$56 = (1)(-15) + (3)(-11) + (6)(-7) + (10)(-3) + (15')(1) + (\bar{21})(5)$	
$60 = (\bar{6})(1) + (6)(-7) + (8)(-3) + (10)(-3) + (15)(1) + (\bar{15})(5)$	
$64 = (3)(-2) + (\bar{3})(2) + (\bar{6})(-2) + (6)(2) + (8)(6) + (8)(-6) + (15)(-2) + (\bar{15})(2)$	
$70 = (1)(-6) + (3)(-2) + (\bar{3})(-10) + (6)(2) + (8)(-6) + (10)(6) + (15)(-2) + (\bar{24})(2)$	
$84 = (1)(0) + (3)(4) + (\bar{3})(-4) + (\bar{6})(-8) + (6)(8) + (8)(0) + (15)(4) + (\bar{15})(-4) + (27)(0)$	
$84' = (3)(-11) + (\bar{3})(-7) + (6)(-7) + (8)(-3) + (10)(-3) + (15)(1) + (15')(1) + (\bar{24})(5)$	
$84'' = (1)(-18) + (3)(-14) + (6)(-10) + (10)(-6) + (15')(-2) + (\bar{21})(2) + (28)(6)$	
SU(4) $\rightarrow$ SU(2) $\times$ SU(2) $\times$ U(1)	
$4 = (2, 1)(1) + (1, 2)(-1)$	
$6 = (1, 1)(2) + (1, 1)(-2) + (2, 2)(0)$	
$10 = (2, 2)(0) + (3, 1)(2) + (1, 3)(-2)$	
$15 = (1, 1)(0) + (2, 2)(2) + (2, 2)(-2) + (3, 1)(0) + (1, 3)(0)$	
$20 = (2, 1)(1) + (2, 1)(-3) + (1, 2)(3) + (1, 2)(-1) + (3, 2)(-1) + (2, 3)(1)$	
$20' = (1, 1)(4) + (1, 1)(0) + (1, 1)(-4) + (2, 2)(2) + (2, 2)(-2) + (3, 3)(0)$	
$20'' = (3, 2)(-1) + (2, 3)(1) + (4, 1)(-3) + (1, 4)(3)$	
$35 = (3, 3)(0) + (4, 2)(2) + (2, 4)(-2) + (5, 1)(4) + (1, 5)(-4)$	
$36 = (2, 1)(1) + (1, 2)(-1) + (3, 2)(3) + (3, 2)(-1) + (2, 3)(1) + (2, 3)(-3) + (4, 1)(1) + (1, 4)(-1)$	
$45 = (2, 2)(2) + (2, 2)(-2) + (3, 1)(4) + (3, 1)(0) + (1, 3)(0) + (1, 3)(-4) + (3, 3)(0) + (4, 2)(2) + (2, 4)(-2)$	
$50 = (1, 1)(6) + (1, 1)(2) + (1, 1)(-2) + (1, 1)(-6) + (2, 2)(4) + (2, 2)(0) + (2, 2)(-4) + (3, 3)(2) + (3, 3)(-2) + (4, 4)(0)$	
$56 = (4, 3)(1) + (3, 4)(-1) + (5, 2)(3) + (2, 5)(-3) + (6, 1)(5) + (1, 6)(-5)$	
$60 = (2, 1)(5) + (2, 1)(1) + (2, 1)(-3) + (1, 2)(3) + (1, 2)(-1) + (1, 2)(-5) + (3, 2)(3) + (3, 2)(-1) + (2, 3)(1) + (2, 3)(-3) + (4, 3)(1) + (3, 4)(-1)$	
$64 = (1, 1)(2) + (1, 1)(-2) + (2, 2)(4) + 2(2, 2)(0) + (2, 2)(-4) + (3, 1)(2) + (3, 1)(-2) + (1, 3)(2) + (1, 3)(-2) + (3, 3)(2) + (3, 3)(-2) + (4, 2)(0) + (2, 4)(0)$	
$70 = (2, 2)(0) + (3, 1)(2) + (1, 3)(-2) + (3, 3)(2) + (3, 3)(-2) + (4, 2)(4) + (4, 2)(0) + (2, 4)(0) + (2, 4)(-4) + (5, 1)(2) + (1, 5)(-2)$	
$84 = (1, 1)(0) + (2, 2)(2) + (2, 2)(-2) + (3, 1)(0) + (1, 3)(0) + (3, 3)(4) + (3, 3)(0) + (3, 3)(-4) + (4, 2)(2) + (4, 2)(-2) + (2, 4)(2) + (2, 4)(-2) + (5, 1)(0) + (1, 5)(0)$	
$84' = (3, 2)(3) + (3, 2)(-1) + (2, 3)(1) + (2, 3)(-3) + (4, 1)(5) + (4, 1)(1) + (1, 4)(-1) + (1, 4)(-5) + (4, 3)(1) + (3, 4)(-1) + (5, 2)(3) + (2, 5)(-3)$	
$84'' = (4, 4)(0) + (5, 3)(2) + (3, 5)(-2) + (6, 2)(4) + (2, 6)(-4) + (7, 1)(6) + (1, 7)(-6)$	

Table A.71: SU(5) Branching Rules

SU(5) $\rightarrow$ SU(4) $\times$ U(1)	
<b>5</b>	$= (\mathbf{1})(-4) + (\mathbf{4})(1)$
<b>10</b>	$= (\mathbf{4})(-3) + (\mathbf{6})(2)$
<b>15</b>	$= (\mathbf{1})(-8) + (\mathbf{4})(-3) + (\mathbf{10})(2)$
<b>24</b>	$= (\mathbf{1})(0) + (\mathbf{4})(5) + (\overline{\mathbf{4}})(-5) + (\mathbf{15})(0)$
<b>35</b>	$= (\mathbf{1})(12) + (\overline{\mathbf{4}})(7) + (\overline{\mathbf{10}})(2) + (\mathbf{20}'')(-3)$
<b>40</b>	$= (\overline{\mathbf{4}})(7) + (\mathbf{6})(2) + (\overline{\mathbf{10}})(2) + (\mathbf{20})(-3)$
<b>45</b>	$= (\mathbf{4})(1) + (\mathbf{6})(6) + (\mathbf{15})(-4) + (\mathbf{20})(1)$
<b>50</b>	$= (\overline{\mathbf{10}})(6) + (\mathbf{20})(1) + (\mathbf{20}')(-4)$
<b>70</b>	$= (\mathbf{1})(-4) + (\mathbf{4})(1) + (\overline{\mathbf{4}})(-9) + (\mathbf{10})(6) + (\mathbf{15})(-4) + (\mathbf{36})(1)$
<b>70'</b>	$= (\mathbf{1})(16) + (\overline{\mathbf{4}})(11) + (\overline{\mathbf{10}})(6) + (\mathbf{20}'')(1) + (\overline{\mathbf{35}})(-4)$
<b>75</b>	$= (\mathbf{15})(0) + (\mathbf{20})(5) + (\overline{\mathbf{20}})(-5) + (\mathbf{20}')(0)$
<b>105</b>	$= (\overline{\mathbf{4}})(11) + (\mathbf{6})(6) + (\overline{\mathbf{10}})(6) + (\mathbf{20})(1) + (\mathbf{20}'')(1) + (\overline{\mathbf{45}})(-4)$
<b>126</b>	$= (\overline{\mathbf{4}})(-5) + (\mathbf{6})(-10) + (\mathbf{15})(0) + (\overline{\mathbf{20}})(-5) + (\mathbf{36})(5) + (\mathbf{45})(0)$
<b>126'</b>	$= (\mathbf{1})(-20) + (\mathbf{4})(-15) + (\mathbf{10})(-10) + (\overline{\mathbf{20}}'')( -5) + (\mathbf{35})(0) + (\mathbf{56})(5)$
<b>160</b>	$= (\mathbf{1})(-8) + (\mathbf{4})(-3) + (\overline{\mathbf{4}})(-13) + (\mathbf{10})(2) + (\mathbf{15})(-8) + (\overline{\mathbf{20}}'')(7) + (\mathbf{36})(-3) + (\mathbf{70})(2)$
<b>175</b>	$= (\mathbf{4})(-3) + (\mathbf{6})(2) + (\mathbf{10})(2) + (\mathbf{15})(-8) + (\mathbf{20})(-3) + (\overline{\mathbf{20}})(7) + (\mathbf{36})(-3) + (\mathbf{64})(2)$
<b>175'</b>	$= (\mathbf{10})(-10) + (\overline{\mathbf{20}})(-5) + (\mathbf{20}')(0) + (\overline{\mathbf{20}}'')( -5) + (\mathbf{45})(0) + (\mathbf{60})(5)$
<b>175''</b>	$= (\overline{\mathbf{20}}'')( -9) + (\mathbf{45})(-4) + (\mathbf{50})(6) + (\mathbf{60})(1)$
SU(5) $\rightarrow$ SU(3) $\times$ SU(2) $\times$ U(1)	
<b>5</b>	$= (\mathbf{1}, \mathbf{2})(-3) + (\mathbf{3}, \mathbf{1})(2)$
<b>10</b>	$= (\mathbf{1}, \mathbf{1})(-6) + (\overline{\mathbf{3}}, \mathbf{1})(4) + (\mathbf{3}, \mathbf{2})(-1)$
<b>15</b>	$= (\mathbf{1}, \mathbf{3})(-6) + (\mathbf{3}, \mathbf{2})(-1) + (\mathbf{6}, \mathbf{1})(4)$
<b>24</b>	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{3}, \mathbf{2})(5) + (\overline{\mathbf{3}}, \mathbf{2})(-5) + (\mathbf{8}, \mathbf{1})(0)$
<b>35</b>	$= (\mathbf{1}, \mathbf{4})(9) + (\overline{\mathbf{3}}, \mathbf{3})(4) + (\overline{\mathbf{6}}, \mathbf{2})(-1) + (\overline{\mathbf{10}}, \mathbf{1})(-6)$
<b>40</b>	$= (\mathbf{1}, \mathbf{2})(9) + (\overline{\mathbf{3}}, \mathbf{1})(4) + (\mathbf{3}, \mathbf{2})(-1) + (\overline{\mathbf{3}}, \mathbf{3})(4) + (\overline{\mathbf{6}}, \mathbf{2})(-1) + (\mathbf{8}, \mathbf{1})(-6)$
<b>45</b>	$= (\mathbf{1}, \mathbf{2})(-3) + (\mathbf{3}, \mathbf{1})(2) + (\overline{\mathbf{3}}, \mathbf{1})(-8) + (\overline{\mathbf{3}}, \mathbf{2})(7) + (\mathbf{3}, \mathbf{3})(2) + (\overline{\mathbf{6}}, \mathbf{1})(2) + (\mathbf{8}, \mathbf{2})(-3)$
<b>50</b>	$= (\mathbf{1}, \mathbf{1})(12) + (\mathbf{3}, \mathbf{1})(2) + (\overline{\mathbf{3}}, \mathbf{2})(7) + (\mathbf{6}, \mathbf{1})(-8) + (\overline{\mathbf{6}}, \mathbf{3})(2) + (\mathbf{8}, \mathbf{2})(-3)$
<b>70</b>	$= (\mathbf{1}, \mathbf{2})(-3) + (\mathbf{3}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{4})(-3) + (\mathbf{3}, \mathbf{3})(2) + (\overline{\mathbf{3}}, \mathbf{3})(-8) + (\mathbf{6}, \mathbf{2})(7) + (\mathbf{8}, \mathbf{2})(-3) + (\mathbf{15}, \mathbf{1})(2)$
<b>70'</b>	$= (\mathbf{1}, \mathbf{5})(12) + (\overline{\mathbf{3}}, \mathbf{4})(7) + (\overline{\mathbf{6}}, \mathbf{3})(2) + (\overline{\mathbf{10}}, \mathbf{2})(-3) + (\overline{\mathbf{15}}', \mathbf{1})(-8)$
<b>75</b>	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{3}, \mathbf{1})(-10) + (\overline{\mathbf{3}}, \mathbf{1})(10) + (\mathbf{3}, \mathbf{2})(5) + (\overline{\mathbf{3}}, \mathbf{2})(-5) + (\overline{\mathbf{6}}, \mathbf{2})(5) + (\mathbf{6}, \mathbf{2})(-5) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{8}, \mathbf{3})(0)$
<b>105</b>	$= (\mathbf{1}, \mathbf{3})(12) + (\overline{\mathbf{3}}, \mathbf{2})(7) + (\mathbf{3}, \mathbf{3})(2) + (\overline{\mathbf{6}}, \mathbf{1})(2) + (\overline{\mathbf{3}}, \mathbf{4})(7) + (\overline{\mathbf{6}}, \mathbf{3})(2) + (\mathbf{8}, \mathbf{2})(-3) + (\overline{\mathbf{10}}, \mathbf{2})(-3) + (\overline{\mathbf{15}}, \mathbf{1})(-8)$
<b>126</b>	$= (\mathbf{1}, \mathbf{3})(0) + (\mathbf{3}, \mathbf{2})(5) + (\overline{\mathbf{3}}, \mathbf{2})(-5) + (\mathbf{3}, \mathbf{3})(-10) + (\mathbf{6}, \mathbf{1})(10) + (\overline{\mathbf{3}}, \mathbf{4})(-5) + (\mathbf{6}, \mathbf{2})(-5) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{8}, \mathbf{3})(0) + (\mathbf{10}, \mathbf{1})(0) + (\mathbf{15}, \mathbf{2})(5)$
<b>126'</b>	$= (\mathbf{1}, \mathbf{6})(-15) + (\mathbf{3}, \mathbf{5})(-10) + (\mathbf{6}, \mathbf{4})(-5) + (\mathbf{10}, \mathbf{3})(0) + (\mathbf{15}', \mathbf{2})(5) + (\overline{\mathbf{21}}, \mathbf{1})(10)$
<b>160</b>	$= (\mathbf{1}, \mathbf{3})(-6) + (\mathbf{3}, \mathbf{2})(-1) + (\mathbf{1}, \mathbf{5})(-6) + (\mathbf{6}, \mathbf{1})(4) + (\mathbf{3}, \mathbf{4})(-1) + (\overline{\mathbf{3}}, \mathbf{4})(-11) + (\mathbf{6}, \mathbf{3})(4) + (\mathbf{8}, \mathbf{3})(-6) + (\mathbf{10}, \mathbf{2})(9) + (\mathbf{15}, \mathbf{2})(-1) + (\mathbf{24}, \mathbf{1})(4)$
<b>175</b>	$= (\mathbf{1}, \mathbf{1})(-6) + (\overline{\mathbf{3}}, \mathbf{1})(4) + (\mathbf{1}, \mathbf{3})(-6) + 2(\mathbf{3}, \mathbf{2})(-1) + (\overline{\mathbf{3}}, \mathbf{2})(-11) + (\overline{\mathbf{3}}, \mathbf{3})(4) + (\mathbf{6}, \mathbf{1})(4) + (\mathbf{3}, \mathbf{4})(-1) + (\overline{\mathbf{6}}, \mathbf{2})(-1) + (\mathbf{8}, \mathbf{1})(-6) + (\mathbf{6}, \mathbf{3})(4) + (\mathbf{8}, \mathbf{2})(9) + (\mathbf{8}, \mathbf{3})(-6) + (\overline{\mathbf{15}}, \mathbf{1})(4) + (\mathbf{15}, \mathbf{2})(-1)$
<b>175'</b>	$= (\mathbf{1}, \mathbf{2})(-15) + (\mathbf{3}, \mathbf{1})(-10) + (\overline{\mathbf{3}}, \mathbf{2})(-5) + (\mathbf{3}, \mathbf{3})(-10) + (\overline{\mathbf{6}}, \mathbf{2})(5) + (\mathbf{6}, \mathbf{2})(-5) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{6}, \mathbf{4})(-5) + (\mathbf{8}, \mathbf{3})(0) + (\mathbf{10}, \mathbf{3})(0) + (\overline{\mathbf{15}}, \mathbf{1})(10) + (\mathbf{15}, \mathbf{2})(5)$
<b>175''</b>	$= (\mathbf{1}, \mathbf{1})(-18) + (\overline{\mathbf{3}}, \mathbf{1})(-8) + (\mathbf{3}, \mathbf{2})(-13) + (\overline{\mathbf{6}}, \mathbf{1})(2) + (\mathbf{6}, \mathbf{3})(-8) + (\mathbf{8}, \mathbf{2})(-3) + (\overline{\mathbf{10}}, \mathbf{1})(12) + (\mathbf{10}, \mathbf{4})(-3) + (\overline{\mathbf{15}}, \mathbf{2})(7) + (\mathbf{15}, \mathbf{3})(2)$



Table A.72: SU(6) Branching Rules

SU(6) $\rightarrow$ SU(5) $\times$ U(1)	
<b>6</b>	$= (1)(-5) + (5)(1)$
<b>15</b>	$= (5)(-4) + (10)(2)$
<b>20</b>	$= (10)(-3) + (\overline{10})(3)$
<b>21</b>	$= (1)(-10) + (5)(-4) + (15)(2)$
<b>35</b>	$= (1)(0) + (5)(6) + (\overline{5})(-6) + (24)(0)$
<b>56</b>	$= (1)(-15) + (5)(-9) + (15)(-3) + (\overline{35})(3)$
<b>70</b>	$= (5)(-9) + (10)(-3) + (15)(-3) + (\overline{40})(3)$
<b>84</b>	$= (5)(1) + (10)(7) + (24)(-5) + (45)(1)$
<b>105</b>	$= (10)(2) + (\overline{10})(8) + (40)(2) + (45)(-4)$
<b>105'</b>	$= (\overline{15})(8) + (40)(2) + (50)(-4)$
<b>120</b>	$= (1)(-5) + (5)(1) + (\overline{5})(-11) + (15)(7) + (24)(-5) + (70)(1)$
<b>126</b>	$= (1)(20) + (\overline{5})(14) + (\overline{15})(8) + (35)(2) + (70')(-4)$
<b>175</b>	$= (50)(6) + (\overline{50})(-6) + (75)(0)$
<b>189</b>	$= (24)(0) + (45)(6) + (\overline{45})(-6) + (75)(0)$
<b>210</b>	$= (40)(7) + (45)(1) + (50)(1) + (75)(-5)$
<b>210'</b>	$= (\overline{5})(14) + (\overline{10})(8) + (\overline{15})(8) + (35)(2) + (40)(2) + (105)(-4)$
<b>252</b>	$= (1)(25) + (\overline{5})(19) + (\overline{15})(13) + (35)(7) + (70')(1) + (\overline{126}')(-5)$
<b>280</b>	$= (\overline{5})(-6) + (\overline{10})(-12) + (24)(0) + (\overline{45})(-6) + (70)(6) + (126)(0)$
<b>315</b>	$= (1)(-10) + (5)(-4) + (\overline{5})(-16) + (15)(2) + (24)(-10) + (\overline{35})(8) + (70)(-4) + (160)(2)$
SU(6) $\rightarrow$ SU(4) $\times$ SU(2) $\times$ U(1)	
<b>6</b>	$= (1, 2)(-2) + (4, 1)(1)$
<b>15</b>	$= (1, 1)(-4) + (4, 2)(-1) + (6, 1)(2)$
<b>20</b>	$= (4, 1)(-3) + (\overline{4}, 1)(3) + (6, 2)(0)$
<b>21</b>	$= (1, 3)(-4) + (4, 2)(-1) + (10, 1)(2)$
<b>35</b>	$= (1, 1)(0) + (1, 3)(0) + (4, 2)(3) + (\overline{4}, 2)(-3) + (15, 1)(0)$
<b>56</b>	$= (1, 4)(-6) + (4, 3)(-3) + (10, 2)(0) + (\overline{20}'', 1)(3)$
<b>70</b>	$= (1, 2)(-6) + (4, 1)(-3) + (4, 3)(-3) + (6, 2)(0) + (10, 2)(0) + (\overline{20}, 1)(3)$
<b>84</b>	$= (1, 2)(-2) + (4, 1)(1) + (\overline{4}, 1)(-5) + (4, 3)(1) + (6, 2)(4) + (15, 2)(-2) + (20, 1)(1)$
<b>105</b>	$= (4, 2)(-1) + (\overline{4}, 2)(5) + (6, 1)(2) + (6, 3)(2) + (\overline{10}, 1)(2) + (15, 1)(-4) + (20, 2)(-1)$
<b>105'</b>	$= (1, 1)(8) + (\overline{4}, 2)(5) + (6, 1)(2) + (\overline{10}, 3)(2) + (20', 1)(-4) + (20, 2)(-1)$
<b>120</b>	$= (1, 2)(-2) + (4, 1)(1) + (1, 4)(-2) + (4, 3)(1) + (\overline{4}, 3)(-5) + (10, 2)(4) + (15, 2)(-2) + (36, 1)(1)$
<b>126</b>	$= (1, 5)(8) + (\overline{4}, 4)(5) + (\overline{10}, 3)(2) + (20'', 2)(-1) + (\overline{35}, 1)(-4)$
<b>175</b>	$= (10, 1)(-6) + (\overline{10}, 1)(6) + (15, 1)(0) + (20, 2)(3) + (\overline{20}, 2)(-3) + (20', 3)(0)$
<b>189</b>	$= (1, 1)(0) + (4, 2)(3) + (\overline{4}, 2)(-3) + (6, 1)(6) + (6, 1)(-6) + (15, 1)(0) + (15, 3)(0) + (20', 1)(0) + (20, 2)(3) + (\overline{20}, 2)(-3)$
<b>210</b>	$= (4, 1)(1) + (\overline{4}, 1)(7) + (6, 2)(4) + (\overline{10}, 2)(4) + (15, 2)(-2) + (20, 1)(1) + (\overline{20}, 1)(-5) + (20', 2)(-2) + (20, 3)(1)$
<b>210'</b>	$= (1, 3)(8) + (\overline{4}, 2)(5) + (\overline{4}, 4)(5) + (6, 3)(2) + (\overline{10}, 1)(2) + (\overline{10}, 3)(2) + (20, 2)(-1) + (20'', 2)(-1) + (\overline{45}, 1)(-4)$
<b>252</b>	$= (1, 6)(10) + (\overline{4}, 5)(7) + (\overline{10}, 4)(4) + (20'', 3)(1) + (\overline{35}, 2)(-2) + (\overline{56}, 1)(-5)$
<b>280</b>	$= (1, 3)(0) + (4, 2)(3) + (\overline{4}, 2)(-3) + (\overline{4}, 4)(-3) + (6, 3)(-6) + (10, 1)(6) + (15, 1)(0) + (15, 3)(0) + (\overline{20}, 2)(-3) + (36, 2)(3) + (45, 1)(0)$
<b>315</b>	$= (1, 3)(-4) + (4, 2)(-1) + (1, 5)(-4) + (4, 4)(-1) + (\overline{4}, 4)(-7) + (10, 1)(2) + (10, 3)(2) + (15, 3)(-4) + (\overline{20}'', 2)(5) + (36, 2)(-1) + (70, 1)(2)$

