GENERAL AND PRACTICAL OPTICS

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BY

LIONEL LAURANCE

THIRD AND REVISED EDITION

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PREFACE

In this third edition of "General and Practical Optics" I trust that some of the faults of previous editions will be found remedied. The subject-matter has been revised, condensed, rearranged, and rewritten, to some extent, and considerable fresh matter has been introduced.

Although primarily intended as a textbook for candidates for the examination of the Worshipful Company of Spectacle Makers, it is written also as a reference book for those engaged in spectacle work, for other students of Optics, and as an introduction to the study of more advanced works.

In previous editions I acknowledged my indebtedness to Dr. George Lindsay Johnson and to Mr. H. Oscar Wood. Here again, and to an increased extent, I repeat my recognition of the invaluable co-operation of Mr. Wood in compiling and writing the subject-matter, and in revising and correcting the work.

As stated before, I have endeavoured to cover, in this book and in "Visual Optics and Sight-Testing," all that is essential for the sight-testing optician.

LIONEL LAURANCE.

COMMERCIAL NOTATION FOR OPERA GLASSES AND SMALL TELESCOPES.

The French inch is divided into 12 lignes or lines, in which the diameter of the object-glass of ordinary opera glasses and small telescopes is expressed.

Lines ...	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
M.M. (approx.)	11	13	15	17	20	22	24	26	29	31	33	36	38	40	43	45	47	49

Lines ...	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
M.M. (approx.)	52	54	56	58	61	63	65	67	70	72	74	76	78	81	3	85	87	90

THE GREEK ALPHABET, WITH THE ENGLISH EQUIVALENTS AND THE PRONUNCIATION OF THE LETTERS.

Α α ... alpha *a*	Ι ι ... iōta ...*i*	Ρ ρ .. rho ...*r*
Β β ... bēta ...*b*	Κ κ ... kappa ...*k*	Σ σ ς ... sigma *s*
Γ γ ... gamma *g* (hard)	Λ λ ... lamda ...*l*	Τ τ ... tau ...*t*
Δ δ ... delta...*d*	Μ μ ... mu ...*m*	Υ υ ... ūpsĭlon *u* or *y*
Ε ε ... epsilon *e* (short)	Ν ν ... nu ...*n*	Φ φ ... phi ...*ph*
Ζ ζ ... zēta ...*z*	Ξ ξ ... xi ...*x*	Χ χ ... chi ...*ch* (hard)
Η η ... ēta ...*e*	Ο ο ... ŏmicron *o* (short)	Ψ ψ ... psi ...*ps*
Θ θ ϑ ... thēta *th*	Π π ... pi ...*p*	Ω ω ... ōmĕga *o* (long)

CONTENTS

OPTICAL SYMBOLS AND ABBREVIATIONS

S. or *Sph.*	Spherical.
C. or *Cyl.*	Cylindrical.
Pr.	Prism.
Ax.	Axis.
Pc. or *Peris.*	Periscopic.
Pcx. or *Pcvx.*	Periscopic convex.
Pcc. or *Pcvc.*	Periscopic concave.
Dcx. or *Dcvx.*	Double convex.
Dcc. or *Dcvc.*	Double concave.
F. or *P.F.*	Principal focal distance or focus.
F_1 and F_2	Anterior and posterior focal distances or foci.
f_1 or u, f_2 or v	Conjugate focal distances or foci.
O.	Object.
I.	Image.
Hor. or *H.*	Horizontal.
Ver. or *V.*	Vertical.
Mer.	Meridian.

D.	Diopter.
+, *Cx.*, or *Cvx.*	Plus, Convex.
−, *Cc.*, or *Cvc.*	Minus, Concave.
°*d*	Degree of deviation.
▵ or *PD.*	Prism diopter.
Δ	Prism power.
∇	Centrad.
$\wedge$	Metran.
∞	Infinity, a distance infinitely great.
⌢	Combined with.
μ or n	The index of refraction.
Δ or δ	The difference between (applied to lines of the spectrum).
ω	The ratio between the dispersion and refraction of a medium.
ν	The ratio between the refraction and dispersion of a medium.

MATHEMATICAL SYMBOLS AND ABBREVIATIONS

M.	Metres.
cm.	Centimetres.
mm.	Millimetres.
μ	Microns.
$\mu\mu$	Micromillimetres.
Ft. or ′	Foot.
In. or ″	Inch.
‴	Line.
°	Degree.
′	Minute.
″	Second.
∞ or $1/0$	Infinity, a number infinitely great.
0 or $1/\infty$	Zero, a number infinitely small.
$\angle$	Angle.
:	Is to.
::	So is.
$\therefore$	Therefore.
$\because$	Because.
$\propto$	Varies as.
$\perp$	Perpendicular to.
$\parallel$	Parallel to.
∟	Right angles to.
π	(Pi) Ratio of circumference to diameter.

r or ρ	Radius.
θ ϕ	Any angles.
$+$	Plus, addition.
$-$	Minus, subtraction.
$\pm$	Either + or −.
$\times$	Multiplied by.
$\div$ or /	Divided by.
$\sim$	The difference between.
$\sqrt{\ }$	The square root of.
$\sqrt[3]{\ }$	The cube root of.
$\sqrt[n]{\ }$	The *n*th root of.
x^2	x squared.
x^3	x cubed.
x^n	x raised to the power of a number equal to n.
$\overline{a+b}$	Bond or vinculum, showing that the numbers are to be taken together. Is the same as $(a+b)$.
$=$	Equal to.
$>$	Greater than.
$<$	Less than.

For ophthalmic abbreviations and symbols, see "Visual Optics and Sight-Testing."

ERRATA.

Page 40, line 13, *for* A′ F′ B′ *read* A′ F B′.

" 41, line 14, *for* $h_2 = h_1 \frac{f_1}{f_2}$ *read* $h_2 = h_1 \frac{f_2}{f_1}$

" 43, line 5, *for* $h_1 \times \frac{10}{5}$ *read* $h_1 \times \frac{5}{10}$

" 55, line 22, *for* $t' \frac{t \cos r}{\mu \cos i}$ *read* $t' = \frac{t \cos r}{\mu \cos i}$

" 65, last line but two, *for* Maddex *read* Maddox.

" 76, line 10, *for* Cx. a *read* a Cx.

" 94, line 5, *for* F = 20 cm *read* F = 40 cm.

" 144, line 3, *for* − 1/10 − (− 1/12) *read* − 1/12 − (−1/10).

" 184, in the formula for P_2, *read*

$$P_2 = \frac{\mu_3 r_2 t (\mu_2 - \mu_1)}{\mu_2 Q}$$

" 193, line 19, *for* wave points *read* wave fronts.

" 229, line 14, *for* Fig. 239 *read* Fig. 242.

" 231, line 19, *for* angle *read* angles.

" 241, line 6, *for* case *read* cases.

" 259, line 6, *for* Fig. 288 and Fig. 289 *read* Fig. 278 and 279.

" 268, last line, *for* instances *read* distances.

" 272, line 40, *for* zero *read* unity.

" 279, line 25, *for* nickel *read* Nicol.

" 322, in Fig. 337, the rays V and R should be shown crossed in the second prism.

GENERAL AND PRACTICAL OPTICS.

GENERAL AND PRACTICAL OPTICS

CHAPTER I

LIGHT

Light.—Everything seen is rendered visible by means of a form of radiant ergy termed light. With the exception of certain manifestations such fluorescence, phosphorescence, etc., all light has its source in bodies which e in a condition of incandescence. The source of light itself may not be sible, but the reflected light by which objects—the sky, moon, trees, houses, .—are seen, can invariably be traced to the sun, or to some artificial urce of incandescence.

It was once supposed that light was something which radiated from the eye the objects seen, and later it was thought to be due to minute corpuscles ich proceeded from a visible object to the eye at great speed, but it is w accepted that light is due to vibrations set up in the luminiferous ether the molecular agitations of an incandescent body.

Ether.—This is a medium believed to occupy the whole universe; it fills estial space, lies between the particles of the earth's atmosphere and ween the molecules and atoms of which solid and liquid bodies are com- sed, so that everything is saturated with it. A vacuum consists of ether. tle is known about its nature, its properties being chiefly negative, since annot be appreciated by any of the senses. It has been concluded, vever, that it possesses density, rigidity and elasticity, properties enabling o propagate transverse undulations or waves, generated by vibrations incandescent material bodies; these waves travel to an infinite distance hout appreciable loss of energy. Ether is the connecting medium of the verse, and it is due to its presence that material bodies are capable of ng on one another at a distance, and by which such forms of radiant energy ight, heat, actinism, magnetism, electricity, etc., are made manifest.

Light Waves and Rays.— Every point L (Fig. 1) of a source of light erates an ethereal oscillation in every direction This forms a tiny ere, and according to the accepted theory of Huyghen, every point on the circumference of this sphere forms a new centre of disturbance which generates a fresh sphere, and each of these spheres again forms fresh ones, and so on. These spheres lie side by side overlapping each other, and, taken collectively, at any distance from the primary centre of disturbance, form a *wave-front* (*a b c d e*). Each wave-front then forms a series of centres for

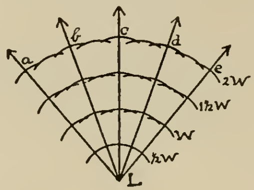

Fig. 1

the formation of a fresh row of spheres, whose diameter is equal to a wave-length. Although it is convenient to consider light as advancing in the form of a simple wave-front which forms part of an ever-enlarging sphere, yet in reality the process is exceedingly complex.

The wave motion of the ether takes place in every plane and is always *transverse*, i.e. at right angles to the direction of propagation of the light. The ether particles themselves do not travel, but merely oscillate, much in the same way as the particles of water bob up and down when ripples are formed on the surface; or as the vibrations travel along a rope when it is shaken.

In every plane each successive wave-front may be considered as the crest (W or $2W$), and the space between it and the next wave-front as the trough of a wave ($\frac{1}{2}W$ or $1\frac{1}{2}W$). The length of these waves varies to some extent, as does also their *frequency*, or number of vibrations per second, such that V=LT, where V is the speed of the wave travel per second, L is the wave-length and T is the frequency. The frequency and the wave-length must be within certain limits in order that light, and not some other form of radiant energy, may result.

The incandescence of the sun is, of course, the principal source from which light on the earth is derived. Impact, friction, electricity, chemical combination, combustion, in fact anything which causes increased molecular motion, also may give rise to light.

Although light is propagated from a luminous point in a series of wave fronts, it is more convenient to consider the direction of propagation of any particular point on the main wave, which can be shown as a straight line. From the luminous point L (Fig. 1) the light radiates in every direction, and any line of propagation such as La, Lb, etc., is termed a *ray* of light. Thus "rays" are really the imaginary radii of the wave-fronts, and as such have no real existence. For diagrammatic purposes, however, their assumption is most useful, since they indicate the directions in which portions of the real wave-front are travelling.

Light, Heat and Actinism.—When the temperature of a body is raised, the increased molecular activity causes a generation of ether waves which constitutes *radiant heat.* Their length is too great and their frequency too low to cause the sensation of light, and they are termed *infra-red.* If the temperature is raised still more, the activity is proportionally increased, so that the waves become shorter and the vibrations more rapid. Thus, when the temperature of a body reaches about 500° centigrade, it not only emits the relatively long waves of *heat,* but also the shorter waves of *light.* The longest light waves give rise to the visual sensation of red. On further raising the temperature of the body, still shorter waves are also produced which cause the sensation of various colours, violet resulting from the shortest visible waves. White is a sensation caused by the combined action of all waves ranging between red and violet, and is produced when the temperature reaches about 1000° C.

Ether waves which are too short and whose vibrations are too rapid to cause the sensation of light are termed *ultra-violet:* they cause chemical action, and are said to be *actinic.* The difference between these three forms of radiant energy exists solely in the length of the waves.

Density of Media.—The speed with which light travels within a certain medium depends on the nature of the latter or, more exactly, on the elasticity of the ether within it; thus light travels more slowly in a dense medium, i.e. one in which its component particles are crowded together like glass, than in a rare one, such as air.

Velocity of Light.—Light travels in space (free ether) at about 186,000 miles or 300,000 kilometres per second; in air its speed is practically the same although actually slightly less. The velocity is lessened in denser media, the decrease being roughly proportional to the density, although this is not invariably the case. Thus, in glass, the rate of progression is about one third less, and in water one fourth less, than it is in air.

186,000 miles is a distance equal to about eight times the circumference of the earth at the equator, a journey travelled by light in one second. From the sun it takes about eight minutes to reach the earth, some 93 million miles distant. At this rate light travels six million million miles in a year, and the distance of a fixed star, being so enormous, is measured in light years, thus expressing the number of years the light from the star takes to reach the earth. A *light year,* termed a *Persac,* is thus six billion miles.

In space—and practically so in air—all light waves travel with the same velocity, and therefore it follows that the short waves must have a higher frequency than the longer waves. It is only when light passes into material bodies, like glass or water, that the velocities of the various waves become unequal.

Measurement of Light-Speed.—Following are four methods by which the velocity of light has been measured.

Römer's Method.—One of Jupiter's moons m (Fig. 2) becomes eclipsed by the planet J every 48½ hours. At a certain period of the earth's annual revolution round the sun it is in opposition to Jupiter. If light were to travel instantaneously, the eclipse, and its observation by an observer on the earth, would occur simultaneously. The light, however, has to travel

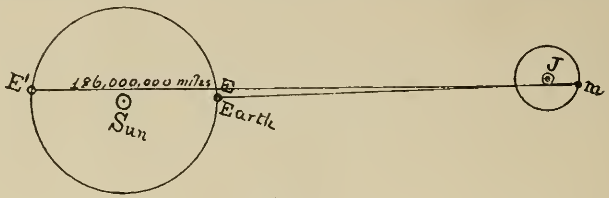

Fig. 2.

from Jupiter to the earth before the eclipse can be seen. Let R and r be respectively the radii of the orbits of Jupiter and the earth round the sun. Then $J\ E$ (i.e. $R-r$) is the distance the light has to travel at a velocity V. This time, therefore, will be $(R-r)/V$ seconds after the eclipse has taken place. After six months the earth and Jupiter will again be in opposition, the earth now being at E' on the other side of the sun. The eclipse will therefore be observed $(R+r)/V$ seconds after the occurrence, the difference between the two observations being equal to $2r=186$ million miles.

Römer observed that, as the earth moved from E to E', the observed time steadily exceeded the calculated time. Thus he found that an eclipse observed when the earth was at E' occurred 995 seconds later than when it was observed at E. Since the diameter of the earth's orbit is 186 million miles, $V=186{,}000{,}000/995=186{,}000$ miles per second (approx.).

Bradley's Method.—The apparent direction of light from a star, owing to the earth's motion, makes an angle with its true direction. As the earth pursues its elliptical orbit round the sun it must move in an opposite direction to that which it took six months before, so that a telescope directed to

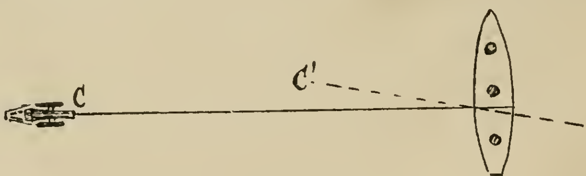

Fig. 3.

a star somewhere along a line at right angles to the earth's motion must be pointed slightly in front of the mean calculated position at the first period of observation, and a similar distance behind at the second observation. The angle which the telescope makes between the calculated and the observed position is called the *aberration* of the star.

Bradley knew the velocity of the earth's motion, he measured the angle of aberration, and from these data he proved the velocity of light to be,

$$V=\frac{\text{velocity of earth}}{\text{tan of angle}}=\frac{18 \text{ miles}}{\tan 20''}=\frac{18}{\cdot 0001}=180{,}000 \text{ miles per sec.}$$

Bradley's method may be illustrated as follows; if a shot from a gun C (Fig. 3) be fired at a ship, moving at right angles to the direction of the shot, the latter will not pass through the ship at right angles to its line of travel, but obliquely as if the shot came in the direction of the dotted line C'.

Fizeau's Method.—Fizeau's method depends on the interruption of a beam of light by the teeth of a revolving wheel. The light from a source S (Fig. 4)—rendered convergent by a lens L—falls on a plane unsilvered mirror m which is inclined at 45° and situated between the lens and its focus F, the latter being at the teeth of the wheel. Another lens L', placed at its principal focal length on the other side of the wheel and in a line with the mirror, renders the light from F parallel. The beam of light is collected by a third lens L'', situated at a distance (say four miles), and is brought to a focus on a spherical mirror M, from which it is reflected, so as to return along the same path, finally forming a real image at F which is viewed by the observer at E through an eyepiece.

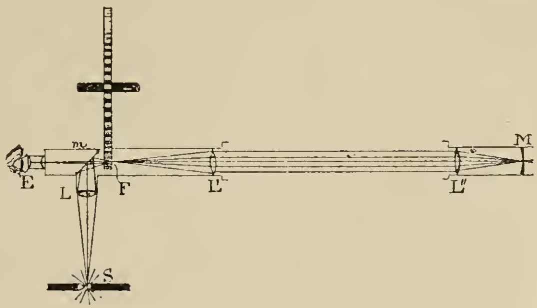

Fig. 4.

Suppose the light escapes through the first gap while the wheel is turning slowly, then it will, after travelling eight miles, pass through the same opening and a flickering image is seen. If the speed is greater the second tooth blocks out the light, but if still greater the light passes through the second gap, the wheel having revolved one tooth while the light travelled eight miles, and so reappears to an observer at E. The speed of the wheel being further increased the light appears and disappears as an additional tooth or gap passes by before the light returns. The speed of the toothed wheel, the size of the teeth, and the distance between m and M being known, Fizeau, and later Cornu, who improved on the apparatus, found the velocity of light in air to be about 300,000 km. per second.

Foucault's Method.—Light (Fig. 5) is passed through a slit S and a lens L on to a plane mirror M_1, whence the light passes to a concave mirror M_2 placed at a distance equal to its radius. From M_2 the light is again reflected back to M_1 and retracing its path is partly reflected by the glass plate M_3 to the eye at T. If M_1 is then rapidly rotated it will have had time to turn through an appreciable angle during the time that the light has travelled from M_1 to M_2 and back again, so that it will not be reflected back to the

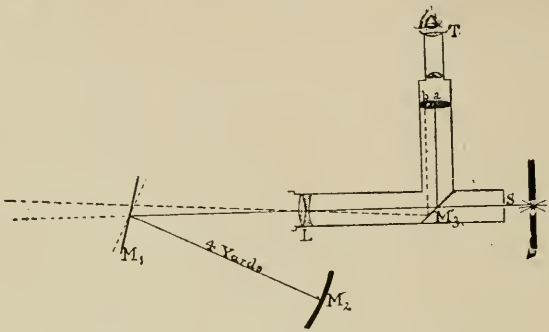

Fig. 5.

same spot on the mirror M_3. Thus the image seen by the observer through the telescope will not be formed on the cross wires at a, but will be found shifted to some point b. If the speed be known at which the mirror M_1 is rotated, and the distance which the light has to travel from M_1 to M_2 and back (which in this case is equal to eight yards) the velocity of light can be calculated by the displacement of the image from a to b as seen through the telescope T.

Solar Light is a combination of seven distinct colours—namely, red, orange, yellow, green, blue, indigo, and violet. Some authorities omit indigo and consider the spectrum to consist of six main colours, and some even omit the yellow. The combination of these colours in correct proportion produces white light.

Sunlight is said to consist of about 50 parts red, 30 parts green, and 20 parts violet in 100, and has about 30 per cent. of luminous rays. Artificial light has a higher proportion of heat and red rays, and the proportion of luminous rays is much smaller, varying from 20 per cent. for electricity (arc), 10 per cent. for oils and coal-gas, to 1 per cent. for alcohol. With the exception of the electric arc and similar sources, artificial light is very deficient in actinic and violet rays.

Cause of Colour.—Ethereal waves of certain length and frequency always produce a mental sensation of a definite colour, in a person of normal colour perception. Whether the length of the wave or its frequency, or both, give rise to the definite sensation, and whether the retina or the mind differentiates between the various waves, are points which are not yet precisely settled. As before stated, red is produced by comparatively long waves of low frequency, the sensation of violet by short waves of high frequency, while the other colours are produced by wave-lengths and frequencies between these two.

The Spectrum.—When sunlight passes through a dense medium, the shorter violet waves are more retarded and, if refracted, are bent to a greater extent than the longer red waves, so that the component colours become separated. The dispersed colours, caused by refraction of white light by a prism, can be seen on a screen as a bright-coloured band, called the spectrum, which contains the seven principal colours above mentioned. The various colours are not sharply separated, but merge so imperceptibly into one another that it is almost impossible to locate where one colour ends and another commences. The space in the spectrum, formed by a prism, occupied by the different colours varies with the refracting medium used for its production. In a solar spectrum, due to refraction by a given prism of flint glass, the red is somewhat crowded and the violet drawn out, but if it be divided into 100 parts the proportional space occupied approximately by the red is 30, by the green 25, and by the violet 45 parts, these three being the main colours seen.

Fraunhöfer's Lines.—When a gas is rendered incandescent, the spectrum of the light emitted by it consists of one or more isolated *bright* lines, on a dark ground, characteristic of the gas in question; this is known as a *line* spectrum. The solar spectrum is *continuous ;* it is a bright-coloured band crossed by *dark* lines known as *the Fraunhöfer lines*, which are very numerous and of varying widths. They show absence of certain wave-lengths.

Line.	*Position in Spectrum.*	*Metal or Gas producing the Line.*	*Wave-Lengths.*
			$\mu\mu$
A	Red	Oxygen (O)	759
a	Red	Water vapour	733
B	Red	Oxygen	686
C	Orange-red	Hydrogen (H)	656
D	Yellow	Sodium (Na)	590
E	Green	Iron (Fe) Calcium (Ca) ..	527
b	Blue-green	Magnesium (Mg)	518
F	Blue	Hydrogen	486
G	Dark blue	Hydrogen Iron	430
H	Violet	Calcium (bright line)	397

The experiments of Kirchhoff, Bunsen and Fraunhöfer have proved that the flame of each element radiates characteristic wave-lengths which produce the bright lines of its spectrum, and that the vapour of this same element at a lower temperature transmits freely all wave lengths except those which it would itself give out if it were incandescent, and these waves it absorbs. Thus sodium or salt, if burnt in a Bunsen flame, emits monochromatic yellow light, and white light from a *hotter* source would be robbed of precisely the same colour, i.e. yellow, on its passage through a sodium flame. The dark absorption lines, of the solar spectrum, correspond to the bright lines of specific substances, and are the result of the absorption of certain wave-lengths from the hot nucleus of the sun by the relatively cooler layers of incandescent gases continually being ejected to form its outer envelope. Some of the Fraunhöfer lines are due to certain unknown substances, while some are said to be due to absorption by the terrestrial atmosphere. Absorption spectra can be produced experimentally.

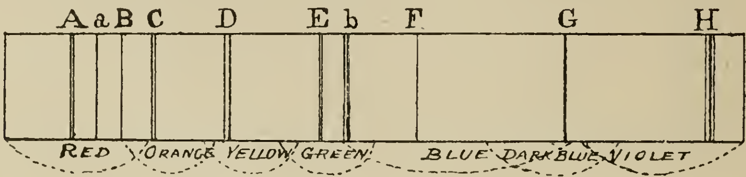

FIG. 6.

TABLE OF WAVE-LENGTHS AND FREQUENCIES.

Wave-Lengths in μμ.	*Number of Vibrations in Billions per Second.*	*Character.*
100,000,000 (100 mm.)		Electrical vibrations (Hertzian waves).
3,000,000 (3 mm.)		Shortest are about 3 mm. Longest 1 metre to several miles.
61,000	4·8 } Infra-red spectrum	Longest heat waves measured by Langley by his bolometer.
8,000	37	Longest heat waves measured by Ruebens and Snow by fluor-spar prism and bolometer.
812	370	Longest waves capable of being seen by the spectroscope, according to Helmholtz.
750	400 } Ordinary visible spectrum	Red.
650	460	Orange.
590	508	Yellow.
530	566	Green.
460	652	Blue.
420	710	Indigo.
375	800	Violet.
330	909 } Ultra-violet spectrum	Shortest waves visible according to Soret.
210	1,430	Shortest waves visible according to Mascart.
185	1,620	Shortest waves photographed through fluor-spar prism alone.
100	3,000	Shortest waves photographed by means of fluor-spar prism, vacuum camera and bromide of silver plate without gelatine.
—	—	X and Röntgen Rays (?).

NOTE.—In Britain a billion is a million millions. A micromillimetre $\mu\mu$ = one-millionth part of a millimetre or the billionth part of a kilometre. A micron μ = one thousandth of a millimetre. In the U.S.A. a billion = 1,000 millions.

The chief Fraunhöfer lines are indicated by letters of the alphabet, and as they always correspond to rays of a definite wave-length, they form a convenient means of identifying any particular part of the spectrum. Fig. 6 shows their approximate positions.

The Visible and Invisible Spectrum.—In general, the spectrum within certain limits consists of the long infra-red (heat) waves, the luminous or visible portion, and the short ultra-violet (chemical) waves. Besides these, there are the long Hertzian (electrical) waves beyond the infra-red, and what are supposed to be the X rays beyond the ultra-violet, as shown in the table of wave-lengths.

In addition to light and heat, it is obvious that light waves possess other properties, especially the chemical actions which occur in photography, bleaching, the generation of carbonic acid, and the formation of chlorophyll necessary for vegetable life, although for the latter, the heat rays may be equally active or even more so.

There is no sharp line of separation between the heat, light and chemical parts of the spectrum; the calorific effect of the red dies away gradually towards the more luminous portions, and as the latter fade away the actinic effect of the blue and violet increases. Only the central part of the spectrum, particularly the yellow, can be said to cause light only.

The existence of the *infra-red* waves may be shown in various ways. Thus a blackened thermometer bulb placed just beyond where the red in the spectrum ceases will show a rise of temperature, proving the existence of heat rays. Again, by employing a lens made of rocksalt, which readily transmits the long heat waves, the latter can be demonstrated when the visible spectrum is cut off.

The existence of the *ultra-violet* waves can be proved by placing beyond the visible violet a screen coated with a substance of a fluorescent nature which glows under the influence of the ultra-violet light. A quartz prism, which is very transparent to the short vibrations, must be used to produce the spectrum.

Speed and Frequency of Light.—The visible spectrum consists of those light waves whose lengths vary approximately between 750 and 400 $\mu\mu$, and whose vibrations respectively vary between 400 and 750 billions per second. The speed of light in air is 300,000 kilometres per second, and if we express the length of the waves in billionths of a kilometre, that is, in $\mu\mu$, and the frequencies in billions per second, then by dividing 300,000 by the wave-length in $\mu\mu$ the number of billions of frequencies per second for any kind of light is obtained. The wave-length multiplied by the frequency of any part of the spectrum is a constant, i.e. $LT = V = 300{,}000$.

In the yellow, which is the most luminous part of the solar spectrum, the number of billionths of a kilometre of the wave-length is approximately equal to the billions of frequencies per second. The mean refractive index of glass, or any other substance, is expressed by that of yellow light (the D line).

Luminous Bodies.—Light is termed *incident* when it falls on a body. A body is *luminous* when it is, in itself, an original source of light. Every visible body, which is not in itself a source of light, is illuminated by the light it receives from a luminous source, but it is convenient to consider that every visible body is luminous, since light is radiated from every point of it. The rays diverging from these points travel without change so long as they are in the same medium.

Transparency, Opacity and Translucency.—A body is *transparent* when light passes freely through it, with a minimum of absorption or reflection, such as clear glass. It is *opaque* when all the rays of light, incident on it, are either absorbed or reflected, so that none traverse it. It is *translucent* when it transmits only a portion of the light, as frosted glass and tortoise-shell. Much of the light incident on such a body is reflected, scattered or absorbed, so that objects cannot be seen clearly through it.

Reflection.—Reflection is the rebound of light waves from the surface, on which they are incident, into the original medium. The reflection is *regular* from a polished surface and *irregular* from a roughened surface. Regularly reflected light causes the image of the original source of light to be seen, the reflecting surface being practically invisible, and is treated in Chapter III.

Irregular Reflection.—When light falls on an unpolished surface such as white paper it is, owing to the irregular nature of the surface, incident at all conceivable angles, at each point of the surface. The incident light is broken up so that each point of the surface, giving rise to a fresh series of waves, becomes a source of light. No image is therefore formed either of the original source, or of any external object, but the diffused light diverging in every direction renders the surface visible, no matter from what direction it is viewed, and it is either coloured or white according as some wave-lengths are, or are not, absorbed.

Relative Transmission, Absorption and Reflection.—No substance is absolutely transparent, the clearest glass or water absorbing some of the incident light. It is estimated that below 50 fathoms the sea is dark, at least to the human eye, and even glass of sufficient thickness is opaque. Again any ordinary opaque object such as stone, metal, etc., may be ground or hammered into a sheet so thin as to permit the passage of some light through it. Thus gold leaf of sufficient thinness is translucent and transmits greenish rays. It follows, therefore, that transparency and opacity are relative, and depend not only on the nature of the medium, but also on its thickness.

A body usually translucent may be rendered transparent by making it less capable of reflection. If a drop of Canada balsam be placed on a camera focussing-screen, and a cover-glass pressed over it, the screen becomes transparent there. The liquid fills the spaces of the rough surface and, being of the same index of refraction, converts the whole into a homogeneous body. Moistening a piece of paper with oil or water makes it much more translucent. The fibres of which the paper is made are optically denser than the air, so that, when the latter is replaced by oil or water, the two are more nearly alike and less light is scattered. The glass tube of a soda-water siphon is visible in the water, but if the water were replaced by oil of the same optical density as that of the tube, the latter would be invisible.

Material, such as tracing paper or frosted glass, which is ordinarily translucent, becomes transparent when an object, such as print, is placed in contact with it, or has formed on it a real image.

The rougher the surface, the greater is the proportion of irregularly reflected light; the smoother the surface, the greater that of regularly reflected light. The proportion of light regularly reflected from a partially roughened surface is increased as the angle of incidence of the light becomes greater, so that a reflected and fairly distinct image may be obtained with very oblique incidence of the light from a body which ordinarily gives no definite reflected image, as, for instance, polished wood.

Total regular reflection never occurs, for even a silvered mirror or highly polished surface of metal fails to reflect all the light falling on it, but the proportion reflected by metallic surfaces does not vary so much with the incidence of the light as it does with glass. Polished silver reflects some 90 per cent., polished steel some 60 per cent., and mirrors reflect from about 70 to 85 per cent. of the incident light. Nor is there ever total irregular reflection; even fresh snow absorbs some of the light it receives.

It is probable that the highest amount of diffusive reflection from a perfectly diffusive surface, as white blotting-paper, is greatest with normal incidence.

Some of the incident light is reflected from the polished surface of a transparent body, and the proportion reflected varies with the nature of the body and with the angle of incidence, it being greater as such angle increases. The proportion reflected is very small (about 8 per cent.) when the light is incident perpendicularly, and it is almost totally reflected if the angle of incidence is nearly 90°.

If with perpendicular incidence practically all the light is transmitted and none reflected, and if with an extremely oblique incidence (nearly 90°) practically none is transmitted and all reflected, there must be some angle of incidence at which half the light is reflected and half transmitted and refracted. This occurs when the light is incident at about 70° with the normal to the point of incidence. Also the proportion reflected increases as the index of refraction of the medium is greater, and *vice versa*. If glass is dusty, the irregularly reflected light is increased and the glass becomes more visible. Scratches on a piece of glass roughen the surface and so tend to destroy its transparency by irregularly reflecting the light. If the scratches be multiplied indefinitely, the glass ceases to be transparent and becomes translucent.

Thus, in the case of every transparent body, some of the incident light is always transmitted, some absorbed and some reflected. Of the light falling from all sides on to a piece of well-polished transparent glass, about

75 per cent. is refracted and transmitted, 15 per cent. is regularly reflected and gives an image of the source from which the light proceeds, about 5 per cent. is irregularly reflected, and so makes the glass itself visible, while the remainder is lost, being absorbed and changed into heat, etc.

Linear Propagation of Light.—The propagation of light is rectilinear, and the familiar instance of sunlight, admitted through a hole in the shutter into a darkened room, illustrates this fact by the illumination of the dust particles in the air along its path. The illuminated dust renders the course of the light visible, for light itself is invisible, unless directly received by the eye so as to cause vision.

Divergent Light.—In nature, light always diverges from luminous points. A sphere may be regarded as the common terminal of a multitude of straight lines diverging from a point. A wave-front as it advances is an arc of a circle of which the luminous point is the centre; the multitude of straight lines contained in the arc are rays of light which, diverging from a luminous point, form a cone, of which the point itself is the vertex, and such a collection of rays is called a *pencil* of light. From a luminant of sensible size an innumerable number of such cones of light diverge, all having as their common base the illuminated object itself.

The divergence of the light is proportional to the angle included between the rays, proceeding from the luminous point, which fall on the outermost edges of the object; consequently the angle of divergence varies inversely with the distance between the source of light and the illuminated body, and directly with the size of the body. The latter is usually ignored, because it is a fixed quantity for any particular case.

If the luminous point be very distant the angle of light divergence becomes so small that it may be ignored and the rays from that point considered parallel to each other; the luminous point is then said to be at infinity. A collection of such parallel rays is called a *beam* of light.

Parallel Light.—If light from a distant point is regarded as parallel, and that from a near point as divergent, there must be some distance at which divergence can be assumed to merge into parallelism. In visual optics 20 feet or 6 metres marks the shortest distance from which light is regarded as parallel, and this distance, or any beyond it, is regarded as infinity, which is written thus: ∞. For some branches of optics a much greater distance is taken as the divergence limit. Thus in photographic optics it may amount to 100 yards or more, while in astronomy the nearest ∞ point may be taken as several miles.

Convergent Light.—Light is never naturally convergent, but can be rendered so by means of a lens or reflector. A collection of convergent rays is also called a *pencil* of light; the apex of the pencil, towards which they are convergent, is the focus. Similarly, therefore, if light is *converging* to a focus a great distance off, it may be considered parallel; for visual purposes, such distance is 6 M. or more.

Light Divergence.—If δ is the angle of divergence, a the aperture of the lens, and d the distance of the source, the angular divergence of light is, with sufficient exactitude, found from $\tan \delta = a/d$. For example, suppose the source of light is at 6 M., and the pupil of the eye to be 3·5 mm. in diameter, then the angle of divergence will be 2′, since

$$\tan \delta = \frac{3\cdot5}{6000} = \cdot0006 = \tan 2'.$$

A divergence of 2′ is so small that it is negligible, and 6 M. considered the same as ∞ in this connection. At 20 cm., with the same pupil, the divergence of the light is 1°.

As before stated, the size of the receiving body is usually ignored, it being taken to be uniform, or as unity, and if circular measure is substituted for angles, then the divergence of light is the reciprocal of the distance of the source, i.e. $1/d$.

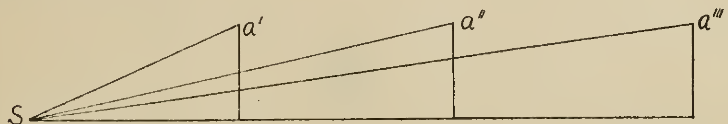

Fig. 7.

This is shown in Fig. 7, where the light from S has a divergence which varies inversely as the receiving surface is at a', a'', or a'''.

Later it will be seen that the divergence of light enters into our calculations for conjugate foci, and it is then expressed as $1/f_1$ or $1/u$.

Object and Image.—The source of the light is, in optics, commonly called the object. An image is the reproduction of the object due solely to light; it may be real, or only imaginary or virtual.

Optics, the science treating of light and vision, includes **Catoptrics,** which deals with reflection from polished surfaces, and **Dioptrics,** which deals with refraction by transparent media.

CHAPTER II

SHADOWS AND PHOTOMETRY

Shadows.—Since light travels in straight lines, any opaque body in their path will arrest their march and cast, on a screen behind it, a negative image of itself, called a *shadow*. When the ground on which the shadow is cast is at right angles to the central line connecting the source, the body and the

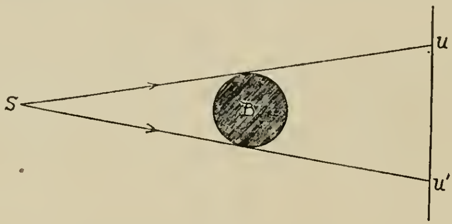

Fig. 8.

ground, the shadow has an outline corresponding to that of the body, because then, as in Fig. 8, the periphery of B cuts off the light equally in every direction. The shape of the shadow otherwise depends on the inclination of the screen to the opaque body and the source of light.

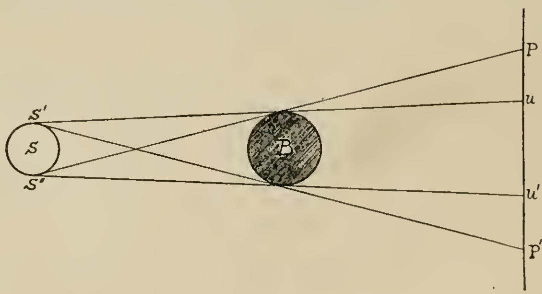

Fig. 9.

In general the shadow (Fig. 9) exhibits a dark centre $u\,u'$ called the *umbra*, from which the light is entirely cut off, and a less black outer portion $Pu, P'u'$, called the *penumbra*, which receives a certain amount of illumination. The space Pu receives light from S', but none from S'', while $P'u'$ receives light from S'', but none from S'. The area $u\,u'$ receives light from neither S' nor S''.

If the light S is, or approximates to, a point, the shadow is mainly umbra and uniformly dark, as $u\ u'$ in Fig. 8. It becomes larger as the shadow is further away.

If S is of definite size (Fig. 9), but smaller than the intercepting body B, both the umbral and penumbral cones are divergent, and become larger as the shadow is further from B.

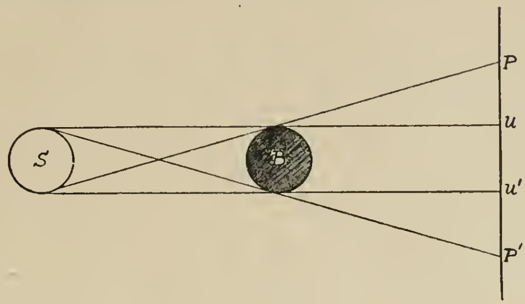

Fig. 10.

When S and B are of equal size (Fig. 10), the umbra does not vary in size with its distance, but the penumbra increases as it is further away, because the penumbral cone diverges.

When S is larger than B (Fig. 11), the umbra decreases with distance, since the umbral cone is convergent, while the penumbra increases, the

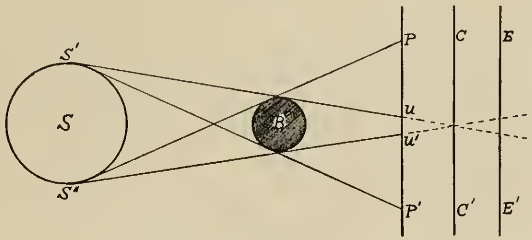

Fig. 11.

penumbral cone being divergent; beyond a certain point there is no umbra, as when the screen is at $C\ C'$ or beyond it at $E\ E'$.

The penumbral cone is *always* divergent, but the umbral cone may be divergent, cylindrical, or convergent according as the source S is smaller than, equal to, or larger than the intercepting body B.

When the hand is held close to a wall, in a well-illuminated room, the projected shadow is almost entirely umbra; as the hand is moved away the umbra decreases and the penumbra increases until, at a certain distance, the whole shadow becomes penumbral. The larger the size of the luminant as compared with that of the intercepting body, the smaller is the umbra, and the larger and fainter the penumbra, and *vice versa*.

Except with a very small source, approximating to a point, the edge of a shadow is never cleanly defined, nor are the umbral and penumbral portions sharply separated, but merge imperceptibly into each other. Generally the brighter the light, the deeper is the shadow cast, for then the contrast between the illuminated ground and the part, from which the light is totally or partially obstructed, is greater than in a dull light, when shadows are barely perceptible.

Calculations of Umbræ and Penumbræ.—The calculations for determining the size of the umbra and penumbra are somewhat complicated and vary with the conditions under which the shadow is cast, so that every case must be worked out on its own merits, and from general principles. But if we assume that the size of the luminant is small compared with its distance from the intercepting body (and this is so in the great majority of cases), the necessary calculations can be much simplified. The angle subtended by the luminant at the intercepting body being small, either the edge or centre of the luminant (Fig. 12) may be *assumed* to be in line with either edge of the body, so that the edge of the geometrical shadow may be regarded as *exactly bisecting the penumbral cone on either side.* By the geometrical shadow is meant an imaginary space on the screen equal in size to the intercepting body.

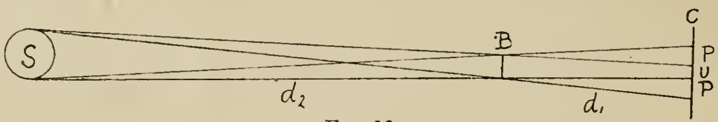

FIG. 12.

In Fig. 12 let U be the size of the umbra, P that of either penumbral cone, T that of the total shadow, S that of the source of light, B that of the intercepting body. Let d_2 be the distance of S to B, and d_1 that of B to the screen C. Now the angle subtended by S at the edge of B equals the angle of the penumbral cone, so that

$$\frac{P}{S}=\frac{d_1}{d_2}, \quad \text{or} \quad P=\frac{Sd_1}{d_2}$$

and
$$U=B-2\frac{P}{2}=B-P$$

so that
$$T=2P+U=B+P$$

If $P=B$ there is no umbra, and if P is greater than B, then U is negative.

As an example, if S be a square window 2 ft. in diameter, the size of P and U on a wall 20 ft. distant, cast by a coin 1 in. in diameter held 1 ft. from the wall, would be thus—

$$P=\frac{24\times 1}{19}=1{\cdot}26'' \qquad U=1-1{\cdot}26=-{\cdot}26''$$

$$T=1+1{\cdot}26=2{\cdot}26''$$

Thus there is no umbra, it being a negative quantity, as on EE', Fig. 11. If the coin were 2 in. in diameter, the other conditions being similar, we should have

$$P=\frac{24\times 1}{19}=1{\cdot}26'' \qquad U=2-1{\cdot}26={\cdot}74''$$

$$T=2+1{\cdot}26=3{\cdot}26''$$

Here the umbra is real or positive, as on $P\,P'$, Fig. 11.

When the angle a subtended by the luminant only is known,

$$P=d_1 \tan a$$

Thus if B is 3″ diameter, and 100″ from a wall, S being the sun subtending an angle of 30′,

$$P=100 \tan 30'=100\times{\cdot}0087={\cdot}87'' \qquad U=3-{\cdot}87=2{\cdot}13''$$

$$T={\cdot}87+3=3{\cdot}87''.$$

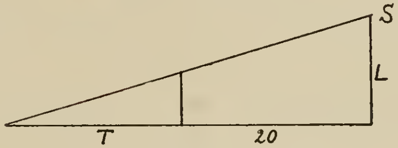

Fig. 13.

Shadows cast on the Ground.—For the length of a shadow cast by a vertical narrow body on to a horizontal plane, generally a simple proportion suffices. For example, what is the length of the shadow cast by a stick 3 ft. long, 20 ft. from a lamp post L whose lamp is 10 ft. above the ground? Then, if the length of the shadow be T (Fig. 13), the horizontal distance to the end of the shadow is $20+T$,

and $$\frac{20+T}{10}=\frac{T}{3}$$

that is, $$60+3T=10T$$

or $$7T=60 \text{ and } T=8{\cdot}57 \text{ ft.}$$

Shadows cast by Lenses.—A lens or prism, when placed between a source and a screen, casts a shadow like an opaque body. The light transmitted by a concave lens is diverged so that a dark area, surrounded by a luminous zone, is thrown on the screen. A convex lens condenses on to a small area all the light passing through it. This bright area is surrounded by a dark one from which all light is excluded. If light be passed through a prism the space on the screen immediately behind it is dark, the light deviated by the prism falling on another part of the screen, which, being also illuminated directly, exhibits there a bright area.

Photometry.—The measurement of the *luminosity* of a light source, or of the *illumination* of a surface, is termed *photometry*, and the instrument or apparatus employed is called a *photometer*.

Luminosity is the illuminating power of a light source; it is expressed in *candle-power* (C.P.), the latter unit being the luminosity of a standard candle, as described later.

A luminous source, unless it be a point, has a definite area which is seldom of equal luminosity throughout. The quantity of light emitted varies at different points, but the sum of the light emitted is the total luminosity, and it is this which is measured or expressed in C.P. The intrinsic intensity of luminosity is the mean quantity of light emitted from a unit of surface, and is expressed by the total amount of light emitted divided by the area of the luminous source.

Illumination is the amount of light received by an illuminated surface from a luminous source. The intensity of illumination is the amount of light which falls on a unit of the illuminated surface, and is expressed in *foot-candles*, the foot-candle (F.C.) being the luminosity of a standard candle at the unit distance of 1 foot. The term "foot-candles" expresses the luminosity of so many standard candles at 1 foot distance.

Intensity of Illumination.—To illustrate how the intensity of illumination varies with the distance between a source of light and an illuminated area, suppose a candle flame to be at the centre of a sphere of one foot radius, and let the intensity of the light at the surface be considered unity. The area of a sphere is equal to $4\pi r^2$, r being the radius. Now if the radius of the spherical envelope be increased from one foot to two feet, i.e. doubled, its area will be quadrupled, and therefore the available light is distributed over 4 times the area and the amount of light received on each point of the sphere is $\frac{1}{4}$ of what it was when the radius was one foot. If the sphere be 5 feet in radius its area will be increased 25 times, and the available light on a given area is but $\frac{1}{25}$ that of the first sphere.

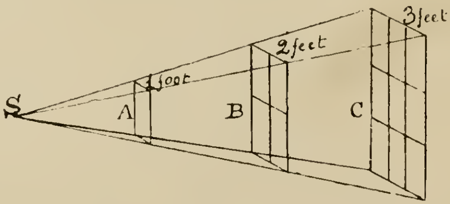

Fig. 14.

The Law of Inverse Squares.—Since a flat surface virtually forms a portion of a sphere having the source of light for its centre, it may also be stated, without much error, *that the illumination of a flat surface varies inversely with the square of its distance from the source of light.*

This is illustrated in Fig. 14, where *S* is the source of light and *A*, *B* and *C* screens subtending equal angles, placed vertically at distances of 1, 2 and

3 feet respectively. The same amount of light from S is received by all, but at B it is spread over 4 times and at C over 9 times the area that it is at A. It follows, therefore, that each unit of area of B receives $\frac{1}{4}$, and of C receives only $\frac{1}{9}$ of the quantity of light received by each similar unit of A, practically the same as in the case of a spherical surface. The formula which expresses the intensity of illumination I in F.C. received by a surface at a given distance d from a source of given candle-power C.P. is

$$I=\frac{C.P.}{d^2}$$

This follows from the law of inverse squares above.

For example, the illumination received by a surface 5 feet from a 100 C.P. lamp is

$$I=\frac{100}{5^2}=\frac{100}{25}=4 \text{ F.C.}$$

Again, at what distance must a 32 C.P. lamp be placed above a table to give an illumination of 2 C.P. directly underneath ?

Here $$2=\frac{32}{d^2}$$

whence $d^2=16$ and $d=4$ feet.

If at 1 foot a certain intensity of illumination is obtained from a lamp, and this be moved to, say, 9 feet, then the intensity becomes $1/9^2=1/81$ of the illumination received at one foot, and it will require 81 such lamps to obtain an equal intensity as at 1 foot.

For equal illumination, the luminosity or C.P. of a source is directly proportional to the square of its distance from the surface.

It should be noted that I does not depend on the colour or nature of the receiving surface, which might reflect much or little of the light it receives.

A standard of illumination termed a *lumen* is that of an area of 1 square foot illuminated with an intensity of 1 foot-candle.

Apparent Exceptions.—The law of inverse squares holds good only for light received directly on a screen, and not if it passes through a lens system so as to form an image, as in a camera or the eye. The light gained by bringing the object nearer is exactly neutralized by spreading it over a proportionately larger area in the image formed on the screen or retina.

A luminous or illuminated surface appears equally bright at whatever angle it is seen, since, although it receives less light, the area perceived is correspondingly diminished.

Obliquity of Surface or of the Light.—The intensity of illumination depends also upon the obliquity of the surface. *It varies as the cosine of the angle of incidence of the light*—that is, the angle that the rays make with the normal or perpendicular to the surface.

In Fig. 15 S is a source, and AB is an illuminated surface tilted through the angle b. The angle of incidence is i, which the light from S makes with the normal NN', and it is obvious that the angles i and b are equal.

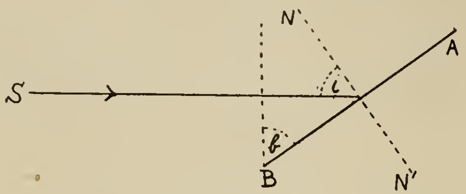

Fig. 15.

Suppose parallel light to fall square upon the surface CB (Fig. 16) and the latter be then tilted into the position AB through the angle b, so that the angle of incidence is i. Then in the latter case only those rays between

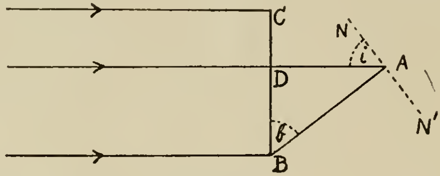

Fig. 16.

D and B will impinge on the screen, as compared with the full number between C and B before the screen is moved. The relative intensity of illumination of any point on AB as compared with CB is then

$$\frac{DB}{CB}=\frac{DB}{AB}=\cos b.$$

But $$b=i$$

So that the full expression for illumination for all conditions is

$$\mathrm{I}=\frac{\mathrm{C.P.}}{\mathrm{d}^2}\cos i$$

For example, if a screen be tilted 60° with respect to the incident light, or, what is the same thing, the light falls upon it at an angle of incidence of 60°, the illumination is exactly one half, because cos 60°=·5.

Calculations on Illumination.—A 30 C.P. lamp is 5 ft. above a table; then on the latter directly underneath

$$\mathrm{I}=\frac{\mathrm{C.P.}}{\mathrm{d}^2}=\frac{30}{5^2}=1{\cdot}2\ \mathrm{F.C.}$$

At a point 4 feet away along the table the oblique distance of the lamp is

$$d=\sqrt{4^2+5^2}=\sqrt{41}=6{\cdot}4 \text{ ft.}$$

and

$$I=\frac{C.P.}{d^2}\cos i=\frac{30}{41}\times\frac{5}{6{\cdot}4}={\cdot}6 \text{ F.C.}$$

Here the actual value of i is not necessary, as its cosine is obvious.

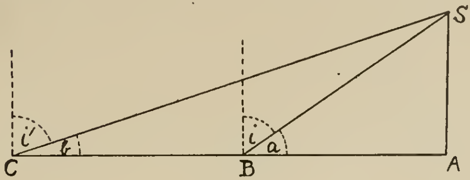

Fig. 17.

In Fig. 17 AC is a floor 20 ft. long, and S is a 100 C.P. lamp 7 ft. above A, the point directly underneath. What is the illumination at A, at B 10 feet from A, and C 20 ft. from A?

At A

$$I=\frac{100}{7^2}=2 \text{ F.C.}$$

The oblique distance $SB=\sqrt{SA^2+AB^2}=\sqrt{7^2+10^2}=12$ ft. (approx.); and since the angle of incidence $i=ASB$

$$\cos i=\frac{AS}{SB}=\frac{7}{12}$$

Therefore at B

$$I=\frac{100}{12^2}\times\frac{7}{12}={\cdot}4 \text{ F.C. (approx.)}$$

and at C

$$I=\frac{100}{7^2+20^2}\times\frac{7}{\sqrt{7^2+20^2}}=\frac{700}{449\times 21}={\cdot}074 \text{ F.C.}$$

In the last example the angle of incidence is $i'=ASC$.

In some cases, notably that in Fig. 17, the more obvious angles of obliquity of the light are a and b, i.e. those between the light and the illuminated surface itself. The same results as in the above examples can be obtained provided $\sin a$ be substituted for $\cos i$, and $\sin b$ for $\cos i'$, the angles a and i, also b and i', being complementary.

Photometric Standards.—The usual standard of luminosity in Great Britain is that given by a sperm candle $\frac{7}{8}$ inch in diameter, $\frac{1}{6}$ of a pound in weight, and burning 120 grains per hour. It has a variation of about 20 per cent. The luminosity of gas, with an ordinary burner, is from 12 to 16 British candles (B.C.).

There are various other photometric units, among them the following:—

The "Pentane" standard. A mixture of pentane gas and air is burnt at the rate of $\frac{1}{2}$ cubic foot per hour; the flame is circular, $2\frac{1}{2}$ inches high and ¼ inch in diameter, and there is neither wick nor chimney. Pentane is a volatile liquid, like naphtha, prepared from petroleum. The form designed by Vernon Harcourt is a 10 candle-power standard, and is largely used in this country. It is said to vary less than 1 per cent.

The German standard is the Hefner-Alteneck lamp, called a "Hefner-lamp" (H), having a cylindrical wick 8 mm. in diameter burning amylacetate, the flame being 40 mm. high. It is correct to about 2 per cent.

The French "Carcel" is a lamp of special construction burning 42 grammes of colza oil per hour.

The "Violle" or absolute unit was the standard invented by M. Violle, and adopted at the International Congress at Paris in 1884. It consists of the light emitted from a square cm. of platinum heated to its melting-point. Of all the standards it is the most exact and reliable, but it is expensive and difficult to apply.

The International Congress of 1890 adopted as the standard the "Bougie-decimale" or decimal candle, the unit illumination of a surface being that produced by one bougie-decimal at one metre.

The British candle and the bougie-decimal have about the same intensities. The "Carcel" equals about 9½ candles, and the "Violle" about 20 candles. 20 bougie-decimals=19·75 B.C.=22·8 Hefner=2·08 Carcel=1 Violle.

Measurement of Light Sources.—Photometry consists of making a comparison of the unknown illuminating power of a source of light with that of a known source or standard. Direct comparison would be difficult, but the stronger light can be placed at a greater distance, where it produces an intensity of illumination equal to that of the standard at some shorter distance. The illuminating powers of the two sources are as the squares of the distances at which, on a given surface, they respectively produce equal intensities of illumination. If the luminosity of the unknown source be L and that of the standard C, and the distance of L be d_2 and of C be d_1 then

$$\frac{L}{C}=\frac{d_2^2}{d_1^2}$$

If a standard candle at 1 ft. and light at 4 ft. give equal intensity of illumination at some common point, then the greater luminant is $1\times4^2=$ 16 C.P., because

$$\frac{1}{C}=\frac{1^2}{4^2} \text{ or } C=\frac{1\times16}{1}=16 \text{ C.P.}$$

Four candles 4 ft. from a screen have the same effect as one candle at 2 ft., for $2^2/4^2=4/16=1/4$.

The Rumford Photometer.—The *shadow* or *Rumford* photometer consists of a vertical white screen before which is placed a rod. The standard is placed (preferably at one foot) in front of the screen and the rod casts a shadow. The luminant (Fig. 18) to be measured is placed so far away that the shadow cast by the rod from its light is of equal intensity to that of the other. The space on the screen, occupied by the candle's shadow, is illuminated only by the light from the lamp, while that occupied by the lamp's shadow is illuminated only by the standard. It is these intensities

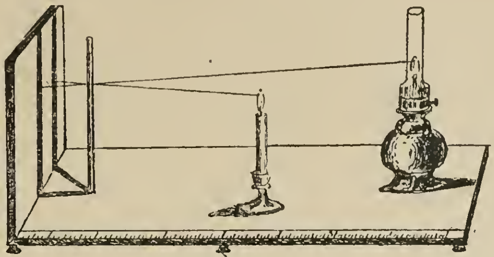

FIG. 18

of illuminations that are actually compared, although apparently it is the shadows themselves. The lights should be placed so that the two shadows lie near to each other without overlapping. The luminant measured is of so many candle-power according to the distance at which the shadow pertaining to it equals in depth that pertaining to the standard, for then $L/d_2^2=C/d_1^2$.

The Bunsen Photometer.—The *grease spot* or *Bunsen* photometer consists of a screen of white paper, suitably mounted in a frame, on which there is a spot rendered semi-transparent by grease or oil. If the paper be viewed on the side remote from a luminant the grease spot looks lighter than the

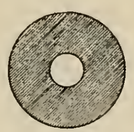

FIG. 19.

FIG. 20

balance of the paper, because more light penetrates (Fig. 19). Viewed from the other side, the grease spot looks darker, because less light is reflected from it than from the rest of the paper (Fig. 20). Used as a photometer, the screen is placed one foot from the standard, the light from which is totally reflected by the ungreased part of the screen and transmitted to a great extent by the grease spot. The luminant to be tested is placed on the other side of the screen at such a distance that the amount of light from it, transmitted by the grease spot, equals that passing the other way; then the paper appears of uniform brightness all over. If there were employed a standard candle at one foot, then the candle-power of the light is equal to the square of its distance in feet from the grease spot. Otherwise, as before, $L/C=d_2^2/d_1^2$.

Instead of moving the lights about these may remain fixed and the screen moved about between them in order to obtain a balance.

The Slab Photometer.—The *paraffin slab* photometer consists of two thick slabs of solid paraffin separated by an opaque layer of tin foil. The two lights are placed one on either side, and their intensities are compared by viewing the sides of the two slabs simultaneously. This is also known as the Joly photometer.

The Lummer-Brodhun Photometer.—This photometer is largely used in scientific laboratories, being accurate to about 1%. Its superiority over the Bunsen and some other photometers is due to the fact that, with these, the two images to be compared cannot be seen simultaneously. With the Lummer-Brodhun instrument one combined image is seen by one eye.

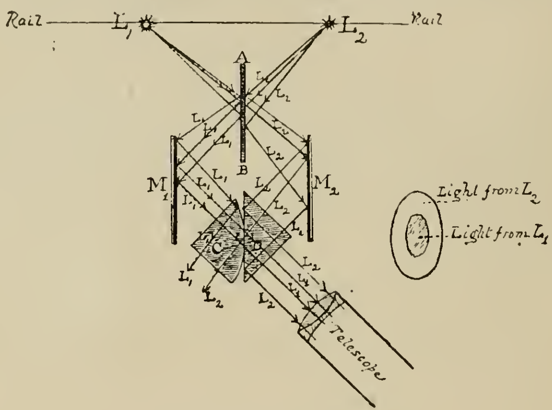

Fig. 21.

The instrument (Fig. 21) consists of a rail on which the two luminants L_1 and L_2 can be made to travel at right angles to the opaque screen A B, which is whitened on both sides. From A B the light is reflected to the two mirrors M_1 and M_2 and thence through the cube of glass C D made of two right-angled prisms cemented together, the hypothenuse of one of which is partly cut away.

The observer looks through a short telescope placed in front of D. The light from L_1 which reaches the telescope passes through the central cemented portion of C and D, while that from L_2 is reflected from the peripheral portion of D. The two lights therefore enter the eye simultaneously as a circle from L_1 and as a ring from L_2, as shown in the figure. The lights are moved to and fro along the rail until the whole disc appears equally bright.

The Simmance-Abady "Flicker" Photometer.—This consists essentially of a white circular disc or wheel, the edge of which is peculiarly bevelled by being "chucked" eccentrically at two positions with the turning tool set obliquely at 45°. Thus the periphery of the wheel, when revolved, presents a bevel of 45° on the one side, say the right, and no bevel on the left, then graduates to a knife edge, and finally to a bevel of 45° on the left and no bevel on the right.

This wheel is so fixed in a box that part only of it projects, and immediately in front of it, but leaving its projecting portions unobscured, there is a sighting tube carrying a Cx. lens for magnifying purposes. The box contains a clockwork arrangement by means of which the wheel is made to revolve at a rapid speed. The box itself is fixed on a bar 60 inches long, scaled in terms of a standard candle, and along which the apparatus can be freely moved.

The two luminants which are to be compared are placed one at each end of the bar, and the light from them falls on that part of the revolving disc which projects from the box. When the light falls on the bevelled edge at 45° it is reflected, and, passing through the sighting tube, is seen by the observer. When incident on the unbevelled part of the disc, the light does not pass through the sighting tube, so that each luminant is alternately *light* and *dark* to the observer's eye, and both are light at the same time when the knife edge is immediately in front of the sighting tube. Then when the intensities are equal the light is absolutely steady, while it flickers when they are not. If there is flickering the apparatus is moved until this disappears, and the position is found where $L/d_2^2=C/d_1^2$. The smallest alteration of the position of the apparatus towards either light causes flicker. The test is made more sensitive, and the point of *balanced intensities* more exactly located, when the speed of revolution of the wheel is lessened. The apparatus can be set obliquely for measuring lights at any angle, and one of its special advantages is mentioned in the next article.

Photometry of Coloured Lights.—One of the great difficulties of photometry is the difference in the nature and colour of various lights; and the comparison or measurement of actually coloured or monochromatic lights is still more difficult, or rather impossible, by ordinary photometry.

The eye, although fairly accurate in judging the difference of hue of two sources, is very deficient in the comparison of the relative intensities of two differently coloured lights. These difficulties seem, however, to be obviated by the Simmance-Abady photometer. Here the rapidly alternating light from the sources does not afford the eye sufficient time to appreciate the difference of colour, but only their difference of intensity, since the flicker depends on intensity of illumination on the two sides of the bevelled disc, and is independent of the colour of these illuminations.

Therefore by the flicker photometer coloured lights, and therefore also the transmissive qualities of coloured and smoked glasses, can be compared and measured. By it also the illuminating power of daylight can be measured, as well as that of different sources of artificial light. Coloured lights may, however, be compared by occlusion, using for the purpose a series of properly graduated smoked glasses.

Calculations in Photometry.—Having by means of a photometer made the intensities of illumination equal, the candle-power of the luminant is calculated from the formula

$$L=\frac{Cd_2^2}{d_1^2}.$$

When d_1 is unity or 1 foot of course no division is necessary, as the square of 1 is 1. Thus if the luminant at 5 feet is equal to a standard candle at 1 foot, the former is of $5^2=25$ C.P.

If the candle is at 2 feet and the luminant at 8 feet

$$L=1\times\frac{8^2}{2^2}=1\times\frac{64}{4}=16 \text{ C.P.}$$

To compare the intensity of two sources L_1 and L_2 of different powers, if L_1 be 30 C.P. placed at 20 feet, while L_2 is 200 C.P. at 70 feet, their relative intensities are

$$L_1=\frac{30}{20^2} \qquad \text{and} \qquad L_2=\frac{200}{70^2}$$

so that

$$\frac{L_2}{L_1}=\frac{200}{4900}\times\frac{400}{30}=\frac{5}{9} \text{ (approx.)}.$$

The relative distances for equality of illumination of two sources of 9 C.P. and 36 C.P. are as $\sqrt{9} : \sqrt{36}=3 : 6$ or as $1 : 2$.

What power lamp at 100 feet would give the same illumination as one of 1,000 C.P. at 30 feet ?

$$\frac{L}{100^2}=\frac{1000}{30^2} \text{ or } L=11,111.$$

At what distance should an arc lamp of 1,200 C.P. be placed so as to give an illumination three times as great as that of an incandescent light of 70 C.P. at 15 feet ?

$$\frac{70}{15^2}\times 3=\frac{1200}{d^2} \text{ or } d^2=\frac{1200\times 225}{70\times 3}=1286$$

therefore

$$d=\sqrt{1286}=36 \text{ feet (approx.)}.$$

Light Transmission and Absorption.—The transmission and absorption of light by smoke glass, frosted glass, etc., can be measured fairly closely by means of a simple photometer such as the Bunsen. Take any two sources of light, A and B, balance them photometrically in the usual way, and measure the distance d_1 in feet or inches of one of them, say B, from the screen. Then interpose the smoke glass to be tested between A and the screen, when it will be found necessary to withdraw B in order to secure a second balance; let this distance be d_2. Then the relative intensities of illumination of A, with and without the smoke glass, are as $d_1^2 : d_2^2$, so that the proportion transmitted is

$$\frac{d_1^2}{d_2^2} \text{ or } \frac{100 d_1^2}{d_2^2} \%.$$

Therefore the proportion cut out is

$$1 - \frac{d_1^2}{d_2^2} \text{ or } \frac{100(d_2^2 - d_1^2)}{d_2^2} \%.$$

If the first distance d_1 is unity there is transmitted by the glass

$$\frac{1}{d_2^2} \text{ or } \frac{100}{d_2^2} \%.$$

And there is cut out $1 - \frac{1}{d_2^2}$ or $\frac{100(d_2^2 - 1)}{d_2^2} \%$.

It will be noticed that only B is moved, A remaining fixed; nor does the actual distance of the latter from the screen affect the result. The smoke glass should be sufficiently large to cover the light completely, and be close to it, although its position between A and screen is immaterial.

Example:—If B, when at 2 ft., balanced A, and had to be moved to 3 ft. when a smoke glass was placed before A, then the light transmitted by the glass is

$$\frac{d_1^2}{d_2^2} = \frac{2^2}{3^2} = \frac{4}{9} \text{ or } 45 \%.$$

And that blocked out is $1 - \frac{4}{9} = \frac{5}{9}$ or 55 %.

Coloured Glasses.—To measure the absorptive or transmissive power of a coloured glass the method described above can be employed, but, for the reason given previously, ordinary artificial lights, which are generally white or yellowish, cannot be employed alone. To overcome this difficulty the following procedure may be followed. Suppose the glass to be measured is green. Place over A and B a green glass lighter in tint than the one to be measured; this renders the light uniform, though duller, and the necessary measurements can then be carried out as for neutral glasses. This subject is further discussed in Chapter XIX.

CHAPTER III

REFLECTION AND MIRRORS

A *normal* is a straight line perpendicular to a given point, as *P C* in Fig. 22. The *angle of incidence* is that which an incident ray makes with the normal at the point of incidence.

Regular Reflection.—When light falls on a smooth surface it is reflected in definite directions according to the following laws:—

(1) The incident and reflected rays are in the same plane as the normal to the point of incidence, but on opposite sides of it.

(2) The angle of reflection is equal to the angle of incidence.

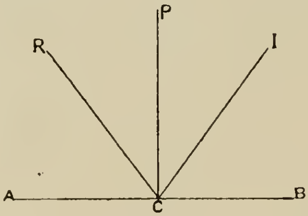

FIG. 22.

Oblique Incidence.—In Fig. 22, *A B* is a reflecting surface at which the ray *I C* is incident at the point *C*, and reflected in the direction *C R*. *P C* is the normal to *A B* at *C*, and the angle of reflection *R C P* is equal to the angle of incidence *I C P*. The perpendicular divides equally the angle *I C R* between the incident and reflected rays, and all three lines are in the plane of the paper.

Perpendicular Incidence.—If the ray be incident in a direction *P C* normal to the surface the angle of incidence is zero, and therefore the angle of reflection is also zero; the ray is thus reflected back along its original path.

Images.—An image of a point is formed when the light, diverging from it, is caused, by reflection or refraction, to converge to, or to appear to diverge from, some other point. An image is said to be *real* or *positive* when the reflected or refracted rays from the original object-point are made to *converge* and *actually meet* in the image-point. If the original rays, after reflection or refraction, are *divergent*, they are referred back by the eye to an imaginary image-point, and the latter is then said to be *virtual* or *negative*. Similarly the real or virtual image of an object is made up of the real or virtual images of its innumerable points.

A real image can be received and seen on a screen, or it can be seen in the air, where it actually exists. A virtual image cannot be formed on a screen; it is only mentally conceived where it appears to be.

Mirror.—A mirror is an opaque body with a highly polished surface. It is usually made of glass backed by a film of mercurial amalgam, or coated with an extremely thin layer of silver. It may also be made of polished metal.

Reflection by Plane Mirror.—If a beam of parallel light falls on a plane mirror, all the rays, having similar angles of incidence, are reflected under equal angles, and are therefore reflected as parallel light. If a pencil of divergent rays be thus incident, after reflection they are equally divergent, and appear to come from a point as far behind the mirror as the original luminous point is situated in front of it. Accordingly, if an object stands in front of a plane mirror the rays, diverging from each point on it, are reflected from the surface of the mirror and enter the eye of an observer as so many cones of light diverging from so many points behind the mirror, and these points, from which the light appears to diverge, constitute the virtual image of the original object. The complete image is erect and corresponds exactly as regards shape, distance, and size to the object itself, the relative directions of the rays from each point on the object being unchanged by reflection.

If the object is parallel to the surface of the mirror the image is also parallel; if the object is oblique to the surface the image forms a similar angle with it.

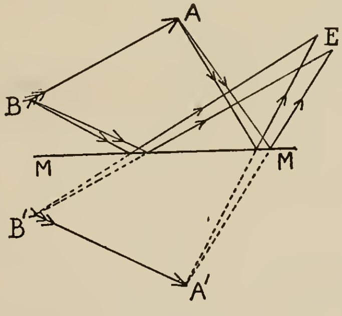

Fig. 23.

Construction of Image.—The image can be graphically constructed by drawing straight lines from the extremities of the object, perpendicular to the mirror or plane of the mirror, and continuing such lines as far behind the mirror as the object-points are in front of it. Thus, in Fig. 23, if a line be drawn from B to B', another from A to A', and B' and A' be connected, the image B' A' is obtained. Rays diverging from A, after reflection, enter the eye E, and are projected to a virtual focus at A', from which point they appear to diverge. Those from *B* are projected to *B′*, so that *A′ B′* is the virtual image of *A B*. *A′* is apparently as far behind *M M* as *A* is in front of it; so also *B* and *B′* are equally distant from *M M*.

Lateral Inversion by Reflection. — The image is, however, laterally inverted, the right hand of a person becoming the left of his image in the mirror, and *vice versa*. If the eye regards *A B* (Fig. 23) directly, *A* is to the right of *A B*, but looking into the mirror *A′* is seen to the left of *A′ B′*. If the top of a page of printed matter be held obliquely downwards against a mirror the letters will be in their true order from left to right, but they will be upside down. If held bottom downwards the print is upright but reversed right for left. Engravers sometimes use a mirror in front of the letters or objects they wish to draw on a wood-block and copy the image as seen in the mirror. On taking an impression of the block the letters or objects are in their right position.

Distance of Image.—If a person stands in front of a plane mirror, say at 2 ft., and looks into it he sees an image of himself at a distance of 4 feet. If an object is placed in contact with a glass mirror its image appears behind the silvered surface, and only twice the thickness of the glass itself separates object and image, although the image appears rather nearer owing to vertical displacement by refraction. If the mirror is of polished metal the two are in contact.

As in the case of any transparent dense medium, the apparent thickness of a mirror is less than its actual thickness, and the apparent thickness is decreased with increased obliquity of view.

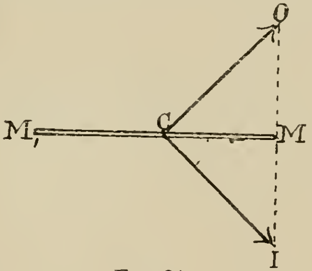

Fig. 24.

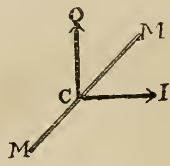

Fig. 25.

Position of Image.—Since the angle *O C M*, between the mirror *M M* and the object *O C* (Fig. 24) and the angle *I C M*, between the mirror and the image *C I*, are equal, it follows that the angle *O C I* between the object and the image is twice as large as either; therefore if the mirror be placed at an angle of 45° with the object, the object and image are at right angles to each other, as is shown in Fig. 25.

Angular Displacement of Image.—If a mirror be turned through any angle the image will move through twice that angle. The angles of incidence and reflection being equal, the total angular distance between the incident and reflected rays is twice the angle of incidence. If the latter be increased or decreased, by rotation of the mirror, any reflected light must turn through twice the angular displacement of the mirror, and travel at twice the angular speed. This fact is allowed for in the construction of the sextant (*q.v.*). In the reflecting galvanometer it doubles the delicacy of the readings.

Size of Mirror.—The smallest plane mirror which will enable a person to see his whole image reflected is one which is about half his height. More precisely the top of the mirror should be on a level with a point midway between the eyes and the top of the head, also it must be half the breadth, one eye being closed, but can be rather less if both are open.

To see in a mirror the whole of a test chart placed over one's head, the size of the mirror should be one half that of the chart in both dimensions; for other distances of object and observer see page 301.

Multiple Images.—When there is but one reflecting surface, as in a metal mirror, there is but one image, but in a glass mirror having two reflecting surfaces, namely, the front surface of the glass *A C* (Fig. 26) and its silvered back surface *B D*, there are multiple images of an object. If a candle flame *O* be held near to a glass mirror a series of images will be seen by an eye *E*; the first image I_1, that nearest to the candle, is formed by direct reflection

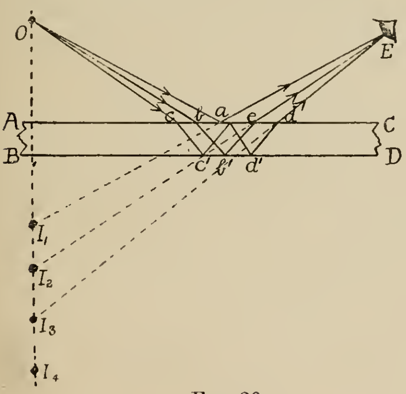

FIG. 26.

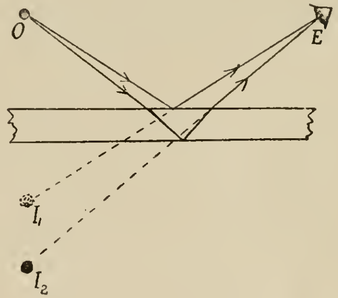

FIG. 27

from the front surface to the glass along *a E*; the second image I_2, which is the brightest, is directly reflected from the silvered surface along *e E*. The other images I_3, I_4, etc., equally distant from each other, are formed by *repeated internal reflection*, between the silvered surface and the front of the glass, but some of the light escaping by refraction at *e*, *d* - - -, after each reflection, the images become progressively fainter.

Ordinarily on looking into a mirror only two images are noticeable, the faint one reflected from the front and the bright one from the back surface (Fig. 27), but the more oblique the line of view and consequently the greater the angle of incidence of the light to the mirror, the greater is the separation and number of images seen. The total number visible also depends, of course, on the luminosity of the flame.

When the surfaces are truly parallel, and the view is direct, only one image is seen, because the multiple images all lie one behind the other in a straight line perpendicular to the mirror.

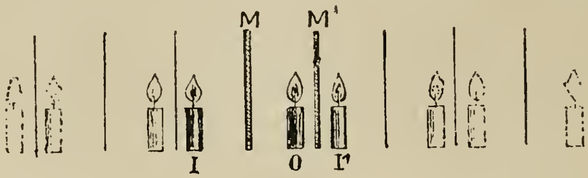

Fig. 28.

Parallel Mirrors.—If two plane mirrors *M* and *M′* (Fig. 28) are parallel to each other, and an object *O* is placed between them, a series of images (the first of which are *I* and *I′*), theoretically infinite in number, is produced by reflection of the light backwards and forwards between the two mirrors. As with the single mirror, the repeatedly reflected light soon becomes feeble and the number of images actually visible depends, therefore, on the brightness of the object. True parallelism of the mirrors is indicated by the images forming a straight line.

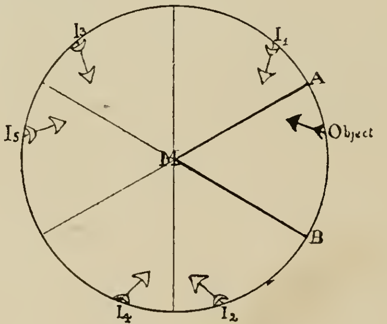

Fig. 29.

Inclined Mirrors.—When two mirrors *A M*, *B M* are mutually inclined (Fig. 29), the multiple images formed are situated on an imaginary circle passing through the object, the radius of which is equal to the distance of the object from the junction of the mirrors. The number *n* of images produced, including the object itself, is

$$n = \frac{360}{\text{angle}}$$

or, the number of images, including the object, being known, the angle can be calculated.

Thus, if the angle is 90° there are four, if 60° there are six, and if 45° there are eight images. A single mirror may be regarded as two inclined to each other at an angle of 180°; there are then two images, or rather, the object itself and its single image in the mirrors. If two mirrors are parallel the angle between them is zero, and the number of images is therefore $360/0 = \infty$, although, as stated, comparatively few are visible to the eye.

When the number of degrees between the mirrors is an exact even divisor of 360, as 45° or 60°, the complete figure is symmetrical from every point of view; if the number is an exact odd divisor of 360, such as 120° and 72°, the figure is symmetrical or not according to the point of view and position of the object. If the number is not an exact divisor of 360, the figure is asymmetrical, as some of the images are either incomplete or overlapping. Whether the mirrors are precisely at right angles, etc., can therefore be determined by the symmetry of the images formed.

The kaleidoscope (*q.v.*) is an instrument formed of inclined mirrors.

Construction of Multiple Images.—To find by construction the images formed by inclined mirrors, let *M A* and *M B* (Fig. 29) be the mirrors at any angle, and *O* the object between them. With *M* as centre and *M O* as radius, describe a circle; measure off $A\ I_1$ equal to *O A*, and $B\ I_2$ equal to *O B*; measure off $A\ I_3$ equal to $A\ I_2$, and similarly $B\ I_4$ equal to $B\ I_1$. Then take $A\ I_5$ equal to $A\ I_4$, and so on until two images coincide or overlap.

Curved Mirrors.

Spherical Mirrors.—A spherical mirror is a portion of a sphere, the cross section of which is an arc of a circle; its centre of curvature is the centre of the sphere of which it forms a part. It may be either concave or convex, and can be considered as made up of an infinite number of minute plane mirrors, each at right angles to one of the radii of the sphere.

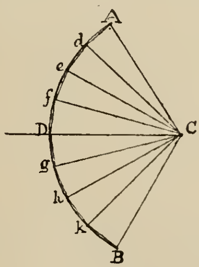

Fig. 30.

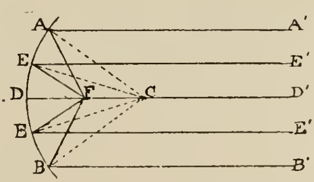

Fig. 31.

Concave Mirror.—Let *A B* be a concave mirror (Fig. 30) and *C* its centre of curvature. Then all straight lines drawn from *C* to any part of *A B* being radii, they are all of equal length and perpendicular or normal to the surface of the mirror. All rays, therefore, starting from *C*, on reaching the surface of the mirror, will be reflected back along the same paths, and form an image at the same point C.

The exposed surface $A\ B$ is the *aperture*, in the centre of which is the *vertex* D. The line passing through C and D is the *principal axis* : all other lines passing through C to the surface are *secondary axes*.

If a luminous point (Fig. 31) be infinitely far away, on the principal axis, the angle of divergence being very small, the rays are considered parallel to each other and to the principal axis. Let $A'A$, $B'B$, $D'D$, etc., be such rays, and let $C\ A$, $C\ B$, and $C\ D$ be joined; then, since these latter are radii, they each form right angles at A, B and D respectively, with the surface of the mirror. Therefore $A\ C$ is a normal to the surface at A, and the ray $A'A$ will be reflected to F, making the angle of reflection $F\ A\ C$ equal to the angle of incidence $A'A\ C$. All the other rays, in the same way, are, provided the aperture is not too great, reflected to F, which is the common image of a single luminous point situated at ∞. F is the *principal focus* of the mirror, and the distance $D\ F$ is the *principal focal distance* or *focal length*, which is equal to *half* the radius $D\ C$.

A Cc. mirror therefore renders parallel light *convergent*, and since the image can be received on a screen, or seen in the air in front of the mirror, the focus of a Cc. mirror is *real* or positive. If light is divergent it is made, by a Cc. mirror, convergent, parallel, or less divergent as the case may be.

The course of a ray can be traced backwards along the same path as that by which it arrived; so that if F be the object-point, the rays $F\ E$, $F\ A$, etc., will be reflected parallel to the axis along the lines $E\ E'$, $A\ A'$, etc. Thus, image and object are interchangeable.

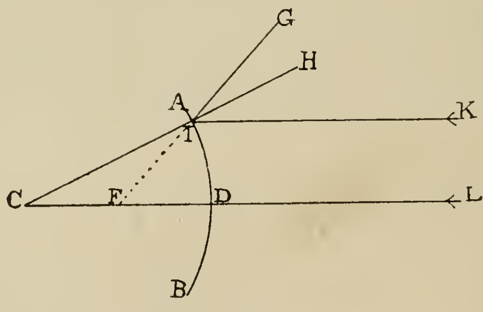

Fig. 32.

Convex Mirror.—Let $A\ B$ (Fig. 32) be a convex mirror, C the centre of curvature, D the vertex, and $C\ D\ L$ the principal axis. Then if the object-point is at ∞ on the principal axis the rays proceeding from it to the mirror are parallel. Let $K\ I$ be one of these rays meeting the mirror at I, and let $C\ H$ be a normal to the surface. The ray $K\ I$ will be reflected at I to $I\ G$, so that the angle of reflection $H\ I\ G$ is equal to the angle of incidence $H\ I\ K$, and the reflected ray $I\ G$, produced backwards, cuts the axis at F, which is the principal focus of the mirror. As with the Cc. mirror the principal focal distance is half the radius of curvature.

A Cx. mirror therefore renders parallel light divergent, and divergent light still more divergent. The image thus formed is imaginary, so that the focus of a convex mirror is *virtual* or negative.

Conjugate Foci.—Conjugate foci may be defined as the relative positions of object and image. If the image is *real* these positions may be reversed, and the object placed where the image previously was, and *vice versa*, the distances of the two conjugate points from the mirror remaining the same. Conjugate focal distances are the distances of the conjugates from the mirror.

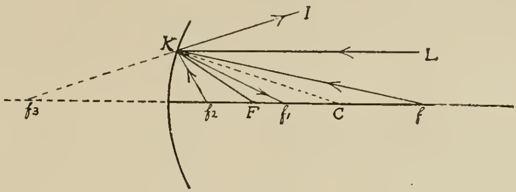

Fig. 33.

Conjugate Focal Distances—Cc. Mirror.—If the object-point (Fig. 33) be on the principal axis between C and ∞, say at f, the image will be at f_1 somewhere between F and C. An object at ∞ will have its image at F, and since the angle of incidence $f\ K\ C$ is smaller than the angle $L\ K\ C$, the angle of reflection $f_1\ K\ C$, which equals the angle of incidence $f\ K\ C$, must be smaller than the angle $F\ K\ C$, so that f_1, the image of f, lies nearer to C than F.

As the object-point approaches C, its real image recedes from the mirror towards C; when the object-point is at C the image will also be at C, the ray $C\ K$ being reflected back along its own path. When the object-point arrives within C at, say, f_1, the real image is beyond C at f, and when it reaches F its image is at ∞.

When the object-point, as f_2, passes F towards the vertex, the reflected ray $K\ I$ lies outside $K\ L$. Then the image is no longer on the same side of the mirror as the object, but is found by prolonging $K\ I$ backwards to f_3 on the other side of the mirror. The image is then not actually formed, but is virtual or negative, existing only in the brain of an observer whose eye is looking into the mirror. As the object-point travels on towards the vertex the image also approaches until the two meet at the mirror.

Conjugate Focal Distances—Cx. Mirror.—As the object-point f (Fig. 34) approaches the mirror from ∞ the angle of incidence increases; thus $f\ K\ I$ is greater than $L\ K\ I$, so that the image f_1 also approaches the mirror from F to D. The nearness of the image increases with the nearness of the object, until at D object and image coincide. Thus the image is always formed behind the mirror either at F, or between it and D, by prolongation backwards of the divergent rays, and is always imaginary or virtual.

Images on Secondary Axes.—In the preceding cases the object is supposed to be on the principal axis, so that the image is also on the principal axis. If the object-point be situated on some secondary axis the image is always on that same secondary axis. Also the object hitherto has been considered as a point; it can now be supposed to have a definite size.

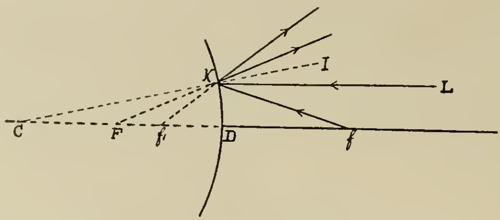

Fig. 34.

Construction of Images—Cc. Mirror.—It is known that—

(1) A ray parallel to the principal axis passes, after reflection, through F the principal focus.

(2) A ray passing through F is parallel, after reflection, to the principal axis.

(3) A ray proceeding through C, the centre of curvature, is reflected back along its original path.

A graphical construction of the image of an object placed in front of a spherical mirror can be made by tracing any two of such rays from each extremity of the object, and their course after reflection. The point where these rays meet (produced backwards, if necessary, as in the case of a virtual image) is the point where all the rays, which diverge from the object-point, also meet, and is therefore the image of that point.

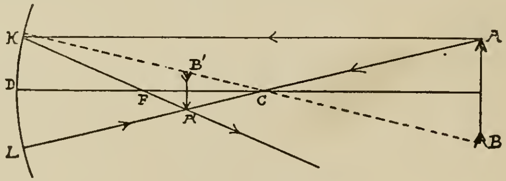

Fig. 35.

The graphical construction when the object is beyond C is as follows:

Let $A\,B$ (Fig. 35) be the object, C the centre of curvature, and F the principal focus. Draw $A\,K$ parallel to the axis, connect $K\,F$, and produce it onward; draw $A\,L$ through C. These two rays cut each other, after reflection, in A', which is therefore the *real* image of A, situated on the secondary axis $A\,C\,L$. If a ray were drawn from A through F to the mirror, its course after reflection would be through A', parallel to the axis.

In the same way, rays drawn from B meet at B', and both B and B' are on the secondary axis $B\ C\ K$. By connecting B' and A' the image of $A\ B$ is obtained, and it is real, inverted, and smaller than the object.

If the object were at $B'\ A'$ between the centre of curvature and F, the image would be $A\ B$, real, inverted, and larger than the object.

The course of any ray other than those mentioned can be constructed by drawing the normal to the point of incidence and making the angle of reflection equal to the angle of incidence.

Graphical construction when the object is between F and the mirror.

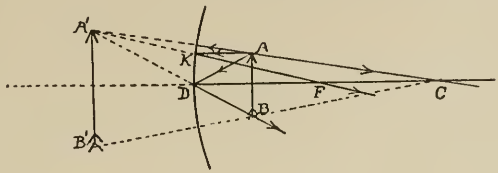

Fig. 36.

Let $A\ B$ (Fig. 36) be the object. Draw $A\ K$ parallel to $D\ C$, connect F and K, and produce towards A'; draw $C\ A$, producing it similarly. These rays are divergent, and by prolongation backwards meet on the secondary axis $C\ A\ A'$ in the point A', which is therefore the *virtual* image-point of A. Any ray $A\ D$ is also reflected as if proceeding from A'. In the same way B' can be shown to be the image of B. By connecting B' and A' the image $B'\ A'$ is obtained. It is virtual, erect, and larger than the object.

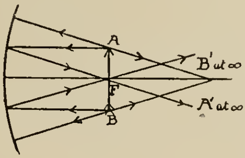

Fig. 37.

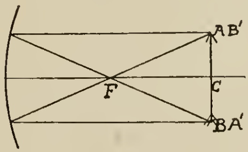

Fig. 38.

The graphical construction of the image of an object at F resolves itself into lines parallel to the secondary axes; the image is at infinity (Fig. 37).

If the object is at C, object and real image coincide, but the image is inverted (Fig. 38).

If the object is at the mirror, no rays can be drawn; object and virtual image are in contact with the mirror and coincide.

Construction of Images—Cx. Mirror.—Draw $A\ K$ (Fig. 39) and connect K with F; join $A\ C$. Where these divergent rays, by prolongation backwards, meet each other at A' is the image of A; it is on the secondary axis $A\ C$. Any other ray from A can be shown to be reflected as if proceeding from A'.

By similar construction the position of B', the image-point of B, is determined on the secondary axis $B\,C$. Connecting $A'\,B'$ the complete image of the object $A\,B$ is obtained.

In the case of a convex mirror the image is always virtual, erect, and smaller than the object.

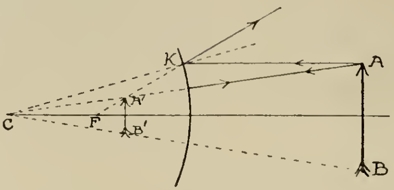

Fig. 39.

The above is based on: (1) a ray parallel to the axis is reflected as if proceeding from F; (2) a ray directed towards C is reflected as if proceeding from C.

Conjugate Foci of Spherical Mirrors.

A list of symbols and abbreviations faces page 1.

If F be the principal focal distance, then 1/F is the reflecting power of the mirror; this is termed the focal power; the two are reciprocals of each other; thus, if F be 10, then 1/F=1/10.

Since $F=r/2$, the power of a mirror can also be written $2/r$.

A Cc. mirror converges incident light, and its power 1/F is positive.

A Cx. mirror diverges incident light; its power is negative and represented by $-1/F$.

Let f_1 be the distance of the object from which light diverges to the mirror; then the divergence of the light is $1/f_1$ and this is considered negative.

If the object is very distant the light divergence is negligible, and therefore equals zero so that the light is reflected to or from F as a result of the power 1/F of the mirror. If, however, the light proceeds from a near object the divergence is appreciable, and the light is reflected to or from some other distance, f_2, which distance is determined by the addition of the divergence of the light to the converging power of the mirror, i.e. $1/F-1/f_1=1/f_2$, where $1/f_2$ is the reciprocal of the distance f_2.

A Cc. mirror being positive, and the light divergence negative, f_2 is positive or negative according as $1/f_1$ is respectively a smaller or greater quantity than 1/F. With a Cx. mirror f_2 is always negative because both 1/F and $1/f_1$ are negative.

If f be a distance, then $1/f$ is convergence to, and $-1/f$ is divergence from, that distance. It is the reflecting power which will cause parallel light respectively to converge to, or to diverge from, that distance f.

Thus the total power of a mirror 1/F is equal to the sum of the powers which represent the distances of the object and image. In other words the reciprocal of the principal focal distance is equal to the sum of the reciprocals of any pair of conjugate foci. Then the formula for conjugate foci for spherical mirrors is:—

$$\frac{1}{F}-\frac{1}{f_1}=\frac{1}{f_2} \quad \text{or} \quad \frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

It can also be written $$\frac{F}{f_1}+\frac{F}{f_2}=1.$$

This formula is one of the most important in optics. In enables us to find the focal length of a mirror if f_1 and f_2 are known; or if F and f_1 are known we can find f_2 (the image). It is universal and holds true for both concave and convex mirrors and, as will be seen, for lenses as well. Since the two fractions $1/f_1+1/f_2$ added together always produce the same sum, it follows that if the one is increased the other is decreased in proportion. Thus if a Cc. mirror be of 20 in. radius, or 10 in. focal length, the sum $1/f_1+1/f_2$ is always 1/10. If a Cx. mirror has F$=-10$ in., the sum $1/f_1+1/f_2$ is always $-1/10$. When f_2 is positive the image is real; when negative the image is virtual.

Size of the Image formed by a Mirror.—The relative sizes of image and object is termed Magnification (M). Now, object and image subtend the same angle at *C*, the centre of curvature, or at the vertex, so that the size of the image bears the same relation to the size of the object, as the distance of the image does to the distance of the object from the centre of curvature *C*, or from the mirror. That is

$$\frac{h_2}{h_1}=\frac{f_2}{f_1} \quad \text{or} \quad h_2=h_1\frac{f_2}{f_1}$$

where h_1 is the size of the object, h_2 that of the image, f_2 the distance of the image, and f_1 that of the object. In the calculation f_1 and f_2 must be in the same terms, then h_2 will be in the same terms as h_1. The formula holds good with both Cx. and Cc. mirrors, and whether the image be real or virtual.

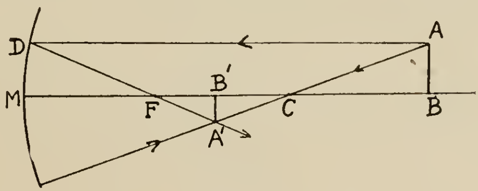

FIG. 40.

Geometrical Proof for Cc. Mirror.—In Fig. 40 *A B* is an object whose image is *B′ A′*. The aperture of the mirror is taken as small, so that *D M* may be considered a straight line.

In the similar triangles *A C B* and *B′ C A′*

$$\frac{AB}{B'A'}=\frac{BC}{B'C}$$

In the similar triangles $D\ F\ M$ and $B'\ F\ A'$

$$\frac{DM}{B'A'}=\frac{MF}{B'F}$$

But $D\ M=A\ B$, so that

$$\frac{MF}{B'F}=\frac{BC}{B'C}$$

Now $M\ F=\mathrm{F}$, $B'\ F=f_2-\mathrm{F}$, $B\ C=f_1-2\ \mathrm{F}$, and $B'\ C=2\ \mathrm{F}-f_2$.

That is
$$\frac{\mathrm{F}}{f_2-\mathrm{F}}=\frac{f_1-2\ \mathrm{F}}{2\mathrm{F}-f_2}$$

whence
$$\mathrm{F}f_1+\mathrm{F}f_2=f_1 f_2$$

or
$$\mathrm{F}(f_1+f_2)=f_1 f_2 \quad \text{and} \quad \frac{1}{\mathrm{F}}=\frac{1}{f_1}+\frac{1}{f_2}.$$

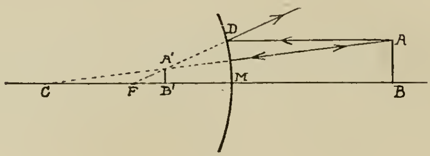

Fig. 41.

Proof for Cx. Mirror.—The aperture being small, $D\ M$ (Fig. 41), as before, may be considered a straight line.

In the similar triangles $A\ C\ B$ and $A'\ C\ B'$

$$\frac{AB}{A'B'}=\frac{BC}{B'C}$$

In the similar triangles $D\ F\ M$ and $A'\ F'\ B'$

$$\frac{DM}{A'B'}=\frac{MF}{B'F}$$

But $D\ M=A\ B$, so that

$$\frac{MF}{B'F}=\frac{BC}{B'C}$$

Now $M\ F=\mathrm{F}$, $B'F=\mathrm{F}-f_2$, $B\ C=f_1+2\ \mathrm{F}$ and $B'C=2\ \mathrm{F}-f_2$.

That is
$$\frac{\mathrm{F}}{\mathrm{F}-f_2}=\frac{f_1+2\ \mathrm{F}}{2\ \mathrm{F}-f_2}$$

whence
$$\mathrm{F}f_2-\mathrm{F}f_1=-f_1 f_2$$

or
$$\mathrm{F}(f_2-f_1)=f_1 f_2 \quad \text{and} \quad -\frac{1}{\mathrm{F}}=\frac{1}{f_1}-\frac{1}{f_2}.$$

Calculations on Conjugate Foci: Examples—Cc. Mirror.—Let a Cc. mirror be of 10 in. F. and let the object be at ∞; then we have

$$\frac{1}{f_2}=\frac{1}{F}-\frac{1}{f_1}=\frac{1}{10}-\frac{1}{\infty}=\frac{1}{10}-0=\frac{1}{10}$$

The image is real and at the principal focal distance, since f_2=F.

In these calculations, any *considerable* distance is regarded as ∞, say, beyond 6 M. or 20 feet, but although a distance may be taken as ∞ for the calculation of f_2, its *definite* distance is needed for calculating h_2.

If the object be at F the calculation is

$$\frac{1}{f_2}=\frac{1}{F}-\frac{1}{F}=0 \qquad \therefore \qquad f_2=\infty$$

F and ∞ are conjugate distances.

If the object is at 30″ and 2″ in height, we get

$$\frac{1}{f_2}=\frac{1}{10}-\frac{1}{30}=\frac{1}{15}$$

Therefore the image is at 15 in. and real.

$$h_2=h_1\frac{f_1}{f_2}=2\times\frac{15}{30}=1''$$

If the object were at 15 in. we find

$$\frac{1}{f_2}=\frac{1}{10}-\frac{1}{15}=\frac{1}{30}$$

15 in. and 30 in. are conjugate foci with respect to a 10 in. Cc. mirror; if the object is at the one distance, the image is at the other.

If h_1 is 1″, then

$$h_2=1\times\frac{30}{15}=2''$$

If the object be at twice F, that is, at the centre of curvature, say 20 in., in front of a 10 in. Cc. mirror, the image is at the same distance, since

$$\frac{1}{f_2}=\frac{1}{10}-\frac{1}{20}=\frac{1}{20} \text{ or } f_2=20 \text{ ins.}$$

If $f_2=f_1$, then $h_2=h_1$; object and image are same size.

When *O* is within F, a higher number than 1/F being deducted from it, the result is negative. Thus if *O*, 1″ high, be 6″ in front of a 10″ Cc. mirror, then

$$\frac{1}{f_2}=\frac{1}{10}-\frac{1}{6}=-\frac{1}{15} \quad \text{and} \quad h_2=1\times\frac{15}{6}=2\tfrac{1}{2}''$$

The virtual image is 15″ behind the mirror and 2·5″ high.

Here −15 in. is the conjugate of 6 in. in respect to a 10 in. Cc. mirror, and 6 in. is the conjugate of −15 in., *but not of* 15 *in.* That is to say, if light incident on the mirror is *convergent* to a point 15 in. behind it, it is reflected so as to come to a focus 6 in. in front of the mirror, for—

$$\frac{1}{f_2}=\frac{1}{10}+\frac{1}{15}=\frac{1}{6}$$

An object 3 yards high is an eighth of a mile from a 30″ mirror.

O being at ∞, the image is at F=30″, and

$$h_2=\frac{3\times 30}{220}=\cdot 4 \text{ inch.}$$

If f_1 or f_2 are not known, M can be found from $M=(f_2-F)/F$ and $M=F/(f_1-F)$ respectively.

The size of image of an inaccessible object can be calculated from the angle it subtends at the centre of curvature; for example, the size of the image of the moon formed by a Cc. mirror of 16 in. F.

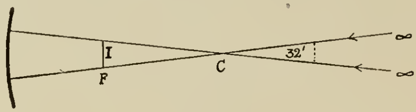

Fig. 42.

O being at ∞ (Fig. 42) the light is parallel, and the image I will be at F =16 ins. The moon subtends an angle of 32′ at C, and therefore

$$h_2=16\times \tan 32'=16\times\cdot 0093=0\cdot 1488 \text{ in., or about } 1/7 \text{ inch.}$$

It will be seen that a real or positive image is obtained with a Cc. mirror so long as O is beyond F, and that I becomes virtual or negative when O is nearer than F. Also that in all cases $1/F=1/f_1+1/f_2$. Thus when the light diverges from 30″ and is converged to 15″ we find $1/10=1/30+1/15$. When it diverges from 6″ before reflection, and from 15″ after reflection, we get $1/10=1/6+(-1/15)$.

The nearer O is to F, the more distant is I; as O recedes from F, the I approaches F, but no positive I can be nearer than F, since no O can be more distant than ∞.

The nearer O is to F, the more distant is the negative I. As O recedes from F and approaches the mirror, I also approaches the mirror, but I is always more distant than O until, when O touches the surface, so also does I.

If light is convergent when incident on a Cc. mirror, the convergence is increased after reflection, and in this case the image is nearer to the mirror than F.

Examples—Cx. Mirror.—Let the mirror be of 10 in. F and the O at ∞. Then

$$\frac{1}{f_2}=\frac{1}{F}-\frac{1}{f_1}=-\frac{1}{10}-\frac{1}{\infty}=-\frac{1}{10}-0=-\frac{1}{10}$$

The image is virtual or negative and at F.

If O be in front of the mirror at a distance equal to F, I is at half F. Thus

$$\frac{1}{f_2}=-\frac{1}{10}-\frac{1}{10}=-\frac{1}{5} \quad \text{and} \quad h_2=h_1\times\frac{10}{5}=\cdot5h_1$$

If O is 30 in. in front of the Cx. mirror and 2″ high

$$\frac{1}{f_2}=-\frac{1}{10}-\frac{1}{30}=-\frac{1}{7\cdot5}\text{in.} \quad \text{and} \quad h_2=\frac{2\times7\cdot5}{30}=\cdot5''$$

The image is virtual, $7\frac{1}{2}$ in. behind the mirror and ·5″ high.

$-7\frac{1}{2}$ in. is conjugate to 30 in., with respect to a 10 in. Cx. mirror, and 30 in. is conjugate to $-7\frac{1}{2}$ in., but not to $7\frac{1}{2}$ in. If light converged to a point $7\frac{1}{2}$ in. behind the surface, the convergence is, by reflection, so much reduced that an image is formed 30 in. in front of the mirror.

If light converges to a point within F the image is real; if convergent to F the light is parallel after reflection, since the convergence of the light and divergence of the mirror neutralize each other. If the light is convergent to a point beyond F the virtual image is also beyond F.

The I of an O formed by a Cx. mirror is therefore always virtual, and cannot be at a greater distance from the mirror than F, the O being then at ∞. When O is nearer than ∞ the I recedes from F towards the mirror, and when O touches the surface I does likewise.

Relationship of Distances from C and from Mirror.

In the foregoing examples the distances from the mirror have been taken for calculating h_2, but the same ratios would exist were they taken from C. Thus 30″ and 15″ from a 10″ Cc. mirror are $30-20=10''$ and $20-15=5''$ from C, and $30/15=10/5$.

Similarly with a 10″ Cx. mirror, 30″ and $7\frac{1}{2}$ in. from it are $30+20=50$ and $20-7\frac{1}{2}=12\frac{1}{2}$ from the centre of curvature, and $30/7\cdot5=50/12\cdot5$.

Unit Magnification.—When object and image are equal in size M=1. The plane of unit magnification for real images is at the centre of curvature of a Cc. mirror—that is, at 2F. Foı virtual images it is at the mirror itself for both Cc. and Cx. The distance from F=F in both cases.

If $M=1$, $f_2=f_1$, and $1/F=2/f_1$ or $2/f_2$—that is, f_1 or $f_2=2F$. If the image is viıtual, f_2 is negative, and f_1 and $f_2=0$.

Newton's Formula for Conjugate Foci.—If the distance of the object from F=A, and that of the image from F=B, then

$$AB=F^2 \text{ or } B=F^2/A$$

Let	$F=10$ and $f_1=30$,	then $A=30-10=20$
and	$B=100/20=5$,	or $f_2=5+10=15$ in.
Let	$F=10$ and $f_1=6$,	then $A=6-10=-4$
and	$B=100/-4=-25$,	or $f_2=-25+10=-15$ in.
Let	$F=-10$ and $f_1=30$,	then $A=30-(-10)=40$
and	$B=100/40=2{\cdot}5$,	or $f_2=2{\cdot}5+(-10)=-7{\cdot}5$ in.

These examples should be compared with those worked by the ordinary formula.

Movement of Image.—If a mirror is rotated a real image moves in the same, and a virtual image in the opposite direction. If the object is moved, its real image moves in the opposite, and its virtual image in the same direction. If the observer's head be moved a real image apparently moves in the opposite, and a virtual image in the same direction. As with a plane mirror, the angular movement of any image is twice that of the mirror itself. Virtual images are also laterally inverted, but a real image is entirely reversed, and therefore not laterally inverted in this sense.

Recapitulation of Conjugates.

Cc. Mirror.—*O* at ∞, *I* is real, inverted, diminished, at *F*.
O between ∞ and 2*F*, *I* is real, inverted, diminished, between 2*F* and *F*.
O at 2*F*, *I* is real, inverted, equal to *O*, at 2*F*.
O between 2*F* and *F*, *I* is real, inverted, enlarged, between 2*F* and ∞.
O at *F*, *I* is infinitely great, at ∞.
O within *F*, *I* is virtual, erect, enlarged, the other side of mirror.
O at mirror, *I* is virtual, erect, equal to *O*, at mirror.

Cx. Mirror.—*O* at ∞, *I* is virtual, erect, diminished, at *F*.
O within ∞, *I* is virtual, erect, diminished, within *F*.
O at the mirror, *I* is virtual, erect, equal to *O*, at mirror.

