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Lie systems: theory, generalisations, and applications

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Abstract

Lie systems form a class of systems of first-order ordinary differential equations whose general solutions can be described in terms of certain finite families of particular solutions and a set of constants, by means of a particular type of mapping: the so-called superposition rule. Apart from this fundamental property, Lie systems enjoy many other geometrical features and they appear in multiple branches of Mathematics and Physics, which strongly motivates their study. These facts, together with the authors' recent findings in the theory of Lie systems, led to the redaction of this essay, which aims to describe such new achievements within a self-contained guide to the whole theory of Lie systems, their generalisations, and applications.

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1. The theory of Lie systems

1.1. Motivation and general scheme of the work. It is a little bit surprising that the theory of *Lie systems* [153, 154, 157, 224], which studies a very specific class of systems of first-order ordinary differential equations, can be employed to investigate a large variety of topics [8, 12, 53, 55, 59, 98, 144, 202, 212]. Indeed, although being a Lie system is rather more an exception than a rule [128], these equations frequently turn up in multiple branches of Mathematics and Physics. For instance, linear systems of first-order differential equations, Riccati equations [86], and matrix Riccati equations [103, 116, 117, 131] are Lie systems that very frequently appear in the literature [62, 98, 112, 141, 207, 212, 234]. This obviously motivates the study of the theory of Lie systems as a means to investigate the properties of various remarkable differential equations and their corresponding applications.

The research on Lie systems involves the analysis of multiple interesting geometric and algebraic problems. For example, the determination of the Lie systems defined in a fixed manifold is related to the existence of finite-dimensional Lie algebras of vector fields over such a manifold [157, 210]. Furthermore, the study of Lie systems leads to the investigation of foliations [35], generalised distributions [38], Lie group actions [141], finite-dimensional Lie algebras [40, 157, 210], etc. As a result of the analysis of the former themes, Lie systems provide methods to study the integrability of systems of first-order differential equations [40], Control Theory [32, 61, 79, 187], geometric phases [98], certain problems in Quantum Mechanics [46, 51], and other topics. Finally, it is remarkable that the theory of Lie systems has been investigated by means of different techniques and approaches, like Galois theory [17, 19] or Differential Geometry [38, 60, 186, 220].

When applying Lie systems to study more general systems of differential equations than merely first-order ones (see for instance [34, 35, 52, 77, 202]), the interest of their analysis becomes even more evident. For example, in the research on systems of second-order differential equations, which very frequently appear in Classical Mechanics, various relevant differential equations can be studied by means of Lie systems. Dissipative Milne–Pinney equations [45], Milne–Pinney equations [52], Caldirola–Kanai oscillators [54], t -dependent frequency harmonic oscillators [55], or second-order Riccati equations [48, 225], are just some examples of such systems of second-order differential equations that have already been analysed successfully through Lie systems.

The relevance of the above studies, along with the determination of new applications of Lie systems, is twofold. On one hand, they allow us to obtain novel results about interesting differential equations. On the other hand, such examples may show us new features or generalisations of the notions appearing in the theory of Lie systems that were not previously determined. Let us

briefly provide a case in point. While studying second-order differential equations by means of Lie systems [52, 53, 202], a new type of ‘superposition-like’ expression describing the general solution of certain systems of second-order differential equations appeared. These essays led to the definition of a possible superposition rule notion for such systems whose main properties are still under analysis [48]. In addition, these works carried out different approaches to analyse second-order differential equations: by means of the SODE Lie system notion [52] and through regular Lagrangians [54]. The relations between these approaches or even the existence of new approaches is still an open question that must be investigated in detail [48].

Apart from the investigation of the above open problems, perhaps the most active field of research into Lie systems is concerned with the development of new generalisations of the Lie system and superposition rule notions. Quasi-Lie systems [34, 35, 42], t -dependent superposition rules [34], PDE Lie systems [38, 172], SODE Lie systems [52], partial superposition rules [38, 153], quantum Lie systems [60], or stochastic Lie–Scheffers systems [144] are just a few generalisations of such concepts that have been carried out in order to analyse non-Lie systems with techniques similar to those ones developed for analysing Lie systems. Indeed, the list of generalisations is much larger and even sometimes the superposition rule term has been used with different, non-equivalent, meanings [198, 215].

In view of the above and many other reasons, the theory of Lie systems, along with its multiple generalisations, can be regarded as a multidisciplinary active field of research which involves the use of techniques from diverse branches of Mathematics and Physics as well as their applications to Control Theory [25, 26, 32, 59, 61, 79, 119, 187, 212], Physics [39, 54, 58, 234], and many other fields [31].

Our work starts by surveying briefly the historical development of the theory of Lie systems and several of their generalisations. In this way, we aim to provide a general overview of the subject, the main authors, trends, and the principal works dedicated to describing most of the results about this theme. Special attention has been paid to provide a complete bibliography, which contains numerous references that cannot be easily found elsewhere. Furthermore, we have detailed a full report containing the works published by the main contributors to the theory of Lie systems: Lie [153]–[157], Vessiot [222]–[227], Winternitz [8, 9, 13, 112, 105, 173, 174, 233, 234, 235, 236], Ibragimov [120]–[125], etc. Additionally, we presented the main contents of some works which have been written in other languages than English, e.g. [153, 222, 223, 225].

After our brief approach to the history of Lie systems, the fundamental notions of this theory and other related topics are presented. More specifically, along with a recently developed differential geometric approach to the investigation of Lie systems [38], results about the application of Lie systems to investigate Quantum Mechanics, partial differential equations (PDEs), systems of second- and higher-order differential equations are discussed. This, together with the previous historical introduction, furnishes a self-contained presentation of the topic which can be used both as an introduction to the subject and as a reference guide to Lie systems.

Later on, in Chapter 2, our survey focuses on detailing the achievements obtained by the authors who described a method to analyse second-order differential equations. Chapter 3 is concerned with various applications of Lie systems in Quantum Mechanics. Subsequently, we describe a theory of integrability of Lie systems in Chapter 4. This theory is employed to investigate some systems of differential equations appearing in Classical Mechanics in Chapter 5 and

various Schrödinger equations in Chapter 6. Finally, Chapters 7 and 8 describe the theory and applications of a new powerful technique, the *quasi-Lie schemes*, developed to apply the methods for studying Lie systems to a much larger set of systems of differential equations. In the same way as Lie systems, this method can straightforwardly be applied to the setting of second- and higher-order differential equations and Quantum Mechanics. Finally, diverse applications of this technique are performed in Chapter 8.

1.2. Historical introduction. It seems that Abel dealt with the superposition rule concept for the first time, while analysing the linearisation of nonlinear operators [128]. Apart from this very early treatment of one of the notions studied within the theory of Lie systems, the fundamentals of this theory were laid down during the end of the XIX century by the Norwegian mathematician Sophus Lie [153, 154, 155, 157] and the French one Ernest Vessiot [222]-[228]. Indeed, Lie systems are also frequently referred to as *Lie–Vessiot systems* in honour to their contributions.

The first study focused on analysing differential equations admitting a superposition rule was carried out by Königsberger [137] in 1883. In his work, he proved that the only first-order ordinary differential equations on the real line admitting a superposition rule that depends algebraically on the particular solutions are (up to a diffeomorphism) Riccati equations, linear and homogeneous linear differential equations. Later on, in 1885, Lie proposed a special class of systems of first-order ordinary differential equations [153, pg. 128] whose general solutions can be worked out of certain finite families of particular solutions and sets of constants [18, 220].

Despite the above mentioned achievements, these pioneering works did not draw too much attention. Nevertheless, the situation changed from 1893. At that time, Vessiot and Guldberg proved, separately, a slightly more general form of Königsberger’s main result. They demonstrated that (up to a diffeomorphism) Riccati equations and linear differential equations are the only differential equations over the real line admitting a superposition rule [108, 124, 128, 222]. This result attracted Lie’s attention [154], who claimed that their contribution is a simple consequence of his previous work [153]. More specifically, he stated that the systems which admit a superposition rule are those ones that he had defined in 1885 [155]. In view of these criticisms, Lie did not recognise the value of Vessiot and Guldberg’s discovery [128]. Nevertheless, some credit to them must be given, as the theory of Lie does not easily lead to the case provided by Vessiot and Guldberg [128].

Lie’s remarks gave rise to one of the most important results about the theory of Lie systems: the today called *Lie Theorem* [157, Theorem 44]. This theorem characterises systems of first-order ordinary differential equations admitting a superposition rule. In addition, it provides some information on the form of such a superposition rule. In [157], Lie and Scheffers presented the first detailed discussion about Lie systems. In recognition of this work, some authors also call Lie–Scheffers systems to Lie systems.

In spite of this important success, Lie Theorem, as stated by Lie, contains some small gaps in its proof as well as a slight lack of rigour about the definition of superposition rule. This was noticed and fixed at the beginning of the XXI century by Cariñena, Grabowski, Marmo, Blázquez, and Morales [18, 38].

After Lie’s reply, Vessiot recognised the importance of Lie’s work and proposed to call *Lie systems* those systems of first-order ordinary differential equations admitting a superposition rule

[224]. Apart from this first ‘trivial result’, Vessiot furnished many new contributions to the theory of Lie systems [223, 224, 226, 228] and he proposed various generalisations [225, 227, 228]. For instance, he showed that a superposition-like expression can be used to analyse particular types of second-order Riccati equations [225]. More specifically, he proved that some of these equations admit their general solutions to be worked out of families of four particular solutions, their derivatives, and two real constants. As far as we know, this constitutes the first result concerning the study of superposition rules for nonlinear second-order differential equations.

After a beginning in which a deep study of superposition rules and Lie systems was carried out [108, 153, 154, 155, 222, 224, 225, 226, 227, 228], the topic was almost forgotten for nearly a century. Just few works were devoted to the study of superposition rules [76, 80, 81, 82, 149, 197]. During the seventies, nevertheless, the interest on the topic revived and many authors focused again on investigating Lie systems, their generalisations, and applications to Mathematics, Physics and Control Theory [127, 130, 175]. Among the reasons that motivated that rebirth of the theory of Lie systems, we can emphasise the importance of the works of Winternitz and Brocket. On one hand, Brocket analysed the interest of Lie systems in Control Theory [25, 26], what initiated a research field that continues until the present [32, 59, 61, 79, 119, 185, 187, 201, 212]. On the other hand, Winternitz and his collaborators made a huge contribution to the theory of Lie systems and their applications to Physics, Mathematics and Control Theory [8, 9, 13, 14, 15, 21, 112, 114, 141, 234, 236].

In view of its important contributions, let us discuss in slight detail some of Winternitz’s results. Using diverse results derived by Lie [156, 157], Winternitz and his collaborators developed and applied a method to derive superposition rules [202, 209, 210]. They also studied the problem of classification of Lie systems through transitive primitive Lie algebras [210], a concept that also appeared in some of his works about the integrability of Lie systems [21, 22]. Winternitz also paid attention to the analysis of discrete problems and numerical approximations of solutions by means of superposition rules [179, 188, 202, 219] and, finally, he, and his collaborators, developed a new generalisation of the superposition rule notion, the so-called *super-superposition rule*, in order to study the general solutions of various types of superequations [12, 13].

Besides their theoretical achievements, Winternitz *et al.* applied their methods to the analysis of multiple discrete and differential equations with applications to Mathematics, Physics and Control Theory. For instance, many superposition rules were derived for Matrix Riccati equations [8, 112, 141, 174, 188, 212], which play an important role in Control Theory, as well as for diverse Lie systems, like projective Riccati equations [21], various superequations [12, 13], or others [9, 14, 15, 99, 114]. Finally, it is also worth mentioning Winternitz’s research on Milne–Pinney equations [202], which represents one of the first papers devoted to analysing second-order differential equations through Lie systems.

Currently, many researches investigate the theory of Lie systems and other closely related topics. Let us merely point out here some of them along with some of their works: Blázquez and Morales [17, 18, 19], Cariñena [34, 37, 38], Clemente [32], Grabowski [37, 38, 39], Ibragimov [120, 121, 123, 124], de Lucas [34, 35, 52], Lázaro-Camí and Ortega [144], Marmo [37, 38, 39], Odziejewicz and Grundland [172], Ramos [40, 59, 62], Rañada [43, 52, 53, 55] and Nasarre [57, 58]. As a result of their contributions, multiple interesting results about the fundamentals, applications, and generalisations of the theory of Lie systems were furnished.

Among the above works, it is interesting to describe briefly the content of [34, 37, 38]. The book [37] presents an instructive geometric introduction to the basic topics of the theory of Lie systems. The second one [38] provides multiple relevant contributions to the comprehension of the theory of Lie systems. First, it fixes a remarkable gap in the proof of Lie Theorem. Additionally, this work establishes that the superposition rule concept amounts to a certain type of flat connection, what substantially clarifies its properties. The furnished demonstration of Lie Theorem shows that the Lie system notion can be naturally extended to the case of PDEs. Finally, this work led, more or less indirectly, to the characterisation of families of systems of first-order differential equations admitting a t -dependent superposition rule [35] and the definition of the mixed and partial superposition rule notions [38, 52]. Finally, we can mention the usefulness of the *Lie scheme* concept provided in [34], which enables us to generalise the Lie system notion and leads to the discovery of new properties for multiple systems of differential equations, including non-Lie systems, appearing in Physics and Mathematics [34, 42, 45, 48, 56].

Let us now turn to discuss some of the authors' contributions that gave rise to the redaction of this work. On one hand, Cariñena and his collaborators investigated the integrability of Lie systems [40, 43, 47, 50, 54, 63], a generalisation of the Wei–Norman method devoted to the study of Lie systems [57], the application of Lie systems techniques to analyse systems of second-order differential equations [48, 49, 52, 53], and other topics like the analysis of certain Schrödinger equations [46, 51, 59]. In this way, they provided a continuation of diverse previous articles dedicated to some of these themes [77, 172, 202, 225] and they opened several research lines [59].

Besides the above contributions, Cariñena and his collaborators also developed numerous applications of Lie systems to Classical Physics [39, 43, 44, 45, 52, 54, 55, 58, 62], Quantum Mechanics [46, 51, 59, 60], Financial Mathematics [31], and Control Theory [60, 61].

Apart from the aforementioned generalisations of the Lie system notion that are related to other works appearing in the literature [7, 172, 202, 225], a new approach to the generalisation of the Lie system and superposition rule notions was carried out by Cariñena, Grabowski and de Lucas: the theory of quasi-Lie schemes [34]. On one hand, this approach provides us with a method to transform differential equations of a certain type into equations of the same type, e.g. Abel equations into Abel equations [56]. This can also be used to transform differential equations into Lie systems [34], what leads to the *quasi-Lie system* notion. Such systems inherit some properties from Lie systems and, for instance, they admit superposition rules showing an explicit dependence on the independent variable of the system [34, 48].

Quasi-Lie schemes admit multiple applications. they can be used not only to analyse the properties of Lie and quasi-Lie systems but also to investigate many other systems, e.g. nonlinear oscillators [34], Emden-Fowler equations [42], Mathews-Lakshmanan oscillators [34], dissipative and non-dissipative Milne–Pinney equations [45], and Abel equations [56] among others. As a consequence, various results about the integrability properties of such equations have been obtained and many others are being analysed at present. Furthermore, the appearance of t -dependent superposition rules led to the examination of the so-called *Lie families*, which cover, as particular cases, Lie systems and quasi-Lie schemes. Additionally, they can be used to analyse the exact solutions of very general families of differential equations [35].

As a result of all the above mentioned achievements, there exists today a vast collection of

methods and procedures to analyse Lie systems from different points of view. All these tools can be used to provide interesting results in Mathematics, Physics, Control Theory, and other fields. At the same time, these applications motivate the development of new techniques, generalisations, and applications of this theory, that presents multiple and interesting topics to be further investigated.

1.3. Fundamental notions about Lie systems and superposition rules. Our main purpose in this section is to review the basic notions and the fundamental results concerning the theory of Lie systems to be employed and analysed throughout our essay. Here, as well as in major part of our essay, we mostly restrict ourselves to analysing differential equations on vector spaces and we assume mathematical objects, e.g. flows of vector fields, to be smooth, real, and globally defined. This will allow us to highlight the key points of our exposition by omitting several irrelevant technical aspects that can be detailed easily from our presentation. Despite this, numerous differential equations over manifolds and diverse technical points will be presented when relevant.

DEFINITION 1.1. Given the projections $\pi : (x, v) \in \mathbb{T}\mathbb{R}^n \mapsto x \in \mathbb{R}^n$ and $\pi_2 : (t, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto x \in \mathbb{R}^n$, a t -dependent vector field X on \mathbb{R}^n is a map $X : (t, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto X(t, x) \in \mathbb{T}\mathbb{R}^n$ such that the diagram

$$\begin{array}{ccc} & & \mathbb{T}\mathbb{R}^n \\ & \nearrow X & \downarrow \pi \\ \mathbb{R} \times \mathbb{R}^n & \xrightarrow{\pi_2} & \mathbb{R}^n \end{array}$$

is commutative, i.e. $\pi \circ X = \pi_2$.

In view of the above definition, it follows that $X(t, x) \in \pi^{-1}(x) = \mathbb{T}_x\mathbb{R}^n$ and hence $X_t : x \in \mathbb{R}^n \mapsto X_t(x) \equiv X(t, x) \in \mathbb{T}\mathbb{R}^n$ is a vector field over \mathbb{R}^n for every $t \in \mathbb{R}$. From here, it is immediate that each t -dependent vector field X is equivalent to a family $\{X_t\}_{t \in \mathbb{R}}$ of vector fields over \mathbb{R}^n .

The t -dependent vector field concept includes, as a particular instance, the standard vector field notion. Indeed, every vector field Y over \mathbb{R}^n can be naturally regarded as a t -dependent vector field X of the form $X_t = Y$ for every $t \in \mathbb{R}$. Conversely, a ‘constant’ t -dependent vector field X over \mathbb{R}^n , i.e. $X_t = X_{t'}$ for every $t, t' \in \mathbb{R}$, can be considered as a vector field $Y = X_0$ over this space.

As vector fields, t -dependent vector fields also admit local integral curves, see [29]. For each t -dependent vector field X over \mathbb{R}^n , this gives rise to defining its *generalised flow* g^X , i.e. the map $g^X : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $g^X(t, x) \equiv g_t^X(x) = \gamma_x(t)$ with $\gamma_x(t)$ being the unique integral curve of X such that $\gamma_x(0) = x$.

DEFINITION 1.2. A t -dependent vector field X over \mathbb{R}^n is said to be *projectable* under a projection $p : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$ if every X_t is projectable, as a usual vector field, under such a map.

The usage of t -dependent vector fields is fundamental in the theory of Lie systems. They provide us with a geometrical object which contains all necessary information to study systems of first-order differential equations. Let us start by showing how systems of first-order differential equations are described by means of t -dependent vector fields.

DEFINITION 1.3. Given a t -dependent vector field

$$X(t, x) = \sum_{i=1}^n X^i(t, x) \frac{\partial}{\partial x^i}, \quad (1.1)$$

over \mathbb{R}^n , its *associated system* is the system of first-order differential equations determining its integral curves, namely,

$$\frac{dx^i}{dt} = X^i(t, x), \quad i = 1, \dots, n. \quad (1.2)$$

Note that there exists a one-to-one correspondence between t -dependent vector fields and systems of first-order differential equations of the form (1.2). That is, every t -dependent vector field has an associated system of first-order differential equations and each system of this type, in turn, determines the integral curves of a unique t -dependent vector field. Taking this into account, we can hereby use X to refer to both a t -dependent vector field and the system of equations describing its integral curves. This simplifies our exposition and it does not lead to confusion as the difference of meaning is clearly noticed from the context.

The following definition and lemma, whose proof is straightforward and it shall not be detailed, notably simplify the statements and proofs of various results about the theory of Lie systems.

DEFINITION 1.4. Given a (possibly infinite) family \mathcal{A} of vector fields on \mathbb{R}^n , we denote by $\text{Lie}(\mathcal{A})$ the smallest Lie algebra V of vector fields on \mathbb{R}^n containing \mathcal{A} .

LEMMA 1.5. *Given a family of vector fields \mathcal{A} , the linear space $\text{Lie}(\mathcal{A})$ is spanned by the vector fields*

$$\mathcal{A}, [\mathcal{A}, \mathcal{A}], [\mathcal{A}, [\mathcal{A}, \mathcal{A}]], [\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{A}]]], \dots$$

where $[\mathcal{A}, \mathcal{B}]$ denotes the set of vector fields obtained through the Lie brackets between elements of the families of vector fields \mathcal{A} and \mathcal{B} .

Throughout this work two different notions of linear independence are used frequently. In order to state a clear meaning of each, we provide the following definition.

DEFINITION 1.6. Let us denote by $\mathfrak{X}(\mathbb{R}^n)$ the space of vector fields over \mathbb{R}^n . We say that the vector fields, X_1, \dots, X_r , on \mathbb{R}^n are *linearly independent over \mathbb{R}* if they are linearly independent as elements of $\mathfrak{X}(\mathbb{R}^n)$ when considered as a \mathbb{R} -vector space, i.e. whenever

$$\sum_{\alpha=1}^r \lambda_{\alpha} X_{\alpha} = 0$$

for certain constants, $\lambda_1, \dots, \lambda_r$, then $\lambda_1 = \dots = \lambda_r = 0$. On the other hand, the vector fields, X_1, \dots, X_r , are said to be *linearly independent at a generic point* if they are linearly independent as elements of $\mathfrak{X}(\mathbb{R}^n)$ when regarded as a $C^{\infty}(\mathbb{R}^n)$ -module. That is, if one has

$$\sum_{\alpha=1}^r f_{\alpha} X_{\alpha} = 0$$

over any open set of \mathbb{R}^n for certain functions $f_1, \dots, f_r \in C^{\infty}(\mathbb{R}^n)$, then $f_1 = \dots = f_r = 0$.

In this essay, we frequently deal with linear spaces of the form $\mathbb{R}^{n(m+1)}$. Such spaces are always considered as a product $\mathbb{R}^n \times \dots \times \mathbb{R}^n$. Each point of $\mathbb{R}^{n(m+1)}$ is denoted by

$(x_{(0)}, \dots, x_{(m)})$, where $x_{(j)}$ stands for a point of the j -th copy of the manifold \mathbb{R}^n within $\mathbb{R}^{n(m+1)}$.

Associated with $\mathbb{R}^{n(m+1)}$, there exists a group of permutations S_{m+1} whose elements, S_{ij} , with $i \leq j = 0, 1, \dots, m$, act on $\mathbb{R}^{n(m+1)}$ by permutating the variables $x_{(i)}$ and $x_{(j)}$. Finally, let us define the projections

$$\text{pr} : (x_{(0)}, \dots, x_{(m)}) \in \mathbb{R}^{n(m+1)} \mapsto (x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^{nm} \quad (1.3)$$

and

$$\text{pr}_0 : (x_{(0)}, \dots, x_{(m)}) \in \mathbb{R}^{n(m+1)} \mapsto x_{(0)} \in \mathbb{R}^n, \quad (1.4)$$

to be employed in various parts of our work.

Once the fundamental definitions and assumptions to be used hereafter have been established, we proceed to introduce the notion of *superposition rule*, which plays a central role in the study of Lie systems.

For each system of first-order ordinary homogeneous linear differential equations on \mathbb{R}^n of the form

$$\frac{dy^i}{dt} = \sum_{j=1}^n A_j^i(t)y^j, \quad i = 1, \dots, n, \quad (1.5)$$

where $A_j^i(t)$, with $i, j = 1, \dots, n$, is a family of t -dependent functions, its general solution, $y(t)$, can be written as a linear combination of the form

$$y(t) = \sum_{j=1}^n k_j y_{(j)}(t), \quad (1.6)$$

with, $y_{(1)}(t), \dots, y_{(n)}(t)$, being a family of n generic (linearly independent) particular solutions, and, k_1, \dots, k_n , being a set of constants. The above expression is called *linear superposition rule* for system (1.5).

Linear superposition rules allow us to reduce the search for the general solution of a linear system to the determination of a finite set of particular solutions. This property is not exclusive for homogeneous linear systems. Indeed, for each linear system

$$\frac{dy^i}{dt} = \sum_{j=1}^n A_j^i(t)y^j + B^i(t), \quad i = 1, \dots, n, \quad (1.7)$$

where $A_j^i(t), B^i(t)$, with $i, j = 1, \dots, n$, are a family of t -dependent functions, its general solution, $y(t)$, can be written as a linear combination of the form

$$y(t) = \sum_{j=1}^n k_j (y_{(j)}(t) - y_{(0)}(t)) + y_{(0)}(t), \quad (1.8)$$

with, $y_{(0)}(t), \dots, y_{(n)}(t)$, being a family of $n + 1$ particular solutions such that $y_{(j)}(t) - y_{(0)}(t)$, with $j = 1, \dots, n$, are linearly independent solutions of the homogeneous problem associated with (1.7), and, k_1, \dots, k_n , being a set of constants.

In a more general way, system (1.5) becomes (generally) a nonlinear system

$$\frac{dx^i}{dt} = X^i(t, x), \quad i = 1, \dots, n, \quad (1.9)$$

through a diffeomorphism $\varphi : \mathbb{R}^n \ni y \mapsto x = \varphi(y) \in \mathbb{R}^n$. In view of the linear superposition rule (1.6), the above system admits its general solution, $x(t)$, to be described in terms of a family of certain particular solutions, $x_{(1)}(t), \dots, x_{(m)}(t)$, as

$$x(t) = \varphi \left(\sum_{j=1}^n k_j \varphi^{-1}(x_{(j)}(t)) \right).$$

This clearly shows that there exist many systems of first-order differential equations whose general solutions can be described, nonlinearly, in terms of certain families of particular solutions and sets of constants. A relevant family of different equations admitting such a property are Riccati equations [4, 64, 102, 112, 170, 189, 212] of the form

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2, \quad (1.10)$$

with $x \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}$. More specifically, for each of such Riccati equations, its general solution, $x(t)$, can be cast into the form

$$x(t) = \frac{x_1(t)(x_3(t) - x_2(t)) - kx_2(t)(x_3(t) - x_1(t))}{(x_3(t) - x_2(t)) - k(x_3(t) - x_1(t))}, \quad (1.11)$$

where, $x_1(t), x_2(t), x_3(t)$, are three particular solutions of the equation and $k \in \bar{\mathbb{R}}$.

It is worth noting that, given a fixed family of three different particular solutions with initial conditions within \mathbb{R} , if we only choose k in \mathbb{R} , the above expression does not recover the whole general solution of the Riccati equation, as $x_2(t)$ cannot be recovered.

The above examples show the existence of a certain type of expression, the so-called *global superposition rule*, which enables us to express the general solution of certain systems of first-order ordinary differential equations in terms of certain families of particular solutions and a set of constants. Let us state a rigorous definition of this notion for systems of differential equations in \mathbb{R}^n .

DEFINITION 1.7. The system of first-order ordinary differential equations

$$\frac{dx^i}{dt} = X^i(t, x), \quad i = 1, \dots, n, \quad (1.12)$$

is said to admit a *global superposition rule* if there exists a t -independent map $\Phi : (\mathbb{R}^n)^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$x = \Phi(x_{(1)}, \dots, x_{(m)}; k_1, \dots, k_n), \quad (1.13)$$

such that its general solution, $x(t)$, can be written as

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)}(t); k_1, \dots, k_n), \quad (1.14)$$

with, $x_{(1)}(t), \dots, x_{(m)}(t)$, being any generic family of particular solutions of system (1.12) and, k_1, \dots, k_n , being a set of n constants to be related to initial conditions.

In order to grasp the meaning of the above definition, it is necessary to understand the sense in which the term ‘generic’ is used in the above statement. Precisely speaking, it is said that expression (1.14) is valid for any generic family of m particular solutions if there exists an open dense subset $U \subset (\mathbb{R}^n)^m$ such that expression (1.14) is satisfied for every set of particular solutions $x_1(t), \dots, x_m(t)$, such that $(x_1(0), \dots, x_m(0))$ lies in U .

Let us now show that the aforementioned examples admit a global superposition rule. Consider the function $\Phi : (\mathbb{R}^n)^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$\Phi(x_{(1)}, \dots, x_{(n)}; k_1, \dots, k_n) = \sum_{j=1}^n k_j x_{(j)}. \quad (1.15)$$

This mapping is a superposition rule for system (1.5). Indeed, note that for each set of particular solutions $x_{(1)}(t), \dots, x_{(n)}(t)$, of (1.5) such that the point $(x_{(1)}(0), \dots, x_{(n)}(0))$ belongs to the open dense subset

$$U = \left\{ (x_{(1)}, \dots, x_{(n)}) \in (\mathbb{R}^n)^n \mid \det \begin{pmatrix} x_{(1)}^1 & \dots & x_{(n)}^1 \\ \dots & \dots & \dots \\ x_{(1)}^n & \dots & x_{(n)}^n \end{pmatrix} \neq 0 \right\},$$

of $(\mathbb{R}^n)^n$, the general solution $x(t)$ of (1.5) can be written in the form (1.6). Likewise, a superposition rule can be now proved to exist for the systems (1.9) obtained from (1.5) by means of a diffeomorphism.

The function $\Phi : (\mathbb{R}^n)^{n+1} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$\Phi(x_{(0)}, \dots, x_{(n)}; k_1, \dots, k_n) = \sum_{j=1}^n k_j (x_{(j)} - x_{(0)}) + x_{(0)}, \quad (1.16)$$

is a superposition function for the system (1.7). In fact, note that for each set of particular solutions, $x_{(0)}(t), \dots, x_{(n)}(t)$, of (1.7) such that the point $(x_{(0)}(0), \dots, x_{(n)}(0))$ belongs to the open dense subset

$$U = \left\{ (x_{(0)}, \dots, x_{(n)}) \in (\mathbb{R}^n)^{n+1} \mid \det \begin{pmatrix} x_{(1)}^1 - x_{(0)}^1 & \dots & x_{(n)}^1 - x_{(0)}^1 \\ \dots & \dots & \dots \\ x_{(1)}^n - x_{(0)}^n & \dots & x_{(n)}^n - x_{(0)}^n \end{pmatrix} \neq 0 \right\},$$

of $(\mathbb{R}^n)^{n+1}$, the general solution $x(t)$ of (1.7) can be put in the form (1.8).

Finally, let us analyse the case of Riccati equations in \mathbb{R} . This example differs a little from previous ones, as it concerns a differential equation defined in the manifold $\mathbb{R} \simeq S^1$. Nevertheless, the generalisation of Definition 1.7 to manifolds is obvious. It is only necessary to replace \mathbb{R}^n by a manifold N . In view of this, the map $\Phi : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$\Phi(x_{(1)}, x_{(2)}, x_{(3)}; k) = \frac{x_{(1)}(x_{(3)} - x_{(2)}) - kx_{(2)}(x_{(3)} - x_{(1)})}{(x_{(3)} - x_{(2)}) - k(x_{(3)} - x_{(1)})} \quad (1.17)$$

is a global superposition rule for Riccati equations in \mathbb{R} . To verify this, it is sufficient to note that given one of these equations with three particular solutions, $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$, such that $(x_{(1)}(0), x_{(2)}(0), x_{(3)}(0)) \in U$, where

$$U = \left\{ (x_{(1)}, x_{(2)}, x_{(3)}) \in \mathbb{R}^3 \mid x_{(1)} \neq x_{(2)}, x_{(1)} \neq x_{(3)} \text{ and } x_{(2)} \neq x_{(3)} \right\},$$

its general solution can be cast into the form (1.11).

The aforementioned superposition rules illustrate that for each permutation of their arguments, $x_{(1)}, \dots, x_{(m)}$, e.g. an interchange of the arguments $x_{(i)}$ and $x_{(j)}$, one has, in general, that

$$\Phi(x_{(1)}, \dots, x_{(i)}, \dots, x_{(j)}, \dots, x_{(m)}; k) \neq \Phi(x_{(1)}, \dots, x_{(j)}, \dots, x_{(i)}, \dots, x_{(m)}; k).$$

Nevertheless, it can be proved (cf. [38]) that there exists a map $\varphi : k \in \mathbb{R}^n \rightarrow \varphi(k) \in \mathbb{R}^n$ such that

$$\Phi(x_{(1)}, \dots, x_{(i)}, \dots, x_{(j)}, \dots, x_{(m)}; k) = \Phi(x_{(1)}, \dots, x_{(j)}, \dots, x_{(i)}, \dots, x_{(m)}; \varphi(k)).$$

It is interesting to note that, if we consider Riccati equations to be defined on the real line, a global superposition rule for such equations would be a map of the form $\Phi : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$. Obviously, expression (1.17) does not give rise to a global superposition of this form. Indeed, if we restrict (1.17) to $\mathbb{R}^3 \times \mathbb{R}$, we will not be able to recover $x_{(2)}(t)$ from a set of different particular solutions, $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$, for any $k \in \mathbb{R}$. Even more, the function (1.17) is not globally defined over $\mathbb{R}^3 \times \mathbb{R}$. Nevertheless, such a function is what in the literature is known as a *superposition rule* for Riccati equations over the real line [108, 157, 222].

In the literature, the superposition rule notion appears as a ‘milder’ version of aforementioned global superposition rule concept. In other words, superposition rules admit almost the same properties as global superposition rules but, for instance, they may fail to recover certain particular solutions. Although it is enough to bear in mind the above example for Riccati equations to understand fully the main difference between both notions, the precise definition of a local superposition rule is very technical (see [18]) and it does not provide, in practice, a much deeper knowledge about Lie systems. That is why, as everywhere else in the literature [37, 108, 124, 125, 153, 157, 222, 223, 234], we will assume hereafter superposition rules to recover general solutions and to be globally defined. This simplifies considerably our theoretical presentation and it highlights the main features of superposition rules and Lie systems. Despite these assumptions, a fully rigorous treatment can be easily carried out and some technical remarks will be discussed when relevant.

A relevant question now arises: which systems of first-order ordinary differential equations admit a superposition rule? Several works have been devoted to investigating this question. Its analysis was accomplished by Königsberger [137], Vessiot [222], and Guldberg [108]. They proved that every system of first-order differential equations defined over the real line admitting a superposition rule is, up to a diffeomorphism, a Riccati equation or a first-order linear differential equation.

Apart from these preliminary results, it was Lie [153, 154, 157] who established the conditions ensuring that a system of first-order differential equations of the form (1.12) admits a superposition rule. His result, the today named *Lie Theorem*, reads in modern geometric terms as follows.

THEOREM 1.8. (Lie Theorem) *A system of first-order ordinary differential equations (1.12) admits a superposition rule (1.13) if and only if its corresponding t -dependent vector field (1.1) can be cast into the form*

$$X(t, x) = \sum_{\alpha=1}^r b_{\alpha}(t) X_{\alpha}(x), \quad (1.18)$$

with, X_1, \dots, X_r , being a family of vector fields over \mathbb{R}^n spanning a r -dimensional real Lie algebra of vector fields V .

Within the proof to his theorem [157, Theorem 44], Lie also claimed that the dimension of the decomposition (1.18) and the number m of particular solutions for the superposition rule are related. More specifically, he proved that the existence of a superposition rule depending on m

particular solutions for a system (1.12) in \mathbb{R}^n implies that there exists a decomposition (1.18) associated with a Lie algebra V obeying the inequality $\dim V \leq m \cdot n$, the referred to as *Lie's condition*. Conversely, given a decomposition of the form (1.18), we can ensure the existence of a superposition rule for system (1.12) whose number of particular solutions obeys the same condition.

Although Lie Theorem solves theoretically the problem of determining whether a system (1.12) admits a superposition rule, it does not provide a solution for many other questions concerning the study of superposition rules. Let us briefly comment on some of these queries.

- From a practical point of view, it is not straightforward, solely in view of Lie Theorem, to prove that a system of first-order differential equations does not admit a superposition rule. Later on in this section, we will sketch a procedure to do so.
- Lie Theorem says nothing about the possible existence of multiple superposition rules for the same system. What is more, it does not explain explicitly how to determine any of such superposition rules (although its proof [157, Theorem 4] furnishes some key hints). These questions are addressed later in this Chapter, where we review a recent geometrical approach to Lie systems developed in [38].
- A system $X(t, x)$ admitting a superposition rule may be written in the form (1.18) in one or, sometimes, several different ways. Each one of these decompositions is related to a different finite-dimensional Lie algebra of vector fields V . Such Lie algebras are generally called the *Vessiot–Guldberg Lie algebras* associated with a system. Lie Theorem does not explain the possible relations amongst all possible Vessiot–Guldberg Lie algebras of a system (1.12). In fact, only Lie's condition suggests that each different Vessiot–Guldberg Lie algebra may be related to different superposition rules. We will discuss these questions, in a more extensive way, later in this section and next.
- Finally, it is worth noting that Lie Theorem cannot be used to characterise straightforwardly systems of first-order differential equations of the form $F^i(t, x, \dot{x}) = 0$, with $i = 1, \dots, n$. Indeed, this is an open question of the research on Lie systems.

The discovery of Lie Theorem [157] in 1893 established definitively the Lie system notion, which, on the other hand, had already been suggested long time ago by Lie [153], and whose name was coined by Vessiot in [224] as a recognition to Lie's success in characterising systems admitting a superposition rule. The definition of this relevant notion goes as follows.

DEFINITION 1.9. A system of the form (1.12) is a *Lie system* if and only if its corresponding t -dependent vector field, namely (1.1), admits a decomposition of the form (1.18).

In view of Lie Theorem, the above definition of Lie system can be rephrased by saying that a system (1.12) is a Lie system if and only if it admits a superposition rule. From here, it is obvious that the systems of first-order differential equations (1.5), (1.7) and (1.10), which admit the global superposition rules (1.15), (1.16) and (1.17), respectively, are Lie systems. Let us analyse in detail such examples. This brings us the opportunity to illustrate diverse characteristics of Lie systems and the Lie Theorem here and in forthcoming sections.

Consider again the homogeneous linear system (1.5). This system describes the integral

curves of the t -dependent vector field

$$X(t, x) = \sum_{i,j=1}^n A^i_j(t) x^j \frac{\partial}{\partial x^i}, \quad (1.19)$$

which is a linear combination of vector fields of the form

$$X(t, x) = \sum_{i,j=1}^n A^i_j(t) X_{ij}(x), \quad (1.20)$$

of the n^2 vector fields

$$X_{ij} = x^j \frac{\partial}{\partial x^i}, \quad i, j = 1, \dots, n. \quad (1.21)$$

Furthermore, one has that

$$[X_{ij}, X_{lm}] = \delta_m^i X_{lj} - \delta_j^l X_{im},$$

where δ_m^i is the Kronecker delta function, i.e. the vector fields (1.21) close on a n^2 -dimensional Vessiot–Guldberg Lie algebra isomorphic to the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$, see [62].

In view of decomposition (1.20), each system (1.5) is a Lie system. This is not a surprise, as each system (1.5) admits the superposition rule (1.15) and Lie Theorem states that every system admitting a superposition rule must be a Lie system. Moreover, in view of Lie's condition, since homogeneous linear systems in \mathbb{R}^n admit a superposition rule depending on n particular solutions, their associated t -dependent vector fields must take values in *some* Lie algebra of dimension lower or equal to n^2 . Indeed, note that decomposition (1.20) shows that $X(t, x)$ takes values in a Lie algebra isomorphic to $\mathfrak{gl}(n, \mathbb{R})$, what clearly obeys the Lie's condition corresponding to the superposition rule (1.15).

Note that we have italicised the last 'some' in the paragraph above. We did it because we wanted to stress that a Lie system can take values in different Lie algebras, some of which do not need to satisfy the same Lie's condition. This will become more clear with the next example.

Let us now turn to analyse an inhomogeneous system of the form (1.7). This system describes the integral curves of the t -dependent vector field

$$X(t, x) = \sum_{i=1}^n \left(\sum_{j=1}^n A^i_j(t) x^j + B^i(t) \right) \frac{\partial}{\partial x^i}, \quad (1.22)$$

which is a linear combination with t -dependent coefficients,

$$X_t = \sum_{i,j=1}^n A^i_j(t) X_{ij} + \sum_{i=1}^n B^i(t) X_i, \quad (1.23)$$

of the vector fields (1.21) and

$$X_i = \frac{\partial}{\partial x^i}, \quad i = 1, \dots, n. \quad (1.24)$$

The above vector fields satisfy the commutation relations

$$[X_i, X_j] = 0, \quad i, j = 1, \dots, n, \quad [X_{ij}, X_l] = -\delta^{lj} X_i, \quad i, j, l = 1, \dots, n.$$

This shows that the vector fields (1.21) and (1.24) span a Lie algebra of vector fields isomorphic to the $(n^2 + n)$ -dimensional Lie algebra of the affine group [62]. Then, in view of decomposition (1.23), systems (1.7) are Lie systems.

As systems (1.7) admit a superposition rule (1.16) depending on $n + 1$ particular solutions, Lie's condition implies that their t -dependent vector fields must take values in some Lie algebra of dimension lower or equal to $n(n + 1)$. In fact, the above results easily show that this is the case.

The previous example allows us to exemplify that a Lie system may admit multiple Vessiot–Guldberg Lie algebras. Recall that every homogeneous linear system (1.5) is related to a t -dependent vector field taking values in a Lie algebra isomorphic to $\mathfrak{gl}(n, \mathbb{R})$. Additionally, as a particular instance of system (1.7), its t -dependent vector field also takes in the above defined $n^2 + n$ -dimensional Lie algebra of vector fields. In other words, linear systems admit, at least, two non-isomorphic Vessiot–Guldberg Lie algebras.

Now, we can illustrate how different superposition rules for the same system may be associated with multiple, non-isomorphic, Vessiot–Guldberg Lie algebras and lead to distinct Lie's conditions. We showed that linear systems admit a linear superposition rule which leads, in view of Lie's condition, to the existence of an associated Vessiot–Guldberg Lie algebra of dimension lower or equal to n^2 , which was determined. Nevertheless, the abovementioned second Vessiot–Guldberg Lie algebra for linear systems does not hold this condition. On the contrary, this second Vessiot–Guldberg Lie algebra shows that there must exist a second superposition rule, namely (1.8), which, along with this Vessiot–Guldberg Lie algebra, satisfies a new Lie's condition.

To sum up, Lie Theorem implies that a system admitting a superposition rule is related to the existence of, at least, one Vessiot–Guldberg Lie algebra satisfying the Lie's condition relative to this superposition. Nevertheless, the system can possess more Vessiot–Guldberg Lie algebras, some of which do not need to obey the Lie's condition for the assumed superposition rule. In that case, the other Vessiot–Guldberg Lie algebras are related to other superposition rules for which, a new Lie's condition is satisfied.

In order to detail the last of the most usual examples of Lie systems admitting a superposition rule, we now consider Riccati equations (1.10). These differential equations determine the integral curves of the t -dependent vector field on $\bar{\mathbb{R}}$ of the form

$$X(t, x) = (b_1(t) + b_2(t)x + b_3(t)x^2) \frac{\partial}{\partial x}. \quad (1.25)$$

As Riccati equations admit a global superposition rule, they must satisfy the assumptions detailed in Lie Theorem. Indeed, note that X is a linear combination with t -dependent coefficients of the three vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x}, \quad X_3 = x^2 \frac{\partial}{\partial x}, \quad (1.26)$$

which close on a three-dimensional Lie algebra with defining relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \quad (1.27)$$

Thus, as it was expected, Riccati equations obey the conditions given by Lie to admit a superposition rule. Moreover, Riccati equations are associated with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since this Lie algebra is three dimensional and Riccati equations admit a superposition rule depending on three particular solutions, it is immediate that the equations (1.10) satisfy the corresponding Lie's condition.

The existence of different Vessiot–Guldberg Lie algebras for a system of first-order ordinary differential equations is an important question because their characteristics determine, among

other features, the integrability by quadratures of Lie systems [31].

Let us now turn our attention to determine when a system (1.12) is *not* a Lie system. In order to analyse this question, it becomes useful to rewrite Lie Theorem in the following, abbreviated, form.

PROPOSITION 1.10. (Abbreviated Lie Theorem) *A system X on \mathbb{R}^n is a Lie system if and only if $\text{Lie}(\{X_t\}_{t \in \mathbb{R}})$ is finite-dimensional.*

In view of the above result, determining that (1.12) is not a Lie system reduces to showing that $\text{Lie}(\{X_t\}_{t \in \mathbb{R}})$ is infinite-dimensional. The standard procedure to prove this consists in demonstrating that there exists an infinite chain, $\{Z_j\}_{j \in \mathbb{N}}$ of linearly independent vector fields over \mathbb{R} obtained through successive Lie brackets of elements in $\{X_t\}_{t \in \mathbb{R}}$. In order to illustrate how this is usually made, consider the particular example based on the study of the Abel equation of the first-type

$$\frac{dx}{dt} = x^2 + b(t)x^3, \quad b(t) \neq 0,$$

where $b(t)$ is additionally a non-constant function. These equations describe the integral curves of the t -dependent vector field

$$X_t = (x^2 + b(t)x^3) \frac{\partial}{\partial x}.$$

Consider the chain of vector fields

$$Z_1 = x^2 \frac{\partial}{\partial x}, \quad Z_2 = x^3 \frac{\partial}{\partial x}, \quad Z_j = [X_1, X_{j-1}], \quad j = 3, 4, 5, \dots$$

Since $Z_j = x^{j+1} \partial / \partial x$, it turns out that $\text{Lie}(\{X_t\}_{t \in \mathbb{R}})$ admits the infinite chain of linearly independent vector fields $\{Z_j\}_{j \in \mathbb{R}}$ and, in consequence, in view of the abbreviated Lie Theorem, Abel equations of the above type are not Lie systems.

There are many other relevant Lie systems associated with important systems of differential equations appearing in the physical and mathematical literature. For instance, a non exhaustive brief list of these Lie systems includes

1. Linear first-order systems and, more specifically, Euler-systems [62, 98].
2. Riccati equations [47, 222, 234] and coupled Riccati equations of projective type [6].
3. Matrix Riccati equations [112, 141, 174, 188, 212, 234].
4. Bernoulli equations, several equations appearing in supermechanics [13], etc.

Apart from the above instances, there are other important systems of differential equations which can be studied through other Lie systems. Several of such Lie systems will be detailed throughout next sections.

The determination of the general solution of any Lie system reduces to deriving a particular solution of a particular type of Lie system defined in a Lie group. Let us analyse in detail this claim.

Consider a Lie system related to a t -dependent vector field (1.18) over \mathbb{R}^n and associated, for simplicity, with a Vessiot–Guldberg Lie algebra V made up of complete vector fields. This gives rise to a Lie group action $\Phi : G \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ whose fundamental vector fields are exactly those of V . Obviously, this implies that the Lie algebra $\mathfrak{g} \simeq T_e G$ is isomorphic to V . Choose now a

basis $\{a_1, \dots, a_r\}$ of \mathfrak{g} such that $\Phi : G \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and

$$\Phi(\exp(-sa_\alpha), x) = g_s^{(\alpha)}(x), \quad \alpha = 1, \dots, r, \quad s \in \mathbb{R}, \quad (1.28)$$

where $g^{(\alpha)} : (s, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto g^{(\alpha)}(s, x) = g_s^{(\alpha)}(x) \in \mathbb{R}^n$ is the flow of the vector field X_α . In this way, each vector field X_α becomes the fundamental vector field corresponding to a_α and the map $\phi : \mathfrak{g} \rightarrow V$ such that $\phi(a_\alpha) = X_\alpha$ for $\alpha = 1, \dots, r$, is a Lie algebra isomorphism.

Let X_α^R be the right-invariant vector field on G with $(X_\alpha^R)_e = a_\alpha$, i.e. $(X_\alpha^R)_g = R_{g*e}a_\alpha$, where $R_g : g' \in G \mapsto g'g \in G$ is the right action of G on itself. Then, the t -dependent right-invariant vector field

$$X^G(t, g) = - \sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(g), \quad (1.29)$$

defines a Lie system on G whose integral curves are the solutions of the system on G given by

$$\frac{dg}{dt} = - \sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(g). \quad (1.30)$$

Applying $R_{g^{-1}*g}$ to both sides of the equation, we see that its general solution, $g(t)$, satisfies that

$$R_{g^{-1}(t)*g(t)}\dot{g}(t) = - \sum_{\alpha=1}^r b_\alpha(t)a_\alpha \in T_e G. \quad (1.31)$$

Note that right-invariance implies that the knowledge of one particular solution of the above equation, e.g. the particular one $g_0(t)$, with $g_0(0) = g_0$, is enough to obtain the general solution of the equation (1.31). Indeed, consider $g'(t) = R_{\bar{g}}g_0(t)$ for a given $\bar{g} \in G$. Such a curve obeys that

$$\frac{dg'}{dt}(t) = R_{\bar{g}*g_0(t)} \left(\frac{dg_0}{dt}(t) \right) \iff \frac{dg'}{dt}(t) = R_{\bar{g}*g_0(t)} \left(- \sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(g_0(t)) \right).$$

Taking into account that $R_{\bar{g}*g_0} X_\alpha^R(g_0) = X_\alpha^R(g_0\bar{g})$, one has that

$$\frac{dg'}{dt}(t) = - \sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(R_{\bar{g}}g_0(t)) = - \sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(g'(t))$$

and $g'(t)$ is another particular solution of (1.29) with initial condition $g'(0) = R_{\bar{g}}g_0$. In consequence, the general solution $g(t)$ for equation (1.31) can be written as

$$g(t) = R_{\bar{g}}g_0(t), \quad \bar{g} \in G.$$

That is, system (1.29) admits a superposition rule and, according to Lie Theorem, it must be a Lie system. This is not surprising, as the vector fields X_α^R span a Lie algebra of vector fields isomorphic to V and, in consequence, system (1.30) describes the integral curves of a t -dependent vector field taking values in a finite-dimensional Lie algebra of vector fields.

The relevance of the Lie system (1.31) relies on the fact that the integral curves of the t -dependent vector field $X(t, x)$ can be obtained from one particular solution of equation (1.31). More explicitly, the general solution $x(t)$ of the Lie system $X(t, x)$ reads $x(t) = \Phi(g_e(t), x_0)$, where x_0 is the initial condition of the particular solution and $g_e(t)$ is the particular solution of equation (1.31) with $g_e(0) = e$.

Note that, in view of Ado's Theorem [2], every finite-dimensional Lie algebra, e.g. the above Vessiot–Guldberg Lie algebra V , admits an isomorphic matrix Lie algebra. Related to this matrix

Lie algebra, there exists a matrix Lie group \tilde{G} . In this way, the system describing the t -dependent vector field (1.18) reduces to solving an equation of the form

$$\dot{A}(t)A^{-1}(t) = - \sum_{\alpha=1}^r b_{\alpha}(t)M_{\alpha} \implies \dot{A} = - \sum_{\alpha=1}^r b_{\alpha}(t)M_{\alpha}A,$$

with $A(t)$ being a curve taking values in the matrix Lie group \tilde{G} and, M_1, \dots, M_r , being a basis closing the same structure constants as the elements, X_1, \dots, X_r . Obviously, the above equation becomes a homogeneous linear differential equation in the coefficients of the matrix A . Consequently, determining the general solution of a Lie system reduces to solving a linear problem.

Although the above process was described for Lie systems associated with Vessiot-Guldberg Lie algebras of complete vector fields, it can be proved that a similar process, with almost identical final results, can be applied to any Lie system $X(t, x)$. Indeed, this can be done by taking the compactification of \mathbb{R}^n in order to make all vector fields complete (as in the case of the Riccati equation) or just by considering that the induced action is just a local one.

A generalisation of the method [57] used by Wei and Norman for linear systems [231, 232] is very useful for solving equations (1.31). Furthermore, there exist reduction techniques that can also be used [40]. Such techniques show, for instance, that Lie systems related to solvable Vessiot–Guldberg Lie algebras are integrable by quadratures ([40], Section 8). Finally, as right-invariant vector fields X^R project onto the fundamental vector fields in each homogeneous space for G , the solution of equation (1.31) enables us to find the general solution for the corresponding Lie system in each homogeneous space. Conversely, the knowledge of particular solutions of the associated system in a homogeneous space gives us a method for reducing the problem to the corresponding isotopy group [40].

1.4. Geometric approach to superposition rules. Let us now turn to review the modern geometrical approach to the theory of Lie systems carried out in [38]. Although we here basically point out the results given in that work, several slight improvements have been included in our presentation.

A fundamental notion in the geometrical description of Lie systems is the so-called *diagonal prolongation* of a t -dependent vector field. Its definition and most important properties are described below.

DEFINITION 1.11. Given a t -dependent vector field over \mathbb{R}^n of the form

$$X(t, x_{(0)}) = \sum_{i=1}^n X^i(t, x_{(0)}) \frac{\partial}{\partial x_{(0)}^i},$$

its *diagonal prolongation* to $\mathbb{R}^{n(m+1)}$ is the t -dependent vector field over this latter space given by

$$\hat{X}(t, x_{(0)}, \dots, x_{(m)}) = \sum_{a=0}^m \sum_{i=1}^n X^i(t, x_{(a)}) \frac{\partial}{\partial x_{(a)}^i}.$$

Recall that every vector field X over \mathbb{R}^n can be regarded as a t -dependent vector field in a natural way. Evidently, it is immediate that the above definition can also be applied to define

diagonal prolongations for vector fields over \mathbb{R}^n . Obviously, such diagonal prolongations turn out to be vector fields over $\mathbb{R}^{n(m+1)}$ as well.

Note that diagonal prolongations can be redefined in an intrinsic, and equivalent, way as follows.

DEFINITION 1.12. Given a t -dependent vector field X over \mathbb{R}^n , its *diagonal prolongation* to $\mathbb{R}^{n(m+1)}$ is the unique t -dependent vector field \widehat{X} over $\mathbb{R}^{n(m+1)}$ such that:

- The t -dependent vector field \widehat{X} is invariant under the action of the symmetry group S_{m+1} over $\mathbb{R}^{n(m+1)}$.
- The vector fields \widehat{X}_t are projectable under the projection pr_0 given by (1.4) and $pr_{0*}\widehat{X}_t = X_t$.

LEMMA 1.13. For every two vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^n)$, it is immediate that $[\widehat{X}, \widehat{Y}] = \widehat{[X, Y]}$. In consequence, given a Lie algebra of vector fields $V \subset \mathfrak{X}(\mathbb{R}^n)$, the prolongations of its elements to $\mathbb{R}^{n(m+1)}$ span an isomorphic Lie algebra of vector fields.

Proof. It is straightforward and it is left to the reader. ■

LEMMA 1.14. Consider a family, X_1, \dots, X_r , of vector fields over \mathbb{R}^n satisfying that their diagonal prolongations to \mathbb{R}^{nm} are linearly independent at a generic point. Given the diagonal prolongations, $\widehat{X}_1, \dots, \widehat{X}_r$, to $\mathbb{R}^{n(m+1)}$, the vector field $\sum_{\alpha=1}^r b_\alpha \widehat{X}_\alpha$, with $b_\alpha \in C^\infty(\mathbb{R}^{n(m+1)})$, is also a diagonal prolongation if and only if the coefficients, b_1, \dots, b_r , are constant.

Proof. Let us write in local coordinates

$$X_\alpha = \sum_{i=1}^n A_\alpha^i(x) \frac{\partial}{\partial x^i}, \quad \alpha = 1, \dots, r,$$

what implies that

$$\widehat{X}_\alpha = \sum_{i=1}^n \sum_{a=0}^m A_\alpha^i(x_{(a)}) \frac{\partial}{\partial x_{(a)}^i}, \quad \alpha = 1, \dots, r.$$

Then,

$$\sum_{\alpha=1}^r b_\alpha(x_{(0)}, \dots, x_{(m)}) \widehat{X}_\alpha = \sum_{\alpha=1}^r \sum_{i=1}^n \sum_{a=0}^m b_\alpha(x_{(0)}, \dots, x_{(m)}) A_\alpha^i(x_{(a)}) \frac{\partial}{\partial x_{(a)}^i},$$

which is a diagonal prolongation if and only if there exist functions $B^j : x \in \mathbb{R}^n \mapsto B^j(x) \in \mathbb{R}$, with $j = 1, \dots, n$, such that for each pair of indexes j and a ,

$$\sum_{\alpha=1}^r b_\alpha(x_{(0)}, \dots, x_{(m)}) A_\alpha^i(t, x_{(a)}) = B^i(x_{(a)}), \quad a = 0, \dots, m, \quad i = 1, \dots, n.$$

In particular, the functions $b_\alpha(x_{(0)}, \dots, x_{(m)})$, with $\alpha = 1, \dots, r$, solve the subsystem of linear equations in the variables, u_1, \dots, u_r , given by

$$\sum_{\alpha=1}^r u_\alpha A_\alpha^i(x_{(a)}) = B^i(x_{(a)}), \quad a = 1, \dots, m, \quad i = 1, \dots, n.$$

The coefficient matrix of the above system of $m \cdot n$ equations with r unknowns has rank $r \leq m \cdot n$ since the $pr_*(\widehat{X}_\alpha)$ are linearly independent. Hence, the solutions, u_1, \dots, u_r , are completely determined in terms of the functions $B^i(x_{(a)})$, with $a = 1, \dots, m$, and $i = 1, \dots, n$, and do

not depend on $x_{(0)}$. But since the prolongations are invariant under the action of the symmetry group S_{m+1} , functions $u_\alpha = b_\alpha(x_{(0)}, \dots, x_{(m)})$, with $\alpha = 1, \dots, r$, must satisfy this symmetry. Consequently, they cannot depend on the variables $x_{(1)}, \dots, x_{(m)}$, and therefore they must be constant. ■

LEMMA 1.15. *For every family of vector fields, $X_1, \dots, X_r \in \mathfrak{X}(\mathbb{R}^n)$ linearly independent over \mathbb{R} , there exists an integer m such that their prolongations to \mathbb{R}^{nm} are linearly independent at a generic point.*

Proof. Denote by \widehat{X}_α^q the diagonal prolongation to \mathbb{R}^{nq} of X_α and define $\sigma(q)$ to be the maximum number of vector fields, among the family \widehat{X}_α^q , linearly independent at a generic point of \mathbb{R}^{nq} .

By reduction to the absurd, we assume that each family, $\widehat{X}_1^q, \dots, \widehat{X}_r^q$, of diagonal prolongations are linearly dependent at a generic point of \mathbb{R}^{nq} , in other words, $1 \leq \sigma(q) < r$ for every q . Therefore, the function $\sigma(q)$ must admit a maximum $p < r$ for a certain integer \bar{m} , i.e. $p = \sigma(\bar{m})$. We can assume, without loss of generality, that, $\widehat{X}_1^{\bar{m}}, \dots, \widehat{X}_p^{\bar{m}}$, are linearly independent at generic point of $\mathbb{R}^{n\bar{m}}$. Moreover, the vector fields, $\widehat{X}_1^{\bar{m}+1}, \dots, \widehat{X}_p^{\bar{m}+1}$, are also linearly independent at a generic point of $\mathbb{R}^{n(\bar{m}+1)}$ and, as $\sigma(\bar{m})$ is a maximum, it must be $\sigma(\bar{m} + 1) = \sigma(\bar{m})$. In consequence, there exist p uniquely defined functions $\bar{f}_1, \dots, \bar{f}_p \in C^\infty(\mathbb{R}^{n(\bar{m}+1)})$ obeying the equation

$$\bar{f}_1 \widehat{X}_1^{\bar{m}+1} + \dots + \bar{f}_p \widehat{X}_p^{\bar{m}+1} = \widehat{X}_{p+1}^{\bar{m}+1}. \quad (1.32)$$

This forces the left-hand side to be a diagonal prolongation. Additionally, since $\widehat{X}_1^{\bar{m}}, \dots, \widehat{X}_p^{\bar{m}}$, are linearly independent in a generic point, Lemma (1.14) applies and it turns out that, $\bar{f}_1, \dots, \bar{f}_p$, must be constant. Then, projecting the above expression by pr_0 , it follows that, X_1, \dots, X_{p+1} , are linearly dependent over \mathbb{R} . This violates our initial assumption and thereby we conclude that our initial premise, i.e. $\sigma(q) < r$ for every q , must be false and there must exist an integer m such that the diagonal prolongations of, X_1, \dots, X_r , to \mathbb{R}^{nm} become linearly independent at a generic point, what proves our lemma. ■

The above lemma already contains the key point to prove the following result.