Table A.72: SU(6) Branching Rules (continued)

SU(6) $\rightarrow$ SU(3) $\times$ SU(3) $\times$ U(1)	
<b>6</b>	$= (\mathbf{3}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{3})(-1)$
<b>15</b>	$= (\bar{\mathbf{3}}, \mathbf{1})(2) + (\mathbf{1}, \bar{\mathbf{3}})(-2) + (\mathbf{3}, \mathbf{3})(0)$
<b>20</b>	$= (\mathbf{1}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{1})(-3) + (\mathbf{3}, \bar{\mathbf{3}})(-1) + (\bar{\mathbf{3}}, \mathbf{3})(1)$
<b>21</b>	$= (\mathbf{3}, \mathbf{3})(0) + (\mathbf{6}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{6})(-2)$
<b>35</b>	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{3}, \bar{\mathbf{3}})(2) + (\bar{\mathbf{3}}, \mathbf{3})(-2) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0)$
<b>56</b>	$= (\mathbf{6}, \mathbf{3})(1) + (\mathbf{3}, \mathbf{6})(-1) + (\mathbf{10}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{10})(-3)$
<b>70</b>	$= (\mathbf{3}, \bar{\mathbf{3}})(-1) + (\bar{\mathbf{3}}, \mathbf{3})(1) + (\mathbf{8}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{8})(-3) + (\mathbf{6}, \mathbf{3})(1) + (\mathbf{3}, \mathbf{6})(-1)$
<b>84</b>	$= (\mathbf{3}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{3})(-1) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})(3) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})(-3) + (\bar{\mathbf{6}}, \mathbf{1})(1) + (\mathbf{1}, \bar{\mathbf{6}})(-1) + (\mathbf{8}, \mathbf{3})(-1) + (\mathbf{3}, \mathbf{8})(1)$
<b>105</b>	$= (\bar{\mathbf{3}}, \mathbf{1})(2) + (\bar{\mathbf{3}}, \mathbf{1})(-4) + (\mathbf{1}, \bar{\mathbf{3}})(4) + (\mathbf{1}, \bar{\mathbf{3}})(-2) + (\mathbf{3}, \mathbf{3})(0) + (\mathbf{3}, \bar{\mathbf{6}})(0) + (\bar{\mathbf{6}}, \mathbf{3})(0) + (\mathbf{8}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \mathbf{8})(2)$
<b>105'</b>	$= (\mathbf{3}, \mathbf{3})(0) + (\mathbf{6}, \mathbf{1})(-4) + (\mathbf{1}, \mathbf{6})(4) + (\mathbf{8}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \mathbf{8})(2) + (\bar{\mathbf{6}}, \bar{\mathbf{6}})(0)$
<b>120</b>	$= (\mathbf{3}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{3})(-1) + (\mathbf{6}, \bar{\mathbf{3}})(3) + (\bar{\mathbf{3}}, \mathbf{6})(-3) + (\mathbf{8}, \mathbf{3})(-1) + (\mathbf{3}, \mathbf{8})(1) + (\mathbf{15}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{15})(-1)$
<b>126</b>	$= (\bar{\mathbf{6}}, \bar{\mathbf{6}})(0) + (\bar{\mathbf{10}}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \bar{\mathbf{10}})(2) + (\bar{\mathbf{15}}', \mathbf{1})(-4) + (\mathbf{1}, \bar{\mathbf{15}}')(4)$
<b>175</b>	$= (\mathbf{1}, \mathbf{1})(6) + (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{1})(-6) + (\mathbf{3}, \bar{\mathbf{3}})(2) + (\mathbf{3}, \bar{\mathbf{3}})(-4) + (\bar{\mathbf{3}}, \mathbf{3})(4) + (\bar{\mathbf{3}}, \mathbf{3})(-2) + (\mathbf{6}, \bar{\mathbf{6}})(-2) + (\bar{\mathbf{6}}, \mathbf{6})(2) + (\mathbf{8}, \mathbf{8})(0)$
<b>189</b>	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{3}, \bar{\mathbf{3}})(2) + (\mathbf{3}, \bar{\mathbf{3}})(-4) + (\bar{\mathbf{3}}, \mathbf{3})(4) + (\bar{\mathbf{3}}, \mathbf{3})(-2) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0) + (\mathbf{6}, \mathbf{3})(-2) + (\mathbf{3}, \mathbf{6})(2) + (\bar{\mathbf{6}}, \bar{\mathbf{3}})(2) + (\bar{\mathbf{3}}, \bar{\mathbf{6}})(-2) + (\mathbf{8}, \mathbf{8})(0)$
<b>210</b>	$= (\mathbf{3}, \mathbf{1})(1) + (\mathbf{3}, \mathbf{1})(-5) + (\mathbf{1}, \mathbf{3})(5) + (\mathbf{1}, \mathbf{3})(-1) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})(3) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})(-3) + (\mathbf{6}, \bar{\mathbf{3}})(-3) + (\bar{\mathbf{3}}, \mathbf{6})(3) + (\mathbf{8}, \mathbf{3})(-1) + (\mathbf{3}, \mathbf{8})(1) + (\mathbf{8}, \bar{\mathbf{6}})(-1) + (\bar{\mathbf{6}}, \mathbf{8})(1)$
<b>210'</b>	$= (\mathbf{3}, \bar{\mathbf{6}})(0) + (\bar{\mathbf{6}}, \mathbf{3})(0) + (\mathbf{8}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \mathbf{8})(2) + (\bar{\mathbf{6}}, \bar{\mathbf{6}})(0) + (\bar{\mathbf{10}}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \bar{\mathbf{10}})(2) + (\bar{\mathbf{15}}', \mathbf{1})(-4) + (\mathbf{1}, \bar{\mathbf{15}}')(4)$
<b>252</b>	$= (\bar{\mathbf{10}}, \bar{\mathbf{6}})(-1) + (\bar{\mathbf{6}}, \bar{\mathbf{10}})(1) + (\bar{\mathbf{15}}', \bar{\mathbf{3}})(-3) + (\bar{\mathbf{3}}, \bar{\mathbf{15}}')(3) + (\mathbf{21}, \mathbf{1})(-5) + (\mathbf{1}, \mathbf{21})(5)$
<b>280</b>	$= (\mathbf{3}, \bar{\mathbf{3}})(2) + (\bar{\mathbf{3}}, \mathbf{3})(-2) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0) + (\mathbf{6}, \mathbf{3})(4) + (\mathbf{6}, \mathbf{3})(-2) + (\mathbf{3}, \mathbf{6})(2) + (\mathbf{3}, \mathbf{6})(-4) + (\mathbf{10}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{10})(0) + (\mathbf{8}, \mathbf{8})(0) + (\mathbf{15}, \bar{\mathbf{3}})(2) + (\bar{\mathbf{3}}, \mathbf{15})(-2)$
<b>315</b>	$= (\mathbf{3}, \mathbf{3})(0) + (\mathbf{6}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{6})(-2) + (\mathbf{10}, \bar{\mathbf{3}})(4) + (\bar{\mathbf{3}}, \mathbf{10})(-4) + (\mathbf{6}, \mathbf{8})(2) + (\mathbf{8}, \mathbf{6})(-2) + (\mathbf{15}, \mathbf{3})(0) + (\mathbf{3}, \mathbf{15})(0) + (\mathbf{24}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{24})(-2)$

Table A.73: SU(7) Branching Rules

SU(7) $\rightarrow$ SU(6) $\times$ U(1)	
<b>7</b>	$(\mathbf{1})(-6) + (\mathbf{6})(1)$
<b>21</b>	$(\mathbf{6})(-5) + (\mathbf{15})(2)$
<b>28</b>	$(\mathbf{1})(-12) + (\mathbf{6})(-5) + (\mathbf{21})(2)$
<b>35</b>	$(\mathbf{15})(-4) + (\mathbf{20})(3)$
<b>48</b>	$(\mathbf{1})(0) + (\mathbf{6})(7) + (\overline{\mathbf{6}})(-7) + (\mathbf{35})(0)$
<b>84</b>	$(\mathbf{1})(-18) + (\mathbf{6})(-11) + (\mathbf{21})(-4) + (\mathbf{56})(3)$
<b>112</b>	$(\mathbf{6})(-11) + (\mathbf{15})(-4) + (\mathbf{21})(-4) + (\mathbf{70})(3)$
<b>140</b>	$(\mathbf{6})(1) + (\mathbf{15})(8) + (\mathbf{35})(-6) + (\mathbf{84})(1)$
<b>189</b>	$(\mathbf{1})(-6) + (\mathbf{6})(1) + (\overline{\mathbf{6}})(-13) + (\mathbf{21})(8) + (\mathbf{35})(-6) + (\mathbf{120})(1)$
<b>196</b>	$(\overline{\mathbf{21}})(10) + (\overline{\mathbf{70}})(3) + (\mathbf{105}')(-4)$
<b>210</b>	$(\overline{\mathbf{15}})(10) + (\mathbf{20})(3) + (\overline{\mathbf{70}})(3) + (\mathbf{105})(-4)$
<b>210'</b>	$(\mathbf{1})(24) + (\overline{\mathbf{6}})(17) + (\overline{\mathbf{21}})(10) + (\overline{\mathbf{56}})(3) + (\mathbf{126})(-4)$
<b>224</b>	$(\mathbf{15})(2) + (\mathbf{20})(9) + (\mathbf{84})(-5) + (\mathbf{105})(2)$
<b>378</b>	$(\overline{\mathbf{6}})(17) + (\overline{\mathbf{15}})(10) + (\overline{\mathbf{21}})(10) + (\overline{\mathbf{56}})(3) + (\overline{\mathbf{70}})(3) + (\mathbf{210}')(-4)$
<b>392</b>	$(\mathbf{35})(0) + (\mathbf{84})(7) + (\overline{\mathbf{84}})(-7) + (\mathbf{189})(0)$
<b>462</b>	$(\mathbf{1})(30) + (\overline{\mathbf{6}})(23) + (\overline{\mathbf{21}})(16) + (\overline{\mathbf{56}})(9) + (\mathbf{126})(2) + (\mathbf{252})(-5)$
<b>490</b>	$(\overline{\mathbf{70}})(9) + (\mathbf{105})(2) + (\mathbf{105}')(-2) + (\mathbf{210})(-5)$
<b>490'</b>	$(\mathbf{105}')(8) + (\mathbf{175})(-6) + (\mathbf{210})(1)$
<b>540</b>	$(\overline{\mathbf{6}})(-7) + (\overline{\mathbf{15}})(-14) + (\mathbf{35})(0) + (\overline{\mathbf{84}})(-7) + (\mathbf{120})(7) + (\mathbf{280})(0)$
SU(7) $\rightarrow$ SU(5) $\times$ SU(2) $\times$ U(1)	
<b>7</b>	$(\mathbf{1}, \mathbf{2})(-5) + (\mathbf{5}, \mathbf{1})(2)$
<b>21</b>	$(\mathbf{1}, \mathbf{1})(-10) + (\mathbf{5}, \mathbf{2})(-3) + (\mathbf{10}, \mathbf{1})(4)$
<b>28</b>	$(\mathbf{1}, \mathbf{3})(-10) + (\mathbf{5}, \mathbf{2})(-3) + (\mathbf{15}, \mathbf{1})(4)$
<b>35</b>	$(\mathbf{5}, \mathbf{1})(-8) + (\overline{\mathbf{10}}, \mathbf{1})(6) + (\mathbf{10}, \mathbf{2})(-1)$
<b>48</b>	$(\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{5}, \mathbf{2})(7) + (\overline{\mathbf{5}}, \mathbf{2})(-7) + (\mathbf{24}, \mathbf{1})(0)$
<b>84</b>	$(\mathbf{1}, \mathbf{4})(-15) + (\mathbf{5}, \mathbf{3})(-8) + (\mathbf{15}, \mathbf{2})(-1) + (\overline{\mathbf{35}}, \mathbf{1})(6)$
<b>112</b>	$(\mathbf{1}, \mathbf{2})(-15) + (\mathbf{5}, \mathbf{1})(-8) + (\mathbf{5}, \mathbf{3})(-8) + (\mathbf{10}, \mathbf{2})(-1) + (\mathbf{15}, \mathbf{2})(-1) + (\overline{\mathbf{40}}, \mathbf{1})(6)$
<b>140</b>	$(\mathbf{1}, \mathbf{2})(-5) + (\mathbf{5}, \mathbf{1})(2) + (\overline{\mathbf{5}}, \mathbf{1})(-12) + (\mathbf{5}, \mathbf{3})(2) + (\mathbf{10}, \mathbf{2})(9) + (\mathbf{24}, \mathbf{2})(-5) + (\mathbf{45}, \mathbf{1})(2)$
<b>189</b>	$(\mathbf{1}, \mathbf{2})(-5) + (\mathbf{1}, \mathbf{4})(-5) + (\mathbf{5}, \mathbf{1})(2) + (\mathbf{5}, \mathbf{3})(2) + (\overline{\mathbf{5}}, \mathbf{3})(-12) + (\mathbf{15}, \mathbf{2})(9) + (\mathbf{24}, \mathbf{2})(-5) + (\mathbf{70}, \mathbf{1})(2)$
<b>196</b>	$(\mathbf{1}, \mathbf{1})(20) + (\overline{\mathbf{5}}, \mathbf{2})(13) + (\overline{\mathbf{10}}, \mathbf{1})(6) + (\overline{\mathbf{15}}, \mathbf{3})(6) + (\mathbf{40}, \mathbf{2})(-1) + (\mathbf{50}, \mathbf{1})(-8)$
<b>210</b>	$(\overline{\mathbf{5}}, \mathbf{2})(13) + (\overline{\mathbf{10}}, \mathbf{1})(6) + (\mathbf{10}, \mathbf{2})(-1) + (\overline{\mathbf{10}}, \mathbf{3})(6) + (\overline{\mathbf{15}}, \mathbf{1})(6) + (\mathbf{40}, \mathbf{2})(-1) + (\mathbf{45}, \mathbf{1})(-8)$
<b>210'</b>	$(\mathbf{1}, \mathbf{5})(20) + (\overline{\mathbf{5}}, \mathbf{4})(13) + (\overline{\mathbf{15}}, \mathbf{3})(6) + (\mathbf{35}, \mathbf{2})(-1) + (\mathbf{70}', \mathbf{1})(-8)$
<b>224</b>	$(\mathbf{5}, \mathbf{2})(-3) + (\mathbf{10}, \mathbf{1})(4) + (\overline{\mathbf{10}}, \mathbf{2})(11) + (\mathbf{10}, \mathbf{3})(4) + (\mathbf{24}, \mathbf{1})(-10) + (\mathbf{40}, \mathbf{1})(4) + (\mathbf{45}, \mathbf{2})(-3)$
<b>378</b>	$(\mathbf{1}, \mathbf{3})(20) + (\overline{\mathbf{5}}, \mathbf{2})(13) + (\overline{\mathbf{5}}, \mathbf{4})(13) + (\overline{\mathbf{10}}, \mathbf{3})(6) + (\overline{\mathbf{15}}, \mathbf{1})(6) + (\overline{\mathbf{15}}, \mathbf{3})(6) + (\mathbf{35}, \mathbf{2})(-1) + (\mathbf{40}, \mathbf{2})(-1) + (\mathbf{105}, \mathbf{1})(-8)$
<b>392</b>	$(\mathbf{1}, \mathbf{1})(0) + (\mathbf{5}, \mathbf{2})(7) + (\overline{\mathbf{5}}, \mathbf{2})(-7) + (\mathbf{10}, \mathbf{1})(14) + (\overline{\mathbf{10}}, \mathbf{1})(-14) + (\mathbf{24}, \mathbf{1})(0) + (\mathbf{24}, \mathbf{3})(0) + (\mathbf{45}, \mathbf{2})(7) + (\overline{\mathbf{45}}, \mathbf{2})(-7) + (\mathbf{75}, \mathbf{1})(0)$
<b>462</b>	$(\mathbf{1}, \mathbf{6})(25) + (\overline{\mathbf{5}}, \mathbf{5})(18) + (\overline{\mathbf{15}}, \mathbf{4})(11) + (\mathbf{35}, \mathbf{3})(4) + (\mathbf{70}', \mathbf{2})(-3) + (\overline{\mathbf{126}'}, \mathbf{1})(-10)$
<b>490</b>	$(\overline{\mathbf{5}}, \mathbf{1})(18) + (\mathbf{10}, \mathbf{1})(4) + (\overline{\mathbf{10}}, \mathbf{2})(11) + (\overline{\mathbf{15}}, \mathbf{2})(11) + (\mathbf{40}, \mathbf{1})(4) + (\mathbf{40}, \mathbf{3})(4) + (\mathbf{45}, \mathbf{2})(-3) + (\mathbf{50}, \mathbf{2})(-3) + (\mathbf{75}, \mathbf{1})(-10)$
<b>490'</b>	$(\overline{\mathbf{15}}, \mathbf{1})(16) + (\mathbf{40}, \mathbf{2})(9) + (\mathbf{45}, \mathbf{1})(2) + (\overline{\mathbf{50}}, \mathbf{1})(-12) + (\mathbf{50}, \mathbf{3})(2) + (\mathbf{75}, \mathbf{2})(-5)$
<b>540</b>	$(\mathbf{1}, \mathbf{3})(0) + (\mathbf{5}, \mathbf{2})(7) + (\overline{\mathbf{5}}, \mathbf{2})(-7) + (\overline{\mathbf{5}}, \mathbf{4})(-7) + (\overline{\mathbf{10}}, \mathbf{3})(-14) + (\mathbf{15}, \mathbf{1})(14) + (\mathbf{24}, \mathbf{1})(0) + (\mathbf{24}, \mathbf{3})(0) + (\mathbf{45}, \mathbf{2})(-7) + (\mathbf{70}, \mathbf{2})(7) + (\mathbf{126}, \mathbf{1})(0)$

Table A.73: SU(7) Branching Rules (continued)

SU(7) $\rightarrow$ SU(4) $\times$ SU(3) $\times$ U(1)
<b>7</b> = (1, 3)(-4) + (4, 1)(3)
<b>21</b> = (1, $\bar{3}$ )(-8) + (4, 3)(-1) + (6, 1)(6)
<b>28</b> = (4, 3)(-1) + (1, 6)(-8) + (10, 1)(6)
<b>35</b> = (1, 1)(-12) + ( $\bar{4}$ , 1)(9) + (4, $\bar{3}$ )(-5) + (6, 3)(2)
<b>48</b> = (1, 1)(0) + (4, $\bar{3}$ )(7) + ( $\bar{4}$ , 3)(-7) + (1, 8)(0) + (15, 1)(0)
<b>84</b> = (4, 6)(-5) + (1, 10)(-12) + (10, 3)(2) + ( $\bar{20}$ '', 1)(9)
<b>112</b> = (4, $\bar{3}$ )(-5) + (6, 3)(2) + (1, 8)(-12) + (4, 6)(-5) + (10, 3)(2) + ( $\bar{20}$ , 1)(9)
<b>140</b> = (1, 3)(-4) + (4, 1)(3) + ( $\bar{4}$ , $\bar{3}$ )(-11) + (1, $\bar{6}$ )(-4) + (6, $\bar{3}$ )(10) + (4, 8)(3) + (15, 3)(-4) + (20, 1)(3)
<b>189</b> = (1, 3)(-4) + (4, 1)(3) + ( $\bar{4}$ , 6)(-11) + (4, 8)(3) + (10, $\bar{3}$ )(10) + (1, 15)(-4) + (15, 3)(-4) + (36, 1)(3)
<b>196</b> = (1, 6)(16) + (6, 3)(2) + ( $\bar{4}$ , 8)(9) + ( $\bar{10}$ , $\bar{6}$ )(2) + (20', 1)(-12) + (20, $\bar{3}$ )(-5)
<b>210</b> = (1, $\bar{3}$ )(16) + ( $\bar{4}$ , 1)(9) + (4, $\bar{3}$ )(-5) + (6, 3)(2) + (6, $\bar{6}$ )(2) + ( $\bar{4}$ , 8)(9) + ( $\bar{10}$ , 3)(2) + (15, 1)(-12) + (20, $\bar{3}$ )(-5)
<b>210'</b> = ( $\bar{4}$ , $\bar{10}$ )(9) + ( $\bar{10}$ , $\bar{6}$ )(2) + (1, $\bar{15}$ '')(16) + (20'', $\bar{3}$ )(-5) + ( $\bar{35}$ , 1)(-12)
<b>224</b> = (1, $\bar{3}$ )(-8) + ( $\bar{4}$ , 1)(-15) + (4, 3)(-1) + (6, 1)(6) + ( $\bar{4}$ , $\bar{3}$ )(13) + (4, $\bar{6}$ )(-1) + ( $\bar{10}$ , 1)(6) + (6, 8)(6) + (15, $\bar{3}$ )(-8) + (20, 3)(-1)
<b>378</b> = (6, $\bar{6}$ )(2) + ( $\bar{4}$ , 8)(9) + ( $\bar{10}$ , 3)(2) + ( $\bar{4}$ , $\bar{10}$ )(9) + ( $\bar{10}$ , $\bar{6}$ )(2) + (1, $\bar{15}$ )(16) + (20, $\bar{3}$ )(-5) + (20'', $\bar{3}$ )(-5) + ( $\bar{45}$ , 1)(-12)
<b>392</b> = (1, 1)(0) + (4, $\bar{3}$ )(7) + ( $\bar{4}$ , 3)(-7) + (6, 3)(14) + (6, $\bar{3}$ )(-14) + (1, 8)(0) + (4, 6)(7) + ( $\bar{4}$ , $\bar{6}$ )(-7) + (15, 1)(0) + (20', 1)(0) + (15, 8)(0) + (20, $\bar{3}$ )(7) + ( $\bar{20}$ , 3)(-7)
<b>462</b> = ( $\bar{4}$ , $\bar{15}$ '')(13) + ( $\bar{10}$ , $\bar{10}$ )(6) + (1, 21)(20) + (20'', $\bar{6}$ )(-1) + ( $\bar{35}$ , $\bar{3}$ )(-8) + ( $\bar{56}$ , 1)(-15)
<b>490</b> = (1, 3)(20) + (4, 3)(-1) + (6, 1)(6) + ( $\bar{4}$ , $\bar{3}$ )(13) + ( $\bar{4}$ , 6)(13) + (6, 8)(6) + (15, $\bar{3}$ )(-8) + ( $\bar{10}$ , 8)(6) + ( $\bar{20}$ , 1)(-15) + (20, 3)(-1) + (20', $\bar{3}$ )(-8) + (20, $\bar{6}$ )(-1)
<b>490'</b> = (1, 1)(24) + (4, 1)(3) + ( $\bar{4}$ , 3)(17) + (6, $\bar{3}$ )(10) + (10, 1)(-18) + ( $\bar{10}$ , 6)(10) + (15, 3)(-4) + ( $\bar{20}$ , $\bar{3}$ )(-11) + (20', $\bar{6}$ )(-4) + (20, 8)(3)
<b>540</b> = (4, $\bar{3}$ )(7) + ( $\bar{4}$ , 3)(-7) + (1, 8)(0) + (4, 6)(7) + (1, 10)(0) + (6, 6)(-14) + (10, 3)(14) + (15, 1)(0) + ( $\bar{4}$ , 15)(-7) + (15, 8)(0) + ( $\bar{20}$ , 3)(-7) + (36, $\bar{3}$ )(7) + (45, 1)(0)