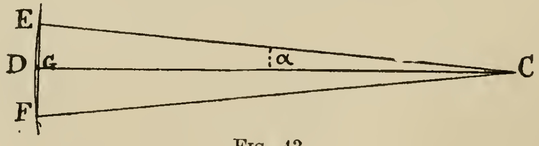

Fig. 43.

Aperture of a Mirror.—In order that a true image of a point may be obtained with a spherical mirror, it is essential that the aperture should be small compared with its radius, subtending an angle of, say, not more than

20° at C, so that the arc of the aperture may be regarded as a straight line. In Fig. 43, $E\ F$ is the actual, and $E\ C\ F$ is the angular aperture of the mirror, $C\ D$ being the principal axis, and C the centre of curvature. Join $E\ F$. Then if the angle $E\ C\ D$ be small (not exceeding 10°) the distance $D\ G$ will also be small, so that $C\ G$ may, without much error, be taken as equal to $C\ E$ or $C\ D$; also $E\ D\ C = E\ G\ C$ may be taken as a right angle.

Now $$\tan a = \frac{E\ G}{G\ C}, \ \sin a = \frac{E\ G}{E\ C}, \text{ and } \cos a = \frac{G\ C}{E\ C}.$$

Since $C\ E = C\ G$, $\sin a = \tan a =$ the arc $E\ D$, and $\cos a = 1/1 = 1$.

Thus all calculations involving mirrors—and, as will be seen later, lenses also—are greatly simplified, since the sine and tangent may be considered equal, and can be replaced by the arc, and the cosine by unity, whenever the angular aperture is small.

In the proofs for conjugates this approximation is introduced, which is permissible, seeing that the portion of the mirror chiefly responsible for the production of the image is that immediately surrounding the vertex. Mirrors of large aperture do not produce true point images of point objects, and are said to suffer from *aberration* (Chapter XXI.).

CHAPTER IV

REFRACTION AND THE REFRACTIVE INDEX

THE fact that the velocity of light is lessened in a dense medium is the cause of refraction.

Normal Incidence.—When a beam of light, from air, is incident normally on a dense medium such as a plate of glass, the whole of the wave-front is retarded simultaneously and equally, and is unchanged in direction during its progression through the denser medium. On reaching the second surface, each part of the wave-front is again incident at the same time, and is equally

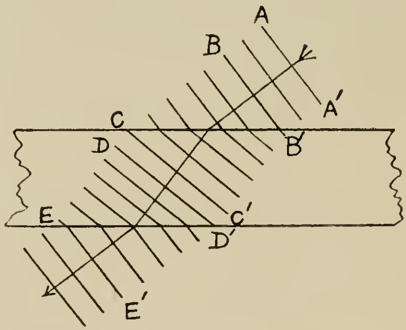

FIG. 44.

increased in speed as it passes again into the rarer medium, so that its line of progression still remains unchanged.

Oblique Incidence of Light.—But if the plane wave-front $A\ A'$ (Fig. 44) be incident on the first surface obliquely, one part B' meets the denser medium sooner than the rest and is retarded, while the others are still in the rarer medium advancing at an undiminished rate of speed. Each wavelet on reaching the glass becomes retarded, one by one, until all have passed into the denser medium, and in consequence the wave-front is changed in direction. When the whole of the wave-front $C\,C'$ is in the denser medium, it travels without further deviation, but at a diminished rate of speed. On reaching the second surface of the glass the wave-front $D\ D'$ is again incident sooner at one point D' than at others. The wavelet at that point increases its speed, while the remainder is still moving less rapidly in the denser medium; then other wavelets emerge and increase their speed until, all having passed into a rarer medium, the entire wave-front $E\ E'$ travels more rapidly, that is, with its original velocity, and in a direction parallel to its original direction.

The angular change of direction depends on the distance that the more rapidly advancing parts of the wave-front travel before their speed is also checked, that is, on the obliquity of the incidence of the light, and on the retardation itself, that is, on the optical density of the medium.

The Laws of Refraction.—When a ray of light is incident (obliquely) on the boundary plane of two media of different optical densities:—

(1) The incident and refracted rays are in the same plane as the normal to the point of incidence, but on opposite sides of it.

(2) A constant ratio exists between the sines of the angles of incidence and refraction.

The ratio in (2) is the mathematical expression of the definition of the *index of refraction* as given in the next article.

From the second law we can deduce the following:

A ray passing obliquely from a rarer into a denser medium is refracted towards the normal at the boundary plane of the two media.

A ray passing obliquely from a denser into a rarer medium is refracted away from the normal at the boundary plane of the two media.

A ray suffers no deviation if, at the point of incidence, it is normal to the boundary plane of the two media.

The Index of Refraction.—The index of refraction between two media is *the ratio of the velocities of the light in these media.* The velocity varies inversely with the optical density; thus if the light travels three miles in the first medium, while, in the same time, it is travelling two miles in the second, then the index is $3/2=1{\cdot}5$; if the direction of the light travel were reversed the index would be $2/3$.

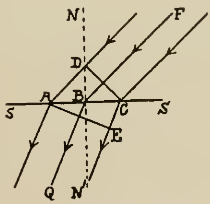

FIG. 45.

Geometrical Proof.—In Fig. 45, let $S\ S$ be the bounding surface between two media, of which the second is the denser, and let the velocities of the light in the two media be respectively V_1 and V_2. Let $D\ C$ be a plane wave-front incident on the surface at the angle $D\ C\ A=i$, which, after refraction, passes into the second medium at the angle of refraction $E\ A\ C=r$.

Then $A\ D$ and $C\ E$ are distances travelled by the extremities of the wave in the same time, D in the rarer and C in the denser medium; $A\ D$ and $C\ E$ therefore represent the two velocities.

A D and *C E* are perpendicular to the wave-front, and the angles *A D C* and *A E C* are right angles. The hypothenuse *A C* is common to the triangles *A D C* and *A E C*, so that *A D* and *C E* are the sines of the angles of incidence and refraction respectively, thus

$$\frac{A\,D}{C\,E}=\frac{\mathrm{V}_1}{\mathrm{V}_2}=\frac{\sin i}{\sin r}$$

That is, the index of refraction is the ratio of the velocities of the light in the two media, or the ratio of the sines of the angles of incidence and refraction.

This proof holds good when the incident wave is curved, since the portion *D C* under consideration is so small that it may be considered plane—we work from the "ray" *F B*, which is really the path taken by the wave-front. Similarly if the medium *S S* is curved, the portion *A C* may also be considered plane.

The angle of incidence of a *wave-front* is that which it makes with the *surface*, as *A C D* in Fig. 45, the corresponding angle of refraction being *C A E*. The angle of incidence of a *ray* is that which it makes with the *normal* as *F B N*, the angle of refraction being *Q B N′*. It is immaterial as to which is taken since *A C D* and *F B N* are equal, as are also *E A C* and *Q B N′*. It is, however, more convenient to treat of the incidence and refraction of rays than of wave-fronts.

Absolute Refractive Index.—It can be taken that the velocity of light is a maximum in free ether (i.e. a vacuum) through which light waves, of every length, travel with equal speed. Its progression through air is very slightly slower, and for all practical purposes no distinction is made. The optical density of air is therefore taken as unity or 1, and the density of any other medium, such as water or glass, is expressed in terms of this unit, and is called the *absolute index of refraction*, generally denoted by the Greek letter μ (mu). Thus if the μ of a certain kind of glass is 1·5, it implies that light travels one and a half times as fast in air as in the glass, or the velocity in the glass is only 1/1·5=2/3 that in air. μ expresses the optical density of a medium, and if μ=1·5, the medium to which it pertains has an optical density which is 1·5 times greater than air. To a certain extent the optical density varies directly with the specific gravity, but there are some exceptions, in transparent media, as for instance with oil and water; oil has the greater optical density, but the lesser specific gravity than water.

When reference is made to the μ of a substance it invariably indicates the *absolute index*. Also, unless otherwise stated, μ refers to *yellow light;* the index varies with the colour of the light, but, for the present, we shall consider light monochromatic, and the wave-length that of the D line.

It is usual, when several media are involved in a calculation, to refer to their indices as μ_1, μ_2, μ_3, etc., but when there are only two media, one of which is air, the index of the denser is denoted simply by μ without any suffix, that of air being, as before stated, taken as 1, although actually it is about 1·000294.

Relative Refractive Index.—The relative index of refraction, μ_r, is the expression of the refractivity when light passes from one dense medium into another, say, from water into glass or *vice versa*. It is found by dividing the absolute index of the medium into which the light passes, by the absolute index of the medium from which it proceeds; thus when light passes from water $\mu=1·333$ into glass $\mu=1·545$ the relative index is

$$\mu_r=\frac{\mu_2}{\mu_1}=\frac{1·545}{1·333}=1·16$$

Again, the sines of the angles of incidence and refraction, as light passes through two such media, are to each other as the velocities of the light in those two media.

The Law of Sines—commonly known as Snell's Law—is that

$$\frac{\sin i}{\sin r}=\frac{\mu_2}{\mu_1}$$

where μ_2 is the optical density of the second medium and μ_1 that of the first. This is the fundamental formula of refraction. In the case of reflection the light returns to the same medium as it comes from, and therefore $\mu_2=\mu_1$, so that $\sin i=\sin r$ or the angle of reflection is the same as that of incidence. The angles i and r are equal in refraction only when light passes obliquely through two different media of the same optical density, as crown glass and Canada balsam.

Reciprocal μ's.—In the case of any two media A and B the index of refraction for light passing from A into B is the reciprocal of the index for light passing from B into A. Thus, when light passes from air into glass, the sines of the angles of incidence and refraction are, say, as 3 : 2, and the index is 3/2. If it passes from glass into air, the sines of the two angles are as 2 : 3 and the index is 2/3. Taking the example above of light passing from water to glass with a relative μ of 1·16; if the light passed from the glass to the water $\mu_r=1·333/1·545=1/1·16=·862$.

The Course of a Ray.—Examples of the application of the law of sines.

Let a ray be incident from air ($\mu_1=1$) at 30° with the normal, to glass of $\mu_2=1·5$, and it is required to find the course of the ray after refraction.

$$\frac{\sin i}{\sin r}=\frac{\mu_2}{\mu_1}, \quad \text{that is} \quad \frac{\sin 30°}{\sin r}=\frac{1·5}{1}$$

therefore $$\sin r=\frac{\sin 30° \times 1}{1·5}=\frac{·5}{1·5}=·333=\sin 19° 30' \text{ (approx.)},$$

so that $r=19° 30'$.

If light passes at an angle of 30° from glass of $\mu_1=1·5$ into air

$$\frac{\sin i}{\sin r}=\frac{\mu_2}{\mu_1}, \quad \text{that is} \quad \frac{\sin 30°}{\sin r}=\frac{1}{1·5}$$

therefore $\sin r = \sin 30° \times 1{\cdot}5 = {\cdot}5 \times 1{\cdot}5 = {\cdot}75 = \sin 48° 30'$ (approx.), so that $r = 48° 30'$.

It should be noted that in the case of light passing into the denser medium it is deviated 10° 30′, and when passing into the rare medium it is deviated 18° 30′ for the same angle of incidence—*i.e.*, that of 30°.

If a ray is incident at 12° to the boundary plane between water ($\mu_1 = 1{\cdot}333$) and glass ($\mu_2 = 1{\cdot}545$), then

$$\frac{\sin i}{\sin r} = \frac{1{\cdot}545}{1{\cdot}333}, \text{ and } \sin r = \frac{{\cdot}2079 \times 1{\cdot}333}{1{\cdot}545}$$

If it passes the other way at the same angle

$$\frac{\sin i}{\sin r} = \frac{1{\cdot}333}{1{\cdot}545}, \text{ and } \sin r = \frac{{\cdot}2079 \times 1{\cdot}545}{1{\cdot}333}$$

r is smaller than i when light passes into a denser medium, and *vice versa.*

Further, the angle of incidence i is *invariably* taken as occurring in the first medium μ_1, and the angle of refraction r in the second medium μ_2.

It is the direction of the light that determines which medium is denoted by μ_1 or μ_2 as the case may be.

The actual deviation which light undergoes, when passing from one medium into another of different density, at a given angle of incidence, depends *on the ratio between the μ's of the two media,* and not on the high value of the μ of the second medium. Thus the refraction is greater when light passes from air into glass of $\mu = 1{\cdot}5$ than when it passes from water into glass of $\mu = 1{\cdot}9$. In the former case the density of the second medium relative to the first is 1·5, and in the latter it is $1{\cdot}9/1{\cdot}33 = 1{\cdot}43$.

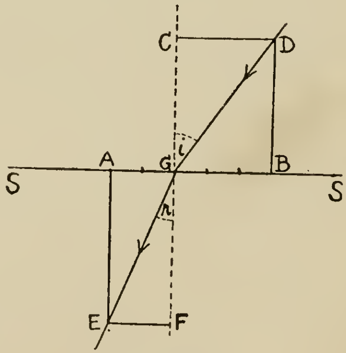

Fig. 46.

Graphical Constructions.—The course of a ray can also be graphically constructed in the following manner: Let $D\,G$ be any ray in air incident at G on the surface $S\,S$ (Fig. 46) of a medium whose index is 1·5 or 3/2. From D drop the normal $D\,B$ and divide $B\,G$ into three equal spaces. Then from G mark off $G\ A$ equal to two such spaces. From A drop a normal and from G draw a line $G\ E$, *equal in length to* $G\ D$, cutting the perpendicular from A in E. Then $G\ E$ is the direction of the refracted ray. In this construction $B\ G$ is equal to and takes the place of $D\ C$, which is sin i, and $A\ G$ is equal to and replaces $E\ F$, which is sin r.

In order to trace the course of a ray of light through any refracting body, with plane or curved surfaces, the procedure is the same, but in the case of a curved surface the tangent to the curve, at the point on which the ray is incident, is taken to represent the refracting plane.

This construction is a graphical representation of the sine law, because in the right-angled triangles $C\ D\ G$ and $F\ E\ G$, the hypothenuses $D\ G$ and $G\ E$ are equal, and $C\ D$ and $E\ F$, the sines of i and r respectively, are numerically in the ratio of 3 and 2. Therefore $G\ E$ must be the direction of the refracted ray.

The construction is applicable to any pair of media. If they are, say, oil of $\mu_1 = 1{\cdot}45$, and water of $\mu_2 = 1{\cdot}33$, $B\ G$ would be as 14·5 to $A\ G$ 13·3. When the ratio is complex, as in this case, it is better to mark off, with a millimetric rule, the spaces $B\ G$ and $A\ G$, and then drop $D\ B$ and $A\ E$, finally filling in $D\ G$ and $G\ E$, which must be of equal length.

It should be observed, as indicated in Fig. 46, that the length $B\ G$, representing μ_2, *is always on the same side of the normal as the incident ray, and that the length* $A\ G$, *representing* μ_1, *is therefore on the side of the refracted ray.*

TABLE OF REFRACTIVE INDICES (FOR THE D LINE).

(*For other Media see Chapter XXVII.*)

Air	1·000
Water	1·336
Alcohol	1·366
Pebble	1·544
Canada balsam	1·535
Tourmaline	1·636
Crown glass .. .. say	1·500 to 1·600
Flint glass .. .. ,,	1·530 to 1·800
Diamond	2·47

The index of glass varies with the materials used in its manufacture, and as a rule the higher the μ the softer is the glass.

Dispersion.—The shorter waves, with rare exceptions, are retarded by a medium, more than the longer waves, so that when white light undergoes refraction its components are refracted to different extents, and the various colours become separated, producing what is known as dispersion or chromatism, which subject is treated later. As before stated THE index of a medium is its *mean* μ, that is, μ_D.

Critical Angle and Total Reflection.—When a ray of light passes from a dense into a rare medium, it is bent away from the normal, with which it makes a larger angle than before refraction. In Fig. 47, let $A\ B$ be the incident and *B C* the refracted ray. As the angle of incidence is increased, so the corresponding angle of refraction becomes still larger. Hence if the ray *A′B* be incident at an angle sufficiently large, the angle of refraction becomes a right angle, and the refracted ray *B C′* will skim along the bounding surface. The angle of incidence in the denser medium which produces

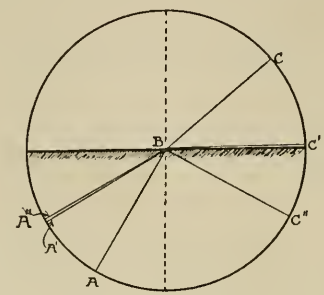

FIG. 47.

this result is termed the *critical angle*, because the slightest further increase of it prevents the ray from passing out of the denser medium. If the incident ray be *A″ B* it is reflected as *B C″*, and *internal reflection* takes place. This is termed *total* to distinguish it from ordinary reflection, which is always accompanied by a certain amount of absorption or transmission.

From the sine formula $\sin i/\sin r=\mu_2/\mu_1$,

For i we can substitute C, the critical angle.

Also $r=90°$, so that $\sin r=1$, and therefore can be omitted.

Then $$\sin C=\frac{\mu_2}{\mu_1}, \text{ or } \sin C=\mu_r.$$

The sine of the critical angle is equal to the relative index from the denser to the rarer medium.

If the rarer medium be air, we have:—

$$\sin C=\frac{1}{\mu}, \text{ or } \mu \sin C=1.$$

Thus for glass of $\mu=1{\cdot}52$ and water 1·33 the critical angle is 61° 18′ because

$$\sin C=1{\cdot}333/1{\cdot}52={\cdot}877=\sin 61° \, 18'.$$

For glass of $\mu=1{\cdot}5$ in air, C=41° 46′ since $\sin C=1/1{\cdot}5={\cdot}666=\sin$ 41° 16′

The critical angle when light passes through several media is the same as that which obtains directly between the first and the last.

TABLE OF CRITICAL OR LIMITING ANGLES.

Medium.	*Index of Refraction.*	*Critical Angle.*
Chromate of lead	2·92	20°
Diamond	2·47	24°
Various precious stones	—	25° to 30°
Flint glass	—	38° to 40°
Crown glass	—	40° to 43°
Pebble	1·54	40°
Water	1·33	48° 30′

It will be seen, from the above, that the critical angle varies inversely with μ. That of glass *in general* is about 40°.

Some Effects of Total Reflection.—On looking upwards through the side of an aquarium tank the upper surface glistens like quicksilver, owing to the light being reflected downwards.

If a tank with water has some benzine on the top, the two liquids do not mix. As the benzine has the higher index, light from above may be totally reflected upwards from the common surface which glistens when viewed obliquely.

A solid bent tube of glass with a strong source of light close to one end transmits the light by internal reflection for illumination purposes in microscopy.

A crack in a thick plate glass demonstrates total reflection at the film of air in the fracture.

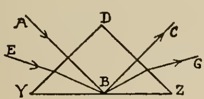

FIG. 48.

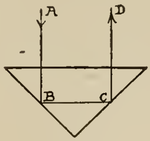

FIG. 49.

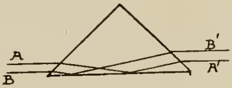

FIG. 50.

Reflecting Prisms.—If the principal angle of a prism exceeds twice the critical angle of the medium of which it is made, total reflection ensues for any angle of incidence of the light, because, even for the largest possible angle of incidence—*i.e.*, 90° at the first surface—the internal angle of incidence at the second surface is greater than half the principal angle, and is therefore greater than the critical angle. Since all glass has a critical angle of less than 45°, if (Fig. 48), a ray AB enters a right-angled prism normally, it is incident at 45° to the surface $Y Z$, and is then totally reflected in the direction $B C$. It is not refracted at the surfaces $D Y$ and $D Z$ because it is normal to each. Thus a right-angled prism serves as a total reflector when the light is incident perpendicularly to the one face, *the direction of the emergent light being at right angles to the original course.*

There is reflection even if the light does not enter at right angles to $D Y$, provided, after refraction, the angle of incidence at YZ is greater than the critical angle. Any dispersion which takes place as the ray enters is reversed as it leaves the prism, so that the emergent ray consists of white light similar to that which entered.

If the light falls normally on the hypothenuse side of a right-angled prism it causes total reflection twice, at B and C, as in Fig. 49, so that the final direction $C\ D$ of the light is parallel to its original course $A\ B$. The forms shown in Figs. 48 and 49, with variations, are extensively employed in prism binoculars, range finders, etc. With them lateral without vertical inversion, or *vice versa*, can be obtained.

By means of a right-angled prism, as indicated in Fig. 50, vertical without lateral inversion may be obtained. This prism is used in process photography, and in projection work. As shown here and in Fig. 48, if a ray is not normal to one of the faces of a right-angled prism, its angle of emergence is the same as that of incidence.

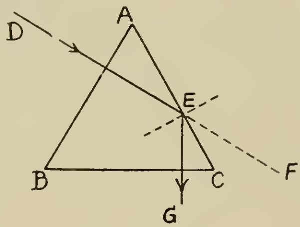

Fig. 51.

With an equilateral prism ABC (Fig. 51) a ray DE incident normally on the surface AB is reflected at the surface AC in the direction EG, and emerges normally from the third surface BC. The deviation of the ray is $FEG = 60°$, that is, the same as the principal angle A.

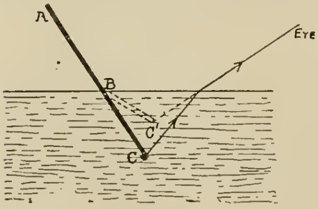

Fig. 52.

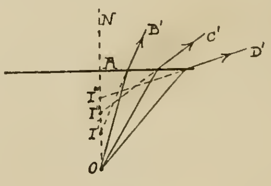

Fig. 53.

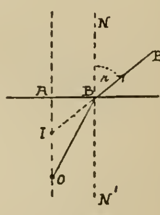

Fig. 54.

Displacement due to Refraction.—In Fig. 52 let C be a luminous point in a dense medium from which, after refraction, rays enter an observer's eye and are projected back to C', the virtual image of C. Thus a stick $A\ B\ C$ partly immersed in water and viewed obliquely appears bent towards the surface, the bend commencing at the level of the water. The lateral displacement and the raising of each image-point is proportional to the depth of the object-point in the denser medium and the μ of the latter.

The apparent position of the object depends also on the obliquity of view; the nearer the eye to the surface, the greater is the refraction of the emergent light, and the greater the apparent raising of the object. If the eye be practically at the surface, the object is also apparently raised to the surface, but is very distorted.

Fig. 53 represents the surface of a dense medium and O the object. Any ray other than $O\ A\ N$, normal to the surface, is bent away from the normal, and when referred back by the eye appear to come from points I', I'', I''', these being the images of O when the eye is at B', C', and D' respectively. The series of images actually form a curve, although in the diagram they are projected on to the normal $O\ N$.

The position of any particular image-point can be calculated as follows:

In Fig. 54, let O be the object and $O\ B$ any ray making an angle with the normal $N\ N'$. After refraction it will take the course $B\ B'$, so that to an eye at B' the object is apparently raised to I. It is required to find the apparent depth of the image, i.e. the distance $A\ I$ in terms of the real depth $A\ O$, μ and r, the angle of view with the normal.

Now the angle A O B=OBN′=i and A I B=NBB′=r

also
$$\text{A O}=\frac{\text{A B}}{\tan i}\text{ and A I}=\frac{\text{A B}}{\tan r},\text{ or }\frac{\text{A I}}{\text{A O}}=\frac{\tan i}{\tan r}$$

Let the real depth A O be t and the apparent depth A I be t', and since $\tan i=\sin i/\cos i$ and $\tan r=\sin r/\cos r$

$$t'=t\frac{\sin i}{\cos i}\times\frac{\cos r}{\sin r}=t\frac{1}{\mu}\times\frac{\cos r}{\cos i}\text{, or } t'=\frac{t\cos r}{\mu\cos i}.$$

Thus, knowing μ and r, the angle of view, we can calculate the value of i, and after that t' from the known depth t.

If the eye be on, or near to, the normal $O\ A$, the angles r and i are very small, and $\cos r$ and $\cos i$ are practically unity. Therefore, without appreciable error we can write for the apparent vertical position of an object in a dense medium, or the apparent depth of a medium, viewed vertically,

$$t'=\frac{t}{\mu}$$

If the medium be water whose index is 4/3, then the apparent depth is 3/4 that of the real depth; with glass $\mu=1{\cdot}5$, $t'=2/3\ t$.

The apparent vertical displacement d is

$$d=t-\frac{t}{\mu}=t\frac{(\mu-1)}{\mu}$$

$(\mu-1)/\mu$ is about 1/4 in the case of water, and 1/3 for glass.

A fish appears nearer the surface than it really is, and when the eye is near the surface, it appears distorted, thinner if lengthways, and stunted if with its head towards the surface. Light from its under portions suffers relatively more deviation than that from the upper, thus giving the idea of vertical compression. Supposing the course of a bullet to be unaltered by the water, one would have to aim, with a rifle, well beneath a fish in order to hit it. To reach a coin at the bottom of a bath one would have to dive towards a point apparently nearer. Again if a coin were hidden from view by the rim of a basin, it may come into view if water be poured into the basin.

On the other hand, if the eye were in a dense medium and viewing an object in air, the apparent position of O would be more distant than it really is, such that $t'=t\mu$.

Refraction by a Parallel Plate.—If a ray $A\ B$ (Fig. 55) be incident on a medium with parallel surfaces such as a plate of glass in air, it is refracted towards the normal at the first surface in the direction $B\ C$, and after refraction at the second surface emerges as $C\ D$ parallel to its original course $A\ B$; the angular deviation is zero.

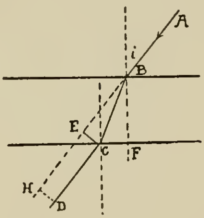

FIG. 55.

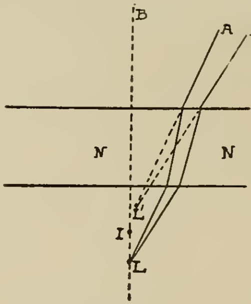

FIG. 56.

The ray, however, as a whole, is laterally displaced over the distance $H\ D$, depending on the angle of incidence i, the μ of the medium, and its thickness t. Let d be the displacement and r the angle of refraction in the plate.

Now $H\ D = E\ C$, the angle $E\ B\ F = i$, $C\ B\ F = r$, and $E\ B\ C = i - r$, while $F\ B = t$ the thickness.

Then $$\mathrm{E\ C = B\ C\ \sin\ E\ B\ C = B\ C\ \sin}\ (i-r)$$

but $$\mathrm{B\ C = B\ F/\cos\ C\ B\ F} = t/\cos r$$

Therefore $$\mathrm{E\ C} = d = t\,\frac{\sin\,(i-r)}{\cos r}$$

The value of r is first found from i and μ, and then d is calculated.

Lateral displacement causes slight distortion of a near object when viewed through a plate, but if the thickness is small, the effect is inappreciable. Distortion disappears for any thickness when the light is parallel.

If the object be viewed through a plate from vertically above, it appears to be nearer, but not laterally displaced. The vertical displacement depends on the thickness of the plate, and is

$$d = t\frac{(\mu - 1)}{\mu}$$

as for an object situated in a dense medium. Thus for glass of index 1·5, the object viewed would appear nearer by about 1/3 the thickness of the plate.

Fig. 56 shows the course of an oblique pencil AA from which arises the apparent vertical displacement of a point L to L' when viewed through a plate N.

Multiple Parallel Media.—If a number of parallel plates of different indices be superposed, their combined action is similar to that of a single plate of uniform index, provided that the first and last media have the same index. The refraction that occurs in this case is such that the angle of emergence at the last surface equals the angle of incidence at the first. When the first and last media have not the same index the deviation suffered by the light, on emergence, is the same as if the light entered from the first medium directly into the last.

Eye in Dense Media.—There is, of course, no critical angle for light passing from a rare into a dense medium, so that to an eye E (Fig. 57) under water, a field of 180° is visible when looking upwards from the dense medium.

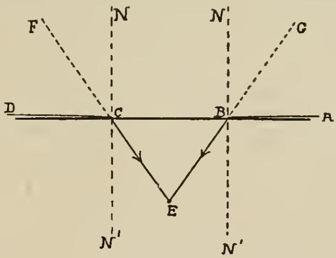

Fig. 57.

Rays $A\ B$ and $D\ C$ from objects level with the surface are practically parallel to the latter and therefore refracted into the water at the critical angle $E\ B\ N'$ and $E\ C\ N'$ and are referred back in the direction G and F. The cone $F\ E\ G$ contains a distorted view of all external objects, and its angle $C\ E\ B$ is equal to the sum of the angles $E\ B\ N'$ and $E\ C\ N'$, that is, to twice the critical angle of water—about 96°. Also, as previously stated, objects directly above appear more distant by an amount equal to about 1/3 their real distance above the water, but those closer to the surface are displaced to a rather greater extent. The distortion and indistinctness are greatest for objects near the surface, and disappear for those directly above.

The circular space into which all external objects appear to be crowded is, owing to dispersion, bordered by a ring of colour, blue on the in- and red on the outside.

CHAPTER V

REFRACTION BY PRISMS

If the two surfaces of a medium are not parallel to each other, all incident light must suffer refraction, since no ray can be perpendicular to both surfaces.

Prism.—An optical prism is a transparent body, usually of glass, having two plane refracting surfaces which are inclined to each other and meet in a line. For special purpose a prism may be of quartz, rocksalt, fluorspar, etc.

Defining Terms.—The line of junction *A B* (Fig. 58) of the two refracting surfaces is the *edge*. *F C D E* is the *base*. *A B D C* and *A B E F* are the two *refracting surfaces*. The angle between them is the *principal angle*. The plane *A B K I* containing the edge and situated symmetrically with respect to the two surfaces is the *base-apex plane :* generally it bisects the base. Any line, as *L M*, at right angles to the edge and lying in the base-apex plane,

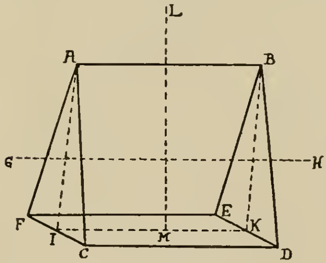

Fig. 58.

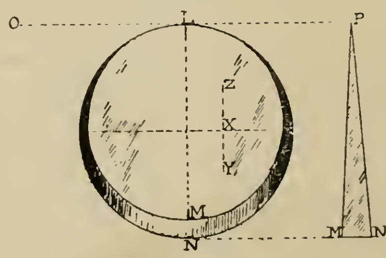

Fig. 59.

is a *base-apex line*. The line *G H*, parallel to the edge and lying in the base-apex plane, midway between the edge and base, is the *axis*. A *principal section* is any section, as *A F C*, cutting it from edge to base perpendicularly to the axis. The point *A*, or any point on the edge *A B*, is the *apex*.

In a circular or oval prism the thinnest part *L* (Fig. 59) is considered the apex. *M N* is the base. The *central* line *L M* connecting the thinnest and thickest parts is the base-apex line, and is usually marked on the circular prisms of the trial case by two small scratches. *O P*, tangent to *L* and perpendicular to *L M*, is the imaginary edge. *P M N* shows a section of such a prism along the base-apex line.

Refraction by a Prism.—*All light refracted by a prism is bent towards the base.* If a ray be incident normally to the first surface, it is undeviated until it reaches the second, when it is bent away from the normal. If incident other than normally, as it passes from the rarer into the denser medium, it is refracted towards the normal at the first surface, and again away from the normal as it passes from the denser into the rarer medium, at the second surface.

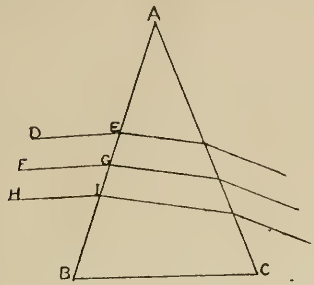

FIG. 60.

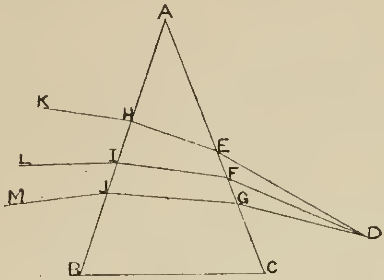

FIG. 61.

If the incident light is parallel (Fig. 60), divergent as from *D* (Fig. 61) or convergent, it is in general the same, respectively, after refraction by the prism. Nevertheless, as will be seen later, the *degree* of divergence or convergence does not remain quite the same.

The Principal Angle.—The principal angle at *A* (Fig. 62), formed by the two refracting surfaces, is sometimes known as the *refracting angle,* or *angle of inclination.* It is indicated in degrees. A prism of 10° is one whose sides enclose that angle.

The Angle of Deviation.—*An object viewed through a prism appears deviated towards the edge.* The incident ray *D E* (Fig. 62) is refracted at *E*, takes the

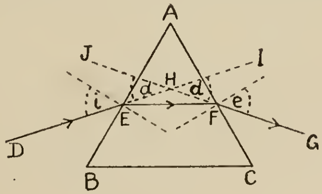

FIG. 62.

direction *E F*, and is again refracted at *F* to pass out in the direction *F G* as if proceeding from *J*. The *angle of deviation d* is, in this case, *I H G*, because *D E*, instead of following the path *H I*, follows the path *H G*. An object at *D*, viewed through the prism, appears as if it were situated at *J*. The deviating angle constitutes the important optical property of a prism, and expresses its power or refracting effect.

The apparent deviation of an object is the result of the refraction suffered at the two surfaces. It is commonly said to be towards the apex. A ray incident at *X* (Fig. 59), from an object beyond the prism, is refracted towards

Y and is referred back towards Z, the plane of deviation $Z X Y$ being parallel to the base-apex line $L M$.

The deviation depends on and varies directly with (1) the angle of the prism, (2) the μ of the medium, and (3) the angle of incidence of the ray.

The last, however, is usually ignored when the power or deviation of a prism is considered, and the incidence is taken to be that resulting in *minimum deviation.*

Minimum Deviation.—For every prism there is one position of the base-apex plane in which an incident ray is less deflected than in any other. From this position, if the prism be rotated around its axis, either way, the image seen through the prism appears still more deviated towards the edge of the prism.

Minimum deviation obtains when the incident and emergent rays are equidistant from the edge, and, as shown in Fig. 62, *the angles of incidence and emergence* (i *and* e) *are also equal.* In this position the ray, as it traverses the prism, forms, with the sides, an isosceles triangle, of which it is the base. In this case the angle of deviation is formed in the base-apex plane, which may be regarded as the *refracting plane* of the prism. In other words, minimum deviation occurs when the refraction is shared *equally between the two surfaces.*

The Prism Formula.—In the prism $B A C$ (Fig. 63) P is the principal angle, d the deviating angle, i the angle of incidence, r the angle of refraction at the first surface, s the angle of incidence at the second surface, e the angle of emergence, and μ the index. The formula given below connecting these factors, when d is minimum, enables us to find the index of refraction of a prism when P and d are known, having being measured by the spectrometer (*q.v.*).

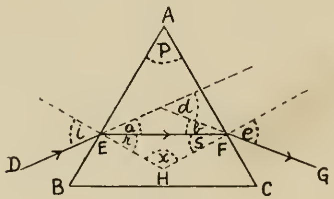

FIG. 63.

Geometrical Proof.—$D E F G$ is a ray (Fig. 63) traversing the prism $B A C$ at minimum deviation. Being symmetrical

$$i=e, \quad r=s, \quad \text{and} \quad a=b.$$

In the quadrilateral $E A F H$, the angles at E and F are right angles, so that $P=180° - x$.

In the triangle $E F H$ the angles $r+s=180-x$

therefore $$P=r+s=2r, \text{ or } r=P/2.$$

The external angle d=the two internal opposite angles a and b, that is,

$$d=a+b=2a, \text{ or } a=d/2$$

so that

$$i=r+a=P/2+d/2=(P+d)/2$$

and as

$$\mu=\frac{\sin i}{\sin r}$$

Therefore

$$\mu=\frac{\sin\left(\frac{P+d}{2}\right)}{\sin\left(\frac{P}{2}\right)}$$

Example.—What is the index of a prism whose angle of minimum deviation is 28° and principal angle 45° ? We have

$$\mu=\frac{\sin\left(\frac{P+d}{2}\right)}{\sin\left(\frac{P}{2}\right)}=\frac{\cdot\sin\left(\frac{45°+28°}{2}\right)}{\sin\left(\frac{45°}{2}\right)}=\frac{\sin 36° 30'}{\sin 22° 30'}=\frac{\cdot 5948}{\cdot 3827}=1{\cdot}554$$

To Calculate the Angle of Deviation.—If μ and P are known, d can be found thus:—

Since $\mu=\frac{\sin\left(\frac{P+d}{2}\right)}{\sin\left(\frac{P}{2}\right)}$ then $\mu \sin\frac{P}{2}=\sin\frac{P+d}{2}$

Let $(P+d)/2$ be called a.

Then $2a=P+d$ and $2a-P=d$.

To find the value of d we require two steps, thus :
(1) Find a from $\sin a=\mu \sin P/2$; (2) then $d=2a-P$.
Expressed as a formula this becomes

$$d=2\,[\sin^{-1}(\mu \sin P/2)]-P$$

that is, d equals twice that angle whose sine is ($\mu \sin P/2$) less P.

Example.—What is the angle of deviation of a prism whose principal angle is 60° and index 1·62 ?

Here $\mu \sin P/2=1{\cdot}62 \sin 30°=1{\cdot}62\times{\cdot}5={\cdot}81$

and ${\cdot}81=\sin 54°$ (nearly)

Therefore $d=(2\times 54)-60=48°$

To Calculate the Principal Angle.—If μ and d are known, P can be found thus:—

$$\mu=\frac{\sin (\mathrm{P}/2+d/2)}{\sin \mathrm{P}/2}$$

$$\mu=\frac{\sin \mathrm{P}/2 \cos d/2+\cos \mathrm{P}/2 \sin d/2}{\sin \mathrm{P}/2}=\cos d/2+\frac{\sin d/2}{\tan \mathrm{P}/2}$$

whence
$$\tan \mathrm{P}/2=\frac{\sin d/2}{\mu-\cos d/2}$$

Example.—What angle must be given to a prism of 36° minimum deviation when $\mu=1{\cdot}586$?

$$\tan \frac{\mathrm{P}}{2}=\frac{\sin 18^\circ}{1{\cdot}586-\cos 18^\circ}=\frac{\cdot 3090}{1{\cdot}586-\cdot 9511}=\frac{\cdot 3090}{\cdot 6349}=\cdot 4882$$

whence
$$\mathrm{P}/2=26^\circ \text{ (nearly), and } \mathrm{P}=52^\circ$$

The Angle of Incidence.—When deviation is a minimum, the angle of refraction r at the first surface is equal to half P, so that:—

$$\sin i=\mu \sin r=\mu \sin \mathrm{P}/2$$

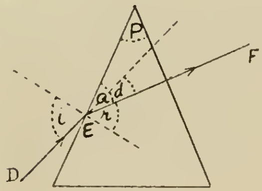

Fig. 64.

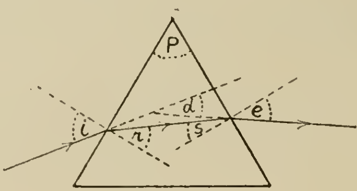

Fig. 65.

Normal Emergence or Incidence.—When a ray $D\ E\ F$ (Fig. 64) enters a prism at such an angle that it emerges normally from the second surface,

$$i=r+d$$

But
$$r+a=90^\circ, \text{ also } \mathrm{P}+a=90^\circ \quad \therefore\ r=\mathrm{P}$$

so that
$$\sin i=\mu \sin r=\mu \sin \mathrm{P}$$

If the light is incident normally, as $F\ E$, all the refraction takes place at the second surface, where r (Fig. 64) is the angle of incidence and $r=\mathrm{P}$; the angle of emergence is then i.

Angles i and e.—With minimum deviation the angles of incidence i and of emergence e are equal. With normal incidence or emergence, the one has zero value. As the one increases the other decreases, but *for any angle of incidence*

$$\mathrm{P}+d=i+e$$

As the angle of incidence departs from that of minimum deviation, the value of d increases, but the increase of d is small compared with the increase of i, or the decrease of e, or *vice versa.*

To trace the course of a ray through a prism at any angle i, when P and μ are known, we have, as in Fig. 65,

$$\sin r = \sin i/\mu \quad \text{and} \quad s = \text{P} - r$$

then
$$\sin e = \mu \sin s = \mu \sin (\text{P} - r)$$

The course of the light might be reversed, e becoming the angle of incidence, and i that of emergence.

General Formula for Given Deviation.—When d is neither minimum, nor that resulting when incidence or emergence is normal, the calculation for i becomes complicated, but Mr. T. Smith, of the National Physical Laboratory, indicates, in a contribution to the Transactions of the Optical Society, a method of finding the angles of incidence and emergence for any value of d with a given prism of angle P. On that article the formula below is based.

Since $i + e = \text{P} + d$ under all conditions, as d varies so i increases and e decreases—or *vice versa*—to the same extent x. Let $(\text{P} + d)/2$ be called a and P/2 be called b.

We have
$$\sin (a + x) = \mu \sin (b + y)$$

and
$$\sin (a - x) = \mu \sin (b - y)$$

Expanding these equations, and then taking half their sum and half their difference, we get two expressions which, when squared and equated, enables y to be eliminated. Further simplification and equation results in

$$\sin^2 x = \frac{(\sin a + \mu \sin b)\,(\sin a - \mu \sin b)}{(\sin a + \cos a \tan b)\,(\sin a - \cos a \tan b)}$$

Then
$$i = (\text{P} + d)/2 + x \text{ and } e = (\text{P} + d)/2 - x$$

Thus with $\text{P} = 30°$ and $\mu = 1{\cdot}5$, we have for minimum deviation $d = 15° \, 42'$ and $i = 22° \, 51'$; for normal emergence $d = 18° \, 36'$ and $i = 48° \, 36'$. There is for an increase of i of $25° \, 45'$ an increase in d of $2° \, 54'$ only.

For d to be equal to 17° we should have—

$$a = (\text{P} + d)/2 = 23° \, 30' \text{ and } b = 15°;$$

whence
$$\sin^2 x = {\cdot}0842 \text{ and } x = 16° \, 54';$$

so that
$$i = 23° \, 30' + 16° \, 54' = 40° \, 24' \text{ and } e = 6° \, 36'.$$

Simplified Formulæ.—By substituting angles for their sines, which can be done without serious error when the angles are small, as in ophthalmic prisms, the formulæ may be greatly simplified as follows. Small angled prisms are those not exceeding, say, 15° P.

$$\mu = \frac{d + \text{P}}{\text{P}} = \frac{d}{\text{P}} + 1$$

whence
$$d = \text{P}\,(\mu - 1)$$

If the refractive index$=1{\cdot}5$, as is practically the case in ophthalmic glass, then $\mu-1=1/2$, and $d=P/2$.

Thus for a prism of 5° the angle d may be taken as 2° 30′.

When a prism is thin, any moderate departure from the position of minimum deviation does not result in any appreciable increase of deviation, so that this factor may also be ignored.

For the angle of incidence, with minimum deviation,

$$i=(P+d)/2=\mu P/2$$

and for normal emergence

$$i=P+d=\mu P$$

Examples.—If the principal angle is 10° and the deviating angle 5·25°, then

$$\mu=\frac{5{\cdot}25}{10}+1=1{\cdot}525.$$

A prism of P=10° and $\mu=1{\cdot}54$, has an angle of deviation of

$$d=10\times{\cdot}54=5{\cdot}4^\circ=5^\circ\ 24'.$$

If a prism of 6·25° d is required, μ being 1·56, the prism angle is

$$P=\frac{6{\cdot}25}{{\cdot}56}=11{\cdot}166 \text{ or } 11^\circ\ 10'$$

If P=10° and $\mu=1{\cdot}5$ the angle of incidence for minimum deviation is

$$i=(1{\cdot}5\times 10)/2=7{\cdot}5^\circ$$

If P=10° and $\mu=1{\cdot}5$ the angle of incidence for normal emergence is

$$i=1{\cdot}5\times 10=15^\circ$$

Neutralising Prisms.—Two prisms of similar angle d will, when placed in opposition, i.e. base to apex, neutralise each other. If they are also of similar P and μ they act as a plate, having parallel surfaces, on light passing through them. If the μ's are unequal, so also must be the angles P.

To calculate the thin prism P_1, made of glass of a certain μ_1, which will neutralise another P_2, whose μ_2 is different, we have

$$P_1(\mu_1-1)=P_2(\mu_2-1)$$

Thus if a crown glass prism of 15°, whose $\mu_1=1{\cdot}54$, has to be neutralised by a flint prism whose $\mu_2=1{\cdot}62$, then from the above

$$P_2=\frac{15\times{\cdot}54}{{\cdot}62}=13^\circ$$

If the prisms are thick, P_1 and P_2 are not then directly proportional to d or to $(\mu-1)$, so that d_1, the deviation of P_1, must first be calculated from the true formula. The other deviation d_2 must equal d_1, and from d_2 the value of P_2 for an index of μ_2 can be obtained.

Displacement by a Prism.—In Fig. 66 A is seen, through the prism, at A'; if the object is at B or at C, its image is seen at B' or C' respectively. The angular displacement by a given prism depends entirely on the angle of

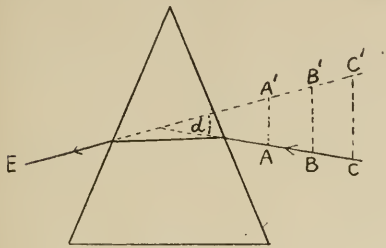

Fig. 66.

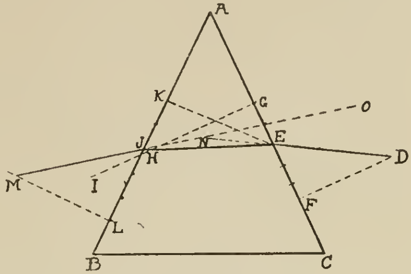

Fig. 67.

deviation and is invariable; but the actual or linear displacement $A\ A'$, $B\ B'$, $C\ C'$, is proportional to the distance of the object, and is represented approximately by the tangent of the angle of deviation.

Construction.—To trace graphically the course of a ray through a prism of given P and μ it is necessary to use a double construction like that of Fig. 46. At both surfaces the larger space, such as HL, is towards the base, and the smaller, as HK, towards the apex, as shown in Fig. 67.

False Images of a Prism.—On looking through a weak prism at the primary image of a bright source, a second and fainter image, due to internal reflection, is seen projected parallel to the base-apex line under an angle five or six times the deviating angle, so that in strong prisms it lies too far away towards the edge to be visible unless specially sought. In "The Clinical Use of Prisms" Dr. Maddox suggests its utilisation for the exact adjustment of the base-apex line.

The Measurement and Notation of Prisms are treated in Chaps. X. and XI.

CHAPTER VI

REFRACTION BY CURVED SURFACES

Curved Surface.—A refracting surface is one which separates two media of different densities, so that, when light passes from the one to the other, refraction takes place. Only one refraction occurs and in this respect a surface differs essentially from a lens, where there are at least two surfaces and two refractions of the light. Unless otherwise stated a curved surface is deemed to be spherical.

Since every line drawn from the centre *C* (Fig. 68) to the circumference of a sphere is a radius of curvature, every point on the circumference may be

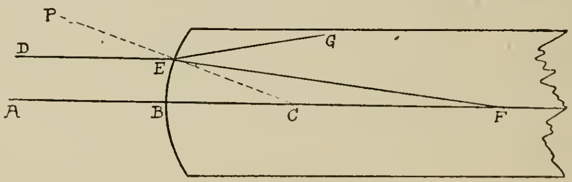

Fig. 68.

regarded as a minute plane at right angles to a radius. Thus *C E* and *C B* are normal to the surface at *E* and *B* respectively, and also when prolonged beyond the circumference.

Any ray *A B* or *P E* directed towards *C* passes into the medium without deviation. A ray *D E*, which is not normal to the surface, is bent towards the

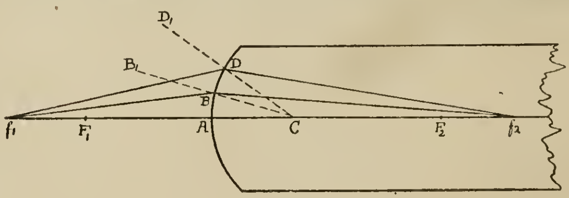

Fig. 69.

normal *P E C* in the direction *E F*, if the medium is of a higher index of refraction, or it is bent away from the normal in the direction *E G*, if of a lower index than that of the medium from which the light proceeds. Both media are supposed to be of indefinite extent.

Cx. and Cc. Surfaces.—In Fig. 69 let a mass of glass have a Cx. surface, and the outer medium be air. The ray f_1 *A* directed towards *A* is at normal incidence, and passes onward without deviation. The rays f_1 B and f_1 D form certain angles with the normals to the surface, and each, on passing into the denser medium, is bent towards the normal to an amount governed by the ratio between the sines of the angles of incidence and refraction. Thus f_1 D is bent more than $f_1 B$, and the two meet the line $f_1 f_2$ at the point f_2. Similarly, all rays diverging from f_1 are refracted to f_2, which is, therefore, the focus or image of the source of light f_1, and the points f_1 and f_2 are conjugate foci. If the object were at f_2, the image would be at f_1.

The focus thus formed by a *convex* surface of a medium of higher μ is *positive or real.* If the medium is of lower μ, light entering it is rendered divergent, and the focus is *negative or virtual.* If the surface is concave, as

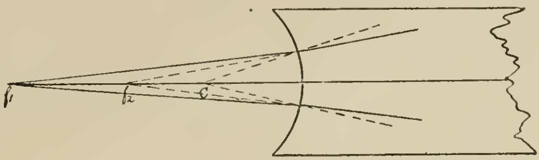

FIG. 70.

in Fig. 70, the reverse is the case, and f_2 is virtual and on the same side of the surface as f_1. The boundary plane between the two media may be regarded either as the convex surface of the one or the concave surface of the other, but it is more convenient to express it as convex or concave to the medium of lower index, which usually is air. Thus for a dense medium having a convex surface in contact with air we could calculate the position of f_2 from the refractivity and curvature of the dense medium, or from those of the rare medium, the result being the same.

Defining Terms.—The line f_1 A f_2 (Fig. 69), is the *principal axis.* It is normal to the surface and passes through the centre of curvature C and the principal foci; all other lines passing through C are *secondary axes.* A C=r is the *radius of curvature,* and C is the *centre of curvature.* f_1 and f_2 are the positions occupied by object and image. F_1 is the *anterior principal focus* formed by the light proceeding from a distant source on the principal axis in the denser medium, and F_2 is the *posterior principal focus* formed by light proceeding from a distant source in the rarer medium. The surface itself is the *refracting plane* and the centre of curvature is also the *optical centre.*

Formulæ connecting f_1 and f_2.—In Fig. 71, let O be an object on the axis of a surface, and I its image formed by refraction of the ray O D incident at D. From C draw the radius C D and let the angle A O D=a, A C D=b and A I D=c. The indices of refraction of the first and second medium are μ_1 and μ_2.

Then $$\mu_1 \sin i = \mu_2 \sin r$$

But $$i = a + b \text{ and } r = b - c$$

Therefore $$\mu_1 \sin (a+b) = \mu_2 \sin (b-c)$$

If the incident pencil be small and axial, the angles a, b and c are also small, so that we can omit sines, and substitute circular measure for the angles themselves.

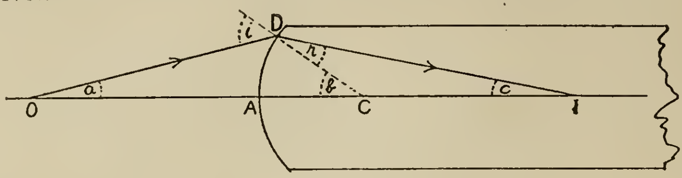

FIG. 71.

Let $O\,A = f_1$, $A\,I = f_2$ and $A\,C = r$ the radius of curvature.*

Then $$\mu_1\left(\frac{1}{f_1}+\frac{1}{r}\right)=\mu_2\left(\frac{1}{r}-\frac{1}{f_2}\right) \quad \text{or} \quad \frac{\mu_1}{f_1}+\frac{\mu_1}{r}=\frac{\mu_2}{r}-\frac{\mu_2}{f_2}$$

Therefore $$\frac{\mu_1}{f_1}+\frac{\mu_2}{f_2}=\frac{\mu_2-\mu_1}{r}$$

The Focal Lengths and Powers of a Curved Surface.—The refractive power $1/F$ of a curved surface depends on its curvature and the refractive index of the medium, so that an increase in either is accompanied by increase of power. The focal length F depends on the refractive power, the one being inversely proportional to the other, *i.e.* the greater the power, the shorter is the focal length.

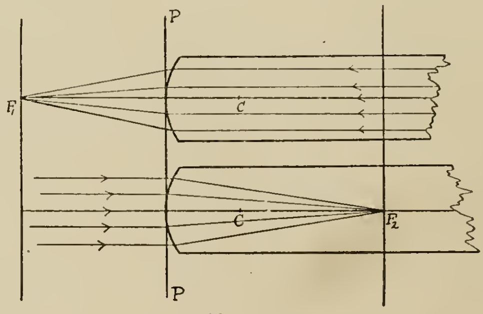

FIG. 72.

In Fig. 72 P is the principal or refracting plane where all refraction takes place. C is the optical centre (or nodal point) because all rays passing through it are unrefracted. If light is parallel to the principal axis in the dense medium, on emergence into the rare medium it is refracted to meet at F_1 situated on the principal axis. $P\,F_1$ is the *anterior focal distance*, and F_1 the *anterior principal focus*. If light diverges from F_1 it is parallel, after refraction, in the dense medium.

* It is so customary to employ r to indicate the angle of refraction, and r also to indicate radius, that this symbol is here retained for both. Although the two may appear in the same calculation, they never do so in a formula, so that no confusion is likely to arise.

If light is parallel to the principal axis in the rare medium, after refraction it meets at F_2 in the denser medium. $P\ F_2$ is the *posterior principal focal distance* and F_2 the *posterior principal focus.* If light diverges from F_2 it is parallel, after refraction, in the rare medium.

In the formula given in the preceding article, if f_2 is at ∞ so that the light in the denser medium may be regarded as parallel, the term μ_2/f_2 becomes $\mu_2/\infty=0$.

Then $$\frac{\mu_1}{f_1}=\frac{\mu_2-\mu_1}{r}$$

But f_1 is now the anterior principal focal length, and is written F_1.

Therefore $$\frac{\mu_1}{F_1}=\frac{\mu_2-\mu_1}{r} \quad \text{or} \quad F_1=\frac{\mu_1 r}{\mu_2-\mu_1}$$

If f_1 is at ∞ so that the light in the rarer medium is parallel, the term μ_1/f_1 becomes $\mu_1/\infty=0$.

Then $$\frac{\mu_2}{f_2}=\frac{\mu_2-\mu_1}{r}$$

But f_2 is now the posterior principal focal length and written F_2.

Therefore $$\frac{\mu_2}{F_2}=\frac{\mu_2-\mu_1}{r} \quad \text{or} \quad F_2=\frac{\mu_2 r}{\mu_2-\mu_1}$$

The planes passing through F_1 and F_2, perpendicular to the principal axis, are respectively the anterior and posterior focal planes.

It must be particularly noted that μ_1 *always pertains to the medium in which the object is situated*, and μ_2 to the medium towards which the light proceeds, but which may or may not be that in which the image is actually formed, since this may be either real or virtual.

If the one medium is air, $\mu_1=1$, so that it can be omitted from the formulæ; the index of the dense medium we can then call μ, and therefore the formulæ become simplified to

$$F_1=\frac{r}{\mu-1} \quad \text{and} \quad F_2=\frac{\mu r}{\mu-1}$$

Examples.—If μ of the dense medium is 1·5 and the other medium is air, for a radius of curvature of 8 in. we have

$$F_1=\frac{8}{1{\cdot}5-1}=16 \text{ in.}$$

$$F_2=\frac{1{\cdot}5\times 8}{(1{\cdot}5-1)}=24 \text{ in.}$$

Thus, with glass, having an index of 1·5, $F_1=2r$ and $F_2=3r$.

If the surface is Cc. to the air, r is negative and its value prefixed in the formulæ by a − sign; F_1 and F_2 become negative, and are situated on the same side of the surface as the source of light, *i.e.* F_2 is in air and F_1 is in the dense medium.

Thus let $r = -8''$ and $\mu = 1{\cdot}5$, then

$$F_1 = \frac{-8}{\cdot 5} = -16 \text{ in.}$$

$$F_2 = \frac{1{\cdot}5 \times (-8)}{\cdot 5} = -24 \text{ in.}$$

Suppose parallel light passes from water of $\mu_1 = 1{\cdot}33$ into glass of $\mu_2 = 1{\cdot}5$ and let $r = 8$ in.; then

$$F_2 = \frac{1{\cdot}5 \times 8}{1{\cdot}5 - 1{\cdot}33} = \frac{12}{\cdot 17} = 70{\cdot}6 \text{ in.}$$

If the light passes from glass into water,

$$F_1 = \frac{1{\cdot}33 \times 8}{1{\cdot}5 - 1{\cdot}33} = \frac{10{\cdot}64}{\cdot 17} = 62{\cdot}6 \text{ in.}$$

In these calculations the relative μ, which equals μ_2/μ_1, can be found, and the calculation then made as if the lower μ were 1.

Relationship of F_1 and F_2.—*The anterior and posterior focal distances measured from the refracting surface are proportional to the indices of refraction of the two media.* Thus in the examples given we have

$$\frac{F_2}{F_1} = \frac{\mu_2}{\mu_1} = \frac{24}{16} = \frac{1{\cdot}5}{1} \quad \text{and} \quad \frac{F_2}{F_1} = \frac{70{\cdot}6}{62{\cdot}6} = \frac{1{\cdot}5}{1{\cdot}33}$$

Also whatever the refractive indices may be, $r = F_2 - F_1$.

In the examples $r = 8 = 24 - 16$ or $70{\cdot}6 - 62{\cdot}6$.

Thus when $\mu_1 = 1$, $F_2 = F_1 + r = F_1\mu$, and $F_1 = F_2 - r = F_2/\mu$.

That F_1 is shorter than F_2 follows from the law of sines. If the two μ's are respectively 1 and 1·5, when light passes into the denser medium sin r is 2/3 sin i, whereas when light passes into the rarer medium sin r is 3/2 sin i; hence the angular deviation is greater when the focus is in the air, it being then about $\frac{1}{2}$ i, than when it is in the dense medium, then it is about $\frac{1}{3}$ i.

In addition to what is stated in the first paragraph of this chapter, a surface differs from a lens in that, with the former, the first and last media being different, F_1 differs from F_2, whereas with a lens $F_1 = F_2$. Also as shown in Fig. 71, the optical centre (or nodal point) C does not coincide with the principal point which marks the refracting plane at A, the apex of the surface.

To find r or μ.—The radius or the refractive index can be found by substituting known values for the symbols given in the above formulæ, and then equating. When the denser medium has a concave surface, care must be taken that the $-$ sign be given to F and to r in all calculations.

Position of an Image-Point.—An object-point situated on the *principal axis* has its image on the *principal axis.* One situated on a *secondary axis* has its image on *that same axis* and the focus is a *secondary focus.* One point only of an object is situated on the principal axis; every other is situated on a different secondary axis, and similarly with the image of the object.

The image of a luminous point being on a line drawn from that point through C, its position on that line can be determined by calculation or construction. It is on the opposite side of the refracting surface if the rays converge after refraction; and on the same side if, after refraction, they diverge. The greater the convergence or divergence the sooner do the rays meet, or appear to meet, and form the image of the object-point from which they originally diverged.

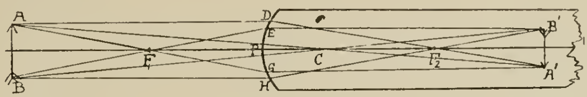

Fig. 73.

Construction of Image—Cx. Surface.—In Fig. 73 $A\,B$ is an object situated in front of the refracting surface $D\,P\,H$, its image $B'\ A'$ can be constructed in the following way.

There are three rays emanating from any point of the object, say A, the course of which can be easily traced, viz.,

(*a*) The secondary ray $A\ C$ passing through C without deviation.

(*b*) The ray $A\ D$ parallel to the principal axis refracted to pass through F_2.

(*c*) The ray $A\ G$ through F_1 refracted parallel to the principal axis.

Where these rays meet at A' is common to all other rays diverging from A and is its image. Similar rays from B form an image at B', and these define the position and size of the real inverted image of the object $A\ B$.

It suffices to draw any two only of the three rays specified.

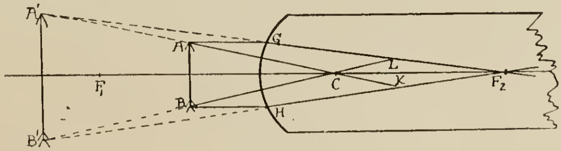

Fig. 74.

Let the object be nearer to the surface than F_1, as $A\,B$ in Fig. 74. From A draw $A\ C$ which passes through C without deviation; draw $A\ G$ parallel to the axis; this passes through F_2. Since $A\ C\ K$ and $G\ F_2$ diverge, they can meet only by being prolonged back to A'. Similarly, $B\ C$ and $B\ H$ are drawn, and produced backwards to B'. Thus $A'B'$ is the virtual erect image of $A\,B$.

Course of any Ray—Cx. Surface.—When the object is situated in the anterior focal plane the rays, diverging from any point D (Fig. 75) on it are, after refraction, parallel to each other and to a secondary axis DC in the denser medium so that the image, in theory, is formed at ∞. Similarly if the object lies in the posterior focal plane the light is parallel in the rarer medium after refraction.

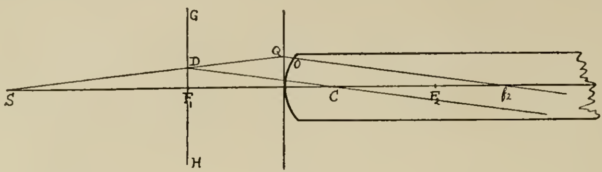

Fig. 75.

Any ray $S\ D\ Q$ incident on the refracting surface from a point S on the principal axis passes through the first focal plane $G\ H$ at D, and through the principal plane at Q, and its course, after refraction at Q, will be parallel to $D\ C$ drawn from D through C; it therefore takes the direction $Q f_2$, and f_2 is the image of S.

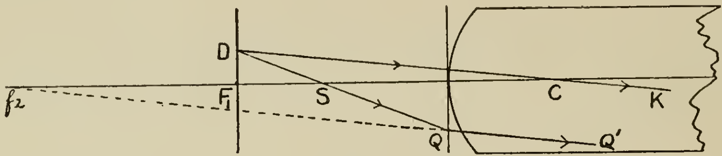

Fig. 76.

If the point S (Fig. 76) is within F_1, draw a line $S\ D$ backwards to D in the first focal plane and forwards to Q in the refracting plane. Draw the secondary axis $D\ C\ K$; then $S\ Q\ Q'$ will be parallel to it after refraction. The latter produced back cuts the axis in f_2, which is the image of S.

The distance a (Fig. 75) between the ray and the principal axis in the refracting plane is equal to the sum of b and d, the distances between the ray and the axis in, respectively, the first and second focal planes; the point f_2 can be located by measuring off on the second focal plane $d=a-b$, and then connecting Q through that point to f_2. When S is within F_1, as in Fig. 76, $d=a+b$.

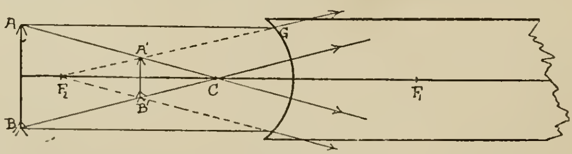

Fig. 77.

Construction of Image—Cc. Surface.—The rays (a) and (b) as given for a Cx. serve also for a Cc. surface, but (b) diverges.

$A\,B$ is the object (Fig. 77). Draw $A\ G$; this, after refraction, diverges as if proceeding from F_2. Draw $A\ C$ through C, whose direction is unchanged by refraction. Now *A C* and *A G* are more divergent in the denser than they were originally in the rarer medium, and when projected backwards meet at *A′*, which is the virtual image of *A*. Similar rays from *B* locate its image at *B′*. Consequently *A′ B′* is the image of the object *A B*, and is *virtual or negative, erect* and *diminished.*

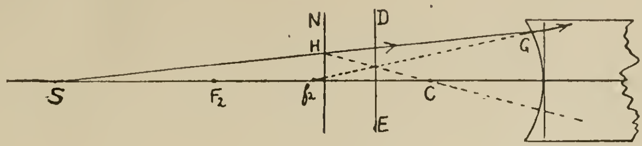

FIG. 78.

If rays diverge from a distance equal to F_1 they have their image in a plane *D E* midway between F_2 and the surface. Let *S G* (Fig. 78) be incident on a Cc. surface. Now *S G* cuts the plane *N* at a distance F_1 from the surface in *H*, and if from *H* we draw the secondary axis *H C* we determine the point where it cuts *D E*. The ray is refracted as if it came in the direction f_2 *G*, so that if *S* is on the principal axis, f_2 is its image.

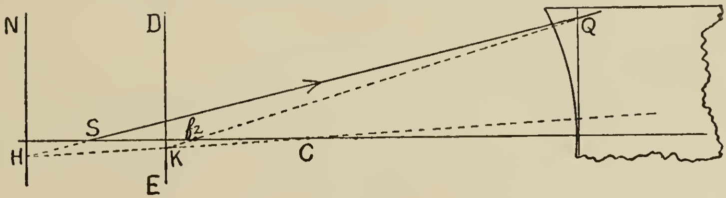

FIG. 79.

If the object-point *S* (Fig. 79) is within *N*, draw any line *H S* backwards to the plane *N* and forwards to *Q* on the surface. Draw *H C* cutting *D E* in *K*; connect *K* with the surface to meet *H S* at *Q* and where *K Q* cuts the axis in f_2, is the image of *S*.

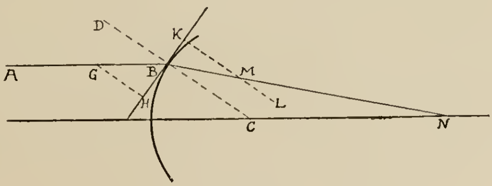

FIG. 80.

General Construction for the Course of a Ray.—This is illustrated in Fig. 80. Let *A B* be the incident ray on a surface whose centre is *C*. Draw *D B C* normal at the point of incidence, and a tangent *K H* to the surface at *B*, and at right angles to *D B C*; then *H K* is the refracting plane, and the procedure is exactly as for a plane surface (Fig. 46).

The Formula for Conjugate Foci.—The object distance being f_1, the image distance f_2, and the first medium being air—

$$\frac{1}{F_1}=\frac{1}{f_1}+\frac{\mu}{f_2} \quad \text{or} \quad \frac{1}{F_2}=\frac{1}{\mu f_1}+\frac{1}{f_2}$$

As will be seen later with lenses, as well as with mirrors, the divergence of the light from the object is added to the power of the surface, and the resultant convergence or divergence gives the position of the image, real or virtual.

Size of Image.—*The sizes of image and object are to each other as their respective distances from the centre of curvature, where the axial rays cross each other.* This is shown in Fig. 81, and whether the image be real or virtual, the object and image always subtend the same angle at *C*.

Let the distance of the image from the surface be f_2 and that of the object f_1, and let r be the distance from the surface to the centre of curvature.

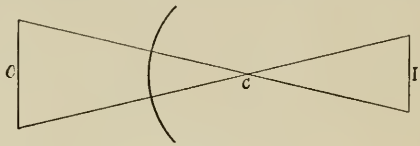

FIG. 81.

Let the size of the image be h_2 and that of the object h_1 and their distances from C respectively $I\ C$ and $O\ C$; then the magnification, in the case of a real image, is

$$M=\frac{h_2}{h_1}=\frac{I\ C}{O\ C}=\frac{f_2-r}{f_1+r}$$

From the original formula

$$\frac{\mu_1}{f_1}+\frac{\mu_2}{f_2}=\frac{\mu_2-\mu_1}{r}=\frac{\mu_2}{r}-\frac{\mu_1}{r} \quad \text{or} \quad \frac{\mu_1}{f_1}+\frac{\mu_1}{r}=\frac{\mu_2}{r}-\frac{\mu_2}{f_2}$$

that is
$$\frac{\mu_1(f_1+r)}{f_1 r}=\frac{\mu_2(f_2-r)}{f_2 r} \quad \text{or} \quad \frac{f_2\mu_1}{f_1\mu_2}=\frac{f_2-r}{f_1+r}$$

so that
$$M=\frac{h_2}{h_1}=\frac{f_2\mu_1}{f_1\mu_2}$$

The linear size of I or O, when the other data are known, is found from

$$h_2=\frac{h_1 f_2 \mu_1}{f_1\mu_2} \quad \text{and} \quad h_1=\frac{h_2 f_1\mu_2}{f_2\mu_1}$$

If f_1 and f_2 are in the same terms, then h_2 is in the same terms as h_1.

Should the distance of f_2 or f_1 not be known, M can be found from $F_1/(f_1-F_1)$ or $(f_2-F_2)/F_2$ respectively.

Unit Magnification.—*The object and its real image are the same size when* they are at equal distances from C, on opposite sides of the surface. They are then in the planes of unit magnification. O is at $2F_1$ and I is at $2F_2$ from the surface, or O is at $2F_1+r$, and I is at $2F_2-r$ from C.

Thus, when M=1, then $f_2=\mu f_1$ or $f_1=f_2/\mu$

$$\frac{1}{F_1}=\frac{1}{f_1}+\frac{\mu}{f_2}=\frac{2}{f_1} \qquad \text{that is, } f_1=2F_1$$

$$\frac{1}{F_2}=\frac{1}{\mu f_1}+\frac{1}{f_2}=\frac{2}{f_2} \qquad \text{that is, } f_2=2F_2$$

If the image is virtual, *with a concave or convex surface*, $M=-1$ can occur only when O and I are both at the surface and therefore equally distant from C.

Since the axial rays cross at C, the I formed at F_2 of a surface equals that formed by a lens whose $F=F_1$, and that formed at F_1 equals that formed by a lens whose $F=F_2$.

Examples.—Let $r=10$ mm., $\mu_2=1{\cdot}5$, $\mu_1=1$, and f_1 be in the air at 100 mm. from the surface. h_1 its height=10 mm.; then

$$\frac{1{\cdot}5}{f_2}=\frac{1{\cdot}5-1}{10}-\frac{1}{100}=\frac{4}{100} \qquad \text{that is, } f_2=\frac{150}{4}=37{\cdot}5 \text{ mm.}$$

$$h_2=\frac{10\times 37{\cdot}5}{100\times 1{\cdot}5}=2{\cdot}5 \text{ mm.}$$

The image is real, inverted, 2·5 mm. high.

Let $r=8$ mm., $\mu_1=1{\cdot}333$, $\mu_2=1$, and the object be in the denser medium at 3·6 mm. from the surface and 2 mm. in size; then

$$\frac{1}{f_2}=\frac{1-1{\cdot}333}{-8}-\frac{1{\cdot}333}{3{\cdot}6}=-\frac{1}{3{\cdot}05} \quad f_2=-3{\cdot}05 \text{ mm.}$$

The image is virtual at 3·05 mm. behind the surface, and

$$h_2=\frac{2\times 3{\cdot}05\times 1{\cdot}33}{3{\cdot}6}=2{\cdot}25 \text{ mm.}$$

The pupil of the eye, if 2 mm. in diameter, and 3·6 mm. from the cornea, appears to be 2·25 mm. in diameter and about 3 mm. behind the cornea.

Suppose $r=-3''$, $\mu_2=1{\cdot}5$, $\mu_1=1$, $f_1=20''$ and $h_1=2''$. Then

$$\frac{1{\cdot}5}{f_2}=\frac{1{\cdot}5-1}{-3}-\frac{1}{20}=-\frac{13}{60} \qquad \text{therefore } f_2=-\frac{90}{13}=-6\tfrac{12}{13}''$$

and

$$h_2=\frac{2\times 6\frac{12}{13}}{20\times 1{\cdot}5}=\frac{6}{13} \text{ in.}$$

Another Expression for Conjugate Foci.—Since

$$\frac{\mu_1}{f_1}+\frac{\mu_2}{f_2}=\frac{\mu_2-\mu_1}{r}=\frac{\mu_1}{F_1} \qquad \text{and} \qquad \frac{\mu_1 F_1}{f_1}+\frac{\mu_2 F_1}{f_2}=\mu_1$$

If $\mu_1=1$ and substituting F_2 for $\mu_2 F_1$

$$\frac{F_1}{f_1}+\frac{F_2}{f_2}=1 \quad \text{or} \quad f_2=\frac{f_1 F_2}{f_1-F_1} \quad \text{and} \quad f_1=\frac{f_2 F_1}{f_2-F_2}$$

Examples.—Suppose the object be 20 inches in front of a Cx. surface where $F_1=6$ in. and $F_2=9$ in.

Then $$f_2=\frac{20\times 9}{20-6}=\frac{180}{14}=12\tfrac{6}{7} \text{ in.}$$

If an object is 5″ in front of a surface of $F_1=6$ in. and $F_2=9$ in.

Then $$f_2=\frac{5\times 9}{5-6}=\frac{45}{-1}=-45 \text{ in.}$$ The image is negative.

If f_1 is situated at F_1 the denominator is 0, so that f_2 is at ∞; if f_1 is at ∞, then f_2 corresponds to F_2.

Let the one conjugate be $12\frac{6}{7}$ in. behind Cx. a surface whose $F_2=9$ in., and $F_1=6$ in., then

$$f=\frac{12\frac{6}{7}\times 6}{12\frac{6}{7}-9}=\frac{77\frac{1}{7}}{3\frac{6}{7}}=20 \text{ in.}$$

This example should be compared with the one previously given, where the object is in front of the refracting surface, 20″ and $12\frac{6}{7}$″ being conjugates for the given surfaces.

An object 20 inches from a Cc. surface, whose F_1 and$_2$ F are respectively -6 and -9 in. has its virtual image at

$$f_2=\frac{20\times(-9)}{20-(-6)}=\frac{-180}{26}=-6\tfrac{12}{13} \text{ in.}$$

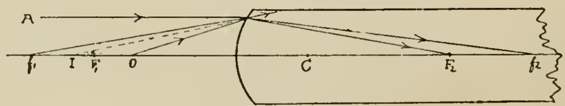

Fig. 82.

Conjugate Focal Distances—Cx. Surfaces.—If the O is at ∞ represented by A (Fig. 82), the light is parallel and, after refraction, meets at F_2. This is the nearest point to the surface at which a real focus can be formed.

If the light diverges from f_1 at a finite distance from the surface, some of the converging power of the medium is required to neutralize the divergence of the light and there is less residual convergence; the light therefore is convergent to a greater distance behind the refracting surface than if the light had been previously parallel; the I in the denser medium is at some point f_2 situated between F_2 and ∞.

As the O approaches from ∞ the I recedes from F_2, and *vice versa*, until when the O is at F_1 the I is at ∞.

If the object is nearer than F_1 as at O, the image is at I on the same side of the surface. As O then further approaches the surface so also does I, and when O touches the surface I does so also.

When O is *within the dense medium*, and the light is parallel, I is at F_1; as O approaches F_2 so I recedes from F_1; thus when O is at f_2 the I is at f_1, and when O is at F_2 the image is at ∞.

When O lies nearer to the surface than F_2 the image is virtual and on the same side of the surface as O and within the dense medium. Thus in Fig. 83

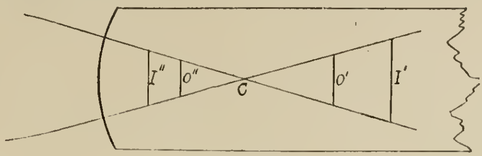

FIG. 83.

if the object is at O', the image is farther away at I'; if the O is at C then the I also is at C, and if the object is at O'' then the image is nearer the surface at I''; when O touches the surface I does so also. If light diverges from a point beyond C it becomes less divergent by refraction at the surface, and if from a point nearer than C it becomes more divergent.

It should be noticed that when the O is at the surface of a Cx. dense medium the I is the same size, and as the O moves away from the surface the I increases in size until, when O is at C, I is there also, but with a magnification equal to μ (the dense medium being assumed to be bounded by air). As the O moves beyond C towards F_2 the image continues to increase in size, until, when O is at F_2, I is at ∞ and infinitely magnified.

Conjugate Focal Distances—Cc. Surfaces.—When the surface of the dense medium is Cc. and the O is at ∞ the I is at F_2. This is the most distant point from the surface at which an image can be formed. If the O is within ∞, the original rays being divergent are rendered still more divergent after refraction than if they had been originally parallel; hence the I is formed nearer to the surface, that is, as O approaches the surface so also does I.

The virtual I is nearer to the surface than the O so long as O is beyond C; when O is at C so also is I, but the latter is diminished μ times; when O is within C then I is beyond O, and when O touches the surface I does so also and M=1.

When the O is in a Cc. dense medium, unit magnification occurs when O touches the surface; as O moves away towards F_1 the I becomes progressively smaller until when O is at ∞, I is at F_1 and proportionately diminished.

Real Conjugates are always interchangeable. If the O is at f_1, its I is at f_2, and the positions could be reversed, as can be seen from the various examples given.

Virtual Conjugates.—Virtual conjugate foci, formed by Cx. or Cc. surfaces, are not interchangeable as are real conjugates, but if the light were directed *converging* towards f_2 the image formed would be at f_1.

Newton's Formulæ for Conjugate Foci and M.—If A and B be respectively the distance of O from F_1 and of I from F_2, then

$$A\,B = F_1\,F_2 \quad \text{and} \quad M = \frac{I}{O} = \frac{F_1}{A} = \frac{B}{F_2}$$

Dioptral Formulæ.—The diopter expresses refracting power and $D = 100/F$, F and r being in cm. The anterior and posterior powers are—

$$D_A = \frac{100\,(\mu_2 - \mu_1)}{r\mu_1} \qquad D_P = \frac{100\,(\mu_2 - \mu_1)}{r\mu_2}$$

$$D_A : D_P \text{ as } \mu_2 : \mu_1.$$

For conjugate foci $D_A = d_1\mu_1 + d_2\mu_2$ or $D_P = d_1/\mu_2 + d_2/\mu_1$

CHAPTER VII

THIN LENSES

As already defined, a real I is formed by the focus of convergent light, and therefore actually exists; a *virtual* I is formed by divergent light, and is imaginary, merely appearing to exist.

Images are seen in the same way as an ordinary object, from which light diverges, and of which a real I is formed on the retina of the eye. To view a virtual image, the eye must look *through* a single lens, whether Cx. or Cc., but to see a real I, formed in the air or on a screen, the eye must be placed a reasonable distance behind the I. The convergent light from the latter crosses and diverges as from an ordinary object, and can therefore be viewed and magnified by other lenses, as in the telescope and microscope.

Similarly, a virtual I is seen by looking *into* a mirror, whether plane, Cx., or Cc.; a real image formed by the convergence of light from a Cc. mirror is seen, as in the case of the Cx. lens, in the air or on a screen.

Position of Object.—It is always taken that an object is in *front* of a lens or mirror, and the image is in front or behind according as it is, respectively, on the same side as, or on the opposite side to, the object.

Optical Signs.—In this work the following convention is followed. Light *divergence* is considered *negative*, and *convergence* is considered *positive*.

Surfaces, mirrors or lenses that cause, or tend to cause, convergence of light, are positive, as also are their focal lengths and powers, and the real images and foci produced by them: to all these the + sign is assigned.

Surfaces, mirrors or lenses that produce, or tend to produce, divergence, together with their focal lengths and powers, and virtual images and foci, are negative, and given the − sign.

Thus when a convex spherical surface of glass is in contact with air, refraction occurs, and this may be taken as due either to the Cx. glass surface or to the Cc. air surface; both are + since both cause convergence of parallel light. If a double Cc. air lens be in water we can consider the resulting converging effect from the Cc. air surfaces, or the Cx. water surfaces. A Cx. surface is not necessarily positive, nor a Cc. negative; when they reflect they are the reverse, as they are, also, when refracting if of lower μ than the adjacent medium. Usually, however, in optics, a Cx. refracting surface is positive and a Cc. negative because it has a higher μ than the adjacent air, but this may not be the case when light passes successively through various media.

Axial and Other Rays.—It is most essential to differentiate between the direction of axial rays and that of the rays from the various points on an object, with reference to their axes.

From each point of the object a pencil of rays diverges and each pencil has an axis, which is the axial ray of that pencil. Axial rays *always converge* to the optical centre of the lens, and their convergence governs the *size of the angle subtended by the object and the image at the lens.*

The rays themselves *always diverge* from the luminous point to the lens, and their divergence governs the *position or distance of the image,* the rays, after refraction being more or less divergent or convergent, according to the original divergence, and the diverging or converging power of the lens. Parallel light is light having a negligible degree of divergence.

These *most important considerations,* for students are apt to confuse the conditions, should be carefully noted. Thus, in a diagram which shows light parallel to the axis, and incident on various parts of the lens surface, the rays are presumed to originate, not in various points, but in one single point on the axis. These considerations apply not only to all lenses, but to surfaces and mirrors as well, and all positions of the object.

Definition of Lens.—A lens is a transparent body usually made of glass, bounded by two surfaces, both of which are curved or the one may be plane. It is usually surrounded by air. This definition covers all forms of convex and concave sphericals, as well as cylindrical and other special forms of lenses.

Prismatic Formation.—If a block of glass is formed by two similar prisms *A C D* and *B C D* base to base, as in Fig. 84, incident rays such as *E* are bent towards the base of the prism *A C D*, and rays such as *F* are bent towards

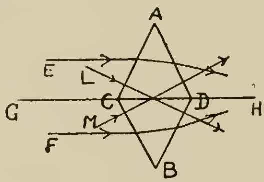

Fig. 84.

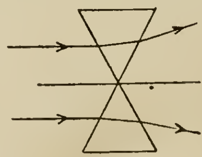
Fig. 85.

the base of the prism *B C D*, so that those refracted by the one prism meet those refracted by the other. One ray, viz., *G C D H* suffers no deviation since it coincides with the base of both prisms; *L* and *M* are incident normally to both surfaces, and are therefore also not deviated.

If the prisms, as in Fig. 85, be edge to edge all rays incident on them, being refracted towards the bases, are deflected from the common edge, except the central ray incident at the junction of the two edges.

What is true of two prisms is also true of any number, and a Cx. or Cc. spherical lens may be considered as if formed of an infinite number of prisms whose bases or apices respectively have a common centre; every meridian may be regarded as if formed of a series of truncated prisms of different angles of inclination, increasing gradually towards the periphery, but having a common base-apex line.

Any two point areas as A and B (Fig. 86) opposite each other constitute a portion of a prism whose base, in the Cx., and whose apex, in the Cc., is turned towards the principal axis of the lens. The areas A and B, near the periphery of the lens, are more inclined towards each other than C and D, situated

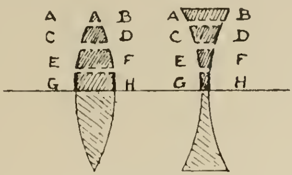

Fig. 86.

nearer to the axis, and the inclination between the surfaces decreases gradually until at G H on the principal axis they are parallel. Since the angle formed by A and B is greater than that formed by C and D, a ray passing through A B is bent to a greater extent than one passing through C D, while the ray which passes along the axis is not deviated at all.

Each zone of a lens, therefore, whether concave or convex, has a refractive power which becomes greater as its distance from the axis is increased, and it is due to this fact that rays diverging from a point, and incident on the lens, are brought to, or appear to diverge from, a common focus practically as a point.

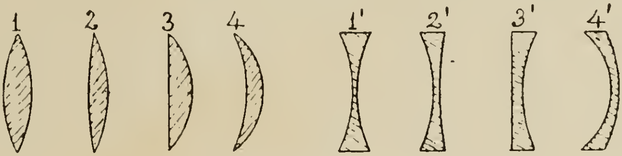

Fig. 87.

Forms of Lenses.—There are (Fig. 87) four forms of thin convex and four of concave spherical lenses:

1. Double Cx. or equi-Cx. having two equally curved Cx. surfaces.
1′. Double Cc. or equi-Cc. ,, ,, ,, Cc. ,,
2. Bi-Cx. having two unequally curved Cx. surfaces.
2′. Bi-Cc. ,, ,, ,, Cc. ,,
3. Plano-Cx. having one Cx. and one plane surface.
3′. Plano-Cc. ,, ,, Cc. ,, ,, ,, ,,
4. Meniscus or periscopic Cx. having one surface Cx. and the other Cc., the Cc. being the weaker power.
4′. Meniscus or periscopic Cc. having one surface Cc. and the other Cx., the Cx. being the weaker power.

Variations of the above, due to increasing the interval between the two surfaces, are treated in the chapter on thick lenses.

Terms of a Lens.—Let Fig. 88 represent a thin Cx. lens; *C C* are the *centres of curvature*, and *O* the *optical centre*. The line *A O B* passing through the two centres of curvature, and the optical centre, is the *principal axis;* it is normal to both surfaces of the lens. The plane *L O L* passing through

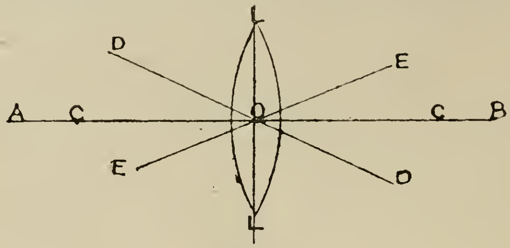

Fig. 88.

O, perpendicular to *A B*, is the *refracting plane*, on which all the refraction effected by both surfaces of a thin lens is presumed to be united. Any lines as *D D*, *E E* directed to *O*, are *secondary axes;* they are presumed to pass, obliquely to the principal axis, through the lens, and without any deviation.

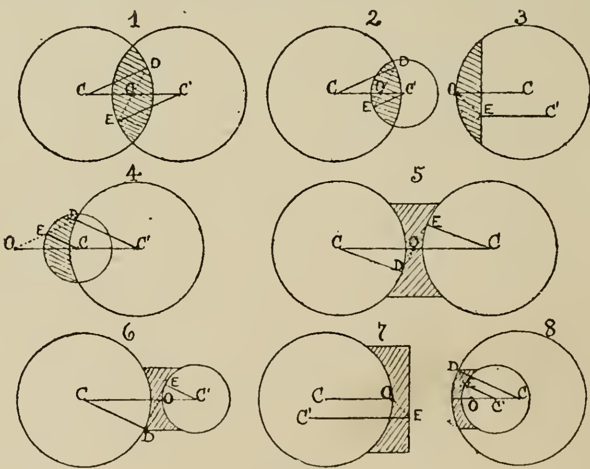

Fig. 89.

Spherical Formation of Lenses.—In each of the diagrams in Fig. 89, which shows the formation of lenses by the intersection or non-intersection of spheres and planes, the radius of curvature is a line drawn from the centre of each sphere to its corresponding surface of the lens. The optical centre in each case is O.

In the equi-Cx. and bi-Cx. (1 and 2), and the equi-Cc. and bi-Cc. (5 and 6) the centres C and C′ are on opposite sides of the lens.

In the plano-Cx. (3) and plano-Cc. (7) the centre of curvature of the plane surface is taken to be at ∞ and therefore on either side.

A Cx. lens consisting of a complete sphere has the centres of its opposite surfaces coincident.

In the periscopic Cx. (4) and periscopic Cc. (8) the centres are on the same side.

The Optical Centre.—In any lens there are innumerable pairs of points on the two surfaces as R and S (Fig. 90), such that tangents drawn to the surfaces at these points are parallel. A ray T R incident at one of these points R emerges from the other S, and its final direction S V is parallel to its initial

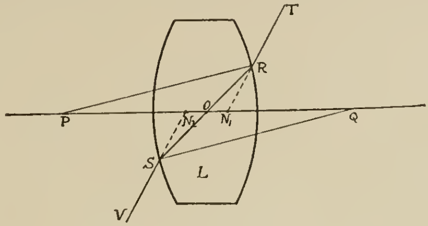

Fig. 90.

course T R, as though it had passed through a parallel plate. The ray is not deviated, but is merely laterally displaced by an amount depending upon the radii, thickness and μ of the lens. There are any number of pairs of points similar to R and S, and they are located by drawing from the centres of curvature any two mutually parallel radii such as P R and Q S. While in the lens the ray takes the direction R S, cutting the principal axis P Q in O, which is a fixed point. O is the *optical centre,* and any line R S passing through it is a *secondary axis.*

The point N_1 on the principal axis, towards which a secondary axial ray is directed, is the *first nodal point,* and N_2, from which it apparently emerges, is the *second nodal point.* From these points the principal and conjugate focal distances are measured, since, as will be shown later in the chapter on thick lenses, it is on planes drawn perpendicular to the axis through N_1 and N_2 that the refraction of the surfaces of the lens is presumed to be united. The nodal points are also referred to as *principal* or *equivalent* points, and the difference between these terms will be explained later. For the present lenses will be regarded as *thin,* the thickness being negligible in comparison with the focal length. All lenses employed in visual optics are considered to be *thin,* as distinct from those whose thickness cannot be disregarded without introducing appreciable error in calculating the power and focal length.

In a thin lens we may assume the interval between the nodal points to be so small that they fuse into the *optical centre.* Similarly the equivalent planes passing through the nodal points are also considered to unite into a single *refracting plane* passing through the optical centre, from which all distances and foci are measured. The refracting plane is the base-apex plane of the component prisms of a symmetrical lens.

Calculation of O.C.—The position of the optical centre depends on the two radii of curvature r_1 and r_2 and t the thickness of the lens, and is calculated from:

$$O = \frac{tr_1}{r_1 + r_2} \text{ from the one surface and } \frac{tr_2}{r_1 + r_2} \text{ from the other.}$$

Thus in a bi-Cx. lens where t = ·2 inch, and r_1 and r_2 are respectively 6 and 10 inches,

$$O = \frac{\cdot 2 \times 6}{6 + 10} = \cdot 075 \text{ in. from } r_1, \quad \text{or} \quad \frac{\cdot 2 \times 10}{6 + 10} = \cdot 125 \text{ in. from } r_2$$

O lies on the axis proportionately nearer to the surface of greater curvature. When both surfaces are Cx. or both Cc., O lies within the lens, but in a periscopic lens O lies outside it on the side of the surface of greater power. If one surface is plane O lies on the vertex of the curved surface. The distance, if positive, is measured from each surface *inwards* towards the other surface, but if *negative* it is measured *outwards.*

Let t = ·2 in., r_1 of the Cx. surface be 9 in., and r_2 of the Cc. − 12 in. Then $r_1 + r_2 = 9 - 12 = -3$ and $O = 1{\cdot}8/-3 = -{\cdot}6$ in.; that is, ·6″ beyond the Cx. surface.

Construction for O. C.—The method of finding the O. C. by construction is described in connection with Fig. 90; and that of any form of lens is shown in Fig. 89. From C, in any of the diagrams, draw a radius C D to the surface. From C′ draw a radius C′ E, to its corresponding surface, parallel to C D. Connect the extremities by the line D E, and where it cuts the principal axis at O, is the O. C.

In (3) and (7) C′ being at ∞, the only radius from C that can be parallel to C′ E, is the principal axis itself.

In (4) and (8) D E must be produced to cut the principal axis.

Characteristics of a Convex or Positive Lens.

(*a*) It is thicker at the centre than at the edge.

(*b*) It forms a magnified image of an object held within the focus.

(*c*) It forms on a screen an inverted real I of, say a distant flame or window.

(*d*) It causes the virtual image of an object, viewed through it, to move in the *contrary* direction as the lens is moved.

Characteristics of a Concave or Negative Lens.

(*a*) It is thinner at the centre than at the edge.

(*b*) It diminishes the apparent size of an object seen through it.

(*c*) No image can be projected by it on to a screen.

(*d*) When moved, an object seen through it appears to move in the *same* direction.

Properties of Lenses.—A Cx. lens has positive refracting power and can form a real focus and image; it renders parallel light convergent, and divergent light less divergent, parallel or convergent as the case may be.

A Cc. lens has negative refracting power and can only form a virtual or negative focus and image; it renders parallel light divergent, and divergent light more divergent.

The general effect of every spherical (and cylindrical) lens is, as with a prism, to bend incident light towards the thickest part. The above apply when the lens has a higher μ than the surrounding medium, otherwise the reverse occurs.

When discussing lenses we take them, unless otherwise stated, to be in air.

The Focus.—A real focus is that point at which rays from a point actually meet after refraction.

A virtual focus is that point from which rays from a point appear to diverge after refraction.

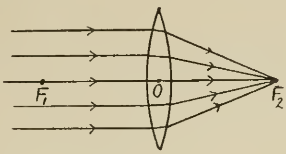

Fig. 91.

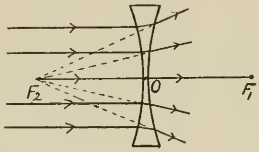

Fig. 92.

The *principal focus* F of a Cx. lens is positive and is on the principal axis on the opposite side of the lens from the source of light.

The distance between the O. C. and F is the principal focal distance of a thin convex lens (Fig. 91).

The *anterior focus* F_1 is on the same side as the light source and is that point from which light must diverge in order to be parallel after refraction.

The *posterior focus* F_2 is on the opposite side to the light source and is that point to which originally parallel rays are converged after refraction. The parallel rays in the figure are presumed to diverge from *a single point* on the principal axis at ∞.

In a lens $F_1 = F_2$ always, provided there is the same medium on both sides of it.

F being the focus of originally parallel rays, it is the *nearest* point to a Cx. lens at which a focus of natural rays can be obtained.

The *principal focus* F of a Cc. lens is negative, and is situated on the principal axis on the same side of the lens as the source of light. The distance between *O* and the focus, Fig. 92, is the principal focal distance, F being the point from which, after refraction, parallel rays appear to diverge. It is the *furthest* point from a concave lens at which a focus can be obtained for natural rays.

A *secondary* focus is one formed on a secondary axis.

A *conjugate* focus denotes one formed of appreciably divergent light, as distinct from a principal focus.

Lens Value.—*The value of a lens* is expressed by its *principal focal length* F, by its *focal power* 1/F, or by its *refractive power* D. These properties depend solely on curvature and μ, since in a thin lens, as stated, the thickness is ignored. F and 1/F, being reciprocals, vary inversely with each other, as the one is increased the other is proportionately diminished.

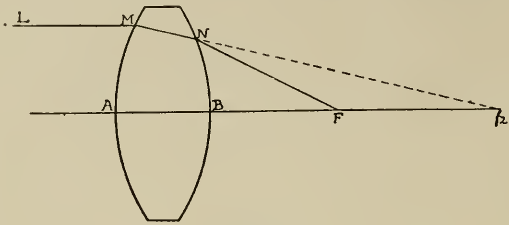

FIG. 93.

Formulæ for F.—Let AB (Fig. 93) be a bi-convex lens of radii r_1 and r_2 and index μ_2, that of the surrounding medium being μ_1. Then if any ray LM parallel to the principal axis be incident at M it will be refracted and tend to focus at f_2, the posterior focal distance of the first surface.

Thus $$f_2 = \frac{\mu_2 r_1}{\mu_2 - \mu_1}$$

Therefore f_2 is virtually an object with respect to the second surface. Since the thickness is disregarded we may take AF as equal to BF. For the second surface, f_1' and f_2' being the conjugates,

$$\frac{\mu_2}{f_1'} + \frac{\mu_1}{f_2'} = \frac{\mu_2 - \mu_1}{r_2}$$

But the image distance f_2 of the first surface is the virtual object distance f_1' of the second surface, so that substituting the former for the latter and using the negative sign, we get

$$\frac{\mu_1}{f_2'} - \frac{\mu_2}{\mu_2 r_1/(\mu_2 - \mu_1)} = \frac{\mu_2 - \mu_1}{r_2}$$

or $$\frac{\mu_1}{f_2'} = \frac{\mu_2 - \mu_1}{r_1} + \frac{\mu_2 - \mu_1}{r_2}$$

The final image distance f_2' is the principal focal distance F ;

$$\therefore \quad \frac{\mu_1}{F} = (\mu_2 - \mu_1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad \text{or} \quad F = \frac{\mu_1 r_1 r_2}{(r_1 + r_2)(\mu_2 - \mu_1)}$$

These are the general formulæ for a thin lens in any medium, but if the outer medium is air, which is usually the case, $\mu_1 = 1$, and can be omitted. Then taking μ as the index of the lens, the above simplify to

$$\frac{1}{F} = (\mu - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad \text{or} \quad F = \frac{r_1 r_2}{(r_1 + r_2)(\mu - 1)}$$

Since $1/r_1$ and $1/r_2$ represent the curvatures of the two surfaces, *the power of a lens is equal to the sum of its curvatures multiplied by the refractivity of the medium of which it is made.*

Convex surfaces of a lens are here always taken as positive and concave surfaces as negative, and the focus is of the same sign.

If one surface is positive and the other negative, the focus will be positive or negative as the one or other predominates in curvature.

Examples.—A Cx. lens of $\mu=1{\cdot}54$, having surfaces of radii of 8 in. and 5 in. F is positive, thus—

$$F=\frac{8\times5}{(8+5)\,(1{\cdot}54-1)}=\frac{40}{7{\cdot}02}=5{\cdot}7 \text{ in.}$$

If the surfaces are concave the negative sign must be prefixed to each; and F also is negative.

$$F=\frac{-8\times(-5)}{(-8-5)\times(1{\cdot}54-1)}=\frac{40}{-7{\cdot}02}=-5{\cdot}7 \text{ in.}$$

In a periscopic Cx. let the two surfaces be respectively -8 in. and $+4$ in. and $\mu=1{\cdot}6$. Here F is positive.

$$F=\frac{-8\times4}{(-8+4)\times{\cdot}6}=\frac{-32}{-2{\cdot}4}=+13{\cdot}3 \text{ in.}$$

In a periscopic Cc., with surfaces of $+8$ in. and -4 in. F is negative.

$$F=\frac{8\times(-4)}{(8-4)\times{\cdot}6}=\frac{-32}{2{\cdot}4}=-13{\cdot}3 \text{ in.}$$

Simplified Formulæ.—If both surfaces have the same radius, *i.e.* $r_1=r_2$, as in an equi-Cx. or equi-Cc. lens, the formula becomes simplified, for

$$F=\frac{r_1r_2}{(r_1+r_2)\,(\mu-1)}=\frac{r^2}{2r\,(\mu-1)}=\frac{r}{2\,(\mu-1)}$$

Thus if r_1 and $r_2=5$ and $\mu=1{\cdot}54$, $F=\dfrac{5}{{\cdot}54\times2}=\dfrac{5}{1{\cdot}08}=4{\cdot}63$ in.

If $\mu=1{\cdot}5$ in equi-Cx. or equi-Cc. lenses $F=r$.

If one surface is plane, then $r_1=\infty$ and $1/r_1=1/\infty=0$, so that it may be ignored and only the curved surface considered. Then

$$F=\frac{r}{\mu-1}$$

If $\mu=1{\cdot}5$, in a plano Cx. or Cc. lens $F=2\,r$.

To find r or μ.—To calculate r_1 or r_2 when that of the other, as well as μ and F, are known, the values of the known quantities are substituted for the symbol in the formula and the equation then worked out, as in the following examples :

If $r_2=8$ in. and $\mu=1{\cdot}5$ what radius should be given to the other surface so that F=6 in. ?

$$F=\frac{r_1 r_2}{(r_1+r_2)(\mu-1)} \qquad 6=\frac{8r_1}{(8+r_1)\times{\cdot}5}$$

then
$$6=\frac{8r_1}{4+{\cdot}5r_1}; \quad \text{or} \quad 24+3r_1=8r_1$$

$$5r_1=24 \quad \text{or} \quad r_1=+4{\cdot}8 \text{ in.}$$

What should be the radius of the Cc. surface of a meniscus when that of the Cx. is 5 in., F being 12 in. and $\mu=1{\cdot}6$?

Then
$$12=\frac{5r_1}{(5+r_1){\cdot}6}; \quad \text{or} \quad 5r_1=12\times(3+{\cdot}6r_1)$$

and
$$r_1=\frac{36}{-2{\cdot}2}=-16{\cdot}36 \text{ in.}$$

The same procedure is followed for finding μ.

If F=24 cm. and the radii are ×6 and −12 cm., then μ is found from

$$24=\frac{6\times-12}{(6-12)(\mu-1)}=\frac{-72}{-6\mu+6}$$

$$\therefore \quad -72=-144\mu+144 \quad \text{and} \quad \mu=1{\cdot}5$$

Distance of F.—In all cases $F_1=F_2$. Whether the one or other side of an equi Cx. or Cc. lens is exposed to the light, F is at the same distance from the back surface since *O* is midway between them; but this is not the case with

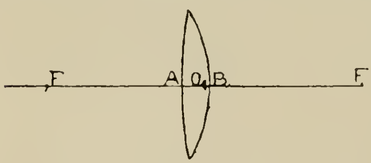

Fig. 94.

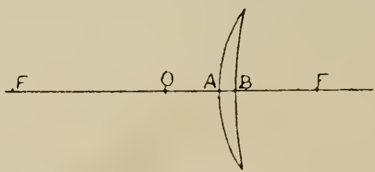

Fig. 95.

other forms of lenses. In Fig. 94 the principal focal distance *O F* of a bi-convex lens being measured from *O*, the distance of *F* behind the posterior surface depends on which surface faces the light. If *A* faces the light *F* lies further from *B* than it does from *A* when *B* faces the light. With the periscopic Cx., as shown in Fig. 95 (or the periscopic Cc.), the difference in the distance of *F* as measured to the right from *B* or to the left from *A* is marked; also, but to less extent, with the plano-Cx. and plano-Cc., *F* being measured from the curved surface, since *O* is situated thereon.

Relative Powers.—With similar radii, the power of a lens in air is proportional, not to μ, but to $(\mu-1)$, the refractivity of the medium. Thus if two lenses A and B be ground with the same radii on glasses of different μ's, the ratio of their powers is as $(\mu_A - 1) : (\mu_B - 1)$, their focal lengths being as $(\mu_B - 1) : (\mu_A - 1)$.

Calculations when μ_1 is not Air.—Let a double Cx. lens of $\mu = 1{\cdot}54$ and 8 cm. radius be in water

Then $$F = \frac{8 \times 8 \times 1{\cdot}33}{(8+8)(1{\cdot}54 - 1{\cdot}33)} = \frac{85{\cdot}12}{3{\cdot}36} = 25{\cdot}33 \text{ cm.}$$

Or the relative index μ_r can be found from μ_2/μ_1, and the formula for thin lenses in air employed. Here $\mu_r = 1{\cdot}54/1{\cdot}33 = 1{\cdot}158$,

Then $$F = \frac{8 \times 8}{(8+8) \times {\cdot}158} = 25{\cdot}33 \text{ cm. as above.}$$

Let a similar lens, but of $\mu = 1{\cdot}33$, be placed in cedar oil of $\mu = 1{\cdot}54$, then

$$F = \frac{8 \times 8 \times 1{\cdot}54}{(8+8)(1{\cdot}33 - 1{\cdot}54)} = \frac{98{\cdot}66}{-3{\cdot}36} = -29{\cdot}33 \text{ cm.}$$

The lens has a negative power, and a Cc. air lens in water has a positive power. Dr. Dudgeon constructed such a lens to enable divers, without helmets, to see under water. It consisted of two watch-glasses of deep curvature cemented into a vulcanite ring. The Cc. surfaces being outward produced two Cx. water surfaces in contact with them, and thus gave the required refractive power.

Let a Cc. air lens be of 10 inch radius on both surfaces be in water.

Then $$F = \frac{-10 \times -10 \times 1{\cdot}33}{(-10-10)(1-1{\cdot}33)} = \frac{100 \times 1{\cdot}33}{-20 \times -{\cdot}33} = \frac{133}{6{\cdot}66} = +20.$$

A Cx. water lens of the same radius in air has F=15 in. The difference, when the conditions are reversed, is similar to that between the anterior and posterior foci of a single surface. If light passes finally into a rare medium F is shorter than when it passes finally into a dense medium.

Dioptral Formulæ, r being in cm.

