

# Many-Body Physics with Ultracold Gases

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This article reviews recent experimental and theoretical progress on many-body phenomena in dilute, ultracold gases. Its focus are effects beyond standard weak-coupling descriptions, like the Mott-Hubbard-transition in optical lattices, strongly interacting gases in one and two dimensions or lowest Landau level physics in quasi two-dimensional gases in fast rotation. Strong correlations in fermionic gases are discussed in optical lattices or near Feshbach resonances in the BCS-BEC crossover.

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A. Bose-Hubbard model	21	<b>I. INTRODUCTION</b>	
B. Superfluid-Mott-Insulator transition	22	The achievement of Bose-Einstein-Condensation (BEC) (Anderson <i>et al.</i> , 1995; Bradley <i>et al.</i> , 1995; Davis <i>et al.</i> , 1995) and of Fermi degeneracy (DeMarco and Jin, 1999; Schreck <i>et al.</i> , 2001; Truscott <i>et al.</i> , 2001) in ultracold, dilute gases has opened a new chapter in atomic and molecular physics in which the particle statistics and their interactions, rather than the study of single atoms or photons, are at center stage. For a number of years, a main focus in this field has been to explore the wealth of phenomena associated with the existence of coherent matter waves. Major examples include the observation of interference of two overlapping condensates (Andrews <i>et al.</i> , 1997), of long range phase coherence (Bloch <i>et al.</i> , 2000) or of quantized vortices and vortex lattices (Abo-Shaeer <i>et al.</i> , 2001; Madison <i>et al.</i> , 2000) and molecular condensates with bound pairs of fermions (Greiner <i>et al.</i> , 2003; Jochim <i>et al.</i> , 2003b; Zwerlein <i>et al.</i> , 2003b). Common to all of these phenomena is the existence of a coherent,	
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macroscopic matter wave in an interacting many-body system, a concept familiar from the classic area of superconductivity and superfluidity. It was the basic insight of Ginzburg and Landau (1950) that, quite independent of a detailed microscopic understanding, an effective description of the coherent many-body state is provided by a complex, macroscopic wave function  $\psi(\mathbf{x}) = |\psi(\mathbf{x})| \exp i\phi(\mathbf{x})$ . Its magnitude squared gives the superfluid density, while the phase  $\phi(\mathbf{x})$  determines the superfluid velocity via  $\mathbf{v}_s = \hbar/M \cdot \nabla\phi(\mathbf{x})$  (see the Appendix for a discussion of these concepts and their subtle connection with the microscopic criterion for BEC). As emphasized by Cummings and Johnston (1966) and by Langer (1968), this picture is similar to the description of laser light by a coherent state (Glauber, 1963). It applies both to the standard condensates of bosonic atoms and also to weakly bound fermion pairs which are the building blocks of the BCS-picture of superfluidity in Fermi systems. In contrast to conventional superfluids like  $^4\text{He}$  or superconductors, where the macroscopic wave function only provides a phenomenological description of the superfluid degrees of freedom, the situation in dilute gases is considerably simpler. In fact, as a result of the weak interactions, the dilute BEC's are essentially pure condensates sufficiently below the transition. The macroscopic wave function is thus directly connected with the microscopic degrees of freedom, providing a complete and quantitative description of both static and time-dependent phenomena in terms of a reversible, nonlinear Schrödinger-equation, the famous Gross-Pitaevskii equation (Gross, 1961; Pitaevskii, 1961). In dilute gases therefore, the many-body aspect of a BEC is reduced to an effective single-particle description, where interactions just give rise to an additional potential proportional to the local particle density. Adding small fluctuations around this zeroth-order picture leads to the well known Bogoliubov theory of weakly interacting Bose gases. Similar to the closely related BCS-superfluid of weakly interacting fermions, the many-body problem is then completely soluble in terms of a set of non-interacting quasiparticles. Dilute, ultracold gases provide a concrete realization of these basic models of many-body physics and many of their characteristic properties have indeed been verified quantitatively. Excellent reviews of this remarkably rich area of research have been given by Dalfovo *et al.* (1999) and by Leggett (2001) and - more recently - in the comprehensive books by Pethick and Smith (2002) and by Pitaevskii and Stringari (2003).

In the past several years, two major new developments have considerably enlarged the range of physics which is accessible with ultracold gases. They are associated with

- the ability to tune the interaction strength in cold gases by Feshbach resonances (Courteille *et al.*, 1998; Inouye *et al.*, 1998) and
- the possibility to change the dimensionality with optical potentials and, in particular, to generate

strong periodic potentials for cold atoms through optical lattices (Greiner *et al.*, 2002a).

Both developments, either individually or in combination, allow to enter a regime, in which the interactions even in extremely dilute gases can no longer be described by a picture based on non-interacting quasi-particles. The appearance of such phenomena is characteristic for the physics of strongly correlated systems. For a long time, this area of research was confined to the dense and strongly interacting quantum liquids of condensed matter or nuclear physics. By contrast, gases - almost by definition - were never thought to exhibit strong correlations.

The use of Feshbach resonances and optical potentials for exploring strong correlations in ultracold gases was crucially influenced by earlier ideas from theory. In particular, Stoof *et al.* (1996) suggested that Feshbach resonances in a degenerate gas of  $^6\text{Li}$ , which exhibits a tunable attractive interaction between two different hyperfine states, may be used to realize BCS-pairing of fermions in ultracold gases. A remarkable idea in a rather unusual direction in the context of atomic physics was the proposal by Jaksch *et al.* (1998) to realize a quantum phase transition from a superfluid to a Mott-insulating state by loading a BEC into an optical lattice and simply raising its depth. Further directions into the regime of strong correlations were opened with the suggestions by Olshanii (1998) and Petrov *et al.* (2000b), to realize a Tonks-Girardeau gas with BEC's confined in one dimension and by Wilkin and Gunn (2000) to explore quantum Hall effect physics in fast rotating gases.

Experimentally, the strong coupling regime in dilute gases was first reached by Cornish *et al.* (2000), using Feshbach resonances for bosonic atoms. Unfortunately, in this case, increasing the scattering length  $a$  leads to a strong decrease in the condensate lifetime due to three-body losses, whose rate on average varies like  $a^4$  (Fedichev *et al.*, 1996b; Petrov, 2004). A quite different approach to the regime of strong correlations, which does not suffer from problems with the condensate lifetime, was taken by Greiner *et al.* (2002a). Loading BEC's into an optical lattice, they observed a quantum phase transition from a superfluid to a Mott-insulating phase even in the standard regime where the average interparticle spacing is much larger than the scattering length. Subsequently, the strong confinement available with optical lattices made possible the achievement of low dimensional systems where new phases can emerge. In fact, the first example of a bosonic Luttinger liquid was obtained with the observation of a (Tonks-Girardeau) hard-core Bose gas in one dimension by Paredes *et al.* (2004) and Kinoshita *et al.* (2004). In two dimensions a Kosterlitz-Thouless crossover between a normal phase and one with quasi-long range order was observed by Hadzibabic *et al.* (2006). The physics of strongly interacting bosons in the lowest Landau level is accessible with fast rotating BEC's (Bretin *et al.*, 2004; Schweikhard *et al.*, 2004), where the vortex lattice is eventually predicted to melt by quantum

fluctuations. Using atoms like  $^{52}\text{Cr}$ , which have a larger permanent magnetic moment, BEC's with strong dipolar interactions have been realized by Griesmaier *et al.* (2005). In combination with Feshbach resonances, this opens the way to tune the nature and range of the interaction (Lahaye *et al.*, 2007), which might, for instance, be used to reach novel many-body states that are not accessible in the context of the fractional Quantum Hall Effect.

In Fermi gases, the Pauli-principle strongly suppresses three-body losses, whose rate in fact decreases with increasing values of the scattering length (Petrov *et al.*, 2004b). Feshbach resonances therefore allow to enter the strong coupling regime  $k_F|a| \gg 1$  in ultracold Fermi gases (Bourdel *et al.*, 2003; O'Hara *et al.*, 2002). In particular, there exist stable molecular states of weakly bound fermion pairs, in highly excited ro-vibrational states (Cubizolles *et al.*, 2003; Strecker *et al.*, 2003). The remarkable stability of fermions near Feshbach resonances allows to explore the crossover from a molecular BEC to a BCS-superfluid of weakly bound Cooper-pairs (Bartenstein *et al.*, 2004; Bourdel *et al.*, 2004; Regal *et al.*, 2004a; Zwierlein *et al.*, 2004). In particular, the presence of pairing due to many-body effects has been probed by spectroscopy of the gap (Chin *et al.*, 2004), or the closed channel fraction (Partridge *et al.*, 2005) while superfluidity has been verified by the observation of quantized vortices (Zwierlein *et al.*, 2005). Recently, these studies have been extended to Fermi gases with unequal densities for the spin-up and spin-down components (Partridge *et al.*, 2006; Zwierlein *et al.*, 2006), where pairing is suppressed by the mismatch of the respective Fermi energies.

Repulsive fermions in an optical lattice allow to realize an ideal and tunable version of the Hubbard model, a paradigm for the multitude of strong correlation problems in condensed matter physics. Experimentally, some of the basic properties of degenerate fermions in periodic potentials like the existence of a Fermi surface and the appearance of a band insulator at unit filling have been observed by Köhl *et al.* (2005a). While it is difficult to cool fermions to temperatures much below the bandwidth in a deep optical lattice these experiments give rise to the hope that eventually magnetically ordered or unconventional superconducting phases of the fermionic Hubbard model will be accessible with cold gases. The perfect control and tunability of the interactions in these systems provide a completely novel approach to study basic problems in many-body physics and, in particular, to enter regimes which have never been accessible in condensed matter or nuclear physics.

The present review aims to give an overview of this rapidly evolving field, covering both theoretical concepts and their experimental realization. It provides an introduction to the strong correlation aspects of cold gases, that is, phenomena which are not captured by weak-coupling descriptions like the Gross-Pitaevskii or Bogoliubov theory. The focus of this review is on examples

which have already been realized experimentally. Even within this limitation, however, the rapid development of the field in recent years makes it impossible to give a complete survey. In particular, important subjects like spinor gases, Bose-Fermi mixtures, quantum spin systems in optical lattices or dipolar gases will not be discussed here (see e.g. Lewenstein *et al.* (2007)). Also, applications of cold atoms in optical lattices for quantum information are omitted completely, for a recent introduction see Jaksch and Zoller (2005).

## A. Scattering of ultracold atoms

For an understanding of the interactions between neutral atoms, first at the two-body level, it is instructive to use a toy model (Gribakin and Flambaum, 1993), in which the van der Waals attraction at large distances is cutoff by a hard core at some distance  $r_c$  on the order of an atomic dimension. The resulting spherically symmetric potential

$$V(r) = \begin{cases} -C_6/r^6 & \text{if } r > r_c \\ \infty & \text{if } r \leq r_c \end{cases} \quad (1)$$

is, of course, not a realistic description of the short range interaction of atoms, however it captures the main features of scattering at low energies. The asymptotic behavior of the interaction potential is fixed by the van der Waals coefficient  $C_6$ . It defines a characteristic length

$$a_c = \left( \frac{2M_r C_6}{\hbar^2} \right)^{1/4} \quad (2)$$

at which the kinetic energy of the relative motion of two atoms with reduced mass  $M_r$  equals their interaction energy. For alkali atoms, this length is typically on the order of several nano-meters. It is much larger than the atomic scale  $r_c$  because alkalis are strongly polarizable, resulting in a large  $C_6$  coefficient. The attractive well of the van der Waals potential thus supports many bound states (of order 100 in  $^{87}\text{Rb}$ !). Their number  $N_b$  may be determined from the WKB-phase

$$\Phi = \int_{r_c}^{\infty} dr \sqrt{2M_r |V(r)|} / \hbar = a_c^2 / 2r_c^2 \gg 1 \quad (3)$$

at zero energy, via  $N_b = [\Phi/\pi + 1/8]$  where  $[\ ]$  means taking the integer part<sup>1</sup>. The number of bound states in this model therefore crucially depends on the precise value of the short range scale  $r_c$ . By contrast, the low energy scattering properties are essentially determined by the van

<sup>1</sup> This result follows from Eq. (5) below by noting that a new bound state is pulled in from the continuum each time the scattering length diverges (Levinson's theorem).

der Waals length  $a_c$ , which is only sensitive to the asymptotic behavior of the potential. Consider the scattering in states with angular momentum  $l = 0, 1, 2, \dots$  in the relative motion (for identical bosons or fermions only even or odd values of  $l$  are possible, respectively). The effective potential for states with  $l \neq 0$  contains a centrifugal barrier whose height is of order  $E_c = \hbar^2/M_r a_c^2$ . Converting this energy into an equivalent temperature, one obtains temperatures around 1 mK for typical atomic masses. At temperatures below that, the energy  $\hbar^2 k^2/2M_r$  in the relative motion of two atoms is typically below the centrifugal barrier. Scattering in states with  $l \neq 0$  is therefore frozen out, unless there exist so-called shape resonances, i.e. bound states with  $l \neq 0$  behind the centrifugal barrier, which may be in resonance with the incoming energy, see Boesten *et al.* (1997); Dürr *et al.* (2005). For gases in the sub-mK regime, therefore, usually lowest angular momentum collisions dominate (*s*-wave for bosons, *p*-wave for fermions), which in fact *defines* the regime of ultracold atoms. In the *s*-wave case, the scattering amplitude is determined by the corresponding phase shift  $\delta_0(k)$  via (Landau and Lifshitz, 1987)

$$f(k) = \frac{1}{k \cot \delta_0(k) - ik} \rightarrow \frac{1}{-1/a + r_e k^2/2 - ik}. \quad (4)$$

At low energies, it is characterized by the scattering length  $a$  and the effective range  $r_e$  as the two single parameters. For the truncated van der Waals potential (1), the scattering length can be calculated analytically as (Gribakin and Flambaum, 1993)

$$a = \bar{a} [1 - \tan(\Phi - 3\pi/8)] \quad (5)$$

where  $\Phi$  is the WKB-phase (3) and  $\bar{a} = 0.478 a_c$  the so-called mean scattering length. The expression (5) shows that the characteristic magnitude of the scattering length is the van der Waals length. Its detailed value, however, depends on the short range physics via the WKB-phase  $\Phi$ , which is sensitive to the hard core scale  $r_c$ . Since the detailed behavior of the potential is typically not known precisely, in many cases neither the sign of the scattering length nor the number of bound states can be determined from ab initio calculations. The toy-model result, however, is useful beyond the identification of  $a_c$  as the characteristic scale for the scattering length. Indeed, if the ignorance about the short range physics is replaced by the (maximum likelihood) assumption of a uniform distribution of  $\Phi$  in the relevant interval  $[0, \pi]$ , the probability for finding a positive scattering length, i.e.  $\tan \Phi < 1$  is 3/4. A repulsive interaction at low energy, which is connected with a positive scattering length, is therefore three times more likely than an attractive one, where  $a < 0$  (Pethick and Smith, 2002). Concerning the effective range  $r_e$  in Eq. (4), it turns out that also  $r_e$  is on the order of the van der Waals or the mean scattering length  $\bar{a}$  rather than the short range scale  $r_c$  as might

have been expected naively<sup>2</sup> (Flambaum *et al.*, 1999). Since  $ka_c \ll 1$  in the regime of ultracold collisions, this implies that the  $k^2$ -contribution in the denominator of the scattering amplitude is negligible. In the low-energy limit, the two-body collision problem is thus completely specified by the scattering length  $a$  as the single parameter and a corresponding scattering amplitude

$$f(k) = \frac{-a}{1 + ika}. \quad (6)$$

As noted by Fermi in the context of scattering of slow neutrons and by Lee, Huang and Yang for the low temperature thermodynamics of weakly interacting quantum gases, Eq. (6) is the exact scattering amplitude at *arbitrary* values of  $k$  for the pseudopotential<sup>3</sup>

$$V(\mathbf{x})(\dots) = \frac{4\pi\hbar^2 a}{2M_r} \cdot \delta(\mathbf{x}) \frac{\partial}{\partial r} (r \dots). \quad (7)$$

At temperatures such that  $k_B T < E_c$ , two-body interactions in ultracold gases may thus be described by a pseudopotential, with the scattering length usually taken as an experimentally determined parameter. This approximation is valid in a wide range of situations, provided no longer range contributions come into play as e.g. in the case of dipolar gases. The interaction is repulsive for positive and attractive for negative scattering length. Now, as shown above, the true interaction potential has many bound states, irrespective of the sign of  $a$ . For the low energy scattering of atoms, however, these bound states are irrelevant as long as no molecule formation occurs via three-body collisions. The scattering amplitude in the limit  $k \rightarrow 0$  is only sensitive to bound (or virtual for  $a < 0$ ) states near zero energy. In particular, within the pseudopotential approximation the amplitude (6) has a single pole  $k = i\kappa$ , with  $\kappa = 1/a > 0$  if the scattering length is positive. Quite generally, poles of the scattering amplitude in the upper complex  $k$ -plane are connected with bound states with binding energy  $\varepsilon_b = \hbar^2 \kappa^2/2M_r$  (Landau and Lifshitz, 1987). In the pseudopotential approximation, only a single pole is captured; the energy of the associated bound state is just below the continuum threshold. A repulsive pseudopotential thus describes a situation where the full potential has a bound state with a binding energy  $\varepsilon_b = \hbar^2/2M_r a^2$  on the order of or smaller than the characteristic energy  $E_c$  introduced above. The associated positive scattering length is then identical with the decay length of the wave function  $\sim \exp -r/a$  of the highest bound state. In the attractive case  $a < 0$ , in turn, there is no bound state within a

<sup>2</sup> This is a general result for deep potentials with a power law decay at large distances, as long as the scattering energy is much smaller than the depth of the potential well.

<sup>3</sup> Due to the delta-function, the last term involving the partial derivative with respect to  $r = |\mathbf{x}|$  can be omitted when the potential acts on a function which is regular at  $r = 0$ .

range  $E_c$  below the continuum threshold, however there is a virtual state just above it.

## B. Weak interactions

For a qualitative discussion of what defines the weak interaction regime in dilute, ultracold gases, it is useful to start with the idealization of no interactions at all. Depending on the two fundamental possibilities for the statistics of indistinguishable particles, Bose or Fermi, the ground state of a gas of  $N$  non-interacting particles is either a perfect BEC or a Fermi sea. In the case of an ideal BEC, all particles occupy the lowest available single particle level, consistent with a fully symmetric many-body wave function. For fermions, in turn, the particles fill the  $N$  lowest single particle levels up to the Fermi-energy  $\epsilon_F(N)$ , as required by the Pauli-principle. At finite temperatures, the discontinuity in the Fermi-Dirac distribution at  $T = 0$  is smeared out, giving rise to a continuous evolution from the degenerate gas at  $k_B T \ll \epsilon_F$  to a classical gas at high temperatures  $k_B T \gtrsim \epsilon_F$ . By contrast, bosons exhibit in 3D a phase transition at finite temperature, where the macroscopic occupancy of the ground state is lost. In the homogeneous gas, this transition occurs when the thermal de Broglie wavelength  $\lambda_T = h/\sqrt{2\pi M k_B T}$  reaches the average interparticle distance  $n^{-1/3}$ . The surprising fact that a phase transition appears even in an ideal Bose gas is a consequence of the correlations imposed by the particle statistics alone, as noted already in Einstein's fundamental paper (Einstein, 1925). For trapped gases, with geometrical mean trap frequency  $\bar{\omega}$ , the transition to a BEC is in principle smooth<sup>4</sup>. Yet, for typical particle numbers in the range  $N \approx 10^4 - 10^7$ , there is a rather sharply defined temperature  $k_B T_c^{(0)} = \hbar\bar{\omega} \cdot (N/\zeta(3))^{1/3}$ , above which the occupation of the oscillator ground state is no longer of order  $N$ . This temperature is again determined by the condition that the thermal de Broglie wavelength reaches the average interparticle distance at the center of the trap (see Eq. (95) and below).

As discussed above, interactions between ultracold atoms are described by a pseudopotential (7), whose strength  $g = 4\pi\hbar^2 a/2M_r$  is fixed by the exact s-wave scattering length  $a$ . Now, for identical fermions, there is no s-wave scattering due to the Pauli-principle. In the regime  $ka_c \ll 1$ , where all higher momenta  $l \neq 0$  are frozen out, a single component Fermi gas thus approaches an ideal, non-interacting quantum gas. To reach the necessary temperatures, however, requires thermalization by elastic collisions. For identical fermions,  $p$ -wave collisions dominate at low temperatures, whose cross section

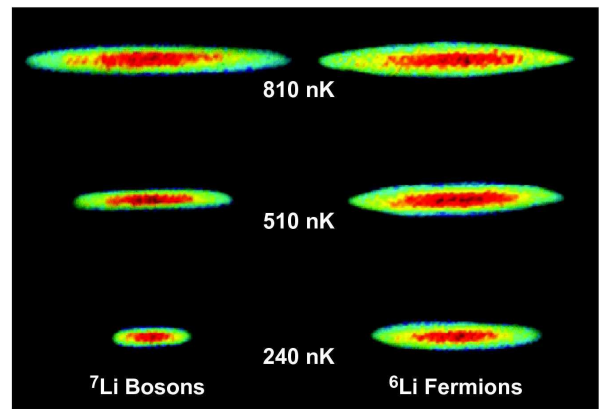


FIG. 1 Simultaneous cooling of a bosonic and fermionic quantum gas of  ${}^7\text{Li}$  and  ${}^6\text{Li}$  to quantum degeneracy. In the case of the Fermi gas, the Fermi pressure prohibits the atom cloud to shrink further in space as quantum degeneracy is approached. Reprinted with permission from Truscott *et al.* (2001).

$\sigma_p \sim E^2$  leads to a vanishing of the scattering rates  $\sim T^2$  (DeMarco *et al.*, 1999). Evaporative cooling therefore does not work for a single component Fermi gas in the degenerate regime. This problem may be circumvented by cooling in the presence of a different spin state which is then removed, or by sympathetic cooling with another atomic species. In this manner, an ideal Fermi gas, which is one of the paradigms of statistical physics, has first been realized by DeMarco and Jin (1999), Truscott *et al.* (2001) and Schreck *et al.* (2001) (see Fig. 1).

In the case of mixtures of fermions in different internal states or for bosons, there is in general a finite scattering length  $a \neq 0$ , which is typically of the order of the van der Waals length Eq. (2). By a simple dimensional argument, interactions are expected to be weak when the scattering length is much smaller than the average interparticle spacing. Since ultracold alkali gases have densities between  $10^{12}$  and  $10^{15}$  particles per  $\text{cm}^{-3}$ , the average interparticle spacing  $n^{-1/3}$  typically is in the range  $0.1 - 1 \mu\text{m}$ . As shown above, the scattering length, in turn, is usually only in the few nm range. Interaction effects are thus expected to be very small, unless the scattering length happens to be large near a zero energy resonance of Eq. (5). In the attractive case  $a < 0$ , however, even small interactions can lead to instabilities. In particular, attractive bosons are unstable towards collapse. However, in a trap, a metastable gaseous state arises for sufficiently small atom numbers (Pethick and Smith, 2002). For mixtures of fermions in different internal states, an arbitrary weak attraction leads to the BCS-instability, where the ground state is essentially a BEC of Cooper pairs (see section VIII). In the case of repulsive interactions, in turn, perturbation theory works in the limit

<sup>4</sup> A Bose gas in a trap exhibits a sharp transition only in the limit  $N \rightarrow \infty$ ,  $\bar{\omega} \rightarrow 0$  with  $N\bar{\omega}^3 = \text{const}$ , i.e. when the critical temperature approaches a finite value in the thermodynamic limit.

$n^{1/3}a \ll 1$ <sup>5</sup>. For fermions with two different internal states, an appropriate description is provided by the dilute gas version of Landau's theory of Fermi liquids. The associated ground state chemical potential is given by (Lifshitz and Pitaevskii, 1980)

$$\mu_{\text{Fermi}} = \frac{\hbar^2 k_F^2}{2M} \left( 1 + \frac{4}{3\pi} k_F a + \frac{4(11 - 2 \ln 2)}{15\pi^2} (k_F a)^2 + \dots \right), \quad (8)$$

where the Fermi wavevector  $k_F = (3\pi^2 n)^{1/3}$  is determined by the total density  $n$  in precisely the same manner as in the non-interacting case. Weakly interacting Bose gases, in turn, are described by the Bogoliubov theory, which has  $\sqrt{na^3}$  as the relevant small parameter. For example, the chemical potential at zero temperature for a homogeneous gas is given by (Lifshitz and Pitaevskii, 1980)

$$\mu_{\text{Bose}} = \frac{4\pi\hbar^2 a}{M} n \left( 1 + \frac{32}{3} \left( \frac{na^3}{\pi} \right)^{1/2} + \dots \right). \quad (9)$$

Moreover, interactions lead to a depletion

$$n_0 = n \left( 1 - \frac{8}{3} \left( \frac{na^3}{\pi} \right)^{1/2} + \dots \right) \quad (10)$$

of the density  $n_0$  of particles at zero momentum compared to the perfect condensate of an ideal Bose gas. The finite value of the chemical potential at zero temperature defines a characteristic length  $\xi$  by the relation  $\hbar^2/2M\xi^2 = \mu_{\text{Bose}}$ . This is the so-called healing length (Pitaevskii and Stringari, 2003) which is the scale, over which the macroscopic wave function  $\psi(\mathbf{x})$  varies near a boundary (or a vortex core, see section VII) where BEC is suppressed. To lowest order in  $\sqrt{na^3}$ , this length is given by  $\xi = (8\pi na)^{-1/2}$ . In the limit  $na^3 \ll 1$ , the healing length is therefore much larger than the average interparticle spacing  $n^{-1/3}$ . In practice, the dependence on the gas parameter  $na^3$  is so weak, that the ratio  $\xi n^{1/3} \sim (na^3)^{-1/6}$  is never very large. On a microscopic level,  $\xi$  is the length associated with the ground state energy per particle by the uncertainty principle. It can thus be identified with the scale, over which bosons may be considered to be localized spatially. For weak coupling BEC's, the atoms are therefore smeared out over distances much larger than the average interparticle spacing.

Interactions also shift the critical temperature for BEC away from its value  $T_c^{(0)}$  in the ideal Bose gas. To lowest order in the interactions, the shift is positive and linear in the scattering length, (Baym *et al.*, 1999)

$$T_c/T_c^{(0)} = 1 + cn^{1/3}a + \dots \quad (11)$$

<sup>5</sup> We neglect the possibility of a Kohn-Luttinger instability (Kohn and Luttinger, 1965) of repulsive fermions to a (typically) p-wave superfluid state, which usually only appears at temperatures very far below  $T_F$ , see Baranov *et al.* (1996).

with a numerical constant  $c \approx 1.32$  (Arnold and Moore, 2001; Kashurnikov *et al.*, 2001). The unexpected increase of the BEC condensation temperature with interactions is due to a reduction of the critical density. While a quantitative derivation of Eq. (11) requires quite sophisticated techniques (Holzmann *et al.*, 2004), the result can be recovered by a simple argument. To leading order, the interaction induced change in  $T_c$  only depends on the scattering length. Compared with the non-interacting case, the finite scattering length may be thought of as effectively increasing the quantum mechanical uncertainty in the position of each atom due to thermal motion from  $\lambda_T$  to  $\bar{\lambda}_T = \lambda_T + a$ . To lowest order in  $a$ , the modified ideal gas criterion  $n\bar{\lambda}_T^3 = \zeta(3/2)$  then gives rise to the linear and positive shift of the critical temperature in Eq. (11) with a coefficient  $\bar{c} \approx 1.45$ , which is not far from the numerically exact value.

In the standard situation of a gas confined in a harmonic trap with characteristic frequency  $\bar{\omega}$ , the influence of weak interactions is quantitatively quite different for temperatures near  $T = 0$  or near the critical temperature  $T_c$ . At zero temperature, the non-interacting Bose gas has a density distribution  $n^{(0)}(\mathbf{x}) = N \cdot |\phi_0(\mathbf{x})|^2$ , which just reflects the harmonic oscillator ground state wave function  $\phi_0(\mathbf{x})$ . Its characteristic width is the oscillator length  $\ell_0 = \sqrt{\hbar/M\bar{\omega}}$  which is on the order of one  $\mu\text{m}$  for typical confinement frequencies. Adding even small repulsive interactions changes the distribution quite strongly. Indeed, in the experimentally relevant limit  $Na \gg \ell_0$ , the density profile  $n(\mathbf{x})$  in the presence of an external trap potential  $U(\mathbf{x})$  can be obtained from a local density approximation (LDA)

$$\mu[n(\mathbf{x})] + U(\mathbf{x}) = \mu[n(0)]. \quad (12)$$

For weakly interacting bosons in an isotropic harmonic trap, the linear dependence  $\mu_{\text{Bose}} = gn$  of the chemical potential on the density in the homogeneous case then leads to a so-called Thomas-Fermi profile  $n(\mathbf{x}) = n(0) (1 - (r/R_{TF})^2)$ . Using the condition  $\int n(\mathbf{x}) = N$ , the associated radius  $R_{TF} = \zeta\ell_0$  considerably exceeds the oscillator length since the dimensionless parameter  $\zeta = (15Na/\ell_0)^{1/5}$  is typically much larger than one (Giorgini *et al.*, 1997)<sup>6</sup>. This broadening leads to a significant decrease in the density  $n(\mathbf{0})$  at the trap center by a factor  $\zeta^{-3}$  compared with the non-interacting case. The strong effect of even weak interactions on the ground state in a trap may be understood from the fact that the chemical potential  $\mu = \hbar\bar{\omega} \cdot \zeta^2/2$  is much larger than

<sup>6</sup> For fermions, the validity of the LDA, which is in fact just a semiclassical approximation (see e.g. Brack and Bhaduri (1997)), does not require interactions. The leading term  $\mu_{\text{Fermi}} \sim n^{2/3}$  of Eq. (8) leads to a density profile  $n(\mathbf{x}) = n(0) (1 - (r/R_{TF})^2)^{3/2}$  with a radius  $R_{TF} = \zeta\ell_0$ . Here  $\zeta = k_F(N)\ell_0 = (24N)^{1/6} \gg 1$  and the Fermi wavevector  $k_F(N)$  in a trap is defined by  $\epsilon_F(N) = \hbar^2 k_F^2(N)/2M$

the oscillator ground state energy. Interactions are thus able to mix in many single particle levels beyond the harmonic trap ground state. Near the critical temperature, in turn, the ratio  $\mu/k_B T_c \simeq (n(\mathbf{0})a^3)^{1/6}$  is small. Interaction corrections to the condensation temperature, which dominate finite size corrections for particle numbers much larger than  $N \simeq 10^4$ , are therefore accessible perturbatively (Giorgini *et al.*, 1997). In contrast to the homogeneous case, where the density is fixed and  $T_c$  is shifted upwards, the dominant effect in a trap arises from the reduced density at the trap center. Near  $T_c$  this effect is small and the corresponding shift of may be expressed in the form  $\Delta T_c/T_c = -\text{const } a/\lambda_{T_c}$  (Davis and Blakie, 2006; Giorgini *et al.*, 1997; Holzmann *et al.*, 2004). A precise measurement of this shift has been performed by Gerbier *et al.* (2004). Their results are in quantitative agreement with mean-field theory, with no observable contribution of critical fluctuations at their level of sensitivity. Quite recently, evidence for critical fluctuations has been inferred from measurements of the correlation length  $\xi \sim (T - T_c)^{-\nu}$  very close to  $T_c$ . The observed value  $\nu = 0.67 \pm 0.13$  (Donner *et al.*, 2007) agrees well with the expected critical exponent of the 3D XY-model.

In spite of the strong deviations in the density distribution compared to the non-interacting case, the one- and two-particle correlations of weakly interacting bosons are well described by approximating the many-body ground state of  $N$  bosons by a product

$$\Psi_{GP}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \prod_{i=1}^N \phi_1(\mathbf{x}_i) \quad (13)$$

in which all atoms are in the identical single particle state  $\phi_1(\mathbf{x})$ . Taking Eq. (13) as a variational ansatz, the optimal macroscopic wave function  $\phi_1(\mathbf{x})$  is found to obey the well known Gross-Pitaevskii equation. More generally, it turns out that for trapped BEC's the Gross-Pitaevskii theory can be derived in a mathematically rigorous manner by taking the limits  $N \rightarrow \infty$  and  $a \rightarrow 0$  in such a way that the ratio  $Na/\ell_0$  is fixed (Lieb *et al.*, 2000). A highly nontrivial aspect of these derivations is that they show explicitly that in the dilute limit interactions enter only via the scattering length. The Gross-Pitaevskii equation thus remains valid e.g. for a dilute gas of hard spheres. Since all the interaction energy is of kinetic origin in this case, the standard mean field derivation of the Gross-Pitaevskii equation via the replacement of the field operators by a classical c-number  $\hat{\Psi}(\mathbf{x}) \rightarrow \sqrt{N}\phi_1(\mathbf{x})$  is thus incorrect in general. From a many-body point of view, the Ansatz Eq. (13), where the ground state is written as a product of optimized single-particle wave functions, is just the standard Hartree-approximation. It is the simplest possible approximation to account for interactions, however it clearly contains no interaction induced correlations between *different* atoms at all. A first step to go beyond that is the well known Bogoliubov theory. This is usually introduced by considering small fluctuations around the Gross-Pitaevskii equation

in a systematic expansion in the number of noncondensed particles (Castin and Dum, 1998). It is more instructive from a many-body point of view, however, to formulate Bogoliubov theory in such a way that the boson ground state is approximated by an optimized product (Leggett, 2001; Lieb, 1963b)

$$\Psi_{Bog.}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \prod_{i < j} \phi_2(\mathbf{x}_i, \mathbf{x}_j) \quad (14)$$

of identical, symmetric *two*-particle wave functions  $\phi_2$ . This allows to include interaction effects beyond the Hartree potential of the Gross-Pitaevskii theory by suppressing configurations in which two particles are close together. The many-body state thus incorporates two-particle correlations which are important e.g. to obtain the standard sound modes and the related coherent superposition of 'particle' and 'hole' excitations. This structure, which has been experimentally verified by Vogels *et al.* (2002), is expected to apply in a qualitative form even for strongly interacting BEC's, whose low energy excitations are exhausted by harmonic phonons (see Appendix).

Quantitatively, however, the Bogoliubov theory is restricted to the regime  $\sqrt{na^3} \ll 1$ , where interactions lead only to a small depletion (10) of the condensate at zero temperature. Going beyond that requires to specify the detailed form of the interaction potential  $V(r)$  and not only the associated scattering length  $a$ . The ground state of a gas of hard sphere bosons, for instance, loses BEC already for  $na^3 \gtrsim 0.24$  by a first order transition to a solid state (Kalos *et al.*, 1974). On a variational level, choosing the two-particle wave functions in (14) of the form  $\phi_2(\mathbf{x}_i, \mathbf{x}_j) \sim \exp -u(|\mathbf{x}_i - \mathbf{x}_j|)$  with an effective two-body potential  $u(r)$ , describes so-called Jastrow wave functions. They allow taking into account strong short range correlations, however they still exhibit BEC even in a regime, where the associated one-particle density describes a periodic crystal rather than a uniform liquid, as shown by Chester (1970). Crystalline order may thus coexist with BEC! For a recent discussion of this issue in the context of a possible supersolid phase of  $^4\text{He}$ , see Clark and Ceperley (2006).

For weakly interacting fermions at  $k_F a \ll 1$ , the variational ground state which is analogous to Eq. (13), is a Slater determinant

$$\Psi_{HF}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \text{Det}[\phi_{1,i}(\mathbf{x}_j)], \quad (15)$$

of optimized single particle states  $\phi_{1,i}(\mathbf{x}_j)$ . In the translational invariant case, they are plane waves  $\phi_{1,i}(\mathbf{x}) = V^{-1/2} \exp i\mathbf{k}_i \mathbf{x}$ , where the momenta  $\mathbf{k}_i$  are filled up to the Fermi-momentum  $k_F$ . Although both the Bose and the Fermi groundstate wave functions consist of symmetrized or anti-symmetrized single-particle states, they describe - of course - fundamentally different physics. In the Bose case, the one particle density matrix  $g^{(1)}(\infty) = n_0/n$  approaches a finite constant at infinite separation, which is the basic criterion for BEC (see Appendix). The many-body wave function is thus sensitive to changes of the

phase at points separated by distances  $r$  which are large compared to the interparticle spacing. By contrast, the Hartree-Fock state (15) for fermions shows no long range phase coherence and indeed, the one particle density matrix decays exponentially  $g^{(1)}(r) \sim \exp -\gamma r$  at any finite temperature (Ismail-Beigi and Arias, 1999). The presence of  $N$  *distinct* eigenstates in Eq. (15), which is a necessary consequence of the Pauli-principle, thus leads to a many-body wave function which may be characterized as *nearsighted*. The notion of nearsightedness, depends on the observable, however. As defined originally by Kohn (1996), it means that a localized external potential around some point  $\mathbf{x}'$  is not felt at a point  $\mathbf{x}$  at a distance much larger than the average interparticle spacing. This requires the density response function  $\chi(\mathbf{x}, \mathbf{x}')$  to be short ranged in position space. In this respect, weakly interacting bosons, where  $\chi(\mathbf{x}, \mathbf{x}') \sim (\exp -|\mathbf{x} - \mathbf{x}'|/\xi)/|\mathbf{x} - \mathbf{x}'|$  decays exponentially on the scale of the healing length  $\xi$  are more nearsighted than fermions at zero temperature, where  $\chi(\mathbf{x}, \mathbf{x}') \sim \sin(2k_F|\mathbf{x} - \mathbf{x}'|)/|\mathbf{x} - \mathbf{x}'|^3$  exhibits an algebraic decay with Friedel oscillations at twice the Fermi wave vector  $2k_F$ . The characterization of many-body wave functions in terms of the associated correlation functions draws attention to another basic point emphasized by Kohn (1999): in situations with a large number of particles the many-body wave function itself is not a meaningful quantity because it cannot be reliably calculated for  $N \gtrsim 100$ . Moreover, physically accessible observables are only sensitive to the resulting one- or two-particle correlations. Cold gases provide a concrete example for the latter statement: the standard time-of-flight technique of measuring the absorption image after a given free expansion time  $t$  provides the one-particle density matrix in Fourier space, while the two-particle density matrix is revealed in the noise correlations of the absorption images (see section III).

### C. Feshbach resonances

The most direct way of reaching the strong interaction regime in dilute, ultracold gases are Feshbach resonances, which allow to increase the scattering length to values beyond the average interparticle spacing. In practice, this method works best for fermions because for them the lifetime due to three-body collisions becomes very large near a Feshbach resonance, in stark contrast to bosons, where it goes to zero. The concept of Feshbach resonances was first introduced in nuclear physics in the context of reactions forming a compound nucleus (Feshbach, 1958). Quite generally, a Feshbach resonance in a two-particle collision appears whenever a bound state in a closed channel is coupled resonantly with the scattering continuum of an open channel. The two channels may correspond, for example, to different spin configurations for the atoms. The scattered particles are then temporarily captured in the quasi-bound state and the associated long time delay gives rise to a Breit-Wigner type reso-

nance in the scattering cross-section. What makes Feshbach resonances in the scattering of cold atoms particularly useful, is the ability to tune the scattering length simply by changing the magnetic field (Tiesinga *et al.*, 1993). This tunability relies on the difference in the magnetic moments of the closed and open channels, which allows to change the position of closed channel bound states relative to the open channel threshold by varying the external, uniform magnetic field. Note that Feshbach resonances can alternatively be induced optically via one- or two-photon transitions, (Bohn and Julienne, 1999; Fedichev *et al.*, 1996a) as realized by Theis *et al.* (2004). The control parameter is then the detuning of the light from atomic resonance. Although more flexible in principle, this method suffers, however, from heating problems for typical atomic transitions, associated with the spontaneous emission processes created by the light irradiation.

On a phenomenological level, Feshbach resonances are described by an effective pseudopotential between atoms in the open channel with scattering length

$$a(B) = a_{\text{bg}} \left( 1 - \frac{\Delta B}{B - B_0} \right). \quad (16)$$

Here  $a_{\text{bg}}$  is the off-resonant background scattering length in the absence of the coupling to the closed channel while  $\Delta B$  and  $B_0$  describe the width and position of the resonance expressed in magnetic field units (see Fig. 2). In this section we outline the basic physics of magnetically tunable Feshbach resonances, providing a connection of the parameters in Eq. (16) with the interatomic potentials. Of course, our discussion only covers the basic background for understanding the origin of large and tunable scattering lengths. A much more detailed presentation of Feshbach resonances can be found in the reviews by Timmermans *et al.* (2001), Duine and Stoof (2004) and by Köhler *et al.* (2006).

*Open and closed channels* We start with the specific example of fermionic  ${}^6\text{Li}$  atoms, which have electronic spin  $S = 1/2$  and nuclear spin  $I = 1$ . In the presence of a magnetic field  $\mathbf{B}$  along the  $z$ -direction, the hyperfine coupling and Zeeman energy lead for each atom to the Hamiltonian

$$\hat{H}' = a_{\text{hf}} \hat{\mathbf{S}} \cdot \hat{\mathbf{I}} + \left( 2\mu_B \hat{S}_z - \mu_n \hat{I}_z \right) B. \quad (17)$$

Here  $\mu_B$  is the standard Bohr magneton and  $\mu_n$  ( $\ll \mu_B$ ) the magnetic moment of the nucleus. This hyperfine-Zeeman Hamiltonian actually holds for any alkali atom, with a single valence electron with zero orbital angular momentum. If  $B \rightarrow 0$  the eigenstates of this Hamiltonian are labeled by the quantum numbers  $f$  and  $m_f$ , giving the total spin angular momentum and its projection along the  $z$  axis, respectively. In the opposite Paschen-Back regime of large magnetic fields ( $B \gg \hbar a_{\text{hf}}/\mu_B \simeq$



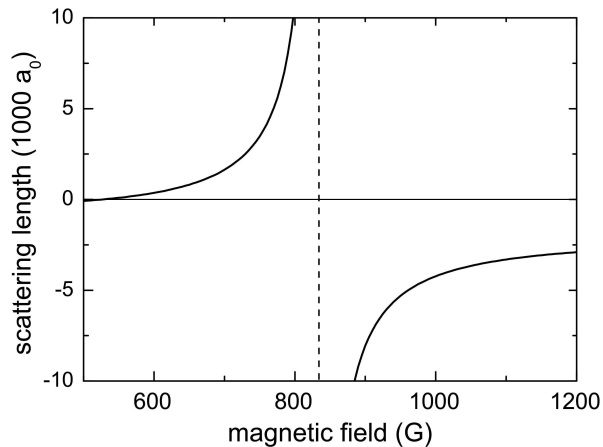


FIG. 2 Magnetic field dependence of the scattering length between the two lowest magnetic sub-states of  ${}^6\text{Li}$  with a Feshbach resonance at  $B_0 = 834$  G and a zero crossing at  $B_0 + \Delta B = 534$  G. The background scattering length  $a_{\text{bg}} = -1405 a_B$  is exceptionally large in this case ( $a_B$  being the Bohr radius).