Table A.74: SU(8) Branching Rules

SU(8) $\rightarrow$ SU(7) $\times$ U(1)	
<b>8</b>	$= (1)(-7) + (7)(1)$
<b>28</b>	$= (7)(-6) + (21)(2)$
<b>36</b>	$= (1)(-14) + (7)(-6) + (28)(2)$
<b>56</b>	$= (21)(-5) + (35)(3)$
<b>63</b>	$= (1)(0) + (7)(8) + (\overline{7})(-8) + (48)(0)$
<b>70</b>	$= (35)(-4) + (\overline{35})(4)$
<b>120</b>	$= (1)(-21) + (7)(-13) + (28)(-5) + (84)(3)$
<b>168</b>	$= (7)(-13) + (21)(-5) + (28)(-5) + (112)(3)$
<b>216</b>	$= (7)(1) + (21)(9) + (48)(-7) + (140)(1)$
<b>280</b>	$= (1)(-7) + (7)(1) + (\overline{7})(-15) + (28)(9) + (48)(-7) + (189)(1)$
<b>330</b>	$= (1)(-28) + (7)(-20) + (28)(-12) + (84)(-4) + (\overline{210})(4)$
<b>336</b>	$= (28)(-12) + (112)(-4) + (\overline{196})(4)$
<b>378</b>	$= (21)(-12) + (35)(-4) + (112)(-4) + (\overline{210})(4)$
<b>420</b>	$= (21)(2) + (35)(10) + (140)(-6) + (224)(2)$
<b>504</b>	$= (35)(3) + (\overline{35})(11) + (210)(3) + (224)(-5)$
<b>630</b>	$= (7)(-20) + (21)(-12) + (28)(-12) + (84)(-4) + (112)(-4) + (\overline{378})(4)$
<b>720</b>	$= (48)(0) + (140)(8) + (\overline{140})(-8) + (392)(0)$
SU(8) $\rightarrow$ SU(6) $\times$ SU(2) $\times$ U(1)	
<b>8</b>	$= (1, 2)(-3) + (6, 1)(1)$
<b>28</b>	$= (1, 1)(-6) + (6, 2)(-2) + (15, 1)(2)$
<b>36</b>	$= (1, 3)(-6) + (6, 2)(-2) + (21, 1)(2)$
<b>56</b>	$= (6, 1)(-5) + (15, 2)(-1) + (20, 1)(3)$
<b>63</b>	$= (1, 1)(0) + (1, 3)(0) + (6, 2)(4) + (\overline{6}, 2)(-4) + (35, 1)(0)$
<b>70</b>	$= (15, 1)(-4) + (\overline{15}, 1)(4) + (20, 2)(0)$
<b>120</b>	$= (1, 4)(-9) + (6, 3)(-5) + (21, 2)(-1) + (56, 1)(3)$
<b>168</b>	$= (1, 2)(-9) + (6, 1)(-5) + (6, 3)(-5) + (15, 2)(-1) + (21, 2)(-1) + (70, 1)(3)$
<b>216</b>	$= (1, 2)(-3) + (6, 1)(1) + (\overline{6}, 1)(-7) + (6, 3)(1) + (15, 2)(5) + (35, 2)(-3) + (84, 1)(1)$
<b>280</b>	$= (1, 2)(-3) + (1, 4)(-3) + (6, 1)(1) + (6, 3)(1) + (\overline{6}, 3)(-7) + (21, 2)(5) + (35, 2)(-3) + (120, 1)(1)$
<b>330</b>	$= (1, 5)(-12) + (6, 4)(-8) + (21, 3)(-4) + (56, 2)(0) + (\overline{126}, 1)(4)$
<b>336</b>	$= (1, 1)(-12) + (6, 2)(-8) + (15, 1)(-4) + (21, 3)(-4) + (70, 2)(0) + (\overline{105}', 1)(4)$
<b>378</b>	$= (6, 2)(-8) + (15, 1)(-4) + (15, 3)(-4) + (20, 2)(0) + (21, 1)(-4) + (70, 2)(0) + (\overline{105}, 1)(4)$
<b>420</b>	$= (6, 2)(-2) + (15, 1)(2) + (15, 3)(2) + (20, 2)(6) + (35, 1)(-6) + (84, 2)(-2) + (105, 1)(2)$
<b>504</b>	$= (15, 2)(-1) + (\overline{15}, 2)(7) + (20, 1)(3) + (20, 3)(3) + (\overline{70}, 1)(3) + (84, 1)(-5) + (105, 2)(-1)$
<b>630</b>	$= (1, 3)(-12) + (6, 2)(-8) + (6, 4)(-8) + (15, 3)(-4) + (21, 1)(-4) + (21, 3)(-4) + (56, 2)(0) + (70, 2)(0) + (\overline{210}', 1)(4)$
<b>720</b>	$= (1, 1)(0) + (6, 2)(4) + (\overline{6}, 2)(-4) + (15, 1)(8) + (\overline{15}, 1)(-8) + (35, 1)(0) + (35, 3)(0) + (84, 2)(4) + (84, 2)(-4) + (189, 1)(0)$

Table A.74: SU(8) Branching Rules (continued)

SU(8) $\rightarrow$ SU(5) $\times$ SU(3) $\times$ U(1)	
8	$= (1, 3)(-5) + (5, 1)(3)$
28	$= (1, \bar{3})(-10) + (5, 3)(-2) + (10, 1)(6)$
36	$= (1, 6)(-10) + (5, 3)(-2) + (15, 1)(6)$
56	$= (1, 1)(-15) + (5, \bar{3})(-7) + (\bar{10}, 1)(9) + (10, 3)(1)$
63	$= (1, 1)(0) + (5, \bar{3})(8) + (\bar{5}, 3)(-8) + (1, 8)(0) + (24, 1)(0)$
70	$= (5, 1)(-12) + (\bar{5}, 1)(12) + (10, \bar{3})(-4) + (\bar{10}, 3)(4)$
120	$= (5, 6)(-7) + (1, 10)(-15) + (15, 3)(1) + (\bar{35}, 1)(9)$
168	$= (5, \bar{3})(-7) + (1, 8)(-15) + (5, 6)(-7) + (10, 3)(1) + (15, 3)(1) + (\bar{40}, 1)(9)$
216	$= (1, 3)(-5) + (5, 1)(3) + (1, \bar{6})(-5) + (\bar{5}, \bar{3})(-13) + (10, \bar{3})(11) + (5, 8)(3) + (24, 3)(-5) + (45, 1)(3)$
280	$= (1, 3)(-5) + (5, 1)(3) + (\bar{5}, 6)(-13) + (5, 8)(3) + (1, 15)(-5) + (15, \bar{3})(11) + (24, 3)(-5) + (70, 1)(3)$
330	$= (5, 10)(-12) + (1, 15')(-20) + (15, 6)(-4) + (\bar{35}, 3)(4) + (\bar{70}', 1)(12)$
336	$= (1, \bar{6})(-20) + (10, \bar{3})(-4) + (5, 8)(-12) + (15, 6)(-4) + (\bar{40}, 3)(4) + (\bar{50}, 1)(12)$
378	$= (1, 3)(-20) + (5, 1)(-12) + (10, \bar{3})(-4) + (\bar{10}, 3)(4) + (5, 8)(-12) + (10, 6)(-4) + (15, \bar{3})(-4) + (\bar{40}, 3)(4) + (\bar{45}, 1)(12)$
420	$= (1, \bar{3})(-10) + (\bar{5}, 1)(-18) + (5, 3)(-2) + (10, 1)(6) + (5, \bar{6})(-2) + (\bar{10}, \bar{3})(14) + (10, 8)(6) + (24, \bar{3})(-10) + (\bar{40}, 1)(6) + (45, 3)(-2)$
504	$= (5, \bar{3})(-7) + (\bar{5}, \bar{3})(17) + (\bar{10}, 1)(9) + (10, 3)(1) + (\bar{15}, 1)(9) + (10, \bar{6})(1) + (\bar{10}, 8)(9) + (24, 1)(-15) + (\bar{40}, 3)(1) + (\bar{45}, \bar{3})(-7)$
630	$= (5, 8)(-12) + (5, 10)(-12) + (10, 6)(-4) + (1, 15)(-20) + (15, \bar{3})(-4) + (15, 6)(-4) + (\bar{35}, 3)(4) + (\bar{40}, 3)(4) + (\bar{105}, 1)(12)$
720	$= (1, 1)(0) + (5, \bar{3})(8) + (\bar{5}, 3)(-8) + (1, 8)(0) + (5, 6)(8) + (\bar{5}, \bar{6})(-8) + (10, 3)(16) + (\bar{10}, \bar{3})(-16) + (24, 1)(0) + (24, 8)(0) + (\bar{45}, \bar{3})(8) + (\bar{45}, 3)(-8) + (\bar{75}, 1)(0)$
SU(8) $\rightarrow$ SU(4) $\times$ SU(4) $\times$ U(1)	
8	$= (4, 1)(1) + (1, 4)(-1)$
28	$= (6, 1)(2) + (1, 6)(-2) + (4, 4)(0)$
36	$= (4, 4)(0) + (10, 1)(2) + (1, 10)(-2)$
56	$= (\bar{4}, 1)(3) + (1, \bar{4})(-3) + (4, 6)(-1) + (6, 4)(1)$
63	$= (1, 1)(0) + (4, \bar{4})(2) + (\bar{4}, 4)(-2) + (15, 1)(0) + (1, 15)(0)$
70	$= (1, 1)(4) + (1, 1)(-4) + (4, \bar{4})(-2) + (\bar{4}, 4)(2) + (6, 6)(0)$
120	$= (10, 4)(1) + (4, 10)(-1) + (\bar{20}'', 1)(3) + (1, \bar{20}'')(3)$
168	$= (4, 6)(-1) + (6, 4)(1) + (10, 4)(1) + (4, 10)(-1) + (\bar{20}, 1)(3) + (1, \bar{20})(-3)$
216	$= (4, 1)(1) + (1, 4)(-1) + (6, \bar{4})(3) + (\bar{4}, 6)(-3) + (15, 4)(-1) + (4, 15)(1) + (20, 1)(1) + (1, 20)(-1)$
280	$= (4, 1)(1) + (1, 4)(-1) + (10, \bar{4})(3) + (\bar{4}, 10)(-3) + (15, 4)(-1) + (4, 15)(1) + (36, 1)(1) + (1, 36)(-1)$
330	$= (10, 10)(0) + (\bar{20}'', 4)(2) + (4, \bar{20}'')(2) + (35, 1)(4) + (1, 35)(-4)$
336	$= (6, 6)(0) + (10, 10)(0) + (20', 1)(4) + (1, 20')(-4) + (\bar{20}, 4)(2) + (4, \bar{20})(-2)$
378	$= (4, \bar{4})(-2) + (\bar{4}, 4)(2) + (6, 6)(0) + (15, 1)(4) + (1, 15)(-4) + (10, 6)(0) + (6, 10)(0) + (\bar{20}, 4)(2) + (4, \bar{20})(-2)$
420	$= (6, 1)(2) + (1, 6)(-2) + (4, 4)(0) + (\bar{4}, \bar{4})(4) + (\bar{4}, \bar{4})(-4) + (\bar{10}, 1)(2) + (1, \bar{10})(-2) + (15, 6)(-2) + (6, 15)(2) + (4, 20)(0) + (20, 4)(0)$
504	$= (\bar{4}, 1)(3) + (\bar{4}, 1)(-5) + (1, \bar{4})(5) + (1, \bar{4})(-3) + (4, 6)(-1) + (6, 4)(1) + (4, \bar{10})(-1) + (\bar{10}, 4)(1) + (15, \bar{4})(-3) + (\bar{4}, 15)(3) + (20, 6)(-1) + (6, 20)(1)$
630	$= (10, 6)(0) + (6, 10)(0) + (10, 10)(0) + (\bar{20}, 4)(2) + (4, \bar{20})(-2) + (\bar{20}'', 4)(2) + (4, \bar{20}'')(2) + (45, 1)(4) + (1, 45)(-4)$
720	$= (1, 1)(0) + (4, \bar{4})(2) + (\bar{4}, 4)(-2) + (6, 6)(4) + (6, 6)(-4) + (15, 1)(0) + (1, 15)(0) + (20', 1)(0) + (1, 20')(0) + (\bar{20}, 4)(-2) + (4, \bar{20})(2) + (20, \bar{4})(2) + (\bar{4}, 20)(-2) + (15, 15)(0)$

Table A.75: SU(9) Branching Rules

SU(9) $\rightarrow$ SU(8) $\times$ U(1)	
<b>9</b>	$= (1)(-8) + (8)(1)$
<b>36</b>	$= (8)(-7) + (28)(2)$
<b>45</b>	$= (1)(-16) + (8)(-7) + (36)(2)$
<b>80</b>	$= (1)(0) + (8)(9) + (\bar{8})(-9) + (63)(0)$
<b>84</b>	$= (28)(-6) + (56)(3)$
<b>126</b>	$= (56)(-5) + (70)(4)$
<b>165</b>	$= (1)(-24) + (8)(-15) + (36)(-6) + (120)(3)$
<b>240</b>	$= (8)(-15) + (28)(-6) + (36)(-6) + (168)(3)$
<b>315</b>	$= (8)(1) + (28)(10) + (63)(-8) + (216)(1)$
<b>396</b>	$= (1)(-8) + (8)(1) + (\bar{8})(-17) + (36)(10) + (63)(-8) + (280)(1)$
<b>495</b>	$= (1)(-32) + (8)(-23) + (36)(-14) + (120)(-5) + (330)(4)$
<b>540</b>	$= (36)(-14) + (168)(-5) + (336)(4)$
<b>630</b>	$= (28)(-14) + (56)(-5) + (168)(-5) + (378)(4)$
<b>720</b>	$= (28)(2) + (56)(11) + (216)(-7) + (420)(2)$
<b>990</b>	$= (8)(-23) + (28)(-14) + (36)(-14) + (120)(-5) + (168)(-5) + (630)(4)$
<b>1008</b>	$= (\bar{56})(13) + (70)(4) + (378)(4) + (504)(-5)$
SU(9) $\rightarrow$ SU(7) $\times$ SU(2) $\times$ U(1)	
<b>9</b>	$= (1, 2)(-7) + (7, 1)(2)$
<b>36</b>	$= (1, 1)(-14) + (7, 2)(-5) + (21, 1)(4)$
<b>45</b>	$= (1, 3)(-14) + (7, 2)(-5) + (28, 1)(4)$
<b>80</b>	$= (1, 1)(0) + (1, 3)(0) + (7, 2)(9) + (\bar{7}, 2)(-9) + (48, 1)(0)$
<b>84</b>	$= (7, 1)(-12) + (21, 2)(-3) + (35, 1)(6)$
<b>126</b>	$= (21, 1)(-10) + (\bar{35}, 1)(8) + (35, 2)(-1)$
<b>165</b>	$= (1, 4)(-21) + (7, 3)(-12) + (28, 2)(-3) + (84, 1)(6)$
<b>240</b>	$= (1, 2)(-21) + (7, 1)(-12) + (7, 3)(-12) + (21, 2)(-3) + (28, 2)(-3) + (112, 1)(6)$
<b>315</b>	$= (1, 2)(-7) + (7, 1)(2) + (\bar{7}, 1)(-16) + (7, 3)(2) + (21, 2)(11) + (48, 2)(-7) + (140, 1)(2)$
<b>396</b>	$= (1, 2)(-7) + (1, 4)(-7) + (7, 1)(2) + (7, 3)(2) + (\bar{7}, 3)(-16) + (28, 2)(11) + (48, 2)(-7) + (189, 1)(2)$
<b>495</b>	$= (1, 5)(-28) + (7, 4)(-19) + (28, 3)(-10) + (84, 2)(-1) + (\bar{210}, 1)(8)$
<b>540</b>	$= (1, 1)(-28) + (7, 2)(-19) + (21, 1)(-10) + (28, 3)(-10) + (112, 2)(-1) + (\bar{196}, 1)(8)$
<b>630</b>	$= (7, 2)(-19) + (21, 1)(-10) + (21, 3)(-10) + (28, 1)(-10) + (35, 2)(-1) + (112, 2)(-1) + (\bar{210}, 1)(8)$
<b>720</b>	$= (7, 2)(-5) + (21, 1)(4) + (21, 3)(4) + (35, 2)(13) + (48, 1)(-14) + (140, 2)(-5) + (224, 1)(4)$
<b>990</b>	$= (1, 3)(-28) + (7, 2)(-19) + (7, 4)(-19) + (21, 3)(-10) + (28, 1)(-10) + (28, 3)(-10) + (84, 2)(-1) + (112, 2)(-1) + (\bar{378}, 1)(8)$
<b>1008</b>	$= (\bar{21}, 2)(17) + (\bar{35}, 1)(8) + (35, 2)(-1) + (\bar{35}, 3)(8) + (\bar{112}, 1)(8) + (210, 2)(-1) + (224, 1)(-10)$

Table A.75: SU(9) Branching Rules (continued)