General formula $$D = \left(\frac{100}{r_1} + \frac{100}{r_2}\right)\left(\frac{\mu_2 - \mu_1}{\mu_1}\right)$$

Lens in air $$D = \frac{100(\mu - 1)(r_1 + r_2)}{r_1 r_2} \quad \text{or} \quad \left(\frac{100}{r_1} + \frac{100}{r_2}\right)(\mu - 1)$$

Equi Cx. or Cc. $$D = \frac{200(\mu - 1)}{r}$$

Plano Cx. or Cc. $$D = \frac{100(\mu - 1)}{r}$$

Course of Light—Cx. Lens.—A beam of rays, shown by the thick lines in Fig. 96, incident on a Cx. lens parallel to the principal axis F_1 F_2 is re-

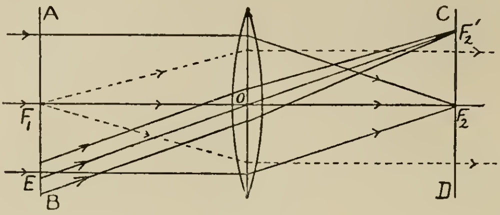

FIG. 96.

fracted to meet at F_2, the second principal focus, situated on that axis. C D drawn through F_2 perpendicular to the axis is the *second focal plane.* If the light diverges from F_1 (shown by the dotted lines), they are parallel after refraction. A B perpendicular to the axis through F_1 is the *first focal plane.*

If the object-point is situated on a secondary axis E F_2', the rays from it focus at F_2', on that same axis.

Construction of I for Cx. Lens.—There are three rays diverging from any point whose course, after refraction, it is easy to follow, viz.:

(*a*) The secondary axial ray which passes through O without deviation.

(*b*) The ray parallel to the principal axis, passing through F_2.

(*c*) The ray through F_1 refracted parallel to the principal axis.

Two only of these rays are needed, since where they meet all other rays diverging from that same point also meet.

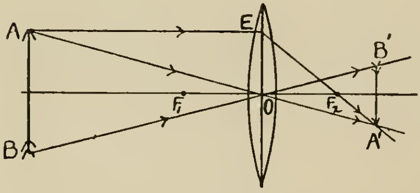

FIG. 97.

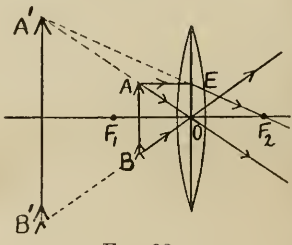

FIG. 98.

Real I.—Draw from A (Fig. 97) a ray A E parallel to the axis and, when refracted, through F_2. Draw the secondary axial ray A O straight through O. These meet at A', the image of A. In the same way B', the image of B, can be constructed. B' A' shows the position and size of the real inverted image of the object A B.

Virtual I.—From A (Fig. 98) draw A E parallel to the axis; E F_2 is the course after refraction. Draw A O passing through the optical centre. Since they are divergent after refraction, they can meet only by being produced backwards to A', the virtual image of A. Similar rays drawn from B

locate its image as B', and $A' B'$ is the complete virtual erect image of the object $A B$.

When the object is in the anterior focal plane, the rays from each point are, after refraction, parallel to each other and to a secondary axis, the image being, in theory, at infinity.

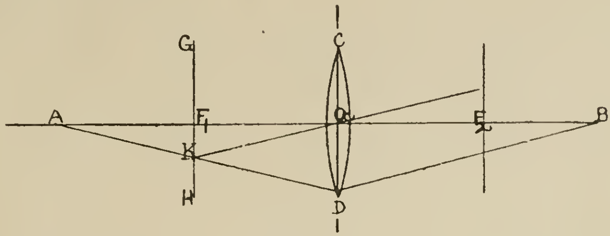

FIG. 99.

The Course of any Ray.—To construct the image of A, a point on the principal axis (Fig. 99), draw any line $A K D$, cutting the first focal plane at K and the refracting plane at D. From K draw a line through O and from D draw $D B$ parallel to $K O$ which cuts the principal axis at B, the image of A.

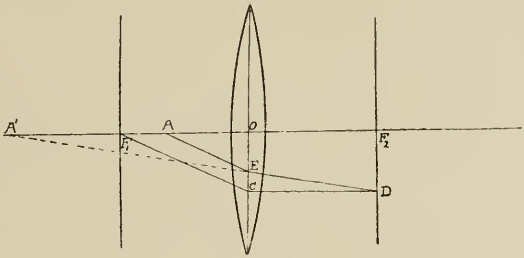

FIG. 100.

If the object-point A is nearer than F_1 (Fig. 100) from A draw any line $A E$, and from F_1 draw $F_1 C$ parallel to $A E$ which takes the direction $C D$ parallel to the principal axis and cuts the second focal plane in D. Connect D and E and produce to A' on the principal axis; then A' is the virtual image of A.

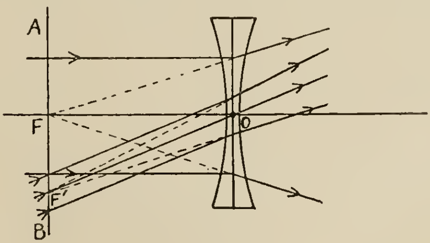

FIG. 101.

Course of Light—Cc. Lens.—If a beam of light parallel to the axis (Fig. 101), is incident on a Cc. lens, they apparently diverge, after refraction, from F, and $A\ B$ perpendicular to the axis through F is the focal plane. A point on a secondary axis as $F'\ O$ has its image on that same axis.

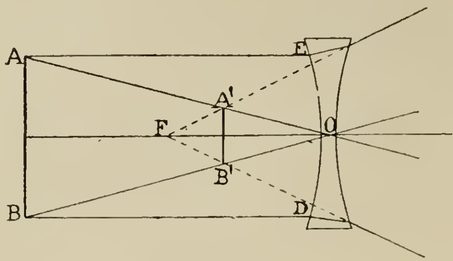

FIG. 102.

Construction of I for Cc. Lens.—The rays (*a*) and (*b*) given for the construction with a Cx. lens serve also for a Cc., but (*b*) diverges. From A (Fig. 102) trace $A\ O$, the secondary axial ray. Draw $A\ E$ parallel to the axis and refracted as if diverging from F. These rays appear to diverge, after refraction, from A' the image of A. Similar rays from B locate B', its image. The complete virtual erect image of AB is $A'B'$.

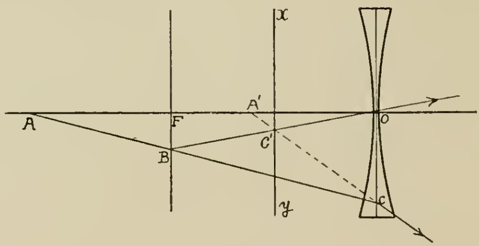

FIG. 103.

The Course of Any Ray.—To construct the image of a point A on the axis (Fig. 103), draw any ray $A\ B\ C$ cutting the focal plane in B and the refracting plane in C. Erect $x\ y$, a plane midway between the focal and refracting planes. From B draw $B\ C'\ O$, connect C with C', and prolong to A' on the principal axis; A' is the image of A.

General Construction for the Course of a Ray.—The principle is shown in Fig. 80, for a surface. For a lens or sphere, the course being determined for the first surface, a second construction is needed for the second.

Numeration of Lenses.—Lenses are numbered by the focal length, commonly termed the *inch* system and by the dioptric or power system.

For the *focal length system* the unit is a lens of one inch focus. Since F varies inversely with power 1/F, a lens which brings parallel light to a focus at 10 ins. or at 20 ins., has respectively 1/10 or 1/20 the power of the unit; while one whose F=1/2 in. has twice the power; thus focal length and power are reciprocals of each other. The abbreviations Cx. and Cc. are commonly employed in conjunction with the focal notation of lenses.

One disadvantage is that the inch, in various countries, differs in value, so that a lens of given F in one country may not be the same as one of similar number in another. Thus one of 18 French (Paris) inches is about equivalent to one of 20 English or American inches. Again, the intervals between the lenses, although regular as to their F's, are irregular as to their powers; thus there is a far greater difference between the powers of a 5 and a 6 inch, than between a 15 and a 16 inch lens. Further, ophthalmic and practical calculations involve the use of vulgar fractions.

For theoretical calculations the F system is superior, and is therefore mainly used in textbooks, although it is entirely superseded by the dioptric system in practical visual optics.

The *dioptric system* is based on the refractive power, and the unit is the diopter, which is that power causing parallel light to focus at 1 metre. The diopter of refraction is a measure of converging or diverging power, while the metre is a unit of linear measurement; yet it is often convenient to express distances in dioptric measure. The symbols + and − are always used with this system.

1/F is commonly termed *focal power* to distinguish it from the *dioptric power* D.

The dioptric system is in practice much more simple than the inch, and is universal. The unit being weak, the power of most other lenses is a whole number, while if fractions are involved they are expressed as decimals. The intervals between the lenses are uniform as regards their refracting powers.

If 1 D has F=1 M, a 4 D lens, having four times as much power, has F=1/4 M. But since the M can be sub-divided into 100 cm. (or 1000 mm.) the F of a 4 D is more conveniently expressed as 100/4=25 cm. A 10 D lens has ten times the power; therefore its F=100/10=10 cm., or 1/10 that of the unit. A 0·50 D has half the power; consequently its F=100/·5=200 cm., or twice that of the standard lens.

Conversion.—Since the + 1 D lens has F=1 M, or 40 inches, it is equal to No. 40 of the inch system, and a 40 D lens is the same as a 1 inch lens. The M=39·37 English inches, and for all practical purposes may be regarded as equivalent to either 40 or 39 inches. Therefore conversion from the one system to the other is effected by dividing 40 or 39 by the known number.

Thus 2·5 D=40/2·5=16 in., 13 D=39/13=3 in.,
2 in.=40/2 =20 D 13 in.=39/13=3 D.

Some numbers do not divide evenly into 40 or 39, but small remainders need not be considered beyond the 1/4, 1/2 and 3/4 in the lower inch numbers, and ·25, ·50, and ·75 in the dioptral numbers. For instance, 3·50 D=No. 11″; 3·25 D=No. 12″; 4·50 D=No. 9″, etc.

To Find F or D.—Dividing 40 or 100 or 1000 by the D number gives F in inches, in cm., or in mm. respectively.

Thus, 5 D lens has F=40/5=8 in., 100/5=20 cm., or 1000/5=200 mm.

If F is known in cm., mm., or inches, the dioptral number is found by dividing respectively into 100 or 1000 or 40.

Thus, if F=200 mm., D=1000/200=5.

If F=20 cm., D=100/40=2·5; if F=160 in., D=40/160=·25.

Obsolete Systems.—Originally lenses were numbered according to the radius of curvature. No. 10 meant a DCx. or DCc. of 10″ radius. Cc. sphericals were formerly numbered by an arbitrary system commencing at 0000—the weakest—and terminating with No. 20—the strongest.

Addition of Lenses.—The combined strength 1/F of the two thin lenses in contact, whose values are indicated by their focal lengths F_1 and F_2 respectively, is obtained by the addition of their focal powers, thus

$$1/F = 1/F_1 + 1/F_2$$

If the two lenses be, say, 24″ Cx. and 10″ Cx.

$$1/F = 1/24 + 1/10 = 34/240 = 1/7 \text{ approx.}$$

The two are equivalent to a lens of 7″ F.

Here convergence has been added to convergence. If the two lenses are Cc., divergence is added to divergence. Thus, if they be 5″ and 6″ Cc.,

$$1/F = -1/5 + (-1/8) = -13/40 = -1/3 \text{ approx.}$$

When the one lens is Cx., and the other Cc., 1/F is positive or negative according as F_1 or F_2 is the shorter. The convergence of the one and the divergence of the other neutralise each other more or less, and the residual power of the stronger is that of the combination. Thus, with a 15 Cx. and a 12 Cc.

$$1/F = 1/15 + (-1/12) = 12/180 - 15/180 = -3/180 = -1/60$$

A 20 Cc. and a 10 Cx. $= 1/10 + (-1/20) = +1/20$, *i.e.* a 20 Cx.

The addition of several lenses is achieved in a similar manner; thus

10 Cx., 16 Cx., 7 Cx., and 5 Cc. make $1/10 + 1/16 + 1/7 - 1/5 = 59/560$, that is, $9\frac{1}{2}$ Cx. approx.

The combined strength D of two dioptral lenses D_1 and D_2 in contact is

$$D = D_1 + D_2$$

Thus, +2 D and +4 D = +6 D; −5·25 D and −2·50 D = −7·75 D; +4 D and −3 D = +1 D; +3 D and −3 D = 0, they neutralise.

$$+7 \text{ D} + 4{\cdot}50 \text{ D} + 1{\cdot}75 \text{ D} - 6{\cdot}50 \text{ D} = +6{\cdot}75 \text{ D}$$

Conjugate Foci.—If f_1 be the distance from the optical centre from which light from the object diverges, then $1/f_1$ represents that divergence; if f_2 is the distance of the image then $1/f_2$ is the ultimate convergence or divergence of the light which produces the image. 1/F is positive or negative according as it pertains to a converging or diverging lens respectively, while $1/f_1$ is always negative. $1/f_2$ *is found by adding algebraically the divergence of the light* $1/f_1$ *to the converging or diverging power of the lens,* that is,

$$\frac{1}{f_2}=\frac{1}{F}+\left(-\frac{1}{f_1}\right) \text{ or } \frac{1}{f_2}=\frac{1}{F}-\frac{1}{f_1}, \text{ whence } \frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

The power of the lens is equal to the sum of the reciprocals of any pair of conjugate foci, or to the sum of its actions on the light.

With a Cx. lens f_2 is positive or negative according as the convergence of the lens $1/F$ is greater or less than the divergence of the light $1/f_1$. With a Cc. lens f_2 is always negative, since the divergence of the light is added to the divergence of the lens. It should be observed that calculations for conjugates are the same as the addition of lenses.

By inverting the formula we get a sometimes useful variation in

$$F=\frac{f_1 f_2}{f_1+f_2} \quad \text{or} \quad f_2=\frac{f_1 F}{f_1-F} \quad \text{and} \quad f_1=\frac{f_2 F}{f_2-F}$$

It can also be written, $F/f_1+F/f_2=1$

Geometrical Proof.—In Figs. 104 and 105, AB is the object and $B'\,A'$ is the image. The triangles $A\,O\,B$ and $A'\,O\,B'$ are similar, as are also the triangles $D\,F\,O$ and $A'\,F\,B'$, also $A\,B=D\,O$.

Then $\frac{AB}{A'\,B'}=\frac{O\,B}{O\,B'}$ and $\frac{D\,O}{A'\,B'}=\frac{O\,F}{B'\,F}$ or $\frac{O\,B}{O\,B'}=\frac{O\,F}{B'\,F}$

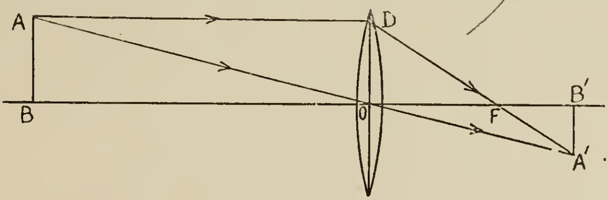

Fig. 104.

For the Cx. $O\,B=f_1$; $O\,B'=f_2$; $O\,F=F$; $B'F=f_2-F$.

Therefore $f_1/f_2=F/(f_2-F)$ or $f_1 f_2-F/f_1=Ff_2$.

Then $f_1 f_2=F(f_1+f_2)$ or $1/F=1/f_1+1/f_2$.

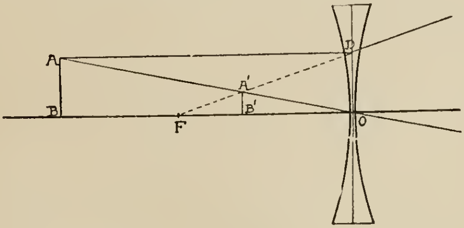

Fig. 105.

For the Cc. $O\,B=f_1$; $O\,B'=f_2$; $O\,F=F$; $B'F=F-f_2$.
Therefore $f_1/f_2=F/(F-f_2)$ or $Ff_1-f_1f_2=Ff_2$.
Then $f_1f_2=F(f_1-f_2)$ or $-1/F=1/f_1-1/f_2$.

Conjugate Foci—Examples—Cx. Lens.—The converging power of the lens is decreased, neutralised, or exceeded by the divergence of the light from the object. The image is real so long as the object is beyond F, and it is virtual when the object is within F. Approach of the object causes the light to be less convergent after refraction, so that any real conjugate focus is more distant than F, which is *the nearest point* at which a real image can be formed from natural rays.

If F=8″ and f_1 is at 40″, then f_2 will be at 10″, for $1/f_2=1/8-1/40=1/10$. This is proved by $1/10+1/40=1/8$.

If a real image is 16″ behind a 7″ Cx., the object is at $12\frac{4}{9}$″.

$$1/f_1=1/7-1/16=9/112.\quad f_1=112/9=12\tfrac{4}{9}.$$

When the object is at ∞, the image is at F since

$$1/f_2=1/\mathrm{F}-1/\infty=1/\mathrm{F}-0=1/\mathrm{F},\ i.e.\ f_2=\mathrm{F}$$

When the object is at F, the image is at ∞ since

$$1/f_2=1/\mathrm{F}-1/\mathrm{F}=0,\ i.e.\ f_2=\infty.$$

Therefore F and ∞ are conjugate focal distances.

The shortest possible distance between the O and its real I is 4 F.

When the object is nearer than F, the light is divergent after refraction, although less so than before. Whereas the light diverged originally from f_1 it appears after refraction to diverge from f_2. The virtual or negative conjugate of f_1 lies on the same side of the lens. Thus let the object be 6 in. from an 8 in. Cx. lens, then

$$1/f_2=1/8-1/6=-1/24.\quad f_2=-24''.$$

If the incident light is *convergent*, the image is nearer than F. Thus if light converges to 24″ behind a 8″ Cx. lens we have the image at 6″.

$$1/f_2=1/8+1/24=1/6.\quad f_2=6 \text{ inches}.$$

Conjugate Foci—Examples—Cc. Lens.—The diverging power of the lens is increased by the divergence of the light; the image is always virtual. When the object is within ∞ the conjugate focus is nearer to the lens than F which is the *most distant* image-point for natural rays.

If F=10″ and f_1 is at 40″, then f_2 is 8″ virtual, for $1/f_2=-1/10-1/40=-1/8$.

When the object is at ∞ the image is at F; when the object is at F the image is at F/2.

If the incident light is convergent, the image is beyond F; thus, if light is convergent towards 15″ behind a 6″ Cc. lens, then

$$1/f_2=-1/6+1/15=-9/90;\ \text{the image is virtual at } 10''.$$

If the light converges to F the light is rendered parallel. If convergent to a point nearer than F, a real image is formed.

The Addition of Conjugates.—

If the conjugates are 5″ and 10″	$1/F=1/5+1/10=3/10$
,, ,, ,, 5″ and – 10″	$1/F=1/5-1/10=1/10$
,, ,, ,, – 5″ and 10″	$1/F=-1/5+1/10=-1/10$

Reciprocity of Conjugates.—Real conjugate foci are interchangeable so that if O is at either of them, I is at the other. Thus when O is at 40″ in front of an 8″ Cx. lens, I is at 10″, and if I were at 10″, O would be at 40″. Virtual conjugates are not interchangeable in this sense. If O is at 6″ from an 8″ Cx. lens, I is at 24″ virtual. O could not be at – 24″, which is a negative distance, and if it were at 24″ actually, I would not be at 6″. These conjugates are interchangeable merely in the sense that if light *converges towards* the virtual focus, in this case 24″, then I would be at the real distance 6″.

The same occurs with the virtual conjugate of a Cc. lens. If O is at 40″ and the lens is – 1/10 the I is at 8″ virtual; light would need to converge towards 8″ behind the lens in order that a real image be formed at 40″.

Dioptral Formulæ for Conjugates.—A say +5 D lens has F=20 cm., and light diverging from 20 cm. is rendered parallel by it, the converging power of the lens just neutralising the divergence of the light. Conversely light from ∞ is brought to a focus at 20 cm.

If the light originates in some point within ∞ it has then a divergence equal to that of a Cc. lens whose F is equal to the distance; the resulting image d_2 is the dioptral result of the addition of the divergence of the light d_1 and the power of the lens D. That is,

$$D-d_1=d_2 \qquad \text{or} \qquad D=d_1+d_2$$

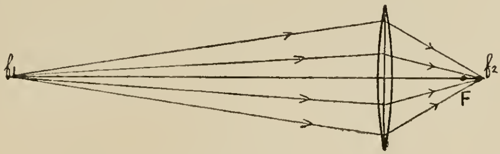

Fig. 106.

Or D, the power of a lens, is equal to the sum of the two conjugates f_1 and f_2 expressed in diopters as d_1 and d_2.

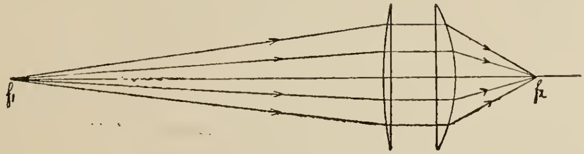

Fig. 107.

Let f_1 be 100 cm. (Fig. 106). The lens has a converging power of 5 D, and the light has a divergence of 1 D. Consequently, after refraction, the light has a convergence of $5-1=4$ D, the I being at 25 cm.

In Fig. 107 the +5 D is shown as if split into two lenses, the +1 D rendering parallel the light diverging from 100 cm., while the +4 D brings the parallel rays to a focus at 25 cm.

The two dioptral distances 100 cm. and 25 cm. are +1 and +4 respectively.

We may therefore write 1+4=5 D=the power of the lens. 5 - 1=4 D= the dioptral distance of I. 5 - 4=1 D=the dioptral distance of O.

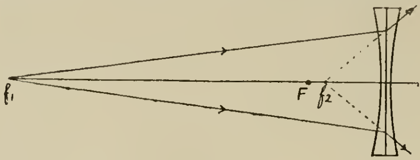

Fig. 108.

If the lens is Cc. we have D negative, and therefore d_2 is also negative, since d_2 is the sum of the divergences D and d_1 but, as with a Cx. lens, the sum of $f_1 + f_2$ expressed in diopters as $d_1 + d_2$=D (Fig. 108).

Examples.—Suppose the object be placed 50 cm. in front of a lens having its image 12·5 cm. behind it, then to find the power of the lens

$$d_1=100/50=2;\ d_2=100/12{\cdot}5=8;\ D=2+8=10.$$

If an object is 200 cm. in front of a +7 D lens, the image is

$$d_1=100/200={\cdot}5;\ d_2=7-{\cdot}5=6{\cdot}5;\ f_2=100/6{\cdot}5=15 \text{ cm.}$$

An image is 22 cm. behind an 8 D lens, where is the object ?

$$d_2=100/22=4{\cdot}5;\ d_1=8-4{\cdot}5=3{\cdot}5;\ f_1=100/3{\cdot}5=28 \text{ cm.}$$

If the object is at ∞, then $d_1=100/\infty=0$; the image is at F.

$$d_2=D-0=D \quad \text{and} \quad 100/D=F.$$

If O is at F, then d_1=100/F=D; the image is ∞.

$$d_2=D-D=0 \quad \text{and} \quad 100/0=\infty.$$

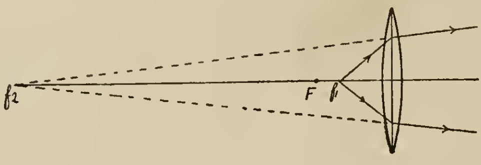

Fig. 109.

Let the lens be +5 D and O be at 14 cm. (Fig. 109) then

$$d_1=100/14=7;\ d_2=5-7=-2, \text{ so that } f_2=50 \text{ cm. virtual.}$$

The lens has a converging power of 5 D, the light has a divergence of 7 D; therefore, after refraction, there is a residual divergence of 2 D.

$$d_1+d_2=D, \text{ that is, } 7+(-2)=+5 \text{ D.}$$

If light converges towards 50 cm. behind a +5 D lens we have

$$d_2 = 5 + 2 = +7 \text{ D}, \; f_2 \text{ is at } 14 \text{ cm.}$$

Let the lens be −5 D and f_1 at 100 cm.; then

$$d_2 = -5 \text{ D} - 1 \text{ D} = -6 \text{ D, and } f_2 = 100/-6 = -16{\cdot}66 \text{ cm.}$$

Light diverging from 100 cm. to a −5 D lens, after refraction is divergent as if from 16·66 cm. If convergent towards a point 16·66 cm. behind a −5 D lens it is, after refraction, convergent to 100 cm.

If the conjugates are			20 and 50 cm.	D=5+2=+7
,,	,,	,,	20 and −50 cm.	D=5−2=+3
,,	,,	,,	−20 and 50 cm.	D=−5+2=−3

As before stated a calculation on conjugate foci is the same as adding two lenses together. This is illustrated in the last examples.

Whether light diverges from 50 cm., or whether parallel light is rendered divergent by an added −2 D lens, the converging effect of, say, a + 5 D lens is equally reduced, and in both cases f_2 is at +5−2=3 D=33 cm. behind the lens.

Similarly whether light diverges from 50 cm. (2 D) to a −5 D lens, or whether a −2 D be added to the −5 D, and the two combined act on parallel light, f_2 in either case is at −5−2=−7 D or 14 cm. negative.

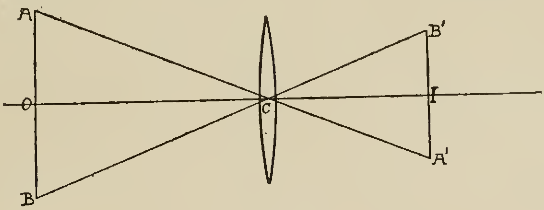

Fig. 110.

Magnification or Relative Sizes of O and I.—In Fig. 110, the object O and the image I *subtend equal angles at* C, the optical centre of the lens, since both are contained between the extreme secondary axes $A\,A'$ and $B\,B'$. The triangles $A\,C\,B$ and $A'\,C\,B'$ are similar.

Therefore

$$M = \frac{I}{O} = \frac{B'\,A'}{A\,B} = \frac{I\,C}{O\,C}$$

The *relative sizes of I and O are proportional to their respective distances from the optical centre of the lens.* This is true for real and virtual images of both Cx. and Cc. lenses.

The ratio $B'\,A'/A\,B$ is the *magnification*, and denotes the linear *increase* or *decrease* in the size of the image with respect to the object. Superficial magnification applies to area, and is the linear magnification squared.

With a Cx. lens, so long as O is beyond 2 F the I must be smaller than O, since it is nearer to the lens. When O is at 2 F the size of I is the same as that of O, because both are at the same distance. When O is within 2 F, I is larger, because it is further from the lens than O.

To calculate the size of I or of O the following formulæ are applicable to all cases.

$$M=\frac{h_2}{h_1}=\frac{f_2}{f_1}, \qquad \text{that is,} \qquad h_2=\frac{h_1 f_2}{f_1} \qquad \text{and} \qquad h_1=\frac{h_2 f_1}{f_2}$$

where f_1 and f_2 are the distances of O and I respectively from the lens, h_1 is the linear size of O, and h_2 that of I. h_1 and f_1 must be in similar terms, but not necessarily that of f_2; h_2 will then be in the same terms as f_2, whether inches, cm., etc. Or h_2 and f_2 must be in the same terms; and h_1 will be in that of f_1.

Thus if O is at 2 M, and I at 25 cm. and ·625 cm. high; then

$$h_1=\frac{\cdot 625\times 200}{25}=5 \text{ cm.}$$

O is eight times the size of I. If O were at 25 cm. and I at 2 M, then I would be eight times the size of O.

Let O, 4 yards long, be $\frac{1}{4}$ mile distant from a +5 D lens; then the object being at ∞, $f_2=20$ cm. and

$$h_2=4\times 20/440=\cdot 18 \text{ cm.}$$

The answer here is in cm., showing that, so long as h_1 and f_1 are in the same terms, O and I need not be.

When the I formed by a Cx. lens is virtual, it is always larger than O, since it is always more distant from the lens. With a Cc. lens the virtual I formed is always smaller than O, since it is always nearer to the lens.

Planes of Unit Magnification.—In order that O and I be equal in size they must be equally distant from the lens, *i.e.* they must be in the planes of unit magnification which, for real images, are the *symmetrical planes*, which cut the axis at twice the principal focal distance; then $h_1=h_2$.

If M=1, then $f_1=f_2$, and we can write—

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_1}=\frac{2}{f_1} \qquad \text{or} \qquad \frac{1}{F}=\frac{1}{f_2}+\frac{1}{f_2}=\frac{2}{f_2}$$

so that $$f_1=2F \quad \text{and} \quad f_2=2F.$$

For a virtual I to be equal in size to O, that is, M= – 1, it must be in contact with the lens. This is true for both Cx. and Cc. lenses, so that the planes of unit magnification for virtual images is zero. Both planes of unit magnification are from F a distance equal to F.

Recapitulation of Conjugates—Cx. Lens.

O at ∞	I real, inverted, diminished, at F.
O between ∞ and 2 F	I real, inverted, diminished, between F and 2 F.
O at 2 F	I real, inverted, equal to O, at 2 F.
O between 2 F and F	I real, inverted, enlarged, between 2 F and ∞.
O at F	I infinitely enlarged, at ∞.
O within F	I virtual, erect, enlarged, same side as O.
O at the lens	I virtual, erect, equal to O, at the lens.

Cc. Lens.

O at ∞	I virtual, erect, diminished, at F.
O within ∞	I virtual, erect, diminished, within F.
O at the lens	I virtual, erect, equal to O, at the lens.

Relationship of Conjugate Distances.—If the distance of the two conjugates f_1 and f_2 of a Cx. lens be measured respectively from F_1 and F_2 they are reciprocals of each other in terms of F. If f_1 is at a distance n F beyond F_1, then f_2 is F/n beyond F_2. Thus, for instance, if the distance of O to F_1 is 2 F, then the distance of I to F_2 is F/2.

Let the distance of O to F_1 be called A, and f_2 to F_2 be called B; then $A\,B=F^2$.

For M (magnification)=1, both conjugates are at F+F. For M=2 the one is at F+2 F, the other being at F+F/2. For M=3 the one is at F+3 F, the other being at F+F/3, and so on.

If the object is n F from the lens, the image, with a Cx. lens, is $n\,F/(n-1)$, and with a Cc. the latter is at $n\,F/(n+1)$. Thus if the distance from a 5″ Cx. lens is 5 × 4=20″, the image is at 5 × 4/3=6·66″; in the case of a 5″ Cc. if the object is at 5 × 4=20″, the image is at 5 × 4/5=4″. Then $n\,F \times F/n=F^2$.

The size of the object is to the real and the virtual image, formed by a given Cx. lens, the same when O is as far beyond F in the first case as it is within F in the second case. Thus, suppose O situated 14 in. and 6 in. respectively in front of a 10 in. Cx. lens, it is in either position 4 in. from F, then the size of the image in each case is 2½ times that of O.

Newton's Formula for Conjugate Foci.—Let the distances A and B be as defined in the last article. Now the ordinary formula for conjugate foci is

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{F+A}+\frac{1}{F+B} \quad \text{then} \quad A\,B=F^2$$

This gives an alternative formula for calculating conjugate foci.

The relative sizes of I and $O=M=\frac{h_2}{h_1}=\frac{F}{A}=\frac{B}{F}$

It is essential to remember that positive quantities are measured forwards from F_1 and backwards from F_2; also that in Cc. lenses F_1 is on the remote side of the lens, and F_2 on the object side. A is always reckoned from F_1 and B from F_2. These points make this otherwise valuable formula difficult of application. To obtain f_1 or f_2 the value of F must be added to A or B respectively.

Examples.—Let f_1 be 50 cm. in front of a Cx. lens of 10 cm. F.
$A=40$; $40\,B=10^2=100$; $B=2{\cdot}5$; and $f_2=2{\cdot}5+10=12{\cdot}5$ cm.
If O is 5 cm. high, $h_2/5=10/40$, so that $40\,h_2=50$, or $h_2=1{\cdot}25$ cm.
If an object 5 cm. high be placed 8 cm. in front of a lens of 10 cm. F,
$A=-2$; $-2\,B=10^2=100$; $B=-50$, and $f_2=-50+10=-40$ cm.
$h_2/5=10/2$, or $h_2=25$ cm. I is negative at 40 cm. and 25 cm. high.
If an object 5 cm. high be 50 cm. in front of a Cc. lens of F=10 cm.,
$A=60$; $B=10^2=100$. $B=1{\cdot}66$ and $f_2=1{\cdot}66+(-10)=-8{\cdot}33$ cm.
$h_2/5=10/60$, or $h_2={\cdot}833$ cm. I is negative at 8·33 cm. and ·833 cm. high.

The Geometrical Proof is the same as shown in Fig. 104.
But $OB=F+A$, $OB'=F+B$, $OF=F$, and $B'F=B$.
Therefore, $(F+A)/(F+B)=F/B$ or $AB=F^2$.

Removal of I.—To move the image from f_2 to some other position f_2' more distant or nearer, there must be added to the lens another Cc. or Cx. respectively whose power is the difference between $1/f_2'$ and $1/f_2$.

Thus, supposing f_2 to be 20 cm. and f_2' to be 25 cm., the required lens $4-5=-1$ D. It is Cc. because f_2' is more distant than f_2.

To place the image at 16 in. behind the lens instead of at f_2, which is 20 in., the added lens must be positive of $1/16-1/20$ or 80 inches F.

Position of the Conjugates for given M.—To find f_1 the position of O for a given magnification M, the ordinary conjugate formula can be used. Since h_1 and h_2 are proportional to f_1 and f_2, we can express f_2 in terms of f_1. Thus, if the I is to be 3 times the size of O, $f_2=3f_1$; if it is to be 1/3 the size of O, $f_2=f_1/3$. Then we have

$$1/F=1/f_1+1/Mf_1$$

For example, the lens is a 6 in. Cx., a photograph is 2 inches long, and it has to be enlarged 4 times. Then

$$1/6=1/f_1+1/4f_1=5/4f_1 \text{ or } 4f_1=30 \text{ and } f_1=7{\cdot}5'' \text{ and } f_2=30''$$

If a reduction is required, M is a fraction.
If I is virtual with a Cx. or Cc. lens, $1/M f_1$ needs the $-$ sign.
Similarly f_2 can be found by expressing it in terms of f_1.

From the above the following formulæ are extracted for finding the position of O and I when the one has to be magnified a certain number of times.

$$a=F(M+1) \qquad \text{and} \qquad b=a/M,$$

a is the longer conjugate and b is the shorter. $(M+1)$ becomes $(M-1)$ when I is virtual, with either a Cx. or a Cc.

When f_1 or f_2 is not known M can be found respectively from

$$M=(f_2-F)/F \qquad \text{and} \qquad M=F/(f_1-F)$$

M is positive for a real, but negative for a virtual, image, and is a fraction when there is diminution.

The Position of Lens for given Distance between O and I.—The calculation necessitates finding two conjugates such that the sum of their reciprocals equals the power of the lens. Let d be the distance between O and I, and x be the one conjugate; then

$$\frac{1}{F}=\frac{1}{x}+\frac{1}{d-x}. \qquad \text{or} \qquad x^2-dx=-dF$$

the solution of which involves a quadratic equation.

With a Cx. lens when d is not less than 4 F, the I is real and may be at either conjugate, and there are two positions for the convex lens, between O and I, which will fulfil the conditions. When d is less than 4 F, the shorter conjugate is positive and is the distance of the O; the greater is negative and is that of the virtual I, d then being a negative quantity.

When the lens is concave, d is positive but F is negative. The greater conjugate is positive and is the distance of the O, while the smaller is negative and is that of the virtual I.

Let F=7 in. and the distance between O and I be 36 in. Then

$$x^2-36\,x=-252 \qquad \text{and} \qquad x^2-36x+324=-252+324=72$$

Therefore $\sqrt{x^2-36x+324}=\sqrt{72}$ or $x-18=\pm 8{\cdot}5$

so that $x=+8{\cdot}5+18=26{\cdot}5$ or $-8{\cdot}5+18=9{\cdot}5$

The lens may be either 9·5 in. or 26·5 from O.

Let F=5 in. and d=16 in.; d is negative. Then

$$x^2+16x=+80 \qquad \text{and} \qquad x^2+16x+64=80+64=144$$

so that $x+8=\pm 12$

and $x=+12-8=+4$ or $-12-8=-20$

The lens is 4 in. from the O and 20 in. from the virtual I.

Let F be 5 in. Cc. and d, as before, 16 in. Then

$$x^2-16x=80 \qquad \text{and} \qquad x^2-16x+64=80+64=144$$

$$x-8=\pm 12 \qquad \text{and} \qquad x=+12+8=+20, \text{ or } -12+8=-4.$$

The lens is 20 in. from O and 4 in. from the virtual I.

If the strength of the lens is expressed in diopters it is better to convert it into focal length for this calculation, but the two distances x and y can be calculated by the method in which two numbers, whose sum and multiple are known, have to be found. d is in cm. Thus from above

$$x+y=d, \qquad \text{and} \qquad x\,y=Fd=100d/D \ ]$$

CHAPTER VIII

CYLINDRICAL LENSES

The Cylinder.—A cylinder is a body (Fig. 111) generated by the revolution of a rectangle about one of its sides as an axis. Such a body consists of two flat circular ends and an intermediate convex surface.

The cylinder possesses no curvature in any plane parallel to the axis $A B$. At right angles to the axis, in any plane parallel to the direction $C\ D$, the curvature is spherical, and has its maximum value. In any other direction, as $E\ F$, the curvature is that of an ellipse.

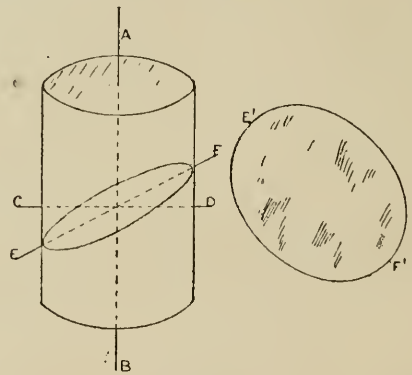

Fig. 111.

Any section of the cylinder at right angles to its axis is a circle whose centre lies on the axis of the cylinder; a section in the plane of the axis is a parallelogram; one anywhere between these two is an ellipse, as $E'\ F'$.

Fig. 112 represents a Cx. cylindrical lens. It is a segment of a cylinder as to the one surface and is plane on the other; it is formed by a cylinder and a plane which intersect each other. The Cc. cylindrical lens (Fig. 113) has a hollowed surface on one side; it is formed by a cylinder and a plane which do not intersect each other.

The Cx. cyl. lens may be conceived as formed of a series of prisms whose bases are directed towards the axis and whose apices are outwards, and the Cc. cyl. as formed of prisms whose apices are towards the axis and whose bases are outwards; in both cases the power of the prisms increases towards the edge of the lens.

Meridian.—The term meridian in connection with lenses signifies a plane passing through the geometrical centre of a lens, as shown in Fig. 114.

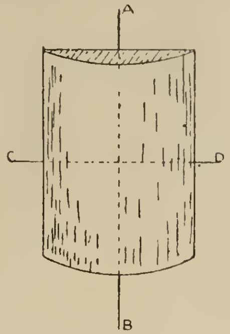

Fig. 112.

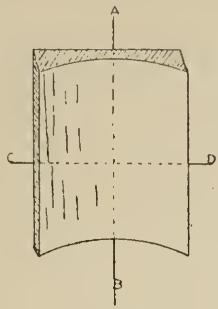

Fig. 113.

Nomenclature.—A lens with a cyl. curvature only is a *plano* or *simple cyl.* One having, at right angles to each other, a cyl. curvature on both surfaces is a *cross-cyl.* One having a sph. curvature on the one surface and a cyl. on the other is a *sphero-cyl.*

The Principal Meridians.—Since in the direction of its axis (Fig. 115) a cyl. lens has no curvature, it has in that direction no refractive power; the directions of maximum curvature *A B C*, *D E F*, *G H K*, are at right angles

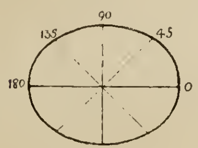

Fig. 114.

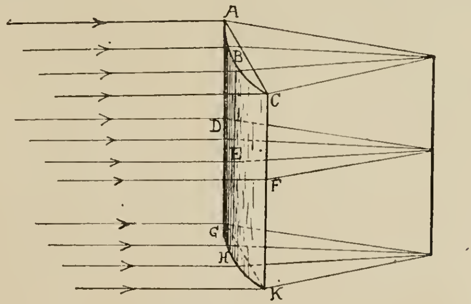

Fig. 115.

to the axis. The meridian of no refraction—*i.e.* the axis—and the meridian of power, at right angles to the axis, are the *two principal meridians*, and these alone need be considered when treating of cyl. lenses.

The other meridians are merely individual elements contributing to the total power of the lens, but they have a dioptric value, as will be shown in a later chapter.

The position of a cylindrical is indicated by the direction of its axis. Its power is expressed, generally in diopters, by the maximum refractivity, the numeration being the same as for spherical lenses.

The Refraction of a Cylindrical.—A sph. lens has equal curvature and therefore similar refractivity in every meridian, so that a point image of a point object is obtained. In a cyl. it is only the meridian at right angles to the axis that can form a focus, so that all the light from an object-point at ∞ refracted by a cyl. meets in a line at the focal distance of the meridian of greatest refraction. This is called the *focal line*, and it is at right angles to the meridian of power and therefore parallel to the axis. Theoretically the cylindrical lens has two focal distances and the image of a point is two lines, but since the one focus is at ∞, it need not be considered.

Using for illustration a +5 D cyl. axis vertical, as shown in Fig. 115, the meridian of power is horizontal and the focal line is vertical and at 20 cm. At any other distance the streak broadens out into a band of light, and a section of the emergent light, at any distance from the lens, is rectangular in outline. If the lens be rotated the line is rotated with it.

Since the image of an object consists of the images of its various points, and each object-point has its own line image, the complete image consists of an infinite number of streaks, parallel to the axis, and narrow as the focal length is short, and *vice versa.* The length of a streak is that of the aperture of the lens in the meridian of its axis.

The refraction of a Cc. cylindrical is similar to that of a Cx. but, of course, the focus and the focal line are virtual, and formed in front of the lens. The shape of the refracted pencil, however, is not so apparent, because the pupil of the eye acts as a very small stop, so that the focal line, on looking through the lens, is so short as to appear little different from the point focus of a concave spherical, unless the cyl. be very strong.

A square seen through a cyl. with vertical axis appears to be a rectangle of natural size in the meridian of the axis, but magnified by a Cx., and diminished by a Cc., across the axis.

If the cyl. is rotated, around its centre, the square takes the form of a rhombus, the obliquity of the sides being due to the fact that the light from each point diverges from, or tends towards, a line parallel to the axis and therefore appears to come from points in space other than the real ones. The series of oblique parallel lines (or ellipses) which constitute the virtual object, of which the retinal image is formed, results in vertical and horizontal lines appearing oblique. This explains also the *dipping* of cross lines as a cyl. is rotated (*vide* Neutralisation). The apparent obliquity is lessened if the lens is near to the object or very near to the eye.

Viewing a circular object, say a shilling, through a Cx. cyl. axis Ver., the image is an oblate ellipse in form, having its minor axis equal to the diameter of the shilling. With a similar Cc. cyl. the image is a prolate ellipse.

The Refraction of a Sphero-Cylindrical.—When a sph. is combined with a cyl., the curvature of the former is ground on the one side of the lens, and that of the latter on the other. Since there is no curvature and consequently no refractive power along the axis of the cyl., there is, in that meridian, only the power of the sph., whereas at right angles to the axis there is the united power of the sph. and the cyl. As with the plano-cyl., these are the two principal Mers. of the combination, which alone need be considered in practice. Further, the lens has two finite focal distances.

The sign ⊂⊃ indicates *combined with.*

Let the lens be +4 D Sph. ⊂⊃ +4 D Cyl. Axis Ver. and let the object be a point at ∞. All the light incident on the lens is so refracted as to pass through a vertical line at 12·5 cm. to which it is converged; thence, expanding horizontally and converging vertically, it meets in a horizontal line at 25 cm. The vertical meridian acts as a plano-Cx. lens, the horizontal as a double Cx. lens, and these principal meridians have true foci. Every ray incident on the lens lies in both principal planes and is, therefore, converged to a certain extent in the vertical, and to a still greater extent in the horizontal, the resultant deviation being intermediate as to direction and extent. The same rays combine to form both the focal lines.

The action of the concave sph.-cyl. is similar to that of the convex, except that the foci and images are virtual.

As with the plano-cyl., the images at the two focal planes of an object of definite size, consist of bands of light, whose width and length depend on the powers of the lens, and its aperture or diameter.

A section of the cone of light emergent from the lens is elliptical except where the focal lines are formed; also at some position between them, where the cone of light has equal diameters in both Mers., producing what is termed *the circle of least confusion.*

Combined Cylindricals.—If two Cx. cyls. of similar power be placed in contact, with their axes corresponding in, say, the vertical meridian, the cyl. power is doubled. If the second cyl. be at right angles to the first, they are *equivalent to a sph. lens* of the same power as either cyl. In this case the power of the one corresponds to the axis of the other, and in all intermediate meridians any deficiency of power in the one is supplied by the other. As the second cyl. is rotated from axis vertical to axis horizontal the original vertical streak image shrinks until, when the two axes are at right angles, it is a point of light, or a complete image as the case may be. When the axes are oblique to one another, the effect is that of some sph.-cyl., whose two principal powers vary with the angle between the axes. Two unlike cyls. are always equivalent to some sph.-cyl. combination no matter what may be the inclination of their axes, except in the case of the axes being parallel, when they constitute a plano-cyl. The effect is the same whether the two cyl. powers be ground on opposite sides of a piece of plass, or whether two plano-cyls. be placed in contact.

The Interval of Sturm.—The two principal focal distances of a sph.-cyl. lens may be indicated by F_1, the first, and F_2, the second, and their powers by D_1 and D_2. The distance between the focal lines is termed *the interval of Sturm* or *the focal interval.*

Let a screen be held close behind a Cx. sph.-cyl., say +4 Sph. ⊂⊃ + 4 Cyl. axis Ver.; then the light from a small bright source, some distance in front of the lens, is cast as a light patch on the screen. If now the latter be gradually drawn away from the lens (Fig. 116), at the distance 12·5 cm., which is equal to the focal length of the combined sph. and cyl. powers, a Ver. line is formed at F_1; as the screen is still slowly receded the line develops gradually into a Ver. (prolate) oval at C, an almost perfect circle at B, a Hor. (oblate) oval at A, and finally into a Hor. line at F_2. The screen is then at 25 cm., which is F of the sph. As the screen is still further removed from the lens, the patch of light takes the form of an ever-enlarging oblate ellipse.

The lengths of the two focal lines L_1 and L_2 depend on F_1 and F_2, their distances from the lens, on d the effective aperture of the lens, and S the length of the interval of Sturm, *i.e.* the distance between F_1 and F_2 (Fig. 116), or the dioptric difference $D_1 - D_2$.

$$L_1 = dS/F_2 = dS/D_1 \qquad L_2 = dS/F_1 = dS/D_2$$

$$L_1 F_2 = L_2 F_1 \qquad L_1 D_1 = L_2 D_2$$

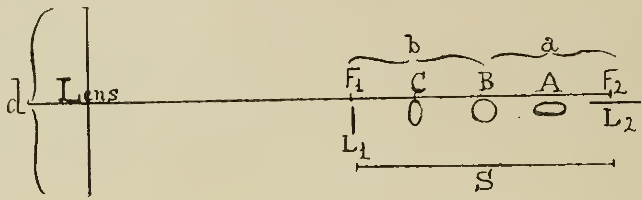

Fig. 116.

The circular disc of confusion B divides S into two parts b and a, which are proportional to F_1 and F_2.

$$S = a + b \qquad \text{and} \qquad b/a = L_1/L_2 = F_1/F_2 = D_2/D_1$$

B is distant from L_1 and L_2 respectively

$$b = \frac{SF_1}{F_1 + F_2} = \frac{SD_2}{D_1 + D_2} \qquad \text{and} \qquad a = \frac{SF_2}{F_1 + F_2} = \frac{SD_1}{D_1 + D_2}$$

B is not midway between L_1 and L_2, but is always nearer to L_1, when both powers are positive or both negative.

Its size is $$B = bL_2/S = aL_1/S$$

The distance of B from the lens is $E = \dfrac{2F_1F_2}{F_1 + F_2} = \dfrac{200}{D_1 + D_2}$

Example with $+4$ D Sph. ⊂ $+2$ D Cyl. having $d = 5$ cm.

$$F_1 = 16{\cdot}66 \text{ cm.} \qquad F_2 = 25 \text{ cm.} \qquad S = 25 - 16{\cdot}66 = 8{\cdot}33 \text{ cm.}$$

$$L_1 = \frac{5 \times 8{\cdot}33}{25} = 1{\cdot}66 \text{ cm.} \qquad L_2 = \frac{5 \times 8{\cdot}33}{16{\cdot}66} = 2{\cdot}5 \text{ cm.}$$

$$b=\frac{8\cdot33\times16\cdot66}{25+16\cdot66}=3\cdot33 \text{ cm.} \qquad a=\frac{8\cdot33\times25}{25+16\cdot66}=5 \text{ cm.}$$

$$B=\frac{3\cdot33\times2\cdot5}{8\cdot33}=\frac{5\times1\cdot66}{8\cdot33}=1 \text{ cm.} \qquad E=\frac{2\times16\cdot66\times25}{16\cdot66+25}=20 \text{ cm.}$$

When the combination is negative the interval of Sturm is also negative; when the combination is *mixed*, it is partly positive, and partly negative.

Thus when $F_1=10''$, $F_2=-20''$, and $d=1''$ we find B behind L_1 and negative. $S=-30$, $L_1=+1\cdot5$, $L_2=-3$, $b=+30$, $a=-60$, $B=-3$, $E=+40$.

If the stronger power is negative, B lies in front of L_1 and is again negative. If the $+$ and $-$ powers are numerically equal, B is at ∞.

CHAPTER IX

TRANSPOSING AND TORICS

Standard Angle Notation for the angular location of the principal meridians of a cyl. lens (Fig. 117A), is the same for both the right and left eyes. The numeration commences on the right hand of the imaginary horizontal line drawn through the lens when looked at *from the front*, that is, the surface remote from the eye of the wearer.

This notation corresponds with the trigonometrical division of the circle into 360 degrees. The upper right quadrant contains the angles between 0° and 90°, and the upper left those between 90° and 180°. The notation is not carried beyond 180° (the half-circle), since a meridian corresponds to a diameter, *i.e.* to two continuous radii—for instance, 45° is in the same meridian as 225°; 10° the same as 190°, etc. The vertical meridian is 90°, and the horizontal is 0° or 180°, but is preferably indicated as 180°.

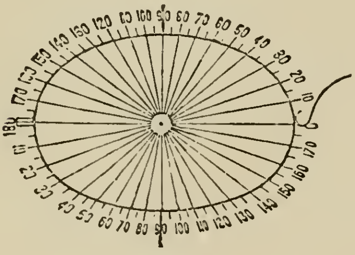

FIG. 117A.

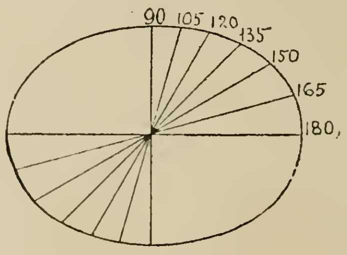

FIG. 117B.

Other Notations.—Some trial frames and prescription forms are notated differently from that shown in Fig. 117A, and it may occur that the optician has to transfer from one notation to another. The most commonly met with are the bi-nasal and the bi-temporal methods, in which the zero is placed at, respectively, the two nasal and the two temporal extremities of the horizontal line of the eye, the numeration running upwards, or conversely running downwards. Sometimes zero is placed in the vertical meridian, the numeration proceeding to the right and left. Indeed, there are many different methods of *notating* the *two eyes*, but it is hardly necessary to attempt to detail them here. Fig. 117B shows a notation reverse to the *standard*. Right eye *bi-nasal* and left eye *bi-temporal* correspond to *standard*.

To translate to standard a prescription written with the indicated cylindrical axis at 125° according to the notation of Fig. 117B, it must be considered how many degrees the required position is from the horizontal or the vertical. In this case 125°, in Fig. 117B, is 35° from the vertical on the right and, therefore, corresponds to 55° of Fig. 117A. If the location of the axis is 40° above the horizontal on the right, it would be 40° in Fig. 117A and 140° in Fig. 117B. The same mode of calculating serves if the cylindrical axis is indicated as so many degrees, say, out and down. This last-mentioned method of axis indication, unless accompanied by a stroke to show the direction, and, sometimes even if so accompanied, does not remove all ambiguity, and may lead to error. Some omit the ⊂ sign, and write a combination thus:

$$\frac{+4\text{ S.}}{+2{\cdot}50\text{ C. Axis }70^\circ}$$

In all methods, however, the direction indicated refers to the front of the lens, or the surface away from the wearer's eye.

Transposition of Sph. Lenses.—A Cx. sph., say, +6 D, can be made in the form of a plano Cx., in which all the power is on the one side; as an equi-Cx., in which the power is equally divided between the two surfaces; as a bi-Cx., in which the powers are unequally divided between the two surfaces; or as a periscopic-Cx., in which the Cx. power on the one side is more than 6 D, but the total is reduced to +6 D by the necessary Cc. curvature of the other surface. Similarly, a Cc. sph. can be made in various forms. The change from one form to another, without altering the refractive power of a lens, is called a *transposition*. The power of the one surface increases as that of the other decreases, so that the number of possible forms for a given sph. power is infinite; the position of the optical centre changes with the different forms of a lens.

The trade periscopic has one surface ±1·25 D, to 2 D, and the trade meniscus ±6 D.

Transposition of Cyl. Lenses.—Lenses which contain a cyl. element are susceptible of only two or three changes of form, and it is to such a change, which does not alter the refractive powers of the two principal meridians, that the term "transposition" is generally applied.

When the two powers have the same sign, they are said to be of *like* nature, or *congeneric*; when they are of opposite signs (the one + and the other –) they are of *unlike* nature, or *contrageneric*.

A *plano-cyl.* possesses no sph. element, but may be regarded as one whose sph. is of infinite focal length, and it will be so treated in this chapter.

A *sph.-cyl.* has sph. and cyl. elements, and may be a *compound cyl.*, having + or – powers in both principal meridians, or a *mixed cyl.*, having a + power in the one and – power in the other.

A *cross-cyl.* has two similar or dissimilar cyls. crossed at right angles.

Powers and Principal Meridians.—The one principal Mer. of a sph.-cyl. corresponds to the axis of the cyl., and its power is that of the sph. alone;

the other is at right angles to the axis of the cyl., and its power is the algebraical sum of the sph. and cyl. Thus the powers of—

+3 S. ⊂⊃ +2 C. Ax. 70° are +3 at 70° and +5 at 160°.
+3 S. ⊂⊃ −1 C. Ax. 110° are +3 at 110° and +2 at 20°.
+3 S. ⊂⊃ −3 C. Ax. 5° are +3 at 5° and 0 at 95°.
+3 S. ⊂⊃ −5 C. Ax. 120° are +3 at 120° and −2 at 30°.

In the cross-cyl. the two principal powers are those of the cyls. themselves, each being in the Mer. corresponding to that of the axis of the other. Thus the powers of—

+2 C. Ax. 40° ⊂⊃ +5 C. Ax. 130° are +2 at 130° and +5 at 40°.
+2 C. Ax. 70° ⊂⊃ −4 C. Ax. 160° are +2 at 160° and −4 at 70°.

Possible Combinations.—A cyl. combination may consist of two different powers of similar nature, as +2 and +5, or −3 and −7, or of two powers of dissimilar nature as +2 and −2, or +3 and −4. It can be made in three forms, viz., a cross-cyl. and two forms of sph.-cyl. If the one power

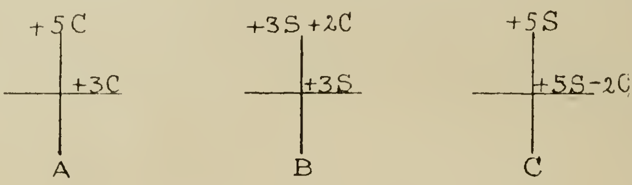

Fig. 118.

is 0 it can be made only as a plano-cyl., and in one form of sph.-cyl. If there are two similar equal powers the possible forms are only a cross-cyl. and a sph.

The Various Forms of a Lens with a Cyl. Element.—Where unequal powers in the principal Mers. are required, as +3 at 180° and +5 at 90°—

(*a*) The +3 needed at 180° (Fig. 118) can be obtained from +3 C. Ax. 90°, and the +5 at 90° from +5 C. Ax. 180°, the axis of each cyl. being at right angles to the direction in which the power is required, as in *A*.

(*b*) The +3 needed at 180° can be obtained from +3 sph., which also supplies 3 of the +5 D needed at 90°, the balance of the latter being obtained from +2 C. Ax. 180°, which gives +2 at 90° and 0 at 180°, as in *B*.

(*c*) The +5 needed at 90° can be obtained from +5 sph., but this not only supplies the +3 needed for 180°, but is 2 D too strong. To reduce the latter to +3 D a −2 C. Ax. 90° is required, this giving −2 at 180° and 0 at 90°, as in *C*.

If the two elements of the forms (*a*) (*b*) or (*c*) be placed over one another, the total combination is, in each case, +5 at 90° and +3 at 180°. The three forms are thus made up by—

(*a*) A cyl. of each of the two powers, the axis of each being at right angles to the meridian where the power is needed.

(*b*) A sph. of the lower power, and a cyl. of the difference between the two powers, the axis corresponding to the meridian of least power. If the lower power is 0, the sph. is also 0.

(*c*) A sph. of the higher power, and a cyl. of the opposite sign and of the difference between the two, the axis being in the meridian of greater power.

Whether the two powers are of like or unlike nature, the number of the cyl. is obtained by the *algebraical subtraction* of the power taken as the sph. from that of the other principal power. Thus in the example the powers are +3 and +5, so that if the sph. is +3, the cyl. is +2; if the sph. is +5 the cyl. is −2. If the two powers are +2 and −3, then, if the sph. is +2, the cyl. is −5; if the sph. is −3, the cyl. is +5.

For −4 at 60° and −7 at 150°, the three forms are:

(*a*) −4 C. Ax. 150° ⊂⊃ −7 C. Ax. 60°.
(*b*) −4 S. ⊂⊃ −3 C. Ax. 60°. (*c*) −7 S. ⊂⊃ +3 C. Ax. 150°.

For −1 at 45° and +5 at 135° they are:

(*a*) −1 C. Ax. 135° ⊂⊃ +5 C. Ax. 45°.
(*b*) −1 S. ⊂⊃ +6 C. Ax. 45°. (*c*) +5 S. ⊂⊃ −6 C. Ax. 135°.

For +3 at 120° and 0 at 30° they are:

(*a*) 0 S. ⊂⊃ +3 C. Ax. 30°. (*b*) +3 S. ⊂⊃ −3 C. Ax. 120°.

Rules.—(1) *To transpose a sph.-cyl. or plano-cyl. into another form of sph.-cyl. or plano-cyl.*

The following apply to all cases, but when the original or the transposed form is a plano-cyl., the one power being 0, the sph. may also be 0.

(*a*) The new sph. is found by adding *algebraically* the power of the sph. to that of the cyl.

(*b*) The new cyl. has the same power as the original cyl., but its sign is changed and its axis is at right angles.

(2) *To transpose a sph.-cyl. into a cross-cyl.*

(*a*) The one cyl. of the new form has the same number and sign as the original sph. with its axis at right angles to that of the original cyl.

(*b*) The other cyl. has its axis in the same Mer. as that of the original cyl. Its sign and number result from the *algebraical addition* of the powers of the original sph. and the original cyl.

(3) *To transpose a cross-cyl. into a sph. cyl.*

(*a*) The sph. of the new form has the number and sign of the first original cyl.

(*b*) The new cyl. has its axis corresponding to that of the second original cyl. and a sign and number which result from the *algebraical subtraction* of the first from the second original cyl.

Since either original cyl. may be taken as the first, there are two forms of sph.-cyls. into which a cross-cyl. can be transposed.

Examples.—The above rules can be better appreciated by studying examples at the same time. In the following, which illustrate all possible combinations, the first is the original, and those following are the forms into which it can be transposed.

(1) +4 S. ⊂ +2 C. Ax. 20°=
+6 S. ⊂ −2 C. Ax. 110°
+4 C. Ax. 110° ⊂ +6 C. Ax. 20°

(2) −2·50 S. ⊂ −1·50 C. Ax. 175°=
−4·00 S. ⊂ +1·50 C. Ax. 85°
−2·50 C. Ax. 85° ⊂ −4·00 C. Ax. 175°

(3) +3·50 S. ⊂ −2·50 C. Ax. 45°=
+1·00 S. ⊂ +2·50 C. Ax. 135°
+1·00 C. Ax. 45° ⊂ +3·50 C. Ax. 135°

(4) +3 S. ⊂ −3 C. Ax. 105°=
+ 3 C. Ax. 15°

(5) +2·50 S. ⊂ −4·50 C. Ax. 115°=
−2·00 S. ⊂ +4·50 C. Ax. 25°
+2·50 C. Ax. 25° ⊂ −2·00 C. Ax. 115°

(6) −1·25 S. ⊂ +1·75 C. Ax. 160°=
+0·50 S. ⊂ −1·75 C. Ax. 70°
−1·25 C. Ax. 70° ⊂ +0·50 C. Ax. 160°

(7) +2·75 C. Ax. 95°=
+2·75 S. ⊂ −2·75 C. Ax. 5°

(8) +2 C. Ax. 80° ⊂ +3 C. Ax. 170°=
+2 S. ⊂ +1 C. Ax. 170°
+3 S. ⊂ −1 C. Ax. 80°

(9) −5·50 C. Ax. 155° ⊂ −2·50 C. Ax. 65°=
−2·50 S. ⊂ −3 C. Ax. 155°
−5·50 S. ⊂ +3 C. Ax. 65°

(10) +2·25 C. Ax. 75° ⊂ −2·25 C. Ax. 165°=
+2·25 S. ⊂ −4·50 C. Ax. 165°
−2·25 S. ⊂ +4·50 C. Ax. 75°

(11) +3·50 C. Ax. 120° ⊂ −0·75 C. Ax. 30°=
+3·50 S. ⊂ −4·25 C. Ax. 30°
−0·75 S. ⊂ +4·25 C. Ax. 120°

(12) −10·00 C. Ax. 180° ⊂ +2 C. Ax. 90°=
+2 S. ⊂ −12 C. Ax. 180°
−10 S. ⊂ +12 C. Ax. 90

(13) +3·50 C. Ax. 90 ⊂ +3·50 C. Ax. 180°
+3·50 S.

(14) −4 S.=
−4 C. ⊂ −4 C. with axes at right angles.

Comparison of Original and Transposed Forms.—The two principal powers and Mers. of the original form of a combination can be extracted and compared with those of the transposed form, and they must be alike if the transposition is correct. Thus, suppose – 3 S. ⊂ +4 C. Ax. 90°. The two principal powers are – 3 at 90° and +1 at 180°. The power of the – 3 Sph. is in both principal meridians, while that of the +4 C. Ax. 90° is only at 180°; its axis, being at 90°, contributes no refractive power to that meridian.

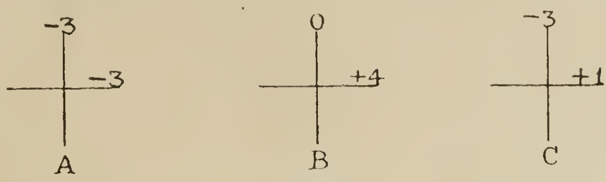

FIG. 119.

The two components separately are represented by *A* and *B* of Fig. 119. When combined they are represented by *C*. The two forms into which the combination can be transposed are

(*a*) +1 S. ⊂ – 4 C. Ax. 180°; (*b*) +1 C. Ax. 90° ⊂ – 3 C. Ax. 180°

Proof by Neutralisation.—Since a transposition simply assigns the needed powers in a different way, as regards the two surfaces of a lens, and does not change the refractive power of the combination, that combination which will neutralise the original form will also neutralise the transposed forms. Thus—

(*a*) +1 S. ⊂ – 4 C. Ax. 180° transposes into
(*b*) – 3 S. ⊂ +4 C. Ax. 90°

(*a*) is neutralised by – 1 S. ⊂ +4 C. Ax. 180°, and these also neutralise (*b*) as can be seen by adding them together thus—

(– 3 S. ⊂ +4 C. Ax. 90°) + (– 1 S. ⊂ +4 C. Ax. 180°)

The 2 sphs. = – 4 S., the 2 cyls. = +4 S.; – 4 S. +4 S. = 0.

Best Form.—It is never required in practice to employ crossed cyls. since the same effect results from a sph.-cyl., at less cost. The best form to employ is usually a +Sph. ⊂ – Cyl., or a – Sph. ⊂ +Cyl., since then a periscopic effect is obtained.

Toric or Toroidal Lenses.—A toric lens is one having two principal powers worked in the same surface with their axes at right angles to each other, as shown in Fig. 120. The curvature of the lens along *A B* is, say, +3 D, while along *C D* it is, say, +5 D. It is, therefore, equal to +3 S. ⊂ +2 C. and has the same optical effects. The name is derived from the tore or arched moulding used at the base of pillars. It has the curvature of a bent tube or rod; the side of an egg or the bowl of a spoon resembles a toric surface.

The curvature of a toroidal surface is spherical in the two principal meridians, and elliptical in the intermediate ones, and can be either Cx. or Cc. Astigmatism of the cornea is due to its toroidal curvature.

Since the possible toric forms of any combination are practically infinite, it is usual to employ tools of a given *base curve.* An assortment can then be

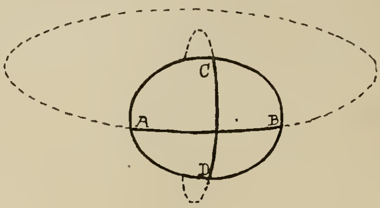

Fig. 120.

kept of toric lenses having the one surface unworked, on which any spherical curve can be ground. The *base power* indicates the standard or *fixed* power of the toric surface. It is usually the lower of the two powers, but may be, and occasionally is, the higher. In the following it will be taken as if it were always the lower toric power.

Toric tools and lenses are usually made to a base of 6 D, and sometimes to 3 D or 9 D, the other power always being *stronger* by an amount *equal to the cylindrical effect* required in the combination. The number of the toric tool or blank is therefore the difference in the principal powers, since the weaker power is always 6, 3, or 9, as the case may be; thus a 2 D tool is one having a 6 D curve in one direction and 8 D at right angles; or if the series is on a 3 D base the curvatures would be 3 and 5.

Therefore if a +1 S. ⊃ +1·5 C. were required in toric form with +6 base, a 1·5 tool would be used, giving on the Cx. surface powers of +6 and +7·5. On the other surface −5 D sph. would be necessary in order to reduce the principal powers to +1 D and +2·5 D, as required in the combination.

The series of tools being in pairs, the toric surface can be made Cc. if the powers of the original lens are not suitable for a Cx. toric. For example, the above sph.-cyl. could be made with −6 D and −7·5 D powers on the one side, the adjusting spherical on the other being +8·5 D.

Advantages of the Toric Lens.—One utility of the toric form of lens is that it permits of the refracting power of a lens to be more equally divided between the two surfaces. Thus if +10 S. ⊃ +1 C. be required, instead of +10 S. on the one surface and +1 C. on the other, it can be made with +4 S. on the one and +6 C. ⊃ +7 C. on the other. Or it can be made with any other Cx. sph. power, the virtual cyls. of the toric surface being accordingly stronger or weaker, but always having 1 D difference between them. Thus, a strong lens as needed in aphakia or high myopia can be made less thick and unsightly and more nearly resembling a Dcx. or Dcc. Another, and still greater, advantage of the toric surface is that, with it, a sph.-cyl. can be made periscopic to a considerably greater extent than is possible with the ordinary sph-.cyl. form. The advantages of highly periscopic lenses are mentioned under *Menisci*, page 248.

Conversion to Toric Form.—Due regard must be paid to the powers of the original lens when selecting the base curve in order to get the best periscopic result. If this is not done the result may be little better than what could be obtained from the ordinary sph.-cyl., and on the other hand it may be so deep as to render the lens clumsy or unsuitable for mounting. Thus – 5 S. ⊃ – 1 C. Ax. 45° on a – 6 base becomes—

$$\frac{+1\text{ S.}}{-6\text{ C. Ax. }135^\circ \supset -7\text{ C. Ax. }45^\circ}$$

This differs but little from the sph.-cyl. form of – 6 S. ⊃ +1 C. Ax. 135°; and on a +6 base, the sph. being – 12 D, the lens would be thicker and heavier and with doubtful advantages over the ordinary sph.-cyl. The toric is most useful, in weak combinations, because then generally only very small periscopic effects can be obtained from the ordinary forms. A toric surface may be expressed as *two powers* in certain meridians, or as the *two virtual cylindricals*, which are contained therein, with their axes at right angles and in certain meridians. The latter method is followed in this article.

Rules for Conversion of a combination into a toric of given base.

(*a*) Convert into *cross-cyls.*
(*b*) Find B. Change its sign and this is the sph.
(*c*) Add B to each original cyl. for the toric cyls., B is as follows:

(1) The difference between the lower power and the base, when the two powers are of same sign and same as the base.
(2) The difference between the higher power and the base, when the two powers are of same sign but opposite to that of base.
(3) The difference between the power opposite to that of the base and the base, when the two powers are not of the same sign.

Example of (1) +5 D Sph. ⊃ +2 D Cyl. Axis 70° to +6 D base.

(*a*) +5 D Cyl. Axis 160° ⊃ +7 D Cyl. Axis 70°.
(*b*) B = +6 – 5 = +1 D. The Sph. is – 1 D.
(*c*) C_1 = +5 + 1 = +6 D Axis 160° and C_2 = +7 +1 = +8 D Axis 70°.

that is,
$$\frac{-1\text{ D Sph.}}{+6\text{ D Cyl. Axis }160^\circ \supset +8\text{ DCyl. Axis }70^\circ}$$

Example of (2) +2 D Sph. ⊃ +3 D Cyl. Axis 135° to – 6 D base.

(*a*) +2 D Cyl. Axis 45° ⊃ +5 D Cyl. Axis 135°.
(*b*) B = – 6 – 5 = – 11 D. The Sph. is +11 D.
(*c*) C_1 = +2 – 11 = – 9 D Axis 45° and C_2 = +5 – 11 = – 6 D Axis 135°.

that is, $\dfrac{+11 \text{ D Sph.}}{-6 \text{ D Cyl. Axis } 135° \supset -9 \text{ D Cyl. Axis } 45°.}$

Example of (3) -1 D Sph. $\supset$ $+4$ D Cyl. Axis 90° to $+3$ D base.

(*a*) -1 D Cyl. Axis 180° $\supset$ $+3$ D Cyl. Axis 90°.

(*b*) $B = +3 - (-1) = +4$ D. The Sph. is -4 D.

(*c*) $C_1 = -1 + 4 = +3$ D Axis 180°. $C_2 = +3 + 4 = +7$ D Axis 90°.

that is, $\dfrac{-4 \text{ D Sph.}}{+3 \text{ D Cyl. Axis } 180° \supset +7 \text{ D Cyl. Axis } 90°.}$

The Toric, Cyl. and Sph.—If a circle is revolved around a chord not passing through its centre a toroid is generated. If the circle is of infinite radius the body generated is a cylinder. It is a sphere if the chord becomes a diameter. Thus a cyl. lens may be regarded as a toric having one power =0, and a sph. lens as one having both powers equal.

CHAPTER X

ANALYSIS AND NEUTRALISATION OF THIN LENSES AND PRISMS

Neutralisation is the process of finding that lens (or lenses) of opposite refraction and of known power (from the test case) which stops the movement caused by the lens to be analysed.

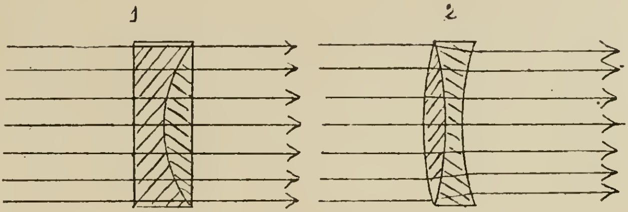

Fig. 121.

A Cx. and a Cc. lens (Fig. 121) of the same power, when placed in contact, have no converging or diverging effect, the convergence of the Cx. being counteracted by the divergence of the Cc., and incident parallel light emerges parallel. When moved in front of the eye, they cause no movement of the image of the object viewed through them, as with a plane glass.

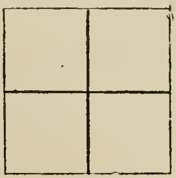

Fig. 122.

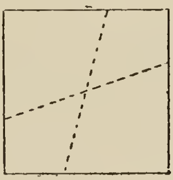

Fig. 123.

Analysing Card.—Analysis and neutralisation are facilitated by the use of an analysing card, as shown in Fig. 122, although, in its absence, any clearly defined straight vertical, or horizontal line, as the framework of a window, serves the purpose. The card should be 18 or 20 inches square, with two crossed black lines about $\frac{1}{4}$ inch in width, running vertically and horizontally, and for most work should be distant not less than, say, 4 feet.

Determination of Nature of a Lens.—The first step in the analysis of a spectacle lens is to learn whether or not it contains a cyl. element. A lens having a sph. power only, on being rotated around its geometrical centre in a plane parallel to the card, does not cause any change in the appearance of the lines of the analysing chart, because its prismatic elements are alike in all meridians. If the lens has a cyl. element the lines become oblique, as shown in Fig. 123, where the dotted lines represent the black lines of the chart as seen when the lens is rotated. This occurs because the prismatic elements are not alike in all meridians, and the subject is treated more fully in Chapter XII.

Determination of Positive and Negative Power.—If an object be viewed through a lens and the latter moved across the line of vision, the virtual image seen moves in the *opposite direction with a Cx. lens*, and in the *same direction with a Cc. lens*. If the lens is displaced, say, downwards, the light from it passes through a peripheral portion of greater deviating power than the centre, and the object appears deviated in the direction of the apices of the virtual prisms of which the lens is formed, that is, towards the edge of a Cx., and towards the centre of a Cc. lens. The degree of deviation and the rapidity of movement of the image are proportional to the strength of the lens, the deviation being greater, as the part of the lens looked through is near the periphery. The apparent motion of the object viewed, as the lens is moved, *is due to the fact that the lens increases gradually in prismatic or deviating power from centre to periphery.*

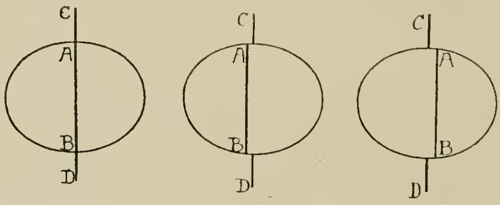

FIG. 124. FIG. 125. FIG. 126.

Let the lens be a sph. or a cyl. whose power is in the horizontal plane. When a vertical line is viewed through the centre of the lens the part *A B* is seen continuous with the parts *C* and *D* beyond its edges (Fig. 124). Then if the lens be moved, say, to the *right, A B* becomes broken away from *C* and *D* to the *left* if the lens is Cx. (Fig. 125), and to the *right* if it is Cc. (Fig. 126). When making this test the lens should be moved slowly in a certain direction, and not rapidly from side to side or up and down. The lens should not be too close to the eyes, for then the line *C* and *D* beyond the edges cannot be seen; the best distance is generally about 10 inches.

If the object be first viewed through, say, the bottom of the lens, and this then moved downwards, the motion of the image is continuously *with* or *against* throughout the journey.

If, instead of the lens, the head is moved, the image moves with the head if the lens is Cx., and in the opposite direction if it is Cc.; movement of the head, say, to the right produces the same effect as movement of the lens to the left.

A plano causes no deviation of the image when it is displaced, nor of the lines when it is rotated.

A prism when displaced causes no further change in the image; its effect on being rotated is treated later.

Magnification.—A square viewed through a sph. appears to be increased in size by a Cx. and decreased by a Cc. equally in every direction and (disregarding distortion) remains a true square. Through a cyl., it is apparently magnified across the axis, by a Cx., and diminished by a Cc.; the size is unaltered in the direction of the axis, so that a square appears rectangular.

Neutralisation of Sphericals.—If a lens has no cyl. element, its nature is next determined, and if Cx., a Cc. is selected from the trial case, as near the necessary power as can be judged, and then the two, held together, are again moved. If the movement is *against*, i.e. still that of a Cx., the power of the neutralising Cc. is insufficient, and a stronger one must be tried. If with the first neutralising lens the movement of the two combined is *with*, i.e. that of a Cc., the neutralising lens is too strong, and a weaker one must be taken. A few trials will enable one to find a lens which, when placed in contact with the unknown lens, causes no displacement of the line; then the number of the neutralised Cx. is that of the neutralising Cc. To find the power of an unknown Cc. lens, a neutralising Cx. must, of course, be used. Practice will soon enable one to judge, by the degree or rapidity of movement, the approximate neutralising power needed, as well as to appreciate such slight movements as occur when neutralisation is nearly, but not quite, effected. When neutralising, the lenses must be in actual contact, because if separated the Cx. acts with increased effect.

As stated, the best distance for neutralising is some 10″, but if the lens is a powerful Cx. it must be held nearer the observer's eyes, or nothing can be seen through it owing to the strong convergence of the light. The nature of such a lens is, however, easily recognised from its form and from the blurred view. Again, if held at a distance well beyond its focal length, for instance, if a 4 in. Cx. be held 10″ from the eye, the apparent movement of the object when the lens is moved is the same as with a Cc. lens, because then an inverted aerial image of the object is seen, and not the ordinary virtual image.

The Principal Meridians of a Cylindrical.—If the lens contains a cyl. element the cross lines of the analysing card are seen continuous within and beyond the edges of the lens, as in Fig. 127, *only when the axis of the lens is horizontal or vertical.* The two principal Mers. then correspond in direction to the lines of the chart. Such a position for a cyl. must be found in order (*a*) to learn whether it is a plano- or a sph.-cyl., (*b*) to determine whether it is Cx. or Cc., and (*c*) to neutralise it. This position being found, the lens is first moved vertically and then horizontally. If no movement is observed in the one direction it is a plano-cyl.; if there is movement in both directions it is a sph.-cyl., or its equivalent, a cross-cyl.

Movement *against* indicates Cx. and movement *with* indicates Cc. power *in that meridian.* If there is movement in both Mers. they may be both *against,* both *with,* or the one *against* and the other *with.* The movement in the one Mer. must differ from that in the other if there is a cyl. element.

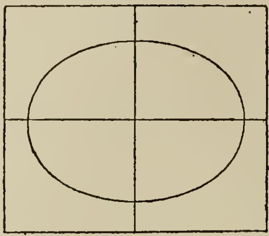

Fig. 127.

The axis of a plano-cyl. lies in the meridian in which there is *no* movement. The axis of the cyl., in a sph.-cyl., which has two positive or two negative powers, is in the principal meridian of *lesser* movement. When there are + and − powers the axis of the cyl. is also *presumed* to be in the principal meridian of *lesser* movement. In all cases the axis of the actual cyl. might be in the meridian of greater movement, because the same principal powers can be obtained in lenses of various forms. (See *Transposing.*)

The angular *position* of the axis is *the same as that of the neutralising cyl.* This can be determined after a little practice, with a fair degree of accuracy, when the lens is held as when in use. With more accuracy its numerical position can be determined by holding the lens against the neutralising lenses when the latter are in a trial frame, with the long diameter of the neutralised lens horizontal. The axis of the trial lens, being marked by a scratch, can be read off from the notation of the frame.

There are several forms of inclinometers or axis-finders—that of Dr. Maddox is an excellent one—designed for the location of the axis of an unknown cyl. lens. A quick and fairly accurate method of locating the axis is by means of the protractor on the " Orthops " rule.

For accuracy the procedure is as follows: Holding the lens and the neutralisers in position, the axis of the cyl. is marked with a grease pencil on the lens by a line coinciding with the axis of the neutralising cyl.; also the optical centre is marked by a dot. The lens is placed on a protractor with the dot at the centre, the long diameter of the lens being exactly horizontal; the angular position of the axis is then indicated on the protractor. Care must be taken that the same meridian is covered by the marked grease line *both above and below the central horizontal line.* When the axis is oblique and the lens is not in a frame, consideration must be given as to which of the two faces of the lens is supposed to be directed outwards, since the position of the axis varies accordingly. The rule is that the Cc. surface of a periscopic or the less Cx., or the more Cc., surface of a lens is placed next to the eye.

Neutralisation of Cylindricals.—For a plano-cyl. the procedure is the same as with a sph., except that cyls. are employed. The lens is rotated until the principal Mers. are vertical and horizontal, the Mer. of *no power* is determined, and the other Mer. is then neutralised with a cyl. of opposite nature. The axis must be *exactly* vertical (or horizontal), therefore *continuity of the crossed lines at the edges of the lens must be looked for, and constantly maintained during the process of neutralisation.*

Care also must be taken that the axis of the neutraliser *precisely* corresponds to that of the lens.

In a sph.-cyl. the lesser movement is that due to the sph. alone, while the greater movement is caused by the united powers of the sph. and the cyl. The lens being held with its axis, say, *vertical,* that sph. of opposite refraction is found which neutralises, in the Ver. meridian, the movement of the Hor. line. This being achieved, the lens and the neutralising sph. are held together, and the cyl. element is then neutralised with a cyl. *axis vertical,* of opposite refraction, in the same manner as if the lens were a plano-cyl. The rapidity and exactitude of the neutralisation depends, as with a plano-cyl., on the care exercised *in keeping the principal meridians exactly parallel to the two lines of the chart, and the axes of the two cyls. exactly corresponding.*

Neutralisation of a sph.-cyl. can also be effected by neutralising each principal meridian separately with a sph., or with a cyl. whose axis is placed at right angles to the meridian that is being neutralised, the two powers thus found being transposed into a sph.-cyl. combination. These methods are, however, not so exact, especially for beginners.

Cross-cyls., torics and obliquely crossed cyls. are all merely special forms of sph.-cyls., and are therefore analysed and neutralised in a similar manner.

Expressing Sphero-Cylindricals.—Since any lens having two principal meridians can be put up in various forms, the neutralising combinations, while correctly indicating the refracting powers of the lens, may not represent the exact form in which it is made. It is always correct to express a combination as a sph.-cyl. *with a sph. of the lower power.*

True Form of a Lens.—This can be learnt by (*a*) ordinary inspection, (*b*) reflection from the surfaces, (*c*) the lens measure or spherometer, (*d*) by a straight-edge which, when in contact, easily shows the difference between Cx. and Cc. curvature.

The Scissors Movement.—On rotating a cyl. in a plane parallel to the analysing chart the lines on the latter appear to make a scissors-like movement, and if the rotation be continued, appear to move back again, the amount of *dipping* being dependent on the strength of the cyl. Each line appears to bend towards the meridian of greatest positive, or least negative, refraction, so that they both rotate towards the axis of a Cc. or away from the axis of a Cx. cyl., and since they incline towards each other, they are never at right angles except when the principal meridians of the lens correspond to them in direction. The inclination of the cross lines is due to the prismatic formation, *the apparent displacement being towards the edges of the virtual prisms contained in the lens.*

The scissors movement with both Cx. and Cc. resemble each other, one end of the horizontal line moving up, the other down; one end of the vertical line moving to the right, the other to the left. For instance, a Cx. cyl. axis Ver. and a Cc. cyl. axis Hor., both rotated, say, clock-wise, cause similar movements of the cross lines. An attempt to neutralise by "stopping" the apparent inclinations might result in selecting for that purpose another cyl. of similar power and nature, instead of a cyl. of opposite nature but equal power, the two together making a sph. lens.

Reversion of a Cyl.—If a cyl. (Fig. 128), having its axis at, say, 60° when the one face is to the front, is turned over so that the other face becomes the front, the axis, is then at 120° (Fig. 129). If the one position were 5°, the other would be 175°. It is only when the axis is vertical or horizontal that no change occurs on reversing the lens. When the one inclination is 45° or at 135°, turning the lens over brings the axis to a position at right angles to the former one. The change in the numerical position of the axis, caused by reversing an oblique cyl., is calculated as so many degrees above or below

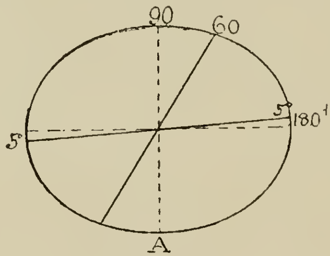

FIG. 128.

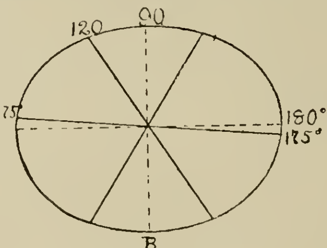

FIG. 129.

the horizontal, or to the right or to the left of the vertical, and assigning its position accordingly; or it is done simply by deducting the numerical position of the axis from 180°. Thus, suppose the axis is at 60°, this is 30° to the right of the vertical; on turning the lens the axis is at 90° + 30° = 120°, *i.e.*, 30° to the left of the vertical, or more simply by 180° − 60° = 120°.

Prisms—The Base-Apex Plane.—When a prism or a lens having a prismatic element, is rotated around its geometrical centre, the base-apex plane and the edge of the prism are similarly rotated. If the cross lines of the chart *A B C D* (Fig. 130) be observed, the junction *Z* of the cross lines being deflected towards the edge of the prism, the vertical line moves horizontally and the horizontal line moves vertically, *but the two always remain at right angles to each other.*

In, however, one certain position, there is *a continuity of one of the lines within and beyond the edges of the glass,* as in Fig. 131, where the vertical line *A B* is continuous. The direction of this line indicates that of the base-apex plane of the prism, or of the prismatic element of the lens. If the Hor. line *C D* is deflected upwards, as to *E F*, the apex is then pointing *up* towards *A*, and the base is *down* towards *B*. If the deflection of *C D* is downwards towards

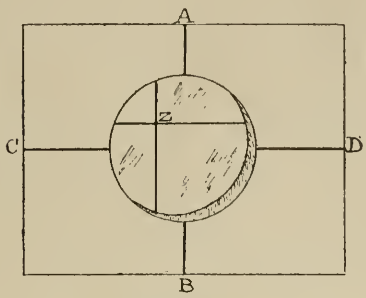

FIG. 130.

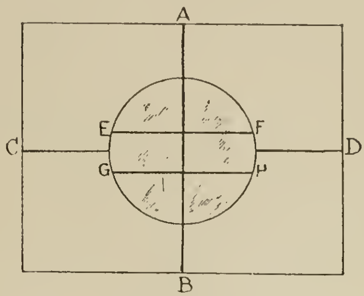

FIG. 131.

G H the edge of the prism is pointing *down*, and the base is *up*. If properly marked, the indicating scratches of circular trial prisms lie over *A B* when that line appears continuous through the prism.

Neutralisation of Prisms.—The strength of a prism can be learnt by neutralisation. The base-apex line being located, the displacement of a bar of the analyser can be neutralised by trying one prism after another from the test case and placing it in opposition to the unknown prism; that is, placing the base of the former over the edge of the latter, until that test prism is found which causes both chart lines to be seen continuous beyond and through the two prisms. The number of the test prism, which neutralises the unknown prism, indicates the value of the latter. The *deviation of the prism is neutralised*, although the neutraliser may be numbered according to its principal angle.

If the prism is combined with a sph. (or a sph.-cyl.) this latter must be first neutralised. With the lens and neutralisers held together, all having their geometrical centres coincident, the prismatic element is located and neutralised. It must be remembered that the decentration, with respect to each other, of neutralising Cx. and Cc. lenses, introduces prismatic effect not actually existing in the lens which is under analysis. Therefore *the geometrical centres of all the lenses should exactly coincide* when any prismatic element is suspected, or is being measured.

If the angular inclination of an oblique prism is needed the base-apex line, when located, should be marked with a grease pencil and the angular position determined on a protractor, as for the axis of a cyl.

Points on Neutralising.—Practice is necessary to neutralise rapidly and correctly, and it is well to commence with sphs., then proceed to plano-cyls. and finally to sph.-cyls.

Holding several lenses together is difficult, but is rendered easier if the adjacent surfaces are Cx. to Cc.

A lens possessing sph., cyl. and prismatic elements should be neutralised in that order.

The front of the lens, *i.e.* the Cx. surface of a periscopic, or the more Cx. or less Cc. surface of any lens, must face the observer.

A simple prism may be mistaken for a plano, since neither causes *movement;* rotation is needed to distinguish between them.

It is necessary to guard against supposing a prismatic element to exist, when it may be produced by holding the neutralising lens out of centre with the lens which is being tested.

If the sph. is strong compared with the cyl. it is difficult to appreciate the latter until the sph. is partly neutralised.

Similarly it is difficult to appreciate a weak sph., when combined with a strong cyl., until the latter is wholly or partly neutralised.

When the two powers of a sph.-cyl. are nearly equal it is not always easy to determine in which Mer. the movement is the lesser, but this becomes easy when the lens is partly neutralised.

A strong Cx. and a strong Cc. of equal powers (say over 10 D) do not properly neutralise owing to the appreciable interval between their optical centres; the Cx. effect predominates. This is further considered in Chapter XV.

When a sph.-cyl. or a toric has to be neutralised by a *double,* there is contact at the centre only and neutralisation at the periphery is then not obtained owing to separation. Attention must be confined to the centre.

A great help in neutralising difficult lenses is a diaphragm made of cardboard, with a quarter inch aperture, held between the two lens. This not only reduces the effective aperture of the lens but, also, renders holding them together much easier.

Holding the lens or lenses farther away, or viewing a more remote object, facilitates the determination of very weak powers, or whether neutralisation is obtained.

Other methods for measuring the powers of lenses and prisms are dealt with in later chapters.

CHAPTER XI

OPHTHALMIC PRISMS

Prism Nomenclature.—A prism placed in a spectacle frame with its base towards the nose is termed + or *base in*, one with its base towards the temple is − or *base out ;* it is *up* if the base is towards the brow, and *down* if towards the cheek. If there is a pair of prisms they are both base *in*, or both base *out*, or the one is *up*, and the other *down*. If they are oblique, the one is, say, *out* and *up*, the other is then *out* and *down*. In all cases they are, with respect to each other, in such positions that, if they were placed over each other, they would neutralise.

A prism is *horizontal, vertical*, or *oblique* according as the base-apex line is Hor., Ver. or oblique respectively.

Ophthalmic prisms are presumed to be *thin*, i.e. not exceeding, say, 20° principal angle.

The indicated power of a prism in any notation is that due to its *minimum deviation*.

The symbol Δ indicates prism power.

Notation by the Principal Angle.—This actually gives the form only, and is somewhat similar to the numeration of lenses according to curvature; the true optical effect is not indicated. Two prisms of, say, 3°, the one of $\mu =$ 1·5, and the other μ 1·54, are both prisms of 3°, but their optical properties are not the same. The unit is a prism of 1° P. (principal angle).

Notation by the Angle of Deviation indicates the true optical value of the prism in angular deviating power. Its value depends on P and μ, with both of which it varies directly. This system has the drawback that the angle itself is inconvenient to measure in practice. The unit is a prism of 1° *d*. (deviation).

Relationship of the °P. and the °*d*.—If $\mu = 1{\cdot}5$, $1° = {\cdot}5°d$, but the °*d* increases with respect to the ° as the μ is higher. The number of degrees in the deviating angle being approximately half that of its principal angle, the °*d* is double the value of the °. Therefore, if two prisms of the same strength be numbered respectively in the two systems, its number in °*d* would be half that in °. If, however, μ is taken as 1·52, the relative values are slightly less than 2 to 1. As here used, the °P is commonly written °.

Notation by the Prism Diopter.—This notation, introduced by Mr. Charles Prentice of New York, is based on the linear deviation, and presents many advantages. The unit is the 1Δ, which causes a deviation of 1 cm. (on a tangent) at a distance of 1 metre. It is, therefore, a 1% linear deviation, and $N^{\Delta}/100 = \tan d$.

Two prisms, numbered in $^{\Delta}$, when placed together are, however, equal to rather more than the sum of their individual powers. Again, the deviation being measured on a flat surface, increase in the linear deviation does not result in a corresponding increase of angular deviation. Thus 1^{Δ} is equal to $34' \, 22\frac{1}{2}'$, but a 10^{Δ} is of less value in $^{\circ}d$ than 10 times that amount. These differences are, however, so very inconsiderable, especially in the weak prisms needed in spectacle work, as to be of no practical importance.

If $\mu = 1{\cdot}575$, the $^{\circ} = {}^{\Delta}$, for $\cdot 575^{\circ} = 34' \, 30''$, the tangent of which is ·01. When $\mu = 1{\cdot}52$, the refractive index of the glass usually employed, the $^{\circ} = \cdot 9^{\Delta}$. These values can, however, be considered true for small angles only such as we find in the optics of spectacle work.

Relative Values.—The relative values of the three units mentioned, in terms of the deviation they cause at the same distance, are

$$\text{the } {}^{\circ}P = \cdot 9; \text{ the } {}^{\Delta} = 1; \text{ the } {}^{\circ}d = 1{\cdot}745, \text{ or say } 1{\cdot}75.$$

The above are also the *constants* employed with each notation, and represent percentage of linear deviation.

Their equivalents are

$$1^{\circ} = \cdot 52^{\circ}d = \cdot 9^{\Delta}$$
$$1^{\Delta} = \cdot 57^{\circ}d = 1{\cdot}1^{\circ}$$
$$1^{\circ}d = 1{\cdot}745^{\Delta} = 1{\cdot}9^{\circ}$$

Calculations in Prism Measurement.—The following, while sufficiently accurate for all practical purposes, are not exact, for the reasons given, and because we take angular and tangential variations to be equal:

Let Δ represent the power of the prism, M its distance in metres from the object viewed, C the deviation in centimetres, and K a constant for each system of prism notation. Then

$$C = \Delta \, M \, K.$$

Thus at 3 metres, the deviation caused by a 4°, a $4^{\circ}d$, and a 4^{Δ} respectively is

$$4 \times 3 \times \cdot 9 = 10{\cdot}8 \text{ cm.}$$
$$4 \times 3 \times 1{\cdot}75 = 21 \text{ cm.}$$
$$4 \times 3 = 12 \text{ cm.}$$

If the deviation caused by a prism at four metres is 5 cm., the prism is

$$\frac{5}{4 \times \cdot 9} = 1{\cdot}4^{\circ}, \quad \text{or} \quad \frac{5}{4 \times 1{\cdot}75} = \cdot 7^{\circ}d, \quad \text{or} \quad \frac{5}{4} = 1{\cdot}25^{\Delta}.$$

The distance at which a prism of 5°, one of $5^{\circ}d$, and one of 5^{Δ} respectively causes a deviation of 15 cm., is

$$\frac{15}{5^{\circ} \times \cdot 9} = 3{\cdot}33 \text{ M.} \qquad \frac{15}{5^{\circ}d \times 1{\cdot}75} = 1{\cdot}75 \text{ M.} \qquad \frac{15}{5^{\Delta}} = 3 \text{ M.}$$

Conversion of Prismatic Values.—For conversion from one system of prism notation to another it is only necessary to remember the relative values of the units. Thus

$$4^\circ = 4 \times \cdot 9 = 3\cdot 6^\Delta, \text{ or } 4 \times \cdot 9/1\cdot 75 = 2\cdot 06^\circ d$$
$$4^\Delta = 4/\cdot 9 = 4\cdot 44^\circ, \text{ or } 4/1\cdot 75 = 2\cdot 28^\circ d$$
$$4^\circ d = 4 \times 1\cdot 75/\cdot 9 = 7\cdot 77^\circ, \text{ or } 4 \times 1\cdot 75 = 7^\Delta.$$

The Centrad is another prism unit, which causes at 1 M. a deviation of 1 cm. on the arc of a circle. The deviation is 1% and the difference between the arc and the tangent of small angles being negligible, the centrad and $^\Delta$ may be considered equal. A given prism notated in $^\Delta$ would be of fractionally higher number than if numbered in centrads. It is, however, much more inconvenient to measure on a curved than on a flat surface, and the centrad has never come into general use. $N\nabla/100$=arc d. Calculations for centrads can be taken as the same as for prism diopters.

The Metran.—Another prism unit, suggested by L. Laurance, is the metran, which causes a deviation of 3 cm. when placed in front of the eye at one metre from the scale. It has, therefore, about 1·75° (or 1° 45′) deviation, and is the same as the metre angle for the average interpupillary distance of $2\frac{3}{8}$ in. or 60 mm. The symbol is thus 4 $\wedge$.

The Measure of the Principal Angle is termed *goniometry*. The *principal angle* of a prism can be found roughly by enclosing it between the legs of a pair of compasses and finding the angle so obtained on a protractor, or by any instrument made for the purpose; also by the goniometer, consisting of a pivoted arm, at one end of which there are two legs which rest on the face of the prism, the other end indicating the angle on a scale. It can also be determined by the *pin* method described in Chapter XXVIII. The really accurate method is that of the *spectrometer* which is given in Chapter XX. Also, without much error for weak ophthalmic prisms, the *tangent scale* can be, and is, employed.

The Measure of the Deviating Angle or Prismetry is accurately made by the *spectrometer* method; an approximate *pin* method is described in Chapter XXVIII. The *tangent scale* and *neutralisation* methods are the practical methods for thin prisms.

The Measure of the Linear Deviation in $^\Delta$ is made by *neutralisation* or the *tangent scale.*

The Tangent Scale.—A tangent scale, shown in Fig. 132, serves as the most convenient method of measuring ophthalmic prisms. It is a card, say, 12 inches wide and 30 inches long, scaled so that the intervals between the divisions represent the tangents of the angles of deviation, and was originally designed by Dr. Maddox. The intervals vary in size with the distance at which the card is used.

The line *A C* (Fig. 133) is looked at through the prism, which is held with its base directed towards A, while the edge points to B. If the line $A\ B$ is displaced upwards or downwards the prism must be rotated in a plane parallel to the card until $A\ B$ is *continuous and seen unbroken through the prism.* The base-apex line is then horizontal, and the deviation is greater than with any other position of the prism in a plane parallel to the card. The number, towards which the deviated part A points, indicates the prismatic power

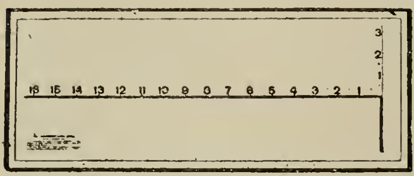

Fig. 132.

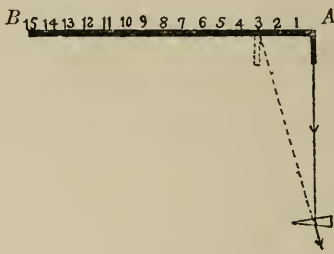

Fig. 133.

of the prism in degrees of deviation, prism diopters or degrees, according to which notation the scale is arranged for. The prism must be held sufficiently low so that the numbers on the scale may be seen over the top of the prism while A is seen through it.

The deviation caused by the prism varies if its position departs from that of minimum deviation; consequently, when $A\,B$ is unbroken, the prism must be rotated on its axis in order to secure minimum deviation, this being the numerical strength of the prism. Thus, in Fig. 133, the prism is presumed to be in the position of minimum deviation, and the indicated number is 3, but if the edge of the prism were turned either towards or away from the scale, the indicated deviation would be greater than 3.

If the prism is combined with sph. or cyl. powers, these must be neutralised before the prismatic power can be measured on a tangent scale, *care being taken that the geometrical centres of neutralising and neutralised lenses exactly coincide ;* otherwise a false measure of the prismatic power is obtained.

A tangent scale arranged for one system could be utilised for others by using it at the proper distance. Thus, the intervals on the "Orthops" scale (Fig. 132) are 3·5 cm., so that used at 2 M. the numbers indicate degrees of deviation, and at 3·5 M. prism diopters. If used at 4 M. it serves for ordinary degrees. The construction of tangent scales is dealt with in Chapter XXVI.

Another Tangent Measurement.—As a modification of the ordinary tangent scale, parallel light is passed through a suitable Cx. cyl. and brought to a sharp focus as a vertical line at the zero of a tangent scale. The prism is then introduced, close to the cyl., with its edge towards the zero and its base-apex plane parallel to the horizontal line; the line of light is then deviated to some number on the scale, which indicates the value of the prism. This method is suggested by Dr. Maddox.

Oblique Prisms.—A prism so changes the direction of light that an object viewed through it appears in a different position from that which it really occupies. The deviation is in the base-apex plane and towards the edge of the prism.

If a cross bar be viewed through a prism held with base-apex plane horizontal, the vertical bar is displaced horizontally to an extent dependent on the strength of the prism, and there is no vertical displacement of the horizontal bar. If, now, the prism be rotated in a plane parallel to the card, so that the base-apex line is oblique to both bars, the horizontal deviation becomes less, and a vertical deviation is introduced (Fig. 130). If the rotation be continued, the horizontal deviation continues to decrease and the vertical to increase, until when the base-apex line is vertical all the deviation is vertical, and there is none in the horizontal plane. The maximum effect d of the prism *is always in the base-apex plane,* and when the latter is oblique, its effect can be divided into V, a vertical, and H, a horizontal component, which are equal when the base-apex line is at 45°.

Resultant Prism.—The power of a prism varies as cos r, the angle between the base-apex plane and a given Mer. (Chapter XXII.). An oblique prism has Hor. and Ver. components, so that if Hor. and Ver. prismatic effects are needed they can be obtained from a single oblique or *resultant* prism $\triangle$, as calculated from the following formulæ, which are simplified forms of those given in Chapter XXII.

$$\triangle=\sqrt{H^2+V^2} \qquad \text{and} \qquad \tan r=V/H$$

Where H is the horizontal and V is the vertical component, and r is the angular distance of the base-apex plane of the resultant prism from the horizontal.

Thus, let a $3°d$ base-apex line Hor., and $2°d$ base-apex line Ver. be required,

then $$\triangle=\sqrt{3^2+2^2}=\sqrt{13}=3{\cdot}6°d=3°\,36'$$

$$\tan r=V/H=2/3={\cdot}666=\tan 33°\,40', \text{ so that } r=33°\,40'$$

The base-apex line is 33° 40′ from the Hor., or 56° 20′ from the Ver.; that is, approximately, $3{\cdot}5°d$ base-apex line at 35°.

If the original Hor. prism were base *in*, and the Ver. prism base *down*, Δ would have its base inwards and downwards; that is, for a right eye, the apex would be at 145° (standard notation). If the prism power were divided into a pair, then each would be of half the value of $\triangle$, and that for the left eye would be at the same inclination with its base inwards and upwards, that is (in standard notation), with the base at 145°.

Fuller discussions on this and other points in connection with prisms will be found in Chapter XXII., and in "Visual Optics and Sight-Testing."

Construction.—For the construction of the resultant of two prisms at right angles, draw a vertical line V (Fig. 134) as many inches (or cm.) long as there are units (degrees, etc.) in the Ver. prism. and a horizontal line H as many inches (or cm.) long as there are units in the horizontal prism. Their

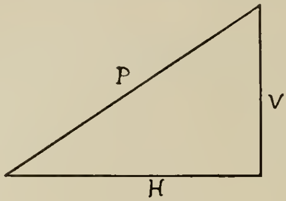

Fig. 134.

extremities are connected by a third line, P, whose length represents the number of units in the resultant prism; the angular distance of P from H or V, as measured by a protractor, is that of the base-apex line of the resultant prism from the Hor. or Ver. respectively.

Practical Method.—Holding the Hor. and Ver. prisms together, find on a tangent scale the maximum deviation; this is the value of the resultant prism, and its direction with respect to the originals is marked with a grease pencil. Or the two original prisms can be put into a trial frame and neutralised by a single prism from the trial case; the power of the neutraliser is that of the resultant prism, and its inclination is indicated.

Rotary or Variable Prism.—A rotary prism consists of two Ver. prisms of equal power conveniently mounted. In the primary position the base of the one coincides with the edge of the other, so that the effect is 0. From this position they are rotated towards the Hor., so that their bases approach each other; thus a gradually increasing Hor. effect is obtained while the Ver. remains 0. The maximum effect is obtained when the two bases coincide in the Hor. Mer. If the primary position is Hor. a varying Ver. effect is obtained while the Hor. remains zero.