30 G in Lithium), the eigenstates are labeled by the quantum numbers  $m_s$  and  $m_I$ , giving the projection on the  $z$  axis of the electron and nuclear spins, respectively. The projection  $m_f = m_s + m_I$  of the total spin along the  $z$  axis remains a good quantum number for any value of the magnetic field.

Consider a collision between two lithium atoms, prepared in the two lowest eigenstates  $|a\rangle$  and  $|b\rangle$  of the Hamiltonian (17) in a large magnetic field. The lowest state  $|a\rangle$  (with  $m_{fa} = 1/2$ ) is  $\approx |m_s = -1/2, m_I = 1\rangle$  with a small admixture of  $|m_s = 1/2, m_I = 0\rangle$ , whereas  $|b\rangle$  (with  $m_{fb} = -1/2$ ) is  $\approx |m_s = -1/2, m_I = 0\rangle$  with a small admixture of  $|m_s = 1/2, m_I = -1\rangle$ . Two atoms in these two lowest states thus predominantly scatter in their triplet state <sup>7</sup>. Quite generally, the interaction potential during the collision can be written as a sum

$$V(r) = \frac{1}{4}(3V_t(r) + V_s(r)) + \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 (V_t(r) - V_s(r)), \quad (18)$$

of projections onto the singlet  $V_s(r)$  and triplet  $V_t(r)$  molecular potentials, where the  $\hat{\mathbf{S}}_i$ 's ( $i = 1, 2$ ) are the spin operators for the valence electron of each atom. These potentials have the same van der Waals attractive behavior at long distances, but they differ considerably at short distances, with a much deeper attractive well for the singlet than for the triplet potential. Now, in a large but finite magnetic field the initial state  $|a, b\rangle$  is not a purely triplet state. Because of the tensorial nature of  $V(r)$ , this spin state will thus evolve during

<sup>7</sup> The fact that there is a nonvanishing  $s$ -wave scattering length for these states is thus connected with the different *nuclear* and not *electronic* spin in this case !

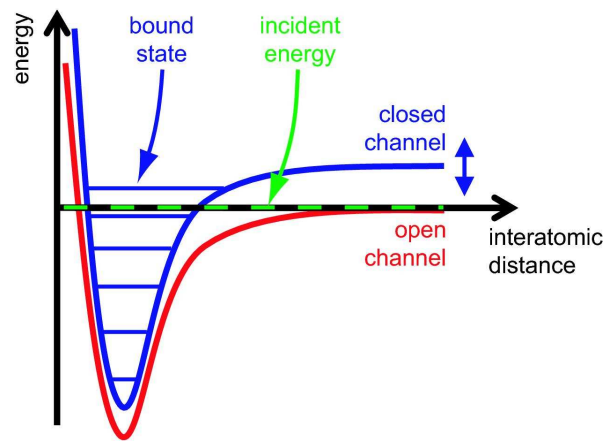


FIG. 3 The two-channel model for a Feshbach resonance. Atoms prepared in the open channel, corresponding to the interaction potential  $V_{\text{op}}(r)$  (in red), undergo a collision at low incident energy. In the course of the collision the open channel is coupled to the closed channel  $V_{\text{cl}}(r)$  (in blue). When a bound state of the closed channel has an energy close to zero, a scattering resonance occurs. The position of the closed channel can be tuned with respect to the open one e.g. by varying the magnetic field  $B$ .

the collision. More precisely, since the second term in Eq. (18) is not diagonal in the basis  $|a, b\rangle$ , the spin state  $|a, b\rangle$  may be coupled to other scattering channels  $|c, d\rangle$ , provided the  $z$  projection of the total spin is conserved ( $m_{fc} + m_{fd} = m_{fa} + m_{fb}$ ). When the atoms are far apart the Zeeman+hyperfine energy of  $|c, d\rangle$  exceeds the initial kinetic energy of the pair of atoms prepared in  $|a, b\rangle$  by an energy on the order of the hyperfine energy. Since the thermal energy is much smaller than that for ultracold collisions, the channel  $|c, d\rangle$  is closed and the atoms always emerge from the collision in the open channel state  $|a, b\rangle$ . However, due to the strong coupling of  $(a, b)$  to  $(c, d)$  via the second term in Eq. (18), which is typically on the order of eV, the effective scattering amplitude in the open channel can be strongly modified.

*Two-channel model* We now present a simple two-channel model which captures the main features of a Feshbach resonance (see Fig. 3). We consider a collision between two atoms with reduced mass  $M_r$ , and model the system in the vicinity of the resonance by the Hamiltonian (Nygaard *et al.*, 2006)

$$\hat{H} = \begin{pmatrix} -\frac{\hbar^2}{2M_r}\nabla^2 + V_{\text{op}}(r) & W(r) \\ W(r) & -\frac{\hbar^2}{2M_r}\nabla^2 + V_{\text{cl}}(r) \end{pmatrix}. \quad (19)$$

Before collision the atoms are prepared in the *open channel*, whose potential  $V_{\text{op}}(r)$  gives rise to the background scattering length  $a_{\text{bg}}$ . Here the zero of energy is chosen such that  $V_{\text{op}}(\infty) = 0$ . In the course of the collision a coupling to the *closed channel* with potential  $V_{\text{cl}}(r)$  ( $V_{\text{cl}}(\infty) > 0$ ) occurs via the matrix element  $W(r)$ , whose

range is on the order of the atomic scale  $r_c$ . For simplicity, we consider here only a *single* closed channel, which is appropriate for an isolated resonance. We also assume that the value of  $a_{\text{bg}}$  is on the order of the van der Waals length (2). If  $a_{\text{bg}}$  is anomalously large, as occurs e.g. for the  ${}^6\text{Li}$  resonance shown in Fig. 2, an additional open channel resonance has to be included in the model, as discussed by Marcelis *et al.* (2004).

We assume that the magnetic moments of the colliding states differ for the open and closed channels, and we denote their difference by  $\mu$ . Varying the magnetic field by  $\delta B$  therefore amounts to shifting the closed channel energy by  $\mu\delta B$  with respect to the open channel. In the following we are interested in the magnetic field region close to  $B_{\text{res}}$  such that one (normalized) bound state  $\phi_{\text{res}}(r)$  of the closed channel potential  $V_{\text{cl}}(r)$  has an energy  $E_{\text{res}}(B) = \mu(B - B_{\text{res}})$  close to 0. It can thus be resonantly coupled to the collision state where two atoms in the open channel have a small positive kinetic energy. In the vicinity of the Feshbach resonance, the situation is now very similar to the well known Breit-Wigner problem (see e.g. Landau and Lifshitz (1987), section 134). A particle undergoes a scattering process in a (single channel) potential with a quasi or true bound state at an energy  $\nu$ , which is nearly resonant with the incoming energy  $E(k) = \hbar^2 k^2 / (2M_r)$ . According to Breit and Wigner, this leads to a resonant contribution

$$\delta_{\text{res}}(k) = -\arctan\left[\frac{\Gamma(k)/2}{E(k) - \nu}\right] \quad (20)$$

to the scattering phase shift, where  $\nu = \mu(B - B_0)$  is conventionally called the *detuning* in this context (for the difference between  $B_{\text{res}}$  and  $B_0$  see below). The associated resonance width  $\Gamma(k)$  vanishes near zero energy, with a threshold behavior linear in  $k = \sqrt{2M_r E}/\hbar$  due to the free particle density of states. It is convenient to define a characteristic length  $r^* > 0$  by

$$\Gamma(k \rightarrow 0)/2 = \frac{\hbar^2}{2M_r r^*} k. \quad (21)$$

The scattering length  $a = -\lim_{k \rightarrow 0} \tan(\delta_{\text{bg}} + \delta_{\text{res}})/k$  then has the simple form

$$a = a_{\text{bg}} - \frac{\hbar^2}{2M_r r^* \nu}. \quad (22)$$

This agrees precisely with Eq. (16) provided the width parameter  $\Delta B$  is identified with the combination  $\mu\Delta B a_{\text{bg}} = \hbar^2 / (2M_r r^*)$  of the two characteristic lengths  $a_{\text{bg}}$  and  $r^*$ .

On a microscopic level, these parameters may be obtained from the two channel Hamiltonian (19) by the standard Green function formalism. In absence of coupling  $W(r)$  the scattering properties of the open channel are characterized by  $G_{\text{op}}(E) = (E - H_{\text{op}})^{-1}$  with  $H_{\text{op}} = P^2 / (2M_r) + V_{\text{op}}(r)$ . We denote by  $|\phi_0\rangle$  the eigenstate of  $H_{\text{op}}$  associated with the energy 0 which behaves

as  $\phi_0(r) \sim (1 - a_{\text{bg}}/r)$  for large  $r$ . In the vicinity of the resonance the closed channel contributes essentially through the state  $\phi_{\text{res}}$ , and its Green function reads

$$G_{\text{cl}}(E, B) \simeq \frac{|\phi_{\text{res}}\rangle\langle\phi_{\text{res}}|}{E - E_{\text{res}}(B)}. \quad (23)$$

With this approximation one can project the eigenvalue equation for the Hamiltonian  $\hat{H}$  onto the background and closed channels. One can then derive the scattering length  $a(B)$  of the coupled channel problem and write it in the form of Eq. (16). The position of the zero energy resonance  $B_0$  is shifted with respect to the ‘bare’ resonance value  $B_{\text{res}}$  by

$$\mu(B_0 - B_{\text{res}}) = -\langle\phi_{\text{res}}|WG_{\text{op}}(0)W|\phi_{\text{res}}\rangle. \quad (24)$$

The physical origin of this resonance shift is that an infinite scattering length requires that the contributions to  $k \cot \delta(k)$  in the total scattering amplitude from the open and the closed channel precisely cancel. In a situation where the background scattering length deviates considerably from its typical value  $\bar{a}$  and where the off-diagonal coupling measured by  $\Delta B$  is strong, this cancellation appears already when the bare closed channel bound state is far away from the continuum threshold. A simple analytical estimate for this shift has been given by Julienne *et al.* (2004)

$$B_0 = B_{\text{res}} + \Delta B \cdot \frac{x(1-x)}{1+(1-x)^2}, \quad (25)$$

where  $x = a_{\text{bg}}/\bar{a}$ . The characteristic length  $r^*$  defined in (21) is determined by the off-diagonal coupling via

$$\langle\phi_{\text{res}}|W|\phi_0\rangle = \frac{\hbar^2}{2M_r} \sqrt{\frac{4\pi}{r^*}}. \quad (26)$$

We shall be mostly interested in the *wide resonance* case, which corresponds to the situation where  $r^* \ll |a_{\text{bg}}|$ . Since the background scattering length is generically on the order of the van der Waals length Eq. (2), this implies that the width  $\mu|\Delta B|$  in the detuning over which the scattering length deviates considerably from its background value is much larger than the characteristic energy  $E_c$  below which collisions are in the ultracold regime. For a quantitative estimate, consider the specific resonances in fermionic  ${}^6\text{Li}$  and  ${}^{40}\text{K}$  at  $B_0 = 834\text{ G}$  and  $B_0 = 202\text{ G}$  respectively, which have been used to study the BCS-BEC crossover with cold atoms (see section VIII). They are characterized by the experimentally determined parameters  $a_{\text{bg}} = -1405 a_B$ ,  $\Delta B = -300\text{ G}$ ,  $\mu = 2\mu_B$  and  $a_{\text{bg}} = 174 a_B$ ,  $\Delta B = 7.8\text{ G}$ ,  $\mu = 1.68\mu_B$  respectively, where  $a_B$  and  $\mu_B$  are the Bohr radius and Bohr magneton. From these parameters, the characteristic length associated with the two resonances turns out to be  $r^* = 0.5 a_B$  and  $r^* = 28 a_B$ , both obeying the wide resonance condition  $r^* \ll |a_{\text{bg}}|$ .

*Weakly bound states close to the resonance* In addition to the control of scattering properties, an important feature of the Feshbach resonance concerns the possibility to resonantly form weakly bound dimers in the region  $a > 0$ . We briefly present below some key properties of these dimers, restricting for simplicity to the wide resonance case and to the vicinity of the resonance  $|B - B_0| \ll |\Delta B|$ , so that the scattering length  $a(B)$  deviates substantially from its background value  $a_{\text{bg}}$ .

To determine the bound state for the two-channel Hamiltonian (19), one considers the Green function  $G(E) = (E - \hat{H})^{-1}$  and looks for the low energy pole at  $E = -\varepsilon_b < 0$  of this function. The corresponding bound state can be written

$$\langle \mathbf{x} | \Psi^{(b)} \rangle = \begin{pmatrix} \sqrt{1-Z} \psi_{\text{bg}}(r) \\ \sqrt{Z} \phi_{\text{res}}(r) \end{pmatrix}, \quad (27)$$

where the coefficient  $Z$  characterizes the closed channel admixture. The values of  $\varepsilon_b$  and  $Z$  can be calculated explicitly by projecting the eigenvalue equation for  $\hat{H}$  on each channel. The binding energy

$$\varepsilon_b = (\mu(B - B_0))^2 / \varepsilon^* + \dots \quad (28)$$

of the weakly bound state vanishes quadratically near the resonance, with characteristic energy  $\varepsilon^* = \hbar^2 / 2M_r (r^*)^2$ . The length  $r^*$  thus provides a direct measure of the curvature in the energy of the weakly bound state as a function of the detuning  $\mu(B - B_0)$ . In an experimental situation which starts from the atom continuum, it is precisely this weakly bound state which is reached upon varying the detuning by an adiabatic change in the magnetic field around  $B_0$ . The closed channel admixture  $Z$  can be written as

$$Z \simeq 2 \frac{r^*}{|a_{\text{bg}}|} \frac{|B - B_0|}{|\Delta B|} = 2 \frac{|\nu|}{\varepsilon^*}. \quad (29)$$

For a wide resonance, where  $r^* \ll |a_{\text{bg}}|$ , this admixture is always much smaller than one over the magnetic field range  $|B - B_0| \lesssim |\Delta B|$ . The bound state near the Feshbach resonance can thus be described in a single-channel picture, without explicitly taking into account the closed channel state.

The bound state  $|\Psi^{(b)}\rangle$  that we just presented should not be confused with the bound state  $|\Phi_{\text{op}}^{(b)}\rangle$ , that exists for  $a_{\text{bg}} > 0$  in the open channel, for a vanishing coupling  $W(r)$ . The bound state  $|\Phi_{\text{op}}^{(b)}\rangle$  has a binding energy  $\sim \hbar^2 / (2M_r a_{\text{bg}}^2)$ , that is much larger than that of Eq. (28) when  $|B - B_0| \ll |\Delta B|$ . For  $|B - B_0| \sim |\Delta B|$  the states  $|\Psi^{(b)}\rangle$  and  $|\Phi_{\text{op}}^{(b)}\rangle$  have comparable energies and undergo an avoided crossing. The universal character of the above results is then lost and one has to turn to a specific study of the eigenvalue problem.

To conclude, Feshbach resonances provide a flexible tool to change the interaction strength between ultracold atoms over a wide range. To realize a proper many-body Hamiltonian with tunable two-body interactions,

however, an additional requirement is that the relaxation rate into deep bound states due to three-body collisions is negligible. As will be discussed in section VIII.A, this is possible for fermions, where the relaxation rate is small near Feshbach resonances (Petrov *et al.*, 2004b, 2005).

## II. OPTICAL LATTICES

In the following, we will discuss how to confine cold atoms by laser light into configurations of a reduced dimensionality or in periodic lattices, thus generating situations in which the effects of interactions are strongly enhanced.

### A. Optical potentials

The physical origin of the confinement of cold atoms with laser light is the dipole force

$$\mathbf{F} = \frac{1}{2} \alpha(\omega_L) \nabla (|\mathbf{E}(\mathbf{r})|^2) \quad (30)$$

due to a spatially varying ac-Stark shift which atoms experience in an off-resonant light field (Grimm *et al.*, 2000). Since the time scale for the center-of-mass motion of the atoms is much slower than the inverse laser frequency  $\omega_L$ , only the time averaged intensity  $|\mathbf{E}(\mathbf{r})|^2$  enters. The direction of the force depends on the sign of the polarizability  $\alpha(\omega_L)$ . In the vicinity of an atomic resonance from the ground  $|g\rangle$  to an excited state  $|e\rangle$  at frequency  $\omega_0$ , the polarizability has the standard form  $\alpha(\omega_L) \approx |\langle e | \hat{d}_{\mathbf{E}} | g \rangle|^2 / \hbar(\omega_0 - \omega_L)$ , with  $\hat{d}_{\mathbf{E}}$  the dipole operator in the direction of the field. Atoms are thus attracted to the nodes or to the anti-nodes of the laser intensity for blue detuned ( $\omega_L > \omega_0$ ) or red detuned ( $\omega_L < \omega_0$ ) laser light respectively. A spatially dependent intensity profile  $I(\mathbf{r})$  therefore creates a trapping potential for neutral atoms. Within a two level model, an explicit form of the dipole potential may be derived by using the rotating wave approximation, which is a reasonable approximation provided that the detuning  $\Delta = \omega_L - \omega_0$  of the laser field is small compared to the transition frequency itself  $|\Delta| \ll \omega_0$ . With  $\Gamma$  as the decay rate of the excited state, one obtains for  $|\Delta| \gg \Gamma$  (Grimm *et al.*, 2000)

$$V_{\text{dip}}(\mathbf{r}) = \frac{3\pi c^2}{2\omega_0^3} \frac{\Gamma}{\Delta} I(\mathbf{r}), \quad (31)$$

which is attractive or repulsive for red ( $\Delta < 0$ ) or blue ( $\Delta > 0$ ) detuning, respectively. Atoms are thus attracted or repelled from an intensity maximum in space. It is important to note that, in contrast to the form suggested in Eq. (30), the light force is not fully conservative. Indeed, spontaneous emission gives rise to an imaginary part of the polarizability. Within a two level approximation, the related scattering rate  $\Gamma_{sc}(\mathbf{r})$  leads to an absorptive contribution  $\hbar\Gamma_{sc}(\mathbf{r})$  to the conservative dipole

potential (31), which can be estimated as (Grimm *et al.*, 2000)

$$\Gamma_{sc}(\mathbf{r}) = \frac{3\pi c^2}{2\hbar\omega_0^3} \left(\frac{\Gamma}{\Delta}\right)^2 I(\mathbf{r}). \quad (32)$$

As the eqs. (31,32) show, the ratio of scattering rate to the optical potential depth vanishes in the limit  $|\Delta| \gg \Gamma$ . A strictly conservative potential can thus be reached in principle by increasing the detuning of the laser field. In practice however, such an approach is limited by the maximum available laser power. For experiments with ultracold quantum gases of alkali atoms, the detuning is typically chosen to be large compared to the excited state hyperfine structure splitting and in most cases even large compared to the fine structure splitting in order to sufficiently suppress spontaneous scattering events.

The intensity profile  $I(r, z)$  of a gaussian laser beam propagating along the  $z$ -direction has the form

$$I(r, z) = \frac{2P}{\pi w^2(z)} e^{-2r^2/w^2(z)}. \quad (33)$$

Here  $P$  is the total power of the laser beam,  $r$  is the distance from the center and  $w(z) = w_0\sqrt{1 + z^2/z_R^2}$  is the  $1/e^2$  radius. This radius is characterized by a beam waist  $w_0$  which is typically around  $100 \mu\text{m}$ . Due to the finite beam divergence, the beam width increases linearly with  $z$  on a scale  $z_R = \pi w_0^2/\lambda$  which is called the Rayleigh length. Typical values for  $z_R$  are in the mm range. Around the intensity maximum a potential depth minimum occurs for a red detuned laser beam, leading to an approximately harmonic potential

$$V_{\text{dip}}(r, z) \approx -V_{\text{trap}} \left\{ 1 - 2 \left(\frac{r}{w_0}\right)^2 - \left(\frac{z}{z_R}\right)^2 \right\}. \quad (34)$$

The trap depth  $V_{\text{trap}}$  is linearly proportional to the laser power and typically ranges from a few kHz up to a MHz (from the Nanokelvin to the Microkelvin regime). The harmonic confinement is characterized by radial  $\omega_r$  and axial  $\omega_z$  trapping frequencies  $\omega_r = (4V_{\text{trap}}/Mw_0^2)^{1/2}$  and  $\omega_z = (2V_{\text{trap}}/Mz_R^2)$ . Optical traps for neutral atoms have a wide range of applications (Grimm *et al.*, 2000). In particular, they are inevitable in situations where magnetic trapping does not work for the atomic states under consideration. This is often the case when the interactions are manipulated via Feshbach resonances, involving high field seeking atomic states.

*Optical Lattices* A periodic potential is simply generated by overlapping two counterpropagating laser beams. Due to the interference between the two laser beams an optical standing wave with period  $\lambda/2$  is formed, in which the atoms can be trapped. More generally, by choosing the two laser beams to interfere under an angle less than  $180^\circ$ , one can also realize periodic potentials with a larger period (Hadzibabic *et al.*, 2004; Peil *et al.*, 2003). The simplest possible periodic optical potential is formed by

overlapping two counterpropagating beams. For a gaussian profile, this results in a trapping potential of the form

$$V(r, z) = -V_0 \cdot e^{-2r^2/w^2(z)} \cdot \sin^2(kz) \quad (35)$$

where  $k = 2\pi/\lambda$  is the wave vector of the laser light and  $V_0$  the maximum depth of the lattice potential. Note that due to the interference of the two laser beams  $V_0$  is four times larger than  $V_{\text{trap}}$  if the laser power and beam parameters of the two interfering lasers are equal.

Periodic potentials in two dimensions can be formed by overlapping two optical standing waves along different, usually orthogonal, directions. For orthogonal polarization vectors of the two laser fields no interference terms appear. The resulting optical potential in the center of the trap is then a simple sum of a purely sinusoidal potential in both directions.

In such a two-dimensional optical lattice potential, the atoms are confined to arrays of tightly confining one-dimensional tubes (see Fig. 4a). For typical experimental parameters the harmonic trapping frequencies along the tube are very weak and on the order of 10-200 Hz, while in the radial direction the trapping frequencies can become as high as up to 100 kHz. For sufficiently deep lattice depths, atoms can thus move only axially along the tube. In this manner, it is possible to realize quantum wires with neutral atoms, which allow to study strongly correlated gases in one dimension, as discussed in Section V. Arrays of such quantum wires have been realized by several groups (Greiner *et al.*, 2001; Kinoshita *et al.*, 2004; Moritz *et al.*, 2003; Paredes *et al.*, 2004; Tolra *et al.*, 2004).

For the creation of a three-dimensional lattice potential, three orthogonal optical standing waves have to be overlapped. The simplest case of independent standing waves, with no cross interference between laser beams of different standing waves can be realized by choosing orthogonal polarization vectors and also by using slightly different wavelengths for the three standing waves. The resulting optical potential is then simply given by the sum of three standing waves. In the center of the trap, for distances much smaller than the beam waist, the trapping potential can be approximated as the sum of a homogeneous periodic lattice potential

$$V_p(x, y, z) = V_0 (\sin^2 kx + \sin^2 ky + \sin^2 kz) \quad (36)$$

and an additional external harmonic confinement due to the gaussian laser beam profiles. In addition to this harmonic confinement, a confinement due to the magnetic trapping is often used, which has to be taken into account as well for the total harmonic confinement of the atom cloud.

For deep optical lattice potentials, the confinement on a single lattice site is approximately harmonic. The atoms are then tightly confined at a single lattice site, with trapping frequencies  $\omega_0$  of up to 100 kHz. The energy  $\hbar\omega_0 = 2E_r (V_0/E_r)^{1/2}$  of local oscillations in the well

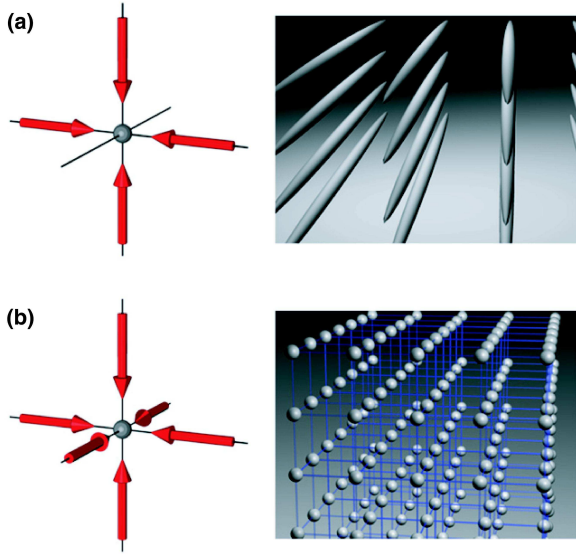


FIG. 4 Two-dimensional (a) and three-dimensional (b) optical lattice potentials formed by superimposing two or three orthogonal standing waves. For a two-dimensional optical lattice, the atoms are confined to an array of tightly confining one-dimensional potential tubes, whereas in the three-dimensional case the optical lattice can be approximated by a three dimensional simple cubic array of tightly confining harmonic oscillator potentials at each lattice site.

is on the order of several recoil energies  $E_r = \hbar^2 k^2 / 2m$ , which is a natural measure of energy scales in optical lattice potentials. Typical values of  $E_r$  are in the range of several kHz for  $^{87}\text{Rb}$ .

*Spin Dependent Optical Lattice Potentials* For large detunings of the laser light forming the optical lattices compared to the fine structure splitting of a typical alkali atom, the resulting optical lattice potentials are almost the same for all magnetic sublevels in the ground state manifold of the atom. However, for more near-resonant light fields, situations can be created for which the different magnetic sublevels can be exposed to vastly different optical potentials (Jessen and Deutsch, 1996). Such spin dependent lattice potentials can e.g. be created in a standing wave configuration formed by two counter-propagating laser beams with linear polarization vectors enclosing an angle  $\theta$  (Brennen *et al.*, 1999; Jaksch *et al.*, 1999; Jessen and Deutsch, 1996; Mandel *et al.*, 2003a). The resulting standing wave light field can be decomposed into a superposition of a  $\sigma^+$  and  $\sigma^-$  polarized standing wave laser field, giving rise to lattice potentials  $V_+(x, \theta) = V_0 \cos^2(kx + \theta/2)$  and  $V_-(x, \theta) = V_0 \cos^2(kx - \theta/2)$ . By changing the polarization angle  $\theta$  one can thereby control the relative separation between the two potentials  $\Delta x = \theta/\pi \cdot \lambda_x/2$ . When increasing  $\theta$ , both potentials shift in opposite directions and overlap again when  $\theta = n \cdot \pi$ , with  $n$  being an integer. Such a configuration has been used to coher-

ently move atoms across lattices and realize quantum gates between them (Jaksch *et al.*, 1999; Mandel *et al.*, 2003a,b). Spin dependent lattice potentials, furthermore offer a convenient way to tune the interactions between two atoms in different spin states. By shifting the spin dependent lattices relative to each other, the overlap of the on-site spatial wave function can be tuned between zero and its maximum value, thus controlling the interspecies interaction strength within a restricted range. Recently, Sebby-Strabley *et al.* (2006) have also been able to demonstrate a novel spin dependent lattice geometry, in which 2D arrays of double well potentials could be realized. Such 'superlattice' structures allow for versatile intra- and inter-well manipulation possibilities (Fölling *et al.*, 2007; Lee *et al.*, 2007; Sebby-Strabley *et al.*, 2007). A variety of lattice structures can be obtained by interfering laser beams under different angles, see e.g. (Grynberg and Robillard, 2001; Jessen and Deutsch, 1996).

## B. Band Structure

We consider in this section the single particle eigenstates in an infinite periodic potential. Any additional potential, that could originate from the intensity profile of the laser beams, or from some magnetic confinement is neglected (for the single particle spectrum in the presence of an additional harmonic confinement see (Hooley and Quintanilla, 2004)). In a simple cubic lattice, the potential is given by Eq. (36), with a tunable amplitude  $V_0$  and lattice constant  $d = \pi/k$ . In the limit  $V_0 \gg E_r$ , each well supports a number of vibrational levels, separated by an energy  $\hbar\omega_0 \gg E_r$ . At low temperatures, the atoms are restricted to the lowest vibrational level at each site. Their kinetic energy is then frozen, except for the small tunneling amplitude to neighboring sites. The associated single-particle eigenstates in the lowest band are Bloch waves with quasi-momentum  $\mathbf{q}$  and energy

$$\varepsilon_0(\mathbf{q}) = \frac{3}{2}\hbar\omega_0 - 2J(\cos q_x d + \cos q_y d + \cos q_z d) + \dots \quad (37)$$

The parameter  $J > 0$  is the gain in kinetic energy due to nearest neighbor tunneling. In the limit  $V_0 \gg E_r$ , it can be obtained from the width  $W \rightarrow 4J$  of the lowest band in the 1D Mathieu-equation

$$J = \frac{4}{\sqrt{\pi}} E_r \left( \frac{V_0}{E_r} \right)^{3/4} \exp -2 \left( \frac{V_0}{E_r} \right)^{1/2}. \quad (38)$$

For lattice depths larger than  $V_0 > 15E_r$  this approximation yields  $J$  to better than 10% accuracy (see Table I). More generally, for any periodic potential  $V_p(\mathbf{r} + \mathbf{R}) = V_p(\mathbf{r})$  which is not necessarily deep and separable, the exact eigenstates are Bloch functions  $\psi_{n,\mathbf{q}}(\mathbf{r})$ . They are characterized by a discrete band index  $n$  and a quasi-momentum  $\mathbf{q}$  within the first Brillouin-zone of the reciprocal lattice (Ashcroft and Mermin, 1976). Since Bloch

functions are multiplied by a pure phase factor  $\exp i\mathbf{q} \cdot \mathbf{R}$  upon translation by one of the lattice vectors  $\mathbf{R}$ , they are extended over the whole lattice. An alternative single-particle basis, which is more useful for describing the hopping of particles among the discrete lattice sites  $\mathbf{R}$ , are the Wannier functions  $w_{n,\mathbf{R}}(\mathbf{r})$ . They are connected with the Bloch functions by a Fourier transform

$$\psi_{n,\mathbf{q}}(\mathbf{r}) = \sum_{\mathbf{R}} w_{n,\mathbf{R}}(\mathbf{r}) e^{i\mathbf{q} \cdot \mathbf{R}} \quad (39)$$

on the lattice. The Wannier functions depend only on the relative distance  $\mathbf{r} - \mathbf{R}$  and, at least for the lowest bands, they are centered around the lattice sites  $\mathbf{R}$  (see below). By choosing a convenient normalization, they obey the orthonormality relation

$$\int d^3r w_n^*(\mathbf{r} - \mathbf{R}) w_{n'}(\mathbf{r} - \mathbf{R}') = \delta_{n,n'} \delta_{\mathbf{R},\mathbf{R}'} \quad (40)$$

for different bands  $n$  and sites  $\mathbf{R}$ . Since the Wannier functions for all bands  $n$  and sites  $\mathbf{R}$  form a complete basis, the operator  $\hat{\psi}(\mathbf{r})$  which destroys a particle at an arbitrary point  $\mathbf{r}$  can be expanded in the form

$$\hat{\psi}(\mathbf{r}) = \sum_{\mathbf{R},n} w_n(\mathbf{r} - \mathbf{R}) \hat{a}_{\mathbf{R},n}. \quad (41)$$

Here,  $\hat{a}_{\mathbf{R},n}$  is the annihilation operator for particles in the corresponding Wannier states, which are not necessarily well localized at site  $\mathbf{R}$ . The Hamiltonian for free motion on a periodic lattice then has the form

$$\hat{H}_0 = \sum_{\mathbf{R},\mathbf{R}',n} J_n(\mathbf{R} - \mathbf{R}') \hat{a}_{\mathbf{R},n}^\dagger \hat{a}_{\mathbf{R}',n}. \quad (42)$$

It describes the hopping in a given band  $n$  with matrix elements  $J_n(\mathbf{R})$ , which in general connect lattice sites at arbitrary distance  $\mathbf{R}$ . The diagonalization of this Hamiltonian by Bloch states (39) shows that the hopping matrix elements  $J_n(\mathbf{R})$  are uniquely determined by the Bloch band energies  $\varepsilon_n(\mathbf{q})$  via

$$\sum_{\mathbf{R}} J_n(\mathbf{R}) \exp i\mathbf{q} \cdot \mathbf{R} = \varepsilon_n(\mathbf{q}). \quad (43)$$

In the case of separable periodic potentials  $V_p(\mathbf{r}) = V(x) + V(y) + V(z)$ , generated by three orthogonal optical lattices, the single particle problem is one-dimensional, and a complete analysis of Wannier functions has been given by Kohn (1959). Choosing appropriate phases for the Bloch functions, there is a unique Wannier function for each band, which is real and exponentially localized. The decay  $\sim \exp -h_n|x|$  is characterized by a decay constant  $h_n$ , which is a decreasing function of the band index  $n$ . For the lowest band  $n = 0$ , where the Bloch function at  $q = 0$  is finite at the origin, the Wannier function  $w(x)$  can be chosen to be symmetric around  $x = 0$  (and correspondingly it is

$V_0/E_r$	$4J/E_r$	$W/E_r$	$J(2d)/J$	$ \langle w \phi \rangle ^2$
3	0.444109	0.451894	0.101075	0.9719
5	0.263069	0.264211	0.051641	0.9836
10	0.076730	0.076747	0.011846	0.9938
15	0.026075	0.026076	0.003459	0.9964
20	0.009965	0.009965	0.001184	0.9975

TABLE I Hopping matrix elements to nearest ( $J$ ) and next nearest neighbors ( $J(2d)$ ), bandwidth  $W$  and overlap between the Wannier function and the local gaussian ground state in 1D optical lattices. Table courtesy of M. Holthaus.

antisymmetric for the first excited band). More precisely, the asymptotic behavior of the 1D Wannier functions and the hopping matrix elements is  $|w_n(x)| \sim |x|^{-3/4} \exp -h_n|x|$  and  $J_n(R) \sim |R|^{-3/2} \exp -h_n|R|$ , respectively (He and Vanderbilt, 2001). In the particular case of a purely sinusoidal potential  $V_0 \sin^2(kx)$  with lattice constant  $d = \lambda/2$ , the decay constant  $h_0$  increases monotonically with  $V_0/E_r$ . In the deep lattice limit  $V_0 \gg E_r$ , it approaches  $h_0 d = \pi \sqrt{V_0/E_r}/2$ . It is important to realize, that even in this limit, the Wannier function does not uniformly converge to the local harmonic oscillator ground state  $\phi$  of each well: the  $w_n(x)$  decay exponentially rather than in a Gaussian manner and they always have nodes in order to guarantee the orthogonality relation (40). Yet, as shown in Table I, the overlap is near one even for shallow optical lattices.

### C. Time-of-flight and adiabatic mapping

*Sudden release* When releasing ultracold quantum gases from an optical lattice, two possible release methods can be chosen. If the lattice potential is turned off abruptly and interaction effects can be neglected, a given Bloch state with quasi-momentum  $q$  will expand according to its momentum distribution as a superposition of plane waves with momenta  $p_n = \hbar q \pm n \times 2\hbar k$ . This is a direct consequence of the fact that Bloch waves can be expressed as a superposition of plane wave states  $\exp i(\mathbf{q} + \mathbf{G}) \cdot \mathbf{r}$  with momenta  $\mathbf{q} + \mathbf{G}$ , which include arbitrary reciprocal lattice vectors  $\mathbf{G}$ . In a simple cubic lattice with lattice spacing  $d = \pi/k$ , the vectors  $\mathbf{G}$  are integer multiples of the fundamental reciprocal lattice vector  $2k$ . After a certain time-of-flight time, this momentum distribution can be imaged using standard absorption imaging methods. If only a single Bloch state is populated, as is the case for a Bose-Einstein condensate with quasi-momentum  $q = 0$ , this results in a series of interference maxima that can be observed after a time-of-flight period  $t$  (see Fig. 5). As will be shown in section III.A below, the density distribution observed after a fixed time-of-flight at position  $\mathbf{x}$ , is nothing but the momentum distribution of the particles trapped in the lattice

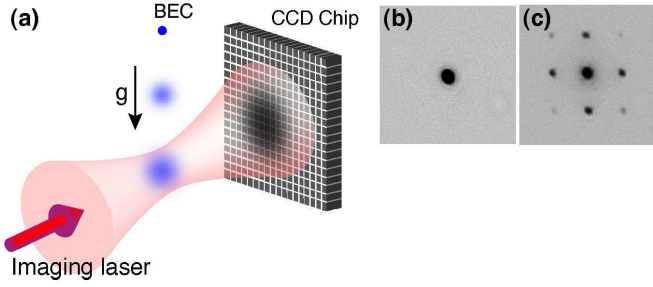


FIG. 5 Schematic setup for absorption imaging after a time-of-flight period (a). Absorption image for a BEC released from a harmonic trap (b). Absorption image for a BEC released from a shallow optical lattice ( $V_0 = 6 E_r$ )(c). Note the clearly visible interference peaks in the image.

$$n(\mathbf{x}) = \left(\frac{M}{\hbar t}\right)^3 |\tilde{w}(\mathbf{k})|^2 \mathcal{G}(\mathbf{k}). \quad (44)$$

Here  $\mathbf{k}$  is related to  $\mathbf{x}$  by  $\mathbf{k} = M\mathbf{x}/\hbar t$  due to the assumption of ballistic expansion while  $\tilde{w}(\mathbf{k})$  is the Fourier transform of the Wannier function. The coherence properties of the many-body state are characterized by the Fourier transform

$$\mathcal{G}(\mathbf{k}) = \sum_{\mathbf{R}, \mathbf{R}'} e^{i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} G^{(1)}(\mathbf{R}, \mathbf{R}') \quad (45)$$

of the one-particle density matrix  $G^{(1)}(\mathbf{R}, \mathbf{R}') = \langle \hat{a}_{\mathbf{R}}^\dagger \hat{a}_{\mathbf{R}'} \rangle$ .

In a BEC, the long range order in the amplitudes leads to a constant value of the first order coherence function  $G^{(1)}(\mathbf{R}, \mathbf{R}')$  at large separations  $|\mathbf{R} - \mathbf{R}'|$  (see Sec. X). The resulting momentum distribution coincides with the standard multiple wave interference pattern obtained with light diffracting off a material grating (see Fig. 5c and section IV.B below). The atomic density distribution observed after a fixed time-of-flight time, thus yields information on the coherence properties of the many-body system. It should be noted, however, that the observed density distribution after time-of-flight can deviate from the in-trap momentum distribution, if interaction effects during the expansion occur, or the expansion time is not so long that the initial size of the atom cloud can be neglected ("far-field approximation") (Gerbier *et al.*, 2007; Pedri *et al.*, 2001). It is important to be aware of these discrepancies and take them into account for an interpretation of the experimental data.

*Adiabatic mapping* One of the advantages of using optical lattice potentials is that the lattice depth can be dynamically controlled by simply tuning the laser power. This opens another possibility for releasing the atoms from the lattice potential e.g. by adiabatically converting a deep optical lattice into a shallow one and eventually completely turning off the lattice potential. Under adiabatic

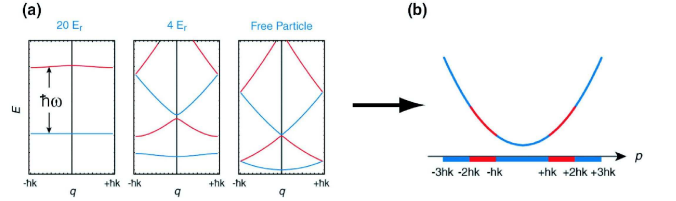


FIG. 6 (a) Bloch bands for different potential depths. During an adiabatic ramp down the quasi momentum is conserved and (b) a Bloch wave with quasi momentum  $q$  in the  $n$ th energy band is mapped onto a free particle with momentum  $p$  in the  $n$ th Brillouin zone of the lattice. Reprinted with permission from Greiner *et al.* (2001).

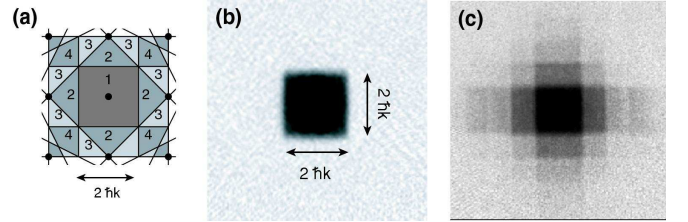


FIG. 7 (a) Brillouin zones of a 2D simple cubic optical lattice. For a homogeneously filled lowest Bloch band, an adiabatic shut off of the lattice potential leads to a homogeneously populated first Brillouin zone, which can be observed through absorption imaging after a time-of-flight expansion (b). If in addition higher Bloch bands were populated, higher Brillouin zones become populated as well (c). Reprinted with permission of Greiner *et al.* (2001).

transformation of the lattice depth, the quasi-momentum  $\mathbf{q}$  is preserved and during the turn off process a Bloch wave in the  $n$ th energy band is mapped onto a corresponding free particle momentum  $\mathbf{p}$  in the  $n$ th Brillouin zone (see Fig. 6) (Greiner *et al.*, 2001; Kastberg *et al.*, 1995; Köhl *et al.*, 2005a).

The adiabatic mapping technique has been used with both bosonic (Greiner *et al.*, 2001) and fermionic (Köhl *et al.*, 2005a) atoms. For the situation of a homogeneous filled lowest energy band, an adiabatic ramp down of the lattice potential leaves the central Brillouin zone - a square of width  $2\hbar k$  - fully occupied (see Fig. 7b). If on the other hand higher energy bands are populated, one also observes populations in higher Brillouin zones (see Fig. 7c). As in this method each Bloch wave is mapped onto a specific free-particle momentum state, it can be used to efficiently probe the distribution of the particles over Bloch states in different energy bands.