LEMMA 1.16. *If $\sigma(q) < r$, then $\sigma(q) < \sigma(q + 1)$.*

Proof. It is immediate that $\sigma(q) \leq \sigma(q + 1)$. Now, by reduction to absurd, if we assume $p = \sigma(q) < r$ and $\sigma(q) = \sigma(q + 1)$, one can pick up, among the \widehat{X}_α^q , a family of p vector fields linearly independent at a generic point of \mathbb{R}^{nq} . We can assume, with no loss of generality, that they are $\widehat{X}_1^q, \dots, \widehat{X}_p^q$. Consequently, as in the above lemma, we can write

$$\bar{f}_1 \widehat{X}_1^{q+1} + \dots + \bar{f}_p \widehat{X}_p^{q+1} = \widehat{X}_{p+1}^{q+1},$$

for certain uniquely defined functions $\bar{f}_1, \dots, \bar{f}_p \in C^\infty(\mathbb{R}^{n(m+1)})$. In a similar way to the proof of the former lemma, this yields that, X_1, \dots, X_{p+1} , are linearly dependent over \mathbb{R} . This is in contradiction with our initial assumption. In consequence, if $p < r$, the vector field \widehat{X}_{p+1}^{q+1} is linearly independent at a generic point with respect to the previous vector fields and $\sigma(q + 1) > \sigma(q)$. ■

Taking into account the above two lemmas, it follows trivially that $\sigma(q)$ grows monotonically until it reaches the maximum r . This gives rise to the following proposition.

PROPOSITION 1.17. *For every family of vector fields $X_1, \dots, X_r \in \mathfrak{X}(\mathbb{R}^n)$ linearly independent over \mathbb{R} , there exists an integer $m \leq r$ such that their prolongations to \mathbb{R}^{nm} are linearly independent at a generic point.*

The above proposition constitutes an explicit proof for vector fields over \mathbb{R}^n of the analog result for vector fields over manifolds pointed out in [38]. Let us now turn to describe a geometric interpretation of the superposition rule notion.

Consider a t -dependent vector field (1.1) associated with the system

$$\frac{dx^i}{dt} = X^i(t, x), \quad i = 1, \dots, n, \quad (1.33)$$

describing its integral curves. Recall that the above system admits a superposition rule if there exists a map $\Phi : \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$ of the form $x = \Phi(x_{(1)}, \dots, x_{(m)}; k_1, \dots, k_n)$ such that its general solution, $x(t)$, can be written as

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)}(t); k_1, \dots, k_n),$$

with, $x_{(1)}(t), \dots, x_{(m)}(t)$, being a generic family of particular solutions and k_1, \dots, k_n , a set of constants associated with each particular solution.

The map $\Phi(x_{(1)}, \dots, x_{(m)}; \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ can be inverted, at least locally around points of an open dense subset of \mathbb{R}^{nm} , to give rise to a map $\Psi : \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$,

$$k = \Psi(x_{(0)}, \dots, x_{(m)}),$$

where we write $x_{(0)}$ instead of x and $k = (k_1, \dots, k_n)$ in order to simplify the notation. Note that the map Ψ is defined so that

$$k = \Psi(\Phi(x_{(1)}, \dots, x_{(m)}; k), x_{(1)}, \dots, x_{(m)}).$$

Hence, the map Ψ defines an n -codimensional foliation on the manifold $\mathbb{R}^{n(m+1)}$.

As the fundamental property of the map Ψ states that

$$k = \Psi(x_{(0)}(t), \dots, x_{(m)}(t)), \quad (1.34)$$

for any $(m+1)$ -tuple of generic particular solutions of system (1.33), the foliation determined by Ψ is invariant under permutations of its $(m+1)$ arguments, $x_{(0)}, \dots, x_{(m)}$. Moreover, when differentiating expression (1.34) with respect to the variable t , we get

$$\sum_{a=0}^m \sum_{j=1}^n X^j(t, x_{(a)}(t)) \frac{\partial \Psi^k}{\partial x_{(a)}^j}(\bar{p}(t)) = \widehat{X}_t \Psi^k(\bar{p}(t)) = 0, \quad k = 1, \dots, n,$$

where $(\Psi^1, \dots, \Psi^n) = \Psi$ and $\bar{p}(t) = (x_{(0)}(t), \dots, x_{(m)}(t))$. Thus, the functions Ψ^1, \dots, Ψ^n are first-integrals for the vector fields $\{\widehat{X}_t\}_{t \in \mathbb{R}}$ defining an n -codimensional foliation \mathfrak{F} over $\mathbb{R}^{n(m+1)}$ such that the vector fields $\{\widehat{X}_t\}_{t \in \mathbb{R}}$ are tangent to its leaves.

The foliation \mathfrak{F} has another important property. Given a leaf \mathfrak{F}_k corresponding to the level set of Ψ determined by $k = (k_1, \dots, k_n) \in \mathbb{R}^n$ and a point $(x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^{mn}$, there exists a unique point $(x_{(0)}, x_{(1)}, \dots, x_{(m)}) \in \mathfrak{F}_k$, namely,

$$(\Phi(x_{(1)}, \dots, x_{(m)}; k), x_{(1)}, \dots, x_{(m)}) \in \mathfrak{F}_k.$$

Consequently, the projection onto the last $m \cdot n$ factors, i.e. the map pr given by (1.3), induces diffeomorphisms between \mathbb{R}^{nm} and each one of the leaves \mathfrak{F}_k . In other words, the foliation \mathfrak{F} is horizontal with respect to the projection pr .

The foliation \mathfrak{F} corresponds to a connection ∇ on the bundle $\text{pr} : \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^{nm}$ with zero curvature. Indeed, the restriction of the projection pr to a leaf gives a one-to-one map that gives rise to a linear map among vector fields on \mathbb{R}^{nm} and ‘horizontal’ vector fields tangent to a leaf.

Note that the knowledge of this connection (foliation) gives us the superposition rule without referring to the map Ψ . If we fix a point $x_{(0)}(0)$ and m particular solutions, $x_{(1)}(t), \dots, x_{(m)}(t)$, then $x_{(0)}(t)$ is the unique point in \mathbb{R}^n such that the point $(x_{(0)}(t), x_{(1)}(t), \dots, x_{(m)}(t))$ belongs to the same leaf as $(x_{(0)}(0), x_{(1)}(0), \dots, x_{(m)}(0))$. Thus, it is only \mathfrak{F} that really matters when the superposition rule is concerned.

On the other hand, if we have a connection ∇ on the bundle

$$\text{pr} : \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^{nm},$$

with zero curvature, i.e. a horizontal distribution ∇ on $\mathbb{R}^{n(m+1)}$ that is involutive and can be integrated to give a foliation on $\mathbb{R}^{n(m+1)}$ such that the vector fields \widehat{X}_t belong to ∇ , then the procedure described above determines a superposition rule for system (1.33). Indeed, let $k \in \mathbb{R}^n$ enumerates smoothly the leaves \mathfrak{F}_k of the foliation \mathfrak{F} , then we can define $\Phi(x_{(1)}, \dots, x_{(m)}; k) \in \mathbb{R}^n$ to be the unique point $x_{(0)}$ of \mathbb{R}^n such that

$$(x_{(0)}, x_{(1)}, \dots, x_{(m)}) \in \mathfrak{F}_k.$$

This gives rise to a superposition rule $\Phi : \mathbb{R}^{nm} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ for the system of first-order differential equations (1.33). To see this, let us observe the inverse relation

$$\Psi(x_{(0)}, \dots, x_{(m)}) = k,$$

which is equivalent to $(x_{(0)}, \dots, x_{(m)}) \in \mathfrak{F}_k$. If we fix k and take a generic family of particular solutions, $x_{(1)}(t), \dots, x_{(m)}(t)$, of equation (1.33), then $x_{(0)}(t)$, defined with the aid of the condition $\Psi(x_{(0)}(t), \dots, x_{(m)}(t)) = k$, satisfies (1.33). In fact, let $x'_{(0)}(t)$ be the solution of (1.33) with initial value $x'_{(0)} = x_{(0)}$. Since the t -dependent vector fields $\widehat{X}(t, x)$ are tangent to \mathfrak{F} , the curve $(x_{(0)}(t), x_{(1)}(t), \dots, x_{(m)}(t))$ lies entirely within a leaf of \mathfrak{F} , so in \mathfrak{F}_k . But a point of a leaf is entirely determined by its projection by pr , then $x'_{(0)}(t) = x_{(0)}(t)$ and $x_{(0)}(t)$ is a solution.

PROPOSITION 1.18. *Giving a superposition rule depending on m generic particular solutions for a Lie system described by a t -dependent vector field X is equivalent to giving a zero curvature connection ∇ on the bundle $\text{pr} : \mathbb{R}^{(m+1)n} \rightarrow \mathbb{R}^{nm}$ for which the vector fields $\{\widehat{X}_t\}_{t \in \mathbb{R}}$ are horizontal vector fields with respect to this connection.*

Although we rejected to investigate in full detail the difference between global superposition rules and superposition rules, it is interesting to comment briefly this theme here. Note that a rigorous analysis of the above discussion shows that a global or ‘simple’ superposition rule gives rise to a zero curvature connection. Nevertheless, on the contrary, a zero curvature connection *only* ensures the existence of a superposition rule. This is due to the connection, which only guarantees the existence of a series of *local* first-integrals that give rise to a superposition rule. In order to ensure the existence of a global superposition rule, some extra conditions on the connection must be required as well (see [18]).

1.5. Geometric Lie Theorem. Let us now prove the classical Lie theorem [157, Theorem 44] from a modern geometric perspective by using the previous results. The following theorem constitutes a review of the geometric version of the Lie Theorem given in [38, Theorem 1]. Our aim

in doing so is to include in our exposition one of the main results of the theory of Lie systems and, at the same time, to furnish a slightly more detailed proof of this theorem.

MAIN THEOREM 1.19. (Geometric Lie Theorem) *A system (1.33) admits a superposition rule depending on m generic particular solutions if and only if the t -dependent vector field X can be written as*

$$X_t = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha, \quad (1.35)$$

where the vector fields, X_1, \dots, X_r , form a basis for an r -dimensional real Lie algebra.

Proof. Suppose that system (1.33) admits a superposition rule (1.14) and let \mathfrak{F} be its associated foliation over $\mathbb{R}^{n(m+1)}$. As the vector fields $\{\widehat{X}_t\}_{t \in \mathbb{R}}$ are tangent to the leaves of \mathfrak{F} , the vector fields of $\text{Lie}(\{\widehat{X}_t\}_{t \in \mathbb{R}})$ span a generalised involutive distribution

$$\mathcal{D}_p = \left\{ \widehat{Y}(t, p) \mid Y \in \text{Lie}(\{\widehat{X}_t\}_{t \in \mathbb{R}}) \right\} \in \text{T}_p \mathbb{R}^{n(m+1)},$$

whose elements are also tangent to the leaves of \mathfrak{F} . Since the Lie bracket of two prolongations is a prolongation, we can choose, among the elements of $\text{Lie}(\{\widehat{X}_t\}_{t \in \mathbb{R}})$, a finite family, $\widehat{X}_1, \dots, \widehat{X}_r$, that gives rise to a local basis of diagonal prolongations for the distribution \mathcal{D} . As the map pr projects each leaf of the foliation \mathfrak{F} into \mathbb{R}^{nm} diffeomorphically, we get that the vector fields $\text{pr}_*(\widehat{X}_\alpha)$, with $\alpha = 1, \dots, r$, are linearly independent at a generic point of \mathbb{R}^{nm} . These vector fields close on the commutation relations

$$[\widehat{X}_\alpha, \widehat{X}_\beta] = \sum_{\gamma=1}^r f_{\alpha\beta\gamma} \widehat{X}_\gamma, \quad \alpha, \beta = 1, \dots, r,$$

for certain functions $f_{\alpha\beta\gamma} \in C^\infty(\mathbb{R}^{n(m+1)})$. In view of Lemma 1.14, these functions must be constant, let us say $f_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}$, and, taking into account the properties of diagonal prolongations, one has that, X_1, \dots, X_r , are linearly independent vector fields obeying the relations

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} X_\gamma, \quad \alpha, \beta = 1, \dots, r.$$

Since, at each time, \widehat{X}_t is spanned by the vector fields, $\widehat{X}_1, \dots, \widehat{X}_r$, there are t -dependent functions $b_\alpha \in C^\infty(\mathbb{R} \times \mathbb{R}^{n(m+1)})$, with $\alpha = 1, \dots, r$, such that

$$\widehat{X}_t = \sum_{\alpha=1}^r b_\alpha \widehat{X}_\alpha.$$

But each \widehat{X}_t is a diagonal prolongation, so, using Lemma 1.14, one gets that the functions, b_1, \dots, b_r , depend only on the time and thus

$$\widehat{X}_t = \sum_{\alpha=1}^r b_\alpha(t) \widehat{X}_\alpha. \quad (1.36)$$

From here, it is immediate that (1.35).

To prove the converse property, assume that the t -dependent vector field X can be put in the form (1.35), where the vector fields, X_1, \dots, X_r , are linearly independent over \mathbb{R} and span a r -dimensional Lie algebra.

As the vector fields, X_1, \dots, X_r , are linearly independent over \mathbb{R} , there exists, in view of Proposition 1.17, a minimal number $m \leq r$, such that their diagonal prolongations to \mathbb{R}^{nm} are linearly independent at a generic point (what yields that $r \leq n \cdot m$). Moreover, the diagonal prolongations, $\hat{X}_1, \dots, \hat{X}_r$, to $\mathbb{R}^{n(m+1)}$ are linearly independent and they form a basis for an involutive distribution \mathcal{D} . This distribution leads to a $(n(m+1) - r)$ -codimensional foliation \mathfrak{F}_0 on $\mathbb{R}^{n(m+1)}$. As the codimension of \mathfrak{F}_0 is at least n , we can consider an n -codimensional foliation \mathfrak{F} whose leaves include those of \mathfrak{F}_0 . The leaves of this foliation project onto the last $m \cdot n$ factors diffeomorphically and they are at least n -codimensional. Hence, according to Proposition 1.18, foliation \mathfrak{F} defines a superposition rule depending on m particular solutions. ■

Note that the converse part of the previous proof shows that all systems described by t -dependent vector fields of the form (1.36) share a common superposition rule. More specifically, all such t -dependent vector fields give rise to the same distribution \mathcal{D} over the same space $\mathbb{R}^{n(m+1)}$, and this straightforwardly ensures the existence of a common superposition rule for all of them. This fact will be analysed more extensively in the second part of our work, where certain families of systems of differential equations that admit a t -dependent common superposition rule, the referred to as *Lie families*, are investigated.

1.6. Determination of superposition rules. Note that the previous geometric demonstration of Lie Theorem also contains information about the superposition rules associated with a Lie system. Let us analyse this fact more carefully.

Consider a Lie system in \mathbb{R}^n associated with a t -dependent vector field X . In view of Lie Theorem, such a t -dependent vector field can be written in the form

$$X(t, x) = \sum_{i=1}^n \sum_{\alpha=1}^r b_{\alpha}(t) X_{\alpha}^i(x) \frac{\partial}{\partial x^i},$$

where the vector fields $X_{\alpha}(x) = \sum_{i=1}^n X_{\alpha}^i(x) \partial / \partial x^i$ span a r -dimensional Lie algebra of vector fields. Now, the geometric proof of Lie Theorem shows that the above decomposition gives rise to a superposition rule depending on m generic particular solutions with $r \leq m \cdot n$. More exactly, the number m coincides with the minimal integer that makes the diagonal prolongations of X_1, \dots, X_r , to \mathbb{R}^{mn} to become linearly independent at a generic point. In different words, the only functions $f_1, \dots, f_r \in C^{\infty}(\mathbb{R}^{nm})$ such that

$$\sum_{\alpha=1}^r f_{\alpha} X_{\alpha}^i(x_{(a)}) = 0, \quad a = 1, \dots, m, \quad i = 1, \dots, n, \quad (1.37)$$

at a generic point $(x_{(1)}, \dots, x_{(k)})$ are $f_1 = \dots = f_r = 0$.

Let us illustrate our above comments by means of a simple example. Consider the Riccati equation

$$\dot{x} = b_1(t) + b_2(t)x + b_3(t)x^2,$$

which describes the integral curves of the t -dependent vector field

$$X_t = b_1(t) \frac{\partial}{\partial x} + b_2(t)x \frac{\partial}{\partial x} + b_3(t)x^2 \frac{\partial}{\partial x}.$$

Recall that the vector fields $\{X_t\}_{t \in \mathbb{R}}$ take values in the three-dimensional Lie algebra V spanned

by the vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x}, \quad X_3 = x^2 \frac{\partial}{\partial x}.$$

Consequently, we can determine the number of particular solutions for a superposition rule for Riccati equations by considering the minimal m such that corresponding system (1.37) admits only the trivial solution. For $m = 2$, this system reads

$$f_1 + f_2 x_{(1)} + f_3 x_{(1)}^2 = 0, \quad f_1 + f_2 x_{(2)} + f_3 x_{(2)}^2 = 0,$$

and it has non-trivial solutions. Nevertheless, the system for the prolongations to \mathbb{R}^3 , that is,

$$f_1 + f_2 x_{(1)} + f_3 x_{(1)}^2 = 0, \quad f_1 + f_2 x_{(2)} + f_3 x_{(2)}^2 = 0, \quad f_1 + f_2 x_{(3)} + f_3 x_{(3)}^2 = 0,$$

does not admit any non-trivial solution because the determinant of the coefficients, i.e.

$$\left| \begin{pmatrix} 1 & x_{(1)} & x_{(1)}^2 \\ 1 & x_{(2)} & x_{(2)}^2 \\ 1 & x_{(3)} & x_{(3)}^2 \end{pmatrix} \right| = (x_{(2)} - x_{(1)})(x_{(2)} - x_{(3)})(x_{(1)} - x_{(3)}),$$

is different from zero when the three points $x_{(1)}$, $x_{(2)}$, and $x_{(3)}$ are different. Thus, we get that $m = 3$ and the superposition rule for the Riccati equation depends on three particular solutions. Obviously, the relations $m \leq \dim V \leq m \cdot n$ are valid in this case.

Once the number m of particular solutions has been determined, the superposition rule can be worked out in terms of first-integrals for the diagonal prolongations, $\widehat{X}_1, \dots, \widehat{X}_r$, over $\mathbb{R}^{n(m+1)}$. Finally, it is worth noting that when the vector fields, $\widehat{X}_1, \dots, \widehat{X}_r$, over $\mathbb{R}^{n(m+1)}$ admit more than n common first-integrals, the system X admits more than one superposition rule (see [38]).

1.7. Mixed superposition rules and constants of the motion. Roughly speaking, a *mixed superposition rule* is a t -independent map describing the general solution of a system of first-order differential equations in terms of a generic family of particular solutions of various systems (generically different ones) of first-order differential equations and a set of constants. Obviously, mixed superposition rules include, as particular instances, the standard superposition rules related to Lie systems.

DEFINITION 1.20. A *mixed superposition rule* for a system of first-order differential equations determined by a t -dependent vector field X over \mathbb{R}^{n_0} is a t -independent map $\Phi : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_0}$ of the form

$$x = \Phi(x_{(1)}, \dots, x_{(m)}; k_1, \dots, k_{n_0}),$$

such that the general solution, $x(t)$, of system X can be written as

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)}(t); k_1, \dots, k_{n_0}),$$

with, $x_{(1)}(t), \dots, x_{(m)}(t)$, being a generic family of curves satisfying that each $x_{(a)}(t)$ is a particular solution of the system determining the integral curves for a t -dependent vector field $X^{(a)}$ over \mathbb{R}^{n_a} , with $a = 1, \dots, m$.

As a particular example of mixed superposition rule, consider the linear system of differential equations

$$\frac{dx^i}{dt} = \sum_{j=1}^n A_j^i(t) x^j + B^i(t), \quad i = 1, \dots, n, \quad (1.38)$$

whose general solution, $x(t)$, can be written as

$$x(t) = y_{(1)}(t) + \sum_{j=1}^n k_j z_{(j)}(t),$$

in terms of one particular solution $y_{(1)}(t)$ of (1.38), any family of n linearly independent particular solutions, $z_{(1)}(t), \dots, z_{(n)}(t)$, of the homogeneous linear system

$$\frac{dz^i}{dt} = \sum_{j=1}^n A_j^i(t) z^j, \quad i = 1, \dots, n,$$

and a set of n constants, k_1, \dots, k_n .

We here aim to give a method to obtain a particular type of mixed superposition rule for a Lie system in terms of particular solutions of another Lie system. Additionally, we relate our results to the commentary given in [38, Remark 5], where it was briefly discussed that the solutions of a certain first-order differential equation on a manifold may be obtained in terms of solutions of other first-order systems by constructing a certain foliation.

Consider the system on \mathbb{R}^{n_0} given by

$$\frac{dx^i}{dt} = \sum_{\alpha=1}^r b_{\alpha}(t) X_{\alpha}^i(x), \quad i = 1, \dots, n_0, \quad (1.39)$$

determining the integral curves of the t -dependent vector field

$$X(t, x) = \sum_{\alpha=1}^r \sum_{i=1}^{n_0} b_{\alpha}(t) X_{\alpha}^i(x) \frac{\partial}{\partial x^i}, \quad (1.40)$$

where the vector fields $X_{\alpha}(x) = \sum_{i=1}^{n_0} X_{\alpha}^i(x) \partial / \partial x^i$, close on a r -dimensional Lie algebra V , i.e. there exist r^3 constants $c_{\alpha\beta\gamma}$ such that

$$[X_{\alpha}, X_{\beta}] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} X_{\gamma}, \quad \alpha, \beta = 1, \dots, r.$$

We here aim to derive a particular type of mixed superposition rule of the form $\Phi : (\mathbb{R}^{n_1})^m \times \mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_0}$ for the above Lie system in such a way that its general solution, $x(t)$, can be expressed as

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)}(t); k_1, \dots, k_n),$$

where, $x_{(1)}(t), \dots, x_{(m)}(t)$, are a generic family of particular solutions of a Lie system determined by a t -dependent vector field $X^{(1)}$ on \mathbb{R}^{n_1} . Let us assume that system $X^{(1)}$ takes the particular form

$$X_t^{(1)} = \sum_{\alpha=1}^r b_{\alpha}(t) X_{\alpha}^{(1)}, \quad (1.41)$$

where the vector fields $X_{\alpha}^{(1)} \in \mathfrak{X}(\mathbb{R}^{n_1})$ obey the same commutation relations as the vector fields X_{α} , that is,

$$[X_{\alpha}^{(1)}, X_{\beta}^{(1)}] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} X_{\gamma}^{(1)}, \quad \alpha, \beta = 1, \dots, r, \quad (1.42)$$

It is important to clarify when such a t -dependent vector field $X^{(1)}$ exists. Let us prove its existence. On one hand, Ado's Theorem states that for every finite-dimensional Lie algebra V , e.g. the one spanned by the vector fields X_α , there exists an isomorphic matrix Lie algebra V_M of $n_1 \times n_1$ square matrices. Now, since the homogeneous linear system

$$\dot{y} = A(t)y,$$

where $A(t)$ takes values in V_M is a Lie system associated with a Lie algebra of vector fields isomorphic to V_M (see [31]), it follows immediately that we can always determine a family of linear vector fields on \mathbb{R}^{n_1} obeying relations (1.42). In terms of this family, we can build up a t -dependent vector field of the form (1.41). Apart from the t -dependent vector field $X_t^{(1)}$ constructed in the aforementioned way, there might exist other ones made of through finite-dimensional Lie algebras of vector fields admitting a basis whose elements obey relations (1.42).

Proposition 1.17 ensures the existence of a minimal m such that the diagonal prolongations of the $X_\alpha^{(1)}$ to $\mathbb{R}^{n_1 m}$ are linearly independent at a generic point. Let us denote such prolongations by

$$\tilde{X}_\alpha = \sum_{a=1}^m X_\alpha^{i(1)}(x_{(a)}) \frac{\partial}{\partial x_{(a)}^i}, \quad \alpha = 1, \dots, r,$$

and define the vector fields on $\tilde{N} = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1 m}$ of the form

$$Y_\alpha = X_\alpha + \sum_{a=1}^m X_\alpha^{i(1)}(x_{(a)}) \frac{\partial}{\partial x_{(a)}^i}, \quad \alpha = 1, \dots, r.$$

where we have considered the vector fields X_α and $X_\alpha^{(1)}$ as vector fields on \tilde{N} in the natural way. From the above definition, one has

$$[Y_\alpha, Y_\beta] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} Y_\gamma, \quad \alpha, \beta = 1, \dots, r.$$

Consequently, the system of differential equations that determines the integral curves of the t -dependent vector field

$$Y_t = \sum_{\alpha=1}^r b_\alpha(t) Y_\alpha,$$

is a Lie system associated with a Vessiot-Guldberg Lie algebra isomorphic to V .

Define the involutive distribution $\tilde{\mathcal{V}}$ on \tilde{N} of the form

$$\tilde{\mathcal{V}}_{\tilde{x}} = \langle (Y_1)_{\tilde{x}}, \dots, (Y_r)_{\tilde{x}} \rangle, \quad \tilde{x} \in \tilde{N},$$

whose rank is r , around a generic point of \tilde{N} . Additionally, as $r \leq m \cdot n_1$, we may choose, at least locally, n_0 common first-integrals of the vector fields, Y_1, \dots, Y_r , giving rise to a n_0 -codimensional local foliation \mathcal{F} over $\mathbb{R}^{n_0} \times \mathbb{R}^{n_1 m}$, whose leaves project diffeomorphically onto $\mathbb{R}^{n_1 m}$ through the projection

$$p : (x, x_{(1)}, \dots, x_{(m)}) \in \tilde{N} \mapsto (x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^{n_1 m}.$$

Additionally, the vector fields Y_α are tangent to the leaves of this foliation.

On one hand, it is immediate that the above results lead to defining a flat connection ∇ on the bundle $p : \tilde{N} \rightarrow \mathbb{R}^{n_1 m}$. On the other hand, as it happened in the case of superposition rules (see Section 1.4), for every point $(x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^{n_1 m}$ and a leave \mathcal{F}_k , with $k = (k_1, \dots, k_{n_0})$,

of the foliation \mathcal{F} , there exists a unique point $x_{(0)}$ in \mathbb{R}^{n_0} such that $(x_{(0)}, x_{(1)}, \dots, x_{(m)}) \in \mathcal{F}_k$. This gives rise to the definition of a map

$$x_{(0)} = \Phi(x_{(1)}, \dots, x_{(m)}; k_1, \dots, k_{n_0}).$$

Mutatis mutandis, the same arguments showed at the end of the Section 1.4 apply here, and it can easily be proved that given a generic set of m particular solutions of system $X^{(1)}$, the general solution of X can be written as

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)}(t); k_1, \dots, k_{n_0}),$$

what shows that Φ is a particular type of mixed superposition rule. In this way, we have also shown that, as claimed in [38, Remark 5], a flat connection ∇ on a bundle of the form $N_0 \times N_1 \times \dots \times N_m \rightarrow N_1 \times \dots \times N_m$ can be used to obtain the solutions of a first-order system in N_0 by means of particular solutions of other first-order systems in N_1, \dots, N_m .

1.8. Differential geometry on Hilbert spaces. In order to provide some basic knowledge to develop the main results of the applications of the theory of Lie systems to Quantum Mechanics, we report in this section some known concepts of the Differential Geometry on infinite-dimensional manifolds. For further details one can consult [51, 60, 138].

As far as Quantum Mechanics is concerned, the separable complex Hilbert space of states \mathcal{H} can be seen as a (infinite-dimensional) real manifold admitting a global chart [23]. Infinite-dimensional manifolds do not enjoy the same geometric properties as finite-dimensional ones, e.g. in the most general case, and given an open $U \subset \mathcal{H}$, there is not a one-to-one correspondence between derivations on $C^\infty(U, \mathbb{R})$ and sections of the tangent bundle TU . Therefore, some explanations must be given before dealing with such manifolds.

On one hand, given a point $\phi \in \mathcal{H}$, a *kinematic tangent vector* with foot point ϕ is a pair (ϕ, ψ) with $\psi \in \mathcal{H}$. We call $T_\phi \mathcal{H}$ the space of all kinematic tangent vectors with foot point ϕ . It consists of all derivatives $\dot{c}(0)$ of smooth curves $c : \mathbb{R} \rightarrow \mathcal{H}$ with $c(0) = \phi$. This fact gives a reason for the name of kinematic.

From the concept of kinematic tangent vector we can provide the definition of smooth kinematic vector fields as follows: A *smooth kinematic vector field* is an element $X \in \mathfrak{X}(\mathcal{H}) \equiv \Gamma(\pi)$, with $T\mathcal{H}$ the so-called *kinematic tangent bundle* and $\pi : T\mathcal{H} \rightarrow \mathcal{H}$ the projection of this bundle. We define a *kinematic vector field* X as a map $X : \mathcal{H} \rightarrow T\mathcal{H}$ such that $\pi \circ X = \text{Id}_{\mathcal{H}}$. Given a $\psi \in \mathcal{H}$, we will denote from now on $X(\psi) = (\psi, X_\psi)$, with X_ψ being the value of $X(\psi)$ in $T_\psi \mathcal{H}$.

Similarly to the Differential Geometry on finite-dimensional manifolds, we say that a kinematic vector field X on \mathcal{H} admits a local flow on an open subset $U \subset \mathcal{H}$ if there exists a map $Fl^X : \mathbb{R} \times U \rightarrow \mathcal{H}$ such that $Fl^X(0, \psi) = \psi$ for all $\psi \in U$ and

$$X_\psi = \frac{d}{ds} \Big|_{s=0} Fl^X(s, \psi) = \frac{d}{ds} \Big|_{s=0} Fl_s^X(\psi),$$

with $Fl_s^X(\psi) = Fl^X(s, \psi)$.

Let us use all these mathematical concepts to study Quantum Mechanics as a geometric theory. Note that the Abelian translation group on \mathcal{H} provides an identification of the tangent space $T_\phi \mathcal{H}$ at any point $\phi \in \mathcal{H}$ with \mathcal{H} itself. Furthermore, through such an identification of \mathcal{H} with $T_\phi \mathcal{H}$ at any $\phi \in \mathcal{H}$, a continuous kinematic vector field is simply a continuous map $X : \mathcal{H} \rightarrow \mathcal{H}$.

Starting with a bounded \mathbb{C} -linear operator A on \mathcal{H} , we can define the kinematic vector field X^A by $X^A_\psi = A\psi \in \mathcal{H} \simeq T_\psi\mathcal{H}$. In other words, we have

$$X^A : \psi \in \mathcal{H} \mapsto (\psi, X\psi) \in T\mathcal{H} \simeq \mathcal{H} \oplus \mathcal{H}.$$

Usually, operators in Quantum Mechanics are neither continuous nor defined on the whole space \mathcal{H} . The most relevant case happens when A is a skew-self-adjoint operator of the form $A = -iH$. The reason is that \mathcal{H} can be endowed with a natural (strongly) symplectic structure, and then such skew-self-adjoint operators are singled out as the linear vector fields that are Hamiltonian. The integral curves of such a Hamiltonian vector field X^A are the solutions of the corresponding Schrödinger equation [23, 51]. Even when A is not bounded, if A is skew-self-adjoint it must be densely defined and, by Stone's Theorem, its integral curves are strongly continuous and defined in all \mathcal{H} .

Additionally, these kinematic vector fields related to skew-self-adjoint operators admit local flows, i.e. any skew-self-adjoint operator A has a local flow

$$Fl_s^A(\psi) = \exp(sA)(\psi) \quad \text{as} \quad \frac{d}{ds}Fl_s^A(\psi) = A\exp(sA)(\psi) = A(Fl_s^A(\psi)). \quad (1.43)$$

We remark that given two constants $\lambda, \mu \in \mathbb{R}$ and two skew-self-adjoint operators A and B , we get that $X^{\lambda A + \mu B} = \lambda X^A + \mu X^B$. Moreover, skew-self-adjoint operators considered as vector fields are fundamental vector fields relative to the usual action of the unitary group $U(\mathcal{H})$ on the Hilbert space \mathcal{H} .

Let us turn to define the Lie bracket of two kinematic vector fields X^A and X^B associated with two skew-self-adjoint operators A and B , correspondingly. In order to simplify the notation, and as it shall be clear from the context, we hereafter denote both the commutator of operators, i.e. $[A, B] = AB - BA$, and the Lie bracket of vector fields $[X^A, X^B]$ in the same way. In view of the previous remarks, we can declare the Lie bracket of vector fields related to skew-self-adjoint operators to be

$$[X^A, X^B] = X^{[B, A]}.$$

It is worth noting that the above formula is equivalent to the standard one

$$[X, Y]_\psi = \left. \frac{1}{2} \frac{d^2}{ds^2} \right|_{t=0} (Fl_{-s}^Y \circ Fl_{-s}^X \circ Fl_s^Y \circ Fl_s^X(\psi)), \quad (1.44)$$

for finite-dimensional Differential Geometry when the right-hand side is properly defined. In-

deed, the above formula yields

$$\begin{aligned}
 [X^A, X^B]_\psi &= \frac{1}{2} \frac{d^2}{ds^2} \Big|_{s=0} \exp(-sB) \exp(-sA) \exp(sB) \exp(sA) (\psi) \\
 &= \frac{1}{2} \frac{d^2}{ds^2} \Big|_{s=0} \left(\sum_{n_1=0}^{\infty} \frac{(-sB)^{n_1}}{n_1!} \right) \left(\sum_{n_2=0}^{\infty} \frac{(-sA)^{n_2}}{n_2!} \right) \\
 &\quad \left(\sum_{n_3=0}^{\infty} \frac{(sB)^{n_3}}{n_3!} \right) \left(\sum_{n_4=0}^{\infty} \frac{(sA)^{n_4}}{n_4!} \right) (\psi) \\
 &= \frac{1}{2} \frac{d^2}{ds^2} \Big|_{s=0} (-s^2 AB + s^2 BA) (\psi) \\
 &= \frac{1}{2} \frac{d^2}{ds^2} \Big|_{s=0} (s^2 [B, A]) (\psi) = [B, A](\psi),
 \end{aligned}$$

when the above expressions are properly defined. From where, we obtain again

$$[X^A, X^B] = -X^{[A, B]}, \quad (1.45)$$

as we defined.

1.9. Quantum Lie systems. The theory of Lie systems can be applied to investigate a particular class of t -dependent Hamiltonians satisfying a specific set of conditions, the so-called *quantum Lie systems*. Let us now precisely define this notion and sketch some of its properties.

We call a t -dependent Hamiltonian $H(t)$ a t -parametric family of self-adjoint operators $H_t : \mathcal{H} \rightarrow \mathcal{H}$.

DEFINITION 1.21. We say that the t -dependent Hamiltonian $H(t)$ is a *quantum Lie system* if it can be written as

$$H(t) = \sum_{\alpha=1}^r b_\alpha(t) H_\alpha, \quad (1.46)$$

where the operators iH_α are a family of skew-self-adjoint operators on \mathcal{H} giving rise to a basis of a real r -dimensional Lie algebra of operators V under the commutator of operators, i.e.

$$[iH_\alpha, iH_\beta] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} iH_\gamma, \quad \alpha, \beta = 1, \dots, r, \quad (1.47)$$

for certain r^3 real structure constants $c_{\alpha\beta\gamma}$. We call V a *quantum Vessiot–Guldberg Lie algebra* associated with $H(t)$.

Each quantum Lie system $H(t)$ leads to a Schrödinger equation

$$\frac{d\psi}{dt} = -iH(t)\psi = -\sum_{\alpha=1}^r b_\alpha(t) iH_\alpha \psi, \quad (1.48)$$

describing the integral curves for the kinematic t -dependent vector field on \mathcal{H} given by

$$X_t = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha,$$

where X_α is the vector field associated with the operator $-iH_\alpha$. In view of the relation (1.45) and the commutation relations (1.47), we obtain

$$[X_\alpha, X_\beta] = -X^{[iH_\alpha, iH_\beta]} = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} X_\gamma, \quad \alpha, \beta = 1, \dots, n. \quad (1.49)$$

In consequence, the vector fields X_α span an r -dimensional Lie algebra of vector fields. In addition, the structure constants for the basis $\{X_\alpha \mid \alpha = 1, \dots, r\}$ coincide with those of the quantum Vessiot–Guldberg Lie algebra for the basis $\{iH_\alpha \mid \alpha = 1, \dots, r\}$.

Given the Lie algebra V , consider an isomorphic Lie algebra \mathfrak{g} corresponding to a connected Lie group G . Choose a basis $\{\mathfrak{a}_\alpha \mid \alpha = 1, \dots, r\}$ of the Lie algebra $T_e G \simeq \mathfrak{g}$ such that the Lie brackets of its elements, denoted by $[\cdot, \cdot]$, obey the relations

$$[\mathfrak{a}_\alpha, \mathfrak{a}_\beta] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} \mathfrak{a}_\gamma, \quad \alpha, \beta = 1, \dots, r. \quad (1.50)$$

It can be proved that there exists a unitary action $\Phi : G \times \mathcal{H} \rightarrow \mathcal{H}$ such that each X_α is the fundamental vector field associated with the element \mathfrak{a}_α , according to the relation (1.50). Indeed, note that, fixed the basis $\{\mathfrak{a}_\alpha \mid \alpha = 1, \dots, r\}$, each element g , in a sufficiently small open U containing the neutral element of G , can be put in a unique way as

$$g = \exp(-\mu_1 \mathfrak{a}_1) \times \dots \times \exp(-\mu_r \mathfrak{a}_r).$$

Now, we define

$$\Phi(\exp(-\mu_\alpha \mathfrak{a}_\alpha), \psi) = \exp(-i\mu_\alpha H_\alpha) \psi, \quad \alpha = 1, \dots, r.$$

As G is connected, every element can be written as a product of elements in U , what, in view of the above relations, gives rise to an action $\Phi : G \times \mathcal{H} \rightarrow \mathcal{H}$.

Similarly to the procedure carried out to show that solving a Lie system reduces to working out a particular solution for an equation in a Lie group (see Section 1.3), it can be proved that solving the Schrödinger equation for a quantum Lie system $H(t)$ reduces to determining the solution of the equation in G given by

$$R_{g^{-1} * g} \dot{g} = - \sum_{\alpha=1}^r b_\alpha(t) \mathfrak{a}_\alpha \equiv \mathfrak{a}(t), \quad g(0) = e.$$

More specifically, the particular solution of the Schrödinger equation (1.48) with initial condition ψ_0 reads $\psi_t = \Phi(g(t), \psi_0)$, where $g(t)$ is the solution of the above equation.

1.10. Superposition rules for second and higher-differential equations. Although the theory of Lie systems is mainly devoted to the study first-order differential equations, it can also be applied to investigate various systems of second-order differential equations, e.g. the so-called SODE Lie systems. This allows us to derive t -dependent and t -independent constants of the motion, exact solutions, superposition rules or mixed superposition rules for these equations, etc. Moreover, our methods to study systems of second-order differential equations can also be generalised to study systems of higher-order differential equations.

Vessiot pioneered the analysis of systems of second-order differential equations by means of the theory of Lie systems [225]. Additionally, this theme was also briefly examined by Winternitz,

Chisholm and Common [77, 202]. Apart from these few works, the analysis of systems of second-order differential equations through the theory of Lie systems was not deeply analysed until the beginning of the XXI century, when the SODE Lie system concept was defined and employed to investigate various systems of second-order differential equations [36, 44, 45, 48, 52, 53]. This allowed us to recover previous results from a new clarifying perspective as well as to obtain some new achievements.

The description of the general solution of systems of second-order differential equations in terms of certain families of particular solutions and sets of constants appears in the study of some systems in Physics and Mathematics [115, 194]. Nevertheless, these results are frequently obtained through *ad hoc* procedures that neither explain their theoretical meaning nor the possibility of their generalisation. This section is concerned with the application of the theory of Lie systems to SODE Lie systems in order to review, through a geometrical unifying approach, some achievements previously obtained in the literature. Not only this provides a deeper theoretical understanding of these works, but it also offers several new achievements concerning these and other related topics.

Recall that the theory of Lie systems initially aimed to study systems of first-order differential equations admitting its general solution to be expressed in terms of certain families of particular solutions and a set of constants. Nevertheless, this property is not exclusive for systems of first-order differential equations. For instance, each second-order differential equation of the form $\ddot{x} = a(t)x$, with $a(t)$ being a t -dependent real function, satisfies that its general solution, $x(t)$, can be cast into the form

$$x(t) = k_1 x_{(1)}(t) + k_2 x_{(2)}(t), \quad (1.51)$$

with, k_1, k_2 , being a set of constants and, $x_{(1)}(t), x_{(2)}(t)$, being a family of particular solutions whose initial conditions $(x_{(1)}(0), \dot{x}_{(1)}(0))$ and $(x_{(2)}(0), \dot{x}_{(2)}(0))$ are two linearly independent vectors of $\mathbb{T}\mathbb{R}$. Note also that such a superposition rule leads to the existence of many other non-linear superposition rules for other systems of second-order differential equations. For instance, the change of variables $y = 1/x$ transforms the previous system into $y\ddot{y} - 2\dot{y}^2 = -a(t)y^2$ admitting, in view of the above linear superposition rule and the above change of variable, its general solution to be written as

$$y(t) = (k_1 y_1^{-1}(t) + k_2 y_2^{-1}(t))^{-1}, \quad (1.52)$$

in terms of certain families, $y_{(1)}(t), y_{(2)}(t)$, of particular solutions and a set of two constants.

Consequently, in view of the previous examples and other ones that can be found, for instance, in [34, 43], it is natural to define superposition rules for second-order differential equations as follows.

DEFINITION 1.22. We say that a second-order differential equation

$$\ddot{x}^i = F^i(t, x, \dot{x}), \quad i = 1, \dots, n, \quad (1.53)$$

on \mathbb{R}^n admits a global superposition rule if there exists a map $\Psi : \mathbb{T}\mathbb{R}^{mn} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ such that its general solution $x(t)$ can be written as

$$x(t) = \Psi(x_{(1)}(t), \dots, x_{(m)}(t), \dot{x}_{(1)}(t), \dots, \dot{x}_{(m)}(t); k_1, \dots, k_{2n}), \quad (1.54)$$

in terms of a generic family, $x_{(1)}(t), \dots, x_{(m)}(t)$, of particular solutions, their derivatives, and a set of $2n$ constants.

In order to understand the previous definition, it is necessary to establish the precise meaning for ‘generic’ in the above statement. Formally, it is said that expression (1.54) is valid for a generic family of particular solutions when it holds for every family of particular solutions, $x_1(t), \dots, x_m(t)$, satisfying that $(x_1(0), \dot{x}_1(0), \dots, x_m(0), \dot{x}_m(0)) \in U$, with U being an open dense subset of $(\mathbb{T}\mathbb{R}^n)^m$.

There exists no characterisation for systems of SODEs of the form (1.53) admitting a superposition rule. In spite of this, there exists a special class of such systems, the so-called *SODE Lie systems* [52], accepting such a property. Even though this fact has been broadly used in the literature, it has been proved very recently [48]. We next furnish the definition of the SODE Lie system along with a proof for showing that every SODE Lie system admits a superposition rule. In addition, some remarks on the interest of this notion and its main properties are discussed.

DEFINITION 1.23. We say that the system of second-order differential equations (1.53) is a SODE Lie system if the system of first-order differential equations

$$\begin{cases} \dot{x}^i = v^i, \\ \dot{v}^i = F^i(t, x, v), \end{cases} \quad i = 1, \dots, n, \quad (1.55)$$

obtained from (1.53) by defining the new variables $v^i = \dot{x}^i$, with $i = 1, \dots, n$, is a Lie system.

PROPOSITION 1.24. *Every SODE Lie system (1.53) admits a superposition rule $\Psi : (\mathbb{T}\mathbb{R}^n)^m \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ of the form $\Psi = \pi \circ \Phi$, where $\Phi : (\mathbb{T}\mathbb{R}^n)^m \times \mathbb{R}^{2n} \rightarrow \mathbb{T}\mathbb{R}^n$ is a superposition rule for the system (1.55) and $\pi : \mathbb{T}\mathbb{R}^n \rightarrow \mathbb{R}^n$ is the projection associated with the tangent bundle $\mathbb{T}\mathbb{R}^n$.*

Proof. Each SODE Lie system (1.53) is associated with a first-order system of differential equations (2.13) admitting a superposition rule $\Phi : (\mathbb{T}\mathbb{R}^n)^m \times \mathbb{R}^{2n} \rightarrow \mathbb{T}\mathbb{R}^n$. This allows us to describe the general solution $(x(t), v(t))$ of system (1.55) in terms of a generic set $(x_a(t), v_a(t))$, with $a = 1, \dots, m$, of particular solutions and a set of $2n$ constants, i.e.

$$(x(t), v(t)) = \Phi(x_1(t), \dots, x_m(t), v_1(t), \dots, v_m(t); k_1, \dots, k_{2n}). \quad (1.56)$$

Each solution, $x_p(t)$, of the second-order system (1.53) corresponds to one and only one solution $(x_p(t), v_p(t))$ of the system of first-order differential equations (1.55) and vice versa. Furthermore, since one has that $(x_p(t), v_p(t)) = (x_p(t), \dot{x}_p(t))$, it turns out that the general solution $x(t)$ of (1.53) can be written as

$$x(t) = \pi \circ \Phi(x_1(t), \dots, x_m(t), \dot{x}_1(t), \dots, \dot{x}_m(t); k_1, \dots, k_{2n}), \quad (1.57)$$

in terms of a generic family $x_a(t)$, with $a = 1, \dots, m$, of particular solutions of (1.53). That is, the map $\Psi = \pi \circ \Phi$ is a superposition rule for the system of SODEs (1.53). ■

Since every autonomous system is related to a one-dimensional Vessiot–Guldberg Lie algebra [34], a corollary follows immediately.

COROLLARY 1.25. *Every autonomous system of second-order differential equations of the form $\ddot{x}^i = F^i(x, \dot{x})$, with $i = 1, \dots, n$, admits a superposition rule.*

The above result is, in practice, almost useless. Actually, the superposition rule ensured by Proposition 1.24 relies on the derivation of a superposition rule for an autonomous first-order system of differential equations. Applying the method sketched in Section 1.6, it is found that determining this superposition rule implies working out all the integral curves of a vector field on

$(\mathbb{T}\mathbb{R}^n)^2$. Although the solution of this problem is known to exist, its explicit description can be as difficult as solving the initial system (indeed, this is usually the case). Consequently, deriving explicitly a superposition rule for the above autonomous system frequently depends on the search of an alternative superposition rule for the associated first-order system.

Many superposition rules for second-order differential equations do not present an explicit dependence on the derivatives of the particular solutions. Consider, for instance, either the linear superposition rule (1.51) for the equation $\ddot{x} = a(t)x$, or the affine one,

$$x(t) = k_1(x_1(t) - x_2(t)) + k_2(x_2(t) - x_3(t)) + x_3(t),$$

for $\ddot{x} = a(t)x + b(t)$. Such superposition rules are called *velocity free superposition rules* or even *free superposition rules*. The conditions ensuring the existence of such superposition rules is an interesting open problem. Let us provide a brief analysis about the existence of such superposition rules.

PROPOSITION 1.26. *Every system of SODEs (1.53) admitting a free superposition rule is a SODE Lie system.*

Proof.

Suppose that system (1.53) admits a superposition rule of the special form

$$x^i = \Phi_x^i(x_1, \dots, x_m; k_1, \dots, k_{2n}), \quad i = 1, \dots, n. \quad (1.58)$$

In such a case, the general solution, $x(t)$, of the system could be expressed as

$$x^i(t) = \Phi_x^i(x_1(t), \dots, x_m(t); k_1, \dots, k_{2n}), \quad i = 1, \dots, n. \quad (1.59)$$

Define $p(t) = (x_1(t), \dots, x_m(t), \dot{x}_1(t), \dots, \dot{x}_m(t))$ and $v^i = \dot{x}^i$ for $i = 1, \dots, n$. Take the time derivative in the above expression. This yields

$$v^i(t) = \dot{x}^i(t) = \sum_{a=1}^m \sum_{j=1}^n \left(v_a^j(t) \frac{\partial \Phi_x^i}{\partial x_a^j}(p(t)) \right), \quad i = 1, \dots, n, \quad (1.60)$$

where we have used that $\partial \Phi_x^i / \partial v_a^j = 0$, for $i, j = 1, \dots, n$, and $a = 1, \dots, m$. Consequently, there exists a function

$$\Phi_v^i(x_1, \dots, x_m, v_1, \dots, v_m) = \sum_{a=1}^m \sum_{j=1}^n \left(v_a^j \frac{\partial \Phi_x^i}{\partial x_a^j} \right), \quad i = 1, \dots, n,$$

such that

$$\begin{cases} x^i(t) = \Phi_x^i(x_1(t), \dots, x_m(t); k_1, \dots, k_{2n}), \\ v^i(t) = \Phi_v^i(x_1(t), \dots, x_m(t), v_1(t), \dots, v_m(t); k_1, \dots, k_{2n}), \end{cases} \quad i = 1, \dots, n.$$

Therefore, system (2.13) admits a superposition rule and (1.53) becomes a SODE Lie system. ■

Apart from the SODE Lie system notion, there exists another method to study certain second-order differential equations admitting a regular Lagrangian, like Caldirola–Kanai oscillators or Milne–Pinney equations [52, 97]. Although this method cannot be used for studying all systems of second-order differential equations, it provides some additional information that cannot be derived by means of SODE Lie systems when it applies, e.g. information on the t -dependent constants of the motion of the system [97].

1.11. Superposition rules for PDEs. The geometrical formulation of the theory of Lie systems enables us to extend the Lie system notion to partial differential equations. Here, we briefly analyse this generalisation and its properties [38, 185].

Consider the system of first-order PDEs of the form

$$\frac{\partial x^i}{\partial t^a} = X_a^i(t, x), \quad x \in \mathbb{R}^n, \quad t = (t^1, \dots, t^s) \in \mathbb{R}^s, \quad (1.61)$$

whose solutions are maps $x(t) : \mathbb{R}^s \rightarrow \mathbb{R}^n$. When $s = 1$, the above system of PDEs becomes the system of ordinary differential equations (1.33). The main difference between these systems is that for $s > 1$ there exists, in general, no solution with a given initial condition. For a better understanding of this problem, let us put (1.61) in a more general and geometric framework.

Let $P_{\mathbb{R}^n}^s$ be the trivial fibre bundle

$$P_{\mathbb{R}^n}^s = \mathbb{R}^s \times \mathbb{R}^n \rightarrow \mathbb{R}^s.$$

A connection \bar{Y} on this bundle is a horizontal distribution over $TP_{\mathbb{R}^n}^s$, i.e. a s -dimensional distribution transversal to the fibres. This distribution may be determined by the horizontal lifts of the vector fields $\partial/\partial t^a$ on \mathbb{R}^s , i.e.

$$\bar{X}_a(t, x) = \frac{\partial}{\partial t^a} + X_a(t, x),$$

where

$$X_a(t, x) = \sum_{i=1}^n X_a^i(t, x) \frac{\partial}{\partial x^i}.$$

The solutions of system (1.61) can be identified with integral submanifolds of the distribution \bar{X} ,

$$(t, X_a(t, x)), \quad t \in \mathbb{R}^s, \quad x \in \mathbb{R}^n.$$

It is now clear that there is a (obviously unique) solution of (1.61) for every initial data if and only if the distribution \bar{Y} is integrable, i.e. the connection has a trivial curvature. This means that

$$[\bar{X}_a, \bar{X}_b] = \sum_{c=1}^r f_{abc} \bar{X}_c$$

for some functions f_{abc} in $P_{\mathbb{R}^n}^s$. But the commutators $[\bar{X}_a, \bar{X}_b]$ are clearly vertical, while \bar{X}_c are linearly independent horizontal vector fields, so $f_{abc} = 0$, which yields the integrability condition in the form of the system of equations $[\bar{X}_a, \bar{X}_b] = 0$, i.e. in local coordinates,

$$\frac{\partial X_b^i}{\partial t^a}(t, x) - \frac{\partial X_a^i}{\partial t^b}(t, x) + \sum_{j=1}^n \left(X_a^j(t, x) \frac{\partial X_b^i}{\partial x^j}(t, x) - X_b^j(t, x) \frac{\partial X_a^i}{\partial x^j}(t, x) \right) = 0. \quad (1.62)$$

Let us assume now that we analyse a system of first-order PDEs of the form (1.61) that satisfies integrability conditions (1.62). Then, for a given initial value, there exists a unique solution of system (1.61). Furthermore, it is immediate that the geometrical interpretation for superposition rules for first-order described in Section (1.4) can be generalised straightforwardly to the case of PDEs. In consequence, Proposition 1.18 takes now the following form.

PROPOSITION 1.27. *Giving a superposition rule for system (1.61) obeying integrability condition (1.62) is equivalent to giving a connection on the bundle $\text{pr} : \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^{nm}$ with a zero curvature such that the family of vector fields $\{(X_a)_t \mid t \in \mathbb{R}^s, a = 1, \dots, s\}$ are horizontal.*

Also the proof of Lie Theorem remains unchanged. Therefore, we get the following analogous of Lie Theorem for PDEs.

THEOREM 1.28. *The system (1.61) of PDEs defined on \mathbb{R}^n and satisfying the integrability condition (1.62) admits a superposition rule if and only if the vector fields $\{(X_a)_t\}$ on \mathbb{R}^n depending on the parameter $t \in \mathbb{R}^s$, can be written in the form*

$$(X_a)_t = \sum_{\alpha=1}^r u_a^\alpha(t) X_\alpha, \quad a = 1, \dots, s, \quad (1.63)$$

where the vector fields X_α span a finite-dimensional real Lie algebra.

Note that the integrability condition for $Y_a(t, x)$ of the form (1.63) can be written as

$$\sum_{\alpha, \beta, \gamma=1}^r \left[(u_b^\gamma)'(t) - (u_a^\gamma)'(t) + u_a^\alpha(t) u_b^\beta(t) c_{\alpha\beta}^\gamma \right] X_\gamma = 0.$$

We now turn to illustrate the above results by means of a particular example. Consider the following system of partial differential equations on \mathbb{R}^2 associated with the $SL(2, \mathbb{R})$ -action on \mathbb{R} ,

$$\begin{aligned} u_x &= a(x, y)u^2 + b(x, y)u + c(x, y), \\ u_y &= d(x, y)u^2 + e(x, y)u + f(x, y). \end{aligned} \quad (1.64)$$

This equation can be written in the form of a ‘total differential equation’

$$(a(x, y)u^2 + b(x, y)u + c(x, y))dx + (d(x, y)u^2 + e(x, y)u + f(x, y))dy = du.$$

The integrability condition only states that the one-form

$$\omega = (a(x, y)u^2 + b(x, y)u + c(x, y))dx + (d(x, y)u^2 + e(x, y)u + f(x, y))dy$$

is closed for an arbitrary function $u = u(x, y)$. If this is the case, there is a unique solution with the initial condition $u(x_0, y_0) = u_0$ and there is a superposition rule giving a general solution as a function of three independent solutions exactly as in the case of Riccati equations:

$$u = \frac{(u_{(1)} - u_{(3)})u_{(2)}k + u_{(1)}(u_{(3)} - u_{(2)})}{(u_{(1)} - u_{(3)})k + (u_{(3)} - u_{(2)})}.$$

2. SODE Lie systems

We already pointed out that the theory of Lie systems is mainly dedicated to the analysis of systems of first-order differential equations. In spite of this, such a theory can also be applied to studying a variety of systems of second-order differential equations. This can be done in several ways that rely, as a last resort, on using some kind of transformation to convert systems of second-order differential equations into first-order ones [52, 54, 77, 100, 202]. A class of such systems that can be investigated by means of these techniques are the referred to as SODE Lie systems, which were theoretically analysed in Section 1.10. In this chapter, we focus on analysing several instances of SODE Lie systems in order to derive t -independent constants of the motion, exact solutions, superposition rules, and other properties. This allows us not only to study the

mathematical properties of such systems, but also to provide tools to analyse the diverse physical or control systems modelled through such equations.

Among the above applications to SODEs, one must be emphasised: the use of the referred to as *mixed superposition rules*. This recently described notion enables us to express the general solution of SODE Lie systems in terms of particular solutions of the same, or other, SODE Lie systems. In this way, this new concept can be employed to analyse the properties of the general solutions of certain SODEs appearing in the Physics and mathematical literature [115, 194]. As a consequence of such an analysis, new results can be obtained and other known ones will be recovered, in a systematic way, which will enhance their understanding.

The following section is dedicated to the application of the theory of Lie systems to SODE Lie systems in order to review, through a geometrical unifying approach, some results previously obtained in the literature by means of *ad hoc* methods and to provide new ones. The whole chapter can be divided into two parts: The first one is devoted to the application of the geometric theory of Lie systems for deriving superposition rules, constants of the motion and exact solutions for various SODE Lie systems. More specifically, we study t -dependent harmonic oscillators, generalised Ermakov systems and Milne–Pinney equations, providing a new superposition rule for the latter. The second part is concerned with the study and application of mixed superposition rules.

2.1. The harmonic oscillator with t -dependent frequency. Perhaps, the one-dimensional t -dependent frequency harmonic oscillator is the most simple SODE which allows us to illustrate the application of the SODE Lie system notion. Let us make use of this fact to show, clearly, how this notion applies and to analyse thoroughly the properties of such a system.

The equation of the motion for a one-dimensional harmonic oscillator with t -dependent frequency $\omega(t)$ takes the form $\ddot{x} = -\omega^2(t)x$. In view of Definition 1.23, this equation is a SODE Lie system if and only if the system of first-order differential equations

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\omega^2(t)x, \end{cases} \quad (2.1)$$

is a Lie system. This feature depends on the properties of the t -dependent vector field over $\mathbb{T}\mathbb{R}$ given by

$$X(t, x, v) = v \frac{\partial}{\partial x} - \omega^2(t)x \frac{\partial}{\partial v},$$

which describes the integral curves of system (2.1). It is immediate that

$$X_t = X_1 + \omega^2(t)X_3, \quad (2.2)$$

where X_1 and X_3 are the vector fields

$$X_1 = v \frac{\partial}{\partial x}, \quad X_3 = -x \frac{\partial}{\partial v}.$$

These vector fields obey the commutation relations

$$[X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3, \quad [X_1, X_2] = X_1, \quad (2.3)$$

with X_2 being the vector field on $\mathbb{T}\mathbb{R}$ given by

$$X_2 = \frac{1}{2} \left(x \frac{\partial}{\partial x} - v \frac{\partial}{\partial v} \right).$$

According to the commutation relations (2.3) and decomposition (2.2), it follows that X_t defines a Lie system associated with a Vessiot–Guldberg Lie algebra $V = \langle X_1, X_2, X_3 \rangle$. Hence, one-dimensional harmonic oscillators with a t -dependent frequency are SODE Lie systems.