Variable prismatic effect may also be obtained by sliding, one over the other, two sph. lenses of equal and opposite power, that is, by their mutual decentrations in opposite directions.

CHAPTER XII

DECENTRATION

The Optical Centre of a sph. lens is situated on the principal axis, and is the point of no prismatic effect. It is, therefore, in the thickest part of a convex, and the thinnest part of a concave lens.

The Geometrical Centre is that point which is equi-distant from the opposite edges. It can be located by inspection, or, more exactly, by drawing a line connecting the two extremities of the long diameter, and another connecting the highest and lowest points; where they cut each other is the geometrical centre.

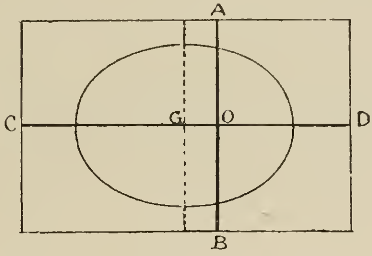

Fig. 135.

To locate the O. C., the lens must be moved about until cross lines seen through it are continuous with the parts of the lines seen beyond the edges, as in Fig. 135, where G is the geometrical centre of the lens. The optical centre O coincides with that point of the lens opposite to the intersection of the cross lines. It can, if necessary, be marked by a dot with a grease pencil or with ink. For strong lenses the test should be made with fine cross lines on a small card placed on the table, the lens being held steadily a short distance above the card, and in a plane parallel to it. Accuracy is enhanced by employing a pinhole, through which the observation is made. The analysing card at a reasonable distance is better for weak lenses.

The same procedure is employed for a sph.-cyl., the principal meridians *being parallel to the lines of the card.* With a plano-cyl., the central point of the axis may be regarded as the O. C.

Centered and Decentered Lenses.—A lens is centered when its optical and geometrical centres coincide, and is decentered when they do not. When an object is viewed through the geometrical centre of a decentered lens the effect is precisely the same as if the lens were combined with a prism. Similarly, if a centered lens is looked through at a point which is not in line with the optical and geometrical centres the effect is the same as if a sphero-prism were substituted.

To learn whether a sph. lens is truly centered the analysing card is viewed through its geometrical centre. If centered (Fig. 136) the junction of the cross lines is seen in line with the exact centre of the lens. If decentered the

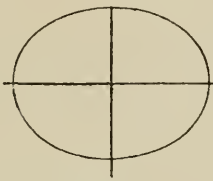

Fig. 136.

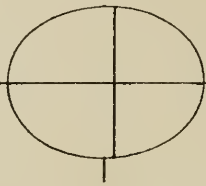

Fig. 137.

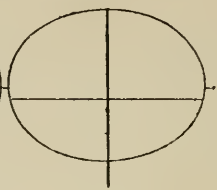

Fig. 138.

junction does not coincide with the centre of the lens and the Ver. line, as in Fig. 137, or the Hor. line as in Fig. 138, is broken at the edges of the lens, or both are broken.

In a sph. lens there is only one *point*, i.e. the optical centre on the principal axis of the lens where there is no prismatic effect.

In a plano-cyl. there is a *line* without prismatic effect along the axis.

In a sph.-cyl. lens there is a *point* of no prismatic effect where the axis of the cyl. cuts that of the sph., and it is therefore at the geometrical centre of a centered lens.

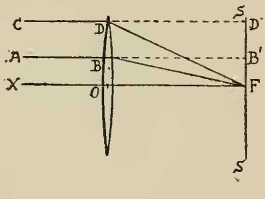

Fig. 139.

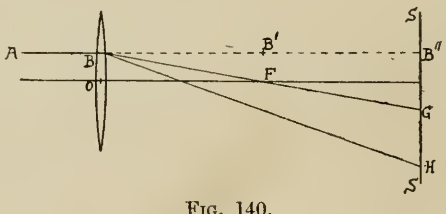

Fig. 140.

The Prismatic Action of Sphs.—A sph. lens, at any point other than the O. C., acts as a prism, *the prismatic power being greater as the part of the lens, through which the ray passes, is more distant from the axis.*

Let S S (Fig. 139) be a screen situated in the focal plane of a $+1$ D lens; the distance O F is therefore 1 M. The ray A B incident at B situated, say, 1 cm. from the axis, instead of falling on the screen at B', as it would if it were unrefracted, is refracted to meet the axis at F. It is, therefore, deviated the distance B' F=B O=1 cm. at 1 M. The ray C D incident, say, 2 cm. from the axis, is deviated the distance D' F=D O=2 cm. at 1 M since it also meets the axis at F. The prismatic effect of the lens at B or at D is, therefore, the same as that of a prism of 1^{Δ} or 2^{Δ} since such prisms cause similar deviations.

In Fig. 140 the lens is +2 D and S S is 1 M from it; a ray $A B$, parallel to the axis, and incident at B, 1 cm. from the axis, meets the latter at F, 50 cm. from the lens, and the screen at G instead of at B''. The ray is deviated a distance B' F=1 cm. at 50 cm., and B'' G=2 cm. at 1 M, so that the prismatic effect is the same as that of a 2^{Δ} acting on a ray unrefracted by the lens. If the lens were +4 D, $A B$ would meet the axis at 25 cm. from the lens and would be there deviated 1 cm. The +2 D at 2 cm. from the axis has the effect of 4^{Δ}, while the 4 D at the same point has the effect of 8^{Δ}.

The prismatic power Δ in prism diopters, of a sph. lens at a point C whose distance from the axis is expressed in cm., is

$$\Delta = \text{D} \ \ \text{C}.$$

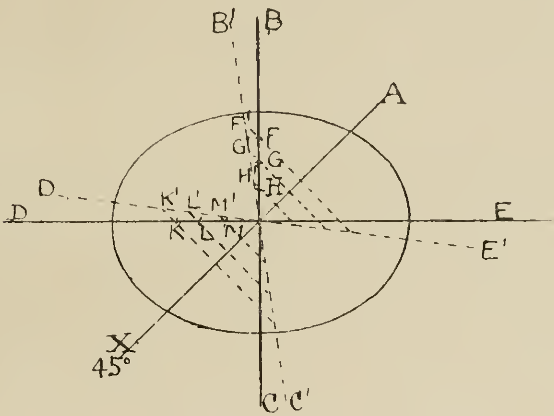

Fig. 141.

The Prismatic Action of Cyls.—In Fig. 141 let the lens be a + cyl., whose axis A X is at 45°, B C is a Ver., and D E a Hor. line. On looking through the lens the points F G H on the vertical line B C are seen deflected by the prismatic action of the cyl. to $F'G'H'$, upwards and to the left, the virtual prisms being base down and to the right. The points K L M on the horizontal line D E are seen deflected to $K'L'M'$, also upwards and to the left, the virtual prisms being base down and to the right. On the other side of the axis the virtual prisms are base up and to the left, and the deflections are downwards and to the right. Thus a convex cylindrical axis, say, 45°, causes a vertical line B C to appear as $B'C'$, and a horizontal line D E to appear as $D'E'$, both being deviated away from the axis.

If another equal + cyl. be placed axis at right angles to the first, the horizontal deviation of the vertical line, and the vertical deviation of the horizontal, are neutralised, but the vertical effect in the vertical meridian and the horizontal effect in the horizontal are doubled, the combination being equivalent to a sph. lens in which the prismatic effects are equal in every meridian.

With a concave cyl. the edges of the virtual prisms are towards the axis, and if a − cyl. Ax. 45° be looked through (Fig. 142) a vertical line *B C* appears

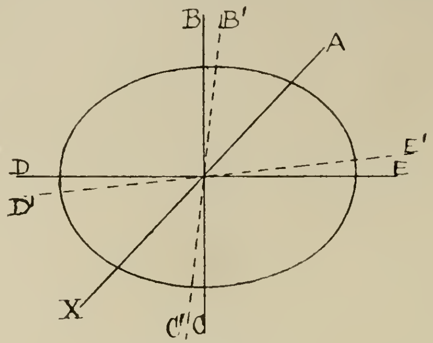

Fig. 142.

as *B′ C′*, and a horizontal line *D E* appears as *D′ E′*, the deviation of these lines being towards the axis of the lens, or towards the apices of the virtual prisms.

In Fig. 143 let the lens be a + sph.-cyl., whose axis *A X* is at 45°. Let *B* be a point situated between the Ver. and the axis. There is, at this point, the effect *O B* of a prism base down to the left derived from the sph. The cyl. contributes a prismatic effect *P B*, the base of the virtual prism being down to the right. Thus there are two Ver. effects both directed upwards,

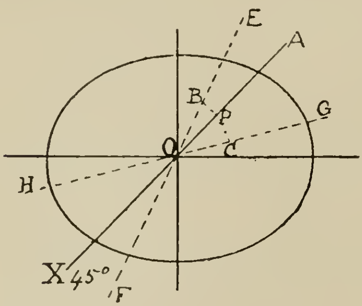

Fig. 143.

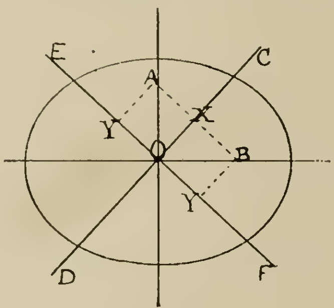

Fig. 144.

and two Hor., the one directed to the left and the other to the right. These latter neutralise each other at some point *B*, and similarly at every point on the line *E F*.

Between the axis and the Hor., at some point *C*, there is the effect *O C* of a prism base down and to the left derived from the sph., and from the cyl.

there is the effect *P C* of a prism base up and to the left. There are thus two Hor. effects both directed to the left, and two Ver., the one up and the other down. At some point *C* the opposing Ver. effects neutralise each other, and similarly we have a neutralising effect all along the line *G H*.

In a Cc. sph.-cyl. there are similar prismatic effects, but in the opposite directions.

Let Fig. 144 be a combination of +Cyl. Ax. 45° ⊃ – Cyl. Ax. 135°, the two being of equal power. *C D* is the axis of the Cx. cyl., and *E F* that of the Cc. At some point *A* the Cx. cyl. has an effect *X A* of a prism base down and to the right, the Cc. has an effect *YA* of one base up and to the right. The up and down Ver. effects neutralise each other, but there is a combined lateral effect. At the point *B* the Cx. acts with an effect *X B* base up and to the left, and the Cc. with an effect *Y B* base up and to the right. The right and left Hor. effects neutralising each other, the combined deviation being Ver. Thus the point *A* would be deviated to the left, and *B* downwards. A Ver. line is seen inclined to the left above, and to the right below; a Hor. line is inclined downwards on the right, and upwards on the left.

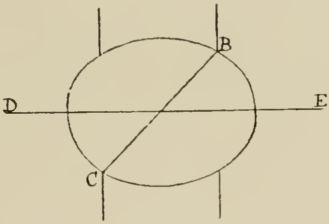

Fig. 145.

Locating the Lines of No Prism Effect.—If an oblique sph.-cyl. be moved horizontally until the oblique image of a vertical line is seen in contact at *B* (Fig. 145), at the upper edge of the lens, with the line itself seen above the lens, and similar contact is then obtained at the lower edge of the lens, say at *C*, the line connecting these two contact points indicates the line of no

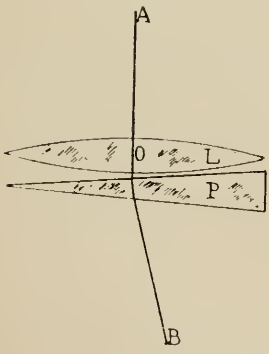

Fig. 146.

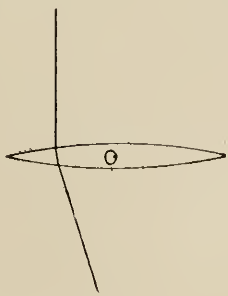

Fig. 147.

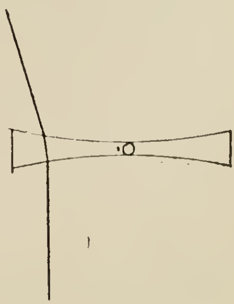

Fig. 148.

Hor. prismatic effect. Similarly the points can be found where, by moving the lens vertically upwards and downwards, a horizontal bar is, at each side, in contact with its image; the line connecting them indicates the line of no Ver. prismatic effect.

Prismatic Action by Decentration.—A ray *A B* (Fig. 146) passing through the O. C. of a Cx. lens *L* and a prism *P*, *base to the right*, is undeviated by the lens and is bent towards the right by the prism. If that ray passed through the *left* portion of a Cx. lens (Fig. 147) or the *right* portion of a Cc. (Fig. 148), it is similarly deviated to the right. Therefore, if the action of a prism, base in a certain direction, is required it can be obtained by utilising the opposite portion of a Cx. and the corresponding portion of a Cc. lens.

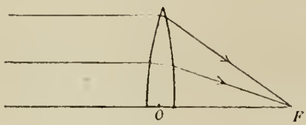

FIG. 149.

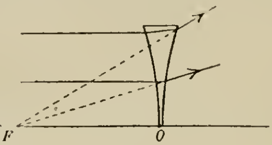

FIG. 150.

If half only of a lens be utilised (Figs. 149 and 150), the image of a point on the axis has its image opposite to one extremity of the lens instead of being opposite to the geometrical centre, as with an ordinary centered lens.

FIG. 151.

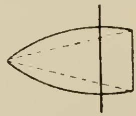

FIG. 152.

There is no difference whatever between combining a prism with a sph., grinding sph. surfaces on to a prism, or at such an angle to each other that a prismatic as well as sph. form results, or by cutting out a lens so that its O. C. is displaced from the centre of the lens, that is, by cutting away part of the lens as in Fig. 151. In all cases the result is as if two plano-Cx. lenses had been cemented to the faces of a prism (Fig. 152).

Formulæ for Decentering.—The prismatic effect obtained by decentration is directly proportional to the amount of decentration and to the strength of the lens, so that the decentration necessary to obtain a desired prismatic effect is directly proportional to the effect required, and inversely proportional to the power of the lens.

Let Δ represent the prismatic effect needed in Δ, D the dioptral number of the lens, and C the decentration *in centimetres*, then

$$\Delta = D\ C$$

This simple relationship between lenses and prisms was first shown by Prentice of New York.

By introducing the necessary constant K, the formula for prism diopters apply also for degrees, whose constant is ·9, and to degrees of deviation, the constant being 1·75, so that

$$\Delta = D\,C/K$$

The constant ·9 for degrees is on the presumption that the prisms used have $\mu=1{\cdot}52$; if $\mu=1{\cdot}5$, then K=·87 and for $\mu=1{\cdot}54$, K=·94.

Direction of Decentration.—To produce the effect of a prism with its base in a certain direction, a Cx. lens must be decentered in *that same direction*, and a Cc. in the *opposite direction.*

How to Decentre.—Decentering is achieved by so cutting the unedged glass disc that the optical centre is nearer than the geometrical centre to one part of the edge of the finished lens. The optical centre *O* (Fig. 153) is located as previously described, and marked by a dot, the required amount of decentration is measured off, and the point *G*, which is to be the geometrical centre of the edged lens, is also marked.

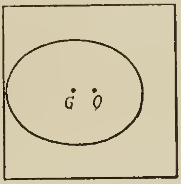

FIG. 153.

The lens is then cut out so that *G* is in the centre, as shown in Fig. 153.

The direction of decentration is indicated *by the position of O, the fixed point of a lens, with reference to G.* The distance *O G* is marked off *contrary to the decentration required.* If *O* has to be *in*, the distance measured off, *in order to mark where G is to be*, is *out* from *O*.

To Measure Decentration.—The geometrical and optical centres are marked, and the distance between them is measured by placing the lens on a metric rule.

Examples.—On a + 4·5 D lens, the effect of 2^{Δ} base *down* is required, the lens must be decentered

C=2/4·5=·444 cm. *down.*

For the effect of ·5°*d* base *in* on a + 6 D

C=·5 × 1·75/6=·15 *in.*

For the effect of 2°*d* base *in* on a − 8 D lens

C=2 × 1·75/8=·44 cm. *out.*

If on a − 5 D lens the effect of 3·5° base *down* is required,

C=3·5 × ·9/5=·63 cm. *up.*

If a +4 D is decentered out ·75 cm., the prismatic effect will then be

$$\Delta = \cdot75 \times 4 = 3^{\Delta} \quad \text{or} \quad \cdot75 \times 4/1\cdot75 = 1\cdot75^{\circ}d, \quad \text{or} \quad \cdot75 \times 4/\cdot9 = 3\cdot3^{\circ}$$

If the O. C. of the lens is ·75 cm. from the geometrical centre, and the prismatic effect, as measured on a tangent scale, is 3^{Δ}, the lens is

$$D = 3/\cdot75 = 4 \text{ D.}$$

In using these formulæ, fractions of degrees should be expressed as decimals, and not as minutes and seconds, and the decentration in cm. and decimals thereof. These rules, while sufficiently accurate for practical spectacle work, especially as no lens can be decentered to a very great extent, are not exact, since the variation of angles has been taken as equivalent to that of their tangents.

Limitations to Decentering.—The smaller the size of the lens required and the larger the disc from which it is cut, the greater is the extent of decentration possible. If the edged lens were to be about as large as the unedged disc in any direction no decentering would be possible. Since the usual finished lens is longer in the horizontal than in the vertical diameter, a greater vertical than horizontal decentration is possible. Thus a lens of No. 1 eye size, when edged, can be decentered about 4 mm. horizontally and 7 mm. vertically; for smaller lenses the extents are greater, while for larger ones they are smaller. The average size of the uncut disc is 40 to 44 mm. diameter.

Another method of decentering is to place in contact with the lens a prism, equal to the effect required, with its base in the opposite direction; mark the O. C. on the lens as then found, and this point is the geometrical centre needed. By this method one can at once see whether the required effect can be obtained by decentering, it being impossible if the marked point be too near the periphery, much less if, as may be the case, it is beyond the lens. It is specially suitable for oblique decentrations. Thus suppose 1^{Δ} *base in* effect is needed on a +4 D lens; a 1^{Δ} *base out* is placed with the lens, the O. C. is then displaced outwards, the lens must be shifted ·25 cm. inwards to find the point of no prism effect, and this indicates the geometrical centre of the finished lens.

Formulæ Involving F.—Where F or 1/F is given, it is easier to convert into diopters, but the calculations can be made by the following, where *both* F and the decentration are expressed either in inches or cm., K being the constant—

$$\text{Decentration} = {}^{\Delta}FK/100$$

Thus, how much should a 4 in. lens be decentered for 1° ?

$$\text{Decentration} = 1 \times 4 \times \cdot9/100 = \cdot036 \text{ in.}$$

More Exact Formulæ.—More accurate formulæ for decentration for $°d$ are as follows, where F and D have the usual significations, and d is the degree of deviation :

$$\tan d = C/F \quad \text{or} \quad C = F \tan d;$$

and

$$\tan d = C\,D/100 \quad \text{or} \quad C = 100 \tan d/D.$$

These formulæ are illustrated in Fig. 139, where $D'\ F$ equals the tangent of the angle of deviation of the ray $C\ D\ F$.

Resultant Decentrations.—As the value of a prism is at its maximum in the base-apex plane, and has a value of $\Delta \cos r$ in any other Mer., a lens if decentered has its maximum prismatic effect in the plane of decentration, and in any other Mer. $\Delta = DC \cos r$. An oblique decentration has always Hor. and Ver. effects and if a lens has to be decentered for both Ver. and Hor. prismatic effects, the two can be obtained by a single oblique decentration.

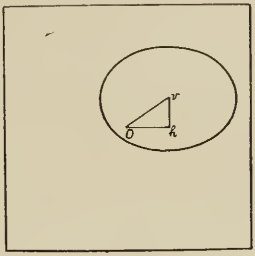

Fig. 154.

Thus in Fig. 154 let O be the optical centre. If the geometrical centre is moved horizontally from O to h and vertically to v, the actual displacement is along $O\ v$. A resultant decentration is calculated by finding the oblique prismatic effect required and its angle r, and then decentering accordingly.

Thus suppose a $+5$ D lens has to be decentered for a Hor. effect of 2^{Δ}, and a Ver. effect of $1{\cdot}5^{\Delta}$; then

$$\Delta = \sqrt{2^2 + 1{\cdot}5^2} = 2{\cdot}5^{\Delta} \quad \text{and} \quad \tan r = 1{\cdot}5/2 = {\cdot}75 = \tan 37° \text{ (approx.)}$$

$$C = 2{\cdot}5/5 = {\cdot}5 \text{ cm.}$$

The two needed prismatic effects are obtained by decentering the lens ·5 cm. along meridian 37°.

The Ver. and Hor. decentrations V and H could be found separately and a resultant decentration then calculated, but the above method is simpler. Thus, in the above example,

$$H = 2/5 = {\cdot}4 \text{ cm.} \quad \text{and} \quad V = 1{\cdot}5/5 = {\cdot}3 \text{ cm.}$$

and

$$\Delta = \sqrt{{\cdot}4^2 + {\cdot}3^2} = \sqrt{{\cdot}25} = {\cdot}5 \text{ cm.}$$

$$\tan r = V/H = {\cdot}3/{\cdot}4 = {\cdot}75 = \tan 37°.$$

The Decentration of Cyls. and Sph.-Cyls.—Those cyls. and sph.-cyls. whose principal Mers. are Ver. and Hor. are *upright*, in contradistinction to those which are *oblique*.

A lens possessing a cyl. element should not be decentered except in its principal meridians, that is to say, upright cyls. ought not to be decentered obliquely, nor should oblique cyls. be decentered horizontally or vertically. Such decentrations can be made, but the results are difficult to calculate owing to the fact that the prismatic elements in a cyl. have their base-apex lines at right angles to the axis and therefore, also, it is impossible to obtain a Hor. or Ver. effect alone.

Upright Cyls.—The effect of decentering a cyl. *across* its axis is the same as decentering a sph. in that direction; *along* the axis there is no effect, since there is no refractive power. Thus a cyl. axis Ver. can be decentered horizontally, but not vertically; a cyl. axis Hor. can only be decentered vertically.

If +4 C. Ax. 90° requires decentration for the effect of 2^{Δ}, base *out*,

$$C=2/4=\cdot 5 \text{ cm. } out.$$

Upright Sph.-Cyls.—Decentering a sph.-cyl. across the axis of the cyl. has the same effect as decentering a sph. whose power is that of the two powers combined; while in the direction of the axis it is the same as decentering the sph. alone.

If -3 S. ⊂⊃ -2 C. Ax. 90° is to be decentered for 2^{Δ} base *in*, the power in the Hor. Mer. is $3+2=5$ D; therefore

$$C=2/5=\cdot 4 \text{ cm. } out.$$

If $+2$ S. ⊂⊃ -5 C. Ax. 180° needs to be decentered for 2^{Δ} base down, the power in the Ver. Mer. being -3 D,

$$C=2/3=\cdot 66 \text{ cm. } up.$$

Oblique Cyls. and Decentrations.—For oblique decentrations of upright cyls. and sph.-cyls., and the Hor. and Ver. decentrations of oblique cyls. and sph.-cyls., and the Hor. and Ver. effects of oblique decentrations see Chapter XXII.

CHAPTER XIII

EFFECTIVITY AND BACK FOCAL DISTANCE

Effect of Altered Position of a Cx. Lens.—The power of a given lens 1/F or D is a fixed quantity, but its effect, in relation to a given plane behind it, varies with its distance from that plane.

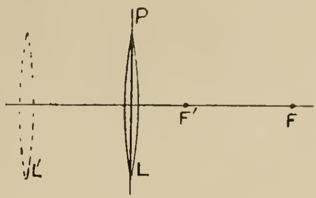

Fig. 155.

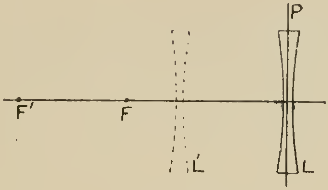

Fig. 156.

Thus, in Fig. 155 let a Cx. lens L of 10″ focal length be in the plane P, so that, if the source of light be distant, F lies 10″ behind P. If the lens be advanced say, 2″, to L' towards the light source, F is similarly advanced to F' and lies 8″ behind P. Then the effect of the advancement is the same as if the 10″ had been replaced by an 8″ lens in the plane P. Or the effect is the same as if $1/8-1/10=1/40$ Cx. lens had been added to the 1/10. The expression of effectivity for a Cx. lens is

$$\frac{1}{F_v}=\frac{1}{F-d} \quad \text{or} \quad F_v=F-d$$

where d is the distance between the lens and the plane of reference.

The increase of effect is due to the fact that the light in the plane of P is more convergent when L is in advance of it.

If $d=F$ then $1/F-d=1/0=\infty$; the converging effect is infinite in a plane when the lens is in advance of it a distance equal to its focal length. If d exceeds F, then $1/F_v$ becomes negative, the light diverging to the plane after coming to a focus in F.

The effect of a Cx. lens on P, when the light is parallel, increases, as d increases, from 1/F to ∞ and then becomes negative.

Effect of Altered Position of a Cc. Lens.—The effect produced by similarly moving a Cc. lens is opposite in character. Let P be the plane (Fig. 156), and L a 10 in. Cc. lens placed in it. F will then be 10″ in front of P if the source is distant. If L be moved forward to L' a distance $d=2″$, for example, then

F is similarly moved to F', so that the effect in P is the same as if a lens of 12″ F had been substituted for the 10″; or the effect is the same as if a $-1/10-(-1/12)=+1/60$ had been added to the $-1/10$. The expression of effectivity for a Cc. lens is the same as for a Cx., but the focal distance, being negative, requires the use of the $-$ sign before the value of F in substitution.

The decrease of effect is due to the fact that the light in the plane of P is less divergent when L is in advance of it. The effect of the lens in P when the light is parallel decreases, as d increases, from 1/F to 0, the latter obtaining when d is infinite.

For distant objects, therefore, a Cx. lens, on movement away from a plane, can never have *less* +, and a Cc. can never have *more* $-$ effect than its own power.

Change of Effect.—The altered effect of a lens when moved from one position to another in front of a plane, or another lens, is the difference between its effectivity in its original, and in its new, position; thus, a 5 in. Cx. lens moved from 1 in. to 2 in. away from a plane, the change is

$$\frac{1}{5-2}-\frac{1}{5-1}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

or an increase of effect equal to that of an added 1/12 Cx.

A 5 in. Cc. similarly moved causes a decrease of effect just as if a 1/42 convex had been added to the concave, as shown by

$$\frac{1}{-5-2}-\frac{1}{-5-1}=-\frac{1}{7}-\left(-\frac{1}{6}\right)=+\frac{1}{42}$$

Cx. Lens and Divergent Light.—Let a 10″ Cx. lens be in a plane P and a source of light be at some finite distance, say, 40″ in front of it. Then f_2 is at (approx.) 13″ behind P. If now the lens be advanced a distance d towards the source f_2 tends to be advanced towards P, but since L approaches the source, f_1 becomes shorter and in consequence f_2 becomes longer and tends to be further behind P. Of these two counteracting effects the one or other predominates obviously according as d is greater or smaller than the recession of f_2 due to the shortening of f_1.

If f_1 the anterior conjugate is shortened by an amount d, f_2 the posterior conjugate is lengthened *less* than d if f_1 exceeds 2F, and *more* than d if f_1 is less than 2F. Therefore we have the rule that:—*Removal of a Cx. lens towards the source of light causes increased effectivity so long as the distance between the Cx. lens and the object f_1 is not less than* 2F. At this distance the lens has for the *given position of the object* with respect to P, the highest possible effectivity, *which is reduced by any further withdrawal of the lens outwards.*

When the Cx. lens is nearer to the object than F, the light after refraction is no longer convergent, but divergent, and when *the lens is in contact with the object the effectivity of the Cx. lens is zero,* because, in this case the divergence of the light from the object is infinite in comparison with the power of the lens.

Cc. Lens and Divergent Light.—Let a 10″ Cc. lens be in a plane *P*, and the sources be near, say, 40″. Then f_2 is at 8″ in front of *P*. When the lens is advanced a distance *d*, f_2 tends to be also advanced, but the increased divergence of the light, owing to the shortening of f_1, tends to cause f_2 to be also shortened. Of these two opposing effects the former always predominates, that is to say, the effectivity is lessened, and becomes *zero when the Cc. lens is in contact with the object.* Again, in the last case, the divergence of light from the object is infinite compared with the power of the lens.

Summary.—When light is parallel, advance of a Cx. lens always results in increased effectivity, and advance of a Cc. in decreased effectivity, with respect to the original plane.

If incident light be divergent an increased effect may be obtained by increasing the distance between a Cx. lens and a plane behind it, but the increase for a given movement is less than if the light were parallel; there will be a decreased effect if f_1 is less than 2 F. With a Cc. lens the resultant effect is always decreased, but the change is smaller as the distance between the object and lens is less.

When a lens, whether Cx. or Cc., is in a given plane its effect there, when the light is parallel, is 1/F, and when the light is divergent it is $1/f_2$. When the lens is in contact with the object the effect is, as before stated, zero.

Dioptral Expression for Effectivity.—If the power of the lens be expressed in diopters, and the source is distant, its effective power D_v in a new position becomes

$$D_v = \frac{100}{F - d} \quad \text{or} \quad \frac{1000}{F - d}$$

F and *d* being expressed in cm. or mm. respectively.

Thus, suppose a + 8 D lens is moved from a given plane to a position 10 mm. further forward, towards the source of light, then F is 1000/8 = 125 mm.

$$D_v = 1000/(125 - 10) = 8{\cdot}7$$

The effect is increased 8·7 − 8 = ·7D.

If the lens were − 8 D, similarly moved

$$D_v = 1000/(-125 - 10) = -7{\cdot}4$$

The effect is decreased ·6D.

If a + 10 D lens be moved from 15 to 20 mm. in front of a given plane, since F = 100 mm., we have

$$\text{at 15 mm.} \quad D_v = 1000/(100 - 15) = 11{\cdot}77$$
$$\text{at 20 mm.} \quad D_v = 1000/(100 - 20) = 12{\cdot}5$$

so that the effect is increased by 12·5 − 11·77 = ·73 D.

Similarly, moving the lens back from 20 to 15 mm. decreases the effectivity to a like extent.

Effectivity of Two Thin Lenses.—The combined power of two thin lenses, placed together, is equal to the sum of their individual powers, thus

$$\frac{1}{F}=\frac{1}{F_1}+\frac{1}{F_2}=\frac{F_1+F_2}{F_1F_2} \quad \text{or} \quad F=\frac{F_1F_2}{F_1+F_2}$$

But if they are separated by an interval d the resultant effect is not the same as if they were in contact. The distance of F behind the back lens, that is, *the back surface or effective focal distance* F_B, is different because the effectivity of the front lens has now become $1/(F_1-d)$ in the plane of the second lens F_2. Therefore

$$\frac{1}{F_B}=\frac{1}{F_1-d}+\frac{1}{F_2} \quad \text{or} \quad F_B=\frac{(F_1-d)\,F_2}{F_1+F_2-d}$$

where F_1 pertains to the front, F_2 to the back lens, and d is the distance between them.

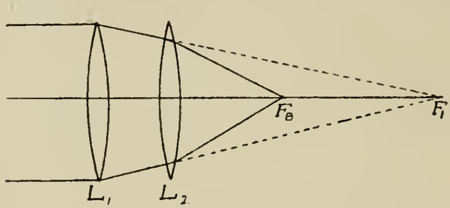

FIG. 157.

Effective F of Two Cx. Lenses.—In Fig. 157 let L_1 and L_2 be two thin Cx. lenses of 10″ and 7″ focal length respectively, separated by 2″, then

$$F_B=\frac{(F_1-d)F_2}{F_1+F_2-d}=\frac{(10-2)\times 7}{10+7-2}=\frac{56}{15}=3\tfrac{11}{15}\text{ in.}$$

The distance of F behind the back lens is shortened. Parallel light incident on L_1 is converged towards F_1, 10 in. behind it, but on its way it meets, at 2 in. from L_1, the 7 in. Cx. lens L_2, and converges towards a point $10-2=8$ in. behind the latter. The effectivity of L_1 in the plane of L_2 is 1/8, or the effect is the same as if an 8 in. lens were in contact with L_2, and the common focus F_B is at $3\frac{11}{15}$ in. instead of $4\frac{2}{17}$ in., where it would be if L_1 were touching L_2. *The separation of Cx. lenses increases the effectivity* of the combination in the plane of L_2 or any plane behind it.

The distance of F_B differs considerably when the two lenses are of different powers, according as the one or the other faces the light. Thus, if the combination were reversed so that the 7 in. Cx. faced the light, and the 10 in. Cx. were 2 in. behind it, $F_B=3\frac{1}{3}$ in. instead of $3\frac{11}{15}$ in. F_B is equally distant from the second (back) lens only when the lenses are equal and of the same nature.

When d=F_1.—If a Cx. lens L_1 (Fig. 158) is placed at its principal focal distance in front of another Cx. lens L_2 the latter has no effect whatever, its power being zero compared with the infinite effectivity of L_1. $F_B=0$ or $1/F_B=\infty$.

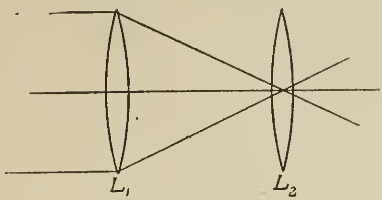

Fig. 158.

When d exceeds F_1 then F_B is negative.

When d=F_1+F_2 of two Cx. lenses the system is *afocal—i.e.* parallel light emerges parallel. Such lenses represent the *Erecting Eye-piece* when the two lenses are *equal* (Fig. 159). They represent the *Astronomical Telescope* when they are *unequal.*

When d exceeds F_1+F_2 the principle of the *Microscope* obtains.

When d=F_1-F_2 the *Huyghen Eye-piece* is represented, and $F_B=F_2/2$.

When $F_1=F_2$ and d=F_1 we have the *Kellner Eye-piece.*

When d=$2F_1/3$ we have the *Ramsden Eye-piece.*

The instruments and eye-pieces above mentioned are treated more fully in Chapter XXVII.

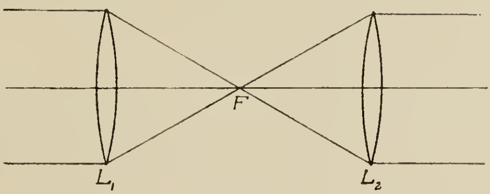

Fig. 159.

Effective F of Two Cc. Lenses.—When two Cc. lenses are separated the focal length becomes lengthened. Thus let L_1 and L_2 be two thin Cc. lenses of, say, 10 in. and 7 in. F respectively. If L_1 is 2 in. in advance of L_2

$$F_B=\frac{(-10-2)\times-7}{-10-7-2}=\frac{+84}{-19}=-4\tfrac{8}{19}\text{ in.}$$

The distance of F as measured from L_2 is longer, owing to the interval between the lenses. Parallel light incident on L_1 is rendered divergent as if proceeding from $F_1=10''$ in front of L_1, and therefore $12''$ in front of L_2. The divergence of L_1 in the plane of L_2 is then 1/12 instead of 1/10, so that the combined focus F_B is at $4\frac{8}{19}''$ instead of at $4\frac{2}{17}$ where it would be if the lenses were close together. *The separation of Cc. lenses lessens the effectivity* in the plane of L_2 or in any plane behind it.

As with Cx. lenses, the distance of F_B from the back lens of a combination of two unequal Cc. lenses, separated by an interval, varies as the one or the other lens faces the light. Thus, if the 7 in. Cc. were 2 in. in front of the 10 in., $F_B=4\frac{14}{19}$ in. from L_2.

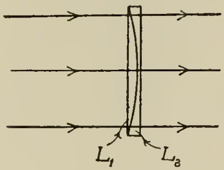

Fig. 160.

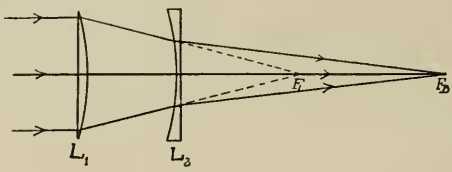

Fig. 161.

Effective F of Cx. and Cc. with Cx. in Front.—In Fig. 160 let L_1 be a 10 in Cx. and L_2 a 10 in. Cc.; in contact with each other they neutralise whichever lens is to the front. If they be separated (Fig. 161) the increased effect of L_1 in the plane of L_2 is such that the combination becomes positive, illustrating the principle of the *Unofocal* photographic objective.

Thus, if $d=4''$

$$F_B=\frac{(10-4)\times -10}{10-10-4}=\frac{-60}{-4}=15 \text{ in.}$$

Parallel light incident on the Cx. is converged to $F_1=10''$ behind it. It then meets the Cc. 4″ further back, so that F_1 is 6″ behind the latter, and the effect of the Cx. in the plane of the Cc. is $1/(10-4)=1/6$; F_B is then lengthened by the Cc. to 15″ as calculated.

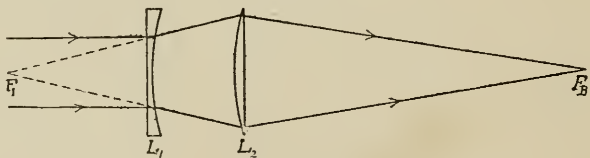

Fig. 162.

If, however, the combination shown in Fig. 161 be reversed (Fig. 162) so that the light is incident first on the Cc. it is rendered divergent as from $10+4=14''$ from the Cx., and

$$1/F_B=+1/10-1/14=1/35;$$

thus F_B is at 35″, or 20″ further from L_2 when the Cc. faces the light than when the Cx. does so. The difference in F_B on the one and other side of a combination is much more pronounced when the lenses are of opposite nature than when they are both Cx. or both Cc. Indeed, with the former F_B may be positive when the light meets the Cc. first and negative when the Cx. is in front, as when d exceeds F of the Cx.

Effective F when the Cx. is the Stronger.—If a Cc. be in contact with a stronger Cx. the combined effect is positive whichever lens faces the light. With the Cx. to the front separation results in still greater positive effect, F_B being shorter than when the lenses are in contact.

If $d = F_1$ the Cc. lens has no influence, so that $F_B = 0$. If d exceeds F_1, the light refracted by the Cx. is brought to a focus, whence it diverges to the Cc., which increases the divergence, so that F_B is negative.

With the weaker Cc. to the front the positive combined effect increases with greater separation, but the minimum value of F_B is F of the Cx., the two being equal when $d = \infty$. Thus, with the Cc. forward, F_B must always be positive.

Effective F when the Cc. is the Stronger.—If the Cc. lens has a shorter focal length than the Cx., and the two are in contact, there is an excess of negative power. If the Cx. be moved outwards it gains in effectivity, but the total effect is still negative until, *when the separation is equal to the sum of their focal lengths*, F_B is infinite. Further separation will give the two lenses a positive effectivity, which is shorter as the separation is increased until when $d = F$ of the Cx., $F_B = 0$. Still further increased separation results in a negative effect.

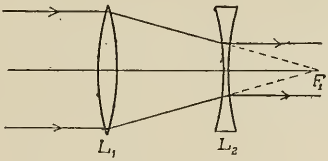

FIG. 163.

Thus, in order that a Cx. and stronger Cc. may neutralise each other, and parallel light emerge parallel from the second lens (Fig. 163) $d = F_1 + F_2$, the principal focus of the Cx. being behind the Cc. as far as that of the latter is in front of it.

Thus, if $F_1 = +6$ and $F_2 = -4$ the separation must be $6 - 4 = 2$ in. Here by calculation

$$F_B = \frac{(6-2) \times 4}{6-4-2} = \frac{16}{0} = \infty$$

whether the one or other lens is to the front. In such a condition the lenses are commonly said to be separated by the algebraical sum of their focal lengths, so as to include the case of two Cx. lenses, already mentioned, from which parallel light emerges also parallel.

Thus, the Cc. being the stronger, and with the combination either way round, there is

Negative effect when $d < F_1 + F_2$.

Afocal effect (*i.e.* neutralisation) when $d=F_1+F_2$. We have here the principle of the *Galilean Telescope* or *Opera Glass.*

Positive effect when $d>F_1+F_2$, but less than the focal length of the Cx. With the latter in front we have here the principle of the *Telephoto* lens.

With the Cx. forward and $d>F_1$ the effect is *negative,* but it is *positive* for *any* separation when the Cc. first receives the light.

Dioptral Formulæ for Effectivity.—The formula for finding the effective dioptral lens D_B of two separated lenses D_1 and D_2 is

$$D_B=D_1+D_2+\frac{D_1^2 d}{100-D_1 d},\ d \text{ being in cm.}$$

To find d for a given F_B.—This can be obtained from the formula

$$d=\frac{F_1F_2-F_B(F_1+F_2)}{F_2-F_B}$$

or it is better calculated from the original formula. Thus, let parallel light fall on a 5″ Cx. in combination with a 2″ Cc., and the effect required be that of a 20 Cx.; then

$$+20=\frac{(5-d)\times(-2)}{5-2-d}=\frac{-10+2d}{3-d}$$

$$60-20d=-10+2d. \qquad 22d=70 \text{ and } d=3\tfrac{2}{11}''$$

If the effect required with the same lenses is that of 20 Cc., then

$$-20=\frac{(5-d)\times(-2)}{5-2-d}=\frac{-10+2d}{3-d}$$

$$-60+20d=-10+2d \qquad 18d=50 \text{ and } d=2\tfrac{7}{9}''$$

The above examples give the distance between the two component lenses of an opera glass when the light emerging from the Cc. is required to have respectively a positive and a negative $F_B=20''$, or in other words, a convergence of 2 D in the first case, and a divergence of 2 D in the second.

It should be noticed that for emergent parallel light $d=F_1+F_2$, in this case $5-2=3''$, and in order that F_B *be positive d is more than* 3″, and for F_B *to be negative d is less than* 3″.

The distance *d* can be deduced from general principles. Thus, using the same lenses as in the foregoing example, suppose we need a resultant convergence of 4 D (to suit the vision of a hypermetrope of 4 D) then the effect of the Cx. in the plane of the Cc. must be +24 D, so that the back focal power =+4 D. Then *d* is the difference between F of the Cx. and F of that lens which would, in the plane of the Cc., have the desired effect. In this case the two lenses being +8D and −20D.

$$d=\frac{100}{8}-\frac{100}{24}=12{\cdot}5-4{\cdot}16=8{\cdot}34 \text{ cm. or } 5-40/24=3\tfrac{1}{3}''$$

For a resultant divergence of 4 D (to suit the vision of a myope of 4 D) the Cx. needs an effect of +16 D in the plane of the Cc., so that

$$d=\frac{100}{8}-\frac{100}{16}=12{\cdot}5-6{\cdot}25=6{\cdot}25 \text{ cm. or } 5-40/16=2\tfrac{1}{2}''$$

If a Cx. lens of F=10″ and another Cx. of 2″ be separated by 12″ they constitute a telescope. For the emergent light to be parallel $d=F_1+F_2=12''$; for the light to be divergent d is less than F_1+F_2; for convergent light d is more than F_1+F_2.

With these lenses for $F_B=+20''$, $d=12\frac{2}{9}''$, and for $F_B=-20$, $d=11\frac{9}{11}''$.

Effectivity when Light is Divergent.—When the incident light is divergent the conditions for neutralisation and certain effectivities with separated lenses differ from those obtaining with parallel light; the conjugate focus of the front lens of the combination must be found in order to determine its effective value in the plane of the second lens. For example, an object is 24 in. in front of an 8 in. Cx. lens behind which, at 2 in., a 5 in. Cx. is placed; where is the image? Now, the first conjugate f_2 is at

$$\frac{1}{f_2}=\frac{1}{8}-\frac{1}{24}=\frac{1}{12} \quad\text{and}\quad \frac{1}{12-2}+\frac{1}{5}=\frac{3}{10} \quad\text{so that } f_2'=3\tfrac{1}{3}\text{ in.}$$

An object is 40 in. in front of a 7 in. Cx., where should a 5 in. Cc. be placed so that the light may be rendered parallel? Now

$$1/f_2=1/7-1/40=33/280$$

the image f_2 is thus $8\frac{16}{33}$ in. behind the Cx., and the Cc. must be placed $8\frac{16}{33}-5=3\frac{16}{33}$ in. behind the Cx.

An object is 40 in. in front of a 7 in. Cx. and a 5 in. Cc. For the image to be 20 in. behind the back lens f_2 is $8\frac{16}{33}$ in. behind the Cx., which must have the effect of $1/5+1/20=1/4$ in the plane of the Cc. Therefore $d=8\frac{16}{33}-4=4\frac{16}{33}$.

Let a +3 D lens be 10 cm. in front of a screen; where must a +6 D be placed in front of the +3 D so that the image of an object 50 cm. from the front lens be on the screen?

The original divergence is 2 D; after refraction by the front lens there is a convergence of 6 − 2=4 D. The convergence needed in the plane of the second lens is 10 D, of which the second lens provides 3 D; therefore,

$$d=\frac{100}{4}-\frac{100}{7}=25-14=11 \text{ cm.}$$

It is important to differentiate clearly between *effectivity* and *equivalence,* dealt with in the next chapter.

CHAPTER XIV

EQUIVALENCE OF THIN LENSES

Equivalence.—Any two or more lenses, whether in contact or separated, can be replaced by a single *equivalent* lens having the same refracting power as the component lenses. Or, to put it in another way, since the size of image is proportional to focal length, any number of lenses can always be replaced *by that single thin lens giving the same magnification.*

If two thin lenses are placed in contact the resultant focal length is the same as that of a single lens situated in the same plane, whose power 1/F is that of the sum of the two components F_1 and F_2. The combined power and F may be written

$$\frac{1}{F}=\frac{1}{F_1}+\frac{1}{F_2} \quad \text{and} \quad F=\frac{F_1F_2}{F_1+F_2}$$

If the lenses are separated by a distance d, we have seen that the effective power and back surface focal distance are

$$\frac{1}{F_B}=\frac{1}{F_1-d}+\frac{1}{F_2}, \text{ and } F_B=\frac{(F_1-d)\,F_2}{F_1+F_2-d}$$

It now remains to find an expression for the equivalent focal length of two thin separated lenses.

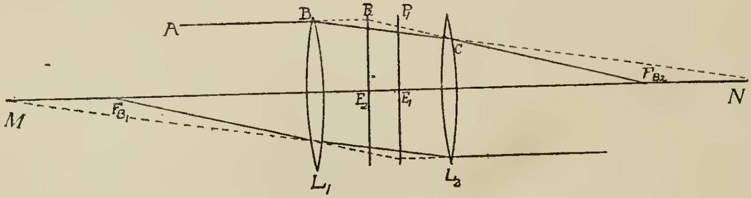

Fig. 164.

Equivalent Lens and Focal Length.—Let L_1 and L_2 (Fig. 164) be two thin Cx. lenses separated by a distance d, and let AB be a ray incident on L_1 parallel to the principal axis MN. This is deviated by L_1, and, were it not intercepted by L_2, would focus at N, but it is refracted still more at C to cross the principal axis in the posterior focus F_{B2}.

Now if the incident ray AB be produced, and the final refracted ray CF_{B2} prolonged backwards, the two will meet in the point P_2. Through P_2 drop the perpendicular P_2E_2. Then if a thin lens of focal length E_2F_{B2} be introduced into the plane P_2E_2 and the other lenses removed, this single lens would give precisely the same result as the combination L_1L_2. For this reason the plane P_2E_2 is called the *second equivalent plane* and the point E_2 the *second equivalent point.*

Similarly if parallel light be incident first on L_2, it will pass through the first back focus F_{B1} and P_1E_1 is located in the same way as P_2E_2. The plane P_1E_1 is called the *first equivalent plane,* and E_1 the *first equivalent point,* and it is here that the equivalent lens must be situated to replace the combination for light coming from the side of L_2.

Thus it will be seen that P_1 or P_2 corresponds to the refracting plane of a single thin lens, since all refraction appears to take place on either P_1 or P_2 depending upon the direction of the light. E_1 and E_2 likewise correspond to the optical centre, because any ray directed towards E_1 will, after refraction, appear to emerge from E_2 in a direction parallel to its initial path. This is illustrated in the next diagram.

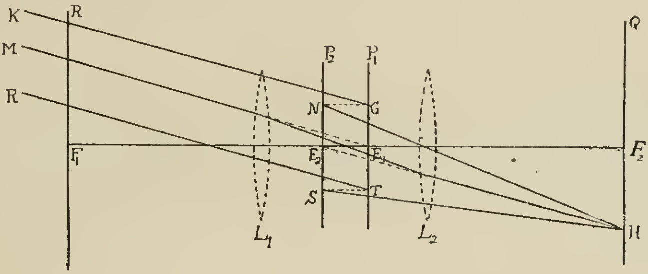

FIG. 165.

In Fig. 165, P_1 and P_2 are the equivalent planes, E_1 and E_2 the equivalent points, F_1 and F_2 the principal foci, $R\ F_1$ and $Q\ F_2$ the focal planes. Let an oblique parallel beam, of which M is the secondary axis, fall on L_1. The ray $M\ E_1$ directed towards E_1 is bent towards the axis by L_1, but is again rendered parallel to its original direction by L_2 such that it appears to proceed from E_2 towards H. Another ray $K\ G$ after refraction by L_1 and L_2 is directed towards H in the posterior focal plane, apparently proceeding from a corresponding point N on P_2 *such that the distances of G and N from the axis are equal.* Similarly $R\ T$ is refracted towards H, the point of emergence on P_2 being S, such that $S\ E_2 = T\ E_1$. Thus H is the image of the point from which the light originally diverged. Conversely rays diverging from H, or any other point in the focal plane, will emerge as a parallel beam.

Since the intrinsic power of a combination is a fixed quantity *the equivalent focal length is the same on each side, and is the distance E_2F_2 or E_1F_1.* The equivalent planes P_1 and P_2 are always situated symmetrically with respect to the focal planes, and with two ordinarily separated convex lenses P_1 and P_2 are invariably crossed such that E_1 lies nearer to F_2 than to F_1, and E_2

nearer to F_1 than to F_2, that is to say, the 2nd equivalent plane lies nearer to the source of light. With combinations other than two Cx. lenses—as also with a single thick Cx. lens—however, the equivalent points and planes are uncrossed. That which is the first equivalent plane when the light is incident on the one lens becomes the second equivalent plane when the lenses are reversed.

The space E_1E_2, over which the light apparently jumps, is called the *optical interval* or *equivalent thickness*. Were the two lenses brought together this interval would vanish, so that E_1 and E_2 merge to form the optical centre of the resultant thin combination, and the united planes P_1 and P_2 becomes the refracting plane.

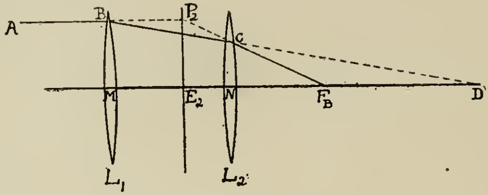

FIG. 166.

Expression for F_E.—In Fig. 166 *AB* is a ray parallel to the axis and is refracted through F_B, the back focus. Let the focal lengths of L_1 and L_2 be F_1 and F_2 respectively, d the separation, and F_E the equivalent focal length. Then we have two pairs of similar triangles $C\,F_BN$ and $P_2F_BE_2$, also $C\,D\,N$ and $B\,D\,M$.

Therefore

$$\frac{E_2F_B}{N\,F_B}=\frac{P_2E_2}{C\,N}=\frac{B\,M}{C\,N}=\frac{M\,D}{N\,D} \quad \text{or} \quad E_2F_B=\frac{M\,D\times N\,F_B}{N\,D}$$

Now

$$E_2F_B=F_E;\quad M\,D=F_1;\quad N\,F_B=\frac{F_2\,(F_1-d)}{F_1+F_2-d};\quad N\,D=F_1-d$$

$$F_E=\frac{F_1\times F_2(F_1-d)}{(F_1+F_2-d)\,(F_1-d)}=\frac{F_1F_2}{F_1+F_2-d}$$

This formula is independent of the direction of the light. The equivalent focal power is $1/F_E=1/F_1+1/F_2-d/F_1F_2$.

The distance of the second equivalent point E_2 from L_2 is found by subtracting the back from the equivalent focal distance, *i.e.* F_E-F_B. Thus

$$E_2=\frac{F_1F_2}{F_1+F_2-d}-\frac{F_2\,(F_1-d)}{F_1+F_2-d}=\frac{F_2d}{F_1+F_2-d}$$

The corresponding distance of E_1 from L_1 is

$$E_1=\frac{F_1F_2}{F_1+F_2-d}-\frac{F_1\,(F_2-d)}{F_1+F_2-d}=\frac{F_1d}{F_1+F_2-d}$$

The *equivalent thickness* or *optical interval* is found from the following equations, the first of which also shows whether the equivalent points are crossed or not:

$$t = d - (E_1 + E_2) = \frac{d^2}{F_1 + F_2 - d}$$

The distance E_1 is measured *backwards* from the first lens, and E_2 *forwards* from the second lens, that is, in each case *towards the other lens*. If, however, either is a negative quantity, it is measured in the opposite direction or *away from the other lens*.

The positions of E_1 and E_2 are unchanged in the combination, no matter which lens faces the light; E_1 is that which *theoretically* is nearer the source, but actually it may not be so. E_2 is that from which the focal length is measured, and if the combination is reversed that which was E_1 then becomes E_2, and *vice versa*. When the one lens faces the light F_E is measured from a certain position, and it is measured from another position if the other lens faces the light.

F_1, F_2, E_1 and E_2 are the *four cardinal points* of a combination.

The Result of Separation.—Separating two Cx. lenses results always in reduced power or longer F_E—indeed if d is great, F_E may become infinite, or even negative. With Cc. lenses the reverse occurs, the power being increased, or F_E shortened. With a Cx. and a Cc. in combination the result varies with the powers of the two components as tabulated later.

In Chapter XIII. it was shown that separation of two Cx. lenses resulted in *increased effect* or *shorter* F_B, while with two Cc. lenses there is a *decreased effect* or *longer* F_B.

It is necessary to distinguish between *true power* and *effect*. Suppose two thin Cx. lenses in contact in a given plane, and the one lens then moved outwards. The resultant refractive action is that of a single weaker lens placed a certain distance out from that plane, so that, although the combination is weakened, the distance of F behind the plane is shortened.

With two Cc. lenses the resultant action, due to separation, is that of a single stronger lens placed in advance of the plane occupied by the two original lenses.

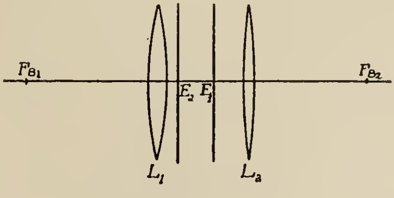

Fig. 167.

Equivalence of Two Cx. Lenses.—Let a 5 in. Cx. lens be 2 in. from a 10 in. Cx. lens. Then (Fig. 167)

$$F_E=\frac{5\times 10}{10+5-2}=\frac{50}{13}=3\tfrac{11}{13}\text{ in.}$$

$E_1=5\times 2/13=\frac{10}{13}$ in. behind L_1 $\quad E_2=10\times 2/13=1\frac{7}{13}$ in. in front of L_2

or $\quad 2-1\frac{7}{13}=\frac{6}{13}$ in. behind L_1

$$t=2-\left(\frac{10}{13}+\frac{20}{13}\right)=-\frac{4}{13}\text{ in.}$$

If the 10 in. lens faces the light, the two equivalent points change places, F_E being the same. Since $d=2$ in., and E_1 is 10/13 in. behind L_1, while E_2 is $1\frac{7}{13}$ in. in front of L_2, the distance t is negative, and the two equivalent planes are crossed by 4/13 in.

Special Cases.—The following special cases occur with two separated Cx. lenses:

(1) *When* $d=F_1-F_2$, then $F_E=F_1/2$, and E_2 is midway between the two lenses. This is the case of the *Huyghen eye-piece* (q.v.). If $F_1=3$ in., $F_2=1$ in. and $d=2$ in.

$$F_E=\frac{3\times 1}{3+1-2}=1\tfrac{1}{2}\text{ in.}$$

E_2 is 1 in. in front of the back lens, midway between the two lenses; E_2, being 3 in. behind the front lens, is 1 in. behind the back lens, and outside the lens system. Here $d>F_2$ but $<F_1$, and if the lens of shorter focus faces the light F lies in front of the back lens.

(2) *When* $d=F_1+F_2$, then $F_E=\infty$, this being the case of the *telescope* (q.v.).

(3) *When* $d>F_1$ *but* $<F_1+F_2$, then F_E is positive, and E_1 and E_2 may be one or both beyond the lenses and crossed (Fig. 168); d being $>F_1$, the light, after refraction by L_1, is divergent to L_2 as if a single lens were placed further than its focus from a given plane.

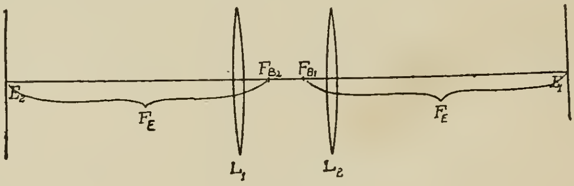

FIG. 168.

Let $F_1=4$ in., $F_2=4$ in., and $d=6$ in. Then

$$F_E=\frac{4\times 4}{4+4-6}=\frac{16}{2}=8\text{ in.}$$

The lenses being equal E_1 or $E_2=4\times 6/2=12$ in.

Parallel light incident on L_1 comes to a focus at 4 in., whence it diverges to L_2, and has its focus, after refraction, at 4 in. in front of L_2, or 8 in. behind E_2. $t = -18$ inches in this case.

Suppose $F_1 = 7$ in., $F_2 = 16$ in., and $d = 9$ in. Then $F_E = 8$ in., E_1 is $4\frac{1}{2}$ in. from L_1, and E_2 is $10\frac{2}{7}$ in. from L_2. The effect is as if an 8 in. lens were placed $10\frac{2}{7}$ in. in front of the plane of L_2. Light, refracted by this system, is converged to 7 in., and, after the second refraction, diverges as if from a point $2\frac{2}{7}$ in. in front of L_2.

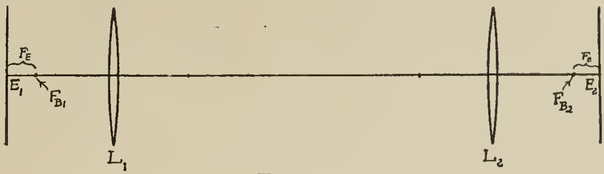

FIG. 169.

(4) *When* $d > F_1 + F_2$, then F_E is negative, and E_1 and E_2 are also negative (Fig. 169). Thus, if two 4 in. lenses are 20 in. apart, we get

$$F_E = \frac{4 \times 4}{4 + 4 - 20} = \frac{16}{-12} = -1\tfrac{1}{3} \text{ in.}$$

The lenses being similar E_1 or $E_2 = 4 \times 20 / -12 = -6\frac{2}{3}$ in.

Here the equivalent points, being negative, are measured outwards instead of inwards, and F_B lies behind L_2, but $1\frac{1}{3}''$ in front of E_2.

(5) *When* $d = F_1$, then $F_E = F_1$, and the system illustrates the *Kellner eye-piece* (q.v.) if the lenses are equal. If the lenses are $3''$ and $1''$ with $d = 3''$, we have $F_E = 3''$, $E_1 = 9''$ and $E_2 = 3''$, that is, in the plane of L_1.

(6) *When* $d = F_2$, then $F_E = F_2$, and $E_1 = d$, the image being the same as if the front lens were not there, but its position is shifted. Thus, with a $10''$ and a $1''$ lens separated by $1''$ we find $F_E = 1''$, $E_1 = 1''$ and $E_2 = \frac{1}{10}''$ which is the distance that the image is shifted. This illustrates the case of a *lens at the anterior focal point of the eye.*

(7) *When* $F_1 = F_2$, then $F_B + F_E = F_1$ or F_2. This is the case of the *Ramsden eye-piece.* Let F_1 and F_2 be each of 4 in. focal length, d being $2\frac{2}{3}$ in. Then

$$F_E = \frac{4 \times 4}{4 + 4 - 2\frac{2}{3}} = \frac{16}{5\frac{1}{3}} = 3'' \qquad E_1 \text{ or } E_2 = 4 \times 2\tfrac{2}{3} / 5\tfrac{1}{3} = 2''$$

F_B is therefore $3 - 2 = 1''$.

Equivalence of Two Cc. Lenses.—Example, $F_1 = -8$ in., $F_2 = -10$ in., and $d = 2$ in., then

$$F_E = \frac{-8 \times (-10)}{-8 - 10 - 2} = \frac{80}{-20} = -4 \text{ in.}$$

$$E_1 = -8 \times 2 / -20 = \tfrac{4}{5} \text{ in.} \qquad E_2 = -10 \times 2 / -20 = 1 \text{ in.}$$

$$t = 2 - (1 + \tfrac{4}{5}\text{in.}) = \tfrac{1}{5}$$

Special Cases.—If $d=F_1 \sim F_2$ then F_E is half that of the stronger lens, and the equivalent point measured from the weaker lens is midway between the two.

If $F_1=F_2$, then $F_B+F_E=F_1$ or F_2.

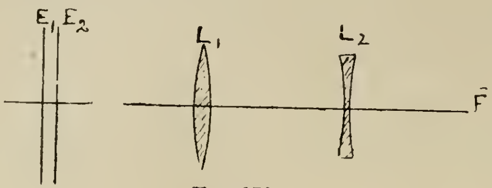

FIG. 170.

Equivalence of a Cx. and a Cc. Lens.—Suppose $F_1=10$ cm., $F_2=-15$ cm., and $d=2$ cm. Then (Fig. 170)

$$F_E=\frac{10\times(-15)}{10-15-2}=\frac{-150}{-7}=21\tfrac{3}{7} \text{ cm.}$$

$$E_1=10\times 2/-7=-2\tfrac{6}{7} \text{ in front of } L_1$$

$$E_2=-15\times 2/-7=4\tfrac{2}{7} \text{ cm. in front of } L_2 \text{ or } 4\tfrac{2}{7}-2=2\tfrac{2}{7} \text{ cm. in front of } L_1$$

$$t=2-(-2\tfrac{6}{7}+4\tfrac{2}{7})=\tfrac{4}{7} \text{ cm.}$$

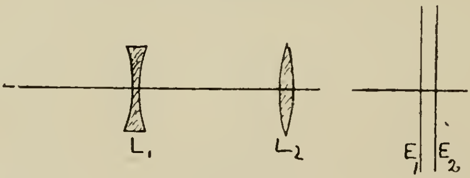

FIG. 171.

If the negative lens is in front (Fig. 171), E_2 is $2\tfrac{6}{7}$ cm. behind the Cx., or $-2\tfrac{6}{7}-2=-4\tfrac{6}{7}$ cm. behind the Cc. In the first case F_E lies $17\tfrac{1}{7}$ cm. behind the back lens, and in the second case $24\tfrac{2}{7}$ cm. behind it. The combination resembles that of a positive meniscus in which the optical centre lies outside the Cx. surface. Whether a combination, such as this, will have a positive or negative focal length depends, not only on the respective powers of the components, but also, and essentially, on the value of d. The weakest Cx. can more than neutralise the strongest Cc. if the separation be great enough.

Special Cases.—*When* $d=F_1+F_2$. If the two lenses are separated by the sum of their focal lengths, the negative being of shorter focus, then $F_E=\infty$, and the lenses neutralise each other. This is the case of the *opera-glass*. Thus, with 9 in. Cx. and a 4 in. Cc. separated by 5 in.

$$F_E=\frac{9\times(-4)}{9-4-5}=\frac{-36}{0}=\infty$$

If $d < F_1 + F_2$, the combination is negative; if $d > F_1 + F_2$ it is positive.

When $F_1 = F_2$.—If the two lenses have equal focal lengths, $F_B - F_E = F_1$ or F_2, and the formula for finding F_E (which is positive) becomes simplified to $F_E = F^2/d$. In this case E_1 is negative and both equivalent planes lie beyond the Cx. lens; $E_1 = F_2$ and $E_2 = F_1$; also $t = d$.

To find d for a given F_E.—To find the distance d which should separate two lenses so that they may have a given F_E the original formula can be employed, or the following—

$$d = F_1 + F_2 - F_1F_2/F_E$$

If d results in a negative quantity, it shows that the desired result is impossible. If both lenses are similar the formula may be written

$$d = 2\,F - F^2/F_E;$$

and if $F_1 = -F_2$ the formula simplifies to $d = F^2/F_E$.

Thus, when F_1 is 10 in. and F_2 is -5 in., for F_E to be 12″

$$d = 10 - 5 - 10 \times (-5)/12 = 5 - (-4\tfrac{1}{6}) = 9\tfrac{1}{6} \text{ in.}$$

For F_E to be $-12''$ we find that

$$d = 10 - 5 - 10 \times (-5)/-12 = 5 - 4\tfrac{1}{6} = \tfrac{5}{6} \text{ in.}$$

Change of F_E for Variation in d.—As d increases with two Cx. lenses, F_E varies directly, and t varies inversely or becomes negative.

As d increases with two Cc. lenses, F_E varies inversely, and t varies directly.

As d increases with one Cx. and the other Cc., the Cx. being the stronger, F_E varies inversely and t varies directly.

As d increases with one Cx. and the other Cc., the Cc. being the stronger, and F_E being negative, F_E varies directly, and t varies inversely or becomes negative.

As d increases with one Cx. and the other Cc., the Cc. being the stronger, and F_E being positive, F_E varies inversely, and t varies inversely.

Conjugate Foci.—The ordinary formulæ for conjugate foci hold good, but the distance of f_1 is measured from E_1, and that of f_2 from E_2, as with thick lenses (q.v.).

The planes of unit magnification—*i.e.* the *symmetrical planes*—are at 2 F_E from E_1 and E_2 respectively.

The position of the image can be worked out on general principles, but unless the equivalent points are known the magnification cannot be calculated. Thus, if an O is 10″ in front of a 5″ Cx. lens, behind which there is another Cx. of 10″ F, at 2″ we could calculate

$$\frac{1}{f_2} = \frac{1}{5} - \frac{1}{10} = \frac{1}{10} \text{ and } \frac{1}{f_2} = \frac{1}{10-2} + \frac{1}{10} = \frac{1}{4\frac{4}{9}}$$

The distance of I behind the back lens is $4\frac{4}{9}''$; or we can find $F_E = \frac{50}{13}$, $E_1 = \frac{10}{13}$ and $E_2 = \frac{20}{13}$.

Then f_1 is $10\frac{10}{13}''$ from E_1 and $\frac{1}{f_2}=\frac{13}{50}-\frac{13}{140}=\frac{117}{700}$

$f_2=5\frac{115}{117}''$ from E_2 or $5\frac{115}{117}-\frac{20}{13}=4\frac{4}{9}''$ from the back lens.

Combination of More than Two Lenses.—When more than two lenses are separated by intervals, the method of finding F_E of the whole system is to obtain that of the first pair of lenses, and then combine this combination with the third lens, or another pair of lenses, and so on. It must be remembered that the distance d between two combinations is that between *the two theoretically most adjacent equivalent points*, that is, between E_2 of the first and E_1 of the second combination; also that the positions of the equivalent points E_1 and E_2 of the whole combination are reckoned respectively from E_1 of the first, and E_2 of the second combination. In fact the calculations are similar to those required for two thick lenses (q.v.).

Dioptral Formulæ.—With dioptral powers, the equivalent power and points of two separated lenses are found from the following formulæ, where D_1 and D_2 are the powers of the two lenses, d is the interval between them expressed in cm., D_E is the equivalent dioptral lens, E_1 and E_2 are respectively the first and second equivalent points, and t is the distance between E_1 and E_2:

$$D_F=D_1+D_2-D_1D_2d/100$$

$$E_1=\frac{D_1D_2d}{D_1(D_1+D_2-D_1D_2d/100)}=\frac{D_1D_2d}{D_1D_E}=\frac{D_2d}{D_E}$$

$$E_2=\frac{D_1D_2d}{D_2(D_1+D_2-D_1D_2d/100)}=\frac{D_1D_2d}{D_2D_E}=\frac{D_1d}{D_E}$$

$$t=d-(E_1+E_2)$$

If d is expressed in terms of a metre, we can write:

$$D_E=D_1+D_2-D_1D_2d$$

If D_1 is positive and equal in power to D_2, which is negative, then

$$D_E=D^2d/100$$

The distance between two dioptral lenses so that they may have a certain equivalent dioptral power is found from

$$d=\frac{100(D_1+D_2-D_E)}{D_1D_2}$$

which, when D_1 and D_2 are equal, simplifies to

$$d=\frac{100\,(2\,D-D_E)}{D^2}$$

If the one lens is positive and the other negative and of equal powers, the formula becomes

$$d=100D_F/D^2$$

CHAPTER XV

THICK LENSES AND COMBINATIONS

HITHERTO we have considered lenses to be thin, that is, to have no appreciable thickness in relation to their focal length, so that the refraction caused by the two surfaces may be presumed to take place at a single refracting plane passing through the optical centre. Further, this plane may be taken as coinciding with the surfaces, and therefore all measurements may be taken from the lens itself, and secondary axes passing through the optical centre assumed to undergo no lateral deviation. With a thick lens, these simplifications are not permissible.

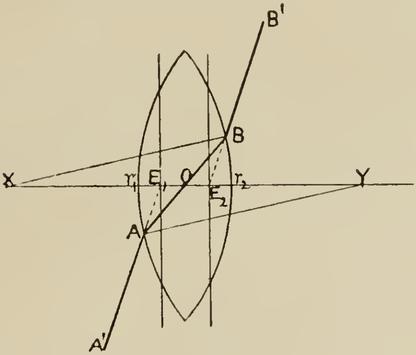

FIG. 172.

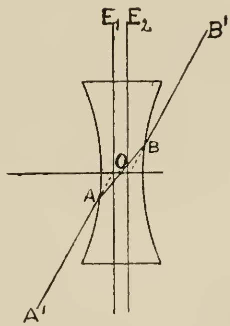

FIG. 173.

Let Fig. 172 represent a thick bi-Cx. lens of which X and Y are the centres of curvature. From X and Y let any two parallel radii, $X B$ and $Y A$ be drawn meeting their respective surfaces in B and A; then tangent planes drawn through A and B are parallel, so that at these points the lens acts as a plate, and there is one ray $A'A$, incident at A, which, after refraction, emerges as BB' parallel to its original course. $A'AOBB'$ is a *secondary axial* ray and the point O where it, and all other secondary axial rays, cut the principal axis is the *optical centre* of the lens. The position of O on $X Y$ depends upon the ratio of the two radii of curvature.

The point E_1, towards which $A'A$ is directed, is the *first equivalent point*, while E_2, from which it apparently emerges, is the *second equivalent point*. E_1 and E_2 have the same properties as in thin lens combinations, *i.e.* they are the points from which the principal and secondary foci are measured, and through which pass the planes where all refraction is presumed to take place. In a single thick Cx. lens, however, E_1 and E_2 are not crossed as they are in a thin Cx. lens combination. Fig. 173 shows the equivalent planes and optical centre of a bi-concave lens.

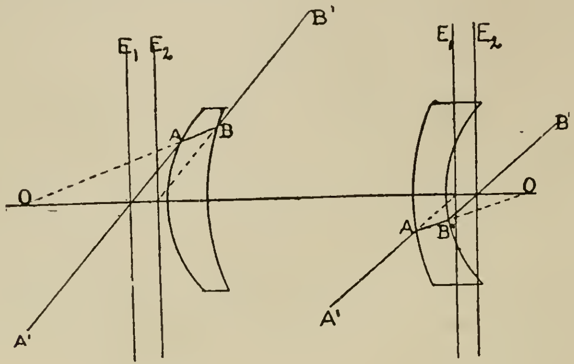

FIG. 174. FIG. 175.

In periscopic Cx. or Cc. lenses (Figs. 174 and 175) both E_1 and E_2 generally lie outside the lens on the Cx. side of the PCx., and on the Cc. side of the PCc., but in some cases the one point may be outside, and the other still within the lens; moreover the optical centre O lies beyond both equivalent points. A ray directed towards E_1 appears, after refraction, to proceed from E_2, its course $A\ B$ within the lens being on a line connecting the optical centre O, the point of incidence A of the ray at the first surface, and the

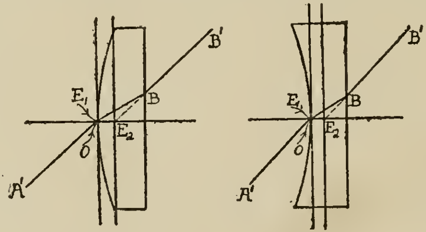

FIG. 176. FIG. 177.

point of emergence B at the second surface. The position of O is therefore virtual and it is determined by producing $B\ A$ to cut the principal axis.

In plano Cx. and Cc. lenses (Figs. 176 and 177) the only point on the curved surface parallel to any point on the plane surface is at the vertex, through which passes the principal axis. Therefore E_1, the first equivalent point, and O, the optical centre, coincide at the curved surface. All the secondary axes proceeding from the various points of a body are directed towards E_1, and after refraction appear to diverge from E_2, so that they cut the principal axis at O.

The terms *nodal* or *principal* points are sometimes applied to the *equivalent points*; but it is better to reserve the latter term for points that possess the functions of both the former, as they do in lenses where the first and last media—usually air—are similar. Nodal and principal points are discussed in the chapter on *Compound Refracting Systems.*

The Effect of Thickness is shown in the foregoing diagrams, and it may be said that a thick lens differs from a thin one in that it has a plate-like power of laterally displacing incident light. A thin Cx. lens can be transformed into a thick lens by splitting it in the refracting plane and cementing the two halves to the opposite sides of a parallel plate. The consequence is similar to that which results from the separation of two thin Cx. lenses in that a thick Cx. has a weaker equivalent power than a thin one of similar curvature and μ, while a thick Cc. has a stronger equivalent power than a thin one.

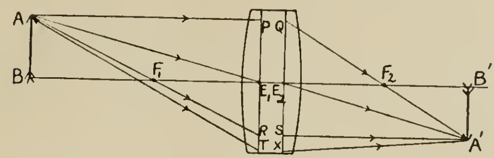

Fig. 178.

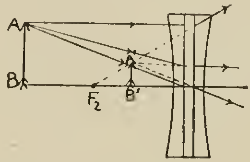

Fig. 179.

The Course of Light through Thick Lenses.—Fig. 178 represents a thick Cx. lens in front of which is the object AB. Any ray AP parallel to the axis, takes the course $Q\,A'$ after refraction, and passes through F_2. The secondary axial ray $A\,E_1$, directed to E_1, is refracted so as to proceed from E_2 parallel to its original course, and a third ray $A\,R$, passing through F_1, is refracted as $S\,A'$ parallel to the axis. All three rays meet in the image-point A', so that $B'\,A'$ is the complete real image of $A\,B$. Any other ray $A\,T$ directed towards the first equivalent plane at T emerges from the second at X, such that $E_1\,T = E_2\,X$.

The construction in the case of thick Cc. is shown in Fig. 179. It needs no explanation.

Direct Formulæ for a Single Thick Lens in Air.—To find the back or effective focal length, the equivalent focal length, and the positions of the equivalent points in terms of the radii, thickness and μ of the lens, let

F_B be the back focal length.
F_E the equivalent focal length.
E_1 and E_2 the first and second equivalent points.
T the distance between E_1 and E_2 (the optical interval).
r_1 and r_2 the radii of curvature of respectively the first and second surfaces.
A and B the first and second surfaces at the principal axis.
μ the index of refraction of the glass.
t the thickness of the lens on the axis.

Fig. 180 represents a thick bi-convex lens; let $R\,Q$ be a ray incident at Q and parallel to the principal axis $A\,B$; this is deviated towards the axis by the first surface, and would, if not intercepted by the second surface, cross $A\,B$ in D, but is brought to a nearer point F_B by the further refracting power of the second surface. Then, from definition, D is the posterior focus of the first surface, F_B the principal focal point of the lens as a whole, and $B\,F_B$

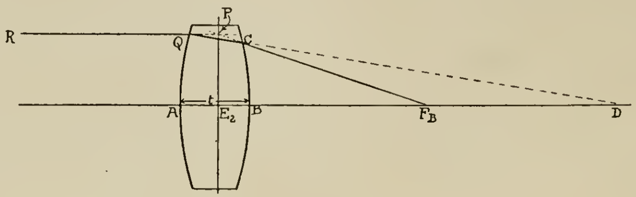

FIG. 180.

the back focal distance. Let F_BC be produced backward to meet $R\,Q$ prolonged in P. Now a plane perpendicular to the axis, dropped through P, will locate the second equivalent plane, and where it cuts the axis, the second equivalent point E_2. All the refraction of incident light from the direction $R\,Q$ (parallel or otherwise) appears to take place on $P\,E_2$. The distance $E_2\,F_B$ is, therefore, the equivalent focal length, since it is the focal length of the single thin lens which, if placed in the plane of E_2, would have the same power as the original thick lens as a whole.

The distance of F_B, the principal focal point of the lens, measured from the second surface B, is determined by the sum of the anterior focal powers $1/F_A$ and $1/F'_A$ of the two surfaces respectively, that of the first being modified by t/μ, the index and the thickness of the lens, which the light has to traverse, before it meets the second surface.

That is,
$$\frac{1}{F_B}=\frac{1}{F_A-t/\mu}+\frac{1}{F'_A}$$

Substituting $r_1/(\mu-1)$ for F_A, and $r_2/(\mu-1)$ for F'_A we get

$$\frac{1}{F_B}=\frac{1}{r_1/(\mu-1)-t/\mu}+\frac{1}{r_2/(\mu-1)}=\frac{(\mu-1)}{r_1-t(\mu-1)/\mu)}+\frac{\mu-1}{r_2}$$
$$=\frac{(\mu-1)\,(r_1+r_2-t\,(\mu-1)/\mu)}{r_2\,(r_1-t\,(\mu-1)/\mu)}$$

so that
$$F_B=B\,F_B=\frac{r_2\,(r_1-t\,(\mu-1)/\mu)}{(\mu-1)\,(r_1+r_2-t\,(\mu-1)/\mu)}$$

Similarly for the other surface the back surface power is

$$\frac{1}{F'_B}=\frac{1}{F'_A-t/\mu}+\frac{1}{F_A}$$

and the back surface focal length is

$$F_B = \frac{r_1 (r_2 - t(\mu - 1)/\mu)}{(\mu - 1)(r_1 + r_2 - t(\mu - 1)/\mu)}$$

In Fig. 180, since the ray $R\,Q$ is presumed to lie close to the axis, the arcs $Q\,A$ and $C\,B$ may be taken as straight lines giving two pairs of similar triangles $C\,F_B\,B$ and $P\,F_B E_2$, $C\,D\,B$ and $Q\,D\,A$. Then it follows that

$$\frac{E_2F_B}{BF_B} = \frac{PE_2}{CB} = \frac{QA}{CB} = \frac{AD}{BD}$$

so that

$$E_2F_B = BF_B \times \frac{AD}{BD}$$

But $E_2\,F_B$ is the equivalent focal distance, $B\,F_B$ is the back focal distance, $A\,D$ the posterior focus of the first surface, and $B\,D$ is this quantity less the thickness t. On substituting these values, therefore, in the above equation we get as the expression for the equivalent focal length,

$$F_E = E_2F_B = \frac{r_2 (r_1 - t(\mu - 1)/\mu)}{(\mu - 1)(r_1 + r_2 - t(\mu - 1)/\mu)} \times \frac{\mu r_1}{\mu - 1} \div \left(\frac{\mu r_1}{\mu - 1} - t\right)$$

which becomes

$$F_E = E_2F_B = \frac{r_1 r_2}{(\mu - 1)(r_1 + r_2 - t(\mu - 1)/\mu)}$$

F_E is the same whichever surface faces the light, but it is measured from the posterior equivalent point.

The distance of E_2 from the pole B of the second surface, is found by subtracting the back from the equivalent focal length, which in terms similar to those already used is

$$E_2 = F_E - F_B = \frac{r_2 t}{\mu (r_1 + r_2 - t(\mu - 1)/\mu)}$$

and the corresponding distance of E_1 from A is

$$E_1 = F_E - F'_B = \frac{r_1 t}{\mu (r_1 + r_2 - t(\mu - 1)/\mu)}$$

The distances of the first and second equivalent points are measured *inwards*—that is, respectively from the first towards the second surface, and from the second towards the first. If, however, by calculation the value is negative, as occurs in some cases, the measurement is *outwards* from the corresponding surface.

If we calculate the quantity Q which enters into the formulæ,

that is

$$Q = r_1 + r_2 - t(\mu - 1)/\mu$$

we have

$$F_E = \frac{r_1 r_2}{(\mu - 1)Q} \qquad E_1 = \frac{r_1 t}{\mu Q} \qquad E_2 = \frac{r_2 t}{\mu Q}$$

The equivalent thickness, or optical interval—*i.e.* the distance between the equivalent points, is

$$T = t - (E_1 + E_2)$$

It should be noted that the formula for F_E is the same as for F of a thin lens, except that the quantity $t(\mu - 1)/\mu$ enters into it.

An approximate formula (accurate when $\mu = 1{\cdot}5$) is

$$F_E = \frac{r_1 r_2}{(\mu - 1)(r_1 + r_2 - t/3)}$$

Example of a Bi-Cx. Lens.—If r_1 and $r_2 = 10$ cm. and 6 cm. respectively, $\mu = 1{\cdot}5$, $t = 3$ cm., then (Fig. 180)

$$F_E = \frac{10 \times 6}{{\cdot}5(10 + 6 - 3 \times {\cdot}5/1{\cdot}5)} = \frac{60}{{\cdot}5 \times (16 - 1)} = \frac{60}{7{\cdot}5} = 8 \text{ cm.}$$

$$E_1 = \frac{10 \times 3}{1{\cdot}5 \times (16 - 1)} = \frac{30}{22{\cdot}5} = 1{\cdot}33 \text{ cm.} \qquad E_2 = \frac{6 \times 3}{1{\cdot}5 \times (16 - 1)} = \frac{18}{22{\cdot}5} = {\cdot}8 \text{ cm.}$$

$$T = 3 - (1{\cdot}333 + {\cdot}8) = {\cdot}86 \text{ cm.}$$

F_B is anteriorly $8 - 1{\cdot}333 = 6{\cdot}66$ from *A*, and posteriorly $8 - {\cdot}8 = 7{\cdot}2$ cm. from *B*. The optical centre is located at

$$3 \times 6/(10 + 6) = 1{\cdot}125 \text{ cm. from B, and } 1{\cdot}875 \text{ cm. from A.}$$

A thin lens of same radii and μ has $F = \frac{10 \times 6}{{\cdot}5 \times (10 + 6)} = \frac{60}{8} = 7{\cdot}5$ cm.

Thus *in a bi-Cx. thick lens the true or equivalent focal length is longer than that of the corresponding thin lens, but its back focal length is shorter.* In the case of the thick lens F=8 cm. from E_2, but 7·2 cm. from B, while if the lens were thin so that $t=0$, F is 7·5 cm. from B. If two Cx. lenses be made of the same glass and similar curvatures, but the one thicker than the other, the thicker lens is *actually the weaker,* although its *effectivity is greater.*

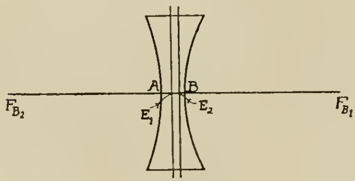

FIG. 181.

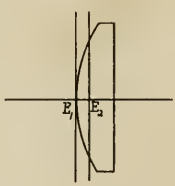

FIG. 182.

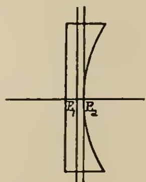

FIG. 183.

Example of a Bi-Cc. Lens.—In Fig. 181 let r_1 and $r_2 = -10$ cm. and -6 cm. respectively, $\mu = 1{\cdot}5$ and $t = 3$ cm.

$$F_E = \frac{-10 \times (-6)}{\cdot 5 \times (-10 - 6 - 3 \times \cdot 5/1\cdot 5)} = \frac{60}{\cdot 5 \times (-17)} = -7\cdot 06 \text{ cm.}$$

$$E_1 = \frac{-10 \times 3}{1\cdot 5 \times (-17)} = 1\cdot 18 \text{ cm.} \qquad E_2 = \frac{-6 \times 3}{1\cdot 5 \times (-17)} = \cdot 7 \text{ cm.}$$

$$T = 3 - (1\cdot 18 - \cdot 7) = 1\cdot 12 \text{ cm.}$$

Although the true focal length is the same on either side, if the surface B faces the light, F lies 7·06 + 1·18=8·24 cm. from A, while if A faces the light F is 7·06 + ·7=7·76 cm. from B. If the lens were thin F=7·5 cm., *so that increased thickness causes a Cc. to have a greater equivalent power or a shorter equivalent F, but a smaller effectivity or longer back surface F.*

Example with a Plano-Cx. Lens.—Let r_1 (Fig. 182) that of the curved surface=6 cm.; r_2 of the plano= ∞; μ=1·5; and t=3 cm. Then since r_2= ∞, and this quantity occurs in the upper and lower part of the formula, we can omit it from our calculations as well as the other quantities in the bracket containing this value. The formula therefore simplifies to that used for a thin lens, viz. $F_E = r_1/(\mu - 1)$

$$E_1 = 6 \times 3/1\cdot 5\ \infty = 0 \qquad E_2 = 3/1\cdot 5 = 2 \text{ cm.}$$
$$F_E = 6/\cdot 5 = 12 \text{ cm.} \qquad T = 3 - 2 = 1 \text{ cm.}$$

E_1 is at the curved surface, and E_2 is 2 cm. in front of the plane surface. In the above example, when the Cx. surface is exposed to the light, F lies 12 – 2 =10 cm. behind the plane, and 13 cm. behind the curved surface. When the plane surface is so exposed, F lies 12 cm. behind the curved and 15 cm. behind the plane surface.

Example with a Plano-Cc. Lens.—If r_1 (Fig. 183) that of the plane surface = ∞, as before stated, it may be neglected. Let r_2, that of the Cc.=6 cm., μ=1·5, and t=3 cm.

$$E_1 = 3/1\cdot 5 = 2 \text{ cm.} \qquad E_2 = 6 \times 3/1\cdot 5\ \infty = 0.$$
$$F_E = -6/\cdot 5 = -12 \text{ cm.} \qquad T = 3 - 2 = 1 \text{ cm.}$$

E_1 is 2 cm. from the plane surface, and E_2 is at the Cc. surface. If the curved surface faces the light F is 12 + 2=14 cm. in front of the plane and 11 cm. from the curved surface. When the light is incident on the plane surface, F lies 12 cm. from the curved and 9 cm. from the plane surface.

Example of a Positive Meniscus.—In a periscopic Cx. lens (Fig. 184) let r_1 and r_2 of the Cx. and Cc. surfaces respectively be + 6 cm. and – 10 cm., μ=1·5 and t=3 cm.

$$F_E = \frac{6 \times (-10)}{\cdot 5\ (6 - 10 - 3 \times \cdot 5/1\cdot 5)} = \frac{-60}{\cdot 5 + (-5)} = 24 \text{ cm.}$$

$$E_1 = 6 \times 3/1\cdot 5 \times -5 = -2\cdot 4 \text{ cm.}, \quad E_2 = -10 \times 3/1\cdot 5 \times -5 = 4 \text{ cm.}$$

$$T = 3 - (-2\cdot 4 + 4) = 1\cdot 4 \text{ cm.}$$

E_1 being negative must be reckoned outwards from the Cx. surface, so that both equivalent points are reckoned the same way, the second being inwards from the Cc. E_1 is 2·4 cm. and E_2 is $4-3=1$ cm. outside the Cx. surface.

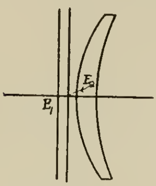

Fig. 184.

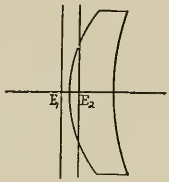

Fig. 185.

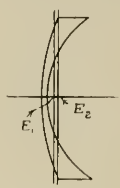

Fig. 186.

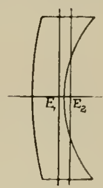

Fig. 187.

In some cases, with a periscopic Cx. lens, as when the Cc. surface has very little curvature, the one equivalent point lies within the Cx. surface, as in Fig. 185. The more nearly equal the two curvatures, the more are E_1 and E_2 displaced beyond the Cx. surface. The distance of F_B varies considerably as the one or the other surface is exposed to the light.

Example of a Negative Meniscus.—In a periscopic Cc., as in Fig. 186, let r_1 and r_2 of the Cx. and Cc. surfaces respectively $=+10$ cm., and -6 cm., $\mu=1{\cdot}5$ and $t=3$ cm.

$$F_E=\frac{10\times(-6)}{\cdot5\,(+10-6-3\times\cdot5/1{\cdot}5)}=\frac{-60}{\cdot5\times(3)}=-40\text{ cm.}$$

$$E_1=10\times3/1{\cdot}5\times3=6{\cdot}66\text{ cm., } E_2=-6\times3/1{\cdot}5\times3=-4\text{ cm.}$$

$$T=3-(6{\cdot}66-4)=\cdot33\text{ cm.}$$

The distance of both equivalent points are reckoned the same way, E_1 inwards from the Cx. surface, and E_2, being negative, outwards from the Cc. surface. The first is $6{\cdot}66-3=3{\cdot}66$ cm. outside the Cc. surface, and the second is 4 cm. outside it.

If the Cc. surface has but little curvature, the one equivalent plane of a Cc. meniscus may lie within the Cc. surface (Fig. 187). Also, the difference in the distance of F_B is marked as E_1 or E_2 is taken as the first equivalent point.

Special Cases—Afocal Lenses.—In a meniscus when r of the Cx. is longer than that of the Cc. surface

$$F_E=\infty \text{ if } r_1+r_2=t\,(\mu-1)/\mu, \text{ or } t=\mu\,(r_1+r_2)/(\mu-1)$$

F_E is positive when r_1+r_2 is less than $t(\mu-1)/\mu$ and negative when r_1+r_2 is greater than $t(\mu-1)/\mu$. That is to say, F is positive or negative according as t is sufficiently great or small respectively; and that, when t is of certain value, the power of the Cx. surface neutralises that of the Cc. This condition obtains when $Q=0$.

Thus, in order that $F=\infty$ when $r_1=-1$, $r_2=+3$, and $\mu=1{\cdot}5$.

$$t=1{\cdot}5\,(-1+3)/\cdot5=6\text{ cm.}$$

This is the principle of the Steinheil cone (Fig. 188), which is practically a fixed focus opera-glass.

If $r_1 = +10$ and $t=3$ when $\mu=1{\cdot}5$, then r_2 must be -9 in order that $F_E = \infty$.

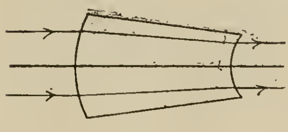

FIG. 188.

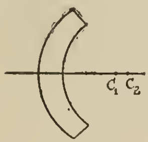

FIG. 189.

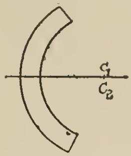

FIG. 190.

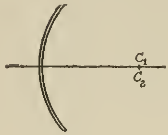

FIG. 191.

Fig. 189 illustrates the form of the *worked* globular or coquille of the optical trade, where an afocal effect is required; the radius of the Cc. surface is shorter than that of the Cx. by an amount equal to approximately a third the thickness of the glass. A thick afocal lens can be obtained, also, when both radii are positive, if the above-mentioned condition be fulfilled. Thus if $\mu=1{\cdot}5$, $r_1=2$ cm. and $r_2=5$ cm., the lens is afocal if $t=3\ (r_1+r_2)$, in this case 21 cm., for then Q=0. When $F=\infty$, the equivalent points and optical centre are also at ∞.

Concentric Lenses.—If $r_1+r_2=t$ (r_2 being negative)—*i.e.* if $r_1-t=r_2$, so that the two centres of curvature coincide, the Cc. surface has the shorter radius and F is negative. Thus (Fig. 190) let $r_1=10$ cm., $r_2=-6$ cm., $t=4$ cm., and $\mu=1{\cdot}5$. Then

$$F_E = \frac{10\times(-6)}{{\cdot}5\ (10-6-4\times{\cdot}5/1{\cdot}5)} = \frac{-60}{{\cdot}5(2{\cdot}666)} = -45 \text{ cm.}$$

$$E_1=r_1=10;\ E_2=r_2=-6;\ T=4-(10-6)=0.$$

The equivalent points coincide at the common centre of curvature.

If the glass be thin (Fig. 191), and the centres coincide, there is a *slight* concave power, as is found in the ordinary *unworked* globular or coquille.

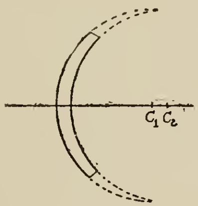

FIG. 192.

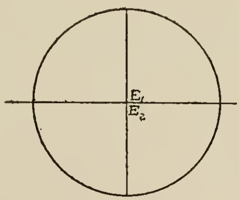

FIG. 193.

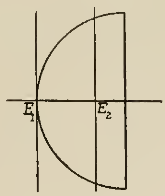

FIG. 194.

Equi-Curved Lenses.—If $r_1=-r_2$ (Fig. 192), F_E is positive. Thus, let $r_1=+10$ cm., $r_2=-10$ cm., $t=3$ cm., and $\mu=1{\cdot}5$. Then

$$F_E = \frac{10\times(-10)}{{\cdot}5\ (10-10-3\times{\cdot}5/1{\cdot}5)} = \frac{-100}{-{\cdot}5} = 200 \text{ cm.}$$

E_1 or $E_2 = r/(\mu-1)$, in this case $10/{\cdot}5=20$ cm.

Other Conditions.—If the radius of the Cx., is shorter than that of the Cc., F_E is positive. If t is greater than F of the first surface, the light is brought to a focus within the dense medium, and, after crossing, diverges to the second surface by which it is converged or diverged according as it is positive or negative respectively.

The Sphere.—In a sphere (Fig. 193), $r_1=r_2$ and t=the diameter=$2r$. Let $\mu=1{\cdot}5$, and $r=6$ cm., so that $t=12$ cm.

$$F_E=\frac{6\times 6}{\cdot 5\ (6+6-12\times\cdot 5/1{\cdot}5)}=\frac{36}{\cdot 5\times 8}=9 \text{ cm.}$$

$$E_1 \text{ or } E_2=6\times 12/1{\cdot}5\times 8=6 \text{ cm.} \qquad T=12-(6+6)=0.$$

Therefore, the equivalent planes of a sphere coincide and pass through the centre of curvature C, as in Fig. 193. The formula, in the case of a sphere, simplifies to

$$F_E=\frac{\mu r}{2\ (\mu-1)} \text{ and } F_B=F_E-r$$

When μ of a sphere is 1·5, $F_E=1{\cdot}5\ r$, and $F_B={\cdot}5\ r$.

Calculations with a sphere are similar to those of any other thick lens, when the object is situated outside the sphere. If, however, the object is within the sphere, they are similar to those connected with a single surface.

The Hemisphere.—With the hemisphere (Fig. 194) the Cx. surface to the front, $F_E=r/(\mu-1)$, $E_1=r_1t/\infty=0$, $E_2=t/\mu$; from the Cx. surface $F_B=F_E=r/(\mu-1)$; from the plane surface $F_B=r/\mu(\mu-1)$.

When $\mu=1{\cdot}5$, $F_E=2r$; $F_B=2r$ from the Cx., and $1\frac{1}{3}r$ from the plane surface.

Other Calculations.—To find μ or one of the radii, when the other data are given, involves substituting values for symbols in the formula and equating.

Thus what radius must be given to a DCx. lens so that $F_E=5$ cm. when $\mu=1{\cdot}5$ and $t={\cdot}75$ cm.? Substituting the known values we have

$$5=\frac{r^2}{\cdot 5\ (2r-\cdot 5\times\cdot 75/1{\cdot}5)}=\frac{r^2}{\cdot 5\ (2r-\cdot 25)}=\frac{r^2}{r-\cdot 125}$$

then $$r^2-5\ r=-\cdot 625.$$

$$r^2-5r+6{\cdot}25=-\cdot 625+6{\cdot}25$$

$$r-2{\cdot}5=\pm 2{\cdot}35$$

so that $$r=4{\cdot}85 \text{ or } \cdot 15.$$

·15 is an impossible answer, therefore the required radius is 4·85 cm.

Calculations of a Thick Lens in Terms of the Foci of its Surfaces.

The value of a thick lens in air can also be calculated from the foci of the two surfaces.

Let f_1 and f_2 represent respectively the anterior and posterior focal distances of the first surface, and f_1' and f_2' represent respectively the anterior and posterior focal distances of the second surface. Let t be the thickness of the lens; then

$$f_1=\frac{r_1}{\mu-1} \qquad f_2=\frac{\mu r_1}{\mu-1} \qquad f_1'=\frac{\mu r_2}{\mu-1} \qquad f_2'=\frac{r_2}{\mu-1}$$

In Fig. 180 D is the virtual object for the second surface B of radius r_2, and the distance $B F_B$, which is the second back focus F_B, is the final image distance with respect to D the virtual object.

Let $B D=u$ and $B F_B=v$. Now the expression connecting the conjugate foci of the second surface B is

$$\frac{1}{v}+\frac{\mu}{u}=\frac{\mu-1}{r_2}$$

But $(\mu-1)/r_2=1/f'_2$, and $u=A D-t=f_2-t$, the latter expression being a negative distance since the object is virtual.

Therefore
$$\frac{1}{v}-\frac{\mu}{f_2-t}=\frac{1}{f_2'}$$

whence
$$\frac{1}{v}=\frac{1}{f_2'}+\frac{\mu}{f_2-t}=\frac{\mu f_2'+f_2-t}{f_2'(f_2-t)}$$

But
$$\mu f_2'=\mu r_2/(\mu-1)=f_1'$$

Therefore
$$v=F_B=\frac{f_2'(f_2-t)}{f_1'+f_2-t}$$

The corresponding back focus from A, by similar reasoning, is

$$F_A=\frac{f_1(f_1'-t)}{f_1'+f_2-t}$$

Equivalent Focus.—In Fig. 180 P is the second equivalent plane and E_2 the second equivalent point. As before we may consider $C B$ as being sensibly straight.

Then
$$\frac{E_2F_B}{BF_B}=\frac{PE_2}{CB}=\frac{QA}{CB}=\frac{AD}{BD}$$

and
$$E_2F_B=\frac{AD\times BF_B}{BD}$$

But $E_2F_B=F_E$, $AD=f_2$, $BF_B=F_B$, and $BD=f_2-t$

Therefore
$$F_E = f_2 \times \frac{f_2'(f_2 - t)}{f_1' + f_2 - t} \times \frac{1}{f_2 - t}$$
$$= \frac{f_2 f_2'}{f_1' + f_2 - t}$$

It is important to notice that $f_2 f_2' = f_1 f_1'$, so that F_E may be taken as

$$\frac{f_1 f_1'}{f'_1 + f_2 - t}$$

when the light is incident first on the surface B.

Equivalent Points.—The distances of E_1 and E_2 from the surfaces A and B are found by deducting from the equivalent focal length the respective back foci. Thus

$$E_1 = F_E - F_B = \frac{f_1 f_1'}{f_1' + f_2 - t} - \frac{f_1(f_1' - t)}{f_1' + f_2 - t} = \frac{f_1 t}{f_1' + f_2 - t}$$

Similarly

$$E_2 = F_E - F_B = \frac{f_2 f_2'}{f_1' + f_2 - t} - \frac{f_2'(f_2 - t)}{f_1' + f_2 - t} = \frac{f_2' t}{f_1' + f_2 - t}$$

Example.—Let $r_1 = 10$ cm., and $r_2 = 6$ cm., $\mu = 1{\cdot}5$, $t = 3$ cm.; then

$$f_1 = \frac{10}{{\cdot}5} = 20 \qquad f_2 = \frac{10 \times 1{\cdot}5}{{\cdot}5} = 30 \qquad f_1' = \frac{6 \times 1{\cdot}5}{{\cdot}5} = 18 \qquad f_2' = \frac{6}{{\cdot}5} = 12$$

When light is incident first on the surface A

$$F_E = \frac{f_2 f_2'}{f_1' + f_2 - t} = \frac{30 \times 12}{18 + 30 - 3} = 8 \text{ cm.}$$

and when incident on B

$$F_E = \frac{f_1 f_1'}{f_1' + f_2 - t} = \frac{20 \times 18}{18 + 30 - 3} = 8 \text{ cm.}$$

the equivalent focus, of course, being the same in either case.

The equivalent points E_1 and E_2 are distant from A and B respectively

$$E_1 = \frac{f_1 t}{f_1' + f_2 - t} = \frac{20 \times 3}{45} = 1{\cdot}33 \text{ cm.}$$

$$E_2 = \frac{f_2' t}{f_1' + f_2 - t} = \frac{12 \times 3}{45} = {\cdot}8 \text{ cm.}$$

Thus the back foci from A and B are respectively $8 - 1{\cdot}33 = 6{\cdot}66$ cm., and $8 - {\cdot}8 = 7{\cdot}2$ cm.

Two Thick Lenses in Combination.—Let A be the first and B the second lens of a combination of two thick convex lenses separated by an interval.

Let r_1 and r_2 be the radii of curvature of A, and r_1' and r_2' those of B.

Let t_1 and t_2 be, respectively, the actual thicknesses of A and B.

Let E_1 and E_2 be, respectively, the first and second equivalent points of A.

Let E_1' and E_2' be, respectively, the first and second equivalent points of B.

Let T_1 and T_2 be, respectively, the equivalent thicknesses of A and B.

Let F_1 and F_2 be, respectively, the focal lengths of A and B.

Let d be their distance apart, this being the distance between their most adjacent equivalent points, *i.e.* the distance between E_2 and E'_1.

Let E and E' be, respectively, the first and second equivalent points of the combination.

Let F be the equivalent focal distance of the combination.

Let T be the equivalent thickness of the combination.

The equivalent focal distance F of two combined lenses is obtained from the formula

$$\mathrm{F}=\frac{\mathrm{F}_1\mathrm{F}_2}{\mathrm{F}_1+\mathrm{F}_2-d}=\frac{\mathrm{F}_1\mathrm{F}_2}{\mathrm{N}}$$

which is the same as for two thin lenses in combination. This illustrates the great utility of the equivalent planes in simplifying all thick lens calculations, since, provided we measure from the equivalent planes, a combination can in every way be treated as a simple system.

Similarly the distance of E, the first equivalent point of the combination. measured from E_1, the first equivalent point of A, is

$$\mathrm{E}=\frac{\mathrm{F}_1 d}{\mathrm{F}_1+\mathrm{F}_2-d}=\frac{\mathrm{F}_1 d}{\mathrm{N}}$$

The distance of E', the second equivalent point of the combination, measured from E_2', the second equivalent point of B, is

$$\mathrm{E}'=\frac{\mathrm{F}_2 d}{\mathrm{F}_1+\mathrm{F}_2-d}=\frac{\mathrm{F}_2 d}{\mathrm{N}}$$

The distance $T=E\,E'$, between the equivalent points of the combination, is determined by the following

$$\mathrm{T}=d+\mathrm{T}_1+\mathrm{T}_2-(\mathrm{E}+\mathrm{E}') \qquad \text{or} \qquad \mathrm{T}=\mathrm{T}_1+\mathrm{T}_2-d^2/\mathrm{N}.$$

As an example let $r_1=10$ cm., $r_2=8$ cm., and $t_1=2$ cm.
$r'_1=9$ cm., $r'_2=7$ cm., and $t_2=2$ cm.
$\mu=1{\cdot}5$ and $d=2{\cdot}5$ cm.