SU(9) $\rightarrow$ SU(6) $\times$ SU(3) $\times$ U(1)	
<b>9</b>	$(\mathbf{1}, \mathbf{3})(-2) + (\mathbf{6}, \mathbf{1})(1)$
<b>36</b>	$(\mathbf{1}, \bar{\mathbf{3}})(-4) + (\mathbf{6}, \mathbf{3})(-1) + (\mathbf{15}, \mathbf{1})(2)$
<b>45</b>	$(\mathbf{1}, \mathbf{6})(-4) + (\mathbf{6}, \mathbf{3})(-1) + (\mathbf{21}, \mathbf{1})(2)$
<b>80</b>	$(\mathbf{1}, \mathbf{1})(0) + (\mathbf{6}, \bar{\mathbf{3}})(3) + (\bar{\mathbf{6}}, \mathbf{3})(-3) + (\mathbf{1}, \mathbf{8})(0) + (\mathbf{35}, \mathbf{1})(0)$
<b>84</b>	$(\mathbf{1}, \mathbf{1})(-6) + (\mathbf{6}, \bar{\mathbf{3}})(-3) + (\mathbf{15}, \mathbf{3})(0) + (\mathbf{20}, \mathbf{1})(3)$
<b>126</b>	$(\mathbf{6}, \mathbf{1})(-5) + (\bar{\mathbf{15}}, \mathbf{1})(4) + (\mathbf{15}, \bar{\mathbf{3}})(-2) + (\mathbf{20}, \mathbf{3})(1)$
<b>165</b>	$(\mathbf{1}, \mathbf{10})(-6) + (\mathbf{6}, \mathbf{6})(-3) + (\mathbf{21}, \mathbf{3})(0) + (\mathbf{56}, \mathbf{1})(3)$
<b>240</b>	$(\mathbf{6}, \bar{\mathbf{3}})(-3) + (\mathbf{1}, \mathbf{8})(-6) + (\mathbf{6}, \mathbf{6})(-3) + (\mathbf{15}, \mathbf{3})(0) + (\mathbf{21}, \mathbf{3})(0) + (\mathbf{70}, \mathbf{1})(3)$
<b>315</b>	$(\mathbf{1}, \mathbf{3})(-2) + (\mathbf{6}, \mathbf{1})(1) + (\mathbf{1}, \bar{\mathbf{6}})(-2) + (\bar{\mathbf{6}}, \bar{\mathbf{3}})(-5) + (\mathbf{6}, \mathbf{8})(1) + (\mathbf{15}, \bar{\mathbf{3}})(4) + (\mathbf{35}, \mathbf{3})(-2) + (\mathbf{84}, \mathbf{1})(1)$
<b>396</b>	$(\mathbf{1}, \mathbf{3})(-2) + (\mathbf{6}, \mathbf{1})(1) + (\bar{\mathbf{6}}, \mathbf{6})(-5) + (\mathbf{6}, \mathbf{8})(1) + (\mathbf{1}, \mathbf{15})(-2) + (\mathbf{21}, \bar{\mathbf{3}})(4) + (\mathbf{35}, \mathbf{3})(-2) + (\mathbf{120}, \mathbf{1})(1)$
<b>495</b>	$(\mathbf{6}, \mathbf{10})(-5) + (\mathbf{1}, \mathbf{15}')(-8) + (\mathbf{21}, \mathbf{6})(-2) + (\mathbf{56}, \mathbf{3})(1) + (\bar{\mathbf{126}}, \mathbf{1})(4)$
<b>540</b>	$(\mathbf{1}, \bar{\mathbf{6}})(-8) + (\mathbf{6}, \mathbf{8})(-5) + (\mathbf{15}, \bar{\mathbf{3}})(-2) + (\mathbf{21}, \mathbf{6})(-2) + (\mathbf{70}, \mathbf{3})(1) + (\bar{\mathbf{105}}', \mathbf{1})(4)$
<b>630</b>	$(\mathbf{1}, \mathbf{3})(-8) + (\mathbf{6}, \mathbf{1})(-5) + (\mathbf{6}, \mathbf{8})(-5) + (\mathbf{15}, \bar{\mathbf{3}})(-2) + (\mathbf{15}, \mathbf{6})(-2) + (\mathbf{20}, \mathbf{3})(1) + (\mathbf{21}, \bar{\mathbf{3}})(-2) + (\mathbf{70}, \mathbf{3})(1) + (\bar{\mathbf{105}}, \mathbf{1})(4)$
<b>720</b>	$(\mathbf{1}, \bar{\mathbf{3}})(-4) + (\bar{\mathbf{6}}, \mathbf{1})(-7) + (\mathbf{6}, \mathbf{3})(-1) + (\mathbf{6}, \bar{\mathbf{6}})(-1) + (\mathbf{15}, \mathbf{1})(2) + (\mathbf{20}, \bar{\mathbf{3}})(5) + (\mathbf{15}, \mathbf{8})(2) + (\mathbf{35}, \bar{\mathbf{3}})(-4) + (\mathbf{84}, \mathbf{3})(-1) + (\mathbf{105}, \mathbf{1})(2)$
<b>990</b>	$(\mathbf{6}, \mathbf{8})(-5) + (\mathbf{6}, \mathbf{10})(-5) + (\mathbf{1}, \mathbf{15})(-8) + (\mathbf{15}, \mathbf{6})(-2) + (\mathbf{21}, \bar{\mathbf{3}})(-2) + (\mathbf{21}, \mathbf{6})(-2) + (\mathbf{56}, \mathbf{3})(1) + (\mathbf{70}, \mathbf{3})(1) + (\bar{\mathbf{210}}', \mathbf{1})(4)$
<b>1008</b>	$(\bar{\mathbf{6}}, \bar{\mathbf{3}})(7) + (\bar{\mathbf{15}}, \mathbf{1})(4) + (\mathbf{15}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{21}}, \mathbf{1})(4) + (\mathbf{20}, \mathbf{3})(1) + (\bar{\mathbf{15}}, \mathbf{8})(4) + (\mathbf{20}, \bar{\mathbf{6}})(1) + (\bar{\mathbf{70}}, \mathbf{3})(1) + (\mathbf{84}, \mathbf{1})(-5) + (\mathbf{105}, \bar{\mathbf{3}})(-2)$
SU(9) $\rightarrow$ SU(5) $\times$ SU(4) $\times$ U(1)	
<b>9</b>	$(\mathbf{1}, \mathbf{4})(-5) + (\mathbf{5}, \mathbf{1})(4)$
<b>36</b>	$(\mathbf{1}, \mathbf{6})(-10) + (\mathbf{5}, \mathbf{4})(-1) + (\mathbf{10}, \mathbf{1})(8)$
<b>45</b>	$(\mathbf{5}, \mathbf{4})(-1) + (\mathbf{1}, \mathbf{10})(-10) + (\mathbf{15}, \mathbf{1})(8)$
<b>80</b>	$(\mathbf{1}, \mathbf{1})(0) + (\mathbf{5}, \bar{\mathbf{4}})(9) + (\bar{\mathbf{5}}, \mathbf{4})(-9) + (\mathbf{1}, \mathbf{15})(0) + (\mathbf{24}, \mathbf{1})(0)$
<b>84</b>	$(\mathbf{1}, \bar{\mathbf{4}})(-15) + (\mathbf{5}, \mathbf{6})(-6) + (\bar{\mathbf{10}}, \mathbf{1})(12) + (\mathbf{10}, \mathbf{4})(3)$
<b>126</b>	$(\mathbf{1}, \mathbf{1})(-20) + (\bar{\mathbf{5}}, \mathbf{1})(16) + (\mathbf{5}, \bar{\mathbf{4}})(-11) + (\bar{\mathbf{10}}, \mathbf{4})(7) + (\mathbf{10}, \mathbf{6})(-2)$
<b>165</b>	$(\mathbf{5}, \mathbf{10})(-6) + (\mathbf{15}, \mathbf{4})(3) + (\mathbf{1}, \bar{\mathbf{20}}'')(-15) + (\bar{\mathbf{35}}, \mathbf{1})(12)$
<b>240</b>	$(\mathbf{5}, \mathbf{6})(-6) + (\mathbf{10}, \mathbf{4})(3) + (\mathbf{5}, \mathbf{10})(-6) + (\mathbf{15}, \mathbf{4})(3) + (\mathbf{1}, \bar{\mathbf{20}})(-15) + (\bar{\mathbf{40}}, \mathbf{1})(12)$
<b>315</b>	$(\mathbf{1}, \mathbf{4})(-5) + (\mathbf{5}, \mathbf{1})(4) + (\bar{\mathbf{5}}, \mathbf{6})(-14) + (\mathbf{10}, \bar{\mathbf{4}})(13) + (\mathbf{5}, \mathbf{15})(4) + (\mathbf{1}, \mathbf{20})(-5) + (\mathbf{24}, \mathbf{4})(-5) + (\mathbf{45}, \mathbf{1})(4)$
<b>396</b>	$(\mathbf{1}, \mathbf{4})(-5) + (\mathbf{5}, \mathbf{1})(4) + (\bar{\mathbf{5}}, \mathbf{10})(-14) + (\mathbf{15}, \bar{\mathbf{4}})(13) + (\mathbf{5}, \mathbf{15})(4) + (\mathbf{24}, \mathbf{4})(-5) + (\mathbf{1}, \mathbf{36})(-5) + (\mathbf{70}, \mathbf{1})(4)$
<b>495</b>	$(\mathbf{15}, \mathbf{10})(-2) + (\mathbf{5}, \bar{\mathbf{20}}'')(-11) + (\mathbf{1}, \mathbf{35})(-20) + (\bar{\mathbf{35}}, \mathbf{4})(7) + (\bar{\mathbf{70}}', \mathbf{1})(16)$
<b>540</b>	$(\mathbf{10}, \mathbf{6})(-2) + (\mathbf{1}, \mathbf{20}')(-20) + (\mathbf{15}, \mathbf{10})(-2) + (\mathbf{5}, \bar{\mathbf{20}})(-11) + (\bar{\mathbf{40}}, \mathbf{4})(7) + (\bar{\mathbf{50}}, \mathbf{1})(16)$
<b>630</b>	$(\mathbf{5}, \bar{\mathbf{4}})(-11) + (\bar{\mathbf{10}}, \mathbf{4})(7) + (\mathbf{10}, \mathbf{6})(-2) + (\mathbf{1}, \mathbf{15})(-20) + (\mathbf{10}, \mathbf{10})(-2) + (\mathbf{15}, \mathbf{6})(-2) + (\mathbf{5}, \bar{\mathbf{20}})(-11) + (\bar{\mathbf{40}}, \mathbf{4})(7) + (\bar{\mathbf{45}}, \mathbf{1})(16)$
<b>720</b>	$(\mathbf{1}, \mathbf{6})(-10) + (\mathbf{5}, \mathbf{4})(-1) + (\bar{\mathbf{5}}, \bar{\mathbf{4}})(-19) + (\mathbf{10}, \mathbf{1})(8) + (\mathbf{1}, \bar{\mathbf{10}})(-10) + (\bar{\mathbf{10}}, \bar{\mathbf{4}})(17) + (\mathbf{10}, \mathbf{15})(8) + (\mathbf{5}, \mathbf{20})(-1) + (\mathbf{24}, \mathbf{6})(-10) + (\mathbf{40}, \mathbf{1})(8) + (\mathbf{45}, \mathbf{4})(-1)$
<b>990</b>	$(\mathbf{10}, \mathbf{10})(-2) + (\mathbf{15}, \mathbf{6})(-2) + (\mathbf{15}, \mathbf{10})(-2) + (\mathbf{5}, \bar{\mathbf{20}})(-11) + (\mathbf{5}, \bar{\mathbf{20}}'')(-11) + (\bar{\mathbf{35}}, \mathbf{4})(7) + (\bar{\mathbf{40}}, \mathbf{4})(7) + (\mathbf{1}, \mathbf{45})(-20) + (\bar{\mathbf{105}}, \mathbf{1})(16)$
<b>1008</b>	$(\mathbf{1}, \bar{\mathbf{4}})(25) + (\bar{\mathbf{5}}, \mathbf{1})(16) + (\mathbf{5}, \bar{\mathbf{4}})(-11) + (\bar{\mathbf{10}}, \mathbf{4})(7) + (\mathbf{10}, \mathbf{6})(-2) + (\bar{\mathbf{15}}, \mathbf{4})(7) + (\mathbf{10}, \bar{\mathbf{10}})(-2) + (\bar{\mathbf{5}}, \mathbf{15})(16) + (\mathbf{24}, \mathbf{1})(-20) + (\bar{\mathbf{10}}, \mathbf{20})(7) + (\mathbf{40}, \mathbf{6})(-2) + (\mathbf{45}, \bar{\mathbf{4}})(-11)$



Table A.76: SU(10) Branching Rules

SU(10) $\rightarrow$ SU(9) $\times$ U(1)	
10	= (1)(-9) + (9)(1)
45	= (9)(-8) + (36)(2)
55	= (1)(-18) + (9)(-8) + (45)(2)
99	= (1)(0) + (9)(10) + ( $\bar{9}$ )(-10) + (80)(0)
120	= (36)(-7) + (84)(3)
210	= (84)(-6) + (126)(4)
220	= (1)(-27) + (9)(-17) + (45)(-7) + (165)(3)
252	= (126)(-5) + ( $\bar{126}$ )(5)
330	= (9)(-17) + (36)(-7) + (45)(-7) + (240)(3)
440	= (9)(1) + (36)(11) + (80)(-9) + (315)(1)
540	= (1)(-9) + (9)(1) + ( $\bar{9}$ )(-19) + (45)(11) + (80)(-9) + (396)(1)
715	= (1)(-36) + (9)(-26) + (45)(-16) + (165)(-6) + (495)(4)
825	= (45)(-16) + (240)(-6) + (540)(4)
SU(10) $\rightarrow$ SU(8) $\times$ SU(2) $\times$ U(1)	
10	= (1, 2)(-4) + (8, 1)(1)
45	= (1, 1)(-8) + (8, 2)(-3) + (28, 1)(2)
55	= (1, 3)(-8) + (8, 2)(-3) + (36, 1)(2)
99	= (1, 1)(0) + (1, 3)(0) + (8, 2)(5) + ( $\bar{8}$ , 2)(-5) + (63, 1)(0)
120	= (8, 1)(-7) + (28, 2)(-2) + (56, 1)(3)
210	= (28, 1)(-6) + (56, 2)(-1) + (70, 1)(4)
220	= (1, 4)(-12) + (8, 3)(-7) + (36, 2)(-2) + (120, 1)(3)
252	= (56, 1)(-5) + ( $\bar{56}$ , 1)(5) + (70, 2)(0)
330	= (1, 2)(-12) + (8, 1)(-7) + (8, 3)(-7) + (28, 2)(-2) + (36, 2)(-2) + (168, 1)(3)
440	= (1, 2)(-4) + (8, 1)(1) + ( $\bar{8}$ , 1)(-9) + (8, 3)(1) + (28, 2)(6) + (63, 2)(-4) + (216, 1)(1)
540	= (1, 2)(-4) + (1, 4)(-4) + (8, 1)(1) + (8, 3)(1) + ( $\bar{8}$ , 3)(-9) + (36, 2)(6) + (63, 2)(-4) + (280, 1)(1)
715	= (1, 5)(-16) + (8, 4)(-11) + (36, 3)(-6) + (120, 2)(-1) + (330, 1)(4)
825	= (1, 1)(-16) + (8, 2)(-11) + (28, 1)(-6) + (36, 3)(-6) + (168, 2)(-1) + (336, 1)(4)
SU(10) $\rightarrow$ SU(7) $\times$ SU(3) $\times$ U(1)	
10	= (1, 3)(-7) + (7, 1)(3)
45	= (1, $\bar{3}$ )(-14) + (7, 3)(-4) + (21, 1)(6)
55	= (1, 6)(-14) + (7, 3)(-4) + (28, 1)(6)
99	= (1, 1)(0) + (1, 8)(0) + (7, $\bar{3}$ )(10) + ( $\bar{7}$ , 3)(-10) + (48, 1)(0)
120	= (1, 1)(-21) + (7, $\bar{3}$ )(-11) + (21, 3)(-1) + (35, 1)(9)
210	= (7, 1)(-18) + (21, $\bar{3}$ )(-8) + (35, 1)(12) + (35, 3)(2)
220	= (1, 10)(-21) + (7, 6)(-11) + (28, 3)(-1) + (84, 1)(9)
252	= (21, 1)(-15) + ( $\bar{21}$ , 1)(15) + (35, $\bar{3}$ )(-5) + (35, 3)(5)
330	= (1, 8)(-21) + (7, $\bar{3}$ )(-11) + (7, 6)(-11) + (21, 3)(-1) + (28, 3)(-1) + (112, 1)(9)
440	= (1, 3)(-7) + (1, $\bar{6}$ )(-7) + (7, 1)(3) + ( $\bar{7}$ , $\bar{3}$ )(-17) + (7, 8)(3) + (21, $\bar{3}$ )(13) + (48, 3)(-7) + (140, 1)(3)
540	= (1, 3)(-7) + (7, 1)(3) + ( $\bar{7}$ , 6)(-17) + (7, 8)(3) + (1, 15)(-7) + (28, $\bar{3}$ )(13) + (48, 3)(-7) + (189, 1)(3)
715	= (1, 15')( -28) + (7, 10)(-18) + (28, 6)(-8) + (84, 3)(2) + ( $\bar{210}$ ', 1)(12)
825	= (1, $\bar{6}$ )(-28) + (7, 8)(-18) + (21, $\bar{3}$ )(-8) + (28, 6)(-8) + (112, 3)(2) + ( $\bar{196}$ , 1)(12)

Table A.76: SU(10) Branching Rules (continued)

SU(10) $\rightarrow$ SU(6) $\times$ SU(4) $\times$ U(1)	
10	$= (\mathbf{1}, \mathbf{4})(-3) + (\mathbf{6}, \mathbf{1})(2)$
45	$= (\mathbf{1}, \mathbf{6})(-6) + (\mathbf{6}, \mathbf{4})(-1) + (\mathbf{15}, \mathbf{1})(4)$
55	$= (\mathbf{6}, \mathbf{4})(-1) + (\mathbf{1}, \mathbf{10})(-6) + (\mathbf{21}, \mathbf{1})(4)$
99	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{6}, \overline{\mathbf{4}})(5) + (\overline{\mathbf{6}}, \mathbf{4})(-5) + (\mathbf{1}, \mathbf{15})(0) + (\mathbf{35}, \mathbf{1})(0)$
120	$= (\mathbf{1}, \overline{\mathbf{4}})(-9) + (\mathbf{6}, \mathbf{6})(-4) + (\mathbf{15}, \mathbf{4})(1) + (\mathbf{20}, \mathbf{1})(6)$
210	$= (\mathbf{1}, \mathbf{1})(-12) + (\mathbf{6}, \overline{\mathbf{4}})(-7) + (\overline{\mathbf{15}}, \mathbf{1})(8) + (\mathbf{15}, \mathbf{6})(-2) + (\mathbf{20}, \mathbf{4})(3)$
220	$= (\mathbf{6}, \mathbf{10})(-4) + (\mathbf{1}, \overline{\mathbf{20}}'')(-9) + (\mathbf{21}, \mathbf{4})(1) + (\mathbf{56}, \mathbf{1})(6)$
252	$= (\mathbf{6}, \mathbf{1})(-10) + (\overline{\mathbf{6}}, \mathbf{1})(10) + (\mathbf{15}, \overline{\mathbf{4}})(-5) + (\overline{\mathbf{15}}, \mathbf{4})(5) + (\mathbf{20}, \mathbf{6})(0)$
330	$= (\mathbf{6}, \mathbf{6})(-4) + (\mathbf{6}, \mathbf{10})(-4) + (\mathbf{15}, \mathbf{4})(1) + (\mathbf{1}, \overline{\mathbf{20}})(-9) + (\mathbf{21}, \mathbf{4})(1) + (\mathbf{70}, \mathbf{1})(6)$
440	$= (\mathbf{1}, \mathbf{4})(-3) + (\mathbf{6}, \mathbf{1})(2) + (\overline{\mathbf{6}}, \mathbf{6})(-8) + (\mathbf{15}, \overline{\mathbf{4}})(7) + (\mathbf{6}, \mathbf{15})(2) + (\mathbf{1}, \mathbf{20})(-3) + (\mathbf{35}, \mathbf{4})(-3) + (\mathbf{84}, \mathbf{1})(2)$
540	$= (\mathbf{1}, \mathbf{4})(-3) + (\mathbf{6}, \mathbf{1})(2) + (\overline{\mathbf{6}}, \mathbf{10})(-8) + (\mathbf{6}, \mathbf{15})(2) + (\mathbf{21}, \overline{\mathbf{4}})(7) + (\mathbf{1}, \mathbf{36})(-3) + (\mathbf{35}, \mathbf{4})(-3) + (\mathbf{120}, \mathbf{1})(2)$
715	$= (\mathbf{6}, \overline{\mathbf{20}}'')(-7) + (\mathbf{21}, \mathbf{10})(-2) + (\mathbf{1}, \mathbf{35})(-12) + (\mathbf{56}, \mathbf{4})(3) + (\overline{\mathbf{126}}, \mathbf{1})(8)$
825	$= (\mathbf{15}, \mathbf{6})(-2) + (\mathbf{1}, \mathbf{20}')(-12) + (\mathbf{6}, \overline{\mathbf{20}})(-7) + (\mathbf{21}, \mathbf{10})(-2) + (\mathbf{70}, \mathbf{4})(3) + (\overline{\mathbf{105}}', \mathbf{1})(8)$
SU(10) $\rightarrow$ SU(5) $\times$ SU(5) $\times$ U(1)	
10	$= (\mathbf{5}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{5})(-1)$
45	$= (\mathbf{5}, \mathbf{5})(0) + (\mathbf{10}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{10})(-2)$
55	$= (\mathbf{5}, \mathbf{5})(0) + (\mathbf{15}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{15})(-2)$
99	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{5}, \overline{\mathbf{5}})(2) + (\overline{\mathbf{5}}, \mathbf{5})(-2) + (\mathbf{24}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{24})(0)$
120	$= (\overline{\mathbf{10}}, \mathbf{1})(3) + (\mathbf{1}, \overline{\mathbf{10}})(-3) + (\mathbf{5}, \mathbf{10})(-1) + (\mathbf{10}, \mathbf{5})(1)$
210	$= (\overline{\mathbf{5}}, \mathbf{1})(4) + (\mathbf{1}, \overline{\mathbf{5}})(-4) + (\mathbf{5}, \overline{\mathbf{10}})(-2) + (\overline{\mathbf{10}}, \mathbf{5})(2) + (\mathbf{10}, \mathbf{10})(0)$
220	$= (\mathbf{15}, \mathbf{5})(1) + (\mathbf{5}, \mathbf{15})(-1) + (\overline{\mathbf{35}}, \mathbf{1})(3) + (\mathbf{1}, \overline{\mathbf{35}})(-3)$
252	$= (\mathbf{1}, \mathbf{1})(5) + (\mathbf{1}, \mathbf{1})(-5) + (\mathbf{5}, \overline{\mathbf{5}})(-3) + (\overline{\mathbf{5}}, \mathbf{5})(3) + (\mathbf{10}, \overline{\mathbf{10}})(-1) + (\overline{\mathbf{10}}, \mathbf{10})(1)$
330	$= (\mathbf{5}, \mathbf{10})(-1) + (\mathbf{10}, \mathbf{5})(1) + (\mathbf{15}, \mathbf{5})(1) + (\mathbf{5}, \mathbf{15})(-1) + (\overline{\mathbf{40}}, \mathbf{1})(3) + (\mathbf{1}, \overline{\mathbf{40}})(-3)$
440	$= (\mathbf{5}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{5})(-1) + (\mathbf{10}, \overline{\mathbf{5}})(3) + (\overline{\mathbf{5}}, \mathbf{10})(-3) + (\mathbf{24}, \mathbf{5})(-1) + (\mathbf{5}, \mathbf{24})(1) + (\mathbf{45}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{45})(-1)$
540	$= (\mathbf{5}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{5})(-1) + (\mathbf{15}, \overline{\mathbf{5}})(3) + (\overline{\mathbf{5}}, \mathbf{15})(-3) + (\mathbf{24}, \mathbf{5})(-1) + (\mathbf{5}, \mathbf{24})(1) + (\mathbf{70}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{70})(-1)$
715	$= (\mathbf{15}, \mathbf{15})(0) + (\overline{\mathbf{35}}, \mathbf{5})(2) + (\mathbf{5}, \overline{\mathbf{35}})(-2) + (\overline{\mathbf{70}}', \mathbf{1})(4) + (\mathbf{1}, \overline{\mathbf{70}}')(-4)$
825	$= (\mathbf{10}, \mathbf{10})(0) + (\mathbf{15}, \mathbf{15})(0) + (\overline{\mathbf{40}}, \mathbf{5})(2) + (\mathbf{5}, \overline{\mathbf{40}})(-2) + (\overline{\mathbf{50}}, \mathbf{1})(4) + (\mathbf{1}, \overline{\mathbf{50}})(-4)$

Table A.77: SU(11) Branching Rules

SU(11) $\rightarrow$ SU(10) $\times$ U(1)	
<b>11</b>	$= (1)(-10) + (10)(1)$
<b>55</b>	$= (10)(-9) + (45)(2)$
<b>66</b>	$= (1)(-20) + (10)(-9) + (55)(2)$
<b>120</b>	$= (1)(0) + (10)(11) + (\overline{10})(-11) + (99)(0)$
<b>165</b>	$= (45)(-8) + (120)(3)$
<b>286</b>	$= (1)(-30) + (10)(-19) + (55)(-8) + (220)(3)$
<b>330</b>	$= (120)(-7) + (210)(4)$
<b>440</b>	$= (10)(-19) + (45)(-8) + (55)(-8) + (330)(3)$
<b>462</b>	$= (210)(-6) + (252)(5)$
<b>594</b>	$= (10)(1) + (45)(12) + (99)(-10) + (440)(1)$
SU(11) $\rightarrow$ SU(9) $\times$ SU(2) $\times$ U(1)	
<b>11</b>	$= (1, 2)(-9) + (9, 1)(2)$
<b>55</b>	$= (1, 1)(-18) + (9, 2)(-7) + (36, 1)(4)$
<b>66</b>	$= (1, 3)(-18) + (9, 2)(-7) + (45, 1)(4)$
<b>120</b>	$= (1, 1)(0) + (1, 3)(0) + (9, 2)(11) + (\overline{9}, 2)(-11) + (80, 1)(0)$
<b>165</b>	$= (9, 1)(-16) + (36, 2)(-5) + (84, 1)(6)$
<b>286</b>	$= (1, 4)(-27) + (9, 3)(-16) + (45, 2)(-5) + (165, 1)(6)$
<b>330</b>	$= (36, 1)(-14) + (84, 2)(-3) + (126, 1)(8)$
<b>440</b>	$= (1, 2)(-27) + (9, 1)(-16) + (9, 3)(-16) + (36, 2)(-5) + (45, 2)(-5) + (240, 1)(6)$
<b>462</b>	$= (84, 1)(-12) + (\overline{126}, 1)(10) + (126, 2)(-1)$
<b>594</b>	$= (1, 2)(-9) + (9, 1)(2) + (\overline{9}, 1)(-20) + (9, 3)(2) + (36, 2)(13) + (80, 2)(-9) + (315, 1)(2)$
SU(11) $\rightarrow$ SU(8) $\times$ SU(3) $\times$ U(1)	
<b>11</b>	$= (1, 3)(-8) + (8, 1)(3)$
<b>55</b>	$= (1, \overline{3})(-16) + (8, 3)(-5) + (28, 1)(6)$
<b>66</b>	$= (1, 6)(-16) + (8, 3)(-5) + (36, 1)(6)$
<b>120</b>	$= (1, 1)(0) + (1, 8)(0) + (8, \overline{3})(11) + (\overline{8}, 3)(-11) + (63, 1)(0)$
<b>165</b>	$= (1, 1)(-24) + (8, \overline{3})(-13) + (28, 3)(-2) + (56, 1)(9)$
<b>286</b>	$= (1, 10)(-24) + (8, 6)(-13) + (36, 3)(-2) + (120, 1)(9)$
<b>330</b>	$= (8, 1)(-21) + (28, \overline{3})(-10) + (56, 3)(1) + (70, 1)(12)$
<b>440</b>	$= (1, 8)(-24) + (8, \overline{3})(-13) + (8, 6)(-13) + (28, 3)(-2) + (36, 3)(-2) + (168, 1)(9)$
<b>462</b>	$= (28, 1)(-18) + (\overline{56}, 1)(15) + (56, \overline{3})(-7) + (70, 3)(4)$
<b>594</b>	$= (1, 3)(-8) + (1, \overline{6})(-8) + (8, 1)(3) + (\overline{8}, \overline{3})(-19) + (8, 8)(3) + (28, \overline{3})(14) + (63, 3)(-8) + (216, 1)(3)$