#### D. Interactions and two-particle effects

So far we have only discussed single particle behavior of ultracold atoms in optical lattices. However, the short-ranged  $s$ -wave interactions between the particles give rise

to an on-site interaction energy, when two or more atoms occupy a single lattice site. Within the pseudopotential approximation, the interaction between bosons has the form

$$\hat{H}' = \frac{g}{2} \int d^3r \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}). \quad (46)$$

Inserting the expansion Eq. (41) leads to interactions involving Wannier states both in different bands and different lattice sites. The situation simplifies, however, for a deep optical lattice and the assumption that only the lowest band is occupied. The overlap integrals are then dominated by the on-site term  $U \hat{n}_{\mathbf{R}}(\hat{n}_{\mathbf{R}} - 1)/2$ , which is non-zero, if two or more atoms are in the same Wannier state. At the two-particle level, the interaction between atoms in Wannier states localized around  $\mathbf{R}$  and  $\mathbf{R}'$  is thus reduced to a local form  $U \cdot \delta_{\mathbf{R},\mathbf{R}'}$  with

$$U = g \int d^3r |w(\mathbf{r})|^4 = \sqrt{\frac{8}{\pi}} ka E_r \left( \frac{V_0}{E_r} \right)^{3/4} \quad (47)$$

(for simplicity, the band index  $n = 0$  is omitted for the lowest band). The explicit result for the on-site interaction  $U$  is obtained by taking  $w(\mathbf{r})$  as the Gaussian ground state in the local oscillator potential. As mentioned above, this is *not* the exact Wannier wave function of the lowest band. In the deep lattice limit  $V_0 \gg E_r$ , however, the result (47) provides the asymptotically correct behavior. Note that the strength  $|U|$  of the on-site interaction *increases* with  $V_0$ , which is due to the squeezing of the Wannier wave function  $w(\mathbf{r})$ .

*Repulsively bound pairs* Consider now an optical lattice at very low filling. An occasional pair of atoms at the same site has an energy  $U$  above or below the center of the lowest band. In the attractive case  $U < 0$ , a two-particle bound state will form for sufficiently large values of  $|U|$ . In the repulsive case, in turn, the pair is expected to be unstable with respect to breakup into two separate atoms at different lattice sites, to save repulsive interaction. This process, however, is forbidden if the repulsion is *above* a critical value  $U > U_c$ . The physical origin for this surprising result is that momentum and energy conservation do not allow the two particles to separate. There are simply no free states available if the energy lies more than  $zJ$  above the band center, which is the upper edge of the tight binding band. Here  $z$  denotes the number of nearest neighbours on a lattice. Two bosons at the same lattice site will thus stay together if their interaction is sufficiently repulsive. In fact, the two-particle bound state above the band for a repulsive interaction is the precise analog of the standard bound state below the band for attractive interactions, and there is a perfect symmetry around the band center.

Such 'repulsively bound pairs' have been observed in a recent experiment by Winkler *et al.* (2006). A dilute gas of  $^{87}\text{Rb}_2$  Feshbach molecules was prepared in the vibrational ground state of an optical lattice. Ramping the

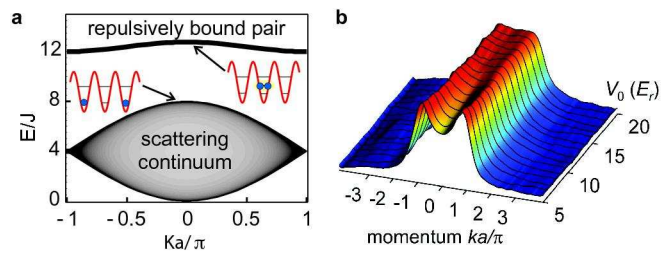


FIG. 8 Repulsively bound atom pairs. (a) Spectrum of energy  $E$  of the 1D Bose-Hubbard Hamiltonian for  $U/J = 8$  as a function of the center of mass quasi-momentum  $K$ . The Bloch band for repulsively bound pairs is located above the continuum of unbound states. (b) Experimentally measured quasi-momentum distribution of repulsively bound pairs vs. lattice depth  $V_0$ . Reprinted with permission from Winkler *et al.* (2006).

magnetic field across a Feshbach resonance to negative  $a$ , these molecules can be adiabatically dissociated and then brought back again to positive  $a$  as repulsive pairs. Since the bound state above the lowest band is built from states in which the relative momentum of the two particles is near the edge of the Brillouin zone, the presence of repulsively bound pairs can be inferred from corresponding peaks in the quasi-momentum distribution observed in a time-of-flight experiment (see Fig. 8) (Winkler *et al.*, 2006). The energy and dispersion relation of these pairs follows from solving the equation  $UG_K(E, 0) = 1$  for a bound state of two particles with center-of-mass momentum  $K$ . In close analogy to Eq. (79) below,  $G_K(E, 0)$  is the local Green function for free motion on the lattice with hopping matrix element  $2J$ . Experimentally, the optical lattice was strongly anisotropic such that tunneling is possible only along one direction. The corresponding bound state equation in one dimension can be solved explicitly, giving (Winkler *et al.*, 2006)

$$E(K, U_1) = 2J \sqrt{(2 \cos Kd/2)^2 + (U_1/2J)^2} - 4J \quad (48)$$

for the energy with respect to the upper band edge. Since  $E(K = 0, U_1) > 0$  for arbitrary small values of  $U_1 > 0$ , there is always a bound state in one dimension. By contrast, in 3D there is a finite critical value, which is  $U_c = 7.9136 J$  for a simple cubic lattice. The relevant on-site interaction  $U_1$  in one dimension is obtained from Eq. (78) below for the associated pseudopotential. With  $\ell_0$  the oscillator length for motion along the direction of hopping, it is given by

$$U_1 = g_1 \int dx |w(x)|^4 = \sqrt{\frac{2}{\pi}} \hbar \omega_\perp \cdot \frac{a}{\ell_0}. \quad (49)$$

Evidently,  $U_1$  has the transverse confinement energy  $\hbar \omega_\perp$  as the characteristic scale, rather than the recoil energy  $E_r$  of Eq. (47) in the 3D case.

*Tightly confined atom pairs* The truncation to the lowest band requires that both the thermal and the on-site



interaction energy  $U$  are much smaller than  $\hbar\omega_0$ . In the deep lattice limit  $V_0 \gg E_r$ , the condition  $U \ll \hbar\omega_0$  leads to  $ka(V_0/E_r)^{1/4} \ll 1$  using (47). This is equivalent to  $a \ll \ell_0$ , where  $\ell_0 = \sqrt{\hbar/M\omega_0} = (E_r/V_0)^{1/4}d/\pi$  is the oscillator length associated with the local harmonic motion in the deep wells of the optical lattice. The assumption of staying in the lowest band in the presence of a repulsive interaction, thus requires the scattering length to be much smaller than  $\ell_0$  which is itself smaller, but of the same order, than the lattice spacing  $d$ . For standard values  $a \approx 5$  nm and  $d \approx 0.5$   $\mu\text{m}$ , this condition is very well justified. In the vicinity of Feshbach resonances, however, the scattering lengths become comparable to the lattice spacing. A solution of the two-particle problem in the presence of an optical lattice for arbitrary values of the ratio  $a/\ell_0$  has been given by Fedichev *et al.* (2004). Neglecting interaction induced couplings to higher bands, they have shown that the effective interaction at energies smaller than the bandwidth is again described by a pseudopotential. For repulsive interactions  $a > 0$ , the associated effective scattering length reaches a bound  $a_{\text{eff}} \approx d$  on the order of the lattice spacing, even if  $a \rightarrow \infty$  near a Feshbach resonance. In the case where the free space scattering length is negative,  $a_{\text{eff}}$  exhibits a geometric resonance which precisely describes the formation of a two-particle bound state at  $|U| = 7.9136J$  discussed above.

This analysis is based on the assumption that the particles remain in a given band even in the presence of strong interactions. Near Feshbach resonances, however, this is usually not the case. In order to address the question of interaction induced transitions between different bands, it is useful to consider the simple problem of two interacting particles in a harmonic well (Busch *et al.*, 1998). Provided the range of the interaction is much smaller than the oscillator length  $\ell_0$ , the interaction of two particles in a single well is still described by a pseudopotential. The ratio of the scattering length  $a$  to  $\ell_0$ , however, may be arbitrary. The corresponding energy levels  $E = \hbar\omega_0(3/2 - \Omega)$  as a function of the ratio  $\ell_0/a$  follow from the transcendental equation

$$\frac{\ell_0}{a} = \frac{\sqrt{2}\Gamma(\Omega/2)}{\Gamma((\Omega-1)/2)} = f_3(\Omega). \quad (50)$$

where  $\Gamma(z)$  is the standard Gamma function. In fact, this is the analytical continuation to an arbitrary sign of the dimensionless binding energy  $\Omega$  in Eq. (82) below, for the case  $n = 3$ , since a harmonic confinement is present in all three spatial directions.

As shown in Fig. 9, the discrete levels for the *relative* motion of two particles form a sequence which, at infinite scattering length, is shifted upwards by precisely  $\hbar\omega_0$  compared to the non-interacting levels at zero angular momentum  $E^{(0)}(n_r) = \hbar\omega_0(2n_r + 3/2)$ . In particular, a change of the scattering length from small positive to small negative values, through a Feshbach resonance where  $a$  diverges, increases the energy by  $2\hbar\omega_0$  while the particles are transferred to the next higher level  $n_r = 1$ . Feshbach resonances can thus be used to switch pairs

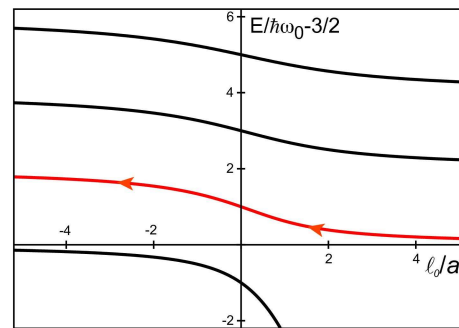


FIG. 9 Energy spectrum of two interacting particles in a 3D harmonic oscillator potential from Eq. (50). The arrows indicate the transfer of a pair in the ground state to the first excited level by sweeping across a Feshbach resonance. There is a single bound state below the lowest oscillator level, whose energy has been measured by Stöferle *et al.* (2006).

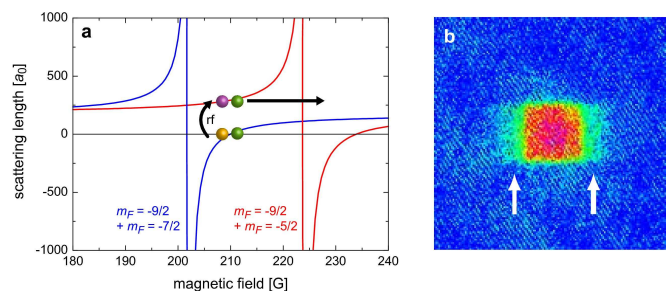


FIG. 10 Experimentally observed interaction induced transitions between Bloch bands. **a** Two Feshbach resonances between the  $|F = 9/2, m_F = -9/2\rangle$  and  $|F = 9/2, m_F = -7/2\rangle$  states (left) and the  $|F = 9/2, m_F = -9/2\rangle$  and  $|F = 9/2, m_F = -5/2\rangle$  states (right) are exploited to tune the interactions in the gas. **b** Quasi-momentum distribution for a final magnetic field of  $B = 233$  G. Arrows indicate the atoms in the higher bands. Reprinted with permission from Köhl *et al.* (2005a).

of particles in individual wells of a deep optical lattice, where tunneling is negligible, to higher bands. Experimentally, this has been studied by Köhl *et al.* (2005a). Starting from a two-component gas of fermionic  $^{40}\text{K}$  at  $a \approx 0$  and unit filling, i.e. with two fermions at each lattice site in the center of the trap, the atoms were transferred to a different hyperfine state and the magnetic field was then increased beyond the associated Feshbach resonance at  $B_0 = 224$  G<sup>8</sup>. The resulting transfer of particles into higher bands is then revealed by observing the quasi-momentum distribution in time-of-flight images after adiabatically turning off the optical lattice, see Fig. 10. It was pointed out by Ho (2006) that such Feshbach sweeps open novel possibilities to create fermionic Mott insulat-

<sup>8</sup> Changing  $a$  from positive to negative values avoids creation of molecules in an adiabatic ramp.

ing states.

### III. DETECTION OF CORRELATIONS

In order to probe interacting many-body quantum states with strong correlations, it is essential to use detection methods that are sensitive to higher order correlations. Here, recent proposals for using analogues of quantum optical detection techniques have proven to be novel tools for analyzing strongly interacting quantum matter (Altman *et al.*, 2004; Duan, 2006; Gritsev *et al.*, 2006; Niu *et al.*, 2006; Polkovnikov *et al.*, 2006; Zhang *et al.*, 2007). Most of these techniques make use of the fact that the quantum fluctuations in many observables, such as e.g. the visibility of the interference pattern between two released quantum gases or the fluctuations in the momentum distribution after release from the trap, contain information of the initial correlated quantum state. Whereas in the usual time-of-flight momentum distributions one essentially probes first order coherence properties of the system, the noise-correlation techniques introduced below will yield information on the second (or higher) order correlation properties and therefore possible long range order in real space. Such correlation techniques in expanding atom clouds have begun to be successfully employed in recent experiments, probing the momentum correlations between atomic fragments emerging from a dissociated molecule (Greiner *et al.*, 2005b), revealing the quantum statistics of bosonic or fermionic atoms in an optical lattice (Fölling *et al.*, 2005; Rom *et al.*, 2006), or in exploring the Kosterlitz-Thouless transition in two-dimensional Bose-Einstein condensates (Hadzibabic *et al.*, 2006). All the correlation techniques for strongly correlated quantum gases can also greatly benefit from efficient single atom detectors that have recently begun to be used in the context of cold quantum gases (Jeltes *et al.*, 2007; Öttl *et al.*, 2005; Schellekens *et al.*, 2005).

#### A. Time-of-flight versus noise correlations

Let us begin by considering a quantum gas released from a trapping potential. After a finite time-of-flight time  $t$ , the resulting density distribution yields a three-dimensional density distribution  $n_{3D}(\mathbf{x})$ <sup>9</sup>. If interactions can be neglected during time-of-flight, the average density distribution is related to the in-trap quantum state via:

$$\begin{aligned} \langle \hat{n}_{3D}(\mathbf{x}) \rangle_{\text{tof}} &= \langle \hat{a}_{\text{tof}}^\dagger(\mathbf{x}) \hat{a}_{\text{tof}}(\mathbf{x}) \rangle_{\text{tof}} \\ &\approx \langle \hat{a}^\dagger(\mathbf{k}) \hat{a}(\mathbf{k}) \rangle_{\text{trap}} = \langle \hat{n}_{3D}(\mathbf{k}) \rangle_{\text{trap}}, \end{aligned} \quad (51)$$

where  $\mathbf{k}$  and  $\mathbf{x}$  are related by the ballistic expansion condition  $\mathbf{k} = M\mathbf{x}/\hbar t$  (a factor  $(M/\hbar t)^3$  from the transformation of the volume elements  $d^3x \rightarrow d^3k$  is omitted, see Eq. (44)). Here we have used the fact that for long time-of-flight times, the initial size of the atom cloud in the trap can be neglected. It is important to realize, that in each experimental image, a single realization of the density is observed, not an average. Moreover, each pixel in the image records on average a substantial number  $N_\sigma$  of atoms. For each of those pixels, however, the number of atoms recorded in a *single realization* of an experiment will exhibit shot noise fluctuations of relative order  $1/\sqrt{N_\sigma}$  which will be discussed below. As shown in Eq. (51), the density distribution after time-of-flight represents a momentum distribution reflecting the first order coherence properties of the in-trap quantum state. This assumption is however only correct, if during the expansion process interactions between the atoms do not modify the initial momentum distribution, which we will assume throughout the text. When the interactions between the atoms have been enhanced, e.g. by a Feshbach resonance, or a high density sample is prepared, such an assumption is not always valid. Near Feshbach resonances one therefore often ramps back to the zero crossing of the scattering length before expansion.

*Density-density correlations in time-of-flight images* Let us now turn to the observation of density-density correlations in the expanding atom clouds (Altman *et al.*, 2004). These are characterized by the density-density correlation function

$$\langle \hat{n}(\mathbf{x}) \hat{n}(\mathbf{x}') \rangle = \langle \hat{n}(\mathbf{x}) \rangle \langle \hat{n}(\mathbf{x}') \rangle g^{(2)}(\mathbf{x}, \mathbf{x}') + \delta(\mathbf{x} - \mathbf{x}') \langle \hat{n}(\mathbf{x}) \rangle \quad (52)$$

which contains the normalized pair distribution  $g^{(2)}(\mathbf{x}, \mathbf{x}')$  and a self correlation term. Relating the operators after time-of-flight expansion to the in-trap momentum operators, using Eq. (51), one obtains:

$$\begin{aligned} \langle \hat{n}_{3D}(\mathbf{x}) \hat{n}_{3D}(\mathbf{x}') \rangle_{\text{tof}} &\approx \langle \hat{a}^\dagger(\mathbf{k}) \hat{a}(\mathbf{k}) \hat{a}^\dagger(\mathbf{k}') \hat{a}(\mathbf{k}') \rangle_{\text{trap}} = \\ &\langle \hat{a}^\dagger(\mathbf{k}) \hat{a}^\dagger(\mathbf{k}') \hat{a}(\mathbf{k}') \hat{a}(\mathbf{k}) \rangle_{\text{trap}} + \delta_{\mathbf{k}\mathbf{k}'} \langle \hat{a}^\dagger(\mathbf{k}) \hat{a}(\mathbf{k}) \rangle_{\text{trap}} \end{aligned} \quad (53)$$

The last term on the rhs of the above equation is the autocorrelation term and will be dropped in the subsequent discussion, as it only contributes to the signal for  $\mathbf{x} = \mathbf{x}'$  and contains no more information about the initial quantum state, than the momentum distribution itself. The first term, however, shows that for  $\mathbf{x} \neq \mathbf{x}'$ , subtle momentum-momentum correlations of the in-trap quantum states are present in the noise-correlation signal of the expanding atom clouds.

Let us discuss the obtained results for two cases that have been analyzed in the experiment: (1) Ultracold atoms in a Mott insulating state or a fermionic band insulating state released from a 3D optical lattice and (2) two interfering one-dimensional quantum gases separated by a distance  $\mathbf{d}$ .

<sup>9</sup> In this section, we denote in-trap spatial coordinates by  $\mathbf{r}$  and spatial coordinates after time-of-flight by  $\mathbf{x}$  for clarity.

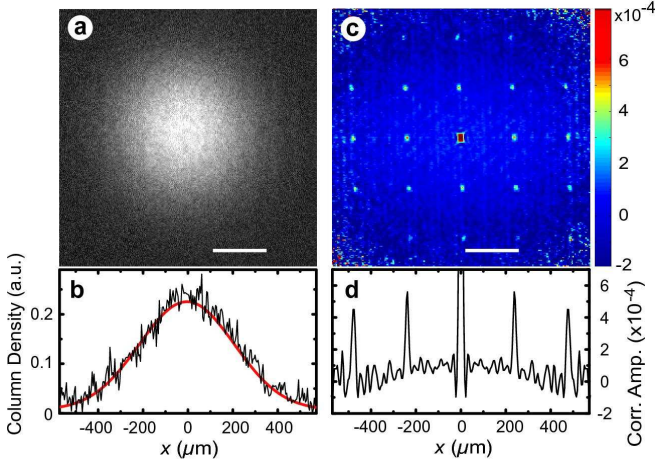


FIG. 11 Noise correlations of a Mott insulator released from a 3D optical lattice. (a) Single shot absorption image of a Mott insulator released from an optical lattice and associated cut through the image (b). A statistical correlation analysis over several independent images such as the one in (a) yields the correlation function (c). A cut through this two-dimensional correlation function reveals a Hanbury-Brown & Twiss type bunching of the bosonic atoms (d). Reprinted with permission from Fölling *et al.* (2005).

## B. Noise correlations in bosonic Mott and fermionic band insulators

Consider a bosonic Mott insulating state or a fermionic band insulator in a three-dimensional simple cubic lattice. In both cases, each lattice site  $\mathbf{R}$  is occupied by a fixed atom number  $n_{\mathbf{R}}$ . Such a quantum gas is released from the lattice potential and the resulting density distribution is detected after a time-of-flight  $t$ . In a deep optical lattice, the (in-trap) field operator  $\hat{\psi}(\mathbf{r})$  can be expressed as a sum over destruction operators  $\hat{a}_{\mathbf{R}}$  of localized Wannier states, by using the expansion (41) and neglecting all but the lowest band. The field operator for destroying a particle with momentum  $\mathbf{k}$  is therefore given by

$$\hat{a}(\mathbf{k}) = \int e^{-i\mathbf{k}\mathbf{r}} \hat{\psi}(\mathbf{r}) d^3r \simeq \tilde{w}(\mathbf{k}) \sum_{\mathbf{R}} e^{-i\mathbf{k}\mathbf{R}} \hat{a}_{\mathbf{R}}, \quad (54)$$

where  $\tilde{w}(\mathbf{k})$  denotes the Wannier function in momentum space.

For the two states considered here, the expectation value in Eq. (53) factorizes into one-particle density matrices  $\langle \hat{a}_{\mathbf{R}}^\dagger \hat{a}_{\mathbf{R}'} \rangle = n_{\mathbf{R}} \delta_{\mathbf{R},\mathbf{R}'}$  with vanishing off-diagonal order. The density-density correlation function after a time-of-flight is then given by (omitting the autocorrelation term of order  $1/N$ )

$$\begin{aligned} \langle \hat{n}_{3D}(\mathbf{x}) \hat{n}_{3D}(\mathbf{x}') \rangle &= |\tilde{w}(M\mathbf{x}/\hbar t)|^2 |\tilde{w}(M\mathbf{x}'/\hbar t)|^2 N^2 \\ &\times \left[ 1 \pm \frac{1}{N^2} \left| \sum_{\mathbf{R}} e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{R}(M/\hbar t)} n_{\mathbf{R}} \right|^2 \right]. \quad (55) \end{aligned}$$

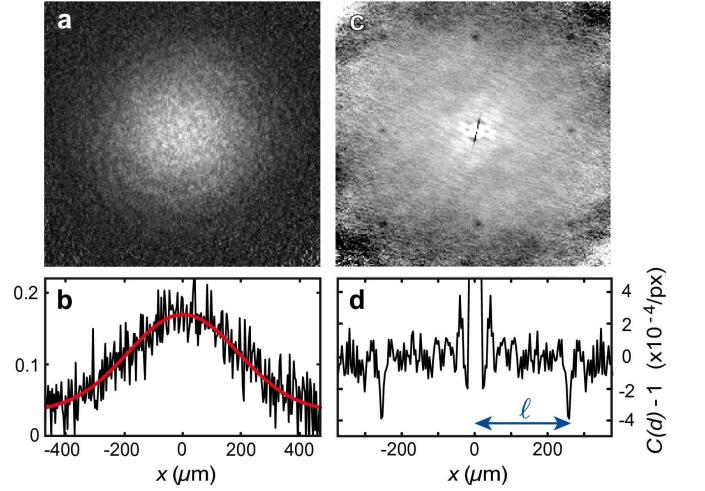


FIG. 12 Noise correlations of a band insulating Fermi gas. Instead of the correlation "bunching" peaks observed in Fig. 11 the fermionic quantum gas shows an HBT type antibunching effect, with dips in the observed correlation function. Reprinted with permission from Rom *et al.* (2006)

The plus sign in the above equation corresponds to the case of bosonic particles and the minus sign to the case of fermionic particles in a lattice. Both in a Mott state of bosons and in a filled band of fermions, the local occupation numbers  $n_{\mathbf{R}}$  are fixed integers. The above equation then shows that correlations or anticorrelations in the density-density expectation value appear for bosons or fermions, whenever the difference  $\mathbf{k} - \mathbf{k}'$  is equal to a reciprocal lattice vector  $\mathbf{G}$  of the underlying lattice. In real space, where the images are actually taken, this corresponds to spatial separations for which

$$|\mathbf{x} - \mathbf{x}'| = \ell = \frac{2\hbar t}{\lambda M}. \quad (56)$$

Such spatial correlations or anticorrelations in the quantum noise of the density distribution of expanding atom clouds can in fact be traced back to the famous Hanbury Brown & Twiss effect (Baym, 1998; Brown and Twiss, 1956a,b) and its analogue for fermionic particles (Henny *et al.*, 1999; Iannuzzi *et al.*, 2006; Jelts *et al.*, 2007; Kiesel *et al.*, 2002; Oliver *et al.*, 1999; Rom *et al.*, 2006). For the case of two atoms localized at two lattice sites this can be readily understood in the following way: there are two possible ways for the particles to reach two detectors at positions  $\mathbf{x}$  and  $\mathbf{x}'$  which differ by exchange. A constructive interference for the case of bosons or a destructive interference for the case of fermions then leads to correlated or anticorrelated quantum fluctuations that are registered in the density-density correlation function (Altman *et al.*, 2004; Baym, 1998).

The correlations for the case of a bosonic Mott insulating state and anticorrelations for the case of a fermionic band insulating state have recently been observed experimentally (Fölling *et al.*, 2005; Rom *et al.*,

2006; Spielman *et al.*, 2007). In these experiments several single images of the desired quantum state are recorded after releasing the atoms from the optical trapping potential and observing them after a finite time-of-flight time (for a single of these images see e.g. Fig. 11a or Fig. 12a). These individually recorded images only differ in the atomic shot noise from each other. A set of such absorption images is then processed to yield the spatially averaged second order correlation function  $g_{\text{exp}}^{(2)}(\mathbf{b})$ :

$$g_{\text{exp}}^{(2)}(\mathbf{b}) = \frac{\int \langle n(\mathbf{x} + \mathbf{b}/2) \cdot n(\mathbf{x} - \mathbf{b}/2) \rangle d^2\mathbf{x}}{\int \langle n(\mathbf{x} + \mathbf{b}/2) \rangle \langle n(\mathbf{x} - \mathbf{b}/2) \rangle d^2\mathbf{x}}. \quad (57)$$

As shown in Fig. 11, the Mott insulating state exhibits long range order in the pair correlation function  $g^{(2)}(\mathbf{b})$ . This order is not connected with the trivial periodic modulation of the average density imposed by the optical lattice after time-of-flight, which is factored out in  $g^{(2)}(\mathbf{x}, \mathbf{x}')$  (see Eq. (52)). Therefore, in the superfluid regime, one expects  $g^{(2)}(\mathbf{x}, \mathbf{x}') \equiv 1$  despite the periodic density modulation in the interference pattern after time-of-flight. It is interesting to note that the correlations or anticorrelations can also be traced back to the enhanced fluctuations in the population of the Bloch waves with quasi momentum  $q$  for the case of the bosonic particles and the vanishing fluctuations in the population of Bloch waves with quasi momentum  $q$  for the case of fermionic particles (Rom *et al.*, 2006).

Note that in general the signal amplitude obtained in the experiments for the correlation function deviates significantly from the theoretically expected value of 1. In fact, one typically observes signal levels of  $10^{-4} - 10^{-3}$  (see Figs. (11,12)). This can be explained by the finite optical resolution when imaging the expanding atomic clouds, thus leading to a broadening of the detected correlation peaks and thereby a decreased amplitude, as the signal weight in each correlation peak is preserved in the detection process. Using single atom detectors with higher spatial and temporal resolution such as the ones used in Schellekens *et al.* (2005) and Jelten *et al.* (2007), one can overcome such limitations and thereby also evaluate higher order correlation functions.

### C. Statistics of interference amplitudes for low-dimensional quantum gases

As a second example, we consider two bosonic one-dimensional quantum gases oriented along the  $z$ -direction and separated by a distance  $d$  along the  $x$ -direction. The density-density correlation function at positions  $\mathbf{x} = (x, y = 0, z)$  and  $\mathbf{x}' = (x', y' = 0, z')$  after time-of-flight is then given by formula:

$$\langle \hat{n}_{3D}(\mathbf{x}) \hat{n}_{3D}(\mathbf{x}') \rangle = \langle \hat{a}_{tof}^\dagger(\mathbf{x}) \hat{a}_{tof}^\dagger(\mathbf{x}') \hat{a}_{tof}(\mathbf{x}) \hat{a}_{tof}(\mathbf{x}') \rangle + \delta_{\mathbf{x}\mathbf{x}'} n(\mathbf{x}) n(\mathbf{x}'). \quad (58)$$

The operators for the creation  $\hat{a}_{tof}^\dagger(\mathbf{x})$  and destruction  $\hat{a}_{tof}(\mathbf{x})$  of a particle after a time-of-flight period at position  $\mathbf{x}$  can be related to the in-trap operators describing the trapped quantum gases 1 and 2. Since the expansion mostly occurs along the initially strongly confined directions  $x$  and  $y$ , we can neglect for simplicity the expansion along the axial direction  $z$  and obtain for  $y = 0$

$$\hat{a}_{tof}(\mathbf{x}) = \hat{a}_1(z) e^{ik_1x} + \hat{a}_2(z) e^{ik_2x}, \quad (59)$$

with  $k_{1,2} = M(x \pm d/2)/\hbar t$ . The interference part

$$\langle \hat{a}_2^\dagger(z_1) \hat{a}_1(z_1) \hat{a}_1^\dagger(z_2) \hat{a}_2(z_2) \rangle \times \left[ e^{ik(x_1-x_2)} + c.c. \right], \quad (60)$$

of the correlation function in Eq. (58) is an oscillatory function, with wavevector  $k = Md/\hbar t$ . In a standard absorption image, with the propagation direction of the imaging beam pointing along the  $z$ -direction, one has to additionally take into account an integration along this direction over a length  $L$  from which a signal is recorded. Using the above equation, one obtains for this case:

$$\langle \hat{n}(x) \hat{n}(x') \rangle_{\text{int}} = \langle |\hat{A}_k|^2 \rangle \left[ e^{ik(x_1-x_2)} + c.c. \right], \quad (61)$$

with the observable

$$\hat{A}_k = \int_{-L/2}^{L/2} dz \hat{a}_1^\dagger(z) \hat{a}_2(z). \quad (62)$$

The above observable characterizes the visibility of an interference pattern obtained in a single run of the experiment. Note that the density-density correlations in the expanding atom clouds are however determined by the expectation value of

$$\langle |\hat{A}_k|^2 \rangle = \int dz_1 \int dz_2 \langle \hat{a}_2^\dagger(z_1) \hat{a}_1(z_1) \hat{a}_1^\dagger(z_2) \hat{a}_2(z_2) \rangle, \quad (63)$$

which can be obtained by a statistical analysis of the visibility of the interference patterns obtained in several runs of the experiment. The basic example in this context is the observation of a pronounced interference pattern in a single realization of two overlapping but independent condensates with fixed particle numbers by Andrews *et al.* (1997). As discussed e.g. by Castin and Dalibard (1997), the detection of particles at certain positions entails a non-vanishing interference amplitude in a single realization, whose typical visibility is determined by  $\langle |\hat{A}_k|^2 \rangle \neq 0$ . Averaging over many realizations, in turn, completely eliminates the interference because  $\langle \hat{A}_k \rangle = 0$  (Leggett, 2001).

For the case of identical (but still independent) quantum gases, one can simplify Eq. (63), to yield:

$$\langle |\hat{A}_k|^2 \rangle \approx L \int_{-L/2}^{L/2} dz \langle \hat{a}^\dagger(z) \hat{a}(0) \rangle^2 = L \int |G^{(1)}(z)|^2 dz. \quad (64)$$

The fluctuations in the interference pattern are thus directly linked to the coherence properties of the one-dimensional quantum systems. For the case of Luttinger liquids, the one-particle density matrix  $G^{(1)}(z) \sim z^{-1/(2K)}$  at zero temperature decays algebraically with an exponent determined by the Luttinger parameter  $K$  (see section V.B). As a result, the interference amplitudes exhibit an anomalous scaling  $\langle |\hat{A}_k|^2 \rangle \propto L^{-1/K}$  (Polkovnikov *et al.*, 2006). By determining higher moments  $\langle |\hat{A}_k|^{2n} \rangle$  of arbitrary order  $n$  of the visibility in an interference experiment, one can characterize the full distribution function of the normalized random variable  $|\hat{A}_k|^2 / \langle |\hat{A}_k|^2 \rangle$ . Full knowledge of the distribution function in fact amounts to a complete characterization of the correlations in the many-body systems, as has been recently shown by Gritsev *et al.* (2006). For the case of 1D Bose-Einstein condensates, this has recently been tested by Hofferbeth *et al.* (2007).

The above analysis for one-dimensional quantum systems can be readily extended to the case of two-dimensional systems (Polkovnikov *et al.*, 2006) and has been directly used to detect a Berezinskii-Kosterlitz-Thouless-transition (Hadzibabic *et al.*, 2006) with ultracold quantum gases (see sec. VI). For the case of lattice based systems it has been shown that noise correlations can be a powerful way to reveal e.g. an antiferromagnetically ordered phase of two-component bosonic or fermionic quantum gases (Altman *et al.*, 2004; Werner *et al.*, 2005), to characterize Bose-Fermi mixtures (Ahufinger *et al.*, 2005; Wang *et al.*, 2005) and quantum phases with disorder (Rey *et al.*, 2006), as well as to detect supersolid phases (Scarola *et al.*, 2006).

#### IV. MANY-BODY EFFECTS IN OPTICAL LATTICES

As a first example, illustrating how cold atoms in optical lattices can be used to study genuine many-body phenomena in dilute gases, we discuss the Mott-Hubbard transition for bosonic atoms. Following the original idea by Jaksch *et al.* (1998), this transition was first observed experimentally by Greiner *et al.* (2002a). The theory of the underlying quantum phase transition is based on the Bose-Hubbard model, originally introduced by Fisher *et al.* (1989) to describe the destruction of superfluidity due to strong interactions and disorder.

##### A. Bose-Hubbard model

A conceptually simple model to describe cold atoms in an optical lattice at finite density is obtained by combining the kinetic energy (42) in the lowest band with the on-site repulsion arising from (46) in the limit of a sufficiently deep optical lattice. More precisely, the Bose-Hubbard model (BHM) is obtained from a general many-body Hamiltonian with a pseudopotential interaction under the assumptions

- both the thermal and the mean interaction energies at a single site are much smaller than the separation  $\hbar\omega_0$  to the first excited band.
- the Wannier functions decay essentially within a single lattice constant.

Under these assumptions, only the lowest band needs to be taken into account in Eq. (41). Moreover, the hopping matrix elements  $J(\mathbf{R})$  are non-negligible only for  $\mathbf{R} = 0$  or to nearest neighbors (NN) in Eq. (42) and the interaction constants are dominated by the on-site contribution (47). This leads to the Bose-Hubbard model (BHM)

$$\hat{H} = -J \sum_{\langle \mathbf{R}, \mathbf{R}' \rangle} \hat{a}_{\mathbf{R}}^\dagger \hat{a}_{\mathbf{R}'} + \frac{U}{2} \sum_{\mathbf{R}} \hat{n}_{\mathbf{R}} (\hat{n}_{\mathbf{R}} - 1) + \sum_{\mathbf{R}} \epsilon_{\mathbf{R}} \hat{n}_{\mathbf{R}}. \quad (65)$$

( $\langle \mathbf{R}, \mathbf{R}' \rangle$  denotes a sum over all lattice sites  $\mathbf{R}$  and its nearest neighbors at  $\mathbf{R}' = \mathbf{R} + \mathbf{d}$ , where  $\mathbf{d}$  runs through the possible nearest neighbor vectors). The hopping matrix element  $J(\mathbf{d}) = -J < 0$  to nearest neighbors is always negative in the lowest band, because the ground state must have zero momentum  $\mathbf{q} = 0$  in a time-reversal invariant situation. For a separable lattice and in the limit  $V_0 \gg E_r$ , it is given by Eq. (38). More generally, the hopping matrix elements are determined by the exact band energy using Eq. (43). An alternative, but more indirect, expression is  $J(\mathbf{R}) = \langle w(\mathbf{R}) | \hat{H}_0 | w(0) \rangle$  (Jaksch *et al.*, 1998).

Since the standard BHM includes next neighbor hopping only, a convenient approximation for  $J$  in Eq. (65) is obtained by simply adjusting it to the given bandwidth. Concerning the on-site repulsion  $U$ , which disfavors configurations with more than one boson at a given site, its precise value as determined by Eq. (47) requires the exact Wannier function. In the low filling  $\bar{n} \sim 1$  regime, it follows from the single-particle Bloch states via Eq. (39). For higher fillings, the mean-field repulsion on each lattice site leads to an admixture of excited states in each well and eventually to a description, where for  $\bar{n} \gg 1$  one has a lattice of coupled Josephson junctions with a Josephson coupling  $E_J = 2\bar{n}J$  and an effective 'charging energy'  $U$  (Cataliotti *et al.*, 2001; Fisher *et al.*, 1989). For intermediate fillings, the Wannier functions entering both the effective hopping matrix element  $J$  and on-site repulsion  $U$  have to be adjusted to account for the mean-field interaction (Li *et al.*, 2006). The change in the on-site interaction energy with filling has been observed experimentally by Campbell *et al.* (2006). In a more detailed description, the effects of interactions at higher filling can be accounted for by a multi-orbital generalization of the Gross-Pitaevskii ansatz (Alon *et al.*, 2005). This leads to effective 'dressed' Wannier states which include higher bands and coupling between different sites. The last term with a variable on-site energy  $\epsilon_{\mathbf{R}} = \tilde{V}(\mathbf{R})$  describes the effect of the smooth trapping potential  $\tilde{V}(\mathbf{r})$ . It includes the constant band center energy, arising from

the  $J(\mathbf{R}=0)$ -term of the hopping contribution (42) and acts like a spatially varying chemical potential.

The BHM describes the competition between the kinetic energy  $J$  which is gained by delocalizing particles over the lattice sites in an extended Bloch state and the repulsive on-site interaction  $U$ , which disfavors having more than one particle at any given site. In an optical lattice loaded with cold atoms, the ratio  $U/J$  between these two energies can be changed easily by varying the dimensionless depth  $V_0/E_r$  of the optical lattice. Indeed, from Eqs. (38) and (47), the ratio  $U/J \sim (a/d) \cdot \exp(2\sqrt{V_0/E_r})$  increases exponentially with the lattice depth. Of course, to see strong interaction effects, the average site occupation  $\langle \hat{n}_{\mathbf{R}} \rangle$  needs to be on the order of one, otherwise the atoms never see each other. This was the situation for cold atoms in optical lattices in the 90's, studied e.g. by Grynberg *et al.* (1993); Hemmerich and Hänsch (1993); Kastberg *et al.* (1995); Westbrook *et al.* (1990).

## B. Superfluid-Mott-Insulator transition

The BHM Eq. (65) is not an exactly soluble model, not even in one dimension, despite the fact that the corresponding continuum model in 1D, the Lieb-Liniger model, is exactly soluble. Nevertheless, the essential physics of the model and, in particular, the existence and properties of the quantum phase transition which the BHM exhibits as a function of  $U/J$  are rather well understood (Fisher *et al.*, 1989). In fact, for the 3D case and effectively unit filling, the existence of a quantum phase transition from a homogeneous BEC to a MI with a nonzero gap has been proven rigorously in a model of hard core bosons in the presence of a staggered field by Aizenman *et al.* (2004). Let us first discuss the limiting cases, which describe the two possible phases within the BHM.

*Superfluid phase* In the trivial limit  $U = 0$ , the many-body ground state is simply an ideal BEC where all  $N$  atoms are in the  $\mathbf{q} = 0$  Bloch-state of the lowest band. Including the normalization factor in a lattice with  $N_L$  sites, this state can be written in the form

$$|\Psi_N\rangle(U=0) = \frac{1}{\sqrt{N!}} \left( \frac{1}{\sqrt{N_L}} \sum_{\mathbf{R}} \hat{a}_{\mathbf{R}}^\dagger \right)^N |0\rangle. \quad (66)$$

In the limit  $U/J \rightarrow 0$  therefore, the ground state of the BHM is a Gross-Pitaevskii type state with a condensate fraction which is trivially equal to one. The critical temperature of the ideal Bose gas in an optical lattice at filling  $\bar{n} = 1$  can be obtained from the condition  $\int d\varepsilon g(\varepsilon) n_B(\beta_c \varepsilon) = 1$ , where  $g(\varepsilon)$  is the density of states in the lowest band and  $n_B(x) = (\exp(x) - 1)^{-1}$  the Bose-Einstein distribution. This gives  $k_B T_c = 5.59 J$ . In the presence of an optical lattice, therefore, the critical temperature for BEC is significantly reduced compared with the free space situation, essentially due to

the increased effective mass  $M^*$  of the particles in the lattice. The relevant parameter, however, is not the temperature but the entropy. Indeed, by starting with a deeply degenerate gas and adiabatically switching on the optical lattice, the degeneracy parameter stays constant and the temperature is essentially reduced by a factor  $M/M^*$  (Blakie and Porto, 2004; Hofstetter *et al.*, 2002; Olshanii and Weiss, 2002).

For a sufficiently large system  $N, N_L \rightarrow \infty$  at fixed (not necessarily integer) density  $N/N_L$  (in the experiment (Greiner *et al.*, 2002a), the total number of occupied lattice sites was about  $10^5$ ), the perfect condensate Eq. (66) becomes indistinguishable in practice from a coherent state

$$\exp(\sqrt{N} \hat{a}_{q=0}^\dagger) |0\rangle = \prod_{\mathbf{R}} \left( \exp \left[ \sqrt{\frac{N}{N_L}} \hat{a}_{\mathbf{R}}^\dagger \right] |0\rangle_{\mathbf{R}} \right). \quad (67)$$

It factorizes into a product of *local* coherent states at every lattice site  $\mathbf{R}$  with average  $\bar{n} = \langle \hat{n} \rangle = N/N_L$  because boson operators at different sites commute. The probability distribution for the number of atoms at any given site for a perfect BEC in an optical lattice is therefore Poissonian with a standard deviation given by  $\sigma(\bar{n}) = \sqrt{\bar{n}}$ . Taking  $N = N_L$ , i.e. an average density such that there is one atom for each lattice site, there is a  $1 - 2/e = 0.27$  probability that any given site is occupied with more than one atom. The kinetic energy minimization requirement that every atom wants to be at all lattice sites with equal amplitude thus necessarily leads to a substantial probability of finding more than one atom on a given site. At finite repulsion  $U > 0$ , such configurations are, of course, disfavoured.