Then, when calculated, we obtain

$\mathrm{F}_1=9{\cdot}23$ cm., $\mathrm{E}_1={\cdot}769$ cm., $\mathrm{E}_2={\cdot}615$ cm., $\mathrm{T}_1={\cdot}616$ cm.
$\mathrm{F}_2=8{\cdot}26$ cm., $\mathrm{E}'_1={\cdot}783$ cm., $\mathrm{E}'_2={\cdot}609$ cm., $\mathrm{T}_2={\cdot}608$ cm.

and for the combination

$$F = \frac{9{\cdot}23 \times 8{\cdot}26}{9{\cdot}23 + 8{\cdot}26 - 2{\cdot}5} = \frac{76{\cdot}2398}{15} = 5{\cdot}08 \text{ cm.}$$

$$E = \frac{9{\cdot}23 \times 2{\cdot}5}{9{\cdot}23 + 8{\cdot}26 - 2{\cdot}5} = \frac{23{\cdot}075}{15} = 1{\cdot}538 \text{ cm.}$$

$$E' = \frac{8{\cdot}26 \times 2{\cdot}5}{9{\cdot}23 + 8{\cdot}26 - 2{\cdot}5} = \frac{20{\cdot}65}{15} = 1{\cdot}377 \text{ cm.}$$

$$T = 2{\cdot}5 + {\cdot}616 + {\cdot}608 - (1{\cdot}538 + 1{\cdot}377) = {\cdot}81 \text{ cm.}$$

or

$$T = {\cdot}616 + {\cdot}608 - 2{\cdot}5^2/15 = 1{\cdot}224 - 6{\cdot}25/15 = {\cdot}81 \text{ cm.}$$

The combination is of 5·08 cm. focal length and its equivalent planes are ·81 cm. apart.

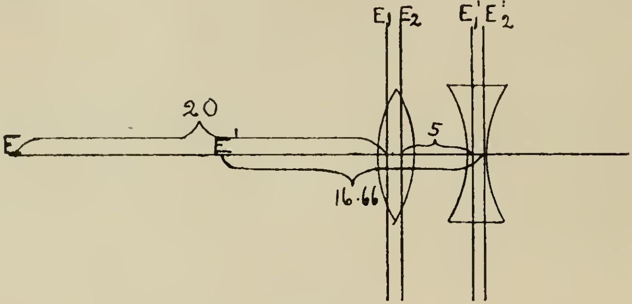

Fig. 195.

Example with a Convex and a Concave Lens.—Let $F_1 = +12$ in.; $F_2 = -10$ in.; $d = 5$ in.; $T_1 = {\cdot}5$ in.; $T_2 = {\cdot}2$ in.; then combined we obtain (Fig. 195)

$$F = \frac{+12 \times (-10)}{+12 - 10 - 5} = \frac{-120}{-3} = +40 \text{ in.}$$

$$E = 12 \times 5/-3 = -20 \text{ in.} \qquad E' = -10 \times 5/-3 = 16{\cdot}66 \text{ in.}$$

$$T = 5 + {\cdot}5 + {\cdot}2 - (-20 + 16{\cdot}66) = 9{\cdot}03 \text{ in.}$$

or

$$T = {\cdot}5 + {\cdot}2 - 5^2/-3 = {\cdot}7 - 25/-3 = {\cdot}7 - (-8{\cdot}33) = 9{\cdot}03 \text{ in.}$$

Coincidence of E and E′.—In order that E and E' should coincide, d can be found, for two thick lenses, by the following formula.

$$d = \frac{\sqrt{(T_1 + T_2)^2 + 4\,(F_1 + F_2)\,(T_1 + T_2)} - (T_1 + T_2)}{2}$$

Taking as an example a combination where $F_1 = 9$ in., $F_2 = 8$ in., $T_1 = {\cdot}2$ in., and $T_2 = {\cdot}3$ in.

$$d=\frac{\sqrt{(\cdot 2+\cdot 3)^2+4\times(9+8)\times(\cdot 2+\cdot 3)}-(\cdot 2+\cdot 3)}{2}$$

$$d=\frac{\sqrt{\cdot 25+34}-\cdot 5}{2}=\frac{5\cdot 8524-\cdot 5}{2}=2\cdot 6762 \text{ in.}$$

When the lenses are 2·6762 in. apart T=0.

To find F_E of More than Two Lenses.—When more than two lenses are in combination the equivalent cardinal points of two of them are determined, and then this combination is again combined with the third lens, or with another combination as the case might be. Thus, if there are four lenses, A B C D, the equivalent of A and B, also of C and D, are found separately, and these two equivalent combinations again merged into a single one, or the focal length of such a combination can be found directly by the Gauss equation given later.

Conjugate Foci.—Calculations of conjugate foci with thick lenses are the same as with thin lenses provided all measurements are taken from the adjacent equivalent planes.

Let O be 20 cm. from the surface A of the lens of F_E=8 cm., E_1=1·33 cm., and E_2=·8 cm. To find the distance of the conjugate image from B, the distance f_1 is 20 cm. from A and therefore 20 + 1·33=21·33 cm. from E_1, and since F is 8 cm. we have $1/f_2$=1/8 – 1/21·33, whence f_2=12·8 cm. Now f_2 is measured from E_2, which is ·8 cm. from B. Therefore the distance of the image from the second surface of the lens is 12·8 – ·8=12 cm. The calculation for the corresponding thin lens would be $1/f_2$=1/8 – 1/20, whence f_2=13·33 cm., and since both surfaces are considered coincident with the optical centre, the distance of the image, in this case, from the lens is 1·33 cm. more than when there is a thickness of 3 cm. Similar calculations can be made for any type of thick lens or lens combination.

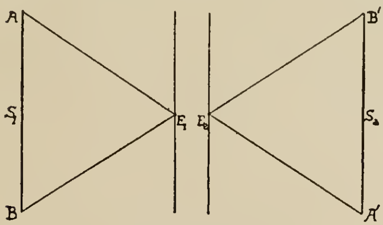

Fig. 196.

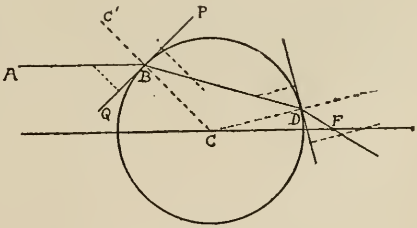

Fig. 197.

Planes of Unit Magnification.—The *symmetrical points* S_1 and S_2 (Fig. 196) lie on the principal axis at a distance equal to 2F measured from E_1 anteriorly, and from E_2 posteriorly; they mark the *symmetrical planes*. An object-point A in the one symmetrical plane has its image A' in the other symmetrical plane and *at an equal distance from the principal axis.* Thus, when an object AB is situated at the one symmetrical plane, its image $B'A'$ is situated at the other, and *the two are of equal size*; these are the planes of unit magnification for real images. The symmetrical planes are sometimes termed *negative equivalent planes.*

The planes of unit *virtual* magnification for thick lenses and lens systems lie in *the equivalent planes* themselves. In other words the equivalent planes are virtual images of each other.

Construction.—The graphical construction of the image has been shown at the commencement of this chapter (Figs. 178 and 179). To trace the course of a ray through a thick lens, suppose the case of a sphere (Fig. 197). Let AB be a ray incident at B; from C draw the normal CC' to B, and the tangent PQ to B. Then PQ may be regarded as the refracting surface, which is divided off as shown for an ordinary plane surface. BD is the course of the ray after the first refraction, and at D the process is repeated, the emergent ray being DF. If the thick lens is other than a sphere, C_1 and C_2, the two centres of curvature will, of course, be at their proper distances from their corresponding surfaces.

To trace the course of any ray or to determine the image of any point the symmetrical planes S_1 and S_2 and the equivalent planes E_1 and E_2 can be made use of (Fig. 198).

Let A be such an object-point on the principal axis. Let a ray from A strike the first principal plane at C, and cutting S_1 at B. It will then appear to proceed after refraction in the direction $C'B'$, such that $E_1C = E_2C'$ on the same side of the axis, and $x'B' = xB$ *on opposite sides of the axis.* The image A' is located where $C'B'$ crosses the principal axis. If the object-point, such as D, lies nearer the lens than S_1, the corresponding ray DC must be prolonged backwards to find the distance xy corresponding to Bx in the first case. After refraction the ray cuts S_2 above the axis.

Dioptral Formulæ—Single Thick Lens from Radii.—Let r_1 be the radius of the first, and r_2 that of the second surface, let D_E be the equivalent dioptric power, and E_1 and E_2 the equivalent points. Then

$$D_E = \frac{100\,(\mu-1)\,(r_1+r_2-t\,(\mu-1)/\mu)}{r_1 r_2} = \frac{100Q\,(\mu-1)}{r_1 r_2}$$

$$E_1 = \frac{r_1 t}{\mu Q} \qquad E_2 = \frac{r_2 t}{\mu Q} \qquad T = t-(E_1+E_2)$$

If the distances are expressed in terms of a metre

$$D_E = \frac{(\mu-1)\,(r_1+r_2-t\,(\mu-1)/\mu)}{r_1 r_2} = \frac{Q\,(\mu-1)}{r_1 r_2}$$

Single Thick Lens from Powers.—Let d_1 be the power of the first surface found from $d=100\ (\mu-1)/r$; d_2 is the power of the second surface; the thickness t being in cm.

$$D_E=d_1+d_2-\frac{d_1 d_2 t}{100\mu}$$

$$E_1=\frac{d_2 t}{D_E \mu} \qquad E_2=\frac{d_1 t}{D_E \mu}$$

If t is expressed in terms of a metre

$$D_E=d_1+d_2-d_1 d_2 t/\mu.$$

Combination of Two Thick Lenses.—Let D_1 and D_2 be the powers of the two lenses, T_1 and T_2 their respective optical thicknesses, and d the distance in cm. between the adjacent equivalent planes of the two lenses.

$$D=D_1+D_2-D_1 D_2 d/100$$

If d is expressed in terms of a metre

$$D=D_1+D_2-D_1 D_2 d$$

The first equivalent plane E is distant from E_1 of the first lens

$$E=\frac{D_1 D_2 d}{D_1(D_1+D_2-D_1 D_2 d/100)}=\frac{D_1 D_2 d}{D_1 D}=\frac{D_2 d}{D}$$

The second equivalent plane E′ is distant from E_2 of the second lens

$$E'=\frac{D_1 D_2 d}{D_2(D_1+D_2-D_1 D_2 d/100)}=\frac{D_1 D_2 d}{D_2 D}=\frac{D_1 d}{D}$$

$$T=d+T_1+T_2-(E+E')$$

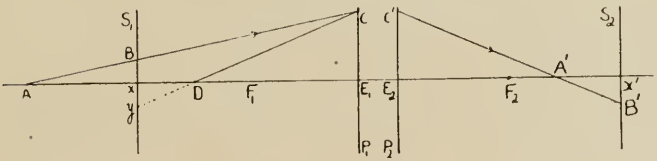

FIG. 198.

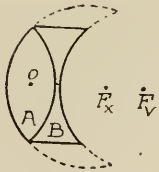

FIG. 199.

Strong Opposite Lenses.—It is difficult to get absolute neutralisation with strong lenses, say over 10 D, there being always some slight positive movement. This is due to the thickness of the Cx., or rather to the interval between the optical centres of the two lenses. As shown in Fig. 199 by the dotted lines, the two lenses actually constitute a weak Cx. meniscus, because with the same radius of curvature, the total lens is one formed of two intersecting circles, or it is an equicurved lens (q.v.).

The thickness of a Cc. lens in the centre, no matter how strong it is, can be ignored, but this is not the case with a strong Cx. If the focal length of the Cx. is equal to that of the Cc., it is clear that F_v of the Cc. cannot coincide with F_x of the Cx. Parallel light incident on B is rendered divergent as if proceeding from F_v, a point outside F_x, and is therefore slightly convergent after refraction by A. If parallel light is incident on A, it is converged to F_x, a point nearer than F_v, so that it is still slightly convergent after refraction.

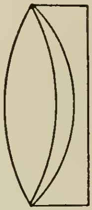

Fig. 200.

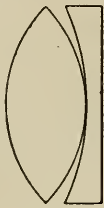

Fig. 201.

Thus a strong DCx. and DCc. lens of exactly equal powers do not actually neutralise each other. Or it can be explained as follows:

If the light be incident on the Cx. it is converged, and the convergence is increased as it traverses the thickness of the two lenses to an extent that the final Cc. surface is unable to neutralise. On the other hand, if the light is incident first on the Cc. surface it is diverged, but in passing through the Cx. some divergence is lost, with the result that the Cx. surface over-neutralises it and produces a slight positive effect. Thus with either lens to the front, the result is the *same* Cx. power, but when the Cx. lens is in advance of the Cc. the *effectivity* of the resultant Cx. power is slightly enhanced; that is, the back surface focus is shorter. Again, if a ray of light originally parallel to the axis traverses first a Cx. and then a Cc. of equal dioptric power, or *vice versa*, its passage *in both cases* is, in the Cx., at a part of the lens more distant from the axis than that of the Cc. and, therefore, where the prismatic element is greater in the former.

Strong Cx. and Cc. lenses may neutralise each other at the centre and not at the periphery or *vice versa*, with excess of either Cx. or Cc. effect at the one or other; or there may be Cc. effect at the centre and Cx. at the periphery, or the reverse. In such cases the lenses are not in contact either at the centre (Fig. 200) or at the periphery (Fig. 201), and the fault is due to the increased effect of a Cx. owing to separation. It is this separation that renders the neutralisation of torics and deep menisci so difficult.

A strong Cc. being thin at its axis, its required radius for a given focal length would be calculated by the formula where thickness is ignored, while that of a strong Cx. would need to be calculated with its thickness considered. The + 20 D from a trial case, being of large diameter, is about ·75 cm. thick in the centre, and its radius would need to be shorter than that of the − 20 D to have equal *equivalent* power. Giving the same radius to each, the *true* or *equivalent* power of the Cx. is weaker than that of the Cc. In order that two strong opposite lenses should neutralise, the Cc. must be the more powerful, the focal length of the Cx. being approximately one third its thickness longer than that of the Cc., which, however, is not the case when the radii of curvature of the two are equal. In short, although a thick Cx. has a longer equivalent focal length than a Cc. of similar radius and μ, it is not sufficiently so for the Cx. to be neutralised by the Cc. For a -20 D whose $F=-5$ cm. to neutralise a Cx. having a thickness of ·75 cm., the Cx. would need have $F=5{\cdot}25$ cm., or $D=+19$, and if $\mu=1{\cdot}5$ would require a radius of curvature of 5·125 cm. In other words the *back foci* of the lenses must be equal if they are to neutralise each other. If, therefore, a Cx. and a Cc. do neutralise, the Cc. is stronger, but the difference is quite inappreciable in weak lenses, and not of importance in spectacle lenses, even if strong.

In modern cases of test lenses the concaves are of their indicated strength, but the convexes are made to neutralise the concaves of similar numerical value. Up to 10 D the difference is negligible, but the nominal + 10 D is only 9·8 D approximately, and the nominal + 20 is +18·75 D approx., the intermediate numbers being of proportional nominal value. Whether there is good reason for this arrangement is somewhat doubtful.

CHAPTER XVI

COMPOUND REFRACTING SYSTEMS

The Nodal Points.—The term is applied to the point or points on the principal axis of any system, through which the secondary axes pass. Thus the optical centre of a thin lens, and the equivalent points of a thick lens or system, bounded on both sides by air or media of the same optical density, have the properties of nodal points. If, however, the first and last media are different, then the equivalent points, although retaining their property of locating the principal planes or planes of refraction, no longer act as the crossing or nodal points of the secondary axes. Instead, we have a second point or pair of points—the *nodal points*—displaced towards the denser medium if the system is positive, and towards the rarer if it is negative. This can be illustrated very well by the case of a single refracting surface. Here the refracting plane is the surface itself, while the nodal point is at the centre of curvature of the surface.

Principal points and *nodal points* coincide when the first and the last media are of the same refractive index; they are then, in this book, termed *equivalent points*, and the focal length of the system is the same on both sides. When the first and last media differ, F_1 and F_2 also differ, and employing the example of the single surface, the difference between them is equal to the radius, that is, to the distance between the nodal point and the principal point; also the ratio of F_1 to F_2 is the ratio of the indices of the first and last media. Refraction depends on *change* in the indices of refraction of two media. So long as light remains in the same medium there is no change in its course, but when it passes into another medium, the alteration of direction is the more violent as the change of indices is great.

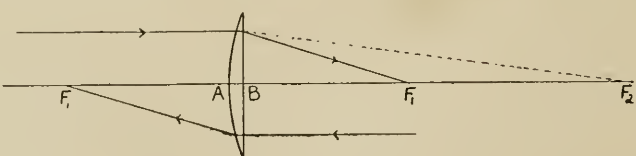

Fig. 202.

The Plano-Cx. Lens.—The formula for a thin plano-Cx. lens, $F = r/(\mu - 1)$, is the same as that for F_1, the anterior focus of a single surface. If parallel light enters at the plane surface of the lens B, all the refraction takes place at the other, where light passes from the dense into the rare medium. If the light is incident at the curved surface A, it is refracted towards a more distant point F_2, since it passes from the air into a dense medium. But at B, the second surface, which the light meets convergently, it is again refracted such that F_2 becomes $F_2/\mu = F_1$ as measured from the surface B. The lens being thin—i.e. t being zero—F is then taken to be the same whether the light enters the one surface or the other.

If the lens is thick, t cannot be ignored. The light on meeting the surface B is converging to $F_2 - t$ beyond that surface, and since $F_2/\mu = F_1$ we have $(F_2 - t)/\mu = F_1 - t/\mu$, which is the back surface F from the plane surface. Thus the shortening of F which results from entry of the light into air is $(F_2 - t)/\mu$ or $F_1 - t/\mu$.

Passage from One Medium to Another.—If light is tending to a focus F in a dense medium, but meets a rare medium before the focus is formed, the surface being plane, F is shortened to F'. The shortening is calculated from d/μ where d is the distance of F beyond the plane surface. If the final medium is denser F is lengthened to $d\mu$. Here d is the $F_2 - t$ of the last article.

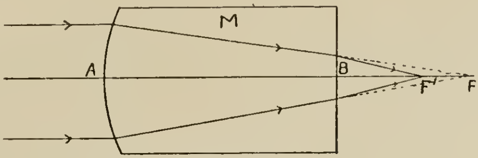

Fig. 203.

Thus in Fig. 203 suppose light is convergent in a medium M, as a small tank of water, and bends to focus at F 10″ behind A the curved surface, but at 6″ meets a plane boundary B beyond which is air.

Then $d = 4''$ and $BF' = 4/1{\cdot}33 = 3''$.

If the outside medium were oil of $\mu = 1{\cdot}46$ instead of air F is lengthened to F''.

Then $BF'' = d \times 1{\cdot}47/1{\cdot}33 = 4{\cdot}4''$

In the case of a hemisphere, for example, F of the curved surface$= r\mu/(\mu - 1)$, but the light meets air at a distance$= r$, so that $d = r\mu/(\mu - 1) - r$, and this divided by μ works down to $r/\mu(\mu - 1)$, which is F_B from the plane surface of a hemisphere.

Change of F in Dense Media.—The change in the power and F of a lens when transferred from air to some denser medium is very marked. It has been shown that F is inversely proportional to $(\mu - 1)$, so that when a lens of μ_2 is immersed in a medium of μ_1, we have $F : F' :: (\mu_r - 1) : (\mu_2 - 1)$, where μ_r is the relative index μ_2/μ_1. The lens has a focal length of F in air, and one of F′ in the medium (Fig. 203), the same as if it were made of a substance of μ_r, and surrounded by air. Thus

$$F'=\frac{F(\mu_2-1)}{(\mu_r-1)} \quad \text{or} \quad D'=\frac{D(\mu_r-1)}{(\mu_2-1)}$$

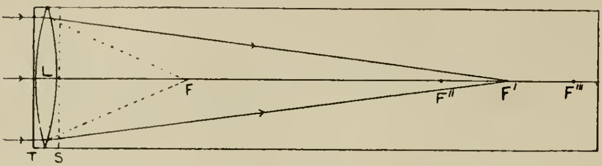

FIG. 204.

For instance, a thin DCx. lens L (Fig. 204) of $\mu=1{\cdot}5$ and $r=8$ in, has in air $F=8$ in. If it be placed into a tank of water where $\mu_r=1{\cdot}5/1{\cdot}33=1{\cdot}13$,

Then $$F'=\frac{8\times{\cdot}5}{{\cdot}13}=30 \text{ in.}$$

or the dioptral change is from $+5$ D to $+1{\cdot}3$ D. Thus a glass lens placed into water has its F increased nearly four times, and its power correspondingly reduced.

If, however, the tank were so small that it only just contains the lens, its boundaries being T and S beyond which is air, F' is shortened to $F''=30/1{\cdot}33=22{\cdot}5$ in. On the other hand if beyond S there were oil of $\mu=1{\cdot}45$, F' becomes lengthened to $F'''=30\times1{\cdot}45/1{\cdot}33=32{\cdot}62$ in.

The crystalline lens of the eye *in situ* has power of about 22 D, in air it has about 125 D. Here the relative index is about 1·09. The lower the relative μ the greater is the change; if a lens is placed into cedar oil all its refracting power is lost since the two μ's are practically equal, and $\mu_r=1$, so that $\mu_r-1=0$.

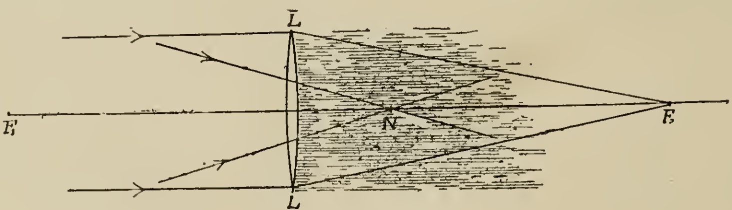

FIG. 205.

Thin Lens bounded by Different Media.—When a thin lens is bounded, say, on one side by air and on the other by some medium denser than air, since the lens is thin all the refraction is presumed to take place in the refracting plane, the position of which is presumed to be unaltered, but the focal lengths become lengthened. The distance of the nodal point from the principal point is $F_2 - F_1$. These are calculated from

$$F_1 = \frac{\mu_1 r_1 r_2}{r_1(\mu_2 - \mu_3) + r_2(\mu_2 - \mu_1)} \quad \text{and} \quad F_2 = \frac{\mu_3 r_1 r_2}{r_1(\mu_2 - \mu_3) + r_2(\mu_2 - \mu_1)}$$

Thus let a thin DCx. lens (Fig. 205) of 10 in. radii and $\mu = 1{\cdot}5$ be bounded in front by air and behind by water. Then

$$F_1 = \frac{1 \times 10 \times 10}{10 \times {\cdot}17 + 10 \times {\cdot}5} = 15 \text{ in.} \quad \text{and} \quad F_2 = \frac{1{\cdot}33 \times 10 \times 10}{10 \times {\cdot}17 + 10 \times {\cdot}5} = 20 \text{ in.}$$

Therefore the distance of the nodal point N through which the secondary axes now pass is $F_2 - F_1 = 20 - 15 = 5''$ behind the refracting plane LL, which remains unchanged.

Cases of Various Media.—When a thin lens of μ_2 separates two media of μ_1 and μ_3—that is, when there are three different media separated by two curved surfaces—the following formula can also be employed:

$$\frac{\mu_3}{F} = \frac{\mu_2 - \mu_1}{r_1} + \frac{\mu_3 - \mu_2}{r_2}$$

If there are four media

$$\frac{\mu_4}{F} = \frac{\mu_2 - \mu_1}{r_1} + \frac{\mu_3 - \mu_2}{r_2} + \frac{\mu_4 - \mu_3}{r_3}$$

The power of any number of surfaces separated by negligible distances can be found by taking the sum of their anterior focal powers and multiplying it by the last μ—*i.e.* by μ_3 or μ_4 as the case may be. If the last medium be air, like the first, we have 1/F equal to the sum of the anterior focal powers of all the media.

It should be particularly noted that in the above the numerator of each fraction is obtained by deducting the preceding μ from the μ following—*e.g.* $\mu_3 - \mu_2$, and that r is positive or negative according as it is respectively Cx. or Cc. towards the direction of the light. Also it should be noted that F is either F_1 or F_2 (as calculated in the preceding article). If the light passes one way instead of the other, F_1 and F_2 change places, as do μ_1 and μ_3, or μ_4 as the case may be; also the signs of the radii change.

We have a case of four media when light passes through a combination formed by a bi-focal made by the insertion of a deeply curved convex segment of high μ into a larger lens of low μ. Such a combination is also formed by the contact of a double Cc. lens of, say, $\mu = 1{\cdot}5$ with a double Cx. lens of, say, $\mu = 1{\cdot}6$, the two being of equal curvature. The focal power can be found by calculating for each lens separately and then adding them together, or by calculating for each surface separately, as indicated above.

Thick Lens bounded by Different Media.—In this case (Fig. 206) the thickness of the lens cannot be ignored, and there are now two principal

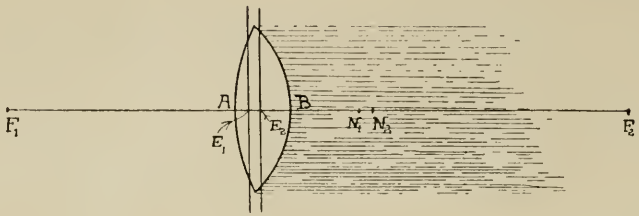

Fig. 206.

and two nodal points in the system such that the distance $P_1N_1 = P_2N_2 = F_2 - F_1$, and $F_1/F_2 = \mu_1/\mu_3$. The cardinal points are calculated from the following formulæ.

$$F_1 = \frac{\mu_1 r_1 r_2}{r_1(\mu_2 - \mu_3) + r_2(\mu_2 - \mu_1) - t(\mu_2 - \mu_1)(\mu_2 - \mu_3)/\mu_2}$$

Let the denominator of the above be called Q, then:

$$F_1 = \frac{\mu_1 r_1 r_2}{Q} \qquad F_2 = \frac{\mu_3 r_1 r_2}{Q}$$

$$P_1 = \frac{\mu_1 r_1 t(\mu_2 - \mu_3)}{\mu_2 Q} \text{ from A.} \qquad P_2 = \frac{\mu_2 r_2 t(\mu_2 - \mu_1)}{\mu_2 Q} \text{ from B.}$$

The back surface focal distances can be obtained by deducting P_1 from F_1 and P_2 from F_2.

As an example, suppose the case of the crystalline lens of the eye with the cornea and aqueous removed (Fig. 206). Let $\mu_1 = 1$, $\mu_2 = 1{\cdot}45$, $\mu_3 = 1{\cdot}33$, $r_1 = 10$ mm., $r_2 = 6$ mm., and t, the thickness of the crystalline, $= 3{\cdot}6$ mm.

Working from the given data we find

$$F_1 = 15{\cdot}93 \quad \text{and} \quad F_2 = 21{\cdot}24$$
$$P_1 = {\cdot}8 \text{ from A} \quad \text{and} \quad P_2 = 2{\cdot}63 \text{ from B.}$$

The distances of the nodal points from the equivalent points are

$$N_1 = F_2 - F_1 = 21{\cdot}24 - 15{\cdot}93 = 5{\cdot}31 \text{ from } P_1$$
$$N_2 = F_2 - F_1 = 21{\cdot}24 - 15{\cdot}93 = 5{\cdot}31 \text{ from } P_2$$

or N_1 is 6·11 mm. from A, and N_2 is 2·68 mm. from B

The equivalent thickness or optical interval T = ·17 mm., and the same interval exists between N_1 and N_2.

Combinations of Two Systems when μ_1 differs from μ_4.—Let F_1 and F_2 be the anterior and posterior focal distances of the first system, and F_1' and F_2', those of the second system. E_1 and E_2 pertain to the first, and E_1' and E_2' to the second system. The distance d between the two systems is that

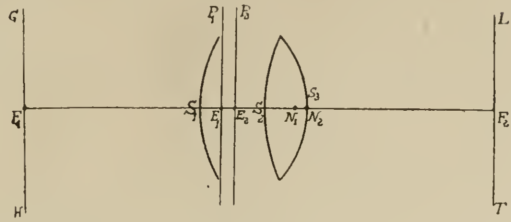

FIG. 207.

between E_2 and E_1', *i.e.* between the two most adjacent points. Let Q be the distance between F_1' and F_2, that is,

$$Q = F_2 + F_1' - d.$$

F_A and F_P are the anterior and posterior focal lengths of the combined system, P_1 and P_2 are the principal points, and N_1 and N_2 are the nodal points.

$$P_1 = \frac{F_1 d}{Q} \text{ from } E_1 \qquad P_2 = \frac{F_2' d}{Q} \text{ from } E'_2 \qquad F_A = \frac{F_1 F_1'}{Q} \qquad F_P = \frac{F_2 F_2'}{Q}$$

$$F_P - F_A = P_1 - N_1 = P_2 - N_2, \qquad P_1 F_A = N_2 F_P, \qquad P_2 F_P = N_1 F_A$$

$$T = d + T_1 + T_2 - (P_1 + P_2)$$

Such a system as the above is found in the eye (Fig. 207), taking the two components independently; or in a lens placed in front of the eye, the latter, as a whole, being the second system.

Calculations concerning the eye are to be found in the chapter on the *Gauss Equation* (q.v.), but they are more fully treated in "Visual Optics and Sight Testing."

Some Recapitulated Points on Compound Systems.—Rays parallel to the principal or a secondary axis in the first medium meet on that *same axis* in the last medium, and *vice versa.* Rays diverging from a point in the focal plane of the first medium are parallel in the last, and *vice versa.*

A ray directed to the one nodal point, after refraction, appears to come from the other, *and its direction is parallel to its original course.*

A ray directed to any point on the one principal plane, appears after refraction, to proceed from a corresponding point situated on the other. These two points are *on the same side of the axis and equally distant from it,* and each is the image of the other.

The distances $P_1 P_2 = N_1 N_2 : N_1 F_1 = P_2 F_2 : N_2 F_2 = P_1 F_1$

$F_2 - F_1 = P_1 N_1 = P_2 N_2 =$ the imaginary equivalent radius of curvature.

$F_2/F_1 = \mu_x/\mu_1$ = the imaginary combined relative μ, where μ_x is that of the last medium.

$$F_1\mu_x = F_2\mu_1 \therefore \frac{F_1\mu_x}{\mu_1} - F_1 = \frac{F_1(\mu_x - \mu_1)}{\mu_1} = P_1N_1 = P_2N_2$$

$$\text{If } \mu_1 = 1,\ F_1(\mu_x - 1) = P_1N_1 \ :\ \text{If } \mu_1 = \mu_x,\ P_1N_1 = 0$$

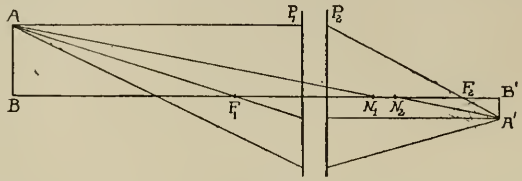

Fig. 208.

Construction of Image.—In Fig. 208 let P_1P_2 be the principal planes, N_1N_2 the nodal points, F_1F_2 the principal foci, and AB any object in the rarer medium, the system being positive. A ray AP_1 parallel to the axis is refracted at P_2 through F_2. A secondary axis $A\,N_1$ passes on emergence from N_2 parallel to its original course. A ray passing through F_1 is, after refraction, parallel to the axis.

Where these rays meet in A' is the image of A, so that $B'\,A'$ is the complete image of $A\,B$. As will be seen the construction, with the exception of the displacement of N_1 and N_2, is the same as for any ordinary thick lens or system in air. Provided the six cardinal points are known, the most complicated system can be reduced to the simplicity of a single lens.

When the first and last media have the same optical density, the equivalent and nodal points coincide, so that the relative sizes of image and object are as their distances from the equivalent points; when the media are different the relative sizes of image and object depend upon their distances from the nodal points. The formulæ for single thin lenses, and single refracting surfaces are applicable for calculating conjugate foci, provided all measurements are taken from the appropriate equivalent points.

Negative System bounded by Different Media.—This does not occur in practice so that no special discussion is necessary. The calculations would be similar to those for a positive system.

CHAPTER XVII

THE GAUSS EQUATION

By the aid of the Gauss equation every optical system can be so simplified that all problems of conjugate foci, etc., may be worked by the formulæ applicable to single thin lenses. The calculations in the case of more than two surfaces are necessarily long, but they always involve the solution of a continued fraction, so that the difficulties are purely arithmetical.

In using the equation, which serves for any number of surfaces, media and thicknesses, the pencils of light are presumed to be axial and small; in other words, *aberration* is neglected. In order to keep the formulæ as symmetrical as possible and avoid a mixture of signs, the following conventions must be observed, namely, (1) all distances measured to the left of a surface are negative, and to the right positive; (2) all thicknesses are considered negative, and therefore, on substituting actual values, it is necessary to use the minus sign.

Thick Lens.—The following formulæ are deduced from the consideration of the lens having positive radii of curvature according to the above convention, *i.e.* a periscopic with the concave surface turned towards the right. Let μ_1 be the refractive index of the surrounding medium, μ_2 that of the lens, t the axial thickness, r_1 the radius of the first surface, and r_2 that of the second. Let u be the object distance, v_1 the image distance formed by refraction at the first surface, and v the final image distance after refraction at the second. The fundamental equation connecting u and v_1 is

$$\mu_2/v_1 - \mu_1/u = (\mu_2 - \mu_1)/r_1$$

but in order to simplify the formulæ $(\mu_2 - \mu_1)/r_1$ is replaced by F_1, while μ_2/v_1 and μ_1/u are replaced by $1/v_1$ and $1/u$ respectively. These last two are termed *reduced* expressions, *i.e.* actual distances divided by the μ*'s* of the media to which they pertain. Similarly in the expression connecting v_1 and v, given later, $(\mu_1 - \mu_2)/r_2$ and μ_1/v are replaced by F_2 and $1/v$ respectively, while t is also employed reduced, being divided by the μ in which it is measured. Consequently the values subsequently found are similarly reduced and must be multiplied by the μ in which each occurs, in order that their true values may be determined.

The fundamental formula reduced becomes

$$1/v_1 - 1/u = F_1, \text{ or } 1/v_1 = F_1 + 1/u$$

whence $$v_1=\frac{1}{F_1+1/u} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The expression connecting v_1 and v is

$$\mu_1/v-\mu_2/(v_1+t)=(\mu_1-\mu_2)/r_2$$

which, in reduced terms, becomes

$$1/v-1/(v_1+t)=F_2, \text{ or } 1/v=F_2+1/(v_1+t)$$

whence $$v=\cfrac{1}{F_2+\cfrac{1}{v_1+t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in (2) the value of v_1 in (1) we have

$$v=\cfrac{1}{F_2+\cfrac{1}{t+\cfrac{1}{F_1+\cfrac{1}{u}}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

On working out this continued fraction in (3) we get

$$v=\frac{u(F_1t+1)+t}{u(F_1F_2t+F_1+F_2)+F_2t+1} \quad . \quad . \quad . \quad . \quad (4)$$

which, for the sake of brevity, is usually written

$$v=\frac{C\,u+D}{A\,u+B} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $A=F_1F_2t+F_1+F_2$; $B=F_2t+1$

$C=F_1t+1$; $D=t$.

No. (5) connects v and u when both are finite distances. If u is at ∞ the quantities D and B disappear and u cancels, so that the focal length measured from the second surface is

$$v=C/A \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The value of v in equation (6) *is the back focal distance as measured from the pole of the second surface.*

If v is at ∞, then $A\ u+B=0$, so that *the back focal distance measured from the pole of the first surface is*

$$u=-B/A \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Before proceeding further an expression for the total magnification M produced by the lens must be found.

Let m_1 be the magnification due to the first surface, and m_2 that due to the second; then the total magnification M is $m_1\times m_2$.

In Fig. 209 let AB be an object in front of the first surface, and $B'A'$ its corresponding image. A ray from A meeting the vertex in x will be

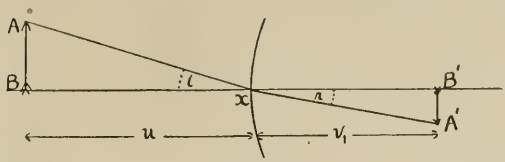

Fig. 209.

refracted to A' such that i and r are the angles of incidence and refraction respectively. Then

$$m_1 = \mathrm{A'B'/A\,B}$$

But i and r being small, AB/u may be considered equal to sin i, and $A'B'/v_1 = \sin r$, and $\sin r/\sin i = \mu_1/\mu_2$.

Therefore $$m_1 = \mathrm{A'B'/A\,B} = \mu_1 v_1/\mu_2 u$$

But u/μ_1 and v_1/μ_2 are *reduced* quantities and therefore to preserve our notation the refractive indices must be omitted, so that

$$m_1 = v_1/u$$

Similarly the magnification m_2 of the second surface is

$$m_2 = v/(v_1 + t)$$

Therefore the total magnification

$$\mathrm{M} = v_1/u \times v/(v_1 + t)$$

But from (1) $$v_1/u = 1/(\mathrm{F}_1 u + 1)$$

And from (1) and (2) $$\frac{v}{v_1 + t} = \frac{1}{\mathrm{F}_2\left(\frac{u}{\mathrm{F}_1 u + 1} + t\right) + 1}$$

Therefore $$\mathrm{M} = \frac{1}{\mathrm{F}_1 u + 1} \times \frac{\mathrm{F}_1 u + 1}{u(\mathrm{F}_1\mathrm{F}_2 t + \mathrm{F}_1 + \mathrm{F}_2) + \mathrm{F}_2 t + 1}$$

$$= \frac{1}{u(\mathrm{F}_1\mathrm{F}_2 t + \mathrm{F}_1 + \mathrm{F}_2) + \mathrm{F}_2 t + 1}$$

$$= \frac{1}{\mathrm{A}\,u + \mathrm{B}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Now let the magnification be $+1$, *i.e.* let *virtual* image and object be equal in size. Then

$$\mathrm{A}\,u + \mathrm{B} = 1$$

whence $$u=P_1=(1-B)/A \quad . \quad . \quad . \quad . \quad . \quad (9)$$

this distance being measured from the first surface.

On substituting this value of u in (5), the corresponding value of v is

$$v=P_2=\frac{C-BC+AD}{A}=\frac{C-1}{A} \quad . \quad . \quad . \quad . \quad (10)$$

because it can be shown that $AD-BC=-1$. This distance is measured from the second surface.

These planes of unit virtual magnification denote the *equivalent planes*, and the points P_1 and P_2 where they cut the axis are the *equivalent points*. If it were possible to place a small object in the one plane, then its virtual image, identical in all respects to the object, would be situated in the other.

If the magnification be -1, then the corresponding values of u and v will locate the *symmetrical planes*, where object and *real* image are equal in size.

To find, therefore, the equivalent focal distances, the values of (9) and (10) must be added to those of u and v in (5); thus

$$v+\frac{C-1}{A}=\frac{C(u+(1-B)/A)+D}{A(u+(1-B)/A)+B}$$

which simplifies to $$A=1/v-1/u \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This expression (11) should be compared with that of a simple thin lens for the focal length in terms u and v. Then if $u=\infty$

$$v=1/A \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and if $v=\infty$ $$u=-1/A \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The princpal focal distances given in (12) and (13) are equal when the first and last μ's are of equal optical density. The values are reduced, and must be multiplied by the μ in which each occurs, so that when in air they are unchanged.

As a simple example, let $r_1=6$, $r_2=8$, $\mu_2=1{\cdot}5$, $\mu_1=1$ (air), and $t=1$; then

$$v=\cfrac{1}{F_2+\cfrac{1}{t+\cfrac{1}{F_1+\cfrac{1}{u}}}} \qquad v=\cfrac{1}{{\cdot}0625+\cfrac{1}{-{\cdot}666+\cfrac{1}{{\cdot}0833+\cfrac{1}{u}}}}$$

which works out to $$v=\frac{{\cdot}9445\,u-{\cdot}666}{{\cdot}1423\,u+{\cdot}9584}$$

Then, if $u=\infty$

$$v=\frac{C}{A}=\frac{{\cdot}9445}{{\cdot}1423}=6{\cdot}63$$

Also
$$u=\frac{-B}{A}=\frac{-\cdot9584}{\cdot1423}=-6\cdot73$$

$$P_1=\frac{1-B}{A}=\frac{1-\cdot9584}{\cdot1423}=+\cdot29$$

$$P_2=\frac{C-1}{A}=\frac{\cdot9445-1}{\cdot1423}=-\cdot39$$

The equivalent focal distance

$$1/A=1/\cdot1423=7\cdot02$$

Multiple Surfaces.—The Gauss equation may be applied to an optical system having any number of surfaces surrounded by corresponding media of different densities and thicknesses. The equation

$$v=\frac{C\,u+D}{A\,u+B}$$

is universal, although the various values become more complicated as the number of surfaces is increased, but the problem always takes this form, involving the solution of a continued fraction.

Suppose the case of the eye having three surfaces, F_1, F_3 and F_5 with thicknesses t_2 and t_4, with the following data $r_1=8$, $r_3=10$, $r_5=6$, $t_2=3\cdot6$, $t_4=3\cdot6$, $\mu_1=1$, $\mu_2=1\cdot333$, $\mu_3=1\cdot45$, $\mu_4=1\cdot333$. Then

$$F_1=\frac{\mu_2-\mu_1}{r_1}=\frac{1\cdot333-1}{8}=\cdot0416$$

$$F_3=\frac{\mu_3-\mu_2}{r_3}=\frac{1\cdot45-1\cdot333}{10}=\cdot0117$$

$$F_5=\frac{\mu_4-\mu_3}{r_5}=\frac{1\cdot333-1\cdot45}{-6}=\cdot0195.$$

The reduced value of

$$t_2=-3\cdot6/1\cdot333=-2\cdot7007$$

and that of
$$t_4=-3\cdot6/1\cdot45=-2\cdot4828$$

Then we have

$$v=\cfrac{1}{F_5+\cfrac{1}{t_4+\cfrac{1}{F_3+\cfrac{1}{t_2+\cfrac{1}{F_1+\cfrac{1}{u}}}}}} \qquad v=\cfrac{1}{\cdot0195+\cfrac{1}{-2\cdot4828+\cfrac{1}{\cdot0117+\cfrac{1}{-2\cdot7007+\cfrac{1}{\cdot0416+\cfrac{1}{u}}}}}}$$

which becomes, when worked out,

$$v=\frac{\cdot 7586\,u-5\cdot 1050}{\cdot 0668\,u+\cdot 8689}$$

That is, $A=\cdot 0668$, $B=\cdot 8689$, $C=\cdot 7586$, $D=-5\cdot 1050$.

The anterior $F=-\mu_1/A=-1/\cdot 0668=-15$ mm.

The posterior $F=\mu_4/A=1\cdot 333/\cdot 0668=20$ mm.

$P_1=\mu_1(1-B)/A=\cdot 1311/\cdot 0668=1\cdot 96$ mm. from r_1

$P_2=\mu_4(C-1)/A=-\cdot 3128/\cdot 0668=-4\cdot 81$ mm. from r_5

or $7\cdot 2-4\cdot 81=2\cdot 39$ mm. from r_1.

The nodal points N_1 and N_2, found by subtraction, are, respectively, 6·96 and 7·39 mm. from r_1. Neglecting the intervals between P_1 and P_2 and that between N_1 and N_2, we have P at 2·2 mm., and N at 7·2 mm. from the apex of the cornea.

When working with the Gauss equation two things must be borne in mind; firstly, the convention as to signs upon which the symmetry of the formulæ depend; and secondly, the use of *reduced* instead of absolute distances in order to simplify the formulæ by the inclusion of the refractive indices in other terms. Thus v, the final image distance, is always multiplied by the index of the last medium to give the absolute values of the second principal focus and the second equivalent point. On the other hand u which, in the final expression, denotes the anterior focus and first principal point is, except in very rare cases, already reduced, the first medium generally being air. In fact, the same may be said of v, as a difference in the indices of the first and last media occurs only in the case of the eye, and in certain instruments as, for instance, the immersion objective of the microscope.

The calculation of a continued fraction for three surfaces being complicated, the results obtained may be checked by the following, which is the continued fraction worked down.

$$v=\frac{uN+R}{u(F_5N+F_1F_3t_2+F_1+F_5)+F_5R+F_3t_2+1}$$

where $N=F_1F_3t_2t_4+F_1t_2+F_1t_4+F_3t_4+1$

and $R=F_3t_2t_4+t_2+t_4$

CHAPTER XVIII

THE CURVATURE SYSTEM

Curvature is a symmetrical departure from straightness of a line or planeity of a surface. Unless otherwise stated, it is presumed to be spherical or circular, but it may be toroidal or of an aspherical nature. Curvature C is the reciprocal of the radius; thus $C=1/r$, and this applies also to figures other than circular, and surfaces other than spherical. In the case of circles and spheres, the curvature is equal at all points, but this is not so with conic curves. The curvature at any point on any refracting or reflecting surface, having a symmetrical axis, is determined by dropping from it a normal to the axis; the length of this line is then the radius of curvature of that particular point.

Curvature can also be expressed in diopters; the unit 1D having a radius of 1 M, and for radius expressed in cm., mm., and inches respectively,

$$D=100/r,\ 1{,}000/r,\ \text{and}\ 40/r.$$

Thus if $r=10''$ then $C=1/10$, or $40/10=4$ D.

Curvature Method.—Formulæ in connection with mirrors, prisms and lenses can be deduced from the paths of the waves. The following are elementary examples of the application of this method, which is by some writers preferred to the "ray" method, since it represents the actual physical change in shape and direction undergone by the wave points when refraction or reflection takes place.

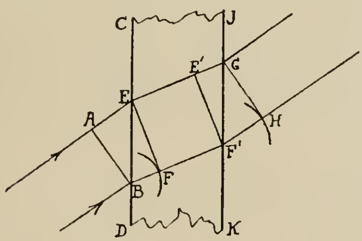

Fig. 210.

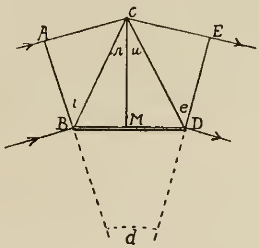

Fig. 211.

Plane Surface.—*A B* (Fig. 210) is a plane wave front incident obliquely on the surface *C D*. If $\mu_1=1$, and $\mu_2=1{\cdot}5$, the part of the wave which enters at *B* travels in the same time to *F* only 2/3 of the distance *A E*. With *B* as centre and $B\ F$ as radius describe a small arc, a tangent $E\ F$ from E showing the inclination of the wave front in the dense medium. At the second surface a similar construction shows the wave front $G\ H$ after emergence, $F'\ H$ being 1·5 times $E'\ G$.

Course of a Wave through a Prism at Minimum Deviation.—Let $C\ B\ D$ (Fig. 211) be a prism on which is incident the plane wave $A\ B$ at an angle of incidence i. The portion B of the wave meeting the base of the prism is retarded to a greater extent than A, the portion in air, so that when the whole wave enters the prism it takes up the position $C\ M$, r being the angle of refraction.

Since the deviation is supposed to be minimum, the total refraction is symmetrical with respect to the surfaces $C\ B$ and $C\ D$, so that $C\ M$ bisects the principal angle. The wave is then incident on the second surface at the angle u, and on emergence it is swung over still more towards the base, so that, when completely clear of the prism, it has the position $E\ D$ making the angle of emergence e with the second surface, e being equal to i.

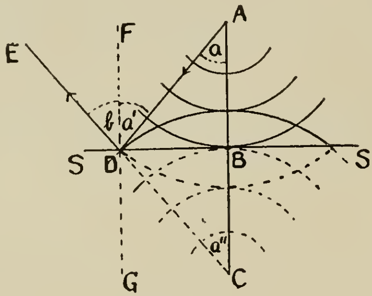

Fig. 211a.

Reflection.—In Fig. 211a let SS be a plane surface, and A a point from which waves diverge. The latter are reflected back with their curvature unaltered, such that they appear to originate from the virtual image C. The distances BC and AB are equal. A ray AD incident at the point D is reflected in the direction E as if from the image C. FG is the normal to the point of incidence, a' is the angle of incidence and b the angle of reflection. From the symmetry of the figure it is obvious that the angles b and a' are equal, proving that the angles of incidence and reflection are equal.

Wave Front—Curvature.—The unit of the dioptric curvature system being one having a radius of 1 metre, the curvature of any wave may be denoted and measured by it. Thus if light diverges from $\frac{1}{2}$ M or 50 cm. the wave is said to have a divergence of 2 D; at some other distance, say 2 M, the curvature would be ·5 D, and so on. Thus if C denote the actual curvature of a wave, it may be expressed either as the reciprocal of its radius r in metres, or simply in diopters D. Then $C=1/r$ or D. Now from the sphero meter formula, for *shallow* curvatures $r=d^2/2s$, where d is the semi-chord, and s the sagitta of the corresponding arc of radius r. In other words, provided the chord remains constant, the radius is inversely proportional to the sag, and *vice versa*, while the curvature of the arc is *directly proportional to the sag on the same chord.* Thus we may say that $1/r$ or $C \propto s$, or $s \propto C$ or $1/r$. It will be seen that the "curvature" formulæ are identical with the "ray" formulæ, only that, with the exception of μ, all the symbols employed in the one are the reciprocals of those used in the other, and *vice versa.*

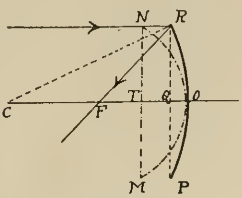

FIG. 212.

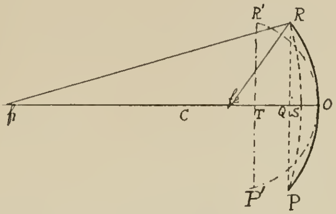

FIG. 213.

Cc. Mirror—Plane Incident Wave.—Let $P\ O\ R$ (Fig. 212) be any Cc. mirror on which is incident the plane wave $P\ Q\ R$. If the aperture be small the points P and R of the wave front first meeting the mirror may be considered to be reflected back to M and N while the central point Q is travelling to the vertex O. When Q has arrived at O the contour of the reflected wave is $M\ O\ N$; it remains to find the curvature of $M\ O\ N$.

Now since $P\ M=O\ Q$, $O\ Q=Q\ T$ and we have $T\ O=2\ O\ Q$. But, since the curvature of an arc may be taken as proportional to the sag for equal chords, $T\ Q$ represents the curvature F of the reflected wave $M\ O\ N$, and $O\ Q$ the curvature C of the mirror. Thus $T\ O = 2\ Q\ O$, or $F = 2\,C$. In other words the curvature of the reflected wave is double that of the mirror, so that the focal distance $O\ F$ is half the radius $O\ C$.

Cc. Mirror—Divergent Wave.—Let $P\ S\ R$ (Fig. 213) be a wave diverging from a near object f_1; then while the vertex S of the wave is travelling to O, the extremities P and R are reflected to P' and R' respectively such that $T\ Q=S\ O$. Then $P'\ O\ R'$ is the reflected wave converging towards f_2. $O\ Q$ is the mirror sag, and since $T\ Q=S\ O$ we have $T\ O=2\ S\ O+Q\ S=2\ Q\ O-Q\ S$.

Let F_1 be the curvature of the object wave $P\ S\ R$, F_2 that of the image wave $P'\ Q\ R'$, while C is the curvature of the mirror, and F its focal curvature or power. Then $Q\ S=F_1$, $O\ Q=C$, $T\ O=F_2$, and $F=2\ C$. Therefore $T\ O=2\ Q\ O-Q\ S$, or $F_2=2\ C-F_1$, so that $F_1+F_2=2\ C=F$, i.e. the focal power of a Cc. mirror is equal to the sum of the object and image curvatures, and this is the formula for expressing conjugate foci. It will be noticed that C, the mirror curvature, is the mean of the object and image curvatures; thus $C=(F_1+F_2)/2$.

Convex Mirror—Plane Incident Wave.—Let *M N* (Fig. 214) be a plane wave incident on the Cx. mirror *P O R*. *O* is now the first incident point, and this is reflected to *Q′*, while *M* and *N* are travelling to *P* and *R*, so that *P Q′ R* is the reflected wave, which can be shown to have a curvature double that of the mirror, as with a Cc. In other words, since QQ′=2 O Q, F=2 C.

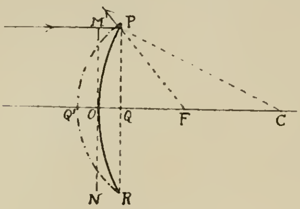

Fig. 214.

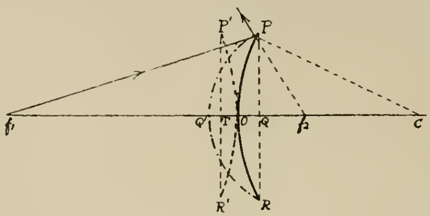

Fig. 215.

Convex Mirror—Divergent Wave.—When the wave is divergent from a near object f_1 (Fig. 215), the incident wave is *P′ O R′*, and the reflected wave *P Q′ R* such that *O Q′=O T+O Q*.

$$\therefore \text{Q Q}'=\text{O Q}+\text{O Q}'=\text{O Q}+(\text{O Q}+\text{O T})=2\text{ O Q}+\text{O T}$$

that is, $F_2=2\,C+F_1$, or $F_2=F+F_1$

In other words the image curvature is equal to the sum of the object and mirror curvatures, because both are divergent in effect. Employing the usual convention as to signs this expression would be written as for a Cc. mirror, *i.e.* $F_1 + F_2 = 2\ C = F$, the negative sign being employed when substituting the value of F.

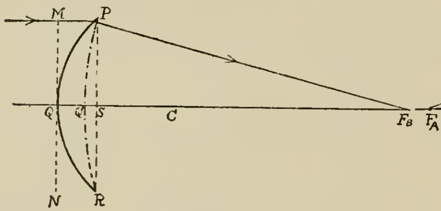

Fig. 216.

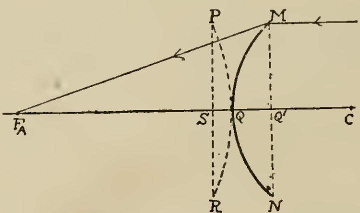

Fig. 217.

Single Surface—Cx.—Let *M N* (Fig. 216) be a plane wave incident on the single Cx. refracting surface *P Q R* such that *P Q′ R* is the refracted wave convergent towards the posterior principal focus F_B. Let *C* be the curvature of the surface, F_B that of the refracted light, and μ the index of the medium, the first being air. Then we have Q S=μ Q Q′. But

$$\text{Q Q}'=\text{C}-\text{F}_B \text{ and Q S}=\text{C}$$

$$\therefore \qquad \text{C}=(\text{C}-\text{F}_B)\mu \qquad \text{or} \qquad \text{F}_B=\text{C}(\mu-1)/\mu$$

Similarly an expression can be found for the anterior principal focus of the same surface $M\ Q\ N$. Here (Fig. 217) the plane wave advances from the denser medium to meet the surface as $M\ N$, the retarded wave convergent towards the anterior focus F_A, being $P\ Q\ R$. Then $Q'\ S = \mu\ Q\ Q'$. But

$$Q'\ S = F_A + C, \text{ and } O\ Q' = C$$

Therefore $F_A + C = \mu\ C$.

and $$F_A = C(\mu - 1)$$

For a *concave surface* the formulæ are the same, C being negative.

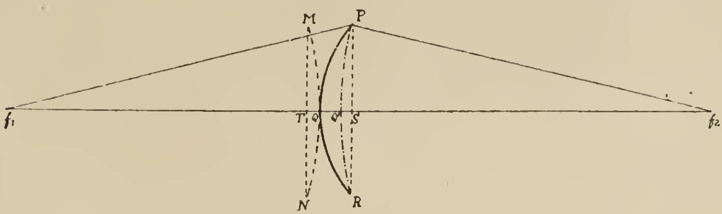

Fig. 218.

Conjugate Foci—Single Cx. Surface.—Let f_1 (Fig. 218) be any near object from which diverges the wave $M\ N$ to the surface $P\ Q\ R$, and let f_2 be the image formed by the image wave $P\ Q'\ R$. Let $TQ = F_1$, $Q\ S = C$ and $Q'\ S = F_2$. Then

$$T\ S = \mu\ Q\ Q' = \mu\ (S\ Q - S\ Q')$$

and $$T\ S = T\ Q + Q\ Q' + Q'\ S$$

$$\therefore\ \mu(C - F_2) = F_1 + (C - F) + F_2 \qquad \text{or} \qquad \mu\ C - \mu\ F_2 = F_1 + C$$

that is $$F_1 + \mu\ F_2 = C\ (\mu - 1)$$

Similar formulæ may be deduced for a *concave surface*, but here C and F_2 are negative.

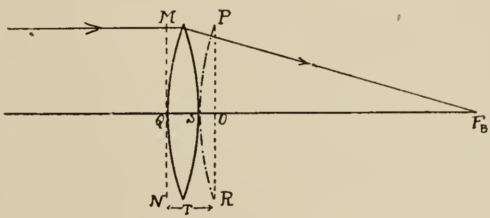

Fig. 219.

Thin Convex Lens.—With a lens the curvature of each surface is likewise represented by their respective sags, so that in the case of a double Cx. (Fig. 219) $Q\ S$ represents the sum of the sags C_1 and C_2. Let $M\ N$ be a plane wave incident on the lens; then, owing to the greater axial thickness, the centre of the wave is retarded more than the periphery, the resulting wave front taking the form *P S R* converging to the focus F_B. Let *T* be the united sags of the lens surface and wave fronts; then $T = C_1 + C_2 + F$. But

$$T = \mu\, Q\, S = \mu\, (C_1 + C_2)$$

$$C_1 + C_2 + F = \mu\, (C_1 + C_2)$$

whence $$F = (C_1 + C_2)\, (\mu - 1)$$

In other words the power F of the lens is the product of the united curvatures and the refractivity of the glass.

Similar formulæ in the case of conjugate foci and for concave lenses can be deduced on similar lines; in numerical examples, of course, C_1 and C_2 of concave lenses are reckoned negative.

CHAPTER XIX

COLOUR

Colour is the result of waves of definite length and frequency.

Primary and Secondary Colours.—There are six or seven distinct colours in the solar spectrum (*vide* Chapter I.), but it was shown by Young, and confirmed by Helmholtz, that every shade of colour in nature can be obtained from the mixture of red, green and violet in certain proportions, whereas these three colours cannot be produced by mixing others. For this reason red, green and blue-violet are termed the *primary* colours, while the other spectrum colours are *secondaries*. Thus red and green, in varying proportions, produce orange or yellow, while green and violet produce blue or indigo.

When a solar spectrum is viewed, there are three main divisions—viz., green in the centre, with red at the one end and violet at the other. Between the centre and the extremities different observers can distinguish, on either or both sides, one or more different colours. Some would describe the spectrum as containing three colours only of varying shades. It should be appreciated, however, that every point in the visible spectrum is due to a different wave length.

A mixture of all the wave-lengths contained in the visible spectrum forms white light; but for white light all of them are not essential provided two complementary colours are contained in the mixture.

Colour Sensation.—According to Young and Helmholtz, there exist in the eye three sets of nerves, each of which conveys to the brain a primary colour sensation. Stimulation of all three produces the sensation of white, of none of them black, or absence of colour and light. By the stimulation of one or more in varying proportions, all colours are mentally appreciated. This subject is more fully treated in, "Visual Optics and Sight-Testing."

Varying Wave-Lengths.—As described in Chapter I, the infra-red, or heat waves, and the ultra-violet, or chemical waves, lie just beyond the two ends of the visible spectrum. They are of the same nature as light waves, but differ from the latter in their effects, and they are, respectively, too long and too short to stimulate the retina. Both heat and actinism are, however, produced to a small extent by visible waves which, therefore, have different properties—namely, that of light on the eye, and of heat or chemical action on bodies. The yellow and green are probably the only waves which are light-producing only.

Complementary Colours.—Two spectrum colours which, when combined, form white light, are complementary to each other. Hence a *complementary colour* may be defined as that which, when united with another, produces white light. The complement of a primary colour is that secondary colour which results from the mixture of the other two primaries; the complement of a secondary colour is that primary colour which is not contained in it.

Spectrum Colour.	*Complement.*
Red.	Green-Blue.
Orange.	Blue.
Yellow.	Blue-Violet.
Green.	Purple-Red.
Blue.	Orange.
Indigo.	Orange-Yellow.
Violet.	Green-Yellow.

The purple-red is not in the visible spectrum, it being a combination of red and violet. A graphical presentation of this table is shown in Fig. 220.

Colours of Light.—Spectrum red and green will, if mixed in certain proportions, produce yellow or orange. If spectrum red, green and blue-violet be mixed in the correct proportions white light is formed. If the wave-lengths of orange-red and blue-green be added together the mean will give the wave-length of yellow, thus, $656+518=1174$, and $1174/2=587$. Taking the wave-lengths of red, green and blue respectively, the sum divided by three will give the wave-length for the brightest part, that is, yellow, which is the nearest approach to white light which the spectrum affords; thus $748+527+486=1761$, and $1761/3=587$.

Colour.	*Violet.*	*Indigo.*	*Cyan Blue.*	*Blue-Green.*	*Green.*	*Greenish-Yellow.*	*Yellow.*
Red	Purple	Dark rose	Light rose	White	Whitish-yellow	Golden-yellow	Orange
Orange	Dark rose	Light rose	White	Light yellow	Yellow	Yellow	—
Yellow	Light rose	White	Light green	Light green	Greenish-yellow	—	—
Greenish-yellow	White	Light green	Light green	Green	—	—	—
Green	Light blue	Sea blue	Blue-green	—	—	—	—
Blue-green	Deep blue	Sea blue	—	—	—	—	—
Cyan blue	Indigo	—	—	—	—	—	—

The quantity of light of one colour necessary to mix with any other to produce white light, or a third colour, does not appear to follow any definite law, but the proportions usually remain the same for different observers.

Certain colours, *e.g.* purple, which do not appear in the spectrum are those formed by a combination of two or more non-adjacent wave-lengths, the resultant effect on the eye being, in general, that colour corresponding to the mean wave-length of the components.

The table on p. 200, according to Helmholtz, shows the effects produced by the addition of any two spectrum colours.

Brightness of Colour.—In a prismatic spectrum the red appears fuller than the violet because the former is more crowded together, while the latter is spread out; this is not the case in a diffraction spectrum, in which the extent of colour is about equal on either side of the yellow. The latter is the brightest part of the spectrum to the human eye, and in general the intensity rises in the prismatic spectrum from zero, at the extreme red, rapidly to the yellow and then, dropping off again, but more slowly, to zero at the extreme violet.

Colours in Pigments.—The primary colours in pigments (dyes, paints or colouring matter) are so-called red, yellow and blue; any other colour is obtained by mixing two primaries.

The primaries and their complements are shown in Fig. 220, from which it will be seen that *the primaries of pigments are the complements of the primaries of light.* Thus 1, 6 and 10 are the primaries of light, and 4, 7 and 12 are the primaries of pigments. Although the primaries of pigments are popularly known as red, yellow and blue, yet the actual tints are not quite those usually associated with the terms.

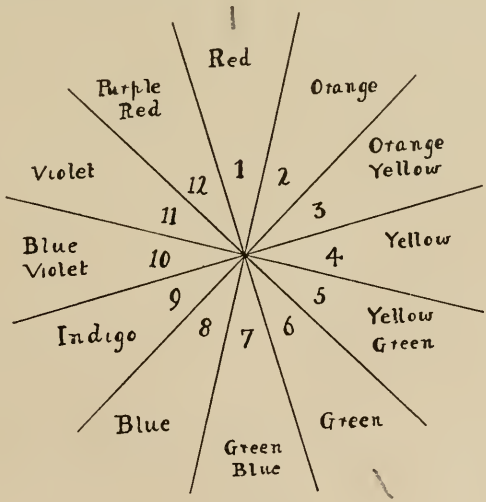

Fig. 220.

Mixing Colours.—The fundamental difference in the results obtained by mixing spectrum lights and pigment colours lies in the fact that the former is an *additive,* and the latter a *subtractive* process. The colouration resulting from mingled lights is due to the mixture of wave-lengths, while the resultant colour of a mixture of pigments is that remaining after each pigment has *absorbed* a certain wave or series of wave-lengths. The tendency of added lights is to give increased illumination and to approximate it to white, while with pigments the mixture tends towards black.

Thus, when the primaries of light, *i.e.* red, green and blue-violet, are mingled—projected, say, from three separate lanterns—the white screen reflects all three impartially to the retina, where their superposition produces the sensation of white. With pigments, however, the final colour is due to that remaining after each pigment, in a certain mixture, has absorbed from the incident white light its own complement. In this way the primary colours of pigments are those capable of absorbing the three primaries of white light, *i.e.* red, green, and blue-violet, whose respective complements are green-blue (peacock blue), purple-red, and yellow. These last three are therefore the primaries of pigments because, when mixed in the right proportion they (theoretically) produce *black*. In practice, however, owing to the natural impurities of pigments, and the impossibility of combining the correct proportions, the result is a dark grey. For the same reasons, it is impossible accurately to match the spectrum colours by means of pigments, and this is especially the case towards the violet end; in fact we cannot imitate spectrum violet by any known pigment or combination of pigment colours.

The additive effect can be roughly imitated by painting yellow and blue sectors alternately on a disc which, when rapidly rotated, gives the impression of white if the proportions of colour are correct. Here the yellow and blue alternately impinge so rapidly on the retina that the sensations caused by alternate sectors have not time to fade away, and therefore mentally become mingled, and give rise to the sensation of white. The experiment must be carried out in white light, but even then the effect is generally far from pure owing to the inevitable muddiness of the pigments. By increasing the number of sectors, and repeating the six spectrum colours in proper proportion all round the disc, a better white is secured.

The result of a pigment mixture may be surprisingly different from the result of mingling lights of corresponding colours. If blue and yellow lights are mingled in the right proportion on a white screen they cause the sensation of white. If blue and yellow pigments are combined, the blue absorbs red, the yellow absorbs violet, so that green is produced by such a mixture. Rose red and blue-green are complementary colours which, added to one another, produce white in the case of coloured lights (additive effect), but neutralise each other, *i.e.* produce black in the case of pigments (subtractive effect).

Additive effects can also be produced by the mixture of pigment or coloured powders, where absorption does not occur, but both pigments or powders reflect light. Especially is this so if the two colours are not complementary, or tending to be so; thus red and yellow combined in pigment make orange as they do in the case of lights. Or using the illustration above of blue and yellow pigments combined making green, a blue pigment reflects violet and green, yellow reflects red and green, so that if the two pigments be mixed there is reflected a certain quantity of violet and of red, and a double quantity of green. The red, the violet, and a portion of the green combine to form white light, so that there is a residue of green light, which gives the nature of the colour to the mixture of the two pigments.

Qualities of Colours.—Colours in pigments possess three qualities, viz., tone, brightness and purity. Tone or hue is that quality which differentiates between the various colours—say, red and orange; it depends on wave-length. Brightness, intensity or luminosity is that quality which represents the strength of a colour; it depends on the amount of light reflected; one which reflects little light is a dark colour, and one which reflects much light is a light colour. Fullness, saturation, tint or purity is that quality which represents the depth of a colour; the less the admixture of white or black the purer is the colour. Red mixed with white forms pink, whereas red mixed with black makes a kind of maroon. Yellow or orange becomes straw or brown according as it is mixed respectively with white or black.

Colours of Bodies.—A substance is said to be of certain colour when it reflects or transmits rays of certain wave-lengths and absorbs the rest of the spectrum. Thus an object which absorbs the violet and green and reflects the red waves appears red; if it absorbs red waves and reflects green and violet it has a blue colour. A green body absorbs all but the green waves; one which is orange in colour reflects red and green and absorbs violet. The colour reflected by a body is usually the same as that which it transmits, but some bodies transmit the complementary colour to that which they reflect.

Others, again, reflect and transmit different colours; thus, gold leaf transmits green. The colour of a body, whether opaque, translucent, or transparent, varies also to some extent, and sometimes greatly, with its thickness.

A body which reflects light of all wave-lengths is called white; a body which has affinity for all the colours, so that all are absorbed and none reflected, is called black. No body, however, is of a nature so chemically pure as to absorb entirely or reflect all the incident light. An absolutely black body does not exist in nature; even those coated with lamp-black and soot reflect some light, which renders them visible, and allows of their form and solidity being recognised; on the blackest velvet still blacker shadows can be cast. Similarly, there is no object which reflects all the light it receives; pure, fresh snow, which is the whitest of all bodies, absorbs some 15 per cent. of the light it receives, and white paper 20 or 30 per cent. Dark colours reflect little light, and slight differences between them are hardly appreciated in dull illumination; similarly, light colours reflect much light, and slight differences are hardly noticed in very bright illumination. The proportion of light reflected varies with the nature and colour of the body. Approximately a coloured body absorbs 50 to 80 per cent. of the light which falls on it.

A yellow body will be seen longest as light is reduced and it can be seen further, although its colour may not be distinguishable. Generally speaking, as a characteristic and recognised colour, red is the most persistent of all; owing to its long wave-length it can be recognised at a greater distance than others, it freely penetrates haze, fog, or smoke glass, while the penetrations of other colours follow more or less in the order of the spectrum. For this reason red is employed as the danger signal, while blue-green is employed as the contrast signal on railways and ships. The sun appears redder at sunrise and sunset than at midday, also in fog, the blue-violet end of the spectrum being absorbed; the colour of light seen through a thick impure smoke glass is generally a brilliant red.

White, grey and black are, in effect, the same, and really represent varying degrees of luminosity, the only difference between them being in the total amount of light reflected. By all three the treatment of the different wavelengths is the same, *i.e.* there is no selective property as with coloured bodies. The extent of the light absorption varies in the three cases, but the proportion of the components in the light reflected remains unaltered.

White, being produced by the mixture of all colours, is the standard in brightness and luminosity, but this standard may be displaced, as when coloured illumination is used, or coloured glasses looked through. Black may be described as absence of colour and of light.

Coloured Bodies and Lights.—The real colour of a body is that which it exhibits in daylight; it often appears of a quite different colour in artificial light in which some particular colour, usually red or orange, predominates, and therefore the mental standard of white is temporarily shifted towards that colour. Thus the exhibited colour of a body is dependent, to a great extent, on the nature of the luminant, and this is the more marked because the pigmentation of a body, from which its colour results is never pure in the sense that it reflects one wave-length only. In order to reflect a certain colour, the object must receive that colour in the light.

The nearer the colour of the luminant approaches to that of the body the whiter will it appear; on the other hand should the colour of the luminant approach the complement of the body, the latter will appear darker than it would if viewed in white light. Should the light be of a colour exactly corresponding to that which the body absorbs, none will be reflected, and the body will consequently appear black.

Of course a white body seen by coloured light is really coloured although it may be interpreted mentally as white. It certainly is so accepted if the colouration of the luminant is not excessive; thus by gas light a white paper is actually reddish-yellow, but we still call it white. Painting and matching colours is always difficult in artificial light; the difference between some blues and greens can barely be distinguished by gas light; and still less by lamp or candle light. Even if an artificial illumination is practically white it is unlikely to radiate all wave-lengths between 750 $\mu\mu$ and 375 $\mu\mu$, so that, in it, coloured bodies would not exhibit their true colours, especially those whose colour is not in the light. In order to remedy the excess of red-orange in artificial lights many absorption screens have been devised, but the latest device for the production of *artificial daylight* is the invention of Mr. G. Sheringham which has been scientifically developed by Mr. L. C. Martin and Major A. Klein. It consists of a reflecting screen, which is coated with selected colours in definite proportion to area. All the light from the source is received by the screen and diffused therefrom, so that the solar light values of coloured bodies are exhibited.

As illumination becomes progressively feeble all bodies lose their distinctive colours, the latter being replaced by shades varying from light grey to black, and in a very dull illumination all appear equally grey.

Coloured Glass.—Pure neutral or smoke glass absorbs part of all the component colours of white light; if not exactly neutral some one colour penetrates it more than the others—generally red and sometimes green—and gives a distinct tint to a light seen through it. A glass of definite colour, as red or green, transmits not only its distinctive colour, but also some of the adjacent colours; thus green transmits some yellow and blue. Spectrum blue blocks out both the red and violet ends of the spectrum, and transmits blue, green and a little yellow. Cobalt-blue transmits blue and red, but blocks out green and yellow. Orange, amber, yellow and green-yellow glass absorb the violet and ultra-violet light. White crown, and still more, flint glass is absorptive for ultra-violet light, while quartz is specially transmissive for it. White glass absorbs also some of the infra-red, and nearly 15% of the visible light.

Coloured Bodies and Glasses.—All colours are profoundly modified or changed when viewed through coloured glass, as they are by coloured lights.

A body viewed through a glass of the same colour appears white, or at least indistinguishable from a white object seen through the same glass. Thus with red letters on a white ground, seen through red glass, the white background appears the same colour as the letters, so that the whole field is of uniform tint; here the colour of the glass is temporarily the *mental standard of white*. On looking at a red object on a green ground, through a piece of red glass, one sees a white object on a black ground. Similar phenomena result with other colours.

If a coloured body be viewed through a coloured glass which absorbs the rays reflected by the body, the latter appears black. Thus a red body appears black through a green glass of the proper shade, the red rays reflected by the body not traversing the glass. If the ground be black, the object is barely distinguishable from the ground, or may not be at all, as in the "FRIEND" test.

Superposition of Coloured Glasses.—Two coloured glasses placed together form an example of the subtractive process similar to the mixture of pigments. The first glass eliminates from incident white light all but its own colour, and if the second glass is the same as the first, no further alteration takes place, except a reduction in intensity. If the second glass is not of the same colour as the first, a certain amount of absorption by *subtraction* takes place in the second, and the more nearly complementary are the two glasses the more nearly will the whole of the incident light be cut off. For example, if a blue-green and a red, or an orange and blue glass, be placed together, the combination is opaque. Cobalt-blue and green glass, on being placed together, transmit original white light as blue, since blue is transmitted by each, but the remaining colours absorbed. Red, green and blue-violet, together absorb all visible rays, but light rose-red, yellow, and blue glasses transmit grey—*i.e.* a dull white.

Transmissiveness of Coloured Glasses.—For the method of measuring this, and the photometry of coloured lights, see Chapter II.

The proportion of incident light transmitted depends on the thickness of the glass, and it is not easy to express variations, but approximately the transmission varies inversely as the square of the thickness.

If a standard No. 6 smoke glass *transmits* 1/5 of the incident light, a second No. 6 placed behind transmits 1/5 of that transmitted by the first—*i.e.* $1/5 \times 1/5 = 1/25$ of the total light, originally incident on the first glass, is transmitted by the two together.

If one glass *absorbs* 20% of the incident light, and another absorbs 30%, the total absorbed can be calculated from that which is transmitted, that is 70% of 80% or

$$\frac{70}{100} \times \frac{80}{100} = \frac{56}{100}, \text{ so that } 44\% \text{ is absorbed.}$$

Coloured glasses are numbered from 1 the lightest to, say, 10. No real standards are in general use, but those of the Optical Society are given below.

	1	2	3	4	5	6	7	8	9	10
Percentage of light transmitted ..	80	60	50	40	30	20	10	5	2·5	1·25

Standards are difficult to establish owing to the part played by thickness, and further on account of the uncertainty attending the product—*i.e.*, melting, etc., of coloured glass.

Since a Cx. or Cc. lens varies in thickness from centre to periphery it cannot, if made of coloured glass, be of uniform tint throughout. The subjects of colour vision, coloured glasses, lenses, etc., are treated in "Visual Optics and Sight-Testing."

CHAPTER XX

CHROMATIC ABERRATION

Dispersion or Chromatism.—When white light suffers *refraction*, this is always accompanied by *dispersion*, because the component waves are deviated to different extents and become separated. The shorter waves, with rare exceptions, are retarded, by the refracting medium, more than the longer waves. Reflection is not accompanied by dispersion. A body is said to be *chromatic* if it causes dispersion, and *achromatic* if it does not.

Chromatism is due to the nature of the light, although its degree varies also with the nature of the refracting body and the kind of material of which it is made. There is no chromatism if the light is *monochromatic*, i.e. of one colour only, nor is there always chromatism if the light is polychromatic as with a plate or a *corrected* refracting body.

Velocity of Light and Colour.—The velocity of light in free ether is the same for all colours, and is taken to be so in air, although this is not quite the case, blue being retarded slightly more than red in its passage through the atmosphere. If V_1 be the velocity in air (about 300,000 km. per second) and V_2 that in a dense medium, then $V_1/V_2=\mu$.

V_2 here refers to the mean of the various wave-lengths which combine to form white light, and is represented by the yellow or D line of the spectrum. But V_2 is greater for red light (line A), and lower for violet (line H), so that μ also represents only the optical density of a medium for the D line. Every other line of the spectrum has a different μ which, for a given ordinary medium, is higher towards the violet, and lower towards the red end of the spectrum. Suppose in a medium $\mu_D=1{\cdot}5$, $\mu_H=1{\cdot}51$, $\mu_A=1{\cdot}49$. Then $V_D=300{,}000/1{\cdot}5=200{,}000$ km., $V_H=300{,}000/1{\cdot}51=<200{,}000$ km., and $V_A=300{,}000/1{\cdot}49=>200{,}000$ km. per sec.

Dispersive Index.—Each refracting medium has an *index of dispersion*, which represents the differences between the indices of refraction of the lines A and H of the spectrum. Thus, water has an index of refraction for the line A of 1·3289, and for the line H of 1·3434; and $1{\cdot}3434-1{\cdot}3289=\cdot 0145$, is the index of dispersion of water.

Partial dispersion is the difference between the μ's of any two given lines of the spectrum.

Mean dispersion is the difference between the indices of refraction of the lines C and F, *i.e.* between orange-red and blue, and is sometimes represented by δ; that is $\delta=\mu_F-\mu_C$.

The dispersion of various kinds of glass differs with the materials used in their manufacture, and is more or less independent of their refracting power; the two examples given in the following table are merely representative. Some media of high mean refraction have low dispersion, and *vice versa;* generally, however, high refractivity and high dispersivity accompany each other.

			Mean Dispersion.	*Total Dispersion.*
Water	$\mu_C=1{\cdot}3317$	$\mu_F=1{\cdot}3378$	0·0061	0·0145
Alcohol	$\mu_C=1{\cdot}3621$	$\mu_F=1{\cdot}3683$	0·0062	0·0149
Pebble	—	—	—	0·014
Canada balsam	—	—	—	0·021
Tourmaline	—	—	—	0·019
Crown glass *if*	$\mu_C=1{\cdot}5376$	$\mu_F=1{\cdot}5462$	0·0086	0·018
Flint glass *if*	$\mu_C=1{\cdot}6199$	$\mu_F=1{\cdot}6335$	0·0136	0·026
Diamond	$\mu_C=2{\cdot}4102$	$\mu_F=2{\cdot}4355$	0·0253	0·056

ν or the Ratio of Refraction to Dispersion.—Since refraction and dispersion are more or less independent of each other, neither the total nor the mean dispersion indicates the optical properties of a medium; for this we must take the ratio between the mean refractivity (μ_D-1) and the mean dispersivity ($\mu_F-\mu_C$), which ratio is termed the *refractive efficiency,* denoted by the symbol ν (nu), and expressed by

$$\nu=\frac{\mu_D-1}{\mu_F-\mu_C}$$

The formula gives a value which, when compared with that of another medium, shows which of the two has the greater power for refracting with equal dispersion. It also enables us to calculate the components of an achromatic prism or lens, and by its aid glasses can be tabulated in the order of their efficiencies, so that selection can be made for the purposes for which they are needed.

A high value of ν denotes high refraction and relatively low dispersion, while a low ν indicates the reverse, *i.e.* a low refraction and relatively high dispersion.

Thus if in a variety of flint glass, $\mu_D=1{\cdot}6$, $\mu_F=1{\cdot}61$, and $\mu_C=1{\cdot}59$ the efficiency is

$$\nu=\frac{1{\cdot}6-1}{1{\cdot}61-1{\cdot}59}=\frac{{\cdot}6}{{\cdot}02}=30.$$

If a certain crown glass has $\mu_D=1{\cdot}525$, $\mu_F=1{\cdot}532$ and $\mu_C=1{\cdot}523$

$$\nu=\frac{{\cdot}525}{{\cdot}009}=60 \text{ (approx.)}$$

These values of ν, *i.e.* 30 and 60, show that in the flint the dispersion is relatively twice as great as in the crown; or, otherwise expressed, the crown has double the mean refraction of the flint for the same amount of dispersion. If two glasses have the same μ, but different dispersions, the one with the lower dispersion has the higher ν. If two glasses have the same dispersion but different μ's, the one with the higher μ has the higher ν.

In general flint refracts more than crown, but disperses still more, so that if a given crown and a given flint were compared it would be found that the flint has the higher $(\mu_D - 1)$, and still higher $(\mu_F - \mu_C)$, so that its ν is lower than that of the crown.

μ_D of water $= 1{\cdot}3336$, and its mean dispersion $= {\cdot}0061$, so that ν is nearly 55. With air $\mu = 1{\cdot}00029$, and the mean dispersion is $\cdot 0000029$, so that $\nu =$ 100 (approx.).

δ and Δ.—As stated above δ represents $\mu_F - \mu_C$, and Δ represents the difference between the ν's of two different media, *i.e.* $\Delta = \nu_1 - \nu_2$.

Expression for ω.—Calculations with respect to chromatism are sometimes based on ω (omega), the *dispersive power*, which is the reciprocal of ν, and therefore

$$\omega = \frac{\mu_F - \mu_C}{\mu_D - 1}.$$

Achromatism of a Plate.—When light is incident obliquely on a plate (parallel plane surfaces), although dispersion occurs at the first surface, it is neutralised at the second.

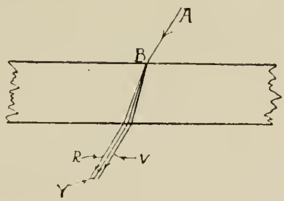

Fig. 221.

Let $A\,B$ (Fig. 221) represent a beam of parallel light incident on, and refracted by, a plate. At B dispersion takes place, violet being deviated the most and red the least, and were it possible for the eye to receive the beam before it leaves the plate, the object would appear deviated and tinged with colour as with prism. At the second surface, however, all the dispersed rays are rendered parallel to each other, and therefore, by their overlapping on the retina, produce the sensation of white. In other words the appearance of the object, so far as dispersion is concerned, is the same as though viewed direct. In order to cause chromatism or dispersion, a medium must have the power of *altering the line of travel of the various colours* with respect to each other.

When a prism or lens is achromatised its action is similar to that of a plate, while the course of light, as a whole, is changed.

Chromatism of a Prism.

Crown and Flint Prisms.—If a prism of crown and one of flint be taken, such that their mean deviations are equal, the spectrum produced by the flint is considerably the longer owing to the greater dispersive effect. If spectra of the same lengths be required, the crown glass prism must be stronger than the flint.

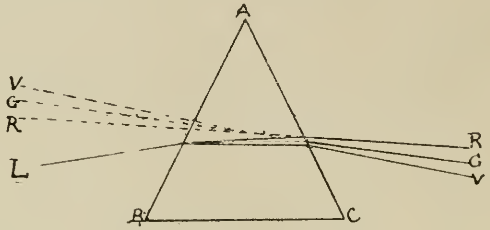

Fig. 222.

Virtual Spectrum.—A prism refracts violet waves most, and red least towards the base, as shown in Fig. 222, where L is the source of light. If the light be received by the eye, the rays are projected back to form a virtual spectrum, and the violet is nearest the edge and the red nearest the base. Thus, a disc of light viewed through a prism, base down, exhibits blue above and red below as shown by the dotted lines in Fig. 222.

If a white body be viewed through a prism, the latter causes a series of separate images of the body to be formed, each characterised by a distinctive spectrum colour. These recombine in the centre so that a white virtual image is seen, but the ultimate displacements of blue at the one end, and of red at the other, cause a fringe of blue to appear on that border nearest the edge of the prism, and a red-orange fringe on that nearest the base.

If the body is black or dark, as compared with its background, the red-orange fringe is towards the edge, and the blue fringe towards the base of the prism, these resulting from the dispersion of the light from the space or body surrounding the black. Thus a window bar viewed in daylight, through a prism base down, is blue at the bottom and reddish-yellow on top, but if viewed by artificial light at night the colours are reversed.

Dispersion of a Prism.—A beam of light, incident on a prism, is refracted towards the base, and since the retardation is greater as the wave-length is shorter the blue is, as stated above, more deviated towards the base than the red, and the components are separated to form the band of colours known as the spectrum.

The extent of the dispersion varies with the medium of which the prism is formed, with the angle of the prism, and with the angle of incidence of the light.

The Refraction Spectrum.—The source should be the sun or a bright artificial luminant and the light should be admitted through a small horizontal aperture A (Fig. 223), preferably about 20 mm. long by ·5 to 1 mm. wide placed parallel to the edge of the prism P. The light, thus admitted, is incident on the prism placed in its path in the position of minimum deviation.

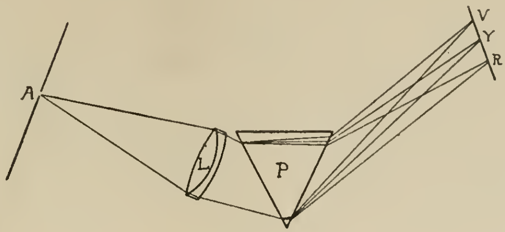

Fig. 223.

The resultant spectrum is received on a screen, but it is *impure* because adjacent colours overlap each other. If, however, an achromatic bi-convex lens L be placed close to the prism, with the aperture and screen at approximately twice its focal distance on either side of it, the various colours are brought to a focus at the screen and a series of coloured real images of the slit are seen, forming together a *pure refraction spectrum* $V\ Y\ R$. If the prism be placed base up the violet is above and the red below, and *vice versa.* Although the different colours are well defined, the red end of the spectrum is crowded, while the blue is spread out. The lens projects a real image of the slit, and the prism produces from this single white image, an innumerable series of others corresponding to every wave-length. Much better results are, however, obtained with the spectroscope used for viewing and comparing spectra, and the spectrometer for measuring the principal, deviating and dispersing angles of a prism. Their construction (*q.v.*) is given in Chapter XXVI.

It is impossible to obtain a theoretically pure spectrum, since the source must be of some definite magnitude, and therefore a certain amount of overlapping takes place between adjacent colours. The purity, however, reaches a high standard in the spectroscope where, in addition to the finest possible slit, the light received by the prism is parallel, so that prismatic distortion is eliminated. The spectrum produced by a given source can be studied and, if necessary, the spectra from two sources can, by suitable arrangement, be formed side by side for comparison.

Spectrometry.—To measure the deviating angle the collimator C and telescope T are brought into line (Fig. 224) so that the image of the slit appears in the centre of the field of view, the objective of the telescope forming a real image of the slit in the focal plane of the ocular, through which it is viewed, and a reading is taken on the circle. The prism is then placed in position, and the telescope must be rotated to T' until the image of the slit can again be seen. The angular distance through which T is moved is the deviating angle of the prism, care being taken that the deviation is a minimum. This can be done by slightly rotating the prism backwards and forwards until a position is found when the slightest movement in *either* direction *increases* the deviation.

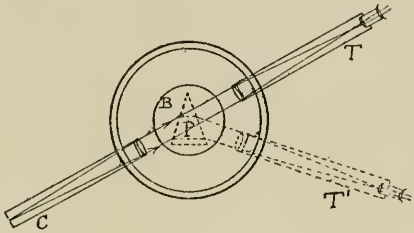

Fig. 224.

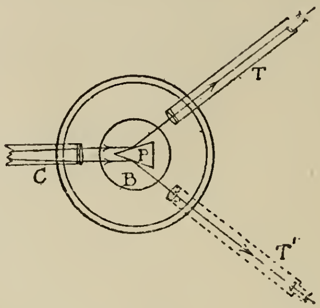

Fig. 225.

The principal angle of a prism is measured by turning the prism until its edge splits into two halves the beam of light issuing from the collimator (Fig. 225). The telescope is rotated to T until the image of the slit is seen reflected from the one surface, and then turned to T' to receive the image from the other surface of the prism. Half the angle through which the telescope has been rotated gives the principal angle of the prism.

When the principal angle and the deviating angles are known, the refractive index and the dispersion of the glass, of which the prism is made, can be calculated by the formulæ given elsewhere.

For very accurate determination of refractive index and dispersion, various incandescent gases are employed, which give line spectra, instead of a white source which produces a continuous spectrum.

The mean deviation is indicated when the yellow of the spectrum lies on the cross wire placed in the focus of the ocular. The deviation of a prism, for any other colour, is determined by bringing that colour on to the cross wire. By this means, the refractive index, or the total, mean or partial dispersion of the medium, of which a prism is made, can be learnt; but the position of minimum deviation of the prism must, also, be found for each colour.

The Diffraction Spectrum, which is purer than that of refraction, is mentioned in Chapter XXV.

Refraction and Dispersion.—As before stated, refraction by a simple medium is, so far as known, always accompanied by dispersion or chromatism. If a number of prisms of different substances, but of the same angle, be taken, those having the higher refractive index usually, but not of necessity, possess the longer spectra. A spectrum can, of course, extend beyond what is visible in the ordinary way; thus the photographic representation of the spectrum of a quartz prism is about three times as long as that of a crown prism.

Different spectra can be made of the same length by altering the angles or the position of the prisms, or the position of the screens. If (Fig. 226) the spectra be placed one under the other so that the *B* lines at the red and the *H* lines at the blue correspond in position, it will be found that the intermediate lines do not do so, rendering it difficult to fix the exact position of lines in the

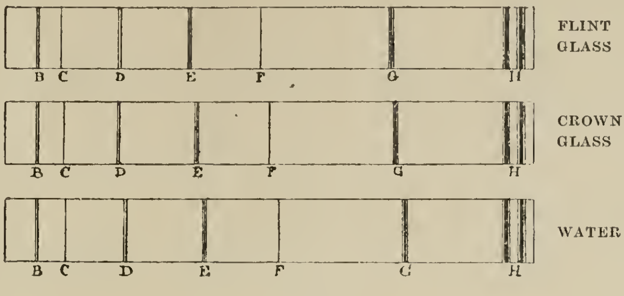

Fig 226.

spectrum. This want of coincidence of all the colours or the irregularity of sequence of the principal colours in any two spectra produced by different media, is called *spectrum irrationality.*

Anomalous Dispersion.—In glass, water and most substances, the order of refrangibility is from the red through the orange, yellow, green, blue, indigo to violet, which is the most refrangible, but certain substances have

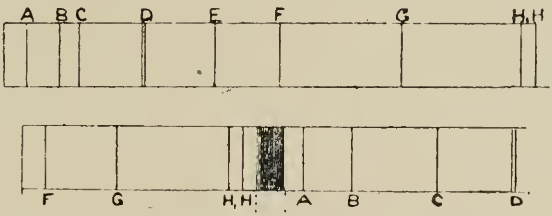

Fig 227.

the property of refracting the normally more refrangible rays less, and the less refrangible more (Fig. 227). This is called *anomalous dispersion.* The substances which exhibit this peculiarity usually possess what is termed surface colour, *i.e.* they have a different colour when viewed by reflected light from what they have by transmitted light. They reflect a certain colour, and the complementary colour is transmitted, and their spectra exhibit an absorption band of more or less considerable dimensions, it being the space which would have been occupied by the reflected colour had it been transmitted. Such substances are termed *dichroic.*

Most metals, except gold and copper, as well as many of the aniline products, possess this abnormal dispersion, the order of the colours being changed. Moreover, Kundt found, in the aniline products, the dispersion abnormally increased on the red side of the band, but diminished on the violet side;

so that in the case of fuchsin, for example, the red end, usually so short, is actually more extended than the violet end.

Recomposition of Dispersed Light.—To recombine the dispersed colours of a prismatic spectrum and again form white light there can be employed—

1. A prism of equal dispersive power placed in the path of the dispersed light, but having its base turned in the opposite direction to that of the first prism. (Newton's method.)

2. A series of plane mirrors arranged so that each receives a different portion of the spectrum; from each the light is reflected to the same part of a screen, where the colours are recombined.

3. The dispersed light is received on a concave mirror, from which it is reflected on to a screen; then by rapidly oscillating the mirror the impression of white light is produced. Or the prism may be oscillated or rotated to produce a similar effect without the interposition of the concave mirror.

Any mechanical arrangement of rotation or oscillation by which the colours of the spectrum, whether produced by dispersion or by transmission through coloured glasses, or by reflection from pigments, are caused to successively enter the eye with sufficient rapidity, produces the impression of white. Fresh colour sensations are caused while others are still existing, and the combination of all results in a sensation of white or grey. Colour tops, or discs, divided into sectors of different colours, are examples of this method.

Angular Dispersion.—The deviating angle of a prism is that for the mean ray (*D* line), and is expressed (in the case of thin prisms) by $d = P(\mu - 1)$, where P is the refracting, and d the deviating angle. Now, since the red ray suffers less, and the blue greater refraction than the *D* line, their angular deviations are respectively

$$d_C = P(\mu_C - 1), \text{ and } d_F = P(\mu_F - 1)$$

d_C and d_F being the deviating angles, and μ_C and μ_F being the indices of refraction for red and blue light respectively.

The angular dispersion of the prism expressed in degrees is then

$$P(\mu_F - 1) - P(\mu_C - 1) = P(\mu_F - \mu_C)$$

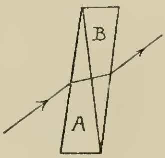

FIG. 228.

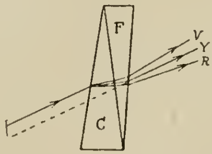

FIG. 229.

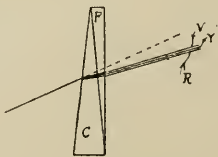

FIG. 230.

If two similar prisms, *A* and *B* (Fig. 228) are placed in opposition—base to edge—their angles, refractive indices, and dispersions being the same, both the deviation and dispersion are neutralised, and all the rays emerge parallel to their original course.

In Fig. 229 let the principal angle of a crown prism of $\mu=1{\cdot}54$ be $11{\cdot}3°$, and that of a flint prism of $\mu=1{\cdot}61$ be $10°$. Their deviating angles are the same, namely, $6{\cdot}1°$.

If in the crown $\mu_C=1{\cdot}534$, $\mu_F=1{\cdot}554$, the dispersion$=11{\cdot}3\times(1{\cdot}554-1{\cdot}534)={\cdot}226°$. If in the flint $\mu_C=1{\cdot}586$, $\mu_F=1{\cdot}62$, the dispersion$=10\times(1{\cdot}62-1{\cdot}586)={\cdot}34°$. The resultant angular dispersion is therefore ${\cdot}34-{\cdot}226={\cdot}114°=6'\ 50''$. Thus, while no deviation of the mean yellow ray occurs, the red and blue are separated by an angle of nearly $7'$.

Achromatic Prism.—If a crown prism of 3°d and $\mu=1{\cdot}54$, and a flint of 2°d and $\mu=1{\cdot}61$ (Fig. 230), having efficiencies of 45 and 30 respectively, be placed in opposition, they neutralise each other's dispersion, while there remains 1° deviation. Such prisms are said to be achromatized, *i.e.* they constitute an achromatic prism, which causes deviation without dispersion. The principal angle P of the crown is $3/{\cdot}54=5{\cdot}55°$, and of the flint $2/{\cdot}61=3{\cdot}28°$. Here, as will be seen from Fig. 230, *every* ray is deviated to the same extent, and the recombination of light is secured, as with a plate.

To Calculate an Achromatic Prism.—Let d be the deviating angle of the required achromatic prism. Let d_1, ν_1 and P_1 be those of the crown, and d_2, ν_2 and P_2 those of the flint components respectively.

Now $d=d_1+d_2$, and since they have to be in opposition d_1 may be regarded as positive and d_2 as negative. For achromatism to result

$$\nu_1/\nu_2=d_1/d_2 \text{ or } d_1\nu_2=-d_2\nu_1 \quad \text{and} \quad d_1\nu_2+d_2\nu_1=0$$

Or the condition for achromatism is

$$P\,(\mu_F-\mu_C) \text{ of the crown}=P\,(\mu_F-\mu_C) \text{ of the flint.}$$

To obtain the values of the two components d_1 and d_2, *the deviating angle d of the required achromatic prism* must be divided in proportion to the values of ν_1 and ν_2, that is

$$d_1=\frac{d\nu_1}{\nu_1-\nu_2}, \quad \text{and} \quad d_2=\frac{d\nu_2}{\nu_2-\nu_1}$$

It should be noticed that the *deviating*, not the principal angles enter into the formulæ. The principal angles of the two components are found from $P=d/(\mu_D-1)$ which, however, holds good only for thin prisms; if strong, P_1 and P_2 must be found by the more complete formulæ previously given.

As an example, an achromatic prism of 5°d is needed, the glasses of the component parts being

Crown	$\mu_D=1{\cdot}53$	$\mu_C=1{\cdot}527$	$\mu_F=1{\cdot}536$.
Flint	$\mu_D=1{\cdot}63$	$\mu_C=1{\cdot}624$	$\mu_F=1{\cdot}644$.

$$\nu_1=\frac{1{\cdot}53-1}{1{\cdot}536-1{\cdot}527}=\frac{{\cdot}53}{{\cdot}009}=58{\cdot}9 \qquad \nu_2=\frac{1{\cdot}63-1}{1{\cdot}644-1{\cdot}624}=\frac{{\cdot}63}{{\cdot}02}=31{\cdot}5$$

$$d_1=\frac{5\times58{\cdot}9}{58{\cdot}9-31{\cdot}5}=\frac{294{\cdot}5}{27{\cdot}4}=10{\cdot}7^\circ\text{d, and } P_1=\frac{10{\cdot}7}{{\cdot}53}=20^\circ$$

$$d_2=\frac{5\times31{\cdot}5}{31{\cdot}5-58{\cdot}9}=\frac{157{\cdot}5}{-27{\cdot}4}=-5{\cdot}7^\circ\text{d, and } P_2=\frac{5{\cdot}7}{{\cdot}63}=9^\circ$$

$$d=10{\cdot}7^\circ-5{\cdot}7^\circ=5^\circ.$$

To find the Achromatising Prism.—The flint prism d_2 of ν_2, which will neutralise the dispersion of a given crown of d_1 and ν_1 is calculated from

$$\nu_1/\nu_2=d_1/d_2 \qquad \text{or} \qquad d_2=d_1\nu_2/\nu_1$$

Thus, let the crown be 10·7°d, $\nu_2=31{\cdot}5$, and $\nu_1=58{\cdot}9$, then

$$d_2=10{\cdot}7\times31{\cdot}5/58{\cdot}9=5{\cdot}7^\circ\text{d}$$

and
$$d=d_1+d_2=10{\cdot}7-5{\cdot}7=5^\circ$$

P_2 can be found directly from $P_2\delta_2=P_1\delta_1$

That is $\quad P_1\,(\mu_F-\mu_C)$ of the crown $=P_2\,(\mu_F-\mu_C)$ of the flint.

Whence
$$P_2=P_1\,\frac{(\mu_F-\mu_C)\text{ of crown}}{(\mu_F-\mu_C)\text{ of flint}}$$

Thus in the example above $\quad 20\times{\cdot}009=9\times{\cdot}02$.

Chromatism of a Lens.—The effect of dispersion, when the refracting body is a lens, is to bring the more refrangible blue and violet to a focus sooner than the less refrangible red and orange. This different focalisation of the various colours is termed *chromatism,* and the confusion of the image caused by it, *chromatic aberration.* The defect is made apparent by a fringe of colour on the edge of the real or virtual image projected by the lens. Indeed, lenses being similar in nature to prisms, produce similar chromatic phenomena.

An ordinary lens cannot be achromatic for a real image; but when it is used as a magnifier the virtual image is really composed of a series of images formed by every different colour, which series, being contained within the same visual angle, combine on the retina to form a single impression. This image, however, appears coloured at the edges, owing rather to the chromatic effects of spherical aberration, which is greater for blue than for red. If spherical aberration is entirely eliminated, the virtual image is colourless.

If a horizontal white line (Fig. 231) be observed through the marginal portion of a convex lens, a blue-violet fringe will be seen on the side towards the edge of the lens, and a red-orange on the other, the blue being projected back beyond the red. Viewed through the periphery of a concave, the colours are reversed. Looking at a black line, the fringes are seen in the opposite order to those on a white line, for the reason given in connection with a prism. The centre of the image, whether virtual or real, of a white object,

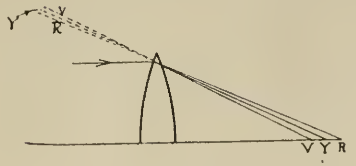

Fig. 231.

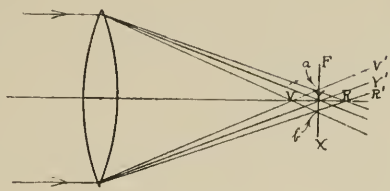

Fig. 232.

appears white, because the different colours are superposed, so that only at the extremities, where certain colours are not combined with others, is chromatism apparent.

Longitudinal and Lateral Aberrations of Colour.—In Fig. 232 a parallel beam of light from a point source is refracted by a convex lens; the various coloured rays meet at different distances behind the lens, the violet focussing at V, the yellow at Y, and the red at R. If a screen be held at V, the diffusion patch has a reddish-yellow fringe; the red and orange rays, being convergent to a more distant point R, impinge on the screen outside the blue and violet. If the screen be placed at R, the fringe becomes blue-violet, since these rays, having already met at V and crossed, impinge on the screen outside the red and orange. The distance VR is the *longitudinal aberration*, and the diameter $a\ b$ of the disc of confusion in the plane where the extreme violet and red rays cross each other, is the *lateral aberration* ; this plane is very nearly that of $F\ X$, where the yellow (most luminous) light is brought to a focus. Here the circle formed by the red and blue discs being practically of the same size is termed the *circle of least confusion.*

F of the Various Colours.—The index of refraction of a given medium refers to that of the D (sodium) line, which is situated in the yellow or most luminous part of the spectrum. With such a medium, if $\mu_D=1{\cdot}54$, the index of refraction for the red (line A) might be 1·53 ($\mu_A=1{\cdot}53$), while for the violet (line H) it might be 1·56 ($\mu_H=1{\cdot}56$). Suppose a thin double convex lens of 10 in. radius, then

$$F=\frac{r_1 r_2}{(r_1+r_2)\ (\mu_D-1)}=\frac{10\times10}{(10+10)\times{\cdot}54}=\frac{100}{10{\cdot}8}=9{\cdot}26 \text{ in.}$$

which is the mean focal length for yellow light.

Instead of $\mu=1{\cdot}54$ we must employ $\mu_A=1{\cdot}53$, and $\mu_H=1{\cdot}56$ to find the focal lengths F_A and F_H for red and violet light respectively; thus

$$F_A=9{\cdot}43 \text{ in. and } F_H=8{\cdot}93 \text{ in.}$$

The difference in the focal lengths of a lens for red and blue light may be illustrated with a cobalt-blue glass (chromatic disc), which transmits red and blue light, but absorbs the central part of the spectrum, or by focussing with a convex lens light which is rendered monochromatic by being passed through respectively standard red and spectrum blue glass. The difference in the focal distances with these two coloured lights is sufficiently well marked to be appreciated.

A positive and a negative lens of different dispersions which neutralise for white or yellow may not neutralise for red or blue light.

Expression for Chromatic Aberration.—Let r_1 and r_2 represent the two radii, and F_D the focal length of a thin lens for the *D* line. Then, if F_A and F_H represent the focal lengths, and μ_A and μ_H the indices for extreme red and violet respectively, the chromatic focal difference may be expressed by

$$F_A - F_H = \frac{r_1 r_2}{(r_1 + r_2)(\mu_A - 1)} - \frac{r_1 r_2}{(r_1 + r_2)(\mu_H - 1)} = \frac{r_1 r_2(\mu_H - \mu_A)}{(r_1 + r_2)(\mu_A - 1)(\mu_H - 1)}$$

If instead of $(\mu_H - 1)(\mu_A - 1)$ there be substituted $(\mu_D - 1)^2$, as may be done without sensible error, then the *longitudinal chromatic aberration* is

$$F_A - F_H = \frac{r_1 r_2 (\mu_H - \mu_A)}{(r_1 + r_2)(\mu_D - 1)^2} = \frac{F_D (\mu_H - \mu_A)}{(\mu_D - 1)} = \frac{F_D}{\nu} = F_D \omega$$

The formulæ for the *refractive efficiency* ν, and for the *dispersive power* ω, being the same as with a prism,

$$\nu = \frac{\mu_D - 1}{\mu_H - \mu_A} \qquad \omega = \frac{\mu_H - \mu_A}{\mu_D - 1}$$

As an example let $F_D = 10$ in., $\mu_A = 1{\cdot}60$; $\mu_D = 1{\cdot}61$ and $\mu_H = 1{\cdot}625$, then

$$F_A - F_H = \frac{10 \times (1{\cdot}625 - 1{\cdot}60)}{1{\cdot}61 - 1} = 10 \times \frac{{\cdot}025}{{\cdot}61} = {\cdot}41 \text{ in.}$$

The lateral chromatic aberration of a lens = diameter of lens/2ν.
Similar calculations can be used for a thick lens.