Determining the general solution of every SODE Lie system reduces to working out the solution of an equation on a Lie group. Unsurprisingly, since the general solution of a SODE Lie system is straightforwardly related to the solution of a Lie system whose solution can be obtained from an equation in a Lie group. Let us illustrate our claim in detail through the example of harmonic oscillators.

Since system (2.1) is a Lie system, its general solution can be worked out by means of the solution of an equation on a certain Lie group (see Section 1.3). Recall that as the elements of V are complete, there exists a Lie group action $\Phi_L : G \times \mathbb{T}\mathbb{R} \rightarrow \mathbb{T}\mathbb{R}$ whose fundamental vector fields are exactly those corresponding to V . It is easy to check that this action can be chosen to be $\Phi_L : SL(2, \mathbb{R}) \times \mathbb{T}\mathbb{R} \rightarrow \mathbb{T}\mathbb{R}$, with

$$\Phi_L \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} x \\ v \end{pmatrix} \right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} \alpha x + \beta v \\ \gamma x + \delta v \end{pmatrix}.$$

Indeed, if we take the basis

$$\mathfrak{a}_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{a}_2 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{a}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (2.4)$$

of the Lie algebra of 2×2 traceless matrices (the usual representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$), its elements satisfy the same commutation relations as the vector fields, X_1, X_2, X_3 . Furthermore, it can be easily verified that the vector fields X_1, X_2 and X_3 are the fundamental vector fields associated with the matrices, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$, according to our convention (1.28).

Once the action Φ_L is determined, it enables us to write the general solution $(x(t), v(t))$ of system (2.1) in the form

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \Phi_L \left(g(t), \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \right), \quad \text{with } \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \in \mathbb{T}\mathbb{R}, \quad (2.5)$$

where $g(t)$ is the solution of the Cauchy problem

$$R_{g^{-1}*} \dot{g} = - \sum_{\alpha=1}^3 b_\alpha(t) \mathfrak{a}_\alpha, \quad g(0) = e,$$

on $SL(2, \mathbb{R})$. This immediately gives us the general solution, $x(t)$, of the equation (2.1) from expression (2.5). Moreover, this process is easily generalised to every SODE Lie system.

Apart from the above Lie group approach, the SODE Lie system notion furnishes us with a second approach to investigate one-dimensional t -dependent frequency harmonic oscillators. This is based on determining a superposition rule for the Lie system (2.1).

Recall that a superposition rule for a Lie system can be worked out by means of a set of first-integrals for certain diagonal prolongations of the vector fields of an associated Vessiot–Guldberg Lie algebra V . As it was discussed in Section 1.6, the way to obtain these first-integrals requires to determine the minimal integer m such that the prolongations to \mathbb{R}^{nm} of the elements of a basis of the Lie algebra V become linearly independent at a generic point. This yields that $\dim V \leq m \cdot n$. Additionally, if we consider the diagonal prolongations of such a basis to

$\mathbb{R}^{n(m+1)}$, these elements are again linearly independent at a generic point and a family of $m \cdot n - r$ first-integrals appears. These first-integrals allow us to determine a superposition rule.

We next illustrate the above process by means of the study of harmonic oscillators. In addition, we analyse in parallel the problem of finding t -independent constants of the motion for systems made of some copies of the initial system. This problem will be proved to be related to the above process and, in addition, will permit us to show interesting properties about harmonic oscillators.

Consider two copies of the same one-dimensional harmonic oscillator, i.e.

$$\begin{cases} \ddot{x}_1 &= -\omega^2(t)x_1, \\ \ddot{x}_2 &= -\omega^2(t)x_2. \end{cases} \quad (2.6)$$

This system of SODEs, which corresponds to a two-dimensional isotropic harmonic oscillator with a t -dependent frequency $\omega(t)$, is related to the following system of first-order differential equations

$$\begin{cases} \dot{x}_1 &= v_1, \\ \dot{x}_2 &= v_2, \\ \dot{v}_1 &= -\omega^2(t)x_1, \\ \dot{v}_2 &= -\omega^2(t)x_2. \end{cases} \quad (2.7)$$

Its solutions are the integral curves of the t -dependent vector field

$$X_t^{2d} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} - \omega^2(t)x_1 \frac{\partial}{\partial v_1} - \omega^2(t)x_2 \frac{\partial}{\partial v_2},$$

which is a linear combination

$$X_t^{2d} = X_1^{2d} + \omega^2(t)X_3^{2d}, \quad (2.8)$$

with X_1^{2d} and X_3^{2d} being the vector fields

$$X_1^{2d} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}, \quad X_3^{2d} = -x_1 \frac{\partial}{\partial v_1} - x_2 \frac{\partial}{\partial v_2},$$

satisfying the commutation relations

$$[X_1^{2d}, X_3^{2d}] = 2X_2^{2d}, \quad [X_2^{2d}, X_3^{2d}] = X_3^{2d}, \quad [X_1^{2d}, X_2^{2d}] = X_1^{2d}, \quad (2.9)$$

where X_2 reads

$$X_2^{2d} = \frac{1}{2} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - v_1 \frac{\partial}{\partial v_1} - v_2 \frac{\partial}{\partial v_2} \right).$$

The previous decomposition of the t -dependent vector field X_t^{2d} has been obtained by considering the new vector fields, $X_1^{2d}, X_2^{2d}, X_3^{2d}$, to be diagonal prolongations to $\mathbb{T}\mathbb{R}^2$ of the vector fields, X_1, X_2, X_3 . In this way, we get that the commutation relations (2.9) are the same as (2.3) and, in view of decomposition (2.8), this t -dependent vector field defines a Lie system related to a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

The distribution associated with the Lie system X_t^{2d} , i.e.

$$\mathcal{V}_p^{2d} = \langle (X_1^{2d})_p, (X_2^{2d})_p, (X_3^{2d})_p \rangle, \quad p \in \mathbb{T}\mathbb{R}^2,$$

has rank lower or equal to the dimension of the Lie algebra V . More specifically, it has rank three in an open dense subset of $\mathbb{T}\mathbb{R}^2$. Hence, there exists a local non-trivial first-integral common to all the vector fields of the above distribution. Furthermore, this first-integral is a t -independent constant of the motion of system (2.7). Let us analyse this statement more carefully. Given a

constant of the motion $F : (x_1, v_1, x_2, v_2) \in \mathbb{T}\mathbb{R}^2 \mapsto F(x_1, v_1, x_2, v_2) \in \mathbb{R}$ of system (2.7), it follows that

$$\frac{dF}{dt}(p(t)) = \sum_{j=1}^2 \left(\frac{dx^i}{dt}(t) \frac{\partial F}{\partial x^i}(p(t)) + \frac{dv^i}{dt}(t) \frac{\partial F}{\partial v^i}(p(t)) \right) = X_t^{2d} I(p(t)) = 0,$$

where $p(t) = (x_1(t), v_1(t), x_2(t), v_2(t))$. If F is a first-integral for the system (2.7), whatever $\omega(t)$ is, then F must be a first-integral of the vector fields of X_1^{2d} , X_3^{2d} and, therefore, of X_2^{2d} .

Consequently, there exists, at least locally, a function F that is a constant of the motion for every system (2.7) and such that dF is incident to the distribution generated by the, X_1^{2d} , X_2^{2d} , X_3^{2d} , i.e. $dF(X_1^{2d}) = dF(X_2^{2d}) = dF(X_3^{2d}) = 0$ in a certain dense open subset U of $\mathbb{T}\mathbb{R}^2$.

Since $X_3^{2d}F = 0$, there is a function $\bar{F}(\xi, x_1, x_2)$ such that $F(x_1, x_2, v_1, v_2) = \bar{F}(\xi, x_1, x_2)$, with $\xi = x_1 v_2 - x_2 v_1$. Next, in view of condition $X_1^{2d}\bar{F} = 0$, we have

$$v_1 \frac{\partial \bar{F}}{\partial x_1} + v_2 \frac{\partial \bar{F}}{\partial x_2} = 0$$

and there exists a function $\widehat{F}(\xi)$ such that $\bar{F}(\xi, x_1, x_2) = \widehat{F}(\xi)$. As $2X_2^{2d} = [X_1^{2d}, X_3^{2d}]$, the conditions $X_1^{2d}\widehat{F} = X_3^{2d}\widehat{F} = 0$ imply $X_2^{2d}\widehat{F} = 0$ and hence $F(x_1, x_2, v_1, v_2) = x_1 v_2 - x_2 v_1$ is a first-integral which physically corresponds to the angular momentum. Additionally, this first-integral allows us to solve the second-order differential equation $\ddot{x} = -\omega^2(t)x$ by means of a particular solution. Actually, if $x_1(t)$ is a non-vanishing solution of this equation, every other particular solution $x_2(t)$ gives rise to a particular solution $(x_1(t), v_1(t), x_2(t), v_2(t))$ of system (2.7). As the first-integral F is constant along this particular solution, we have that $x_2(t)$ obeys the equation

$$x_1(t) \frac{dx_2}{dt} = k + \dot{x}_1(t)x_2,$$

whose solution reads

$$x_2(t) = k' x_1(t) + k x_1(t) \int^t \frac{d\zeta}{x_1^2(\zeta)}, \quad (2.10)$$

what gives us the general solution to the t -dependent frequency harmonic oscillator in terms of a particular solution.

In order to look for a superposition rule, we must consider a system made of some copies of (2.1) and obtain at least as many t -independent constants of the motion as the dimension of the initial manifold. Also, it must be possible to obtain the variables of the initial manifold explicitly in terms of the other variables and such constants. Recall that the number m of particular solutions to obtain a superposition rule satisfies that the diagonal prolongations of the vector fields X_1 , X_2 and X_3 to \mathbb{R}^{nm} are linearly independent in a generic point.

In the case of two copies of the t -dependent harmonic oscillator, the condition on the prolongations of the vector fields, X_1, X_2, X_3 , that is, $\lambda_1 X_1^{2d} + \lambda_2 X_2^{2d} + \lambda_3 X_3^{2d} = 0$, implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Therefore, the one-dimensional oscillator admits a superposition rule involving two particular solution and, in view of our previous results, we need to study three copies of the t -dependent harmonic oscillator (2.1) so as to obtain a superposition rule. Consider therefore

the system of first-order ordinary differential equations

$$\begin{cases} \dot{x}_1 = v_1, \\ \dot{v}_1 = -\omega^2(t)x_1, \\ \dot{x}_2 = v_2, \\ \dot{v}_2 = -\omega^2(t)x_2, \\ \dot{x} = v, \\ \dot{v} = -\omega^2(t)x, \end{cases} \quad (2.11)$$

whose solutions are the integral curves for the t -dependent vector vector field

$$X_t^{3d} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v \frac{\partial}{\partial x} - \omega^2(t)x_1 \frac{\partial}{\partial v_1} - \omega^2(t)x_2 \frac{\partial}{\partial v_2} - \omega^2(t)x \frac{\partial}{\partial v},$$

which is a linear combination, $X_t^{3d} = X_1^{3d} + \omega^2(t)X_3^{3d}$, with X_1^{3d} and X_3^{3d} being the vector fields

$$X_1^{3d} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v \frac{\partial}{\partial x}, \quad X_3^{3d} = -x_1 \frac{\partial}{\partial v_1} - x_2 \frac{\partial}{\partial v_2} - x \frac{\partial}{\partial v},$$

obeying the commutation relations

$$[X_1^{3d}, X_3^{3d}] = 2X_2^{3d}, \quad [X_2^{3d}, X_3^{3d}] = X_3^{3d}, \quad [X_1^{3d}, X_2^{3d}] = X_1^{3d},$$

where the vector field X_2^{3d} is defined by

$$X_2^{3d} = \frac{1}{2} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x \frac{\partial}{\partial x} - v_1 \frac{\partial}{\partial v_1} - v_2 \frac{\partial}{\partial v_2} - v \frac{\partial}{\partial v} \right).$$

We can determine the first-integrals F for these three vector fields as solutions of the system of PDEs $X_1^{3d}F = X_3^{3d}F = 0$, because $2X_2^{3d} = [X_1^{3d}, X_3^{3d}]$ and the previous relations automatically imply the condition $X_2^{3d}F = 0$. This last condition yields that there exists a function $\bar{F} : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ such that $F(x_1, x_2, x, v_1, v_2, v) = \bar{F}(\xi_1, \xi_2, x_1, x_2, x)$ with $\xi_1(x_1, x_2, x, v_1, v_2, v) = xv_1 - x_1v$ and $\xi_2(x_1, x_2, x, v_1, v_2, v) = xv_2 - x_2v$. In view of this, the condition $X_1^{3d}F = 0$ transforms into

$$v_1 \frac{\partial \bar{F}}{\partial x_1} + v_2 \frac{\partial \bar{F}}{\partial x_2} + v \frac{\partial \bar{F}}{\partial x} = 0,$$

i.e. the functions ξ_1 and ξ_2 are first-integrals (Of course, $\xi = x_1v_2 - x_2v_1$ is also a first-integral). They produce a superposition rule, because from

$$\begin{cases} xv_2 - x_2v = k_1, \\ x_1v - v_1x = k_2, \end{cases}$$

we get the expected superposition rule for two solutions

$$x = c_1 x_1 + c_2 x_2, \quad v = c_1 v_1 + c_2 v_2, \quad c_i = \frac{k_i}{k}, \quad k = x_1v_2 - x_2v_1.$$

2.2. Generalised Ermakov system. Let us now turn to study the so-called generalised Ermakov system, i.e.

$$\begin{cases} \ddot{x} = \frac{1}{x^3}f(y/x) - \omega^2(t)x, \\ \ddot{y} = \frac{1}{y^3}g(y/x) - \omega^2(t)y, \end{cases} \quad (2.12)$$

which has been broadly studied in [104, 191, 192, 193, 194, 205, 206]. Although this system is, in general, more complex than the standard Ermakov system, which will be discussed later, its analysis is easier from our point of view and it is therefore studied now. More exactly, our aim is to recover by means of our methods its known constant of motion, which is used next to study the Milne–Pinney equation and to obtain a superposition rule.

For the sake of simplicity, let us consider the generalised Ermakov system on \mathbb{R}_+^2 . This system can be written as a system of first-order differential equations

$$\begin{cases} \dot{x} = v_x, \\ \dot{y} = v_y, \\ \dot{v}_x = -\omega^2(t)x + \frac{1}{x^3}f(y/x), \\ \dot{v}_y = -\omega^2(t)y + \frac{1}{y^3}g(y/x), \end{cases} \quad (2.13)$$

in $\mathbb{T}\mathbb{R}_+^2$ by introducing the new variables $v_x = \dot{x}$ and $v_y = \dot{y}$. Therefore, we can study its solutions as the integral curves for a t -dependent vector field X_t on $\mathbb{T}\mathbb{R}_+^2$ of the form

$$X_t = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \left(-\omega^2(t)x + \frac{1}{x^3}f(y/x) \right) \frac{\partial}{\partial v_x} + \left(-\omega^2(t)y + \frac{1}{y^3}g(y/x) \right) \frac{\partial}{\partial v_y},$$

which can be written as a linear combination

$$X_t = N_1 + \omega^2(t) N_3,$$

where N_1 and N_3 are the vector fields

$$N_1 = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \frac{1}{x^3}f(y/x) \frac{\partial}{\partial v_x} + \frac{1}{y^3}g(y/x) \frac{\partial}{\partial v_y} \quad N_3 = -x \frac{\partial}{\partial v_x} - y \frac{\partial}{\partial v_y}.$$

Note that these vector fields generate a three-dimensional real Lie algebra with the third generator

$$N_2 = \frac{1}{2} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - v_x \frac{\partial}{\partial v_x} - v_y \frac{\partial}{\partial v_y} \right).$$

In fact, as

$$[N_1, N_3] = 2N_2, \quad [N_1, N_2] = N_1, \quad [N_2, N_3] = N_3,$$

they generate a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and thus the generalised Ermakov system is a SODE Lie system.

As Lie system (2.13) is associated with an integrable distribution of rank three in a generic point of a four-dimensional manifold, there exists, at least locally, a first-integral, $F : \mathbb{T}\mathbb{R}_+^2 \rightarrow \mathbb{R}$, for any $\omega^2(t)$. Such a first-integral F satisfies $N_i F = 0$ for $i = 1, 2, 3$, but as $[N_1, N_3] = 2N_2$ it is sufficient to impose $N_1 F = N_3 F = 0$ to get $N_2 F = 0$. Then, if $N_3 F = 0$ we have

$$x \frac{\partial F}{\partial v_x} + y \frac{\partial F}{\partial v_y} = 0,$$

and the associated system of characteristics is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dv_x}{x} = \frac{dv_y}{y}.$$

In view of this, we conclude that there exists a function $\bar{F} : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $F(x, y, v_x, v_y) = \bar{F}(x, y, \xi = xv_y - yv_x)$ and, taking this into account, the condition $N_1 F = 0$ reads

$$v_x \frac{\partial \bar{F}}{\partial x} + v_y \frac{\partial \bar{F}}{\partial y} + \left(-\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x) \right) \frac{\partial \bar{F}}{\partial \xi} = 0.$$

We can therefore consider the associated system of characteristics

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{d\xi}{-\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x)},$$

and using that

$$\frac{-y dx + x dy}{\xi} = \frac{dx}{v_x} = \frac{dy}{v_y},$$

we arrive to

$$\frac{-y dx + x dy}{\xi} = \frac{d\xi}{-\frac{y}{x^3} f\left(\frac{y}{x}\right) + \frac{x}{y^3} g\left(\frac{y}{x}\right)},$$

i.e.

$$-\frac{y^2 d\left(\frac{x}{y}\right)}{\xi} = \frac{d\xi}{-\frac{y}{x^3} f\left(\frac{y}{x}\right) + \frac{x}{y^3} g\left(\frac{y}{x}\right)}$$

and integrating we obtain the following first-integral

$$\frac{1}{2} \xi^2 + \int^u \left[-\frac{1}{\zeta^3} f\left(\frac{1}{\zeta}\right) + \zeta g\left(\frac{1}{\zeta}\right) \right] d\zeta = C, \quad (2.14)$$

with $u = x/y$. This first-integral allows us to determine, by means of quadratures, a solution of one subsystem in terms of a solution of the other equation.

2.3. Milne–Pinney equation. We call Milne–Pinney equation the second-order ordinary nonlinear differential equation [163, 182]

$$\ddot{x} = -\omega^2(t)x + \frac{k}{x^3}, \quad (2.15)$$

where k is a non-zero constant. This equation describes the t -evolution of an isotonic oscillator [28, 181] (also called pseudo-oscillator), i.e. an oscillator with an inverse quadratic potential [204]. This oscillator shares with the harmonic one the property of having a period independent of the energy [68], i.e. they are isochronous systems and, in the quantum case, they have an equispaced spectrum [10]. The equation (2.15) appears in the study of certain Friedmann–Lemaître–Robertson–Walker spaces [85], certain scalar field cosmologies [115], and many other works in Physics and Mathematics (see [147] and references therein).

The Milne–Pinney equation is defined on $\mathbb{R}^* \equiv \mathbb{R} - \{0\}$ and it is invariant under parity, i.e. if $x(t)$ is a solution, then $-x(t)$ is a solution too. That means that it is sufficient to restrict ourselves to analysing this equation in \mathbb{R}_+ .

As usual, we can relate the Milne–Pinney equation to a system of first-order differential equations on $T\mathbb{R}_+$

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= -\omega^2(t)x + \frac{k}{x^3}, \end{cases}$$

by introducing a new auxiliary variable $v \equiv \dot{x}$. Then, the t -dependent vector field on $\mathbb{T}\mathbb{R}_+$ describing its integral curves reads

$$X_t = v \frac{\partial}{\partial x} + \left(-\omega^2(t)x + \frac{k}{x^3} \right) \frac{\partial}{\partial v}.$$

This is a Lie system because X_t can be written as $X_t = L_1 + \omega^2(t)L_3$, where the vector fields L_1 and L_3 are given by

$$L_1 = v \frac{\partial}{\partial x} + \frac{k}{x^3} \frac{\partial}{\partial v}, \quad L_3 = -x \frac{\partial}{\partial v},$$

and satisfy

$$[L_1, L_3] = 2L_2, \quad [L_1, L_2] = L_1, \quad [L_2, L_3] = L_3,$$

with

$$L_2 = \frac{1}{2} \left(x \frac{\partial}{\partial x} - v \frac{\partial}{\partial v} \right),$$

i.e. they span a 3-dimensional real Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Let us choose the basis (2.4) for $\mathfrak{sl}(2, \mathbb{R})$, which satisfies the same commutation relations as the vector fields, L_1, L_2, L_3 . Actually, it is possible to show that each L_α is the fundamental vector field corresponding to \mathfrak{a}_α with respect to the action $\Phi : (A, (x, v)) \in SL(2, \mathbb{R}) \times \mathbb{T}\mathbb{R}_+ \mapsto (\bar{x}, \bar{v}) \in \mathbb{T}\mathbb{R}_+$ given by

$$\begin{cases} \bar{x} = \sqrt{\frac{k + [(\beta v + \alpha x)(\delta v + \gamma x) + k(\delta\beta/x^2)]^2}{(\delta v + \gamma x)^2 + k(\delta/x)^2}}, \\ \bar{v} = \kappa \sqrt{(\delta v + \gamma x)^2 + \frac{k\delta^2}{x^2} \left(1 - \frac{x^2}{\delta^2 \bar{x}^2} \right)}, \end{cases} \quad \text{with } A \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where κ is ± 1 or 0 , depending on the initial point (x, v) and the element of the group $SL(2, \mathbb{R})$ that acts on it. In order to obtain an explicit expression for κ in terms of A and (x, v) , we can use the below decomposition for every element of the group $SL(2, \mathbb{R})$

$$A = \exp(-\alpha_1 \mathfrak{a}_1) \exp(\alpha_3 \mathfrak{a}_3) \exp(-\alpha_2 \mathfrak{a}_2) = \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_3 & 1 \end{pmatrix} \begin{pmatrix} e^{\alpha_2/2} & 0 \\ 0 & e^{-\alpha_2/2} \end{pmatrix},$$

from where we obtain that $\alpha_3 = \gamma\delta$ and $\alpha_1 = \beta/\delta$. As we know that

$$\Phi(\exp(-\alpha_2 \mathfrak{a}_2), (x, v))$$

is the integral curve of the vector field L_2 starting from the point (x, v) parametrised by α_2 , it is straightforward to check that

$$(x_1, v_1) \equiv \Phi(\exp(-\alpha_2 \mathfrak{a}_2), (x, v)) = (\exp(\alpha_2/2)x, \exp(-\alpha_2/2)v),$$

and in a similar way

$$(x_2, v_2) \equiv \Phi(\exp(\alpha_3 \mathfrak{a}_3), (x_1, v_1)) = (x_1, \alpha_3 x_1 + v_1).$$

Finally, we want to obtain $(\bar{x}, \bar{v}) = \Phi(\exp(-\alpha_1 \mathfrak{a}_1), (x_2, v_2))$, and taking into account that the integral curves of L_1 satisfy that

$$\frac{x^3 dv}{k} = \frac{dx}{v} = d\alpha_1, \quad (2.16)$$

it turns out that when $k > 0$ we have $\bar{v}^2 + k/\bar{x}^2 = v_2^2 + k/x_2^2 \equiv \lambda$ with $\lambda > 0$. Thus, using this fact and (2.16) we obtain

$$\frac{k^{1/2}dv}{(\lambda - v^2)^{3/2}} = d\alpha_1,$$

and integrating v between v_2 and \bar{v} ,

$$\frac{\bar{v}}{(\lambda - \bar{v}^2)^{1/2}} = \alpha_1 \frac{\lambda}{k^{1/2}} + \frac{v_2}{(\lambda - v_2^2)^{1/2}} = \frac{1}{k^{1/2}} (\alpha_1 \lambda + v_2 |x_2|).$$

As $\kappa = \text{sign}[\bar{v}]$, we see that κ is given by

$$\kappa = \text{sign}[\alpha_1 \lambda + v_2 |x_2|] = \text{sign} \left[\frac{\beta}{\delta} (x\gamma + v\delta)^2 + \frac{k\delta\beta}{x^2} + \frac{|x|}{\delta} (v\delta + x\gamma) \right].$$

System (2.15) has no non-trivial first-integrals independent of $\omega(t)$, i.e. there is no function $I : U \subset \mathbb{T}\mathbb{R}_+ \rightarrow \mathbb{R}$ such that $X_t I = 0$ for X determined by any function $\omega(t)$. This is equivalent to $dI(L_\alpha) = 0$ on an open U , with $\alpha = 1, 2, 3$. Thus, the first-integrals we are looking for hold that dI_p is incident to the involutive distribution $\mathcal{V}_p \simeq \langle (L_1)_p, (L_2)_p, (L_3)_p \rangle$ generated by the fundamental vector fields L_α in U . In almost any point we obtain that $\mathcal{V}_p = \mathbb{T}_p \mathbb{T}\mathbb{R}_+$. Then, as $dI_p = 0$ in a generic point $p \in U \subset \mathbb{T}\mathbb{R}_+$, the only possibility is $dI = 0$ and therefore I is a constant first-integral.

2.4. A new superposition rule for the Milne–Pinney equation. Our aim now is to show that there exists a superposition rule for the Milne–Pinney equation (2.15) for the case $k > 0$ [53, 163, 182] in terms of a pair of its particular solutions [44]. The case $k < 0$ can be analogously described.

In fact, one sees from the first-integral (2.14) that in the particular case of $f = g = k$, if a particular solution x_1 is known, there is a t -dependent constant of motion for the Milne–Pinney equation given by (see e.g. [53]):

$$I_1 = (x_1 \dot{x} - \dot{x}_1 x)^2 + k \left[\left(\frac{x}{x_1} \right)^2 + \left(\frac{x_1}{x} \right)^2 \right], \quad (2.17)$$

If another particular solution x_2 of the equation (2.15) is given, then we have another t -dependent constant of motion

$$I_2 = (x_2 \dot{x} - \dot{x}_2 x)^2 + k \left[\left(\frac{x}{x_2} \right)^2 + \left(\frac{x_2}{x} \right)^2 \right], \quad (2.18)$$

Moreover, the two solutions x_1 and x_2 provide a function of t which is a constant of the motion and generalises the Wronskian W of two solutions of the equation (2.15)

$$I_3 = (x_1 \dot{x}_2 - x_2 \dot{x}_1)^2 + k \left[\left(\frac{x_2}{x_1} \right)^2 + \left(\frac{x_1}{x_2} \right)^2 \right]. \quad (2.19)$$

Remark that for any real number α the inequality $(\alpha - 1/\alpha)^2 \geq 0$ implies

$$\alpha^2 + \frac{1}{\alpha^2} \geq 2,$$

and the equality sign is valid if and only if $|\alpha| = 1$,

$$\alpha^2 + \frac{1}{\alpha^2} = 2 \iff |\alpha| = 1.$$

Therefore, as we have considered $k > 0$, we see that $I_i \geq 2k$, for $i = 1, 2, 3$. Moreover, as the solutions $x_1(t)$ and $x_2(t)$ are different solutions of the Milne–Pinney equation, it turns out that $I_3 > 2k$.

The knowledge of the two first-integrals I_1 and I_2 , together with the constant value of I_3 for a pair of solutions of equation (2.15), can be used to obtain the superposition rule for the Milne–Pinney equation. In fact, given two particular solutions x_1 and x_2 , the first-integral (2.18) allows us to write an explicit expression for \dot{x} in terms of x , x_2 and I_2

$$\dot{x} = \dot{x}_2 \frac{x}{x_2} \pm \sqrt{-k \frac{x^2}{x_2^4} + I_2 \frac{1}{x_2^2} - k \frac{1}{x_2^2}},$$

and using such an expression with the first-integral (2.17), we see, after a careful computation, that x satisfies the following fourth degree equation

$$(I_2^2 - 4k^2)x_1^4 - 2(I_1I_2 - 2I_3k)x_1^2x_2^2 + (I_1^2 - 4k^2)x_2^4 - 2((I_2I_3 - 2I_1k)x_1^2 + (I_1I_3 - 2I_2k)x_2^2)x^2 + (I_3^2 - 4k^2)x^4 = 0, \quad (2.20)$$

where we have used that I_3 is constant along pairs of solutions, $x_1(t)$, $x_2(t)$, of the Milne–Pinney equation.

Hence, we can obtain from the condition (2.20) the expression for the square of the solutions of the Milne–Pinney equation in terms of any pair of its particular positive solutions by means of a superposition rule

$$x^2 = k_1x_1^2 + k_2x_2^2 \pm 2\sqrt{\lambda_{12}[-k(x_1^4 + x_2^4) + I_3x_1^2x_2^2]}, \quad (2.21)$$

where the constants k_1 and k_2 are given by

$$k_1 = \frac{I_2I_3 - 2I_1k}{I_3^2 - 4k^2}, \quad k_2 = \frac{I_1I_3 - 2I_2k}{I_3^2 - 4k^2},$$

and λ_{12} is a constant which reads as follows

$$\lambda_{12} = \lambda_{12}(k_1, k_2; I_3, k) = \frac{k_1k_2I_3 + k(-1 + k_1^2 + k_2^2)}{I_3^2 - 4k^2} = \varphi(I_1, I_2; I_3, k),$$

where the function φ is given by

$$\varphi(I_1, I_2; I_3, k) = \frac{I_1I_2I_3 - (I_1^2 + I_2^2 + I_3^2)k + 4k^3}{(I_3^2 - 4k^2)^2}.$$

It is important to remark that if $k_1 < 0$ then $k_2 > 0$ and if $k_2 < 0$ then $k_1 > 0$, i.e. if $k_1 < 0$ then $I_2I_3 < 2I_1k$, and thus $I_2 < 2kI_1/I_3$. Therefore, $\lambda_2(I_3^2 - 4k^2) = I_1I_3 - 2kI_2 > I_1I_3 - 4k^2I_1/I_3 = I_1(I_3^2 - 4k^2) > 0$, and thus, as $I_3 > 2k$, $k_2 > 0$. Similarly we obtain that $k_2 < 0$ implies $k_1 > 0$.

The parity invariance of (2.15) is displayed by (2.21), which gives us the solutions

$$x^2 = k_1x_1^2 + k_2x_2^2 \pm 2\sqrt{\lambda_{12}[-k(x_1^4 + x_2^4) + I_3x_1^2x_2^2]}. \quad (2.22)$$

In order to ensure that the right-hand term of the above formula is positive, which gives rise to a real solution of the Milne–Pinney equation, the constants k_1 and k_2 in the preceding expression

should satisfy some additional restrictions. In particular, they must obeying

$$\lambda_{12}[-k(x_1^4(0) + x_2^4(0)) + I_3 x_1^2(0)x_2^2(0)] \geq 0$$

and

$$k_1 x_1^2(0) + k_2 x_2^2(0) \pm 2\sqrt{\lambda_{12}[-k(x_1^4(0) + x_2^4(0)) + I_3 x_1^2(0)x_2^2(0)]} > 0.$$

If these conditions are satisfied, then, differentiating expression (2.22) in $t = 0$ for $x_1 = x_1(t)$ and $x_2 = x_2(t)$ solutions of the Milne–Pinney equation (2.15), it can be checked that $\dot{x}(0)$ is also a real constant. As $x(t)$ is a solution with real initial conditions, then $x(t)$ given by (2.22) is real in an interval of t and thus all the obtained conditions are valid in an interval of t .

If we take into account that we have considered $x_2 > 0$, we can simplify the study of such restrictions by writing (2.22) in terms of the variables x_2 and $z = (x_1/x_2)^2$ as

$$x^2 = x_2^2 \left(k_1 z + k_2 \pm 2\sqrt{\lambda_{12}[-k(z^2 + 1) + I_3 z]} \right),$$

and the preceding conditions turn out to be $\lambda_{12}[-k(z^2 + 1) + I_3 z] \geq 0$ and $k_1 z + k_2 \pm 2\sqrt{\lambda_{12}[-k(z^2 + 1) + I_3 z]} > 0$.

Next, in order to get $\lambda_{12}[-k(z^2 + 1) + I_3 z] \geq 0$, we first notice that this expression is not definite because its discriminant is $\lambda_{12}^2(I_3^2 - 4k^2) \geq 0$, and this restricts the possible values of k_1 and k_2 for a given z . With this aim we define the polynomial $P(z)$ given by

$$P(z) = -k(z^2 + 1) + I_3 z,$$

with roots

$$z = z_{\pm} = \frac{I_3 \pm \sqrt{I_3^2 - 4k^2}}{2k},$$

which can be written in terms of the variable $\alpha_3 = I_3/2k$ as

$$z_{\pm} = \alpha_3 \pm \sqrt{\alpha_3^2 - 1}.$$

As $\alpha_3 > 1$, then $\alpha_3 > \sqrt{\alpha_3^2 - 1} > 0$ and thus $z_{\pm} > 0$. The sign of the polynomial $P(z)$ is displayed in Fig. 1.

The region $\mathbb{R}_+ \times \mathbb{R}_+$ splits into three regions,

$$A = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x_1 > \sqrt{z_+} x_2\} \cup \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x_1 < \sqrt{z_-} x_2\},$$

$$B = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \sqrt{z_-} x_2 < x_1 < \sqrt{z_+} x_2\}$$

separated by the region

$$C = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x_1 = \sqrt{z_+} x_2\} \cup \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x_1 = \sqrt{z_-} x_2\}$$

of the straight lines $x_1 = \sqrt{z_+} x_2$ and $x_1 = \sqrt{z_-} x_2$. The condition to make $\lambda_{12}P(z)$ non-negative in region A , where the polynomial P takes negative values, is to choose k_1 and k_2 so that $\lambda_{12}(k_1, k_2, I_3, k) \leq 0$. Similarly, as P is positive in region B we have to choose k_1 and k_2 such that $\lambda_{12}(k_1, k_2, I_3, k) \geq 0$. Finally, as P vanishes in region C , there is no restriction on the coefficients k_1 and k_2 .

Once we have stated the conditions for $\lambda_{12}P(z)$ to be non-negative we still have to impose the condition

$$k_1 z + k_2 \pm 2\sqrt{\lambda_{12}[-k(z^2 + 1) + I_3 z]} > 0. \quad (2.23)$$

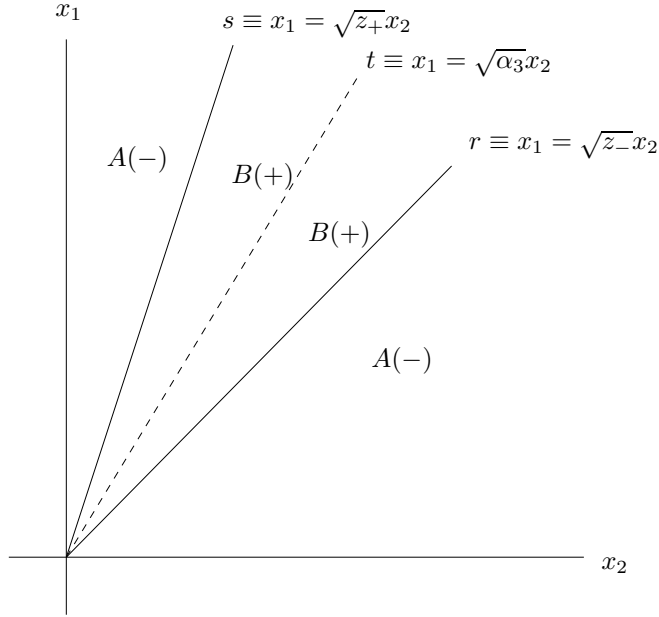


Fig. 1. Sign of the polynomial $P(x_1, x_2)$.

In order to study these conditions, we study the sign of the polynomial

$$\begin{aligned}
 P_{I_3, k}(z, k_1, k_2) &= (k_1 z + k_2)^2 - 4\lambda_{12}[-k(z^2 + 1) + I_3 z] \\
 &= \frac{4P(z)I_3}{I_3^2 - 4k^2} + (ak_1 + bk_2)^2,
 \end{aligned}$$

where

$$a = \sqrt{-\frac{4P(z)k}{I_3^2 - 4k^2} + z^2}, \quad b = \sqrt{1 - \frac{4P(z)k}{I_3^2 - 4k^2}}.$$

As we remarked before, the constants k_1, k_2 cannot be both negative. Let K denote the set

$$K = \mathbb{R}^2 - \{(k_1, k_2) \in \mathbb{R}^2 \mid k_1 < 0, k_2 < 0\}$$

and consider three cases:

1. If $(x_1, x_2) \in A$, then as $P(z) \leq 0$, it must be $\lambda_{12} \leq 0$ in order to satisfy $\lambda_{12}P(z) \geq 0$. In this case, if K_1 and K_2 are the sets

$$\begin{aligned}
 K_1 &= \left\{ (k_1, k_2) \in K; \sqrt{-\frac{4P(z)I_3}{I_3^2 - 4k^2}} > |ak_1 + bk_2| \right\}, \\
 K_2 &= \left\{ (k_1, k_2) \in K; \sqrt{-\frac{4P(z)I_3}{I_3^2 - 4k^2}} < |ak_1 + bk_2| \right\}.
 \end{aligned}$$

We find the following particular cases

- (a) If $(k_1, k_2) \in K_1$, then $P_{I_3, k}(z, k_1, k_2) > 0$.
- (b) If $(k_1, k_2) \in K_2$ then $P_{I_3, k}(z, k_1, k_2) < 0$,

that can be summarised by means of Figure 2.

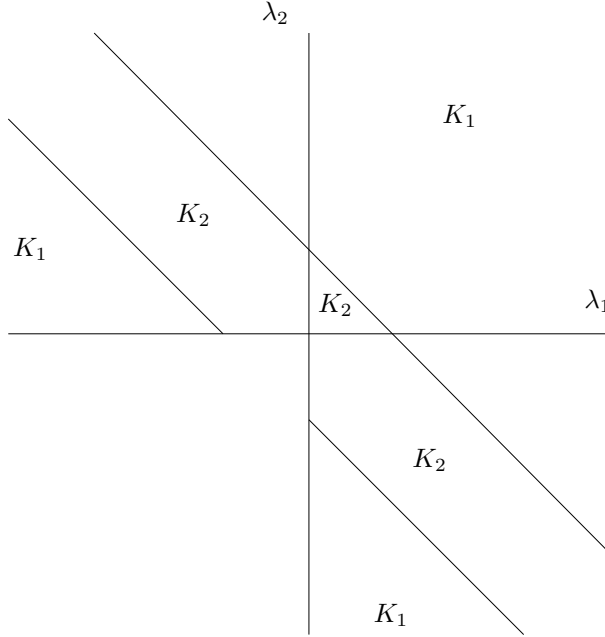


Fig. 2. Sign of the polynomial $P_{I_3,k}(z, k_1, k_2)$ in K .

2. If $(x_1, x_2) \in B$, as $P(z)$ is positive, then λ_{12} must also be positive, $\lambda_{12} \geq 0$. Thus for $(k_1, k_2) \in K_1 \cup K_2$, $P_{I_3,k}(z, k_1, k_2) > 0$.
3. If $(x_1, x_2) \in C$, then for $(k_1, k_2) \in K_1 \cup K_2$, $P_{I_3,k}(z, k_1, k_2) > 0$.

In those cases in which $P_{I_3,k}(z, k_1, k_2) > 0$, we can assert that

$$|k_1 z + k_2| > 2\sqrt{\lambda_{12}[-k(z^2 + 1) + I_3 z]}$$

but we still have to impose that $\lambda_1 z + \lambda_2 > 0$ for (2.23) to be positive. Nevertheless, this is very simple, because if the pair (k_1, k_2) does not satisfy $k_1 z + k_2 > 0$, the pair of opposite elements $(-k_1, -k_2)$ does it, while the other conditions are invariant under the change $k_i \rightarrow -k_i$ with $i = 1, 2$.

In those cases in which $P_{I_3,k}(z, k_1, k_2) < 0$ we can assert that

$$|k_1 z + k_2| < 2\sqrt{\lambda_{12}[-k(x_1^4 + x_2^4) + I_3 x_1^2 x_2^2]}$$

and in this case the unique valid superposition rule is

$$x = |x_2| \left(k_1 z + k_2 + 2\sqrt{\lambda_{12}[-k(z^2 + 1) + I_3 z]} \right)^{1/2},$$

which is equivalent to

$$x = \left(k_1 x_1^2 + k_2 x_2^2 + 2\sqrt{\lambda_{12}[-k(x_1^4 + x_2^4) + I_3 x_1^2 x_2^2]} \right)^{1/2}.$$

Note that if we had considered no restriction on k_1, k_2 , we would have obtained real and imaginary solutions of the Milne–Pinney equation.

Expression (2.22) provides us with a superposition rule for the positive solutions of the Pinney equation (2.15) in terms of two of its independent particular positive solutions. Therefore, once two particular solutions of the equation (2.15) are known, we can write its general solution. Note also that, because of the parity symmetry of (2.15), the superposition (2.22) can be used with both positive and negative solutions. In all these ways we obtain non-vanishing solutions of (2.15) when $k > 0$. *Mutatis mutandis*, the above procedure can also be applied to analyse Milne–Pinney equations when $k < 0$.

A similar superposition rule works for negative solutions of Milne–Pinney equation (2.15):

$$x = - \left(k_1 x_1^2 + k_2 x_2^2 \pm 2 \sqrt{\lambda_{12} (-k(x_1^4 + x_2^4) + I_3 x_1^2 x_2^2)} \right)^{1/2}, \quad (2.24)$$

where once again x_1 and x_2 are arbitrary solutions.

2.5. Painlevé–Ince equations and other SODE Lie systems. In this section we show a new relevant instance of SODE Lie systems including, as particular instances, some Painlevé–Ince equations [93]. In the process of analysing that this particular case of Painlevé–Ince is a SODE Lie system, we find a much larger family of SODE Lie systems which frequently occur in the mathematical and physical literature.

Consider the family of differential equations

$$\ddot{x} + 3x\dot{x} + x^3 = f(t), \quad (2.25)$$

with $f(t)$ being any t -dependent function. The interest in these equations is motivated by their frequent appearance in Physics and Mathematics [66, 71, 134]. The different properties of these equations have been deeply analysed since their first analysis by Vessiot and Wallenberg [224, 229] as a particular case of second-order Riccati equations. For instance, these equations appear in [106] in the study of the Riccati chain. There, it is stated that such equations can be used to derive solutions for certain PDEs. In addition, equation (2.25) also appears in the book by Davis [86], and the particular case with $f(t) = 0$ has recently been treated through geometric methods in [41, 66].

The results described in previous sections can be used to study differential equations (2.25). Let us first show that the above differential equations are SODE Lie systems and, in view of Proposition 1, they admit a superposition rule that is derived. According to definition 1.53, equation (2.25) is a SODE Lie system if and only if the system

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -3xv - x^3 + f(t), \end{cases} \quad (2.26)$$

determining the integral curves of the t -dependent vector field of the form

$$X_{PI}(t, x, v) = X_1(x, v) + f(t)X_2(x, v), \quad (2.27)$$

with

$$X_1 = v \frac{\partial}{\partial x} - (3xv + x^3) \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial v},$$

is a Lie system.

In view of the decomposition (2.27), all equations (2.25) are SODE Lie systems if the vector fields X_1 and X_2 are included in a finite-dimensional real Lie algebra of vector fields V . This happens if and only if $\text{Lie}(\{X_1, X_2\})$ span a finite-dimensional linear space. We consider the family of vector fields on $\mathbb{T}\mathbb{R}$ given by

$$\begin{aligned} X_1 &= v \frac{\partial}{\partial x} - (3xv + x^3) \frac{\partial}{\partial v}, & X_2 &= \frac{\partial}{\partial v}, \\ X_3 &= -\frac{\partial}{\partial x} + 3x \frac{\partial}{\partial v}, & X_4 &= x \frac{\partial}{\partial x} - 2x^2 \frac{\partial}{\partial v}, \\ X_5 &= (v + 2x^2) \frac{\partial}{\partial x} - x(v + 3x^2) \frac{\partial}{\partial v}, & X_6 &= 2x(v + x^2) \frac{\partial}{\partial x} + 2(v^2 - x^4) \frac{\partial}{\partial v}, \\ X_7 &= \frac{\partial}{\partial x} - x \frac{\partial}{\partial v}, & X_8 &= 2x \frac{\partial}{\partial x} + 4v \frac{\partial}{\partial v}, \end{aligned} \quad (2.28)$$

where $X_3 = [X_1, X_2]$, $-3X_4 = [X_1, X_3]$, $X_5 = [X_1, X_4]$, $X_6 = [X_1, X_5]$, $X_7 = [X_2, X_5]$, $X_8 = [X_2, X_6]$, and then the vector fields, X_1, \dots, X_8 , are linearly independent over \mathbb{R} . Their commutation relations read

$$\begin{aligned} [X_1, X_2] &= X_3, & [X_1, X_3] &= -3X_4, & [X_1, X_4] &= X_5, & [X_1, X_5] &= X_6, \\ [X_1, X_6] &= 0, & [X_1, X_7] &= \frac{1}{2}X_8, & [X_1, X_8] &= -2X_1, & [X_2, X_3] &= 0, \\ [X_2, X_4] &= 0, & [X_2, X_5] &= X_7, & [X_2, X_6] &= X_8, & [X_2, X_7] &= 0, \\ [X_2, X_8] &= 4X_2, & [X_3, X_4] &= -X_7, & [X_3, X_5] &= -\frac{1}{2}X_8, & [X_3, X_6] &= -2X_1, \\ [X_3, X_7] &= -2X_2, & [X_3, X_8] &= 2X_3, & [X_4, X_5] &= -X_1, & [X_4, X_6] &= 0, \\ [X_4, X_7] &= X_3, & [X_4, X_8] &= 0, & [X_5, X_6] &= 0, & [X_5, X_7] &= -3X_4, \\ [X_5, X_8] &= -2X_5, & [X_6, X_7] &= -2X_5, & [X_6, X_8] &= -4X_6, & [X_7, X_8] &= 2X_7, \end{aligned} \quad (2.29)$$

In other words, the vector fields, X_1, \dots, X_8 , span an eight-dimensional Lie algebra of vector fields V containing X_1 and X_2 . Therefore, equation (2.25) is a SODE Lie system. Moreover, the elements of the following family of traceless real 3×3 matrices

$$\begin{aligned} M_1 &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, & M_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ M_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, & M_4 &= -\frac{1}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ M_5 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, & M_6 &= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ M_7 &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, & M_8 &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \end{aligned}$$

obey the same commutation relations as the corresponding vector fields, X_1, \dots, X_8 , i.e. the linear map $\rho : \mathfrak{sl}(3, \mathbb{R}) \rightarrow V$, such that $\rho(M_\alpha) = X_\alpha$, with $\alpha = 1, \dots, 8$, is a Lie algebra isomorphism. Consequently, the finite-dimensional Lie algebra of vector fields V is isomorphic

to $\mathfrak{sl}(3, \mathbb{R})$ and the systems of differential equations describing the integral curves for the t -dependent vector fields

$$X(t, x, v) = \sum_{\alpha=1}^8 b_{\alpha}(t) X_{\alpha}(x, v), \quad (2.30)$$

are Lie systems related to a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$.

Many instances of the family of Lie systems (2.30) are associated with interesting SODE Lie systems with applications to Physics or related to remarkable mathematical problems. In all these cases, the theory of Lie systems can be applied to investigate these second-order differential equations, recover some of their known properties, and, possibly, provide new results. Let us illustrate this assertion by means of a few examples.

Another equation appearing in the Physics literature [71, 72, 218] which can be analysed by means of our methods is

$$\ddot{x} + 3x\dot{x} + x^3 + \lambda_1 x = 0, \quad (2.31)$$

which is a special kind of Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$, with $f(x) = 3x$ and $g(x) = x^3 + \lambda_1 x$. The above equation can also be related to a generalised form of an Emden equation occurring in the thermodynamical study of equilibrium configurations of spherical clouds of gas acting under the mutual attraction of their molecules [88].

As in the study of equations (2.25), by considering the new variable $v = \dot{x}$, equation (2.31) becomes the system

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -3xv - x^3 - \lambda_1 x, \end{cases} \quad (2.32)$$

describing the integral curves of the vector field $X = X_1 - \lambda_1/2(X_7 + X_3)$ included in the family (6.13).

Finally, we can also treat the equation

$$\ddot{x} + 3x\dot{x} + x^3 + f(t)(\dot{x} + x^2) + g(t)x + h(t) = 0, \quad (2.33)$$

describing, as particular cases, all the previous examples [134]. The system of first-order differential equations associated with this equation reads

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -3xv - x^3 - f(t)(v + x^2) - g(t)x - h(t). \end{cases} \quad (2.34)$$

Hence, this system describes the integral curves of the t -dependent vector field

$$X_t = X_1 - h(t)X_2 - \frac{1}{4}f(t)(X_8 - 2X_4) - \frac{1}{2}g(t)(X_7 + X_3).$$

Therefore, equation (2.33) is a SODE Lie system and the theory of Lie systems can be used to analyse its properties.

Some particular cases of system (2.33) were pointed out in [72, 134]. Additionally, the case with $f(t) = 0$, $g(t) = \omega^2(t)$ and $h(t) = 0$ was studied in [71] and it is related to harmonic oscillators. The case with $g(t) = 0$ and $h(t) = 0$ appears in the catalogue of equations possessing the Painlevé property [126]. Additionally, our result generalises Vessiot's contribution [225] describing the existence of an expression determining the general solution of a system like (2.33)

(but with constant coefficients) in terms of four of their particular solutions, their derivatives and two constants.

Finally, it is worth noting that the second-order differential equation (2.33) is a particular case of second-order Riccati equations [66, 106]. Such equations were analysed through Lie systems in [77]. The approach carried out in that paper is based on the use of certain *ad hoc* changes of variables which transform second-order Riccati equations into some Lie systems. The advantage of our approach here is that it allows us to study equations (2.33) without using, as it was performed in [77], any *ad hoc* transformations. In addition, our presentation along with the theory of quasi-Lie schemes can be used to perform a quite complete study of second-order Riccati equations in a systematic way [48].

2.6. Mixed superposition rules and Ermakov systems. Let us now turn to show how the theory developed in Section 1.7 for mixed superposition rules works. By adding some, probably different, Lie systems to an initial one, we get new Lie systems that admit constants of motion which do not depend on the t -dependent coefficients of these systems and relate the different solutions of the constituting Lie systems. Moreover, if we add enough copies, these constants of the motion can be used to construct a mixed superposition rule.

We here investigate Ermakov systems. These systems are formed by a second-order homogeneous linear differential equation and a Milne–Pinney equation, i.e.

$$\begin{cases} \ddot{x} &= -\omega^2(t)x + \frac{k}{x^3}, \\ \ddot{y} &= -\omega^2(t)y, \end{cases} \quad (x, y) \in \mathbb{R}_+^2.$$

These systems have been broadly studied in Physics and Mathematics since its introduction until the present day. In Physics they appear in the study of Bose-Einstein condensates and cosmological models [109, 115, 152] and in the solution of t -dependent harmonic or anharmonic oscillators [87, 96, 101, 150, 192, 204]. A lot of works have also been devoted to the usage of Hamiltonian or Lagrangian structures in the study of such systems, see e.g. [194]. Here we recover a constant of the motion, the so-called *Lewis-Ermakov invariant* [150], which appears naturally.

In order to use the theory of Lie systems to analyse Ermakov systems, consider the system of ordinary first-order differential equations [87, 146]

$$\begin{cases} \dot{x} = v_x, \\ \dot{y} = v_y, \\ \dot{v}_x = -\omega^2(t)x + \frac{k}{x^3}, \\ \dot{v}_y = -\omega^2(t)y, \end{cases} \quad (2.35)$$

defined over $\mathbb{T}\mathbb{R}_+^2$ and built by adding the new variables $\dot{x} = v_x$ and $v_y = \dot{y}$ to the Ermakov systems and satisfying the conditions explained in Section 1.7. Its solutions are the integral curves for the t -dependent vector field

$$X_t = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \left(-\omega^2(t)x + \frac{k}{x^3} \right) \frac{\partial}{\partial v_x} - \omega^2(t)y \frac{\partial}{\partial v_y},$$

which is a linear combination with t -dependent coefficients, $X_t = X_1 + \omega^2(t)X_3$, of

$$X_1 = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \frac{k}{x^3} \frac{\partial}{\partial v_x}, \quad X_3 = -x \frac{\partial}{\partial v_x} - y \frac{\partial}{\partial v_y}.$$

Taking into account the vector field

$$X_2 = \frac{1}{2} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - v_x \frac{\partial}{\partial v_x} - v_y \frac{\partial}{\partial v_y} \right),$$

the vector fields X_1, X_2 and X_3 span a three dimensional Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In this way, this system is a SODE Lie system related to a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

The vector fields, L_1, L_2, L_3 , associated with the Milne–Pinney equation (see Section 2.3) span a distribution of rank two on TRR_+ . Consequently, there is no local first-integral I such that $(L_1 + \omega(t)^2(t)L_2)I = 0$ for any given $\omega(t)$. In other words, Milne–Pinney equations do not admit a common t -independent constant of the motion.

By adding the other $\mathfrak{sl}(2, \mathbb{R})$ linear Lie system appearing in the Ermakov system, i.e. the harmonic oscillator with t -dependent angular frequency $\omega(t)$, the distribution spanned by X_1, X_2 and X_3 has rank three over a dense open subset of TRR_+^2 . Therefore, there is a local first-integral. This one can be obtained from $X_1 F = X_3 F = 0$. But $X_3 F = 0$ implies that there exists a function $\bar{F} : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $F(x, y, v_x, v_y) = \bar{F}(x, y, \xi)$, with $\xi = yv_x - xv_y$, and then $X_1 F = 0$ is written

$$v_x \frac{\partial \bar{F}}{\partial x} + v_y \frac{\partial \bar{F}}{\partial y} + k \frac{y}{x^3} \frac{\partial \bar{F}}{\partial \xi},$$

and we obtain the associated system of characteristics

$$k \frac{y dx - x dy}{\xi} = \frac{x^3 d\xi}{y} \implies \frac{d(y/x)}{\xi} + \frac{x d\xi}{ky} = 0.$$

From here, the following first-integral is found [150]

$$\psi(x, y, v_x, v_y) = k \left(\frac{y}{x} \right)^2 + \xi^2 = k \left(\frac{y}{x} \right)^2 + (yv_x - xv_y)^2,$$

which is the well-known Ermakov–Lewis invariant [87, 146, 192].

Once we have obtained a first-integral, we can obtain new constants by adding new copies of any of the systems we have already used. For instance, consider the system of first-order differential equations

$$\begin{cases} \dot{x} = v_x, \\ \dot{y} = v_y, \\ \dot{z} = v_z, \\ \dot{v}_x = -\omega^2(t)x + \frac{k}{x^3}, \\ \dot{v}_y = -\omega^2(t)y, \\ \dot{v}_z = -\omega^2(t)z, \end{cases} \quad (2.36)$$

which corresponds to the vector field

$$X_t = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + \frac{k}{x^3} \frac{\partial}{\partial v_x} - \omega^2(t) \left(x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial v_y} + z \frac{\partial}{\partial v_z} \right).$$

The t -dependent vector field X_t can be expressed as $X_t = N_1 + \omega^2(t)N_3$ where N_1 and N_3 are

$$N_1 = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + \frac{k}{x^3} \frac{\partial}{\partial v_x}, \quad N_3 = -x \frac{\partial}{\partial v_x} - y \frac{\partial}{\partial v_y} - z \frac{\partial}{\partial v_z}.$$

These vector fields generate a three-dimensional real Lie algebra with the vector field N_2 given by

$$N_2 = \frac{1}{2} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - v_x \frac{\partial}{\partial v_x} - v_y \frac{\partial}{\partial v_y} - v_z \frac{\partial}{\partial v_z} \right).$$

In fact, they span a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ because

$$[N_1, N_3] = 2N_2, \quad [N_1, N_2] = N_1, \quad [N_2, N_3] = N_3.$$

The distribution spanned by these fundamental vector fields has rank three in an open dense subset of $\mathbb{T}\mathbb{R}_+^3$. Thus, there exist three local first-integrals for all the vector fields of the latter distribution. In other words, system (2.36) admits three t -independent constants of the motion which turn out to be the Ermakov invariant I_1 of the subsystem involving variables x and y , the Ermakov invariant I_2 of the subsystem involving variables x and z , i.e.

$$I_1 = \frac{1}{2} \left((yv_x - xv_y)^2 + k \left(\frac{y}{x} \right)^2 \right), \quad I_2 = \frac{1}{2} \left((xv_z - zv_x)^2 + k \left(\frac{z}{x} \right)^2 \right),$$

and the Wronskian $W = yv_z - zv_y$ of the subsystem involving variables y and z . They define a foliation with three-dimensional leaves. We can use this foliation to obtain in terms of it a superposition rule. That is reached by describing x in terms of y, z and the integrals I_1, I_2, W , i.e.

$$x = \frac{\sqrt{2}}{|W|} \left(I_2 y^2 + I_1 z^2 \pm \sqrt{4I_1 I_2 - kW^2} yz \right)^{1/2}. \quad (2.37)$$

This can be interpreted, as pointed out by Pinney [182], as saying that there is a superposition rule allowing us to express the general solution of the Milne–Pinney equation in terms of two independent solutions of the corresponding harmonic oscillator with the same t -dependent angular frequency.

2.7. Relations between the new and the known superposition rule. We can now compare the known superposition rule for the Milne–Pinney equation

$$x(t) = \frac{\sqrt{2}}{|W|} \left(I_2 y_1^2(t) + I_1 y_2^2(t) \pm \sqrt{4I_1 I_2 - kW^2} y_1(t)y_2(t) \right)^{1/2}, \quad (2.38)$$

where $y_1(t)$ and $y_2(t)$ are two independent solutions of

$$\ddot{y} = -\omega^2(t)y, \quad (2.39)$$

and (2.22) and check that actually the latter reduces to the former when x_1 and x_2 are obtained from solutions y_1 and y_2 of the associated harmonic oscillator equation.