*Mott insulating phase* To understand the behavior in the opposite limit  $U \gg J$ , it is useful to consider the case of unit filling, i.e. the number  $N$  of atoms is precisely equal to the number  $N_L$  of lattice sites. In the limit  $U \gg J$ , hopping of the atoms is negligible and the obvious ground state

$$|\Psi_{N=N_L}\rangle(J=0) = \left( \prod_{\mathbf{R}} \hat{a}_{\mathbf{R}}^\dagger \right) |0\rangle \quad (68)$$

is a simple product of local Fock-states with precisely one atom per site. With increasing  $J$ , the atoms start to hop around, which necessarily involves double occupancy, increasing the energy by  $U$ . Now as long as the gain  $J$  in kinetic energy due to hopping is smaller than  $U$ , the atoms remain localized. For any  $J \neq 0$ , however, the ground state is no longer a simple product state as in Eq. (68). Once  $J$  becomes of order or larger than  $U$ , the gain in kinetic energy outweighs the repulsion due to double occupancies. The atoms then undergo a transition to a superfluid, in which they are delocalized over the whole lattice. This is a sharp quantum phase transition in the thermodynamic limit, because the state (66), in contrast to (68), exhibits off-diagonal long range order, which cannot disappear in a continuous manner. By

contrast, the evolution between these two states is completely smooth for say two particles in two wells, where a simple crossover occurs from a state with a well defined relative phase at  $J \gg U$  to one with a well defined particle number in each well at  $J \ll U$ .

*Phase diagram* The zero temperature phase diagram of the homogeneous BHM is shown schematically in Fig. 13a as a function of  $J/U$ , with the density controlled by a chemical potential  $\mu$ . At  $U/J \rightarrow 0$ , the kinetic energy dominates and the ground state is a delocalized superfluid, described by Eq. (67) to lowest order. At large values of  $U/J$ , interactions dominate and one obtains a series of Mott-insulating (MI) phases with fixed integer filling  $\bar{n} = 1, 2, \dots$ . These states are *incompressible*, implying that their density remains unchanged upon varying the chemical potential. In fact, it is the property  $\partial n / \partial \mu = 0$ , which is *the* defining property of a MI, and not the existence of local Fock states which only exist at  $J = 0$ . The transition between the SF and MI phases is associated with the loss of long range order in the one-particle density matrix  $g^{(1)}(\mathbf{x})$ . In the 3D case, the order parameter of the SF-MI transition is therefore the condensate fraction  $n_0/n$ , which drops continuously from one at  $U/J \ll 1$  to zero at  $(U/J)_c$ . The continuous nature of the SF-MI quantum phase transition in any dimension follows from the fact that the effective field theory for the complex order parameter  $\psi$  is of that of a  $d+1$ -dimensional XY-model (Fisher *et al.*, 1989; Sachdev, 1999). More precisely, this is valid only for the special transition at integer density, which is driven by phase fluctuations only. By contrast, crossing the SF-MI phase boundary by a change in the chemical potential, the associated change in the density gives rise to a different critical behavior (Fisher *et al.*, 1989). For instance, the excitation gap in the MI phase vanishes linearly with the distance from the boundary of the Mott lobe in this more generic case.

Within a mean-field approximation, the critical value for the transition from a MI to a SF in a three dimensional optical lattice, is given by  $(U/J)_c = 5.8z$  for  $\bar{n} = 1$  and  $(U/J)_c = 4\bar{n}z$  for  $\bar{n} \gg 1$  (Fisher *et al.*, 1989; van Oosten *et al.*, 2001; Sheshadri *et al.*, 1993). Here  $z$  is the number of nearest neighbors and thus  $2zJ$  is the total bandwidth of the lowest Bloch band, which is the relevant parameter which has to be compared with  $U$ . Recently, precise Quantum Monte-Carlo simulations by Capogrosso-Sansone *et al.* (2007) have determined the critical value for the  $\bar{n} = 1$  transition in a simple cubic lattice to be at  $(U/J)_c = 29.36$  with an accuracy of about 0.1%. In one dimension, the SF-MI transition is of the Kosterlitz-Thouless type, with a finite jump of the superfluid density at the transition. Precise values for the critical coupling are available from DMRG calculations, giving  $(U_1/J)_c = 3.37$  (Kollath *et al.*, 2004; Kühner *et al.*, 2000) for the  $\bar{n} = 1$  transition. For  $\bar{n} \gg 1$ , the BHM is equivalent to a chain of Josephson junctions with coupling energy  $E_J = 2\bar{n}J$ . The SF-MI transition

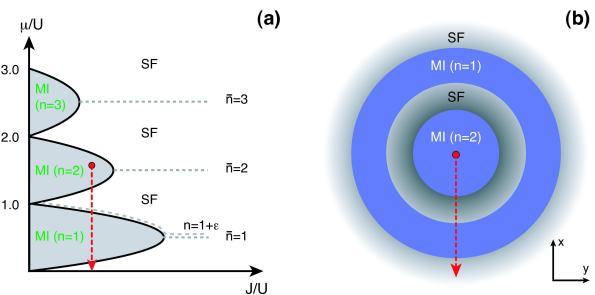


FIG. 13 Schematic zero temperature phase diagram of the Bose-Hubbard model. The dashed lines of constant integer density  $\langle \hat{n} \rangle = 1, 2, 3$  in the SF hit the corresponding MI phases at the tips of the lobes at a critical value of  $J/U$ , which decreases with density  $\bar{n}$ . For  $\langle \hat{n} \rangle = 1 + \varepsilon$  the line of constant density stays outside the  $\bar{n} = 1$  MI because a fraction  $\varepsilon$  of the particles remains superfluid down to the lowest values of  $J$ . In an external trap with a  $\bar{n} = 2$  MI phase in the center, a series of MI and SF regions appear by going towards the edge of the cloud, where the local chemical potential has dropped to zero

is then described by the 1 + 1-dimensional  $O(2)$ -model, which gives  $(U_1/J)_c = 2.2\bar{n}$  (Hamer and Kogut, 1979; Roomany and Wyld, 1980).

From Eqs. (38) and (47), the critical value of the dimensionless lattice depth  $V_0/E_r$  for rather deep lattices is obtained from

$$(V_0/E_r)_c = \frac{1}{4} \ln^2 \left( \frac{\sqrt{2}d}{\pi a} \cdot (U/J)_c \right) \quad (69)$$

Using the experimental parameters  $d = 426$  nm and  $a = 5.7$  nm (Greiner *et al.*, 2002a), the precise result for  $(U/J)_c$  in a simple cubic lattice gives a critical value  $V_0/E_r|_c = 11.89$  for the SF-MI transition with  $\bar{n} = 1$ . Given that Eq. (38), on which the above estimate for the critical lattice depth is based, is not very precise in this regime, this result is in reasonable agreement with the lattice depth of  $V_0 = 12 - 13E_r$ , where the transition is observed experimentally (Gerber *et al.*, 2005b; Greiner *et al.*, 2002a).

Consider now a filling with  $\langle \hat{n} \rangle = 1 + \varepsilon$  which is slightly larger than one. For large  $J/U$  the ground state has all the atoms delocalized over the whole lattice and the situation is hardly different from the case of unit filling. Upon lowering  $J/U$ , however, the line of constant density remains slightly above the  $\bar{n} = 1$  'Mott-lobe', and stays in the SF regime down to the lowest  $J/U$  (see Fig. 13). For any noninteger filling, therefore, the ground state remains SF as long as the atoms can hop at all. This is a consequence of the fact, that even for  $J \ll U$  there is a small fraction  $\varepsilon$  of atoms which remain SF on top of a frozen MI-phase with  $\bar{n} = 1$ . Indeed this fraction can still gain kinetic energy by delocalizing over the whole lattice without being blocked by the repulsive interaction  $U$  because two of those particles will never be at the same place. The same argument applies to holes when  $\varepsilon$  is negative. As a result, in the homogeneous system, the

quantum phase transition from a SF to a MI only appears if the density is equal to a commensurate, integer value.

*In-trap density distribution* Fortunately, the situation is much less restrictive in the presence of a harmonic trap. Indeed, within a local density approximation, the inhomogeneous situation in a harmonic trap is described by a spatially varying chemical potential  $\mu_{\mathbf{R}} = \mu(0) - \epsilon_{\mathbf{R}}$  with  $\epsilon_{\mathbf{R}} = 0$  at the trap center. Assuming e.g. that the chemical potential  $\mu(0)$  at trap center falls into the  $\bar{n} = 2$  'Mott-lobe', one obtains a series of MI domains separated by a SF by moving to the boundary of the trap where  $\mu_{\mathbf{R}}$  vanishes (see Fig. 13b). In this manner, all the different phases which exist for given  $J/U$  below  $\mu(0)$  are present simultaneously! The SF phase has a finite compressibility  $\kappa = \partial n / \partial \mu$  and a gapless excitation spectrum of the form  $\omega(q) = cq$  because there is a finite superfluid density  $n_s$ . By contrast, in the MI-phase both  $n_s$  and  $\kappa$  vanish. As predicted by Jaksch *et al.* (1998), the incompressibility of the MI phase allows to distinguish it from the SF by observing the local density distribution in a trap. Since  $\kappa = 0$  in the MI, the density stays constant in the Mott phases, even though the external trapping potential is rising. In the limit of  $J \rightarrow 0$  the SF regions vanish and one obtains a 'wedding cake' type density profile, with radii  $R_n$  of the different Mott insulating regions, given by  $R_n = \sqrt{(2[\mu(0) - nU]/M\omega^2)}$  (DeMarco *et al.*, 2005).

The existence of such wedding-cake like density profiles of a Mott insulator has been supported by Monte-Carlo (Batrouni *et al.*, 2002; Kashurnikov *et al.*, 2002; Rigol *et al.*, 2006; Wessel *et al.*, 2004) and DMRG (Kollath *et al.*, 2004) calculations in one, two, and three dimensions. Very recently number state resolved, in-trap density profiles have been detected experimentally by Campbell *et al.* (2006) and Fölling *et al.* (2006). In the latter case it has been possible to directly observe the wedding cake density profiles and thus confirm the incompressibility of the Mott insulating regions of the atomic gas in the trapping potential. A sharp drop in the radii of the  $n = 2$  occupied regions has been observed when the crossing the transition point (Fölling *et al.*, 2006). It should be noted that the in-trap density profiles can be used as a sensitive thermometer for the strongly interacting quantum gas. For typical experimental parameters, one finds that for temperatures around  $T^* \gtrsim 0.2U/k_B$ , the wedding cake profiles become completely washed out (Gerbier, 2007). Within the strongly interacting regime, the superfluid shells accommodate most of the entropy of the system and can turn already into a normal thermal gas at a lower temperature  $T_c \sim zJ$  with the Mott insulating shells still intact (Capogrosso-Sansone *et al.*, 2007; Gerbier, 2007; Ho and Zhou, 2007). In order to reach the lowest temperatures in this regime, it is advantageous to keep the external harmonic confinement as low as possible, or even decrease it during an increase of the lattice depth (Gerbier, 2007; Ho and Zhou, 2007).

*Phase coherence across the SF-MI transition* The disappearance of superfluidity (or better of BEC) at the SF-

MI transition was initially observed experimentally by a time-of-flight method (Greiner *et al.*, 2002a). The corresponding series of images is shown in Fig. 14 for different values of  $V_0$ , ranging between  $V_0 = 0$  (a) and  $V_0 = 20E_r$  (h). One observes a series of interference peaks around the characteristic 'zero-momentum' peak of a condensate in the absence of an optical lattice. With increasing  $V_0$  these peaks become more pronounced. Beyond a critical lattice depth around  $V_0 \approx 12 - 13E_r$  (e), which agrees very well with the above estimate for the SF-MI transition for one Boson per site, this trend is suddenly reversed, however, and the interference peaks eventually disappear completely. In order to understand why these pictures indeed provide a direct evidence for a SF to MI transition predicted by the Bose-Hubbard model, it is useful to consider the idealized situation of a perfect periodic lattice in the absence of any trapping potential. From Eq. (44) the observed density at position  $\mathbf{x}$  reflects the momentum distribution at  $\mathbf{k} = M\mathbf{x}/\hbar t$ . Factoring out the number of lattice sites, it is proportional to the lattice Fourier transform

$$n(\mathbf{k}) \sim |\tilde{w}(\mathbf{k})|^2 \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} G^{(1)}(\mathbf{R}). \quad (70)$$

of the one-particle density matrix  $G^{(1)}(\mathbf{R})$  at separation  $\mathbf{R}$ . For optical lattice depths below the critical value, the ground state in a 3D situation is a true BEC, where  $G^{(1)}(|\mathbf{R}| \rightarrow \infty) = n_0$  approaches a finite value at large separation. For the MI phase, in turn,  $G^{(1)}(\mathbf{R})$  decays to zero exponentially. The SF phase of cold atoms in a homogeneous optical lattice is thus characterized by a momentum distribution which exhibits sharp peaks at the reciprocal lattice vectors  $\mathbf{k} = \mathbf{G}$  (defined by  $\mathbf{G} \cdot \mathbf{R} = 2\pi$  times an integer, see e.g. (Ashcroft and Mermin, 1976)) plus a smooth background from the short range correlations. The fact that the peaks in the momentum distribution at  $\mathbf{k} = \mathbf{G}$  initially grow with increasing depth of the lattice potential is a result of the strong decrease in spatial extent of the Wannier function  $w(\mathbf{r})$ , which entails a corresponding increase in its Fourier transform  $\tilde{w}(\mathbf{k})$  at higher momenta. In the MI regime, where  $G^{(1)}(\mathbf{R})$  decays to zero, remnants of the interference peaks still remain (see e.g. Fig. 14f) as long as  $G^{(1)}(\mathbf{R})$  extends over several lattice spacings, because the series in Eq. (70) adds up constructively at  $\mathbf{k} = \mathbf{G}$ . A more detailed picture for the residual short range coherence features beyond the SF-MI transition is obtained by considering perturbations deep in the Mott insulating regime at  $J = 0$ . There,  $G^{(1)}(\mathbf{R})$  vanishes beyond  $\mathbf{R} = 0$  and the momentum distribution is a structureless Gaussian, reflecting the Fourier transform of the Wannier wave function (see Fig. 14h). With increasing tunneling  $J$ , the Mott state at  $J/U \rightarrow 0$  is modified by a coherent admixture of particle-hole pairs. However due the presence of a gapped excitation spectrum, such particle hole pairs cannot spread out and are rather tightly bound to close distances. They do, however, give rise to a significant degree of short range coherence. Using first order perturbation theory with the



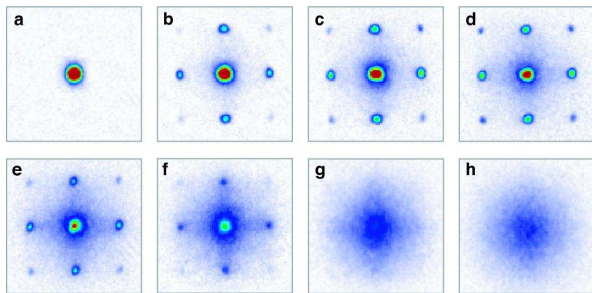


FIG. 14 Absorption images of multiple matter wave interference patterns after releasing the atoms from an optical lattice potential with a potential depth of **a**  $0 E_r$ , **c**  $3 E_r$  **c**  $7 E_r$  **d**  $10 E_r$  **e**  $13 E_r$  **f**  $14 E_r$ , **g**  $16 E_r$  and **b**  $20 E_r$ . The ballistic expansion time was 15 ms. Reprinted with permission from Greiner *et al.* (2002a).

tunneling operator as a perturbation on the dominating interaction term, one finds that the amplitude of the coherent particle hole admixtures in a Mott insulating state is proportional to  $J/U$ :

$$|\Psi\rangle_{U/J} \approx |\Psi\rangle_{U/J \rightarrow \infty} + \frac{J}{U} \sum_{\langle \mathbf{R}, \mathbf{R}' \rangle} \hat{a}_{\mathbf{R}}^\dagger \hat{a}_{\mathbf{R}'} |\Psi\rangle_{U/J \rightarrow \infty}. \quad (71)$$

Close to the transition point, higher order perturbation theory or a Green function analysis can account for coherence beyond nearest neighbors and the complete liberation of the particle-hole pairs, which eventually leads to the formation of long range coherence in the superfluid regime. The coherent particle hole admixture and its consequence on the short range coherence of the system have been investigated theoretically and experimentally in (Gerber *et al.*, 2005a,b; Sengupta *et al.*, 2005).

The SF-MI quantum phase transition therefore shows up directly in the interference pattern. For the homogeneous system it reveals the existence or not of off-diagonal long range order in the one-particle density matrix. The relevant order parameter is the condensate fraction. Of course the actual system is not homogeneous and a numerical computation of the interference pattern is necessary for a quantitative comparison with experiment. This has been done e.g. for the 3D case in (Kashurnikov *et al.*, 2002) and for 1D case in (Batrouni *et al.*, 2002; Kollath *et al.*, 2004). Due to the finite size and the fact that different MI phases are involved, the pattern evolves continuously from the SF to the MI regime. While the critical values for  $J/U$  are different for the MI phases with  $\bar{n} = 1$  and  $\bar{n} = 2$  which are present in the experiment (Greiner *et al.*, 2002a), the transition seen in the time-of-flight images occurs rather rapidly with increasing lattice depth. Indeed, from Eq. (69), the experimental control parameter  $V_0/E_r$  depends only logarithmically on the relevant parameter  $U/J$  of the BHM. The small change from  $V_0 = 13E_r$  in (e) to  $V_0 = 14E_r$  in (f) thus covers a range in  $J/U$  wider than that, which would be required to distinguish the

$\bar{n} = 1$  from the  $\bar{n} = 2$  transition. For a quantitative evaluation of the interference patterns, one must also take into account the broadening mechanism during time-of-flight expansion, as discussed in Sec. II C.

When approaching the SF-MI transition from the superfluid regime, the increasing interactions tend to increase the depletion of the condensate and thereby reduce the long range phase coherent component with increasing  $U/J$  (Hadzibabic *et al.*, 2004; Orzel *et al.*, 2001; Schori *et al.*, 2004; Xu *et al.*, 2006). For increasing lattice depth, the condensate density as a measure of the long range coherent fraction, then decreases continuously and vanishes at the transition point. The visibility of the interference pattern in general, however, evolves smoothly across the SF-MI transition, due to the presence of a strong short range coherent fraction in the MI just across the transition point (see discussion above). Above the transition point the visibility of the momentum distribution can also show kinks as the lattice depth is increased, which have been attributed to the beginning formation of shell structures in the MI state (Gerber *et al.*, 2005a,b; Sengupta *et al.*, 2005).

*Excitation spectrum* A second signature of the SF-MI transition is the appearance of a finite excitation gap  $\Delta \neq 0$  in the Mott insulator. Deep in the MI phase, this gap has size  $U$ , which is just the increase in energy if an atom tunnels to an already occupied adjacent site (note that  $U$  is much smaller than the gap  $\hbar\omega_0$  for the excitation of the next vibrational state). The existence of a gap has been observed experimentally by applying a potential gradient in the MI (Greiner *et al.*, 2002a) or by using a modulation spectroscopy method (Stöferle *et al.*, 2004) and measuring the resulting excitations. Recent calculations indicate that such measurements simultaneously probe global (Huber *et al.*, 2007; Iucci *et al.*, 2006) and local properties of the system. In particular, e.g. a peaked excitation spectrum can also appear in a strongly interacting superfluid regime, where  $U > J$  (Kollath *et al.*, 2006). A way to probe global features of the many-body excitation spectrum, also close to the transition point, might be achieved by employing Bragg spectroscopy techniques as proposed in (van Oosten *et al.*, 2005; Pupillo *et al.*, 2006; Rey *et al.*, 2005).

In the SF regime, there is no excitation gap. Instead, the homogeneous system exhibits a sound like mode with frequency  $\omega(q) = cq$ . As shown in the appendix, the associated sound velocity  $c$  is determined by  $Mc^2 = n_s/\kappa$  and thus gives information about the superfluid density  $n_s$ . The existence of a sound like excitation even in the presence of an underlying lattice which explicitly breaks translation invariance is a consequence of long range phase coherence in the SF. Its observation would therefore directly probe superfluidity, in contrast to the peaks in the interference pattern, which measure BEC.

*Number statistics* Associated with the transition from a superfluid to a Mott insulating state is a profound change

in the atom number statistics per lattice site. As noted above, in the homogeneous system the ground state in the extreme MI limit ( $J/U \rightarrow 0$ ) is a product of Fock states with an integer number  $\bar{n}$  of particles at each site. At finite hopping  $J \neq 0$ , this simple picture breaks down because the atoms have a finite amplitude to be at different sites. The many-body ground state can then no longer be written as a simple product state. In the opposite limit  $U \rightarrow 0$ , the ground state is a condensate of zero quasi-momentum Bloch states. In the limit  $N, N_L \rightarrow \infty$  at fixed (not necessarily integer) density  $\bar{n} = N/N_L$ , the associated perfect condensate is a product of coherent states on each lattice site

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (72)$$

with  $\alpha$  describing the amplitude and phase of the coherent matter wave field. This corresponds to a Poissonian atom number distribution on each lattice site with average  $|\alpha|^2 = \bar{n}$ .

A remarkable consequence of the representation (67) is that, at least for integer densities  $\bar{n} = 1, 2, \dots$ , the many-body ground state may be factorized into a product over single sites

$$|\Psi_{GW}\rangle = \prod_{\mathbf{R}} \left( \sum_{n=0}^{\infty} c_n |n\rangle_{\mathbf{R}} \right) \quad (73)$$

in both limits  $J \rightarrow 0$  and  $U \rightarrow 0$ . The associated atom number probability distribution  $p_n = |c_n|^2$  is either a pure Fock or a full Poissonian distribution. It is now very plausible to use the factorized form in Eq. (73) as an *approximation* for arbitrary  $J/U$ , taking the coefficients  $c_n$  as variational parameters which are determined by minimizing the ground state energy (Rokhsar and Kotliar, 1991; Sheshadri *et al.*, 1993). As pointed out by Rokhsar and Kotliar (1991), this is effectively a Gutzwiller ansatz for bosons. Beyond being very simple computationally, this ansatz describes the SF to MI transition in a mean-field sense, becoming exact in infinite dimensions. In addition, it provides one with a very intuitive picture of the transition to a MI state, which occurs precisely at the point, where the local number distribution becomes a pure Fock distribution. Indeed, within the Gutzwiller approximation, the expectation value

$$\langle \Psi_{GW} | \hat{a}_{\mathbf{R}} | \Psi_{GW} \rangle = \sum_{n=1}^{\infty} \sqrt{n} c_{n-1}^* c_n \quad (74)$$

of the local matter wave field vanishes if and only if the probability for finding different particle numbers at any given site is equal to zero. It is important, however, to emphasize that the Gutzwiller ansatz fails to account for the nontrivial correlations between different sites present at any finite  $J$ . These correlations imply that the one particle density matrix  $G^{(1)}(\mathbf{R})$  is different from zero at finite distance  $|\mathbf{R}| \neq 0$ , becoming long ranged at the

transition to a SF. By contrast, in the Gutzwiller approximation, the one particle density matrix has no spatial dependence at all: it is zero at any  $|\mathbf{R}| \neq 0$  in the MI and is completely independent of  $\mathbf{R}$  in the SF. Moreover, in the Gutzwiller approximation, the phase transition is directly reflected in the local number fluctuations, with the variance of  $n_{\mathbf{R}}$  vanishing throughout the MI phase. In reality, however, local variables like the on-site number distribution will change in a smooth manner near the transition and the variance of the local particle number will only vanish in the limit  $J \rightarrow 0$ .

Crossing the SF-MI transition, therefore, the number statistics evolves rather smoothly from a Poissonian distribution to Fock states on each lattice site. Recent experimental progress has allowed measurements of the number distribution in the optical lattice via microwave spectroscopy exploiting collisional frequency shifts (Campbell *et al.*, 2006) or spin changing collisions (Gerber *et al.*, 2006). When crossing the SF-MI transition, Campbell *et al.* (2006) were able to observe the emergence of a discrete excitation spectrum with Hz resolution. In the second experiment, the change in atom number statistics from Poissonian to Fock states could be revealed (Gerber *et al.*, 2006). Another possibility to observe the number squeezing of the initially Poissonian atom number distribution in the weakly interacting regime due to increasing interatomic interactions has been to use a matter wave beam splitter and observe the timescale of the collapse in the ensuing phase diffusion dynamics (Greiner *et al.*, 2002b; Jo *et al.*, 2007; Sebby-Strabley *et al.*, 2007), which is discussed in the following paragraph.

### C. Dynamics near quantum phase transitions

One of the major advantages of cold atoms in studying many-body phenomena is the possibility to change the parameters characterizing the relative strength of the kinetic and interaction energy dynamically. This opens the possibility to study the real time dynamics of strongly correlated systems in a controlled manner. As a simple example, we discuss the quench of the system from the superfluid into the Mott insulating regime. This issue has been investigated in an experiment, observing collapses and revivals of the matter wave due to the coherent superposition of states with different atom numbers in the SF (Greiner *et al.*, 2002b). In the weakly interacting regime of a BEC in an optical lattice potential, the ground state (67) is a product of coherent states on each lattice site with a Poissonian atom number distribution. If the lattice depth is now suddenly increased to a parameter regime, where the ground state of the system is a Mott insulating state, the initial atom number fluctuations of the coherent state will be frozen out, as the system is not given enough time to redistribute towards the novel many-body ground state. The evolution with time of such a coherent state can be evaluated by tak-

ing into account the time evolution of the different Fock states forming the coherent state:

$$|\alpha\rangle(t) = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\frac{1}{2}U n(n-1)t/\hbar} |n\rangle. \quad (75)$$

The coherent matter wave field  $\psi$  on each lattice site can then simply be evaluated through  $\psi = \langle\alpha(t)|\hat{a}|\alpha(t)\rangle$ , which exhibits an intriguing dynamical evolution (Castin and Dalibard, 1997; Dunningham *et al.*, 1998; Imamoglu *et al.*, 1997; Sols, 1994; Wright *et al.*, 1996; Yurke and Stoler, 1986). At first, the different phase evolutions of the atom number states lead to a collapse of  $\psi$ . However, at integer multiples in time of  $h/U$  all phase factors in the above equation re-phase modulo  $2\pi$  and thus lead to a revival of the initial coherent state. In fact, precise revivals appear as long as the initial state can be written in the factorized form of Eq. (73). Since the time evolution operator  $\exp -i\hat{H}t/\hbar$  factorizes into a product of on-site terms  $\exp -in(n-1)Ut/2\hbar$ , the time dependence is perfectly periodic with period  $t_{\text{rev}} = h/U_f$ , where  $U_f$  is the value of the on-site repulsion after the quench. Clearly the period is independent of the precise form of the initial number distribution  $|c_n|^2$ . The collapse time  $t_c \approx t_{\text{rev}}/\sigma_n$  in turn, depends on the variance  $\sigma_n^2 = \langle n^2 \rangle - \langle n \rangle^2$  of the local number distribution. Its measurement thus provides information about how the coherent superposition of different particle numbers in the SF state is eventually destroyed by approaching the MI regime (Greiner *et al.*, 2002b).

The collapse and revival of the coherent matter wave field of a BEC is reminiscent to the collapse and revival of the Rabi oscillations in the interaction of a single atom with a single mode electromagnetic field in cavity quantum electrodynamics (Brune *et al.*, 1996; Rempe *et al.*, 1987). There, the nonlinear atom-field interaction induces the collapse and revival of the Rabi oscillations whereas here the nonlinearity due to the interactions between the atoms themselves leads to the series of collapse and revivals of the matter wave field. It should be pointed out that such a behavior has also been theoretically predicted to occur for a coherent light field propagating in a nonlinear medium (Yurke and Stoler, 1986) but to our knowledge has never been observed experimentally. Such a dynamical evolution of the atomic quantum state due to the nonlinear interactions between the particles is also known as quantum phase diffusion and has been detected in (Greiner *et al.*, 2002b) for low atom numbers on each site. For larger atom numbers, the initial time evolution of the quantum phase diffusion could be recently observed by Jo *et al.* (2007) in a double well scenario.

The simple single site description is valid only in the limits  $U_i \ll J$  of a nearly perfect SF in the initial state and  $U_f \gg J$  of negligible tunneling in the final state. To determine the dynamics in a more general situation, is a complicated non-equilibrium many-body problem. Numerical results for arbitrary values of  $U_i$  and  $U_f$  have been obtained for the 1D BHM by Kollath *et al.* (2007),

using the time-dependent density matrix renormalization group (Schollwöck, 2005).

In a related scenario it has been proposed that when jumping from an initial Mott insulating state into the superfluid regime, one should observe oscillations of the superfluid order parameter (Altman and Auerbach, 2002; Polkovnikov *et al.*, 2002). For large filling factors, oscillating coherence has been observed after a quench from a deep to a shallow lattice by Tuchman *et al.* (2006). The formation of a superfluid from an initial Mott insulating phase poses a general problem of interest in the context of the dynamics of strongly correlated quantum systems. Both experiment (Greiner *et al.*, 2002a) and theory (Clark and Jaksch, 2004) have confirmed that the emergence of coherence in the system can occur rather rapidly on timescales of a few tunneling times  $\hbar/J$ . It is an open question, however, whether off-diagonal long range order in the one-particle density matrix indeed sets in within such a short time and what length scales are relevant over which order has established in order to observe coherence in a time-of-flight picture.

#### D. Bose-Hubbard model with finite current

The SF-MI transition discussed in IV.B above is a continuous phase transition in the ground state of a many-body Hamiltonian. The observation from the time-of-flight images, that long range phase coherence is lost beyond a critical value of  $U/J$ , provides a signature for the disappearance of BEC. The expected simultaneous loss of *superfluidity* across this transition may be studied by considering the phase boundary, where stationary states with a finite current lose their stability. Such stationary out-of-equilibrium states may be created experimentally by boosting the condensate to a finite momentum state (Fallani *et al.*, 2004), or by inducing a center-of-mass oscillation in the trap (Fertig *et al.*, 2005). The question, what happens to the equilibrium SF-MI transition in a situation with a finite current has been addressed by Polkovnikov *et al.* (2005). For a given number  $\bar{n}$  of bosons per site, the kinetic energy term in the BHM (65) gives rise to a Josephson coupling energy  $E_J = 2\bar{n}J$  due to next neighbor-tunneling which favors a vanishing relative phase between adjacent lattice sites. In the limit  $E_J \gg U$ , there is non-vanishing matter wave field  $\psi_{\mathbf{R}} = \langle \hat{a}_{\mathbf{R}} \rangle$ . In the ground state, all bosons have zero momentum and  $\psi_{\mathbf{R}}$  is uniform. States with a finite current, in turn, are BEC's in which single particle states with non-zero momentum  $\mathbf{q}$  are macroscopically occupied. To zeroth order in  $U/E_J$ , their energy is the Bloch band energy Eq. (37). The associated current per particle  $\mathcal{J} = (2J/\hbar) \sin q_x d$  for motion along the  $x$ -direction has a maximum at  $p = q_x d = \pi/2$ . States with a larger momentum are unstable in a linear stability analysis (Polkovnikov *et al.*, 2005). This instability was observed experimentally by Fallani *et al.* (2004). A moving optical lattice is created by two counterpropagating beams

at frequencies which differ by a small detuning  $\delta\nu$ . Averaged over the optical frequencies, this gives rise to an interference pattern which is a standing wave moving at velocity  $v = \lambda\delta\nu/2$ . Adiabatically switching on such a lattice in an existing BEC then leads to a condensate in a state with quasi-momentum  $q = Mv/\hbar$ . Its lifetime shows a very rapid decrease for momenta near the critical value  $q_c$ .

In the strongly interacting regime near the SF-MI transition, such a single particle picture is no longer valid. At the mean-field level, the problem may be solved by using the field theoretical description of the SF-MI transition. The SF phase is then characterized by a nonzero complex order parameter  $\psi$ , whose equilibrium value  $|\psi| \sim \xi^{-1}$  vanishes like the inverse of the correlation length  $\xi$  (this relation holds in the mean-field approximation, which is appropriate for the transition at integer densities in 3D). The stationary solutions of the dimensionless order parameter equation  $\nabla^2\psi + \xi^{-2}\psi = |\psi|^2\psi$  with *finite* momentum are of the form  $\psi(x) = \sqrt{\xi^{-2} - p^2} \exp(ipx)$ . Evidently, such solutions exist only if  $|p| < 1/\xi$ . The critical value of the dimensionless momentum  $p$ , where current carrying states become unstable, thus approaches zero continuously at the SF-MI transition (Polkovnikov *et al.*, 2005). In fact, the same argument can be used to discuss the vanishing of the critical current in superconducting wires near  $T_c$ , see Tinkham (1996). The complete mean field phase diagram, shown in Fig. 15 interpolates smoothly between the classical instability at  $p_c = \pi/2$  and  $p_c \rightarrow 0$  in the limits  $U \rightarrow 0$  and  $U \rightarrow U_c$  respectively. In contrast to the equilibrium transition at  $p = 0$  which is continuous, the dynamical transition is of *first order*. Crossing the phase boundary at any nonzero current is therefore connected with an irreversible decay of the current to zero. Experimentally, the decrease of the critical momentum near the SF-MI transition has been observed by Mun *et al.* (2007). Their results are in good agreement with the phase diagram shown in Fig. 15.

In the mean-field picture, states of a SF with non-zero momentum have an infinite lifetime. More precisely, however, such states can only be *metastable*, because the ground state of any time-reversal invariant Hamiltonian necessarily has zero current. The crucial requirement for SF in practice, therefore, is that current carrying states have lifetimes which by far exceed experimentally relevant scales. This requires these states to be separated from the state with vanishing current by energy barriers, which are much larger than the thermal or relevant zero point energy<sup>10</sup>. The rate for phase slips near the critical line in Fig. 15 has been calculated by Polkovnikov *et al.*

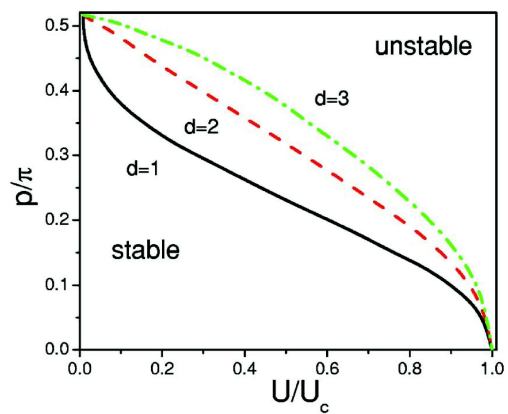


FIG. 15 Mean-field phase diagram separating stable and unstable motion of condensate regions. The vertical axis denotes the condensate momentum in inverse lattice units and the horizontal axis denotes the normalized interaction. Reprinted with permission from (Polkovnikov *et al.*, 2005).

(2005). It turns out, that the mean-field transition survives fluctuations in 3D, so in principle it is possible to locate the equilibrium SF-MI transition by extrapolating the dynamical transition line to zero momentum. In the experiments of Fertig *et al.* (2005), the system still showed sharp interference peaks even in the 'overdamped' regime where the condensate motion was locked by the optical lattice (see Fig. 16). This may be due to localized atoms at the sample edges, which block the dipole oscillation even though the atoms in the center of the trap are still in the SF regime. A theoretical study of the damped oscillations of 1D bosons has been given by (Gea-Banacloche *et al.*, 2006).

A different method to drive a SF-MI transition dynamically has been suggested by Eckardt *et al.* (2005). Instead of a uniformly moving optical lattice, it employs an oscillating linear potential  $K \cos(\omega t) \cdot \hat{x}$  along one of the lattice directions (in a 1D BHM  $\hat{x} = \sum_j j \hat{n}_j$  is the dimensionless position operator). For modulation frequencies such that  $\hbar\omega$  is much larger than the characteristic scales  $J$  and  $U$  of the unperturbed BHM, the driven system behaves like the undriven one, however with a renormalized tunneling matrix element  $J_{\text{eff}} = J \cdot \mathcal{J}_0(K/(\hbar\omega))$ , where  $\mathcal{J}_0(x)$  is the standard Bessel function. Since  $U$  is unchanged in this limit, the external perturbation completely suppresses the tunneling at the zero's of the Bessel function. Moreover, it allows to invert the sign of  $J_{\text{eff}}$  to negative values, where e.g. the superfluid phase corresponds to a condensate at finite momentum  $q = \pi$ . It has been shown numerically (Eckardt *et al.*, 2005) that by a slow variation of the driving amplitude  $K$  from zero to  $K = 2.4\hbar\omega$  (where  $J_{\text{eff}} \approx 0$ ) and back to zero allows to adiabatically transform a superfluid into a Mott insulator and then back to a superfluid. In recent experiments by Lignier *et al.* (2007), the dynamical suppression of tunneling with increasing driving  $K$  has been observed

<sup>10</sup> This is different from the well known Landau criterion of superfluid flow below a finite critical velocity (Pitaevskii and Stringari, 2003). Indeed, the existence of phase slips implies that the critical velocity is always zero in a strict sense.

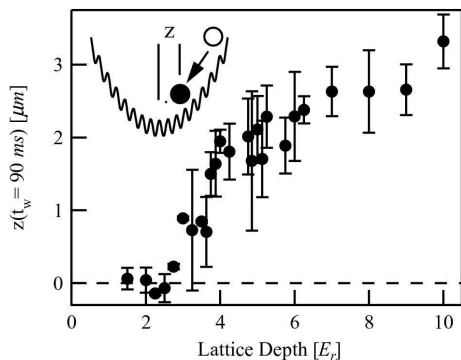


FIG. 16 Inhibition of transport in a one-dimensional bosonic quantum system with an axial optical lattice. For lattice depths above approx.  $2 E_r$ , an atom cloud displaced to the side of the potential minimum (see inset) is stuck at this position and does not relax back to the minimum. Reprinted with permission from Fertig *et al.* (2005).

through a measurement of the expansion velocity along the direction of the optical lattice after switching off the axial confinement.

### E. Fermions in optical lattices

In this section we would like to focus on fermions in 3D optical lattice potentials and experimental results that have been obtained in these systems (see also Giorgini *et al.* (2007)). Interacting fermions in an periodic potential can be described by the Hubbard hamiltonian. Let us for now restrict the discussion to the case of atoms confined to the lowest energy band and to two possible spin states  $|\uparrow\rangle, |\downarrow\rangle$  for the fermionic particles. The single band Hubbard hamiltonian thus reads:

$$H = -J \sum_{\langle \mathbf{R}, \mathbf{R}' \rangle, \sigma} \left( \hat{c}_{\mathbf{R}, \sigma}^\dagger \hat{c}_{\mathbf{R}', \sigma} + h.c. \right) + U \sum_{\mathbf{R}} \hat{n}_{\mathbf{R}\uparrow} \hat{n}_{\mathbf{R}\downarrow} + \frac{1}{2} M \omega^2 \sum_{\mathbf{R}, \sigma} \mathbf{R}^2 \hat{n}_{\mathbf{R}, \sigma} \quad (76)$$

As in the case of bosonic particles, the phase diagram depends strongly on the interaction strength vs. kinetic energy of the atoms in the optical lattices. An important difference between the bosonic and fermionic Hubbard hamiltonian can also be seen in the form of the interaction term, where only two particles of different spin states are allowed to occupy the same lattice site, giving rise to an interaction energy  $U$  between the atoms.

*Filling factor and Fermi surfaces* A crucial parameter in the fermionic Hubbard model is the filling factor of the atoms in the lattice. Due to the overall harmonic confinement of the atoms (last term in Eq. (76)), this filling fraction changes over the cloud of trapped atoms.

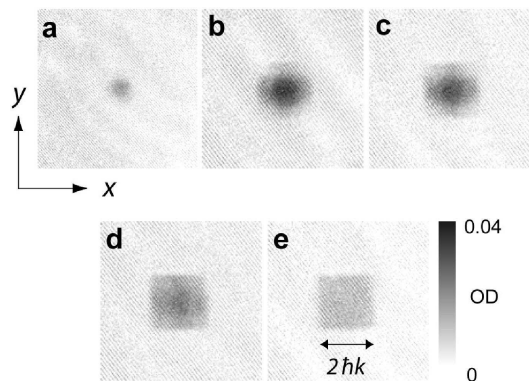


FIG. 17 Fermi surfaces vs band-filling for ultracold fermionic  $^{40}\text{K}$  atoms in a three-dimensional simple cubic lattice potential. From (a) to (e) the filling factor has been continuously increased, bringing the system from a conducting to a band insulating state. Reprinted with permission from Köhl *et al.* (2005a).

One can however specify an average characteristic filling factor,

$$\rho_c = \frac{N_F d^3}{\zeta^3}, \quad (77)$$

with  $\zeta = \sqrt{2J/M\omega^2}$  describing the typical delocalization length of the single particle wave-functions in the combined periodic lattice and external harmonic trapping potential (Köhl *et al.*, 2005a; Rigol and Muramatsu, 2004). The characteristic filling factor can be controlled experimentally, by either increasing the total number of fermionic atoms  $N_F$ , by reducing  $J$  via an increase of the lattice depth or via an increase of the overall harmonic confinement. The latter case however has the disadvantage that a strong harmonic confinement will lead to a decoupling of the independent lattice sites, as the tunnel coupling  $J$  will not be large enough to overcome the potential energy offset due to the overall harmonic confinement. One is then left with an array of uncoupled, independent harmonic oscillators. The characteristic filling factor of the system can be revealed experimentally by observing the population of the different Bloch states via adiabatic band mapping, introduced in Sec. II.C. By changing the atom number or the lattice depth, Köhl *et al.* (2005a) could thus observe a change from a low filling to a band insulating state, where the single species particles are completely localized to individual lattice sites (see Fig. 17).

*Thermometry and pair correlations* An important question regarding fermionic quantum gases in optical lattices is the temperature of the many-body system. It has been shown that for non-interacting 50/50 spin mixtures of fermions, the number of doubly occupied spin-states can be used to determine the temperature of the system. For zero temperature and deep optical lattices, one would expect all lattice sites to be occupied by a spin-up and

spin-down atom equally. For finite temperatures, however, atoms could be thermally excited to higher lying lattice sites at the border of the system, thus reducing the number of doubly occupied lattice sites. By converting doubly occupied sites into fermionic molecules, it has been possible to determine the number of doubly vs. singly occupied sites and obtain an estimate for the temperature of the spin-mixture (Köhl, 2006; Stöferle *et al.*, 2006). Another possibility to determine the temperature of the system even for a single species fermionic quantum gas in the lattice has been provided by the use of quantum noise correlations, as introduced in Sec. III. For higher temperatures of the quantum gas, the atoms tend to spread out in the harmonic confinement and thus increase the spatial size of the trapped atom cloud. As the shape of each noise-correlation peak essentially represents the Fourier transform of the in-trap density distribution, but with fixed amplitude of 1 (see Eq. (55)), an increase in the size of the fermionic atom cloud by temperature will lead to a decrease in the observed correlation signal, as has recently been detected in (Rom *et al.*, 2006).