Table A.77: SU(11) Branching Rules (continued)

SU(11) $\rightarrow$ SU(7) $\times$ SU(4) $\times$ U(1)	
<b>11</b>	$= (1, 4)(-7) + (7, 1)(4)$
<b>55</b>	$= (1, 6)(-14) + (7, 4)(-3) + (21, 1)(8)$
<b>66</b>	$= (7, 4)(-3) + (1, 10)(-14) + (28, 1)(8)$
<b>120</b>	$= (1, 1)(0) + (7, \bar{4})(11) + (\bar{7}, 4)(-11) + (1, 15)(0) + (48, 1)(0)$
<b>165</b>	$= (1, \bar{4})(-21) + (7, 6)(-10) + (21, 4)(1) + (35, 1)(12)$
<b>286</b>	$= (7, 10)(-10) + (1, \bar{20}'')(-21) + (28, 4)(1) + (84, 1)(12)$
<b>330</b>	$= (1, 1)(-28) + (7, \bar{4})(-17) + (21, 6)(-6) + (\bar{35}, 1)(16) + (35, 4)(5)$
<b>440</b>	$= (7, 6)(-10) + (7, 10)(-10) + (1, \bar{20})(-21) + (21, 4)(1) + (28, 4)(1) + (112, 1)(12)$
<b>462</b>	$= (7, 1)(-24) + (\bar{21}, 1)(20) + (21, \bar{4})(-13) + (\bar{35}, 4)(9) + (35, 6)(-2)$
<b>594</b>	$= (1, 4)(-7) + (7, 1)(4) + (\bar{7}, 6)(-18) + (1, 20)(-7) + (7, 15)(4) + (21, \bar{4})(15) + (48, 4)(-7) + (140, 1)(4)$
SU(11) $\rightarrow$ SU(6) $\times$ SU(5) $\times$ U(1)	
<b>11</b>	$= (1, 5)(-6) + (6, 1)(5)$
<b>55</b>	$= (6, 5)(-1) + (1, 10)(-12) + (15, 1)(10)$
<b>66</b>	$= (6, 5)(-1) + (1, 15)(-12) + (21, 1)(10)$
<b>120</b>	$= (1, 1)(0) + (6, \bar{5})(11) + (\bar{6}, 5)(-11) + (1, 24)(0) + (35, 1)(0)$
<b>165</b>	$= (1, \bar{10})(-18) + (6, 10)(-7) + (15, 5)(4) + (20, 1)(15)$
<b>286</b>	$= (6, 15)(-7) + (21, 5)(4) + (1, \bar{35})(-18) + (56, 1)(15)$
<b>330</b>	$= (1, \bar{5})(-24) + (6, \bar{10})(-13) + (\bar{15}, 1)(20) + (15, 10)(-2) + (20, 5)(9)$
<b>440</b>	$= (6, 10)(-7) + (15, 5)(4) + (6, 15)(-7) + (21, 5)(4) + (1, \bar{40})(-18) + (70, 1)(15)$
<b>462</b>	$= (1, 1)(-30) + (\bar{6}, 1)(25) + (6, \bar{5})(-19) + (\bar{15}, 5)(14) + (15, \bar{10})(-8) + (20, 10)(3)$
<b>594</b>	$= (1, 5)(-6) + (6, 1)(5) + (\bar{6}, 10)(-17) + (15, \bar{5})(16) + (6, 24)(5) + (35, 5)(-6) + (1, 45)(-6) + (84, 1)(5)$

Table A.78: SU(12) Branching Rules

SU(12) $\rightarrow$ SU(11) $\times$ U(1)	
<b>12</b>	$= (\mathbf{1})(-11) + (\mathbf{11})(1)$
<b>66</b>	$= (\mathbf{11})(-10) + (\mathbf{55})(2)$
<b>78</b>	$= (\mathbf{1})(-22) + (\mathbf{11})(-10) + (\mathbf{66})(2)$
<b>143</b>	$= (\mathbf{1})(0) + (\mathbf{11})(12) + (\overline{\mathbf{11}})(-12) + (\mathbf{120})(0)$
<b>220</b>	$= (\mathbf{55})(-9) + (\mathbf{165})(3)$
<b>364</b>	$= (\mathbf{1})(-33) + (\mathbf{11})(-21) + (\mathbf{66})(-9) + (\mathbf{286})(3)$
<b>495</b>	$= (\mathbf{165})(-8) + (\mathbf{330})(4)$
<b>572</b>	$= (\mathbf{11})(-21) + (\mathbf{55})(-9) + (\mathbf{66})(-9) + (\mathbf{440})(3)$
<b>780</b>	$= (\mathbf{11})(1) + (\mathbf{55})(13) + (\mathbf{120})(-11) + (\mathbf{594})(1)$
<b>792</b>	$= (\mathbf{330})(-7) + (\mathbf{462})(5)$
SU(12) $\rightarrow$ SU(10) $\times$ SU(2) $\times$ U(1)	
<b>12</b>	$= (\mathbf{1}, \mathbf{2})(-5) + (\mathbf{10}, \mathbf{1})(1)$
<b>66</b>	$= (\mathbf{1}, \mathbf{1})(-10) + (\mathbf{10}, \mathbf{2})(-4) + (\mathbf{45}, \mathbf{1})(2)$
<b>78</b>	$= (\mathbf{1}, \mathbf{3})(-10) + (\mathbf{10}, \mathbf{2})(-4) + (\mathbf{55}, \mathbf{1})(2)$
<b>143</b>	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{10}, \mathbf{2})(6) + (\overline{\mathbf{10}}, \mathbf{2})(-6) + (\mathbf{99}, \mathbf{1})(0)$
<b>220</b>	$= (\mathbf{10}, \mathbf{1})(-9) + (\mathbf{45}, \mathbf{2})(-3) + (\mathbf{120}, \mathbf{1})(3)$
<b>364</b>	$= (\mathbf{1}, \mathbf{4})(-15) + (\mathbf{10}, \mathbf{3})(-9) + (\mathbf{55}, \mathbf{2})(-3) + (\mathbf{220}, \mathbf{1})(3)$
<b>495</b>	$= (\mathbf{45}, \mathbf{1})(-8) + (\mathbf{120}, \mathbf{2})(-2) + (\mathbf{210}, \mathbf{1})(4)$
<b>572</b>	$= (\mathbf{1}, \mathbf{2})(-15) + (\mathbf{10}, \mathbf{1})(-9) + (\mathbf{10}, \mathbf{3})(-9) + (\mathbf{45}, \mathbf{2})(-3) + (\mathbf{55}, \mathbf{2})(-3) + (\mathbf{330}, \mathbf{1})(3)$
<b>780</b>	$= (\mathbf{1}, \mathbf{2})(-5) + (\mathbf{10}, \mathbf{1})(1) + (\overline{\mathbf{10}}, \mathbf{1})(-11) + (\mathbf{10}, \mathbf{3})(1) + (\mathbf{45}, \mathbf{2})(7) + (\mathbf{99}, \mathbf{2})(-5) + (\mathbf{440}, \mathbf{1})(1)$
<b>792</b>	$= (\mathbf{120}, \mathbf{1})(-7) + (\mathbf{210}, \mathbf{2})(-1) + (\mathbf{252}, \mathbf{1})(5)$
SU(12) $\rightarrow$ SU(9) $\times$ SU(3) $\times$ U(1)	
<b>12</b>	$= (\mathbf{1}, \mathbf{3})(-3) + (\mathbf{9}, \mathbf{1})(1)$
<b>66</b>	$= (\mathbf{1}, \overline{\mathbf{3}})(-6) + (\mathbf{9}, \mathbf{3})(-2) + (\mathbf{36}, \mathbf{1})(2)$
<b>78</b>	$= (\mathbf{1}, \mathbf{6})(-6) + (\mathbf{9}, \mathbf{3})(-2) + (\mathbf{45}, \mathbf{1})(2)$
<b>143</b>	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0) + (\mathbf{9}, \overline{\mathbf{3}})(4) + (\overline{\mathbf{9}}, \mathbf{3})(-4) + (\mathbf{80}, \mathbf{1})(0)$
<b>220</b>	$= (\mathbf{1}, \mathbf{1})(-9) + (\mathbf{9}, \overline{\mathbf{3}})(-5) + (\mathbf{36}, \mathbf{3})(-1) + (\mathbf{84}, \mathbf{1})(3)$
<b>364</b>	$= (\mathbf{1}, \mathbf{10})(-9) + (\mathbf{9}, \mathbf{6})(-5) + (\mathbf{45}, \mathbf{3})(-1) + (\mathbf{165}, \mathbf{1})(3)$
<b>495</b>	$= (\mathbf{9}, \mathbf{1})(-8) + (\mathbf{36}, \overline{\mathbf{3}})(-4) + (\mathbf{84}, \mathbf{3})(0) + (\mathbf{126}, \mathbf{1})(4)$
<b>572</b>	$= (\mathbf{1}, \mathbf{8})(-9) + (\mathbf{9}, \overline{\mathbf{3}})(-5) + (\mathbf{9}, \mathbf{6})(-5) + (\mathbf{36}, \mathbf{3})(-1) + (\mathbf{45}, \mathbf{3})(-1) + (\mathbf{240}, \mathbf{1})(3)$
<b>780</b>	$= (\mathbf{1}, \mathbf{3})(-3) + (\mathbf{1}, \overline{\mathbf{6}})(-3) + (\mathbf{9}, \mathbf{1})(1) + (\overline{\mathbf{9}}, \overline{\mathbf{3}})(-7) + (\mathbf{9}, \mathbf{8})(1) + (\mathbf{36}, \overline{\mathbf{3}})(5) + (\mathbf{80}, \mathbf{3})(-3) + (\mathbf{315}, \mathbf{1})(1)$
<b>792</b>	$= (\mathbf{36}, \mathbf{1})(-7) + (\mathbf{84}, \overline{\mathbf{3}})(-3) + (\overline{\mathbf{126}}, \mathbf{1})(5) + (\mathbf{126}, \mathbf{3})(1)$

Table A.78: SU(12) Branching Rules (continued)

SU(12) $\rightarrow$ SU(7) $\times$ SU(5) $\times$ U(1)	
<b>12</b>	$= (\mathbf{1}, \mathbf{5})(-7) + (\mathbf{7}, \mathbf{1})(5)$
<b>66</b>	$= (\mathbf{1}, \mathbf{10})(-14) + (\mathbf{7}, \mathbf{5})(-2) + (\mathbf{21}, \mathbf{1})(10)$
<b>78</b>	$= (\mathbf{7}, \mathbf{5})(-2) + (\mathbf{1}, \mathbf{15})(-14) + (\mathbf{28}, \mathbf{1})(10)$
<b>143</b>	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{7}, \overline{\mathbf{5}})(12) + (\overline{\mathbf{7}}, \mathbf{5})(-12) + (\mathbf{1}, \mathbf{24})(0) + (\mathbf{48}, \mathbf{1})(0)$
<b>220</b>	$= (\mathbf{1}, \overline{\mathbf{10}})(-21) + (\mathbf{7}, \mathbf{10})(-9) + (\mathbf{21}, \mathbf{5})(3) + (\mathbf{35}, \mathbf{1})(15)$
<b>364</b>	$= (\mathbf{7}, \mathbf{15})(-9) + (\mathbf{28}, \mathbf{5})(3) + (\mathbf{1}, \overline{\mathbf{35}})(-21) + (\mathbf{84}, \mathbf{1})(15)$
<b>495</b>	$= (\mathbf{1}, \overline{\mathbf{5}})(-28) + (\mathbf{7}, \overline{\mathbf{10}})(-16) + (\mathbf{21}, \mathbf{10})(-4) + (\overline{\mathbf{35}}, \mathbf{1})(20) + (\mathbf{35}, \mathbf{5})(8)$
<b>572</b>	$= (\mathbf{7}, \mathbf{10})(-9) + (\mathbf{7}, \mathbf{15})(-9) + (\mathbf{21}, \mathbf{5})(3) + (\mathbf{28}, \mathbf{5})(3) + (\mathbf{1}, \overline{\mathbf{40}})(-21) + (\mathbf{112}, \mathbf{1})(15)$
<b>780</b>	$= (\mathbf{1}, \mathbf{5})(-7) + (\mathbf{7}, \mathbf{1})(5) + (\overline{\mathbf{7}}, \mathbf{10})(-19) + (\mathbf{21}, \overline{\mathbf{5}})(17) + (\mathbf{7}, \mathbf{24})(5) + (\mathbf{1}, \mathbf{45})(-7) + (\mathbf{48}, \mathbf{5})(-7) + (\mathbf{140}, \mathbf{1})(5)$
<b>792</b>	$= (\mathbf{1}, \mathbf{1})(-35) + (\mathbf{7}, \overline{\mathbf{5}})(-23) + (\overline{\mathbf{21}}, \mathbf{1})(25) + (\mathbf{21}, \overline{\mathbf{10}})(-11) + (\overline{\mathbf{35}}, \mathbf{5})(13) + (\mathbf{35}, \mathbf{10})(1)$
SU(12) $\rightarrow$ SU(6) $\times$ SU(6) $\times$ U(1)	
<b>12</b>	$= (\mathbf{6}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{6})(-1)$
<b>66</b>	$= (\mathbf{6}, \mathbf{6})(0) + (\mathbf{15}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{15})(-2)$
<b>78</b>	$= (\mathbf{6}, \mathbf{6})(0) + (\mathbf{21}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{21})(-2)$
<b>143</b>	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{6}, \overline{\mathbf{6}})(2) + (\overline{\mathbf{6}}, \mathbf{6})(-2) + (\mathbf{35}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{35})(0)$
<b>220</b>	$= (\mathbf{6}, \mathbf{15})(-1) + (\mathbf{15}, \mathbf{6})(1) + (\mathbf{20}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{20})(-3)$
<b>364</b>	$= (\mathbf{21}, \mathbf{6})(1) + (\mathbf{6}, \mathbf{21})(-1) + (\mathbf{56}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{56})(-3)$
<b>495</b>	$= (\overline{\mathbf{15}}, \mathbf{1})(4) + (\mathbf{1}, \overline{\mathbf{15}})(-4) + (\mathbf{6}, \mathbf{20})(-2) + (\mathbf{20}, \mathbf{6})(2) + (\mathbf{15}, \mathbf{15})(0)$
<b>572</b>	$= (\mathbf{6}, \mathbf{15})(-1) + (\mathbf{15}, \mathbf{6})(1) + (\mathbf{21}, \mathbf{6})(1) + (\mathbf{6}, \mathbf{21})(-1) + (\mathbf{70}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{70})(-3)$
<b>780</b>	$= (\mathbf{6}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{6})(-1) + (\mathbf{15}, \overline{\mathbf{6}})(3) + (\overline{\mathbf{6}}, \mathbf{15})(-3) + (\mathbf{35}, \mathbf{6})(-1) + (\mathbf{6}, \mathbf{35})(1) + (\mathbf{84}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{84})(-1)$
<b>792</b>	$= (\overline{\mathbf{6}}, \mathbf{1})(5) + (\mathbf{1}, \overline{\mathbf{6}})(-5) + (\mathbf{6}, \overline{\mathbf{15}})(-3) + (\overline{\mathbf{15}}, \mathbf{6})(3) + (\mathbf{15}, \mathbf{20})(-1) + (\mathbf{20}, \mathbf{15})(1)$

A.3.2.  $SO(N)$

Table A.79:  $SO(7)$  Branching Rules

$SO(7) \rightarrow SU(4)$
$7 = 1 + 6$
$8 = 4 + \bar{4}$
$21 = 6 + 15$
$27 = 1 + 6 + 20'$
$35 = 10 + \bar{10} + 15$
$48 = 4 + \bar{4} + 20 + \bar{20}$
$77 = 1 + 6 + 20' + 50$
$105 = 6 + 15 + 20' + 64$
$112 = 20 + \bar{20} + 36 + \bar{36}$
$112' = 20'' + \bar{20}'' + 36 + \bar{36}$
$168 = 4 + \bar{4} + 20 + \bar{20} + 60 + \bar{60}$
$168' = 20' + 64 + 84$
$182 = 1 + 6 + 20' + 50 + 105$
$189 = 10 + \bar{10} + 15 + 45 + \bar{45} + 64$
$294 = 35 + \bar{35} + 70 + \bar{70} + 84$
$330 = 6 + 15 + 20' + 50 + 64 + 175$
$378 = 45 + \bar{45} + 64 + 70 + \bar{70} + 84$
$378' = 1 + 6 + 20' + 50 + 105 + 196$
$448 = 4 + \bar{4} + 20 + \bar{20} + 60 + \bar{60} + 140' + \bar{140}'$
$512 = 20 + \bar{20} + 36 + \bar{36} + 60 + \bar{60} + 140 + \bar{140}$
$560 = 20'' + \bar{20}'' + 36 + \bar{36} + 84' + \bar{84}' + 140 + \bar{140}$
$616 = 10 + \bar{10} + 15 + 45 + \bar{45} + 64 + 126 + \bar{126} + 175$
$672 = 56 + \bar{56} + 120 + \bar{120} + 160 + \bar{160}$
$693 = 20' + 50 + 64 + 84 + 175 + 300$
$714 = 1 + 6 + 20' + 50 + 105 + 196 + 336$
$720 = 60 + \bar{60} + 140 + \bar{140} + 160 + \bar{160}$
$819 = 6 + 15 + 20' + 50 + 64 + 105 + 175 + 384$
$825 = 50 + 175 + 300 + 300'$
$1008 = 84' + \bar{84}' + 120 + \bar{120} + 140 + \bar{140} + 160 + \bar{160}$
$1008' = 4 + \bar{4} + 20 + \bar{20} + 60 + \bar{60} + 140' + \bar{140}' + 280' + \bar{280}'$
$1254 = 1 + 6 + 20' + 50 + 105 + 196 + 336 + 540'''$
$1386 = 35 + \bar{35} + 70 + \bar{70} + 84 + 140'' + \bar{140}'' + 256 + \bar{256} + 300$
$1386' = 84'' + \bar{84}'' + 189 + \bar{189} + 270 + \bar{270} + 300'$
$1512 = 20 + \bar{20} + 36 + \bar{36} + 60 + \bar{60} + 140 + \bar{140} + 140' + \bar{140}' + 360 + \bar{360}$
$1560 = 10 + \bar{10} + 15 + 45 + \bar{45} + 64 + 126 + \bar{126} + 175 + 280 + \bar{280} + 384$
$1617 = 45 + \bar{45} + 64 + 70 + \bar{70} + 84 + 126 + \bar{126} + 175 + 256 + \bar{256} + 300$
$1728 = 20'' + \bar{20}'' + 36 + \bar{36} + 84' + \bar{84}' + 140 + \bar{140} + 224 + \bar{224} + 360 + \bar{360}$
$1750 = 6 + 15 + 20' + 50 + 64 + 105 + 175 + 196 + 384 + 735$
$1911 = 20' + 50 + 64 + 84 + 105 + 175 + 300 + 384 + 729$
$2016 = 4 + \bar{4} + 20 + \bar{20} + 60 + \bar{60} + 140' + \bar{140}' + 280' + \bar{280}' + 504 + \bar{504}$
$2079 = 126 + \bar{126} + 175 + 256 + \bar{256} + 270 + \bar{270} + 300 + 300'$
$2079' = 1 + 6 + 20' + 50 + 105 + 196 + 336 + 540''' + 825'$
$2310 = 140'' + \bar{140}'' + 189 + \bar{189} + 256 + \bar{256} + 270 + \bar{270} + 300 + 300'$

Table A.80: SO(8) Branching Rules

SO(8) $\rightarrow$ SO(7)
$8_s = 1 + 7$
$8_v = 8$
$8_c = 8$
$28 = 7 + 21$
$35_v = 35$
$35_c = 35$
$35_s = 1 + 7 + 27$
$56_s = 21 + 35$
$56_v = 8 + 48$
$56_c = 8 + 48$
$112_s = 1 + 7 + 27 + 77$
$112_v = 112'$
$112_c = 112'$
$160_s = 7 + 21 + 27 + 105$
$160_v = 48 + 112$
$160_c = 48 + 112$
$224_{vs} = 35 + 189$
$224_{cs} = 35 + 189$
$224_{cv} = 112 + 112'$
$224_{sv} = 8 + 48 + 168$
$224_{vc} = 112 + 112'$
$224_{sc} = 8 + 48 + 168$
$294_v = 294$
$294_c = 294$
$294_s = 1 + 7 + 27 + 77 + 182$
$300 = 27 + 105 + 168'$
$350 = 21 + 35 + 105 + 189$
$567_v = 189 + 378$
$567_c = 189 + 378$
$567_s = 7 + 21 + 27 + 77 + 105 + 330$
$672_{vc} = 294 + 378$
$672_{cv} = 294 + 378$
$672_{cs} = 112' + 560$
$672_{sc} = 8 + 48 + 168 + 448$
$672_{vs} = 112' + 560$
$672_{sv} = 8 + 48 + 168 + 448$
$672'_s = 1 + 7 + 27 + 77 + 182 + 378'$
$672'_v = 672$
$672'_c = 672$
$840_s = 105 + 168' + 189 + 378$
$840_v = 48 + 112 + 168 + 512$
$840_c = 48 + 112 + 168 + 512$
$840'_s = 168' + 294 + 378$
$840'_c = 35 + 189 + 616$
$840'_v = 35 + 189 + 616$



Table A.80: SO(8) Branching Rules (continued)