Let y_1 and y_2 be two solutions of (2.39) and W its Wronskian. Consider the two particular positive solutions of the Milne–Pinney-equation $x_1(t)$ and $x_2(t)$ given by

$$\begin{aligned} x_1(t) &= \frac{\sqrt{2}}{|W|} \sqrt{C_1 y_1^2(t) + C_2 y_2^2(t)}, \\ x_2(t) &= \frac{\sqrt{2}}{|W|} \sqrt{C_2 y_1^2(t) + C_1 y_2^2(t)}, \end{aligned} \quad (2.40)$$

where $C_1 < C_2$ and we additionally impose

$$4C_1C_2 = kW^2. \quad (2.41)$$

The t -dependent constant of the motion I_3 given by (2.19) for the two particular solutions of the Milne–Pinney equation can then be expressed as a function of the solutions y_1 and y_2 of the t -dependent harmonic oscillator and its Wronskian W . After a long computation I_3 turns out to be

$$I_3 = \frac{4(C_1^2 + C_2^2)}{W^2}, \quad (2.42)$$

and then using the explicit form (2.40) of the particular solutions and taking into account the constant (2.42) in (2.22) we obtain that

$$\begin{aligned} k_1x_1^2 + k_2x_2^2 \pm 2\sqrt{\lambda_{12}(-k(x_1^4 + x_2^4) + I_3x_1^2x_2^2)} &= \frac{2}{W^2}(C_1k_1 + C_2k_2)y_1^2 \\ &+ (C_1k_2 + C_2k_1)y_2^2 \pm \frac{2}{W^2}\sqrt{4(C_1k_1 + C_2k_2)(C_1k_2 + C_2k_1) - kW^2}y_1y_2. \end{aligned} \quad (2.43)$$

Consequently, from the superposition rule (2.22), we recover expression (2.37):

$$x = \frac{\sqrt{2}}{|W|} \sqrt{\mu_1y_1^2 + \mu_2y_2^2 \pm \sqrt{4\mu_1\mu_2 - kW^2}y_1y_2}, \quad (2.44)$$

where

$$\begin{cases} \mu_1 = (C_1k_1 + C_2k_2), \\ \mu_2 = (C_1k_2 + C_2k_1). \end{cases}$$

Once we have stated the superposition rule, we still have to analyse the possible values of λ_1 and λ_2 that we can use in this case. If we use the expression (2.42) we obtain after a short calculation the following values z_{\pm}

$$z_+ = \frac{4C_2^2}{kW^2}, \quad z_- = \frac{4C_1^2}{kW^2}. \quad (2.45)$$

Now if we write y_1^2 and y_2^2 in terms of x_1^2, x_2^2 and W from the system (2.40) we obtain

$$\frac{1}{C_1^2 - C_2^2} \begin{pmatrix} C_1 & -C_2 \\ -C_2 & C_1 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix}. \quad (2.46)$$

Therefore, as $C_2 > C_1$ the condition of y_1^2 and y_2^2 being positive is

$$\begin{cases} C_1x_1^2 \leq C_2x_2^2 \\ C_2x_1^2 \geq C_1x_2^2 \end{cases} \quad (2.47)$$

and it is satisfied if $x_1^2/x_2^2 \leq C_2/C_1 = 4C_2^2/kW^2 = z_+$ and $x_1^2/x_2^2 \geq C_1/C_2 = 4C_1^2/kW^2 = z_-$, because of (2.41). Thus, $(x_1, x_2) \in B$ and therefore the only restrictions for k_1, k_2 are $\lambda_{12} \geq 0$ and $k_1x_1^2 + k_2x_2^2 \geq 0$. Obviously, by means of the change of variables (2.40) this last expression is equivalent to $\mu_1y_1^2 + \mu_2y_2^2 \geq 0$ and thus μ_1 and μ_2 cannot be simultaneously negative. Furthermore, $\lambda_{12}(I_3^2 - 4k^2) = 4\mu_1\mu_2 - kW^2$. As we have said that $\lambda_{12} \geq 0$ then $4\mu_1\mu_2 \geq kW^2$, i.e. $\mu_1\mu_2$ is positive and thus, μ_1 and μ_2 are positive. In this way we recover the usual constants of the known superposition rule of the Milne–Pinney equation in terms of solutions of an harmonic oscillator.

2.8. A new mixed superposition rule for the Pinney equation. In this section we derive a mixed superposition rule for the Milne–Pinney equation in terms of a Riccati equation. Consider again the t -dependent Riccati equation

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2 \quad (2.48)$$

which has been studied in [50, 63] from the perspective of the theory of Lie systems. We have already mentioned that this Riccati equation can be considered as the differential equation determining the integral curves for the t -dependent vector field (1.25). This vector field is a linear combination with t -dependent coefficients of the three vector fields, X_1, X_2, X_3 , given by (1.26), which close on a three-dimensional real Lie algebra with defining relations (1.27). Consequently, this Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Note also that the commutation relations (1.27) are the same as (2.3).

Take now the following particular case of Riccati equation

$$\frac{dx}{dt} = 1 + \omega^2(t)x^2.$$

This Riccati equation reads in terms of the X_i as the equation of the integral curves of the t -dependent vector field $X_t = X_1 + \omega^2(t)X_3$. Thus, we can apply the procedure of the Section 1.7 and consider the following differential equation in $\mathbb{R}^3 \times \mathbb{T}\mathbb{R}_+$

$$\begin{cases} \dot{x}_1 = 1 + \omega^2(t)x_1^2, \\ \dot{x}_2 = 1 + \omega^2(t)x_2^2, \\ \dot{x}_3 = 1 + \omega^2(t)x_3^2, \\ \dot{x} = v, \\ \dot{v} = -\omega^2(t)x + \frac{k}{x^3}, \end{cases}$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$, $x \in \mathbb{R}_+$ and $(x, v) \in T_x\mathbb{R}_+$. According to our general recipe, consider the following vector fields

$$\begin{aligned} M_1 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + v \frac{\partial}{\partial x} + \frac{k}{x^3} \frac{\partial}{\partial v}, \\ M_2 &= x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + \frac{1}{2} \left(x \frac{\partial}{\partial x} - v \frac{\partial}{\partial v} \right), \\ M_3 &= x_1^2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2} + x_3^3 \frac{\partial}{\partial x_3} - x \frac{\partial}{\partial v}, \end{aligned}$$

that, by construction, satisfy same commutation relations as before, i.e.

$$[M_1, M_3] = 2M_2, \quad [M_1, M_2] = M_1, \quad [M_2, M_3] = M_3,$$

and the full system of differential equations can be understood as the system of differential equations for the determination of the integral curves of the t -dependent vector field $M(t) = M_1 + \omega^2(t)M_3$. The distribution associated with this Lie system has rank three in almost any point and then there exist locally two first-integrals. As $2M_2 = [M_1, M_3]$, it is enough to find the common first-integrals for M_1 and M_3 , i.e. a function $F : \mathbb{R}^5 \rightarrow \mathbb{R}$ such that $M_1F = M_3F = 0$.

We first look for first-integrals independent of x_3 . i.e. we suppose that F depends just on x_1, x_2, x and v . Using the method of characteristics, the condition $M_3F = 0$ implies that the

characteristics system is

$$\frac{dx_1}{x_1^2} = \frac{dx_2}{x_2^2} = \frac{dv}{-x} = \frac{dx}{0}$$

That means that for such a first-integral for M_3 , which depends on x_1, x_2, x and v , there is a function $\bar{F} : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $F(x_1, x_2, x, v) = \bar{F}(I_1, I_2, I_3)$, with I_1, I_2 and I_3 given by

$$I_1 = \frac{1}{x_1} - \frac{1}{x_2}, \quad I_2 = \frac{1}{x_1} - \frac{v}{x}, \quad I_3 = x.$$

Now, in terms of \bar{F} , the condition $M_1 F = M_1 \bar{F} = 0$ implies

$$v \left(-\frac{2I_1}{I_3} \frac{\partial \bar{F}}{\partial I_1} - \frac{2I_2}{I_3} \frac{\partial \bar{F}}{\partial I_2} + \frac{\partial \bar{F}}{\partial I_3} \right) + (I_1 - 2I_2) I_1 \frac{\partial \bar{F}}{\partial I_1} - \left(I_2^2 + \frac{k}{I_3^4} \right) \frac{\partial \bar{F}}{\partial I_2} = 0. \quad (2.49)$$

Thus the linear term on v and the other one must vanish independently. The method of characteristics applied to the first term implies that there exists a map $\hat{F} : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\bar{F}(I_1, I_2, I_3) = \hat{F}(K_1, K_2)$ where

$$K_1 = \frac{I_1}{I_2}, \quad K_2 = I_2 I_3^2.$$

Finally, taking into account the last result in $M_1 \hat{F} = 0$, we get

$$\left(-K_1^2 - K_1 + \frac{kK_1}{K_2^2} \right) \frac{\partial \hat{F}}{\partial K_1} - \left(K_2 + \frac{k}{K_2} \right) \frac{\partial \hat{F}}{\partial K_2} = 0,$$

and by means of the method of characteristics expression (2.49) involves

$$\frac{dK_1}{dK_2} = \frac{K_1^2 + K_1 - \frac{kK_1}{K_2^2}}{K_2 + \frac{k}{K_2}}$$

which gives us the first-integral

$$C_1 = K_2 + \frac{k + K_2^2}{K_1 K_2},$$

that in terms of the initial variables reads

$$C_1 = \left(x_2 - \frac{v}{x} \right) x^2 + \frac{k + \left(x_2 - \frac{v}{x} \right)^2 x^4}{(x_1 - x_2) x^2}.$$

If we repeat this procedure with the assumption that the integral does not depend on x_2 we obtain the following first-integral

$$C_2 = \left(x_3 - \frac{v}{x} \right) x^2 + \frac{k + \left(x_3 - \frac{v}{x} \right)^2 x^4}{(x_1 - x_3) x^2}.$$

It is a long but easy calculation to check that both are first-integrals of M_1, M_2 and M_3 . We can obtain now the general solution x of the Milne–Pinney equation in terms of x_1, x_2, x_3, C_1, C_2 , as

$$x = \sqrt{\frac{(C_1(x_1 - x_2) - C_2(x_1 - x_3))^2 + k(x_2 - x_3)^2}{(C_2 - C_1)(x_2 - x_3)(x_2 - x_1)(x_1 - x_3)}},$$

where C_1 and C_2 are constants such that, once $x_1(t), x_2(t)$ and $x_3(t)$ have been fixed, they make $x(0)$ given by the latter expression be real.

Thus we have obtained a new mixed superposition rule which enables us to express the general solution of the Pinney equation in terms of three solutions of Riccati equations and, of course, two constants related to initial conditions which determine each particular solution.

3. Applications of quantum Lie systems

In Sections 1.9 and 1.8, it is proved that we can make use of the geometric theory of Lie systems to treat a certain kind of Schrödinger equations, those related to the so-called quantum Lie systems. In this section we use this point of view to investigate Quantum Mechanics.

First, we develop the geometric theory of reduction for quantum Lie systems. Reduction techniques have already been put into practice to study Lie systems [40, 47, 50, 63]. In these works, a variety of reduction methods and other closely related topics are analysed. Most of these methods are based on the properties of a special type of Lie system in a Lie group associated with the Lie system under study. As quantum Lie systems can also be related to such a type of Lie system in a similar way as any Lie system, we can apply most of the methods developed in the aforementioned works to analyse Quantum Lie systems. This is the main purpose of the present section.

In detail, we start by analysing the reduction technique for quantum Lie systems and we complete some previous classic achievements about the topic. We next show that the interaction picture can be explained from this geometrical point of view in terms of this reduction technique. Furthermore, the method of unitary transformations is analysed from our perspective to exemplify that quantum Lie systems associated with solvable Lie algebras of linear operators, in similarity with the classical case, can be exactly solved. On the other hand, systems related to non-solvable Lie algebras can be solved in particular cases. Both cases can be analysed to reproduce some results on the method of unitary transformations in particular cases found in the literature.

3.1. The reduction method in Quantum Mechanics. We here review the reduction techniques explained, for example, in [40, 51, 63]. While in some previous works certain sufficient conditions to perform a reduction process were explained [40, 63], here we show that these conditions are also as necessary [51]. Additionally, we use the geometric reduction technique to explain the interaction picture used in Quantum Mechanics and we review, from a geometric point of view, the method of unitary transformations.

In Section 1.3 it was shown that the study of Lie systems can be reduced to that of finding the solution of the equation

$$R_{g^{-1}*g}\dot{g} = - \sum_{\alpha=1}^r b_{\alpha}(t)a_{\alpha} \equiv a(t) \in T_eG \quad (3.1)$$

with $g(0) = e$.

The reduction method developed in [40] shows that given a solution $\tilde{x}(t)$ of a Lie system on a homogeneous space G/H , the solution of the Lie system in the group G , and therefore the general solution in the given homogeneous space, can be reduced to that of a Lie system in the subgroup H . More specifically, if the curve $\tilde{g}(t)$ in G is such that $\tilde{x}(t) = \Phi(\tilde{g}(t), \tilde{x}(0))$, with Φ being the given action of G in the homogeneous space, then $g(t) = \tilde{g}(t)g'(t)$, where $g'(t)$ turns out to be a curve in H which is a solution of a Lie system in the Lie subgroup H of G . Actually, once the curve $\tilde{g}(t)$ in G has been fixed, the curve $g'(t)$, that takes values in H , satisfies

the equation [40]

$$R_{g'^{-1} *_g g'} \dot{g}' = -\text{Ad}(\tilde{g}^{-1}) \left(\sum_{\alpha=1}^r b_{\alpha}(t) a_{\alpha} + R_{\tilde{g}^{-1} *_g \dot{\tilde{g}}} \right) \equiv a'(t) \in T_e H. \quad (3.2)$$

This transformation law can be understood in the language of the theory of connections. It has been shown in [40, 60] that Lie systems can be related to connections in a bundle and that the group of curves in G , which is the group of automorphisms of the principal bundle $G \times \mathbb{R}$ [60], acts on the left on the set of Lie systems on G , and defines an induced action on the set of Lie systems in each homogeneous space for G . More specifically, if $x(t)$ is a solution of a Lie system in a homogeneous space N defined by the curve $a(t)$ in \mathfrak{g} , then for each curve $\bar{g}(t)$ in G such that $\bar{g}(0) = e$ we see that $x'(t) = \Phi(\bar{g}(t), x(t))$ is a solution of the Lie system defined by the curve

$$a'(t) = R_{\bar{g}^{-1} *_g \dot{\bar{g}}} + \text{Ad}(\bar{g})a(t), \quad (3.3)$$

which is the transformation law for a connection.

In conclusion, the aim of the reduction method is to find an automorphism $\bar{g}(t)$ such that the right-hand side in (3.3) belongs to $T_e H \equiv \mathfrak{h}$ for a certain Lie subgroup H of G . In this way, the papers [40, 60] gave a sufficient condition for obtaining this result. In this section we study the above geometrical development in Quantum Mechanics and we determine a necessary condition for the right-hand side in (3.3) to belong to \mathfrak{h} .

Quantum Lie systems are those t -dependent self-adjoint Hamiltonians such that

$$H(t) = \sum_{\alpha=1}^r b_{\alpha}(t) H_{\alpha}, \quad (3.4)$$

with iH_{α} closing under the commutator of operators on a finite-dimensional real Lie algebra of skew-self-adjoint operators V . Therefore, by regarding these operators as fundamental vector fields of a unitary action of a connected Lie group G with Lie algebra \mathfrak{g} isomorphic to V , we can relate the Schrödinger equation to a differential equation in G determined by curves in $T_e G$ given by $a(t) = -\sum_{\alpha=1}^r b_{\alpha}(t) a_{\alpha}$ by considering $-iH_{\alpha}$ as fundamental vector fields of the basis of \mathfrak{g} given by $\{a_{\alpha} \mid \alpha = 1, \dots, r\}$.

Now, the preceding methods enable us to transform the problem into a new one in the same group G , for each choice of the curve $\bar{g}(t)$ but with a new curve $a'(t)$. The action of G on \mathcal{H} is given by a unitary representation U , and therefore the t -dependent vector field determined by the original t -dependent Hamiltonian $H(t)$ becomes a new one with t -dependent Hamiltonian $H'(t)$. Its integral curves are the solutions of the equation

$$\frac{d\psi'}{dt} = -iH'(t)\psi',$$

where

$$-iH'(t) = -iU(\bar{g}(t))H(t)U^{\dagger}(\bar{g}(t)) + \dot{U}(\bar{g}(t))U^{\dagger}(\bar{g}(t))$$

That is, from a geometric point of view, we have related a Lie system on the Lie group G to certain curve $a(t)$ in $T_e G$ and the corresponding system in \mathcal{H} determined by a unitary representation of G to another one with different curve $a'(t)$ in $T_e G$ and its associated one in \mathcal{H} .

Let us choose a basis of $T_e G$ given by $\{c_{\alpha} \mid \alpha = 1, \dots, r\}$ with $r = \dim \mathfrak{g}$, such that $\{c_{\alpha} \mid \alpha = 1, \dots, s\}$ be a basis of $T_e H$, where $s = \dim \mathfrak{h}$, and denote $\{c^{\alpha} \mid \alpha = 1, \dots, r\}$ the

dual basis of $\{c_\alpha \mid \alpha = 1, \dots, r\}$. In order to find \bar{g} such that the right-hand term of (3.3) belongs to $T_e H$ for all t , the condition for \bar{g} is

$$c^\alpha (\text{Ad}(\bar{g})a(t) + R_{\bar{g}^{-1}*\bar{g}}\dot{\bar{g}}) = 0, \quad \alpha = s+1, \dots, r.$$

Now, if θ^α is the left invariant 1-form on G induced from c^α , the previous equation implies

$$\theta_{\bar{g}^{-1}}^\alpha \left(R_{\bar{g}^{-1}*e}a(t) - \frac{d\bar{g}^{-1}}{dt} \right) = 0, \quad \alpha = s+1, \dots, r.$$

Let be $\tilde{g} = \bar{g}^{-1}$, the latter expression implies that $R_{\tilde{g}*e}a(t) - \dot{\tilde{g}}$ is generated by left invariant vector fields on G from the elements of \mathfrak{h} . Then, given $\pi^L : G \rightarrow G/H$, the kernel of π_*^L is spanned by the left invariant vector fields on G generated by the elements of \mathfrak{h} . Then it follows

$$\pi_{*\tilde{g}}^L (R_{\tilde{g}*e}a(t) - \dot{\tilde{g}}) = 0. \quad (3.5)$$

Therefore, if we use that $\pi_*^L \circ X_\alpha^R = -X_\alpha^L \circ \pi^L$, where X_α^L denotes the fundamental vector field of the action of G in G/H and X_α^R denotes the right-invariant vector field in G whose value in e is a_α , we can prove that $\pi^L(\tilde{g})$ is a solution on G/H of the equation

$$\frac{d\pi^L(\tilde{g})}{dt} = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha^L(\pi^L(\tilde{g})). \quad (3.6)$$

Thus, we obtain that given a certain solution $g'(t)$ in \mathfrak{h} related to the initial $g(t)$ by means of $\tilde{g}(t)$ according to $g(t) = \tilde{g}(t)g'(t)$, then the projection to G/H of $\tilde{g}(t)$, i.e. $\pi^L(\tilde{g}(t))$, is a solution of (3.6). This result shows that whenever $g'(t)$ is a curve in H , then $\tilde{g}(t)$ satisfies equation (3.6). Moreover, as it has been shown in [40], if $\tilde{g}(t)$ satisfies (3.6), then $g'(t)$ is a curve in H satisfying (3.2). The previous result shows that such a condition for obtaining (3.2) is not only sufficient but necessary too. Thus, we provide a new result which completes that one found in [40].

Finally, it is worth noting that even when this last proof has been developed for Quantum Mechanics, it can also be applied to ordinary differential equations, because it appears as a consequence of the group structure of Lie systems which is the same for both quantum and ordinary Lie systems.

3.2. Interaction picture and Lie systems. As a first application of the reduction method for Lie systems, we analyse here how this theory can be applied to explain the interaction picture used in Quantum Mechanics. This picture has been proved to be very effective in the developments of perturbation methods. It plays a rôle when the t -dependent Hamiltonian can be written as a linear combination with t -dependent coefficients of a simpler Hamiltonian H_1 and a perturbation $V(t)$. In the framework of Lie systems, we can analyse what happens when the t -dependent Hamiltonian is

$$H(t) = H_1 + V(t) = H_1 + \sum_{\alpha=2}^r b_\alpha(t) H_\alpha = \sum_{\alpha=1}^r b_\alpha(t) H_\alpha, \quad b_1(t) = 1,$$

where the set of skew-self-adjoint operators $\{-iH_\alpha \mid \alpha = 1, \dots, r\}$ is closed under commutation and generates a finite dimensional real Lie algebra. The situation is very similar to the case of control systems with a drift term (here H_1) that are linear in the control functions. The functions $b_\alpha(t)$ correspond to the control functions.

According to the theory of Lie systems, take a basis $\{a_\alpha \mid \alpha = 1, \dots, r\}$ of the Lie algebra with corresponding associated fundamental vector fields $-iH_\alpha$. The equation to be studied in T_eG is the one (3.1) and whether we define $g'(t) = \bar{g}(t)g(t)$, where $\bar{g}(t)$ is a previously chosen curve, it obeys a similar equation for $g'(t)$ given by (3.3).

If, in particular, we choose $\bar{g}(t) = \exp(a_1 t)$, we find the new equation in T_eG

$$R_{g'^{-1} * g'} \dot{g}' = -\text{Ad}(\exp(a_1 t)) \left(\sum_{\alpha=2}^r b_\alpha(t) a_\alpha \right) = -\exp(\text{ad}(a_1)t) \left(\sum_{\alpha=2}^r b_\alpha(t) a_\alpha \right). \quad (3.7)$$

Correspondingly, the action of G on \mathcal{H} by a unitary representation defines a transformation on \mathcal{H} in which the state ψ_t transforms into $\psi'_t = \exp(iH_1 t)\psi_t$ and its dynamical evolution is given by the vector field corresponding to the right-hand side of (3.7). In particular, if $\{a_2, \dots, a_r\}$ span an ideal of the Lie algebra \mathfrak{g} , the problem reduces to the corresponding normal subgroup in G .

3.3. The method of unitary transformations. A second application of the theory of Lie systems in Quantum Mechanics and, in particular, of the reduction method is to obtain information about how to proceed to solve a quantum Lie Hamiltonian. Let us discuss here a general procedure to accomplish this task.

Every Schrödinger equation of Lie type is determined by a Lie algebra \mathfrak{g} , a unitary representation of its connected and simply connected Lie group G on \mathcal{H} , and a curve $a(t)$ in T_eG . Depending on \mathfrak{g} , there are two cases. If \mathfrak{g} is solvable, we can use the reduction method in Quantum Mechanics to obtain the general solution. If \mathfrak{g} is not solvable, it is not known how to integrate the problem in terms of quadratures in the most general case. Nevertheless, it is possible to solve the problem completely for some specific curves as for instance it happens for the Caldirola–Kanai Hamiltonian [118]. A way of dealing with such systems consist in trying to transform the curve $a(t)$ into another one $a'(t)$, easier to handle, as it has been done in the previous section for the interaction picture. In a more general case than the interaction picture, although any two curves $a(t)$ and $a'(t)$ are always connected by an automorphism, the equation determining the transformation can be as difficult to solve as the initial problem. Because of this, it is interesting to look for a curve that:

1. It determines an easily solvable equation.
2. It can be transformed through an explicitly known transformation into the curve associated with our initial problem.

This is the topic of next three sections, where conditions for such Schrödinger equations are analysed. In any case, we can always express the solution of the initial problem in terms of a solution of the equation determining the transformation. In certain cases, for an appropriate choice of the curve $\bar{g}(t)$ the new curve $a'(t)$ belongs to T_eH for all t , where H is a solvable Lie subgroup of G . In this case we can reduce the problem from \mathfrak{g} to a certain solvable Lie subalgebra \mathfrak{h} of \mathfrak{g} . Of course, in order to do this, a solution of the equation of reduction is needed, but once this is known we can solve the problem completely in terms of it. Other methods have alternatively been used in the literature, like the Lewis-Riesenfeld (LR) method. However, this method seems to offer a complete solution only if \mathfrak{g} is solvable. If \mathfrak{g} is not solvable, the LR method offers a solution which depends on a solution of a system of differential equations, like in the method of reduction.

To sum up, given a Lie system associated with a Lie algebra \mathfrak{g} , whose Lie group G acts, by unitary operators, on \mathcal{H} , and determined by a curve $a(t)$ in T_eG , the systematic procedure to be used is the following:

- If \mathfrak{g} is solvable, we can solve the problem easily by quadratures as it appears in [94, 107].
- If \mathfrak{g} is not solvable, we can try to solve the problem for a given curve like in the Caldirola–Kanai Hamiltonian in [118], by choosing a curve $\bar{g}(t)$ transforming the curve $a(t)$ into another one easier to solve, like in the interaction picture. If this does not work we can try to reduce the problem to an integrable case like in the t -dependent mass and frequency harmonic oscillator or quadratic one dimensional Hamiltonian in [52, 96, 211, 238].

3.4. t -dependent operators for quantum Lie systems. In this section we apply our methods to obtain the t -dependent evolution operators of several problems found in the Physics literature in an algorithmic way.

We first provide a simple example in order to illustrate the main points of our theory. Next, we analyse t -dependent quadratic Hamiltonians. These Hamiltonians describe a very large class of physical models. Sometimes, one of these physical models is described by a certain family of quadratic Hamiltonians associated with a Lie subalgebra of operators of the one given for general quadratic Hamiltonians. If this Lie subalgebra is solvable, the differential equations related to it through the Wei–Norman methods are solvable too and the t -evolution operator can be explicitly obtained. In these cases, we can find the explicit solution of these problems in the literature using different methods for each case. We also describe some approaches to study these quantum Lie systems in the non-solvable cases.

3.5. Initial examples. We start our investigation by studying the motion of a particle with a t -dependent mass under the action of a t -dependent linear potential term. The Hamiltonian describing this physical case is

$$H(t) = \frac{P^2}{2m(t)} + S(t)X.$$

The Lie algebra associated with this example is a central extension of the Heisenberg Lie algebra. A basis for the Lie algebra of vector fields related to this physical model is

$$Z_1 = i\frac{P^2}{2}, \quad Z_2 = iP, \quad Z_3 = iX, \quad Z_4 = iI,$$

which closes on a Lie algebra under the commutation relations

$$\begin{aligned} [Z_1, Z_2] &= 0, & [Z_1, Z_3] &= 2Z_2, & [Z_1, Z_4] &= 0, \\ [Z_2, Z_3] &= Z_4, & [Z_2, Z_4] &= 0, \\ [Z_3, Z_4] &= 0. \end{aligned}$$

This Lie algebra is solvable, and then, the related equations obtained through the Wei–Norman method, can be solved by quadratures for any pair of t -dependent coefficients $m(t)$ and $S(t)$. The solution of the associated Wei–Norman system allows us to obtain the t -evolution operator and the wave function solution of the t -dependent Schrödinger equation.

This t -dependent Hamiltonian has been studied in [221] for some particular cases using *ad-hoc* methods and in general in [94]. Here, we investigate it through the Wei–Norman method. Its

equation in the group G with $T_e G \simeq V$, is

$$R_{g^{-1}*} \dot{g} = -\frac{1}{m(t)} a_1 - S(t) a_3 \equiv a_{MS}(t),$$

where the a_1, \dots, a_4 , are a basis of \mathfrak{g} closing on the same commutation relations as the operators, Z_1, \dots, Z_4 . The factorisation

$$g(t) = \exp(v_2(t)a_2) \exp(-v_3(t)a_3) \exp(-v_4(t)a_4) \exp(-v_1(t)a_1),$$

allows us to solve the equation in G by the Wei–Norman method to get

$$\begin{aligned} \dot{v}_1 &= \frac{1}{m(t)}, \\ \dot{v}_2 &= \frac{v_3}{m(t)}, \\ \dot{v}_3 &= S(t), \\ \dot{v}_4 &= -S(t)v_2 - \frac{v_3^2}{2m(t)}, \end{aligned}$$

with initial conditions $v_1(0) = v_2(0) = v_3(0) = v_4(0) = 0$. The solution of this system can be expressed using quadratures because the related group is solvable

$$\begin{aligned} v_1(t) &= \int_0^t \frac{du}{m(u)}, \\ v_2(t) &= \int_0^t \frac{du}{m(u)} \left(\int_0^u S(v) dv \right), \\ v_3(t) &= \int_0^t S(u) du, \\ v_4(t) &= -\int_0^t S(u) \left(\int_0^u \frac{dv}{m(v)} \left(\int_0^v S(w) dw \right) \right) du - \int_0^t \frac{du}{2m(u)} \left(\int_0^u S(v) dv \right)^2, \end{aligned} \tag{3.8}$$

and the t -evolution operator is

$$\begin{aligned} U(g(t)) &= \exp(v_2(t)Z_2) \exp(-v_3(t)Z_3) \exp(-v_4(t)Z_4) \exp(-v_1(t)Z_1) \\ &= \exp(iv_2(t)P) \exp(-iv_3(t)X) \exp(-iv_4(t)I) \exp(-iv_1(t)\frac{P^2}{2}). \end{aligned}$$

3.6. Quadratic Hamiltonians. After dealing with an easy example before, we can proceed now in a similar way in order to treat the t -dependent quadratic Hamiltonian given by [237] (see [59])

$$H(t) = \alpha(t) \frac{P^2}{2} + \beta(t) \frac{X P + P X}{4} + \gamma(t) \frac{X^2}{2} + \delta(t)P + \epsilon(t) X + \phi(t)I, \tag{3.9}$$

where X and P are the position and momentum operators satisfying the commutation relation

$$[X, P] = iI.$$

It is important to solve this quantum quadratic Hamiltonian because it frequently appears in Quantum Mechanics.

In order to prove that (3.9) is a quantum Lie system, we must check that this t -dependent Hamiltonian can be written as a sum with t -dependent coefficients of some self-adjoint Hamiltonians closing on a real finite-dimensional Lie algebra of operators.

As we can write

$$H(t) = \alpha(t) H_1 + \beta(t) H_2 + \gamma(t) H_3 - \delta(t) H_4 + \epsilon(t) H_5 + \phi(t) H_6,$$

with the Hamiltonians

$$H_1 = \frac{P^2}{2}, \quad H_2 = \frac{1}{4}(XP + PX), \quad H_3 = \frac{X^2}{2}, \\ H_4 = -P, \quad H_5 = X, \quad H_6 = I,$$

satisfying the commutation relations

$$[iH_1, iH_2] = iH_1, \quad [iH_2, iH_3] = iH_3, \quad [iH_3, iH_4] = iH_5, \quad [iH_4, iH_5] = -iH_6, \\ [iH_1, iH_3] = 2i H_2, \quad [iH_2, iH_4] = -\frac{i}{2} H_4, \quad [iH_3, iH_5] = 0, \\ [iH_1, iH_4] = 0, \quad [iH_2, iH_5] = \frac{i}{2} H_5, \\ [iH_1, iH_5] = -iH_4,$$

and $[iH_\alpha, iH_6] = 0$, $\alpha = 1, \dots, 5$, we get that $H(t)$ is a quantum Lie system.

This means that the skew-self-adjoint operators iH_α generate a six-dimensional real Lie V algebra of operators. Now, we can relate them to the basis $\{a_1, \dots, a_6\}$ for an abstract real Lie algebra isomorphic to the one spanned by the $-iH_\alpha$. This basis is chosen in such a way that

$$[a_1, a_2] = a_1, \quad [a_2, a_3] = a_3, \quad [a_3, a_4] = a_5, \quad [a_4, a_5] = -a_6, \quad [a_5, a_6] = 0, \\ [a_1, a_3] = 2a_2, \quad [a_2, a_4] = -\frac{1}{2}a_4, \quad [a_3, a_5] = 0, \quad [a_4, a_6] = 0, \\ [a_1, a_4] = 0, \quad [a_2, a_5] = \frac{1}{2}a_5, \quad [a_3, a_6] = 0, \\ [a_1, a_5] = -a_4, \quad [a_2, a_6] = 0, \\ [a_1, a_6] = 0.$$

This six-dimensional real Lie algebra is a semidirect sum of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ spanned by $\{a_1, a_2, a_3\}$ and the Heisenberg–Weyl Lie algebra generated by $\{a_4, a_5, a_6\}$, which is an ideal.

In order to find the t -evolution provided by the t -dependent Hamiltonian (3.9) we should find the curve $g(t)$ in G , with $T_e G \simeq V$, such that

$$R_{g^{-1}*} \dot{g} = - \sum_{\alpha=1}^6 b_\alpha(t) a_\alpha, \quad g(0) = e,$$

with

$$b_1(t) = \alpha(t), \quad b_2(t) = \beta(t), \quad b_3(t) = \gamma(t), \quad b_4(t) = -\delta(t), \quad b_5(t) = \epsilon(t), \quad b_6(t) = \phi(t).$$

This can be carried out by using the generalised Wei–Norman method, i.e. by writing the curve $g(t)$ in G in terms of a set of second class canonical coordinates. For instance,

$$g(t) = \exp(-v_4(t)a_4) \exp(-v_5(t)a_5) \exp(-v_6(t)a_6) \times \\ \times \exp(-v_1(t)a_1) \exp(-v_2(t)a_2) \exp(-v_3(t)a_3), \quad (3.10)$$

and a straightforward application of the above mentioned Wei–Norman method technique leads

to the system

$$\begin{cases} \dot{v}_1 = b_1 + b_2 v_1 + b_3 v_1^2, & \dot{v}_4 = b_4 + \frac{1}{2} b_2 v_4 + b_1 v_5, \\ \dot{v}_2 = b_2 + 2 b_3 v_1, & \dot{v}_5 = b_5 - b_3 v_4 - \frac{1}{2} b_2 v_5, \\ \dot{v}_3 = e^{v_2} b_3, & \dot{v}_6 = b_6 - b_5 v_4 + \frac{1}{2} b_3 v_4^2 - \frac{1}{2} b_1 v_5^2, \end{cases} \quad (3.11)$$

with $v_1(0) = v_2(0) = v_3(0) = v_4(0) = v_5(0) = v_6(0) = 0$.

If we consider the following vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial v_1} + v_5 \frac{\partial}{\partial v_4} - \frac{1}{2} v_5^2 \frac{\partial}{\partial v_6}, \\ X_2 &= v_1 \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} + \frac{1}{2} v_4 \frac{\partial}{\partial v_4} - \frac{1}{2} v_5 \frac{\partial}{\partial v_5}, \\ X_3 &= v_1^2 \frac{\partial}{\partial v_1} + 2v_1 \frac{\partial}{\partial v_2} + e^{v_2} \frac{\partial}{\partial v_3} - v_4 \frac{\partial}{\partial v_5} + \frac{1}{2} v_4^2 \frac{\partial}{\partial v_6}, \\ X_4 &= \frac{\partial}{\partial v_4}, \\ X_5 &= \frac{\partial}{\partial v_5} - v_4 \frac{\partial}{\partial v_6}, \\ X_6 &= \frac{\partial}{\partial v_6}, \end{aligned} \quad (3.12)$$

we can check that these vector fields satisfy the same commutation relations as the corresponding $\{a_\alpha \mid \alpha = 1, \dots, 6\}$ and thus, system (3.11) is a Lie system related to the same Lie algebra as the t -dependent Hamiltonian (3.9) or its corresponding equation in a Lie group.

Now, once the functions $v_\alpha(t)$, with $\alpha = 1, \dots, 6$, have been determined, the t -evolution of any state is given by

$$\begin{aligned} \psi_t &= \exp(-v_4(t)iH_4) \exp(-v_5(t)iH_5) \exp(-v_6(t)iH_6) \times \\ &\quad \times \exp(-v_1(t)iH_1) \exp(-v_2(t)iH_2) \exp(-v_3(t)iH_3) \psi_0, \end{aligned}$$

and thus

$$\begin{aligned} \psi_t &= \exp(v_4(t)iP) \exp(-v_5(t)iX) \exp(-v_6(t)iI) \times \\ &\quad \times \exp\left(-v_1(t)i\frac{P^2}{2}\right) \exp\left(-v_2(t)i\frac{PX + XP}{4}\right) \exp\left(-v_3(t)i\frac{X^2}{2}\right) \psi_0. \end{aligned} \quad (3.13)$$

3.7. Particular cases. t -dependent quadratic Hamiltonians describe a very large class of physical models. Sometimes, one of these physical models is described by a certain family of quadratic Hamiltonians that can be regarded as a quantum Lie system related to a Lie subalgebra of the one given for general quadratic Hamiltonians. If they are associated with a Lie solvable subalgebra, then the system of differential equations related to it through the Wei–Norman method is solvable too and the t -evolution operator can be explicitly obtained. In this section we treat some instances

of this case through a unified approach. In these instances, we can also find the explicit solutions of these problems in the literature, but by different *ad hoc* methods.

Once we have obtained the solution for a generic quadratic Hamiltonian $H(t)$, we can review the solution for a system with constant mass and linear potential given by

$$H(t) = \frac{P^2}{2m} + S(t)X, \quad (3.14)$$

to obtain, in view of equations (3.11),

$$\begin{aligned} v_1(t) &= \frac{t}{m}, \\ v_2(t) &= 0, \\ v_3(t) &= 0, \\ v_4(t) &= \frac{1}{m} \int_0^t \left(\int_0^u S(v) dv \right) du, \\ v_5(t) &= \int_0^t S(u) du, \\ v_6(t) &= -\frac{1}{m} \int_0^t \left(S(u) \int_0^u \left(\int_0^v S(w) dw \right) dv \right) du - \frac{1}{2m} \int_0^t \left(\int_0^u S(v) dv \right)^2 du, \end{aligned}$$

which give the t -evolution operator if we substitute them into the t -evolution operator (3.13).

Now we can consider particular instances of this t -dependent Hamiltonian. For example, for the curves with constant mass m and $S(t) = q\epsilon_0 + q\epsilon \cos(\omega t)$, studied in [107], we obtain

$$\begin{aligned} v_1(t) &= \frac{t}{m}, \quad v_2(t) = 0, \quad v_3(t) = 0, \\ v_4(t) &= \frac{q}{2m\omega^2} (2\epsilon + \epsilon_0\omega^2 t^2 - 2\epsilon \cos(\omega t)), \quad v_5(t) = \frac{q}{\omega} (\epsilon_0\omega t + \epsilon \sin(\omega t)), \end{aligned}$$

and

$$\begin{aligned} v_6(t) &= \frac{-q^2}{12m\omega^3} (4\epsilon_0^2\omega^3 t^3 - 3\epsilon(\epsilon - 4\epsilon_0)\omega t + \\ &\quad 3\epsilon(4\epsilon + 2\epsilon_0(\omega^2 t^2 - 2) - 3\epsilon \cos(\omega t)) \sin(\omega t)) . \end{aligned}$$

The procedure to obtain a solution with arbitrary non-constant mass and $S(t) = q\epsilon_0 + q\epsilon \cos(\omega t)$ was pointed out in [107] and solved in [94]. From our point of view, the most general solution comes straightforwardly from expression (3.8), because all cases in the literature are particular instances of our approach with general functions $m(t)$ and $S(t)$.

Now, we can obtain the wave function solution of this system. We know that the wave function solution ψ_t with initial condition ψ_0 is

$$\begin{aligned} \psi_t(x) &= U(g(t))\psi(x, 0) \\ &= \exp(iv_6(t)) \exp(-v_4(t)iP) \exp(-v_5(t)iX) \exp\left(-v_1(t)i\frac{P^2}{2}\right) \psi_0(x). \end{aligned}$$

However, if we express the initial wave function $\psi_0(x)$ in the momentum space as $\phi_0(p)$, the solution will take a similar form as before but with $U(g(t))$ in the momentum representation. In

this case the solution with initial condition $\phi_0(p)$ is

$$\begin{aligned}
\phi_t(p) &= U(g(t))\phi_0(p) \\
&= \exp(-iv_6(t)) \exp(v_4(t)iP) \exp(-v_5(t)iX) \exp\left(-iv_1(t)\frac{P^2}{2}\right) \phi_0(p) \\
&= \exp(-iv_6(t)) \exp(v_4(t)iP) \exp(-v_5(t)iX) \exp\left(-iv_1(t)\frac{p^2}{2}\right) \phi_0(p) \\
&= \exp(-iv_6(t)) \exp(v_4(t)iP) \exp\left(-iv_1(t)\frac{(p+v_5(t))^2}{2}\right) \phi_0(p+v_5(t)) \\
&= \exp\left(-iv_6(t) + iv_4(t)p - iv_1(t)\frac{(p+v_5(t))^2}{2}\right) \phi_0(p+v_5(t)).
\end{aligned}$$

3.8. Non-solvable Hamiltonians and particular instances. In the preceding section the differential equations associated with the t -dependent quantum Hamiltonians were Lie systems related to a solvable Lie algebra. Thus, it was proved that the differential equations obtained were integrable by quadratures through the Wei–Norman method. If this does not happen, it is not easy to obtain a general solution. Now, we describe some examples of ‘non-solvable’ t -dependent quadratic Hamiltonians. In general we do not obtain a general solution in terms of the t -dependent functions of the quadratic Hamiltonians. Nevertheless, we show that for some instances of them, whose coefficients satisfy certain integrability conditions [52, 54], the differential equations can be integrated.

As a first case, consider the Hamiltonian for a forced harmonic oscillator with t -dependent mass and frequency given by

$$H(t) = \frac{P^2}{2m(t)} + \frac{1}{2}m(t)\omega^2(t)X^2 + f(t)X.$$

This case, either with or without t -dependent frequency, has been studied in [78, 107, 238]. The equations describing the solutions of this Lie system by the method of Wei–Norman are

$$\begin{aligned}
\dot{v}_1 &= \frac{1}{m(t)} + m(t)\omega^2(t)v_1^2, \\
\dot{v}_2 &= 2m(t)\omega^2(t)v_1, \\
\dot{v}_3 &= e^{v_2}m(t)\omega^2(t), \\
\dot{v}_4 &= \frac{1}{m(t)}v_5, \\
\dot{v}_5 &= f(t) - m(t)\omega^2(t)v_4, \\
\dot{v}_6 &= \frac{1}{2}m(t)\omega^2(t)v_4^2 - f(t)v_4 - \frac{1}{2m(t)}v_5^2,
\end{aligned}$$

with initial conditions $v_1(0) = v_2(0) = v_3(0) = v_4(0) = v_5(0) = v_6(0) = 0$, where the factorisation (3.10) has been used. The solution of this system cannot be obtained by quadratures in the general case because the associated Lie algebra is not solvable. Nevertheless, we can consider a particular instance of this kind of Hamiltonian, the so-called Caldirola–Kanai Hamiltonian [118]. In this case, for the particular t -dependence $m(t) = e^{-rt}m_0$, $\omega(t) = \omega_0$ and $f(t) = 0$ the

Hamiltonian reads

$$H(t) = \frac{P^2}{2m_0} e^{rt} + \frac{1}{2} m_0 e^{-rt} \omega_0^2 X^2.$$

In this case the solution is completely known and is given by

$$\begin{aligned} v_1(t) &= \frac{2e^{rt}}{m_0(r + \bar{\omega}_0 \coth(\frac{t}{2}\bar{\omega}_0))}, \\ v_2(t) &= rt + 2 \log \bar{\omega}_0 - 2 \log \left(r \sinh \left(\frac{t}{2} \bar{\omega}_0 \right) + \bar{\omega}_0 \cosh \left(\frac{t}{2} \bar{\omega}_0 \right) \right), \\ v_3(t) &= \frac{2m_0\omega_0^2}{r + \bar{\omega}_0 \coth(\frac{t}{2}\bar{\omega}_0)}, \\ v_4(t) &= 0, \quad v_5(t) = 0, \quad v_6(t) = 0, \end{aligned}$$

where $\bar{\omega}_0 = \sqrt{r^2 - 4\omega_0^2}$. This example shows that the problem may also be exactly solved for particular instances of curves in \mathfrak{g} of Lie systems with non solvable Lie algebras. Another example is the following one

$$H(t) = \frac{P^2}{2m} + \frac{m\omega_0^2}{2(t+k)^2} X^2,$$

for which the solution of the Wei–Norman system reads

$$\begin{aligned} v_1(t) &= \frac{2(k+t)((k+t)^{\bar{\omega}_0} - k^{\bar{\omega}_0})}{m(k^{\bar{\omega}_0}(\bar{\omega}_0 - 1) + (k+t)^{\bar{\omega}_0}(\bar{\omega}_0 + 1))}, \\ v_2(t) &= (1 + \bar{\omega}_0) \log(k+t) - (1 + \bar{\omega}_0) \log k + 2 \log(2k^{\bar{\omega}_0}\bar{\omega}_0) \\ &\quad - 2 \log(k^{\bar{\omega}_0}(\bar{\omega}_0 - 1) + (k+t)^{\bar{\omega}_0}(\bar{\omega}_0 + 1)), \\ v_3(t) &= \frac{2m\omega_0^2}{k} \frac{(k+t)^{\bar{\omega}_0} - k^{\bar{\omega}_0}}{k^{\bar{\omega}_0}(\bar{\omega}_0 - 1) + (k+t)^{\bar{\omega}_0}(\bar{\omega}_0 + 1)} \\ v_4(t) &= 0, \quad v_5(t) = 0, \quad v_6(t) = 0, \end{aligned}$$

where now $\bar{\omega}_0 = \sqrt{1 - 4\omega_0^2}$.

Other examples of Hamiltonians, which can be studied by our method, can be found in [118]. We just mention two examples which can be completely solved

$$\begin{aligned} H_1(t) &= \frac{P^2}{2m_0} + \frac{1}{2} m_0 (U + V \cos(\omega_0 t)) X^2, \\ H_2(t) &= \frac{P^2}{2m_0} e^{rt} + \frac{1}{2} m_0 e^{-rt} \omega_0^2 X^2 + f(t) X. \end{aligned}$$

The first one corresponds to a Paul trap which has been studied in [95] and admits a solution in terms of Mathieu's functions. The second one is a damped Caldirola–Kanai Hamiltonian analysed in [221].

3.9. Reduction in Quantum Mechanics. Quite often, when a quantum Lie system is related to a non-solvable Lie algebra, it is interesting to solve it in terms of (unknown) solutions of differential equations. Next, we study some examples of how to proceed with the method of reduction in order to deal with problems in this way. So, we obtain that the reduction method can

be applied not only to analyse systems of differential equations but also enables to solve certain quantum problems in an algorithmic way.

Consider a harmonic oscillator with t -dependent frequency whose Hamiltonian is given by

$$H(t) = \frac{P^2}{2} + \frac{1}{2}\Omega^2(t)X^2.$$

As a particular case of the Hamiltonian described in Section 1.8, this example is related to an equation in the connected Lie group associated with the semidirect sum of $\mathfrak{sl}(2, \mathbb{R})$, spanned by the elements $\{a_1, a_2, a_3\}$, with the Heisenberg Lie algebra generated by the ideal $\{a_4, a_5, a_6\}$

$$R_{g^{-1}*}g\dot{g} = -a_1 - \Omega^2(t)a_3, \quad g(0) = e. \quad (3.15)$$

Since the solution of this equation starts from the identity and $\{a_1, a_2, a_3\}$ close on a $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra, then the t -dependent Hamiltonian $H(t)$ is related to the group $SL(2, \mathbb{R})$.

As a particular application of the reduction technique we will perform the reduction from $G = SL(2, \mathbb{R})$ to the Lie group related to the Lie subalgebra $\mathfrak{h} = \langle a_1 \rangle$. To obtain such a reduction, we have shown in Section 3.1 that we have to solve an equation in G/H , namely

$$\frac{d\pi^L(\tilde{g})}{dt} = \sum_{\alpha=1}^3 b_\alpha(t)X_\alpha^L(\pi^L(\tilde{g})) \quad (3.16)$$

where X_α^L are the fundamental vector fields of the action λ of G on G/H . Now, we are going to describe this equation in a set of local coordinates. First, in an open neighbourhood U of $e \in G$ we can write any element of this open in a unique way as

$$g = \exp(-c_3a_3)\exp(-c_2a_2)\exp(-c_1a_1), \quad (3.17)$$

where the matrices a_α , with $\alpha = 1, 2, 3$, are given by (2.4).

This decomposition allows us to establish a local diffeomorphism between an open neighbourhood $V \subset G/H$ and the set of matrices given by $\exp(-c_3a_3)\exp(-c_2a_2)$. Now, the decomposition (3.17) reads in matrix terms as

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -c_3 & 1 \end{pmatrix} \begin{pmatrix} e^{c_2/2} & 0 \\ 0 & e^{-c_2/2} \end{pmatrix} \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{c_2/2} & 0 \\ -c_3e^{c_2/2} & e^{-c_2/2} \end{pmatrix} \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

If we express c_1, c_2, c_3 in terms of α, β, γ and δ , we obtain that $c_3 = -\gamma/\alpha$, $c_2 = \log \alpha^2$, and $c_1 = \beta/\alpha$. Consequently, we get

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma/\alpha & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix}.$$

Thus, we can define the projection $\pi^L : U \subset G \rightarrow G/H$ given by

$$\pi^L \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} H, \quad (3.18)$$

which allows us to represent the elements of G/H , locally, as the 2×2 lower triangular matrices with determinant one. Now, given $\lambda_g : g'H \in G/H \mapsto gg'H \in G/H$ as $\lambda_g \circ \pi^L = \pi^L \circ L_g$, the fundamental vector fields defined in G/H by a_1 and a_3 through the action $\lambda : (g, g'H) \in$

$G \times G/H \mapsto \lambda_g(g'H) \in G/H$ are given by

$$\begin{aligned} X_1^L(\pi^L(g)) &= \frac{d}{dt} \Big|_{t=0} \pi^L \left(\exp(-ta_1) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma/\alpha^2 \end{pmatrix}, \\ X_3^L(\pi^L(g)) &= \frac{d}{dt} \Big|_{t=0} \pi^L \left(\exp(-ta_3) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}, \end{aligned}$$

and the equation on $V \subset G/H$ is described by

$$\begin{pmatrix} \dot{\alpha} & 0 \\ \dot{\gamma} & -\dot{\alpha}\alpha^{-2} \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ -\Omega^2(t)\alpha & -\gamma\alpha^{-2} \end{pmatrix}.$$

Therefore, we need to obtain a solution of the system

$$\begin{cases} \ddot{\alpha} = -\Omega^2(t)\alpha, \\ \gamma = \dot{\alpha}. \end{cases} \quad (3.19)$$

Then, taking into account (3.18), if α_1 is a solution of the system (3.19), the curve $\tilde{g}(t)$ that satisfies $g(t) = \tilde{g}(t)h(t)$, where $h(t)$ is a solution of an equation defined on the Lie group with Lie algebra $\mathfrak{h} = \langle a_1 \rangle$, reads

$$\tilde{g}(t) = \begin{pmatrix} \alpha_1 & 0 \\ \dot{\alpha}_1 & \alpha_1^{-1} \end{pmatrix} = \begin{pmatrix} e^{c_2/2} & 0 \\ -c_3 e^{c_2/2} & e^{-c_2/2} \end{pmatrix} = \exp\left(\frac{\dot{\alpha}_1}{\alpha_1} a_3\right) \exp(-2 \log \alpha_1 a_2),$$

and the curve which acts on the initial equation in $SL(2, \mathbb{R})$ to transform it into one in the mentioned Lie subalgebra is given by $\bar{g}(t) = \tilde{g}^{-1}(t)$,

$$\bar{g}(t) = \exp(2 \log \alpha_1 a_2) \exp\left(-\frac{\dot{\alpha}_1}{\alpha_1} a_3\right).$$

This curve transforms the initial equation in the group given by (3.15) into the new one given by (3.3), i.e.

$$a'(t) = -\frac{a_1}{\alpha_1^2(t)},$$

which corresponds to the t -dependent Hamiltonian $H'(t) = P^2/(2\alpha_1^2(t))$. The induced transformation in the Hilbert space \mathcal{H} that transforms $H(t)$ into $H'(t)$ is

$$\exp\left(i \frac{\log \alpha_1}{2} (PX + XP)\right) \exp\left(-i \frac{\dot{\alpha}_1}{2\alpha_1} X^2\right).$$

Both results can be found in [96].

There are other possibilities of choosing different Lie subalgebras of \mathfrak{g} in order to perform the reduction, however the results are always given in terms of a solution of a differential equation.

4. Integrability conditions for Lie systems

The main aim of this chapter is concerned with the description of the main aspects of the integrability theory for Lie systems detailed in [47] and based on the geometrical understanding of Riccati equations.

The Riccati equation can be considered as the simplest nonlinear differential equation [40, 50]. It is, basically, the only first-order ordinary differential equation admitting a nonlinear superposition rule [157, 234]. In spite of its apparent simplicity, its general solution cannot be

described by means of quadratures with the exception of some very particular cases [63, 132, 169, 183, 214, 239].

The relevance of Riccati equation becomes evident when we take into account its frequent appearance in many fields of Mathematics and Physics [57, 159, 176, 184, 203, 207, 216, 234]. This also implies the necessity of a theory of integrability providing all those integrable cases that might lead to solvable physical models.

4.1. Integrability of Riccati equations. In order to provide a first insight into the study of integrability conditions for Riccati equations, we review here some very well-known results about this topic.

Recall that Riccati equations are first-order differential equations of the form

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2. \quad (4.1)$$

A first particular example of Riccati equation integrable by quadratures is the one with $b_3 = 0$. In fact, in such a case, Riccati equation reduces to an inhomogeneous linear equation, which can be explicitly integrated by means of two quadratures.

Additionally, the change of variable $w = -1/x$ transforms the above Riccati equation into the new one

$$\frac{dw}{dt} = b_1(t)w^2 - b_2(t)w + b_3(t).$$

Consequently, if we suppose that $b_1 = 0$ in equation (4.1), that is, if we consider a Bernoulli equation, the mentioned change of variable leads to an integrable linear equation.

Another known property on the integrability of Riccati equations establishes that given a particular solution $x_1(t)$ of (4.1), the change $x = x_1(t) + z$ permits us to transform a Riccati equation into a new one for which the coefficient of the term independent of z is zero, i.e.

$$\frac{dz}{dt} = (2b_3(t)x_1(t) + b_2(t))z + b_3(t)z^2,$$

and, as we pointed out previously, this equation reduces to an inhomogeneous linear equation with the change of variables $z = -1/u$. Consequently, the knowledge of a particular solution of a Riccati equation allows us to find its general solution by means of two quadratures. It is worth recalling that this property can be more generally understood by means of the theory of Lie systems. Indeed, this theory states that the knowledge of a particular solution of a Lie system enables us to reduce the initial equation into a ‘simpler’ one, see Section 1.2 or [40].

If we know two particular solutions, $x_1(t)$ and $x_2(t)$, of equation (4.1), its general solution can be determined with one quadrature. Indeed, the change of variable $z = (x - x_1(t))/(x - x_2(t))$ transforms the original equation into a homogeneous linear differential equation and, hence, the general solution can be immediately found.

Finally, giving three particular solutions, $x_1(t)$, $x_2(t)$, $x_3(t)$, the general solution can be written, without making use of any quadrature, in terms of the superposition rule (1.11).

The simplest case of Riccati equation, i.e. the one with b_1 , b_2 and b_3 being constant, has been fully studied and it is integrable by quadratures, see in example [64]. This can be viewed as the consequence of the existence of a constant (maybe complex) solution, permitting us to reduce the equation into an inhomogeneous linear one. Note also that, in a similar way, separable Riccati

equations of the form

$$\frac{dx}{dt} = \varphi(t)(c_1 + c_2 x + c_3 x^2),$$

with $\varphi(t)$ being a non-vanishing function, are integrable, because they admit a constant solution again, which enables us to transform the equation into a linear inhomogeneous one again. On the other hand, the integrability of the above equation can also be related to the existence of a t -reparametrisation, reducing the problem to an autonomous one.

4.2. Transformation laws of Riccati equations. We here describe an important property of Lie systems, in the particular case of Riccati equations, playing a relevant rôle for establishing integrability criteria: *The group \mathcal{G} of curves in a Lie group G associated with a Lie system acts on the set of the related Lie systems.*

More explicitly, consider a family X_1, X_2, X_3 , of vector fields on $\overline{\mathbb{R}}$, e.g. the set given in (1.26), spanning the Vessiot-Guldberg Lie algebra of vector fields associated with Riccati equations and isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In terms of this family, each Riccati equation (4.1) is related to a t -dependent vector field $X_t = b_1(t)X_1 + b_2(t)X_2 + b_3(t)X_3$, which can be considered as a curve $(b_1(t), b_2(t), b_3(t))$ in \mathbb{R}^3 . Each element \bar{A} of the group of smooth curves in $SL(2, \mathbb{R})$, i.e. $\bar{A} \in \mathcal{G} \equiv \text{Map}(\mathbb{R}, SL(2, \mathbb{R}))$, transforms every curve $x(t)$ in $\overline{\mathbb{R}}$ into a new one $x'(t) = \Phi(\bar{A}(t), x(t))$ by means of the action $\Phi : (A, x) \in SL(2, \mathbb{R}) \times \overline{\mathbb{R}} \mapsto \Phi(A, x) \in \overline{\mathbb{R}}$ of the form:

$$\Phi(A, x) = \begin{cases} \frac{\alpha x + \beta}{\gamma x + \delta} & x \neq -\frac{\delta}{\gamma}, \quad x \neq \infty, \\ \alpha/\gamma & x = \infty, \\ \infty & x = -\frac{\delta}{\gamma}, \end{cases} \quad \text{where } A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (4.2)$$

Moreover, the above t -dependent change of variables transforms the Riccati equation (4.1) into a new one with t -dependent coefficients b'_1, b'_2, b'_3 given by

$$\begin{cases} b'_3 = \delta^2 b_3 - \delta\gamma b_2 + \gamma^2 b_1 + \gamma\dot{\delta} - \delta\dot{\gamma}, \\ b'_2 = -2\beta\delta b_3 + (\alpha\delta + \beta\gamma) b_2 - 2\alpha\gamma b_1 + \delta\dot{\alpha} - \alpha\dot{\delta} + \beta\dot{\gamma} - \gamma\dot{\beta}, \\ b'_1 = \beta^2 b_3 - \alpha\beta b_2 + \alpha^2 b_1 + \alpha\dot{\beta} - \beta\dot{\alpha}. \end{cases} \quad (4.3)$$

Indeed, the above expressions define an affine action of the group \mathcal{G} on the set of Riccati equations. In other words, given the elements $A_1, A_2 \in \mathcal{G}$, transforming the coefficients of a general Riccati equation by means of two successive transformations of the above type, e.g. first by A_1 and then by A_2 , gives exactly the same result as doing only one transformation with the element $A_2 \cdot A_1$ of \mathcal{G} , see [63, 151].

The group \mathcal{G} also acts on the set of equations of the form (1.31) on $SL(2, \mathbb{R})$. In order to show this, note first that \mathcal{G} acts on the left on the set of curves in $SL(2, \mathbb{R})$ by left translations, i.e. given two curves $\bar{A}(t), A(t) \subset SL(2, \mathbb{R})$, the curve $\bar{A}(t)$ transforms the curve $A(t)$ into a new one $A'(t) = \bar{A}(t)A(t)$. Moreover, if $A(t)$ is a solution of equation (1.31), then the curve $A'(t)$ satisfies a new equation like (1.31) but with a different right hand side $a'(t)$. Differentiating the relation $A'(t) = \bar{A}(t)A(t)$ and taking into account the form of (1.31), we get that, in view of the

basis (2.4), the relation between the curves $a(t)$ and $a'(t)$ in $\mathfrak{sl}(2, \mathbb{R})$ is

$$a'(t) = \bar{A}(t)a(t)\bar{A}^{-1}(t) + \dot{\bar{A}}(t)\bar{A}^{-1}(t) = -\sum_{\alpha=1}^3 b'_\alpha(t)a_\alpha, \quad (4.4)$$

which yields the expressions (4.3). Conversely, if $A'(t) = \bar{A}(t)A(t)$ is the solution for the equation corresponding to the curve $a'(t)$ given by the transformation rule (4.4), then $A(t)$ is the solution of the equation (1.31) determined by the curve $a(t)$.