When increasing a red-detuned optical lattice on top of a fermionic atom cloud, this usually also leads to an increased overall harmonic confinement of the system. It has been shown that for such a case, the density of states of the system can be significantly modified, thus leading to an adiabatic heating of the fermionic system by up to a factor of two for a strong overall harmonic confinement (Köhl, 2006). In order to reach low temperatures for the fermionic system, it would thus be advantageous to keep the harmonic confinement as low as possible or even decrease it as the optical lattice depth is increased. Such a configuration is e.g. possible with a blue-detuned optical lattice in conjunction with a red-detuned optical dipole trap.

## V. COLD GASES IN ONE DIMENSION

In the following it is shown that the confinement of cold atoms in a quantum wire geometry which can be achieved via strong optical lattices (Kinoshita *et al.*, 2004; Paredes *et al.*, 2004) provides a means of reaching the strong interaction regime in dilute gases. This opens the possibility to realize both bosonic and fermionic Luttinger liquids and a number of exactly soluble models in many-body physics.

### A. Scattering and bound states

Consider cold atoms subject to a strong 2D optical lattice in the  $y, z$ -plane which confines the motion to the (axial)  $x$ -direction. In the regime, where the excitation energy  $\hbar\omega_\perp$  for motion in the radial  $y, z$ -directions is much larger than the chemical potential, only the lowest transverse eigenmode is accessible. In terms of the oscillator length  $\ell_\perp = \sqrt{\hbar/M\omega_\perp}$  for the transverse mo-

tion, this requires the 1D density  $n_1 = n\pi\ell_\perp^2$  to obey  $n_1 a \ll 1$ . The effective interaction of atoms confined in such a geometry has first been discussed by Olshanii (1998). In the realistic case, where  $\ell_\perp$  is much larger than the effective range  $r_e$  of the atom-atom interaction<sup>11</sup>, the two-particle scattering problem at low energies can again be described by a (3D) pseudopotential. However the asymptotic scattering states are now of the form  $\phi_0(y, z) \exp \pm ikx$ , where  $\phi_0(y, z)$  is the Gaussian ground state wave function for the transverse motion and  $k$  the wavevector for motion along the axial direction. Now, away from Feshbach resonances, the 3D scattering length is much smaller than the transverse oscillator length. In this weak confinement limit  $|a| \ll \ell_\perp$ , the effective 1D interaction is simply obtained by integrating the 3D pseudopotential  $g \int d^3x |\phi_0(y, z)|^2 \delta(\mathbf{x})$  over the ground state density of the transverse motion. The resulting effective 1D interaction is then of the form  $V(x) = g_1 \delta(x)$  with

$$g_1(|a| \ll \ell_\perp) = g|\phi_0(0, 0)|^2 = 2\hbar\omega_\perp \cdot a. \quad (78)$$

Trivially, an attractive or repulsive pseudopotential lead to a corresponding sign of the 1D interaction. In order to discuss what happens near Feshbach resonances, where the weak confinement assumption breaks down, we consider the question of possible bound states of two particles in a strong transverse confinement. Starting with an attractive 3D pseudopotential  $a < 0$ , the effective 1D potential (78) has a bound state with binding energy  $\varepsilon_b = M_r g_1^2 / 2\hbar^2$ . With increasing magnitude of  $a$ , this binding energy increases, finally diverging at a Feshbach resonance. In turn, upon crossing the resonance to the side where  $a > 0$ , the effective potential (78) becomes repulsive. The bound state, therefore, has disappeared even though there is one in the 3D pseudopotential for  $a > 0$ . Obviously, this cannot be correct. As pointed out above, the result (78) only applies in the weak confinement limit  $|a| \ll \ell_\perp$ . In the following we present a treatment of the scattering properties of confined particles for arbitrary values of the ratio  $|a|/\ell_\perp$ . To this end, we will first consider the issue of two-particle bound states with a pseudopotential interaction in quite general terms by mapping the problem to a random walk process. Subsequently, we will derive the low energy scattering amplitudes by analytic continuation.

*Confinement induced bound states* Quite generally, two-particle bound states may be determined from the condition  $V\hat{G} = 1$  of a pole in the exact T-matrix. In the case of a pseudopotential with scattering length  $a$ , the matrix

<sup>11</sup> For a typical frequency  $\omega_\perp = 2\pi \cdot 67$  kHz, the transverse oscillator length is equal to 41.5 nm for <sup>87</sup>Rb

elements  $\langle \mathbf{x} | \dots | 0 \rangle$  of this equation lead to

$$\frac{1}{4\pi a} = \frac{\partial}{\partial r} (r G(E, \mathbf{x}))_{r=0} = \lim_{r \rightarrow 0} \left( G(E, \mathbf{x}) + \frac{1}{4\pi r} \right), \quad (79)$$

in units, where  $\hbar = 2M_r = 1$ . Here,  $G(E, \mathbf{x}) = \langle \mathbf{x} | (E - \hat{H}_0)^{-1} | 0 \rangle$  is the Green function of the free Schrödinger equation and  $\hat{H}_0$  includes both the kinetic energy and the harmonic confining potential. As  $r \rightarrow 0$ , the Green function diverges like  $-(4\pi r)^{-1}$ , thus providing the regularization for the pseudopotential through the second term of Eq. (79). For energies  $E$  below the bottom of the spectrum of  $\hat{H}_0$ , the resolvent  $(E - \hat{H}_0)^{-1} = -\int_0^\infty dt \exp(E - \hat{H}_0)t$  can be written as a time integral. Moreover, by the Feynman-Kac formulation of quantum mechanics, the (imaginary time) propagator  $P(\mathbf{x}, t) = \langle \mathbf{x} | \exp -\hat{H}_0 t | 0 \rangle$  can be interpreted as the sum over Brownian motion trajectories from  $\mathbf{x} = 0$  to  $\mathbf{x}$  in time  $t$  weighted with  $\exp -\int_0^t U[\mathbf{x}(t')]$ , where  $U(\mathbf{x})$  is the external potential in the free Hamiltonian  $\hat{H}_0$ . Similarly, the contribution  $(4\pi r)^{-1} = \int_0^\infty dt P^{(0)}(\mathbf{x}, t)$  can be written in terms of the probability density of free Brownian motion with, again, diffusion constant  $D = 1$  in units where  $\hbar = 2M_r = 1$ . Taking the limit  $r \rightarrow 0$ , Eq. (79) finally leads to the exact equation

$$\frac{1}{4\pi a} = \int_0^\infty dt \left[ P^{(0)}(t) - e^{Et} P(t) \right] \quad (80)$$

for the bound state energies  $E$  of two particles with a pseudopotential interaction and scattering length  $a$ . Here  $P^{(0)}(t) = (4\pi t)^{-3/2}$  is the probability density at the origin of a free random walk after time  $t$ , starting and ending at  $\mathbf{x} = 0$ , while  $P(t)$  is the same quantity in the presence of an additional confining potential. Note that in the formulation of Eq. (80), the regularization of the  $1/r$ -singularity in  $G(E, \mathbf{x})$  is accounted for by the cancellation of the short time divergence due to  $P(t \rightarrow 0) = (4\pi t)^{-3/2}$ , because the random walk does not feel the confinement as  $t \rightarrow 0$ . For two particles in free space, where  $P(t) = P^{(0)}(t)$  at all times, Eq. (80) gives the standard result that a bound state at  $E = -\varepsilon_b$  below the continuum at  $E = 0$  exists only for  $a > 0$ , with the standard value  $\varepsilon_b = \hbar^2/2M_r a^2$ . In the presence of an additional confinement however, there is a bound state for arbitrary sign of the scattering length. The quasi-bound state in the continuum at  $a < 0$  thus becomes a true bound state, i.e. it is shifted upwards by *less* than the continuum threshold  $E_c$ . The physics behind this is the fact that the average time  $\sim \int dt P(t) \exp E_c t$  spent near the origin is infinite for the *confined* random walk. This provides an intuitive understanding for why an infinitesimally small (regularized) delta potential is able to bind a state.

For harmonic confinement, the probability density  $P(t) = ((4\pi)^3 \det \hat{J}(t))^{-1/2}$  can be calculated from the determinant  $\det \hat{J}(t)$  for small fluctuations around the

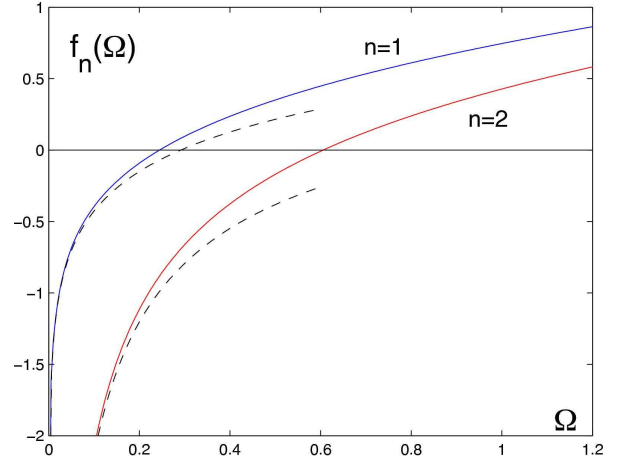


FIG. 18 The functions  $f_n(\Omega)$  defined in Eq. (82) for  $n = 1, 2$ . The limiting dependence  $f_1(\Omega) = \ln(\pi\Omega/B)/\sqrt{2\pi}$  and  $f_2(\Omega) = -1/\sqrt{\Omega} + A$  on the dimensionless binding energy  $\Omega$  for  $\Omega \ll 1$  is indicated by the dashed lines.

trivial Brownian path  $\mathbf{x}(t') \equiv 0$ . Here,  $\hat{J}(t)$  obeys the simple 3 by 3 matrix equation (Schulman, 1981)

$$(-\partial_t^2 + \hat{\omega}^2) \hat{J}(t) = 0 \text{ with } \hat{J}(0) = 0 \quad \partial_t \hat{J}(t)|_{t=0} = 1, \quad (81)$$

where  $\hat{\omega}^2$  is the (diagonal) matrix of the trap frequencies. The fluctuation determinant thus is equal to  $\det \hat{J}(t) = t(\sinh(\omega_\perp t)/\omega_\perp)^2$  for the situation with two confining directions and to  $\det \hat{J}(t) = t^2 \sinh(\omega_z t)/\omega_z$  in the case of a pancake geometry. Since the continua start at  $\hbar\omega_\perp$  and  $\hbar\omega_z/2$ , respectively, we write the bound state energies as  $E = \hbar\omega_\perp - \varepsilon_b$  or  $E = \hbar\omega_z/2 - \varepsilon_b$ . The dimensionless binding energy  $\Omega = \varepsilon_b/\hbar\omega_{\perp,z}$  in the presence of confinement then follows from the transcendental equation

$$\frac{\ell_{\perp,z}}{a} = \int_0^\infty \frac{du}{\sqrt{4\pi u^3}} \left( 1 - \frac{e^{-\Omega u}}{[(1 - \exp -2u)/2u]^{n/2}} \right) = f_n(\Omega) \quad (82)$$

where  $n = 1, 2$  is the number of confined directions.

The functions  $f_{1,2}$  are shown in Fig. 18. For small binding energies  $\Omega \ll 1$ , their asymptotic behavior is  $f_1(\Omega) = \ln(\pi\Omega/B)/\sqrt{2\pi}$  and  $f_2(\Omega) = -1/\sqrt{\Omega} + A$  with numerical constants  $A = 1.036$  and  $B = 0.905$ . In the range  $\Omega \lesssim 0.1$ , where these expansions are quantitatively valid, the resulting bound state energies are

$$\varepsilon_b = \hbar\omega_z \cdot \frac{B}{\pi} \exp \left( -\sqrt{2\pi} \ell_z / |a| \right) \quad (83)$$

in a 2D pancake geometry or

$$\varepsilon_b = \frac{\hbar\omega_\perp}{(\ell_\perp/|a| + A)^2} \quad (84)$$

in a 1D waveguide. These results were obtained, using different methods, by Petrov and Shlyapnikov (2001) and by Bergeman *et al.* (2003), respectively. With increasing

values of  $|a|$  the binding energy increases, reaching finite, universal values  $\varepsilon_b = 0.244 \hbar\omega_z$  or  $\varepsilon_b = 0.606 \hbar\omega_\perp$  precisely at the Feshbach resonance for one or two confined directions. Going beyond the Feshbach resonance, where  $a > 0$ , the binding energy increases further, finally reaching the standard 3D result in the weak confinement limit  $a \ll \ell_\perp$ . Here the binding is unaffected by the nature of confinement and  $f_n(\Omega \gg 1) = \sqrt{\Omega} = f_0(\Omega)$ .

Experimentally, confinement induced bound states have been observed by rf spectroscopy in an array of 1D quantum wires using a mixture of fermionic  $^{40}\text{K}$  atoms in their two lowest hyperfine states  $m_F = -9/2$  and  $m_F = -7/2$  (Moritz *et al.*, 2005). The different states allow for a finite s-wave scattering length which may be tuned using a Feshbach resonance at  $B_0 = 202$  G. In the absence of the optical lattice a finite binding energy appears only for magnetic fields below  $B_0$ , where  $a > 0$ . In the situation with a strong transverse confinement, however, there is a nonzero binding energy both below and above  $B_0$ . Using Eq. (16) for the magnetic field dependence of the scattering length, its magnitude is in perfect agreement with the result obtained from Eq. (82). In particular, the prediction of a universal value  $\varepsilon_b = 0.606 \hbar\omega_\perp$  of the binding energy right at the Feshbach resonance has been verified by changing the confinement frequency  $\omega_\perp$ .

*Scattering amplitudes for confined particles* Consider now two-particle scattering in the continuum, i.e. for energies  $E = \hbar\omega_\perp + \hbar^2 k^2 / 2M_r$  above the transverse ground state energy. Quite generally, the effective 1D scattering problem is described by a unitary S-matrix  $\hat{S} = \begin{pmatrix} t & r \\ r & t \end{pmatrix}$ , with reflection and transmission amplitudes  $r$  and  $t$ . They are related to the even and odd scattering phase shifts  $\delta_{e,o}(k)$  by the eigenvalues  $\exp(2i\delta_{e,o}(k)) = t \pm r$  of the S-matrix. In analogy to the 3D case, the corresponding scattering amplitudes are defined as  $f_{e,o}(k) = (\exp(2i\delta_{e,o}(k)) - 1) / 2$ . For the particular case of a delta-function interaction, the odd scattering amplitude and phase shift vanish identically. The relevant dimensionless scattering amplitude  $f_e(k) = r(k) = t(k) - 1$  is thus simply the standard reflection amplitude. In the low energy limit, a representation analogous to Eq. (4) allows to define a 1D scattering length  $a_1$  by <sup>12</sup>

$$f(k) = \frac{-1}{1 + i \cot \delta(k)} \rightarrow \frac{-1}{1 + ika_1} \quad (85)$$

with corrections of order  $k^3$  because  $\cot \delta(k)$  is odd in  $k$ . The universal limit  $f(k=0) = -1$  reflects the fact that finite 1D potentials become impenetrable at zero energy. For a delta-function potential  $V(x) = g_1 \delta(x)$ , the low energy form of Eq. (85) holds for arbitrary  $k$ ,

with a scattering length  $a_1 \equiv -\hbar^2 / M_r g_1$ , which approaches  $a_1 \rightarrow -\ell_\perp^2 / a$  in the weak confinement limit. More generally, the exact value of  $a_1$  for an arbitrary ratio  $a / \ell_\perp$  may be determined by using the connection  $\varepsilon_b = \hbar^2 \kappa^2 / 2M_r$  between the bound state energy and a pole at  $k = i\kappa = i/a_1$  of the 1D scattering amplitude (85) in the upper complex  $k$ -plane. Using Eq. (84) for the bound state energy in the regime  $a < 0$  and the fact that  $a_1$  must be positive for a bound state, one finds  $a_1(a) = -\ell_\perp^2 / a + A\ell_\perp$ . Since the effective 1D pseudopotential is completely determined by the scattering length  $a_1$ , two-particle scattering is described by an interaction of the form

$$V_{1D}(x) = g_1 \delta(x) \quad \text{with} \quad g_1(a) = \frac{2\hbar\omega_\perp \cdot a}{1 - Aa/\ell_\perp}. \quad (86)$$

From its derivation, this result appears to be valid only for  $g_1$  small and negative, such that the resulting dimensionless binding energy  $\Omega \lesssim 0.1$  is within the range of validity of Eq. (84). Remarkably, however, the result (86) is exact, as far as scattering of two particles above the continuum is concerned, at *arbitrary* values of  $a / \ell_\perp$ . This is a consequence of the fact that Eq. (86) is the *unique* pseudopotential consistent with the behavior of  $f(k)$  up to order  $k^3$ . The exact scattering length thus is fixed by the binding energy (84) at small values of  $\Omega$ . Moreover, the result uniquely extends into the regime where  $a_1$  becomes negative, i.e. the pseudopotential ceases to support a bound state. The associated change of sign in  $g_1$  at  $a = \ell_\perp / A$  is called a confinement induced resonance (Olshanii, 1998). It allows to change the sign of the interaction between atoms by purely geometrical means! As shown by Bergeman *et al.* (2003), it can be understood as a Feshbach resonance where a bound state in the closed channel drops below the continuum of the ground state. Now, in a 1D waveguide, the separation between energy levels of successive transverse eigenstates with zero angular momentum is exactly  $2\hbar\omega_\perp$ , independent of the value of  $a$  (Bergeman *et al.*, 2003). The confinement induced resonance thus appears precisely when the exact bound state energy  $\varepsilon_b$  - which *cannot* be determined from the pseudopotential (86) unless  $\Omega < 0.1$  - reaches  $2\hbar\omega_\perp$ . An analogous confinement induced resonance appears in a 2D pancake geometry, see Petrov *et al.* (2000a).

## B. Bosonic Luttinger-liquids, Tonks-Girardeau gas

To describe the many-body problem of bosons confined to an effectively 1D situation, the basic microscopic starting point is a model due to Lieb and Liniger (1963)

$$H = -\frac{\hbar^2}{2M} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + g_1 \sum_{i < j} \delta(x_i - x_j). \quad (87)$$

It is based on pairwise interactions with a pseudopotential  $\sim g_1 \delta(x)$ , as given in Eq. (86). This is a valid description of the actual interatomic potential provided that the

<sup>12</sup> As in the 3D case, the scattering length is finite only for potentials decaying faster than  $1/x^3$ . Dipolar interactions are therefore marginal even in one dimension.



two-body scattering amplitude has the low energy form of Eq. (85) for all relevant momenta  $k$ . In the limit  $\mu \ll \hbar\omega_\perp$  of a single transverse mode, they obey  $k\ell_\perp \ll 1$ . Remarkably, in this regime, the pseudopotential approximation is always applicable. Indeed, it follows from the leading correction  $\sim \Omega$  in the low binding energy expansion  $f_2(\Omega) = -1/\sqrt{\Omega} + A + b\Omega + \dots$  of the function defined in Eq. (82), that the denominator of the 1D scattering amplitude (85) has the form  $1 + ik a_1 - ib(k\ell_\perp)^3 + \dots$  at low energies with a numerical coefficient  $b = 0.462$ . Since  $a_1 \approx -\ell_\perp^2/a$  in typical situations where  $a \ll \ell_\perp$ , the condition  $k^2 < 1/(a\ell_\perp)$  for a negligible cubic term  $\sim k^3$  in the scattering amplitude is always obeyed in the regime  $k\ell_\perp \ll 1$  of a single transverse mode. In this limit, the interaction between cold atoms in a 1D tube is therefore generically described by the integrable Hamiltonian (87). In the homogeneous case, stability requires  $g_1$  to be positive<sup>13</sup>, i.e. the 3D scattering length obeys  $\ell_\perp > Aa > 0$ . At a given 1D density  $n_1 = N/L$  the strength of the interactions in Eq. (87) is characterized by a single dimensionless parameter

$$\gamma = \frac{g_1 n_1}{\hbar^2 n_1^2 / M} = \frac{2}{n_1 |a_1|} \approx \frac{2a}{n_1 \ell_\perp^2} \quad \text{if } \ell_\perp \gg a. \quad (88)$$

In marked contrast to the 3D situation, the dimensionless interaction strength  $\gamma$  scales inversely with the 1D density  $n_1$  (Petrov *et al.*, 2000b). In one dimension, therefore it is the *low* density limit where interactions dominate. This rather counterintuitive result can be understood physically by noting that the scattering amplitude Eq. (85) approaches  $-1$  as  $k \rightarrow 0$ . Since, at a given interaction strength  $g_1$ , the low energy limit is reached at low densities, the atoms in this regime are perfectly reflected by the repulsive potential of the surrounding particles. For  $\gamma \gg 1$ , therefore, the system approaches a gas of impenetrable bosons where all the energy is kinetic<sup>14</sup>. In particular, as was shown by Girardeau (1960), at  $\gamma = \infty$ , the hard-core condition of a vanishing wave function whenever two particle coordinates coincide is fulfilled by a wave function

$$\Psi_B(x_1 \dots x_N) = \prod_{i < j} |\sin[\pi(x_j - x_i)/L]| \quad (89)$$

which coincides with the absolute value of the wave function of a non-interacting spinless Fermi gas. Strongly interacting bosons in 1D thus acquire a fermionic character, a fact well known from the exact solution of a hard core Bose or spin 1/2 system on a 1D lattice in terms of non-interacting fermions by the Jordan-Wigner transformation, see e.g. Wen (2004).

*Crossover diagram in a harmonic trap* In the presence of an additional harmonic confinement  $V(x) = M\omega_0^2 x^2/2$  along the axial direction, the relative interaction strength depends, in addition to the parameter  $\gamma$  introduced above, also on the ratio  $\alpha = \ell_0/|a_1| \approx 2a\ell_0/\ell_\perp^2$  between the oscillator length  $\ell_0 = \sqrt{\hbar/M\omega_0}$  and the magnitude of the 1D scattering length. For a tight radial confinement  $\ell_\perp \approx 40\text{nm}$  and typical values  $\ell_0 \approx 2\mu\text{m}$  for the axial oscillator length, one obtains  $\alpha \approx 12$  for <sup>87</sup>Rb. This is in fact the interesting regime, since for  $\alpha \ll 1$  the typical relative momenta  $k \approx 1/\ell_0$  of two particles are so large that the strong interaction limit  $k|a_1| \ll 1$  cannot be reached at all. The conditions for realizing the Tonks-Girardeau (TG) limit in a trap have been discussed by Petrov *et al.* (2000b) using a quantum hydrodynamic description (see also Ho and Ma (1999) and the review by Petrov *et al.* (2004a) on trapped gases in low dimensions). Using Eq. (160) in the appendix, the phase fluctuations in a 1D Bose gas behave like  $\delta\phi^2(x) = (\ln(|x|/\xi))/K$  at zero temperature. From the Lieb-Liniger solution discussed below, both the characteristic healing length  $\xi \approx K(\gamma)/n_1$  and the dimensionless so-called Luttinger parameter  $K = \pi\hbar\kappa c$  are monotonically decreasing functions of the interaction strength. In particular,  $K(\gamma) \rightarrow \sqrt{\pi/\gamma}$  is much larger than one in the limit  $\gamma \ll 1$ . In this limit, the 1D Bose gas loses its phase coherence on a scale  $\ell_\phi(T=0) = \xi \exp K$  (defined by  $\delta\phi^2(x = \ell_\phi) \approx 1$ ), which exceeds the healing length by an exponentially large factor. The gas behaves like a true condensate as long as its size  $R$  is smaller than  $\ell_\phi$ . Applying the Gross-Pitaevskii equation plus the local density approximation, the radius of the associated Thomas-Fermi profile  $n_1(x)$  is  $R_{\text{TF}} \simeq (N\alpha)^{1/3}\ell_0$  (Petrov *et al.*, 2000b). The condition  $R_{\text{TF}} \gg \ell_0$  for the validity of the Thomas-Fermi approximation is thus always obeyed if  $\alpha \gtrsim 1$ . In contrast to the analogous situation in 3D, however, the weak coupling regime requires *high* densities. The local value  $n_1(0)\xi \approx K(\gamma(0))$  of the Luttinger parameter at the trap center must thus be large compared to one. This requires  $\gamma(0) \ll 1$  or, equivalently,  $n_1(0)\ell_0 \approx N\ell_0/R_{\text{TF}} \gg \alpha$ . As a result, the Thomas-Fermi profile becomes invalid if  $N < N_* = \alpha^2 \gg 1$ . For particle numbers below  $N_*$ , the trapped gas reaches the TG-regime. The density distribution is eventually that of a free Fermi gas, with a chemical potential  $\mu = N\hbar\omega_0$  and a cloud size  $R_{\text{TG}} = \sqrt{2N}\ell_0$ . The continuous evolution of the density profile and cloud size between the weak coupling and the TG-limit has been discussed by Dunjko *et al.* (2001).

At finite temperatures, the dominant phase fluctuations are *thermal* and give rise to a linear increase  $\delta\phi^2(x) = |x|/\ell_\phi(T)$  with distance on a scale  $\ell_\phi(T) = \hbar^2 n_s / M k_B T$ , which only depends on the 1D superfluid density  $n_s$ <sup>15</sup>. In a trap of size  $R$ , these fluctuations are

<sup>13</sup> For a possible extension to a metastable 'super-Tonks' regime at  $g_1 < 0$  see Astrakharchik *et al.* (2005a)

<sup>14</sup> In fact, it follows from the Lieb-Liniger solution discussed below, that the ratio of the interaction and kinetic energy per particle diverges like  $1/\sqrt{\gamma}$  for  $\gamma \ll 1$  and decreases monotonically to zero as  $1/\gamma$  for  $\gamma \gg 1$

<sup>15</sup> To simplify the notation, the dimensionality is not indicated in

negligible if  $\ell_\phi > R$ . Using  $n_s \approx N/R$  and the zero temperature result for the Thomas-Fermi radius at  $N > N_*$ , this translates into  $k_B T \lesssim \hbar\omega_0 (N/N_*)^{1/3}$ . In this range, the trapped gas is effectively a true BEC with a Thomas-Fermi density profile and phase coherence extending over the full cloud size. With increasing temperature, phase fluctuations are nonnegligible, however density fluctuations become only relevant if  $T$  exceeds the degeneracy temperature  $T_d = N\hbar\omega_0$  (Petrov *et al.*, 2000b). Defining a characteristic temperature  $T_\phi = T_d/(N\alpha)^{2/3} \ll T_d$  below which phase fluctuations are irrelevant over the system size, there is a wide range  $T_\phi < T < T_d$  in which the density profile is still that of a BEC, however phase coherence is lost. The system can be thought of as a collection of independently fluctuating local BEC's and is called a *quasi-condensate* (Petrov *et al.*, 2000b). At higher temperatures  $k_B T \gtrsim N\hbar\omega_0$  the gas eventually evolves into a non-degenerate regime of a Boltzmann gas. The complete crossover diagram is shown in Fig. 19.

Experimentally, the presence of strong phase fluctuations in a 1D situation already shows up in very elongated 3D condensates which still have a Thomas-Fermi density profile in the radial direction (i.e.  $\mu \gg \hbar\omega_\perp$  such that many transverse modes are involved). In a trap, the strong interaction prevents that local velocity fields due to phase fluctuations show up in the density profile. Switching off the trap, however, the interaction becomes negligible after a certain expansion time and then phase fluctuations are indeed converted into density fluctuations. These have been seen as stripes in absorption images of highly elongated BEC's by Dettmer *et al.* (2001). The linear increase of the phase fluctuations  $\delta\phi^2(x)$  at finite temperature leads to an exponential decay of the first order coherence function. The resulting Lorentzian momentum distribution was observed experimentally by Richard *et al.* (2003) and by Hellweg *et al.* (2003), using Bragg spectroscopy. This enabled a quantitative measurement of  $\ell_\phi(T)$ . Moreover, at very low temperatures, no significant density fluctuations were present, thus confirming the quasi-BEC picture in which  $\langle n(\mathbf{x})^2 \rangle \approx \langle n(\mathbf{x}) \rangle^2$ .

*Lieb-Liniger solution* As shown by Lieb and Liniger (1963) the model Eq. (87) can be solved by the Bethe ansatz. The essential physical property which lies behind the possibility of this exact solution is the fact that in one dimension interactions only give rise to scattering but not to diffraction. All eigenstates of the many-body problem in the domain  $0 \leq x_1 \leq x_2 \dots \leq x_N \leq L$  can thus be written as a sum

$$\Psi_B(x_1 \dots x_N) = \sum_P a(P) \exp i \sum_{l=1}^N k_{P(l)} x_l \quad (90)$$

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the superfluid or quasi-condensate densities  $n_s$  and  $\tilde{n}_0$ .

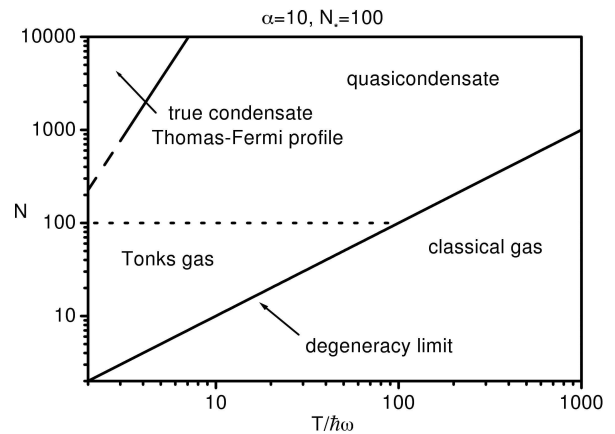


FIG. 19 Phase diagram for a 1D Bose gas in a harmonic trap with  $\alpha = 10$ . The Tonks-Girardeau regime is reached for small particle numbers  $N < N_* = \alpha^2$  and temperatures below the degeneracy limit  $N\hbar\omega_0$ . Reprinted with permission from Petrov *et al.* (2000b).

of plane waves with  $N$  distinct wave vectors  $k_l$ . They are combined with the coordinates  $x_l$  in all  $N!$  possible permutations  $k_{P(l)}$  of the  $k_l$ . The associated amplitude  $a(P) = \prod_{(ij)} e^{i\theta_{ij}}$  factorizes into two-particle scattering phase shifts  $\theta_{ij} = \pi + 2 \arctan(k_i - k_j)a_1/2$ , with  $a_1$  the 1D scattering length associated with the pseudopotential  $g_1\delta(x)$ . Here, the product  $(ij)$  runs over all permutations of two wavenumbers which are needed to generate a given permutation  $P$  of the  $k_l$  from the identity. Due to  $a_1 \sim -1/g_1$ , the two-particle phase shifts  $\theta_{ij} \sim g_1/(k_i - k_j)$  are singular as a function of the momenta in the limit  $g_1 \rightarrow 0$  of an eventually ideal Bose gas. In the limit  $\gamma \gg 1$ , the phase shifts approach  $\theta_{ij} = \pi$  for all momenta. Thus  $a(P)_{\gamma=\infty} = (-1)^{|P|}$  is just the parity of the permutation  $P$  and the Bethe ansatz wave function Eq. (90) is reduced to the free fermion type wave function Eq. (89) of a Tonks-Girardeau gas. For arbitrary coupling, both the ground state energy per particle  $E_0/N = n_1^2 \hbar^2 / 2M \cdot e(\gamma)$  and the chemical potential  $\mu = \partial E_0 / \partial N$  are monotonically increasing functions of  $\gamma$  at fixed density  $n_1$ . For weak interactions  $\gamma \ll 1$ , the chemical potential  $\mu = g_1 n_1$  follows the behavior expected from a mean field approach, which is valid here for *high* densities  $n_1 |a_1| \gg 1$ . In the low density regime  $\gamma \gg 1$ , in turn,  $\mu = \hbar^2 (\pi n_1)^2 / 2M$  approaches a coupling independent value which is just the Fermi energy associated with  $k_F = \pi n_1$ . The energy per particle in this regime is of completely kinetic origin, independent of the interaction strength  $\gamma$ . This remarkable property of the Tonks-Girardeau gas has been observed experimentally by Kinoshita *et al.* (2004) (see Fig. 20). They have measured the axial expansion energy of an array of 1D Bose gases as a function of the strength  $U_0$  of the transverse confinement. Since  $\omega_\perp \sim \sqrt{U_0}$ , the dimensionless coupling  $\gamma$  is monotonically increasing with  $U_0$  at fixed density  $n_1$ . In the weak confinement limit, the expansion energy scales linearly with  $\sqrt{U_0}$ , reflecting the

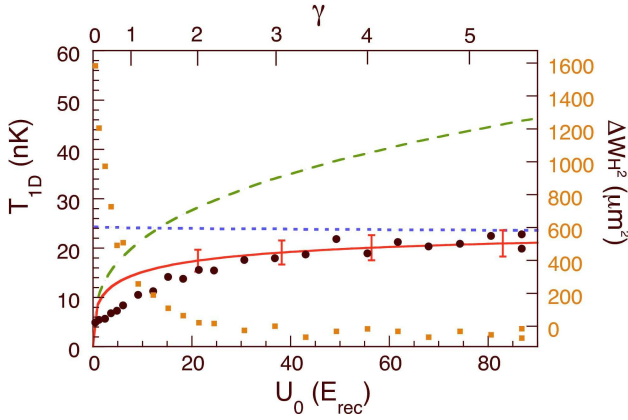


FIG. 20 Average axial energy per particle and equivalent temperature  $T_{1D}$  as a function of the transverse confinement  $U_0$ . With increasing values of  $U_0$ , the energy crosses over from a weakly interacting Bose gas (long-dashed line) to a Tonks-Girardeau gas (short-dashed line), where  $T_{1D}$  is independent of  $U_0$ . Reprinted with permission from Kinoshita *et al.* (2004).

mean field behavior  $e(\gamma) = \gamma(1 - 4\sqrt{\gamma}/3\pi + \dots)$  of the average energy per particle. By contrast, for large values of  $\sqrt{U_0}$ , where  $|a_1|$  becomes shorter than the average interparticle spacing, the energy  $e(\gamma) = \pi^2(1 - 4/\gamma + \dots)/3$  saturates at a value which is fixed by the density.

The low lying excitations of the Lieb-Liniger model have been obtained exactly by Lieb (1963a). Surprisingly, it turned out that there are *two* types of excitations, both with a linear spectrum  $\omega = cq$  at low momenta. One of them has a Bogoliubov like dispersion, linear at  $q\xi \ll 1$  and quadratic at  $q\xi \gg 1$ . The crossover from collective to single-particle behavior occurs at a characteristic length  $\xi$ , which is related to the chemical potential in the ground state via  $\mu = \hbar^2/(2M\xi^2)$ . In the limit  $\gamma \ll 1$ , the crossover length can be expressed in the form  $\xi n_1 = \gamma^{-1/2} \gg 1$ . Similar to the situation in three dimensions, the weak coupling regime can therefore be characterized by the fact that the healing length is much larger than the average interparticle spacing  $n_1^{-1}$ . By contrast, for strong coupling  $\gamma \gg 1$ , where the chemical potential approaches the Fermi energy of a spinless, non-interacting Fermi gas at density  $n_1$ , the healing length  $\xi$  is essentially identical with the average interparticle distance. The sound velocity  $c$  turns out to coincide with the simple thermodynamic formula  $Mc^2 = \partial\mu/\partial n$  which is obtained from the quantum hydrodynamic Hamiltonian Eq. (158) under the assumption that the superfluid density at  $T = 0$  coincides with the full density. The ground state of the Lieb-Liniger gas is in fact fully superfluid at arbitrary values of  $\gamma$  in the sense of Eq. (156), despite the fact that phase fluctuations destroy plain BEC even at zero temperature. The sound velocity increases monotonically with  $\gamma$ , approaching the finite value  $c(\infty) = v_F = \hbar\pi n_1/M$  of an ideal Fermi gas in the Tonks-Girardeau limit. The second type of excitations found by Lieb (1963a) also has a linear dispersion at small

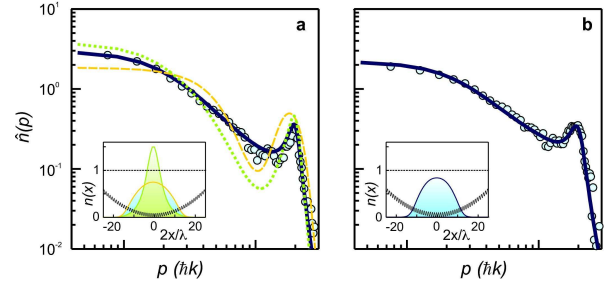


FIG. 21 Axial momentum distribution of a lattice based one-dimensional bosonic quantum gas for (a)  $\gamma_L \approx 14$  and (b)  $\gamma_L \approx 24$ . The solid curve is the theoretical momentum distribution based on fermionization and the short and long-dashed curves in (a) denote the expected values for a non-interacting Bose gas and a non-interacting Fermi gas, respectively. The insets show the corresponding in-trap density distributions. Reprinted with permission from Paredes *et al.* (2004).

$q$  with the same velocity. However, in contrast to the Bogoliubov-like spectrum discussed before, it is restricted to a finite range  $|q| \leq \pi n_1$  and terminates with a vanishing group velocity. It turns out, that these excitations are precisely the solitons of the nonlinear Schrödinger equation (Ishikawa and Takayama, 1980), for a discussion in the cold gas context see Jackson and Kavoulakis (2002).

*Momentum distribution in the Luttinger liquid regime* To obtain the momentum distribution of a strongly correlated 1D Bose gas, it is convenient to start from a quantum hydrodynamic description of the one-particle density matrix. At zero temperature, the logarithmic increase  $\delta\phi^2(x) = (\ln|x|/\xi)/K$  of the phase fluctuations with distance (see Eq. (160)), leads to an algebraic decay of  $g^{(1)}(x)$  at scales beyond the healing length  $\xi$ . The associated exponent  $1/(2K)$  is rather small in the weak coupling regime and approaches its limiting value  $1/2$  in the TG-limit. The resulting momentum distribution thus exhibits a power-law divergence  $\tilde{n}(k) \sim k^{-(1-(1/2K))}$  for  $k\xi \ll 1$ . At any finite temperature, however, this divergence is cutoff due to thermal phase fluctuations. Indeed, these fluctuations increase linearly with distance  $\delta\phi^2(x) = |x|/\ell_\phi(T)$  on a scale  $\ell_\phi(T)$ , implying an exponential decay of the one-particle density matrix for  $|x| > \ell_\phi$ . This leads to a rounding of the momentum distribution at small  $k \simeq 1/\ell_\phi$ .

Experimentally, the momentum distribution of a Tonks-Girardeau gas has been observed for ultracold atoms in a 2D optical lattice by Paredes *et al.* (2004). There a weak optical lattice along the one-dimensional quantum gases was applied in order to tune the system into the strongly interacting regime, where  $K$  is close to one. Indeed, for the low filling factors  $f \ll 1$  used in the experiment, there is no Mott insulating phase. The 1D Bose-Hubbard model at  $f \ll 1$  describes a bosonic Luttinger liquid with  $K \approx 1 + 4\pi f/\gamma_L$  (Cazalilla, 2004b),

where  $\gamma_L = U/J$  is the effective coupling parameter on a lattice. Increasing the axial lattice depth, this ratio becomes very large, of order  $\gamma_L = 5 - 200$ . As shown in Fig. 21, the resulting momentum distributions exhibit a power-law decay over a wide momentum range. They are in very good agreement with a fermionization-based calculation of a hard-core Bose gas in a harmonic confinement. The momentum distributions at finite values of  $U$  have been studied by Quantum Monte Carlo calculations, noting that although the final hardcore limit is only reached for large values of  $\gamma_L \rightarrow \infty$  (Pollet *et al.*, 2004), the deviations in the momentum distribution compared to fermionized bosons are less than a few percent already for  $\gamma_L > 5$  (Wessel *et al.*, 2005). In the experiment, a significant deviation from the limiting value of  $1 - 1/(2K) \rightarrow 1/2$  of the exponent in the momentum distribution at small values of  $k$  is found, which is masked by finite temperature effects and finite size cut-offs. For larger momenta, the momentum distribution of the quantum gas is determined by short-range correlations between the particles, which can increase the coefficient of the power-law decay in the momentum distribution above  $1/2$  in the experiment. In fact, it has been shown by Olshanii and Dunjko (2003), that the momentum distribution of a homogeneous 1D Bose gas at large momenta  $k\xi \gg 1$  should behave like  $1/k^4$  as long as  $kr_e \ll 1$ .

The parameter  $K = \pi\hbar kc$  determines the asymptotic behavior not only of the one-particle density matrix but in fact of *all* correlation functions. This may be understood from Haldane's description of 1D Bose liquids<sup>16</sup> in terms of their long wavelength density oscillations (Haldane, 1981). In its most elementary form, this is just the 1D version of the quantum hydrodynamic Hamiltonian Eq. (158). Introducing a field  $\hat{\theta}(x)$  which is related to the small fluctuations around the average density by  $\delta\hat{n}_1(x) = \partial_x\hat{\theta}(x)/\pi$ , the effective Hamiltonian describing the low lying excitations is of the form

$$\hat{H} = \frac{\hbar c}{2\pi} \int dx \left[ K(\partial_x\hat{\phi})^2 + \frac{1}{K}(\partial_x\hat{\theta})^2 \right] \quad (91)$$

with sound velocity  $c$ . The low energy physics of a 1D Bose liquid is thus completely determined by the velocity  $c$  and the dimensionless parameter  $K$ . In the translation invariant case,  $K = \pi\hbar n_1/Mc$  is fixed by  $c$  and the average density. Moreover, for interactions which may be described by a 1D pseudopotential,  $K$  may be expressed in terms of the microscopic coupling constant  $\gamma$  by using the Lieb-Liniger solution. The resulting value of  $K = v_F/c$  decreases monotonically from  $K(\gamma) = \pi/\sqrt{\gamma} \gg 1$  in the weak interaction, high density limit to  $K(\gamma) = 1 + 4/\gamma + \dots$  in the Tonks-Girardeau limit (see Cazalilla (2004a)). The property  $K > 1$  for repulsive

bosons is valid quite generally for interactions which decay faster than  $1/x^3$  such that the 1D scattering length  $a_1$  is finite. Formally, the algebraic decay of  $g^{(1)}(x)$  in a 1D gas at *zero* temperature is very similar to the situation in 2D at *finite* temperatures, where  $g^{(1)}(r)$  exhibits a power law decay with exponent  $\eta$  (see Eq. (100)). Apart from the different nature of the phase fluctuations (quantum versus thermal), there is, however, an important difference between both situations. In the 2D case, superfluidity is lost via the BKT-transition once  $\eta > 1/4$  (see Eq. (101)). By contrast, in one dimension, there is no such restriction on the exponent and superfluidity still persists if  $K < 2$ . The origin of this difference is related to the fact that phase slips in one dimension require a non-zero modulation of the potential, e.g. by a weak optical lattice (Büchler *et al.*, 2003). Formally, it is related to a Berry phase term beyond Eq. (91), which confines vortex-antivortex pairs in this case, see Wen (2004).