Achromatic Lens.—Chromatism can be remedied by making the lens a combination of two different kinds of glass, so chosen that, while the dispersion of the positive component is neutralised by that of the negative, there still remains some positive converging power, so that a real image may be formed. Such a combination is termed an *achromatic* lens, and usually consists of a flint concave and a crown convex. If a negative achromatic lens is required, as occurs sometimes in practice—for instance, in the telephoto lens—then the concave is of crown and the convex of flint.

Spectrum Lines Combined.—By an achromatic lens two selected lines of the spectrum, usually the *C* and *F* (orange-red and blue) are brought to a focus at the same distance; by uniting these with a third component a third line could also be focussed at the same distance, but for all practical purposes if the C and F lines, which lie near the more central and luminous part of the spectrum, are combined, the combination is one in which chromatism does not cause any appreciable blurring of the image, at least for visual purposes, in which critical definition is not essential. In photographic lenses the lines D and G, or D and H are usually selected in order to unite the violet, which is the most chemically active part of the visible spectrum, with the visual focus. For astro-photographic purposes, in which vision is of little consequence, the lines F and H (or beyond) are brought together.

Formulæ for an Achromatic Combination.—To calculate an achromatic combination for two lenses in contact, let F and C be the two lines of the spectrum which have to be brought to a common focus. F is the focal length of the required combination. F_1 and ν_1 pertain to the crown component, and F_2 and ν_2 to the flint; F_2 is negative, and $1/F = 1/F_1 + 1/F_2$.

In order that chromatism be eliminated

$$\frac{1}{F_{1_H}} - \frac{1}{F_{1_A}} = \frac{1}{F_{2_H}} - \frac{1}{F_{2_A}} \quad \text{or} \quad \frac{F_{1_A} - F_{1_H}}{F_{1_A} F_{1_H}} = \frac{F_{2_A} - F_{2_H}}{F_{2_A} F_{2_H}}$$

But $F_{1_A} - F_{1_H} = F_1/\nu_1$, and $F_{2_A} - F_{2_H} = F_2/\nu_2$, so without serious error, $F_{1_A} F_{1_H} = F_1^2$, and $F_{2_A} F_{2_H} = F_2^2$.

The last equation can then be written

$$\frac{F_1}{\nu_1 F_1^2} = \frac{F_2}{\nu_2 F_2^2} \quad \text{or} \quad \frac{1}{\nu_1 F_1} = \frac{1}{\nu_2 F_2}.$$

That is, $\nu_1 F_1 = -\nu_2 F_2$ and $F_1 \nu_1 + F_2 \nu_2 = 0$

The two components $1/F_1$ and $1/F_2$ are obtained by dividing $1/F$ proportionally to the two efficiencies ν_1 and ν_2; that is,

$$\frac{1}{F_1} = \frac{1}{F} \times \frac{\nu_1}{\nu_1 - \nu_2} = \frac{\nu_1}{F(\nu_1 - \nu_2)} \quad \text{and} \quad \frac{1}{F_2} = \frac{1}{F} \times \frac{\nu_2}{\nu_2 - \nu_1} = \frac{\nu_2}{F(\nu_2 - \nu_1)}$$

or

$$F_1 = F\frac{\nu_1 - \nu_2}{\nu_1} \quad \text{and} \quad F_2 = F\frac{\nu_2 - \nu_1}{\nu_2}.$$

Example: a positive achromatic lens of $6\frac{1}{2}$ in. F is required; the indices of refraction for the various lines are:—

For the crown $\mu_C = 1{\cdot}527$, $\mu_D = 1{\cdot}53$, $\mu_F = 1{\cdot}536$.

For the flint $\mu_C = 1{\cdot}630$, $\mu_D = 1{\cdot}635$, $\mu_F = 1{\cdot}648$.

$$\nu_1 = \frac{1{\cdot}530 - 1}{1{\cdot}536 - 1{\cdot}527} = \frac{{\cdot}530}{{\cdot}009} = 58{\cdot}89 \quad \text{and} \quad \nu_2 = \frac{1{\cdot}635 - 1}{1{\cdot}648 - 1{\cdot}630} = \frac{1{\cdot}635}{{\cdot}018} = 35{\cdot}28$$

$$\nu_1 \sim \nu_2 = 58{\cdot}89 \sim 35{\cdot}28 = \pm 23{\cdot}61$$

Then $F_1 = 6{\cdot}5 \times \frac{23{\cdot}61}{58{\cdot}89} = +2{\cdot}61$ in. and $F_2 = 6{\cdot}5 \times \frac{-23{\cdot}61}{35{\cdot}28} = -4{\cdot}358$ in.

$$\frac{1}{F} = \frac{1}{2{\cdot}61} - \frac{1}{4{\cdot}358} = \frac{1}{6{\cdot}5} \quad \text{or} \quad F = 6\tfrac{1}{2} \text{ in.}$$

To find the Achromatising Cc.—The F of a Cc. of ν_2 which, with a given Cx. of ν_1, will make the combination achromatic, is found from

$$\frac{F_2}{F_1}=\frac{\nu_1}{\nu_2} \quad \text{or} \quad F_2=\frac{F_1\nu_1}{\nu_2}$$

Taking the same figures as in the previous example, if the Cx. has F= 2·61 in., then

$$F_2=2{\cdot}61\times 58{\cdot}89/35{\cdot}28=4{\cdot}358.$$

Dioptral Formulæ.—Let D represent the power of the combination, D_1 and D_2 the powers respectively of the Cx. and Cc., ν_1 and ν_2 the respective efficiencies of the crown and flint lenses, ν_2 being negative.

In order to achromatise each other the relationship must be

$$D_1\,\nu_2=-D_2\,\nu_1 \quad \text{or} \quad D_1\,\nu_2+D_2\,\nu_1=0, \quad \text{and} \quad D=D_1+D_2.$$

To obtain D_1 and D_2, we must divide D proportionally to ν_1 and ν_2, that is,

$$D_1=\frac{D\,\nu_1}{\nu_1-\nu_2} \qquad \text{and} \qquad D_2=\frac{D\,\nu_2}{\nu_2-\nu_1}$$

Taking the same glasses as in the previous example, where $\nu_1=58{\cdot}89$, $\nu_2=35{\cdot}28$, and $F=6\tfrac{1}{2}''$, or D=6

$$D_1=6\times 58{\cdot}89/23{\cdot}61=14{\cdot}97, \text{ and } D_2=6\times 35{\cdot}28/-23{\cdot}61=-8{\cdot}97$$

$$D=+14{\cdot}97-8{\cdot}97=+6$$

These dioptral formulæ are similar to those for achromatic prisms, and may be preferred to those involving F.

To find the Achromatising Cc.—Since the powers of the two component lenses are proportional to their efficiencies, if $\nu_2=60$, and $\nu_2=50$, a + 6 D and a − 5 D will together make an achromatic + 1 D.

If D, the power of the convex, is known, and it is needed to calculate the concave required to make it achromatic, the formulæ are

$$\frac{D_1}{D_2}=\frac{\nu_1}{\nu_2} \quad \text{or} \quad D_2=D_1\frac{\nu_2}{\nu_1}$$

As in the foregoing example, if the crown is + 14·97, the flint is

$$D_2=14{\cdot}97\times 35{\cdot}28/58{\cdot}88=8{\cdot}97$$

Example.—Given an equi-cx. lens, of crown glass, of radius 10 in., there is needed to calculate the radius of curvature of a flint Cc. so that the two combined make an achromatic combination.

For the crown $\mu_D=1{\cdot}5175$; $\mu_F-\mu_C={\cdot}0087$, and $\nu_1=\dfrac{\mu_D-1}{\mu_F-\mu_C}=\dfrac{{\cdot}5175}{{\cdot}0087}=59.$

For the flint $\mu_D=1{\cdot}571$; $\mu_F-\mu_C={\cdot}01327$, and $\nu_2=\dfrac{\mu_D-1}{\mu_F-\mu_C}=\dfrac{{\cdot}571}{{\cdot}01327}=43.$

Now $\frac{1}{F_1}=\cdot 5175\times\left(\frac{1}{10}+\frac{1}{10}\right)=\frac{1}{9\cdot 662}$ for the Cx.

and $\frac{1}{F_2}=\frac{1}{9\cdot 662}\times\frac{43}{59}=\frac{43}{570\cdot 058}=\frac{1}{13\cdot 257}$ for the Cc.

then $\frac{1}{F}=\frac{1}{9\cdot 662}-\frac{1}{13\cdot 257}=\frac{3\cdot 595}{128\cdot 089}=\frac{1}{35\cdot 63}$ or $F=35\cdot 63$ ins.

Now, the radii of curvature of the two adjacent surfaces must be equal, that is, 10 in. Therefore r, the second radius of the Cc., is found from

$$-13\cdot 257=\frac{-10r}{(-10+r)\cdot 571}\text{ so that }r=-31\cdot 15\text{ in.}$$

Example.—A plano-Cx. achromatic combination is required of F=20 in. Let the glasses be

$$\mu_C=1\cdot 535,\ \mu_D=1\cdot 54,\ \mu_F=1\cdot 555\text{ for the crown,}$$

$$\mu_C=1\cdot 59,\ \mu_D=1\cdot 60,\ \mu_F=1\cdot 63\text{ for the flint.}$$

Then $\nu_1=\cdot 54/\cdot 02=27$, and $\nu_2=\cdot 60/\cdot 04=15$

$$F_1=20\times\frac{12}{27}=8\cdot 88\text{ Cx. and }F_2=20\times\frac{-12}{15}=16\text{ Cc.}$$

The combination has $F=\frac{-16\times 8\cdot 88}{-16+8\cdot 88}=+20$ in.

If the one surface of the Cc. is plano, the other is

$$r=-16\times(1\cdot 6-1)=-9\cdot 6\text{ in.}$$

The Cx. must have one surface of radius 9·6 in., and the other—

$$8\cdot 88=\frac{r\times 9\cdot 6}{\cdot 54\times(r+9\cdot 6)}=\frac{9\cdot 6\ r}{\cdot 54\ r+5\cdot 184}\text{ or }r=9\cdot 6''\text{ (approx.).}$$

The positive lens is a double Cx.; it is combined with a plano-Cc.

Illustrating the Dioptral Formulæ.—With data as above, since 20 in. = 2 D we have

$$D_1=2\times 27/12=4\cdot 5\text{ and }D_2=2\times -15/12=-2\cdot 5$$

The Cc. having one surface plano, the other surface has

$$r=\frac{100\times(1\cdot 6-1)}{-2\cdot 5}=-24\text{ Cm.}$$

The one surface of the Cx. has $r=24$ Cm., and since

$$D=\frac{100\ (\mu-1)\ (r+r')}{r\ r'}\text{ we have }4\cdot 5=\frac{100\times\cdot 54\times(24+r)}{24r}$$

Whence $r=24$ Cm. As before, there is a DCx. lens of $r=24$ Cm. and a plano-Cc. of $r=-24$ Cm.

Chromatic Difficulties.—A combination may bring different coloured rays to the same focus, but the images may not be of the same size.

A combination achromatic for an axial pencil of light may not be so for oblique pencils.

A combination achromatised for light proceeding from a given plane may not be so for light proceeding from other planes.

Irrationality of Dispersion.—If with an achromatic lens, the C and F (or D and H) lines coincide, other lines do not. This is called *irrationality of dispersion,* and the dispersion which thus remains in an achromatic lens is *residual* or *secondary,* but as stated, it suffices, for practical purposes, to unite two certain lines of the spectrum according to the use to which the lens is put. With modern glasses, and by careful selection, it is possible to unite practically three spectrum lines with two glasses, but a real common focus for all colours is impossible at present.

Apochromatic Lens.—A combination which actually unites three lines of the spectrum is termed *apochromatic;* for such a lens at least three different sorts of glasses must be employed, the residual spectrum still left being so small as to be negligible. For such a combination

$$\frac{1}{F}=\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_3} \quad \text{and} \quad F_1\nu_1+F_2\nu_2+F_3\nu_3=0.$$

Lens Combinations.—A combination of lenses, having one achromatised component, is not perfectly achromatic; in order that it may be so the achromatised component must be suitably overcorrected. When lenses are separated the conditions for achromatising are different.

Two lenses of the same material may be achromatic, for virtual images, as is the case with Huyghen's eye-piece. Those rays which pass through the thin part of the field lens pass through a thicker part of the eye lens; the violet, being relatively nearer to the axis than the red, is less refracted, and all the components of white light emerge under the same visual angle. Thus, two Cx. lenses of equal ν separated by a distance equal to $(F_1+F_2)/2$ form an achromatic combination for virtual images.

CHAPTER XXI

ABERRATIONS OF FORM

Prismatic Aberrations of Form.

Small Light Pencils.—A pencil of light parallel before refraction is parallel after refraction by a prism; if divergent or convergent, it is taken to be similarly divergent or convergent after refraction, provided the pencil of light is small, and the axial ray suffers minimum deviation. Also the prism itself must be of comparatively small angle.

Large Light Pencils.—In Fig. 233, which is purposely exaggerated for the sake of clearness, let a wide pencil of light diverge from a point *L*, of which *L E* is the central ray, presumed to suffer minimum deviation, and *L D*, *L F*, are extreme rays incident on the surface of the prism in the base-apex plane.

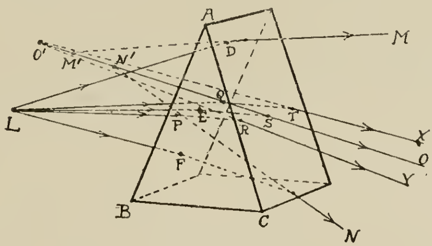

Fig. 233.

These rays suffer unequal deviation towards *M*, *O* and *N* respectively, and if *L E* is at minimum deviation, the others cannot be so and are deviated relatively more than *L E*, which is travelling in the direction *O′O*. Both, when produced backwards, cut *O′O* nearer than *O′*, but *L F*, being more deviated cuts it in *N′*, while *L D* cuts it in *M′*.

The incidence of rays *L Q* and *L P* in a plane parallel to the axis, is also greater than that of *L E*, but the difference is much less than in the base-apex plane; the refracted rays *T X* and *R Y*, when produced backwards, meet in *O′*, which is nearer to the prism than the original source *L*.

Thus rays in the pencil emanating from a point do not have a point-image, there being two focal lines, the one nearer the prism being parallel to the axis, and the other parallel to the base-apex plane of the prism. A circle of least confusion, which lies between *O′* and *N′*, may be regarded as the image. These defects blur the image and cause it to appear nearer than it actually is, and if the prism is in such a position that *L F* or *L D* suffers minimum deviation, the whole of the pencil is rendered still more divergent and the image is still more blurred and nearer. In consequence the position of minimum deviation for a near object differs from that for a distant one.

Aberrations of a Prism.—Although, in a prism of small angle, the effects of aberration due to its form can be ignored, considerable distortion of the image is produced by a strong prism.

The distortion of image, seen through a prism, is due in general to (*a*) varying incidence of the light in a plane, (*b*) varying incidence in one plane as compared with another, and (*c*) varying incidence of different pencils. Chromatism also plays a part in this connection. (*a*) is the origin of *coma* in a lens, (*b*) is the genesis of *radial astigmatism*, (*b*) and (*c*) of curvature, and (*c*) of distortion.

Distortion increases with the *nearness of the object*, when light is very divergent; if the object is distant the light is parallel, so that (*a*) above disappears and (*b*) is not so marked.

Distortion increases with *the size of the object*. If a narrow pencil of light from the centre of an object enters the eye through a prism, and suffers minimum deviation and but little aberration, the pencils from other points cannot do so; the peripheral portions of a large object are blurred compared with the centre. Size of object is also the main factor in *thickness*, which follows.

Distortion is caused by *the greater thickness of the prism* through which oblique pencils pass from the extremities of an object. These pencils suffer more deviation than the central pencils, and therefore appear to come from points relatively more distant from the centre than those nearer to the centre of the object viewed. Thus a straight line, parallel to the edge, appears curved with its convexity towards the base. A square object has its two sides, which are parallel to the edge of the prism, concave to the latter direction.

Distortion is caused, or increased, by *abnormal position of the base-apex plane* which causes pencils to have varying obliquity. If the base-apex line is vertical, say edge up, the vertical magnitude of a square object, whether near or distant, appears increased when the edge of the prism is nearer to it than the base, while the vertical dimension is lessened if the base is nearer the object. The effect increases with the inclination, as does also the *total* deviation of the image in both cases, unless the object is relatively near.

If the prism be rotated around its base-apex line—*i.e.* if, say, a vertical prism, edge upwards, be rotated so that one side of the prism is nearer the object than the other—the image is lengthened diagonally, being drawn out more towards the edge than the base and more on one side than the other, so that a square object appears as a distorted parallelogram.

Lens Aberrations of Form.

Apart from chromatism, the image formed by a spherical lens suffers from five aberrations due to its shape, and these must be severally corrected before the lens is capable of forming a geometrically perfect image of an object. The first is *spherical aberration*, the second *coma*, the third *radial astigmatism*, the fourth *curvature of the field*, the fifth *distortion*.

The first three errors mentioned are *point* aberrations, and lenses corrected for them are called *stigmatic* as distinct from *astigmatic*; a lens corrected for spherical aberration is termed *aplanatic*. The last two errors are aberrations of a plane, and lenses free from them are termed *rectilinear* or *orthoscopic*. Form aberrations, in general, are lessened by (*a*) employing a stop or diaphragm, (*b*) using some special form of single lens, or (*c*) using a combination instead of a single lens.

Spherical Aberration.—Since a lens may be regarded as consisting of an infinite number of prisms whose angles of inclination increase with the distance from the axis, it follows that the deviation effected by the various zones of a lens depends on this distance. In a Cx. lens the varying inclination of opposite points on the two surfaces, in each meridian, causes parallel light to converge, in theory, to a point, but actually, the refraction of a spherical lens is such that light from a point is not brought to a focus at a single point, the rays transmitted by the marginal zones of the lens meeting sooner than those transmitted nearer the centre, as depicted in Fig. 234.

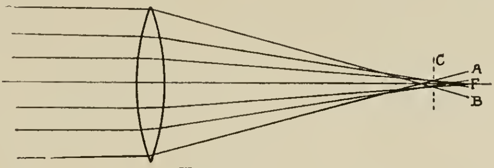

Fig. 234.

Each zone of a lens has its own focal length, varying from the principal focus *F*, for rays refracted in the zones immediately surrounding the principal axis, to a point where rays *A* and *B* passing through the most external zones, meet the axis. The inability to unite in a single point all the rays diverging from an object-point on the principal axis is called *spherical aberration*, which is due, not to the fact that the deviating power is greater towards

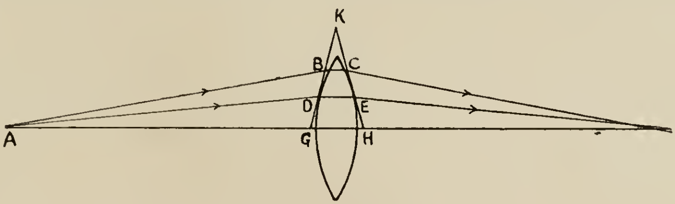

Fig. 235.

the periphery, for that is a natural property of a lens, but to the fact that the deviating power *increases too rapidly* towards the periphery, with the result that wave fronts are not truly spherical after refraction.

In Fig. 235 the opposite points *D* and *E* of the lens constitute a portion of a prism *G K H*, and the ray *A D*, incident such that its point of incidence *D* and its point of emergence *E* are equidistant from the edge, therefore suffers minimum deviation, this latter occurring when the refraction of the ray is shared equally between the two surfaces. The deviation of the ray *A B C* proceeding from the same point, and refracted by the prismatic element *B C*, is not minimum, and is relatively more bent from its course than the ray *A D*. It is mainly owing to the departure from minimum deviation incidence of the light at the periphery that the deviating power there is unduly increased and spherical aberration produced. For parallel light the angle of incidence increases as the tangent of the angle, whereas for *aplanatic* refraction it should increase as the arc.

Central and Peripheral Refraction.—When the peripheral part of a lens (Fig. 236) is blocked out only the central area of the lens is effective, and parallel rays, as a whole, meet slightly within *F*. When the central portion of the lens is covered (Fig. 237) and only the periphery acts on the light, the latter, as a whole, meets still further within *F*.

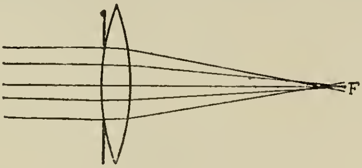

Fig. 236.

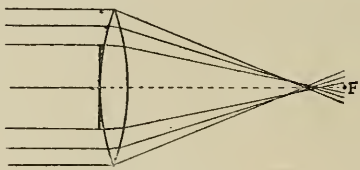

Fig. 237.

Circle of Least Confusion.—When the whole lens is exposed to the light (Fig. 234), the converging circles of confusion from the central, and the diverging circles from the peripheral area, are of about the same mean diameter at *C*, the circle of least confusion; here the disc of light is of minimum size. At any point either nearer or further the disc is larger than at *C*, but the greatest concentration of light occurs at *F*, where the image of a luminous point is a bright spot, surrounded by a halo caused by the diverging light from the periphery of the lens. The true focus *F* is the most distant of the series of foci shown in Fig. 234, and is that due to refraction by that area of the lens immediately surrounding the principal axis.

The distance of the image from a Cx. lens in the three cases where the periphery only, the centre only, or the whole of the lens is effective, can be shown by experiment, the object being a bright source placed behind a small aperture covered by a piece of yellow glass in order to make the light more or less monochromatic.

Longitudinal and Lateral Aberration.—The distance between the extreme foci is the *longitudinal* aberration; the diameter of the confusion disc *A B* (Fig. 234), when the screen is in the theoretical focus, is the *lateral* aberration. The lateral aberration increases more rapidly than the longitudinal with an increase in the aperture of a lens, the latter varying as the square of the aperture, and the former as the cube of the aperture.

Influencing Factors.—The definition of an image depends on the smallness of the circles of confusion of which it is constituted, and these circles are dependent on the degree of spherical aberration. The latter varies with the incidence of the light, the aperture, form, index, and thickness of the lens; as these factors are changed spherical aberration is increased or decreased.

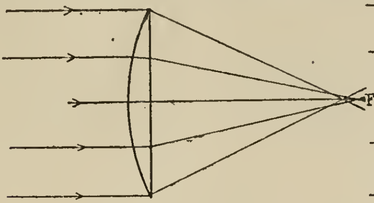

FIG. 238.

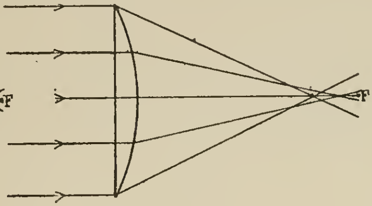

FIG. 239.

Best Form of Single Lens.—Spherical aberration is least when the rays in general are, after refraction at the first surface, more nearly parallel to the bases of the virtual prisms of which the lens is formed, so that the total refraction is approximately *equally divided between the two surfaces.*

As a general rule, for parallel light, the more curved the front, and the less curved the back surface of the lens, the smaller is the spherical aberration (Fig. 238); as the object is nearer the lens and the light becomes more and more divergent less curvature is needed for the front, and more for the back surface. In these cases an approach to minimum deviation at the periphery of the lens is obtained. A very high degree of spherical aberration results if the less curved surface is exposed to parallel light (Fig. 239), or the more curved surface to light diverging from the focus of the lens, for then a considerable departure from minimum deviation for peripheral rays occurs. Since the incidence varies with the distance of the object, spherical aberration depends not only on the form of a lens, but also on the distance of the source from that lens.

The plano-convex, or better, the *crossed* lens (having surface powers in the ratio of about 6 to 1), with its more curved surface turned to the light, is the form of single lens which gives the best definition for objects at extreme distances. The same lens turned the other way is the best for very near objects, while the double convex is the best when the incident rays diverge from twice the focal distance, for then, object and image being equidistant from the lens, the incident and emergent rays form equal angles with the two surfaces. If used for all distances the double Cx. is perhaps the best form of single lens.

To obtain minimum spherical aberration in a single lens the radii of the two surfaces should be in the ratio of $1 + 2\mu$, and $1 - 2\mu + 4/\mu$. These quantities, when $\mu = 1{\cdot}5$, are as 6 is to 1, giving the *crossed* lens previously mentioned. When $\mu = 1{\cdot}686$, the value of $1 - 2\mu + 4/\mu$ is 0, so that the one surface should be plano, and if the index is higher, this quantity being negative, the lens must be a meniscus.

A Numerical Expression for Longitudinal Aberration is sometimes given, as below, for parallel light and thin lenses, where $\mu=1{\cdot}5$. The values are in terms of d^2 F, where d is the semi-diameter of the lens.

A crossed Cx. with the more curved surface to the light	1·07
A plano-Cx., with the curved surface to the light	1·17
An equi-Cx.	1·67
A crossed Cx. with the less curved surface to the light	2·07
A plano-Cx. with the plane surface to the light	4·5

These values vary with the index of refraction with the thickness of the lens, and for different distances of the source of light.

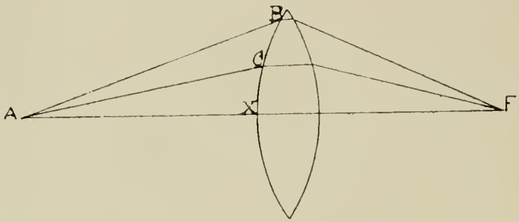

FIG. 240.

Least Time.—Since light travels in a straight line it takes the least possible time to reach a given point, and this principle of *least time* holds good for refraction. Thus, various rays diverging from a point in air and passing into another denser medium must arrive at the same point, at the same time, if a point focus is to be obtained. With a lens, disregarding spherical aberration, this occurs because, although the distance from A to B, and thence to F, is greater than from A to C and F (Fig. 240), yet the distance traversed in the denser medium is greater in the case of $A\ C\ F$. The law of refraction $\mu_1 \sin i=\mu_2 \sin r$ is in accordance with the principle of least time. If a lens is corrected for spherical aberration all rays diverging from an object-point must reach the same image-point and in the same time, no matter what course they take. In other words the *optical length* (which is the actual distance of travel multiplied by the μ of the medium in which this takes place) must be the same for all rays between the object and image points.

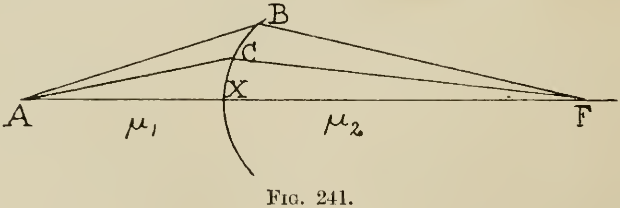

FIG. 241.

Let the distance of A from any point on the refracting surface (Fig. 241) be d_1, and the corresponding distance of F be d_2; then $d_1\mu_1+d_2\mu_2$ is the optical length of any ray diverging from A and refracted to F, so that for $A\ X$, $A\ B$ and $A\ C$ to meet at F it would be necessary that $d_1\,\mu_1 + d_2\,\mu_2$ be a constant for any incidence of the light, i.e. $A\ B\,\mu_1 + B\ F\,\mu_2 = A\ C\,\mu_1 + C\ F\,\mu_2 = A\ X\,\mu_1 + X\ F\,\mu_2$. As this cannot occur with spherical surfaces, spherical aberration may be said to be due to the fact that all the rays diverging from a point on the axis cannot reach the same point in a given time, or rather that, within a given time, the rays reach different points of the axis.

In the case of a lens the influence of the two surfaces has to be considered, since each ray travels in three different media. If d_1 be its course in the first medium μ_1, d_2 its course in the second medium μ_2, and d_3 its course in the third medium μ_3, then $d_1\,\mu_1 + d_2\,\mu_2 + d_3\,\mu_3$ would need to be the same for each ray in order that all rays diverging from an object-point may meet, after refraction, at a single image-point.

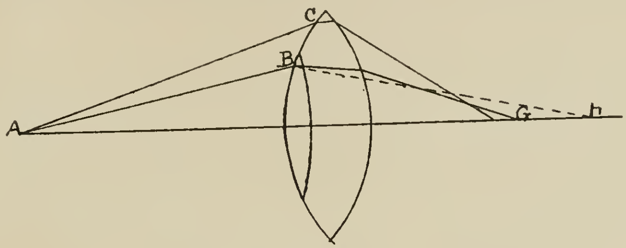

Fig. 242.

Influence of Thickness.—A ray $A\ B$ traversing a thick lens (Fig. 239) is retarded in the denser medium, and can only reach, in a given time, a point G on the principal axis which lies nearer to the lens than H, the point reached by a similar ray passing through a thin lens. Thus, spherical aberration increases with the thickness of the lens.

Remedies.—A theoretical remedy would be found if the speed of the light could be increased, or the refractive power of the lens decreased at the periphery. This would necessitate the lens being made of a medium whose index of refraction decreases as the distance from the principal axis increases, which occurs in the crystalline lens of the eye. Or the lens would need to have less curvature at the periphery than at the centre, *i.e.* one having some curve other than spherical.

The practical remedy for spherical aberration of a single lens is the employment of a stop or diaphragm used in combination with the lens, the marginal rays are then cut off and spherical aberration is consequently lessened. The same result is obtained by making the lens of small diameter.

In general, spherical aberration is less as the number of surfaces sharing the refraction is increased. Thus two positive lenses may be employed in the place of one, for the same refractive power, or the positive and negative components of a system separated by an interval, a principle sometimes made use of in photographic and microscope objectives. As a rule an achromatic combination has little spherical aberration, or it may be eliminated altogether, the positive and negative aberrations of the component lenses neutralizing each other, as do their dispersions. In practice this method affords the only true means of correction.

Aplanatic Lens.—A lens, or lens combination, corrected for spherical aberration is termed aplanatic, but no combination can be rendered entirely aplanatic for all distances of the object, nor can it be for other than monochromatic light; but by employing a stop, as is done in most optical instruments, and a judicious choice of combination and of the form of the individual components, it may be rendered so for practical purposes. A single surface may be aplanatic, also a single spherical lens, but only for one distance of object (*vide* Aplanatism at end of this Chapter).

Positive and Negative Aberration.—*Positive* aberration obtains when the marginal rays come to a focus before the central, *negative* aberration if the central rays come to a focus before the marginal.

Under and Over Correction.—A lens combination which partially neutralises the positive aberration is *under-corrected*, and if it more than neutralizes the positive, it produces *negative* aberration, and is said to be *over-corrected*. In photographic lenses it may happen that spherical aberration is completely eliminated for the axis and periphery, while it may still occur in the intermediate zones.

The Oblique Aberrations.—A beam of light diverging from a point on the principal axis would, on passing through a lens corrected for spherical and central chromatic aberration, meet again as a point on the principal axis. When, however, the luminous point is situated on a secondary axis, further aberrations are introduced by the oblique incidence of the light, these being the point aberrations *coma* and *radial astigmatism*, and the plane and line aberrations *curvature of field* and *distortion*.

If a small bright source be placed obliquely below the axis of a lens and a white screen moved behind it, the image is blurred at all distances, assuming various comet-shaped, cup-shaped, and pear-shaped figures, which are the result of coma. If coma be reduced by placing a fairly small diaphragm in front of the lens and the screen is held within the focus, and slowly drawn away, the image is seen to form a symmetrical ellipse, and then successively a horizontal line, a horizontal ellipse, an irregular circle, a vertical ellipse, a vertical line, and finally broadens out into a blurred patch. These lines result from radial astigmatism.

Coma is an aberration produced by *the unequal refracting effect of the different parts of the various meridians* of a lens, on an oblique pencil of light; it is spherical aberration for oblique light and is, of course, more pronounced as the axis of the incident pencil is more oblique.

Instead of a point-image of a point-object, situated on a secondary axis, there results a blurred pear or comet shaped halo of confusion partly surrounding a bright point, the latter being directed towards the axis. The halo spreads away from the axis because the aberration is caused mainly by the part of the lens nearest the source.

The confusion disc produced by coma is asymmetrical, whereas the confusion disc of spherical aberration is symmetrical with respect to the axis of the beam of light.

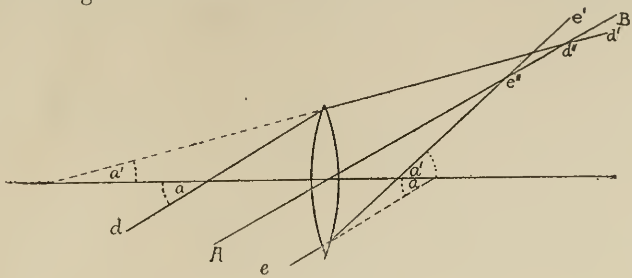

Fig. 243.

Let d and e (Fig. 243) be rays proceeding from a distant point on an oblique axis $A\ B$. The ray e meets the surface of the lens sooner than d, and since e departs more from minimum deviation than does d, the ray $e\ e'$ cuts the axial ray at e'' sooner than does $d\ d'$ at d''.

Influencing Factors.—Coma is increased in direct proportion to the aperture of the lens. It varies also with the form and, in general, whatever tends to increase spherical aberration tends also to increase coma.

Remedies.—Coma is reduced by the use of plano-Cx. and meniscus lenses, for the reason that less refraction takes place at the second surface. The chief remedy is the employment of a stop, especially if this be placed a short distance from the concave surface of a meniscus. The effective aperture of the lens is thus reduced, the portions producing the aberration being practically cut out.

The Sine Condition.—For coma to be eliminated, the sines of the angle, a and a' formed by an incident ray with the axis, before and after refraction should have a constant ratio; *i.e.* $\sin a / \sin a' =$ a constant (Fig. 243).

Radial Astigmatism is an aberration which results from *the unequal refraction of different meridians* of a lens on an oblique pencil of light; it is, naturally, more marked as the axis of the incident pencil is more oblique.

Instead of a point-image of a point-object situated on a secondary axis, there are two line-foci through which pass all the rays contained in the pencil.

For every oblique axis there are two *principal meridians* or planes; the *first* is that containing the oblique axis and the principal axis; the *second* is that which, while containing the secondary axis, is at right angles to the *first*.*

* The terms *sagittal* and *meridional*, *tangential* and *radial* used in the previous edition of "General and Practical Optics," have been dropped in favour of *first* and *second* because the former terms were sometimes confused.

In Fig. 244 let a pencil of light be incident on a lens from a distant point *A*, situated on the secondary axis *A X*; then *S S′* is the *first* and *M M′* the *second* plane, in this case *S S′* being vertical and *M M′* horizontal. All rays in a *first* plane as *a* and *c*, *b* and *d*, meet in points along the line *T T′*, which is the *first* (*tangential*) focal line, whence, diverging in one direction and converging in the other, they continue to the *second* (*radial*) focal line *R R′*.

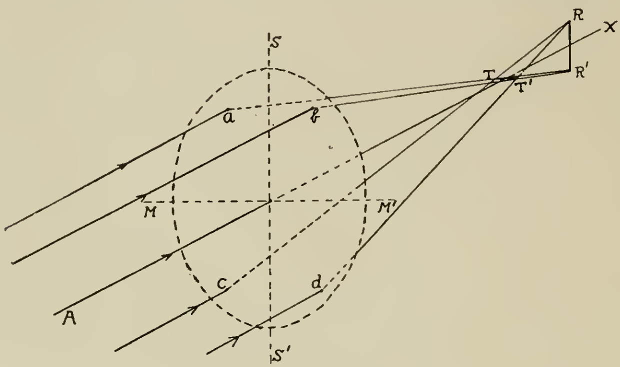

FIG. 244.

Thus the *first* line is the focus of the *first* plane, while the *second* line is that of the *second* plane, each focal line being at right angles to the plane of which it is the focus, and therefore corresponds in direction to the other principal plane.

T is the meeting-point of *a* and *c*, while *T′* is that of *b* and *d* in the first focal line; *R* is the meeting-point of *c* and *d*, while *R′* is that of *a* and *b* in the second focal line.

In the second plane the light has to traverse a greater thickness of the lens than does an axial pencil; and the greater obliquity increases the angles of incidence so that the light is rendered more convergent, and has its focus nearer the lens than the focal plane. In the first plane the light has a still greater thickness to traverse, and is still more oblique as a whole than in the second plane, and therefore its focus is still nearer the lens. Consequently *radial astigmatism is due to the increased angles of incidence of oblique light, and increased effective thickness of the lens.*

The astigmatism is essentially the *interval* between the focal lines produced by the *difference* in the effective powers of the lens in the *first* and *second* planes of incidence. Between the two focal lines there is a position where the cross-section of the refracted light is most nearly circular, and this—the circle of least confusion—may be regarded as the mean focus of the oblique pencil of light. The calculation for the distances of *T T′* and *R R′* are shown in Chapter XXII., where they are termed F_1 and F_2 respectively.

To illustrate oblique refraction, let Fig. 245 represent the focal plane of a Cx. lens viewed from behind. Rays parallel to the principal axis, and directed before refraction to the points *a b c* and *d*, are refracted towards and meet in the point *F*. If the rays are parallel to an oblique axis, as in Fig. 244, *a* meets *c* in *T* and *b* in *R′*, while *d* meets *c* in *R* and *b* in *T′*, but *T T′*, as stated, lies nearer the lens than *R R′*, and both are nearer than *F*.

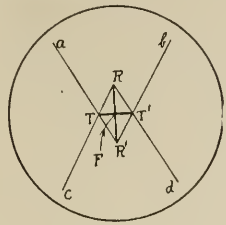

Fig. 245.

Radial astigmatism has been illustrated with light diverging from a point on the lower edge of an object, so that the resulting *first* (*tangential*) focal line is horizontal and the *second* (*radial*) line is vertical. If the luminous point is to the right or left of the object, the first line is vertical and the second line horizontal; if the first plane is oblique both lines are oblique, there being a pair of astigmatic lines at right angles for each secondary axis.

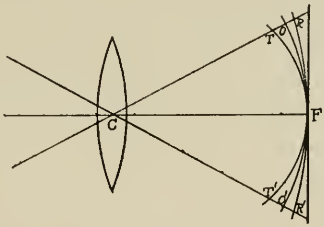

Fig. 246.

The first and second focal lines on the numberless secondary axes constitute two curved surfaces *T T′* and *R R′* respectively and (Fig. 246) both are nearer than *F*; but at the principal axis they meet and form a point in the focal plane, where the two focal lines fuse into a point-image. The field of *circles of least confusion* form a surface *O O′*, concave towards the lens, lying between *R R′* and *T T′*, and this may be regarded as the focal plane of an ordinary lens (see *Curvature of Field*).

Influencing Factors.—Radial astigmatism is greater as the lens aperture is larger. It varies also with the form of the lens, and the more nearly this is of double Cx. form the more marked it is.

Remedies.—Radial astigmatism is lessened by employing a plano-Cx. or a meniscus, especially the latter, combined with a stop placed at some little distance—about a fifth the focal length—on the concave side. The stop cuts off extreme peripheral rays and so makes both focal lines shorter in length and more nearly equal. They are also receded so that the circle of least confusion lies more nearly in the focal plane. By still further displacing the stop away from the lens both lines may even be thrown behind the focal plane.

By combining glasses of high refractive power and low dispersion with those of opposite quality certain conditions are fulfilled which, besides eliminating chromatism, correct astigmatism over a wide area. With the newer varieties of optical glass a degree of correction is secured which was not possible with the older kinds, wherein refractivity and dispersion were more or less proportional.

Sphero-Cyl. Lens.—The difference between the astigmatism of a sphero-cylindrical and the radial astigmatism of a spherical lens, is that the former is due to the varying curvature of the lens, the focal lines being formed on the principal axis; while the latter is due to the oblique incidence of the light, the focal lines lying on a secondary axis. Otherwise the general result, that is, the production of two focal lines instead of a focal point, is very similar in the two cases.

Curvature of the Field.—As stated under *Radial Astigmatism*, if A be a point on the lower extremity of an object the light diverging from it, after refraction by the lens, forms two focal lines, and between them is situated the circle of least confusion, which may be regarded as the focus of the rays diverging from A. On the surface containing the circles of least confusion of all the object-points, the sharpest image of the periphery of the object is formed, and since the effective power of a lens is greater as the light is more oblique, this surface forms a portion of a sphere with its concave surface towards the lens. This curvature of the field is partly due to radial astigmatism (see Figs. 244 and 246).

Radial astigmatism is, however, not the sole cause of curvature, for if $T\ T'$ (Fig. 246), the field of the first lines, were made to coincide with $R\ R'$, that of the second, the combined surface would still be curved.

Again, even if all the peripheral foci were at the same distance from the optical centre (or second equivalent point) as the focus on the principal axis they would together form a portion of a spherical surface whose radius is equal to F of the lens, so that curvature would still remain. Thus, a sphere is entirely free from astigmatism, but the field is nevertheless curved. Therefore, if the image of an ordinary object formed by a lens is projected on to a flat screen, either the centre or the periphery may be focussed, but it is impossible to obtain a good definition of both at the same time. The image of a convex object would be still more curved than that of a flat one, but a concave object might be so placed as to have a flat image.

Remedies.—To flatten the field, radial astigmatism should be eliminated, and the oblique foci lengthened.

For the former the remedies are as for radial astigmatism—namely, a meniscus lens with a stop on the concave side.

A stop, by narrowing the beam on a secondary axis, causes the focus, represented by the disc of least confusion, to be formed at a distance, dependent on the form of the lens, at which curvature is a minimum. This, for single lenses, is generally about one-fifth the focal length on the Cc. side as for radial astigmatism.

If a Cx. and a Cc. of equal power be separated to have convex effect, the distance can be so adjusted as to make the image flat. The oblique rays, after refraction by the convex, meet the concave nearer to the periphery, and the convergence is thereby lessened; therefore the final convergence is to a point further away, for oblique pencils, than would be the case after refraction by a single Cx. lens, whose power is equal to that of the combination.

An almost perfectly flat *virtual* image is obtained with the Ramsden eye-piece where two equal plano-convex lenses have their convex surfaces facing each other. Or by the Huyghen eye-piece formed of two plano-convex lenses, whose respective focal distances are as 1 and 3, both curved surfaces facing the same way.

Curvature is said to be under-corrected, or *positive*, when the image is concave towards the lens, and *negative* if, by over-correction, the image becomes convex.

The Petzval Condition.—In order that a combination of two lenses may form a flat image, the condition which must be satisfied is that $F_1 \mu_1 + F_2 \mu_2 = 0$, where μ_1 and F_1 refer to the crown, and μ_2 and F_2 to the flint components respectively. In order that this shall not controvert the condition for achromatism, which is $F_1 \nu_1 + F_2 \nu_2 = 0$, the crown, with less dispersion, must have a higher refractive index than the flint, a condition already referred to in the section on radial astigmatism. In this case $\nu_1/\nu_2 = \mu_1/\mu_2$.

A Flat Image obtains, if the focal length of each oblique pencil is equal to $F/\cos e$, where e is the angle which the oblique axis makes with the principal axis.

Distortion is an aberration in the magnification of the image. It is chiefly due to spherical aberration which causes peripheral image-points to be formed relatively nearer to the principal axis than their corresponding object-points.

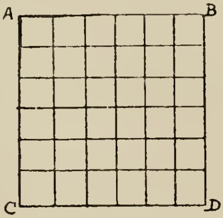

FIG. 247.

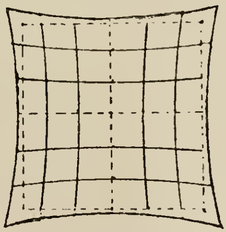
FIG. 248.

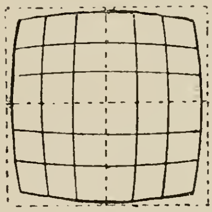
FIG. 249.

Distortion increases as the obliquity of the light is increased, that is to say, the size of the image is relatively more out of proportion to that of the object as the object is larger and, therefore, its extreme points further from the centre.

The real image formed by an ordinary Cx. lens is said to suffer from *negative* or *barrel* distortion, thus the image of a square (Fig. 249) is compressed relatively more at the corners, which lie furthest from the principal axis, than at the sides. The *virtual* image of a square seen through a Cx. lens is said to suffer from *positive* or *pincushion* distortion, as in Fig. 248, this being the reversal of the barrel distortion of the real image. The virtual image of a square seen through a Cc. lens has barrel distortion. Any arrangement of the stop, or separation of the components of a lens system, may produce distortion not otherwise apparent.

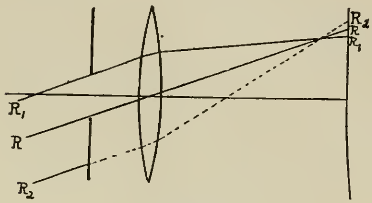

Fig. 250.

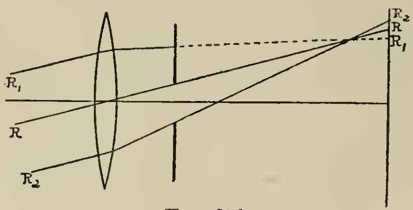

Fig. 251.

Influence of a Stop.—A diaphragm used with a lens, or combination, diminishes spherical aberration, coma, astigmatism and curvature of field. This accentuates and brings into prominence distortion, so that rectilinear lines of the object near the margin appear curved in the image.

When a stop is in front of a Cx. lens the effective area of the lens for an oblique pencil lies mainly on the opposite side of the principal axis to that of the object-point, so that the mean focus lies between R and R_1 (Fig. 250) nearer to the axis than if the whole lens were effective. Thus the natural negative distortion of a Cx. lens is enhanced.

When a stop is behind the lens (Fig. 251) the effective area of the latter for an oblique pencil is chiefly on the same side of the lens as the object-point, so that the mean focus lies between R and R_2 more distant from the principal axis than if there were no stop. The consequence is that the natural negative distortion of the lens is not only corrected, but positive distortion is produced.

The distortion is due to the lens and not to the stop, for if a combination be corrected for distortion the stop may be in front of the lenses, between or behind them, and no distortion ensues.

Remedies.—Distortion is eliminated by employing a combination of lenses with the stop placed between the two components. Then those oblique rays which pass through the one side of the front element must pass through the other side of the back element, and *vice versa*, so that the *negative* distorting effect of the front lens is neutralised by *positive* distortion of the back lens.

Separation of the component parts of a lens system can be utilised for the correction of distortion, and in single lenses it may be reduced somewhat by altering the thickness and curves of the lens.

The Tangent Condition.—A *chief ray* $X\ C$ or $Y\ C$ (Fig. 252) is one which passes through C, the *centre of the stop.* If it be prolonged forwards and, after refraction, produced backwards, the point of intersection p is a *chief point.*

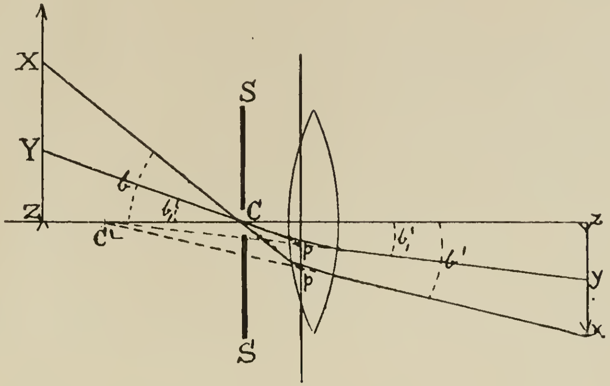

FIG. 252.

When all the chief points thus formed lie in a plane perpendicular to the axis, *i.e.* the refracting plane, the refracted chief rays, produced back to the axis, meet in a single point C'. The lens is then said to be spherically corrected with regard to the stop.

Each chief ray makes with the principal axis, before refraction, some angle b, and, after refraction, some angle b', and when the foregoing conditions obtain, $\tan b'/\tan b$=a constant for every chief ray; the image will then be uniformly magnified throughout, *i.e.* the image will be free from distortion when the *tangent condition* is fulfilled, as in Fig. 252.

Aberrations in General.

In the brief description of the aberrations contained in the foregoing articles certain points are worthy of special note. All the aberrations of a single lens are reduced by the use of a stop with the exception of distortion, which is generally increased, or anyhow made more apparent. The construction of a good lens also largely depends upon the use of meniscus components, without which a wide stigmatic field would be impossible, while, needless to say, crown and flint glasses are essential for achromatism. The nature of the corrections depends largely upon the use to which the lens is to be put, but on the whole, the designer of a photographic objective has a harder task than the maker of telescope and microscope objectives. Of all, perhaps, the photographic objective must be the most generally perfect since it is required to produce a flat, stigmatic and undistorted image over a wide field whose diameter is not infrequently equal to the focal length of the lens. To secure this a kind of compromise must be effected between central and peripheral definition, since the type of lens—the crossed and plano—giving the best central correction for spherical aberration and chromatism, is useless for eliminating the oblique aberrations.

If a first-class photographic lens designed for wide-angled work be examined, it will be found to contain at least one deeply periscopic component, and in all rectilinear objectives both are of meniscus shape. For extreme wide-angle work the periscope type must be still further deepened, until we find, in the Hypergon of Busch, a lens consisting of two thin hemispheres with a stop at their common centre. Generally, therefore, the smaller the angular field the flatter are the curves required to produce it.

In the telescope, prism binocular and opera glass only a narrow angular field—not exceeding a few degrees—is required, and therefore the oblique aberrations may be comparatively ignored, and all the attention centred on the correction of spherical aberration and chromatism, which may be done to an exceedingly high degree of perfection. Thus any good telescope or opera-glass objective will be found to be, as a whole, either plano-Cx. or bi-Cx. with the greater curvature towards the light, which is practically parallel in all cases.

Rather more care must be bestowed on the microscope objective, since here some correction must be given to flatness of field and coma, so that it may be said to occupy an intermediate position between the telescope and photographic objectives, and, the object being near F, the bottom component of the objective is plano-convex, having its plano surface directed outwards.

A plano-Cx. condenser is turned the one way or the other, according as the source is near or distant, and according as the beam of light projected is large or not.

Spherical aberration is a defect of the image on the principal axis, and, therefore, for best definition it is necessary to distinguish between point-objects and objects of definite size. Thus the eye lens of an ocular is always plano-Cx. with the curved surface towards the object to be viewed, which is the real image formed by the objective, and notwithstanding that this object lies in the focal plane of the eye lens. This is because the object viewed is of definite size, and not merely a point on the axis, that is, spherical aberration must be sacrificed to a small extent to allow of correction for distortion, which is really spherical aberration of peripheral pencils *as a whole.*

Again, in visual optics, the deep periscopic spherical and toric are now recognised as being far superior to the doubles in that the field of sharp definition is greatly extended by the elimination of most of the oblique aberrations.

Aberrations of a Cc. Lens.—Although in the foregoing articles Cx. lenses have been used in diagrams and examples, it must not be forgotten that Cc. lenses suffer from precisely similar aberrations. They are, of course, opposite to what would be produced in the virtual image of a Cx., *e.g.* the distortion of the virtual image with a Cc. is barrel, whereas it is pincushion with a Cx., so that when two lenses are neutralised in the ordinary way their aberrations are also practically neutralised, unless the lenses are thick and together take a deep periscopic form.

Aplanatic Refraction.—Refraction at a spherical surface is always accompanied by more or less spherical aberration; it is possible, however, to conceive surfaces that are *aplanatic*, i.e. capable of producing a point-image of a point-object situated on the principal axis.

It is convenient to apply the principle of *least time* in each particular case. As pointed out, the optical length of any ray is its actual distance multiplied by the μ of the medium in which it is travelling and the condition to be fulfilled for aplanatism is that all the rays diverging from an object-point must reach the image-point at the same time. However, surfaces which are aplanatic under certain conditions or for light of a certain wave-length are not so under other conditions or for other wave-lengths.

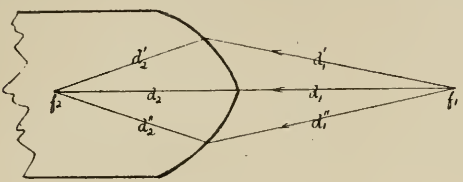

FIG. 253.

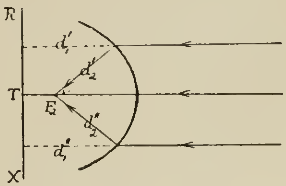

FIG. 254.

Aplanatic Cx. Surface.—Let f_1 (Fig. 253) be a near object-point in air; its real image f_2 in the denser medium will be aplanatic if the distances $d_1' + \mu\, d_2'$, $d_1'' + \mu\, d_2''$ etc. be equal. The light travels along d_1' d_1'' etc. at a velocity V_1, while it travels along d_2', d_2'' etc. at a lessened velocity V_2. The curvature of the surface, where $d_1 + \mu\, d_2$ is a constant, is that of a *Cartesian oval.* If f_2 were the object in the dense medium, and f_1 the image in air the same conditions apply.

If the object-point f_1 (Fig. 254) be at ∞, again the condition for aplanatism is that $d_1' + \mu\, d_2'$, $d_1'' + \mu\, d_2''$ etc. be a constant; the curve must then be that of an *ellipsoid*, i.e. all points on a plane wave $R\ T\ X$ must be retarded so that they reach the focus F_2 in the same time that each point would have travelled to $R\ T\ X$ if uninterrupted. If the object be at F_2, so that the light is projected parallel, the same surface is required.

Aplanatic Cc. Surface.—If f_2 is the virtual image of f_1 (Fig. 255) it is aplanatic if $d_1' - \mu\, d_2'$, $d_1'' - \mu\, d_2''$ etc. is a constant, and the curvature of the surface for this condition is also that of a Cartesian oval. If, however, $d_1 = \mu\, d_2$ the curve is spherical.

Aplanatic Cx. Surface.—If a luminous point be situated within the dense medium of a convex refracting surface (Fig. 256), a position on the axis can be found such that the virtual image is aplanatic. The distance of f_1 from the surface must be $r + r/\mu$, or $r\ (\mu + 1)/\mu$, and therefore the image f_2 is formed at $r+\mu\, r$, or $r\,(\mu+1)$. As the distances of f_1 and f_2 are respectively r/μ and μr from C, the magnification is $f_2/f_1 = \mu^2$. This principle is made use of in Abbé's homogeneous immersion objective employed in high-power microscopes.

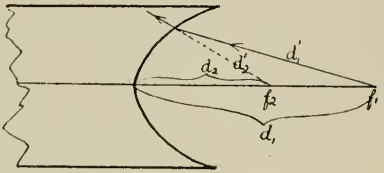

Fig. 255.

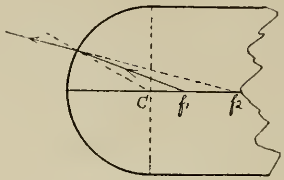

Fig. 256.

In this case the bottom lens of the objective is a hemisphere whose plane surface is towards the object, and, when immersed in cedar oil of the same index as that of the glass, the whole forms a single refracting body as shown in Fig. 256. The object is then placed at f_1, and its aplanatic image is formed at f_2, which in turn serves as an object for the remainder of the objective components.

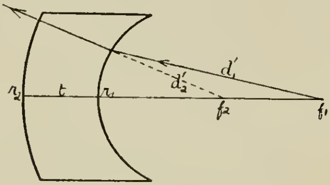

Fig. 257.

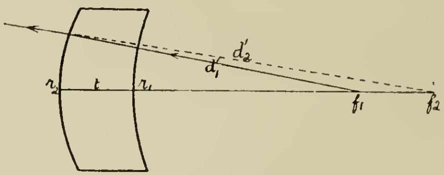

Fig. 258.

Aplanatic Lens.—If, in a Cc. periscopic lens (Fig. 257), the object at f_1 faces the concave surface, the virtual image at f_2 is aplanatic when $d'_1 = \mu\, d_2'$. In this case r_1 the radius of the first surface must be $f_1/(\mu+1)$, while that of the second surface must be $(f_1+t)/\mu$ (t being the thickness), for then f_2 lies in the centre of curvature of the second surface; $\mu f_2 = f_1$, both measured from the first surface.

Example.—Let $f_1 = 15$ cm., $t = 2$ cm., and $\mu = 1{\cdot}5$; then

$$r_1 = 15/(1{\cdot}5+1) = 6 \text{ cm., and } r_2 = (15+2)/1{\cdot}5 = 11{\cdot}33 \text{ cm.}$$

After refraction at the first surface we have

$$1{\cdot}5/f_2 = -{\cdot}5/6 - 1/15 = -4{\cdot}5/30$$

Therefore $$-4{\cdot}5 f_2 = 45, \text{ or } f_2 = -10$$

An aplanatic Cx. meniscus results when the object faces the Cc. surface (Fig. 258) if $r_1 = f_1$ and $r_2 = \mu(f_1+t)/(\mu+1)$. In this case the rays from f_1 are normal to the first surface, and d_1 is constant, as is also d_2, for all rays; f_2 lies in the aplanatic point of the second surface corresponding to the value of f_1 given in Fig. 256 illustrating the case of the single surface.

Example.—Let $f_1=15$ cm., $t=2$ cm., and $\mu=1{\cdot}5$; then $r_1=-15$ cm.,

$$r_2=1{\cdot}5(15+2)/(1{\cdot}5+1)=25{\cdot}5/2{\cdot}5$$

so that $$r_2=10{\cdot}2 \text{ cm.}$$

If there are two unknown quantities r_2 and t, values must be found for them so that $\mu f_1=f_2$, both measured from the second surface.

These are the only case where aplanatism can be obtained with lenses; there is no case for parallel light, nor for double Cx. and Cc. lenses, but, as explained under spherical aberration, this can be minimised by employing certain forms of lenses and a stop.

Reflection.

Aberrations of a Mirror.—If the angular aperture of a spherical mirror be large, rays which diverge from a point O on the principal axis do not meet in a single conjugate image-point after reflection, owing to *spherical aberration.* Mirrors suffer also from coma, radial astigmatism, curvature of the field, and distortion, but not from chromatic aberration.

Since reflection, from a spherical surface, is much more powerful than refraction by it, the spherical aberration of a mirror is greater than that of a surface having the same curvature.

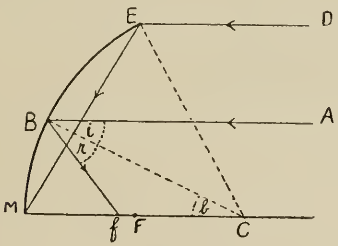

FIG. 259.

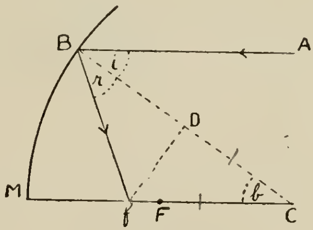

FIG. 260.

As stated in the chapter on Reflection, the aperture of a mirror should not exceed 20° at C, the centre of curvature if a point image is to be obtained of a distant point object. A parallel ray $A\ B$ (Fig. 259) incident on a Cc. mirror at a point beyond such aperture is reflected to cut the axis at f, nearer to the mirror than F. As the angle of incidence i increases so f approaches M, and when $i=60°$, as with the ray $D\,E$, it is reflected to the vertex M, the aperture being then 120°.

Let a be the semi-aperture of the mirror, and if taken as straight

$$a/r=\sin b=\sin i \quad \text{and} \quad a=r \sin i.$$

Thus, if $i=30°$, 45°, and 60°, $a=\cdot5r$, $\cdot7r$, and $\cdot866r$ respectively.

In Fig. 260, $b=i$ and $C\ D=\cdot 5r$, then $\cdot 5r/Cf=\cos i$, so that the distance Mf, at which any ray parallel to the axis cuts the latter, is in terms of i—

$$Mf=r=r/2\cos i=2F-F/\cos i.$$

From the above $a/r=\sin i=\sqrt{1-\cos^2 i};$

whence $\cos i=\sqrt{\frac{r^2-a^2}{r^2}}=\sqrt{\frac{4F^2-a^2}{4F^2}}.$

Substituting these values in the previous formulæ we get Mf in terms of a, the semi-aperture

$$Mf=r-r^2/2\sqrt{r^2-a^2}=2F-2F^2/\sqrt{4F^2-a^2}.$$

The longitudinal aberration $=\frac{2F^2}{\sqrt{4F^2-a^2}}-F.$

The two astigmatic focal lines of a small oblique pencil of parallel light at an angle of incidence i, are distant $F\cos i$ and $F/\cos i$ from the mirror.

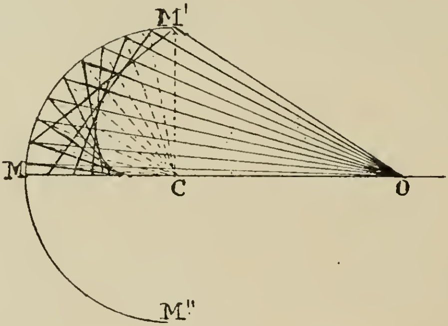

Fig. 261.

Caustic Curve.—Those rays diverging from a point which are within a few degrees of the principal axis, unite in a point, which is taken as the focus. Those rays having larger and larger angles of incidence (Fig. 261), are reflected to cut the principal axis nearer and nearer to the vertex of the mirror, and their intersection gives rise to a series of points of increased illumination which, together, form what is known as a *caustic curve*, the cause of which is spherical aberration.

Caustics may, also, be *virtual* but are not noticed because the pupil of the eye acts as a small stop and limits the divergence of the rays.

Aplanatic Reflection is simpler than refraction because the light before and after contact with the surface is in the same medium.

Aplanatic Mirror.—For a mirror (Fig. 262) capable of producing an aplanatic image, f_2 of some point f_1 within ∞ on the principal axis, if d_1' d_1'' . . . and d_2' d_2'' . . . be the incident and reflected rays, $d_1' + d_2'$ must equal $d_1'' + d_2''$, and likewise for any other incident reflected ray. Thus, in general, $d_1 + d_2 =$ a constant, so that *the mirror must be an ellipsoid of revolution with f_1 and f_2 as the foci.* The object could, of course, be at f_2 and the image at f_1.

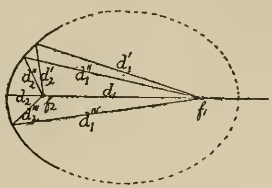

Fig. 262.

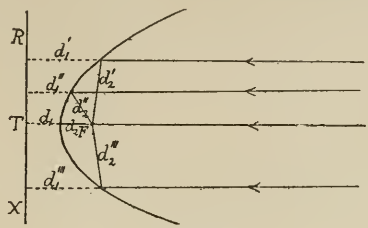

Fig. 263.

The mirror is, however, aplanatic only for these two points, aberration appearing immediately the object-point is displaced from either. For every pair of conjugates a different curve is needed, so that ellipsoidal mirrors have no practical utility, as their limited application never occurs.

Spherical mirrors of any aperture are aplanatic if the light diverges from, or converges to, a point at the centre of curvature.

If the object-point be at ∞ (Fig. 263) *the curve of the reflecting surface becomes that of a parabola* of which F is the focus. Here the directrix $R\ T\ X$ represents a plane wave interrupted by the mirror, and in order that all points on such a wave may meet at a single point, they must be converged to F in precisely equal times, so that, as before, $d_1' + d_2' = d_1'' + d_2''$, etc. If the object point be at F, the light is reflected as a parallel beam.

Parabolic mirrors are employed in reflecting telescopes for bringing rays from an infinitely distant object, such as a star, to a sharp focus. Also for projecting a parallel beam of light, as in lighthouse and optical lanterns, searchlights, microscopic reflectors, etc. Such mirrors possess the advantage

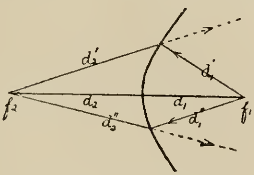

Fig. 264.

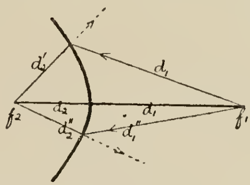

Fig. 265.

over refractors in that all light waves are equally projected, and therefore chromatic aberration does not occur. For this reason also they are preferred to refractors for the photography of celestial bodies. A Cx. spherical mirror cannot project a large beam which even approximates to parallelism.

Let f_1 (Fig. 264) be the object-point, and f_2 its virtual image; the latter is then aplanatic if $d_1' - d_2'$, $d_1'' - d_2''$ etc. be a constant. This results if the curvature of the mirror *is that of a hyperbola*, f_1 and f_2 being the foci. If the virtual object-point be at f_2 and the image at f_1 the same curvature is required. Like the ellipsoidal, the hyperbolic mirror is of no practical value.

An aplanatic *convex* reflecting surface for a near object must be hyperbolic (Fig. 265), while for parallel light it must be parabolic.

CHAPTER XXII

OBLIQUE REFRACTION AND DEVIATION

Oblique Sphericals.

Direct and Oblique Refraction.—Let Fig. 266 represent the face of a Cx. lens placed normally to the light. Let the effect of the refraction in the vertical plane be ignored and that of the horizontal considered by itself. Rays of light parallel to the axis passing through *c c′* would meet in a point behind, and in line with, O; similar rays incident at *d d′* and *e e′* would meet in corresponding points behind the lens, forming a vertical line focus parallel to *B B′*.

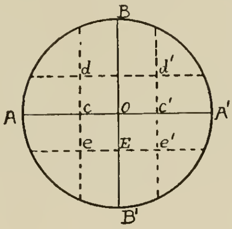

Fig. 266.

Again, considering the vertical plane by itself, the refracting effect is to produce a horizontal line focus parallel to *A A′*. The vertical line meeting its corresponding horizontal line, in the focal plane, they combine to form point foci for rays parallel to the principal axis. This is true also for any two planes at right angles to each other.

When, however, the incidence is oblique the two lines do not combine, so that a point source gives rise to two focal lines. This, as an aberration, is called *radial astigmatism* (q.v.).

Obliquity of Lens.—In Fig. 267 *L* is a Cx. lens whose principal axis is *A D F*. The line *C D* represents the axis of an oblique pencil whose image is $F_1 \ldots F_2$. The *first plane* contains the oblique axis and the principal axis; the *second plane* contains the oblique axis and is at right angles to the first. F_1 the *first line* is the focus of the first plane, and lies in the second plane. F_2, the *second* line, is the focus of the second plane and lies in the first plane. Both lines are nearer to the lens than *F*, and F_1 is nearer than F_2. The two lines are at right angles to each other, and each one is at right angles to that meridian of the lens to which the *plane* refers. In Fig. 267 the two planes are that of the paper, and that passing through $C\ D$, perpendicular to the paper.

Now the conditions obtaining for an oblique pencil of light and an upright lens as in Fig. 267, are the same for a direct pencil and an oblique lens as in

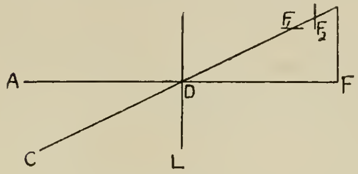

FIG 267.

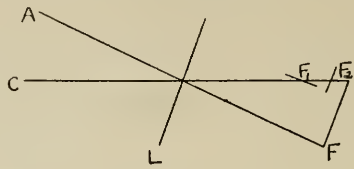

FIG. 268.

Fig. 268. That is to say, if a spherical lens acts with an astigmatic effect on an oblique pencil of light, a spherical held obliquely to the incident light similarly acts as if it were a sphero-cylindrical lens.

When a spherical lens is held upright, parallel to a screen, and at its focal distance, a luminous point on the axis will have a point-image on the screen. If now the lens be rotated around, say, a horizontal axis, the image becomes confused and drawn out as if a cylindrical had been added to the spherical. Two focal lines are formed on the screen hold at the proper distance for each. The second focal line is, in this case, vertical, and slightly within the focus of the lens; the first line is horizontal and still nearer to the lens. *Thus the effect produced by obliquity of a spherical is that of a slightly stronger spherical combined with a cylindrical whose axis corresponds to the axis of rotation.* The refraction is therefore increased in both meridians, but mostly in that at right angles to the axis of rotation.

The increased power in the meridian of the rotation is owing to the fact that the light passes through a rather greater thickness of lens when the latter is oblique than when placed normally. The increased power in the meridian at right angles to the axis of rotation is due partly to the same cause, but still more to the increase in the angles of incidence of the light in that meridian.

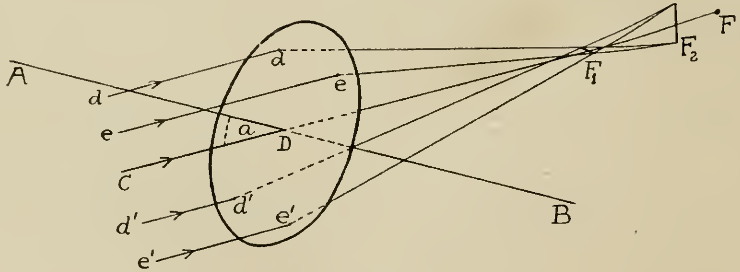

FIG. 269.

Tilted Sph.—In Fig. 269 let a be the angle of rotation of the lens, F the focal length, and F_1 and F_2 the first and second oblique foci. $A\ B$ is the principal axis of the lens, and $C\,D$ is the secondary axial ray on which the focal lines are formed. The pencil of incident light is presumed to be parallel to $C\,D$, so that rays, as d and e, or d' and e', incident in planes parallel to the axis of rotation, meet each other to form the second focal line F_2. Rays as d and d', or e and e', incident in planes at right angles to the first, meet each other to form the first focal line F_1. The angle of rotation a is that between $C\,D$ and $A\,B$, and b is the angle of refraction at the first surface (*i.e.* $\mu \sin b = \sin a$).

The distances of F_1 and F_2 are found from the following formulæ, for the derivation of which see the works of Dennis Taylor and Percival.

$$\mathrm{F}_2 = \mathrm{F} \times \frac{\sin a - \sin b}{\sin (a - b)} = \mathrm{F} \times \frac{(\mu - 1) \sin a}{\sin a \cos b - \sin b \cos a}$$

$$\mathrm{F}_2 = \frac{\mathrm{F}\,(\mu - 1)}{\mu \cos b - \cos a} \quad \text{and} \quad \mathrm{F}_1 = \frac{\mathrm{F}\,(\mu - 1) \cos^2 a}{\mu \cos b - \cos a} = \mathrm{F}_2 \cos^2 a$$

or $$\mathrm{D}_2 = \frac{\mathrm{D}\,(\mu \cos b - \cos a)}{\mu - 1} \quad \text{and} \quad \mathrm{D}_1 = \frac{\mathrm{D}\,(\mu \cos b - \cos a)}{(\mu - 1) \cos^2 a} = \frac{\mathrm{D}_2}{\cos^2 a}$$

For small angles of obliquity and $\mu = 1{\cdot}5$, the following approximate formulæ can be employed—

$$\mathrm{F}_2 = \mathrm{F}\left(1 - \frac{\sin^2 a}{3}\right) \quad \mathrm{F}_1 = \mathrm{F}_2 \cos^2 a \quad \mathrm{D}_2 = \mathrm{D} \Big/ \left(1 - \frac{\sin^2 a}{3}\right) \quad \mathrm{D}_1 = \frac{\mathrm{D}_2}{\cos^2 a}$$

Examples.—A 10″ Cx. lens is rotated 20°, then—

$$\mathrm{F}_2 = 10 \times (1 - {\cdot}117/3) = 9{\cdot}61'' \qquad \mathrm{F}_1 = 9{\cdot}61 \times {\cdot}883 = 8{\cdot}48''.$$

A + 6 D lens is rotated 30°, then—

$$\mathrm{D}_2 = \frac{6}{1 - {\cdot}25/3} = 6{\cdot}54 \qquad \mathrm{D}_1 = \frac{6{\cdot}54}{{\cdot}75} = 8{\cdot}72$$

Since D_2 does not vary greatly from D, the increased or cylindrical effect produced by obliquity of a spherical lens is

$$\mathrm{C} = \mathrm{D}_1 - \mathrm{D}_2 = \frac{\mathrm{D}}{\cos^2 a} - \mathrm{D} = \frac{\mathrm{D} \sin^2 a}{\cos^2 a} = \mathrm{D} \tan^2 a$$

If the two focal distances be measured, the angle of rotation of the lens can be found from the equations

$$\mathrm{F}_1/\mathrm{F}_2 = \mathrm{D}_2/\mathrm{D}_1 = \cos^2 a$$

Table of the Sphero-Cylindrical Effects of Oblique Sphericals.

The following table gives the approximate effects obtained by rotating a 1 D Sph. lens; the effect on other lenses is proportional. The rotation is supposed to be around a horizontal axis. The effect increases rapidly with a greater obliquity.

Angle of Rotation.	F_1 *Ver. Mer.*	F_2 *Hor. Mer.*	D_1 *Ver. Mer.*	D_2 *Hor. Mer.*	*Sph.-Cyl. Combination.*
5°	99	100	1·01	1·00	1·00 ⊂ 0·01
10°	96	99	1·04	1·01	1·01 ⊂ 0·03
15°	91	98	1·09	1·02	1·02 ⊂ 0·07
20°	84	96	1·20	1·04	1·04 ⊂ 0·16
25°	77	94	1·30	1·06	1·06 ⊂ 0·24
30°	70	91	1·45	1·09	1·09 ⊂ 0·36
35°	59	88	1·70	1·13	1·13 ⊂ 0·57
40°	50	86	2·00	1·16	1·16 ⊂ 0·84
45°	42	83	2·40	1·20	1·20 ⊂ 1·20

Tilted Cyl. or Sph.-Cyl.—The effective power of a cyl. rotated around its axis is the same as D_1 and F_1; it is, in effect, a stronger cyl. If rotated across its axis its effect also is that of a very slightly stronger cyl. If a sph.-cyl., both powers being of similar nature, be rotated round the axis of the cyl., the cyl. effect is increased because $D_1 - D_2$ is increased. If rotated round its meridian of greatest power, the sph. effect is increased, and the cyl. decreased because $D_1 - D_2$ is lessened. If the two powers are of opposite nature rotation in either principal Mer. increases the cyl. effect. A rotation oblique to the principal Mers. results in a new combination altogether.

Wide-angle or Meniscus Lenses.—The form of lens which allows of best vision over a fair range of, say, 50° or 60°, i.e. 25° or 30° on each side of the axis, is one which eliminates radial astigmatism and produces a flat field; the two do not necessarily accompany each other, and the former is the more important.

The subject has been treated by Ostwalt and Wollaston, and more recently by Dr. Percival in his "Prescribing of Spectacles" and by Mr. A. Whitwell in the *Optician*. The calculations, which are complicated, are based on motion of the eye about the centre of rotation some 27 mm. behind the plane of the lens. The actual best form, as to the curvature of the two surfaces, varies with the power of the lens, with the μ, with the distance of vision, and with the distance of the lens from the eye; it is, however, nearly always a deep meniscus, except for high power concaves. For ordinary power lenses, made of ordinary crownglass, a good form does not differ much from the ordinary commercial meniscus which has a surface of −6D on the convex and a +6D on the concave. Another series of *best form* has extremely high curvatures of some +20D or −20D.

Oblique Cylindricals.

Meridional Refraction of a Cyl.—Fig. 270 represents a Cx. plano-Cyl. lens, axis horizontal. If a lens measure be placed in contact with the maximum meridian *M* it shows the highest possible curvature of that cylindrical. Along the axis the instrument would indicate 0, and between these two the recorded power varies. Suppose the two fixed legs are at *d* and *d*, then the sag of the central leg indicates the power, which is based on the formula $r=d^2/2\ S$ (vide *The Spherometer*), and the curvature $C=2\ S/d^2$, where d is half the distance $d\ d$. Let the instrument be turned so that the legs lie on the meridian M' at an angle b with M. If now the sag were the same as before

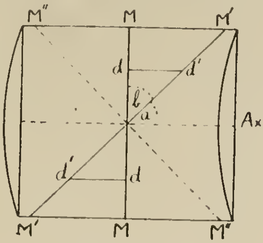

FIG. 270.

it is because the distance between the legs is greater, the curvature C' in the Mer. at M' bearing to the curvature C in Mer. M the relationship $C'/C=d^2/d'^2$, where d' is the new distance between the central and one of the fixed legs in the meridian M'. But $d'=d/\cos b$, so that

$$\frac{C'}{C}=\frac{d^2}{d'^2}=\frac{d^2\cos^2 b}{d^2}=\cos^2 b \quad \text{or} \quad C'=C\cos^2 b$$

Now the dioptric powers D at M, and D′ at M' are directly proportional to the curvatures C and C' respectively, so that in the meridian M' the power of the lens $D'=D\cos^2 b$, or what is the same, $D'=D\sin^2 a$, where a is the angle between M' and the axis. Similarly it can be shown that in the meridian M'' at right angles to M' the power $D''=D\sin^2 b$, or $D\cos^2 a$. If we consider the angle a between a given meridian and the axis of a cyl., its power varies as $\sin^2 a$; if we consider the angle b between it and the maximum meridian, the power varies as $\cos^2 b$.

Although we thus refer to the refractive power of a cyl. in any oblique meridian, yet this latter does not cause a point focus. A cyl. brings incident light to a line focus, parallel to the axis, and if the meridian of maximum power be isolated by means of a stenopœic slit, so that the oblique meridians are cut off, the line is reduced to a point, because the curvature in that meridian is spherical, and the effective aperture of the lens, at right angles to the slit, being reduced practically to zero, the result is similar to that of an ordinary spherical lens. If the slit be slowly rotated the meridians successively uncovered are elliptical in curvature, and the point focus, first obtained, gradually widens into a line parallel to the axis, showing that, although the effective part of the lens is oblique, the effective *curvature* is always that of the maximum meridian. If the rotation be continued until the slit is parallel to the axis, the line reaches its maximum length just as though the whole lens were uncovered. Thus the only meridian capable of producing a tiue focus is the maximum principal meridan, which has a spherical curvature. It is, howevei, useful to assume that the oblique meridians of a cyl. have certain powers relative to the maximum which contribute to the general power of the lens, and the following gives the necessary calculations; a distinction must, however, be drawn between the incomplete line-foci of such powers, and the point-foci produced by spherical curvatures.

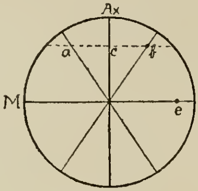

FIG. 271.

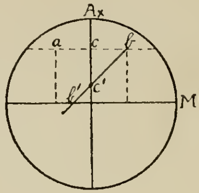

FIG. 272.

Oblique Powers of a Cyl.—Fig. 271 represents a Cx. cyl. lens whose axis *Ax.* is vertical, and maximum power *M* horizontal. Let this lens be a +5 D, and the object be a point at ∞. Any ray of light incident in the meridian *Ax.*, central to the meridian *M*, suffers no deviation, it being normal to the lens at both surfaces. Any ray incident in Mer. *M* is refracted to an extent governed by its distance from the central point of Ax., such that it meets all other rays, incident in that meridian, in a point in line with, and 20 cm. behind it. Any ray, as *b*, incident in an intermediate meridian, say that of 70°, is refracted to meet all other rays, incident in the plane *b c a*, in a point in line with *c*, and also 20 cm. distant. The deviation suffered by the ray *b*, refracted in an intermediate meridian, is less than that which occurs when refracted as *e* in meridian *M*, both being equidistant from the central point of Ax. The total image is a Ver. line.

In the case of a sph.-cyl. (Fig. 272) a ray incident at *b* in an oblique meridian is refracted by the sph. to a point on the principal axis, and by the cyl. to a point in line with *c*, with the resultant oblique deflection in the direction *b′*, so that it meets rays incident in a plane *b c a* parallel to *M* in *b′*, and those incident in a plane parallel to *Ax.* in *c′*; or if the lens be regarded as consisting of crossed cyls., the deviation is towards both axes, resulting in an oblique deviation towards *c′* in the first, and *b′* in the second focal line.

Let D be the maximum power of a cyl., D′ the power in a given Mer., and D″ that at right angles to D′; let *a* be the angle between the axis and the Mer. of D′. Then the powers of a cyl. in any pair of given opposite meridians are, as previously stated, found from

$$D' = D \sin^2 a \qquad \text{and} \qquad D'' = D \cos^2 a$$

The power along the axis is 0, and at right angles it is D, so that the total power of this pair of opposite Mers. is $D + 0 = D$. Likewise the sum of the powers of any pair of opposite Mers. is equal to D, for $\sin^2 a + \cos^2 a = 1$, so that $D \sin^2 a + D \cos^2 a = D' + D'' = D$.

Thus the powers of a + 3 D. Cyl. Ax. 180°, at 20° and 110°, are—

$D'=3\times\cdot11696=\cdot35$ D. at 20°; $D''=3\times\cdot88303=2\cdot65$ D. at 110°.

Let *A Y* and *A Z* (Fig. 273) represent the forces exerted, respectively, in the Hor. and Ver. Mers. by, say, a 3·5 D. Cyl. Ax. 60°. Let H be the horizontal and V the vertical effect. Now $XY=AZ=\sin 60°$, and $XZ=AY=\cos 60°$, whence $H=D\sin^2 a=3\cdot5\times\cdot75=2\cdot625$, $V=D\cos^2 a=3\cdot5\times\cdot25=\cdot875$, and $2\cdot625+\cdot875=3\cdot5=D$.

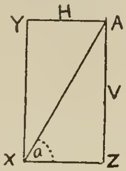

Fig. 273.

In these calculations it is merely necessary to find either D′ or D″ since the other can be obtained by subtraction from D. Thus if V=·875, H=3·5 − ·875=2·625, and *vice versa.*

Following are the approximate powers of unit cyl. in different Mers. calculated as shown above.

Degrees from Ax.	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
Proportional power ..	0	·01	·03	·07	·12	·18	·25	·33	·42	·50	·58	·67	·75	·82	·88	·93	·97	·99	1·0

Obliquely crossed Cylindricals.—If two cyls. D and D′ are placed with their axes corresponding in the Ver. Mer. their combined Ver. power=0, and the Hor.=D+D′. If one or both cyls. be rotated, they are equivalent to a combination of some two other principal powers. When the two axes are at right angles the combination is equal to a sph. if D=D′, and to an ordinary cross-cyl. if D and D′ are unequal. It should be particularly noted that, *with any obliquity of the axes, two (or more) cyls. are always equivalent to some other cross-cyl. whose axes are at right angles, and are, therefore, also equivalent to some sph.-cyl.* The sum of the two principal powers D_1 and D_2 is always equal to the sum of the individual maximum powers D and D′, that is,

$$D+D'=D_1+D_2$$

Not only the powers of the principal Mers., but also *the sum of the powers of any pair of Mers. at right angles to each other*=D+D′. Rotation of the axis of one or both cyls. merely locates the refraction in varying quantities as regards each of any pair of opposite meridans, and does not alter the total power.

Let b (Fig. 274) be the angle between the axes of two cylindricals D and D', of which D is the higher of the two. Let D_1 and D_2 be the two resulting powers, D_1 being the higher. Let c be the angle which the axis of D_1 makes

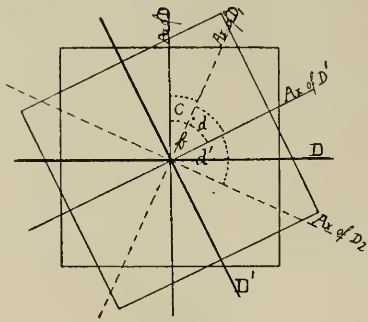

Fig. 274.

with that of D, and let d be the angle it makes with that of D′. Then angle $b=c+d$. Now D_1 corresponds with the axis of D_2, and D_2 with the axis of D_1. From the foregoing we have $D+D'=D_1+D_2$

and $D \sin^2 c+D' \sin^2 d=D_2$, also $D \cos^2 c+D' \cos^2 d=D_1$

Multiplying these together we get

$$D_1D_2=D^2 \sin^2 c \cos^2 c+D'^2 \sin^2 d \cos^2 d+D\,D' \sin^2 c \cos^2 d + D\,D' \sin^2 d \cos^2 c$$

Now $$D^2 \sin^2 c \cos^2 c=D'^2 \sin^2 d \cos^2 d$$

$$\therefore\ D^2 \sin^2 c \cos^2 c+D'^2 \sin^2 d \cos^2 d=2D\,D' \sin c \cos c \sin d \cos d$$

so that

$$D_1D_2=D\,D' (\sin^2 c \cos^2 d+\sin^2 d \cos^2 c)+2D\,D' \sin c \cos c \sin d \cos d$$
$$=D\,D' (\sin c \cos d+\sin d \cos c)^2$$

but $$\sin c \cos d+\sin d \cos c=\sin (c+d)=\sin b$$

therefore $$D_1D_2=D\,D' \sin^2 b$$

and since $$D_1+D_2=D+D'$$

we can, knowing the *multiple* and the *sum* of the two numbers, arrive at their difference C, thus

$$C=D_1-D_2=\sqrt{(D+D')^2-4D\,D' \sin^2 b}$$

Then we get in the resultant combination

The higher power $$D_1=\frac{D+D'+C}{2}$$

and the lower power $$D_2=\frac{D+D'-C}{2}$$

D_1 is the spherical + the cylindrical; D_2 is the spherical; C is the cylindrical.

The following relationships exist:

$$D_1 - D = D' - D_2 = \frac{C + D' - D}{2} \quad \text{and} \quad C \sin c \cos c = D' \sin b \cos b$$

also
$$\frac{D^2}{\sin^2 d \cos^2 d} = \frac{D'^2}{\sin^2 c \cos^2 c} = \frac{C^2}{\sin^2 b \cos^2 b}$$

Now $C^2 \sin^2 c \cos^2 c = D'^{\,2} \sin^2 b \cos^2 b = D' \sin^2 b\,(D' - D' \sin^2 b)$

but
$$D' = D_1 + D_2 - D$$

so that $C^2 \sin^2 c \cos^2 c = D' \sin^2 b\,(D_1 + D_2 - D - D' \sin^2 b)$

$$= D_1 D' \sin^2 b + D_2 D' \sin^2 b - D\,D' \sin^2 b - (D' \sin^2 b)^2$$

Substituting $D_1 D_2$ for $D\,D' \sin^2 b$ we get

$$C^2 \sin^2 c \cos^2 c = D_1 D' \sin^2 b + D_2 D' \sin^2 b - D_1 D_2 - (D' \sin^2 b)^2$$

Or $C \sin^2 c \times C \cos^2 c = (D_1 - D' \sin^2 b)\,(D' \sin^2 b - D_2)$

also since
$$\sin^2 c + \cos^2 c = 1,$$

$$C = C \sin^2 c + C \cos^2 c = (D_1 - D' \sin^2 b) + (D' \sin^2 b - D_2)$$

Then we deduce that

$$C \sin^2 c = D' \sin^2 b - D_2 \quad \text{and} \quad C \cos^2 c = D_1 - D' \sin^2 b$$

Now
$$\tan c = \frac{C \sin^2 c}{C \sin c \cos c} = \frac{D' \sin^2 b - D_2}{D' \sin b \cos b} = \tan b - \frac{D_2}{D' \sin b \cos b}$$

Substituting for D_2 its equivalent $\dfrac{D\,D' \sin^2 b}{D_1}$

$$\tan c = \tan b - \frac{D\,D' \sin^2 b}{D_1\,D' \sin b \cos b} = \tan b - \frac{D}{D_1} \tan b$$

So that
$$\tan c = \frac{(D_1 - D) \tan b}{D_1}$$

c is the angular distance of D_1, the stronger resultant cyl., from that of D, the stronger original. We could find a formula for d, but it is unnecessary, since $d = b - c$. The distance d' of the axis of D_2, the weaker resultant, from that of D', the weaker original cyl., is found from

$$\tan d' = \frac{(D_2 - D') \tan b}{D_2}$$

Since $(D_2 - D') = (D_1 - D)$, it is easy to confirm calculations, but care must be taken with the − signs. When D and D' are of similar signs d' is negative. The two resultant axes must be 90° apart—*i.e.* $b - (c - d') = 90°$. A positive measurement is towards the other axis, and a negative one is away from it.

To find b the angle between two cyls. D and D' in order to produce any two effects D_1 and D_2, we have $\sin^2 b = D_1 D_2 / D\,D'$, but of course it is possible only when $D_1 + D_2 = D + D'$.

Unlike Cyls.—When the one cyl. is positive and the other negative, the same formulæ apply, but c is negative, as also is d'. The calculation is rather more involved, and it is better, as suggested by Mr. A. Jameson, to convert the combination into one with similar signs. This is done by adding to the combination a Cx. sph., whose power is equal to that of the Cc. cyl., thus converting the − cyl. into a + cyl. Then ignoring, for the moment, the added sph. the calculation is made for the resultant of the two obliquely crossed Cx. cyls., and from the result the added sph. is finally deducted. This is illustrated in an example given below.

Two Equal Like Cyls.—Here the calculation is simplified, for when $D=D'$, $c=d$, so that it is unnecessary to calculate C or c. Thus

$$D_1=2\ D\ \sin^2\frac{b}{2} \qquad D_2=2\ D\ \cos^2\frac{b}{2} \qquad c=b/2.$$

Two Equal Unlike Cyls.—Here also the calculation is simplified, for

$$D=-D' \qquad D+D'=0 \qquad D_1+D_2=0$$

$$D\ D'\ \sin^2 b=D_1D_2=-D^2\sin^2 b=D_1^2 \text{ or } D_2^2$$

therefore $\quad D\sin b=D_1$, and $-D'\sin b=-D_2$.

$\tan c=\dfrac{\cos b}{1+\sin b}$ measured negatively from the Cx., for the resultant Cx., or from the Cc. for the resultant Cc.

or $\tan c=\dfrac{1+\sin b}{\cos b}$ measured from the Cx. positively for the resultant Cc., or from the Cc. for the resultant Cx.

The two measurements=90°. C need not be calculated.

Examples.—+3 C. Ax. 70° ⊂ +2 C. Ax. 20°, $D+D'=+5$, $b=50°$.

$$C=\sqrt{(3+2)^2-4\times3\times2\times\cdot5868}=\sqrt{13\cdot92}=3\cdot30.$$

$$D_1=\frac{5+3\cdot30}{2}=4\cdot15 \qquad D_2=\frac{5-3\cdot30}{2}=\cdot85.$$

$$\tan c=\frac{(4\cdot15-3)\times1\cdot1918}{4\cdot15}=\cdot3302=\tan 18°\ 18'.$$

The combination is +·85 S. ⊂ +3·30 C. Ax. 51°42′.

$$DD'\sin^2 b=D_1D_2,\ i.e.\ 3\times2\times\cdot5868=4\cdot15\times\cdot85=3\cdot52.$$

The sum of the maximum powers of the two original cyls., in this example +5 D, is not changed by altering the position of the two axes with respect to each other, for the sum of the two principal meridians of the resultant cyls. is similarly +5 D. That is, $D_1+D_2=4\cdot15+\cdot85=5$ D.

Two Unlike Cyls.—+4 C. Ax. 20° ⊂ −2·75 C. Ax. 65°, D+D′=+1·25, b=45°.

$$C=\sqrt{(4-2{\cdot}75)^2-4\times 4\times -2{\cdot}75\times{\cdot}5}=\sqrt{23{\cdot}5625}=4{\cdot}85.$$

$$D_1=\frac{1{\cdot}25+4{\cdot}85}{2}=3{\cdot}05 \qquad D_2=\frac{1{\cdot}25-4{\cdot}85}{2}=-1{\cdot}80.$$

$$\text{Tan } c=\frac{(3{\cdot}05-4)\times 1}{3{\cdot}05}=-{\cdot}311=\tan 17^\circ\ 15''.$$

The combination is

−1·80 S. ⊂ +4·85 C. Ax. 2° 45′, or +3·05 S. ⊂ −4·85 C. Ax. 92° 45′.

$D_1+D_2=+3{\cdot}05-1{\cdot}80=+1{\cdot}25.$

Here by calculation tan c is a minus quantity, and the angle is measured from the axis of D away from the axis of D′ instead of towards it.

By Jameson's suggested method, adding +2·75 D sph., we have +4 C. Ax. 20° ⊂ +2·75 C. Ax. 155° to deal with.

$$C=\sqrt{(4+2{\cdot}75)^2-4\times 4\times 2{\cdot}75\times{\cdot}5}=\sqrt{23{\cdot}5625}=4{\cdot}85.$$

$$D_1=\frac{6{\cdot}75+4{\cdot}85}{2}=5{\cdot}80 \qquad D_2=\frac{6{\cdot}75-4{\cdot}85}{2}={\cdot}95.$$

$$\text{Tan } c=\frac{(5{\cdot}80-4)\times 1}{5{\cdot}80}={\cdot}311=\tan 17^\circ\ 15''.$$

The combination is +·95 S. ⊂ +4·85 C. Ax. 2° 45′.

Then, subtracting +2·75 sph. we get as above—

−1·80 S. ⊂ +4·85 C. Ax. 2° 45′.

It should be noticed that C is measured towards the axis of the +2·75 C., which is 25° below the horizontal on the right.

Two Equal Like Cyls.—+4 D C. Ax. 10° ⊂ +4 D C. Ax. 60°. c=25°.

$$D_2=2\times 4\times{\cdot}1786=1{\cdot}4288 \qquad D_1=2\times 4\times{\cdot}8214=6{\cdot}5712.$$

The combination = +1·4288 S. ⊂ +5·1424 C. Ax. 35°.

Two Equal Unlike Cyls.—+4 D C. Ax. 60° ⊂ −4 D C. Ax. 120°. b=60°.

$$D_1=-D_2=4\times{\cdot}866=3{\cdot}464.$$

The combination = +3·464 C. Ax. 45° ⊂ −3·464 C. Ax. 135°.

Graphical Illustration of the Formulæ.

Draw $A\ D$ (Fig. 275) in units of length=D, and $A\ D'=D'$, making the angle $D'\ A\ D=b$. On $A\ D$ mark off $A\ H=D_1-D$, and prolong $A\ D$ a distance $D\ F=A\ H=D_1-D$ so that $A\ F=D_1$. From E drop $E\ H$ normal to H, and $E\ H=(D_1-D)\tan b$. From D draw $D\ G$ equal and parallel to $A\ E$;

connect *E G* and from *G* drop the normal *G F* to *F* so that *G F*=*E H*. Connect *A G*. Then G A F=c, and G F=tan $c=\dfrac{(D_1-D)\tan b}{D_1}$.

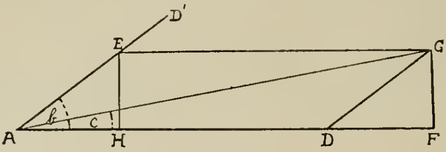

FIG. 275.

Oblique Prisms.

Oblique Powers of a Prism.—The power of a prism lies in the base-apex plane. At right angles thereto—in the axial plane—it possesses no deviating power, and at any Mer. intermediate to these two the power Δ' is—

$$\Delta'=\Delta \cos r,$$

where Δ is the power of the prism, and r the angle between the given meridian and the base-apex plane. Thus the effect at 40° of a 4° prism whose base-apex line is vertical, r being 50°,

$$\Delta'=4\times \cdot 6427=2\cdot 57°,$$

the power Δ'' in the Mer. at right angles thereto is—

$$\Delta''=\Delta \sin r.$$

Following are the approximate powers of unit prism at different Mers. calculated as shown above.

Degrees from base-apex plane ..	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
Proportional power	1	·99	·98	·97	·94	·91	·87	·82	·77	·71	·64	·57	·5	·44	·34	·26	·17	·09	0

Indirect Prismatic Action.—If the base-apex plane is oblique the prism has not only its main power in that plane, but a Ver. and a Hor. effect as well. Let (Fig. 276) V represent the vertical and H=H′ the horizontal effect of an oblique prism. Let Δ=P=O Q be the power of the prism, and r the angular distance of its base-apex line from the horizontal. Then, since $\sin r = V/\Delta$ and $\cos r = H/\Delta$,

$$V=\Delta \sin r \quad \text{and} \quad H=\Delta \cos r.$$

Thus let the base-apex line of a 5°d prism be at 20° from the horizontal. Then $V=5\times\cdot 3420=1\cdot 71°$, and $H=5\times\cdot 9397=4\cdot 698°$.

If the base-apex line is at 45°, a 6Δ has

$$V=6\times\cdot 7071=4\cdot 24\Delta \text{ and } H=6\times\cdot 7071=4\cdot 24\Delta.$$

Given a 4°d prism, the position of the base-apex line so that the Ver. effect be 1°d is sin $r=1/4=\cdot25=$sin 14° 29′ from the horizontal.

Then $V=4\times\cdot25=1°$d and $H=4\times\cdot9681=3\cdot872°$d.

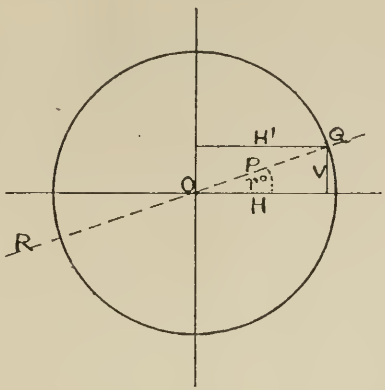

Fig. 276.

If with a 6^Δ a Hor. effect of 3^Δ is needed, cos $r=3/6=\cdot5=$cos 60°, so that the base-apex line must be at 60°, V being $\cdot6\times\cdot866=5\cdot2^\Delta$.

If the angular distance$=r'$ of the base-apex line be taken from the Ver.,

$$V=\Delta\cos r' \text{ and } H=\Delta\sin r'.$$

If a prism of 8^Δ has its base at 30° left eye, its components are—

$$V=8\times\cdot5=4^\Delta \text{ base up;}$$
$$H=8\times\cdot866=7^\Delta \text{ (approx.) base out.}$$

Obliquely Crossed Prisms.—The resultant effect of two (or more) prisms whose base-apex lines are oblique to each other is found as follows:

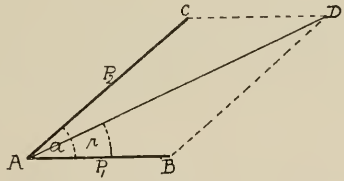

Fig. 277.

In Fig. 277 let $A\,B$ and $A\,C$ be of such lengths that they are proportional to, and represent the deviations caused by two prisms of $\Delta_1=P_1$ and $\Delta_2=P_2$, whose base-apex lines are crossed at an angle a. To construct graphically the resultant deviation the rhombus $A\,B\,C\,D$ is completed by drawing $C\,D$ equal and parallel to $A\,B$, and $B\,D$ equal and parallel to $A\,C$. Then AD represents in units of length the resultant deviation, and r is the angle it makes with the weaker of the original prisms. If a third prism Δ_3 were introduced, a similar construction between *A D* and Δ_3 would give the single resultant of the three prisms Δ_1, Δ_2 and Δ_3, and so on for any further number.

In Fig. 277 $AD^2 = AB^2 + AC^2 + 2AB \cdot AC \cos a$.

But A D is the resultant prism Δ, and A B and A C the original prisms Δ_1 and Δ_2 respectively, so that the formula giving the resultant Δ of two prisms Δ_1 and Δ_2 obliquely crossed at an angle a, can be written—

$$\Delta = \sqrt{\Delta_1^2 + \Delta_2^2 + 2\Delta_1\Delta_2 \cos a},$$

and $\tan r = \dfrac{\Delta_2 \sin a}{\Delta_1 + \Delta_2 \cos a}$, r being measured from the weaker original.

Examples.—Two prisms of 6°d and 8°d respectively whose base-apex lines are 30° apart; then—

$$\Delta = \sqrt{6^2 + 8^2 + 2 \times 6 \times 8 \times \cdot 866} = \sqrt{183 \cdot 136} = 13 \cdot 53°,$$

and $\tan r = \dfrac{8 \times \cdot 5}{6 + (8 \times \cdot 866)} = \cdot 3091 = \tan 17°\ 11'$ from the 6°d.

When $\Delta_1 = \Delta_2$ the formulæ simplify to

$$\Delta = 2\Delta \cos \frac{a}{2} \quad \text{and} \quad r = \frac{a}{2}.$$

When the two prisms are at right angles to each other, the one Hor. and the other Ver., the angle $a = 90°$, $\sin 90° = 1$, and $\cos 90° = 0$, so that the formulæ simplify to

$$\Delta = \sqrt{H^2 + V^2} \quad \text{and} \quad \tan r = V/H$$

as given in Chapter XI.

Oblique Decentrations and Decentrations of Oblique Cyls.

As shown in Chapter XII., a Cx. lens causes the action of a prism whose base lies in the direction of decentration, and a Cc. lens has the reversed action.

The prismatic power of a Sph. (Fig. 278) is governed, in any Mer., by the distance from the axis *O*; thus the zone indicated by the dotted circle has,

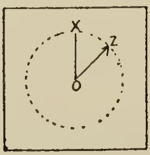

Fig. 278.

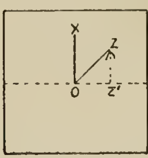

Fig. 279.

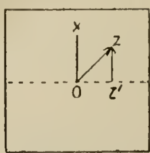

Fig. 280.

at any point, the same prismatic power. A decentration from *O* to *X* or from *O* to *Z* has a prismatic action of $\Delta = D\ C$, where C is the distance in cm.

Fig. 279 shows a Cyl. axis Hor. Its prismatic power lies in the plane at right angles to the axis. At a point X, its action is the same as at X in Fig. 278, but if decentered from O to Z, its action is only $Z'Z$ which is less than $O\,Z$. Therefore the result of a decentration equal to $O\,X$ in the different Mers. varies from the maximum D C in the power plane to zero in the axial plane. Fig. 280 shows a combination of Sph. and Cyl. axis Hor. Here, at the point Z, we have the combined actions of $O\,Z$ in Fig. 288 and $Z'Z$ in Fig. 289. The two effects are of the same nature if the two components are also of the same nature, or contrary to each other if the one is Cx. and the other Cc.

The necessary constants can be introduced into the following formulæ when the prismatic effects are required in degrees or degrees of deviation, or the effects in $^{\Delta}$ can be converted by the usual methods.

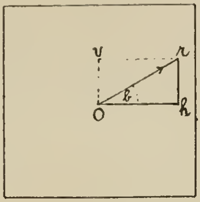

FIG. 281.

The Effects of Oblique Decentration of a Sph.—When a Sph. is decentered obliquely we have, in the meridian of decentration, Δ=D C, where Δ is the prismatic effect, D is the dioptral number of the lens, and C the decentration in cm. In Fig. 281 *or* represents an oblique decentration in a plane at an angle b from the Hor. When O is moved to r, the Hor. displacement is oh, and the Ver. is $ov=hr$, and since $oh=\cos b$ and $ov=hr=\sin b$, the Hor. effect H, and the Ver. effect V, of an oblique decentration Δ, are found by the equations

$$H=\Delta \cos b=D\,C \cos b \quad \text{and} \quad V=\Delta \sin b=D\,C \sin b.$$

Thus, if a +7 D sph. be decentered ·6 cm. at 30°,

$$\Delta=7\times\cdot6=4\cdot2^{\Delta},\ H=7\times\cdot6\times\cdot866=3\cdot637^{\Delta},\ \text{and}\ V=7\times\cdot6\times\cdot5=2\cdot1^{\Delta}.$$

When a Sph. is decentered in any plane M_1 its prismatic action in any other plane M_2 is D C cos b, where b is the angle between M_1 and M_2. If $b=90°$ there is no effect in M_2.

Oblique Cyls.—If a plano-cyl., axis oblique, be moved horizontally, vertically or in any direction, *an object viewed through it will appear to move in a direction across the axis*, thus showing that the prismatic action is always at right angles to the axis, or in the meridian of maximum power. Indeed this result is only to be expected, since the virtual prisms in a Cyl. have their base-apex lines at right angles to the axis. Therefore no matter what decentration be made obliquely to that plane, the resultant effect is as though a smaller decentration had been made in the principal Mer. at right angles to the axis —that is, in the *power plane*.

Oblique Cyls. Decentered in Prin. Mer.—The decentration of an oblique cyl. *along* the axis has no effect whatever.