Summarising, we have shown that it is possible to associate each Riccati equation with an equation on the Lie group $SL(2, \mathbb{R})$ and to define an infinite-dimensional group of transformations acting on the set of Riccati equations. Additionally, this process can be easily derived in a similar way for any Lie system, see [47].

4.3. Lie structure of an equation of transformation of Lie systems. Let us construct a Lie system describing the curves in $SL(2, \mathbb{R})$ which transform the Riccati equation associated with an equation on $SL(2, \mathbb{R})$ characterised by the curve $a(t) \subset \mathfrak{sl}(2, \mathbb{R})$ into the Riccati equation associated with the curve $a'(t) \subset \mathfrak{sl}(2, \mathbb{R})$. By means of this Lie system, we later explain the results derived in [47] in order to describe, from a unified point of view, the developments of the works [40, 50].

Multiply equation (4.4) on the right by $\bar{A}(t)$ to get

$$\dot{\bar{A}}(t) = a'(t)\bar{A}(t) - \bar{A}(t)a(t). \quad (4.5)$$

If we consider the above equation as a system of first-order differential equations in the coefficients of the curve $\bar{A}(t)$ in $SL(2, \mathbb{R})$, with

$$\bar{A}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}, \quad \alpha(t)\delta(t) - \beta(t)\gamma(t) = 1,$$

then system (4.5) reads

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} \frac{b'_2 - b_2}{2} & b_3 & b'_1 & 0 \\ -b_1 & \frac{b'_2 + b_2}{2} & 0 & b'_1 \\ -b'_3 & 0 & -\frac{b'_2 + b_2}{2} & b_3 \\ 0 & -b'_3 & -b_1 & -\frac{b'_2 - b_2}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \quad (4.6)$$

The solutions $y(t) = (\alpha(t), \beta(t), \gamma(t), \delta(t))$ of the above system relating two given Riccati equations are associated with curves in $SL(2, \mathbb{R})$, i.e. they are such that, at any time, $\alpha\delta - \beta\gamma = 1$. Nevertheless, we can drop such a restriction for the time being as it can be implemented by a restraint on the initial conditions for the solutions and, hence, we can treat the variables, $\alpha, \beta, \gamma, \delta$, in the system (4.6) as being independent. In this case, this linear system can be regarded as a Lie system linked to a Lie algebra of vector fields isomorphic to $\mathfrak{gl}(4, \mathbb{R})$. Nevertheless, it may also be understood as a Lie system related to a Lie algebra of vector fields isomorphic to a Lie subalgebra of $\mathfrak{gl}(4, \mathbb{R})$. Indeed, consider the vector fields

$$\begin{aligned} N_1 &= -\alpha \frac{\partial}{\partial \beta} - \gamma \frac{\partial}{\partial \delta}, & N'_1 &= \gamma \frac{\partial}{\partial \alpha} + \delta \frac{\partial}{\partial \beta}, \\ N_2 &= \frac{1}{2} \left(\beta \frac{\partial}{\partial \beta} + \delta \frac{\partial}{\partial \delta} - \alpha \frac{\partial}{\partial \alpha} - \gamma \frac{\partial}{\partial \gamma} \right), & N'_2 &= \frac{1}{2} \left(\alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} - \gamma \frac{\partial}{\partial \gamma} - \delta \frac{\partial}{\partial \delta} \right), \\ N_3 &= \beta \frac{\partial}{\partial \alpha} + \delta \frac{\partial}{\partial \gamma}, & N'_3 &= -\alpha \frac{\partial}{\partial \gamma} - \beta \frac{\partial}{\partial \delta}, \end{aligned}$$

spanning a Vessiot–Guldberg Lie algebra of vector fields isomorphic to $\mathfrak{g} \equiv \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{gl}(4, \mathbb{R})$. Consequently, the linear system of differential equation (4.6) is a Lie system on \mathbb{R}^4 associated with a Lie algebra of vector fields isomorphic to \mathfrak{g} , see [47].

If we denote $y \equiv (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$, system (4.6) is a differential equation on \mathbb{R}^4 of the form

$$\frac{dy}{dt} = N(t, y), \quad (4.7)$$

with N being the t -dependent vector field

$$N_t = \sum_{\alpha=1}^3 (b_\alpha(t)N_\alpha + b'_\alpha(t)N'_\alpha).$$

The vector fields $\{N_1, N_2, N_3, N'_1, N'_2, N'_3\}$ span a regular distribution \mathcal{D} with rank three in almost any point of \mathbb{R}^4 and thus there exists, at least locally, a first-integral for all the vector fields in the distribution \mathcal{D} . The method of characteristics allows us to determine that this first-integral can be

$$I : y = (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \mapsto \det y \equiv \alpha\delta - \beta\gamma \in \mathbb{R}.$$

Moreover, this first-integral is related to the determinant of a matrix $\bar{A} \in SL(2, \mathbb{R})$ with coefficients given by the components of $y = (\alpha, \beta, \gamma, \delta)$. Therefore, if we have a solution of the system (4.6) with initial condition such that $\det y(0) = \alpha(0)\delta(0) - \beta(0)\gamma(0) = 1$, then $\det y(t) = 1$ at any time t and the solution can be understood as a curve in $SL(2, \mathbb{R})$. Summarising, we have proved the following theorem.

THEOREM 4.1. *The curves in $SL(2, \mathbb{R})$ transforming equation (1.31) into a new equation of the same form characterised by a curve $a'(t) = -\sum_{\alpha=1}^3 b'_\alpha(t)a_\alpha$ are described through the solutions of the Lie systems*

$$\frac{dy}{dt} = N(t, y) = \sum_{\alpha=1}^3 b_\alpha(t)N_\alpha(y) + \sum_{\alpha=1}^3 b'_\alpha(t)N'_\alpha(y). \quad (4.8)$$

such that $\det y(0) = 1$. Furthermore, the above Lie system is related to a non-solvable Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.

A consequence of the above Theorem is the following corollary, whose proof is omitted and left to the reader.

COROLLARY 4.2. *Given two Riccati equations associated with curves $a'(t)$ and $a(t)$ in $\mathfrak{sl}(2, \mathbb{R})$, there always exists a curve $\bar{A}(t)$ in $SL(2, \mathbb{R})$ transforming the Riccati equation related to $a(t)$ into the new one associated with $a'(t)$. Furthermore, if $\bar{A}(0) = I$, this curve is uniquely defined.*

Even if we know that given two equations on the Lie group $SL(2, \mathbb{R})$ there always exists a transformation relating both, in order to obtain such a curve we need to solve the differential equation (4.7) which, unfortunately, is Lie system related to a non-solvable Vessiot–Guldberg. Consequently, it is not easy to find its solutions in general as, for instance, it is not integrable by quadratures.

Nevertheless, many known and new properties on integrability conditions for Riccati equations can be determined by means of Theorem 4.1. Furthermore, the procedure to obtain the Lie system (4.7) can be generalised to deal with any Lie system related to a Lie group G with Lie algebra \mathfrak{g} (cf. [47]).

4.4. Description of some known integrability conditions. Note that Lie systems on G of the form (1.31) determined by a constant curve, $a = -\sum_{\alpha=1}^3 c_{\alpha} a_{\alpha}$, are integrable and, therefore, the same happens for curves of the form $a(t) = -D \left(\sum_{\alpha=1}^3 c_{\alpha} a_{\alpha} \right)$, where $D = D(t)$ is a non-vanishing function, as a t -reparametrisation reduces the problem to the previous one.

Our aim now is to determine the curves $\bar{A}(t)$ in $SL(2, \mathbb{R})$ transforming the equation on $SL(2, \mathbb{R})$ characterised by a curve $a(t)$ into the equation on $SL(2, \mathbb{R})$ characterised by $a'(t) = -D(c_1 a_1 + c_2 a_2 + c_3 a_3)$, with $D = D(t)$ a non-vanishing function and $c_1 c_3 \neq 0$. As the final equation is associated with a solvable one-dimensional Vessiot-Guldberg Lie algebra, the transformation establishing the relation to such a final integrable equation allows us to find by quadratures the solution of the initial equation and, therefore, the solution for its associated Riccati equation. In order to get the transformation between the Riccati equations linked to the before equations on $SL(2, \mathbb{R})$, we look for particular curves $\bar{A}(t)$ in $SL(2, \mathbb{R})$ satisfying certain conditions in order to get an integrable equation (4.6). Nevertheless, under the assumed restrictions, we may obtain a system of differential equations which does not admit any solution. In such a case, the conditions ensuring the existence of solutions will describe integrability conditions. As an application we show that many known achievements about the integrability of Riccati equations can be recovered and explained in this way.

We have already showed that Riccati equations (4.1), with $b_1 b_3 \equiv 0$, are reducible to linear differential equations and therefore they are always integrable [57]. Hence, they are not interesting in the study of integrability conditions and we can focus our attention on reducing Riccati equations with $b_1 b_3 \neq 0$ into integrable ones by means of the action of a curve in $SL(2, \mathbb{R})$. With this aim, consider the family of curves with $\beta = 0$ and $\gamma = 0$, i.e. take curves in $SL(2, \mathbb{R})$ of the form

$$A(t) = \begin{pmatrix} \alpha(t) & 0 \\ 0 & \delta(t) \end{pmatrix} \in SL(2, \mathbb{R}), \quad \alpha(t)\delta(t) = 1.$$

The curve $\bar{A}(t)$ in $SL(2, \mathbb{R})$ determines a t -dependent change of variables in $\bar{\mathbb{R}}$ given by $x'(t) = \Phi(\bar{A}(t), x)$. In view of the action (4.2), and as $\alpha\delta = 1$, we get that the previous change of variables reads

$$x' = \alpha^2(t)x = G(t)x, \quad G(t) \equiv \frac{\alpha(t)}{\delta(t)} > 0. \tag{4.9}$$

In view of the relations (4.3), the initial Riccati equations is transformed, by means of the curve $\bar{A}(t)$, into the new Riccati equation with t -dependent coefficients

$$b'_1 = \alpha^2 b_1, \quad b'_2 = \alpha \delta b_2 + \dot{\alpha} \delta - \alpha \dot{\delta}, \quad b'_3 = \delta^2 b_3.$$

Moreover, the functions $\alpha(t)$ and $\delta(t)$ are solutions of the system (4.7), which in this case reduces to

$$\begin{pmatrix} \dot{\alpha} \\ 0 \\ 0 \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} \frac{b'_2 - b_2}{2} & b_3 & b'_1 & 0 \\ -b_1 & \frac{b'_2 + b_2}{2} & 0 & b'_1 \\ -b'_3 & 0 & -\frac{b'_2 + b_2}{2} & b_3 \\ 0 & -b'_3 & -b_1 & -\frac{b'_2 - b_2}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \delta \end{pmatrix}. \tag{4.10}$$

The existence of solutions for the above system related to elements of $SL(2, \mathbb{R})$ that satisfy the required conditions determines the integrability of a Riccati equation by the method described.

Thus, let us analyse the existence of such solutions to get these integrability conditions.

From some of the relations of the above system, we get that

$$-b_1\alpha + b'_1\delta = 0, \quad -b'_3\alpha + b_3\delta = 0.$$

As $\alpha(t) = 1$, these relations imply that $b'_1 b'_3 = b_1 b_3$ and

$$\alpha^2 = \frac{b'_1}{b_1} = \frac{b_3}{b'_3} \equiv G > 0.$$

Hence, the transformation formulas (4.3) reduce to

$$b'_3 = \alpha^{-2} b_3, \quad b'_2 = b_2 + 2\frac{\dot{\alpha}}{\alpha}, \quad b'_1 = \alpha^2 b_1. \quad (4.11)$$

Then, in order to exist a t -dependent function D and two real constants c_1 and c_3 , with $c_1 c_3 \neq 0$, such that $b'_3 = Dc_3$ and $b'_1 = Dc_1$, the function D must be given by

$$D^2 c_1 c_3 = b_1 b_3 \implies D = \pm \sqrt{\frac{b_1 b_3}{c_1 c_3}},$$

where we have used that $b'_1 b'_3 = b_1 b_3$. On the other hand, as $b'_1/b_1 = \alpha^2 > 0$, we have to fix the sign κ of the function D in order to satisfy this relation, i.e. $\text{sg}(c_1 D) = \text{sg}(b_1)$. Therefore,

$$\kappa = \text{sg}(D) = \text{sg}(b_1/c_1).$$

Also, as $b_1 b_3 = b'_1 b'_3$, we get that $\text{sg}(b_1 b_3) = \text{sg}(c_1 c_3)$. Furthermore, in view of relations (4.11), α is determined, up to a sign, by

$$\alpha = \sqrt{\frac{Dc_1}{b_1}} = \left(\frac{c_1 b_3}{c_3 b_1}\right)^{1/4}. \quad (4.12)$$

and therefore the change of variables (4.9) reads

$$x' = \frac{D(t)c_1}{b_1(t)} x. \quad (4.13)$$

Finally, as a consequence of (4.11), in order for b'_2 to be the product $b'_2 = c_2 D$, we see that

$$b_2 + 2\frac{\dot{\alpha}}{\alpha} = \kappa c_2 \sqrt{\frac{b_1 b_3}{c_1 c_3}}. \quad (4.14)$$

Using (4.12) and the above equality, we see that the integrability condition is

$$\sqrt{\frac{c_1 c_3}{b_1 b_3}} \left[b_2 + \frac{1}{2} \left(\frac{\dot{b}_3}{b_3} - \frac{\dot{b}_1}{b_1} \right) \right] = \kappa c_2.$$

Conversely, if the above integrability condition is valid and $D^2 c_1 c_3 = b_1 b_3$, the change of variables (4.13) transforms the Riccati equation (4.1) into $dx'/dt = D(t)(c_1 + c_2 y' + c_3 y'^2)$, with $c_1 c_3 \neq 0$. To sum up, we have proved the following theorem.

THEOREM 4.3. *The necessary and sufficient conditions for the existence of a transformation*

$$x' = G(t)x, \quad G(t) > 0,$$

relating the Riccati equation

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2, \quad b_1 b_3 \neq 0,$$

to an integrable one given by

$$\frac{dx'}{dt} = D(t)(c_1 + c_2x' + c_3x'^2), \quad c_1c_3 \neq 0, \quad D(t) \neq 0, \quad (4.15)$$

where c_1, c_2, c_3 are real numbers and $D(t)$ is non-vanishing functions, are

$$D^2c_1c_3 = b_1b_3, \quad \left(b_2 + \frac{1}{2} \left(\frac{\dot{b}_3}{b_3} - \frac{\dot{b}_1}{b_1} \right) \right) \sqrt{\frac{c_1c_3}{b_1b_3}} = \kappa c_2. \quad (4.16)$$

where $\kappa = \text{sg}(D) = \text{sg}(b_1/c_1)$. The transformation is then uniquely defined by

$$x' = \sqrt{\frac{b_3(t)c_1}{b_1(t)c_3}} x.$$

From previous results, the following corollary follows.

COROLLARY 4.4. *A Riccati equation (4.15) with $b_1b_3 \neq 0$ can be transformed into a Riccati equation of the form (4.15) by a t -dependent change of variables $y' = G(t)y$, with $g(t) > 0$, if and only if*

$$\frac{1}{\sqrt{|b_1b_3|}} \left(b_2 + \frac{1}{2} \left(\frac{\dot{b}_3}{b_3} - \frac{\dot{b}_1}{b_1} \right) \right) = K, \quad (4.17)$$

for a certain real constant K . In such a case, the Riccati equation (4.1) is integrable by quadratures.

In view of Theorem 4.3, if we start with the integrable Riccati equation (4.15), we can obtain the set of all Riccati equations that can be reached from it by means of a transformation of the form (4.9).

COROLLARY 4.5. *Given an integrable Riccati equation*

$$\frac{dx}{dt} = D(t)(c_1 + c_2x + c_3x^2), \quad c_1c_3 \neq 0, \quad D(t) \neq 0,$$

with $D(t)$ a non-vanishing function, the set of Riccati equations which can be obtained with a transformation $x' = G(t)x$, with $G(t) > 0$, are those of the form

$$\frac{dx'}{dt} = b_1(t) + \left(\frac{\dot{b}_1(t)}{b_1(t)} - \frac{\dot{D}(t)}{D(t)} + c_2D(t) \right) x' + \frac{D^2(t)c_1c_3}{b_1(t)} x'^2,$$

and the function G is then given by

$$G = \frac{Dc_1}{\sqrt{b_1}}.$$

Therefore, starting with an integrable equation, we can generate a family of solvable Riccati equations whose coefficients are parametrised by a non-vanishing function b_1 . Moreover, the integrability condition to check if a Riccati equation belongs to this family can be easily verified.

The previous results can now be used for a better comprehension of some integrability conditions found in the literature. Let us illustrate this claim by reviewing some well-known integrability conditions through our methods.

- *The case of Allen and Stein*

The main achievements of the article [4] can be recovered through our more general approach. In that work, a Riccati equation (4.1), with $b_1 b_3 > 0$ and b_0, b_2 being differentiable functions satisfying the condition

$$b_2 + \frac{1}{2} \left(\frac{\dot{b}_3}{b_3} - \frac{\dot{b}_1}{b_1} \right) \frac{1}{\sqrt{b_1 b_3}} = C, \quad (4.18)$$

where C is a real constant, was transformed into the integrable one

$$\frac{dx'}{dt} = \sqrt{b_1(t)b_3(t)} (1 + Cx' + x'^2), \quad (4.19)$$

through a t -dependent linear transformation of the form

$$x' = \sqrt{\frac{b_3(t)}{b_1(t)}} x.$$

If a Riccati equation obeys the integrability condition (4.18), such an equation also satisfies the assumptions of Corollary 4.4 and, therefore, the integrability condition given in Theorem 4.3 with

$$c_1 = c_3 = 1, \quad c_2 = C, \quad D = \sqrt{b_1 b_3}.$$

Consequently, the corresponding t -dependent change of variables described by Theorem 4.3 reads

$$x' = \sqrt{\frac{b_3(t)}{b_1(t)}} x,$$

showing that the transformation in [4] is a particular case of our results. This is not surprising, as Theorem 4.3 shows that if such a t -dependent change of variables is used to transform a Riccati equation (4.1) into one of the form (4.15), this change of variables must be one of the form (4.13) and the initial Riccati equation must satisfy integrability conditions (4.16).

- *The case of Rao and Ukidave:*

Rao and Ukidave stated in their work [190] that a Riccati equation (4.1), with $b_1 b_3 > 0$, can be transformed into an integrable one

$$\frac{dx'}{dt} = \sqrt{c b_1 b_3} \left(1 + kx' + \frac{1}{c} x'^2 \right),$$

through a t -dependent linear transformation

$$x' = \frac{1}{v(t)} x,$$

if there exist two real constants c and k such that the following integrability condition is satisfied

$$b_3 = \frac{b_1}{c v^2}, \quad (4.20)$$

with $v(t)$ being a solution of the differential equation

$$\frac{dv}{dt} = k b_1(t) + b_2(t) v. \quad (4.21)$$

Note that, in view of (4.20), necessarily $c > 0$ and if the integrability conditions (4.20) and (4.21) hold with constants c and k and a negative solution $v(t)$, the same conditions are valid for the constants $c, -k$ and a positive solution $-v(t)$. Consequently, we can restrict ourselves to

studying the integrability conditions (4.20) and (4.21) for positive solutions $v(t) > 0$. In such a case, the above method uses a t -dependent linear change of coordinates of the form (4.9) and the final Riccati equation are of the type described in our work (4.15). Therefore, the integrability conditions derived by Rao and Ukidave have to be a particular instance of the integrable cases described by Theorem 4.3.

Using the value of $v(t)$ in terms of the constant c and the functions b_1 and b_3 obtained with the aid of the formula (4.20) and equation (4.21), we get that

$$\frac{1}{\sqrt{|b_1 b_3|}} \left(b_2 + \frac{1}{2} \left(\frac{\dot{b}_3}{b_3} - \frac{\dot{b}_1}{b_1} \right) \right) = -k \operatorname{sgn}(b_0) \sqrt{c}.$$

Hence, the Riccati equations holding conditions (4.20) and (4.21) satisfy the integrability conditions of Corollary 4.5. Moreover, if we choose

$$D^2 = cb_1 b_3, \quad c_1 = 1, \quad c_2 = -k, \quad c_3 = c^{-1},$$

then $D = \sqrt{cb_1 b_3}$ and the only possible transformation (4.9) given by Theorem 4.3 reads

$$x' = \alpha^2(t)x = \sqrt{\frac{cb_3(t)}{b_1(t)}}x,$$

and hence,

$$\frac{1}{v} = \sqrt{\frac{cb_3}{b_1}}.$$

In this way, we recover one of the results derived by Rao and Ukidave in [190].

In short, many integrability conditions found in the literature can be described by our more general methods.

4.5. Integrability and reduction. Now we develop a similar procedure to the one derived above, but now we assume the solutions of system (4.6) to be included within a two-parameter subset of $SL(2, \mathbb{R})$. As a result, we recover some known integrability conditions and review, from a more general point of view, the integrability method described in [40].

As we did previously, let us try to relate the Riccati equation (4.1) to an integrable one associated, as a Lie system, with a curve $a'(t) = -D(t)(c_1 a_1 + c_2 a_2 + c_3 a_3)$, with $c_3 \neq 0$ and a non-vanishing function $D = D(t)$. Nevertheless, we consider solutions of system (4.7) with $\gamma = 0$, $\alpha > 0$, and related to a curve in $SL(2, \mathbb{R})$, i.e. we analyse transformations

$$x' = \frac{\alpha(t)}{\delta(t)}x + \frac{\beta(t)}{\delta(t)} = \alpha^2(t)x + \frac{\beta(t)}{\delta(t)}.$$

In this case, using the expression in coordinates (4.6) of system (4.8), we get that

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ 0 \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} \frac{b'_3 - b_2}{2} & b_3 & b'_1 & 0 \\ -b_1 & \frac{b'_2 + b_2}{2} & 0 & b'_1 \\ -b'_3 & 0 & -\frac{b'_2 + b_2}{2} & b_3 \\ 0 & -b'_3 & -b_1 & -\frac{b'_2 - b_2}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 0 \\ \delta \end{pmatrix}, \quad (4.22)$$

where $b'_j = D c_j$ and $c_j \in \mathbb{R}$ for $j = 1, 2, 3$. As we suppose $b'_3 \neq 0$, the third equation of the

above system yields

$$\frac{\alpha}{\delta} = \frac{b_3}{b'_3} = \frac{b_3}{Dc_3}.$$

Since $\alpha\delta = 1$ so that the solution of (4.8) is related to an element of $SL(2, \mathbb{R})$, and $b'_3 = Dc_3$, the above expression implies

$$\alpha^2 = \frac{b_3}{Dc_3}. \quad (4.23)$$

Therefore, α is determined by the values of $b_3(t)$, D and c_3 . Additionally, the first differential equation of system (4.22) determines β in terms of α and the coefficients of the initial and final Riccati equations, i.e.

$$\beta = \frac{1}{b_3} \left(\dot{\alpha} - \frac{b'_2 - b_2}{2} \alpha \right).$$

Taking into account the relation (4.23) and as $\alpha\delta = 1$, we can define $M = \beta/\alpha$ and rewrite the above expression as follows

$$\frac{dD}{dt} = \left(b_2(t) + \frac{\dot{b}_3(t)}{b_3(t)} \right) D - c_2 D^2 - 2b_3(t)MD.$$

Considering the differential equation in $\dot{\beta}$ in terms of M , we get the equation

$$\frac{dM}{dt} = -b_1(t) + \frac{c_1 c_3}{b_3(t)} D^2 + b_2(t)M - b_3(t)M^2.$$

Finally, as $\delta\alpha = 1$ is a first-integral of system (4.8), if the system for the variables M and D and all the abovementioned conditions are satisfied, the value $\delta = \alpha^{-1}$ obeys its corresponding differential equations of the system (4.22). Summarising, we have stated the following theorem.

THEOREM 4.6. *Given a Riccati equation (4.1) there exists a transformation*

$$x' = G(t)x + H(t), \quad G(t) > 0,$$

relating it to the integrable equation

$$\frac{dx'}{dt} = D(t)(c_1 + c_2 x' + c_3 x'^2), \quad (4.24)$$

with $c_3 \neq 0$, and D a non-vanishing function, if and only if there exist functions D and M satisfying the system

$$\begin{cases} \frac{dD}{dt} = \left(b_2(t) + \frac{\dot{b}_3(t)}{b_3(t)} \right) D - c_2 D^2 - 2b_3(t)MD, \\ \frac{dM}{dt} = -b_1(t) + \frac{c_1 c_3}{b_3(t)} D^2(t) + b_2(t)M - b_3(t)M^2. \end{cases}$$

The transformation is then given by

$$x' = \frac{b_3(t)}{D(t)c_3} (x + M(t)). \quad (4.25)$$

If we consider $c_1 = 0$ in equation (4.24), the system determining the curve in $SL(2, \mathbb{R})$ which performs the transformation of Theorem 4.6 reads

$$\begin{cases} \frac{dD}{dt} = \left(b_2(t) + \frac{\dot{b}_3(t)}{b_3(t)} \right) D - c_2 D^2(t) - 2b_3(t)MD, \\ \frac{dM}{dt} = -b_1(t) + b_2(t)M - b_3(t)M^2. \end{cases} \quad (4.26)$$

Note that this system does not involve any integrability condition, since there always exists a solution for every initial condition. Nevertheless, finding such solutions can be as difficult as solving the initial Riccati equation. Therefore, we need to assume some simplification in order to find a particular solution. Let us put, for instance, $M = b_1/b_2$. In this case, the first differential equation of the above system does not depend on M and reduces to

$$\frac{dD}{dt} = \left(-b_2(t) + \frac{\dot{b}_3(t)}{b_3(t)} \right) D - c_2 D^2$$

whose solutions read

$$D(t) = \frac{\exp\left(\int_0^t A(t')dt'\right)}{C + c_2 \int_0^t \exp\left(\int_0^{t''} A(t')dt'\right) dt''}, \quad A(t) = \left(-b_2(t) + \frac{\dot{b}_3(t)}{b_3(t)} \right).$$

Meanwhile, as $M = b_2/b_3$ must satisfy the second equation in (4.26), we obtain that

$$\frac{d}{dt} \left(\frac{b_2}{b_3} \right) = -b_1,$$

which gives rise to an integrability condition. This summarises one of the integrability conditions considered in [189].

Let us recover, from our point of view, the result that establishes that the knowledge of a particular solution of the Riccati equation allows us to obtain its general solution. In fact, under the change of variables $M = -x$, the system (4.26) becomes

$$\begin{cases} \frac{dD}{dt} = \left(b_2(t) + \frac{\dot{b}_3(t)}{b_3(t)} \right) D - c_2 D^2 + 2b_3(t)xD, \\ \frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2. \end{cases} \quad (4.27)$$

Each particular solution of the previous system takes the form $(D_p(t), x_p(t))$, with $x_p(t)$ being a particular solution of the Riccati equation (4.1). Therefore, given such a particular solution $x_p(t)$, the function $D_p = D_p(t)$, corresponding to $(D_p(t), x_p(t))$, satisfies the equation

$$\frac{dD_p}{dt} = \left(b_2(t) + \frac{\dot{b}_3(t)}{b_3(t)} + 2b_3(t)x_p(t) \right) D_p - c_2 D_p^2, \quad (4.28)$$

which is a Bernoulli equation and, therefore, is integrable by quadratures. Consequently, the knowledge of a particular solution $x_p(t)$ of the Riccati equation (4.1) allows us to determine a particular solution $(D_p(t), x_p(t))$ of system (4.27) and, in view of the change of variables $x = -M$, a particular solution $(D_p(t), M_p(t)) = (D_p(t), -x_p(t))$ of system (4.26). Finally, the functions $M_p(t)$ and $D(t)$ lead to the change of variables (4.25) described in Theorem 4.6 which

transforms the initial Riccati equation (4.1) into another one related to a solvable Lie algebra of vector fields.

The above process describes a reduction process similar to the one derived in [40], but our method allows us to obtain a direct reduction into an integrable Riccati equation (4.24) through a particular solution.

There exist many ways to impose conditions on the coefficients of the second equation in (4.27) to obtain a particular solution easily. For instance, if there exists a real constant c such that for the t -dependent functions b_1 , b_2 and b_3 we have that $b_1 + b_2c + b_3c^2 = 0$, then c is a particular solution, for example:

1. $b_1 + b_2 + b_3 = 0$ implies that $c = 1$ is a particular solution.
2. $k_2^2b_1 + k_2k_3b_2 + k_3^2b_3 = 0$ means that $c = k_3/k_2$ is a particular solution.

This sketches some cases found in [40, 214].

As a first application of the above method, we can integrate the Riccati equation

$$\frac{dx}{dt} = -\frac{n}{t} + \left(1 + \frac{n}{t}\right)x - x^2. \quad (4.29)$$

related to Hovy's equation [200]. This Riccati equation admits the particular constant solution $x_p(t) = 1$. Using such a particular solution in equation (4.28) and taking, for instance, $c_1 = 0$ and $c_2 = 0$, we can obtain a particular solution for equation (4.28), e.g. $D_p(t) = t^n e^{-t}$. Hence, $(t^n e^{-t}, 1)$ is a particular solution of system (4.27) related to equation (4.29) and $(t^n e^{-t}, -1)$ is a solution of the system (4.26). In this way, Theorem 4.6 states that the transformation (4.25), determined by the $D_p(t) = t^n e^{-t}$ and $M_p(t) = -1$, of the form

$$x' = -t^{-n} e^t c_3^{-1} (x - 1), \quad (4.30)$$

relates the solutions of equation (4.29) to those of the integrable equation

$$\frac{dx'}{dt} = e^{-t} t^n c_3 x'^2.$$

If we fix $c_3 = 1$, the solution of the above equation reads

$$x'(t) = \frac{1}{K - \Gamma(1 + n, t)},$$

where K is an integration constant and $\Gamma(a, b)$ is the incomplete Euler's Gamma function

$$\Gamma(a, t) = \int_t^\infty t'^{a-1} e^{-t'} dt'.$$

In view of the change of variables (4.30), the solutions $x(t)$ of the Riccati equation (4.29) and $x'(t)$ are related through the expression $x'(t) = -t^{-n} e^t c_3^{-1} (x(t) - 1)$. Therefore, if we substitute the general solution $x'(t)$ in this expression, we can derive the general solution for the Riccati equation (4.29), that is,

$$x(t) = 1 - \frac{e^{-t} t^n}{\Gamma(n + 1, t) + K}.$$

4.6. Linearisation of Riccati equations. To finish this chapter, we shall analyse the problem of the linearisation of Riccati equations through the linear fractional transformations (4.9). As a main result, we establish various integrability conditions ensuring that a Riccati equation can

be transformed into a linear one by means of a diffeomorphism on $\overline{\mathbb{R}}$ associated with a linear fractional transformation of a certain class.

As a first insight in the linearisation process, notice that Corollary 4.2 states that there exists a curve in $SL(2, \mathbb{R})$, and therefore a t -dependent linear fractional transformation on $\overline{\mathbb{R}}$, transforming each given Riccati equation into any other one (and, in particular, into a linear one). This clearly implies that Riccati equations are always linearisable by means of this class of transformations. Nevertheless, as Lie system (4.7) describing such transformations is related to a non-solvable Lie algebra of vector fields, determining such a transformation can be as difficult as solving the Riccati equation to be linearised.

Let us try to transform a given Riccati equation into a linear differential equation by means of a linear fractional transformation (4.2) determined by a constant vector $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$ with $\alpha\delta - \beta\gamma = 1$. In this case, determining the conditions ensuring the existence of solutions of system (4.7) performing such a transformation is an easy task. Moreover, as solving system (4.7) also becomes straightforward, we can determine some linearisability conditions and, when these conditions hold, specify the corresponding change of variables.

Note that as $(\alpha, \beta, \gamma, \delta)$ is a constant, we have $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = \dot{\delta} = 0$ and, in view of (4.6), the diffeomorphism on $\overline{\mathbb{R}}$ performing the transformation is related to a vector in the kernel of the matrix

$$B = \begin{pmatrix} \frac{b'_2 - b_2}{2} & b_3 & b'_1 & 0 \\ -b_1 & \frac{b'_2 + b_2}{2} & 0 & b'_1 \\ 0 & 0 & -\frac{b'_2 + b_2}{2} & b_3 \\ 0 & 0 & -b_1 & -\frac{b'_2 - b_2}{2} \end{pmatrix}, \quad (4.31)$$

where we assume $b_1 \neq 0, b_3 \neq 0$. We omit the study of the case $b_1(t)b_3(t) = 0$ in an open interval because, as it was shown in Section 4.1, this case is known to be integrable.

The necessary and sufficient condition for $\ker B$ to be non-trivial is $\det B = 0$. Therefore, a short calculation shows that $\dim \ker B > 0$ if and only if $-b_2^2 + b_3^2(t) + 4b_1b_3)^2 = 0$. Thus, $b'_3 = \pm\sqrt{b_2^2 - 4b_1b_3}$ and b'_3 is fixed, up to a sign, by the values of b_1, b_2 and b_3 . Let us study the kernel of the matrix B in the positive and negative cases for b'_3 .

• **Positive case:** The kernel of matrix (4.31) is given by the vectors

$$\left(\delta \frac{b'_1}{b_1} + \beta \frac{b_2 + \sqrt{b_2^2 - 4b_1b_3}}{2b_1}, \beta, -\delta \frac{-b_2 + \sqrt{b_2^2 - 4b_1b_3}}{2b_1}, \delta \right), \quad \delta, \beta \in \mathbb{R}.$$

Recall that we are only considering the constant elements of $\ker B$, therefore there should be two real constants K_1 and K_2 such that

$$K_1 = \delta \frac{b'_1}{b_1} + \beta \frac{b_2 + \sqrt{b_2^2 - 4b_1b_3}}{2b_1}, \quad K_2 = \frac{-b_2 + \sqrt{b_2^2 - 4b_1b_3}}{2b_1}, \quad (4.32)$$

Moreover, in order to relate these vectors to elements in $SL(2, \mathbb{R})$, we have to impose the condition $\det(K_1, \beta, -\delta K_2, \delta) = \delta(K_1 + \beta K_2) = 1$.

The second condition in (4.32) imposes a restriction on the coefficients of the initial Riccati equation to be linearisable by a constant linear fractional transformation (4.2). Then, if this is satisfied, we can choose β, γ, K_1 and b'_2 to satisfy the other conditions. Thus, the only linearisation condition is the second one in (4.32).

• **Negative case:** In this case, $\ker B$ reads

$$\left(\frac{\delta b'_1}{b_1} + \beta \frac{b_2 - \sqrt{b_2^2 - 4b_1 b_3}}{2b_1}, \beta, -\delta \frac{-b_2 - \sqrt{b_2^2 - 4b_1 b_3}}{2b_1}, \delta \right), \quad \delta, \beta \in \mathbb{R},$$

and now the new conditions reduce to the existence of two real constants K_1 and K_2 such that

$$K_1 = \frac{\delta b'_1}{b_1} + \beta \frac{b_2 - \sqrt{b_2^2 - 4b_1 b_3}}{2b_1}, \quad K_2 = \frac{-b_2 - \sqrt{b_2^2 - 4b_1 b_3}}{2b_1},$$

with $\delta(K_1 + \beta K_2) = 1$. If the second expression of the above conditions is satisfied, we can proceed in a similar fashion as for the positive case to obtain the transformation that performs the linearisation of the initial Riccati equation.

Summarising:

THEOREM 4.7. *The necessary and sufficient condition for the existence of a diffeomorphism on $\overline{\mathbb{R}}$ of linear fractional type associated with a transformation on $SL(2, \mathbb{R})$ transforming the Riccati (4.1) into a linear differential equation is the existence of a real constant K such that*

$$K = \frac{-b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}}{2b_1}. \quad (4.33)$$

As a Riccati equation (4.1) satisfies the above condition if and only if K is a constant particular solution, we get the following corollary:

COROLLARY 4.8. *A Riccati equation can be linearised by means of a diffeomorphism on $\overline{\mathbb{R}}$ of the form (4.2) if and only if it admits a constant particular solution.*

Ibragimov showed that a Riccati equation (4.1) is linearisable by means of a change of variables $z = z(x)$ if and only if the Riccati equation admits a constant solution [125]. Additionally, we have proved that in such a case, the change of variables can be described by means of a transformation of the type (4.2).

5. Lie integrability in Classical Physics

In spite of their apparent simplicity, the methods developed throughout the previous chapter reduce the analysis of certain integrability conditions for Riccati equations to studying integrability conditions for an equation on $SL(2, \mathbb{R})$. Moreover, these methods can also be applied to any other Lie system related to the same equation on $SL(2, \mathbb{R})$. For instance, we here use the results on integrability of Riccati equations to study t -dependent (frequency and/or mass) harmonic oscillators (TDHOs), which are associated with the same kind of equations on $SL(2, \mathbb{R})$ as Riccati equations. As a particular application of our results, we supply t -dependent constants of the motion for certain one-dimensional TDHOs and the solutions for a two-dimensional TDHO. Also, our approach provides a unifying framework which allows us to apply our developments to all Lie systems associated with equations in $SL(2, \mathbb{R})$ and generalise our methods to study any Lie system.

5.1. TDHO as a SODE Lie system. Let us prove that every TDHO is a SODE Lie systems (see [37, 43, 52]). Each TDHO is described by a t -dependent Hamiltonian of the form

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2}F(t)\omega^2 x^2,$$

whose Hamilton equations read

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m(t)}, \\ \dot{p} = -\frac{\partial H}{\partial x} = -F(t)\omega^2 x. \end{cases} \quad (5.1)$$

The solutions of the above system are integral curves for the t -dependent vector field

$$X_t = p \frac{\partial}{\partial x} - F(t)\omega^2 x \frac{\partial}{\partial p},$$

over $T^*\mathbb{R}$. Let X_1^{HO} , X_2^{HO} and X_3^{HO} be the vector fields

$$X_1^{HO} = p \frac{\partial}{\partial x}, \quad X_2^{HO} = \frac{1}{2} \left(x \frac{\partial}{\partial x} - p \frac{\partial}{\partial p} \right), \quad X_3^{HO} = -x \frac{\partial}{\partial p}, \quad (5.2)$$

which satisfy the commutation relations

$$[X_1^{HO}, X_3^{HO}] = 2X_2^{HO}, \quad [X_1^{HO}, X_2^{HO}] = X_1^{HO}, \quad [X_2^{HO}, X_3^{HO}] = X_3^{HO},$$

and therefore span a Lie algebra of vector fields V^{HO} isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Then, the t -dependent vector field X^{HO} associated with system (5.1) can be written as

$$X^{HO}(t) = F(t)\omega^2 X_3^{HO} + \frac{1}{m(t)} X_1^{HO}, \quad (5.3)$$

i.e. it is a linear combination with t -dependent coefficients

$$X^{HO}(t) = \sum_{\alpha=1}^3 b_\alpha(t) X_\alpha^{HO}, \quad (5.4)$$

with $b_1(t) = 1/m(t)$, $b_2(t) = 0$ and $b_3(t) = F(t)\omega^2$. Hence, TDHOs are SODE Lie systems.

Consider the basis $\{a_1, a_2, a_3\}$ for $\mathfrak{sl}(2, \mathbb{R})$ given in (2.4). Its elements satisfy the same commutation relations as the vector fields X_α^{HO} . Denote by $\Phi^{HO} : SL(2, \mathbb{R}) \times T^*\mathbb{R} \rightarrow T^*\mathbb{R}$ the action that associates each a_α with the fundamental vector field X_α^{HO} , i.e. each one-parameter subgroup $\exp(-ta_\alpha)$ acts on $T^*\mathbb{R}$ with infinitesimal generator X_α^{HO} . It can be verified that this action reads

$$\Phi^{HO} \left(\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right), \left(\begin{array}{c} x \\ p \end{array} \right) \right) = \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \left(\begin{array}{c} x \\ p \end{array} \right).$$

Obviously, the linear map $\rho^{HO} : \mathfrak{sl}(2, \mathbb{R}) \rightarrow V^{HO}$ that maps each a_α to X_α is a Lie algebra isomorphism.

The action Φ^{HO} allows us to relate (5.1) to an equation on $SL(2, \mathbb{R})$ given by

$$R_{A^{-1}*A} \dot{A} = - \sum_{\alpha=1}^3 b_\alpha(t) a_\alpha, \quad A(0) = I. \quad (5.5)$$

Thus, if $A(t)$ is the solution of (5.5) and we denote $\xi = (x, p) \in T^*\mathbb{R}$, then the solution starting from $\xi(0)$ is $\xi(t) = \Phi^{HO}(A(t), \xi(0))$ (see e.g. [40]). In summary, system (5.1) is a Lie system

in $T^*\mathbb{R}$ related to an equation on $SL(2, \mathbb{R})$ and the solution of equation (5.5) allows us to obtain the solutions of (5.1) in terms of the initial condition by means of the action Φ^{HO} .

5.2. Transformation laws of Lie equations on $SL(2, R)$. Each t -dependent harmonic oscillator (5.1) can be considered as a curve in \mathbb{R}^3 of the form $(b_1(t), b_2(t), b_3(t))$ through the decomposition (5.4). Then, we can transform each curve $\xi(t)$ in $T^*\mathbb{R}$, by an element $\bar{A}(t)$ of \mathcal{G} as follows:

$$\text{If } \bar{A}(t) = \begin{pmatrix} \bar{\alpha}(t) & \bar{\beta}(t) \\ \bar{\gamma}(t) & \bar{\delta}(t) \end{pmatrix} \in \mathcal{G}, \quad \Theta(\bar{A}, \xi)(t) = \begin{pmatrix} \bar{\alpha}(t)x(t) + \bar{\beta}(t)p(t) \\ \bar{\gamma}(t)x(t) + \bar{\delta}(t)p(t) \end{pmatrix}. \quad (5.6)$$

The above change of variables transforms the TDHO (5.1) into an analogous TDHO with new coefficients b'_1, b'_2, b'_3 given by

$$\begin{cases} b'_3 = \bar{\delta}^2 b_3 - \bar{\delta}\bar{\gamma} b_2 + \bar{\gamma}^2 b_1 + \bar{\gamma}\dot{\bar{\delta}} - \bar{\delta}\dot{\bar{\gamma}}, \\ b'_2 = -2\bar{\beta}\bar{\delta} b_3 + (\bar{\alpha}\bar{\delta} + \bar{\beta}\bar{\gamma}) b_2 - 2\bar{\alpha}\bar{\gamma} b_1 + \delta\dot{\bar{\alpha}} - \bar{\alpha}\dot{\bar{\delta}} + \bar{\beta}\dot{\bar{\gamma}} - \bar{\gamma}\dot{\bar{\beta}}, \\ b'_1 = \bar{\beta}^2 b_3 - \bar{\alpha}\bar{\beta} b_2 + \bar{\alpha}^2 b_1 + \bar{\alpha}\dot{\bar{\beta}} - \bar{\beta}\dot{\bar{\alpha}}. \end{cases}$$

The solutions of the transformed TDHO are of the form $\Theta(\bar{A}(t), \xi(t))$, with $\xi(t)$ being a solution of the initial TDHO. Additionally, the above expressions define an affine action (see e.g. [151] for the general definition of this concept) of the group \mathcal{G} on the set of TDHOs [63]. This means that in order to transform the coefficients of a TDHO by means of two transformations of the above type, first through A_1 and then by means of A_2 , it suffices to do the transformation induced by the product $A_2 A_1$.

The result of this action of \mathcal{G} can also be studied from the point of view of the equations in $SL(2, \mathbb{R})$. First, \mathcal{G} acts on the left on the set of curves in $SL(2, \mathbb{R})$ by left translations, i.e. a curve $\bar{A}(t)$ transforms the curve $A(t)$ into a new one $A'(t) = \bar{A}(t)A(t)$. Therefore, if $A(t)$ is a solution of (5.5), characterised by a curve $\mathfrak{a}(t) \in \mathfrak{sl}(2, \mathbb{R})$, then the new curve satisfies a new equation like (5.5) but with a different right-hand side, $\mathfrak{a}'(t)$, and thus it corresponds to a new equation on $SL(2, \mathbb{R})$ associated with a new TDHO. Of course, $A'(0) = \bar{A}(0)A(0)$, and if we want $A'(0) = \text{Id}$, we have to impose the additional condition $\bar{A}(0) = \text{Id}$. In this way \mathcal{G} acts on the set of curves in $T_I SL(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R})$. It can be shown that the relation between both curves $\mathfrak{a}(t)$ and $\mathfrak{a}'(t)$ in $\mathfrak{sl}(2, \mathbb{R})$ is given by [40]

$$\mathfrak{a}'(t) = - \sum_{\alpha=1}^3 b'_\alpha(t) \mathfrak{a}_\alpha = \bar{A}(t)\mathfrak{a}(t)\bar{A}^{-1}(t) + \dot{\bar{A}}(t)\bar{A}^{-1}(t). \quad (5.7)$$

Summarising, it has been shown that it is possible to associate, in a one-to-one way, any TDHO with an equation in the Lie group $SL(2, \mathbb{R})$ and to define a group \mathcal{G} of transformations on the set of such TDHOs induced by the natural linear action of $SL(2, \mathbb{R})$.

Recall that, in view of Theorem 4.1, system (5.7) can be regarded as a system of first-order ordinary differential equations in the coefficients of the curve in $SL(2, \mathbb{R})$ of the form

$$\bar{A}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}.$$

Moreover, we can enunciate the following results, which are a straightforward application to TDHOs of Theorem 4.1 and Corollary 4.2 formulated for the analysis of certain Lie systems on $SL(2, \mathbb{R})$ related to Riccati equations.

THEOREM 5.1. *The curves in $SL(2, \mathbb{R})$ transforming a TDHO related to an equation on this Lie group determined by a curve $a(t)$ into a new TDHO associated with an equation on $SL(2, \mathbb{R})$ determined by the curve $a'(t)$, with*

$$a'(t) = - \sum_{\alpha=1}^3 b'_\alpha(t) a_\alpha, \quad a(t) = - \sum_{\alpha=1}^3 b_\alpha(t) a_\alpha,$$

are given by the integral curves of the t -dependent vector field

$$N(t) = \sum_{\alpha=1}^3 (b_\alpha(t) N_\alpha + b'_\alpha(t) N'_\alpha), \quad (5.8)$$

such that $\det \bar{A}(0) = 1$. This system is a Lie system associated with a non-solvable Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. Moreover, such curves also transform the TDHO related to the curve $a(t)$ into the new one linked to $a'(t)$.

COROLLARY 5.2. *Given two TDHOs associated with the curves $a(t)$ and $a'(t)$ in $\mathfrak{sl}(2, \mathbb{R})$, there always exists a curve in $SL(2, \mathbb{R})$ transforming the first TDHO into the second one.*

We must remark that even if we know that given two equations in the Lie group $SL(2, \mathbb{R})$ there always exists a transformation relating both, in order to find such a curve we need to solve the system of differential equations providing the integral curves of (5.8). This is the solution of a system of differential equations that is a Lie system related to a non-solvable Lie algebra in general. Hence, it is not easy to find its solutions, i.e. it may not be integrable by quadratures.

The result of Theorem 5.1, i.e. that the system of differential equations describing the transformations of Lie systems on $SL(2, \mathbb{R})$ is a matrix Riccati equation associated, as a Lie system, with a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, suggests us a method to find some sufficiency conditions for integrability of the TDHOs to be explained next.

5.3. Description of some known integrability conditions. We now study some cases when it is possible to find curves $\bar{A}(t)$ in $SL(2, \mathbb{R})$ transforming a given TDHO related to an equation on $SL(2, \mathbb{R})$ characterised by a curve $a(t)$ into a new TDHO associated with an equation on $SL(2, \mathbb{R})$ characterised by a curve of the type $a'(t) = -D(t)(c_1 a_1 + c_2 a_2 + c_3 a_3)$. This is possible if the system determined by (5.8) can be solved easily. The transformation establishing the relation to such a TDHO allows us to find the solution of the given equation by quadratures. We first restrict ourselves to studying cases in which the curve $\bar{A}(t)$ lies in a one-parameter subset of $SL(2, \mathbb{R})$. The results we show next are a direct translation to the framework of TDHO of Theorem 4.1 describing certain integrability properties of Riccati equations (see also [50]).

THEOREM 5.3. *The necessary and sufficient conditions for the existence of a transformation*

$$\xi' = \Phi^{HO}(\bar{A}_0(t), \xi), \quad \xi = \begin{pmatrix} x \\ p \end{pmatrix},$$

with

$$\bar{A}_0(t) = \begin{pmatrix} \alpha(t) & 0 \\ 0 & \alpha^{-1}(t) \end{pmatrix}, \quad \alpha(t) > 0, \quad (5.9)$$

relating the TDHO associated with the t -dependent vector field

$$X_t = b_1(t)X_1 + b_2(t)X_2 + b_3(t)X_3, \quad (5.10)$$

where $b_1(t)b_3(t)$ has a constant sign, i.e. $b_1(t)b_3(t) \neq 0$, to another integrable one given by

$$X'(t) = D(t)(c_1X_1 + c_2X_2 + c_3X_3), \quad (5.11)$$

with, c_1, c_2, c_3 , being real numbers such that $c_1c_3 \neq 0$, are

$$D^2(t)c_1c_3 = b_1(t)b_3(t), \quad b_2(t) + \frac{1}{2} \left(\frac{\dot{b}_3(t)}{b_3(t)} - \frac{\dot{b}_1(t)}{b_1(t)} \right) = c_2 \sqrt{\frac{b_1(t)b_3(t)}{c_1c_3}}.$$

Then, the transformation is uniquely defined by

$$\bar{A}_0(t) = \begin{pmatrix} \left(\frac{b_3(t)c_1}{b_1(t)c_3} \right)^{1/4} & 0 \\ 0 & \left(\frac{b_3(t)c_1}{b_1(t)c_3} \right)^{-1/4} \end{pmatrix}.$$

Note that one coefficient, either c_1 or c_3 , can be reabsorbed with a redefinition of the function D . As a straightforward application of the preceding theorem, which can be found in a similar way as those in [50], we obtain the following corollaries:

COROLLARY 5.4. A TDHO (5.1) with $b_1(t)b_3(t) \neq 0$ is integrable by a t -dependent change of variables

$$\xi' = \Phi^{HO}(\bar{A}_0(t), \xi),$$

with \bar{A}_0 given by (5.9), if and only

$$\sqrt{\frac{c_1c_3}{b_1(t)b_3(t)}} \left[b_2(t) + \frac{1}{2} \left(\frac{\dot{b}_3(t)}{b_3(t)} - \frac{\dot{b}_1(t)}{b_1(t)} \right) \right] = c_2, \quad (5.12)$$

for certain real constants c_1, c_2 , and c_3 . In this case

$$D(t) = \sqrt{\frac{b_1(t)b_3(t)}{c_1c_3}},$$

and the new system is

$$\frac{d\xi'}{dt} = D(t) \begin{pmatrix} c_2/2 & c_1 \\ -c_3 & -c_2/2 \end{pmatrix} \xi'. \quad (5.13)$$

COROLLARY 5.5. Given an integrable TDHO characterised by a t -dependent vector field (5.11), the set of TDHOs which can be obtained through a t -dependent transformation

$$\xi' = \Phi^{HO}(\bar{A}_0(t), \xi),$$

with \bar{A}_0 given by (5.9), are those of the form

$$X_t = b_1(t)X_1 + \left(\frac{\dot{b}_1(t)}{b_1(t)} - \frac{\dot{D}(t)}{D(t)} + c_2D(t) \right) X_2 + \frac{D^2(t)c_1c_3}{b_1(t)} X_3. \quad (5.14)$$

Thus, $\bar{A}_0(t)$ reads

$$\bar{A}_0(t) = \begin{pmatrix} \left(\frac{b_3(t)c_1}{b_1(t)c_3} \right)^{1/4} & 0 \\ 0 & \left(\frac{b_3(t)c_1}{b_1(t)c_3} \right)^{-1/4} \end{pmatrix}.$$

Therefore, starting from an integrable system we can find the family of t -dependent vector fields (5.14) describing solvable TDHO systems whose coefficients are parametrised by $b_1(t)$. Given a TDHO, it is easy to check whether it belongs to such a family and can be easily integrated.

The integrability conditions we have described here arise as requirements on the initial t -dependent functions b_α that allow us to solve the initial TDHO exactly by a t -dependent transformation of the form

$$\xi' = \Phi^{HO}(\exp(\Psi(t)v), \xi),$$

with some $v \in \mathfrak{sl}(2, \mathbb{R})$ and $\Psi(t)$, in such a way that the initial TDHO system (5.1) in the variable ξ is transformed into another one in the variable ξ' associated, as a Lie system, with a Vessiot–Guldberg Lie algebra isomorphic to an appropriate Lie subalgebra of $\mathfrak{sl}(2, \mathbb{R})$ in such a way that the equation in ξ' can be integrated by quadratures and, consequently, the equation in ξ is solvable too.

5.4. Some applications of integrability conditions to TDHOs. As a first application, we show that the usual approach to the solution of the classical Caldirola–Kanai Hamiltonian [27, 133] can be explained through our method (the solution of the quantum case can be obtained in a similar way). Next, we will also apply our approach to get integrable TDHOs.

The Hamiltonian of a t -dependent harmonic oscillator is

$$H(t) = \frac{1}{2} \frac{p^2}{m(t)} + \frac{1}{2} m(t) \omega^2(t) x^2. \quad (5.15)$$

For instance, a harmonic oscillator with a damping term [27, 133] with equation of motion

$$\frac{d}{dt}(m_0 \dot{x}) + m_0 \mu \dot{x} + kx = 0, \quad k = m_0 \omega^2,$$

admits a Hamiltonian description, with a t -dependent Hamiltonian

$$H(t) = \frac{p^2}{2m_0} \exp(-\mu t) + \frac{1}{2} m_0 \exp(\mu t) \omega^2 x^2,$$

i.e. $m(t)$ in (5.15) corresponds to $m(t) = m_0 \exp(\mu t)$. In this case equations (5.1) are

$$\begin{cases} \dot{x} &= \frac{\partial H}{\partial p} = \frac{1}{m_0} \exp(-\mu t) p, \\ \dot{p} &= -\frac{\partial H}{\partial x} = -m_0 \exp(\mu t) x, \end{cases} \quad (5.16)$$

and the t -dependent coefficients of the associated Lie system read

$$b_1(t) = \frac{1}{m_0} \exp(-\mu t), \quad b_2(t) = 0, \quad b_3(t) = m_0 \omega^2 \exp(\mu t).$$

Therefore, as $b_1(t)b_3(t) = \omega^2$, $b_2 = 0$ and

$$\frac{\dot{b}_3}{b_3} - \frac{\dot{b}_1}{b_1} = 2\mu,$$

we see that (5.12) holds if we set $c_1 = c_3 = 1$, $c_2 = \mu/\omega$ and the function D is a constant, $D = \omega$. Hence, this example reduces to the system

$$\frac{d}{dt} \begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} \mu/2 & \omega \\ -\omega & -\mu/2 \end{pmatrix} \begin{pmatrix} x' \\ p' \end{pmatrix},$$

which can be easily integrated. If we put $\bar{\omega}^2 = (\mu^2/4) - \omega^2$, we get

$$\begin{pmatrix} x'(t) \\ p'(t) \end{pmatrix} = \begin{pmatrix} \cosh(\bar{\omega}t) + \frac{\mu}{2\bar{\omega}} \sinh(\bar{\omega}t) & \frac{\omega}{\bar{\omega}} \sinh(\bar{\omega}t) \\ -\frac{\omega}{\bar{\omega}} \sinh(\bar{\omega}t) & \cosh(\bar{\omega}t) - \frac{\mu}{2\bar{\omega}} \sinh(\bar{\omega}t) \end{pmatrix} \begin{pmatrix} x'(0) \\ p'(0) \end{pmatrix}$$

and, in terms of the initial variables, we obtain

$$x(t) = \frac{e^{-\mu t/2}}{\sqrt{m_0\omega}} \left(\left(\cosh(\bar{\omega}t) + \frac{\mu}{2\bar{\omega}} \sinh(\bar{\omega}t) \right) \sqrt{m_0\omega} x_0 + \frac{\omega}{\bar{\omega}} \sinh(\bar{\omega}t) \frac{p_0}{\sqrt{m_0\omega}} \right).$$

We can also study a TDHO described by the t -dependent Hamiltonian

$$H(t) = \frac{1}{2}p^2 + \frac{1}{2}F(t)\omega^2 x^2, \quad F(t) > 0,$$

where we assume, for simplicity, $m = 1$. The t -dependent vector field X is

$$X_t = p \frac{\partial}{\partial x} - F(t)\omega^2 x \frac{\partial}{\partial p},$$

which is a linear combination

$$X_t = F(t)\omega^2 X_3^{HO} + X_1^{HO},$$

i.e. the t -dependent coefficients in (5.10) are

$$b_1(t) = 1, \quad b_2(t) = 0, \quad b_3(t) = F(t)\omega^2,$$

and the condition for F to satisfy (5.12) is

$$\frac{1}{2} \frac{\dot{F}}{F} = c_2 \omega \sqrt{F}.$$

Therefore, F must be of the form

$$F(t) = \frac{1}{(L - c_2\omega t)^2}$$

and the Hamiltonian, which can be exactly integrated, is

$$H(t) = \frac{p^2}{2} + \frac{1}{2} \frac{\omega^2}{(L - c_2\omega t)^2} x^2.$$

The corresponding Hamilton equations are

$$\begin{cases} \dot{x} = p, \\ \dot{p} = -\frac{\omega^2}{(L - c_2\omega t)^2} x, \end{cases}$$

and the t -dependent change of variables to perform is

$$\begin{cases} x' = \sqrt{\frac{\omega}{L - c_2\omega t}} x, \\ p' = \sqrt{\frac{L - c_2\omega t}{\omega}} p. \end{cases}$$

In consequence,

$$\begin{cases} \frac{dx'}{dt} = \frac{\omega}{L - c_2\omega t} \left(\frac{c_2}{2} x' + p' \right), \\ \frac{dp'}{dt} = \frac{\omega}{L - c_2\omega t} \left(-x' - \frac{c_2}{2} p' \right), \end{cases} \quad (5.17)$$

and, under the t -reparametrisation,

$$\tau(t) = \int_0^t \frac{\omega dt'}{L - c_2\omega t'} = \frac{1}{c_2} \ln \left(\frac{K'}{L - c_2\omega t} \right),$$

the system (5.17) becomes

$$\begin{cases} \frac{dx'}{d\tau} = \frac{c_2}{2}x' + p', \\ \frac{dp'}{d\tau} = -x' - \frac{c_2}{2}p', \end{cases}$$

which is equivalent to a transformed Caldirola–Kanai differential equation through the change $\tau \mapsto \omega t$ and $c_2 \mapsto \mu/\omega$. In any case, the solution is

$$x'(\tau) = \left(\cosh(\tilde{\omega}\tau) + \frac{c_2}{2\tilde{\omega}}\sinh(\tilde{\omega}\tau) \right) x'(0) + \frac{1}{\tilde{\omega}}\sinh(\tilde{\omega}\tau)p'(0),$$

where $\tilde{\omega} = \sqrt{\frac{c_2^2}{4} - 1}$. Finally,

$$x(\tau(t)) = \sqrt{\frac{L - c_2\omega t}{\omega}} \left[\left(\cosh(\tilde{\omega}\tau(t)) + \frac{c_2}{2\tilde{\omega}}\sinh(\tilde{\omega}\tau(t)) \right) x'(0) + \frac{1}{\tilde{\omega}}\sinh(\tilde{\omega}\tau(t))p'(0) \right].$$

Let us analyse another integrability condition that, as the preceding one, arises as a compatibility condition for a restricted case of the system describing the integral curves of (5.8). Nevertheless, this time, the solution is restricted to a one-parameter set of matrices of $SL(2, \mathbb{R})$ that is not a group in general.