*Two- and three-particle correlations* An intuitive understanding of the evolution from a weakly interacting quasi-condensate to a Tonks-Girardeau gas with increasing values of  $\gamma$  is provided by considering the pair distribution function  $g^{(2)}(x)$ . It is defined by the density correlation function  $\langle \hat{n}(x)\hat{n}(0) \rangle = n_1\delta(x) + n_1^2 g^{(2)}(x)$  and is a measure of the probability to find two particles to be separated by a distance  $x$ . For an *ideal* BEC in three dimensions, the pair distribution function is equal to  $g^{(2)}(\mathbf{x}) \equiv 1$  at arbitrary distances. Above  $T_c$ , it drops monotonically from  $g^{(2)}(0) = 2$  to the trivial limit  $g^{(2)}(\infty) = 1$  of any homogeneous system on the scale of the thermal wavelength  $\lambda_T$ . For cold atoms in 3D, these basic results on bosonic two-particle correlations have been verified experimentally by Öttl *et al.* (2005) and Schellekens *et al.* (2005). For the Lieb-Liniger gas, the local value of the pair correlation  $g^{(2)}(0) = de(\gamma)/d\gamma$  can be obtained directly from the derivative of the dimensionless ground state energy (Gangardt and Shlyapnikov, 2003). The exact result for  $e(\gamma)$  then gives  $g^{(2)}(0) = 1 - 2\sqrt{\gamma}/\pi + \dots$  and  $g^{(2)}(0) = (2\pi/\sqrt{3}\gamma)^2 \rightarrow 0$  in the limits  $\gamma \ll 1$  and  $\gamma \gg 1$ , respectively. For weak coupling, therefore, there is only a small repulsive correlation hole around each particle. By contrast, in the strong coupling limit, the probability to find two bosons at the same point vanishes like  $1/\gamma^2$ . In the Tonks-Girardeau limit, the equivalence of density correlations to that of a free Fermi gas allows to determine the full pair distribution function exactly as  $g^{(2)}(x) = 1 - (\sin(\pi n_1 x)/\pi n_1 x)^2$ . For low densities, therefore, the zero range repulsion strongly suppresses configurations in which two bosons come closer than their mean interparticle distance. Note that the pair correlation exhibits appreciable oscillations with wave vector  $2k_F = 2\pi n_1$  even though the momentum distribution is completely continuous at  $k_F$ . Experimentally, the local value  $g^{(2)}(0)$  of the pair correlation function has been determined by Kinoshita *et al.* (2005), using photo-association. As suggested by Gangardt and Shlyapnikov

<sup>16</sup> In the context of cold gases the notion of a quantum *liquid* is - of course - purely conventional. These systems are stable only in the gaseous phase, yet may be strongly interacting.

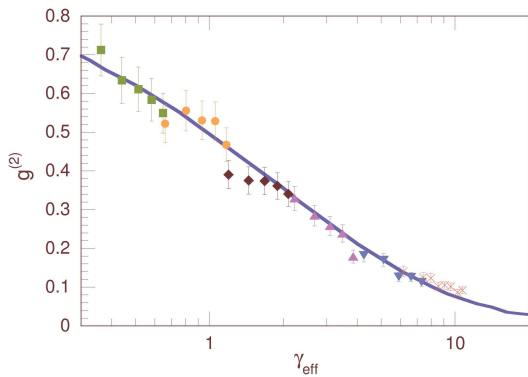


FIG. 22 Local pair correlation function from photo-association measurements as a function of the interaction parameter  $\gamma_{\text{eff}}$  averaged over an ensemble of 1D Bose gases. The theoretical prediction is shown as a solid line. Reprinted with permission from Kinoshita *et al.* (2005).

(2003), the rate  $K_1 = K_3 \cdot g^{(2)}(0)$  for stimulated transitions in which two atoms in a continuum state are transferred to a bound molecule is simply reduced by a factor  $g^{(2)}(0)$  from the corresponding value in 3D, provided the transfer occurs locally on a length scale much less than the transverse oscillator length. The photo-association rate in a one dimensional situation is thus strongly reduced at  $\gamma \gg 1$  due to the much smaller probability for two atoms to be at the same point. From measurements of the atom loss after a variable time of photo-association in an array of several thousand 1D traps with particle numbers in the range  $40 < N < 240$ , Kinoshita *et al.* (2005) have thus extracted the value of  $g^{(2)}(0)$ . As shown in Fig. 22 the results are in very good agreement with theory over a wide range of interaction constants up to  $\gamma \approx 10$ .

The local value of three-body correlation function  $g^{(3)}(0)$  has also been calculated by Gangardt and Shlyapnikov (2003). It behaves like  $g^{(3)}(0) = 1 - 6\sqrt{\gamma}/\pi + \dots$  and  $g^{(3)}(0) \sim (\pi/\gamma)^6$  for small or large  $\gamma$  respectively. The predicted suppression of three-body recombination losses has been observed by Tolra *et al.* (2004), using the strong confinement in a 2D optical lattice.

In-situ measurements of density fluctuations have been performed by Esteve *et al.* (2006). They observe a crossover from an effectively high temperature regime at low densities  $n_1 \ll (|a_1|/\lambda_T^4)^{1/3}$ , where the number fluctuations exceed the shot noise level due to bunching in an essentially ideal Bose gas. At high densities, a quasi-condensate regime is reached with  $n_1 a \simeq 0.7$ , close to the limit of a single transverse mode. There, the number fluctuations are strongly suppressed and may be described by a 1D Bogoliubov description of quasi-condensates (Mora and Castin, 2003).

*Weak optical lattices and coupled 1D gases* The problem of a 1D Bose gas in a *weak* optical lattice has been discussed

by Büchler *et al.* (2003). Using the extension in 1D of the phase-density representation (159) of the field operator in a quantum hydrodynamic description which accounts for the discrete nature of the particles (Haldane, 1981), a periodic potential commensurate with the average particle density gives rise to an additional nonlinear term  $\cos 2\theta(x)$  in (91). This is the well known sine-Gordon model (Giamarchi, 2004), which exhibits a transition at a critical value  $K_c = 2$ . For  $K > 2$  the ground state of the Lieb-Liniger gas remains superfluid in a weak optical lattice. For  $(1 <)K < 2$ , in turn, the atoms are locked in an incompressible Mott state even in an arbitrary weak periodic lattice. From the exact Lieb-Liniger result for  $K(\gamma)$ , the critical value  $K_c = 2$  is reached at  $\gamma_c \approx 3.5$ .

In a configuration using 2D optical lattices, a whole array of typically several thousand parallel 1D gases are generated. For a very large amplitude of the optical lattice  $V_{\perp} \gtrsim 30 E_r$ , hopping between different 1D gases is negligible and the system decouples into independent 1D tubes. By continuously lowering  $V_{\perp}$ , however, it is possible to study the crossover from a 1D to a 3D situation. The equilibrium phase diagram of an array of 1D tubes with an adjustable transverse hopping  $J_{\perp}$  has been studied by (Ho *et al.*, 2004). It exhibits a fully phase coherent BEC for sufficiently large values of  $J_{\perp}$  and the 1D Luttinger parameter  $K$ . For a detailed discussion of weakly coupled 1D gases see Cazalilla *et al.* (2006).

### C. Repulsive and attractive fermions

As mentioned in subsection A, ultracold Fermi gases in a truly 1D regime  $\varepsilon_F \ll \hbar\omega_{\perp}$  have been realized using strong optical lattices. In the presence of an additional axial confinement with frequency  $\omega_0$ , the Fermi energy is  $\varepsilon_F = N\hbar\omega_0$ . The requirement that only the lowest transverse mode is populated therefore requires small particle numbers  $N \ll \omega_{\perp}/\omega_0$ . Typical temperatures in these gases are around  $k_B T \approx 0.2\varepsilon_F$  (Moritz *et al.*, 2005), which is not small enough, to observe non-trivial many-body effects. We will therefore discuss the basic phenomena which may be studied with ultracold fermions in one dimension only briefly.

In a spin-polarised 1D Fermi gas, only p-wave interactions are possible. As shown by Granger and Blume (2004), the corresponding Feshbach resonances are shifted due to the confinement in a similar way as in (86) above. This was confirmed experimentally by Günter *et al.* (2005). In the case of two different states, s-wave scattering dominates in the ultracold limit. For repulsive interactions one obtains a fermionic Luttinger liquid, which is basically a two component version of the quantum hydrodynamic Hamiltonian (91). It has a twofold linear excitation spectrum for fluctuations of the total density and the density difference ('spin density') respectively. Generically, the velocities of 'charge' and spin excitations are different. This is the most elementary form of 'spin-charge separation', which has

been verified experimentally in semiconductor quantum wires (Auslaender *et al.*, 2005). In the context of ultra-cold Fermi gases in a harmonic trap, spin-charge separation effects show up in collective excitation frequencies (Recati *et al.*, 2003) or in the propagation of wavepackets (Kollath *et al.*, 2005). A genuine observation of spin-charge separation, however, requires to study single particle correlations and cannot be inferred from collective excitations only.

For *attractive* interactions, a spin 1/2 Fermi gas which is described by the basic Hamiltonian Eq. (87) is a so-called Luther-Emery liquid. Its fluctuations in the total density still have a linear spectrum  $\omega(q) = cq$ , however there is a finite gap for spin excitations (Giamarchi, 2004). The origin of this gap is the appearance of bound pairs of fermions with opposite spin. The spectrum  $\omega_s(q) = \sqrt{(\Delta_s/2\hbar)^2 + (v_s q)^2}$  for small oscillations of the spin density is similar to that of quasiparticles in the BCS theory. In analogy to Eq. (88) for the bosonic problem, the dimensionless coupling constant  $\gamma = -2/n_1 a_1 < 0$  is inversely proportional to both the 1D scattering length  $a_1 = -\ell_\perp^2/a + Al_\perp$  (now for fermions in different states, note that attractive interactions  $a < 0$  imply a *positive* 1D scattering length  $a_1$ ) and to the total 1D density  $n_1 = n_{1\uparrow} + n_{1\downarrow}$ . As shown by Gaudin (1967) and Yang (1967), the model is exactly soluble by the Bethe ansatz. For weak coupling,  $|\gamma| \lesssim 1$ , the spin gap  $\Delta_s \equiv 2\Delta_{\text{BCS}}$

$$\Delta_s(\gamma) = \frac{16\varepsilon_F}{\pi} \sqrt{\frac{|\gamma|}{\pi}} e^{-\pi^2/2|\gamma|}. \quad (92)$$

has a form similar to that in BCS theory, except for an interaction dependent prefactor  $\sim \sqrt{|\gamma|}$ . Note, however, that the weak coupling regime is reached at *high* densities  $n_1|a_1| \gg 1$ , in contrast to 3D, where  $k_F|a| \ll 1$  in the BCS limit. At low densities, where  $1/\gamma \rightarrow 0^-$ , the spin gap approaches the two-body bound state energy  $\Delta_s \rightarrow \varepsilon_b$ , which was measured by rf-spectroscopy (Moritz *et al.*, 2005), as discussed above. In this regime, the tightly bound fermion pairs behave like a hard core Bose gas. The strong coupling BEC limit of attractive fermions in 1D thus appears to be a Tonks-Girardeau gas, very different from the nearly ideal Bose gas expected in a 3D situation (see section VIII). However, in the presence of a harmonic waveguide, the associated transverse oscillator length  $\ell_\perp \equiv \sqrt{\hbar/M\omega_\perp}$  defines an additional length, not present in a strictly 1D description. As shown in section V.A, the exact solution of the scattering problem for two particles in such a waveguide always exhibits a two-body bound state, whatever the sign and magnitude of the scattering length  $a$ . Its binding energy right at the confinement induced resonance is equal to  $\varepsilon_b = 2\hbar\omega_\perp$ . Since  $\hbar\omega_\perp \gg \varepsilon_F$  in the limit of a singly occupied transverse channel, the two-particle bound state energy  $\varepsilon_b$  is the largest energy scale in the problem beyond this point. In the regime after the confinement induced resonance, where  $\gamma$  becomes positive, the appropriate degrees of freedom are therefore no longer the single atoms, but instead are strongly bound fermion pairs. An exact solution of

the four-body problem in a quasi 1D geometry with tight harmonic confinement shows, that these dimers have a *repulsive* interaction (Mora *et al.*, 2005). Attractive fermions in 1D thus continuously evolve from a fermionic Luther-Emery liquid to a gas of repulsive bosons. As realized by Fuchs *et al.* (2004) and Tokatly (2004), the 1D BCS-BEC crossover problem can be solved exactly by the Bethe ansatz, connecting the Gaudin-Yang model on the fermionic side with the Lieb-Liniger model on the bosonic side.

The problem of attractive Fermi gases in one dimension at different densities  $n_\uparrow \neq n_\downarrow$  of the two components has recently been solved by Hu *et al.* (2007b) and Orso (2007). The superfluid ground state with equal densities becomes unstable above a critical field  $h_c = \Delta_s/2$ . This is the analog of the Clogston-Chandrasekhar limit (Chandrasekhar, 1962; Clogston, 1962), where the paired ground state is destroyed by the paramagnetic, or Pauli, mechanism. In 3D this has been observed by Zwierlein *et al.* (2006). In contrast to the 3D case, however, the transition is continuous in 1D and the result  $h_c = \Delta_s/2$  holds for arbitrary coupling strengths. Moreover, since the gap becomes large at *low* densities, the SF phase with zero density imbalance appears at the trap *edge*. The phase in the trap center, in turn, is a partially polarized phase which still has superfluid correlations. As argued by Yang *et al.* (2001) using Bosonization, this phase exhibits an oscillating superfluid order parameter similar to that predicted by Fulde and Ferrell (1964); Larkin and Ovchinnikov (1965) in a narrow range above the Clogston-Chandrasekhar limit. In contrast to the 3D situation, non-conventional superfluid order is thus expected in a rather wide range of parameters.

## VI. TWO-DIMENSIONAL BOSE GASES

The two-dimensional Bose gas is a system which presents many interesting features from a many-body physics perspective. The first question that arises concerns the possibility to reach Bose-Einstein condensation in a uniform system. The answer to this question is negative, both for the ideal and the interacting gas. Indeed in reduced dimensionality long range order is destroyed by thermal fluctuations at any finite temperature. However in an interacting 2D gas the destruction of order is only marginal and superfluidity can still occur below a finite critical temperature  $T_c$ . Above  $T_c$ , the quasi-long range order is destroyed via the mechanism that was elucidated by Berezinskii (1971), and Kosterlitz and Thouless (1973) and which consists in the breaking of pairs of vortices with opposite circulations. As shown by Nelson and Kosterlitz (1977) this scenario

implies a jump in the superfluid density<sup>17</sup> from a finite and universal value  $n_s(T_c)/(k_B T_c) = 2M/(\pi\hbar^2)$  to zero, right at  $T_c$ . Equivalently, the thermal wavelength  $\lambda_T$  obeys  $n_s(T_c)\lambda_T^2 = 4$ . This prediction has been experimentally tested using helium films (Bishop and Reppy, 1978, 1980).

Quantum atomic gases provide a new system where this concept of a quasi-long range order can be experimentally tested. However the addition of a harmonic potential to confine the gas in the plane changes the problem significantly. For example conventional Bose-Einstein condensation of an ideal gas is possible in a 2D harmonic potential. For an interacting gas, the situation is more involved; a true BEC is still expected at extremely low temperature. At slightly higher temperature phase fluctuations may destabilize it and turn it into a quasi-condensate phase, which is turned into a normal gas above the degeneracy temperature. We review in this section the main features of atomic 2D gases and we discuss the experimental results obtained so far.

We start with an ideal gas of  $N$  bosons at temperature  $T$ , confined in a square box of size  $L^2$ . Using the Bose-Einstein distribution and assuming a smooth variation of the population of the various energy states, we can take the thermodynamic limit  $N, L \rightarrow \infty$  in such a way that the density  $n = N/L^2$  stays constant. We then find a relation between the density  $n$ , the thermal wavelength  $\lambda_T = h/(2\pi M k_B T)^{1/2}$  and the chemical potential  $\mu$ :

$$n\lambda_T^2 = -\ln\left(1 - e^{\mu/(k_B T)}\right). \quad (93)$$

This relation allows one to derive the value of  $\mu$  for any value of the degeneracy parameter  $n\lambda_T^2$ . It indicates that no condensation takes place in 2D, contrarily to the 3D case. In the latter case the relation between  $n_{3D}\lambda_T^3$  and  $\mu$  ceases to admit a solution above the critical value  $n_{3D}\lambda_T^3 = 2.612$ , which is the signature for BEC.

Consider now  $N$  bosons confined by the potential  $V(r) = M\omega^2 r^2/2$  in the  $xy$  plane. The presence of a trap modifies the density of states and BEC is predicted to occur for an ideal gas when the temperature is below the critical temperature  $T_0$  (Bagnato and Kleppner, 1991):

$$N = \frac{\pi^2}{6} \left(\frac{k_B T_0}{\hbar\omega}\right)^2. \quad (94)$$

However it should be pointed out that the condensation remains a very fragile phenomenon in a 2D harmonic potential. To show this point we calculate the spatial density  $n(\mathbf{x})$  using the local density and semiclassical approximations, which amounts to replacing  $\mu$  by  $\mu - V(\mathbf{x})$  in Eq. (93)

$$n(\mathbf{x})\lambda_T^2 = -\ln\left(1 - e^{(\mu - V(\mathbf{x}))/k_B T}\right). \quad (95)$$

Taking  $\mu \rightarrow 0$  to reach the condensation threshold and integrating over  $\mathbf{x}$ , we recover the result (94). But we also note that  $n_{\max}(0) = \infty$ , which means that the condensation in a 2D harmonic potential occurs only when the 2D spatial density at the center of the trap is infinite. This should be contrasted with the result for a 3D harmonically trapped Bose gas, where condensation occurs at a central density  $n_{3D,\max}(0) = \zeta(3/2)/\lambda_T^3$ , which is equal (in the semi-classical approximation) to the threshold density in a homogenous system (see e.g. Pitaevskii and Stringari (2003)).

### A. The uniform Bose gas in two dimensions

We now turn to the more realistic case of a system with repulsive interactions and we consider in this section the case of a uniform gas. We will restrict here to well established results, the main goal being to prepare the discussion of the trapped gas case, that will be addressed in the next section. The first task is to model the atom interaction in a convenient way. As it has been done for a 1D system, it is tempting to use a contact term  $g_2 \delta(\mathbf{x})$ , which leads to a chemical potential  $\mu = g_2 n$  in a mean-field approximation. However two-dimensional scattering has very peculiar properties and we will see that in general it is not possible to describe interactions in 2D by a coupling constant  $g_2$ , and that one has to turn to an energy dependent coefficient. We will then discuss the many-body state expected at low temperature, and present briefly the Berezinskii-Kosterlitz-Thouless transition.

We start by some considerations concerning quantum scattering in two dimensions. We consider two particles of mass  $M$  moving in the  $xy$  plane and we restrict here to low energy motion where the scattering is isotropic. The scattering state can be written (Adhikari, 1986)

$$\psi_{\mathbf{k}}(\mathbf{x}) \sim e^{i\mathbf{k}\cdot\mathbf{x}} - \sqrt{\frac{i}{8\pi}} f(k) \frac{e^{ikr}}{\sqrt{kr}} \quad (96)$$

where  $\mathbf{k}$  is the incident wave vector and  $f(k)$  the dimensionless scattering amplitude for the relative energy  $E = \hbar^2 k^2/M$ . At low energy, one gets for the scattering amplitude the following variation

$$f(k) = \frac{4}{-\cot\delta_0(k) + i} \rightarrow \frac{4\pi}{2\ln(1/ka_2) + i\pi}, \quad (97)$$

which defines the 2D scattering length  $a_2$ . Taking for instance a square well interaction potential of depth  $V_0$  and diameter  $b$  it is equal to  $a_2 = bF(k_0 b)$ , with  $F(x) = \exp[J_0(x)/(xJ_1(x))]$  and  $k_0 = \sqrt{2MV_0}/\hbar$ . Note that, in contrast to the situation in 3D, where  $\lim_{k \rightarrow 0} f(k) = -a$ , in 2D  $f(k)$  tends to 0 when  $k \rightarrow 0$ . The total cross-section  $\lambda = |f(k)|^2/4k$  (dimension of a length), however, tends to infinity.

Since the coupling coefficient  $g_2$  is directly related to the scattering amplitude, it appears that 2D systems are

<sup>17</sup> Note that, unless explicitly indicated, all densities in this section are *areal* and not volume densities

peculiar in the sense that the coupling coefficient is intrinsically energy dependent, by contrast to 1D and 3D systems. In addition to the scattering amplitude  $f(k)$ , one may need the off-shell  $T$ -matrix when addressing many-body problems. It has been calculated for 2D hard disks by Morgan *et al.* (2002). The extension of the notion of a zero-range interaction potential to the two-dimensional case is discussed by Olshanii and Pricoupenko (2002).

We now turn to a macroscopic assembly of bosonic particles and we first address the  $T = 0$  situation. The case of a gas of hard disks of diameter  $b$  and surface density  $n$  has been studied by Schick (1971). The conclusion is that Bose-Einstein condensation is reached with a large condensate fraction, provided  $(\ln(1/nb^2))^{-1} \ll 1$ . This constitutes the small parameter of the problem, to be compared with  $\sqrt{na^3}$  in 3D. The chemical potential is then  $\mu \simeq 4\pi\hbar^2 n / [M \ln(1/nb^2)]$ , indicating that the proper choice for  $g_2$  is (within logarithmic accuracy)  $g_2 = \hbar^2 \tilde{g}_2 / M$ , where the dimensionless number  $\tilde{g}_2$  is equal to the scattering amplitude  $f(k)$  taken for the energy  $E = 2\mu$ . Corrections to the result of Schick (1971) for more realistic densities have been calculated by Andersen (2002); Pilati *et al.* (2005); Pricoupenko (2004).

In the finite temperature case, the impossibility for 2D BEC already mentioned for an ideal gas remains valid for an interacting gas with repulsive interactions. This was anticipated by Peierls (1935) in the general context of long range order in low dimensional systems. It was shown rigorously for interacting bosons by Hohenberg (1967), based on arguments by Bogoliubov (1960). A completely equivalent argument was given for lattice spin systems by Mermin and Wagner (1966). To prove this result one can make a *reductio ad absurdum*. Suppose that the temperature is small, but finite ( $T \neq 0$ ) and that a condensate is present in the mode  $\mathbf{k} = 0$ , with a density  $n_0$ . By the Bogoliubov inequality the number of particles  $\tilde{n}_{\mathbf{k}}$  in state  $\mathbf{k} \neq 0$  satisfies

$$\tilde{n}_{\mathbf{k}} + \frac{1}{2} \geq \frac{k_B T}{\hbar^2 k^2 / M} \frac{n_0}{n}. \quad (98)$$

In the thermodynamic limit, the number of particles  $N'$  in the excited states is

$$N' = \sum_{\mathbf{k}} \tilde{n}_{\mathbf{k}} = \frac{L^2}{4\pi^2} \int \tilde{n}_{\mathbf{k}} d^2k. \quad (99)$$

When  $k$  tends to zero the dominant term in the lower bound given above varies as  $1/k^2$ . In 2D, this leads to a logarithmically diverging contribution of the integral originating from low momenta. This means that the starting hypothesis (existence of a condensate in  $\mathbf{k} = 0$ ) is wrong in 2D.

Even though there is no BEC for a homogeneous, infinite 2D Bose gas, the system at low temperature can be viewed as a quasi-condensate, i.e. a condensate with a fluctuating phase (Kagan *et al.*, 1987; Popov, 1987). The state of the system is well described by a wave function

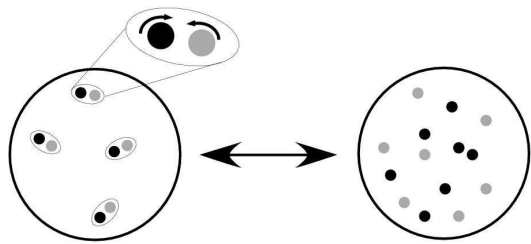


FIG. 23 Microscopic mechanism at the origin of the superfluid transition in the uniform 2D Bose gas. Below the transition temperature vortices only exist in the form of bound pairs formed by two vortices with opposite circulation. Above the transition temperature free vortices proliferate, causing an exponential decay of the one-body correlation function  $g_1(r)$ .

$\psi(\mathbf{x}) = \sqrt{\tilde{n}_0(\mathbf{x})} e^{i\phi(\mathbf{x})}$  and the two dimensional character is revealed by the specific statistical behavior of the spatial correlation functions of the phase  $\phi(\mathbf{x})$  and the quasi-condensate density  $\tilde{n}_0(\mathbf{x})$ . Actually repulsive interactions tend to reduce the density fluctuations and one can in first approximation focus on phase fluctuations only. The energy arising from these phase fluctuations has two contributions. The first one originates from phonon-type excitations, where the phase varies smoothly in space. The second one is due to quantized vortices, i.e. points where the density is zero and around which the phase varies by a multiple of  $2\pi$ . For our purpose it is sufficient to consider only singly charged vortices, where the phase varies by  $\pm 2\pi$  around the vortex core.

Berezinskii (1971) and Kosterlitz and Thouless (1973) have identified how a phase transition can occur in this system, when the temperature is varied (Fig. 23). At low temperature the gas has a finite superfluid density  $n_s$ . The one-body correlation function decays algebraically at large distance:

$$T < T_c : \quad ng^{(1)}(r) = \langle \hat{\psi}(\mathbf{x}) \hat{\psi}(0) \rangle \propto r^{-\eta} \quad (100)$$

with  $\eta = (n_s \lambda_T^2)^{-1}$ . The remarkable fact that there is an exact relation between the coherence properties of the system and the superfluid density is explained in the appendix. Free vortices are absent in this low temperature phase and vortices exist only in the form of bound pairs, formed by two vortices with opposite circulations. At very low temperatures the contribution of these vortex pairs to the correlation function  $g^{(1)}$  is negligible, and the algebraic decay of  $g^{(1)}$  is dominated by phonons. When  $T$  increases, bound vortex pairs lead to a renormalisation of  $n_s$ , which remains finite as long as  $T$  is lower than the critical temperature  $T_c$  defined by

$$T = T_c : \quad n_s \lambda_T^2 = 4. \quad (101)$$

Above  $T_c$  the decay of  $g^{(1)}(r)$  is exponential or even Gaussian, once the temperature is large enough that the gas is close to an ideal system. With increasing temperature, therefore, the superfluid density undergoes a jump at the



critical point from  $4/\lambda_T^2(T_c)$  to 0 (Nelson and Kosterlitz, 1977). Note that Eq. (101) is actually an implicit equation for the temperature since the relation between the superfluid density  $n_s(T)$  and the total density  $n$  remains to be determined. The physical phenomenon at the origin of the Berezinskii-Kosterlitz-Thouless phase transition is related to the breaking of the pairs of vortices with opposite circulation. For  $T$  slightly above  $T_c$ , free vortices proliferate and form a disordered gas of phase defects, which are responsible for the exponential decay of  $g^{(1)}$ . For higher temperatures the gas eventually exhibits strong density fluctuations and the notion of vortices becomes irrelevant.

The value given above for the transition temperature can be recovered rather simply by evaluating the likelihood to have a free vortex appearing in a superfluid occupying a disk of radius  $R$  (Kosterlitz and Thouless, 1973). One needs to calculate the free energy  $F = E - TS$  of this state. The energy  $E$  corresponds to the kinetic energy of the superfluid; assuming that the vortex is at the center of the disk, the velocity field is  $v = \hbar/(Mr)$ , hence  $E = \pi n_s \int v^2(r) r dr = \pi \hbar^2/M \ln(R/\xi)$ , where we set the lower bound of the integral equal to the healing length  $\xi$ , since it gives approximately the size of the vortex core. The entropy associated with positioning the vortex core of area  $\pi \xi^2$  in the superfluid disk of area  $\pi R^2$  is  $k_B \ln(R^2/\xi^2)$ , hence the expression of the free energy:

$$\frac{F}{k_B T} = \frac{1}{2} (n_s \lambda_T^2 - 4) \ln(R/\xi). \quad (102)$$

For  $n_s \lambda_T^2 > 4$  the free energy is large and positive for a large system ( $R \gg \xi$ ), indicating that the appearance of a free vortex is very unlikely. On the opposite for  $n_s \lambda_T^2 < 4$ , the large and negative free energy signals the proliferation of free vortices. The critical temperature estimated above from a single vortex picture turns out to coincide with the temperature where pairs of vortices with opposite circulation dissociate. Such pairs have a finite energy even in an infinite system (Kosterlitz and Thouless, 1973).

The question remains how to relate the various spatial densities appearing in this description, such as the total density  $n$ , and the superfluid density  $n_s$ . In the Coulomb gas analogy where positive and negative charges correspond to clockwise and counterclockwise vortices (see e.g. Minnhagen (1987)) these two quantities are related by  $n_s/n = 1/\varepsilon(T)$ , where  $\varepsilon(T)$  is the dielectric constant of the 2D Coulomb gas. For an extremely dilute Bose gas the relation between  $n$  and  $n_s$  has been addressed by Fisher and Hohenberg (1988). Their treatment is valid in the limit of ultra weak interactions  $\epsilon = 1/\ln(\ln(1/(na_2^2))) \ll 1$ , where  $a_2$  is the 2D scattering length. They obtain the result  $n_s/n \sim \epsilon$  on the low temperature side of the transition point. Using Monte-Carlo calculation, Prokof'ev *et al.* (2001) have studied the case of weak, but more realistic interactions. Denoting  $\hbar^2 \tilde{g}_2/m$  the effective long wavelength interaction constant, they obtain the following result for the total

density at the critical point:  $n \lambda_T^2 = \ln(C/\tilde{g}_2)$  where the dimensionless number  $C = 380 \pm 3$ . A typical value for  $\tilde{g}_2$  in cold atom experiments is in the range 0.01–0.2, which leads to a total phase space density at the critical point  $n \lambda_T^2$  in the range 7.5–10.5. Prokof'ev *et al.* (2001) also evaluate the reduction of density fluctuation with respect to the expected result  $\langle n^2 \rangle = 2\langle n \rangle^2$  for an ideal gas. They observe that these fluctuations are strongly reduced at the transition point for the domain of coupling parameters relevant for atomic gases. These high precision Monte-Carlo methods also allow one to study the fluctuation region around the transition point (Prokof'ev and Svistunov, 2005).

The BKT mechanism has been the subject of several studies and confirmations in various branches of condensed matter physics (for a review see Minnhagen (1987)). In the context of Bose fluids, Bishop and Reppy (1978) performed an experiment with helium films adsorbed on an oscillating substrate. The change in the moment of inertia of the system gave access to the superfluid fraction and showed a clear evidence for the BKT transition. In an experiment performed with atomic hydrogen adsorbed on superfluid helium, Safonov *et al.* (1998) observed a rapid variation of the recombination rate of the 2D hydrogen gas when the phase space density was approaching the critical value Eq. (101). However it is still a matter of debate whether one can reach a quantitative agreement between these experimental observations and the theoretical models (Andersen *et al.*, 2002; Kagan *et al.*, 2000; Stoof, 1994).

## B. The trapped Bose gas in 2D

The recent progress concerning the manipulation, cooling and trapping of neutral atomic gases with electromagnetic fields has naturally opened the way to the study of planar Bose gases. In order to prepare 2D atomic gases one freezes the motion along the  $z$  direction using either light induced forces or magnetic forces. This confining potential  $V(z)$  has to be strong enough so that all relevant energies for the gas (chemical potential, temperature) are well below the excitation energy from the ground state to the first excited state in  $V(z)$ . The two other directions  $x, y$  are much more weakly confined. The potential in the  $xy$  plane is harmonic in all experiments so far. Here we first review the main experimental schemes that have been implemented. We then discuss the new features that appear because of the harmonic confinement in the  $xy$  plane and we present the current status of experimental investigations concerning the coherence properties of these trapped 2D gases.

*Experimental realizations of a 2D gas* The conceptually simplest scheme to produce a 2D gas is to use a sheet of light with a red detuning with respect to the atomic resonance. The dipole potential then attracts the atoms towards the locations of high light intensity, and ensures

a strong confinement in the direction perpendicular to the light sheet. This technique has been implemented at MIT by Görlitz *et al.* (2001) for sodium atoms. A 1064 nm laser was focused using cylindrical lenses, and provided a trapping frequency  $\omega_z/(2\pi)$  around 1000 Hz along the  $z$  direction. The red-detuned light sheet also ensured harmonic trapping in the  $xy$  plane, with much smaller frequencies (30 and 10 Hz along the  $x$  and  $y$  directions, respectively). An adjustable number of atoms, varying between  $2 \times 10^4$  and  $2 \times 10^6$ , was loaded in the dipole trap starting with a 3D condensate. The measurements were essentially devoted to the size of the atom cloud after free ballistic expansion. For small numbers of atoms (below  $10^5$ ) it was observed that the  $z$  motion was indeed frozen, with a release energy essentially equal to the kinetic energy of the ground state  $\hbar\omega_z/4$ . For larger atom numbers, the interaction energy exceeded  $\hbar\omega_z$  and the gas was approaching the 3D Thomas-Fermi limit.

Another way of implementing a 2D trap consists in using an evanescent wave propagating at the surface of a glass prism. In 2004 the group of R. Grimm in Innsbruck loaded such a trap with a condensate of cesium atoms (Rychtarik *et al.*, 2004). The light was blue detuned from resonance, so that the atoms levitated above the light sheet, at a distance  $\sim 4 \mu\text{m}$  from the horizontal glass surface ( $\omega_z/(2\pi) \sim 500$  Hz). The confinement in the horizontal  $xy$  plane was provided by an additional hollow laser beam, which was blue detuned from the atomic resonance and propagating vertically. This provided an isotropic trapping with a frequency  $\omega_\perp/(2\pi) \sim 10$  Hz. As in the MIT experiment, a time-of-flight technique revealed that for small atom numbers the vertical expansion energy was approximately equal to  $\hbar\omega_z/4$ , meaning that the  $z$  motion was frozen. The number of atoms was decreased together with temperature, and a signature of a rapid increase of the spatial density, causing an increase of losses due to 3-body recombination, was observed when the gas approached quantum degeneracy. These data were consistent with the formation of a condensate or a quasi-condensate at the bottom of the trap.

A hybrid trap has been investigated in Oxford, where a blue detuned, single node, Hermite Gaussian laser beam trapped Rb atoms along the  $z$  direction, whereas the confinement in the  $xy$  plane was provided by a magnetic trap (Smith *et al.*, 2005). This allowed to achieve a very large anisotropy factor ( $\sim 700$ ) between the  $z$  axis and the transverse plane. Here also the 2D regime was reached for a degenerate gas with  $\sim 10^5$  atoms.

Trapping potentials that are not based on light beams have also been investigated. One possibility discussed by Hinds *et al.* (1998) consists in trapping paramagnetic atoms just above the surface of a magnetized material, producing an exponentially decaying field. The advantage of this technique lies in the very large achievable frequency  $\omega_z$ , typically in the MHz range. One drawback is that the optical access in the vicinity of the magnetic material is not as good as with optically generated trapping potentials. Another appeal-

ing technique to produce a single 2D sheet of atoms uses the so-called radio-frequency dressed state potentials (Zobay and Garraway, 2001). The atoms are placed in an inhomogeneous static magnetic field, superimposed with a radio-frequency field, whose frequency is of the order of the energy splitting between two consecutive Zeeman sublevels. The dressed states are the eigenstates of the atomic magnetic moment coupled to the static and radio-frequency fields. Since the magnetic field is not homogeneous, the exact resonance occurs on a 2D surface. There, one dressed state (or possibly several, depending on the atom spin) has an energy minimum, and the atoms prepared in this dressed state can form a 2D gas. This method was implemented experimentally for a thermal gas by Colombe *et al.* (2004), but no experiment has yet been performed in the degenerate regime.

Finally a 1D optical lattice setup, formed by the superposition of two running laser waves, is a very convenient way to prepare stacks of 2D gases (Burger *et al.*, 2002; Köhl *et al.*, 2005b; Morsch and Oberthaler, 2006; Orzel *et al.*, 2001; Spielman *et al.*, 2007). The 1D lattice provides a periodic potential along  $z$  with an oscillation frequency  $\omega_z$  that can easily exceed the typical scale for chemical potential and temperature (a few kHz). The simplest lattice geometry is formed by two counter-propagating laser waves, and it provides the largest  $\omega_z$  for a given laser intensity. One drawback of this geometry is that it provides a small lattice period ( $\lambda/2$  where  $\lambda$  is the laser wavelength) so that many planes are simultaneously populated. Therefore practical measurements only provide averaged quantities. Such a setup has been successfully used to explore the transition between a superfluid and a Mott insulator in a 2D geometry (Köhl *et al.*, 2005b; Spielman *et al.*, 2007). Another interesting geometry consists in forming a lattice with two beams crossing at an angle  $\theta$  smaller than  $180^\circ$  (Hadzibabic *et al.*, 2004). In this case the distance  $\lambda/(2 \sin(\theta/2))$  between adjacent planes is adjustable, and each plane can be individually addressable if this distance is large enough (Schrader *et al.*, 2004; Stock *et al.*, 2005). Furthermore the tunneling matrix element between planes can be made completely negligible, which is important if one wants to achieve a true 2D geometry and not a modulated 3D situation.

*From 3D to 2D scattering* In section VI.A we discussed the properties of a 2D gas consisting of hard disks. Cold atomic gases, however, interact through van der Waals forces, and one has to understand how to switch from the 3D coupling constant to the 2D case. The confining potential along  $z$  is  $V(z) = M\omega_z^2 z^2/2$ , and we assume that  $\mu, k_B T \ll \hbar\omega_z$  such that the single atom motion along the  $z$  direction is frozen into the gaussian ground state.

The scattering amplitude in this regime was calculated by Petrov *et al.* (2000a), and Petrov and Shlyapnikov (2001). Quite generally, low energy scattering in 2D is described by a scattering amplitude of the form Eq. (97).

Since  $f(k)$  has a pole at  $k = i/a_2$ , the relation between  $a_2$  and the basic scattering length  $a$  of the 3D pseudopotential may be determined from the bound state energy  $\varepsilon_b = \hbar^2/(2M_r a_2^2)$  in a 2D confined geometry. This has been calculated in section V.A for arbitrary values of the ratio between the 3D scattering length  $a$  and the confinement length  $\ell_z$ . Using Eq. (83) in the limit of small binding energies, the 2D scattering length is related to its 3D counterpart and the confinement scale  $\ell_z = [\hbar/(M\omega_z)]^{1/2}$  by

$$a_2(a) = \ell_z \sqrt{\frac{\pi}{B}} \exp\left(-\sqrt{\frac{\pi}{2}} \frac{\ell_z}{a}\right). \quad (103)$$

with  $B = 0.905$  (Petrov and Shlyapnikov, 2001). As in 1D, the scattering length for particles in the continuum is determined uniquely by the two-particle binding energy in the limit  $\varepsilon_b \ll \hbar\omega_z$ . The fact that  $a_2(a)$  is positive, independent of the sign of  $a$ , shows that for a 3D interaction described by a pseudo potential, a two-particle bound state exists for an arbitrary sign and strength of the ratio  $a/\ell_z$  as discussed in section V.A. Note that for realistic parameters  $\ell_z \sim 100$  nm and  $a$  of the order of a few nm, the 2D scattering length is incredibly small. This is compensated for by the logarithmic dependence of the scattering amplitude on  $a_2(a)$ . Indeed, from Eqs. (97) and (103), the effective low energy scattering amplitude of a strongly confined 2D gas is given by

$$f(k) = \frac{4\pi}{\sqrt{2\pi} \ell_z/a + \ln(B/(\pi k^2 \ell_z^2)) + i\pi}. \quad (104)$$

When the binding along  $z$  is not very strong,  $\ell_z$  is much larger than  $a$  so that the logarithm and the imaginary term in (104) are negligible. This weak confinement limit corresponds to the relevant regime for the experiments performed up to now. The resulting scattering amplitude

$$f(k) \simeq \sqrt{8\pi} \frac{a}{\ell_z} \equiv \tilde{g}_2 \ll 1 \quad (105)$$

is independent of energy and the dimensionless coupling parameter  $\tilde{g}_2 = Mg_2/\hbar^2$  is much smaller than one. This implies that the gas is in the weakly interacting regime in the sense that, at the degeneracy point where  $n\lambda_T^2 = 1$ , the chemical potential  $\mu \sim g_2 n$  is much smaller than the temperature ( $\mu/(k_B T) = \tilde{g}_2/(2\pi)$ ). An important feature of the  $D = 2$  dimensional gas is that the criterion for distinguishing the weakly and strongly interacting regime does not depend on density. Indeed the analog of the ratio  $\gamma$  given in Eq. (88) is simply equal to  $\tilde{g}_2$ . In analogy to Eq. (78) in the 1D case, the result (105) can be recovered simply by integrating the 3D pseudopotential over the  $z$  oscillator ground state. One often refers to a gas in this collisional regime as a quasi-2D system in the sense that it can be considered as a 2D system from the statistical physics point of view, but the dynamics of binary collision remains governed by 3D properties; in particular the 3D scattering length  $a$  remains a relevant parameter.