If it is decentered in the power plane, *across* the axis, the effect is, in the principal meridians, the same as with the Ver. and Hor. decentration of upright cyls. This, however, produces Hor. and Ver. effects as well, because any single oblique action can be resolved into two components at right angles to each other.

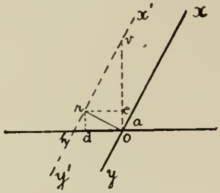

FIG. 282.

In Fig. 282 let xy be the axis of a cyl. at an angle a with the horizontal, and let $x'y'$ be its position when the lens is decentered from o to r in the meridian of maximum refraction. The distance or=C, and the angle b, which the power plane makes with the Hor., is the complement of a. The Hor. power of the lens is D $\cos^2 b$, and the distance $oh=c/\cos b$. The Ver. power is D $\sin^2 b$ and the distance $ov=c/\sin b$. Therefore Δ, the main effect, H the horizontal, and V the vertical, are as follows:

$$\Delta = \text{DC}.$$
$$\text{H} = \text{D}\cos^2 b \times \text{C}/\cos b = \text{D C}\cos b.$$
$$\text{V} = \text{D}\sin^2 b \times \text{C}/\sin b = \text{D C}\sin b,$$

or we can resolve the single effect or into two components od and $oe=dr$ lying in the Hor. and Ver. planes respectively.

Thus $\quad rod=b,\ od=\cos b,\ oe=rd=\sin b.$

Then $\quad \text{H}=\text{D C}\cos b \quad \text{and} \quad \text{V}=\text{D C}\sin b.$

Thus, let +4 D. Cyl. axis 60° be decentered ·4 cm. at 150°; then $b=30°$ and

$$\Delta = 4 \times {\cdot}4 = 1{\cdot}6^{\Delta}$$

$$\text{H} = 4 \times {\cdot}4 \times {\cdot}866 = 1{\cdot}386^{\Delta},\ \text{V} = 4 \times {\cdot}4 \times {\cdot}5 = {\cdot}8^{\Delta}.$$

Oblique Cyl. Decentered Hor. or Ver.—Similar prismatic effects are obtained if an oblique Cyl. be decentered horizontally or vertically, *the maximum effect Δ lying in the power plane of the Cyl.*

If the decentration is Hor. from o to h=C (Fig. 282), then $or=\cos b$, and $ov=\cot b$, so that

$$\text{H}=\text{D}\cos^2 b\ \text{C} \qquad \Delta=\text{D C}\cos b.$$
$$\text{V}=\text{D}\sin^2 b\ \text{C}\cot b=\text{D}\sin b\ \text{C}\cos b.$$

For a Ver. decentration

$$V = D \sin^2 b\, C \qquad \Delta = D\, C \sin b.$$
$$H = D \cos^2 b\, C \tan b = D \cos b\, C \sin b.$$

If a +4 C. Ax. 60° is decentered ·462 cm. horizontally, $b=30$

$$\Delta = 4 \times \cdot 462 \times \cdot 866 = 1 \cdot 6^{\Delta}$$
$$H = 4 \times \cdot 75 \times \cdot 462 = 1 \cdot 386^{\Delta},\ V = 4 \times \cdot 5 \times \cdot 462 \times \cdot 866 = \cdot 8^{\Delta}.$$

If a +4 D Cyl. Ax. 60° be decentered ·8 cm. vertically, $b=30°$

$$\Delta = 4 \times \cdot 8 \times \cdot 5 = 1 \cdot 6^{\Delta}$$
$$H = 4 \times \cdot 866 \times \cdot 8 \times \cdot 5 = 1 \cdot 386^{\Delta},\ V = 4 \times \cdot 25 \times \cdot 8 = \cdot 8^{\Delta}.$$

These results, if compared with the last previous one, show that a 4 D Cyl. Axis 60° decentered ·4 cm. in the power plane or ·462 cm. horizontally or ·8 cm. vertically, has precisely similar prismatic actions.

It should be noted that *a horizontal or vertical effect alone can never be obtained by decentering an oblique cyl.*, and that is why it is inadvisable to decenter such lenses.

Only if the power plane nearly corresponds to the line of decentering can the effects in other meridians be ignored. Indeed it may occur that the Hor. decentering of an oblique cyl. results in a greater Ver. effect, and *vice versa.* This is shown in the table below.

The maximum Ver. effect of a Hor. displacement, and *vice versa,* results when the axis is at 45°. Also since sin 45°=cos 45°, the effect is equal in both directions, no matter how decentered.

The Ver. effect of a Hor. decentration of a cyl. whose axis is at, say, 30° is the same as when the axis is at 60°, because although the Ver. displacement is less in the first case, the Ver. power is greater.

To illustrate these effects let a +1 D C. be decentered horizontally 1 cm., the axis being respectively at 45°, 30° and 60°. Then

With axis at 45°	$\Delta = \cdot 7^{\Delta}$.	$H = \cdot 5^{\Delta}$	$V = \cdot 5^{\Delta}$
,, ,, 30°	$\Delta = \cdot 5^{\Delta}$	$H = \cdot 25^{\Delta}$	$V = \cdot 43^{\Delta}$
,, ,, 60°	$\Delta = \cdot 86^{\Delta}$	$H = \cdot 75^{\Delta}$	$V = \cdot 43^{\Delta}$

Hor. or Ver. Cyl. Decentered Obliquely.—Let Δ' be the effect in the power plane, and Δ'' be that in the plane of decentration, then—

$$\Delta' = D\, C \cos b \qquad \Delta'' = D \cos^2 b\, C.$$

In the axial plane there is, of course, no effect.

The Decentration of a Sph.-Cyl.—Here we have to combine the actions of the two components as already indicated. In the following D is the power of the Sph. and D′ that of the Cyl.

Oblique Sph.-Cyl. Decentered in the Prin. Mers.—The effect of decentering an oblique sph.-cyl. in the principal meridians is the same as with Hor. and Ver. decentration when the axis is Hor. and Ver. respectively.

If the decentration is in the power plane of the cyl. there are, besides the principal effect Δ, certain Hor. and Ver. effects introduced due to the oblique decentering of the sph. When o (Fig. 283) is moved to r there is a Hor.

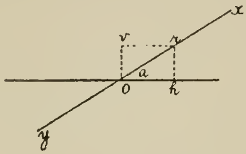

Fig. 283.

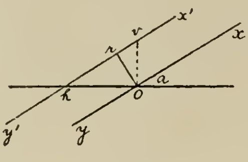

Fig. 284.

decentration $oh=\cos a$, and a Ver. one $ov=rh=\sin a$. In this case a corresponds to b in Fig. 281, so that we can write as before—

$$\Delta=\text{D C} \qquad \text{H}=\text{D C}\cos b \qquad \text{V}=\text{D C}\sin b.$$

Suppose a + 3 S. ⊂ + 2 C. Ax. 30° is decentered ·4 cm. at 30°

$$\Delta=3\times\cdot4=1\cdot2^{\Delta}$$

$$\text{H}=3\times\cdot4\times\cdot866=1\cdot04^{\Delta}, \qquad \text{V}=3\times\cdot4\times\cdot5=\cdot6^{\Delta}$$

When the decentration is in the power plane of the cyl. (Fig. 284) we have in the Mer. of decentration, and in the Hor. and Ver. Mers. the effects of both the sph. and cyl. In this case angle a is the complement of b.

$$\Delta'=\text{D C}+\text{D}'\text{ C}=(\text{D}+\text{D}')\text{ C}.$$
$$\text{H}=\text{D C}\cos b+\text{D}'\text{ C}\cos b=(\text{D}+\text{D}')\text{ C}\cos b.$$
$$\text{V}=\text{D C}\sin b+\text{D}'\text{ C}\sin b=(\text{D}+\text{D}')\text{ C}\sin b.$$

Here Δ' is used in place of Δ because, as will be shown later, the true prismatic action is not yet disclosed.

Thus suppose + 3 Sph. ⊂ + 2 Cyl. Ax. 30° be decentered ·4 cm. at 120°

$$\Delta'=(3+2)\times\cdot4=2^{\Delta}.$$

$$\text{H}=(3+2)\times\cdot4\times\cdot5=1^{\Delta}, \qquad \text{V}=(3+2)\times\cdot4\times\cdot866=1\cdot732^{\Delta}$$

Oblique Sph.-Cyl. Decentered Hor. or Ver.—If the lens is decentered horizontally, the sph. causes no Ver. effect, but the cyl. acts as does the plano-cyl. Let Δ' be the effect in the power plane of the latter. Then

$$\text{H}=\text{D C}+\text{D}'\cos^2 b\text{ C}=(\text{D}+\text{D}'\cos^2 b)\text{ C}.$$
$$\Delta'=\text{D C}\cos b+\text{D}'\text{ C}\cos b=(\text{D}+\text{D}')\text{ C}\cos b.$$
$$\text{V}=\text{D}\cos 90^\circ+\text{D}'\sin^2 b\text{ C}\cot b=\text{D}'\sin b\text{ C}\cos b.$$

If the decentration is Ver. the sph. causes no Hor. effect, but the cyl. acts as when not combined with a sph.

$$\text{V}=\text{D C}+\text{D}'\sin^2 b\text{ C}=(\text{D}+\text{D}'\sin^2 b)\text{ C}.$$
$$\Delta'=\text{D C}\sin b+\text{D}'\text{ C}\sin b=(\text{D}+\text{D}')\text{ C}\sin b.$$
$$\text{H}=\text{D}\cos 90^\circ+\text{D}'\cos^2 b\text{ C}\tan b=\text{D}'\cos b\text{ C}\sin b.$$

Thus + 3 Sph. ⊂ +2 Cyl. axis 30° decentered ·8 cm. Hor. $b=60°$.

$$\Delta'=(3+2)\times{\cdot}8\times{\cdot}5=2^{\Delta}$$

$$H=(3+2\times{\cdot}25)\times{\cdot}8=2{\cdot}8^{\Delta}, \qquad V=2\times{\cdot}866\times{\cdot}8\times{\cdot}5={\cdot}7^{\Delta}.$$

The same lens decentered ·462 cm. Ver. $b=60°$.

$$\Delta'=(3+2)\times{\cdot}462\times{\cdot}866=2^{\Delta}$$

$$H=2\times{\cdot}5\times{\cdot}462\times{\cdot}866={\cdot}4^{\Delta}, \qquad V=(3+2\times{\cdot}75)\times{\cdot}462=2{\cdot}08^{\Delta}.$$

While a Hor. or Ver. prismatic effect can never be obtained with an oblique plano-cyl., this is possible with a sphero-cyl. by adjustment of the decentering so as to neutralise the unneeded effects introduced. Practically this is best achieved, if it be possible, by employing a prism for the marking as described in Chapter XII. It is always possible if the sph. is strong compared with the cyl.

When the plane of decentration, although not precisely corresponding, does not differ much from that of the axis of the Cyl., the latter may be ignored as, indeed, it may also be if the Cyl. is very weak compared with the Sph.

Hor. or Ver. Sph.-Cyl. Decentered Obliquely.—In the axial plane there is an effect Δ''' from the Sph. only. Let Δ' be the effect in the power plane of the cyl., and Δ'' that in the plane of decentration.

$$\Delta'=D\ C\cos b+D'\ C\cos b=(D+D')\ C\cos b.$$

$$\Delta''=D\ C+D'\cos^2 b\ C=(D+D'\cos^2 b)\ C.$$

$$\Delta'''=D\ C\sin b+D'\cos 90°=D\ C\sin b.$$

To Calculate C.—If a certain prismatic effect is required, regardless of other effects produced, by a decentration, C can be found by equating the formula given for the given conditions.

Actual Resultant Prismatic Effects.—An oblique cyl. decentered horizontally or vertically or in any plane M′ always has its greatest prismatic effect in the power plane M, because if b is the angle between the two, the power diminishes from M to M′ as $\cos^2 b$, although the decentration increases from M′ to M as $1/\cos b$, so that the result in M is $D\ C\cos^2 b\times 1/\cos b = D\ C\cos b$. Thus a 1 D lens decentered ·4 cm. in M′ a plane 60° from M, we have in M′ $(1\times{\cdot}25)\times{\cdot}4$ and in M we have $1\times({\cdot}4\times{\cdot}5)$.

With an oblique sph.-cyl. the effect in the power plane of the cyl. might, or might not, be greater than in that meridian in which the displacement is made, this depending on the relative powers of the two components ; and the *actual resultant prismatic power lies between the two*, if the powers of the sph. and cyl. are of the same sign, and outside them if they are of opposite signs.

The Hor. and Ver. effects of a decentration of an oblique sph.-cyl. having been calculated the actual resultant effect Δ can be obtained from the formulæ.

$$\Delta=\sqrt{H^2+V^2} \quad\text{and}\quad \tan r=V/H.$$

Suppose $H=2{\cdot}8^{\Delta}$ and $V={\cdot}7^{\Delta}$; then

$$\Delta=\sqrt{2{\cdot}8^2+{\cdot}7^2}=\sqrt{8{\cdot}33}=2{\cdot}88^{\Delta} \qquad \tan r={\cdot}7/2{\cdot}8={\cdot}25=\tan 14^\circ$$

General Principles.—Although definite formulæ have been given in the case of obliquely decentered cyls. and sph.-cyls., and for finding the Hor. and Ver. components and the main effects, they can be worked, in each case, from first principles, as indicated in the following. This may be necessary if the decentering is neither Hor. nor Ver. nor in the principal meridians.

Decentration in any Mer.—If a cyl. be decentered in any Mer. its effect in the power plane can be calculated, and from this the component Hor. and Ver. effects.

Thus suppose a +2 D Cyl. Ax. 30° be decentered ·2 cm. in Mer. 70°; then $b=50^\circ$ and

$$\Delta=D'\,C\cos b=2\times{\cdot}2\times{\cdot}6428={\cdot}26^{\Delta}\text{ at }120^\circ.$$

Then $\quad H={\cdot}26^{\Delta}\cos 60^\circ \qquad V={\cdot}26^{\Delta}\cos 30^\circ.$

If + 6 S. ⊂ + 2 C. Ax. 30° be decentered 2 mm. upwards in meridian 70°, the prismatic effect due to the sph. is $6\times{\cdot}2=1{\cdot}2^{\Delta}$ base up at 70°, while that due to the cyl., as from above, is $\cdot26^{\Delta}$ base in Mer. 120°.

Therefore there are two prismatic effects crossed at 50° (120°–70°) and the resultant of these can be found from the formulæ given on page 258. In this case

$$\Delta=\sqrt{1{\cdot}2^2+{\cdot}26^2+2\times1{\cdot}2\times{\cdot}26\times{\cdot}6428}=\sqrt{1{\cdot}9087}=1{\cdot}38$$

$$\tan r=\frac{{\cdot}26\times{\cdot}766}{1{\cdot}2+{\cdot}26\times{\cdot}6428}={\cdot}14=\tan 8^\circ$$

So that the effect of decentering + 6 S. ⊂ + 2 C. Ax. 30° 2 mm. up in meridian 70°, is $1{\cdot}38^{\Delta}$ base up at 78°.

Further, this oblique prismatic effect may be resolved into its Hor. and Ver. components from the formulæ given previously.

Any possible case of decentration can be worked from general principles, as in the examples just given, provided, of course, that proper attention be paid to signs, etc., but, as can be seen, the procedure is complicated.

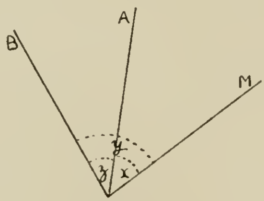

Fig. 285.

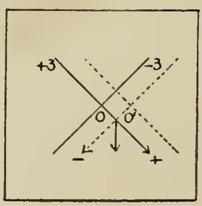

Fig. 286.

Universal Formulæ.—The effect produced in any Mer. *B* by decentering a cyl. in another Mer. *A* can be found as follows: *M* is the Mer. of power,

x is the angle between M and A, y is that between M and B, and z is that between A and B (Fig. 285).

Now for a displacement C in A, that in $M=C \cos x$, and in B it is C cos $x/\cos y$. The power in B is D′ $\cos^2 y$, therefore—

$$\Delta_B = D' \cos^2 y \times C \cos x/\cos y = D \cos y \; C \cos x.$$

For a sph. the effect in B is D C cos z.

Opposite Powers.—It is comparatively easy to learn the prismatic actions when both powers are of the same nature, but when they are of opposite natures this becomes complicated; they may tend to augment or neutralise each other.

Thus, suppose the lens in Fig. 286 be +3 C Ax. 45° ⊂ – 3 C Ax. 135°, and it be decentered to the right from O to O'; the virtual base of the + component is to the left downward, and that of the – is to the right downward, as shown by the two arrow heads. Indeed, in such a combination a Hor.

+ *Cyl. whose power is*	*Decentered.*	*Right.*	*Left.*	*Up.*	*Down.*
Hor.	—	R	L	—	—
5° to 85°	—	R U	L D	U L	D R
Ver.	—	—	—	U	D
95° to 175°	—	R D	L U	U R	D L
– *Cyl. whose power is*					
Hor.	—	L	R	—	—
5° to 85°	—	L D	R U	D R	U L
Ver.	—	—	—	D	U
95° to 175°	—	L U	R D	D L	U R
Equal + and – Cyls. + *Power.*	– *Power.*				
Hor.	Ver.	R	L	D	U
0° to 45°	90° to 135°	R U	L D	D L	U R
45°	135°	U	D	L	R
45° to 90°	135° to 180°	L U	R D	U L	D R
Ver.	Hor.	L[1]	R	U	D
90° to 135°	0° to 45°	L D	R U	U R	D L
135°	45°	D	U	R	L
135° to 180°	45° to 90°	R D	L U	D R	U L

decentration produces no Hor. effect whatever, but merely a Ver. one, and *vice versa*. The arrow *C* shows the direction of the resultant effect in the case cited.

In order to be certain of the position of the base of the actual resultant virtual prism it is better to mark out that of each component separately, when the one power is + and the other −.

The table on p. 265 gives the direction of the virtual base, R, L, U, and D indicating respectively Right, Left, Up and Down. The directions of the Cyl. are those of the powers, as shown in Fig. 286.

CHAPTER XXIII

MAGNIFYING POWER OF LENSES

Apparent Magnification.—Hitherto we have dealt only with the ratio between the actual sizes of object and image (real and virtual), which ratio may vary to an indefinite extent depending upon the position of the object with respect to the lens. Here we deal with what is known as the *apparent* magnification of the object—or rather, its image—when viewed through a Cx. lens used as a simple microscope, loupe or reader. Apparent magnification is not subject to such great variations as the magnification mentioned above.

Magnification is expressed by linear increase of size, the superficial magnification being the square of the linear. Thus $\times$ 3 implies an increase of three diameters, while $\times$ 1/3 expresses a corresponding reduction.

The Visual Angle is that subtended by an object at the nodal point of the eye, and the retinal image subtends there a similar angle. It is, however, more convenient to consider distances from F_1, the anterior focal point of the eye, where the object, and the retinal image projected to the refracting plane of the eye, subtend equal angles, as shown in Fig. 287.

Distance of Most Distinct Vision.—When a person views a near object so as to get the best possible general view of it, he unconsciously holds it, not at his near point, where the demand on accommodation would be very great, but at the most convenient distance called *the distance of most distinct vision.* This distance varies considerably in different individuals, depending upon age, length of eyeball, etc. Thus it is theoretically farther away in

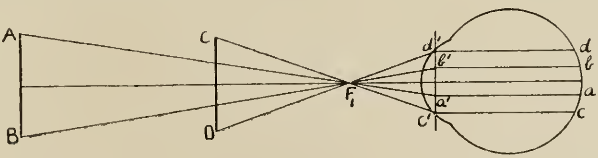

FIG. 287.

hyperopia and nearer in myopia; it is decidedly nearer in youth, when Ac. is active, than in old age. Consequently, for the purpose of establishing formulæ, a conventional value of 10″ is taken as the *average* or *standard* distance of most distinct vision. In the following articles the distance will be reckoned as from F_1, some 15 mm. from the refracting plane of the eye.

Magnification.—In Fig. 287, *A B* is an object at a distance *d* from F_1, *b a* is its retinal image, and *b′ a′* the projection of the latter on to the refracting plane. *C D* is the same object at a shorter distance d_1, *d c* is its retinal image, and *d′ c′* the corresponding projection on to the refracting plane. It is obvious that the sizes of *d′ c′* and *b′ a′* are proportional to the angles they subtend at F_1, which are also those subtended by the object in its two positions. These angles, in turn, are as $d : d_1$, so that—

$$M=\frac{d' c'}{b' a'}=\frac{d}{d_1}$$

Thus if $d=10''$ and $d_1=2''$, M=5, provided the object could be seen clearly without accommodation at both distances. This, however, is not the case, because, in order to see an object clearly at any distance the emmetrope must exert accommodation, for example, 4 D at 10″, which, by increasing the refraction of the eye, reduces the size of the retinal image as compared with that obtained if no accommodation were used. At 2″ much more Ac. is needed, namely, 20 D, and the total refraction of the eye would be still further increased, and the retinal image still further diminished. Further, no ordinary eye could possibly exert anything like 20 D of accommodation, so that an object at a very near distance is quite indistinct, owing to the extreme divergence of the light from its various points. It is to overcome this light divergence that a Cx. lens is employed as a simple microscope.

Thus, when a watchmaker fixes a 2″ lens in front of his eye at the anterior focus, he sees a near object apparently larger than he would, without the lens, at 10″, because, not only does the object subtend a greater angle, but also it is seen without accommodation. It might be said that a powerful convex lens does not magnify in the popular sense of the word, but merely allows the object to be brought within the limits of accommodation. If accommodation were possible at both points, *i.e.* 4 D for 10″ and 20 D for 2″, M would be about 4·5.

The Pinhole.—If an O be viewed at a certain distance by means of accommodation, and is then viewed through a pinhole disc, placed at F_1 of the eye, its apparent size is increased because the diminishing effect of accommodation is, to a great extent, eliminated. Again, if a very small object be held quite close to the eye it is either not seen at all, or very blurred. If, now, a pinhole be introduced between it and the eye, it becomes clearly visible because the pinhole cuts down the retinal confusion circles, and makes accommodation unnecessary. There is, of course, enormous loss of light, and definition may be impaired by diffraction, but the object is seen with fair distinctness.

Let an O at, say, 10 inches be viewed through a pinhole held close to the eye, and then gradually be brought quite close; it is seen distinctly at all instances, without accommodation, and the apparent change of size due to that of the retinal image is easily observed. In these circumstances the ideal condition illustrated in Fig. 287 is realised, and the expression

$$M=\frac{d}{d_1}$$

rendered exact.

The Formula for Magnification.—To obtain the true apparent M. due to a Cx. lens we must compare the retinal image of the object, without and with the lens, *with equal accommodation used in both cases.* In other words, it is necessary to find the distance f_1 of the object from the lens such that its virtual image be projected to 10″ from F_1 of the eye or from the lens. Therefore the lens is presumed to be at the anterior focus of the eye.

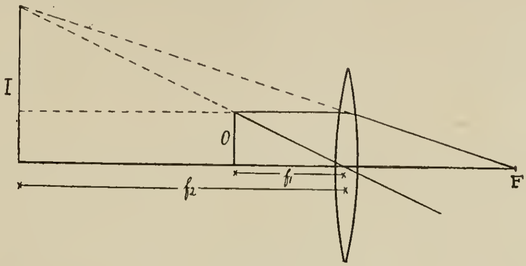

FIG. 288.

Thus in Fig. 288 O is the object at a distance from the lens f_1 which is slightly less than F, and I is its virtual image at a distance f_2 which is, by convention, 10″. The lens being at F_1, we have—

$$\frac{1}{f_1}=\frac{1}{F}+\frac{1}{10}=\frac{10+F}{10\,F} \quad \text{or} \quad f_1=\frac{10\,F}{10+F};$$

and

$$M=\frac{I}{O}=\frac{10}{f_1},$$

substituting for f_1 its value in terms of F—

$$M=\frac{10(10+F)}{10\,F}=\frac{10+F}{F}=1+\frac{10}{F} \quad \text{or} \quad 1+\frac{D}{4}.$$

This is the usually accepted formula to express the magnifying power when the lens is placed so that its optical centre coincides with the anterior focus of the eye.

In this case the accommodation used is the same without and with the lens. That is to say, if the object is at 10″ from F_1 a certain amount of accommodation is brought into action in order to see it clearly. With the lens in position at F_1 the object is at f_1 and the light from it, after refraction by the lens, diverges as if proceeding from 10″. The object at f_1 and its virtual image at 10″ subtend the same angle at F_1 so that this formula expresses the *ratio between the angles subtended by the image and object when both are at* 10″, as shown by the projection of the object back to the plane of the image in Fig. 288. There is no magnification due to the lens itself because, when at F_1, no matter what its strength, *it cannot alter the size of the retinal image.* The only effect a lens, when used as a simple microscope, can have is *to enable the object to be seen under a larger angle* by overcoming the extreme divergence of the light from a very near object.

Thus with a + 2″ lens $M=1+10/2=6$.

When the lens is very strong the formula may be simplified to

$$M=10/F \quad \text{or} \quad D/4$$

A 1/4″ lens has $M=10/\frac{1}{4}=40$ instead of $1+40=41$.

Since the distance of most distinct vision governs the magnifying power of any lens it is smaller for a myope whose distance of distinct vision is shorter than 10″ and greater for the hypermetrope whose position of most acute vision is greater than 10″. Therefore, in the preceding formulæ, instead of 10″, there would be substituted some other figure if needed. Thus, for a hypermetrope, where $d=16$, $M=1+16/2=9$. For a myope, where $d=6$, $M=1+6/2=4$.

Combinations and Cc. Lenses.—The foregoing formulæ apply equally to a combination of lenses like that found in an ordinary eyepiece, provided the equivalent focal length and position of the equivalent points be known.

As magnification results from vision of a near object through a Cx. lens, because the angle under which the image is clearly seen is then larger, diminution is obtained with a Cc. lens because the angle under which the image is seen is then correspondingly smaller.

General Formula.—Now the most comfortable view of an object is obtained when accommodation is at rest. The emmetrope, when using a Cx. lens for magnifying purposes, places the object at F of the lens, so that the light may be rendered parallel. The hypermetrope places it beyond F, so that he may receive convergent light, and the myope places it within F, so that the light may be divergent after refraction. In all cases the conjugate focus of the object distance is the conjugate focus of the retina with accommodation passive—that is to say, the image formed by the lens must coincide with the far point of the eye.

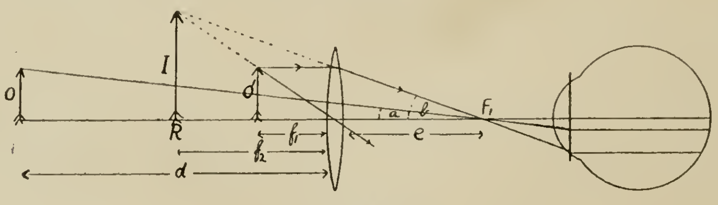

FIG. 289.

In Fig. 289 let e be the distance of the lens from F_1, and let f_1 and f_2 be respectively the object and image distance from the lens. Let d be the conventional distance of 10″ from F_1, and O the object at 10″. O' is the same object so placed that its image I is at R, the far point of the eye. R also is measured from F_1, that is to say, it indicates the nominal error of refraction of the eye. a is the angle subtended by O, and b is that subtended by I at F_1. Then

$$\mathrm{M}=\frac{b}{a}=\frac{\dfrac{\mathrm{I}}{f_2+e}}{\dfrac{\mathrm{O}}{d}}=\frac{\dfrac{\mathrm{I}}{f_2+e}}{\dfrac{\mathrm{O}'}{d}}=\frac{\mathrm{I}d}{\mathrm{O}'(f_2+e)}=\frac{f_2 d}{f_1(f_2+e)}$$

But

$$f_1=\frac{\mathrm{F}f_2}{\mathrm{F}+f_2}$$

$$\therefore\quad \mathrm{M}=\frac{f_2 d(\mathrm{F}+f_2)}{\mathrm{F}f_2(f_2+e)}=\frac{d(\mathrm{F}+f_2)}{\mathrm{F}(f_2+e)}$$

But

$$f_2=\mathrm{R}-e.$$

$$\therefore\quad \mathrm{M}=\frac{d(\mathrm{F}+\mathrm{R}-e)}{\mathrm{FR}}=\frac{10(\mathrm{F}+\mathrm{R}-e)}{\mathrm{FR}}$$

The value of R is positive in myopia and negative in hypermetropia.

When the lens is at F_1, $e=0$, so that—

$$\mathrm{M}=\frac{10(\mathrm{F}+\mathrm{R})}{\mathrm{FR}}.$$

In emmetropia $R=\infty$, so that—

$$\mathrm{M}=\frac{10}{\mathrm{F}}$$

for all distances of the lens from F_1.

In emmetropia, when $\mathrm{F}>10''$, M is less than unity, any apparent magnification being due to suppression of accommodation, the result being an increase in the retinal image between that obtained *without* and *with* accommodation.

A farsighted person whose $\mathrm{R}=-20''$ (H.2D) uses a 2″ lens at 1″ beyond F_1; then

$$\mathrm{M}=\frac{10(2-20-1)}{2\times-20}=\frac{-190}{-40}=4{\cdot}75$$

with the same conditions, the person being shortsighted with $\mathrm{R}=20''$ (M.2D),

$$\mathrm{M}=\frac{10(2+20-1)}{2\times20}=\frac{210}{40}=5{\cdot}25$$

For the emmetrope, M=10/2=5.

It will be noticed that when $e=\mathrm{F}$—that is, when the lens is at a distance equal to its own focal length from F_1—the expression again simplifies to M=10/F. The magnification for any state of refraction is then constant and equal to that obtained by the emmetrope.

The variation in magnification is the same for similar degrees of hypermetropia and myopia, as represented by the correcting lens at F_1, which is to be expected seeing that the decrease and increase, respectively, in axial length of the eye is the same for equal degrees of nominal ametropia.

If the object be within the focus of the lens, and the eye withdrawn from the latter, the retinal image becomes smaller, but when the O is beyond F, *i.e.* adapted for a hyperope, the retinal image increases in size as the eye is drawn back. Should the object be exactly in the focal plane, the retinal image undergoes no change, since the emergent light is parallel. In all three cases, however, the field of view is reduced, less being seen of the object than when the eye is close to the lens.

M for O at a Fixed and Lens at a Variable Distance.—When an object is held, not near to the eye, but at, say, the reading distance, some 16 inches, M varies as a Cx. lens is moved to and fro between the eye and the object. The calculation is based on the optical system of the eye and a Cx. lens at a variable distance, and is shown in "Visual Optics and Sight Testing." The formula is—

$$M=\frac{FF_1}{Fx-xy+y^2},$$

where F pertains to the lens, F_1 is the anterior focus of the eye, x is the distance of object, and y that of the lens, both measured from F_1 of the eye.

This is the expression that applies when a *hand glass* or *reader* is employed, and the lens moved about, as distinct from a simple microscope, when the object is moved in order to get the best view.

This formula shows that $M=1$—that is to say, the lens has no magnifying effect—when $y=o$, so that the lens is at F_1 of the eye. Also $M=1$ when $y=x$, so that the lens touches the object viewed. In both cases the formula simplifies to F_1/x, which is the same as when there is no lens employed.

The formula also shows that the denominator has its minimum value, M being at its maximum, when $y=x/2$—that is, when the lens is midway between F_1 and the object viewed.

The same serves for a Cc. lens, which also has no effect on magnification when at F_1 or at the object. The denominator has its maximum value when $y=x/2$, showing that the maximum diminution occurs when a Cc. lens is midway between F_1 and the object.

Therefore, for any position of the object, and for any Cx. lens, withdrawal of the lens towards the object at first increases the magnification, which reaches a maximum when half-way between the anterior focus and the object; M then decreases until, when the lens touches the object, the magnification is the same as what it was when the lens was coincident with the anterior focus, this being zero. This maximum, when a Cx. lens is about midway between eye and object, holds good in *all* cases, but is quite independent of the *clearness* of the image, which may either be blurred or sharp, depending upon the strength of the lens. Similarly the greatest diminution occurs when any concave lens is midway between eye and object, but in this case, provided there is sufficient accommodative power, the image is clear. These facts explain some of the phenomena in connection with spectacle lenses.

Best Distance.—For an object to be seen at its best through a hand glass, it should be placed slightly within the focus. Firstly, because, owing to the curvature of the field, especially in strong lenses, only the central portions are clearly defined, the object having to be moved nearer to bring the peripheral parts into focus, whereas, if the edges are rendered clear by bringing the object within the focus, a slight effort of accommodation will render also the centre sharp. Secondly, it is difficult to view a near object without involuntary accommodation, and therefore its exertion to a slight extent renders the observation more comfortable. The same applies, more or less, to the simple microscope, but with the latter the object viewed is generally very small, and therefore the peripheral definition is not of so much importance; rather the highest magnification possible is sought, combined with the most comfortable conditions—that is, with accommodation at rest.

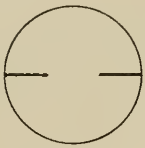
Fig. 290.

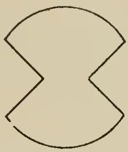
Fig. 291.

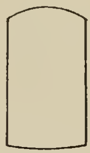
Fig. 292.

Special Forms of Magnifiers in which spherical aberration is reduced.

The Wollaston (Fig. 290) is a sphere cut into halves; these are reunited with a stop (of about F/5) between them.

The Coddington (Fig. 291) is a sphere with a deep V-shaped groove so that the central area is reduced to about F/5, as by a stop.

The Stanhope (Fig. 292) is a cylinder spherically curved at both ends; the object end being less curved than the eye end.

To measure Virtual Magnification.—This is indefinite because it depends, with a given lens, on the distance it is held both from the eye and the object. It can be done roughly by viewing an object at 10 inches by the one eye and a similar object close to the eye through the magnifier. If the lens is after the style of a Coddington, which is not held close to the eye, the same object can be viewed at the same time through the lens and directly. The object should be, in both cases, a series of parallel horizontal lines some 5 mm. apart. Then the magnified image space will appear equal to so many spaces seen by the unaided eye.

CHAPTER XXIV

POLARISATION AND PEBBLES

Polarised Light.—The beam of light transmitted by a homogeneous medium, such as air or glass, is ordinary in the sense that it consists of waves whose transverse vibrations lie in every direction across the line of travel, whereas the vibrations of polarised light are confined to certain directions only. The polarisation of light may be plane, circular, or elliptical. The plane of polarisation of plane-polarised light is that from which the vibrations are eliminated, the latter being executed at right angles to the plane of polarisation. (Sometimes the plane of polarisation is taken as that in which the vibrations *are* executed.)

Suppose a rope attached to a wall and vibrated at the free end; vibrations or waves will run along the rope in any plane. If, however, the rope be passed between two upright sticks all vibrations will be stopped except those in the vertical plane. The former illustrates ordinary unpolarised waves, and the latter plane-polarised light waves.

Polarisation is said to be a proof of the truth of the generally accepted theory of the transverse wave motion of light.

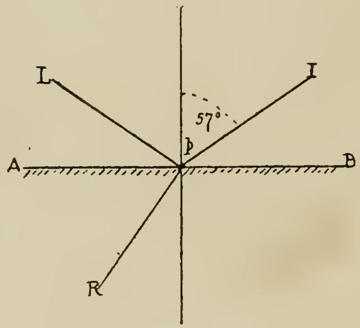

FIG. 293.

Polarisation by Reflection.—At a certain angle of incidence, which varies with the μ of the medium, the reflected and refracted beams L and R (Fig. 293) from the glass surface $A\ B$ are at right angles to each other. The vibrations of the incident light, which are perpendicular to the surface, penetrate it and are transmitted, while some of those parallel to the surface are reflected. The reflected beam is polarised, the vibrations being confined to a plane parallel to the reflecting surface, while the plane of polarisation (in accordance with the definition above) is perpendicular to the surface, and is therefore the same as the plane of incidence of the light.

Only a small portion of the total incident light is reflected, but the amount is increased by increasing the number of surfaces, that reflected from the under surface of a plate being polarised in the same manner as that from the top surface.

The angle of incidence necessary to obtain polarisation of the reflected beam is found by the equation $\mu = \tan p$, where p is the polarising angle. Thus p differs with the optical density of a medium, the polarising angle of water being 53°, that of glass about 57°, and that of a diamond 68°. Differently coloured rays have different polarising angles, so that white light is never completely polarised by reflection. The polished surfaces of metal have no polarising effect.

Polarised reflected light can be best obtained from a sheet of black glass, and, of course, suitably placed with respect to the incidence of the light. The blackening prevents double reflection if a single beam of polarised light is required.

Polarisation by Refraction.—The light, incident at the polarising angle on a transparent body, which is refracted and transmitted at right angles to the reflected beam, is partially polarised, the plane of polarisation being at right angles to that of the polarised reflected light. Pure polarised refracted light can only be obtained when a beam is transmitted obliquely through a bundle of thin glass plates bound together, so that, by repeated reflection, all light polarised in the opposite direction is got rid of. The proportion of light transmitted is less as the number of plates increases, while the proportion reflected is greater.

Double Refraction.—Most crystals polarise light owing to double refraction, notably calcite (Iceland spar), quartz, and tourmaline. A light wave in air or in any homogeneous body vibrates in every direction across its line of propagation, and its velocity is uniform and inversely proportional to what is termed the optical density of the medium. In a crystal, owing to its molecular structure, the retardation of waves, when incident obliquely to the axis of crystallisation, is greater in one direction than in another, so that the rays are transmitted along two separate paths, the one ray being called *ordinary*, and the other the *extraordinary* ray.

The *ordinary* ray consists of spherical waves which obey the ordinary laws of refraction of light in homogeneous media, but the waves of the *extraordinary* ray are elliptical, and conform to no fixed law. The extraordinary ray is not at right angles to the wave front, nor does it lie in the same plane as the incident ray and the normal to the point of incidence. Both rays are polarised in planes at right angles to each other and travel at unequal speeds, except in the direction of what is known as the *optic axis*, where both waves have the same velocity and where no double refraction occurs. In planes at right angles to the optic axis there is also no double refraction in the ordinary sense, but the waves are retarded unequally, the one travelling more slowly behind the other. Thus doubly refracting crystals have two refractive indices, one for the ordinary, and one for the extraordinary ray.

Rock Crystal or Pebble.—Rock crystal or quartz is a pure, usually colourless, crystalline variety of silica, which occurs in nature in the form of a hexagonal (six-sided) prism, terminating in a six-sided pyramid. Its mean index of refraction (μ=1·54) is about the same as that of ordinary crown glass, but lower than that of flint glass, its dispersion ($\mu_H - \mu_A$)=·014 being lower than either. When cut into a slab or ground to form a lens, it is usually styled a *pebble.* It is much harder than glass, more brittle, a better conductor of heat, and it transmits much more readily than glass the ultra-violet rays which lie outside the visible spectrum. Its density is 2·65, that of glass being from about 2·4 to 3·4.

The relative scarcity and greater difficulty of working pebble makes it comparatively expensive. Its freedom from liability to become scratched is its sole advantage, so that pebble is not so good as crown glass for spectacle lenses, although perhaps for simple spherical convex lenses, which are frequently put on and off and therefore specially liable to become scratched in the centre, it is sometimes to be preferred. As lenses, the pebble should be quite clear and free from striæ, specks and flaws, and should be axis cut.

Axis-Cut Pebble.—Axis-cut pebble is that which is cut into slabs at right angles to its line of crystallisation, so that when the surfaces receive their spherical curvatures, the axis of the crystal coincides with the principal axis of the lens. Axis-cut is more expensive than non-axis-cut pebble, because in cutting it there is not so good an opportunity of utilising those parts of the crystal which are free from flaws, as when the slabs are cut without regard to any particular direction.

To Recognise Pebble.—Pebble is recognised by (*a*) feeling colder to the tongue than glass, (*b*) by the fact that on account of its hardness a file makes no impression on it, and (*c*) by the polariscope test. By the latter the difference between axis-cut and ordinary pebble can also be seen. As supplied to the optical trade pebble is usually quite colourless, and when in the form of a lens it has a sharper ring than glass.

Double Refraction in Pebble.—Pebble possesses the property of double refraction, the refractive index for the ordinary ray being 1·548 and for the extraordinary ray 1·558, and since the index is higher for the extraordinary than for the ordinary ray, pebble is described as a *positive* crystal. It is because the difference in the μ's of the two rays is so small that double refraction by a pebble spectacle lens is not appreciable, the images being too close together to be seen double, the more so since the substance of the lens is thin.

Tourmaline.—Tourmaline cut parallel to its axis reduces an incident beam of light to two sets of polarised waves, the one in the plane of the axis of the crystal, the other at right angles to it. By a curious property of tourmaline the former (the ordinary ray) is absorbed almost immediately, and the latter (the extraordinary ray) only is transmitted, so that all the emergent plane-polarised light is vibrating in the plane parallel to the axis. The plane of polarisation of a tourmaline plate can be determined by analysing the light polarised by reflection from a plate of glass, as mentioned on p. 278.

The Tourmaline Polariscope.—The simple polariscope consists of two plates of tourmaline cut parallel to their axes and suitably mounted. These plates are sometimes fitted to the ends of a wire spring like a pair of sugar tongs crossed, and called a *pincette*. If the two plates are placed in such a position that their axes are parallel, the plane-polarised beam of light transmitted by the first plate will traverse the second, and if a polariscope, so fixed, is looked through, green or brown light—due to the colour of the tourmaline—can be seen. The combination looks much more opaque than would pieces of glass of the same intensity of colour, because half the light received by it is quenched. The outer plate which polarises the light is called the *polariser*, and the second plate—the one near the eye—is called the *analyser*. If, now, the analyser be rotated, while still looking through the instrument, the light will be found to become less and less bright, until, when it has been turned through a quarter-circle, the two axes being then at right angles to one another, the plane-polarised beam transmitted by the polariser is stopped by the analyser. If the axis of the polariser is, say, horizontal, it can transmit only waves whose vibrations are horizontal, while the analyser can transmit only those whose direction is vertical; consequently all the light is blocked out. So long as the two axes are oblique to one another, some light passes through both plates.


Fig. 294.

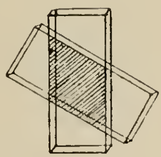
Fig. 295.

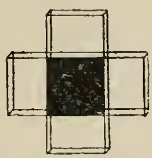
Fig. 296.

It is in the position of *extinction* of the two plates that the polariscope serves as a pebble tester, so that if required for that purpose, it should be looked through and the one plate rotated until the *darkness* is complete. Unless this is done it is useless for the work, although even if it cannot be made quite *dark* there is an appreciable difference in the quantity of light transmitted by glass and by pebble placed between the plates, as explained in the following article. Fig. 294 shows the two tourmalines with their axes parallel, Fig. 295 with their axes oblique, and Fig. 296 with their axes at right angles.

A polariscope can be formed by two plates of black glass, arranged in planes at right angles to each other. The first receives the incident light at an angle of 57°, and the second receives the light thus polarised by reflection also at an angle of 57°. If the original light consists of a parallel beam total extinction at the second glass reflector will occur.

A simple polariscope can also be formed by one piece of black glass and a single plate of tourmaline. Polarised light reflected at the proper angle from the glass may be intercepted and quenched by the tourmaline when the latter is held with its axis perpendicular to the plane of the glass, and at right angles to the reflected ray. In both cases, however, monochromatic light must be used if total extinction is to be obtained; with white light, owing to its constituents of varying wave-length, complete darkness cannot be secured.

Recognition of Pebble by Polariscope.—If an ordinary glass lens, being homogeneous in nature, is placed between the two plates of the tourmaline polariscope, it has no effect on the plane-polarised beam of light transmitted by the polariser, and nothing can be seen through the instrument. A pebble placed in the instrument, by virtue of its double refracting nature, so twists or rearranges the vibrations of the beam transmitted by the first tourmaline plate that the light is incident on the second plate in directions other than at right angles to its axis, and part of it is transmitted. Hence with a pebble tester a pebble can be distinguished from glass, since, when a pebble is placed between the tourmalines, light is seen, while none is seen when glass is so placed. Also a slice of quartz, like most other crystals, when viewed through a polariscope, presents arrangements of colour which are characteristic of it.

If a pebble cut parallel to the axis of the crystal (non-axis-cut) is placed between the dark tourmalines and rotated there are found two positions in which no light passes; the one is where the axis of the pebble is parallel to, or in the same line with, the axis of the polariser, and the other is where it bears the same relation to the axis of the analyser. In either case, the polarised beam of light received by the pebble cannot be made to vibrate so as to be transmitted by the analyser.

When the pebble is *axis-cut* a clear centre, surrounded by a series of coloured rings, is seen, and the light cannot be blocked out, no matter the position of the crystal between the two tourmalines.

When the pebble is cut nearly, but not quite, perpendicular to its axis, coloured arcs of circles (incomplete rings) are seen; the light also cannot be blocked out, no matter what its position between the plates of tourmaline, because the axis cannot be made parallel to that of either the polariser or analyser. The intensities of the colours and the sizes of the arcs are both dependent on the nearness of the section of the pebble to that of right angles to the axis, *i.e.* on its nearness to axis-cut.

Advantage of Axis-Cut Pebble.—Rock crystal which is axis-cut is preferable for lenses to that which is non-axis-cut, because in the former there is no double refraction for light parallel to the axis.

Iceland Spar.—Spar or calcite, like quartz and tourmaline, has the power of double refraction, but since the ordinary wave has a higher index than the extraordinary, it is termed a *negative* crystal. The index of the ordinary ray is 1·659 and that of the extraordinary 1·486, so that the comparatively large difference between the indices causes a corresponding high degree of double refraction, enabling the doubling of objects to be plainly seen through slabs only a few mm. in thickness.

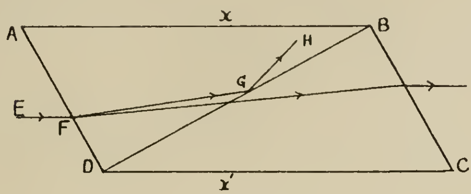

FIG. 297.

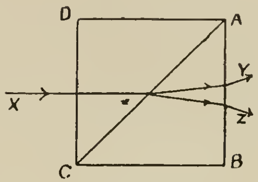

FIG. 298.

The Nicol Prism Polariscope.—This is a device whereby a beam of pure polarised light is obtained by transmission through an arrangement of Iceland spar. The latter is cleaved obliquely to its axis, and the two segments re-cemented by balsam whose index of refraction is 1·54, or about midway between the indices of the two rays. Now the angle of cleavage with the axis is so arranged that, when the ordinary ray is incident on the layer of balsam, it does so at an angle greater than the critical angle for indices of 1·659 and 1·540, and is therefore totally reflected to one side. On the other hand, the extraordinary ray, whose index 1·486 is lower than 1·54, that of the balsam, is transmitted, and constitutes a plane-polarised beam of light which is, however, only half the intensity of the original beam. The polariscope consists of two Nicol prisms, the one being the polariser and the other the analyser. On account of the scarcity of spar, Nicol prisms are now expensive to make, and are largely replaced by reflecting polariscopes of some form or other.

Fig. 297 shows a rough outline of a nickel prism. *ABD* and *BDC* are the two portions cemented together by the layer of Canada balsam *BD*; the axis is xx'. Light entering in the direction *EF* is divided into ordinary waves, which are totally reflected in the direction *GH*, while the extraordinary traverses the prism.

The Wollaston Prism is used, as in the ophthalmometer, to produce double images. It is formed of two right-angled prisms of quartz *ABC* and *ADC* (Fig. 298), having their hypothenuse sides cemented together, thus forming a rectangle. The one prism *ABC* has its axis of crystallisation parallel to its edge *A*, while the other *ADC* has its axis at right angles to its edge *C*, and parallel to its surface *DC*. Thus the axes of crystallisation are at right angles to each other, and for a ray of light X, incident normally on DC, the relative μ of the two prisms at the surface AC is greater than unity in the one plane and lower than unity in the other, with the consequence that the ray is divided and bent partly towards the base BC and partly towards the edge A, the deviation in both directions being symmetrical and further increased at the surface AB towards Y and Z.

The Colours of Polarised Light.—Circular and elliptically polarised light may be regarded as the result of two plane vibrations which are at right angles to each other, the circular having its vibrations equal and the elliptical having them unequal. The vibrations are not in the same plane, but the one is in advance of the other, and since the interval differs for various wave-lengths, the component colours of white light become separated and give rise to the colours seen when a crystal is viewed in the polariscope. The arrangement of the colours for each form of crystal is characteristic of it.

A ray of light transmitted by quartz cut perpendicular to its axis (axis-cut pebble) is not bifurcated, but it possesses the property of *rotating* the plane of polarisation, so that the vibrations transmitted from the polariser are no longer at right angles to the axis of the analyser. The amount of twisting undergone by the plane of polarisation is proportional to the thickness of the quartz, and, provided monochromatic light is used, extinction could again be obtained by rotating the analyser through a sufficient angle. With white light, however, this is impossible, as the rotation of the plane of polarisation depends also upon the wave-length, *i.e.* colour of the incident ray, and therefore the angle of extinction differs for each wave-length. The plane of polarisation is rotated more for the short than for the long waves, and the analyser blocks out those whose plane of polarisation is at right angles to its own, but transmits the complementary colour; consequently the arrangement of colours changes as the analyser is rotated. In addition, some of the light transmitted by the polariscope and the pebble must be oblique so that it suffers double refraction owing to the unequal oblique distances travelled by the two rays. A kind of interference is set up between the ordinary and extraordinary rays, and a series of brightly coloured rings, somewhat similar to Newton's rings, are seen (if white light be used) crossed by two dark brushes at right angles to each other. If the analyser be now turned so that its axis is parallel to that of the polariser, the rings will be seen to change to their complementary colours, and clear spaces are substituted for the dark brushes previously formed. A white cloud is the best source in these experiments.

Unannealed Glass.—Glass which is unannealed, or has been subjected to pressure, strain, or twisting, polarises light and therefore acts in the polariscope somewhat similarly to a pebble, in that light is transmitted; but the effects produced by unannealed glass can never be mistaken for those of crystal since the patterns, even if not irregular, as is generally the case, are totally unlike those caused by any kind of crystal.

CHAPTER XXV

PHENOMENA OF LIGHT

Interference.—If from two adjacent point sources P_1 and P_2 (Fig. 299) waves of light are propagated, the crests and troughs of the waves from P_1 will coincide with those from P_2 along certain lines marked *B*, and reinforce each other, thus causing increased wave motion (amplitude). Along other lines, marked *D*, the crests from the one source coincide with the troughs of the waves from the other, with the result that the wave motion is neutralised owing to the *interference* of the one set of waves with the other. If the light is monochromatic, alternate lines of light and darkness, known as interference bands or fringes, are in this way produced. The light bands are along lines so situated that any point on them is a whole number of wave-lengths from P_1 and P_2. The dark bands are along lines so situated that any point on them

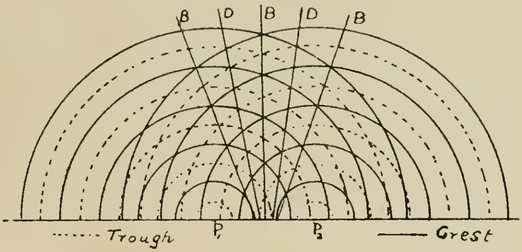

FIG. 299.

is one half wave-length farther from the one source than the other. The shorter the waves which interfere with each other, the less is the distance between the light and the dark bands. If, as in white light, there are waves of different lengths, the interference bands, instead of being alternately light and dark, are alternately red, blue and white, the latter occurring where the various colour bands coincide.

In order to secure interference between the light from two sources, the latter must be *exactly similar*, giving out waves of precisely the same length, amplitude and sequence. For preference the sources should be the duplicated images of a single source and the smaller they are the finer are the interference bands, also they must not be separated by too great a distance.

The colours of thin films, such as soap bubbles, layers of grease on water, etc., are due to interference. Part of the light is reflected from the outer, and part from the inner, surface of the film, and the light reflected from the two surfaces is not in the same phase, a wave reflected from the inner surface has to travel over a greater distance than one from the outer surface. If the thickness of the film, therefore, be such that the inner wave emerges half a wave-length, or any odd number of half wave-lengths, behind the outer wave, they will interfere. Should the inner wave emerge in the same phase, *i.e.* a whole wave-length or any number of wave-lengths behind the outer wave, reinforcement takes place.

Newton's Rings.—When two plane, or two similarly curved, surfaces, the one convex and the other concave, are placed in contact, the film of air contained between them is of equal thickness, but if the one surface is not truly plane, or of exactly similar curvature to the other, the film of air is of varying thickness, and colours, due to interference, as explained above, are exhibited. If a convex surface is placed in contact with a plane or another convex surface, the film of air contained between them is of gradually increasing thickness. At the centre the film is very thin, and, seen by reflected light, there is a central black spot surrounded by a series of alternately dark and bright rings if monochromatic light is employed, or by coloured rings if the light is white. If viewed by transmitted light the centre is bright and the surrounding rings are alternately dark and bright, or of colours which are complementary to those seen by reflected light. These are termed *Newton's Rings.*

The width and regularity of the rings afford a delicate test for similarity between two curves, and is made use of for testing the surfaces of the components of high-class photographic objectives, etc. The standard curve is called a *test plate* on to which is placed the surface to be tested. The absence of coloured rings shows true contact over the whole of the surfaces, but the presence of rings proves a difference in curvature; complete absence of any rings is, however, rare, and the surface is considered satisfactory if the rings are wide and of dull colour.

Diffraction.—When light reaches the edge of a body some of the waves, owing to their undulatory motion, bend round the edge of the obstacle and penetrate the shadow cast by it. This phenomenon is known as *diffraction.* If monochromatic light is admitted through a small aperture there is a series of alternate light and dark bands or rings, parallel to the edge of the shadow. These bands become less and less distinct as they are progressively farther away from the aperture, and they are broader in proportion to the length of the waves. If the light is white, the diffraction fringes of the different colours overlap and a series of coloured fringes are seen. The aperture must be narrow, or small, otherwise the diffraction effects are lost in the general penumbra.

Diffraction bands can be seen by looking at the sky through a pinhole, or through a narrow slit at, say, the filament of an electric lamp, parallel to it. If a hair or thin wire be placed between the light and a screen, a series of fringes can be seen both within and beyond the geometrical shadow. If the obstacle be circular, such as a small round patch on a piece of clear glass, the shadow is seen surrounded by alternate light and dark rings, or, if the source be sunlight, by a series of spectra which encroach on the shadow, at the centre of which a bright dot can be seen. Favourable conditions must be chosen to view diffraction bands. A star seen through a perfectly corrected telescope, and small objects seen by the microscope, appear bordered by one or more faint rings. Owing to diffraction, there is a limit to the possible magnifying power of a microscope, since the higher the power of the objective, the smaller the lenses, and consequently the more marked the diffraction phenomena.

The colours of many beetles and of mother-of-pearl are caused by diffraction and interference phenomena, and are not due to pigmentation; here the wing-cases of the beetles, or the mother-of-pearl, are very finely striated, which causes them to act like irregular diffraction gratings.

Diffraction Grating.—A large number of very fine equidistant lines—some thousands to the inch—ruled parallel to each other on a plate of glass or metal forms a *diffraction grating.*

Diffraction Spectrum.—Dispersion can be obtained by reflection from, or transmission through, a glass diffraction grating, or by reflection from a metal grating; the transmitted or reflected light forms a series of spectra which can be thrown on a screen, or be examined by a telescope, and the finer and closer the lines the purer will be the spectrum obtained.

The lines of the grating scatter a portion of the original waves into fresh and regular series, of which some are quenched by interference. Unlike the spectrum obtained by prismatic refraction, the colours as the direct result of interference are evenly distributed in accordance with their wave-lengths. The red end is not condensed, nor the violet end extended, so that the red

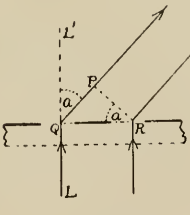

FIG. 300.

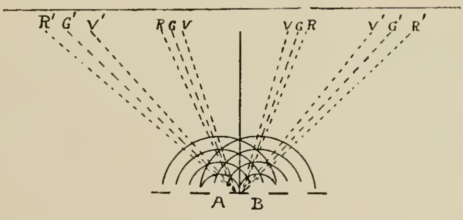

FIG. 301.

and orange occupy more, and the blue or violet occupy less space than in a refraction spectrum; also the most luminous part is more nearly in the centre. Such diffraction gratings afford an accurate means by which to measure the wave-lengths of light and the relative positions of the Fraunhofer lines.

Fig. 300 represents a portion of a highly magnified section of a glass grating, *Q* and *R* being the clear spaces between the lines. The distance *Q R*, equal to one ruling and one space, forms a *grating element.*

Imagine parallel light falling on the grating from the direction L; the bulk of the light passes through uninterrupted, so that an eye placed near L' will see the original source very much as it would through a piece of plane glass. On moving the eye to one side, so that the direction of view is oblique to the grating, colours will commence to appear, these being in the regular spectrum sequence from violet, which makes the smallest angle with the surface, to red, which makes the greatest. A short interval with no colour will occur after the red, but on increasing the obliquity of the eye to the grating, a second series of colours, in the same order as the first, but more drawn out and fainter, will be observed. This is shown diagrammatically in Fig. 301. The first, $V\,G\,R$, is the *primary* spectrum; $V'\,G'\,R'$ is the *secondary* spectrum, beyond which are others, provided the grating is not too fine; usually only the primary and secondary spectra can be seen from a grating having about 15,000 lines to the inch. As previously stated, it is by the reinforcement of the wavelets diverging from the grating spaces along certain lines oblique to the surface, aided by interference, that the spectra are produced.

In Fig. 300 consider a certain direction $Q\,P$ oblique to the normal $L\,L'$, making with the latter the angle a; or, conversely, suppose the grating itself be tilted through that angle with respect to the incident light. Then the wavelets diverging from Q and R will either reinforce or interfere with each other according as *$Q\,P$ is an even or odd number of half wave-lengths*—that is, as the *difference* in the paths of travel of the wavelets is an even or odd number of half wave-lengths. Let $P\,Q$ be equal to the smallest possible even number, i.e. *two*, of half wave-lengths. Then in the direction $P\,Q$ there will be reinforcement giving rise, in the eye or observing telescope, to an image of the original source if the light be monochromatic, or to a spectrum colour, if white light be employed. Now

$$\mathrm{P\,Q}=\mathrm{Q\,R}\sin a, \qquad \text{or} \qquad w=\mathrm{E}\sin a,$$

where $w=\mathrm{P\,Q}$ is one wave-length of the light in question and $\mathrm{E}=\mathrm{Q\,R}$ is a grating element. The element E is known, and the angle a can be found by means of a revolving telescope as in the spectrometer (*q.v.*); therefore the wave-length w can be calculated from the above formula.

Example.—Let the grating have 15,000 lines to the inch, and suppose the angle a for a particular part of the spectrum, say the yellow (D) line, to be 20°. 15,000 lines to the inch corresponds to 25·4/15,000 mm. to every grating element E, and sin 20°=·342. Therefore

$$w=25{\cdot}4\times{\cdot}342/15{,}000={\cdot}000579=579\ \mu\mu.$$

If the secondary spectrum be employed, w will represent *two* wave-lengths, so that $w=\mathrm{E}\sin a/2$, the result being the same as in the example given, but a would be rather more than 40°.

It should be observed that no spectrum is formed when the eye or the observing telescope is normal to the grating, the various reinforcing and interfering wavelets overlapping to form white. The number of spectra formed is smaller as E is smaller, *i.e.* as the number of lines to the inch is greater, and *vice versa.*

The most suitable source is a fine slit, brightly illuminated, placed parallel to the rulings, the spectrum consisting of an innumerable number of diffracted images of the slit ranged side by side, and representing practically a separate image for every wave-length. By the employment of metal gratings, specially in the form of concave mirrors which focus the spectra direct on to a screen or photographic plate, increased intensity of light is secured.

Luminescence is the general name given to the property of a body by which, without sensible rise of temperature, it becomes luminous.

The luminosity of phosphorus, fungi and decaying vegetable matter is caused by oxidation. Chemical or physiological action is usually the cause of the light emitted by shell and deep-sea fishes, fire flies, glow worms, beetles, insects, animalculæ, and the bacteria found in putrefying vegetable and animal matter. The brilliant light observed on tropical seas at night is due to numberless luminescent organisms. The light emitted by various insects is found of almost every colour in one or other species. Luminescence can also be produced by heating fluorspar, quinine, etc., by applying friction to quartz or cane-sugar in the dark, or by cleaving a slab of mica. Fused boric acid or even water when rapidly crystallised or frozen may exhibit this phenomenon.

When a high-tension current is passed through a vacuum tube, Röntgen rays are produced, and the walls of the tube emit a greenish luminescence which is assumed to be due to minute electrified particles striking the wall of the tube with immense velocity and producing light and heat by their impact, the colour of the luminescence depending on the nature of the glass. Radium is found to shine perpetually in the dark, and bodies exposed to the radiation of radium become themselves radio-active, *i.e.* luminescent for a time. Luminescence also includes the following phenomena :

Phosphorescence is the term frequently given to the foregoing phenomena of luminescence, but it is more properly applied to the property of a body of being *luminous in the dark after exposure to light.* Some diamonds, fluorspar and various minerals possess this property; chloride or sulphide of calcium or barium, preserved from air in a sealed glass, will shine brilliantly for a long time.

Phosphorescence is excited by rays of shorter wave-length than those which produce the phosphorescent light, although the latter may be found of every colour of the spectrum. It is supposed to be due to the absorption of light, and its later radiation, as light of longer wave-length, after the exciting action has been removed. When phosphorescence results from exposure to sunlight, the latter is termed *insolation.*

Fluorescence.—Fluorescence is the property possessed by certain bodies of absorbing ultra-violet waves, invisible to the eye, and of emitting, by radiation, light of longer wave-lengths by which they appear self-luminous.

This property was first discovered by Stokes in fluorspar, and so named by him fluorescence. The emission of light ceases immediately the original source of light is cut off, and in this fluorescence differs from phosphorescence.

The phenomenon is not confined to the ultra-violet rays, for if a solution of chlorophyll be placed in a dark room and a beam of white light allowed to fall on it, the surface of the solution emits a red fluorescent light. A solution of quinine emits a pale bluish colour in the presence of daylight. The fluorescence increases if the solution is held in the violet end of the spectrum, and is visible when held beyond the limits of the visible spectrum, the invisible ultra-violet rays exciting fluorescence and becoming changed into visible blue-violet rays. Similar effects may be seen with uranium glass, which fluoresces a brilliant green when placed in ultra-violet light. A thick plate of violet glass placed in front of a beam of light from the electric arc will cause the same phenomenon. Æsculine (the juice of the horse-chestnut bark), platino-cyanide of barium, and many other substances are fluorescent, and so are also the cornea, crystalline lens, and bacillary layer of the retina.

It would appear as if sometimes the radiated light is of shorter wave-length than the original, so that fluorescence is generally taken to be the absorption of invisible light and its radiation as visible light *while the exciting cause is present*. It is said that the ozone of the atmosphere is fluorescent.

Calorescence is the name given by Tyndall to the conversion of the invisible infra-red waves into visible light. This he achieved by focussing an electric light, by a reflector, on to some platinum foil after passing it through substances opaque to visible, but transparent to infra-red light.

Blueness of Sky.—If the air were absolutely transparent and of uniform density, light from the sun would reach the earth without any loss, and the sun, moon and stars would be set in a sky which would appear black both during the daytime and at night. The air, however, contains a great quantity of aqueous vapour, and the blue colour of the sky is said to be due to reflection from the minute particles of this vapour suspended in the higher layers of the atmosphere, and of so-called cosmic dust also held in suspension in the air. Tyndall showed that when mastic is thrown into water the minute insoluble particles of the mastic emit a deep-blue colour similar to that of the unclouded sky. If a cloud of smoke be blown into the air, the smoke particles reflect the short blue waves more freely, and the cloud assumes a blue tint, and if a white screen be held, in bright sunlight, behind the smoke, the screen assumes a reddish-brown hue.

By some the blue of the sky is said to be due to polarisation by oblique reflection from particles of vapour, salt, etc., in the air; by others it is thought to be caused by fluorescence of the ozone.

Aerial Perspective.—If two objects, one light, and the other dark, be seen at a considerable distance, they lose some of their contrast, the light object becoming darker by absorption of its reflected light by the intervening air, and the dark object becoming lighter by the superadded light diffused through the air. This causes what is known as aerial perspective. If the air is clear and the added light is blue, distant hills throw deep shadows of a purple-blue colour in bright sunshine.

Eclipses.—A total eclipse of the sun occurs when the moon is so situated that some portion of the earth lies in the umbra of the shadow cast by it; the eclipse is partial to those portions of the earth in the penumbra of the shadow. An eclipse of the moon occurs when the moon lies in the shadow cast from the sun by the earth.

Light Streams.—The stream of light seen reflected from the surface of the sea, or other body of water in motion, in bright sun or moon light, is due to a series of imperfectly formed images, of the sun or moon, reflected from the ripples of water so that they enter the eye proceeding from different points. If the water is quite smooth a definite image of the luminant is perceived.

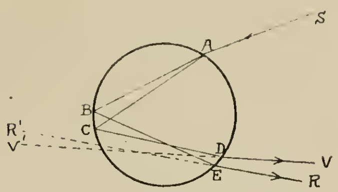

FIG. 302.

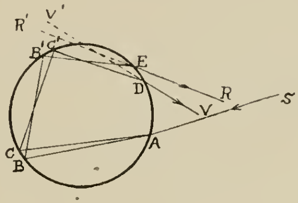

FIG. 303.

The Rainbow.—A rainbow is visible when the sun is behind the observer and a shower of rain in front of him, or it may be seen in the spray of a waterfall. Since the sun's rays falling on the raindrops are parallel, the course of light through all the drops must be the same, and it is therefore sufficient to trace the course of a ray through a single drop. Let a pencil of rays from the sun meet the drop at A (Fig. 302). On entering it is refracted and dispersed towards B and C at the back of the drop, thence reflected to D E, where it is refracted to emerge in the directions V R which make an angle with the entering ray. The emergent dispersed light thus diverges to the observer's eye, and the various colours, being unequally refracted, are projected back as R' V', so that the outside of the bow is red and the inside blue-violet. The extent of the bow depends on the position of the sun; when the latter is at the horizon the bow forms a semicircle to an observer at sea-level. As the sun rises the arc sinks so that its centre is below the horizon and is smaller.

A secondary larger, broader and fainter rainbow is generally seen concentric with the primary. The rays from the sun to a point A (Fig. 303) undergo refraction and are reflected twice at B C and B' C', and again refracted at D E. In the emergent light violet is below and red above ; these being reversed on projection, as V' and R', the secondary bow is blue on the outside and red within.

In Fig. 304 E is the eye and H the horizon, R is the rainbow. The semicircular arc of the primary bow subtends at the eye an angle of some 42° for the red and 40° for the violet. The angles subtended by the secondary rainbow are about 54° for the violet and 51° for the red.

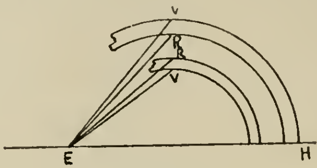

FIG. 304.

The Horizon.—When the sun is low down on the horizon its light has to pass through a thicker layer of atmosphere filled with dust particles and moisture; more of its blue and violet rays are absorbed or reflected, and it thus appears reddish, as for the same reason it appears red in a fog.

Near the horizon, the sun and moon appear larger than when higher in the heavens because they are mentally projected beyond the horizon, as compared with terrestrial objects, whereas when seen in the zenith this cannot be done, as they stand alone; they are not really larger, as measurements with a telescope show. They also appear slightly flattened vertically, when near the horizon, and appear a trifle higher up than they really are, owing to the refraction of the air and the greater obliquity of the light from their lower edges.

Refraction diminishes the dip of the horizon and so slightly increases its apparent distance. The distance of the horizon can be computed approximately from $d=\sqrt{1{\cdot}5h}$, where h is the height in feet of the observer above the sea or earth level, and d is the distance in miles. For nautical miles $d=\sqrt{1{\cdot}3\,h}$. The derivation of this formula is shown in "Simple Calculations."

Mirage (Fata Morgana).—If the layers of the air are of markedly unequal density, as is sometimes the case in hot climates, especially on a desert where the warmest layers are the lowest, the phenomenon known as the *mirage* may be seen. Light from objects above, on its passage to the earth, traverses layers of air which become gradually less refracting, the angles of

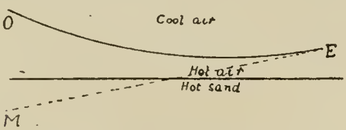

FIG. 305

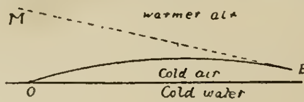

FIG. 306.

incidence accordingly increasing so that the light becomes more and more parallel to the surface, until at length the critical angle is reached, beyond which refraction changes to reflection. The light is then reflected in the contrary direction, and ascends to reach the observer's eye as if proceeding from a point below the ground, and objects appear inverted. This is shown in Fig. 305, where light from an object O, on reaching the eye at E, appears to come from M below the level of the ground.

If the lowest strata of air are the densest, as in Fig. 306, they give rise to the same phenomenon, but the mirage M is in the contrary direction, so that a landscape, or a ship at sea, may appear above the horizon. This occurs in very cold climates.

Scintillation.—The twinkling of a star is due to irregularities in the atmosphere causing variations in the path of the waves, which partially interfere. This produces variations in the apparent brightness and colour of a source of light, subtending a very small angle at the eye, such as a star. It is not observed in the case of a planet, because this has a real magnitude.

CHAPTER XXVI

MISCELLANEOUS

Refraction and Reflection Compared.—A spherical mirror may be regarded as a dioptric system in which $\mu_1=\mu_2$ and therefore $F_1=F_2$. It resembles a single refracting surface in that the principal point is at the vertex of the curve, and the nodal point or optical centre is at the centre of curvature, but it resembles a lens in that F is equally distant from the principal and nodal points. F is midway between them in a mirror, and they are united in a lens. Also the mirror resembles the lens in that the first and last media have equal μ's, the source and its image are both in air, and therefore $F_1=F_2$.

If light is incident on a plane-polished surface it is partly reflected and partly refracted. If r is the angle of refraction and r' that of reflection we have

$$\mu_1 \sin i=\mu_2 \sin r=\mu_2 \sin r',$$

but for the reflection $\mu_2=\mu_1$, and therefore $r'=i$. The relationship between the angles of reflection and refraction is

$$\frac{\sin r'}{\sin r}=\frac{\sin i}{\sin r}=\frac{\mu_2}{\mu_1}.$$

If $\mu_2=1{\cdot}5$, suppose a ray with an angle of incidence of 9°; then $r'=9°$ and $r=6°$ approximately. The angle between the reflected and refracted rays is $180-(r+r')$ which becomes 180° when incidence is normal, and 90° when incidence is that of complete polarisation.

With a curved surface by reflection and refraction respectively, we have

$$\frac{1}{F}=\frac{1}{r/2}=\frac{2}{r} \quad \text{and} \quad \frac{1}{2r},$$

therefore reflection is four times as powerful as refraction, or six times as powerful, if compared with the posterior power of a surface.

If the surface of a Cc. lens is used as a reflector, to find F, μ being taken as 1·5, the dioptric F of that surface is four times as long as the catoptric F.

If a Sph. mirror be placed against a lens measure scaled in diopters, F is 1/4 that shown by the scale. Thus if a mirror shows 2·5 D, its F is $100/(2{\cdot}5\times4)=10$ cm.

Figs. 307 and 308 show the difference in F when an incident beam of light is reflected from, or refracted by, a thin plano Cx. or Cc. glass lens of $\mu=1{\cdot}5$. of which C is the centre of curvature. Light parallel to the axis, if reflected, meets at R, which is half the distance of C from the pole P; if refracted it meets at F, which is twice the distance of C from P. The thick lines represent

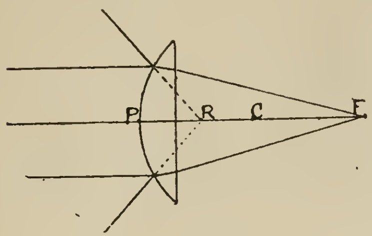

FIG. 307.

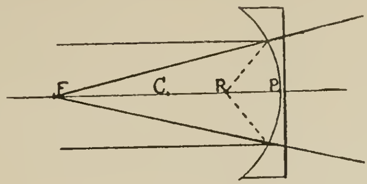

FIG. 308.

the refracted rays and the dotted lines the reflected rays. A Cx. surface is a positive refractor and a negative reflector, while a Cc. surface is a negative refractor and a positive reflector, when it is adjacent to air.

Refracting Reflector.—A lens silvered on its second surface so that refraction and reflection occur is termed a *Mangin mirror*.

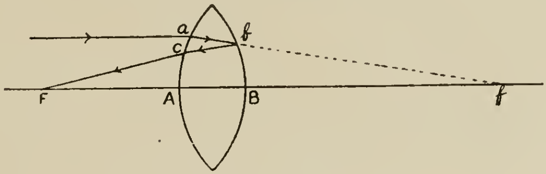

FIG. 309.

In Fig. 309 a lens of index μ has two surfaces A and B of radii of r_1 and r_2 respectively; the surface B is silvered. A ray of light a, parallel to the axis, and incident on A, is refracted towards f, the posterior focus of that surface such that

$$\frac{1}{f}=\frac{\mu-1}{\mu r_1}.$$

The ray on arriving at b on the second surface B is reflected and the power of the mirror is $2/r_2$, so that the convergence of the light becomes

$$\frac{1}{f'}=\frac{\mu-1}{\mu r_1}+\frac{2}{r_2}=\frac{2\mu r_1+r_2(\mu-1)}{\mu r_1 r_2}.$$

This expression is the conjugate virtual object power $1/f_1$ with respect to the first surface A towards which the light is now directed, and from the conjugate formula for a spherical surface, when f_1 is in the dense medium and f_2 is in air,

$$\frac{1}{f_2}=\frac{\mu-1}{r_1}-\frac{\mu}{f_1}.$$

Now f_1 is positive, since the light in this case is convergent, so that, substituting for $1/f_1$ the expression $1/f'$ above

$$\frac{1}{f_2}=\frac{\mu-1}{r_1}+\mu\left(\frac{2\mu r_1+r_2(\mu-1)}{\mu r_1 r_2}\right)$$
$$=\frac{2r_2(\mu-1)+2\mu r_1}{r_1 r_2}$$

The value of $1/f_2$ is the ultimate convergence of the light after refraction at c, and f_2 is the focus of parallel light due to the Mangin mirror, and therefore can be written 1/F. That is,

$$\frac{1}{F}=\frac{2r_2(\mu-1)+2\mu r_1}{r_1 r_2}=\frac{2(\mu-1)}{r_1}+\frac{2\mu}{r_2}=\frac{2(\mu-1)}{r_1}+\frac{2(\mu-1)}{r_2}+\frac{2}{r_2}$$

Now $\frac{2(\mu-1)}{r_1}=\frac{1}{F_1}$ and $\frac{2(\mu-1)}{r_2}=\frac{1}{F_2}$ also $\frac{2(\mu-1)}{r_1}+\frac{2(\mu-1)}{r_2}=\frac{2}{F'}$

where $1/F_1$ and $1/F_2$ are the anterior focal powers of the two surfaces and $1/F'$ is the power of the whole lens, so that the power of the Mangin mirror is—

$$\frac{1}{F}=\frac{2}{F'}+\frac{2}{r_2} \quad \text{or} \quad \frac{1}{F}=\frac{2}{F_1}+\frac{2}{F_2}+\frac{2}{r_2}.$$

These last formulæ show that the convergence—or divergence—of the light received from a Mangin mirror, is precisely the same as if the metal constituted a mirror separated from the back surface of the lens, so that the light passes through, and is refracted by, both surfaces, is then reflected by the mirror and again refracted by the two surfaces of the lens a second time.

The result is the same whether (*a*) the back surface is silvered, (*b*) the lens is backed by a mirror having a curvature exactly the same as that of the second surface, (*c*) if the lens is neither silvered nor has a backing mirror; in this last case, however, only a very small proportion of the light is reflected back.

If the form of the lens, F′ and μ are known, r_2 has to be calculated and F varies, if the two surfaces are unequal, according to which is silvered.

For a double Cx. or Cc. $\frac{1}{F}=\frac{4\mu-2}{r}$.

For a plano-Cx. or Cc. with the plane surface silvered $\frac{1}{F}=\frac{2\mu-2}{r}$.

For a plano-Cx. or Cc. with the curved surface silvered $\frac{1}{F}=\frac{2\mu}{r}$.

The second surface of a lens is positive or negative, whether it acts as a refractor or a reflector, because, if Cx. towards the air it is Cc. towards the glass, and *vice versa;* and since the reflective power of a surface is four times as great as its anterior refractive power when $\mu=1{\cdot}5$, we can write for the Mangin mirror

$$\frac{1}{F}=\frac{2}{F_1}+\frac{6}{F_2} \quad \text{or} \quad D=2\,D_1+6\,D_2,$$

where F_1 and D_1 pertain to the first surface, and F_2 and D_2 to the second.

For a double Cx. all powers are positive and for a double Cc. they are all negative, and D of the Mangin mirror is equal to 4 D of the lens.

Thus with a lens of $D_1=+1$ and $D_2=+2$, we have $D=+14$.

If $D_1=+4$ and $D_2=-2$, we get $D=-4$; if this lens were turned the other way $D=+20$.

The effect of a Cx. or Cc. periscopic may be positive or negative as the one or the other surface faces the light.

In order that incident parallel light emerge as parallel it is necessary that

$$r_2=\frac{-r_1\mu}{\mu-1} \quad \text{or} \quad -r_2=\frac{r_1\mu}{\mu-1}$$

or approximately $r_2=-3r_1$, *i.e.* when the radius of the reflecting surface is equal to the posterior focal length of the first surface, parallel light retraces its own course; or the second surface must be of opposite nature to that of the first and of one third its dioptric power.

The ordinary Cc. glass mirror is silvered on the back surface and has a positive $F=r/2$, the same as if the metal itself were exposed to the light. The light is diverged twice at the front surface, but a preponderating convergence takes place at the second.

Optical Glass.—Glass is a hard, generally transparent or translucent substance, made by the fusion of silica with potash, soda, lime, lead and other substances, such as pearlash, arsenic, manganese, saltpetre, chalk, etc. It is brittle, sonorous, ductile when heated, and fusible only at a very high temperature. It is usually not soluble, but is acted on by hydrofluoric acid, and is a very bad conductor of heat. There are many varieties of glass, and the process of manufacture, as regards the ingredients used and the treatment after complete fusion of the various components, depends on the nature of the glass to be produced.

If suddenly cooled, glass becomes extremely brittle owing to the state of tension produced by the cooling of the outer portions while the inner are still in a molten condition; annealing tends to reduce brittleness. Glass used for optical purposes must be homogeneous, *i.e.* of equal density and refractive power throughout, and perfectly transparent; it is therefore carefully mixed and gradually cooled. It should also be free from air bubbles, striæ and colour for spectacle lenses, although a few air bubbles, if small, may be of little or no consequence in a camera lens. The solid block of glass is usually polished on two sides, so as to allow of the detection of defects, and from it clear discs of appropriate size are cut.

To detect strain a polariscope (*q.v.*) is required. To detect striæ, bubbles, etc., the glass should be examined by the eye in the focus of a combination such as is constituted by an erecting eyepiece. The light diverges from F_1 of the first lens, is rendered parallel, passes through the plate to be examined, and is converged by the second lens to the examiner's eye. Striæ, etc., are disclosed by patches or streaks which spoil what should be a uniformly bright field.

Lenses are made of crown glass, which contains lime, or of flint glass, which contains lead. Flint has generally a higher refractivity and chromativity; the greater the proportion of lead in the glass the greater, usually, are the refractive and dispersive powers. It is denser, heavier and softer than crown, and is almost perfectly colourless. Crown glass has the advantage of lower dispersion and is harder, so that it does not so easily become scratched, but it is more brittle than flint. It has sometimes a decided greenish tint, due to the presence of iron. The pinkish tint found in some glass results from the admixture of manganese.

According to its component ingredients and manufacture, the indices of refraction of glass vary for the various lines of the spectrum, the mean μ of crowns being, say, 1·52, and of flints 1·62.

The following may be taken as very rough examples of the proportions of the materials entering in the manufacture of optical glass:—

Flint Glass (100 parts).—Silica 50, lead 30, potash 10, other ingredients 10.

Crown Glass (100 parts).—Silica 70, soda 10, lime 10, other ingredients 10.

In the following table some examples (not actual kinds) are given to illustrate the refraction, dispersion and specific gravity of different kinds of optical glass, and the method generally employed in arranging them in the order of their ν values or efficiencies.

TABLE OF OPTICAL GLASSES.

Description.	μ_D	$\nu = \frac{\mu_D - 1}{\delta\mu}$	*Dispersion.*				*Specific Gravity.*
			Medium. $C - F = \delta\mu$	$A - D$	$D - F$	$F - G$	
Very light crown ..	1·48	66	0 0073	0·0050	0·0055	0·0040	2·25
Light ,, ..	1·50	62	0·0081	0·0055	0·0065	0·0045	2·50
Ordinary ,, ..	1·52	60	0·0087	0·0060	0·0070	0·0050	2·75
Heavy ,, ..	1·56	55	0·0102	0·0065	0·0075	0·0055	3
Very heavy ,, ..	1·60	52	0·0115	0·0070	0·0085	0·0065	3·5
Very light flint ..	1·54	48	0·0123	0·0075	0·0090	0·0070	3
Light ,, ..	1·58	43	0·0135	0·0085	0·0095	0·0080	3·25
Ordinary ,, ..	1·62	40	0·0155	0·0095	0·0115	0·0100	3·50
Heavy ,, ..	1·68	35	0·0194	0·0105	0·0130	0·0110	4
Very heavy ,, ..	1·85	24	0·0354	0·0185	0·0280	0·0250	5·5

REFRACTIVE INDICES OF VARIOUS MEDIA.

Air	$\mu_D = 1{\cdot}000$
Ice	$\mu_D = 1{\cdot}310$
Water (distilled)	$\mu_D = 1{\cdot}336$
Sea-water	$\mu_D = 1{\cdot}343$
Blood	$\mu_E = 1{\cdot}354$
Albumen	$\mu_E = 1{\cdot}360$
Absolute alcohol	$\mu_D = 1{\cdot}366$
Oil of bergamot	$\mu_D = 1{\cdot}464$
Olive oil	$\mu_E = 1{\cdot}470$
Glycerine	$\mu_D = 1{\cdot}460$
Gum arabic	$\mu_E = 1{\cdot}512$
Spermaceti	$\mu_E = 1{\cdot}444$
Bisulphide of carbon	$\mu_D = 1{\cdot}687$
Alum	$\mu_D = 1{\cdot}457$
Sugar	$\mu_D = 1{\cdot}535$
Rock salt	$\mu_D = 1{\cdot}555$
Salt solution	$\mu_E = 1{\cdot}375$
Phosphorus	$\mu_D = 2{\cdot}224$
Diamond	$\mu_D = 2{\cdot}470$
Chromate of lead	$\mu_D = 2{\cdot}500$ to $2{\cdot}970$
Canada balsam (liquid)	$\mu_D = 1{\cdot}520$
Canada balsam (hard)	$\mu_D = 1{\cdot}535$
Oil of cassia	$\mu_E = 1{\cdot}618$
Oil of fennel	$\mu_D = 1{\cdot}544$
Anilin oil	$\mu_D = 1{\cdot}580$
Oil of cloves	$\mu_D = 1{\cdot}533$
Oil of cinnamon	$\mu_E = 1{\cdot}508$
Cedar oil (lens immersion oil)	$\mu_D = 1{\cdot}512$
Naphtha	$\mu_E = 1{\cdot}475$
Turpentine	$\mu_E = 1{\cdot}478$
Rock crystal, pebble (ordinary ray)	$\mu_D = 1{\cdot}544$
Rock crystal, pebble (extraordinary)	$\mu_D = 1{\cdot}553$
Tourmaline (ordinary ray)	$\mu_D = 1{\cdot}636$
Tourmaline (extraordinary)	$\mu_D = 1{\cdot}620$
Iceland spar or calcite (ordinary ray)	$\mu_D = 1{\cdot}659$
Iceland spar or calcite (extraordinary)	$\mu_D = 1{\cdot}486$
Felspar	$\mu_E = 1{\cdot}764$
Fluorspar	$\mu_D = 1{\cdot}434$

REFRACTIVE INDICES OF SOME METALS (KUNDT).

	Red.	*Yellow (D).*	*Blue.*
Silver	—	0·27	—
Gold	0·38	0·58	1·00
Copper	0·45	0·65	0·95
Platinum	1·76	1·64	1·44
Iron	1·81	1·73	1·54
Nickel	2·17	2·01	1·85
Cobalt	2·61	2·26	2·16

Discs for Lenses.—When a lens of certain power and diameter has to be worked, the thickness of the blank should be such as to avoid undue grinding down (if too thick) or failure to obtain the finished lens (if too thin). The necessary calculation is based on the simplified spherometer formula, writing

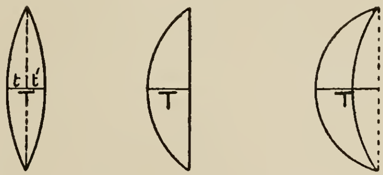

Fig. 310.

t for the required thickness in place of S, the sag in the original. Then $t = d^2/2r$ for each surface, where d is *half the longer diameter* of the finished lens. For the whole lens, t for the one surface must be added to that of the other, as in Fig. 310.

With sufficient accuracy the radius of a plano-Cx. or Cc. lens is half the focal length, and that of a double Cx. or Cc. is equal to the focal length. If, say, a lens of 10 cm. F were needed in plano-Cx. form, $r=5$ cm.; if the lens were double Cx. each $r=10$ cm.; therefore T, the total thickness, is precisely the same for both forms of lenses. Also we can use C, the total diameter of the lens, in place of d, the semi-diameter; that is, $d^2/2r=C^2/8r$, and since we take $2r=F$ we can write as a general formula for thickness of disc

$$T=\frac{C^2}{4F}.$$

This formula serves for all lenses no matter how the powers are distributed *provided both surfaces are Cx. or both Cc.* For periscopics the surface of greater power only need be reckoned for. The long diameter of an oval lens must be taken for C and F and C must be in the same terms, preferably mm. The total T, thus obtained, is the minimum and allows only for a *knife-edge* Cx. or a *wafer thin* Cc.; generally an additional thickness of 1 mm. is needed for the bevel of a Cx. lens or the central thickness of a Cc. For diopters $T=C^2D/400$, all terms being in cm. T for a cylindrical surface is calculated as for the same power spherical.

Thus for a + 20 D. DCx. lens of 37·5 mm. diameter, that is, ordinary test case size. Then F=50 mm. and

$$T=\frac{37{\cdot}5^2}{4\times 50}=6 \text{ mm.} \quad \text{and} \quad 6+1=7 \text{ mm.}$$

For a curved protector of $r=20$ cm. and diameter 30 mm.

$$T=\frac{30^2}{200\times 8}=\cdot 56 \text{ mm. or say } 2 \text{ mm.}$$

For ordinary spectacle lenses the approximate thicknesses are

1 eye ..	1+300/F, or 1+·3 D.	0 eye ..	1+350/F, or 1+·35 D.
00 eye ..	1+400/F, or 1+·4 D.	000 eye ..	1+450/F, or 1+·45 D.

Lens Sizes.—American standard eyes, with their axes, and length of wire needed to make a standard eye wire in mm., are given in the following:

No.	*Axes.*	*Wire.*	*No.*	*Axes.*	*Wire.*	*No.*	*Axes.*	*Wire.*
4	33·8×24·5	93·5	0	37·8×28·8	107·5	000½	42·5×33·5	122·3
3	34 ×26	95·9	00	39·7×30·7	113·8	0000	44·3×36	128·2
2	35 ×26	98·6	000	41 ×32	117·5	Jumbo	46 ×38	134·3
1	36·5×27·5	103·5						

The numeration, based on peripheral measurement, as given in the next table applies to all shapes, the ratio of the long to the short axis of the *oval* being approximately 1·3 to 1, and that of the *long oval* 1·5 to 1.

OPTICAL SOCIETY STANDARDS.

	Length of Periphery.	*Corresponding American.*	*Long Diameters.*					
No.		*No.*	*Oval.*	*Long Oval.*	*Round Oval.*	*Pantos.*	*½ Oval.*	*Round.*
1	92·5 mm.	4	33·5	35	31	34	36	29·5
2	94·5 ,, = 92·5+2	3	34	35·5	31·5	34·5	36·5	30
3	97·5 ,, = 94·5+3	2	35	36·5	32·5	35·5	37·5	31
4	101·5 ,, = 97·5+4	1	36·5	38	34	37	—	32·5
5	106·5 ,, =101·5+5	0	38	39·5	35·5	38·5	—	34
6	112·5 ,, =106·5+6	00	40	41·5	37·5	40·5	—	36

Power of Cement Bifocals.—The power of the segment or wafer in a cement bifocal is that which, added to the main lens, gives the power required for reading. The index of refraction of the Canada balsam, by means of which the wafer is joined to the main lens, is practically the same as that of the glass, so that it need not be considered.

The *free* surface of the segment must be the total reading power less the power of the free surface of the main lens. The power of the contact surface of the segment must be that of the contact surface of the main lens, but of opposite nature, so that they neutralise each other. Suppose the two powers be +2 for distance and +3 for reading (Fig. 311). If the main lens

FIG. 311.

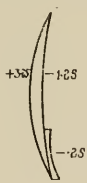

FIG. 312.

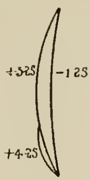

FIG. 313.

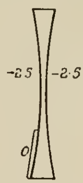

FIG. 314.

is double Cx. with +1 on each surface, the segment would need to be −1 on the contact and +2 on the free surface. If for the same powers the main lens is periscopic Cx. the two surfaces would probably be +3·25 and −1·25 (Fig. 312). The wafer would then be +1·25 on the contact surface, and −·25 on the other, the wafer being placed on the Cc. side of the main lens. If placed on the Cx. side, the contact surface of the wafer must be −3·25, and the free surface + 4·25 D (Fig. 313).

If the main lens is −5 D Cc. and the reading power is −2·5, then the segment requires to be a + 2·5 on the contact surface and plano on the other (Fig. 314). If the main lens is −7 periscopic Cc. with, say, +1·25 on the one surface, the segment, if placed on the Cc. side, is +8·25 on the contact, and −6·25 on the free surface, for a reading power of −5 D.

When the main lens is a plano-cyl. the segment is attached to the plane surface. When the main lens is a sph.-cyl. the segment is attached to the spherical surface. Thus with, say, +3 Sph. ⊂ +2 Cyl. with an addition of +2 for reading, the wafer must have powers of −3 and +5.

Centering of Cement Bifocals.—The added segment is always Cx., the lower part being weaker if the upper is Cc., and stronger if the upper is Cx. If the wafer is itself centered, the prismatic effect due to decentration of the main lens remains. For a properly centered lower, the segment of the bifocal must have a prismatic effect contrary to that of the main lens where they are united. This is obtained by decentering the segment to the requisite extent. When the main lens is Cx. the prismatic effect of its lower portion is *base up*, so that the wafer must be base down, its thick part being at the edge of the main lens. If the latter is Cc. its prismatic effect is base down, and therefore the segment must be base up, *i.e.* its thin part must be at the edge of the main lens.

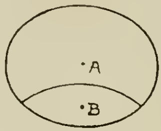

Fig. 315.

In Fig. 315 *A* is the geometrical and optical centre of the main lens, and *B* is the optical centre of the reading position; the distance *A B* is usually about 8 mm., but may vary. Let D_1 be the total power of the main lens, and C_1 be the distance *A B*. Let D_2 be the power of the segment by itself, and C_2 its needed decentration in cm. Now, in order that there be no prismatic effect at *B* it is necessary that $D_1 C_1 = D_2 C_2$, so that the formula for calculating the needed decentration of the segment is

$$C_2 = D_1 C_1 / D_2$$

D_1 is the power of the spherical, or the vertical power of a cyl., or sphero-cyl., whose principal meridians are vertical and horizontal.

Let the upper be +4·5 D. and the lower +6; the segment is +1·5, so that

$$C_2 = 4{\cdot}5 \times {\cdot}8/1{\cdot}5 = 2{\cdot}4 \text{ cm., the thick part } \textit{down}.$$

Let the upper be −3·5 and the lower −1, the segment being +2·5; then

$$C_2 = 3{\cdot}5 \times {\cdot}8/2{\cdot}5 = 1{\cdot}1 \text{ cm., the thick part } \textit{up}.$$

The amount of decentering is often very large, and demands either that the blank from which the segment is taken be of extra large dimensions, or the segment be ground on a prism.

It is necessary to place the optical centres of the *lowers* each 1·5 mm., or so, inwards in order to allow for convergence when reading. If the main lenses are Cx. their prism action is base *out*, and that of Cc.'s *in*. To neutralise this the segments must be decentered *in* if the main lenses are Cx., and decentered *out* if they are Cc., such horizontal decentration being considered for the centres of the reading portions which are *in* from those of the uppers. The difference between the position of the optical centre of each eye for distance and for reading varies, but 1·5 mm. is a good average. In all cases the actual amount of decentering of the wafer required, so that the lowers may not be decentered when used for near work—the required position should be marked by a dot—can be obtained by sliding the segment over the main lens while viewing the small crossbar as described for centering.

Inset or Fused Bifocals are made by inserting a segment of high curvature and high μ into a depression made in a main lens of low μ.

To calculate the curvature of the segment, let D_1 be the distance power of the whole lens, and D_2 the reading power; let μ_1 be the index of the main lens, and μ_2 the higher index of the segment. The radius of curvature of a surface, separating two dense media, when the focus is finally in air, is

$$r=F(\mu_2-\mu_1) \quad \text{or} \quad r=100(\mu_2-\mu_1)/D$$

It is necessary to find the tool which, made for producing a certain dioptric power D when the index is μ_1, shall give to the internal surface of the segment of μ_2 the necessary power D_2, after allowing for the powers obtained from the two outer surfaces D_4 and D_5 (Fig. 316). Let D_3 be the outer power of the surface containing the segment, D_4 the outer power of the segment, D_5 the

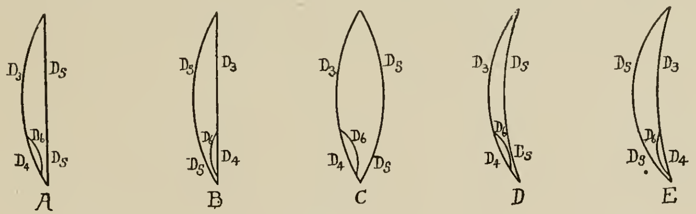

Fig. 316.

power of the surface not containing the segment, and D_6 the power of the internal segment surface between the two glasses. Then

$$D_6=D_2-(D_4+D_5). \quad \text{and} \quad D_4=D_3(\mu_2-1)/(\mu_1-1).$$

The lens may be of various forms, as shown in Fig. 316.

$$D=\frac{D_6\,(\mu_1-1)}{\mu_2-\mu_1}=\frac{(D_2-D_4-D_5)\,(\mu_1-1)}{\mu_2-\mu_1}$$

Now D_4 being of higher μ, although of the same curvature as D_3, which is known, is of greater power such that $D_4=D_3\,(\mu_2-1)/(\mu_1-1)$. Therefore

$$D=\left[D_2-D_5-\frac{D_3\,(\mu_2-1)}{\mu_1-1}\right]\left(\frac{\mu_1-1}{\mu_2-\mu_1}\right)=\frac{(D_2-D_5)\,(\mu_1-1)-D_3\,(\mu_2-1)}{\mu_2-\mu_1}.$$

The values of D_2, D_3 and D_5 and those of the two μ's being known, this equation serves for all forms, whether Cx. or Cc., shown in Fig. 316. For a sph.-cyl. form *A* is used and D_5 disappears from the equation; for a plano-cyl. form *B* is employed and D_3 disappears. Thus for a plano-cyl. of form *A* having $D_1 = -1{\cdot}5$, and $D_2 = 0$, the μ's being 1·52 and 1·65, we find

$$D = \frac{0 \times (1{\cdot}52 - 1) - [-1{\cdot}5\ (1{\cdot}65 - 1)]}{1{\cdot}65 - 1{\cdot}52} = \frac{+{\cdot}975}{{\cdot}13} = 7{\cdot}5.$$

Let $D_1 = +{\cdot}5$ and $D_2 = +2{\cdot}25$, made periscopic so that $D_3 = +1{\cdot}25$ and $D_5 = -{\cdot}75$; using form (*D*) we get

$$D = \frac{[+2{\cdot}25 - (-{\cdot}75)]\,{\cdot}52 - (1{\cdot}25 \times {\cdot}65)}{1{\cdot}65 - 1{\cdot}52} = \frac{{\cdot}7475}{{\cdot}13} = 5{\cdot}75.$$

If the μ's are 1·52 and 1·65, $D = 4\ (D_2 - D_5) - 5\ D_3$ for forms *C*, *D* and *E*; $D = 4\ D_2 - 5\ D_3$ for form *A*; $D = 4\ (D_2 - D_5)$ for form *B*.

If the μ's are 1·5 and 1·6, $D = 5\ (D_2 - D_5) - 6\ D_3$ for forms *C*, *D* and *E*; $D = 5\ D_2 - 6\ D_3$ for form *A*; $D = 5\ (D_2 - D_5)$ for form *B*.

The disc selected must be rather thicker than for ordinary lenses, especially if the segment is of high power.

Having insets of known powers the selection of a suitable blank and the curvatures of the two outer surfaces are as follows: Let the two μ's be 1·65 and 1·52 so that for a given curvature producing D_3 we have $D_4 = 5\ D_3/4$, *i.e.* D_4 is 1/4 stronger than D_3, and $D_4 - D_3 = D_3/4$. Now part of the additional power for reading is obtained from $D_4 - D_3$, and part from D_6, *i.e.* $D_2 - D_1 = (D_4 - D_3) + D_6$; therefore the powers needed on the two surfaces are $D_3 = 4\ (D_2 - D_1 - D_6)$, and $D_5 = D_1 - D_3$.

It is preferable to select a disc such that D_6 is higher than the addition needed for reading, and in that case D_3 is Cc. if D_1 is Cx.

In all cases it is advisable to calculate two or three combinations in order to arrive at the most suitable.

If a Cc. curvature is given to the surface of D_5 there is danger of working through to the segment.

If $D_6 = D_2 - D_1$ the surface D_3 is plano; therefore for sph.-cyls. select $D_6 = D_2 - 1{\cdot}25\ D_1$, the cyl. being to the side of D_5, and the sph. to that of D_3.

The proportional increase of power of D_4 over D_3 is found from

$$\frac{(\mu_2 - \mu_1)}{(\mu_1 - 1)}$$

so that if the two μ's are other than those given above, the factor 4 in the value of D_3 would vary accordingly.

As examples, for $D_1 = +2{\cdot}25$, and $D_2 = +3{\cdot}5$ select $D_6 = 1{\cdot}5$; then $D_3 = 4 \times (1{\cdot}25 - 1{\cdot}5) = -1$, and $D_5 = 2{\cdot}25 + 1 = +3{\cdot}25$.

For $D_1 = -3{\cdot}5$, and $D_2 = -2{\cdot}25$ select $D_6 = 2{\cdot}5$; then $D_3 = 4 \times (1{\cdot}25 - 2{\cdot}5) = -5$ and $D_5 = -3{\cdot}5 + 5 = +1{\cdot}5$.

For +6 S. ⌓ −2 C. with +8 S. for reading, $D_6 = +8 - 1{\cdot}25 \times 6 = {\cdot}5$.

For −10 S. ⌓ −3 C. with −7 S. for reading, $D_6 = -7 + 1{\cdot}25 \times 10 = 5{\cdot}5$.

To Construct Test Types after Snellen.—Each letter is a square block and at a certain distance d for which it is designed, it subtends an angle of 5′. Each limb of each letter is one fifth of the total diameter and subtends an angle of 1′.

The general formula is $S = d \tan V$, where S is the size of the letter, d is the distance in mm., and V is the visual angle. Tan 5′=·001455 and tan 1′=·000291, so that the diameter of each letter is

$$S = 1000 \times {\cdot}001455 = 1{\cdot}455\, d \text{ (} d \text{ being in M. and S. in mm.).}$$

The diameter of each limb is similarly obtained from ·291 d, but the letter dimension divided by 5 gives the limb dimension.

The diameter of any letter in inches=12×·00145=·0174 d (d being in feet).

Thus for 6 M. the types are 6×1·455=8·75 mm., those for 12 M. are 17·5 mm., and so on for every other distance.

If the visual angle is other than 5′ the size in mm. is

$$S = {\cdot}000291 \times \text{visual angle in minutes of arc} \times \text{distance.}$$

The size of the types can also be calculated from circular measure. The radian=57·3°=3438′; if an angle is smaller the arc, subtending it, is proportionately smaller, so that

$$V/3438 = S/d \quad \text{or} \quad S = V\, d/3438.$$

Suppose the types be required for 18 M. under a visual angle of 4′; then

$$S = 4 \times 18000/3438 = 21 \text{ mm. (approx.).}$$

Mirror for Reversed Test Types.—The necessary size S of the mirror depends on the size C of the chart, and the distances d' between the mirror and the chart, and d, that between the subject and the mirror. The mirror should be just large enough to be filled entirely by the image of the chart.

$$\frac{S}{C} = \frac{d}{d+d'} \quad \text{or} \quad S = \frac{Cd}{d+d'}.$$

If, as is generally the case, $d + d' = 6$ M., the subject and the chart being at the same distance, *i.e.* 3 M. from the mirror, the latter is just one half the size of the chart in both dimensions.

To Construct Tangent Scales.—With quite sufficient accuracy the spacing S of a tangent scale for use at a distance d is

$$S = dK/100$$

where K=1 for $^{\Delta}$, K=1·75 for °d and K=·9 for degrees of prism. Thus for prism diopters the card must be scaled so that each division shall be 1 cm. for each M distance at which it is used. Each division is for 20 ft., a one hundredth part of 20 ft. multiplied by the constant pertaining to the notation employed.

On a scale, shown in Fig. 132, used at 2 M. the numbers indicate degrees of deviation, at 3·5 M. they indicate prism diopters, the divisions being each 3·5 cm. Thus if a given prism at 2 M. indicates, say, 4 it is a 4°d; if held at 3·5 M. it will indicate 7^{Δ}, which is the equivalent of 4°d.

In order that equal divisions should indicate accurately equal increase of angular deviating power of prisms, the scale should be on an arc at the centre of which the prism is held. This is the basis of the *Centrad* notation which, however, owing to the inconvenience of such an arrangement, did not come into general use.

On a flat surface the divisions should be d tan 1°, d tan 2°, etc., where d is the distance at which the chart is used; that is, the successive spaces should increase in size, since tangents increase more rapidly than do the angles of deviation. For small angles, however, it is sufficiently accurate to make each division in cm.$=1·75\,d$, where d is the distance in metres. Thus for use at 3 M. each division would be $3\times1·57=5·25$ cm. approx.

If μ is taken as 1·5 the divisions for degrees should be ·875 cm. for each M.; if $\mu=1·52$, they should be ·9 cm.; if $\mu=1·54$ they should be ·94 cm. In practice the $^{\Delta}$ scale serves for degrees.

Artificial Sun.—Practically a collimator (*q.v.*) constitutes an artificial sun, but the best conditions are obtained if the aperture subtends at the lens an angle of 30′. The aperture is at the focal distance of the lens employed, so that its aperture $a=\frac{F\times·5}{57}=F/114$. That is, a: F as ·5°: a radian. If the lens is of 150 mm. F (6 inches) the best aperture is of 1·3 mm. diameter.

Pinhole Apertures.—Since light travels in straight lines, if that from a candle flame be allowed to pass through a small aperture on to a white screen, an inverted image of the flame is formed on the latter. The relative sizes of image and object are as their respective distances from the aperture; thus they are equal in size when the two are equidistant from the aperture. The image is smaller if the screen be brought nearer to the aperture, or if the candle be moved farther away, and *vice versa*. Generally the smaller the aperture, the sharper but less bright is the image.