In this way, we deal with a family of transformations

$$\bar{A}_0(t) = \begin{pmatrix} \frac{1}{V(t)} & 0 \\ -u_1 & V(t) \end{pmatrix}, \quad V(t) > 0, \quad (5.18)$$

where u_1 is a constant, i.e. we want to relate the t -dependent vector field

$$X_t = X_1^{HO} + F(t)\omega^2 X_3^{HO},$$

characterised by the coefficients in (5.10)

$$b_1 = 1, \quad b_2 = 0, \quad b_3 = F(t)\omega^2,$$

to an integrable one characterised by b'_1, b'_2 and b'_3 , or more explicitly, to the t -dependent vector field

$$X_t = D(t)(c_1 X_1 + c_3 X_3),$$

i.e. $b'_1 = Dc_1, b'_2 = 0$, and $b'_3 = Dc_3$. Moreover, if $c_1 \neq 0$, we can reabsorb its value redefining D and assuming $c_1 = 1$.

Under the action of (5.18), the original system transforms into the following system

$$\begin{cases} b'_3 &= V^2 b_3 + u_1 V b_2 + u_1^2 b_1 - u_1 \dot{V}, \\ b'_2 &= b_2 + 2\frac{u_1}{V} b_1 - 2\frac{\dot{V}}{V}, \\ b'_1 &= \frac{1}{V^2} b_1. \end{cases}$$

As $b_2 = b'_2 = 0$ and $b_1 = 1$, the second equation yields $\dot{V} = u_1$, i.e. $V(t) = u_1 t + u_0$ with $u_0 \in \mathbb{R}$. Moreover, using this condition on the first equation together with $b_1 = 1$, we get $b'_3 = V^2 b_3$. Then, as the third equation gives us the value of D as $D = b'_1 = 1/V^2$, we see that

$b'_3 = Dc_3 = V^2 F(t)\omega^2$. Therefore, F has to be proportional to $(u_1 t + u_0)^{-4}$,

$$F(t) = \frac{k}{(u_1 t + u_0)^4}, \quad k = \frac{c_3}{\omega^2}.$$

Let assume $k = 1$ and thus, $c_3 = \omega^2$. Then, the t -dependent transformation $\bar{A}_0(t)$ performing this reduction is

$$\begin{cases} x' = \frac{x}{V(t)}, \\ p' = -u_1 x + V(t)p. \end{cases}$$

Under this transformation, the initial system becomes

$$\begin{cases} \frac{dx'}{dt} = \frac{p'}{V^2(t)}, \\ \frac{dp'}{dt} = -\frac{\omega^2 x'}{V^2(t)}. \end{cases}$$

Using the t -reparametrisation

$$\tau(t) = \int_0^t \frac{dt'}{V^2(t')} = \frac{1}{u_1} \left(\frac{1}{u_0} - \frac{1}{V(t)} \right),$$

we get the following autonomous linear system

$$\begin{cases} \frac{dx'}{d\tau} = p', \\ \frac{dp'}{d\tau} = -\omega^2 x', \end{cases}$$

whose solution is

$$\begin{pmatrix} x'(\tau) \\ p'(\tau) \end{pmatrix} = \begin{pmatrix} \cos(\omega\tau) & \frac{\sin(\omega\tau)}{\omega} \\ -\omega \sin(\omega\tau) & \cos(\omega\tau) \end{pmatrix} \begin{pmatrix} x'(0) \\ p'(0) \end{pmatrix}.$$

Thus, we obtain that

$$x(t) = V(t) \left(\cos(\omega \tau(t)) \frac{x_0}{u_0} + \frac{1}{\omega} \sin(\omega \tau(t)) (-u_1 x_0 + u_0 p_0) \right).$$

5.5. Integrable TDHOs and t -dependent constants of the motion. The autonomisations of the transformed integrable systems obtained above enable us to construct t -dependent constants of the motion. Indeed, in previous cases, a TDHO was transformed into a Lie system related to an equation on $SL(2, \mathbb{R})$

$$R_{A^{-1} * A} \dot{A} = -D(t) (c_1 M_0 + c_2 a_1 + c_3 a_1),$$

associated with a TDHO determined by the t -dependent vector field

$$X_t = D(t)(c_1 X_1 + c_2 X_2 + c_3 X_3).$$

Each t -dependent first-integral $I(t)$ of this differential equation satisfies

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + X_t I = 0.$$

Thus, the function I is a first-integral of the vector field on $\mathbb{R} \times \mathbb{T}^* \mathbb{R}$

$$\bar{X}_t = c_1 X_1(t) + c_2 X_2(t) + c_3 X_3(t) + \frac{1}{D(t)} \frac{\partial}{\partial t}.$$

As $\mathbb{R} \times \mathbb{T}^*\mathbb{R}$ is a three-dimensional manifold and the differential equation we are studying is determined by a distribution of dimension one, there exist (at least locally) two independent first-integrals. Next, we will analyse some integrable cases and their corresponding constants of the motion.

- Case $F(t) = (u_1 t + u_0)^{-2}$:

In this case we obtain that, according to Theorem 5.3, the t -dependent vector field of the initial TDHO is transformed into the following one,

$$X_t = \frac{\omega}{u_1 t + u_0} \left(X_1^{HO} - \frac{u_1}{\omega} X_2^{HO} + X_3^{HO} \right)$$

and thus, using the method of characteristics, we obtain the following constants of the motion for this TDFHO:

$$I_1 = -\frac{u_1}{\omega} p' x' + x'^2 + p'^2, \quad I_2 = \frac{(u_1 + u_0 t)^{\omega/u_1}}{\left(\left(\frac{u_1}{\omega} x' - 2p' \right) + 2\bar{\omega} x' \right)^{\frac{1}{\bar{\omega}}}},$$

with $\bar{\omega} = \pm \sqrt{\frac{u_1^2}{4\omega^2} - 1}$.

- Case $F(t) = (u_1 t + u_0)^{-4}$:

In this case we see that the t -dependent vector field of the initial TDHO is transformed into

$$X_t = \frac{1}{V^2(t)} \left(X_1^{HO} + \omega^2 X_3^{HO} \right),$$

and thus, using the method of characteristics, we get the following t -dependent constants of the motion for the initial TDHO

$$\begin{aligned} I_1 &= \left(\frac{x\omega}{V(t)} \right)^2 + (V(t)p - u_1 x)^2, \\ I_2 &= \arcsin \left(\frac{x\omega}{V(t)\sqrt{I_1}} \right) + \frac{\omega}{u_1 V(t)}. \end{aligned} \tag{5.19}$$

As we have two t -dependent constants of the motion over $\mathbb{R} \times \mathbb{T}^*\mathbb{R}$ and the solutions in this space are of the form $(t, x(t), p(t))$, we can obtain the solutions for our initial system.

5.6. Applications to two-dimensional TDHO's. In this section we apply our previous geometrical methods to analyse the following two-dimensional t -dependent harmonic oscillator

$$H(t, x_1, x_2, p_1, p_2) = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{\omega_1^2 x_1^2 + \omega_2^2 x_2^2}{2V^4(t)},$$

with ω_1 and ω_2 being constant and $V(t) = u_1 t + u_0$. Nevertheless, our approach is also valid for the corresponding generalisation to a n -dimensional TDHOs. This Hamiltonian is related to an uncoupled pair of TDHOs and therefore the same development of the last section applies again. In this way, we obtain that its Hamilton equations read

$$\begin{cases} \dot{x}_i = p_i, \\ \dot{p}_i = -\frac{\omega_i^2}{V^4(t)} x_i, \end{cases} \quad i = 1, 2,$$

and can be transformed into the system

$$\begin{cases} \frac{dx'_i}{dt} = \frac{1}{V^2(t)}p'_i, \\ \frac{dp'_i}{dt} = -\frac{\omega_i^2}{V^2(t)}x'_i, \end{cases} \quad i = 1, 2,$$

by means of the t -dependent change of variables

$$\begin{cases} x'_i = \frac{x_i}{V(t)}, \\ p'_i = -u_1x_i + V(t)p_i, \end{cases} \quad i = 1, 2.$$

The solutions of the latter system are integral curves of a t -dependent vector field in the distribution generated by the vector field

$$X = -\omega_1^2x'_1\frac{\partial}{\partial p'_1} + p'_1\frac{\partial}{\partial x'_1} - \omega_2^2x'_2\frac{\partial}{\partial p'_2} + p'_2\frac{\partial}{\partial x'_2}.$$

If we consider the problem as a differential equation in $\mathbb{T}^*\mathbb{R}^2$, the constants of the motion are first-integrals for the vector field $X + \partial/\partial t$ over $\mathbb{R} \times \mathbb{T}^*\mathbb{R}^2$. Then, as we have a distribution of rank one over a five-dimensional manifold, there exist, at least locally, four functionally independent first-integrals. Additionally, three of them can be chosen to be t -independent ones (in terms of the variables x'_1, x'_2, p'_1, p'_2). The constants of the motion for the initial TDHO corresponding to some of such first-integrals read

$$I_i = \left(\frac{\omega_i x_i}{V(t)}\right)^2 + (V(t)p_i - u_1x_i)^2, \quad i = 1, 2,$$

and

$$I_{12} = \frac{1}{\omega_1} \arcsin\left(\frac{x_1\omega_1}{V(t)\sqrt{I_1}}\right) - \frac{1}{\omega_2} \arcsin\left(\frac{x_2\omega_2}{\sqrt{V(t)I_2}}\right).$$

This first-integral is constant along the solutions. Nevertheless, in order for the function to be correctly defined, ω_1/ω_2 needs to be rational. Finally, with the aid of (5.19), we can obtain two t -dependent constants of the motion of the form

$$\bar{I}_i = \frac{\omega_i}{V(t)u_1} + \arcsin\left(\frac{x'_i\omega_i}{\sqrt{I_i}}\right) \quad i = 1, 2.$$

As a consequence, we can explicitly obtain the t -evolution of the system. Indeed, either from \bar{I}_1 or \bar{I}_2 , we reach the following solutions for x_1 and x_2

$$x_i(t) = \frac{V(t)\sqrt{I_i}}{\omega_i} \sin\left(\bar{I}_i - \frac{\omega_i}{V(t)u_1}\right), \quad i = 1, 2.$$

The properties of these solutions become clearer when we write them in the following way

$$x_i(t) = \frac{V(t)\sqrt{I_i}}{\omega_i} \sin\left(\bar{I}_i - \frac{\omega_i}{u_1(u_1t + u_0)}\right), \quad i = 1, 2,$$

and we realise that the quotient $x_1(t)/x_2(t)$ is a t -independent constant of the motion if ω_1/ω_2 is rational.

These two equations can be considered as the parametric representation of a curve on the configuration space $Q = \mathbb{R}^2$. In the general case x_1 and x_2 evolve in an independent way and

the behaviour of the curve becomes blurred. In the rational case, the evolutions of x_1 and x_2 are correlated in such a way that the t -dependent coupling function I_{12} is preserved. The particular form of this curve will depend on the relation between u_1 and u_0 . If $u_1 = 0$ it will be a Lissajous curve. If $u_1 \neq 0$ it can be considered as a curve obtained by the addition of growing amplitudes to the oscillations of the corresponding Lissajous curve. We can refer to them as ‘ t -dependent Lissajous’ figures. Nevertheless, it is not totally clear whether this term is appropriate, since these new curves are ‘not closed’.

6. Integrability in Quantum Mechanics

Some papers have recently been devoted to applying the theory of Lie systems [38, 157, 234] to Quantum Mechanics [51, 60]. As a result, it has been proved that the theory of Lie systems can be used to treat some types of Schrödinger equations, the so-called quantum Lie systems, to obtain exact solutions, t -evolution operators, etc. One of the fundamental properties found is that quantum Lie systems can be investigated by means of equations in a Lie group. Through this equation we can analyse the properties of the associated Schrödinger equation, e.g. the type of Lie group allows us to know if a Schrödinger equation can be integrated [51].

Lately, a lot of attention has also been dedicated to the study of integrability of Lie systems and, in particular, of Riccati equations [40, 47, 50]. In these papers, as in previous sections, it has been shown that integrability conditions for Lie systems, in the case of Riccati equations, appear related to some transformation properties of the associated equations in $SL(2, \mathbb{R})$. Nevertheless, as we have pointed out in this work and it was shown in [47], the same procedure used to investigate Riccati equations can be applied to deal with any Lie system.

Therefore, in the case of a quantum Lie system, there exists an equation on a Lie group associated with it [51]. The transformation properties investigated in the theory of integrability of Lie systems can be used to study integrability conditions for quantum Lie systems. All results obtained in Chapter 4, can be generalised to apply to the quantum case and some non-trivial integral models can be obtained. The aim of this chapter is to show how to apply the theory of integrability of Lie systems so as to investigate quantum Lie systems. All our results are illustrated by means of the analysis of several types of spin Hamiltonians.

We must stress the practical importance of this method: It enables us to obtain non-trivial exactly solvable t -dependent Schrödinger equations. This fact allows us to investigate physical models by means of non-trivial exact solutions. It also provides a procedure to avoid using numerical methods for studying Schrödinger equations in many cases.

6.1. Spin Hamiltonians. In this section we investigate a particular quantum mechanical system whose dynamics is given by Schrödinger–Pauli equation [39]. We first prove that this Hamiltonian corresponds to a quantum Lie system and we next apply the theory of integrability of Lie systems to such a system to recover some exact known solutions and prove some new ones.

The system under study is described by the t -dependent Hamiltonian

$$H(t) = B_x(t)S_x + B_y(t)S_y + B_z(t)S_z,$$

with S_x, S_y and S_z being the spin operators. Let us denote $S_1 = S_x, S_2 = S_y$ and $S_3 = S_z$, then

the t -dependent Hamiltonian $H(t)$ is a quantum Lie system, because the spin operators are such that

$$[iS_j, iS_k] = - \sum_{l=1}^3 \epsilon_{jkl} iS_l, \quad j, k = 1, 2, 3, \quad (6.1)$$

with ϵ_{jkl} being the components of the fully skew-symmetric Levi-Civita tensor and where we have assumed $\hbar = 1$. The Schrödinger equation corresponding to this t -dependent Hamiltonian is

$$\frac{d\psi}{dt} = -iB_x(t)S_x(\psi) - iB_y(t)S_y(\psi) - iB_z(t)S_z(\psi), \quad (6.2)$$

which can be seen as a differential equation determining the integral curves of the t -dependent vector field in a (maybe infinite-dimensional) Hilbert space \mathcal{H} given by

$$X_t = B_x(t)X_1^{SH} + B_y(t)X_2^{SH} + B_z(t)X_3^{SH},$$

with

$$(X_1^{SH})_\psi = -iS_x(\psi), \quad (X_2^{SH})_\psi = -iS_y(\psi), \quad (X_3^{SH})_\psi = -iS_z(\psi).$$

The t -dependent vector field X can be written as a linear combination

$$X_t = \sum_{k=1}^3 b_k(t)X_k^{SH},$$

of the vector fields X_k^{SH} , with $b_1(t) = B_x(t)$, $b_2(t) = B_y(t)$ and $b_3(t) = B_z(t)$, and therefore our Schrödinger equation is a Lie system related to a quantum Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{su}(2)$.

Take the basis for $\mathfrak{su}(2)$ given by the following skew-self-adjoint 2×2 matrices

$$\mathbf{v}_1 \equiv \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{v}_2 \equiv \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{v}_3 \equiv \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

These matrices satisfy the commutation relations

$$[\mathbf{v}_j, \mathbf{v}_k] = - \sum_{l=1}^3 \epsilon_{jkl} \mathbf{v}_l, \quad j, k = 1, 2, 3,$$

which are similar to (6.1). Hence, we can define an action $\Phi^{SH} : SU(2) \times \mathcal{H} \rightarrow \mathcal{H}$ such that

$$\Phi^{SH}(\exp(c_k \mathbf{v}_k), \psi) = \exp(c_k i H_k)(\psi), \quad k = 1, 2, 3,$$

for any real constants c_1, c_2 and c_3 . Moreover,

$$\left. \frac{d}{dt} \right|_{t=0} \Phi^{SH}(\exp(-it \mathbf{v}_k), \psi) = \left. \frac{d}{dt} \right|_{t=0} \exp(-it H_k)(\Phi) = -i H_k(\psi) = (X_k^{SH})_\psi,$$

getting that each X_k^{SH} is the fundamental vector field associated with \mathbf{v}_k . Thus, the equation on $SU(2)$ related, by means of Φ^{SH} , to the Schrödinger equation (6.2) is

$$R_{g^{-1}*} \dot{g} = - \sum_{k=1}^3 b_k(t) \mathbf{v}_k \equiv \mathbf{a}(t) \in \mathfrak{su}(2), \quad g(0) = e. \quad (6.3)$$

It was shown in [51], and previously in our work, that the group \mathcal{G} of curves in the group of a Lie system, in this case $\mathcal{G} = \text{Map}(\mathbb{R}, SU(2))$, acts on the set of Lie systems associated with an

equation in the Lie group G in such a way that, in a similar way to what happened in [40], a curve $\bar{g} \in \mathcal{G}$ transforms the initial equation (6.3) into the new one characterised by the curve

$$a'(t) \equiv -\text{Ad}(\bar{g}) \left(\sum_{k=1}^3 b_k(t) v_k \right) + R_{\bar{g}^{-1} * \bar{g}} \frac{d\bar{g}}{dt} = - \sum_{k=1}^3 b'_k(t) v_k, \quad (6.4)$$

Once again, this new equation is related to a new Schrödinger equation in \mathcal{H} determined by a new Hamiltonian

$$H'(t) = \sum_{k=1}^3 b'_k(t) S_k.$$

Additionally, the curve $\bar{g}(t)$ in $SU(2)$ induces a t -dependent unitary transformation $\bar{U}(t)$ on \mathcal{H} transforming the initial t -dependent Hamiltonian $H(t)$ into $H'(t)$.

Summarising, the theory of Lie systems reduces the problem of determining the solution of Schrödinger equations related to spin Hamiltonians $H(t)$ to solving certain equations in the Lie group $SU(2)$. Then, the transformation properties of the equations in $SU(2)$ describe the transformation properties of $H(t)$ by means of certain t -dependent unitary transformations described by curves in $SU(2)$.

Note that the theory here developed for spin Hamiltonians can be straightforwardly employed to analyse any quantum Lie system. In this case, our procedure remains essentially the same. It is only necessary to replace $SU(2)$ by the new Lie group G associated with the quantum Lie system under study.

6.2. Lie structure of an equation of transformation of Lie systems. Our aim now is to prove that the curves in $SU(2)$ relating the equations defined by two curves $a(t)$ and $a'(t)$ in $T_T SU(2)$, respectively, can be found as solutions of a Lie system of differential equations.

Recall that the matrices of $SU(2)$ are of the form

$$\bar{g} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad a, b \in \mathbb{C}, \quad (6.5)$$

with $|a|^2 + |b|^2 = 1$ and that the elements of $\mathfrak{su}(2)$ are traceless skew-Hermitian matrices, namely, real linear combinations of the matrices $\{v_k \mid k = 1, 2, 3\}$. Then, the equation (6.4) becomes a matrix equation that can be written

$$\frac{d\bar{g}}{dt} \bar{g}^{-1} = - \sum_{k=1}^3 b'_k(t) v_k + \sum_{k=1}^3 b_k(t) \bar{g} v_k \bar{g}^{-1}. \quad (6.6)$$

Multiplying both sides of this equation by \bar{g} on the right, we get

$$\frac{d\bar{g}}{dt} = - \sum_{k=1}^3 b'_k(t) v_k \bar{g} + \sum_{k=1}^3 b_k(t) \bar{g} v_k. \quad (6.7)$$

If we consider a reparametrisation of the t -dependent coefficients of \bar{g}

$$\begin{aligned} a(t) &= x_1(t) + i y_1(t), \\ b(t) &= x_2(t) + i y_2(t), \end{aligned}$$

for real functions x_j and y_j , with $j = 1, 2$, a straightforward computations shows that (6.7) is a linear system of differential equations in the new variables x_1, x_2, y_1 and y_2 that can be written

as follows

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & b'_2 - b_2 & -b_3 + b'_3 & -b_1 + b'_1 \\ b_2 - b'_2 & 0 & -b_1 - b'_1 & b_3 + b'_3 \\ b_3 - b'_3 & b'_1 + b_1 & 0 & -b_2 - b'_2 \\ b_1 - b'_1 & -b_3 - b'_3 & b_2 + b'_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}. \quad (6.8)$$

Only the solutions of the above system obeying that $x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1$ describe curves in $SU(2)$ and, consequently, are related to solutions of system (6.7). Nevertheless, we can forget such a restriction for the time being, because it can be automatically implemented later in a more suitable way. Therefore, we can deal with the four variables in the preceding system of differential equations (6.8) as if they were independent. This linear system of differential equations is a Lie system associated with a Lie algebra of vector fields $\mathfrak{gl}(4, \mathbb{R})$, but the solutions with initial condition related to a matrix in the subgroup $SU(2)$ always remain in such a subgroup. In fact, consider the set of vector fields

$$\begin{aligned} N_1 &= \frac{1}{2} \left(-y_2 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial y_2} \right), \\ N_2 &= \frac{1}{2} \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2} \right), \\ N_3 &= \frac{1}{2} \left(-y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_1} - x_2 \frac{\partial}{\partial y_2} \right), \\ N'_1 &= \frac{1}{2} \left(y_2 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_1} - x_1 \frac{\partial}{\partial y_2} \right), \\ N'_2 &= \frac{1}{2} \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2} \right), \\ N'_3 &= \frac{1}{2} \left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial y_1} - x_2 \frac{\partial}{\partial y_2} \right), \end{aligned}$$

for which the non-zero commutation relations are given by:

$$\begin{aligned} [N_1, N_2] &= -N_3, & [N_2, N_3] &= -N_1, & [N_3, N_1] &= -N_2, \\ [N'_1, N'_2] &= -N'_3, & [N'_2, N'_3] &= -N'_1, & [N'_3, N'_1] &= -N'_2. \end{aligned}$$

Note that $[N_j, N'_k] = 0$, for $j, k = 1, 2, 3$, and therefore the system of linear differential equations (6.8) is a Lie system on \mathbb{R}^4 associated with a Lie algebra of vector fields isomorphic to $\mathfrak{g} \equiv \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, i.e. the Lie algebra decomposes into a direct sum of two Lie algebras isomorphic to $\mathfrak{su}(2, \mathbb{R})$, the first one is generated by $\{N_1, N_2, N_3\}$ and the second one by $\{N'_1, N'_2, N'_3\}$.

If we denote $y \equiv (x_1, x_2, y_1, y_2) \in \mathbb{R}^4$, the system (6.8) can be written as a system of differential equation in \mathbb{R}^4 :

$$\frac{dy}{dt} = N(t, y), \quad (6.9)$$

with N_t being the t -dependent vector field given by

$$N(t, y) = \sum_{k=1}^3 (b_k(t)N_k(y) + b'_k(t)N'_k(y)).$$

The vector fields $\{N_1, N_2, N_3, N'_1, N'_2, N'_3\}$ span a distribution of rank three in almost any point of \mathbb{R}^4 and consequently there exists, at least locally, a first-integral for all the vector fields

(6.9). It can be verified that such a first-integral is globally defined and reads $I(y) = x_1^2 + x_2^2 + y_1^2 + y_2^2$. Hence, given a solution $y(t)$ of system (6.9) with an initial condition $I(y(0)) = x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1$, then $I(y(t)) = 1$ at any time t and this solution describes a curve in $SU(2)$. Therefore, we have found that the curves in $SU(2)$ relating two different equations on $SU(2)$ associated with two Schrödinger equations of the form (6.2) can be described by means of the solutions $y(t)$ of (6.9) with $I(y(0)) = 1$, and vice versa:

THEOREM 6.1. *The curves in $SU(2)$ relating two equations on the group $SU(2)$ characterised by the curves in $\mathfrak{su}(2)$ of the form*

$$a'(t) = -\sum_{k=1}^3 b'_k(t)v_k, \quad a(t) = -\sum_{k=1}^3 b_k(t)v_k$$

are the solutions, $y(t)$, of the system

$$\frac{dy}{dt} = N(t, y),$$

with

$$N(t, y) = \sum_{k=1}^3 (b_k(t)N_k(y) + b'_k(t)N'_k(y)),$$

satisfying that $I(y(0)) = 1$. This is a Lie system related to a Lie algebra of vector fields isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

COROLLARY 6.2. *Given two Schrödinger equations corresponding to two spin Hamiltonians, there always exists a curve in $SU(2)$ transforming one of them into the other.*

Although the above corollary ensures the existence of a t -dependent unitary transformation mapping a given Spin Hamiltonian into any other one, obtaining such a transformation involves solving system (6.9) explicitly. This Lie system is related to a non-solvable Lie algebra and, consequently, it is not easy to find its solutions in general. In view of this, it becomes interesting to determine integrability conditions which allow us to solve this system and obtain the corresponding transformation. This illustrates the interest of the integrability conditions derived in next sections, which will be used to derive exact solutions for some physical problems involving Spin Hamiltonians.

6.3. Integrability conditions for $SU(2)$ Schrödinger equations. Let $\bar{g}(t)$ be a curve in $SU(2)$ transforming the equation on $SU(2)$ defined by the curve $a(t)$ into another characterised by $a'(t)$ according to the rule (6.6). If $g'(t)$ is the solution of the equation in $SU(2)$ characterised by $a'(t)$, then $g(t) = \bar{g}^{-1}(t)g'(t)$ is a solution for the equation in $SU(2)$ characterised by $a(t)$.

If $a'(t)$ lies in a solvable Lie subalgebra of $\mathfrak{su}(2)$, we can derive $g'(t)$ in many ways [40] and, once $g'(t)$ is obtained, the knowledge of the curve $\bar{g}(t)$ transforming the curve $a(t)$ into $a'(t)$ provides the curve $g(t)$ solution of the equation on $SU(2)$ determined by $a(t)$.

Therefore, starting from a curve $a'(t)$ in a solvable Lie subalgebra of $\mathfrak{su}(2)$ and using (6.9), with curves in a restricted family of curves in $SU(2)$, we can relate $a'(t)$ to other possible curves $a(t)$, finding, in this way a family of equations on $SU(2)$, and thus spin Schrödinger equations on \mathcal{H} , that can be exactly solved.

Let us assume some restrictions on the family of solution curves of the system (6.9), e.g. we choose $b = 0$. Consequently, there are instances of this system which do not admit a solution under these restrictions, i.e. it is not possible to connect the curves $a(t)$ and $a'(t)$ by a curve satisfying the assumed restrictions. This gives rise to some compatibility conditions for the existence of one of these special solutions, either algebraic and/or differential ones, between the t -dependent coefficients of $a'(t)$ and $a(t)$ satisfied by explicitly solvable models found in the literature. Therefore, our approach is useful to provide exactly integrable models found in the literature and, as we will see next, to derive new ones.

The two main ingredients to be taken into account in the following sections are:

1. *The equations which are characterised by a curve $a'(t)$ for which the solution can be obtained.* We here consider that $a'(t)$ is associated with a one-dimensional Lie subalgebra of $\mathfrak{su}(2)$.
2. *The restriction on the set of curves considered as solutions of the equation (6.9).* We next look for solutions of (6.9) related to curves in a one-parameter subset of $SU(2)$.

Consider the below example: suppose that we want to connect a given $a(t)$ with a final family of curves of the form $a'(t) = -D(t)(c_1v_1 + c_2v_2 + c_3v_3)$, with c_1, c_2, c_3 , being real numbers. In this case, system (6.9), which describes the curves $\bar{g}(t) \subset SU(2)$ that transform the equation described by $a(t)$ into the equation determined by $a'(t)$, reads

$$\frac{dy}{dt} = \sum_{k=1}^3 b_k(t)N_k(y) + D(t) \sum_{k=1}^3 c_k N'_k(y) = N(t, y). \quad (6.10)$$

Note that the vector field

$$N' = \sum_{k=1}^3 c_k N'_k,$$

satisfies that

$$[N_k, N'] = 0, \quad k = 1, 2, 3.$$

Hence, Lie system (6.10) is related to a Lie algebra of vector fields isomorphic to $\mathfrak{su}(2) \oplus \mathbb{R}$. As this Lie system is associated with a non-solvable Vessiot-Guldberg Lie algebra, it is not integrable by quadratures and the solution cannot be easily found in the general case. Nevertheless, it is worth noting that (6.10) always has a solution.

In this way, we can consider some instances of (6.10) for which the resulting system of differential equations can be integrated by quadratures. We can consider that x is related to a one-parameter family of elements of $SU(2)$. Such a restriction implies that (6.10) not always has a solution, because sometimes it is not possible to connect $a(t)$ and $a'(t)$ by means of the chosen one-parameter family. This fact imposes differential and algebraic restrictions on the initial t -dependent functions b_k , with $k = 1, 2, 3$. These restrictions will describe known integrability conditions and other new ones. So, we can develop the ideas of [50, 55] in the framework of Quantum Mechanics. Moreover, from this point of view, we can find new integrability conditions that can be used to obtain exact solutions.

6.4. Application of integrability conditions in a $SU(2)$ Schrödinger equation. In this section we restrict ourselves to the case $a'(t) = -D(t)v_3$, i.e.

$$b'_1(t) = 0, \quad b'_2(t) = 0, \quad b'_3(t) = D(t). \quad (6.11)$$

Hence, the system of differential equations (6.8) describing the curves \bar{g} relating a Schrödinger equation to $H'(t) = D(t)S_z$ is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -b_2 & -b_3 + D & -b_1 \\ b_2 & 0 & -b_1 & b_3 + D \\ b_3 - D & b_1 & 0 & -b_2 \\ b_1 & -b_3 - D & b_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}. \quad (6.12)$$

We see that, according to the result of Theorem 6.1, the t -dependent vector field corresponding to such a system of differential equations can be written as a linear combination with t -dependent coefficients of the vector fields N_1, N_2, N_3 and N'_3 :

$$N(t, y) = \sum_{k=1}^3 b_k(t)N_k(y) + D(t)N'_3(y).$$

Thus, system (6.12) is associated with a Lie algebra of vector fields isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$. This Lie algebra is smaller than the initial one (6.8), but it is not solvable and the system is as difficult to solve as the initial Schrödinger equation. Therefore, in order to get exact solvable cases, we need to perform some kind of simplification once again, e.g. by means of the imposition of some extra assumptions on the variables. This may result in a system of differential equations whose solutions are incompatible with our additional conditions. The necessary and sufficient conditions on the t -dependent functions $b_1, b_2, b_3, b'_1, b'_2$ and b'_3 ensuring the existence of a solution compatible with the assumed restrictions on the variables give rise to integrability conditions for spin Hamiltonians.

For instance, suppose that we impose on the solutions to be in the one-parametric subset $A_\gamma \subset SU(2)$ given by

$$A_\gamma = \left\{ \begin{pmatrix} \cos \frac{\gamma}{2} & -e^{-bi} \sin \frac{\gamma}{2} \\ e^{bi} \sin \frac{\gamma}{2} & \cos \frac{\gamma}{2} \end{pmatrix} \mid b \in [0, 2\pi) \right\}. \quad (6.13)$$

where γ is a fixed real constant such that $\gamma \neq 2\pi n$, with $n \in \mathbb{Z}$, because in such a case $A_\gamma = \pm \text{Id}$. In view of the definition of the sets A_γ and in terms of the parametrisation (6.5), we have

$$x_1 = \cos \frac{\gamma}{2}, \quad y_1 = 0, \quad x_2 = -\sin \frac{\gamma}{2} \cos b, \quad y_2 = \sin \frac{\gamma}{2} \sin b. \quad (6.14)$$

The elements of A_γ are matrices in $SU(2)$ and the system of differential equations we obtain reads

$$\begin{pmatrix} 0 \\ \dot{x}_2 \\ 0 \\ \dot{y}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -b_2 & -b_3 + D & -b_1 \\ b_2 & 0 & -b_1 & b_3 + D \\ b_3 - D & b_1 & 0 & -b_2 \\ b_1 & -b_3 - D & b_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ y_2 \end{pmatrix}. \quad (6.15)$$

and then we get two integrability conditions for the system (6.15):

$$\begin{aligned} 0 &= -b_2 x_2 - b_1 y_2, \\ 0 &= (b_3 - D)x_1 + b_1 x_2 - b_2 y_2. \end{aligned} \quad (6.16)$$

We can write the components $(B_x(t), B_y(t), B_z(t))$ of the magnetic field in polar coordinates,

$$\begin{aligned} B_x(t) &= B(t) \sin \theta(t) \cos \phi(t), \\ B_y(t) &= B(t) \sin \theta(t) \sin \phi(t), \\ B_z(t) &= B(t) \cos \theta(t), \end{aligned}$$

with $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$.

The first algebraic integrability condition reads, in polar coordinates, as follows:

$$B(t) \sin \theta(t) \sin \frac{\gamma}{2} (\cos \phi(t) \sin b(t) - \sin \phi(t) \cos b(t)) = 0$$

and thus,

$$B(t) \sin \theta(t) \sin \frac{\gamma}{2} \sin(b(t) - \phi(t)) = 0,$$

from where we see that $b(t) = \phi(t)$. In such a case, the second algebraic integrability condition in (6.16) reduces to

$$(B_z - D) \cos \frac{\gamma}{2} - B \sin \frac{\gamma}{2} \sin \theta = 0$$

and then, the t -dependent coefficient D is

$$D = \frac{B}{\cos \frac{\gamma}{2}} \cos \left(\frac{\gamma}{2} + \theta \right). \quad (6.17)$$

Finally, we have to take into account the differential integrability condition

$$\dot{x}_2 = \frac{1}{2} \left(b_2 \cos \frac{\gamma}{2} + (b_3 + D) \sin \frac{\gamma}{2} \sin b \right),$$

which after some algebraic manipulation leads to

$$\dot{\phi} = \frac{B}{2} \left(\frac{\sin(\theta + \frac{\gamma}{2})}{\sin \frac{\gamma}{2}} + \frac{\cos(\frac{\gamma}{2} + \theta)}{\cos \frac{\gamma}{2}} \right),$$

and then

$$\dot{\phi}(t) = B(t) \frac{\sin(\theta(t) + \gamma)}{\sin \gamma}, \quad (6.18)$$

which is a far larger set of integrable Hamiltonians than the one of the exactly solvable Hamiltonians of this type found in the literature. As a particular example, when θ and B are constant, we find

$$\dot{\phi} = B \frac{\sin(\theta + \gamma)}{\sin \gamma} \equiv \omega \quad (6.19)$$

and consequently,

$$\phi = \omega t + \phi_0.$$

In this way, we get that the t -dependent spin Hamiltonian $H(t)$ determined by the magnetic vector field

$$\mathbf{B}(t) = B(\sin \theta \cos(\omega t), \sin \theta \sin(\omega t), \cos \theta).$$

is integrable.

Another interesting integrable case is that given by $\theta = \frac{\pi}{2}$, that is, the magnetic field moves in the XZ plane, see [20, 139, 140]. In such a case, in view of the integrability conditions (6.19), the angular frequency $\dot{\phi}$ reads

$$\dot{\phi} = B \cotan \gamma = \omega.$$

The last one of the most known integrable cases of Spin Hamiltonian is given by a magnetic field in a fixed direction, i.e. $\mathbf{B}(t) = B(t)(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Obviously, this case satisfies integrability condition (6.19) for $\gamma = -\theta$.

Apart from the previous cases, the integrability condition (6.18) describes more, as far as we know, new integrable cases. For instance, consider the case with θ fixed and B non-constant. In this case, the corresponding $H(t)$ is integrable if

$$\frac{\dot{\phi}}{B(t)} = \frac{\sin(\theta + \gamma)}{\sin \gamma},$$

that is, if we fix $\gamma = \pi/2$ we have that

$$\omega = \dot{\phi} = B(t) \cos \theta \implies \phi(t) = \cos \theta \int^t B(t') dt'.$$

Furthermore, we can consider $\theta(t) = t$ and B constant. In this case, we get that the t -dependent Hamiltonian $H(t)$ is integrable if the $\phi(t)$ holds the condition

$$\dot{\phi} = B \cos t \implies \phi(t) = B \sin t.$$

Indeed, note that in this case the integrability condition (6.18) trivially follows for $\gamma = -1/2$.

To sum up, we have shown that there exists a large family of t -dependent integrable spin Hamiltonians that includes, as particular cases, many integrable cases known up to now. Additionally, it is easy to check whether a t -dependent spin Hamiltonian satisfies the integrability condition (4.33) and then, it can be integrated.

6.5. Applications to Physics. Let us use the above results in order to solve a t -dependent spin Hamiltonian

$$H(t) = \mathbf{B}(t) \cdot \mathbf{S},$$

which broadly appears in Physics: the one characterised by a magnetic field

$$\mathbf{B}(t) = B(\sin \theta \cos(\omega t), \sin \theta \sin(\omega t), \cos \theta), \quad (6.20)$$

that is, a magnetic field with a constant modulus rotating along the OZ axis with a constant angular velocity ω . Such Hamiltonians have been applied, for instance, to analyse the spin precession in a transverse t -dependent magnetic field [208], investigate the adiabatic approximation and the unitary of the t -evolution operator through such an approximation [160, 178], etc.

In the previous section we showed that this t -dependent Hamiltonian is integrable. Indeed, the integrability condition (6.19) can be written as

$$\tan \gamma = \frac{\sin \theta}{\frac{\dot{\phi}}{B} - \cos \theta}, \quad (6.21)$$

where we recall that γ has to be a real constant. In the case of our particular magnetic field (6.20) the angular frequency, $\omega = \dot{\phi}$, the angle θ and the modulus B are constants. Therefore γ is a properly defined constant, the integrability condition (6.19) holds and the value of γ is given by equation (6.21) in terms of the parameters B , θ and ω , which characterise the magnetic vector field (6.20).

We have already shown that if $B(t)$ satisfies (6.19), then $H(t)$ is integrable, because it can be transformed by means of a t -dependent change of variables determined by a curve $g(t)$ in

the set A_γ into a straightforwardly integrable Schrödinger equation determined by a t -dependent Hamiltonian $H'(t) = D(t)S_z$. For simplicity, let us parametrise the elements of A_γ in a new way. Consider $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\mathbf{n} \in \mathbb{R}^3$, where the matrices σ_i are the Pauli matrices, $\sigma_x, \sigma_y, \sigma_z$. We have

$$e^{i\sigma \cdot \mathbf{n}\phi} = \text{Id} \cos \phi + i\sigma \cdot \mathbf{n} \sin \phi.$$

So, for $\mathbf{n} = \frac{(\alpha_1, \alpha_2, 0)}{\sqrt{\alpha_1^2 + \alpha_2^2}}$ with real constants α_1, α_2 and taking into account that $v_1 = \frac{i\sigma_x}{2}, v_2 = \frac{i\sigma_y}{2}$ and $v_3 = \frac{i\sigma_z}{2}$, we get

$$\exp(\alpha_1 v_1 + \alpha_2 v_2) = \exp\left(i\frac{\delta}{2}\sigma \cdot \mathbf{n}\right) = \begin{pmatrix} \cos\frac{\delta}{2} & -e^{-i\varphi} \sin\frac{\delta}{2} \\ e^{i\varphi} \sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix} \quad (6.22)$$

with $\delta = \sqrt{\alpha_1^2 + \alpha_2^2}$ and $-e^{-i\varphi} = (i\alpha_1 + \alpha_2)/\sqrt{\alpha_1^2 + \alpha_2^2}$. In terms of δ and φ the variables α_1 and α_2 can be written $\alpha_1 = \delta \sin \varphi$ and $\alpha_2 = -\delta \cos \varphi$. Hence, in view of (6.22), we see that we can describe the elements of A_γ as

$$\begin{pmatrix} \cos\frac{\gamma}{2} & -e^{-bi} \sin\frac{\gamma}{2} \\ e^{bi} \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} = \exp(\gamma \sin b v_1 - \gamma \cos b v_2), \quad (6.23)$$

where b and γ are real constants. For magnetic vector fields (6.20), the t -dependent change of variables transforming the initial $H(t)$ into an integrable $H'(t) = D(t)S_z$ is determined by a curve in A_γ with γ determined by equation (6.19) and $b(t) = \phi(t)$. Thus, such a curve in A_γ takes the form

$$t \mapsto \exp(\gamma \sin(\omega t) v_1 - \gamma \cos(\omega t) v_2) \quad (6.24)$$

We want to emphasise that the above t -dependent change of variables in $SU(2)$ transforms the equation in $SU(2)$ determined by the initial curve

$$\mathbf{a}(t) = -B_x(t)v_1 - B_y(t)v_2 - B_z(t)v_3,$$

into and a new equation in $SU(2)$ determined by a curve $\mathbf{a}'(t) = -D(t)v_3$. Such a t -dependent transformation in $SU(2)$ induces a t -dependent unitary change of variables in \mathcal{H} transforming the initial Schrödinger equation determined by the t -dependent Hamiltonian $H(t)$, i.e.

$$\frac{\partial \psi}{\partial t} = -iH(t)(\psi),$$

into the new Schrödinger equation

$$\frac{\partial \psi'}{\partial t} = -iH'(t)(\psi') = -iD(t)S_z(\psi'). \quad (6.25)$$

The relation between ψ and ψ' is given by the corresponding t -dependent change of variables in \mathcal{H} induced by curve (6.24), i.e.

$$\psi' = \exp(\gamma \sin(\omega t) iS_x - \gamma \cos(\omega t) iS_y)\psi. \quad (6.26)$$

In view of expression (6.17), we see that

$$D = B(\cos \theta - \tan \frac{\gamma}{2} \sin \theta),$$

and from (6.21) and the relations

$$\tan \gamma = \frac{2 \tan \frac{\gamma}{2}}{1 - \tan^2 \frac{\gamma}{2}} \Rightarrow \tan \frac{\gamma}{2} = \frac{-1 \pm \sqrt{1 + \tan^2 \gamma}}{\tan \gamma},$$

we obtain

$$\tan \frac{\gamma}{2} = \frac{1}{\sin \theta} \left(-\frac{\omega}{B} + \cos \theta \pm \sqrt{\frac{\omega^2}{B^2} - 2\frac{\omega}{B} \cos \theta + 1} \right).$$

If we substitute the above expression in the latter expression for D , it turns out that

$$D = \omega \pm \sqrt{\omega^2 - 2\omega B \cos \theta + B^2}.$$

That is, D becomes a constant. Thus, the general solution ψ'_t for the Schrödinger equation (6.25) with initial condition ψ'_0 is

$$\psi'(t) = \exp(-itDS_z) \psi'_0,$$

and the solution for the initial Schrödinger equation with initial condition ψ_0 can be obtained undoing the t -dependent change of variables (6.26) to get

$$\psi_t = \exp(-i\gamma \sin \omega t S_x + i\gamma \cos \omega t S_y) \exp(-iDtS_z) \psi_0.$$

7. The theory of quasi-Lie schemes and Lie families

7.1. Introduction. Several important systems of first-order ordinary differential equations can be studied through the theory of Lie systems. Moreover, this theory was recently applied to study SODE Lie systems, quantum Lie systems, some partial differential equations, etc. These last successes allow us to recover, from a unifying point of view, several results disseminated throughout the literature and to prove multiple new properties of systems of differential equations appearing in Physics and Mathematics. Apart from these successes, there are still some reasons to go further in the generalisation of the theory of Lie systems:

- *Lie systems are important but rather exceptional.* The theory of Lie systems investigates very interesting equations with many applications, e.g. t -dependent frequency harmonic oscillators, Milne–Pinney equations, Riccati equations, etc. Nevertheless, it fails to study many other (nonautonomous) interesting systems, like nonlinear oscillators, Abel equations, or Emden equations.
- *The theory of Lie systems does not allow us to investigate superposition rules involving an explicit t -dependence* which appears in various interesting systems, e.g. dissipative Milne–Pinney equation, Emden–Fowler equations [42], second-order Riccati equations [48, 126], whose properties are worth analysing.
- Lie systems have an associated group of t -dependent changes of variables enabling us to transform each particular Lie system into a new one of the same class, e.g. the group of curves in $SL(2, \mathbb{R})$ transforms a Riccati equation into a new Riccati equation. A similar property frequently applies to integrate differential equations, like Abel equations [74]. A natural question arises: Is there any kind of systems of differential equations more general than Lie systems admitting an analogue property?

The theory of quasi-Lie schemes [34] and the Generalised Lie Theorem [35], which gives rise to the *Lie family* notion, provide an answer to these problems. More specifically, quasi-Lie schemes, quasi-Lie systems and Lie families are interesting because:

- *The theory of quasi-Lie schemes and the Generalised Lie Theorem permit us to investigate a very large family of differential equations including Lie systems.* More specifically, this family includes, for instance, the following non-Lie systems: Emden–Fowler equations [34, 42], nonlinear oscillators [34], dissipative Milne–Pinney equations [34, 45], second-order Riccati equations [48], Abel equations [35], etc. Moreover, not only quasi-Lie schemes and Lie families can be applied to investigate systems of first-order ordinary differential equations, but they can also be employed, for instance, to investigate second-order differential equations [42, 45].
- *The theory of quasi-Lie schemes and the Generalised Lie Theorem treat, in a natural way, systems admitting a t -dependent superposition rule.* These theories show that many differential equations admit a t -dependent superposition rule, e.g. Abel equations [35], dissipative Milne–Pinney equations [34], Emden-Fowler equations [42], second-order Riccati equations [48], etc.
- *The quasi-Lie scheme concept permits us to transform a differential equation within a fixed family, e.g. a first-order Abel equation into a new one with different t -dependent coefficients.* This feature generalises the transformation properties of Lie systems and enables us to derive integrability conditions for differential equations from a unified point of view.

Consequently, the theory of quasi-Lie schemes and the Generalised Lie Theorem represent powerful methods to study first- and higher-order differential equations.

7.2. Generalised flows and t -dependent vector fields. Recall that a nonautonomous system of first-order ordinary differential equations on \mathbb{R}^n is represented in modern differential geometric terms by a t -dependent vector field $X = X(t, x)$ on such a space. On a non-compact manifold, the vector field $X_t(x) = X(t, x)$, for a fixed t , is generally not defined globally, but it is well defined on a neighbourhood of every point $x_0 \in \mathbb{R}^n$ for sufficiently small t . It is convenient to add the variable t to the manifold and to consider the *autonomisation* of our system, i.e. the vector field

$$\overline{X}(t, x) = \frac{\partial}{\partial t} + X(t, x),$$

defined on a neighbourhood U^X of $\{0\} \times \mathbb{R}^n$ in $\mathbb{R} \times \mathbb{R}^n$. The vector field X_t is then defined on the open set of \mathbb{R}^n ,

$$U_t^X = \{x_0 \in \mathbb{R}^n \mid (t, x_0) \in U^X\},$$

for all $t \in \mathbb{R}$. If $U_t^X = \mathbb{R}^n$ for all $t \in \mathbb{R}$, we speak about a *global t -dependent vector field*. The system of differential equations associated with the t -dependent vector field $X(t, x)$ is written in local coordinates

$$\frac{dx^i}{dt} = X^i(t, x), \quad i = 1, \dots, n,$$

where $X(t, x) = \sum_{i=1}^n X^i(t, x) \partial / \partial x^i$ is locally defined on the manifold for sufficiently small t .

A solution of this system is represented by a curve $s \mapsto \gamma(s)$ in \mathbb{R}^n (integral curve) whose tangent vector $\dot{\gamma}$ at t , so at the point $\gamma(t)$ of the manifold, equals $X(t, \gamma(t))$. In other words,

$$\dot{\gamma}(t) = X(t, \gamma(t)). \tag{7.1}$$

It is well-known that, at least for smooth X we work with, for each x_0 there is a unique maximal solution $\gamma_{X^0}^{x_0}(t)$ of system (7.1) with the initial value x_0 , i.e. satisfying $\gamma_{X^0}^{x_0}(0) = x_0$. This solution

is defined at least for t 's from a neighbourhood of 0. In case $\gamma_X^{x_0}(t)$ is defined for all $t \in \mathbb{R}$, we speak about a *global t -solution*.

The collection of all maximal solutions of the system (7.1) gives rise to a (local) generalised flow g^X on \mathbb{R}^n . By a *generalised flow* g on \mathbb{R}^n we understand a smooth t -dependent family g_t of local diffeomorphisms on \mathbb{R}^n , $g_t(x) = g(t, x)$, such that $g_0 = \text{id}_{\mathbb{R}^n}$. More precisely, g is a smooth map from a neighbourhood U^g of $\{0\} \times \mathbb{R}^n$ in $\mathbb{R} \times \mathbb{R}^n$ into \mathbb{R}^n , such that g_t maps diffeomorphically the open submanifold $U_t^g = \{x_0 \in \mathbb{R}^n \mid (t, x_0) \in U^g\}$ onto its image, and $g_0 = \text{id}_{\mathbb{R}^n}$. Again, for each $x_0 \in \mathbb{R}^n$ there is a neighbourhood U_{x_0} of x_0 in \mathbb{R}^n and $\epsilon > 0$ such that g_t is defined on U_{x_0} for $t \in (-\epsilon, \epsilon)$ and maps U_{x_0} diffeomorphically onto $g_t(U_{x_0})$.

If $U_t^g = \mathbb{R}^n$ for all $t \in \mathbb{R}$, we speak about a *global generalised flow*. In this case $g : t \in \mathbb{R} \mapsto g_t \in \text{Diff}(\mathbb{R}^n)$ may be viewed as a smooth curve in the diffeomorphism group $\text{Diff}(\mathbb{R}^n)$ with $g_0 = \text{id}_{\mathbb{R}^n}$.

Here it is also convenient to *autonomise* the generalised flow g extending it to a single local diffeomorphism

$$\bar{g}(t, x) = (t, g(t, x)) \tag{7.2}$$

defined on the neighbourhood U^g of $\{0\} \times \mathbb{R}^n$ in $\mathbb{R} \times \mathbb{R}^n$. The generalised flow g^X induced by the t -dependent vector field X is defined by

$$g^X(t, x_0) = \gamma_X^{x_0}(t). \tag{7.3}$$

Note that, for $g = g^X$, equation (7.3) can be rewritten in the form:

$$X_t = X(t, x) = \dot{g}_t \circ g_t^{-1}. \tag{7.4}$$

In the above formula, we understood X_t and \dot{g}_t as maps from \mathbb{R}^n into $T\mathbb{R}^n$, where $\dot{g}_t(x)$ is the vector tangent to the curve $s \mapsto g(s, x)$ at $g(t, x)$. Of course, the composition $\dot{g}_t \circ g_t^{-1}$, called sometimes the *right-logarithmic derivative* of $t \mapsto g_t$, is only defined for those points $x_0 \in \mathbb{R}^n$ for which it makes sense. But this is always the case for sufficiently small t , at least locally.

Let us observe that equation (7.4) defines, in fact, a one-to-one correspondence between generalised flows and t -dependent vector fields modulo the observation that the domains of $\dot{g}_t \circ g_t^{-1}$ and X_t need not to coincide. In any case, however, $\dot{g}_t \circ g_t^{-1}$ and X_t coincide in a neighbourhood of any point for sufficiently small t . One can simply say that the *germs* of X and $\dot{g}_t \circ g_t^{-1}$ coincide, where the germ in our context is understood as the class of corresponding objects that coincide on a neighbourhood of $\{0\} \times \mathbb{R}^n$ in $\mathbb{R} \times \mathbb{R}^n$.

Indeed, for a given g , the corresponding t -dependent vector field is defined by (7.4). Conversely, for a given X , the equation (7.4) determines the germ of the generalised flow $g(t, x)$ uniquely, as for each $x = x_0$ and for small t equation (7.4) implies that $t \mapsto g(t, x_0)$ is the solution of the system defined by X with the initial value x_0 . In this way we get the following.

THEOREM 7.1. *Equation (7.4) defines a one-to-one correspondence between the germs of generalised flows and the germs of t -dependent vector fields on \mathbb{R}^n .*

Any two generalised flows g and h can be composed: by definition $(g \circ h)_t = g_t \circ h_t$, where, as usual, we view $g_t \circ h_t$ as a local diffeomorphism defined for points for which the composition is properly defined. It is important to emphasise that in a neighbourhood of any point it really makes sense for sufficiently small t . As generalised flows correspond to t -dependent vector fields, this gives rise to an action of a generalised flow h on a t -dependent vector field X , giving rise to

$h_{\star}X$, defined by the equation

$$g^{h_{\star}X} = h \circ g^X. \quad (7.5)$$

To obtain a more explicit form of this action, let us observe that

$$(h_{\star}X)_t = \frac{d(h \circ g^X)_t}{dt} \circ (h \circ g^X)_t^{-1} = \left(\dot{h}_t \circ g_t^X + Dh_t(\dot{g}_t^X) \right) \circ (g^X)_t^{-1} \circ h_t^{-1},$$

and therefore

$$(h_{\star}X)_t = \dot{h}_t \circ h_t^{-1} + Dh_t(\dot{g}_t^X \circ (g^X)_t^{-1}) \circ h_t^{-1},$$

i.e.

$$(h_{\star}X)_t = \dot{h}_t \circ h_t^{-1} + (h_t)_*(X_t), \quad (7.6)$$

where $(h_t)_*$ is the standard action of diffeomorphisms on vector fields. In a slightly different form, this can be written as an action of t -dependent vector fields on t -dependent vector fields:

$$(g_{\star}^Y X)_t = Y_t + (g_t^Y)_*(X_t). \quad (7.7)$$

For global t -dependent vector fields on compact manifolds, the latter defines a group structure in global t -dependent vector fields. This is an infinite-dimensional analogue of a group structure on paths in a finite-dimensional Lie algebra, which has been used as a source for a nice construction of the corresponding Lie group in [90]. Since every generalised flow has an inverse, $(g^{-1})_t = (g_t)^{-1}$, so generalised flows, or better to say, the corresponding germs, form a group and the formula (7.7) allows us to compute the t -dependent vector field (right-logarithmic derivative) X_t^{-1} associated with the inverse. It is the t -dependent vector field

$$X_t^{-1} = -(g_t^X)_*^{-1}(X_t). \quad (7.8)$$

For t -independent vector fields $X_t = X_0$ for all t we have $(g_t^X)_*X = X$ and also we get the well-known formula

$$X^{-1} = -X.$$

Note that, by definition, the integral curves of $h_{\star}X$ are of the form $h_t(\gamma(t))$, where $\gamma(t)$ are integral curves of X . We can summarise our observation as follows.

THEOREM 7.2. *The equation (7.6) defines a natural action of generalised flows on t -dependent vector fields. This action is a group action in the sense that*

$$(g \circ h)_{\star}X = g_{\star}(h_{\star}X).$$

The integral curves of $h_{\star}X$ are of the form $h_t(\gamma(t))$, for $\gamma(t)$ being an arbitrary integral curve for X .

The above action of generalised flows on t -dependent vector fields can also be defined in an elegant way by means of the corresponding autonomisations. It is namely easy to check the following.

THEOREM 7.3. *For any generalised flow h and any t -dependent vector field X on a manifold \mathbb{R}^n , the standard action $\bar{h}_*\bar{X}$ of the diffeomorphism \bar{h} , being the autonomisation of h , on the vector field \bar{X} , being the autonomisation of X , is the autonomisation of the t -dependent vector field $h_{\star}X$:*

$$\bar{h}_*\bar{X} = \overline{h_{\star}X}.$$

7.3. Quasi-Lie systems and schemes. By a *quasi-Lie system* we understand a pair (X, g) consisting of a t -dependent vector field X on a manifold \mathbb{R}^n (the *system*) and a generalised flow g on \mathbb{R}^n (the *control*) such that $g_\star X$ is a Lie system.

Since for the Lie system $g_\star X$ we are able to obtain the general solution out of a number of known particular solutions, the knowledge of the control makes possible the application of a similar procedure for our initial system possible. Indeed, let $\Phi = \Phi(x_1, \dots, x_m; k_1, \dots, k_n)$ be a superposition function for the Lie system $g_\star X$, so that, knowing m solutions $\bar{x}_{(1)}, \dots, \bar{x}_{(m)}$, of $g_\star X$, we can derive the general solution of the form

$$\bar{x}_{(0)} = \Phi(\bar{x}_{(1)}, \dots, \bar{x}_{(m)}; k_1, \dots, k_n).$$

If we now know m independent solutions, $x_{(1)}, \dots, x_{(m)}$, of X , then, according to Theorem 7.3, $\bar{x}_a(t) = g_t(x_a(t))$ are solutions of $g_\star X$, producing a general solution of $g_\star X$ in the form $\Phi(\bar{x}_{(1)}, \dots, \bar{x}_{(m)}; k_1, \dots, k_n)$. It is now clear that

$$x_{(0)}(t) = g_t^{-1} \circ \Phi(g_t(x_{(1)}(t)), \dots, g_t(x_{(m)}(t)); k_1, \dots, k_n) \tag{7.9}$$

is a general solution of X . In this way we have obtained a *t-dependent superposition rule* for the system X . We can summarise the above considerations as follows.

THEOREM 7.4. *Any quasi-Lie system (X, g) admits a t-dependent superposition rule of the form (7.9), where Φ is a superposition function for the Lie system $g_\star X$.*

Of course, the above t -dependent superposition rule is practically useless for finding the general solution of a system X only if the generalised flow g is explicitly known. An alternative abstract definition of a quasi-Lie system as a t -dependent vector field X for which there exists a generalised flow g such that $g_\star X$ is a Lie system does not have much sense, as every X would be a quasi-Lie system in this context. For instance, given a t -dependent vector field X , the pair $(X, (g^X)^{-1})$ is a quasi-Lie system because $(g^X)^{-1}_t \circ g^X_t = \text{id}_{\mathbb{R}^n}$, thus $(g^X)^{-1}_\star X = 0$, which is a Lie system trivially. On the other hand, finding $(g^X)^{-1}$ is nothing but solving our system X completely, so we just reduce to our original problem. In practice, it is therefore crucial that the control g comes from a system which can be integrated effectively. There are, however, many cases when our procedure works well and provides a geometrical interpretation of many *ad hoc* methods of integration. Consider, for instance, the following scheme that can lead to ‘nice’ quasi-Lie systems.

Take a finite-dimensional real vector space V of vector fields on \mathbb{R}^n and consider the family, $V|_{\mathbb{R}}$, of all t -dependent vector fields X on \mathbb{R}^n such that X_t belongs to V on its domain, i.e. $X_t \in V|_{U_t^X}$ or, in short, $X \in V|_{\mathbb{R}}$. We will say that these are t -dependent vector fields taking values in V . The t -dependent vector fields of $V|_{\mathbb{R}}$ depend on a finite family of control functions. For example, take a basis $\{X_1, \dots, X_r\}$ of V and consider a general t -dependent system with values in V determined by $b = b(t) = (b_1(t), \dots, b_r(t))$ as

$$(X^b)_t = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha.$$

On the other hand, the nonautonomous systems of differential equations associated with $X \in V|_{U_t^X}$ are not Lie systems in general, if V is not a Lie algebra itself. If we additionally have a finitely parametrised family of local diffeomorphism, say $\underline{g} = \underline{g}(a_1, \dots, a_k)$, then any curve $a = a(t) = (a_1(t), \dots, a_k(t))$ in the control parameters, defined for small t , gives rise to a

generalised flow $g_t^a = \underline{g}(a(t))$. Let us additionally assume that there is a Lie algebra V_0 of vector fields contained in V . We can look for control functions $a(t)$ such that for certain $b(t)$ we get that $g_{\star}^a X^b$ has values in V_0 for each t . Let us denote this as

$$g_{\star}^a X^b \in V_0(\mathbb{R}). \tag{7.10}$$

Consequently, each pair (X^b, g^a) becomes a quasi-Lie system and we can get a t -dependent superposition rule for the corresponding system X^b .