More generally, since the relevant energy for relative motion is twice the chemical potential, the momentum  $k = \sqrt{2M\mu}/\hbar$  in Eq. (104) is just the inverse healing length  $\xi$ . At very low energies therefore, the effective interaction in 2D is always repulsive, independent of the sign of the 3D scattering length (Petrov and Shlyapnikov, 2001). This result, however, is restricted to a regime, where  $\ln(\xi/\ell_z) \gg \ell_z/a$ . The logarithmic correction in (104) is therefore significant in the case of a strongly confining potential, when  $\ell_z$  and  $a$  are comparable. One then recovers a variation for  $f(k)$  which is formally similar to that of a pure 2D square well (97) with scattering length  $a_2 \simeq \ell_z$ . This regime could be relevant in a situation where the 3D scattering length  $a$  is enhanced by a Feshbach resonance (Kestner and Duan, 2006; Rajagopal *et al.*, 2004; Wouters *et al.*, 2003). If the 3D scattering length  $a$  is positive, the logarithmic correction in (104) is a mere reduction of the scattering amplitude. On the other hand for a negative  $a$ , this correction can lead to a strong increase of the amplitude for a particular value of  $\ell_z$  (Petrov *et al.*, 2000a), leading to a confinement-induced resonance similar to those that we encountered in the 1D case.

*Is there a true condensation in a trapped 2D Bose gas?* This question has been strongly debated over the last decade as two opposing lines of reasoning could be proposed. On the one hand we recall that for an ideal gas the presence of a trap modifies the density of states so that Bose-Einstein condensation becomes possible in 2D. One could thus expect that this remains valid in the presence of weak interactions. On the other hand in the presence of repulsive interactions, the extension of the (quasi-)condensate in the trap must increase with the number of atoms  $N$ . When  $N$  is large, a local density approximation entails that the correlation function  $g^{(1)}(r)$  decays algebraically as in (100) over a domain where the density is approximately uniform. This prevents from obtaining long range order except for extremely low temperatures. A related reasoning uses the fact that for the ideal gas, condensation is reached when the spatial density calculated semi-classically becomes infinite (see Eq. (95)), which cannot occur in presence of repulsive interactions. The fragility of the condensation of the ideal Bose gas in 2D is further illustrated by the existence at any temperature of a non-condensed Hartree-Fock solution, for arbitrarily small repulsive interactions (Bhaduri *et al.*, 2000). However for very low temperature this solution is not the absolute minimizer of the free energy, as shown using the Hartree-Fock-Bogoliubov method by Fernández and Mullin (2002), Gies and Hutchinson (2004) and Gies *et al.* (2004).

Currently the converging answer, though not yet fully tested experimentally, is the following: at ultra-low temperature one expects a true BEC, i.e. a system that is phase coherent over its full extension. The ground state energy and density of a 2D Bose gas in the limit  $T = 0$  can be obtained using the Gross-Pitaevskii equation, as shown rigorously by Lieb *et al.* (2001) (see

also Cherny and Shanenko (2001); Kim *et al.* (2000); Lee *et al.* (2002) and Posazhennikova (2006)). The cross-over from a three dimensional gas to a two dimensional gas at  $T = 0$  has been addressed by Tanatar *et al.* (2002) and by Hechenblaikner *et al.* (2005).

When the temperature increases one meets the quasi-condensate, superfluid regime, where phase fluctuations due to phonons dominate. The scenario is then very reminiscent of the uniform case and it has been thoroughly analyzed by Petrov *et al.* (2004a). The function  $g^{(1)}(r)$  decays algebraically and vortices are found only in the form of bound pairs. Finally at larger temperature these vortex pairs break and the system becomes normal. A BKT transition is still expected in the thermodynamic limit  $N \rightarrow \infty$ ,  $\omega \rightarrow 0$ ,  $N\omega^2$  constant, but the jump in the total superfluid mass is suppressed because of the inhomogeneity of the atomic density profile (Holzmann *et al.*, 2005). Indeed the energy for breaking a vortex pair depends on the local density, and superfluidity will probably be lost gradually from the edges of the quasi-condensate to the center as the temperature increases. Assuming that the atomic distribution is well approximated by the Hartree-Fock solution at the transition point, Holzmann *et al.* (2005) predict that the BKT transition temperature for a trapped gas is slightly lower than the ideal BEC transition temperature (94), by an amount related to the (small) dimensionless coupling parameter  $\tilde{g}_2 = Mg_2/\hbar^2$ .

We focus for a moment on the quasi-condensate regime. It is described by a macroscopic wave function  $\psi(\mathbf{x}) = \sqrt{\tilde{n}_0(\mathbf{x})} \exp i\phi(\mathbf{x})$ , and the density and phase fluctuations can be analyzed using a Bogoliubov analysis. We refer the reader to the work of Mora and Castin (2003) and Castin (2004) for a thorough discussion of the extension of Bogoliubov theory to quasi-condensates. As for the uniform gas (Prokof'ev *et al.*, 2001), repulsive interactions strongly reduce the density fluctuations for  $k_B T \lesssim \mu$  and  $n\lambda_T^2 \gg 1$ , so that  $\langle \tilde{n}_0^2(\mathbf{x}) \rangle \simeq (\langle \tilde{n}_0(\mathbf{x}) \rangle)^2$ . For large atom numbers ( $N\tilde{g}_2 \gg 1$ ) the equilibrium shape of the gas can be derived using a Thomas-Fermi approximation, as for a true condensate. The kinetic energy plays a negligible role, and the density profile results from the balance between the trapping potential and the repulsive interatomic potential. It varies as an inverted parabola

$$\tilde{n}_0(\mathbf{x}) = \tilde{n}_0(0) \left( 1 - \frac{r^2}{R^2} \right), \quad \frac{\hbar^2}{M} \tilde{g}_2 \tilde{n}_0(0) = \mu, \quad (106)$$

where the chemical potential  $\mu$  and the radius of the clouds  $R$  are:

$$\mu = \hbar\omega (N\tilde{g}_2/\pi)^{1/2}, \quad R = \sqrt{2} a_\perp (N\tilde{g}_2/\pi)^{1/4}, \quad (107)$$

with  $a_\perp = \sqrt{\hbar/(M\omega)}$ .

The parabolic Thomas-Fermi profile appears on the top of a broader background formed by the atoms out of the (quasi-)condensate. Such a profile has first been experimentally observed by Görlitz *et al.* (2001) and

Rychtarik *et al.* (2004). A precise measurement of the onset at which a pure thermal distribution turns into a bimodal (Thomas-Fermi + thermal) profile has recently been performed by Krüger *et al.* (2007). The experiment was performed with a rubidium gas confined in a 1D optical lattice, such that  $\tilde{g}_2 = 0.13$ . The phase space density at which bimodality arises was found in good agreement with the prediction of Prokof'ev *et al.* (2001) for the BKT threshold  $n(0)\lambda_T^2 = \ln(C/\tilde{g}_2) \simeq 8.0$ , which is relevant here if the local density approximation is valid at the center of the trap. At the critical point, the total number of atoms in each plane significantly exceeded the result (94) expected in the ideal case. In this experiment two to three planar gases were actually produced simultaneously, and they could interfere with each other when overlapping during time-of-flight, provided their spatial coherence was large enough. It was observed that the onset of bimodality coincides (within experimental accuracy) with the onset of clearly visible interferences.

It is important to stress that since the expected Thomas-Fermi profile is identical for a true and a quasi-condensate, its observation cannot be used to discriminate between the two situations. The phase fluctuations have been calculated by Petrov and Shlyapnikov (2001) and Petrov *et al.* (2004a) in the regime  $\mu \lesssim k_B T$  and  $n\lambda_T^2 \gg 1$  (see Eq. (160) in the appendix)

$$\delta\phi^2(\mathbf{x}) = \langle (\phi(0) - \phi(\mathbf{x}))^2 \rangle \simeq \frac{2}{\tilde{n}_0(0)\lambda_T^2} \ln(r/\xi). \quad (108)$$

This expression, which is reminiscent of the uniform result (100), is valid for points  $\mathbf{x}$  inside the condensate. The healing length  $\xi = \hbar/\sqrt{2M\mu}$  satisfies  $\xi R = a_\perp^2$ . Therefore it is only at a temperature much below the degeneracy temperature, such that  $\Delta\phi(R) \lesssim \pi$ , that one recovers a quasi uniform phase over the whole sample, hence a true condensate.

*Experimental investigations of phase fluctuations* A convenient way to access experimentally the phase coherence of quasi-2D gases is the matter-wave heterodyning technique. It consists in studying the statistical properties of the matter wave interference pattern which forms when two independent, parallel 2D Bose gases are released from the trap and overlap (figure 24a). A detailed analysis of these patterns has been given by Polkovnikov *et al.* (2006) (see section III.C for the 1D case). Assume that the two gases have the same uniform amplitude  $\psi_0$  and fluctuating phases  $\varphi_a(x, y)$  and  $\varphi_b(x, y)$ . The interference signal  $S(x, z)$  is recorded by sending an imaging beam along the  $y$  direction, which integrates the atomic density over a length  $L_y$ :

$$S(x, z) \propto 2\psi_0^2 + e^{2i\pi z/D} c(x) + e^{-2i\pi z/D} c^*(x) \quad (109)$$

with

$$c(x) = \frac{\psi_0^2}{L_y} \int_{-L_y/2}^{L_y/2} e^{i(\varphi_a(x, y) - \varphi_b(x, y))} dy. \quad (110)$$

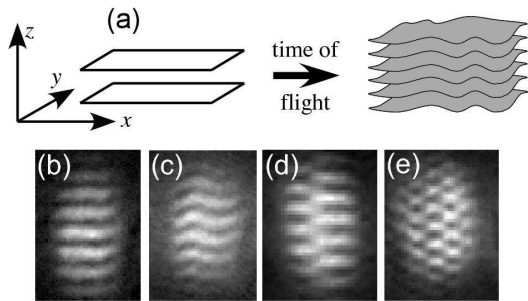


FIG. 24 Matter-wave heterodyning of 2D gases. (a) Principle of the method: two planar Bose gases are released from the trap, expand and overlap, giving rise to an interference pattern that is probed by absorption imaging. (b-e) Examples on experimental interference patterns obtained well below (b) and in the vicinity (c) of the degeneracy temperature. Some patterns show one (d) or several (e) dislocations, revealing the presence of vortices in one of the gases. Reprinted with permission from Hadzibabic *et al.* (2006).

The period  $D$  of the interference pattern is  $D = 2\pi\hbar t/(Md)$ , where  $d$  is the initial distance between the two planes and  $t$  the expansion time. We now integrate the coefficient  $c(x)$  appearing in (109) over a variable length  $L_x$ :

$$C(L_x) = \frac{1}{L_x} \int_{-L_x/2}^{L_x/2} c(x) dx \quad (111)$$

and average  $|C(L_x)|^2$  over many images recorded in the same conditions. Using the fact that the phases  $\varphi_a$  and  $\varphi_b$  are uncorrelated, we obtain for  $L_x \gg L_y$

$$\begin{aligned} \langle |C(L_x)|^2 \rangle &= \frac{1}{L_x^2} \iint \langle c(x) c^*(x') \rangle dx dx' \quad (112) \\ &= \frac{1}{L_x} \int_{-L_x/2}^{L_x/2} |g^{(1)}(x, 0)|^2 dx \propto \left( \frac{1}{L_x} \right)^{2\alpha} \end{aligned}$$

where we have assumed that the two gases have the same statistical properties. The long-range physics is then captured in a single parameter, the exponent  $\alpha$ . It is straightforward to understand the expected values of  $\alpha$  in some simple cases. In a system with true long-range order,  $g^{(1)}$  would be constant and the interference fringes would be perfectly straight. In this case  $\alpha = 0$ , corresponding to no decay of the contrast upon integration. In the low temperature regime, where  $g^{(1)}$  decays algebraically (see Eq. (100)) the exponent  $\alpha$  coincides with the exponent  $\eta(T)$  which describes the quasi-long range order in  $g^{(1)}$ . In the high temperature case, where  $g^{(1)}$  decays exponentially on a length scale much shorter than  $L_x$ , the integral in (112) is independent of  $L_x$ . In this case  $\alpha = 0.5$ , corresponding to adding up local interference fringes with random phases. The BKT mechanism corresponds to a transition between a power law with exponent  $1/(n_s \lambda_T^2) \leq 0.25$  to an exponential decay of  $g^{(1)}$ .

It should thus manifest itself as a sudden jump of  $\alpha$  from 0.25 to 0.5 when the temperature varies around  $T_c$ .

This method has been implemented at ENS with two rubidium planar gases, forming two parallel, elongated strips ( $L_x = 120\mu\text{m}$ ,  $L_y = 10\mu\text{m}$ ) (Hadzibabic *et al.*, 2006) (figure 24b-e). The experimental results confirm the expected behavior, at least qualitatively. At relatively large temperature the fitted exponent  $\alpha$  is close to 0.5. When the temperature decreases a rapid transition occurs and  $\alpha$  drops to  $\sim 0.25$ . At the transition the estimated phase space density of the quasi-condensate is  $\tilde{n}_0(0)\lambda_T^2 \sim 6$ . Note that for a quantitative comparison between experiments and theory, one should account for density fluctuations which are likely to play an important point near the transition, in contrast to the situation in superfluid liquid helium. Also the geometry effects in these elongated samples ( $R_x \sim 12R_y$ ) may be significant.

In addition to the rapid variation of the exponent  $\eta$  characterizing the decay of  $g^{(1)}$ , these experiments also gave evidence for isolated vortices (Hadzibabic *et al.*, 2006; Stock *et al.*, 2005) ((figure 24d-e). A vortex appears as a dislocation of the fringes (Chevy *et al.*, 2001b; Inouye *et al.*, 2001), and these dislocations indeed proliferate on the high temperature side of the transition. Using a theoretical analysis based on classical field method, Simula and Blakie (2006) obtained phase patterns of quasi-condensates close to the critical temperature that indeed exhibit an isolated, free vortex, in good agreement with experimental observation. The probability for observing a vortex pair in a similar configuration has been calculated by Simula *et al.* (2005).

The Berezinskii-Kosterlitz-Thouless mechanism has also been recently investigated using a two-dimensional periodic array of  $\sim 200$  Josephson coupled Bose-Einstein condensates (Schweikhard *et al.*, 2007). Each tube-like condensate contains a few thousands atoms, and has a length  $\sim 35\mu\text{m}$  along the  $z$  direction. The condensates are localized at the sites of a 2D hexagonal optical lattice of period  $4.7\mu\text{m}$  in the  $xy$  plane, and the coupling  $J$  between adjacent sites can be tuned by varying the optical lattice intensity. The phase properties of the ensemble are probed by ramping down the lattice, and recording the density profile in the  $xy$  plane when the wavefunctions from the various sites overlap. Vortices appear as holes in the atomic density distribution, and the vortex surface density is measured as a function of the Josephson coupling  $J$  and the temperature  $T$ . A universal vortex activation curve is obtained as a function of the parameter  $J/T$ , showing vortex proliferation for  $J/T \lesssim 1$  in good agreement with the predictions of the BKT-mechanism.

*Breathing mode of a 2D gas* In the previous subsection we have been mostly interested in the static properties of 2D Bose gases. Here we point out a remarkable dynamical property of these systems in an isotropic harmonic potential, when the interaction potential between particles is such that  $V(\lambda r) = V(r)/\lambda^2$ . Pitaevskii and Rosch (1997) showed that when the gas is prepared in an arbitrary out-of-equilibrium state, the quantity  $\langle r^2 \rangle$  oscil-

lates at the frequency  $2\omega$  without any damping, respectively of the strength of the interaction. They also proved that this property originates from the presence of a hidden symmetry, described by the two-dimensional Lorentz group  $SO(2,1)$ . In fact precisely the same symmetry shows up in the case of a unitary gas in 3D, as will be discussed in section VIII.B.

The Dirac distribution in 2D,  $\delta^{(2)}(r)$ , belongs to the class of functions satisfying  $V(\lambda r) = V(r)/\lambda^2$ . It makes this  $SO(2,1)$  symmetry relevant for trapped neutral atoms at low energies, when the range of interaction is small compared to all other scales. However a true contact interaction is singular in 2D, and leads to logarithmic ultraviolet divergences that are cut off by the finite range of the real interatomic potential. Therefore one cannot hope to observe a fully undamped breathing mode in atomic systems, but rather a very weakly damped dynamics. It was pointed out by Fedichev *et al.* (2003) that vortex pair nucleation could actually play a role in the residual expected damping of this breathing mode. Note that the difficulties with the contact interaction do not arise at the level of the Gross-Pitaevskii equation, where the same property has been predicted (Kagan *et al.*, 1996; Pitaevskii, 1996).

A precursor of this long lived breathing mode has been observed in a 3D, quasi cylindrical geometry by Chevy *et al.* (2001a). The transverse breathing mode of the cylinder was found to oscillate at a frequency very close to  $2\omega$  with an extremely small damping (quality factor of the mode  $> 2000$ ). The damping and shift of the oscillation frequency could be calculated theoretically with a good precision by Jackson and Zaremba (2002) (see also Guilleumas and Pitaevskii (2003)). In this case part of the damping is due to the nucleation of pairs of phonons propagating along  $\pm z$  (Kagan and Maksimov, 2003), a mechanism that is of course absent in a pure 2D geometry. This breathing mode has also been observed in a fast rotating gas by Stock *et al.* (2004). Its frequency was also  $\sim 2\omega$ , with a small correction due to the non-harmonicity of the trapping potential that was necessary to stabilize the center-of-mass motion of the atom cloud in the fast rotating regime (see section VII.B).

## VII. BOSE GASES IN FAST ROTATION

The investigation of rotating gases or liquids is a central issue in the study of superfluidity (Donnelly, 1991). It is relevant for the study of liquid helium, rotating nuclei, neutron stars and pulsars, and for the behavior of superconductors in a magnetic field. During the recent years, several experiments using rotating Bose-Einstein condensates have provided a spectacular illustration of the notion of quantized vortices (Abo-Shaer *et al.*, 2001; Hodby *et al.*, 2001; Madison *et al.*, 2000; Matthews *et al.*, 1999). Depending on the rotation frequency  $\Omega$  of the gas, a single vortex or several vortices can be observed experimentally. When

the number of vortices is large compared to 1, they form a Abrikosov lattice, i.e. a triangular array with a surface density  $n_v = M\Omega/(\pi\hbar)$ . Since the circulation of the velocity around a single charged vortex is  $h/M$ , this ensures that the velocity field of the condensate, when calculated after coarse graining over adjacent vortices, is equal to the orthoradial, rigid body velocity field  $v = \Omega r$  (Feynman, 1955).

For a gas confined in a harmonic potential, the fast rotation regime corresponds to stirring frequencies  $\Omega$  of the order of the trapping frequency  $\omega$  in the plane perpendicular to the rotation axis (hereafter denoted  $z$ ). From a classical point of view the transverse trapping and centrifugal forces then compensate each other, and the motion of the particles in the  $xy$  plane is only driven by Coriolis and interatomic forces. This situation is similar to that of an electron gas in a magnetic field, since Lorentz and Coriolis forces have the same mathematical structure. The single particle energy levels are macroscopically degenerate, as the celebrated Landau levels obtained for the quantum motion of a single charge in a magnetic field. When interactions between atoms are taken into account the fast rotation regime presents a strong analogy with Quantum Hall physics. One can distinguish two limiting cases in this fast rotation regime. Firstly, when the number of vortices inside the fluid  $N_v$  remains small compared to the number of atoms  $N$ , the ground state of the system is still a Bose-Einstein condensate described by a macroscopic wave function  $\psi(\mathbf{x})$ . This situation has been referred to as ‘mean field Quantum Hall regime’ (Fischer and Baym, 2003; Ho, 2001). Secondly, when  $\Omega$  tends to  $\omega$ , the number of vortices reaches values comparable to the total number of atoms  $N$ . The description by a single macroscopic wave function breaks down, and one expects a strongly correlated ground state, such as that of an electron gas in the fractional quantum Hall regime (Cooper *et al.*, 2001).

In this section we start by setting the Lowest Landau Level (LLL) framework for the discussion of the fast rotation regime, and discuss the main properties of a fast rotating condensate, when the mean-field description remains valid. We then present recent experimental results where the LLL regime has indeed been reached. Finally we review some theoretical proposals to reach beyond mean field physics, that present a close analogy with the physics of the fractional quantum Hall effect. We do not discuss here the physics of slowly rotating system, where one or a few vortices are involved. We refer the interested reader to the review article of Fetter and Svidzinsky (2001) and to the recent book by Aftalion (2006). Note that a rigorous derivation of the Gross-Pitaevskii energy functional in the slowly rotating case has been given by Lieb and Seiringer (2006).

### A. The Lowest Landau Level formalism

*a. The Landau Levels.* We consider first a single particle confined in a two-dimensional isotropic harmonic potential of frequency  $\omega$  in the  $xy$  plane. We are interested here in the energy level structure in the frame rotating at angular frequency  $\Omega$  ( $> 0$ ) around the  $z$  axis, perpendicular to the  $xy$  plane. The hamiltonian of the particle is

$$H^{(1)} = \frac{p^2}{2M} + \frac{M\omega^2 r^2}{2} - \Omega L_z = \frac{(\mathbf{p} - \mathbf{A})^2}{2M} + \frac{1}{2}M(\omega^2 - \Omega^2)r^2 \quad (113)$$

with  $r^2 = x^2 + y^2$ ,  $\mathbf{A} = M\Omega \hat{\mathbf{x}} \wedge \mathbf{x}$ ;  $L_z$  is the  $z$  component of the angular momentum. Eq. (113) is formally identical to the hamiltonian of a particle of unit charge placed in a uniform magnetic field  $2m\Omega \hat{\mathbf{z}}$ , and confined in a potential with a spring constant  $M(\omega^2 - \Omega^2)$ . A common eigenbasis of  $L_z$  and  $H$  is the set of (not normalized) Hermite functions

$$\phi_{j,k}(\mathbf{x}) = e^{-r^2/(2a_\perp^2)} (\partial_x + i\partial_y)^j (\partial_x - i\partial_y)^k \left( e^{-r^2/a_\perp^2} \right) \quad (114)$$

where  $j$  and  $k$  are non-negative integers and  $a_\perp = \sqrt{\hbar/M\omega}$ . The eigenvalues are  $\hbar(j - k)$  for  $L_z$  and

$$E_{j,k} = \hbar\omega + \hbar(\omega - \Omega)j + \hbar(\omega + \Omega)k \quad (115)$$

for  $H$ . For  $\Omega = \omega$ , these energy levels group in series of states with a given  $k$ , corresponding to the well known, infinitely degenerate, Landau levels. For  $\Omega$  slightly smaller than  $\omega$ , this structure in terms of Landau levels labeled by the index  $k$  remains relevant, as shown in Fig. 25. Two adjacent Landau levels are separated by  $\sim 2\hbar\omega$ , whereas the distance between two adjacent states in a given Landau level is  $\hbar(\omega - \Omega) \ll \hbar\omega$ . It is clear from these considerations that the rotation frequency  $\Omega$  must be chosen smaller than the trapping frequency in the  $xy$  plane. Otherwise the single particle spectrum (115) is not bounded from below. Physically, this corresponds to the requirement that the expelling centrifugal force  $M\Omega^2 r$  must not exceed the trapping force in the  $xy$  plane  $-M\omega^2 r$ .

We now consider an assembly of cold identical bosons rotating at a frequency  $\Omega$  close to  $\omega$ . Since the effective trapping potential in (113) becomes weaker as  $\Omega$  increases, we expect that as  $\Omega \rightarrow \omega$  the equilibrium size of the atom cloud increases indefinitely, and the interaction energy and the chemical potential  $\mu$  tend to zero. We define the Lowest Landau Level regime as the situation where  $\mu, k_B T \ll \hbar\omega$ , so that the state of the system can be accurately described in terms of Hermite functions  $\phi_{j,k}$  with  $k = 0$  only. Each basis function  $\phi_{j,0}(\mathbf{x})$  is proportional to  $(x + iy)^j e^{-r^2/(2a_\perp^2)}$  and takes significant values on a ring centered on 0 with an average radius  $\sqrt{j} a_\perp$  and a width  $\sim a_\perp$ . Any function  $\psi(\mathbf{x})$  of the LLL is a linear combination of the  $\phi_{j,0}$ 's and can be cast in the form:

$$\psi(\mathbf{x}) = e^{-r^2/(2a_\perp^2)} P(u) \quad (116)$$

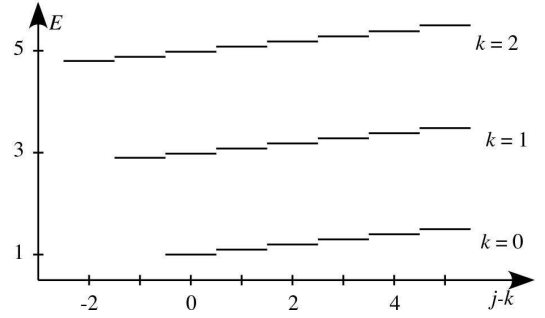


FIG. 25 Single particle energy spectrum for  $\Omega = 0.9\omega$ . The index  $k$  labels the Landau levels. The energy is expressed in units of  $\hbar\omega$ . For  $\Omega = \omega$  the Landau levels are infinitely degenerate.

where  $u = x + iy$  and  $P(u)$  is a polynomial (or an analytic function) of  $u$ . When  $P(u)$  is a polynomial of degree  $n$ , an alternative form of  $\psi(\mathbf{x})$  is

$$\psi(\mathbf{x}) = e^{-r^2/(2a_\perp^2)} \prod_{j=1}^n (u - \zeta_j) \quad (117)$$

where the  $\zeta_j$  ( $j = 1 \dots n$ ) are the  $n$  complex zeroes of  $P(u)$ . Each  $\zeta_j$  is the position of a single-charged, positive vortex, since the phase of  $\psi(\mathbf{x})$  changes by  $+2\pi$  along a closed contour encircling  $\zeta_j$ . Therefore in the LLL, there is a one-to-one correspondence between atom and vortex distributions, contrarily to what happens for slower rotation frequencies. This has interesting consequences on the hydrodynamics of the gas, which cannot be described by conventional Bernoulli and continuity equations (Bourne *et al.*, 2006).

*b. Equilibrium shape of a fast rotating BEC.* We now address the question of the distribution of particles and vortices in the case of fast rotation, assuming for the moment that a mean field description is valid. We suppose that the motion along the rotation axis  $z$  is frozen in a way similar to what we considered in the previous section devoted to static 2D gases.

Consider first the case of an ideal gas. At zero temperature all atoms accumulate in the  $j = k = 0$  ground state. At low but finite temperature ( $k_B T \ll 2\hbar\omega$ ) the occupied states belong to the LLL. The gas can be described at any time by a Hartree wave function of the type (116), where the coefficients  $c_m$  of the polynomial  $P(u) = \sum c_m u^m$  are random independent variables. This fast rotating ideal gas can thus be viewed as a physical realization of a random polynomial (Castin *et al.*, 2006). A measurement of the density distribution of the gas will reveal the presence of the vortices, i.e. the roots of  $P(u)$ . Although the gas is ideal, one can show that the positions of the vortices are correlated and exhibit a strong anti-bunching phenomenon (see Castin *et al.* (2006) and refs. in).

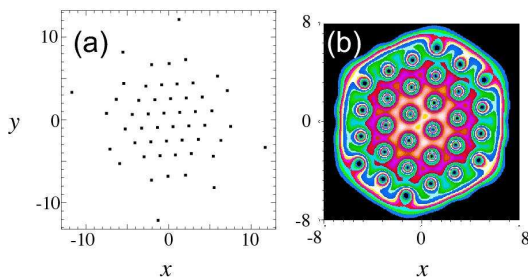


FIG. 26 Calculated structure of the ground state of a rotating Bose-Einstein condensate described by an LLL wave function, showing vortex locations (a) and atomic density profile (b). The parameters of the calculation correspond to 1000 rubidium atoms confined in a trap with frequency  $\omega/(2\pi) = 150$  Hz and rotating at a frequency  $\Omega = 0.99\omega$ . The unit for the positions  $x$  and  $y$  is  $[\hbar/(m\omega)]^{1/2}$ . Reprinted with permission from from (Aftalion *et al.*, 2005).

The case of a fast rotating condensate with repulsive interactions has been analyzed by several authors (Aftalion *et al.*, 2005; Cooper *et al.*, 2004; Ho, 2001; Watanabe *et al.*, 2004) and we will now review the main results. In all this section we will assume that the pairwise interaction between atoms  $i$  and  $j$  can be described by the contact term  $g_2 \delta(\mathbf{x}_i - \mathbf{x}_j)$ . We will furthermore assume that the 3D scattering length  $a$  is much smaller than the extension  $\ell_z$  of the ground state of the motion along  $z$ , so that  $g_2 \simeq \hbar^2 \tilde{g}_2 / M$ , with  $\tilde{g}_2 = \sqrt{8\pi} a / \ell_z \ll 1$  (see Eq. (105)). Note that the restriction of the contact interaction to the LLL subspace is a regular operator: it does not lead to the same mathematical difficulties as the ones encountered by considering the contact interaction in the whole Hilbert space of 2D wave functions. In fact, quite generally, interactions in the LLL are described by the Haldane pseudopotentials  $V_m$  (Haldane., 1983). For a pseudopotential with scattering length  $a$ , the resulting 2D contact interaction has  $V_m = \sqrt{2/\pi} \hbar \omega \cdot a / \ell_z$  for  $m = 0$  and zero otherwise. In the fermionic case, where only odd values of  $m$  are allowed, the analog of this interaction is a hard core model, where  $V_m \neq 0$  only for  $m = 1$ . The fact that the Laughlin states, to be discussed in subsection C below, are exact eigenstates for such pseudopotentials has been realized by Trugman and Kivelson (1985).

We start with a gas rotating exactly at the trap frequency ( $\Omega = \omega$ ), with an infinite number of particles, but a finite spatial density. In this case the numerical minimization of the Gross-Pitaevskii energy functional indicates that the vortices form an infinite regular triangular lattice. We turn now to a gas with a finite number of particles, rotating at a frequency  $\Omega$  slightly below  $\omega$ . The initial treatment of Ho (2001) assumed an infinite, regular triangular vortex lattice also in this case. The total energy of the system was minimized by varying the spacing of the vortex lattice. When injected in (117) this led to the prediction of a Gaussian atom distribution after coarse-graining over the vortex lattice spacing. A more detailed analysis has recently been performed,

where the position  $\zeta_j$  of each vortex is taken as a variational parameter (Aftalion *et al.*, 2005; Cooper *et al.*, 2004; Watanabe *et al.*, 2004) (see also the work of Anglin (2002) and Sheehy and Radzihovsky (2004a,b) for the case of a slower rotation). One spans in this way the whole LLL subspace. These studies have shown that the vortex distribution that minimizes the total energy is nearly regular with the density  $M\Omega/(\pi\hbar)$  close to the center of the condensate, but it is strongly deformed on the edges, with a rarefaction of vortices. For large atom numbers, the predicted coarse-grained density distribution is not gaussian as for a uniform vortex lattice, but it approaches a Thomas-Fermi distribution  $n_2(\mathbf{x}) \propto R^2 - r^2$  similar to (106), for the effective trapping potential  $M(\omega^2 - \Omega^2)r^2/2$ . This Thomas-Fermi prediction is in good agreement with the results obtained in the experiments described later in this section. The cloud radius is

$$R \simeq a_{\perp} \left( \frac{2b}{\pi} \frac{N\tilde{g}_2}{1 - \Omega/\omega} \right)^{1/4} \quad (118)$$

and diverges for  $\Omega \rightarrow \omega$ , as expected from the compensation of the trapping force by the centrifugal one. The dimensionless coefficient  $b$  is the Abrikosov parameter  $b = 1.1596$  (Kleiner *et al.*, 1964) for a triangular lattice. It expresses the fact that due to the restriction to the LLL and to the presence of vortices, the energy and the size of the condensate are actually slightly larger than what one would expect for a static trap with spring constant  $m(\omega^2 - \Omega^2)$  and a smooth equilibrium distribution. The chemical potential is

$$\mu \simeq \hbar\omega \left( \frac{2b}{\pi} N\tilde{g}_2(1 - \Omega/\omega) \right)^{1/2}$$

so that the condition  $\mu \ll \hbar\omega$  for the validity of the LLL approach reads  $1 - \Omega/\omega \ll 1/(N\tilde{g}_2)$ . It is also instructive to calculate the number  $N_v$  of ‘visible’ vortices, i.e. those which sit in the disk of area  $\pi R^2$ . Using  $\Omega \simeq \omega$  so that  $n_v \simeq m\omega/(\pi\hbar)$ , we get

$$\frac{N_v}{N} \simeq \left( \frac{\tilde{g}_2}{N(1 - \Omega/\omega)} \right)^{1/2}.$$

As we will see further, the mean field approach is valid only if  $N_v \ll N$  so that the validity domain of the mean field LLL approach corresponds to the interval:

$$\text{mean field LLL: } \frac{\tilde{g}_2}{N} \ll 1 - \frac{\Omega}{\omega} \ll \frac{1}{N\tilde{g}_2} \quad (119)$$

Note that the total number of vortices, including those sitting outside the Thomas-Fermi radius, can be shown to be infinite for the wave function that minimizes the energy in the LLL subspace (Aftalion *et al.*, 2006).

It is interesting to compare the behavior of a fast rotating BEC with that of a fast rotating bucket of superfluid liquid helium, or a type II superconductor in a large magnetic field (Fischer and Baym, 2003). In the latter cases,



the size of the sample is constant and the vortex density increases as the rotation frequency (or the magnetic field) increases. Since the size of a vortex core  $\ell_c$  depends only on the spatial density of the fluid ( $\ell_c \sim$  the healing length  $\xi$ ), it stays constant as  $\Omega$  increases and one reaches eventually a point where the cores of adjacent vortices overlap. This corresponds to a loss of superfluidity or superconductivity. For superfluid liquid helium, the rotation frequency  $\Omega_{c2}$  where this phenomenon should happen is out of reach for realistic experiments. For superconductors on the contrary, the critical field  $H_{c2}$  where the superconductivity is lost is a relevant experimental parameter. For fast rotating, harmonically trapped gases, the scenario is very different: (i) the vortex density saturates to a constant value  $n_v = M\omega/(\pi\hbar) = 1/(\pi a_\perp^2)$  when  $\Omega$  approaches  $\omega$ ; (ii) the size  $\ell_c$  of the vortex core for a wave function of the type (117) is no longer dictated by interactions that would lead to  $\ell_c \sim \xi$  as for an incompressible fluid, but it is on the order of the vortex spacing  $a_\perp$ . Therefore the fractional area  $\tilde{n}_0 \ell_c^2$  occupied by vortices tends to a finite value, as the trapped BEC rotates faster and faster. The cross-over between the standard to the LLL regime is studied in detail by Baym and Pethick (2003) and Cozzini *et al.* (2006).

## B. Experiments with fast rotating gases

The most intuitive way to rotate a trapped atomic gas is to superpose a rotating anisotropic potential to the axi-symmetric trapping potential  $V(r) = M\omega^2 r^2/2$ . The stirring anisotropy can be written  $\delta V(\mathbf{x}, t) = \epsilon M\omega^2 (X^2 - Y^2)/2$  where the coordinates  $(X, Y)$  are deduced from the static ones  $(x, y)$  by a rotation of angle  $\Omega t$ . The dimensionless parameter  $\epsilon$  characterizes the strength of the stirring potential with respect to the trapping one. In practice, because of experimental limitations,  $\epsilon$  has to be on the order of at least a few percents. Indeed it must overcome by a significant factor the residual static anisotropy of the trapping potential, that is typically in the  $10^{-3} - 10^{-2}$  range (Guéry-Odelin, 2000).

The stirring potential can be created by a modulated laser beam (Abo-Shaer *et al.*, 2001; Madison *et al.*, 2000) or by a rotating magnetic field (Haljan *et al.*, 2001; Hodby *et al.*, 2001). The stirring method has been successfully used to nucleate single vortices as well as large vortex arrays in rotating BECs. However it is not fully appropriate to approach the fast rotating regime of a harmonically trapped gas. Indeed the center-of-mass motion of the atom cloud is dynamically unstable when the rotation frequency  $\Omega$  is set in the interval  $[\omega\sqrt{1-\epsilon}, \omega\sqrt{1+\epsilon}]$  (Rosenbusch *et al.*, 2002). A precise description of the rotating system at the edge of the instability region  $\Omega = \omega\sqrt{1-\epsilon}$  has been given by Sinha and Shlyapnikov (2005) (see also Fetter (2007)) who showed that the gas forms in this case a novel elongated quantum fluid, with a roton-maxon excitation spectrum. Excitation modes with a zero energy appear above a critical interaction

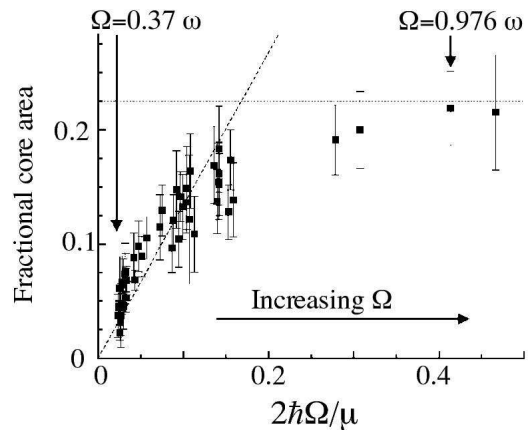


FIG. 27 Fraction of the condensate surface area occupied by the vortex cores, as a function of  $2\hbar\Omega/\mu$ . The vortex radius  $r_v$  is defined as the *r.m.s.* radius of the Gaussian function giving the best fit to the density dip at the vortex location. The dashed line is the predicted 3D bulk value  $r_v = 1.94 \xi$ , where  $\xi$  is the healing length. For fast rotation the vortex core area deviates from this prediction and it saturates at a value close to the prediction by Baym (2003) (continuous line). Reprinted with permission from Schweikhard *et al.* (2004).

strength, leading to the creation of rows of vortices.

A possible way to circumvent the center-of-mass expulsion occurring at  $\Omega \sim \omega$  consists in adding an extra trapping potential, that provides a stronger than quadratic confinement. This method has been explored experimentally by Bretin *et al.* (2004). In this experiment the dipole potential created by a strongly focused laser beam provided a quartic confinement, in addition to the usual quadratic one. It was then possible to explore the critical region  $\Omega \sim \omega$  and to approach the LLL regime  $\mu \sim 2\hbar\omega$ . A striking observation was a strong decrease of the visibility of the vortex pattern in this region. Its origin is not fully understood yet, but it may be related to the fact that the rotating gas was not in the 2D regime. The shape of the rotating cloud was close to spherical, and the vortex lines may have undergone a strong bending with respect to the trap axis, which made them hardly visible in the imaging process. This explanation is favored by the theoretical study by Aftalion and Danaila (2004): when looking for the ground state of the system using imaginary time evolution of the Gross-Pitaevskii equation, it was found that much longer times were required for  $\Omega \simeq \omega$  to reach a well-ordered vortex lattice.

Note that the addition of a quartic potential brings some interesting and novel aspects to the vortex dynamics in the trap, with the possibility to nucleate ‘giant’ vortices. This was initially explored by Fetter (2001), Lundh (2002), Kasamatsu *et al.* (2002). The mean-field description of the dynamics of a BEC in non harmonic potentials has been recently the subject of an important theoretical activity, and we refer the interested reader to the work of Cozzini *et al.* (2006) and references in.

Another successful method to reach the fast rotation regime is the evaporative spinup technique, developed by Engels *et al.* (2002). The cloud is first set in rotation at a frequency  $\Omega$  notably below  $\omega$  using a magnetic stirrer, that is subsequently switched off. Then a nearly one dimensional radio-frequency evaporation along the axis of rotation cools the cloud. Simultaneously the rotation speed of the gas increases since the evaporated atoms carry less angular momentum than average. With this tool the Boulder group has succeeded in producing a gas rotating at  $\Omega > 0.99\omega$  with a purely harmonic confinement. Thanks to the centrifugal deformation, the radius of the gas in the  $xy$  plane increases whereas the thickness along  $z$  shrinks to the size of the ground state  $\sqrt{\hbar/m\omega_z}$ , setting the gas well inside the 2D regime. As the total volume of the gas increases, interactions are reduced: the chemical potential  $\mu \sim 10$  Hz drops below the splitting between two Landau levels  $2\hbar\omega$  (17 Hz), and the LLL regime is reached (Schweikhard *et al.*, 2004). With this setup, the Boulder group has been able to test the prediction that the fractional core area of the vortices saturates to a value of the order of 0.2 (Coddington *et al.*, 2004; Schweikhard *et al.*, 2004), as predicted theoretically (Baym, 2003) (figure 27). In addition to the expected distortion of the vortex lattice with respect to an ideal triangular array could be detected experimentally on the edge of the rotating condensate (Coddington *et al.*, 2004). Following a suggestion by Anglin and Cressimanno (2002), another interesting investigation performed on this system dealt with the Tkachenko oscillations (Tkachenko, 1966) (for a review see Sonin (1987)), i.e. the long-wavelength transverse excitations of the vortex lattice (Coddington *et al.*, 2003). The Tkachenko waves could be directly imaged and their frequency could be measured with a good precision. The theoretical analysis of these oscillations has recently been performed within the mean-field approximation by several authors (Baym (2003, 2004), Gifford and Baym (2004), Choi *et al.* (2003), Baksmaty *et al.* (2004), Woo *et al.* (2004), Mizushima *et al.* (2004), Sonin (2005a,b), Cozzini *et al.* (2004) and Chevy (2006)).

Fast rotation of a BEC can also be achieved by stirring the gas with a potential that is more elaborate than a quadratic one. One can use in particular a rotating optical lattice that creates a rotating, spatially periodic pattern on the gas. This has been explored recently by Tung *et al.* (2006), who superimposed to a rotating BEC a set of columnar pinning sites created by a two-dimensional, co-rotating optical lattice. For a sufficiently large laser intensity the optical lattice can impose its structure to the vortex lattice; Tung *et al.* (2006) studied in particular the transition from the usual triangular Abrikosov lattice to a square configuration imposed by light. Theoretical investigations of this problem were carried out by Pu *et al.* (2005) and Reijnders and Duine (2004, 2005), who found that a rich variety of structural phases can emerge in this geometry, from the competition

between vortex-vortex and vortex-optical lattice interactions.