$1296_s = 21 + 35 + 105 + 189 + 330 + 616$
$1296_v = 112 + 112' + 512 + 560$
$1296_c = 112 + 112' + 512 + 560$
$1386_v = 1386'$
$1386_c = 1386'$
$1386_s = 1 + 7 + 27 + 77 + 182 + 378' + 714$
$1400_s = 27 + 77 + 105 + 168' + 330 + 693$
$1400_v = 168 + 512 + 720$
$1400_c = 168 + 512 + 720$
$1568_s = 7 + 21 + 27 + 77 + 105 + 182 + 330 + 819$
$1568_v = 560 + 1008$
$1568_c = 560 + 1008$
$1680_{vs} = 294 + 1386$
$1680_{cs} = 294 + 1386$
$1680_{cv} = 672 + 1008$
$1680_{sv} = 8 + 48 + 168 + 448 + 1008'$
$1680_{vc} = 672 + 1008$
$1680_{sc} = 8 + 48 + 168 + 448 + 1008'$
$1925 = 77 + 330 + 693 + 825$
$2400_{sv} = 35 + 189 + 616 + 1560$
$2400_{sc} = 35 + 189 + 616 + 1560$
$2400_{vc} = 672 + 720 + 1008$
$2400_{vs} = 112' + 560 + 1728$
$2400_{cv} = 672 + 720 + 1008$
$2400_{cs} = 112' + 560 + 1728$
$2640_s = 1 + 7 + 27 + 77 + 182 + 378' + 714 + 1254$
$2640_v = 2640$
$2640_c = 2640$
$2800_{vs} = 189 + 378 + 616 + 1617$
$2800_{cs} = 189 + 378 + 616 + 1617$
$2800_{cv} = 512 + 560 + 720 + 1008$
$2800_{sv} = 48 + 112 + 168 + 448 + 512 + 1512$
$2800_{vc} = 512 + 560 + 720 + 1008$
$2800_{sc} = 48 + 112 + 168 + 448 + 512 + 1512$
$3675_v = 294 + 378 + 1386 + 1617$
$3675_c = 294 + 378 + 1386 + 1617$
$3675_s = 21 + 35 + 105 + 189 + 330 + 616 + 819 + 1560$
$3696_v = 1386 + 2310$
$3696_c = 1386 + 2310$
$3696_s = 7 + 21 + 27 + 77 + 105 + 182 + 330 + 378' + 819 + 1750$
$3696'_{vc} = 1386' + 2310$
$3696'_{cv} = 1386' + 2310$
$3696'_{cs} = 672 + 3024$
$3696'_{sc} = 8 + 48 + 168 + 448 + 1008' + 2016$
$3696'_{vs} = 672 + 3024$
$3696'_{sv} = 8 + 48 + 168 + 448 + 1008' + 2016$

Table A.81: SO(9) Branching Rules

SO(9) $\rightarrow$ SO(8)
$9 = 1 + 8_v$
$16 = 8_s + 8_c$
$36 = 8_v + 28$
$44 = 1 + 8_v + 35_v$
$84 = 28 + 56_v$
$126 = 35_c + 35_s + 56_v$
$128 = 8_s + 8_c + 56_c + 56_s$
$156 = 1 + 8_v + 35_v + 112_v$
$231 = 8_v + 28 + 35_v + 160_v$
$432 = 56_c + 56_s + 160_s + 160_c$
$450 = 1 + 8_v + 35_v + 112_v + 294_v$
$495 = 35_v + 160_v + 300$
$576 = 8_s + 8_c + 56_c + 56_s + 224_{vs} + 224_{vc}$
$594 = 28 + 56_v + 160_v + 350$
$672 = 112_s + 112_c + 224_{sc} + 224_{cs}$
$768 = 160_s + 160_c + 224_{sc} + 224_{cs}$
$910 = 8_v + 28 + 35_v + 112_v + 160_v + 567_v$
$924 = 35_c + 35_s + 56_v + 224_{cv} + 224_{sv} + 350$
$1122 = 1 + 8_v + 35_v + 112_v + 294_v + 672'_v$
$1650 = 160_v + 300 + 350 + 840_v$
$1920 = 8_s + 8_c + 56_c + 56_s + 224_{vs} + 224_{vc} + 672_{vs} + 672_{vc}$
$1980 = 300 + 840_v + 840'_v$
$2457 = 28 + 56_v + 160_v + 350 + 567_v + 1296_v$
$2508 = 1 + 8_v + 35_v + 112_v + 294_v + 672'_v + 1386_v$
$2560 = 56_c + 56_s + 160_s + 160_c + 224_{vs} + 224_{vc} + 840_c + 840_s$
$2574 = 35_v + 112_v + 160_v + 300 + 567_v + 1400_v$
$2772 = 224_{cv} + 224_{sv} + 350 + 567_c + 567_s + 840_v$
$2772' = 294_c + 294_s + 672_{cs} + 672_{sc} + 840'_v$
$2772'' = 8_v + 28 + 35_v + 112_v + 160_v + 294_v + 567_v + 1568_v$
$3900 = 35_c + 35_s + 56_v + 224_{cv} + 224_{sv} + 350 + 840'_s + 840'_c + 1296_v$
$4004 = 112_v + 567_v + 1400_v + 1925$
$4158 = 567_c + 567_s + 672_{cs} + 672_{sc} + 840_v + 840'_v$
$4608 = 112_s + 112_c + 224_{sc} + 224_{cs} + 672_{sv} + 672_{cv} + 1296_s + 1296_c$
SO(9) $\rightarrow$ SU(4) $\times$ SU(2)
$9 = (1, 3) + (6, 1)$
$16 = (4, 2) + (\bar{4}, 2)$
$36 = (1, 3) + (6, 3) + (15, 1)$
$44 = (1, 1) + (1, 5) + (6, 3) + (20', 1)$
$84 = (1, 1) + (6, 3) + (10, 1) + (\bar{10}, 1) + (15, 3)$
$126 = (6, 1) + (10, 3) + (\bar{10}, 3) + (15, 1) + (15, 3)$
$128 = (4, 2) + (\bar{4}, 2) + (4, 4) + (\bar{4}, 4) + (20, 2) + (\bar{20}, 2)$
$156 = (1, 3) + (6, 1) + (1, 7) + (6, 5) + (20', 3) + (50, 1)$
$231 = (1, 3) + (1, 5) + (6, 1) + (6, 3) + (6, 5) + (15, 3) + (20', 3) + (64, 1)$
$432 = (4, 2) + (\bar{4}, 2) + (4, 4) + (\bar{4}, 4) + (20, 2) + (\bar{20}, 2) + (20, 4) + (\bar{20}, 4) + (36, 2) + (\bar{36}, 2)$

Table A.81: SO(9) Branching Rules (continued)

$$\begin{aligned}
450 &= (1, 1) + (1, 5) + (6, 3) + (1, 9) + (6, 7) + (20', 1) + (20', 5) + (50, 3) + (105, 1) \\
495 &= (1, 1) + (1, 5) + (6, 3) + (6, 5) + (15, 3) + (20', 1) + (20', 5) + (64, 3) + (84, 1) \\
576 &= (4, 2) + (\overline{4}, 2) + (4, 4) + (\overline{4}, 4) + (4, 6) + (\overline{4}, 6) + (20, 2) + (\overline{20}, 2) + (20, 4) + (\overline{20}, 4) + (60, 2) + (\overline{60}, 2) \\
594 &= (1, 3) + (6, 1) + (6, 3) + (6, 5) + (10, 3) + (\overline{10}, 3) + (15, 1) + (15, 3) + (15, 5) + (20', 3) + (45, 1) + (\overline{45}, 1) + (64, 3) \\
672 &= (20, 2) + (\overline{20}, 2) + (20'', 4) + (\overline{20}'', 4) + (36, 2) + (\overline{36}, 2) + (36, 4) + (\overline{36}, 4) \\
768 &= (4, 2) + (\overline{4}, 2) + (20, 2) + (\overline{20}, 2) + (20'', 2) + (\overline{20}'', 2) + (20, 4) + (\overline{20}, 4) + (36, 2) + (\overline{36}, 2) + (36, 4) + (\overline{36}, 4) \\
910 &= (1, 3) + (1, 5) + (1, 7) + 2(6, 3) + (6, 5) + (6, 7) + (15, 1) + (15, 5) + (20', 1) + (20', 3) + (20', 5) + (50, 3) + (64, 3) + (175, 1) \\
924 &= (6, 3) + (10, 1) + (\overline{10}, 1) + (10, 3) + (\overline{10}, 3) + (10, 5) + (\overline{10}, 5) + (15, 1) + 2(15, 3) + (15, 5) + (20', 1) + (45, 3) + (\overline{45}, 3) + (64, 1) + (64, 3) \\
1122 &= (1, 3) + (6, 1) + (1, 7) + (6, 5) + (1, 11) + (6, 9) + (20', 3) + (20', 7) + (50, 1) + (50, 5) + (105, 3) + (196, 1) \\
1650 &= (1, 3) + (6, 1) + (6, 3) + (6, 5) + (10, 3) + (\overline{10}, 3) + (15, 1) + (15, 3) + (15, 5) + (20', 3) + (20', 5) + (45, 3) + (\overline{45}, 3) + (64, 1) + (64, 3) + (64, 5) + (70, 1) + (\overline{70}, 1) + (84, 3) \\
1920 &= (4, 2) + (\overline{4}, 2) + (4, 4) + (\overline{4}, 4) + (4, 6) + (\overline{4}, 6) + (4, 8) + (\overline{4}, 8) + (20, 2) + (\overline{20}, 2) + (20, 4) + (\overline{20}, 4) + (20, 6) + (\overline{20}, 6) + (60, 2) + (\overline{60}, 2) + (60, 4) + (\overline{60}, 4) + (140', 2) + (\overline{140}', 2) \\
1980 &= (1, 1) + (6, 3) + (10, 1) + (\overline{10}, 1) + (15, 3) + (20', 1) + (20', 5) + (35, 1) + (\overline{35}, 1) + (45, 3) + (\overline{45}, 3) + (64, 3) + (64, 5) + (70, 3) + (\overline{70}, 3) + (84, 1) + (84, 5) \\
2457 &= (1, 1) + (1, 5) + 2(6, 3) + (10, 1) + (\overline{10}, 1) + (6, 5) + (6, 7) + (10, 5) + (\overline{10}, 5) + 2(15, 3) + (15, 5) + (20', 1) + (15, 7) + (20', 3) + (20', 5) + (45, 3) + (\overline{45}, 3) + (50, 3) + (64, 1) + (64, 3) + (64, 5) + (126, 1) + (\overline{126}, 1) + (175, 3) \\
2508 &= (1, 1) + (1, 5) + (6, 3) + (1, 9) + (6, 7) + (1, 13) + (6, 11) + (20', 1) + (20', 5) + (20', 9) + (50, 3) + (50, 7) + (105, 1) + (105, 5) + (196, 3) + (336, 1) \\
2560 &= (4, 2) + (\overline{4}, 2) + 2(4, 4) + 2(\overline{4}, 4) + (4, 6) + (\overline{4}, 6) + 2(20, 2) + 2(\overline{20}, 2) + 2(20, 4) + 2(\overline{20}, 4) + (20, 6) + (\overline{20}, 6) + (36, 2) + (\overline{36}, 2) + (36, 4) + (\overline{36}, 4) + (60, 2) + (\overline{60}, 2) + (60, 4) + (\overline{60}, 4) + (140, 2) + (\overline{140}, 2) \\
2574 &= (1, 3) + (1, 5) + (6, 1) + (1, 7) + (6, 3) + 2(6, 5) + (6, 7) + (15, 3) + (15, 5) + 2(20', 3) + (20', 5) + (20', 7) + (50, 1) + (50, 5) + (64, 1) + (64, 3) + (64, 5) + (84, 3) + (175, 3) + (300, 1) \\
2772 &= (6, 3) + (10, 1) + (\overline{10}, 1) + (10, 3) + (\overline{10}, 3) + (10, 5) + (\overline{10}, 5) + (15, 1) + 2(15, 3) + (15, 5) + (20', 3) + (45, 1) + (\overline{45}, 1) + (45, 3) + (\overline{45}, 3) + (45, 5) + (\overline{45}, 5) + (64, 1) + 2(64, 3) + (64, 5) + (70, 3) + (\overline{70}, 3) + (84, 1) + (84, 3) \\
2772' &= (20', 1) + (35, 5) + (\overline{35}, 5) + (45, 3) + (\overline{45}, 3) + (64, 1) + (64, 3) + (70, 3) + (\overline{70}, 3) + (70, 5) + (\overline{70}, 5) + (84, 1) + (84, 3) + (84, 5) \\
2772'' &= (1, 3) + (1, 5) + (6, 1) + (1, 7) + (6, 3) + (1, 9) + 2(6, 5) + (6, 7) + (6, 9) + (15, 3) + (15, 7) + 2(20', 3) + (20', 5) + (20', 7) + (50, 1) + (50, 3) + (50, 5) + (64, 1) + (64, 5) + (105, 3) + (175, 3) + (384, 1) \\
3900 &= (6, 1) + (6, 5) + 2(10, 3) + 2(\overline{10}, 3) + (10, 5) + (\overline{10}, 5) + (15, 1) + (10, 7) + (\overline{10}, 7) + 2(15, 3) + 2(15, 5) + (15, 7) + (20', 3) + (45, 1) + (\overline{45}, 1) + (45, 3) + (\overline{45}, 3) + (45, 5) + (\overline{45}, 5) + (50, 1) + (64, 1) + 2(64, 3) + (64, 5) + (126, 3) + (\overline{126}, 3) + (175, 1) + (175, 3) \\
4004 &= (1, 3) + (1, 7) + (6, 3) + (6, 5) + (6, 7) + (15, 1) + (15, 5) + (20', 3) + (20', 5) + (20', 7) + (50, 3) + (50, 7) + (64, 3) + (64, 5) + (84, 3) + (175, 1) + (175, 5) + (300', 1) + (300, 3) \\
4158 &= (6, 1) + (10, 3) + (\overline{10}, 3) + (15, 1) + (15, 3) + (20', 3) + (35, 3) + (\overline{35}, 3) + (45, 1) + (\overline{45}, 1) + (45, 3) + (\overline{45}, 3) + (45, 5) + (\overline{45}, 5) + (64, 1) + 2(64, 3) + (64, 5) + (70, 1) + (\overline{70}, 1) + (70, 3) + (\overline{70}, 3) + (70, 5) + (\overline{70}, 5) + 2(84, 3) + (84, 5) \\
4608 &= (20, 2) + (\overline{20}, 2) + (20'', 2) + (\overline{20}'', 2) + (20, 4) + (\overline{20}, 4) + (20'', 4) + (\overline{20}'', 4) + (20'', 6) + (\overline{20}'', 6) + 2(36, 2) + 2(\overline{36}, 2) + 2(36, 4) + 2(\overline{36}, 4) + (36, 6) + (\overline{36}, 6) + (60, 2) + (\overline{60}, 2) + (84', 4) + (\overline{84}', 4) + (140, 2) + (\overline{140}, 2) + (140, 4) + (\overline{140}, 4)
\end{aligned}$$

Table A.82: SO(10) Branching Rules

SO(10) $\rightarrow$ SU(5) $\times$ U(1)
<b>10</b> = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2)$
<b>16</b> = $(\mathbf{1})(-5) + (\overline{\mathbf{5}})(3) + (\mathbf{10})(-1)$
<b>45</b> = $(\mathbf{1})(0) + (\mathbf{10})(4) + (\overline{\mathbf{10}})(-4) + (\mathbf{24})(0)$
<b>54</b> = $(\mathbf{15})(4) + (\overline{\mathbf{15}})(-4) + (\mathbf{24})(0)$
<b>120</b> = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2) + (\mathbf{10})(-6) + (\overline{\mathbf{10}})(6) + (\mathbf{45})(2) + (\overline{\mathbf{45}})(-2)$
<b>126</b> = $(\mathbf{1})(10) + (\mathbf{5})(2) + (\overline{\mathbf{10}})(6) + (\mathbf{15})(-6) + (\overline{\mathbf{45}})(-2) + (\mathbf{50})(2)$
<b>144</b> = $(\mathbf{5})(7) + (\overline{\mathbf{5}})(3) + (\mathbf{10})(-1) + (\mathbf{15})(-1) + (\mathbf{24})(-5) + (\mathbf{40})(-1) + (\overline{\mathbf{45}})(3)$
<b>210</b> = $(\mathbf{1})(0) + (\mathbf{5})(-8) + (\overline{\mathbf{5}})(8) + (\mathbf{10})(4) + (\overline{\mathbf{10}})(-4) + (\mathbf{24})(0) + (\mathbf{40})(4) + (\overline{\mathbf{40}})(-4) + (\mathbf{75})(0)$
<b>210'</b> = $(\mathbf{35})(-6) + (\overline{\mathbf{35}})(6) + (\mathbf{70})(2) + (\overline{\mathbf{70}})(-2)$
<b>320</b> = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2) + (\mathbf{40})(-6) + (\overline{\mathbf{40}})(6) + (\mathbf{45})(2) + (\overline{\mathbf{45}})(-2) + (\mathbf{70})(2) + (\overline{\mathbf{70}})(-2)$
<b>560</b> = $(\mathbf{1})(-5) + (\overline{\mathbf{5}})(3) + 2(\mathbf{10})(-1) + (\overline{\mathbf{10}})(-9) + (\mathbf{24})(-5) + (\mathbf{40})(-1) + (\mathbf{45})(7) + (\overline{\mathbf{45}})(3) + (\overline{\mathbf{50}})(3) + (\overline{\mathbf{70}})(3) + (\mathbf{75})(-5) + (\mathbf{175})(-1)$
<b>660</b> = $(\mathbf{70}')(-8) + (\overline{\mathbf{70}}')(8) + (\mathbf{160})(4) + (\overline{\mathbf{160}})(-4) + (\mathbf{200})(0)$
<b>672</b> = $(\mathbf{1})(15) + (\mathbf{5})(7) + (\overline{\mathbf{10}})(11) + (\mathbf{15})(-1) + (\overline{\mathbf{35}})(-9) + (\overline{\mathbf{45}})(3) + (\mathbf{50})(7) + (\mathbf{126})(-5) + (\overline{\mathbf{175}}'')(3) + (\mathbf{210})(-1)$
<b>720</b> = $(\mathbf{15})(-1) + (\overline{\mathbf{15}})(-9) + (\mathbf{24})(-5) + (\mathbf{35})(-1) + (\mathbf{40})(-1) + (\overline{\mathbf{45}})(3) + (\mathbf{70})(7) + (\overline{\mathbf{70}})(3) + (\overline{\mathbf{105}})(3) + (\overline{\mathbf{126}})(-5) + (\mathbf{175})(-1)$
<b>770</b> = $(\mathbf{1})(0) + (\mathbf{10})(4) + (\overline{\mathbf{10}})(-4) + (\mathbf{24})(0) + (\mathbf{50})(-8) + (\overline{\mathbf{50}})(8) + (\mathbf{75})(0) + (\mathbf{175})(4) + (\overline{\mathbf{175}})(-4) + (\mathbf{200})(0)$
<b>945</b> = $(\mathbf{10})(4) + (\overline{\mathbf{10}})(-4) + (\mathbf{15})(4) + (\overline{\mathbf{15}})(-4) + 2(\mathbf{24})(0) + (\mathbf{40})(4) + (\overline{\mathbf{40}})(-4) + (\mathbf{45})(-8) + (\overline{\mathbf{45}})(8) + (\mathbf{75})(0) + (\mathbf{126})(0) + (\overline{\mathbf{126}})(0) + (\mathbf{175})(4) + (\overline{\mathbf{175}})(-4)$
<b>1050</b> = $(\mathbf{5})(12) + (\overline{\mathbf{5}})(8) + (\mathbf{10})(4) + (\mathbf{15})(4) + (\mathbf{24})(0) + (\overline{\mathbf{35}})(-4) + (\mathbf{40})(4) + (\overline{\mathbf{40}})(-4) + (\overline{\mathbf{45}})(8) + (\mathbf{70})(-8) + (\mathbf{75})(0) + (\mathbf{126})(0) + (\overline{\mathbf{175}})(-4) + (\overline{\mathbf{175}}')(0) + (\mathbf{210})(4)$
<b>1200</b> = $(\mathbf{5})(7) + (\overline{\mathbf{5}})(3) + (\mathbf{10})(-1) + (\overline{\mathbf{10}})(11) + (\mathbf{15})(-1) + (\mathbf{24})(-5) + (\mathbf{40})(-1) + (\overline{\mathbf{40}})(-9) + (\mathbf{45})(7) + 2(\overline{\mathbf{45}})(3) + (\mathbf{50})(7) + (\mathbf{75})(-5) + (\mathbf{126})(-5) + (\mathbf{175})(-1) + (\mathbf{210})(-1) + (\overline{\mathbf{280}})(3)$
<b>1386</b> = $(\mathbf{15})(4) + (\overline{\mathbf{15}})(-4) + (\mathbf{24})(0) + (\mathbf{105})(-8) + (\overline{\mathbf{105}})(8) + (\mathbf{126})(0) + (\overline{\mathbf{126}})(0) + (\mathbf{160})(4) + (\overline{\mathbf{160}})(-4) + (\mathbf{175})(4) + (\overline{\mathbf{175}})(-4) + (\mathbf{200})(0)$
<b>1440</b> = $(\mathbf{1})(-5) + (\mathbf{5})(-13) + (\overline{\mathbf{5}})(3) + (\mathbf{10})(-1) + (\overline{\mathbf{10}})(-9) + (\overline{\mathbf{15}})(11) + (\mathbf{24})(-5) + (\mathbf{40})(-1) + (\overline{\mathbf{40}})(-9) + (\mathbf{45})(7) + (\overline{\mathbf{50}})(3) + (\overline{\mathbf{70}})(3) + (\mathbf{75})(-5) + (\mathbf{105})(7) + (\mathbf{175})(-1) + (\overline{\mathbf{175}}')(-5) + (\overline{\mathbf{280}})(3) + (\mathbf{315})(-1)$
<b>1728</b> = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2) + (\mathbf{10})(-6) + (\overline{\mathbf{10}})(6) + (\mathbf{15})(-6) + (\overline{\mathbf{15}})(6) + (\mathbf{24})(10) + (\mathbf{24})(-10) + (\mathbf{40})(-6) + (\overline{\mathbf{40}})(6) + 2(\mathbf{45})(2) + 2(\overline{\mathbf{45}})(-2) + (\mathbf{50})(2) + (\overline{\mathbf{50}})(-2) + (\mathbf{70})(2) + (\overline{\mathbf{70}})(-2) + (\mathbf{105})(2) + (\overline{\mathbf{105}})(-2) + (\mathbf{175})(-6) + (\overline{\mathbf{175}})(6) + (\mathbf{280})(2) + (\overline{\mathbf{280}})(-2)$
<b>1782</b> = $(\mathbf{126}')(10) + (\overline{\mathbf{126}}')(-10) + (\mathbf{315}')(-6) + (\overline{\mathbf{315}}')(6) + (\mathbf{450}')(2) + (\overline{\mathbf{450}}')(-2)$
<b>2640</b> = $(\overline{\mathbf{35}})(-1) + (\mathbf{35})(11) + (\mathbf{70})(7) + (\overline{\mathbf{70}})(3) + (\overline{\mathbf{70}}')(3) + (\overline{\mathbf{105}})(3) + (\overline{\mathbf{126}})(-5) + (\mathbf{160})(-1) + (\overline{\mathbf{160}})(-9) + (\mathbf{175})(-1) + (\mathbf{200})(-5) + (\mathbf{224})(-5) + (\mathbf{280}')(7) + (\mathbf{450})(-1) + (\overline{\mathbf{480}})(3)$
<b>2772</b> = $(\mathbf{1})(20) + (\mathbf{5})(12) + (\overline{\mathbf{10}})(16) + (\mathbf{15})(4) + (\overline{\mathbf{35}})(-4) + (\overline{\mathbf{45}})(8) + (\mathbf{50})(12) + (\overline{\mathbf{70}})(-12) + (\mathbf{126})(0) + (\overline{\mathbf{175}}'')(8) + (\mathbf{210})(4) + (\mathbf{280}')(-8) + (\mathbf{490})(4) + (\overline{\mathbf{560}})(-4) + (\mathbf{700}')(0)$
<b>2970</b> = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2) + (\mathbf{10})(-6) + (\overline{\mathbf{10}})(6) + (\mathbf{40})(-6) + (\overline{\mathbf{40}})(6) + 2(\mathbf{45})(2) + 2(\overline{\mathbf{45}})(-2) + (\mathbf{50})(2) + (\overline{\mathbf{50}})(-2) + (\mathbf{70})(2) + (\overline{\mathbf{70}})(-2) + (\mathbf{75})(10) + (\overline{\mathbf{75}})(-10) + (\mathbf{175})(-6) + (\overline{\mathbf{175}})(6) + (\mathbf{210})(-6) + (\overline{\mathbf{210}})(6) + (\mathbf{280})(2) + (\overline{\mathbf{280}})(-2) + (\mathbf{480})(2) + (\overline{\mathbf{480}})(-2)$