In order that a distinct image of a flame may be seen on a screen, it is necessary that the rays from each point of the luminous body should not, on the screen, overlap those from adjacent points of the source. This may be said to occur practically when the light passes through a minute aperture, because then only a very narrow pencil of light—the cross section of which is similar in shape to that of the aperture—from each point can reach the screen, and for the same reason the image thus formed is faint.

The shape of the small aperture does not materially affect the shape of the image, nor its distinctness. Thus when the sun shines through the gaps in the foliage of a tree, each of these gaps varies in size and shape, but the luminous images of the sun form bright discs on the ground, all identical in shape unless the gaps are large.

If the number of apertures be increased, the number of images will similarly increase with the number of holes, until the images will so overlap one another that it is impossible to distinguish them separately, and there is a general illumination of the screen.

Although the smaller the pinhole the better is the image defined, yet if the aperture be too small the image is blurred by diffraction. Hence the aperture should be theoretically that diameter which is too small for diffusion and too large for diffraction to blur the image. The aperture is found from $\sqrt{4f_2\lambda}$, where f_2 is the distance of the screen from the aperture, and λ is the wave-length, this being ·0004 for photographic and ·0006 for visual effect. A (the aperture), f_2 and λ are expressed in mm. If λ be a constant ·0004, and f_2 be in inches, we can simplify the above to $A = \cdot 2\sqrt{f_2}$, the value of A being in mm. The intensity of the light is A/f_2. The respective sizes of object and image $O/I = f_1/f_2$, where f_1 is the distance of the object.

Transmission and Opacity.—The transmissiveness of various transparent media to different parts of the visible and invisible spectrum varies considerably. Thus crown and flint glass are comparatively opaque to heat rays and equally transparent to light rays, but while crown is rather opaque to the ultra-violet, flint is still more so. Most crystals, as fluorspar and pebble, are exceedingly transparent to the ultra-violet, and fluorspar also to the infra-red rays. Rock salt and iodine are very transparent, while alum is very opaque, to the infra-red rays. Crookes' glass absorbs infra-red and ultra-violet, and the darker shades some of the visible light as well.

The cause of opacity may be said to be due to the restraining influence exerted by bodies—or rather, their composition—on the passage through them of waves of certain lengths. The light is not, however, lost, but is converted into some other form of energy—perhaps generally heat—but the rise in temperature would be slight. Moreover, a rise due to opacity to ethereal vibration must be distinguished from that caused by the nature of the surface, *i.e.* its absorptive power, which has a much more powerful influence in raising the temperature of a body. It is due to the infra-red or heat radiations accompanying light that an opaque body becomes markedly heated when exposed to general radiation. Thus polished and blackened metal may be equally opaque, but the latter would be rendered much the hotter by free absorption of light. It would be difficult to eliminate the factor of absorption in the measurement of a rise of temperature produced by opacity to light.

Some bodies transmit light and not heat or chemical rays, and others the reverse. Bodies which transmit the invisible heat rays without becoming quickly warmed themselves are termed *diathermanous ;* those which do not are termed *athermanous* or adiathermanous.

Confusion Discs.—The size of a disc of confusion C (Fig. 317) bears the same relationship to that of the lens aperture A as its distance b from F does to f, the distance of the lens from F; thus $C/A = b/f$, or $C = A\, b/f$.

For instance, with a +4 D lens the disc of confusion C at 15 cm. from the lens, is 25 – 15=10 cm. from *F*, therefore *C*=10/25 of *A*. It would be the same size at *C′* if 35 cm. from the lens, and also 10 cm. from *F*. If *C′* is 40

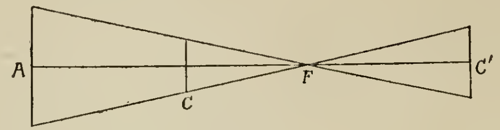

Fig. 317.

cm. from *A*, then *C′*/*A*=15/25. The source of light is presumed to be distant so that the light is parallel; if it is near, the conjugate distance f_2 must be taken instead of *F*.

If a screen be held close behind a Cx. lens facing a distant bright source, the emergent light is similar in size to the lens aperture, and it becomes smaller as the screen is receded, the minimum being reached at the focus, after which it again increases in size.

With cylindrical lenses the two diameters must be calculated, the confusion disc being elliptical. These two dimensions *C* and *C′* at any distance are found from

$$C = A\,a/F_1 \qquad \text{and} \qquad C' = A\,b/F_2$$

where a and b are the distances respectively from F_1 and F_2.

Thus the size of the confusion disc formed at 30 cm. by a +4 S. ⊂ +2 C. Ax. 90°, the diameter of the lens *A* being 5 cm.? Now F_1=16·66, and a=30 – 16·66=13·33 cm.; F_2=25, and b=30 – 25=5 cm., so that

$$C = 5 \times 13{\cdot}33/16{\cdot}66 = 4 \text{ cm.} \qquad \text{and} \qquad C' = 5 \times 5/25 = 1 \text{ cm.}$$

The disc is 4 cm. horizontally and 1 cm. vertically. It is difficult to show the two different diameters in one diagram, but a separate one for each clearly shows the principles involved. For the combination above we find at 20 cm. both dimensions to be 1 cm.

For a near point source, the confusion disc, in the focal plane, $d = a(f_2 - F)/f_2 = aF/f_1$, or $d = aF_1/f_1$ where F_1 and F_2 differ.

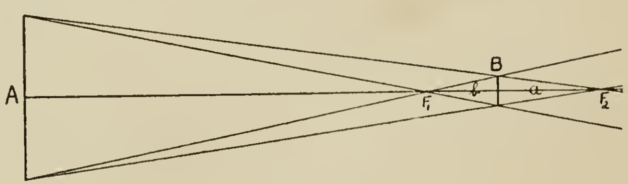

Fig. 318.

To find the circle of least confusion (Fig. 318), as in spherical or chromatic aberration, when F_1 and F_2 are the foci of the periphery and centre, or of the blue and red light, respectively. The calculation, similar to that on the interval of Sturm, is as follows:

$$B/A=b/F_1=a/F_2;\; F_2-F_1=a+b;\; F_1/F_2=b/a$$

$$a=F_2-F_1-b=F_2-F_1-aF_1/F_2 \quad \text{or} \quad a(F_1+F_2)=F_2(F_2-F_1)$$

That is, $a=\frac{F_2(F_2-F_1)}{F_1+F_2}$ Similarly, $b=\frac{F_1(F_2-F_1)}{F_1+F_2}$

Then $B=\frac{Aa}{F_2}=\frac{A\,(F_2-F_1)}{F_1+F_2}$ and the distance $AB=F_1+b=\frac{2F_1F_2}{F_1+F_2}$

Thus, let a lens be of 3″ aperture, F_1 the focus of the periphery is 19″, while that of the centre F_2 is 20″. The disc of least confusion is

$$B=\frac{3\times(20-19)}{20+19}=\frac{1''}{13} \text{ and its distance is } \frac{2\times20\times19}{20+19}=19{\cdot}48''$$

Images Formed by Cyl. Lenses and Mirrors.—The formation of images by cyls. has only been considered so far as the production of focal lines from point sources is concerned. Nevertheless, a plano-cyl. can produce an image of sorts, although naturally very ill defined and distorted, from an ordinary object; such images, even when real, are best examined by the eye, because the pupil of the latter acts as a stop, and cuts down the excessive confusion caused by the absence of point-foci.

The real image produced on a screen by a plano-Cx. cyl. is made up of focal lines approximately equal to the axial diameter of the lens; in consequence the real image is an infinite number of streaks parallel to the axis. Viewed from behind by the eye, the image seen is partly real, partly virtual, it is not altered in size along the axis, and may be diminished, magnified, or of the same size across the axis. With the axis, say, horizontal, the image is not laterally reversed, but is inverted; with the axis vertical the image is reversed but not inverted. When the object is within F the image is wholly virtual, there is neither reversal nor inversion, there being unit magnification along the axis, and enlargement across it.

In the case of a Cc. cylindrical the image is always virtual, is equal in size to the object along the axis, and diminished across the axis. There is neither reversal nor inversion.

A Cc. cyl. mirror acts similarly to a Cx. cyl. lens, and a Cx. cyl. mirror to a Cc. cyl. lens. What has been said above with regard to lenses applies equally to mirrors.

The Flame.—A flame (Fig. 319) consists of three cone-shaped portions, viz.:—

(A) The dark central portion surrounding the wick is the cone of generation or obscure cone. It is of low temperature and composed of gaseous products holding in suspension fine carbon particles which have not yet become incandescent.

(B) The luminous part surrounding A is the cone of decomposition or luminous cone, in which the carbon is in a state of intense incandescence, and in which luminosity is greatest.

(C) The thin external envelope, light yellow towards the summit and light blue at the base, is the cone of complete combustion giving but little light, and is the main source of heat. Here the temperature is high and combustion complete on account of the free access of the oxygen of the air.

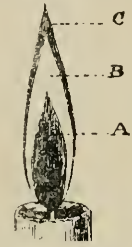

Fig. 319.

The flame in general is brighter at the top where the light predominates, and darker towards the base where heat is in excess. The outer envelope, being mixed with oxygen, is called the oxidizing element, while the inner cone, consisting mainly of unconsumed gas, is called the reducing element of the flame, since there metals may be reduced from their compounds.

A flame is produced by the incandescence of carbon particles which have been brought to a high temperature, the combustion, when once started, being continued owing to the heat produced by the chemical action itself. In a lamp or candle flame the material consumed is drawn up by capillarity through the wick.

Heat being produced by combustion, and luminosity being the result of the incandescence of unconsumed particles of carbon, the luminosity of a flame is low when combustion is complete, as is the case with the flame of some gases and of alcohol. It is high in a coal-gas flame, or in that produced by the combustion of oils and fats, where a considerable quantity of incandescent carbon is present. If the combustion be intensified by the introduction and intimate mixture of a sufficient supply of oxygen, as is done in the ordinary blow-pipe or Bunsen burner in which coal-gas is consumed, luminosity is decreased and heat is increased: the flame produced is then of a faint blue instead of the usual yellowish colour. The oxyhydrogen flame also gives very great heat, and yet is of a pale bluish colour and almost invisible; but when made to impinge on a lime cylinder, it renders it white hot at the point of contact, giving rise to an intensely brilliant spot of light, so that the temperature of a flame is neither indicated by the luminosity nor by the colour alone. To obtain maximum luminosity the supply of air must be neither too large nor too small. If too large the carbon is consumed too quickly, and if too small the carbon passes off unconsumed as soot.

On the other hand, although the *temperature* of the Bunsen flame, or any other source of complete combustion, is very much higher than that of luminous or incandescent sources, yet its power of *radiation* is considerably less. This can be illustrated by means of an experiment with a Bunsen burner and a thermopile, the latter being an apparatus exceedingly sensitive to radiant heat and its detection when placed some distance from a source.

With the complete combustion flame practically no rise in temperature is indicated by the thermopile, but when the oxygen is cut off and the flame becomes luminous, the index of the pile immediately shows a higher reading. Thus, for the production of radiant heat, the source must consist of rapidly vibrating incandescent particles capable of transferring their energy to the surrounding ether. For local heat, from conduction and convection air currents, the highest temperature is produced by complete combustion, where little energy is wasted in agitating the surrounding ether.

CHAPTER XXVII

OPTICAL AND OTHER INSTRUMENTS

The Spherometer is a mechanical instrument for determining the radius of a spherical surface. The most usual form consists of three fixed legs, arranged so that their points describe an equilateral triangle around a fourth leg in the centre which moves up and down, by means of a fine screw. The head of the screw supports a round horizontal plate, which has its edge almost touching a vertical scale divided into mm. or ·5 mm. as the case may be, and the plate itself is usually divided into 100 parts. The elevation or depression, therefore, of the central leg, from the plane of the other three, can be read with considerable accuracy. Generally the pitch of the thread is so arranged that two complete revolutions of the plate lowers or raises the central leg 1 mm., and as the plate itself is divided into 100 parts, the elevation or depression of the leg can be read to an accuracy of ·005 mm.

Fig. 321 shows a plan of the instrument, C being the central leg, and X, Y and Z the three fixed legs.

S, the *sagitta*, or sag (Fig. 320) of the curve for any particular chord $A\ B$, is measured by the central leg of the spherometer, and d by the distance between the central leg and an outside leg. The radius of curvature, r, is found from the formula below.

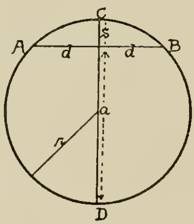

FIG. 320.

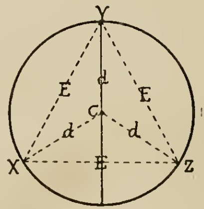

FIG. 321.

If two chords of a circle intersect at right angles the products of their respective parts are equal. Thus in Fig. 320 $A\ B$ and $C\ D$ are at right angles, and the line $A\ B$ is divided into two equal parts d and d, so that

$$S \times a = d \times d = d^2$$

But $a = 2r - S$, so that $d^2 = S\ (2r - S)$

Whence $$r = \frac{d^2 + S^2}{2\ S}$$

The distances $C X$, $C Y$, $C Z$ (Fig. 321)$=d$, and the angles $X C E$ and $Z C E$ are each 60°. Let E be the distance between any two of the fixed legs, say X and Z; then

$$E/2=d \sin 60°, \text{ or } E=\frac{2d\sqrt{3}}{2}=d\sqrt{3} \quad \text{or} \quad d=\frac{E}{\sqrt{3}}.$$

Substituting in the previous formula the value of d in terms of E we get

$$r=\frac{(E/\sqrt{3})^2+S^2}{2\,S}=\frac{E^2/3+S^2}{2\,S}=\frac{E^2+3\,S^2}{6\,S}$$

This latter formula serves when the distance E between two adjacent fixed legs is taken instead of d, the distance between a fixed and the central leg.

When the sagitta S is very small compared with r (as is nearly always the case), the quantity involving S^2 in the formulæ may be neglected, and they become respectively

$$r=d^2/2S \qquad \text{and} \qquad E^2/6S$$

As an example, suppose the distance between the movable and a fixed leg be 24 mm., and all four legs are in contact with a Cx. surface when the central leg is elevated 2·5 mm. Then

$$r=\frac{24^2+2{\cdot}5^2}{2\times 2{\cdot}5}=\frac{582{\cdot}25}{5}=116{\cdot}45 \text{ mm.}$$

or neglecting S^2

$$r=\frac{24^2}{2\times 2{\cdot}5}=115{\cdot}2 \text{ mm.}$$

The Lens Measure, used in the optical trade, is a mechanical instrument based on the construction of the spherometer. Projecting from the top of a small watch-like case are three metal pins, the central one projecting beyond the other two, and is movable. This latter acts on a spring connected with a pointer which indicates on a dial the dioptral number (or F) of a lens. The dial is graduated from known curves whose powers are calculated on a given index of, say, 1·52.

When the surface of a lens is pressed on the pins, until arrested by the two side ones, the central pin becomes depressed, and causes the pointer to revolve and indicate the power of the lens (as represented by its curvature) in diopters. Care must be taken that the plane of the lens is at right angles to the plane of the pins. The surface is sph. if, on rotating the lens, while pressed against the pins, the index remains stationary, and it is a plano if zero is indicated by the pointer.

If the index moves to different positions, when the lens is rotated, it indicates a cyl. or toroidal surface, the maximum power being shown by the highest number attained. The axis of a cyl. is indicated when the index points to zero, while the base-curve of a toric is indicated by the lowest power registered. The maximum curvature of a cyl., and the highest and lowest curvatures of a toric, are, of course, spherical; the intermediate curvatures, although elliptical, are indicated as if they were spherical.

If the lens has a cyl. element, the power of each surface is distinct from the other. When both surfaces are sph. the power of the one is added to that of the other to obtain the dioptral number of the lens; thus with -3 D on each surface, the lens is -6 D. If the one surface is $+2{\cdot}75$ and the other -1, the lens is $+1{\cdot}75$ D sph.

Surfacing tools or discs are those employed for grinding the curvature of lenses; they must, of necessity, be gauged for some given refractive index, usually, 1·52.

Accuracy of Spherometer, Tools, etc.—The pointer of a spherometer or lens measure should indicate zero when a plane glass is applied to it.

To test the accuracy of the scaling, lenses or discs of *known curvature* can be measured by the instruments.

To test the accuracy of surfacing tools, templates of known curvature are employed.

Changed μ.—A lens measure or a surfacing tool can be gauged for one refractive index only, and should the measure be used on a lens not having an index for which it is graduated, the power registered will be wrong. Similarly, if a surfacing tool is used on glass whose index is higher or lower than that for which it is calculated, the lens produced will be respectively stronger or weaker than the indicated power.

Let μ_1 be the index for which the measure or tool is made, and μ_2 be the index of the glass employed. Let D be the power indicated by the measure or tool, which would be correct if the index were μ_1, and let D′ be the true power of the lens, the index being μ_2. Then

$$D' = D\frac{(\mu_2 - 1)}{(\mu_1 - 1)}$$

Thus suppose the reading is 5 D on a lens measure made to $\mu = 1{\cdot}52$, but the lens made of glass of $\mu = 1{\cdot}56$. Then the lens is stronger than 5 D, and is

$$D' = 5 \times \frac{\cdot 56}{\cdot 52} = 5{\cdot}4 \text{ (approximately).}$$

If a $+5$ D surface is ground on a tool made for $\mu = 1{\cdot}52$ and the glass employed has $\mu = 1{\cdot}56$, the power produced is 5·4 D as above.

The dioptral tool D that should be employed to produce a given surface power D′, when the glass is of μ_2 and the tool is gauged for μ_1, is

$$D = D'\frac{(\mu_1 - 1)}{(\mu_2 - 1)}$$

Thus suppose the tools are made for glass of $\mu_1=1{\cdot}52$, and a lens of 10 D has to be made of glass of $\mu_2=1{\cdot}54$, we should employ a tool of

$$10\times{\cdot}52/{\cdot}54=9{\cdot}5 \text{ D}.$$

If focal lengths are indicated we have $F(\mu_1-1)=F'(\mu_2-1)$.

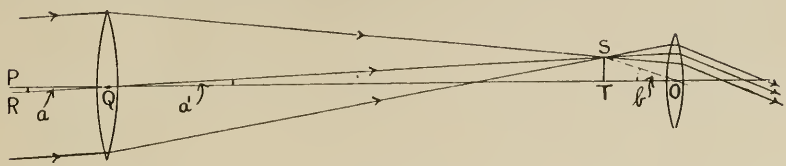

FIG. 322.

Telescopes.

The Astronomical Telescope consists of two unequal convex lenses (Fig. 322), separated by a distance d equal to the sum of their focal lengths, so that parallel light incident on the one lens emerges parallel from the other, but inverted and reversed. The lens of longer focus—the *objective*—receives the light, from the object, and forms a real inverted image in the focal plane of the second lens—the *eyepiece*—so that a normal (emmetropic) eye behind the latter sees, without accommodation, a magnified, inverted image of the object. As the name implies, this instrument is used for viewing celestial bodies, where the inversion of the image is of no importance. Light-gathering power is the essential feature of the Astronomical Telescope.

When the separation $d=F_1+F_2$ the telescope is said to be *in normal adjustment*. For a hypermetrope, who requires convergent light, d is slightly more than F_1+F_2, and for a myope, who requires divergent light, d is slightly less than F_1+F_2.

In Fig. 322 let Q be the optical centre of the objective, and $P\ R$ the extreme axial rays of the object at ∞, the one extremity P being assumed to be on the principal axis $Q\ O$ of the telescope. Then $P\ Q\ R=a$, the angle subtended by the object at Q, and $S\ T$ is the real image formed in the focal plane of the eyepiece, of which O is the optical centre. The angle under which this image is seen is b. The magnification therefore is the ratio between the angle b, under which the image is seen when the telescope is in use, and the angle a, under which the object would be seen by the naked eye, Thus

$$M=b/a=b/a'$$

Now b and a' are small and subtended by the common perpendicular $S\ T$, so that

$$M=b/a'=TQ/TO.$$

But $T\ Q=F_1$, that of the objective, and $T\ O=F_2$, that of the eyepiece Therefore

$$M=F_1/F_2$$

Thus if $F_1=5$ in., $F_2=2$ in., and $d=7$ in., $M=5/2=2\frac{1}{2}$. If the combination is reversed, so that the stronger lens faces the light, $M=2/5$.

The Ramsden Circle *B′ A′* (Fig. 323) may be defined as *the real image of the aperture A B of the objective formed by the eyepiece*, and represents that area through which passes all the light which traverses the telescope. It is sometimes called the *exit pupil*, and may be seen by turning the telescope towards

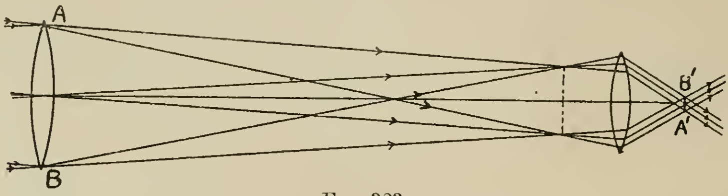

FIG. 323.

a bright source, as a lamp, and focussing the circle on a piece of translucent paper placed close to the eyepiece. It marks the position where the pupil of the observer's eye must be placed in order to obtain the largest possible field of view; the latter is contracted if the eye be withdrawn or advanced an appreciable distance from that position. Usually the Ramsden circle is within an inch or so of the last surface of the eyepiece, and consequently the telescope must be placed close to the eye. Sometimes this is impracticable or dangerous, as with sighting telescopes on ordnance, the recoil of which may injure the eye of the layer, in which case the instrument is so designed that the circle is farther back at, say, 3 inches from the eyepiece.

The significance of the Ramsden circle is the same for the **terrestrial telescope** and for the **prism binocular,** described later.

The aperture of the objective is sometimes called the *entrance pupil*, and the following ratio holds good :

$$M=\frac{\text{Diameter of entrance pupil}}{\text{Diameter of exit pupil}}=\frac{A\,B}{B'A'}=\frac{F_1}{F_2}$$

The Field of View varies directly with the aperture of the eyepiece, and inversely with the magnification. It is independent of the diameter of the objective as such and, as stated, is a maximum when the pupil coincides with the Ramsden circle. The field of view is about 1° to 3° for small instruments.

The Illumination varies directly with the diameter of the objective, and inversely with the magnification. To observe faint celestial bodies, such as stars of tenth or twelfth magnitude, telescopes with very large objectives are essential, the whole of the light falling on the latter being passed through the Ramsden circle and into the eye, provided the pupil is at least as large as the circle itself. In this sense the effect of the telescope is virtually to enlarge the pupil of the eye to an extent equal to the magnification.

Correction of Aberration, on account of the comparatively small angle of view, is necessary only for spherical and chromatic errors, for which, however, the objective must be well corrected. The correction of the eyepiece is carried out in a manner described later.

Resolving Power—that is, the ability of a telescope or other instrument to render fine details apparent—depends upon the diameter of the objective. Thus with a certain telescope a particular star may appear as a single point of light, whereas with an instrument having a larger objective the star proves to be double. It is a question of diffraction.

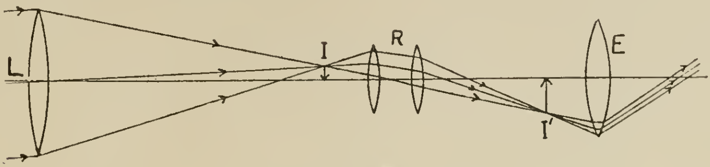

Fig. 324.

The Terrestrial Telescope demands an erect image. For reinverting the inverted image I (Fig. 324) formed by the objective L, of a distant object, an *erector* R is inserted between I and the eyepiece E. By means of the erector a second real image I' is formed in the focal plane of E and this is erect. Then the final virtual image, seen by the observer, is also erect.

The erector consists of two strong Cx. lenses between which, usually, a stop is inserted so as to reduce spherical and chromatic aberration. According to its position and formation, the erector may increase, decrease or leave unchanged the magnification. In this connection it may be taken as part of either the objective or the eyepiece and the equivalent points and focal length of the combination would need to be calculated in order to apply $M=F_1/F_2$ in the ordinary way.

Compared with the **astronomical** telescope, the illumination of the **terrestrial** is poor, owing to the introduction of four extra refracting surfaces, and perhaps of a stop.

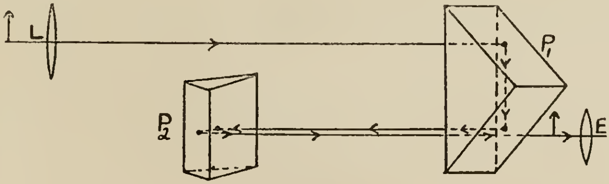

Fig. 325.

The Prism Binocular is identical in principle with that of the terrestrial telescope, except as to the method of causing erection of the image. Total reflection prisms are used for this purpose, and Fig. 325 shows a typical arrangement of the optical parts.

L is the objective, E is the eyepiece, and P_1 and P_2 are the reflecting prisms.

The convergent beam of light from L falls on P_1 which is a right-angled isosceles prism having its edge horizontal. The light is totally reflected, in the vertical plane, at the two inclined surfaces successively, and so it is vertically reversed as if proceeding from an inverted object. A second similar prism P_2, but having its edge in the vertical plane, and therefore at right angles to that of P_1, then receives the light and reflects it twice in the horizontal plane without affecting the vertical inversion caused by P_1. A real upright image is thus formed in the focal plane of the eyepiece E.

By this arrangement the volume of light transmitted is little inferior to that of the astronomical telescope. Also the magnification, which depends on the distance between the objective and ocular, is very high for the length of the instrument, which is about one third that of a telescope having the same magnifying power; the transmitted light traverses the body three times before reaching the eyepiece.

There is theoretically no limit to the magnification of the prism binocular, but for all-round use the best M is 6 or 8 times; if higher, it has to be held very firmly indeed to prevent unsteadiness of the image, while if only a low magnification is required the Galilean is perhaps to be preferred.

The instrument is compact, easily made in binocular form, and presents a single image whose stereoscopic effect is enhanced by placing the objectives wider apart than the natural interpupillary distance.

The field of view varies from about 3° to 12°.

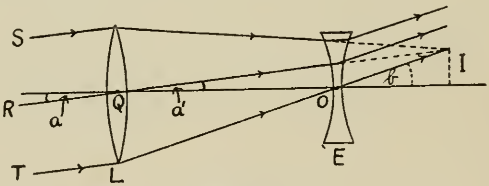

FIG. 326.

The Galilean Telescope or **Opera-Glass** consists of a Cx. lens, L (Fig. 326), placed in front of a Cc., E, of higher power, at a distance equal to the algebraical sum of their focal lengths, so that the lenses neutralise each other by separation. Although the *rays* of each pencil from a distant object emerge parallel after refraction by both lenses, yet the *pencils themselves* are deviated so that the image is seen under a larger angle.

Let R be the axial ray of a beam from the extreme point of a distant object subtending the angle a. Were E not interposed a real inverted image I would be formed, but with E placed at a distance from I equal to its own focal length the pencil is again refracted as a parallel beam. The angle under which the image is seen is b, so that

$$M = b/a = b/a'.$$

The angles a' and b are small, and I is the common perpendicular, so that

$$M = b/a' = IQ/IO.$$

But $IQ = F_1$, that of the objective L, and $IO = F_2$, that of the eyepiece E. Therefore, as with the astronomical telescope,

$$M = F_1/F_2.$$

Thus, if $F_1 = 5$ in., $F_2 = 2$ in., and $d = 5 - 2 = 3$ in., $M = 5/2 = 2\frac{1}{2}$. If the combination is reversed so that the Cc. is to the front, $M = 2/5$.

The Ramsden Circle A′ B′ (Fig. 327) in the opera-glass is the *virtual* image of the aperture of the objective and is therefore situated in front of the eyepiece.

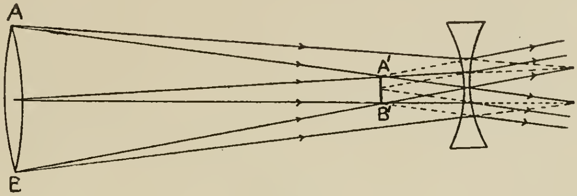

Fig. 327.

The Field of View is therefore contracted, as the pupil cannot be placed in the imaginary circle, and on looking through an opera-glass the field is bounded by the blurred Ramsden circle, which moves in the same direction as the observer's eye, as this is displaced from side to side, thus disclosing different portions of the field.

The field varies directly with the aperture of the objective and inversely with the magnification, and so marked is the contraction in the higher powers that the glass is rendered practically useless. Thus M is rarely greater than 5, and more usually is 4 to 2. The utmost diameter of objective cannot exceed some 63 mm., which is the average distance between the eyes. The field varies from about 2° to 6°. For the notation of small telescopes see p. vi.

The Illumination is good because, the image being upright, no erector is needed and the refracting surfaces are a minimum. The night-glasses used for marine work are Galilean telescopes having the largest possible objectives, and of moderate magnification.

Correction of Aberration is only for spherical and chromatic errors, the angle of view being even smaller than in the telescope or prism binocular for the same magnification. In high quality glasses both the objective and eyepiece are corrected, but in the medium and lower qualities only the objective.

The distance d between the lenses is equal to $F_1 + F_2$ when the emergent light is parallel, as foı a normal eye. For divergent light $d < F_1 + F_2$, and for convergent light $d > F_1 + F_2$, as is needed by the myope and hypermetrope respectively.

Comparison of Galilean and Prism Binoculars.—The Galilean is superior in illumination; its great disadvantage is the contracted field of view for equal magnification.

With no instrument is the intrinsic brightness of the image increased, but rather reduced by reflection at the various lens surfaces, except in the case of a star, which has no magnitude under any conditions.

Measurement of Magnification.—A practical method for measuring the magnification, of any form of telescope, is to view a fairly distant object having regular spaces, such as a brick wall, through the telescope with the one eye and directly with the other eye. With a little manipulation the two images can be made to overlap, and the magnification is given by the number of bricks, seen by the unaided eye, that are contained in a single brick seen through the telescope.

Since the *entrance* and *exit* pupils of a telescope, having a positive eyepiece, can be measured, the ratio between their sizes gives the magnifying power.

The Reflecting Telescope, sometimes used for astronomical purposes, consists of a long focus parabolic concave mirror of large aperture forming the objective, and the usual eyepiece to which the light is reflected by a small secondary mirror or prism. Its entire natural freedom from chromatic aberration renders it very valuable for some purposes.

The Compound Microscope (Fig. 328) is used to obtain a magnified view of a small near object. It consists of two unequal strong Cx. lenses, the front one L, the objective, being a very strong combination, while the second, E, the eyepiece, is also strong but less so than L. The distance d between them is much greater than F_1+F_2 and is governed by the available length of the instrument which is usually 10 in.—the conventional distance of most distinct vision. Some instruments, however, have a tube length of 8 in. or even 6 in.

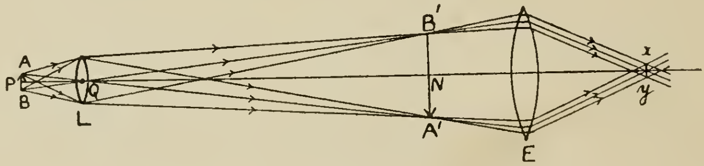

FIG. 328.

The object to be viewed, $A\,B$ in Fig. 328, is placed just beyond F of the objective L and a real inverted highly magnified image $B'\,A'$ of it is formed in the focal plane of the eyepiece E when the microscope is in normal adjustment. An eye placed behind E then sees an enlarged virtual image of the first real image. Hence there is magnification due to both the objective and to the ocular, and it can be very approximately calculated as follows: Let N denote the position of $B'\,A'$ on the axis.

The magnification due to the objective is

$$M_1 = B'\,A'/A\,B = Q\,N/Q\,P.$$

But $Q\ N$ may be taken as the tube length of the microscope, which is usually 10 in., while $Q\,P$ is practically equal to F_1—that of the objective. Thus M_1 may be taken as tube length/F_1=10/F_1.

The magnification M_2 of the eyepiece can be expressed as $d/F_2=10/F_2$, where F_2 is the focal length of the eyepiece, and 10 in. is the conventional distance of visual projection which, in this case, corresponds approximately to the actual distance of the original object.

The total magnification is, therefore,

$$M=M_1\,M_2=\frac{\text{Tube length}}{F_1}\times\frac{\text{Projection distance}}{F_2}=\frac{10\times10}{F_1F_2}=\frac{100}{F_1F_2}$$

where all terms are expressed in inches. Thus suppose $F_1=\frac{1}{4}$ in. and $F_2=2$ in., then M=100/·5=200. If all terms are in cm. the numerator of the above term becomes 625.

The above formula is only approximate but is quite sufficiently exact for practical purposes. $Q\ P$ is not actually equal to F_1, nor is the distance of projection always 10 in. While N is taken to be at F_2 for the emmetrope, it would be beyond for the hypermetrope, who needs convergent light, and within for the myope, who needs divergent light. The microscope has a coarse adjustment so as to vary the distance between L and E, to suit the individual eye and distance of object, and a fine adjustment for the more exact disposition of these parts in order to secure the clearest possible view.

The Ramsden circle has the same significance as in the astronomical telescope, and is denoted by $x\ y$ in Fig. 328.

Aberrations—spherical and chromatic—must be fully corrected, especially with regard to the objective, which, if of high class, is a complicated combination of lenses. Coma also over a certain area must be corrected. The positive eyepieces are the same as used in telescopes.

Resolving power is a most important consideration in microscopy, and depends entirely upon the angular aperture of the objective. For the examination of very small objects, such as bacteria, an objective of very large aperture must be used. A special type is the immersion objective, with which a drop of cedar oil produces a homogeneous medium from the object to the lower lens.

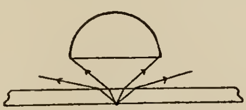

FIG. 329.

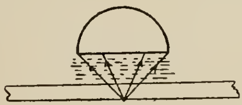

FIG. 330.

The *Angular Aperture a* is that of the cone of light from the object that can enter the objective. The *numerical aperture* N.A. expresses the light gathering power and is given by N.A.=μ sin $a/2$ where μ pertains to the medium in the space between the object, or the cover plate, and the bottom of the objective. This medium (Fig. 329), with a dry objective, is air, and $\mu=1$, so that the N.A. cannot exceed 1.

With an immersion objective (Fig. 330), however, the medium is water of $\mu=1{\cdot}33$, or cedar oil of $\mu=1{\cdot}512$. Thus the N.A. or the *effective* aperture of the objective, on which the resolving powei depends, is increased by the employment of an oil immersion objective.

Eyepieces.

Eyepieces generally consist of two uncorrected separated lenses so arranged that refraction is shared more or less equally between all four surfaces. Spherical aberration for the narrow pencils produced by the objective is thus practically eliminated, and as the final image is virtual the effects of chromatic aberration are inappreciable.

The Ramsden eyepiece consists of two *equal* plano-Cx. lenses having their curved surfaces facing each other, and the separation is usually 2/3 F of either lens.

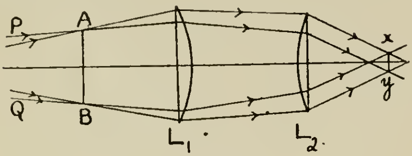

Fig. 331.

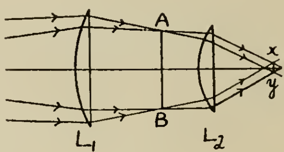

Fig. 332.

In Fig. 331, showing the Ramsden system, L_1 is the *field-lens* and L_2 the *eye-lens*. P and Q are convergent pencils from the objective forming the real image $A\,B$ from which they diverge and, after refraction at both lenses, finally emerge as parallel beams. The Ramsden circle is $x\ y$.

This eyepiece is said to be *positive* because its principal focal plane is outside the lenses—*i.e.* $A\,B$ in Fig. 331. It is therefore a convenient type where cross wires, micrometer scales, etc., are required to be used, as these can then be placed in the plane of $A\,B$, the magnifying effect of the whole eyepiece being equally exerted on both wires and image.

The Huyghen eyepiece is usually employed where high magnification is required. It consists of two plano-Cx. lenses, each curved surface being turned the same way in the direction of the objective. The eye-lens is the more powerful, such that $F_1=3F_2$, the separation being $(F_1+F_2)/2$.

In Fig. 332 L_1 is the field-lens, L_2 the eye-lens, and convergent pencils fall upon L_1 *before* the real image is formed at $A\,B$, the focal plane of the eye-lens. For this reason the Huyghen eyepiece is said to be negative, the principal focal distance on the one side being situated between the lenses.

The Kellner eyepiece, now little used, consists of two equal plano-Cx. lenses with both curved surfaces turned towards the objective, and separated by a distance equal to F of either. The real image from the objective is formed, therefore, in the plane of L_1, which is the focal plane of L_2. In this eyepiece

L_1 has no influence at all on the magnification, and its great disadvantage is that any dust or scratches on L_1 are as conspicuous as the image itself, being situated in the same plane.

In every eyepiece the utility of L_1, the field-lens, is to increase the field of view.

Other Optical Instruments.

The Camera consists, in principle, of a light-tight box, having a positive lens at one end, and a sensitive screen—the plate or film—at the other, upon which is received and recorded the real image of an external object.

The Aperture-Ratio or F/No. is an important factor in the photographic objective, because on it depends the duration of exposure and, to some extent, the definition of the image over a certain area. The No. is found by dividing the equivalent focal length by the effective aperture of the lens; for example, a lens of F=6″ with an aperture of $1\frac{1}{2}$ in. is said to work at F/4. This figure 4 denotes the *maximum* ratio for which the objective is designed—it may work at any smaller aperture down as far as F/64 by the use of a variable iris diaphragm. A certain series of these F/Nos. usually are engraved on the mount, and are so arranged that a change from one to the next higher generally requires a doubling of the exposure.

Principal Types.—The lenses employed in photography may be roughly divided into four main groups: (1) for portraits, (2) for landscapes, (3) for architecture and copying, and (4) for all-round work.

Portraiture demands critical central definition over a relatively small area, the more peripheral portions of the picture being immaterial as regards sharpness. Spherical and chromatic aberration must therefore be fully corrected, the oblique aberrations being relatively neglected. The working aperture must be large in order that indoor photography with artificial light may be done without prolonged exposure. The focal length should also be long in order that distortion of perspective may be as small as possible. The Petzval system is a good example of the portrait lens, working at an aperture of, say, F/4.

For landscapes fair general definition over the whole plate is necessary, but a high degree of correction for any aberration is unnecessary because the size of aperture is not important. For the latter F/8 to F/16 is generally sufficient. The lens may be of the single achromatic form, or any of the other types, except the purely portrait objective, may be employed. The front or back component of a modern high-class lens may be suitable for this work.

Where freedom from distortion is essential, as in architectural studies, or in copying and process work, what is known as the *rectilinear* lens is largely employed. This consists of two equal components having a diaphragm placed midway between them so that any negative distortion of the front lens is neutralised by the equal positive distortion of the back lens. Such a lens is usually designed for an aperture of about F/6 to F/8, rapidity of exposure not being an important factor.

The all-round type of lens, such as the anastigmat, really combines all the advantage of the others; it is one especially well corrected for astigmatism, and has a large flat field. Its working aperture is large, from F/4·5 to F/6, and it is essential where rapid exposure is a necessity, and a large variety of subjects has to be covered. Such lenses are used in high-class hand cameras. It represents, perhaps, the highest skill of the optical designer, and one or both components may, in many cases, be used separately for some classes of work.

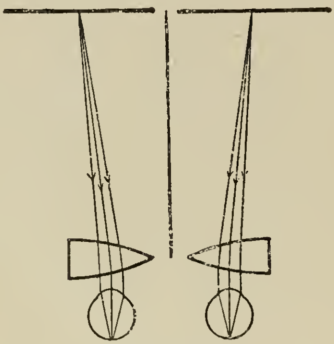

Fig. 333.

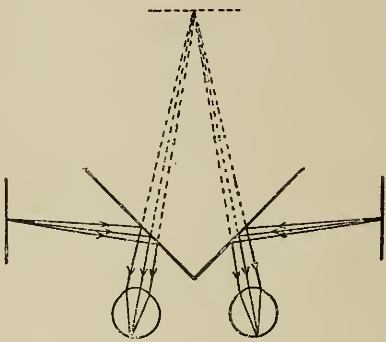

Fig. 334.

The Stereoscope.—The stereoscope consists of a box so arranged that there is presented to each eye a view, of the same object, obtained by photography from slightly different positions.

The photographs are taken simultaneously by a camera, having twin-lenses, placed slightly farther apart than is the distance between the two eyes, and the photographs obtained are reversed, so that the right and left eye sees the picture taken by the right and left lens respectively. The two pictures are mentally fused, and the single picture seen with stereoscopic effect.

The most usual form of stereoscope is that of Brewster (Fig. 333). Each division has a Cx. spherical lens whose F is equal to the normal depth of the box, so that light from the pictures enters the eyes in parallel beams. The lenses are combined with prisms whose bases are *out* so that light from the centres of the pictures, which are farther apart than the eyes, may enter the latter without their having to make any movement.

In Wheatstone's stereoscope (Fig. 334) there are two plane mirrors at right angles to each other. The two pictures are to the side of the box and are seen by reflection from the mirrors, which can be adjusted so that the two images may be fused. This adjustability takes the place of the prisms in the Brewster model, while the greater distance of the pictures renders Cx. lenses unnecessary.

The Optical Lantern is employed for projecting on to a screen a highly magnified real image of, usually, a transparent or translucent object such as a positive, printed from an ordinary photographic negative. The slide, as it is called, is strongly illuminated from behind, and placed just beyond F of a positive lens, resulting in a real, magnified and distant image inverted with respect to the slide. The slide is therefore placed in the carrier upside down, so that the image projected on to the screen may be upright.

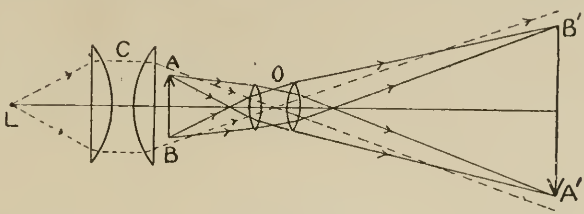

Fig. 335.

Fig. 335 illustrates diagrammatically the optical principles. L is a powerful source, such as an electric arc, and C a large compound condenser which concentrates a wide pencil of light on to the slide $A\,B$ placed just outside F of O, the objective, $B'A'$ being the resultant real image.

C is designed to produce the least possible spherical aberration, and that shown in Fig. 335 is known as the double D condenser. A still better result may be obtained with a combination of double and meniscus Cx., the latter being placed with its concave surface facing the source. It should be observed that to secure the most intense and even illumination the real conjugate focus of L formed by C is coincident with the centre of the objective O. The modern kinematograph projector is based on the same principles.

The objective must be highly corrected for chromatic and spherical aberration, but the field of critical definition need not be large. For this reason portrait lenses lend themselves very well to projection work, the Petzval system in particular being used.

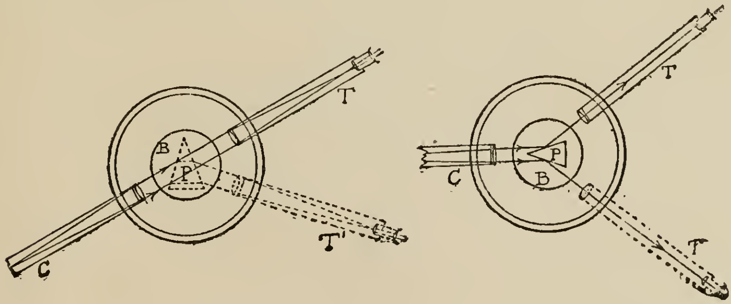

Fig. 336.

The Spectroscope (Fig. 336) is used for viewing and comparing spectra produced by prisms, and consists of a horizontal circle, mounted on a stand, to which are attached a telescope T and a collimator C, both of which can be rotated around the circle. The collimator is a tube having at one end a Cx. lens and at the other a narrow slit parallel to the refracting edge of the prism *P*. The distance between the slit and the collimator lens is equal to F of the latter, so that light, from the slit, is rendered parallel by the lens before reaching the prism. In the centre of the circle there is a small table *B* on which the prism is placed.

The Spectrometer is a spectroscope with the addition of a scale of degrees on which the position of the movable telescope can be indicated, and to which, for accurate readings, a vernier and reading microscope is attached. This enables the principal angle, the deviating angle, and the dispersion of a prism to be measured.

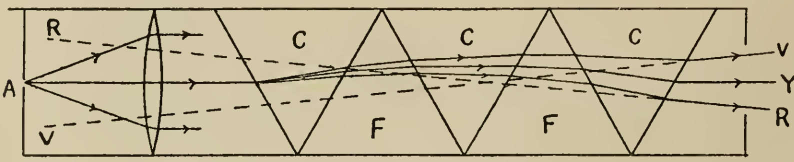

Fig. 337.

A Direct Vision Spectroscope (Fig. 337) is formed of a train of, say, three crown and two flint prisms mounted in opposition in a tube. The prisms are such that there is no ultimate deviation for yellow light, while there is a considerable separation of the red and violet. *A* is a slit aperture parallel to the prism edge, and at the other end is the eye aperture at which a telescopic arrangement is sometimes used.

The Sextant (Fig. 338) is used to measure the angle subtended at the eye by the sun and the horizon, from which the angular elevation of the sun

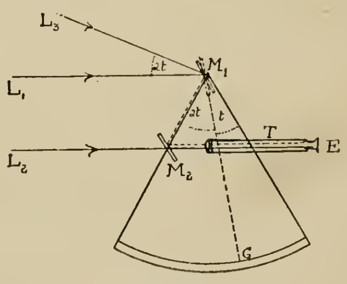

Fig. 338.

can be calculated. It also serves to measure the angle between any two in accessible objects.

A small mirror M_1 revolves about a horizontal axis to which is attached a pointer *G* moving over a scale of degrees. M_2 is a small fixed mirror of which one half is silvered and the other half is clear, and is so inclined that when M_1 and M_2 are parallel the pointer indicates zero on the scale. T is a small telescope so directed forwards that it receives at the same time light from the horizon by direct transmission through the clear part of M_2, and by reflection, from the silvered part, of light which has been reflected to M_2 from M_1.

Let L_3 be a ray emanating from the sun, and L_2 a ray from the horizon. Then to an eye E the image of the sun along the path L_3 will apparently coincide with the image of the horizon seen directly along L_2. The angle which L_3 makes with L_1, which is parallel to L_2, is the angular distance between the sun and the horizon, but G, the pointer, only moves through t, which is half this angle; therefore the scale over which G moves is divided into half degree spaces, which, however, are numbered as whole degrees in order that direct readings may be taken from the scale, to which also a vernier (*q.v.*) is attached for greater accuracy.

An artificial horizon is formed by a bowl of mercury whose surface becomes a truly horizontal plane. The angle between the position of a telescope when an object is seen through it, and its position when an image of the object is seen by reflection from the mercury, is twice the angular altitude of the object above the horizon.

The Kaleidoscope.—The principle of the kaleidoscope depends on the multiple reflection caused by two inclined mirrors. The mirrors are placed lengthways in a tube, which is closed at one end by a disc of transparent glass, beyond which is one of frosted glass. Between these two glass discs there are a number of small coloured objects, or fragments of coloured glass. Looking through the open end of the tube an image is seen consisting of a certain number of images, the whole forming a more or less symmetrical figure. The usual form of kaleidoscope has three mirrors inclined to each other at 60°, and the figure is symmetrically hexagonal. The whole central figure, as seen in a kaleidoscope, is surrounded by others formed by repeated reflections of the light.

The Vernier.

The Vernier is an attachment to instruments where great precision of linear or angular measurement is required, and it obviates the necessity of the division of the main scale into very minute parts. It consists of a short scale V (Fig. 339) which slides along the main scale S to which it is attached.

V is the same length as a definite number of divisions of S, but contains one division more, so that if V is divided into 10 parts, these equal nine divisions of S, or if V has 30 divisions they correspond to 29 of S. Thus each division of V is smaller than each of S by a fraction whose denominator is the number of divisions of V, viz., 1/10th or 1/30th, respectively, in the examples quoted. The greater the number of divisions of V the more accurate are the readings, but also the more difficult is its use.

The scale itself may be divided into whole terms of measurement, as mm. or degrees, or more commonly into main fractions of such terms as

$\frac{1}{2}$ mm. or $\frac{1}{2}$ degrees. Such whole terms, or main fractions thereof, are read from S itself, the measurement being the last beyond which the zero of V has passed. The minute measurement is obtained from V by finding *that division mark of V corresponding to, or in exact line with, a division mark of S.*

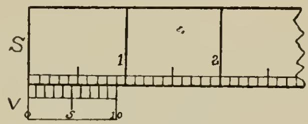

Fig. 339.

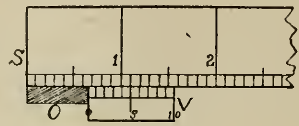

Fig. 340.

Thus, if 10 V=9 S, and the third division mark of V is in line with one of S, the exact measurement is 3/10 more than the whole number indicated on S itself. If V has 60 parts and the 33rd is in line with an S division, the fractional reading is $\frac{33}{60}$ plus the reading on S.

Fig. 340 illustrates a reading on a scale S directly divided to inches and tenths of inches with a vernier V whose 10 divisions=9 of the scale. The length of an object O whose one extremity is at zero of S is ·65 in., the 5th division of V coinciding with a division of the scale. The ·60 in. is read from the scale itself, where the right-hand extremity of O lies between the 6th and 7th divisions of S; the balance ·05 in. is read from the V. The limit of accuracy is $\frac{1}{100}$ in.

As another example, let the scale be divided to inches and tenths of inches, and let 25 V=24 S. If the zero of V showed 5 in. and six spaces of $\frac{1}{10}$ in., plus a certain distance when the fourth division of V is in line with a scale mark, the total measurement would be $5+\frac{6}{10}+\frac{4}{250}$ or 5·616 in. The accuracy of the reading is carried to $\frac{1}{250}$ in.

For an instrument such as the altitude barometer, the vernier is made with the V divisions longer than those of S, so that, say, 9 V=10 S. The V divisions are then on the near side of the zero, and are read backwards.

Verniers for fine straight rules are usually made so that 10 V=9 S, thus measuring to $\frac{1}{10}$ mm. For box sextants and small surveying instruments 30 V=29 S, so that $\frac{1}{2}°$ divisions are subdivided to minutes. For barometers the readings are usually taken to $\frac{1}{10}$ mm. when 10 V=9 S, or to $\frac{1}{250}$ in. when 25 V=24 S. For marine sextants and theodolites 60 V=59 S, measurements being taken to 1/60 of 20′ or of 10′, giving limits of accuracy of, respectively, 20″ or 10″ in the case of these two instruments.

CHAPTER XXVIII

PRACTICAL AND EXPERIMENTAL WORK

In order to grasp the various formulæ and the theories underlying them, the student should perform for himself the simpler experiments connected with general optics. Most of the following can be done with quite rough or improvised apparatus, and a complete optical bench, meeting all requirements, can be obtained at a very moderate cost.

The Optical Bench.—An optical bench should preferably be scaled in cm. and mm. and be about 2 M long, thus enabling fairly weak lenses, mirrors, etc., to be tested. There should be—

(1) A frosted lamp at the zero end of the scale.

(2) A collimator consisting of a pinhole fixed in the focal plane of a Cx. lens, the lamp being placed behind the pinhole when in use.

(3) A screen of ground glass and another, interchangeable with it, of opaque stiff white card having a central aperture equal in diameter at least to the collimating lens. The latter is used with mirrors.

(4) A plate with an aperture of definite size, say 20 mm., with fine cross wires, to serve as an object, when the lamp is behind it.

(5) Three or four carriers for lenses and mirrors. One should be universal and capable of holding any lens from the smallest up to say, 3″ diameter.

(6) Two or three clips on a single stand capable of taking lenses in contact or combinations of separated lenses. This should also be capable of a *horizontal* rotation round the support as a vertical axis.

(7) A small horizontal astronomical telescope with adjustable eyepiece.

All should be on movable stands and adjustable as to height, since axial alignment is essential in most experiments.

Parallax is the term applied to the apparent displacement of an object due to the observer's position. We generally employ the term to indicate the apparent change in the position of one object, in relation to that of another, when the observer changes his point of view. Let an object A be in front of an upright pencil P, and another object B be behind P, and all three in the same straight line in front of the observer. Now on moving the head to, say, the *right*, a gap will be visible between P and A and another between P and B; also A will be to the *left* of P, and B to the *right* of P—that is to say, a near object moves apparently *against* and a distant one *with* the observer's head.

Parallax Test.—If there are two comparatively near small objects P and X, seen close together in the same direct line, the distance of the one P being known, if the head be moved sideways—(*a*) X is actually in the same plane as P, *i.e.* coincident with it, if no gap between them results; (*b*) X is *nearer* than P if X has apparently moved in the *opposite* direction to the observer's head; (*c*) X is more *remote* than P if X has moved in the same direction. By placing P respectively nearer, or farther away, a position can be found for it such that parallax between them is said to be destroyed, since no apparent separation results from any degree of movement on the part of the observer; the distance of P then equals that of X. This principle is utilised for locating the position of virtual images formed by mirrors and lenses and will be referred to in some of the following articles.

Plane Surfaces

Movement.—A plano-spectacle glass can be determined with sufficient accuracy by observing an object (preferably crossed lines) through it while rotating and moving the glass. If the glass has no power due to curvature the image will appear stationary; moreover, if the surfaces be true planes no distortion or irregular movements can be detected. If there is no prismatic power, there is no rotation of the cross on rotating the glass. If the glass be held obliquely to the eye, so that the direction of vision forms a small angle with the surface, any unevenness of the surface becomes more apparent.

Contact.—If one surface be a plane, this can be determined by applying to it a straight-edge, or another plano-glass, and observing whether there is contact throughout when holding the applied surfaces against a bright background. Real contact between two surfaces is also quite easily felt, and they will adhere to each other if slightly moistened by breathing on one of them.

Spherometer.—A plane surface is shown by the spherometer or lens-measure.

Whitworth Plane.—By contact with a Whitworth true plane surface, which has been smeared with some red putty powder, and observing whether all portions have or have not taken an impression.

Newton's Rings.—The absence of interference phenomena between a known plane surface and one tested is the most accurate method.

See also **Reflection** tests and **Telescopic** tests.

Reflection Tests.

A plane surface can be distinguished from a curved one by viewing the reflected image from a bright source of light. If a plane, it acts precisely as a plane mirror, while if a sph. or cyl., the image is altered in size or distorted. If the object viewed is a square, then a Cx. surface will cause it to appear compressed vertically, *i.e.* in the direction of view, so that it has the appearance of a horizontal rectangle, while a Cc. surface causes vertical extension, giving the appearance of a vertical rectangle. In every instance the lens should be held as close and oblique to the eye as possible.

As the lens is rotated, while still viewing the reflected image, there is no change in the appearance of the latter if the surface is sph. or plane, whereas if cyl. the image does change. If the object viewed be of some definite shape, say a vertical window bar, it is seen quite distinctly when the axis of the cyl. is in line with the direction of view, whereas it is indistinct when the axis is oblique to, and most indistinct when the axis is at right angles to, the line of vision, the general image being drawn out if the surface is Cc., and compressed if Cx., as with sph. surfaces. This is an extremely delicate test for locating the axis of a cyl.

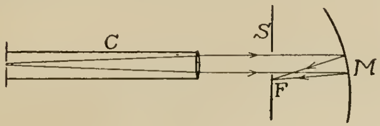

Fig. 341.

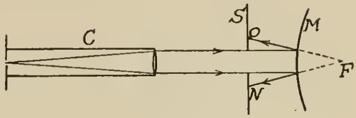

Fig. 342

Focal Length of Cc. Mirrors.

Direct Focalisation.—On the optical bench parallel light is obtained from the collimator *C* (Fig. 341), and passed through the perforated screen *S* on to the mirror *M* whose focal length is to be measured. The mirror is slightly tilted and moved to and fro until the image of the pinhole is thrown sharply on to the screen at *F*. The distance *M F* is the required focal length.

Conjugate Focalisation.—If the cross wires be substituted for the collimator such that a real *conjugate* image be formed on the screen *S*, we have $1/F=1/f_1+1/f_2$, where f_1 is the distance of the cross wires, and f_2 is the conjugate distance *M S* of the screen, to the mirror. This being the same as for Cx. lenses, the examples given serve equally well for Cc. mirrors.

Symmetrical Planes.—An especially rapid and accurate way to find *F* is to use the cross wires and the disc containing them as both object and screen. The mirror is advanced towards *S* until the image of the wires appears sharply on the surrounding disc, which must then be at the centre of curvature. The radius of curvature is thus directly measured, and equals 2 F, *i.e.* $F=r/2$.

Parallax.—If an object be placed within F, the *virtual* image can be located as described under convex mirrors, and the focal length found from the conjugates, care being taken to reckon the distance of the image as a negative quantity.

The Spherometer.—See this method for Cx. mirrors.

Focal Length of Cx. Mirrors.

Projection Method.—Arrange a collimator and perforated screen (Fig. 342) as for a Cc. mirror, the screen being between *C* and *M*. On *S* describe a circle *O N* concentric with the central aperture and of twice the diameter of the collimator lens. The action of the mirror being divergent it will reflect the parallel beam as a cone apparently diverging from *F*. Move the mirror to and fro until the projected area of illumination on *S* exactly fills the circle *O N*. Then the distance of screen to *M* equals *F* of the mirror.

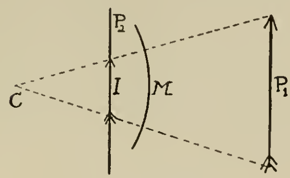

FIG. 343.

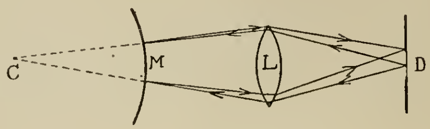

FIG. 344.

Parallax Method.—Take two stiff wires or knitting needles (Fig. 343) and place one P_1 represented by the arrow in front of *M* such that its virtual image is *I*, seen on looking into the mirror from the same side as P_1. Behind *M* place a second needle P_2 such that it approximately coincides with *I* seen *in* the mirror. Now move the head from side to side, and if there is apparent separation between the virtual image *I* of P_1 and the actual pin P_2 the latter must be moved towards or from the mirror until all parallax disappears. Then if $P_1\ M$ be f_1, and $P_2\ M$ be f_2, we have, since f_2 is a negative quantity, $1/F = 1/f_1 + (-1/f_2)$.

Convergence towards C. of C.—Set up the cross wire *D* (Fig. 344) and in front of it place any convex lens *L* so that the latter projects a real image at a distance *L C* greater than the radius of the mirror; the distance *L C* is measured. On interposing *M* and moving it to and fro a position will be found where the image of the wires is received back on to the disc *D*. When such is the case the convergent light from *L* must be incident on *M* directed towards the centre of curvature *C* because it has returned along its own path. Then the radius of the mirror is the distance *M C*, between the mirror and the real image formed by the lens, and $M\ C = L\ C - L\ M$. *M* must be slightly tilted to throw the image to the one side of the disc containing the cross wires.

Spherometer (q.v.).—The radius of the reflecting surface of a Cx. (or Cc.) glass mirror can be found approximately with the spherometer, but the results are uncertain on account of the coating. If, however, it has parallel surfaces and is thin, the radius of the front surface may be taken to be that of the second or reflecting surface. The latter is slightly shorter in a Cx. (and longer in a Cc.) mirror than the front surface measured.

Bench Focalisation of Thin Lenses.

Focalisation is *direct* when the unknown lens is measured by itself; it is *indirect* when another lens is combined with it, for the purpose of focalisation.

Indirect Focalisation.—The procedure is to combine, with the *unknown* lens D, another *known* lens D′; find the power D″ *of the combination*, and then deduct D′ from D″. Thus

$$D=D''-D' \quad \text{or} \quad \frac{1}{F}=\frac{1}{F''}-\frac{1}{F'}$$

where F″ and D″ are, respectively, the focal length and the power of the two lenses combined, F′ and D′ those of the added lens, and F and D are those of the unknown lens. The approximate power to be added can be found experimentally. For great accuracy it is better to divide this power between a pair of lenses, placing one on either side of the lens to be measured.

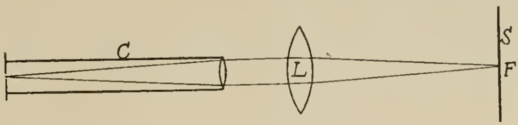

FIG. 345.

Cx. Sph.—The power of an unknown *Cx. sph.* lens can be obtained by measuring the distance between the lens and its principal focus. Set up the collimator *C* (Fig. 345), from which parallel light emerges, and in front of, and near to it, place the unknown lens *L*. On the other side of *L* place the screen *S*, and move the latter to and fro until the image *F* of the collimator aperture is sharpest possible; then *L F* is the principal focal length; the image is a point. This method serves for Cx. lenses between, say, 2D and 10D; if weaker or stronger *indirect* focalisation is to be preferred or becomes necessary.

$$\text{If } F=30 \text{ cm., } D=\frac{100}{30}=3{\cdot}25.$$

Cx. Cyl.—If the lens is a *Cx. Cyl.* the procedure is the same, but the image on the screen is, for the *plano-Cyl.*, a line parallel to the axis. With the *Sph.-Cyl.*, there are two lines at different distances, and at right angles to each other. The more distant one is at the focal distance of the sph., the nearer one is at the focal distance of the united powers of the sph. and cyl.; and is parallel to the cyl. axis. By finding these two lines, and measuring the distance between the lens and the screen for each, the focal length and powers of the two principal meridians of the lens can be learnt.

If F=16 cm., the lens is a +6·25 D Cyl.

If $F_1=16$ cm. and $F_2=30$ cm., $D_1=6{\cdot}25$ and $D_2=3{\cdot}25$—that is, +3·25 D Sph. ⊂ +3 D. Cyl.

Weak Cx. Lens.—The image formed by a weak Cx. lens is large and difficult to decide *where sharp*; also it may be so distant as to be beyond the limits of the bench. Therefore, whether Sph. or Cyl. or Sph.-Cyl., an additional Cx. lens of, say, 2 D or 3 D should be employed for its focalisation.

The added lens $D' = +3$; the combination D'' has $F = 27$ cm.

Then $$D = D'' - D' = \frac{100}{27} = 3{\cdot}75 - 3 = +{\cdot}75.$$

$D' = +3$ and $D_1'' = 25$ cm. and $D_2'' = 30$ cm.

Then $$D_1 = 4 - 3 = +1 \quad \text{and} \quad D_2 = 3{\cdot}25 - 3 = +{\cdot}25.$$

Strong Cx. Lens.—If F is very short, the exact distance is hard to determine with accuracy; for instance, whether $F = 2$ in. or $2\frac{1}{8}$ in.; but if the lens be focalised with, say, a 3 in. Cc., the difference between the one and the other is then about 1 in. Therefore, for its measurement, a strong Cx. lens should be combined with a Cc. lens of sufficient power to lengthen the focal distance to a reasonable extent.

The added lens $D' = -13$; the combination D'' has $F = 20$ cm.

Then $$D = D'' - D' = \frac{100}{20} = 5 - (-13) = +18.$$

Cc. Sph.—Since a Cc. lens does not form a real image, it must be combined, for its indirect focalisation, with a stronger Cx.-Sph.; for preference one which is about 3 or 4 D stronger.

The added lens $D' = +8$; the combination has $F = 40$ cm.

Then $$D = D'' - D' = \frac{100}{40} = 2{\cdot}5 - 8 = -5{\cdot}5.$$

Cc. Cyls.—With a negative cyl. a Cx.-sph., of sufficient power, must be added to render the combination positive so that the two principal powers may be found.

$D' = +8$; the combination has $F_1 = 12{\cdot}5$ cm. and $F_2 = 16$ cm.

Then $$D_2 = \frac{100}{12{\cdot}5} - 8 = 0 \quad \text{and} \quad D_1 = \frac{100}{16} - 8 = -1{\cdot}75.$$

$D' = +8$; the combination has $F_1 = 16$ cm. and $F_2 = 30$ cm.

$$\text{Then } D_1 = \frac{100}{16} - 8 = -1{\cdot}75 \text{ and } D_2 = \frac{100}{30} - 8 = -4{\cdot}75.$$

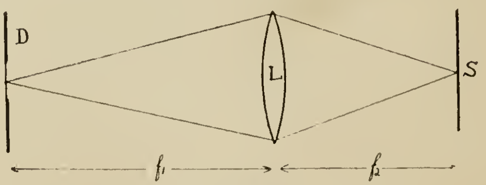

Fig. 346.

Conjugate Focalisation.—Instead of finding on the optical bench the principal focal distance F, it is often more convenient to find the power of a lens from a pair of real conjugate foci. The latter may be used to check the former and two or three pairs of conjugate distance can be found to check one another.

The *object* is the cross-wires D (Fig. 346) placed, in front of the lamp, at zero of the scale of the optical bench. Alternatively the *object* may be the small aperture of the collimator, the latter being reversed and placed with the lens towards the lamp of the bench. The aperture constitutes a small brilliant source from which the light diverges.

In Fig. 346, *D* is the cross-wires and *L* is a Cx. lens placed at a reasonable distance from it, so that a real conjugate image may be formed on the screen *S*. If the distance of the object from the lens be f_1, and the distance of its image on the opposite side be f_2, then the focal power of the lens is

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}.$$

It is, however, much more convenient to use dioptric measure—that is, the distances f_1 and f_2 are converted into diopters so that

$$D=d_1+d_2.$$

Thus if $f_1=25$ cm. and $f_2=20$ cm., instead of calculating that

$$\frac{1}{F}=\frac{1}{20}+\frac{1}{25}$$

we write
$$D=d_1+d_2=\frac{100}{20}+\frac{100}{25}=5+4=9D.$$

This method can be employed *directly* with medium power Cx. lenses, the image on *S* being, when the aperture object is used—

A small circle if the lens is a Sph.

A line parallel to the axis if it is a Cyl.

Two lines at different distances if it is a Sph.-Cyl.

To use the cross-wires with a Cyl. lens it is *essential* that the principal meridians of the lens should be *exactly Hor. and Ver.* This is achieved by rotating the lens, in the carrier, until the confusion disc of light formed on the screen is *exactly Hor. or Ver.* Then *one* of the cross-wires will be seen sharply at F_1, and the *other* at F_2. If the principal meridians do not correspond to the cross-wires no definite images are obtained with a Cyl. lens.

Indirect conjugate focalisation is necessary for a strong Cx. lens, adding a Cc. for its measurement; also for a weak Cx. or for a Cc. lens, adding a suitable Cx. lens.

Symmetrical Planes (Donders).—This method depends on the principle that when image and object are identical in size, the distance of each from the lens is 2 F, and the total distance between them is four times the focal length of a thin lens. It is a special case of *conjugate focalisation.*

The lens is placed between *D* and *S*, which are moved towards or away from the lens until the image on the screen is sharp and of the *same size as the aperture of D.* The experiment is made more accurate if the screen is scaled. If the lens is weak it should be placed between a *pair* of Cx. lenses, if very strong between a pair of Cc. lenses, in order to obtain the symmetrical conjugate foci. The calculation is as given in *indirect focalisation.*

It should be remembered that 4 F is the shortest possible distance between an object and its real image.

Notes on Focalisation.—To focalise *a periscopic* Cx. lens, the distance from the lens to the screen should be taken first with the one face, and then with the other, turned towards the source of light. The mean of the two distances is the true F measured from the optical centre. The distance between the symmetrical planes divided by 4 gives F of a thin periscopic Cx. lens.

With ordinary periscopic spectacle lenses, however, the distance of F, from the lens itself, is sufficiently exact in practice.

Instead of the collimator, any distant bright source, as a window or artificial light, can be employed for fairly strong Cx. sphericals, but this is uncertain for weak or very strong lenses, and practically useless for cyls.

On the ordinary optical bench the easiest conjugates to measure are those of about 3 to 5 D.

Sometimes the one power of a Sph.-Cyl. is more easily measured *without* and the other *with* an added lens.

In *conjugate focalisation* the one conjugate d_1 should be selected as a whole number—that is to say, the lens should be placed at, say, 33 cm. or 25 cm. from the object, thus making $d_1 = 3$ D or 4 D respectively. The lens should not be placed so that d_1 has a fractional power as it would have if it were placed at 30 cm. or 22 cm. The procedure is as in the following example.

The lens being weak an added + 7 D is employed, and the lens is placed 25 cm. from the cross-wire.

(*a*) $d_1 = 4$ and d_2 is at 15 cm. $= 6{\cdot}5$ D.

$$D_1 = 4 + 6{\cdot}5 = 10{\cdot}5 - 7 = +3{\cdot}5 \text{ D.}$$

(*b*) $d_1 = 4$ and d_2 is at 45 cm. $= 2{\cdot}25$ D.

$$D_2 = 4 + 2{\cdot}25 = 6{\cdot}25 - 7 = -{\cdot}75 \text{ D.}$$

(*c*) The combination is $-{\cdot}75$ D $\supset$ $+3{\cdot}5$ D which, by transposition, is—

$$-{\cdot}75 \text{ D Sph.} \supset +4{\cdot}25 \text{ D Cyl.}$$

or

$$+3{\cdot}5 \text{ D Sph.} \supset -4{\cdot}25 \text{ D Cyl.}$$

Alternatively D_2 is the Sph. and $D_1 - D_2$ is the Cyl.

Thus $+6{\cdot}25$ D. Sph. $\supset$ $(+10{\cdot}5 - 6{\cdot}25 =)$ $+4{\cdot}25$ D. Cyl., and from the Sph. the added lens is subtracted. Thus the Sph. $= 6{\cdot}25 - 7 = -{\cdot}75$ D.

Separation Method—Cc. Lens.—This is an optical bench measurement and is very useful if a strong Cc. has to be measured, and there is only a weak Cx. Sph. available. Light from the collimator is refracted by a Cx.

lens L (Fig. 347) to come to a focus on a screen at F. The lens should be shifted about so that F is at some definite position, say the 100 cm. mark on the bench; the screen is then moved some distance back. In the cone of convergent light $A\ F\ B$ the convergence is, at any point C, equal to the dioptral

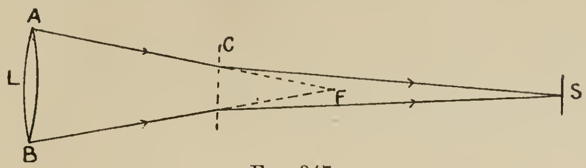

FIG. 347.

distance $C\ F$. Therefore if another lens is placed at C it is as if it were combined with a Cx. lens of $F = C\ F$. Let C be 8 cm. from F, then the convergence equals 12·5 D; if, now, a Cc. lens be placed at C, the light is rendered less convergent and the screen is moved about until the image is sharp thereon at, say, 114 cm. Then $C\ S = 22$ cm. $= 4·5$ D. Therefore the Cc. lens is, expressed in diopters,

$$\text{D} = \text{C S} - \text{C F} = 4·5 - 12·5 = -8 \text{ D}.$$

The cross-wires or the luminous point can be used instead of parallel light. This method serves also for Cyl. lenses, the image being a line or two lines as the case may be.

The distance $C\ S$ can be fixed and screen and lens moved about until the image is sharp but, of course, the selected distance must be appropriate. The combination of separated Cx. and Cc. lenses causes the image on S to be large so that it is difficult to decide when it is sharp.

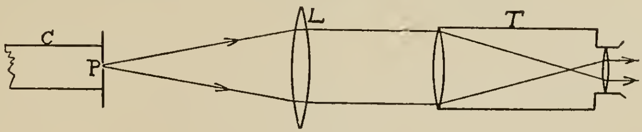

FIG. 348.

Telescope Tests.—More accurate results can be obtained with lenses if the telescope be employed in their focalisation. This is really the reverse of the usual procedure, as will be seen from Fig. 348. The collimator C is reversed, so that its lens faces the lamp and the pinhole P is away from it. The telescope is adjusted for parallel light by pulling the eyepiece well out, and gradually pushing it in, until some *distant* object is seen sharply through it; the eyepiece is then fixed and the telescope T replaced on the bench. The lens to be measured is placed in a clip between P and T and moved to and fro until the image of P is seen sharply through T. Then the distance $L\ P$, from pinhole to lens, will be the focal length of the lens, since only parallel light can have emerged from L to enter the telescope and give rise to a sharp image therein. With a cyl. the image will be a line; with a sph.-

cyl. there will be two line images at different distances. As in other tests, a known sph. must be added, if the unknown lens is too strong, too weak, is negative, or the difference between the principal powers insufficiently marked to give accurate results. The smaller the pinhole used in this experiment the sharper will be the lines obtained.

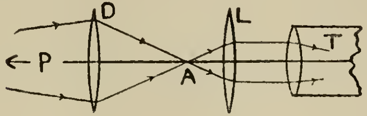

FIG. 349.

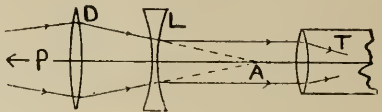

FIG. 350.

Another method. Light from the pinhole P is brought to a focus at A by D, a Cx. lens. A is marked. The telescope T adjusted for ∞ is placed a convenient distance behind A. The lens L to be measured is then introduced between T and D and moved about until the image of P is seen sharply. Then the distance $L\ A$ is F of the lens L which, if Cx., must be placed between the marked point A and T (Fig. 349), and if Cc. between A and D (Fig. 350). This method is good only for fairly strong lenses.

For plane surfaces the telescope is adjusted for infinity. A beam of light rendered parallel by a collimator is allowed to fall obliquely on the surface to be tested and is, after reflection, received in the telescope. If now, on looking through the telescope, the image seen of the source of light is sharp, the surface is a plane. If the surface is Cx., the eyepiece of the telescope must be pulled out, and if Cc., pushed in, in order to get a sharp image. If the surface is irregular, a sharp image cannot be obtained at any spot. The presence of *astigmatism*, whereby one portion of the image is better defined than the other, is the surest proof of convexity or concavity of a surface.

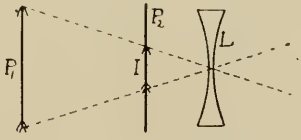

FIG. 351.

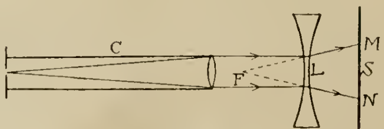

FIG. 352.

Focal Length of Cc. Lenses.

Parallax Method.—This is similar to the method for Cx. mirrors, except that object and image are on the *same* side of the lens, while the observer is on the *opposite* side. A pin P_1 (Fig. 351) is set up, and its virtual image I is observed through the lens. A second *locating* longer pin P_2 is now taken and moved to and fro until, on moving the head, there is an absence of parallax between them, P_2, seen above the lens, apparently coinciding with I seen through the lens. Then, if P_1 to L be f_1, and P_2 to L be f_2, the latter being negative, $1/F = 1/f_1 + (-1/f_2)$.

Locating the virtual image with a Cc. lens is more difficult and confusing than with a Cx. mirror because the observer sees two objects and two images. It should be remembered that the *more distant image must be made to coincide with the nearer object pin.* The pin P_1 seen *above* the lens, and the image of P_2 seen *through* it, must be ignored.

Projection Method.—This is similar to the projection method for Cx. mirrors. A parallel beam from C (Fig. 352) is allowed to fall on the unknown Cc. lens, and is diverged by the latter as if proceeding from F. If now S be moved back until the luminous area exactly fills the marked circle $M\ N$—which is twice the diameter of C—then the focal length of the lens is equal to $L\ S$, the distance of lens to screen.

This method can be utilised for a sph.-cyl., the luminous area being twice the diameter of the lens, in each principal meridian measured separately.

Reflection Method.—The Cc. surfaces of a negative lens may be employed as positive mirrors for measuring their radii of curvature. The object is the cross-wires, and its image is reflected back to the disc containing them, so that the distance of lens to disc is equal to the radius of curvature. Each surface should be calculated separately since the lens may not be a double Cc. sph. The refractive F of a lens surface is approximately $= 2r$.

If parallel light is employed we get F by reflection *for each surface*, and it is $\frac{1}{4}$ of F by refraction. For example, if with parallel light F is found to be 4″ for the one surface, and 8″ for the other, the lens is $1/16 + 1/32 = 1/10$ nearly.

This method can be employed for a Cc.-cyl. surface, the projected real image being a line.

F of Thick Lenses and Combinations.

The position of the image, formed by a Cx. lens., is more accurately determined by using as the screen a thin transparent glass plate, having a small dark spot on its front surface. The real image is formed on the screen and viewed from behind. The image and spot are in the same plane if, on moving the head sideways, there is absence of parallax. Adjustment to this position is obtained by moving the screen to and fro. The test is improved by using a fixed magnifying glass.

Thin Lens Method—Positive Combination.—If a single thin lens is found which gives on a screen an image equal in size to that formed by a combination, the focal distance of the former is that of the latter; also the place at which the single lens is situated determines the second equivalent point of the combination. If the latter is turned so that the original back lens faces the light, the spot at which the single thin lens must be placed in order to give an image similar to that of the combination fixes the position of the first equivalent point.

Thin Lens Method—Negative Combination.—Put up a, say, 6 inch Cx. lens and, at some 10 inches behind it, a screen. The unknown Cc. combination is placed some short distance in front of the Cx. and moved about until a sharp image of a bright distant object, as a window, is formed on the screen. The size of the image is carefully marked on the screen, the Cc. combination is removed, and a known thin Cc. is found that gives an image of equal size. F of the combination equals F of the thin Cc.

Symmetrical Plane Method — Positive Combination. — To find experimentally the equivalent focal length of a thick Cx. lens or combination, it is necessary to locate the equivalent planes, since the focal distances are the distances between these planes and the principal foci.

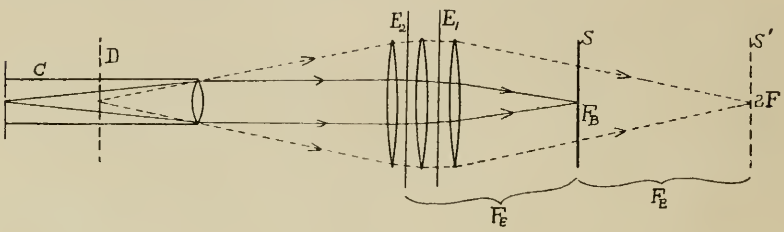

Fig. 353.

Let the system of lenses be suitably mounted (Fig. 353). Parallel light from the collimator C is refracted by it, and the principal focus F_B is formed on the screen S, *whose position on the bench is noted.* Now substitute the cross-wires D for the collimator and move them about until its image formed on S, drawn back to S', is the same size. Then S' is the second symmetrical plane, and is therefore at $2F$ *from some plane—the 2nd equivalent plane—not yet located.* But the distance between $2F$ and F_B, i.e. *the difference in the bench readings of the position of S' and S*, is F, the equivalent focal length. Therefore measuring from S' towards the lens a distance equal to $2F$, the second equivalent plane E_2 is located. Then a similar measurement of $2F$ from the cross-wires determines the position of E_1, the first equivalent plane. In some combinations E_1 and E_2 are crossed, as illustrated in Fig. 353.

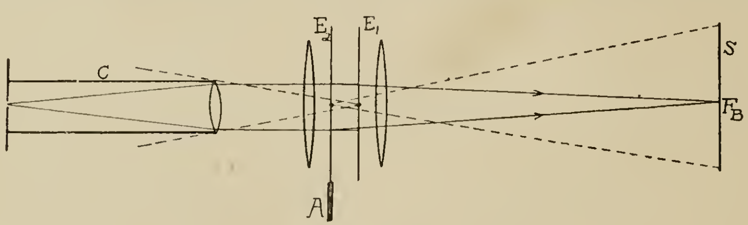

Fig. 354.

Rotation Method—Positive Combination.—This is, perhaps, the quickest and is a most accurate method for finding the equivalent focal length and equivalent points of a combination, such as a photographic objective.

The secondary axes govern the position and size of the image, and since they pass through the second equivalent point, if the combination be rotated horizontally around a vertical axis immediately beneath E_2 *the image from originally parallel light will remain stationary.* If the system be rotated around any point other than E_2, the image will move. Light from the collimator (Fig. 354) falls on the lens so that the screen S locates F_B. The combination is mounted in a special carrier capable of longitudinal adjustment from and towards S, and also rotation round the vertical axis A. Then, by lateral swing and longitudinal movement of the lens, a position is found where the image on S is motionless. Adjust S to secure the sharpest possible image; then the distance from A to S on the bench is the equivalent focal length, and the prolongation of A upwards locates the second equivalent plane E_2. By reversing the combination in the carrier E_1 can be similarly found. It may be necessary in some cases to rotate the combination or lens on a point outside, as with periscopic single lenses.

This method is especially suitable for photographic objectives having a wide angle of sharp definition; with uncorrected lenses, however, only a small rotation is possible before the image becomes confused from oblique aberration.

Rotation Method—Negative Combination.—Rotation also serves for a negative combination, but in this case the *virtual* image formed of originally parallel light must be observed. The combination is placed between the telescope and the collimator. Focus carefully on the virtual image formed by the lens by drawing out the eyepiece, and get the image on the vertical cross-wire of the telescope. Rotate the combination as described for a positive combination until the image seen through the telescope is stationary and sharp. Remove the combination from the carrier and bring up some object until its image is also seen clearly in the telescope. Then the distance of this object to the standard which originally held the Cc. system is the focal length of the latter.

Approximate F of Small Strong Cx. Lenses.—Put up a scaled screen at 10 inches from the lens and on the other side a small object of known diameter. Adjust this so that its real image is formed on the screen, and note the magnification. Then F of the lens=10/M. Thus if M=5, F=2″; if M=40, F=1/4, etc.

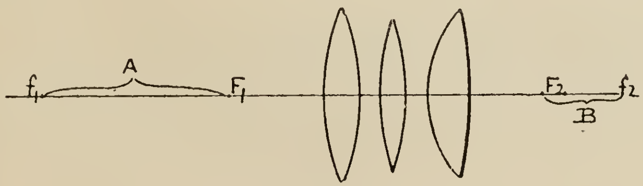

FIG. 355.

Conjugate Method—Positive Combination.—Since $F^2=A\ B$, where A and B are the distance of O and I beyond F, respectively, on the one and the other side of the lens system, this enables the focal length to be experimentally determined. Thus focus parallel light on the screen, and mark F_2 (Fig. 355); repeat the process on the other side and similarly mark F_1. Then place the cross-wires at a convenient distance f_1 and its image is at f_2; measure $F_2 f_2 = B$ also $f_1 F_1 = A$, then $\mathrm{F} = \sqrt{\mathrm{A\,B}}$.

This should be checked by finding a second pair of conjugates A′ and B′.

Laurance's Method—Positive Combination.—Focus sharply for parallel light to locate the principal focus F_B; then move the screen back to f_2 (Fig. 356) which is n inches from F (say 1/3 of its focal length). Move the cross

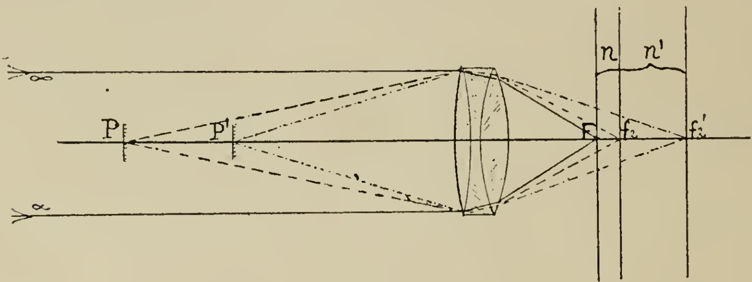

Fig. 356.

wires in front of the lens until its image is sharply focussed on the screen at f_2 and mark its position P. Again withdraw the screen to f_2', which is exactly one (or more) inches farther back, so that it is now n' inches from F; shift the wires to P' until the image is again in focus at f_2'. Measure the distance PP' through which the object has been moved; call it d.

Then $$\mathrm{F} = \sqrt{\frac{d\,n\,n'}{n' - n}}$$

If n' be exactly 1 in. longer than n, then $n' - n = 1$, and therefore need not be regarded. Further, if $n = 1$ and $n' = 2$, the calculation simplifies to $\mathrm{F} = \sqrt{2\,d}$. This is the true focal length, since it is independent of the position of the equivalent planes, which can be found by measuring the focal distance backwards from the principal focus. Thus supposing $d = 3{\cdot}5''$, $\mathrm{F} = \sqrt{2 \times 3{\cdot}5} = 2{\cdot}65''$ approx.

The Gauss Method for a Positive Combination.—Let u be the angle subtended at the lens by any two distant objects (Fig. 357) A and B, one of which B is situated on the principal axis. This angle can be measured by means of a theodolite, and therefore the angle u' subtended by the image $B'\ A'$ at the second equivalent point C is also known, since it is equal to u. Then

$$\tan u' = h'/\mathrm{C\,B'} \qquad \text{or} \qquad \mathrm{C\,B'} = \mathrm{F} = h'/\tan u'$$

The image h' can be measured directly on the screen. Since this method is independent of the position of the equivalent planes, these are not shown in the figure, C being the 2nd equivalent point. If $u=45°$ (Fig. 358), then $\tan 45=1$, and $F=h'$, i.e. the size of the image $B'\ A'$ is equal to the focal length of the lens.

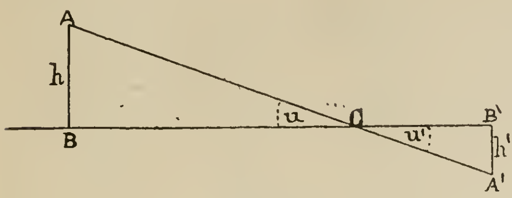

FIG. 357.

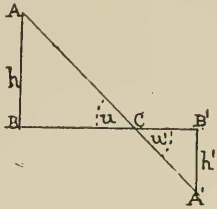

FIG. 358.

Dallmeyer's Method for a Negative Thick Lens.—Take an achromatic positive lens and focus the image of the cross-wires on a screen; measure the size of the image formed and let it be m (Fig. 359). Place the negative lens,

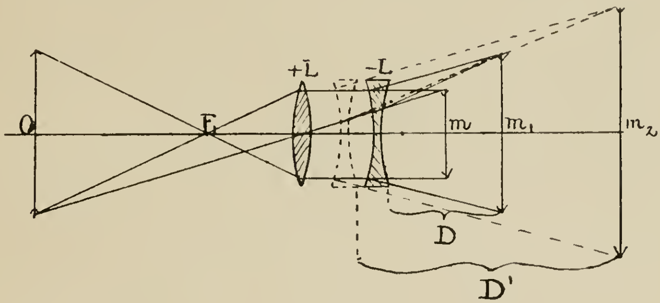

FIG. 359.

whose focus is to be found, a short distance within the convergent beam of the positive lens, *i.e.* between it and the screen. Focus the image formed by the combination and measure its distance D from the back surface or flange of the negative lens; measure the size m_1 of the image formed. The size of m_1 compared with the size of the image produced by the positive lens alone is $M=m_1/m$.

Now move the negative a little nearer the positive lens (which latter must be kept in a fixed position) and focus a second time on the screen; measure the distance D' of the screen from the back of the negative lens or its flange. The size of the image m_2 compared with the size of m is $M'=m_2/m$. Then the focal length F of the negative lens is

$$F=\frac{D'-D}{M'-M}$$

If M can be made equal to 2, and M' to 3, then $F=D'-D$.

This equation is independent of the position of the equivalent planes, and therefore will hold true for any negative combination of lenses.

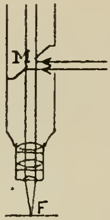

Fig. 360.

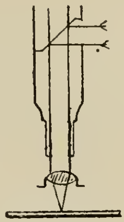

Fig. 361.

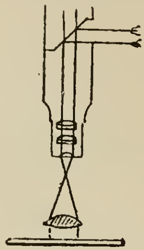

Fig. 362.

Fig. 363.

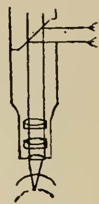

Fig. 364.

Small Strong Positive Lens—Dr. C. V. Drysdale's Method.—The focal length of small lenses can be found by means of a microscope which has a portion removed from the tube so that light, from a distant source placed at the side, enters the aperture and falls on a transparent reflecting surface *M* inclined at 45°, so that part of the light is transmitted down the tube as shown in Fig. 360. The eyepiece is arranged for parallel light by separation of the components, the adjustment being made by turning the reflector so that the light admitted is reflected towards the eyepiece.

Employing no objective in the microscope and a plane mirror behind the lens to be tested, this mirror is moved to and fro until the image is sharp in the field of the eyepiece. The mirror is then at the focal length of the lens, the light converged by the latter being reflected back and refracted again as parallel. The lower focal point is thus found, as in Fig. 361.

Replacing the objective (Fig. 362), the lens is moved farther back to such a position that it is at its focal length behind the focal point of the objective. Then the light converged by the objective and refracted by the lens is parallel, and falling on the mirror, is again reflected as parallel, to be refracted by the lens to meet at the focal point of the objective, by which it is again refracted as parallel light. The image is sharp in the field of the eyepiece, and the upper focal point is found.

The two focal points being marked, the back surface focal lengths are obtained. If, now, the mirror be moved a given distance A downwards, and the objective moved upwards by a distance B until the image is clear, we obtain the equivalent focal length from $F_E = \sqrt{A\,B}$, where A and B are the distances of the conjugates beyond F_B on each side.

Dr. Drysdale has also made an experimental microscope in which the lens under examination can be oscillated around its second equivalent point. This enables the focal length to be determined, and further, by this means, aberrations can be easily detected.

Curved Surfaces.

Dr. C. V. Drysdale's Method.—Dr. Drysdale's method of determining the radius of curvature of small surfaces as follows: Part of the light, received in the tube of the microscope, as described in the last article, is reflected downwards towards, and through, the objective, by which it is brought to a focus at *F* as in Fig. 360. If, then, the reflecting surface of a mirror or lens is placed at the focus of the objective, the light is reflected back and seen by the observer in the field of the eyepiece, as an image of the source. This position or distance of the objective from the reflecting surface is then marked on some part of the microscope. The tube of the latter must be racked upwards, if the surface examined is Cc. (Fig. 363) or downwards if Cx. (Fig. 364), until the image can again be clearly seen. The focus of the objective now coincides with the centre of curvature of the reflecting surface, for the light passing through the objective is incident on the reflecting surface normally and is reflected back along its original course. The distance between the first and second positions of the microscope objective, when the image is clearly seen, is the radius of curvature. The curvature of any zone of the surface can be obtained by using a suitable diaphragm.

A later improvement made by Dr. Drysdale on the arrangement of the instrument used in the above method consists of an illuminator immediately above the microscope objective and a lens above the illuminator, which serves as the objective of the telescope and obviates the necessity of separating the eyepiece lenses.

Telescope Method.—If the object be sufficiently distant compared with that of the image, as is the case with mirrors of small radius, when the object is, say, a metre distant, then the radius r of the curved surface bears to the distance of the image from the pole of the mirror, the relationship of $r=2F$, where F is the focal distance and the distance of the image. Let h_1 and h_2 be the sizes of, respectively, the object and the image, and f_1 the distance of the object from the mirror, while f_2 is its focal length. Then

$$f_2=f_1h_2/h_1, \text{ and } r=2f_2$$

The radius of curvature, of strongly curved lenses and mirrors, whether Cx. or Cc., can be measured by employing an instrument like the ophthalmometer. The distance between the two objects being known, that between the two images can be measured by a micrometer scale placed in the focus of the eyepiece of the telescope. f_1 is the distance of the objects from the curved surface, h_1 is the distance between them, h_2 is here the distance between the two images, as measured by the micrometer, and F is the distance between the objective and the micrometer. The relative size of the image formed at f_2 and that formed at the micrometer is as f_1: F, so that the above formula must be multiplied by f_1/F, and we then obtain

$$r=\frac{2f_1{}^2h_2}{h_1F}$$

Reflection Method.—On the Cx. surface (Fig. 365) to be measured, mark two small spots *A* and *B*, a convenient and known distance apart. At some convenient distance *D E* arrange two small white objects *P* and *Q*, so that

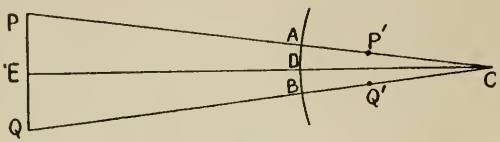

Fig. 365.

on sighting directly over each their virtual images *P′* and *Q′* are aligned, respectively, with *A* and *B*. Then the radius of the surface *C A*, *C B*, can be calculated from the following—

$$\frac{\mathrm{DC}}{\mathrm{EC}}=\frac{\mathrm{DC}}{\mathrm{ED}+\mathrm{DC}}=\frac{\mathrm{AB}}{\mathrm{PQ}} \quad \text{or} \quad \mathrm{DC}=\frac{\mathrm{ED}\times\mathrm{AB}}{\mathrm{PQ}-\mathrm{AB}}$$

Gauges.—The radius of small convex lenses can be determined by accurately made gauges, or more generally by glass cups of known curvature, usually known as test-plates. When the curvature of the lens does not correspond to that of the cup, interference rings are exhibited, while these are not shown if the two curves exactly correspond; or they are faint, and of slight brilliancy of colour, if the curves nearly correspond. A total absence of colour is, however, in practice, rarely found.

The Refractive Index of Solids.

The Spectrometer Method is the most accurate, that of the microscope being fairly exact; the others are more or less approximate.

Spectrometer Method.—P the principal and *d* the deviating angles being measured, as described in Chapter XX.,

$$\mu=\frac{\sin\{(\mathrm{P}+d)/2\}}{\sin(\mathrm{P}/2)}$$

d must be the minimum angle of deviation. If P is not too large and the light falls normally on to one surface, the formula becomes simplified to $\mu=\sin(\mathrm{P}+d)/\sin\mathrm{P}$. If white light be used a spectrum will be formed, but the index for any particular colour can be obtained by bringing the cross-wire of the telescope over that particular colour. The mean index is calculated from the yellow (*D* line) and the mean dispersion from the difference between the indices of blue-violet (*F* line) and orange-red (*C* line).

For example, a prism whose principal angle P is 59° 57′ and whose angle of minimum deviation *d* for the *D* line is 40° 21′, then

$$\frac{\mathrm{P}+d}{2}=\frac{59^\circ\,57'+40^\circ\,21'}{2}=50^\circ\,9' \quad \text{and} \quad \frac{\mathrm{P}}{2}=29^\circ\,58'$$

so that

$$\mu=\frac{\sin 50^\circ\,9'}{\sin 29^\circ\,58'}=\frac{\cdot7677}{\cdot4995}=1{\cdot}536.$$

Microscope Method.—With a thin plate and a low-power microscope, a fine line is focussed and the plate is then placed above the line. Now the microscope must be raised in order that the line be clearly seen, since the rays proceeding from it are divergent as if from a point nearer to the objective. The distance that the microscope objective has to be raised equals the distance between the real position of the line and its apparent position when seen through the plate. Let t be the thickness of the glass, and d the distance that the objective has to be raised; then $\mu = t/(t-d)$. The necessary measurements can be made fairly accurately by means of a mm. scale, some point on the tube being taken as the index pointer. A fixed scale, with a vernier, attached to the microscope, or the scale on the millhead of the fine adjusting screw, gives more exact readings. Thus, if the thickness of the plate be 1 mm. and the object-glass has to be raised ·38 mm., $\mu = 1/{\cdot}62 = 1{\cdot}61$.

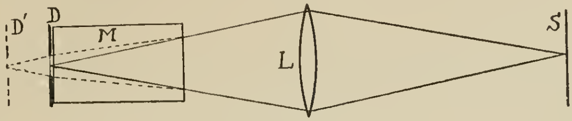

Fig. 366.

Bench Method.—The medium being in the form of a block with two parallel surfaces. Take any Cx. lens L (Fig. 366) of convenient strength and project an image of the cross-wires D on to the screen S, such that S and D occupy, approximately, symmetrical planes; note the position of D. Then introduce the medium M, whose index is to be found, between L and D, when the image on S will be found out of focus owing to the apparent vertical displacement of D. In order again to secure a sharp focus on S the disc must be drawn back to some point D' whose position is also noted. Then, if t be the thickness of the medium and d the distance between D and D'—the apparent displacement—we have $\mu = t/(t-d)$. The image on S must be well defined, and therefore an achromatic lens should be used.

Block Method.—The refractive index of a transparent body, such as glass, can be roughly found as follows:—Make a dot d (Fig. 367) on the back of the block of glass; its image is d', nearer to S; then find such a position for a pin P, placed vertically in front of the glass, that on moving one's head from side to side the virtual image P' of the pin, by reflection from the front surface, appears to be behind that surface at such a distance that, owing to absence of parallax, P' coincides with d'. Then the apparent thickness of the glass is $P'S = PS$, and $\mu = dS/PS$.

Plate Method.—A parallel plate (Fig. 368) of the medium, say glass, is placed on a sheet of white paper on a drawing-board or other smooth surface. A pin P_1 is then stuck in any position and a second pin P_2 is placed close to the plate and sufficiently to the left of P_1 so that a line P_1P_2 makes a fairly large angle i with the normal $N N'$. Now observe, through the plate, the pins

P_1P_2, which will appear displaced towards the right. Stick two more pins P_3 and P_4 in the board such that all four appear in one straight line. Draw the trace of the plate with a fine pencil, remove it and the pins, and with a compass, with P_2 as centre describe any circle—the larger the better, provided

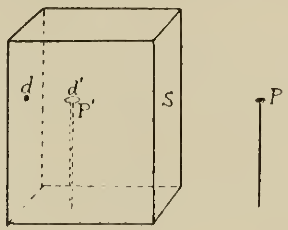

FIG. 367.

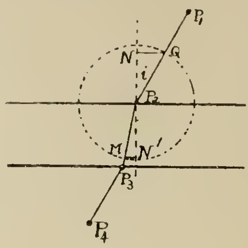

FIG. 368.

it falls within P_3. Where this cuts the course of the ray in Q and M drop the perpendiculars $Q\ N$ and $M\ N'$, which are the sines of i and r respectively; then $\mu = Q\ N/M\ N'$. This method is only approximate unless carefully done and therefore three or four readings for different values of i should be taken and the mean result extracted.

Critical Angle.—Should it be possible to measure C, the critical angle of a medium—generally this is neither easy nor accurate—$\mu = 1/\sin C$. Such a method might be suitable for a substance like butter for which other methods are not. A special apparatus—a refractometer—is required.

Polarising Angle.—This may be the most convenient method for an opaque body, for $\mu = \tan p$.

The polarising angle can be found by an arrangement of a small source of light and a piece of tourmaline from a pebble tester, the axis being vertical. Then on raising or lowering equally both source and tourmaline from the surface of the medium to be tested a position will be found where the reflected image of the source is cut off. Measure the distance d from the point of reflection on the surface of the medium to the point on the surface immediately beneath the tourmaline, also the height h of the latter above the surface. Then $\mu = \tan p = d/h$.

Prism Method.—If a spectrometer is not available, the value of P and d can be found roughly as follows: Place the prism (Fig. 369) on the drawing-board with the apex towards a window; look into the surface $A\,B$, which acts as a plane mirror, and select the image of a vertical window bar; get the image *as near as possible to the apex* A and put the pin P_1 in position so that it is in line with A. Do the same with the other surface $A\,C$. Make a trace of the prism, remove it and the pins; then the angle formed by the lines P_1A and $P_2\,A$ (i.e. $P_1\,A\,P_2$) is twice P, the principal angle. $P_1\,A\,P_2$ is measured with a protractor.

To find the deviating angle d, erect, in any convenient position, two pins P_1 and P_2 (Fig. 370), place the prism with one side in contact with P_2; then on looking through the prism somewhere in the direction V, the pins will appear displaced towards A. Secure minimum deviation by rotating

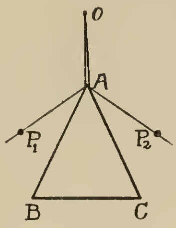

FIG. 369.

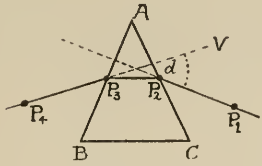

FIG. 370.

the prism both ways, and finally erect two other pins P_3 and P_4 such that all four appear in one line. Then, by making the necessary tracings and connections with a fine-pointed pencil, the angle of minimum deviation d can be marked and measured on a protractor.

Lens Method.—If the medium be in the form of a thin lens, its F and radii can be measured, and μ calculated from the lens formula.

The μ of Metals.—By making exceedingly thin prisms of less than one minute of arc, Kundt successfully determined the refractive indices of a number of the metals. The results showed that silver, gold, copper, magnesium, and sodium have an index less than that of a vacuum, and this, no doubt, accounts for the absence of a polarising angle in these substances. The red rays in some cases were found to be more refracted than the blue, so that metals form good examples of anomalous dispersion. The refractive indices of the metals were found to be proportional to their electric conductivities, *i.e.* those metals which were good conductors have a low refractive index, and *vice versa*.

The Refractive Index of Liquids.

The μ of Liquids.—In general, the methods for solids can be employed for liquids, but the arrangements differ in some instances.

Spectrometer Method.—The liquid is placed in a hollow glass prism whose sides are thin and with quite parallel surfaces.

Microscope Method.—First focus the bottom of a small tank, and then its image when the liquid has been poured in. Finally focus the surface of the liquid, which should have some conspicuous dusk specks floating on it. The difference between the third and first readings gives the real depth, and that between the second and third the apparent depth.

The *critical* and *polarising* angle methods are the same. The *bench, block, plate,* and *prism* methods are also applicable if the liquid be enclosed in a suitable plane glass box or prism. For the *block* method (*q.v.*) the liquid can be in a bowl with d at the bottom and P held above the liquid.

Lens Method.—Take a small quantity of the liquid and place it between a thin plate of glass and a Cx. lens of known radius and focal length F_1; the liquid then forms a plano-Cc. lens. If now F of the combination be found, that of the Cc. F_2 can be learnt from $1/F_2 = 1/F - 1/F_1$. Its radius is also known, it being that of the Cx. lens, so μ can be calculated from $\mu - 1 = r/F$.

INDEX

Those numbers in heavy type indicate chapters or main articles on particular subjects. The subheadings to the articles serve as further guides.

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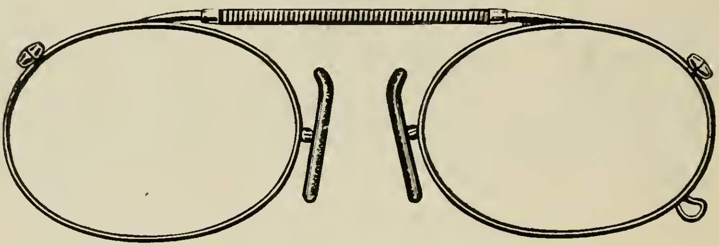

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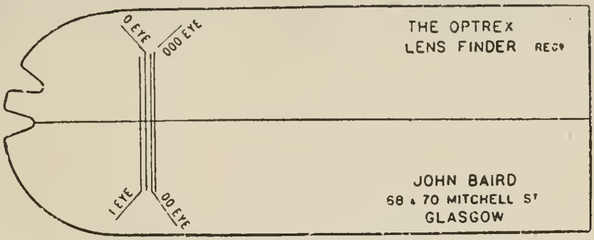

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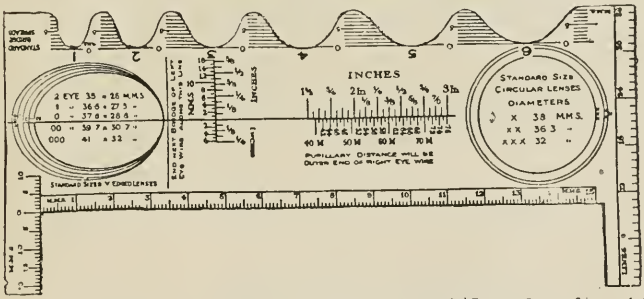

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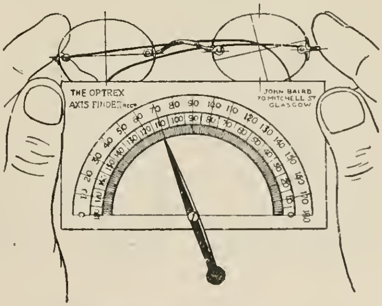

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