Let us observe that in the case when all the generalised flows g^a preserve V , i.e. for each t -dependent vector field $X^b \in V(\mathbb{R})$ also $g_{\star}^a X^b \in V(\mathbb{R})$, the inclusion (7.10) becomes a differential equation for the control functions $a(t)$ in terms of the functions $b(t)$. This situation is not so rare, as it may seem at first sight. Suppose, for instance, that we find a Lie algebra $W \subset V$ such that $[W, V] \subset V$ and that the t -dependent vector fields with values in W can be effectively integrated to generalised flows. In this case, any t -dependent vector field Y^a with values in W gives rise to a generalised flow g^a which, in view of transformation rule (7.7), preserves the set of t -dependent vector fields with values in V . For each $b = b(t)$ the inclusion (7.10) becomes therefore a differential equation for the control function $a = a(t)$ which often can be effectively solved.

DEFINITION 7.5. Let W, V be finite-dimensional real vector spaces of vector fields on \mathbb{R}^n . We say that they form a *quasi-Lie scheme* $S(W, V)$ if the following conditions are satisfied

1. W is a vector subspace of V .
2. W is a Lie algebra of vector fields, i.e. $[W, W] \subset W$.
3. W normalises V , i.e. $[W, V] \subset V$.

If V is a Lie algebra of vector fields, we simply call the quasi-Lie scheme $S(V, V)$ a *Lie scheme* $S(V)$.

NOTE 7.6. Although the normaliser of V in V is the largest Lie algebra of vector fields that we can use as W , for practical purposes it is sometimes useful to consider smaller Lie subalgebras.

DEFINITION 7.7. We call the *group of the scheme* $S(W, V)$ the group $\mathcal{G}(W)$ of generalised flows corresponding to the t -dependent vector fields with values in W .

MAIN THEOREM 7.8. (**Main property of a scheme**) Given a quasi-Lie scheme $S(W, V)$, then $g_{\star} X \in V(\mathbb{R}^n)$ for every t -dependent vector field $X \in V(\mathbb{R})$ and each generalised flow $g \in \mathcal{G}(W)$.

The proof for this is obvious and follows straightforwardly from the fact that if g^Y is the generalised flow of a t -dependent vector field $Y \in W(\mathbb{R})$ and X takes values in V , then, according to the formula (7.7), $g_{\star}^Y X$ takes values in V as well, as $[W, V] \subset V$ and V is finite-dimensional.

In some applications, it turns out to be interesting to use a more general class of transformations than those described by $\mathcal{G}(W)$. Nevertheless, such transformations keep the main property of the generalised flows $\mathcal{G}(W)$, namely, for a given scheme $S(W, V)$ they transform elements of $V(\mathbb{R})$ into elements of this space.

Recall that given a Lie algebra of vector fields $W \subset \mathfrak{X}(\mathbb{R}^n)$, there always exists, at least locally in \mathbb{R}^n , a group action $\Phi : G \times U \rightarrow U$, with G a Lie group with Lie algebra \mathfrak{g} , whose fundamental vector fields are those of W (cf. [144] and Section 1.2). For simplicity, we shall

suppose, as usual, that this action is globally defined on \mathbb{R}^n , and we will write $\Phi : G \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and define the restriction map $\Phi_g : x \in \mathbb{R}^n \mapsto \Phi_g(x) = \Phi(g, x) \in \mathbb{R}^n$ for every $g \in G$.

LEMMA 7.9. *Given a scheme $S(W, V)$, an element $g \in \exp(\mathfrak{g})$, and a vector field $X \in V(\mathbb{R})$, then $\Phi_{g*}X \in V(\mathbb{R})$.*

Proof. As $g \in \exp(\mathfrak{g})$, there exists an element $a \in \mathfrak{g}$ such that $g = \exp(a)$. Consider the curve $h : s \in [0, 1] \mapsto \exp(sa) \in G$. By means of the action $\Phi : G \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, whose fundamental vector fields are the Lie algebra of vector fields W , the curve $h(s)$ induces the generalised flow $h_s^Y : x \in \mathbb{R}^n \mapsto \Phi(\exp(sa), x) \in \mathbb{R}^n$ of the vector field

$$Y(x) = \frac{\partial}{\partial s} \Big|_{s=0} h_s^Y(x) = \frac{\partial}{\partial s} \Big|_{s=0} \Phi(\exp(sa), x)$$

and, obviously, $Y \in W$. Taking into account the relation [1, p. 91]

$$\frac{\partial}{\partial s} h_{-s*}^Y X = h_{-s*}^Y [Y, X],$$

we define, for each s , the vector field $Z_{-s}^{(0)} = h_{-s*}^Y X$ to get

$$(h_{-s*}^Y X)_x = X_x + \int_0^s \frac{\partial}{\partial s'} Z_{-s'}^{(0)}(x) ds' = X_x + \int_0^s (h_{-s'*}^Y [Y, X])_x ds'.$$

If we call $Z_{-s}^{(1)} = h_{-s*}^Y ([Y, X])$ and apply the above expression to $[Y, X]$, we get

$$\begin{aligned} (h_{-s*}^Y [Y, X])_x &= [Y, X]_x + \int_0^s \frac{\partial}{\partial s'} Z_{-s'}^{(1)}(x) ds' \\ &= [Y, X]_x + \int_0^s (h_{-s'*}^Y [Y, [Y, X]])_x ds'. \end{aligned}$$

Defining $Z_{-s}^{(k)}$ in an analogous way and applying all these results to the initial formula for $h_{-s*}^Y X$ we obtain

$$(h_{-s*}^Y X)_x = X_x + [Y, X]_x s + \frac{1}{2!} [Y, [Y, X]]_x s^2 + \frac{1}{3!} [Y, [Y, [Y, X]]]_x s^3 + \dots$$

By means of the properties of the scheme, we obtain that each term belongs to $V(\mathbb{R})$, i.e.

$$[Y, [Y, \dots, [Y, X] \dots]] \in V(\mathbb{R}),$$

and therefore

$$\Phi_{g*}X = h_{1*}^Y X \in V(\mathbb{R}).$$

■

Note that every curve $g(t)$ in G determines a diffeomorphism on $\mathbb{R} \times \mathbb{R}^n$ of the form $\overline{\Phi}_{g(t)} : (t, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto (t, \Phi_{g(t)}x) \in \mathbb{R} \times \mathbb{R}^n$. Therefore, given a t -dependent vector field $X \in \mathfrak{X}_t(\mathbb{R}^n)$ and a curve $g(t)$, this curve transforms X into a new vector field X' such that $X' = \overline{\Phi}_{g(t)} X$. For the sake of simplicity, we hereby denote $X' = g_{\star} X$ and $g_t : x \in \mathbb{R}^n \mapsto \Phi_{g(t)}x \in \mathbb{R}^n$. Obviously, in similarity with equation (7.6), we have $(g_{\star} X)_t = \dot{g}_t \circ g_t^{-1} + g_{t*}(X)$ and the set of curves in G makes up an infinite-dimensional group acting on $\mathfrak{X}_t(\mathbb{R}^n)$.

PROPOSITION 7.10. *Given a scheme $S(W, V)$, a curve $g(t)$ in G , and a t -dependent vector field $X \in V(\mathbb{R})$, then $g_{\star} X \in V(\mathbb{R})$.*

Proof. As formula (7.6) remains valid for the action of curves $g(t)$ included in $\exp(\mathfrak{g})$, proving that $g_\star X$ belongs to $V(\mathbb{R})$ can be reduced to checking that the corresponding terms $\dot{g}_t \circ g_t^{-1}$ and $g_{t\star} X$ are in $V(\mathbb{R})$. On one hand, $\dot{g}_t \circ g_t^{-1} \in W(\mathbb{R}) \subset V(\mathbb{R})$ and, by means of Lemma 7.9, we get that $g_{t\star} X \in V(\mathbb{R})$ for each t . Consequently, we see that $g_\star X \in V(\mathbb{R})$. Since every curve $g(t) \subset G$ decomposes as a product $g = g_1 \cdot \dots \cdot g_p$ of curves $g_j \subset \exp(\mathfrak{g})$, with $j = 1, \dots, p$, it follows that $g_\star X \in V(\mathbb{R})$ for every curve $g(t) \subset G$. ■

DEFINITION 7.11. Given a scheme $S(W, V)$, we call symmetry group of the scheme, $\text{Sym}(W)$, the set of t -dependent transformations $\Phi_{g(t)}$ induced by the curves $g(t)$ in G and an action Φ associated with the Lie algebra of vector fields W .

In order to simplify the notation, we hereby denote the t -dependent transformation $\Phi_{g(t)}$ with the curve g .

DEFINITION 7.12. Given a quasi-Lie scheme $S(W, V)$ and a t -dependent vector field $X \in V(\mathbb{R})$, we say that X is a *quasi-Lie system with respect to $S(W, V)$* if there exists a t -dependent transformation $g \in \text{Sym}(W)$ and a Lie algebra of vector fields $V_0 \subset V$ such that

$$g_\star X \in V_0(\mathbb{R}).$$

We emphasise that if X is a quasi-Lie system with respect to the scheme $S(W, V)$, it automatically admits a t -dependent superposition rule in the form given by (7.9).

7.4. t -dependent superposition rules. Minor modifications in the geometric approach to Lie systems detailed in Section 1.5 allow us to derive a new theory, based on the so-called *Lie family* concept, in order to treat a much larger family of systems of differential equations including Lie and quasi-Lie systems. Roughly speaking, Lie families are sets of systems of differential equations admitting a common superposition rule with t -dependence. This theory clearly generalises the superposition rule notion and provides a characterisation, described by the so-called *Generalised Lie Theorem*, of families of systems admitting such a property. Next, we provide a brief description of this theory and summarise its main results. For further details, see [35].

Consider the family of nonautonomous systems of first-order ordinary differential equations on \mathbb{R}^n , parametrised by the elements d of a set Λ , of the form

$$\frac{dx^i}{dt} = Y_d^i(t, x), \quad i = 1, \dots, n, \quad d \in \Lambda. \tag{7.11}$$

describing the integral curves of the family of t -dependent vector fields $\{Y_d\}_{d \in \Lambda}$ given by

$$Y_d(t, x) = \sum_{i=1}^n Y_d^i(t, x) \frac{\partial}{\partial x^i}.$$

Let us state the fundamental concept to be studied along this section:

DEFINITION 7.13. We say that the family of nonautonomous systems (7.11) admits a *common t -dependent superposition rule* if there exists a map $\Phi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$, i.e.

$$x = \Phi(t, x_{(1)}, \dots, x_{(m)}; k_1, \dots, k_n), \tag{7.12}$$

such that the general solution, $x(t)$, of any system Y_d of the family (7.11) can be written, at least for sufficiently small t , as

$$x(t) = \Phi(t, x_{(1)}(t), \dots, x_{(m)}(t); k_1, \dots, k_n),$$

with $\{x_{(a)}(t) \mid a = 1, \dots, m\}$ being any generic family of particular solutions of Y_d and the set $\{k_1, \dots, k_n\}$ being n arbitrary constants to be associated with each particular solution. A family of systems (7.11) admitting a common t -dependent superposition is called a *Lie family*.

DEFINITION 7.14. Given a t -dependent vector field $Y = \sum_{i=1}^n Y^i(t, x) \partial / \partial x^i$ on \mathbb{R}^n , we define its prolongation to $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ as the vector field on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ given by

$$Y^\wedge(t, x_{(0)}, \dots, x_{(m)}) = \sum_{a=0}^m \sum_{i=1}^n Y^i(t, x_{(a)}) \frac{\partial}{\partial x^i_{(a)}}$$

and its autonomisation, \tilde{Y} , as the vector field on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ of the form

$$\tilde{Y}(t, x_{(0)}, \dots, x_{(m)}) = \frac{\partial}{\partial t} + \sum_{a=0}^m \sum_{i=1}^n Y^i(t, x_{(a)}) \frac{\partial}{\partial x^i_{(a)}}$$

The Implicit Function Theorem states that, given a common t -dependent superposition rule $\Phi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$ of a Lie family $\{Y_d\}_{d \in \Lambda}$, the map $\Phi(t, x_{(1)}, \dots, x_{(m)}; \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, which reads $x_{(0)} = \Phi(t, x_{(1)}, \dots, x_{(m)}; k)$, can be inverted to give rise to a map $\Psi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$ given by

$$k = \Psi(t, x_{(0)}, \dots, x_{(m)}),$$

with $k = (k_1, \dots, k_n)$ being the only point in \mathbb{R}^n such that

$$x_{(0)} = \Phi(t, x_{(1)}, \dots, x_{(m)}; k).$$

As the fundamental property of the map Ψ says that $\Psi(t, x_{(0)}(t), \dots, x_{(m)}(t))$ is constant for any $(m+1)$ -tuple of particular solutions of any system of the family (7.11), the foliation determined by Ψ is invariant under the permutation of its $(m+1)$ arguments $\{x_{(a)} \mid a = 0, \dots, m\}$ and differentiating the preceding expression we get

$$\frac{\partial \Psi^j}{\partial t} + \sum_{a=0}^m \sum_{i=1}^n Y_d^i(t, x_{(a)}(t)) \frac{\partial \Psi^j}{\partial x^i_{(a)}} = 0, \quad j = 1, \dots, n, \quad d \in \Lambda, \quad (7.13)$$

with $\Psi = (\Psi^1, \dots, \Psi^n)$.

The relation (7.13) shows that the functions of the set $\{\Psi^i \mid i = 1, \dots, n\}$ are first-integrals for the vector fields \tilde{Y}_d , that is, $\tilde{Y}_d \Psi^i = 0$, with $i = 1, \dots, n$. Therefore, they generically define an n -codimensional foliation \mathfrak{F} on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ such that the vector fields \tilde{Y}_d are tangent to the leaves \mathfrak{F}_k of this foliation, with $k \in \mathbb{R}^n$.

The foliation \mathfrak{F} has another important property. Given the level set \mathfrak{F}_k of the map Ψ corresponding to $k = (k_1, \dots, k_n) \in \mathbb{R}^n$ and a generic point $(t, x_{(1)}, \dots, x_{(m)})$ of $\mathbb{R} \times \mathbb{R}^{nm}$, there is only one point $x_{(0)} \in \mathbb{R}^n$ such that $(t, x_{(0)}, x_{(1)}, \dots, x_{(m)}) \in \mathfrak{F}_k$. Then, the projection onto the last $m \cdot n$ coordinates and the time

$$\pi : (t, x_{(0)}, \dots, x_{(m)}) \in \mathbb{R} \times \mathbb{R}^{n(m+1)} \mapsto (t, x_{(1)}, \dots, x_{(m)}) \in \mathbb{R} \times \mathbb{R}^{nm},$$

induces local diffeomorphisms on the leaves \mathfrak{F}_k of \mathfrak{F} into $\mathbb{R} \times \mathbb{R}^{nm}$.

This property can also be seen as the fact that the foliation \mathfrak{F} corresponds to a zero curvature connection ∇ on the bundle $\pi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R} \times \mathbb{R}^{nm}$. Indeed, the restriction of the projection π to a leaf gives a one-to-one map. In this way, we get a linear map among vector fields on $\mathbb{R} \times \mathbb{R}^{nm}$ and ‘horizontal’ vector fields tangent to a leaf.

Note that the knowledge of this connection (foliation) gives us the *common t -dependent superposition rule* without referring to the map Ψ . If we fix the point $x_{(0)}(0)$ and m particular solutions, $x_{(1)}(t), \dots, x_{(m)}(t)$, for a system of the family, then $x_{(0)}(t)$ is the unique curve in \mathbb{R}^n such that

$$(t, x_{(0)}(t), x_{(1)}(t), \dots, x_{(m)}(t)) \in \mathbb{R} \times \mathbb{R}^{nm}$$

belongs to the same leaf as the point $(0, x_{(0)}(0), x_{(1)}(0), \dots, x_{(m)}(0))$. Thus, it is only the foliation \mathfrak{F} what really matters when the *common t -dependent superposition rule* is concerned.

On the other hand, if we have a zero curvature connection ∇ on the bundle

$$\pi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R} \times \mathbb{R}^{nm},$$

i.e. if we have an involutive horizontal distribution ∇ on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ that can be integrated to give a foliation \mathfrak{F} on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ and such that the vector fields \tilde{Y}_d are tangent to the leaves of the foliation, then the procedure described above determines a *common t -dependent superposition rule* for the family of nonautonomous systems of first-order differential equations (7.11).

Indeed, let $k \in \mathbb{R}^n$ enumerate smoothly the leaves \mathfrak{F}_k of \mathfrak{F} , i.e. there exists a smooth map $\iota : \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R}^{n(m+1)}$ such that $\iota(\mathbb{R}^n)$ intersects every \mathfrak{F}_k in a unique point. Then, if $x_{(0)} \in \mathbb{R}^n$ is the unique point such that

$$(t, x_{(0)}, x_{(1)}, \dots, x_{(m)}) \in \mathfrak{F}_k,$$

this fact gives rise to a *t -dependent superposition rule*

$$x_{(0)} = \Phi(t, x_{(1)}, \dots, x_{(m)}; k)$$

for the family of nonautonomous systems of first-order ordinary differential equations (7.11). To see this, let us observe that the Implicit Function Theorem shows that there exists a function $\Psi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}$ such that

$$\Psi(t, x_{(0)}, \dots, x_{(m)}) = k,$$

which is equivalent to say that $(t, x_{(0)}, \dots, x_{(m)}) \in \mathfrak{F}_k$. If we fix a certain $k \in \mathbb{R}^n$ and take certain solutions, $x_{(1)}(t), \dots, x_{(m)}(t)$, of a particular instance of (7.11), then $x_{(0)}(t)$ defined by means of the condition $\Psi(t, x_{(0)}(t), \dots, x_{(m)}(t)) = k$ also satisfies such an instance. Indeed, let $x'_{(0)}(t)$ be the solution with initial value $x'_{(0)}(0) = x_{(0)}$. Since the vector fields \tilde{Y}_d are tangent to \mathfrak{F} , the curve

$$t \mapsto (t, x_{(0)}(t), x_{(1)}(t), \dots, x_{(m)}(t))$$

lies entirely in a leaf of \mathfrak{F} , so in \mathfrak{F}_k . But the point of one leaf is entirely determined by its projection π , so $x'_{(0)}(t) = x_{(0)}(t)$ and $x_{(0)}(t)$ is a solution.

PROPOSITION 7.15. *Giving a t -dependent superposition rule (7.12) for a family of systems of differential equations (7.11) is equivalent to give a zero curvature connection on the bundle $\pi : \mathbb{R} \times \mathbb{R}^{(m+1)n} \rightarrow \mathbb{R} \times \mathbb{R}^{nm}$ for which the \tilde{Y}_d are ‘horizontal’ vector fields.*

In general it is difficult to determine whether a family of differential equations admits a common t -dependent superposition rule by means of the above Proposition. It is therefore interesting to find a characterisation of Lie families by means of a more convenient criterion, e.g. through an easily verifiable condition based on the properties of the t -dependent vector fields $\{Y_a\}_{a \in \Lambda}$.

Finding such a criterion is the main result of the theory of Lie families. It is formulated as Generalised Lie Theorem and based on the following lemmas given below. The first two ones are straightforward, a complete detailed proof for the third can be found in [35].

LEMMA 7.16. *Given two t -dependent vector fields X and Y on \mathbb{R}^n , the commutator $[\tilde{X}, \tilde{Y}]$ on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ is the prolongation of a t -dependent vector field Z on \mathbb{R}^n , $[\tilde{X}, \tilde{Y}] = Z^\wedge$.*

LEMMA 7.17. *Given a family of t -dependent vector fields, X_1, \dots, X_r , on \mathbb{R}^n , their autonomisations satisfy the relations*

$$[\bar{X}_j, \bar{X}_k](t, x) = \sum_{l=1}^r f_{jkl}(t) \bar{X}_l(t, x), \quad j, k = 1, \dots, r,$$

for some t -dependent functions $f_{jkl} : \mathbb{R} \rightarrow \mathbb{R}$, if and only if their t -prolongations to $\mathbb{R} \times \mathbb{R}^{n(m+1)}$, $\tilde{X}_1, \dots, \tilde{X}_r$, obey analogous relations

$$[\tilde{X}_j, \tilde{X}_k](t, x) = \sum_{l=1}^r f_{jkl}(t) \tilde{X}_l(t, x), \quad j, k = 1, \dots, r.$$

Moreover, $\sum_{l=1}^r f_{jkl}(t) = 0$ for all $j, k = 1, \dots, r$.

LEMMA 7.18. *Consider a family of t -dependent vector fields, Y_1, \dots, Y_r , with t -prolongations $\tilde{Y}_1, \dots, \tilde{Y}_r$ to $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ such that their projections $\pi_*(\tilde{Y}_j)$ are linearly independent at a generic point in $\mathbb{R} \times \mathbb{R}^{nm}$. Then, $\sum_{j=1}^r b_j \tilde{Y}_j$, with $b_j \in C^\infty(\mathbb{R} \times \mathbb{R}^{nm})$, is of the form Y^\wedge (resp. \tilde{Y}) for a t -dependent vector field Y on \mathbb{R}^n , if and only if the functions b_j only depend on the variable t , that is, $b_j = b_j(t)$, and $\sum_{j=1}^r b_j = 0$ (resp., $\sum_{j=1}^r b_j = 1$).*

MAIN THEOREM 7.19. (**Generalised Lie Theorem**) *The family of systems (7.11) admits a common t -dependent superposition rule if and only if the vector fields $\{\bar{Y}_d\}_{d \in \Lambda}$ can be written in the form*

$$\bar{Y}_d(t, x) = \sum_{\alpha=1}^r b_{d\alpha}(t) \bar{X}_\alpha(t, x), \quad d \in \Lambda,$$

where $b_{d\alpha}$ are functions of the single variable t such that $\sum_{\alpha=1}^r b_{d\alpha} = 1$ and, X_1, \dots, X_r , are t -dependent vector fields satisfying

$$[\bar{X}_\alpha, \bar{X}_\beta](t, x) = \sum_{\gamma=1}^r f_{\alpha\beta\gamma}(t) \bar{X}_\gamma(t, x), \quad \alpha, \beta = 1, \dots, r, \quad (7.14)$$

for certain functions $f_{\alpha\beta\gamma} : \mathbb{R} \rightarrow \mathbb{R}$.

The denomination of the above theorem comes from the following proposition, which shows that each Lie system can be embedded into a Lie family. In order to formulate this result, let us denote by $S_g(W, V; V_0)$ the set of quasi-Lie systems of the scheme $S(W, V)$ such that there exists a g satisfying that $g_\star X \in V_0(\mathbb{R})$ with V_0 a Lie algebra of vector fields included in V . Again, complete proof of this proposition can be found in [35].

PROPOSITION 7.20. *The family of quasi-Lie systems $S_g(W, V; V_0)$ is a Lie family admitting the common t -dependent superposition rule of the form*

$$\bar{\Phi}_g(t, x_{(1)}, \dots, x_{(m)}, k) = g_t^{-1} \circ \Phi(g_t(x_{(1)}, \dots, g_t)x_{(m)}, k),$$

for any t -independent superposition rule Φ associated with the Lie algebra of vector fields V_0 by Lie Theorem.

8. Applications of quasi-Lie schemes and Lie families

The theory of quasi-Lie schemes, quasi-Lie systems [34] and the theory of Lie families [35] can be used to investigate a very large set of differential equations, namely, nonlinear oscillators [34], dissipative Milne–Pinney equations [34, 35, 45], second-order Riccati equations [48], Abel equations [35], Emden equations [34, 42], etc. As we showed in the previous section, these theories enable us to obtain t -dependent superposition rules, constants of the motion, exact solutions, integrability conditions, etc. The main aim in this chapter is to show that the possibilities of application of these methods are very wide and we can obtain a very large set of results from a unified point of view.

More exactly, in previous sections it was proved that Milne–Pinney could be studied by means of the theory of Lie systems (see also [43]). Nevertheless, there exist dissipative Milne–Pinney equations that cannot straightforwardly be studied through this theory. In this section, we provide a quasi-Lie scheme to treat these dissipative Milne–Pinney equations. Then, we use this quasi-Lie scheme to relate these equations to usual Milne–Pinney equations. By means of this relation, we obtain a t -dependent superposition rule for dissipative Milne–Pinney equations.

Apart from dissipative Milne–Pinney equations, we also investigate nonautonomous nonlinear oscillators. We show that some of these differential equations can be transformed into autonomous nonlinear oscillators. This result was already derived by Perelomov [180], but here we recover it from a more general point of view. More specifically, we obtain that the nonautonomous nonlinear oscillators analysed by Perelomov can be seen as differential equations obeying an integrability condition derived by means of a quasi-Lie scheme.

As a last application of the quasi-Lie scheme notion, we extensively analyse Emden equations. We provide a quasi-Lie scheme to obtain t -dependent constants of the motion by means of particular solutions that obey an integrability condition. The method developed also enables us to obtain Emden equations with a fixed t -dependent integral of motion. Kummer–Liouville transformations are also obtained by means of our scheme and many other properties are recovered.

Finally, in the last two sections of this chapter, we apply common t -dependent superposition rules to study some first- and second-order differential equations. In this way, we will show how they can be used to analyse equations which cannot be studied by means of the usual theory of Lie systems. Additionally, some new results for the study of Abel and Milne–Pinney equations are provided.

8.1. Dissipative Milne–Pinney equations. In this section, we study the so-called dissipative Milne–Pinney equations. We show that the first-order ordinary differential equations associated with these second-order ones in the usual way, i.e. by considering velocities as new variables, are not Lie systems. However, the theory of quasi-Lie schemes can be used to deal with such first-order systems. Here we provide a scheme which enables us to transform a certain kind of dissipative Milne–Pinney equations, considered as first-order systems, into some first-order

Milne–Pinney equations already studied by means of the theory of Lie systems [53]. As a result we get a t -dependent superposition rule for some of these dissipative Milne–Pinney equations.

Let us establish the problem under study. Consider the family of dissipative Milne–Pinney equations of the form

$$\ddot{x} = a(t)\dot{x} + b(t)x + c(t)\frac{1}{x^3}. \quad (8.1)$$

We are mainly interested in the case $c(t) \neq 0$, so we assume that $c(t)$ has a constant sign for the set of values of t that we analyse.

Usually, we associate such a second-order differential equation with a system of first-order differential equations by introducing a new variable v and relating the differential equation (8.1) to the system of first-order differential equations

$$\begin{cases} \dot{x} = v, \\ \dot{v} = a(t)v + b(t)x + c(t)\frac{1}{x^3}. \end{cases} \quad (8.2)$$

Let us search for a quasi-Lie scheme to handle the above system. Remember that we need to find linear spaces W_{DisM} and V_{DisM} of vector fields such that

1. $W_{\text{DisM}} \subset V_{\text{DisM}}$.
2. $[W_{\text{DisM}}, W_{\text{DisM}}] \subset W_{\text{DisM}}$.
3. $[W_{\text{DisM}}, V_{\text{DisM}}] \subset V_{\text{DisM}}$.

Also, in order to treat system (8.2) through this scheme, we have to ensure that the t -dependent vector field

$$X_t = v\frac{\partial}{\partial x} + \left(a(t)v + b(t)x + \frac{c(t)}{x^3} \right) \frac{\partial}{\partial v},$$

whose integral curves are solutions for the system (8.2), is such that $X_t \in V_{\text{DisM}}$ for every t in an open interval of \mathbb{R} .

Consider the vector space V_{DisM} spanned by the vector fields

$$X_1 = v\frac{\partial}{\partial v}, \quad X_2 = x\frac{\partial}{\partial v}, \quad X_3 = \frac{1}{x^3}\frac{\partial}{\partial v}, \quad X_4 = v\frac{\partial}{\partial x}, \quad X_5 = x\frac{\partial}{\partial x}$$

and the two-dimensional vector subspace $W_{\text{DisM}} \subset V_{\text{DisM}}$ generated by

$$Y_1 = X_1 = v\frac{\partial}{\partial v}, \quad Y_2 = X_2 = x\frac{\partial}{\partial v}.$$

It can be seen that W_{DisM} is a Lie algebra,

$$[Y_1, Y_2] = -Y_2,$$

and, additionally, as

$$\begin{aligned} [Y_1, X_3] &= -X_3, & [Y_1, X_4] &= X_4, & [Y_1, X_5] &= 0, \\ [Y_2, X_3] &= 0, & [Y_2, X_4] &= X_5 - X_1, & [Y_2, X_5] &= -X_2, \end{aligned}$$

the linear space V_{DisM} is invariant under the action of the Lie algebra W_{DisM} on V_{DisM} , i.e. $[W_{\text{DisM}}, V_{\text{DisM}}] \subset V_{\text{DisM}}$. Thus, the vector spaces

$$V_{\text{DisM}} = \langle X_1, \dots, X_5 \rangle \quad \text{and} \quad W_{\text{DisM}} = \langle Y_1, Y_2 \rangle$$

of vector fields form a quasi-Lie scheme $S(W_{\text{DisM}}, V_{\text{DisM}})$. Let us observe that

$$X_t = a(t)X_1 + b(t)X_2 + c(t)X_3 + X_4$$

and thus $X \in V_{\text{DisM}}(\mathbb{R})$.

We stress that the vector space V_{DisM} is not a Lie algebra, because the commutator $[X_3, X_4]$ does not belong to V_{DisM} . Moreover, $V''' = \langle X_1, \dots, X_4 \rangle$ is not a Lie algebra of vector fields due to a similar reason, i.e. $[X_3, X_4] \notin V'''$. Additionally, there exists no finite-dimensional real Lie algebra V' containing V''' . Thus, system (8.2) is not a Lie system, but we can use the quasi-Lie scheme $S(W_{\text{DisM}}, V_{\text{DisM}})$ to investigate it.

The key tool provided by the scheme $S(W_{\text{DisM}}, V_{\text{DisM}})$ is the infinite-dimensional group $\mathcal{G}(W_{\text{DisM}})$ of generalised flows for the t -dependent vector fields with values in W , i.e. $\alpha_1(t)Y_1 + \alpha_2(t)Y_2$, which leads to the group of t -dependent changes of variables

$$\mathcal{G}(W_{\text{DisM}}) = \left\{ g(\alpha(t), \beta(t)) = \begin{cases} x &= x' \\ v &= \alpha(t)v' + \beta(t)x' \end{cases} \mid \alpha(t) > 0, \beta(0) = 0, \alpha(0) = 1 \right\}.$$

According to the general theory of quasi-Lie schemes, these previous t -dependent changes of variables enable us to transform system (8.2) into a new one taking values in V_{DisM} ,

$$X'_t = a'(t)X_1 + b'(t)X_2 + c'(t)X_3 + d'(t)X_4 + e'(t)X_5. \tag{8.3}$$

The new coefficients are

$$\begin{cases} a'(t) = a(t) - \beta(t) - \frac{\dot{\alpha}(t)}{\alpha(t)}, \\ b'(t) = \frac{b(t)}{\alpha(t)} + a(t)\frac{\beta(t)}{\alpha(t)} - \frac{\beta^2(t)}{\alpha(t)} - \frac{\dot{\beta}(t)}{\alpha(t)}, \\ c'(t) = \frac{c(t)}{\alpha(t)}, \\ d'(t) = \alpha(t), \\ e'(t) = \beta(t). \end{cases}$$

The integral curves for the t -dependent vector field (8.3) are solutions of the system

$$\begin{cases} \frac{dx'}{dt} &= \beta(t)x' + \alpha(t)v', \\ \frac{dv'}{dt} &= \left(\frac{b(t)}{\alpha(t)} + a(t)\frac{\beta(t)}{\alpha(t)} - \frac{\beta^2(t)}{\alpha(t)} - \frac{\dot{\beta}(t)}{\alpha(t)} \right) x' + \\ &+ \left(a(t) - \beta(t) - \frac{\dot{\alpha}(t)}{\alpha(t)} \right) v' + \frac{c(t)}{\alpha(t)} \frac{1}{x'^3}. \end{cases} \tag{8.4}$$

As it was said in Section 7.3, we use schemes to transform the corresponding systems of first-order differential equations into Lie ones. So, in this case, we must find a Lie algebra of vector fields $V_0 \subset V_{\text{DisM}}$ and a generalised flow $g \in \mathcal{G}(W_{\text{DisM}})$ such that $g_\star X \in V_0(\mathbb{R})$. This leads to a system of ordinary differential equations for the functions $\alpha(t)$, $\beta(t)$ and some integrability conditions on the initial functions $a(t)$, $b(t)$ and $c(t)$ for such a t -dependent change of variables to exist.

In order to find a proper Lie algebra of vector fields $V_0 \subset V$, note that Milne–Pinney equations studied in [53] are Lie systems in the family of differential equations defined by systems (8.2) and therefore it is natural to look for the conditions needed to transform a given system of (8.2), described by the t -dependent vector field X_t , into one of these first-order Milne–Pinney

equations of the form

$$\begin{cases} \dot{x} &= f(t)v, \\ \dot{v} &= -\omega(t)x + f(t)\frac{k}{x^3}, \end{cases} \quad (8.5)$$

where k is a constant, i.e. a system describing the integral curves for a t -dependent vector field with values in the Lie algebra of vector fields [53]

$$V_0 = \langle X_4 + k X_3, X_2, \frac{1}{2}(X_5 - X_1) \rangle.$$

As a result, we get that $\beta = 0$, $\alpha = f$ and, furthermore, the functions α , a and c must satisfy

$$k\alpha^2 = c, \quad \dot{\alpha} - a\alpha = 0, \quad (8.6)$$

which yield that c and k have the same sign. The second condition is a differential equation for α and the first one determines c in terms of α . Therefore, both conditions lead to a relation between c and a providing the integrability condition

$$c(t) = k \exp\left(2 \int a(t) dt\right) \quad (8.7)$$

and showing, in view of (8.4), (8.5) and (8.6), that

$$\alpha(t) = \exp\left(\int a(t) dt\right) \quad \text{and} \quad \omega(t) = -b(t) \exp\left(-\int a(t) dt\right),$$

where we choose the constants of integration in order to get $\alpha(0) = 1$ as required.

Summarising the preceding results, under the integrability condition (8.7), the first-order Milne–Pinney equation

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= a(t)v + b(t)x + c(t)\frac{1}{x^3}, \end{cases}$$

can be transformed into the system

$$\begin{cases} \frac{dx'}{dt} &= \exp\left(\int a(t) dt\right) v', \\ \frac{dv'}{dt} &= b(t)\exp\left(-\int a(t) dt\right) x' + \exp\left(\int a(t) dt\right) \frac{k}{x'^3}, \end{cases}$$

by means of the t -dependent change of variables

$$g\left(\exp\left(\int a(t) dt\right), 0\right) = \begin{cases} x' &= x, \\ v' &= \exp\left(\int a(t) dt\right) v. \end{cases}$$

We stress the fact that the previous change of variables is a particular instance of the so-called Liouville transformation [164].

The final Milne–Pinney equation can be rewritten through the t -reparametrisation

$$\tau(t) = \int \exp\left(\int a(t) dt\right) dt,$$

as

$$\begin{cases} \frac{dx'}{d\tau} &= v', \\ \frac{dv'}{d\tau} &= \exp\left(-2 \int a(t) dt\right) b(t(\tau))x' + \frac{k}{x'^3}. \end{cases}$$

These systems were analysed in [50] and there it was shown through the theory of Lie systems that they admit the constant of the motion

$$I = (\bar{x}v' - \bar{v}x')^2 + k \left(\frac{\bar{x}}{x'} \right)^2,$$

where (\bar{x}, \bar{v}) is a solution of the system

$$\begin{cases} \frac{d\bar{x}}{d\tau} = \bar{v}, \\ \frac{d\bar{v}}{d\tau} = \exp\left(-2 \int a(t) dt\right) b(t) \bar{x}, \end{cases}$$

which can be written as a second-order differential equation

$$\frac{d^2\bar{x}}{d\tau^2} = \exp\left(-2 \int a(t) dt\right) b(t) \bar{x}.$$

If we invert the t -reparametrisation, we obtain the following differential equation

$$\ddot{\bar{x}} - a(t)\dot{\bar{x}} - b(t)\bar{x} = 0, \quad (8.8)$$

which is the linear differential equation associated with the initial Milne–Pinney equation.

As it was shown in [53], we can obtain, by means of the theory of Lie systems, the following superposition rule

$$x' = \frac{\sqrt{2}}{|\bar{x}_1\bar{v}_2 - \bar{v}_1\bar{x}_2|} \left(I_2\bar{x}_1^2 + I_1\bar{x}_2^2 \pm \sqrt{4I_1I_2 - k(\bar{x}_1\bar{v}_2 - \bar{v}_1\bar{x}_2)^2} \bar{x}_1\bar{x}_2 \right)^{1/2},$$

and as the t -dependent transformation performed does not change the variable x , we get the t -dependent superposition rule

$$x = \frac{\sqrt{2}\alpha(t)}{|\bar{x}_1\dot{\bar{x}}_2 - \dot{\bar{x}}_1\bar{x}_2|} \left(I_2\bar{x}_1^2 + I_1\bar{x}_2^2 \pm \sqrt{4I_1I_2 - \frac{k}{\alpha^2(t)}(\bar{x}_1\dot{\bar{x}}_2 - \dot{\bar{x}}_1\bar{x}_2)^2} \bar{x}_1\bar{x}_2 \right)^{1/2},$$

in terms of a set of solutions of the second-order linear system (8.8).

Summing up, the application of our scheme to the family of dissipative Milne–Pinney equations

$$\ddot{x} = a(t)\dot{x} + b(t)x + \exp\left(2 \int a(t) dt\right) \frac{k}{x^3}$$

shows that this family admits a t -dependent superposition principle:

$$x = \frac{\sqrt{2}\alpha(t)}{|y_1\dot{y}_2 - y_2\dot{y}_1|} \left(I_2y_1^2 + I_1y_2^2 \pm \sqrt{4I_1I_2 - \frac{k}{\alpha^2(t)}(y_1\dot{y}_2 - y_2\dot{y}_1)^2} y_1y_2 \right)^{1/2},$$

in terms of two independent solutions y_1, y_2 for the differential equation

$$\ddot{y} - a(t)\dot{y} - b(t)y = 0.$$

So, we have fully detailed a particular application of the theory of quasi-Lie schemes to dissipative Milne–Pinney equations. As a result, we provide a t -dependent superposition rule for a family of such systems. Another paper dealing with such an approach to dissipative Milne–Pinney equations and explaining some of their properties can be found in [45].

8.2. Non-linear oscillators. As a second application of our theory, we use quasi-Lie schemes to deal with a certain kind of nonlinear oscillators. The main objective of this section is to explain several properties of a family of t -dependent nonlinear oscillators studied by Perelomov in [180]. We also furnish a, as far as we know, new constant of the motion for these systems.

Consider the subset of the family of nonlinear oscillators investigated in [180]:

$$\ddot{x} = b(t)x + c(t)x^n, \quad n \neq 0, 1.$$

The cases $n = 0, 1$, are omitted because they can be handled with the usual theory of Lie systems. As in the section above, we link the above second-order ordinary differential equation to the first-order system

$$\begin{cases} \dot{x} = v, \\ \dot{v} = b(t)x + c(t)x^n. \end{cases} \quad (8.9)$$

Let us provide a quasi-Lie scheme to deal with systems (8.9). Consider the vector space V_{NO} spanned by the linear combinations of the vector fields

$$X_1 = x \frac{\partial}{\partial v}, \quad X_2 = x^n \frac{\partial}{\partial v}, \quad X_3 = v \frac{\partial}{\partial x}, \quad X_4 = v \frac{\partial}{\partial v}, \quad X_5 = x \frac{\partial}{\partial x}$$

on $\mathbb{T}\mathbb{R}$ and take the vector subspace $W_{NO} \subset V_{NO}$ generated by

$$Y_1 = X_4 = v \frac{\partial}{\partial v}, \quad Y_2 = X_1 = x \frac{\partial}{\partial v}, \quad Y_3 = X_5 = x \frac{\partial}{\partial x}.$$

Therefore, W_{NO} is a solvable Lie algebra of vector fields,

$$[Y_1, Y_2] = -Y_2, \quad [Y_1, Y_3] = 0, \quad [Y_2, Y_3] = -Y_2,$$

and taking into account that

$$\begin{aligned} [Y_1, X_2] &= -X_2, & [Y_1, X_3] &= X_3, & [Y_2, X_2] &= 0, \\ [Y_2, X_3] &= X_5 - X_4, & [Y_3, X_2] &= nX_2, & [Y_3, X_3] &= -X_3, \end{aligned}$$

we see that V_{NO} is invariant under the action of W_{NO} , i.e. $[W_{NO}, V_{NO}] \subset V_{NO}$. In this way we get the quasi-Lie scheme $S(W_{NO}, V_{NO})$.

Now, we have to go over whether the solutions of system (8.9) are integral curves for a t -dependent vector field $X \in V_{NO}(\mathbb{R})$. In order to check this, we realise that the system (8.9) describes the integral curves for the t -dependent vector field

$$X_t = v \frac{\partial}{\partial x} + (b(t)x + c(t)x^n) \frac{\partial}{\partial v},$$

which can be written as

$$X_t = b(t)X_1 + c(t)X_2 + X_3. \quad (8.10)$$

Note also that $[X_2, X_3] \notin V_{NO}$ and $V'' = \langle X_1, X_2, X_3 \rangle$ is not only a Lie algebra of vector fields, but also there is no finite-dimensional Lie algebra V' including V'' . Thus, X cannot be considered as a Lie system and we conclude that the first-order nonlinear oscillator

$$\begin{cases} \dot{x} = v, \\ \dot{v} = b(t)x + c(t)x^n. \end{cases}$$

describing integral curves of the t -dependent vector field (which is not a Lie system)

$$X_t = b(t)X_1 + c(t)X_2 + X_3$$

can be described by means of the quasi-Lie scheme $S(W_{NO}, V_{NO})$.

Now, the group of generalised flows $\mathcal{G}(W_{NO})$ associated with $S(W_{NO}, V_{NO})$ is made of the t -dependent transformations

$$g(\alpha(t), \beta(t), \gamma(t)) = \begin{cases} x = \gamma(t)x' \\ v = \beta(t)v' + \alpha(t)x' \end{cases} \quad \beta(t), \gamma(t) > 0, \beta(0) = \gamma(0) = 1, \alpha(0) = 0.$$

Let us restrict ourselves to the case $\alpha(t) = \dot{\gamma}(t)$ and $\beta(t) = 1/\gamma(t)$ and apply these transformations to the system (8.9). The theory of quasi-Lie systems tells us that

$$g(\alpha(t), \beta(t), \gamma(t)) \star X \in V_{NO}(\mathbb{R}).$$

Indeed, these t -dependent transformations lead to the systems

$$\begin{cases} \frac{dx'}{dt} = \frac{1}{\gamma^2(t)} v', \\ \frac{dv'}{dt} = (\gamma^2(t)b(t) - \ddot{\gamma}(t)\gamma(t))x' + c(t)\gamma^{n+1}(t)x'^n, \end{cases} \quad (8.11)$$

which are related to the second-order differential equations

$$\gamma^2(t)\ddot{x}' = -2\gamma(t)\dot{\gamma}(t)\dot{x}' + (\gamma^2(t)b(t) - \ddot{\gamma}(t)\gamma(t))x' + c(t)\gamma^{n+1}(t)x'^n.$$

But the theory of quasi-Lie schemes is based on the search of a generalised flow $g \in \mathcal{G}(W_{NO})$ such that $g \star X$ becomes a Lie system, i.e. there exists a Lie algebra of vector fields $V_0 \subset V_{NO}$ such that $g \star X \in V_0(\mathbb{R})$. For instance, we can try to transform a particular instance of the systems (8.11) into a first-order differential equation associated with a nonlinear oscillator with a zero t -dependent angular frequency, for example, into the first-order system

$$\begin{cases} \frac{dx'}{dt} = f(t)v', \\ \frac{dv'}{dt} = f(t)c_0x'^n, \end{cases} \quad (8.12)$$

related to the nonlinear oscillator

$$\frac{d^2x'}{d\tau^2} = c_0x'^n,$$

with $d\tau/dt = f(t)$.

The conditions ensuring such a transformation are

$$\gamma(t)b(t) - \ddot{\gamma}(t) = 0, \quad c(t) = c_0\gamma^{-(n+3)}(t), \quad (8.13)$$

with $f(t) = \gamma_1^{-2}(t)$, where γ_1 is a non-vanishing particular solution for $\gamma(t)b(t) - \ddot{\gamma}(t) = 0$. We must emphasise that just particular solutions with $\gamma_1(0) = 1$ and $\dot{\gamma}_1(0) = 0$ are related to generalised flows in $\mathcal{G}(W_{NO})$. Nevertheless, any other particular solution can also be used to transform a nonlinear oscillator into a Lie system as we stated. The Lie system (8.12) is the system associated with the t -dependent vector field

$$X_t = \frac{1}{\gamma_1^2(t)} \left(v' \frac{\partial}{\partial x'} + c_0 x'^n \frac{\partial}{\partial v'} \right).$$

As a consequence of the standard methods developed for the theory of Lie systems [52], we join two copies of the above system in order to get the first-integrals

$$I_i = \frac{1}{2}v_i'^2 - \frac{c_0}{n+1}x_i'^{n+1}, \quad i = 1, 2,$$

and

$$I_3 = \frac{x_1'}{\sqrt{I_1}} \text{Hyp} \left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_1'^{n+1}}{I_1(n+1)} \right) - \frac{x_2'}{\sqrt{I_2}} \text{Hyp} \left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_2'^{n+1}}{I_2(n+1)} \right),$$

where $\text{Hyp}(a, b, c, d)$ denotes the corresponding hypergeometric functions. In terms of the initial variables these first-integrals for $g_\star X$ read

$$I_i = \frac{1}{2}(\gamma_1(t)\dot{x}_i - \dot{\gamma}_1(t)x_i)^2 - \frac{c_0}{\gamma_1^{n+1}(t)(n+1)}x_i^{n+1}, \quad i = 1, 2, \tag{8.14}$$

and

$$I_3 = \frac{1}{\gamma_1(t)} \left(\frac{x_1}{\sqrt{I_1}} \text{Hyp} \left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_1^{n+1}}{\gamma_1^{n+1}(t)I_1(n+1)} \right) - \frac{x_2}{\sqrt{I_2}} \text{Hyp} \left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_2^{n+1}}{\gamma_1^{n+1}(t)I_2(n+1)} \right) \right). \tag{8.15}$$

As a particular application of conditions (8.13), we can consider the following example of [180], where the t -dependent Hamiltonian

$$H(t) = \frac{1}{2}p^2 + \frac{\omega^2(t)}{2}x^2 + c^2\gamma_1^{-(s+2)}(t)x^s,$$

with γ_1 being such that $\ddot{\gamma}_1(t) + \omega^2(t)\gamma_1(t) = 0$, is studied. The Hamilton equations for the latter Hamiltonian are

$$\begin{cases} \dot{x} = p, \\ \dot{p} = -sc^2\gamma_1^{-(s+2)}(t)x^{s-1} - \omega^2(t)x, \end{cases} \tag{8.16}$$

which are associated with the second-order differential equation for the variable x given by

$$\ddot{x} = -sc^2\gamma_1^{-(s+2)}(t)x^{s-1} - \omega^2(t)x. \tag{8.17}$$

Note that here the variable p plays the same role as v in our theoretical development and the latter differential equation is a particular case of our Emden equations with

$$b(t) = -\omega^2(t), \quad c(t) = -sc^2\gamma_1^{-(s+2)}(t), \quad n = s - 1. \tag{8.18}$$

Let us prove that the above coefficients satisfy the conditions (8.13):

1. By assumption, $\omega^2(t)\gamma_1(t) + \ddot{\gamma}_1(t) = 0$. As $\omega^2(t) = -b(t)$, then $\gamma_1(t)b(t) - \ddot{\gamma}_1(t) = 0$.
2. If we fix $c_0 = -sc^2$, in view of conditions (8.18), we obtain $c(t) = c_0\gamma_1^{-(n+3)}(t)$.

Therefore, we get that the t -dependent frequency nonlinear oscillator (8.17) can be transformed into a new one with zero frequency, i.e.

$$\frac{d^2x'}{d\tau^2} = -sc^2x'^{s-1},$$

with

$$\tau = \int \frac{dt}{\gamma_1^2(t)},$$

reproducing the result given by Perelomov [180]. The choice of the t -dependent frequencies is such that it is possible to transform the initial t -dependent nonlinear oscillator into the final autonomous nonlinear oscillator. Then, we recover here such frequencies as a result of an integrability condition. Moreover, in view of the expressions (8.14), (8.15) and (8.18), we get a, as far as we know, new t -dependent constants of the motion for these nonlinear oscillators.

8.3. Dissipative Mathews–Lakshmanan oscillators. In this section we provide a simple application of the theory of quasi-Lie schemes to investigate the t -dependent dissipative Mathews–Lakshmanan oscillator

$$(1 + \lambda x^2)\ddot{x} - F(t)(1 + \lambda x^2)\dot{x} - (\lambda x)\dot{x}^2 + \omega(t)x = 0, \quad \lambda > 0. \quad (8.19)$$

More specifically, we supply some integrability conditions to relate the above dissipative oscillator to the Mathews–Lakshmanan one [65, 67, 142, 161]

$$(1 + \lambda x^2)\ddot{x} - (\lambda x)\dot{x}^2 + kx = 0, \quad \lambda > 0, \quad (8.20)$$

and by means of such a relation we get a, as far as we know, new t -dependent constant of the motion.

Consider the system of first-order differential equation related to equation (8.19) in the usual way, i.e.

$$\begin{cases} \dot{x} = v, \\ \dot{v} = F(t)v + \frac{\lambda xv^2}{1 + \lambda x^2} - \omega(t)\frac{x}{1 + \lambda x^2}, \end{cases} \quad (8.21)$$

and determining the integral curves for the t -dependent vector field

$$X_t = \left(F(t)v + \frac{\lambda xv^2}{1 + \lambda x^2} - \omega(t)\frac{x}{1 + \lambda x^2} \right) \frac{\partial}{\partial v} + v \frac{\partial}{\partial x}.$$

Let us provide a scheme to handle the system (8.21). Consider the vector space V spanned by the vector fields

$$X_1 = v \frac{\partial}{\partial x} + \frac{\lambda xv^2}{1 + \lambda x^2} \frac{\partial}{\partial v}, \quad X_2 = \frac{x}{1 + \lambda x^2} \frac{\partial}{\partial v}, \quad X_3 = v \frac{\partial}{\partial v}, \quad (8.22)$$

and the linear space $W = \langle X_3 \rangle$. The commutation relations

$$[X_3, X_1] = X_1, \quad [X_3, X_2] = -X_2,$$

imply that the linear spaces W, V make up a quasi-Lie scheme $S(W, V)$. As the t -dependent vector field X_t reads in terms of the basis (8.22)

$$X_t = F(t)X_3 - \omega(t)X_2 + X_1,$$

we get that $X_t \in V(\mathbb{R})$.

The integration of X_3 shows that

$$\mathcal{G}(W) = \left\{ g(\alpha(t)) = \left\{ \begin{array}{l} x = x', \\ v = \alpha(t)v'. \end{array} \middle| \alpha(t) > 0, \alpha(0) = 1 \right. \right\},$$

and the t -dependent changes of variables related to the controls of $\mathcal{G}(W)$ transform the system (8.21) into

$$\begin{cases} \dot{x}' = \alpha(t)v', \\ \dot{v}' = \left(F(t) - \frac{\dot{\alpha}(t)}{\alpha(t)}\right)v' - \frac{\omega(t)}{\alpha(t)}\frac{x'}{1 + \lambda x'^2} + \alpha(t)\frac{\lambda x'v'^2}{1 + \lambda x'^2}. \end{cases}$$

Suppose that we fix $\dot{\alpha} - F(t)\alpha = 0$. Hence, the latter becomes

$$\begin{cases} \dot{x}' = \alpha(t)v', \\ \dot{v}' = -\frac{\omega(t)}{\alpha(t)}\frac{x'}{1 + \lambda x'^2} + \alpha(t)\frac{\lambda x'v'^2}{1 + \lambda x'^2}. \end{cases}$$

Let us try to search conditions for ensuring the above system to determine the integral curves for a t -dependent vector field of the form $X(t, x) = f(t)\bar{X}(x)$ with $\bar{X} \in V$, e.g.

$$\begin{cases} \dot{x}' = f(t)v', \\ \dot{v}' = f(t)\left(\frac{x'}{1 + \lambda x'^2} + \frac{\lambda x'v'^2}{1 + \lambda x'^2}\right). \end{cases}$$

In such a case, $\alpha(t) = f(t)$, $\omega(t) = -\alpha^2(t)$ and therefore $\dot{\omega}(t) = -\exp(2\int F(t)dt)$. The t -reparametrisation $d\tau = f(t)dt$ transforms the previous system into the autonomous one

$$\begin{cases} \frac{dx'}{d\tau} = v', \\ \frac{dv'}{d\tau} = \frac{x'}{1 + \lambda x'^2} + \frac{\lambda x'v'^2}{1 + \lambda x'^2}. \end{cases}$$

determining the integral curves for the vector field $X = X_1 + X_2$ and related to a Mathews–Lakshmanan oscillator (8.20) with $k = 1$. The method of characteristics shows, after brief calculation, that this system has a first-integral

$$I(x', v') = \frac{1 + \lambda x'^2}{1 + \lambda v'^2},$$

that reads in terms of the initial variables and the variable t as a t -dependent constant of the motion

$$I(t, x, v) = \frac{\alpha^2(t) + \lambda\alpha^2(t)x^2}{\alpha^2(t) + \lambda v^2},$$

for the t -dependent dissipative Mathews–Lakshmanan oscillator (8.19) getting a, as far as we know, new t -dependent constant of the motion.

8.4. The Emden equation. In this and following sections we analyse, from the perspective of the theory of quasi-Lie schemes, the so-called Emden equations of the form

$$\ddot{x} = a(t)\dot{x} + b(t)x^n, \quad n \neq 1. \quad (8.23)$$

These equations can be associated with the system of first-order differential equations

$$\begin{cases} \dot{x} = v, \\ \dot{v} = a(t)v + b(t)x^n. \end{cases} \quad (8.24)$$

This system was already studied in [34, 42] by means of quasi-Lie schemes. We hereafter summarise some of the results of these papers, which concern the determination of t -dependent

constants of the motion by means of particular solutions, reducible particular cases of Emden equations, etc.

Consider the real vector space, V_{Emd} , spanned by the vector fields

$$X_1 = x \frac{\partial}{\partial v}, \quad X_2 = x^n \frac{\partial}{\partial v}, \quad X_3 = v \frac{\partial}{\partial x}, \quad X_4 = v \frac{\partial}{\partial v}, \quad X_5 = x \frac{\partial}{\partial x}.$$

The t -dependent vector field determining the dynamics of system (8.24) can be written as a linear combination

$$X_t = a(t)X_4 + X_3 + b(t)X_2.$$

Moreover, the linear space $W_{\text{Emd}} \subset V_{\text{Emd}}$ spanned by the complete vector fields,

$$Y_1 = X_4 = v \frac{\partial}{\partial v}, \quad Y_2 = X_1 = x \frac{\partial}{\partial v}, \quad Y_3 = X_5 = x \frac{\partial}{\partial x},$$

is a three-dimensional real Lie algebra of vector fields with respect to the ordinary Lie Bracket because these vector fields satisfy the relations

$$[Y_1, Y_2]_{LB} = -Y_2, \quad [Y_1, Y_3]_{LB} = 0, \quad [Y_2, Y_3]_{LB} = -Y_2.$$

Also $[W_{\text{Emd}}, V_{\text{Emd}}]_{LB} \subset V_{\text{Emd}}$ because

$$\begin{aligned} [Y_1, X_2]_{LB} &= -X_2, & [Y_1, X_3]_{LB} &= X_3, & [Y_2, X_2]_{LB} &= 0, \\ [Y_2, X_3]_{LB} &= X_5 - X_4, & [Y_3, X_2]_{LB} &= nX_2, & [Y_3, X_3]_{LB} &= -X_3. \end{aligned}$$

So we get a quasi-Lie scheme $S(W_{\text{Emd}}, V_{\text{Emd}})$ which can be used to treat the Emden equations (8.24). This suggests that if we perform the t -dependent change of variables associated with this quasi-Lie scheme, namely,

$$\begin{cases} x &= \gamma(t)x', \\ v &= \beta(t)v' + \alpha(t)x', \end{cases} \quad \gamma(t)\beta(t) > 0, \forall t, \quad (8.25)$$

the original system transforms into

$$\begin{cases} \frac{dx'}{dt} = \left(\frac{\alpha(t)}{\gamma(t)} - \frac{\dot{\gamma}(t)}{\gamma(t)} \right) x' + \frac{\beta(t)}{\gamma(t)} v', \\ \frac{dv'}{dt} = \left(a(t) - \frac{\alpha(t)}{\gamma(t)} - \frac{\dot{\beta}(t)}{\beta(t)} \right) v' + \frac{\alpha(t)}{\beta(t)} \left(a(t) - \frac{\alpha(t)}{\gamma(t)} - \frac{\dot{\alpha}(t)}{\alpha(t)} + \frac{\dot{\gamma}(t)}{\gamma(t)} \right) x' \\ \quad + \frac{b(t)\gamma^n(t)}{\beta(t)} x'^n. \end{cases} \quad (8.26)$$

The key point of our method is to choose appropriate functions, α , β and γ , in such a way that the system of differential equations (8.26) becomes a Lie system. A possible way to do so, consists in choosing α , β and γ so that the above system becomes determined by a t -dependent vector field $X_t = f(t)\bar{X}$, where \bar{X} is a true vector field and $f(t)$ is a non-vanishing function (on the interval of t under study). As it is shown in next section, this cannot always be done and some conditions must be imposed on the initial t -dependent functions, α , β and γ , ensuring the existence of such a transformation. These restrictions lead to integrability conditions.