Table A.82: SO(10) Branching Rules (continued)

SO(10) $\rightarrow$ SU(2) $\times$ SU(2) $\times$ SU(4)
10 = (2, 2, 1) + (1, 1, 6)
16 = (2, 1, 4) + (1, 2, $\bar{4}$ )
45 = (3, 1, 1) + (1, 3, 1) + (2, 2, 6) + (1, 1, 15)
54 = (1, 1, 1) + (3, 3, 1) + (2, 2, 6) + (1, 1, 20')
120 = (2, 2, 1) + (3, 1, 6) + (1, 3, 6) + (1, 1, 10) + (1, 1, $\bar{10}$ ) + (2, 2, 15)
126 = (1, 1, 6) + (3, 1, $\bar{10}$ ) + (1, 3, 10) + (2, 2, 15)
144 = (2, 1, 4) + (1, 2, $\bar{4}$ ) + (2, 3, 4) + (3, 2, $\bar{4}$ ) + (2, 1, 20) + (1, 2, $\bar{20}$ )
210 = (1, 1, 1) + (2, 2, 6) + (2, 2, 10) + (2, 2, $\bar{10}$ ) + (1, 1, 15) + (3, 1, 15) + (1, 3, 15)
210' = (2, 2, 1) + (1, 1, 6) + (4, 4, 1) + (3, 3, 6) + (2, 2, 20') + (1, 1, 50)
320 = (2, 2, 1) + (4, 2, 1) + (2, 4, 1) + (1, 1, 6) + (3, 1, 6) + (1, 3, 6) + (3, 3, 6) + (2, 2, 15) + (2, 2, 20') + (1, 1, 64)
560 = (2, 1, 4) + (1, 2, $\bar{4}$ ) + (2, 3, 4) + (3, 2, $\bar{4}$ ) + (4, 1, 4) + (1, 4, $\bar{4}$ ) + (2, 1, 20) + (1, 2, $\bar{20}$ ) + (2, 3, 20) + (3, 2, $\bar{20}$ ) + (2, 1, 36) + (1, 2, $\bar{36}$ )
660 = (1, 1, 1) + (3, 3, 1) + (2, 2, 6) + (5, 5, 1) + (4, 4, 6) + (1, 1, 20') + (3, 3, 20') + (2, 2, 50) + (1, 1, 105)
672 = (2, 1, 20) + (1, 2, $\bar{20}$ ) + (4, 1, 20'') + (1, 4, $\bar{20}''$ ) + (2, 3, 36) + (3, 2, $\bar{36}$ )
720 = (2, 1, 4) + (1, 2, $\bar{4}$ ) + (2, 3, 4) + (3, 2, $\bar{4}$ ) + (4, 3, 4) + (3, 4, $\bar{4}$ ) + (2, 1, 20) + (1, 2, $\bar{20}$ ) + (2, 3, 20) + (3, 2, $\bar{20}$ ) + (2, 1, 60) + (1, 2, $\bar{60}$ )
770 = (1, 1, 1) + (3, 3, 1) + (5, 1, 1) + (1, 5, 1) + (2, 2, 6) + (4, 2, 6) + (2, 4, 6) + (3, 1, 15) + (1, 3, 15) + (1, 1, 20') + (3, 3, 20') + (2, 2, 64) + (1, 1, 84)
945 = (3, 1, 1) + (1, 3, 1) + (3, 3, 1) + 2(2, 2, 6) + (4, 2, 6) + (2, 4, 6) + (2, 2, 10) + (2, 2, $\bar{10}$ ) + (1, 1, 15) + (3, 1, 15) + (1, 3, 15) + (3, 3, 15) + (3, 1, 20') + (1, 3, 20') + (1, 1, 45) + (1, 1, $\bar{45}$ ) + (2, 2, 64)
1050 = (2, 2, 6) + (2, 2, 10) + (2, 2, $\bar{10}$ ) + (4, 2, $\bar{10}$ ) + (2, 4, 10) + (1, 1, 15) + (3, 1, 15) + (1, 3, 15) + (3, 3, 15) + (1, 1, 20') + (3, 1, $\bar{45}$ ) + (1, 3, 45) + (2, 2, 64)
1200 = (2, 1, 4) + (1, 2, $\bar{4}$ ) + (2, 3, 4) + (3, 2, $\bar{4}$ ) + (2, 1, 20) + (1, 2, $\bar{20}$ ) + (2, 1, 20'') + (1, 2, $\bar{20}''$ ) + (2, 3, 20) + (3, 2, $\bar{20}$ ) + (4, 1, 20) + (1, 4, $\bar{20}$ ) + (2, 1, 36) + (1, 2, $\bar{36}$ ) + (2, 3, 36) + (3, 2, $\bar{36}$ )
1386 = (3, 1, 1) + (1, 3, 1) + (3, 3, 1) + (5, 3, 1) + (3, 5, 1) + 2(2, 2, 6) + (4, 2, 6) + (2, 4, 6) + (4, 4, 6) + (1, 1, 15) + (3, 3, 15) + (1, 1, 20') + (3, 1, 20') + (1, 3, 20') + (3, 3, 20') + (2, 2, 50) + (2, 2, 64) + (1, 1, 175)
1440 = (2, 1, 4) + (1, 2, $\bar{4}$ ) + (2, 1, 20) + (1, 2, $\bar{20}$ ) + (2, 3, 20) + (3, 2, $\bar{20}$ ) + (2, 3, 20'') + (3, 2, $\bar{20}''$ ) + (2, 1, 36) + (1, 2, $\bar{36}$ ) + (2, 3, 36) + (3, 2, $\bar{36}$ ) + (4, 1, 36) + (1, 4, $\bar{36}$ )
1728 = (2, 2, 1) + (1, 1, 6) + (3, 1, 6) + (1, 3, 6) + (1, 1, 10) + (1, 1, $\bar{10}$ ) + (3, 3, 6) + (3, 1, 10) + (3, 1, $\bar{10}$ ) + (1, 3, 10) + (1, 3, $\bar{10}$ ) + (3, 3, 10) + (3, 3, $\bar{10}$ ) + 3(2, 2, 15) + (4, 2, 15) + (2, 4, 15) + (2, 2, 20') + (2, 2, 45) + (2, 2, $\bar{45}$ ) + (1, 1, 64) + (3, 1, 64) + (1, 3, 64)
1782 = (2, 2, 1) + (1, 1, 6) + (4, 4, 1) + (3, 3, 6) + (6, 6, 1) + (5, 5, 6) + (2, 2, 20') + (4, 4, 20') + (1, 1, 50) + (3, 3, 50) + (2, 2, 105) + (1, 1, 196)
2640 = (2, 1, 4) + (1, 2, $\bar{4}$ ) + (2, 3, 4) + (3, 2, $\bar{4}$ ) + (4, 3, 4) + (3, 4, $\bar{4}$ ) + (4, 5, 4) + (5, 4, $\bar{4}$ ) + (2, 1, 20) + (1, 2, $\bar{20}$ ) + (2, 3, 20) + (3, 2, $\bar{20}$ ) + (4, 3, 20) + (3, 4, $\bar{20}$ ) + (2, 1, 60) + (1, 2, $\bar{60}$ ) + (2, 3, 60) + (3, 2, $\bar{60}$ ) + (2, 1, 140') + (1, 2, $\bar{140}'$ )
2772 = (1, 1, 20') + (5, 1, $\bar{35}$ ) + (1, 5, 35) + (3, 1, $\bar{45}$ ) + (1, 3, 45) + (2, 2, 64) + (4, 2, $\bar{70}$ ) + (2, 4, 70) + (3, 3, 84)
2970 = (2, 2, 1) + (4, 2, 1) + (2, 4, 1) + (1, 1, 6) + (3, 1, 6) + (1, 3, 6) + 2(3, 3, 6) + (5, 1, 6) + (1, 5, 6) + (3, 1, 10) + (3, 1, $\bar{10}$ ) + (1, 3, 10) + (1, 3, $\bar{10}$ ) + 2(2, 2, 15) + (4, 2, 15) + (2, 4, 15) + (2, 2, 20') + (4, 2, 20') + (2, 4, 20') + (2, 2, 45) + (2, 2, $\bar{45}$ ) + (1, 1, 64) + (3, 1, 64) + (1, 3, 64) + (3, 3, 64) + (1, 1, 70) + (1, 1, $\bar{70}$ ) + (2, 2, 84)

Table A.82: SO(10) Branching Rules (continued)

SO(10) → SO(9)
<b>10</b> = <b>1</b> + <b>9</b>
<b>16</b> = <b>16</b>
<b>45</b> = <b>9</b> + <b>36</b>
<b>54</b> = <b>1</b> + <b>9</b> + <b>44</b>
<b>120</b> = <b>36</b> + <b>84</b>
<b>126</b> = <b>126</b>
<b>144</b> = <b>16</b> + <b>128</b>
<b>210</b> = <b>84</b> + <b>126</b>
<b>210'</b> = <b>1</b> + <b>9</b> + <b>44</b> + <b>156</b>
<b>320</b> = <b>9</b> + <b>36</b> + <b>44</b> + <b>231</b>
<b>560</b> = <b>128</b> + <b>432</b>
<b>660</b> = <b>1</b> + <b>9</b> + <b>44</b> + <b>156</b> + <b>450</b>
<b>672</b> = <b>672</b>
<b>720</b> = <b>16</b> + <b>128</b> + <b>576</b>
<b>770</b> = <b>44</b> + <b>231</b> + <b>495</b>
<b>945</b> = <b>36</b> + <b>84</b> + <b>231</b> + <b>594</b>
<b>1050</b> = <b>126</b> + <b>924</b>
<b>1200</b> = <b>432</b> + <b>768</b>
<b>1386</b> = <b>9</b> + <b>36</b> + <b>44</b> + <b>156</b> + <b>231</b> + <b>910</b>
<b>1440</b> = <b>672</b> + <b>768</b>
<b>1728</b> = <b>84</b> + <b>126</b> + <b>594</b> + <b>924</b>
<b>1782</b> = <b>1</b> + <b>9</b> + <b>44</b> + <b>156</b> + <b>450</b> + <b>1122</b>
<b>2640</b> = <b>16</b> + <b>128</b> + <b>576</b> + <b>1920</b>
<b>2772</b> = <b>2772'</b>
<b>2970</b> = <b>231</b> + <b>495</b> + <b>594</b> + <b>1650</b>

SO(10) → SU(2)×SO(7)
<b>10</b> = <b>(3, 1)</b> + <b>(1, 7)</b>
<b>16</b> = <b>(2, 8)</b>
<b>45</b> = <b>(3, 1)</b> + <b>(3, 7)</b> + <b>(1, 21)</b>
<b>54</b> = <b>(1, 1)</b> + <b>(5, 1)</b> + <b>(3, 7)</b> + <b>(1, 27)</b>
<b>120</b> = <b>(1, 1)</b> + <b>(3, 7)</b> + <b>(3, 21)</b> + <b>(1, 35)</b>
<b>126</b> = <b>(1, 21)</b> + <b>(3, 35)</b>
<b>144</b> = <b>(2, 8)</b> + <b>(4, 8)</b> + <b>(2, 48)</b>
<b>210</b> = <b>(1, 7)</b> + <b>(3, 21)</b> + <b>(1, 35)</b> + <b>(3, 35)</b>
<b>210'</b> = <b>(3, 1)</b> + <b>(1, 7)</b> + <b>(7, 1)</b> + <b>(5, 7)</b> + <b>(3, 27)</b> + <b>(1, 77)</b>
<b>320</b> = <b>(3, 1)</b> + <b>(5, 1)</b> + <b>(1, 7)</b> + <b>(3, 7)</b> + <b>(5, 7)</b> + <b>(3, 21)</b> + <b>(3, 27)</b> + <b>(1, 105)</b>
<b>560</b> = <b>(2, 8)</b> + <b>(4, 8)</b> + <b>(2, 48)</b> + <b>(4, 48)</b> + <b>(2, 112)</b>
<b>660</b> = <b>(1, 1)</b> + <b>(5, 1)</b> + <b>(3, 7)</b> + <b>(9, 1)</b> + <b>(7, 7)</b> + <b>(1, 27)</b> + <b>(5, 27)</b> + <b>(3, 77)</b> + <b>(1, 182)</b>
<b>672</b> = <b>(2, 112)</b> + <b>(4, 112')</b>
<b>720</b> = <b>(2, 8)</b> + <b>(4, 8)</b> + <b>(6, 8)</b> + <b>(2, 48)</b> + <b>(4, 48)</b> + <b>(2, 168)</b>
<b>770</b> = <b>(1, 1)</b> + <b>(5, 1)</b> + <b>(3, 7)</b> + <b>(5, 7)</b> + <b>(3, 21)</b> + <b>(1, 27)</b> + <b>(5, 27)</b> + <b>(3, 105)</b> + <b>(1, 168')</b>
<b>945</b> = <b>(3, 1)</b> + <b>(1, 7)</b> + <b>(3, 7)</b> + <b>(5, 7)</b> + <b>(1, 21)</b> + <b>(3, 21)</b> + <b>(5, 21)</b> + <b>(3, 27)</b> + <b>(3, 35)</b> + <b>(3, 105)</b> + <b>(1, 189)</b>
<b>1050</b> = <b>(3, 21)</b> + <b>(1, 35)</b> + <b>(3, 35)</b> + <b>(5, 35)</b> + <b>(1, 105)</b> + <b>(3, 189)</b>
<b>1200</b> = <b>(2, 8)</b> + <b>(2, 48)</b> + <b>(4, 48)</b> + <b>(2, 112)</b> + <b>(2, 112')</b> + <b>(4, 112)</b>

Table A.82: SO(10) Branching Rules (continued)

1386	= (3, 1) + (5, 1) + (7, 1) + 2(3, 7) + (5, 7) + (7, 7) + (1, 21) + (5, 21) + (1, 27) + (3, 27) + (5, 27) + (3, 77) + (3, 105) + (1, 330)
1440	= (2, 48) + (2, 112) + (2, 112') + (4, 112) + (4, 112')
1728	= (3, 7) + (1, 21) + (3, 21) + (5, 21) + (1, 27) + (1, 35) + 2(3, 35) + (5, 35) + (3, 105) + (1, 189) + (3, 189)
1782	= (3, 1) + (1, 7) + (7, 1) + (5, 7) + (11, 1) + (9, 7) + (3, 27) + (7, 27) + (1, 77) + (5, 77) + (3, 182) + (1, 378')
2640	= (2, 8) + (4, 8) + (6, 8) + (8, 8) + (2, 48) + (4, 48) + (6, 48) + (2, 168) + (4, 168) + (2, 448)
2772	= (1, 168') + (5, 294) + (3, 378)
2970	= (3, 1) + (1, 7) + (3, 7) + (5, 7) + (1, 21) + (3, 21) + (5, 21) + (3, 27) + (5, 27) + (3, 35) + (1, 105) + (3, 105) + (5, 105) + (3, 168') + (3, 189) + (1, 378)

Table A.83: SO(14) Branching Rules

SO(14) → SU(2) × SU(2) × SO(10)	
14	= (2, 2, 1) + (1, 1, 10)
64	= (2, 1, 16) + (1, 2, $\overline{16}$ )
91	= (3, 1, 1) + (1, 3, 1) + (2, 2, 10) + (1, 1, 45)
104	= (1, 1, 1) + (3, 3, 1) + (2, 2, 10) + (1, 1, 54)
364	= (2, 2, 1) + (3, 1, 10) + (1, 3, 10) + (2, 2, 45) + (1, 1, 120)
546	= (2, 2, 1) + (4, 4, 1) + (1, 1, 10) + (3, 3, 10) + (2, 2, 54) + (1, 1, 210')
832	= (2, 1, 16) + (1, 2, $\overline{16}$ ) + (2, 3, 16) + (3, 2, $\overline{16}$ ) + (2, 1, 144) + (1, 2, $\overline{144}$ )
896	= (2, 2, 1) + (4, 2, 1) + (2, 4, 1) + (1, 1, 10) + (3, 1, 10) + (1, 3, 10) + (3, 3, 10) + (2, 2, 45) + (2, 2, 54) + (1, 1, 320)
1001	= (1, 1, 1) + (2, 2, 10) + (3, 1, 45) + (1, 3, 45) + (2, 2, 120) + (1, 1, 210)
1716	= (1, 1, 120) + (3, 1, $\overline{126}$ ) + (1, 3, 126) + (2, 2, 210)
2002	= (1, 1, 10) + (2, 2, 45) + (3, 1, 120) + (1, 3, 120) + (1, 1, 126) + (1, 1, $\overline{126}$ ) + (2, 2, 210)
2275	= (1, 1, 1) + (3, 3, 1) + (5, 5, 1) + (2, 2, 10) + (4, 4, 10) + (1, 1, 54) + (3, 3, 54) + (2, 2, 210') + (1, 1, 660)
3003	= (1, 1, 45) + (2, 2, 120) + (2, 2, 126) + (2, 2, $\overline{126}$ ) + (1, 1, 210) + (3, 1, 210) + (1, 3, 210)
3080	= (1, 1, 1) + (3, 3, 1) + (5, 1, 1) + (1, 5, 1) + (2, 2, 10) + (4, 2, 10) + (2, 4, 10) + (3, 1, 45) + (1, 3, 45) + (1, 1, 54) + (3, 3, 54) + (2, 2, 320) + (1, 1, 770)
4004	= (3, 1, 1) + (1, 3, 1) + (3, 3, 1) + 2(2, 2, 10) + (4, 2, 10) + (2, 4, 10) + (1, 1, 45) + (3, 1, 45) + (1, 3, 45) + (3, 3, 45) + (3, 1, 54) + (1, 3, 54) + (2, 2, 120) + (2, 2, 320) + (1, 1, 945)
4928	= (2, 1, 16) + (1, 2, $\overline{16}$ ) + (2, 3, 16) + (3, 2, $\overline{16}$ ) + (4, 1, 16) + (1, 4, $\overline{16}$ ) + (2, 1, 144) + (1, 2, $\overline{144}$ ) + (2, 3, 144) + (3, 2, $\overline{144}$ ) + (2, 1, 560) + (1, 2, $\overline{560}$ )
5265	= (3, 1, 1) + (1, 3, 1) + (3, 3, 1) + (5, 3, 1) + (3, 5, 1) + 2(2, 2, 10) + (4, 2, 10) + (2, 4, 10) + (4, 4, 10) + (1, 1, 45) + (3, 3, 45) + (1, 1, 54) + (3, 1, 54) + (1, 3, 54) + (3, 3, 54) + (2, 2, 210') + (2, 2, 320) + (1, 1, 1386)
5824	= (2, 1, 16) + (1, 2, $\overline{16}$ ) + (2, 3, 16) + (3, 2, $\overline{16}$ ) + (4, 3, 16) + (3, 4, $\overline{16}$ ) + (2, 1, 144) + (1, 2, $\overline{144}$ ) + (2, 3, 144) + (3, 2, $\overline{144}$ ) + (2, 1, 720) + (1, 2, $\overline{720}$ )

Table A.84: SO(18) Branching Rules

SO(18) $\rightarrow$ SO(8) $\times$ SO(10)
<b>18</b> = $(8_v, 1) + (1, 10)$
<b>153</b> = $(8_v, 10) + (28, 1) + (1, 45)$
<b>170</b> = $(1, 1) + (8_v, 10) + (35_v, 1) + (1, 54)$
<b>256</b> = $(8_s, \overline{16}) + (8_c, 16)$
<b>816</b> = $(28, 10) + (8_v, 45) + (56_v, 1) + (1, 120)$
<b>1122</b> = $(8_v, 1) + (1, 10) + (35_v, 10) + (8_v, 54) + (112_v, 1) + (1, 210')$
<b>1920</b> = $(8_v, 1) + (1, 10) + (28, 10) + (35_v, 10) + (8_v, 45) + (8_v, 54) + (160_v, 1) + (1, 320)$
<b>3060</b> = $(35_c, 1) + (35_s, 1) + (56_v, 10) + (28, 45) + (8_v, 120) + (1, 210)$
<b>4352</b> = $(8_s, \overline{16}) + (8_c, 16) + (56_s, \overline{16}) + (56_c, 16) + (8_s, \overline{144}) + (8_c, 144)$

Table A.85: SO(22) Branching Rules

SO(22) $\rightarrow$ SO(12) $\times$ SO(10)
<b>22</b> = $(1, 10) + (12, 1)$
<b>231</b> = $(12, 10) + (1, 45) + (66, 1)$
<b>252</b> = $(1, 1) + (12, 10) + (1, 54) + (77, 1)$
<b>1024</b> = $(\overline{32}, 16) + (32, \overline{16})$
<b>1540</b> = $(12, 45) + (66, 10) + (1, 120) + (220, 1)$
<b>3520</b> = $(1, 10) + (12, 1) + (12, 45) + (12, 54) + (66, 10) + (77, 10) + (1, 320) + (560, 1)$
<b>7315</b> = $(66, 45) + (12, 120) + (1, 210) + (220, 10) + (495, 1)$

Table A.86: SO(26) Branching Rules

SO(26) $\rightarrow$ SO(16) $\times$ SO(10)
<b>26</b> = $(1, 10) + (16, 1)$
<b>325</b> = $(16, 10) + (1, 45) + (120, 1)$
<b>2600</b> = $(16, 45) + (1, 120) + (120, 10) + (560, 1)$
<b>4096</b> = $(128, \overline{16}) + (\overline{128}, 16)$
<b>5824</b> = $(1, 10) + (16, 1) + (16, 45) + (16, 54) + (120, 10) + (135, 10) + (1, 320) + (1344, 1)$
<b>14950</b> = $(16, 120) + (120, 45) + (1, 210) + (560, 10) + (1820, 1)$
<b>52325</b> = $(16, 10) + (1, 45) + (120, 1) + (16, 120) + (120, 45) + (120, 54) + (135, 45) + (16, 320) + (560, 10) + (1, 945) + (1344, 10) + (7020, 1)$
<b>65780</b> = $(1, 126) + (1, \overline{126}) + (16, 210) + (120, 120) + (560, 45) + (1820, 10) + (4368, 1)$



A.3.3. Exceptional Algebras

Table A.87:  $E_6$  Branching Rules

$E_6 \rightarrow SO(10) \times U(1)$
$27 = (1)(-4) + (10)(2) + (16)(-1)$
$78 = (1)(0) + (16)(3) + (\overline{16})(-3) + (45)(0)$
$351 = (10)(2) + (\overline{16})(5) + (16)(-1) + (45)(-4) + (120)(2) + (144)(-1)$
$351' = (1)(8) + (10)(2) + (\overline{16})(5) + (54)(-4) + (126)(2) + (144)(-1)$
$650 = (1)(0) + (10)(6) + (10)(-6) + (16)(3) + (\overline{16})(-3) + (45)(0) + (54)(0) + (144)(3) + (\overline{144})(-3) + (210)(0)$
$1728 = (1)(-4) + (10)(2) + 2(16)(-1) + (\overline{16})(-7) + (45)(-4) + (120)(2) + (\overline{126})(2) + (\overline{144})(5) + (144)(-1) + (210)(-4) + (320)(2) + (560)(-1)$
$2430 = (1)(0) + (16)(3) + (\overline{16})(-3) + (45)(0) + (126)(-6) + (\overline{126})(6) + (210)(0) + (560)(3) + (\overline{560})(-3) + (770)(0)$
$2925 = (16)(3) + (\overline{16})(-3) + 2(45)(0) + (120)(6) + (120)(-6) + (144)(3) + (\overline{144})(-3) + (210)(0) + (560)(3) + (\overline{560})(-3) + (945)(0)$
$3003 = (1)(-12) + (10)(-6) + (16)(-9) + (54)(0) + (\overline{126})(-6) + (\overline{144})(-3) + (210')(6) + (\overline{672})(-3) + (720)(3) + (\overline{1050})(0)$
$5824 = (10)(-6) + (\overline{16})(-3) + (16)(-9) + (45)(0) + (54)(0) + (120)(-6) + (\overline{126})(-6) + (144)(3) + 2(\overline{144})(-3) + (210)(0) + (320)(6) + (560)(3) + (720)(3) + (945)(0) + (\overline{1050})(0) + (\overline{1200})(-3)$
$7371 = (10)(2) + (\overline{16})(5) + (16)(-1) + (45)(8) + (45)(-4) + (54)(-4) + 2(120)(2) + (126)(2) + (\overline{144})(5) + 2(\overline{144})(-1) + (\overline{144})(-7) + (210)(-4) + (320)(2) + (\overline{560})(5) + (560)(-1) + (720)(-1) + (945)(-4) + (\overline{1200})(-1) + (1728)(2)$
$7722 = (1)(-4) + (10)(2) + (10)(-10) + (16)(-1) + (\overline{16})(-7) + (45)(-4) + (54)(8) + (54)(-4) + (\overline{126})(2) + (\overline{144})(5) + (\overline{144})(-1) + (\overline{144})(-7) + (210)(-4) + (210')(2) + (320)(2) + (560)(-1) + (\overline{720})(5) + (\overline{720})(-1) + (\overline{1050})(-4) + (\overline{1440})(-1) + (1728)(2)$
$17550 = (10)(2) + (\overline{16})(5) + (16)(-1) + (45)(-4) + 2(120)(2) + (126)(2) + (\overline{126})(2) + (\overline{144})(5) + 2(\overline{144})(-1) + (210)(8) + (210)(-4) + (320)(2) + (\overline{560})(5) + 2(560)(-1) + (\overline{560})(-7) + (\overline{770})(-4) + (945)(-4) + (\overline{1050})(-4) + (\overline{1200})(5) + (\overline{1200})(-1) + (1728)(2) + (\overline{2970})(2) + (\overline{3696})(-1)$
$E_6 \rightarrow SU(6) \times SU(2)$
$27 = (6, 2) + (\overline{15}, 1)$
$78 = (1, 3) + (20, 2) + (35, 1)$
$351 = (6, 2) + (\overline{15}, 3) + (\overline{21}, 1) + (84, 2) + (\overline{105}, 1)$
$351' = (\overline{15}, 1) + (\overline{21}, 3) + (84, 2) + (\overline{105}', 1)$
$650 = (1, 1) + (20, 2) + (35, 1) + (35, 3) + (70, 2) + (\overline{70}, 2) + (189, 1)$
$1728 = (6, 2) + (6, 4) + (\overline{15}, 1) + (\overline{15}, 3) + (84, 2) + (\overline{105}, 1) + (\overline{105}, 3) + (120, 2) + (210, 2) + (\overline{384}, 1)$
$2430 = (1, 1) + (1, 5) + (20, 2) + (20, 4) + (35, 3) + (175, 3) + (189, 1) + (405, 1) + (540, 2)$
$2925 = (1, 3) + (20, 2) + (20, 4) + (35, 1) + (35, 3) + (70, 2) + (\overline{70}, 2) + (175, 1) + (189, 3) + (280, 1) + (\overline{280}, 1) + (540, 2)$
$3003 = (56, 4) + (70, 2) + (189, 1) + (280, 3) + (\overline{490}, 1) + (560, 2)$
$5824 = (20, 2) + (35, 1) + (35, 3) + (56, 2) + (70, 2) + (\overline{70}, 2) + (70, 4) + (189, 1) + (189, 3) + (280, 1) + (280, 3) + (540, 2) + (560, 2) + (896, 1)$
$7371 = (6, 2) + (\overline{15}, 1) + (\overline{15}, 3) + (\overline{21}, 1) + (\overline{21}, 3) + 2(84, 2) + (84, 4) + (\overline{105}, 1) + (\overline{105}, 3) + (\overline{105}', 3) + (120, 2) + (\overline{210}, 1) + (210, 2) + (336, 2) + (\overline{384}, 1) + (\overline{384}, 3) + (840, 2) + (\overline{1050}, 1)$
$7722 = (6, 2) + (\overline{15}, 1) + (84, 2) + (\overline{105}, 1) + (\overline{105}', 1) + (\overline{105}, 3) + (120, 2) + (120, 4) + (210, 2) + (\overline{210}', 3) + (\overline{384}, 1) + (\overline{384}, 3) + (420, 2) + (840, 2) + (\overline{1176}, 1)$
$17550 = (6, 2) + (6, 4) + (\overline{15}, 1) + (\overline{15}, 3) + (\overline{15}, 5) + (\overline{21}, 3) + 2(84, 2) + (84, 4) + (\overline{105}, 1) + (\overline{105}', 1) + 2(\overline{105}, 3) + (120, 2) + (210, 2) + (210, 4) + (\overline{315}, 1) + (336, 2) + (\overline{384}, 1) + (\overline{384}, 3) + (840, 2) + (840', 2) + (\overline{1050}, 1) + (\overline{1050}, 3) + (1260, 2) + (\overline{1701}, 1)$



Table A.88: E<sub>7</sub> Branching Rules

E <sub>7</sub> → SU(8)	
56	= 28 + $\overline{28}$
133	= 63 + 70
912	= 36 + $\overline{36}$ + 420 + $\overline{420}$
1463	= 1 + 70 + 336 + $\overline{336}$ + 720
1539	= 63 + 378 + $\overline{378}$ + 720
6480	= 28 + $\overline{28}$ + 420 + $\overline{420}$ + 1280 + $\overline{1280}$ + 1512 + $\overline{1512}$
7371	= 1 + 70 + 720 + 1232 + 1764 + 3584
8645	= 63 + 378 + $\overline{378}$ + 945 + $\overline{945}$ + 2352 + 3584
24320	= 28 + $\overline{28}$ + 1512 + $\overline{1512}$ + 2520' + $\overline{2520'}$ + 8100 + $\overline{8100}$
27664	= 36 + $\overline{36}$ + 420 + $\overline{420}$ + 1176 + $\overline{1176}$ + 1280 + $\overline{1280}$ + 2100 + $\overline{2100}$ + 8820 + $\overline{8820}$
E <sub>7</sub> → SO(12)×SU(2)	
56	= (12, 2) + (32, 1)
133	= (1, 3) + ( $\overline{32}$ , 2) + (66, 1)
912	= (12, 2) + (32, 3) + (220, 2) + (352, 1)
1463	= (66, 1) + (77, 3) + ( $\overline{352}$ , 2) + (462, 1)
1539	= (1, 1) + ( $\overline{32}$ , 2) + (66, 3) + (77, 1) + ( $\overline{352}$ , 2) + (495, 1)
6480	= (12, 2) + (12, 4) + (32, 1) + (32, 3) + (220, 2) + (352, 1) + (352, 3) + (560, 2) + (792, 2) + (1728, 1)
7371	= (1, 1) + (1, 5) + ( $\overline{32}$ , 2) + ( $\overline{32}$ , 4) + (66, 3) + ( $\overline{462}$ , 3) + (495, 1) + (1638, 1) + ( $\overline{1728}$ , 2)
8645	= (1, 3) + ( $\overline{32}$ , 2) + ( $\overline{32}$ , 4) + (66, 1) + (66, 3) + ( $\overline{352}$ , 2) + ( $\overline{462}$ , 1) + (495, 3) + ( $\overline{1728}$ , 2) + (2079, 1)
24320	= ( $\overline{352'}$ , 4) + (560, 2) + (1728, 1) + (2112, 3) + (4224, 1) + (4752, 2)
27664	= (12, 2) + (32, 1) + (32, 3) + (220, 2) + (220, 4) + (352, 1) + (352, 3) + (560, 2) + (792, 2) + (1728, 3) + (2112, 1) + ( $\overline{4928'}$ , 1) + (4928, 2)
E <sub>7</sub> → SU(6)×SU(3)	
56	= (6, 3) + ( $\overline{6}$ , $\overline{3}$ ) + (20, 1)
133	= (1, 8) + (15, $\overline{3}$ ) + ( $\overline{15}$ , 3) + (35, 1)
912	= (6, 3) + ( $\overline{6}$ , $\overline{3}$ ) + (6, $\overline{6}$ ) + ( $\overline{6}$ , 6) + (20, 8) + (70, 1) + ( $\overline{70}$ , 1) + (84, 3) + ( $\overline{84}$ , $\overline{3}$ )
1463	= (1, 1) + (15, $\overline{3}$ ) + ( $\overline{15}$ , 3) + (21, 6) + ( $\overline{21}$ , $\overline{6}$ ) + (35, 1) + (35, 8) + (105, $\overline{3}$ ) + ( $\overline{105}$ , 3) + (175, 1)
1539	= (1, 1) + (1, 8) + (15, $\overline{3}$ ) + ( $\overline{15}$ , 3) + (15, 6) + ( $\overline{15}$ , $\overline{6}$ ) + (21, $\overline{3}$ ) + ( $\overline{21}$ , 3) + (35, 1) + (35, 8) + (105, $\overline{3}$ ) + ( $\overline{105}$ , 3) + (189, 1)
6480	= 2(6, 3) + 2( $\overline{6}$ , $\overline{3}$ ) + (6, $\overline{6}$ ) + ( $\overline{6}$ , 6) + 2(20, 1) + (6, 15) + ( $\overline{6}$ , $\overline{15}$ ) + 2(20, 8) + (70, 1) + ( $\overline{70}$ , 1) + (70, 8) + ( $\overline{70}$ , 8) + 2(84, 3) + 2( $\overline{84}$ , $\overline{3}$ ) + (84, $\overline{6}$ ) + ( $\overline{84}$ , 6) + (120, 3) + ( $\overline{120}$ , $\overline{3}$ ) + (210, 3) + ( $\overline{210}$ , $\overline{3}$ ) + (540, 1)
7371	= (1, 1) + (1, 8) + (15, $\overline{3}$ ) + ( $\overline{15}$ , 3) + (15, 6) + ( $\overline{15}$ , $\overline{6}$ ) + (1, 27) + (15, $\overline{15}$ ) + ( $\overline{15}$ , 15) + (35, 1) + (35, 8) + (105, $\overline{3}$ ) + ( $\overline{105}$ , 3) + (105', 6) + ( $\overline{105'}$ , $\overline{6}$ ) + (189, 1) + (189, 8) + (384, $\overline{3}$ ) + ( $\overline{384}$ , 3) + (405, 1)
8645	= (1, 1) + (1, 8) + (1, 10) + (1, $\overline{10}$ ) + 2(15, $\overline{3}$ ) + 2( $\overline{15}$ , 3) + (15, 6) + ( $\overline{15}$ , $\overline{6}$ ) + (21, $\overline{3}$ ) + ( $\overline{21}$ , 3) + (15, $\overline{15}$ ) + ( $\overline{15}$ , 15) + (35, 1) + 2(35, 8) + (105, $\overline{3}$ ) + ( $\overline{105}$ , 3) + (105', $\overline{3}$ ) + ( $\overline{105'}$ , 3) + (105, 6) + ( $\overline{105}$ , $\overline{6}$ ) + (189, 1) + (189, 8) + (280, 1) + ( $\overline{280}$ , 1) + (384, $\overline{3}$ ) + ( $\overline{384}$ , 3)
24320	= (6, 3) + ( $\overline{6}$ , $\overline{3}$ ) + 2(20, 1) + (56, 10) + ( $\overline{56}$ , $\overline{10}$ ) + (70, 8) + ( $\overline{70}$ , 8) + (84, 3) + ( $\overline{84}$ , $\overline{3}$ ) + (84, $\overline{6}$ ) + ( $\overline{84}$ , 6) + (120, 3) + ( $\overline{120}$ , $\overline{3}$ ) + (120, 15) + ( $\overline{120}$ , $\overline{15}$ ) + (210, 3) + ( $\overline{210}$ , $\overline{3}$ ) + (336, $\overline{6}$ ) + ( $\overline{336}$ , 6) + (540, 1) + (540, 8) + (840', 3) + ( $\overline{840'}$ , $\overline{3}$ ) + (980, 1)
27664	= 2(6, 3) + 2( $\overline{6}$ , $\overline{3}$ ) + 2(6, $\overline{6}$ ) + 2( $\overline{6}$ , 6) + 2(20, 1) + (6, 15) + ( $\overline{6}$ , $\overline{15}$ ) + 2(20, 8) + (20, 10) + (20, $\overline{10}$ ) + (56, 1) + ( $\overline{56}$ , 1) + (70, 1) + ( $\overline{70}$ , 1) + 2(70, 8) + 2( $\overline{70}$ , 8) + 3(84, 3) + 3( $\overline{84}$ , $\overline{3}$ ) + (84, $\overline{6}$ ) + ( $\overline{84}$ , 6) + (84, 15) + ( $\overline{84}$ , $\overline{15}$ ) + (120, 3) + ( $\overline{120}$ , $\overline{3}$ ) + (120, $\overline{6}$ ) + ( $\overline{120}$ , 6) + (210, 3) + ( $\overline{210}$ , $\overline{3}$ ) + (210, $\overline{6}$ ) + ( $\overline{210}$ , 6) + (336, 3) + ( $\overline{336}$ , $\overline{3}$ ) + (540, 1) + (540, 8) + (560, 1) + ( $\overline{560}$ , 1) + (840, 3) + ( $\overline{840}$ , $\overline{3}$ )

Table A.89:  $E_8$  Branching Rules

$E_8 \rightarrow SO(16)$	
248	$= 120 + 128$
3875	$= 135 + 1820 + 1920$
27000	$= 1 + 128 + 1820 + 5304 + \overline{6435} + 13312$
30380	$= 120 + 1920 + 7020 + 8008 + 13312$
$E_8 \rightarrow SU(9)$	
248	$= 80 + 84 + \overline{84}$
3875	$= 80 + 240 + \overline{240} + 1050 + \overline{1050} + 1215$
27000	$= 1 + 80 + 84 + \overline{84} + 1050 + \overline{1050} + 1215 + 1944 + 2520 + \overline{2520} + 5346 + \overline{5346} + 5760$
30380	$= 1 + 80 + 84 + \overline{84} + 240 + \overline{240} + 1050 + \overline{1050} + 1215 + 1540 + \overline{1540} + 3402 + \overline{3402} + 5346 + \overline{5346} + 5760$
$E_8 \rightarrow E_7 \times SU(2)$	
248	$= (1, 3) + (56, 2) + (133, 1)$
3875	$= (1, 1) + (56, 2) + (133, 3) + (912, 2) + (1539, 1)$
27000	$= (1, 1) + (1, 5) + (56, 2) + (56, 4) + (133, 3) + (1463, 3) + (1539, 1) + (6480, 2) + (7371, 1)$
30380	$= (1, 3) + (56, 2) + (56, 4) + (133, 1) + (133, 3) + (912, 2) + (1463, 1) + (1539, 3) + (6480, 2) + (8645, 1)$
$E_8 \rightarrow E_6 \times SU(3)$	
248	$= (1, 8) + (27, 3) + (\overline{27}, \overline{3}) + (78, 1)$
3875	$= (1, 1) + (1, 8) + (27, 3) + (\overline{27}, \overline{3}) + (27, \overline{6}) + (\overline{27}, 6) + (78, 8) + (351, 3) + (\overline{351}, \overline{3}) + (650, 1)$
27000	$= (1, 1) + (1, 8) + (1, 27) + (27, 3) + (\overline{27}, \overline{3}) + (27, \overline{6}) + (\overline{27}, 6) + (27, 15) + (\overline{27}, \overline{15}) + (78, 1) + (78, 8) + (351, 3) + (\overline{351}, \overline{3}) + (351', 6) + (\overline{351}', 6) + (650, 1) + (650, 8) + (1728, 3) + (\overline{1728}, \overline{3}) + (2430, 1)$
30380	$= (1, 1) + (1, 8) + (1, 10) + (1, \overline{10}) + 2(27, 3) + 2(\overline{27}, \overline{3}) + (27, \overline{6}) + (\overline{27}, 6) + (27, 15) + (\overline{27}, \overline{15}) + (78, 1) + 2(78, 8) + (351, 3) + (\overline{351}, \overline{3}) + (351', 3) + (\overline{351}', \overline{3}) + (351, \overline{6}) + (\overline{351}, 6) + (650, 1) + (650, 8) + (1728, 3) + (\overline{1728}, \overline{3}) + (2925, 1)$
$E_8 \rightarrow SU(5) \times SU(5)$	
248	$= (5, 10) + (\overline{10}, 5) + (10, \overline{5}) + (\overline{5}, \overline{10}) + (24, 1) + (1, 24)$
3875	$= (1, 1) + (5, 10) + (\overline{10}, 5) + (10, \overline{5}) + (\overline{5}, \overline{10}) + (5, 15) + (\overline{15}, 5) + (15, \overline{5}) + (\overline{5}, \overline{15}) + (24, 1) + (1, 24) + (5, 40) + (\overline{40}, 5) + (40, \overline{5}) + (\overline{5}, 40) + (24, 24) + (45, 10) + (\overline{10}, 45) + (10, 45) + (45, 10) + (75, 1) + (1, 75)$
27000	$= 2(1, 1) + 2(5, 10) + 2(\overline{10}, 5) + 2(10, \overline{5}) + 2(\overline{5}, \overline{10}) + (5, 15) + (\overline{15}, 5) + (15, \overline{5}) + (\overline{5}, \overline{15}) + 2(24, 1) + 2(1, 24) + (5, 40) + (\overline{40}, 5) + (40, \overline{5}) + (\overline{5}, 40) + 2(24, 24) + 2(45, 10) + 2(\overline{10}, 45) + 2(10, 45) + 2(45, \overline{10}) + (50, 15) + (\overline{15}, 50) + (15, \overline{50}) + (\overline{50}, \overline{15}) + (75, 1) + (1, 75) + (70, 10) + (\overline{10}, 70) + (10, \overline{70}) + (\overline{70}, \overline{10}) + (45, 40) + (\overline{40}, 45) + (40, 45) + (45, 40) + (24, 75) + (75, 24) + (5, 175) + (\overline{175}, 5) + (175, \overline{5}) + (\overline{5}, \overline{175}) + (200, 1) + (1, 200)$
30380	$= 2(1, 1) + 3(5, 10) + 3(\overline{10}, 5) + 3(10, \overline{5}) + 3(\overline{5}, \overline{10}) + (5, 15) + (\overline{15}, 5) + (15, \overline{5}) + (\overline{5}, \overline{15}) + 2(24, 1) + 2(1, 24) + 2(5, 40) + 2(\overline{40}, 5) + 2(40, \overline{5}) + 2(\overline{5}, 40) + 3(24, 24) + 2(45, 10) + 2(\overline{10}, 45) + 2(10, 45) + 2(45, \overline{10}) + (45, 15) + (\overline{15}, 45) + (15, 45) + (45, \overline{15}) + (50, 10) + (\overline{10}, 50) + (10, 50) + (\overline{50}, \overline{10}) + (75, 1) + (1, 75) + (70, 10) + (\overline{10}, 70) + (10, \overline{70}) + (\overline{70}, \overline{10}) + (45, 40) + (\overline{40}, 45) + (40, 45) + (\overline{45}, 40) + (24, 75) + (75, 24) + (126, 1) + (\overline{126}, 1) + (1, 126) + (1, \overline{126}) + (5, 175) + (\overline{175}, 5) + (175, \overline{5}) + (\overline{5}, \overline{175})$

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