Suppose that, for the time being, this is the case. Therefore, the system (8.26) is

$$\begin{cases} \frac{dx'}{dt} = f(t)(c_{11}x' + c_{12}v'), \\ \frac{dv'}{dt} = f(t)(c_{22}x'^n + c_x x' + c_{21}v') \end{cases} \quad (8.27)$$

and it is determined by the t -dependent vector field

$$X_t = f(t)\bar{X},$$

with

$$\bar{X} = (c_{11}x' + c_{12}v')\frac{\partial}{\partial x'} + (c_{22}x'^n + c_x x' + c_{21}v')\frac{\partial}{\partial v'}.$$

Under the t -reparametrisation,

$$\tau = \int^t f(t')dt',$$

system (8.27) is autonomous. The new autonomous system of differential equations is determined by the vector field \bar{X} on $\mathbb{T}\mathbb{R}$ and therefore there exists a first integral. This can be obtained by means of the method of characteristics, which provides the characteristic curves where the first-integrals for such a vector field \bar{X} are constant. These characteristic curves are determined by

$$\frac{dx'}{c_{11}x' + c_{12}v'} = \frac{dv'}{c_{21}v' + c_x x' + c_{22}x'^n},$$

which can be written as

$$(c_{21}v' + c_x x' + c_{22}x'^n)dx' - (c_{11}x' + c_{12}v')dv' = 0. \quad (8.28)$$

This expression can be straightforwardly integrated if

$$\frac{\partial}{\partial v'}(c_{21}v' + c_x x' + c_{22}x'^n) = -\frac{\partial}{\partial x'}(c_{11}x' + c_{12}v') \implies c_{21} = -c_{11}. \quad (8.29)$$

Under this condition we obtain the integral of the motion for (8.28), namely

$$I = -c_{12}\frac{v'^2}{2} + c_x\frac{x'^2}{2} + c_{21}v'x' + c_{22}\frac{x'^{n+1}}{n+1}. \quad (8.30)$$

Finally, if we write the latter expression in terms of the initial variables x, v and t , we get a constant of the motion for the initial differential equation.

If we do not wish to impose condition (8.29), we can alternatively integrate equation (8.28) by means of an integrating factor, i.e. we look for a function, $\mu(x', v')$, such that

$$\frac{\partial}{\partial v'}(\mu(c_{21}v' + c_x x' + c_{22}x'^n)) = \frac{\partial}{\partial x'}(-\mu(c_{11}x' + c_{12}v')).$$

Thus the integrating factor satisfies the partial differential equation

$$\frac{\partial \mu}{\partial v'}(c_{21}v' + c_x x' + c_{22}x'^n) + \frac{\partial \mu}{\partial x'}(c_{11}x' + c_{12}v') = -\mu(c_{11} + c_{21}).$$

If $c_{11} + c_{21} = 0$, the integral factor can be chosen to be $\mu = 1$ and we get the latter first-integral (8.30). On the other hand, if $c_{11} + c_{21} \neq 0$, we can still look for a solution for the partial differential equation for μ and obtain a new first-integral.

8.5. t -dependent constants of the motion and particular solutions for Emden equations. The main purpose of this section is to show that the knowledge of a particular solution of the Emden equation allows us to transform it into a Lie system and to derive a t -dependent constant of the motion.

If we restrict ourselves to the case $\alpha(t) = 0$ in the system of differential equation (8.26), it reduces to

$$\begin{cases} \frac{dx'}{dt} = -\frac{\dot{\gamma}(t)}{\gamma(t)}x' + \frac{\beta(t)}{\gamma(t)}v', \\ \frac{dv'}{dt} = \left(a(t) - \frac{\dot{\beta}(t)}{\beta(t)}\right)v' + \frac{b(t)\gamma^n(t)}{\beta(t)}x'^n. \end{cases} \quad (8.31)$$

In order to transform the original Emden–Fowler differential equation into a Lie system by means of our quasi-Lie scheme, we try to write the transformed differential equation in the form

$$\begin{cases} \frac{dx'}{dt} = f(t)(c_{11}x' + c_{12}v'), \\ \frac{dv'}{dt} = f(t)(c_{22}x'^n + c_{21}v'), \end{cases} \quad (8.32)$$

where the c_{ij} are constants. This system of differential equations can be reduced to an autonomous one as, under the t -dependent change of variables,

$$\tau = \int^t f(t')dt',$$

the latter differential equation becomes

$$\begin{cases} \frac{dx'}{d\tau} = c_{11}x' + c_{12}v', \\ \frac{dv'}{d\tau} = c_{22}x'^n + c_{21}v'. \end{cases} \quad (8.33)$$

In order for system (8.31) to be similar to system (8.32), we look for functions α , β and γ satisfying the conditions,

$$\begin{cases} f(t)c_{11} = -\frac{\dot{\gamma}(t)}{\gamma(t)}, & f(t)c_{12} = \frac{\beta(t)}{\gamma(t)}, \\ f(t)c_{22} = b(t)\frac{\gamma^n(t)}{\beta(t)}, & f(t)c_{21} = a(t) - \frac{\dot{\beta}(t)}{\beta(t)}. \end{cases} \quad (8.34)$$

The conditions in the first line lead to

$$\beta(t) = -\frac{c_{12}}{c_{11}}\dot{\gamma}(t), \quad (8.35)$$

and using this equation in the last relation we obtain

$$f(t) = \frac{a(t)}{c_{21}} - \frac{1}{c_{21}}\frac{\ddot{\gamma}(t)}{\dot{\gamma}(t)}. \quad (8.36)$$

On the other hand from the three first relations in (8.34) we get

$$f(t) = -\frac{b(t)c_{11}}{c_{22}c_{12}}\frac{\gamma^n(t)}{\dot{\gamma}(t)}. \quad (8.37)$$

The equality of the right-hand sides of (8.36) and (8.37) leads to the following equation for the function γ :

$$\ddot{\gamma} = a(t)\dot{\gamma} + \frac{c_{11}c_{21}}{c_{22}c_{12}}b(t)\gamma^n.$$

Suppose that we make the choice, with $c_{21} = -c_{11}$ as indicated in (8.29),

$$c_{22} = -1, \quad c_{11} = 1, \quad c_{21} = -1, \quad c_{12} = 1 \tag{8.38}$$

and thus $(c_{11}c_{22})/(c_{21}c_{12}) = 1$. Therefore we find that γ must be a solution of the initial equation (8.23). In other words, if we suppose that a particular solution $x_p(t)$ of the Emden equation is known, we can choose $\gamma(t) = x_p(t)$. Then, according to the expression (8.35) and our previous choice (8.38), the corresponding function β turns out to be

$$\beta(t) = -\dot{x}_p(t).$$

Finally, in view of conditions (8.34), we get that

$$\frac{-\dot{\gamma}(t)}{c_{11}\gamma(t)} = b(t)\frac{\gamma^n(t)}{c_{22}\beta(t)}$$

and taking into account our choice (8.38) and $\gamma(t) = x_p(t)$, we obtain the condition satisfied by the particular solution:

$$x_p^{n+1}(t) = \dot{x}_p^2(t). \tag{8.39}$$

The system of differential equations (8.32) for such a choice (8.38) of the constants $\{c_{ij} \mid i, j = 1, 2\}$ is the equation for the integrals curves for the t -dependent vector field

$$X_t = f(t) \left((x' + v') \frac{\partial}{\partial x'} - (v' + x'^n) \frac{\partial}{\partial v'} \right).$$

The method of the characteristics can be used to find the following first-integral for this vector field and, in view of (8.30), we get

$$\begin{cases} I(x', v') = \frac{1}{n+1}x'^{n+1} + \frac{1}{2}v'^2 + x'v', & n \notin \{-1, 1\}, \\ I(x', v') = \log x' + \frac{1}{2}v'^2 + x'v', & n = -1, \end{cases}$$

and, if we express this integral of motion in terms of the initial variables and t , we obtain a, as far as we know, new t -dependent constant of the motion for the initial Emden equation

$$\begin{cases} I(t, x, v) = \frac{x^{n+1}}{(n+1)x_p^{n+1}(t)} + \frac{v^2}{2\dot{x}_p^2(t)} - \frac{xv}{x_p(t)\dot{x}_p(t)}, & n \notin \{-1, 1\}, \\ I(t, x, v) = \log \left(\frac{x}{x_p(t)} \right) + \frac{v^2}{2\dot{x}_p^2(t)} - \frac{xv}{x_p(t)\dot{x}_p(t)}, & n = -1. \end{cases} \tag{8.40}$$

So, the knowledge of a particular solution for the Emden equation enables us first to obtain a constant of the motion and then to reduce the initial Emden equation into a Lie system. Thus, all Emden equations are quasi-Lie systems with respect to the above mentioned scheme.

8.6. Applications of particular solutions to study Emden equations. This section is devoted to illustrating the usefulness of the previous theory about Emden equations. More specifically, we detail several Emden equations for which one is able to find a particular solution satisfying an integrability condition and use is made of such a solution in order to derive t -dependent constants of the motion. In this way we recover several results appearing in the literature about Emden–Fowler equations from a unified point of view [42].

We start with a particular case of the Lane-Emden equation

$$\ddot{x} = -\frac{2}{t}\dot{x} - x^5. \quad (8.41)$$

The more general Lane-Emden equation is generally written as

$$\ddot{x} = -\frac{2}{t}\dot{x} + f(x)$$

and the example here considered corresponds to $f(x) = -x^n$, $n \neq 1$, which is one of the most interesting cases, together with that of $f(x) = -e^{-\beta x}$. Equation (8.41) appears in the study of the thermal behaviour of a spherical cloud of gas [135] and also in astrophysical applications. A particular solution for (8.41) satisfying (8.39) is $x_p(t) = (2t)^{-1/2}$. If we substitute this expression for $x_p(t)$ and the corresponding one for $\dot{x}_p(t)$ into the t -dependent constant of the motion (8.40), we get that

$$I'(t, x, v) = \frac{4t^3x^6}{3} + 4t^3v^2 + 4t^2xv$$

is a t -dependent constant of the motion proportional to (8.40) and also proportional to the t -dependent constants of the motion found in [11, 34, 158].

We study from this new perspective other Emden equations investigated in [145]. Consider the particular instance

$$\ddot{x} = -\frac{5}{t+K}\dot{x} - x^2.$$

A particular solution for this Emden equation satisfying (8.39) is

$$x_p(t) = \frac{4}{(t+K)^2}.$$

In this case a t -dependent constant of the motion is

$$I'(t, x, v) = \frac{1}{3}x^3(t+K)^6 + \frac{1}{2}v^2(t+K)^6 + 2xv(t+K)^5,$$

which is proportional to the one found by Leach in [145].

Now another Emden equation found in [145],

$$\ddot{x} = -\frac{3}{2(t+K)}\dot{x} - x^9,$$

admits the particular solution

$$x_p(t) = \frac{1}{\sqrt{2}(t+K)^{1/4}},$$

which satisfies (8.39). The corresponding t -dependent constant of the motion is given by

$$I'(t, x, v) = (K+t)^{3/2}(10(K+t)v^2 + 5vx + 2(K+t)x^{10})$$

which is proportional to that given in [145].

Let us turn now to consider the Emden equation

$$\ddot{x} = -\frac{5}{3(t+K)}\dot{x} - x^7,$$

which admits a particular solution of the form

$$x_p(t) = \frac{1}{3^{1/3}(t+K)^{1/3}},$$

which obeys (8.39) and leads to the t -dependent constant of the motion

$$I'(t, x, v) = (K + t)^{5/3}(12(K + t)v^2 + 8vx + 3x^8(K + t)).$$

Finally we apply our development to obtain a t -dependent constant of the motion for the Emden equation

$$\ddot{x} = -\frac{1}{K_1 + K_3 t} \dot{x} - x^n \tag{8.42}$$

with

$$K_3 = \frac{n-1}{n+3}.$$

We can find a particular solution of the form

$$x_p(t) = \frac{K_2}{(K_1 + K_3 t)^\nu}, \quad \nu \neq 0.$$

In order for $x_p(t)$ to be a particular solution we must have the following relation

$$\frac{(\nu + 1)\nu K_2 K_3^2}{(K_1 + K_3 t)^{\nu+2}} = \frac{\nu K_2 K_3}{(K_1 + K_3 t)^{\nu+2}} - \frac{K_2^n}{(K_1 + K_3 t)^{n\nu}}$$

and thus

$$\nu + 2 = n\nu \quad \text{and} \quad \nu(\nu + 1)K_3^2 K_2 = \nu K_2 K_3 - K_2^n.$$

From these equations we get

$$\nu = \frac{2}{n-1}, \quad K_2^{n-1} = \frac{2^2}{(n+3)^2}.$$

Under these conditions it can be easily verified that $\dot{x}_p^2(t) = x_p^{n+1}(t)$. Thus, a t -dependent constant of the motion is

$$I'(t, x, v) = (K_1 + K_3 t)^{2(n+1)/(n-1)} \left(\frac{x^{n+1}}{n+1} + \frac{v^2}{2} \right) + (K_1 + K_3 t)^{(n+3)/(n-1)} \frac{2vx}{n+3}, \tag{8.43}$$

which can also be found in [145].

Another advantage of our method is that it allows us to obtain Emden equations admitting a previously fixed t -dependent constant of the motion.

Suppose that we want to construct an Emden equation admitting a previously chosen particular solution, $x_p(t)$, satisfying $\dot{x}_p^2(t) = x_p^{n+1}(t)$ for certain $n \in \mathbb{Z} - \{1, -1\}$. We can integrate this equation to get all possible particular solutions which can be used by means of our method, i.e.

$$x_p(t) = \left(K + \frac{1-n}{2} t \right)^{-\frac{2}{n-1}}.$$

We consider functions $a(t)$ and $b(t)$ such that

$$\ddot{x}_p = a(t)\dot{x}_p + b(t)x_p^n.$$

For the sake of simplicity, we can assume that $b(t) = -1$. Then we get

$$a(t) = \frac{\ddot{x}_p + x_p^n}{\dot{x}_p}.$$

If we substitute the chosen particular solution in the above expression, we obtain

$$a(t) = \frac{3+n}{2(K + \frac{1-n}{2}t)}.$$

which leads to an Emden equation equivalent to (8.42) and the t -dependent constant of the motion for this equation is again (8.43). In this way we recover the cases studied in this section.

8.7. The Kummer-Liouville transformation for a general Emden-Fowler equation. As far as we know, the most general form of the Emden–Fowler equation considered nowadays is

$$\ddot{x} + p(t)\dot{x} + q(t)x = r(t)x^n. \quad (8.44)$$

This generalisation arises naturally as a consequence of our scheme. Indeed, the above second-order differential equation is associated with the system of first-order differential equations

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -p(t)v - q(t)x + r(t)x^n, \end{cases} \quad (8.45)$$

which determines the integral curves for the t -dependent vector field

$$X_t = -p(t)X_4 - q(t)X_1 + r(t)X_2 + X_3.$$

This t -dependent vector field is a generalisation of the one studied in previous sections. Under the set of transformations (8.25), the initial system (8.45) becomes the new system

$$\begin{cases} \frac{dx'}{dt} = \left(\frac{\alpha(t)}{\gamma(t)} - \frac{\dot{\gamma}(t)}{\gamma(t)} \right) x' + \frac{\beta(t)}{\gamma(t)} v', \\ \frac{dv'}{dt} = \left(-p(t) - \frac{\alpha(t)}{\gamma(t)} - \frac{\dot{\beta}(t)}{\beta(t)} \right) v' + \frac{\alpha(t)}{\beta(t)} \left(-p(t) - \frac{\alpha(t)}{\gamma(t)} - \frac{\dot{\alpha}(t)}{\alpha(t)} + \right. \\ \left. + \frac{\dot{\gamma}(t)}{\gamma(t)} - q(t) \frac{\gamma(t)}{\alpha(t)} \right) x' + \frac{r(t)\gamma^n(t)}{\beta(t)} x'^n. \end{cases}$$

If we choose $\alpha = \dot{\gamma}$, the system reduces to

$$\begin{cases} \frac{dx'}{dt} = \frac{\beta(t)}{\gamma(t)} v', \\ \frac{dv'}{dt} = \left(-p(t) - \frac{\dot{\gamma}(t)}{\gamma(t)} - \frac{\dot{\beta}(t)}{\beta(t)} \right) v' + \frac{\dot{\gamma}(t)}{\beta(t)} \left(-p(t) - \frac{\dot{\gamma}(t)}{\gamma(t)} - q(t) \frac{\gamma(t)}{\dot{\gamma}(t)} \right) x' \\ + \frac{r(t)\gamma^n(t)}{\beta(t)} x'^n. \end{cases}$$

When the function $\gamma(t)$ is chosen in such a way that $\ddot{\gamma} = -q(t)\gamma - p(t)\dot{\gamma}$, i.e. γ is a solution of the associated linear equation, we obtain

$$\begin{cases} \frac{dx'}{dt} = \frac{\beta(t)}{\gamma(t)} v', \\ \frac{dv'}{dt} = \left(-p(t) - \frac{\dot{\gamma}(t)}{\gamma(t)} - \frac{\dot{\beta}(t)}{\beta(t)} \right) v' + \frac{r(t)\gamma^n(t)}{\beta(t)} x'^n. \end{cases} \quad (8.46)$$

Finally, if the function $\beta(t)$ is such that

$$-p(t) - \frac{\dot{\gamma}(t)}{\gamma(t)} - \frac{\dot{\beta}(t)}{\beta(t)} = 0,$$

we obtain

$$\begin{cases} \frac{dx'}{dt} = \frac{\beta(t)}{\gamma(t)}v', \\ \frac{dv'}{dt} = \frac{r(t)\gamma^n(t)}{\beta(t)}x'^n, \end{cases} \quad (8.47)$$

which is related to the second-order differential equation

$$\frac{\delta^2 x'}{d\tau^2} = r(t) \frac{\gamma^{n+1}(t)}{\beta^2(t)} x'^n,$$

with

$$\tau(t) = \int^t \frac{\beta(t')}{\gamma(t')} dt'.$$

The new form of the differential equation is called the canonical form of the generalised Emden–Fowler equation.

This fact is obtained by means of an appropriate Kummer–Liouville transformation in the previous literature, but we obtain it here as a straightforward application of the properties of transformation of quasi-Lie schemes thereby underscoring the theoretical explanation of such a Kummer–Liouville transformation.

8.8. Constants of the motion for sets of Emden-Fowler equations. In this section we show that under certain assumptions on the t -dependent coefficients $a(t)$ and $b(t)$ the original Emden equation can be reduced to a Lie system and then we can obtain a first-integral which provides us with a t -dependent constant of the motion for the original system.

In fact consider the system of first-order differential equations

$$\begin{cases} \frac{dx'}{dt} = \left(\frac{\alpha(t)}{\gamma(t)} - \frac{\dot{\gamma}(t)}{\gamma(t)} \right) x' + \frac{\beta(t)}{\gamma(t)} v', \\ \frac{dv'}{dt} = \left(a(t) - \frac{\alpha(t)}{\gamma(t)} - \frac{\dot{\beta}(t)}{\beta(t)} \right) v' + \frac{\alpha(t)}{\beta(t)} \left(a(t) - \frac{\alpha(t)}{\gamma(t)} - \frac{\dot{\alpha}(t)}{\alpha(t)} + \frac{\dot{\gamma}(t)}{\gamma(t)} \right) x' \\ \quad + \frac{b(t)\gamma^n(t)}{\beta(t)} x'^n. \end{cases}$$

This system describes all the systems of differential equations that can be obtained by means of the set of t -dependent transformations we got through the scheme $S(W_{Emd}, V_{Emd})$. We recall that the t -dependent change of variable which we use to relate the Emden equation (8.24) with the latter system of differential equation is

$$\begin{cases} x = \gamma(t)x', \\ v = \beta(t)v' + \alpha(t)x'. \end{cases}$$

As in previous papers on this topic, we try to relate the latter system of differential equations to a Lie system determined by a t -dependent vector field of the form $X'(t, x) = f(t)\bar{X}(x)$ and we suppose $f(t)$ to be non-vanishing in the interval we study. So the system of differential equations determining the integrals curves for this t -dependent vector field is a Lie system and we can use the theory of Lie systems to analyse its properties.

As a first example we can consider that we just use the set of transformations with $\gamma(t) = 1$ and $\alpha(t) = 0$. In this case system (8.25) is

$$\begin{cases} \frac{dx'}{dt} = \beta(t)v' \\ \frac{dv'}{dt} = \left(a(t) - \frac{\dot{\beta}(t)}{\beta(t)} \right) v' + \frac{b(t)}{\beta(t)} x'^n. \end{cases}$$

We fix $\beta(t)$ to be such that

$$a(t) - \frac{\dot{\beta}(t)}{\beta(t)} = 0,$$

i.e. $\beta(t)$ is (proportional to)

$$\beta(t) = \exp \left(\int^t a(t') dt' \right).$$

Therefore we get

$$\begin{cases} \frac{dx'}{dt} = \exp \left(\int^t a(t') dt' \right) v', \\ \frac{dv'}{dt} = b(t) \exp \left(- \int^t a(t') dt' \right) x'^n. \end{cases}$$

In order to get the last system of differential equations to describe the integral curves for a t -dependent vector field, $X'(t, x) = f(t)\bar{X}(x)$, for a given function $a(t)$ a necessary and sufficient condition is

$$b(t) \exp \left(-2 \int^t a(t') dt' \right) = K,$$

with K being a real constant. Under this assumption the last system becomes

$$\begin{cases} \frac{dx'}{dt} = \exp \left(\int^t a(t') dt' \right) v', \\ \frac{dv'}{dt} = \exp \left(\int^t a(t') dt' \right) K x'^n. \end{cases}$$

We introduce the t -reparametrisation

$$\tau(t) = \int^t \exp \left(\int^{t'} a(t'') dt'' \right) dt'$$

and the latter system becomes

$$\begin{cases} \frac{dx'}{d\tau} = v', \\ \frac{dv'}{d\tau} = K x'^n, \end{cases}$$

which admits a first-integral

$$I = \frac{1}{2} v'^2 - K \frac{x'^{n+1}}{n+1}.$$

In terms of the initial variables, the corresponding t -dependent constant of the motion is

$$I = \exp \left(-2 \int^t a(t') dt' \right) \left(\frac{1}{2} \dot{y}^2 - b(t) \frac{x^{n+1}}{n+1} \right),$$

which is similar to that found in [16].

Suppose that we restrict the transformations (8.25) to the case $\alpha(t) = 0$. In this case the system of first-order differential equations (8.26) becomes

$$\begin{cases} \frac{dx'}{dt} = -\frac{\dot{\gamma}(t)}{\gamma(t)}x' + \frac{\beta(t)}{\gamma(t)}v', \\ \frac{dv'}{dt} = \left(a(t) - \frac{\dot{\beta}(t)}{\beta(t)}\right)v' + \frac{b(t)\gamma^n(t)}{\beta(t)}x'^n. \end{cases}$$

In order for this system of differential equations to determine the integral curves for a t -dependent vector field of the form $X'(t, x) = f(t)\bar{X}(x)$ we need that

$$\begin{cases} c_{11}f(t) = -\frac{\dot{\gamma}(t)}{\gamma(t)}, & c_{12}f(t) = \frac{\beta(t)}{\gamma(t)}, \\ c_{21}f(t) = a(t) - \frac{\dot{\beta}(t)}{\beta(t)}, & c_{22}f(t) = \frac{b(t)\gamma^n(t)}{\beta(t)}. \end{cases} \quad (8.48)$$

From these relations, or more exactly from those of the first row, we get $f(t)$ as

$$f(t) = -\frac{1}{c_{11}}\frac{\dot{\gamma}(t)}{\gamma(t)} = \frac{1}{c_{12}}\frac{\beta(t)}{\gamma(t)}$$

and therefore

$$\dot{\gamma}(t) = -\frac{c_{11}}{c_{12}}\beta(t).$$

We choose $c_{11} = -1$ and $c_{12} = 1$ so that

$$\beta(t) = \dot{\gamma}(t). \quad (8.49)$$

In view of this and using the third and second relations from (8.48) we get

$$\frac{c_{21}}{c_{12}}\frac{\beta(t)}{\gamma(t)} = a(t) - \frac{\dot{\beta}(t)}{\beta(t)}$$

and thus, as a consequence of (8.49), the last differential equation becomes

$$\frac{c_{21}}{c_{12}}\frac{\dot{\gamma}(t)}{\gamma(t)} = a(t) - \frac{\ddot{\gamma}(t)}{\dot{\gamma}(t)}$$

and, as $c_{12} = 1$ and fixing $c_{21} = 1$, we obtain

$$\frac{d}{dt}\log(\dot{\gamma}) = a(t),$$

which can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\gamma^2(t) = \exp\left(\int^t a(t')dt'\right).$$

Hence we have

$$\gamma(t) = \sqrt{2\int^t \exp\left(\int^{t'} a(t'')dt''\right) dt'}$$

and in view of (8.49)

$$\beta(t) = \frac{1}{\sqrt{2\int^t \exp\left(\int^{t'} a(t'')dt''\right) dt'}} \exp\left(\int^t a(t')dt'\right).$$

So far we have only used three of the four relations we found. The fourth and second relations lead to the integrability condition: there exist a constant $c_{22} = K$ such that

$$K \frac{\beta(t)}{\gamma(t)} = \frac{b(t)\gamma^n(t)}{\beta(t)}.$$

Therefore, using the above expressions for $\gamma(t)$ and $\beta(t)$, we get

$$b(t) \exp\left(-2 \int^t a(t') dt'\right) \left(2 \int^t \exp\left(\int^{t'} a(t'') dt''\right)\right)^{(n+3)/2} = K. \quad (8.50)$$

So under this assumption we have connected the initial Emden equation with the Lie system,

$$\begin{cases} \frac{dx'}{dt} = f(t)(-x' + v'), \\ \frac{dv'}{dt} = f(t)(v' + Kx'^n), \end{cases}$$

and then the method of characteristics shows that it admits the first-integral

$$I' = -\frac{1}{2}v'^2 + \frac{K}{n+1}x'^{n+1} + v'x'.$$

In terms of the initial variables the corresponding constant of the motion is

$$I = \left(\frac{1}{2}\dot{x}^2 - \frac{b(t)}{n+1}x^{n+1}\right) \exp\left(-2 \int^t a(t') dt'\right) \int^t \exp\left(\int^{t'} a(t'') dt''\right) dt' - \frac{1}{2}x\dot{x} \exp\left(-\int^t a(t') dt'\right) \quad (8.51)$$

and in this way we recover the result found in [16]. If we now consider the particular case $n = -3$ we get that the integrability condition (8.50) implies that there is a constant K such that

$$b(t) \exp\left(-2 \int^t a(t') dt'\right) = K,$$

and the corresponding t -dependent constant of the motion is then given by

$$I = \left(\frac{1}{2}\dot{x}^2 + \frac{b(t)}{2}x^{-2}\right) \exp\left(-2 \int^t a(t') dt'\right) \int^t \exp\left(\int^{t'} a(t'') dt''\right) dt' - \frac{1}{2}x\dot{x} \exp\left(-\int^t a(t') dt'\right),$$

which is equivalent to that one found in [16].

8.9. A t -dependent superposition rule for Abel equations. Let us now turn to illustrate the results of our theory of Lie families by deriving a common t -dependent superposition rule for a Lie family of Abel equations, whose elements do not admit a standard superposition rule except for a few particular instances. In this way, we single out that our theory provides new tools for investigating solutions of nonautonomous systems of differential equations than cannot be investigated by means of the theory of Lie systems.

With this aim, we analyse the so-called Abel equations of the first-type [24, 74], i.e. the differential equations of the form

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3, \quad (8.52)$$

with $a_3(t) \neq 0$. Abel equations appear in the analysis of several cosmological models [73, 111, 148] and other different fields in Physics [70, 84, 91, 92, 177, 240]. Additionally, the study of integrability conditions for Abel equations is a research topic of current interest in Mathematics and multiple studies have been carried out in order to analyse the properties of the solutions of these equations [5, 69, 74, 75, 215].

Note that, apart from its inherent mathematical interest, the knowledge of particular solutions of Abel equations allows us to study the properties of those physical systems that such equations describe. Thus, the expressions enabling us to easily obtain new solutions of Abel equations by means of several particular ones, like common t -dependent superposition rules, are interesting to study the solutions of these equations and, therefore, their related physical systems.

Unfortunately, all the expressions describing the general solution of Abel equations presently known can only be applied to study autonomous instances and, moreover, they depend on families of particular conditions satisfying certain extra conditions, see [75, 215]. Taking this into account, common t -dependent superposition rules represent an improvement with respect to these previous expressions, as they enable us to treat nonautonomous Abel equations and they do not require the usage of particular solutions obeying additional conditions.

Recall that, according to Theorem 7.19, the existence of a common t -dependent superposition rule for a family of t -dependent vector fields $\{Y_d\}_{d \in \Lambda}$ requires the existence of a system of generators, i.e. a certain set of t -dependent vector fields, X_1, \dots, X_r , satisfying relations (7.14). Conversely, given such a set, the family of t -dependent vector fields Y whose autonomisations can be written in the form

$$\bar{Y}_c(t, x) = \sum_{j=1}^r b_{cj}(t) \bar{X}_j(t, x), \quad \sum_{j=1}^r b_{cj}(t) = 1,$$

admits a common t -dependent superposition rule and becomes a Lie family.

Consequently, a Lie family of Abel equations can be determined, for instance, by finding two t -dependent vector fields of the form

$$\begin{aligned} X_1(t, x) &= (b_0(t) + b_1(t)x + b_2(t)x^2 + b_3(t)x^3) \frac{\partial}{\partial x}, \\ X_2(t, x) &= (b'_0(t) + b'_1(t)x + b'_2(t)x^2 + b'_3(t)x^3) \frac{\partial}{\partial x}, \quad b'_3(t) \neq 0, \end{aligned} \quad (8.53)$$

such that

$$[\bar{X}_1, \bar{X}_2] = 2(\bar{X}_2 - \bar{X}_1). \quad (8.54)$$

Let us analyse the existence of such two t -dependent vector fields X_1 and X_2 with commutation relations (8.54). In coordinates, the Lie bracket $[\bar{X}_1, \bar{X}_2]$ reads

$$\begin{aligned} &[(b'_3b_2 - b'_2b_3)x^4 + (2(b'_3b_1 - b_3b'_1) - \dot{b}_3 + \dot{b}'_3)x^3 + (-3(b'_0b_3 - b_0b'_3) + (b'_2b_1 - b_2b'_1) \\ &\quad - \dot{b}_2 + \dot{b}'_2)x^2 + (-2b'_0b_2 + 2b_0b'_2 - \dot{b}_1 + \dot{b}'_1)x - b'_0b_1 + b_0b'_1 - \dot{b}_0 + \dot{b}'_0] \frac{\partial}{\partial x}. \end{aligned}$$

Hence, in order to satisfy condition (8.54), $b'_3 b_2 - b'_2 b_3 = 0$, e.g. we may fix $b_2 = b_3 = 0$. Additionally, for the sake of simplicity, we assume $b'_3 = 1$. In this case, the previous expression takes the form

$$[2b_1 x^3 + (3b_0 + b'_2 b_1 + \dot{b}'_2) x^2 + (2b_0 b'_2 - \dot{b}_1 + \dot{b}'_1) x - b'_0 b_1 + b_0 b'_1 - \dot{b}_0 + \dot{b}'_0] \frac{\partial}{\partial x},$$

and, taking into account the values chosen for b_2, b_3 and b'_3 , assumption (8.54) yields $b_1 = 1$ and

$$\begin{cases} b'_2 = 3b_0 + \dot{b}'_2, \\ 2(b'_1 - 1) = 2b_0 b'_2 + \dot{b}'_1, \\ 2(b'_0 - b_0) = -b'_0 + b_0 b'_1 - \dot{b}_0 + \dot{b}'_0. \end{cases}$$

As this system has more variables than equations, we can try to fix some values of the variables in order to simplify it and obtain a particular solution. In this way, taking $b_0(t) = t$, the above system reads

$$\begin{cases} \dot{b}'_2 = b'_2 - 3t, \\ \dot{b}'_1 = 2(b'_1 - 1) - 2tb'_2, \\ \dot{b}'_0 = 2(b'_0 - t) + b'_0 - tb'_1 + 1. \end{cases}$$

This system is integrable by quadratures and one can check that it admits the particular solution

$$b'_2(t) = 3(1 + t), \quad b'_1(t) = 3(1 + t)^2 + 1, \quad b'_0(t) = (1 + t)^3 + t.$$

Summing up, we have proved that the t -dependent vector fields

$$\begin{cases} X_1(t, x) = (t + x) \frac{\partial}{\partial x}, \\ X_2(t, x) = ((1 + t)^3 + t + (3(1 + t)^2 + 1)x + 3(1 + t)x^2 + x^3) \frac{\partial}{\partial x}, \end{cases} \tag{8.55}$$

satisfy (8.54) and, therefore, the family of t -dependent vector fields

$$Y_{b(t)}(t, x) = (1 - b(t))X_1(x) + b(t)X_2(x)$$

is a Lie family. The corresponding family of Abel equations is

$$\frac{dx}{dt} = (t + x) + b(t)(1 + t + x)^3. \tag{8.56}$$

According to the results proved in Section 1.5, in order to determine a common t -dependent superposition rule for the above Lie family, we have to determine a first-integral for the vector fields of the distribution \mathcal{D} spanned by the t -prolongations \tilde{X}_1 and \tilde{X}_2 on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ for a certain m so that the t -prolongations of X_1 and X_2 to $\mathbb{R} \times \mathbb{R}^{nm}$ are linearly independent at a generic point. Taking into account expressions (8.55), the prolongations of the vector fields X_1 and X_2 to $\mathbb{R} \times \mathbb{R}^2$ are linearly independent at a generic point and, in view of (8.54), the t -prolongations \tilde{X}_1 and \tilde{X}_2 to $\mathbb{R} \times \mathbb{R}^3$ span an involutive generalised distribution \mathcal{D} with two-dimensional leaves in a dense subset of $\mathbb{R} \times \mathbb{R}^3$. Finally, a first-integral for the vector fields in the distribution \mathcal{D} will provide us a common t -dependent superposition rule for the Lie family (8.56).

Since, in view of (8.54), the vector fields \tilde{X}_1 and \tilde{X}_2 span the distribution \mathcal{D} , a function $G : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a first-integral of the vector fields of the distribution \mathcal{D} if and only if G is a first-integral of \tilde{X}_1 and $\tilde{X}_1 - \tilde{X}_2$, i.e. $\tilde{X}_1 G = (\tilde{X}_2 - \tilde{X}_1)G = 0$.

The condition $\tilde{X}_1 G = 0$ reads

$$\frac{\partial G}{\partial t} + (t + x_0) \frac{\partial G}{\partial x_0} + (t + x_1) \frac{\partial G}{\partial x_1} = 0,$$

and, using the method of characteristics [129], we note that the curves on which G is constant, the so-called *characteristics*, are solutions of the system

$$dt = \frac{dx_0}{t + x_0} = \frac{dx_1}{t + x_1} \Rightarrow \frac{dx_i}{dt} = t + x_i, \quad i = 0, 1,$$

which read $x_i(t) = \xi_i e^t - t - 1$, with $i = 0, 1$ and $\xi_0, \xi_1 \in \mathbb{R}$. Furthermore, these solutions are determined by the implicit equations $\xi_0 = e^{-t}(x_0 + t + 1)$ and $\xi_1 = e^{-t}(x_1 + t + 1)$. Therefore, there exists a function $G_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $G(t, x_0, x_1) = G_2(\xi_0, \xi_1)$. In other words, each first-integral G of \tilde{X}_1 depends only on ξ_0 and ξ_1 .

Taking into account the previous fact, we look for simultaneous first-integrals of the vector field $\tilde{X}_2 - \tilde{X}_1$ and \tilde{X}_1 , that is, for solutions of the equation $(\tilde{X}_2 - \tilde{X}_1)G = 0$ with G depending on ξ_0 and ξ_1 . Using the expression of $\tilde{X}_2 - \tilde{X}_1$ in the system of coordinates $\{t, \xi_0, \xi_1\}$, we get that

$$(\tilde{X}_2 - \tilde{X}_1)G = \xi_0^3 \frac{\partial G_2}{\partial \xi_0} + \xi_1^3 \frac{\partial G_2}{\partial \xi_1} = 0,$$

and, applying again the method of characteristics, we obtain that there exists a function $G_3 : \mathbb{R} \rightarrow \mathbb{R}$ such that $G(t, x_0, x_1) = G_2(\xi_0, \xi_1) = G_3(\Delta)$, where $\Delta = e^{2t}((x_0 + t + 1)^{-2} - (x_1 + t + 1)^{-2})$. Finally, using this first-integral, we get that the common t -dependent superposition rule for the Lie family (8.56) reads

$$k = e^{2t}((x_0 + t + 1)^{-2} - (x_1 + t + 1)^{-2}),$$

with k being a real constant. Therefore, given any particular solution $x_1(t)$ of a particular instance of the family of first-order Abel equations (8.58), the general solution, $x(t)$, of this instance is

$$x(t) = ((x_1(t) + t + 1)^{-2} + k e^{-2t})^{-1/2} - t - 1.$$

Note that our previous procedure can be straightforwardly generalised to derive common t -dependent superposition rules for generalised Abel equations [166], i.e. the differential equations of the form

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 + \dots + a_n(t)x^n, \quad n \geq 3.$$

Actually, their study can be approached by analysing the existence of two vector fields of the form

$$\begin{aligned} Y_1(t, x) &= (b_0(t) + b_1(t)x + \dots + b_n(t)x^n) \frac{\partial}{\partial x}, \\ Y_2(t, x) &= (b'_0(t) + b'_1(t)x + \dots + b'_n(t)x^n) \frac{\partial}{\partial x}, \quad b'_n(t) \neq 0, \end{aligned}$$

obeying the relation $[\bar{Y}_1, \bar{Y}_2] = 2(\bar{Y}_2 - \bar{Y}_1)$ and following a procedure similar to the one developed above.

8.10. Lie families and second-order differential equations. Common t -dependent superposition rules describe solutions of nonautonomous systems of first-order differential equations. Nevertheless, we shall now illustrate how this new kind of superposition rules can also be applied to

analyse families of second-order differential equations. More specifically, we shall derive a common t -dependent superposition rule in order to express the general solution of any instance of a family of Milne–Pinney equations [30, 75, 195, 196] in terms of each generic pair of particular solutions, two constants, and the variable t , i.e. the time. In this way, we provide a generalization to the setting of dissipative Milne–Pinney equations of the expression previously derived to analyse the solutions of Milne–Pinney equations in [44].

Consider the family of dissipative Milne–Pinney equations [89, 195, 196, 217] of the form

$$\ddot{x} = -\dot{F}\dot{x} + \omega^2 x + e^{-2F} x^{-3}, \quad (8.57)$$

with a fixed t -dependent function $F = F(t)$, and parametrised by an arbitrary t -dependent function $\omega = \omega(t)$. The physical motivation for the study of dissipative Milne–Pinney equations comes from its appearance in dissipative quantum mechanics [3, 113, 171, 213], where, for instance, their solutions are used to obtain Gaussian solutions of non-conservative t -dependent quantum oscillators [171]. Moreover, the mathematical properties of the solutions of dissipative Milne–Pinney equations have been studied by several authors from different points of view as well as for different purposes [34, 44, 45, 83, 110, 195, 196, 230]. As relevant instances, consider the works [45, 195] which outline the state-of-the-art of the investigation of dissipative and non-dissipative Milne–Pinney equations. One of the main achievements on this topic (see [195, Corollary 5]) is concerned with an expression describing the general solution of a particular class of these equations in terms of a pair of generic particular solutions of a second-order linear differential equations and two constants. Recently, the theory of quasi-Lie schemes and the theory of Lie systems has enabled us to recover this latter result and other new ones from a geometric point of view [34, 52].

Note that introducing a new variable $v \equiv \dot{x}$, we transform the family (8.57) of second-order differential equations into a family of first-order ones

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\dot{F}v + \omega^2 x + e^{-2F} x^{-3}, \end{cases} \quad (8.58)$$

whose dynamics is described by the family of t -dependent vector fields on $\mathbb{T}\mathbb{R}$ parametrised by ω of the form

$$Y_\omega = \left(-\dot{F}v + e^{-2F} x^{-3} + \omega^2 x \right) \frac{\partial}{\partial v} + v \frac{\partial}{\partial x}, \quad \omega \in \Lambda = C^\infty(t).$$

Let us show that the above family is a Lie family whose common superposition rule can be used to analyse the solutions of the family (8.57).

In view of Theorem 7.19, if the family of systems related to the above family of t -dependent vector fields is a Lie family, that is, it admits a common t -dependent superposition rule in terms of m particular solutions, then the family of vector fields on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ given by $\text{Lie}(\{Y_\omega\}_{\omega \in \Lambda})$ spans an involutive generalised distribution with leaves of rank $r \leq n \cdot m + 1$.

Note that the distribution spanned by all \tilde{Y}_ω is generated by the vector fields \tilde{Y}_1 and \tilde{Y}_2 , with

$$Y_1 = \left(-\dot{F}v + e^{-2F} x^{-3} + x \right) \frac{\partial}{\partial v} + v \frac{\partial}{\partial x}, \quad Y_2 = \left(-\dot{F}v + e^{-2F} x^{-3} \right) \frac{\partial}{\partial v} + v \frac{\partial}{\partial x},$$

since $\tilde{Y}_\omega = (1 - \omega^2)\tilde{Y}_2 + \omega^2\tilde{Y}_1$. The prolongation $[\tilde{Y}_1, \tilde{Y}_2]$ is not spanned by \tilde{Y}_1 and \tilde{Y}_2 and, so

we have to include the prolongation $Y_3^\wedge = [\tilde{Y}_1, \tilde{Y}_2]$ to the picture, where

$$Y_3 = x \frac{\partial}{\partial x} - (v + x\dot{F}) \frac{\partial}{\partial v}.$$

In the case $m = 0$, the distribution spanned by the vector fields, $\tilde{Y}_1, \tilde{Y}_2, Y_3^\wedge$, does not admit a non-trivial first-integral. In the case $m > 0$, the vector fields, $\tilde{Y}_1, \tilde{Y}_2, Y_3^\wedge$, do not span the linear space $\text{Lie}(\{\tilde{Y}_\omega\}_{\omega \in \Lambda})$ and we need to add a new prolongation $Y_4^\wedge = [\tilde{Y}_1, [\tilde{Y}_1, \tilde{Y}_2]]$ to the previous set, with

$$Y_4 = (2v + x\dot{F}) \frac{\partial}{\partial x} + (2e^{-2F}x^{-3} - 2x - \dot{F}(v + x\dot{F}) - x\ddot{F}) \frac{\partial}{\partial v}.$$

The vector fields, $\tilde{Y}_1, \tilde{Y}_2, Y_3^\wedge, Y_4^\wedge$, satisfy the commutation relations

$$\begin{aligned} [\tilde{Y}_1, \tilde{Y}_2] &= Y_3^\wedge, \\ [\tilde{Y}_1, Y_3^\wedge] &= Y_4^\wedge, \\ [\tilde{Y}_1, Y_4^\wedge] &= (4 + \dot{F}^2 + 2\ddot{F})Y_3^\wedge - (\dot{F}\ddot{F} + \ddot{F})(\tilde{Y}_1 - \tilde{Y}_2), \\ [\tilde{Y}_2, Y_3^\wedge] &= 2(\tilde{Y}_1 - \tilde{Y}_2) + Y_4^\wedge, \\ [\tilde{Y}_2, Y_4^\wedge] &= (2 + \dot{F}^2 + 2\ddot{F})Y_3^\wedge - (\dot{F}\ddot{F} + \ddot{F})(\tilde{Y}_1 - \tilde{Y}_2), \\ [Y_3^\wedge, Y_4^\wedge] &= -2Y_4^\wedge - 2(\tilde{Y}_1 - \tilde{Y}_2)(4 + \dot{F}^2 + 2\ddot{F}). \end{aligned}$$

Consequently, the vector fields $\tilde{Y}_1, \tilde{Y}_2, Y_3^\wedge, Y_4^\wedge$ span the linear space $\text{Lie}(\{\tilde{Y}_\omega\}_{\omega \in \Lambda})$. Adding \tilde{Y}_1 to each prolongation of the previous set, that is, by considering the vector fields $\tilde{X}_1 = \tilde{Y}_1$, $\tilde{X}_2 = \tilde{Y}_2$, $\tilde{X}_3 = \tilde{Y}_1 + Y_3^\wedge$, and $\tilde{X}_4 = \tilde{Y}_1 + Y_4^\wedge$, we get the family of t -prolongations, $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4$, which spans the vector fields of the family $\text{Lie}(\{\tilde{Y}_\omega\}_{\omega \in \Lambda})$. The commutation relations among them read

$$\begin{aligned} [\tilde{X}_1, \tilde{X}_2] &= \tilde{X}_3 - \tilde{X}_1, \\ [\tilde{X}_1, \tilde{X}_3] &= \tilde{X}_4 - \tilde{X}_1, \\ [\tilde{X}_1, \tilde{X}_4] &= -(\dot{F}\ddot{F} + \ddot{F} + 4 + \dot{F}^2 + 2\ddot{F})\tilde{X}_1 + (\dot{F}\ddot{F} + \ddot{F})\tilde{X}_2 + (4 + \dot{F}^2 + 2\ddot{F})\tilde{X}_3, \\ [\tilde{X}_2, \tilde{X}_3] &= 2\tilde{X}_1 - 2\tilde{X}_2 - \tilde{X}_3 + \tilde{X}_4, \\ [\tilde{X}_2, \tilde{X}_4] &= -(1 + \dot{F}^2 + 2\ddot{F} + \dot{F}\ddot{F} + \ddot{F})\tilde{X}_1 + (\dot{F}\ddot{F} + \ddot{F})\tilde{X}_2 + (1 + \dot{F}^2 + 2\ddot{F})\tilde{X}_3, \\ [\tilde{X}_3, \tilde{X}_4] &= -3\tilde{X}_4 + (4 + \dot{F}^2 + 2\ddot{F})\tilde{X}_3 + (8 + \ddot{F} + \dot{F}\ddot{F} + 2\dot{F}^2 + 4\ddot{F})\tilde{X}_2 + \\ &\quad + (-9 - 3\dot{F}^2 - 6\ddot{F} - \dot{F}\ddot{F} - \ddot{F})\tilde{X}_1. \end{aligned}$$

As a consequence of Lemma 7.17, we get that the vector fields $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ and \tilde{X}_4 satisfy the same commutation relations as the vector fields $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4$. Hence, in view of Theorem 7.19, the family (8.58) is a Lie family and the knowledge of non-trivial first-integrals of the vector fields of the distribution \mathcal{D} spanned by $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4$ provides us with a common t -dependent superposition rule.

Let us now turn to determine the aforementioned common t -dependent superposition rule.

As the vector fields \tilde{X}_1 , $\tilde{X}_1 - \tilde{X}_2$ and their successive Lie brackets span the whole distribution \mathcal{D} , a function $G : \mathbb{R} \times \mathbb{T}\mathbb{R}^3 \rightarrow \mathbb{R}$ is a first-integral for the vector fields of such a distribution if and only if it is a first-integral for the vector fields \tilde{X}_1 and $\tilde{X}_2 - \tilde{X}_1$. Therefore, we can reduce the problem of finding first-integrals for the vector fields of the distribution \mathcal{D} to finding common first-integrals G for the vector fields \tilde{X}_1 and $\tilde{X}_1 - \tilde{X}_2$.

Let us analyse the implications of G being a first-integral of the vector field

$$\tilde{X}_1 - \tilde{X}_2 = \sum_{i=0}^2 x_i \frac{\partial}{\partial v_i}.$$

The characteristics of the above vector field are the solutions of the system

$$\frac{dv_0}{x_0} = \frac{dv_1}{x_1} = \frac{dv_2}{x_2}, \quad dx_0 = 0, \quad dx_1 = 0, \quad dx_2 = 0, \quad dt = 0,$$

that is, the solutions are curves in $\mathbb{R} \times \mathbb{T}\mathbb{R}^3$ of the form $s \mapsto (t, x_0, x_1, x_2, v_0(s), v_1(s), v_2(s))$, with $\xi_{02} = x_0 v_2(s) - x_2 v_0(s)$ and $\xi_{12} = x_1 v_2(s) - x_2 v_1(s)$ for two real constants ξ_{02} and ξ_{12} . Thus, there exists a function $G_2 : \mathbb{R}^6 \rightarrow \mathbb{R}$ such that $G(p) = G_2(t, x_0, x_1, x_2, \xi_{02}, \xi_{12})$, with $p \in \mathbb{R} \times \mathbb{T}\mathbb{R}^3$, $\xi_{02} = x_0 v_2 - x_2 v_0$, and $\xi_{12} = x_1 v_2 - x_2 v_1$. In other words, G is a function of $t, x_0, x_1, x_2, \xi_{02}, \xi_{12}$.

The function G also satisfies the condition $\tilde{X}_1 G = 0$ which, in terms of the coordinate system $\{t, x_0, x_1, x_2, \xi_{02}, \xi_{12}, v_2\}$, reads

$$\begin{aligned} \tilde{X}_1 G &= \frac{\partial G}{\partial t} + \frac{(x_0 v_2 - \xi_{02})}{x_2} \frac{\partial G}{\partial x_0} + \frac{(x_1 v_2 - \xi_{12})}{x_2} \frac{\partial G}{\partial x_1} + v_2 \frac{\partial G}{\partial x_2} - \\ &- \left[\dot{F} \xi_{12} + e^{-2F} \left(\frac{x_2}{x_1^3} - \frac{x_1}{x_2^3} \right) \right] \frac{\partial G}{\partial \xi_{12}} - \left[\dot{F} \xi_{02} + e^{-2F} \left(\frac{x_2}{x_0^3} - \frac{x_0}{x_2^3} \right) \right] \frac{\partial G}{\partial \xi_{02}} = 0. \end{aligned}$$

That is, defining the vector fields

$$\begin{aligned} \Xi_1 &= \frac{\partial}{\partial t} - \frac{\xi_{12}}{x_2} \frac{\partial}{\partial x_1} - \frac{\xi_{02}}{x_2} \frac{\partial}{\partial x_0} + \left[-\dot{F} \xi_{12} - e^{-2F} \left(\frac{x_2}{x_1^3} - \frac{x_1}{x_2^3} \right) \right] \frac{\partial}{\partial \xi_{12}} \\ &+ \left[-\dot{F} \xi_{02} - e^{-2F} \left(\frac{x_2}{x_0^3} - \frac{x_0}{x_2^3} \right) \right] \frac{\partial}{\partial \xi_{02}}, \\ \Xi_2 &= \frac{x_0}{x_2} \frac{\partial}{\partial x_0} + \frac{x_1}{x_2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \end{aligned}$$

the condition $\tilde{X}_1 G = 0$ implies that $\Xi_1 G_2 + v_2 \Xi_2 G_2 = 0$ and, as G_2 does not depend on v_2 , the function G must simultaneously be a first-integral for Ξ_1 and Ξ_2 , i.e. $\Xi_1 G = 0$ and $\Xi_2 G = 0$.

Applying the method of characteristics to the vector field Ξ_2 , we get that F can just depend on the variables $t, \xi_{02}, \xi_{12}, \Delta_{02} = x_0/x_2$ and $\Delta_{12} = x_1/x_2$. In other words, there exists a function $G_3 : \mathbb{R}^5 \rightarrow \mathbb{R}$ such that $G(t, x_0, x_1, x_2, v_0, v_1, v_2) = G_3(t, x_0, x_1, x_2, \xi_{02}, \xi_{12}) = G_3(t, \xi_{02}, \xi_{12}, \Delta_{02}, \Delta_{12})$.

We are left to check the implications of the equation $\Xi_1 G = 0$. With the aid of the coordinate system $\{t, \xi_{02}, \xi_{12}, \Delta_{02}, \Delta_{12}, v_2, x_2\}$, the previous equation can be cast into the form $\Xi_1 G =$

$\frac{1}{x_2} \Upsilon_1 G_3 + \Upsilon_2 G_3 = 0$, where

$$\begin{aligned} \Upsilon_1 &= \sum_{i=0}^1 \left(-\xi_{i2} \frac{\partial}{\partial \Delta_{i2}} - e^{-2F} (\Delta_{i2}^{-3} - \Delta_{i2}) \frac{\partial}{\partial \xi_{i2}} \right), \\ \Upsilon_2 &= -\dot{F} \xi_{12} \frac{\partial}{\partial \xi_{12}} - \dot{F} \xi_{02} \frac{\partial}{\partial \xi_{02}} + \frac{\partial}{\partial t}. \end{aligned}$$

As G_3 only depends on the variables, $t, \Delta_{02}, \Delta_{12}, \xi_{12}, \xi_{02}$, we have that $\Upsilon_1 G = 0$ and $\Upsilon_2 G = 0$. Repeating *mutatis mutandis* the previous procedures in order to determine the implications of being a first-integral of Υ_1 and Υ_2 , we finally get that the first-integrals of the distribution \mathcal{D} are functions of I_1, I_2 and I , with

$$I_i = e^{2F} (x_0 v_i - x_i v_0)^2 + \left[\left(\frac{x_0}{x_i} \right)^2 + \left(\frac{x_i}{x_0} \right)^2 \right], \quad i = 1, 2,$$

and

$$I = e^{2F} (x_1 v_2 - x_2 v_1)^2 + \left[\left(\frac{x_1}{x_2} \right)^2 + \left(\frac{x_2}{x_1} \right)^2 \right].$$

Defining $\bar{v}_2 = e^F v_2, \bar{v}_1 = e^F v_1$ and $\bar{v}_0 = e^F v_0$, the above first-integrals read

$$I_i = (x_0 \bar{v}_i - x_i \bar{v}_0)^2 + \left[\left(\frac{x_0}{x_i} \right)^2 + \left(\frac{x_i}{x_0} \right)^2 \right], \quad i = 1, 2,$$

and

$$I = (x_1 \bar{v}_2 - x_2 \bar{v}_1)^2 + \left[\left(\frac{x_1}{x_2} \right)^2 + \left(\frac{x_2}{x_1} \right)^2 \right].$$

Note that these first-integrals have the same form as the ones considered in [52] for $k = 1$. Therefore, we can apply the procedure done there to obtain that

$$x_0 = \sqrt{k_1 x_1^2 + k_2 x_2^2 + 2\sqrt{\lambda_{12}[-(x_1^4 + x_2^4) + I x_1^2 x_2^2]}}, \quad (8.59)$$

with λ_{12} being a function of the form

$$\lambda_{12}(k_1, k_2, I) = \frac{k_1 k_2 I + (-1 + k_1^2 + k_2^2)}{I^2 - 4},$$

and where the constants k_1 and k_2 satisfy special conditions in order to ensure that x_0 is real [44].

Expression (8.59) permits us to determine the general solution, $x(t)$, of any instance of family (8.57) in the form

$$x(t) = \sqrt{k_1 x_1^2(t) + k_2 x_2^2(t) + 2\sqrt{\lambda_{12}[-(x_1^4(t) + x_2^4(t)) + I x_1^2(t) x_2^2(t)]}}, \quad (8.60)$$

with

$$I = e^{2F(t)} (x_1(t) \dot{x}_2(t) - x_2(t) \dot{x}_1(t))^2 + \left[\left(\frac{x_1(t)}{x_2(t)} \right)^2 + \left(\frac{x_2(t)}{x_1(t)} \right)^2 \right],$$

in terms of two of its particular solutions, $x_1(t), x_2(t)$, its derivatives, the constants k_1 and k_2 , and the variable t (included in the constant of the motion I).

Note that the role of the constant I in expression (8.60) differs from the roles played by k_1 and k_2 . Indeed, the value of I is fixed by the particular solutions $x_1(t), x_2(t)$ and its derivatives,

while, for every pair of generic solutions $x_1(t)$ and $x_2(t)$, the values of k_1 and k_2 range within certain intervals ensuring that $x(t)$ is real.

It is clear that the method illustrated here can also be applied to analyse solutions of any other family of second-order differential equations related to a Lie family by introducing the new variable $v = \dot{x}$. Additionally, it is worth noting that in the case $F(t) = 0$ the family of dissipative Milne–Pinney equations (8.57) reduces to a family of Milne–Pinney equations broadly appearing in the literature (see [147] and references therein), and the expression (8.60) takes the form of the expression obtained in [44] for these equations.

9. Conclusions and outlook

Apart from providing a quite self-contained introduction to the theory of Lie systems, this essay describes most of the results concerning this theory and its generalisations developed by the authors and other collaborators along very recent years. In this way, our work presents a state-of-art of the subject and establishes the foundations for our present research activity. Let us here discuss some of the topics which we aim to analyse in a close future and their relations to the contents of this essay.

The theory of superposition rules for second- and higher-order differential equations has just been initiated [48, 49, 52, 77, 202, 225] and many questions about this topic must still be clarified. As an example, we can point out that there exist several approaches to study systems of second-order differential equations by means of the theory of Lie systems nowadays. For instance, one can use the SODE Lie system notion [52], which allows us to study a particular type of systems of second-order differential equations. In addition, if a second-order differential equation admits a regular Lagrangian, the corresponding Hamiltonian formulation can lead to a system of first-order differential equations which can also be a Lie system [54]. Analysing the relations between the results obtained through both approaches is still an open problem.

As a consequence of the above considerations, it became interesting to study a class of Lie systems describing the Hamilton equations of a certain type of t -dependent Hamiltonians. These systems are defined in a symplectic manifold and this structure provides us with new tools for investigating such Lie systems. In addition, these tools can be employed to study the integrability and super-integrability of these particular Lie systems. Our aim is to analyse such relations in depth in the future.

After analysing the Lie systems defined in symplectic manifolds, a natural question arises: What are the properties of those Lie systems describing the solutions of a system in a Poisson manifold $(N, \{\cdot, \cdot\})$ of the form

$$\frac{dx}{dt} = \{x, h_t\}, \quad x \in N,$$

where, for every $t \in \mathbb{R}$, the function $h_t : N \rightarrow \mathbb{R}$ belongs to a finite-dimensional Lie algebra of functions (with respect to the Poisson bracket). This challenging question has led to the analysis of the properties of such Lie systems by means of the Poisson structure of the manifold, what represents an interesting topic of research.

In [12, 13] Winternitz *et al.* proposed, for the first time, a new type of superposition rules, the referred to as *super-superposition rules*, that describe the general solution of a particular family

of systems of first-order differential equations in supermanifolds. These articles gave rise to many interesting unanswered questions. Although it seems that the geometric theory developed in [38] could easily be generalised to describe the properties of *super-superposition rules*, multiple non-trivial technical problems arise. We hope to solve such problems in the future and to develop a geometric theory of Lie systems in graded manifolds.

In [38, Remark 5], it was proposed to accomplish the study of Bäcklund transformations through a slight modification of the methods carried out to analyse superposition rules geometrically, i.e., by means of a certain type of flat connection. This topic deserves a further analysis in order to determine more exactly its relevance and applications.

Since their first appearance in [34], quasi-Lie schemes have been employed to investigate multiple systems of differential equations: nonlinear oscillators [34], Mathews-Lakshmanan oscillators [34], Emden equations [42], Abel equations [56], dissipative Milne–Pinney equations [45], etc. There are still many other applications to be performed, e.g. we expect to apply this theory to study Abel equations in depth. In addition, it would be interesting to continue the analysis of the theory of quasi-Lie schemes and, for instance, to develop new generalisations of this theory. Indeed, we are already investigating a generalisation for the analysis of certain quantum systems, e.g. the quantum Calogero-Moser system. In addition, it would be interesting to study the generalisations of this theory to analyse stochastic Lie-Scheffers systems [144] or Control Lie systems [79].

As we pointed out at the beginning of this essay, being a Lie system is rather more an exception than a rule. In addition, just a few, but relevant, Lie systems are known to have applications in Physics, Mathematics and other branches of science. Consequently, one of our main purposes remains to find new instances of Lie systems with remarkable applications. It seems to us that there still exist multiple applications of Lie systems and, in the future, we aim to determine some of them.

To finish, we hope to have succeeded in showing that the theory of Lie systems, after more than a century of existence, is still an active and interesting field of research.

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