### C. Beyond the mean field regime

In the mean-field description of a fast rotating gas, the macroscopic wave function  $\psi(\mathbf{x})$  is a solution of the non-linear Gross-Pitaevskii equation and corresponds to a vortex lattice. The radius of the atom cloud, given in (118), is a measure of the number  $j_{\max}$  of single particle LLL states  $\phi_{j,0}$  that have a significant population. Recalling that  $\phi_{j,0}$  is maximum for a radius  $\sqrt{j} a_{\perp}$ , we find

$$j_{\max} \simeq (R/a_{\perp})^2 \simeq \left( \frac{N\tilde{g}_2}{1 - \Omega/\omega} \right)^{1/2}. \quad (120)$$

The filling factor  $\nu = N/j_{\max}$  gives the average number of particles in each occupied single particle state. When  $\nu \gg 1$  one expects the mean field treatment to be valid, and  $j_{\max}$  is equal to the number  $N_v$  of visible vortices sitting in the atom disk. On the opposite when  $\Omega$  tends to  $\omega$ ,  $\nu$  becomes on the order of unity or below, the number of vortices  $N_v$  exceeds the number of atoms  $N$ , and one has to turn to a full many-body treatment of the problem. This breakdown of mean-field approximation occurs when

$$\text{non mean-field:} \quad 1 - \frac{\Omega}{\omega} \lesssim \frac{\tilde{g}_2}{N}. \quad (121)$$

The analysis of this ultra-fast rotating regime presents strong analogies with the studies of the fractional quantum Hall effect (FQH). In the latter case one is interested in the correlated state of a 2D electron gas with Coulomb interaction when it is placed in a strong magnetic field. In both cases the states of interest are restricted to the LLL and one looks for specific filling factors where ground states with specific properties can emerge. Note, that for bosons the filling factor can be arbitrarily large within the LLL, while for fermions it is restricted to  $\nu < 1$ . One remarkable feature is incompressibility, meaning that the ground state is separated from all excited states by a ‘macroscopic’ energy gap scaling as  $g_2$ . Moreover it leads to the appearance of edge states near the boundary of the system, similar to the wedding-cake structure of the Mott-insulating state of the Bose-Hubbard model. These edge states are crucial for understanding the quantization of the Hall conductance, see MacDonald (1994) and - on a more mathematical level - Fröhlich and Studer (1993).

*Numerical studies.* So far the ultra-fast rotation regime has not been reached experimentally. Even for the fastest rotations realized in the laboratory, the filling factor  $\nu$  is  $\sim 10^3$  ( $N \sim 10^5$ ,  $N_v \sim 10^2$ ), well inside the mean field regime. Therefore the results obtained so far originate from an exact numerical diagonalization of the many-body hamiltonian and from the connection with known features of the fermionic FQH.

Most studies are performed considering states with a given total angular momentum  $L_z$ , so that the problem essentially consists in finding the eigenstates of the interaction energy

$$V_{\text{int}} = \frac{\hbar^2 \tilde{g}_2}{M} \sum_{i < j} \delta(\mathbf{x}_i - \mathbf{x}_j). \quad (122)$$

All states considered hereafter belong to the LLL subspace, so their functional form in the  $xy$  plane is

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = P(u_1, \dots, u_N) \exp\left(-\sum_{j=1}^N r_j^2 / 2a_{\perp}^2\right) \quad (123)$$

where  $u_j = x_j + iy_j$ ,  $r_j^2 = x_j^2 + y_j^2$ , and where  $P(u_1, \dots, u_N)$  is a symmetric polynomial. The  $z$  motion is expected to be ‘frozen’ to its ground state and it is not explicitly written in what follows. If one is interested only in the bulk properties of the vortex liquid state, it is convenient to replace the inhomogeneous disk geometry of a real experimental setup by a compact, homogeneous geometry. Both torus (Cooper *et al.*, 2001) and spherical (Nakajima and Ueda, 2003; Regnault and Jolicoeur, 2003) manifolds have been considered. The LLL is then a space of finite dimension  $d_{\text{LLL}}$ , proportional to the area  $\mathcal{A}$  of the torus or the sphere:  $d_{\text{LLL}} = \mathcal{A}/(\pi a_{\perp}^2)$ . This allows to define in a non ambiguous way the filling factor  $\nu$  ( $\nu = N/d_{\text{LLL}}$ ) even for values on the order of 1 or below, where the notion of visible vortices becomes dubious.

*Melting of the vortex lattice.* When increasing the rotation speed of the gas, the first expected deviation from the mean-field regime is the quantum melting of the vortex lattice. This has been observed by Cooper *et al.* (2001) in exact numerical calculations for filling factors  $\nu = N/N_v \sim 6$  to 10. This value can be recovered by calculating the quantum fluctuations  $\Delta$  of vortex positions and applying the Lindemann criterion  $\Delta_{\text{melt}} \sim \ell/10$ , where  $\ell$  is the vortex spacing (Sinova *et al.*, 2002). For  $\nu$  smaller than the melting threshold, one meets for the ground state of the many-body system a series of strongly correlated ground states that we now briefly discuss.

*The Laughlin state and its daughter states.* Since increasing the angular momentum spreads out the atoms in space, one expects the interaction energy  $E(L_z)$  of the ground state for a given  $L_z$  to decrease as  $L_z$  increases (see e.g. Jackson and Kavoulakis (2000)). This decrease stops when one reaches the celebrated Laughlin wave function, adapted here for bosonic particles:

$$P_{\text{Lau.}}(u_1, u_2, \dots, u_N) = \prod_{i < j} (u_i - u_j)^2 \quad (124)$$

Indeed since the probability to get two particles at the same point vanishes, this state has the remarkable prop-

erty to be an eigenstate of the contact interaction potential with eigenvalue 0 (Trugman and Kivelson, 1985). Increasing  $L_z$  beyond this point cannot reduce further  $E(L_z)$ . The total angular momentum  $L_z/\hbar$  of this state is equal to the degree  $N(N-1)$  of each term of the polynomial  $P_{\text{Lau.}}$  and this state expands over all LLL single particle wavefunctions  $\phi_{j,0}$  from  $j = 0$  to  $j_{\text{max}} = 2(N-1)$ , i.e. a filling factor  $\nu = 1/2$ . The Laughlin state is incompressible and the gap to the first excited state is  $\sim 0.1 \tilde{g}_2 \hbar \omega$  (Regnault and Jolicoeur, 2003, 2004). The Laughlin state is characterized by a quasi-uniform density of particles over the circle of radius  $a_{\perp} \sqrt{2N}$ . The two-body correlation function for this state shows a strong anti-bunching  $g^{(2)}(r \rightarrow 0) \sim r^2$ . This correlation function has been calculated numerically for a number of bosons  $N$  up to 8 by Barberán *et al.* (2006).

For  $L_z/\hbar > N(N-1)$ , any state  $P(u_1, \dots, u_N) = P_{\text{Lau.}}(u_1, \dots, u_N) Q(u_1, \dots, u_N)$ , where  $Q$  is a arbitrary symmetric polynomial, is a ground state of the system with interaction energy 0. Depending on the total degree  $d_Q$  of  $Q$ , the physical interpretation of the state can be: (i) for  $d_Q \sim 1$ , edge excitations of the Laughlin state (Cazalilla, 2003; Cazalilla *et al.*, 2005), (ii) for  $d_Q = N$ , quasi-holes at a given point  $U_0$  are obtained by taking  $Q = \prod_j (u_j - U_0)$  (Paredes *et al.*, 2001, 2002), (iii) for  $d_Q \sim N^2$ , Laughlin-type wave functions with smaller filling factors, by replacing the exponent 2 by 4, 6, ... in (124).

*The composite Fermion sequence.* For filling factors between the melting point  $\nu \sim 10$  and the Laughlin state  $\nu = 1/2$ , it is not possible to give an exact analytical expression of the ground state at any given  $\nu$ . One finds however a strong overlap between the numerically determined ground states and some relevant states for the physics of electronic FQH. An example is the composite fermion sequence which presents strong analogies with the Jain principal sequence for fermions (Jain, 1989). A first evidence for this sequence was found by Cooper and Wilkin (1999), and it has been studied in detail by Regnault and Jolicoeur (2003, 2004). The physics at the origin of the states of this sequence is reminiscent of that explored in the section on 1D gases, where the problem of bosonic particles with repulsive interaction is mapped onto the properties of an assembly of non-interacting fermionic particles. Here one considers that the gas is formed with fermionic composite entities, each resulting from the attachment of a fermionic particle with a vortex carrying one unit of statistical flux. These composite fermions can be viewed as independent particles which occupy various Landau levels. When they occupy exactly  $n$  Landau levels they form an incompressible state. This occurs when the filling fraction in the initial state is  $\nu = n/(n+1)$ . From a more quantitative point of view, the composite fermion ansatz corresponds

to a wave function of the type:

$$P(u_1, \dots, u_N) = \mathcal{P}_{\text{LLL}} \left[ Q_n(u_1, \dots, u_N) \prod_{i < j} (u_i - u_j) \right] \quad (125)$$

$\mathcal{P}_{\text{LLL}}$  describe the projector onto the LLL subspace (for its precise definition see e.g. Chang *et al.* (2004)). The first term in the bracket  $Q_n(u_1, \dots, u_N)$  is a Slater determinant giving the state of  $N$  fictitious fermions filling exactly  $n$  Landau levels. The second term involving products of  $u_i - u_j$  (Jastrow factor) corresponds to the attachment of a vortex to each fermion. Since both terms in the bracket are antisymmetric in the exchange of two particles, their product is a symmetric wave function, suitable for the description of our  $N$  identical bosons. Numerical evidence for such states was obtained for  $\nu = 2/3$  and  $3/4$  by Regnault and Jolicœur (2003, 2004) and Chang *et al.* (2004). The surface waves of the vortex liquids whose wave functions can be described by the composite fermion ansatz have been studied by Cazalilla (2003), Regnault and Jolicœur (2004) and Cazalilla *et al.* (2005).

*The Read-Moore state and the Read-Rezayi series.* For the filling factor  $\nu = 1$  yet another type of approximate ground state have been identified (Cooper *et al.*, 2001), the Moore-Read state, or Pfaffian. Assuming that  $N$  is even the expression of this state is

$$P(u_1, \dots, u_N) = \mathcal{S} \left[ \prod_{i < j \leq N/2} (u_i - u_j)^2 \prod_{N/2 < l < n} (u_l - u_n)^2 \right] \quad (126)$$

where  $\mathcal{S}$  indicates symmetrization over all indices. The total degree of each term of this polynomial is  $N(N-2)/2$  and the state expands over single particle LLL wavefunctions from  $k = 0$  up to  $k_{\text{max}} = N - 2$ , corresponding to a filling factor  $\nu = 1$ . As for  $\nu = 1/2$  the ground state is incompressible with a gap  $\sim 0.05 \tilde{g}_2 \hbar \omega$  (Chang *et al.*, 2004). It is noteworthy that for this state, the probability is zero to have three particles at the same location in space.

For even larger filling factors ( $\nu$  between 1 and 10) an analysis performed in a torus geometry suggested that the system has incompressible ground states belonging to a family containing clusters of  $k$  particles (Cooper *et al.*, 2001) and corresponding to integer or half integer filling factors. This so-called Read-Rezayi series (Read and Rezayi, 1999) is constructed by taking symmetrized products of  $k$  Laughlin states of the type  $\prod_{i < j \leq N/k} (u_i - u_j)^2$  (assuming that  $N$  is a multiple of  $k$ ). The Laughlin and the Moore-Read states corresponds to  $k = 1$  and  $k = 2$ , respectively. The filling factor associated to these states is  $k/2$  and the total angular momentum is  $L_z/\hbar \sim N^2/k$ . Further calculations performed in the spherical geometry could not

draw any conclusion concerning the survival of the incompressibility of such states at the thermodynamic limit (Regnault and Jolicœur, 2004).

*Possible detection schemes for fractional quantum Hall effects.* We now review some possible ways for observing experimentally effects related to fractional quantum Hall physics. We first note that it is unlikely that condensed-matter-type techniques, based on transport properties with specific conductance plateaus can be implemented with rotating atomic gases, at least in the near future. We also point out that the condition (121) giving the threshold for the observation of mean-field effects imposes *de facto* to work with very small atomic samples. Consider for example a purely harmonic trap. Due to residual trap imperfections it seems unlikely that one can achieve rotation frequencies  $\Omega$  larger than  $0.999 \omega$  (rotation frequencies  $\sim 0.99 \omega$  have been achieved by Schweikhard *et al.* (2004)). Taking  $\tilde{g}_2 \sim 0.1$ , this sets an upper bound of  $\sim 100$  on the atom number. Working with such small atom numbers is not untractable, but it immediately makes this type of experiment quite challenging from a technical point of view.

A first possible experimental signature of the Laughlin and Read-more states could lie in the fact that 3 particles can never sit at the same location in space for these wave functions. As 3-body recombination is often the main source of atom losses one can expect that the achievement of these states could be revealed by a spectacular increase of the lifetime of the gas, as it has been the case for 1D physics (Tolra *et al.*, 2004).

We now turn to more quantitative studies of these incompressible states. A ‘simple’ experimental evidence for a state such as the Laughlin wave function could come from its specific density profile, which is uniform over a disk of radius  $a_{\perp} \sqrt{2N}$ , and zero elsewhere. This flat profile with density  $1/(2\pi a_{\perp}^2)$  is notably different from the parabolic density profile expected in the mean-field regime, and its observation would constitute a clear signature of a beyond mean-field effect. Usually one does not measure directly the in-trap density profile, because the relevant distances are on the order of a few micrometers only, which is too small to be detected with a good accuracy by optical means. The standard procedure to circumvent this problem is to use a time-of-flight technique, where the potential confining the atoms is suddenly switched off so that the atoms fly away ballistically during an adjustable time before being detected. For an arbitrary initial state the functional form of the density distribution is modified during the time-of-flight. For an LLL wavefunction this modification is a mere scaling, at least when interactions between atoms are negligible during the time-of-flight (Read and Cooper, 2003). In particular the disk shape structure associated with the Laughlin state should remain invariant in ballistic expansion.

If the number of atoms is larger than the one required

to form a Laughlin wave function at the particular frequency  $\Omega$  that is used, one expects a ‘wedding cake’ structure for the atomic density (Cooper *et al.*, 2005). This result is obtained within a local density approximation and is very reminiscent of the structure appearing in an atom Mott insulator confined in a harmonic trap. At the center of the trap where the density is the largest, the atoms may form an incompressible fluid corresponding for example to the filling factor  $2/3$ , which is one of the composite fermion states identified above. The atomic density is then expected to be constant and equal to  $2/(3\pi a_{\perp}^2)$  over a central disk of radius  $R_1$ . For  $r = R_1$  the gas switches abruptly (over a distance  $\sim a_{\perp}$ ) to the Laughlin state with filling factor  $1/2$  and the density drops to the value  $1/(2\pi a_{\perp}^2)$ . It then stays constant over a ring of outer radius  $R_2$  ( $R_2 > R_1$ ) and for  $r > R_2$  the density drops to zero. The values of  $R_1$  and  $R_2$  can be obtained from a simple energy minimization (Cooper *et al.*, 2005). For larger atom numbers, several plateaus, with decreasing densities corresponding to the various filling factors of the incompressible states, are expected.

The possibility to add an optical lattice along the  $z$  direction adds an interesting degree of freedom in the problem, and brings experimentalists the hope to be allowed to work with a notably larger atom number. In this configuration one deals with a stack of  $N_d$  parallel disks, all rotating at the same frequency  $\Omega$  along the  $z$  axis. Each disk is coupled to its neighbors by tunnelling across the lattice barrier, with a strength that can be adjusted. For a large coupling the situation is similar to a bulk 3D problem; when the coupling is reduced the system evolves to the quasi-2D regime. This raises interesting questions even at the level of a single vortex motion, as pointed out by Martikainen and Stoof (2003). The melting of the vortex lattice in a stack of  $N_d$  layers has been investigated by Cooper *et al.* (2005) and Snoek and Stoof (2006a,b). For smaller filling factors both the density profiles along  $z$  and in the  $xy$  plane should show the wedding cake structure characteristic of incompressible states (Cooper *et al.*, 2005).

More elaborate techniques have been proposed to test the anyonic nature of the excitations of incompressible states. Paredes *et al.* (2001) have investigated the possibility of creating anyons in a Laughlin state by digging a hole in the atom gas with a tightly focused laser. By moving adiabatically the hole inside the cloud it should accumulate a phase, that could subsequently be measured by an interference experiment. The accumulated phase should then reveal the anyonic structure of the hole-type excitation (see also Paredes *et al.* (2002)).

#### D. Artificial gauge fields for atomic gases

As shown in Eq. (113), rotating a neutral particle system is equivalent to giving these particles a unit charge and placing them in a magnetic field proportional to the rotation vector  $\boldsymbol{\Omega}$  (see Eq. (113)). Other possibilities

have been suggested to apply an artificial gauge field on a neutral gas. The common idea to these proposals is to exploit the Berry’s phase (Berry, 1984) that arises when the atomic ground level is split (e.g. by an electromagnetic field) in several space-dependent sublevels, and the atoms follow adiabatically one of them. We consider the one-body problem, and label by  $\{|n_{\mathbf{x}}\rangle\}$  the local energy basis of the atomic ground level, with the associated energies  $E_n(\mathbf{x})$ . The most general state of the atom is a spinor  $\sum_n \psi_n(\mathbf{x})|n_{\mathbf{x}}\rangle$  which evolves under the Hamiltonian  $p^2/(2m) + \sum_n E_n(\mathbf{x})|n_{\mathbf{x}}\rangle\langle n_{\mathbf{x}}|$ . Suppose now that the atom is prepared in a given sublevel  $n$ , and that the motion of the atomic center of mass is slow enough to neglect transitions to other internal sublevels  $n'$ . This is in particular the case in a magnetic trap where the index  $n$  simply labels the various Zeeman substates. One can then write a Schrödinger equation for the component  $\psi_n(\mathbf{x})$  and the corresponding hamiltonian reads  $H = (\mathbf{p} - \mathbf{A}_n(\mathbf{x}))^2/(2m) + V_n(\mathbf{x})$ . The vector potential  $\mathbf{A}_n$  is related to the spatial variation of the sublevel  $|n\rangle$

$$\mathbf{A}_n(\mathbf{x}) = i\hbar \langle n_{\mathbf{x}} | \nabla n_{\mathbf{x}} \rangle \quad (127)$$

and the scalar potential  $V_n$  is

$$V_n(\mathbf{x}) = E_n(\mathbf{x}) + \frac{\hbar^2}{2m} (\langle \nabla n_{\mathbf{x}} | \nabla n_{\mathbf{x}} \rangle - |\langle n_{\mathbf{x}} | \nabla n_{\mathbf{x}} \rangle|^2) . \quad (128)$$

If the spatial variation of the sublevels  $|n_{\mathbf{x}}\rangle$  is such that  $\nabla \wedge \mathbf{A}_n \neq 0$ , this gauge field can in principle have the same effect as rotating the atomic gas.

The simplest occurrence for an artificial gauge field happens in a Ioffe-Pritchard magnetic trap (Ho and Shenoy, 1996). The sublevels  $|n_{\mathbf{x}}\rangle$  are the various Zeeman substates, which are space-dependent because the direction of the trapping magnetic field is not constant over the trap volume. However the gauge field  $\mathbf{A}_n$  that is generated in this configuration is too small to initiate the formation of vortices. The addition of a strong electric field could increase the magnitude of the artificial gauge field, as shown by Kailasvuori *et al.* (2002). An alternative and promising line of research takes advantage of the concept of *dark states*, where two sublevels of the atomic ground state are coupled by two laser waves to the same excited state. If the laser frequencies are properly chosen, there exists a linear combination of the two sublevels that is not coupled to the light, and the spatial evolution of atoms prepared in this dark state indeed involves the vector and scalar potentials given above (Dum and Olshanii, 1996; Dutta *et al.*, 1999; Visser and Nienhuis, 1998). Possible spatial profile of the laser waves optimizing the resulting artificial rotation field have been discussed by Juzeliunas and Öhberg (2004), Juzeliunas *et al.* (2005, 2006) and by Zhang *et al.* (2005). These proposals have not yet been implemented experimentally.

Similar effects have also been predicted in a lattice geometry by Jaksch and Zoller (2003), where atoms with two distinct internal ground state sublevels are trapped

in different columns of the lattice. Using a two-photon transition between the sublevels, one can induce a non-vanishing phase of particles moving along a closed path on the lattice. Jaksch and Zoller (2003) showed that one can reach in this way a ‘high magnetic field’ regime that is not experimentally accessible for electrons in metals, characterized by a fractal band structure (Hofstadter butterfly). The connection between quantum Hall effect and the lattice geometry in presence of an artificial gauge potential has been analyzed by Mueller (2004) and Sørensen *et al.* (2005). One can also generalize the Berry’s phase approach to the case where several energy states are degenerate (Wilczek and Zee, 1984). Non Abelian gauge fields emerge in this case, and possible implementations on cold atom systems have been investigated theoretically by Osterloh *et al.* (2005) and Ruseckas *et al.* (2005).

### VIII. BCS-BEC CROSSOVER

One of the basic many-body problems which has been brought into focus by the study of ultracold atoms is that of a two component attractive Fermi gas near a resonance of the  $s$ -wave scattering length. The ability of tuning the interaction through a Feshbach resonance allows to explore the crossover from a BCS superfluid, when the attraction is weak and pairing only shows up in momentum space, to a Bose-Einstein condensate of tightly bound pairs in real space. Here we will discuss the problem in the spin-balanced case, which - in contrast to the situation at finite imbalance - is now well understood.

#### A. Molecular condensates and collisional stability

The experimental study of the BCS-BEC crossover problem with ultracold atoms started with the realization of Fermi gases in the regime of resonant interactions  $k_F|a| \gg 1$  by O’Hara *et al.* (2002). They observed an anisotropic expansion, characteristic for hydrodynamic behavior. Typically, this is associated with superfluidity because ultracold gases above the condensation temperature are in the collisionless regime. Near a Feshbach resonance, however, a hydrodynamic expansion is observed both above and below the transition temperature. It is only through the observation of stable vortices that superfluid and collision dominated hydrodynamics can be distinguished. The BEC side of the crossover was first reached by creating ultracold molecules. This may be done either by direct evaporative cooling on the positive  $a$  side (Jochim *et al.*, 2003a), where the weakly bound molecules are formed by inelastic three-body collisions. Alternatively, molecules can be generated in a perfectly reversible manner by using a slow ramp of the magnetic field through a Feshbach resonance (Cubizolles *et al.*, 2003; Regal *et al.*, 2003). This

allows to convert a quasi-bound state of two fermions at  $a < 0$  into a true bound state at  $a > 0$  (for a review of this technique see Köhler *et al.* (2006)). Subsequently, a BEC of those molecules has been realized both by direct evaporative cooling (Jochim *et al.*, 2003b; Zwierlein *et al.*, 2003b) for  $a > 0$ , or by converting a sufficiently cold attractive Fermi gas at  $a < 0$  to a molecular condensate, using an adiabatic ramp across the Feshbach resonance (Greiner *et al.*, 2003). The experiments are done with an equal mixture of the two lowest hyperfine states of  $^6\text{Li}$  or of  $^{40}\text{K}$  confined optically in a dipole trap. This allows to change the scattering length by a magnetically tunable Feshbach resonance at  $B_0 = 835\text{ G}$  or  $B_0 = 202\text{ G}$  respectively. On the BEC side, the fact that the molecules are condensed can be verified experimentally by observing a bimodal distribution in a time-of-flight experiment. Probing superfluidity in Fermi gases on the BCS side of the crossover, however, is much more difficult. In particular, a time-of-flight analysis of the expanding cloud does not work here. Indeed, due to the factor  $\exp(-\pi/(2k_F|a|))$  in the critical temperature (see Eq. (136) below), superfluidity is lost upon expansion at constant phase space density in contrast to the situation in BEC’s<sup>18</sup>. As discussed below, this problem may be circumvented by a rapid ramp back into the BEC regime before the expansion. A major surprise in the study of strongly interacting Fermi gases was the long lifetime of the molecules near a Feshbach resonance (Cubizolles *et al.*, 2003; Jochim *et al.*, 2003a; Strecker *et al.*, 2003), in stark contrast to the situation encountered with bosonic atoms (Dürr *et al.*, 2004; Herbig *et al.*, 2003). The physics behind this was clarified by Petrov *et al.* (2004b) who have solved the problem of scattering and relaxation into deeply bound states of two fermions in the regime where the scattering length  $a$  is much larger than the characteristic range of the interaction  $r_e$ . As shown in section I.A, this range is essentially the van der Waals length Eq. (2), which is much smaller than  $a$  in the vicinity of a Feshbach resonance. The basic physics which underlies the stability of fermionic dimers in contrast to their bosonic counterparts is the fact that relaxation into deep bound states is strongly suppressed by the Pauli-principle. Indeed, the size of the weakly bound dimer states is just the scattering length  $a$ , while that of the deep bound states is  $r_e \ll a$ . By energy and momentum conservation, a relaxation into a deep bound state requires that at least three fermions are at a distance of order  $r_e$ . Since two of them are necessarily in the same internal state and their typical momenta are of order  $k \approx 1/a$ , the probability of

<sup>18</sup> In this respect, the situation in two dimensions, where pair binding appears for arbitrary values of the scattering length  $a$ , is much more favorable because the two-particle binding energy (83) is obviously density independent. Since  $T_c \sim \sqrt{\varepsilon_b \varepsilon_F} \sim n^{1/2}$ , the superfluid transition can thus be reached by an adiabatic expansion at constant  $T/T_F$ , see Petrov *et al.* (2003).

a close three (or four) body encounter is suppressed by a factor  $(kr_e)^2 \sim (r_e/a)^2$  due to the antisymmetrization of the corresponding wave function. From a detailed calculation (Petrov *et al.*, 2005), the relaxation into deeply bound states has a rate constant (in units  $\text{cm}^3/\text{sec}$ )

$$\alpha_{rel} = C \frac{\hbar r_e}{m} \cdot \left(\frac{r_e}{a}\right)^s, \quad (129)$$

which vanishes near a Feshbach resonance with a nontrivial power law. The exponent  $s = 2.55$  or  $s = 3.33$  and the dimensionless prefactor  $C$  depend on whether the relaxation proceeds via dimer-dimer or dimer-atom collisions. From experimental data the coefficient of the dominant dimer-dimer relaxation is  $C \approx 20$  (Bourdel *et al.*, 2004; Regal *et al.*, 2004b). Its value depends on short range physics on the scale  $r_e$  and thus cannot be calculated within a pseudopotential approximation. At a finite density, the power law dependence  $a^{-s}$  holds only as long as the scattering length is smaller than the average interparticle distance. The actual relaxation rate  $n\alpha_{rel}$  in fact stays finite near a Feshbach resonance and is essentially given by replacing the factor  $r_e/a$  in Eq. (129) by  $k_{Fr_e} \ll 1$  (Petrov *et al.*, 2004b). In practice, the measured lifetimes are on the order of 0.1 s for  $^{40}\text{K}$  and up to about 30 s for  $^6\text{Li}$ . This long lifetime of fermionic atoms near a Feshbach resonance is essential for the possibility to study the BCS-BEC crossover, because it allows to reduce the physics near the resonance to an idealized, conservative many-body problem in which relaxational processes are negligible.

The issue of dimer-dimer collisions has an additional aspect, which is important for the stability of the strongly attractive Fermi gas. Indeed, molecules consisting of two bound fermions also undergo purely elastic scattering. It is obvious, that a molecular condensate will only be stable if the associated interaction of these effectively bosonic dimers is repulsive. From an exact solution of the four-particle Schrödinger equation with pseudopotential interactions, Petrov *et al.* (2004b) have shown that in the limit where the distance  $R$  (denoted by  $R/\sqrt{2}$  in their paper) between the centers of mass of two dimers is much larger than the dimer size  $a$  and at collision energies much smaller than their respective binding energies  $\hbar^2/2M_r a^2$ , the wave function has the asymptotic form

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{R}) = \varphi_0(r_1)\varphi_0(r_2)(1 - a_{dd}/R) \quad (130)$$

with  $a_{dd} \simeq 0.60 a$ . Here  $\varphi_0(r) \sim \exp(-r/a)$  is the bound state wave function of an individual dimer and  $\mathbf{x}_{1,2}$  are the respective interparticle distances between the two distinguishable fermions which they are composed of. It follows from Eq. (130) that the effective dimer-dimer interaction at low energies is characterized by a positive scattering length, which is proportional to the scattering length between its fermionic constituents. This guarantees the stability of molecular condensates and also implies that there are no four-particle bound states for

zero-range interactions<sup>19</sup>. Experimentally, the dimer-dimer scattering length can be inferred from the radius  $R = \ell_0(15Na_{dd}/\ell_0)^{1/5}$  of a molecular condensate with  $N$  dimers in a trap. The value found in fact agrees well with the prediction  $a_{dd} = 0.60 a$  (Bartenstein *et al.*, 2004; Bourdel *et al.*, 2004). Physically, the repulsion between dimers can be understood as a statistical interaction due to the Pauli-principle of its constituents. Within a phenomenological Ginzburg-Landau description of the molecular condensate by a complex order parameter  $\psi(\mathbf{x})$ , it is simply related to a positive coefficient of the  $|\psi|^4$ -term. In fact, the repulsive interaction between dimers was first derived from a coherent state functional integral representation of the crossover problem (Drechsler and Zwerger, 1992; Sá de Melo *et al.*, 1993). These results, however, were restricted to a Born approximation of the scattering problem, where  $a_{dd}^{(B)} = 2a$  (Sá de Melo *et al.*, 1993). A derivation of the exact result  $a_{dd} = 0.60 a$  from diagrammatic many-body theory has been given by Brodsky *et al.* (2006) and Levinsen and Gurarie (2006). It is important to note that the stability of attractive fermions along the BCS-BEC crossover relies crucially on the fact that the range of the attractive interaction is much smaller than the interparticle spacing. For more general interactions, where this is not the case, instabilities may arise, as discussed by Fregoso and Baym (2006).

## B. Crossover theory and Universality

For a description of the many-body physics of the BCS-BEC crossover, a natural starting point is a two-channel picture in which fermions in an open channel couple resonantly to a closed channel bound state. The resulting Hamiltonian

$$\hat{H}_{BF} = \int d^3x \left[ \sum_{\sigma} \hat{\psi}_{\sigma}^{\dagger} \left( -\frac{\hbar^2}{2M} \nabla^2 \right) \hat{\psi}_{\sigma} + \hat{\psi}_B^{\dagger} \left( -\frac{\hbar^2}{4M} \nabla^2 + \nu \right) \hat{\psi}_B + \tilde{g} \left( \hat{\psi}_B^{\dagger} \hat{\psi}_{\uparrow} \hat{\psi}_{\downarrow} + \text{h.c.} \right) \right] \quad (131)$$

defines the Bose-Fermi resonance model. It was introduced in this context by Holland *et al.* (2001) and by Timmermans *et al.* (2001) and has been used subsequently e.g. by Ohashi and Griffin (2002) and Drummond and Kheruntsyan (2004). Here  $\hat{\psi}_{\sigma}(\mathbf{x})$  are fermionic field operators describing atoms in the open channel. The two different hyperfine states are labelled by a formal spin variable  $\sigma = \uparrow, \downarrow$ . The bound state in the closed channel is denoted by the bosonic field operator  $\hat{\psi}_B$ . Its energy is detuned by  $\nu$  with respect to

<sup>19</sup> Such states are discussed in nuclear physics, where alpha particles in a nucleus may appear due to pairing correlations, see e.g. Röpke *et al.* (1998)

the open channel continuum and  $\tilde{g}$  is the coupling constant for the conversion of two atoms into a closed channel state and vice-versa. It is caused by the off-diagonal potential  $W(r)$  in Eq. (19) whose range is of order the atomic dimension  $r_c$ . As a result, the conversion is point-like on scales beyond  $r_e$  where a pseudopotential description applies. The magnitude of  $\tilde{g} = \langle \phi_{\text{res}} | W | \phi_0 \rangle$  is determined by the matrix element of the off-diagonal potential between the closed and open channel states. Using Eq. (26), its value is directly connected with the characteristic scale  $r^*$  introduced in Eq. (21), such that  $(2M_r \tilde{g} / \hbar^2)^2 = 4\pi / r^*$  (Bruun and Pethick, 2004). For simplicity, the background scattering between fermions is neglected, i.e. there is no direct term quartic in the fermionic fields. This is justified close enough to resonance  $|B - B_0| \ll |\Delta B|$ , where the scattering length is dominated by its resonant contribution.

*Broad and narrow Feshbach resonances* As was discussed in section I.C, the weakly bound state which appears at negative detuning, has always a vanishing closed channel admixture near resonance. For the experimentally relevant case  $|a_{bg}| \gg r^*$ , the virtual or real bound states within the range  $|\nu| < \mu |\Delta B|$  of the detuning may therefore be effectively described as a *single channel* zero-energy resonance. This criterion is based on two-body parameters only. In order to justify a single channel model for describing the physics of the crossover at a *finite* density  $n = k_F^3 / 3\pi^2$  of fermions, it is necessary that the potential resonance description is valid in the relevant regime  $k_F |a| \gtrsim 1$  of the many-body problem. Now, the range in the detuning where  $k_F |a| \gtrsim 1$  is given by  $|\nu| \lesssim \sqrt{\varepsilon_F \varepsilon^*}$ . Since the closed channel contribution is negligible as long as  $\nu \ll \varepsilon^*$ , a single-channel description applies if  $\varepsilon_F \ll \varepsilon^*$  or  $k_F r^* \ll 1$  (Bruun and Pethick, 2004; Diener and Ho, 2004). This is the condition for a 'broad' Feshbach resonance, which only involves the *many-body* parameter  $k_F r^*$ . In quantitative terms, the Fermi wavelength  $\lambda_F = 2\pi / k_F$  of dilute gases is of order  $\mu\text{m}$  while  $r^*$  is typically on the order of or even smaller than the effective range  $r_e$  of the interaction. The condition  $k_F r^* \ll 1$  is therefore very well obeyed unless one is dealing with exceptionally narrow Feshbach resonances. Physically, the assumption of a broad resonance implies that the bosonic field in Eq. (131), which gives rise to the resonant scattering, is so strongly coupled to the open channel that the relative phase between both fields is perfectly locked, i.e. the closed channel molecules condense simultaneously with the particles in the open channel. In contrast to the two-particle problem, therefore, there is a finite  $Z$ -factor precisely on resonance, as verified experimentally by Partridge *et al.* (2005). An important point to realize is that this situation is precisely opposite to that encountered in conventional superconductors, where the role of  $\varepsilon^*$  is played by the Debye energy  $\hbar\omega_D$ . The ratio  $\hbar\omega_D / \varepsilon_F$  is very small in this case, on the order of the sound velocity divided by the Fermi velocity. Effec-

tively, this corresponds to the case of *narrow* resonances, where  $k_F r^* \gg 1$ . The effective Fermi-Fermi interaction is then retarded and the Bose field in Eq. (131) is basically unaffected by the condensation of the fermions. On a formal level, this case can be treated by replacing the closed channel field by a c-number, giving rise to a reduced BCS model with a mean-field order parameter  $\Delta = \tilde{g} \langle \hat{\psi}_B \rangle$  (DePalo *et al.*, 2005; Sheehy and Radzihovsky, 2006).

There is an essential simplification in describing the crossover problem in the limit  $k_F r^* \ll 1$ . This is related to the fact that the parameter  $r^*$  can be understood as an effective range for interactions of fermions at energies below the continuum, i.e. at  $k = i\kappa$ . Indeed, consider the resonant phase shift for two-body scattering as given in Eq. (20). At zero detuning  $\nu = 0$  and small  $k$  the associated scattering amplitude can be shown to be precisely of the form (4) with an effective range  $r_e = -2r^*$ . Therefore, in the limit  $k_F r^* \ll 1$ , the two-body interaction near resonance is described by the scattering amplitude Eq. (6) of an ideal pseudopotential even at  $k = k_F$ . As a result, the Fermi energy is the only energy scale in the problem right at unitarity. As pointed out by Ho (2004), the thermodynamics of the unitary Fermi gas is then universal, depending only on the dimensionless temperature  $\theta = T / T_F$ . In fact, as found by Nikolic and Sachdev (2007), the universality is much more general and is tied to the existence of an unstable fixed point describing the unitary, balanced gas at zero density. As a result, by a proper rescaling, the complete thermodynamics and phase diagram of low density Fermi gases with short range attractive interactions is a universal function of temperature  $T$ , detuning  $\nu$ , chemical potential  $\mu$  and the external field  $h$  conjugate to a possible density imbalance.

*Universality* The universality provides considerable insight into the problem even without a specific solution of the relevant microscopic Hamiltonian. For simplicity, we focus on the so-called unitary Fermi gas right at the Feshbach resonance and the spin-balanced case of an equal mixture of both hyperfine states which undergo pairing. This problem has in fact first been discussed in nuclear physics as a parameter free model of low density neutron matter (Baker, 1999; Heiselberg, 2001). By dimensional arguments, at  $a = \infty$ , the particle density  $n$  and the temperature  $T$  are the only variables on which the thermodynamics depends. The free energy per particle, which has  $n$  and  $T$  as its natural variables, thus acquires a universal form

$$F(T, V, N) = N \varepsilon_F \cdot f(\theta) \quad (132)$$

with  $\varepsilon_F \sim n^{2/3}$  the bare Fermi energy and  $\theta = T / T_F$  the dimensionless temperature. The function  $f(\theta)$  is monotonically *decreasing*, because  $s = -f'(\theta)$  is just the entropy per particle. As will be shown below, the fact that the ground state is superfluid implies that  $f(0) - f(\theta)$  vanishes proportional to  $\theta^4$  as the temperature approaches zero, in contrast to a Fermi gas (or liquid),



where the behavior is  $\sim \theta^2$ . Physically, this is due to the fact that the low lying excitations are sound modes and not fermionic quasiparticles. By standard thermodynamic derivatives, the function  $f$  determines both the dimensionless chemical potential according to

$$\frac{\mu}{\varepsilon_F} = \frac{5}{3}f(\theta) - \frac{2}{3}\theta f'(\theta) =: \xi(\theta) \quad (133)$$

and the pressure via  $p/n\varepsilon_F = \mu/\varepsilon_F - f(\theta)$ , consistent with the Gibbs-Duhem relation  $\mu N = F + pV$  for a homogeneous system. Moreover, the fact that  $-f'(\theta)$  is the entropy per particle, immediately implies that  $3pV = 2(F + TS)$ . The internal energy  $u$  per volume is therefore connected with pressure and density by the simple identity  $p = 2u/3$ , valid at all temperatures (Ho, 2004). Naively, this appears like the connection between pressure and energy density in a non-interacting quantum gas. In the present case, however, the internal energy has a nonvanishing contribution  $\langle \hat{H}' \rangle$  from interactions. A proper way of understanding the relation  $p = 2u/3$  is obtained by considering the quantum virial theorem  $2\langle \hat{H}_0 \rangle - k\langle \hat{H}' \rangle = 3pV$  for a two-body interaction  $V(\mathbf{x}_i - \mathbf{x}_j) \sim |\mathbf{x}_i - \mathbf{x}_j|^k$ , which is a homogeneous function of the interparticle distance. It implies that  $p = 2u/3$  is valid for an interacting system if  $k = -2$ . The pressure of fermions at unitarity is thus related to the energy density as if the particles had a purely inverse square interaction. An important consequence of this is the virial theorem (Thomas *et al.*, 2005)

$$\langle \hat{H}_{\text{tot}} \rangle = 2\langle \hat{H}_{\text{trap}} \rangle = 2 \int d^3x U_{\text{trap}}(\mathbf{x})n(\mathbf{x}) \quad (134)$$

for a harmonically trapped unitary gas, which allows to determine the thermodynamics of the unitary gas from its equilibrium density profile  $n(\mathbf{x})$ . The relation (134) follows quite generally from the quantum virial theorem with  $k = -2$  and the fact that the contribution  $3pV$  of the external forces to the virial in the case of a box with volume  $V$  is replaced by  $2\langle \hat{H}_{\text{trap}} \rangle$  in the presence of an external harmonic potential. It is therefore valid for finite temperature and arbitrary trap anisotropy, An alternative derivation of Eq. (134) has been given by Werner and Castin (2006). They have noted that the unitary Fermi gas in 3D exhibits a scale invariance which is related to a hidden  $SO(2, 1)$  symmetry. In fact, since the interaction potential at unitarity effectively obeys  $V(\lambda r) = V(r)/\lambda^2$ , the situation is precisely analogous to that discussed at the end of section VI for the 2D Bose gas with a pseudopotential interaction. In particular, the scale invariance implies a simple evolution of arbitrary initial states in a time dependent trap and the existence of undamped breathing modes with frequency  $2\omega$  (Werner and Castin, 2006).

At zero temperature, the ground state properties of the unitary gas are characterized by a *single* universal number  $\xi(0) = 5f(0)/3$ , which is often denoted by  $1+\beta$  in this context. It is smaller than one (i.e.  $\beta < 0$ ), because the

attractive interaction leads to a reduction of the chemical potential at unitarity from its non-interacting value  $\mu^{(0)} = \varepsilon_F$  to  $\mu = \xi(0)\varepsilon_F$ <sup>20</sup>. Experimentally, the most direct way of measuring the universal number  $\xi(0)$  is obtained from in-situ, absorption imaging of the density distribution  $n(\mathbf{x})$  in a trap. Indeed, within the local density approximation Eq. (12), free fermions in an isotropic trap exhibit a density profile  $n(\mathbf{x}) = n(0)(1 - r^2/R_{TF}^2)^{3/2}$  with a Thomas-Fermi radius  $R_{TF}^{(0)} = (24N)^{1/6}\ell_0$ . Since  $\mu \sim n^{2/3}$  at unitarity has the same dependence on density than for non-interacting fermions, with a prefactor reduced just by  $\xi(0) < 1$ , the profile at unitarity is that of a free Fermi gas with a rescaled size. For a given total particle number  $N$  and mean trap frequency  $\bar{\omega}$ , the resulting Thomas-Fermi radius at zero temperature, is therefore reduced by a factor  $\xi^{1/4}(0)$ . Ideally, the value  $R_{TF}^{(0)}$  would be measured by sweeping the magnetic field to the zero crossing of the scattering length at  $B = B_0 + \Delta B$ , where an ideal Fermi gas is realized. In practice, e.g. for <sup>6</sup>Li, there is appreciable molecule formation and subsequent decay processes at this field and it is more convenient to ramp the field to values far on the BCS side, where the thermodynamics is again essentially that of an ideal Fermi gas. Results for the universal parameter  $\xi(0)$  at the lowest attainable temperatures of around  $\theta \approx 0.04$  have been obtained in this way by Bartenstein *et al.* (2004), with the result  $\xi(0) = 0.32 \pm 0.1$ . This value is considerably smaller than that inferred from more recent in situ measurements of the trap radius by Partridge *et al.* (2006), where  $\xi(0) = 0.46 \pm 0.05$  is found. Alternatively, the parameter  $\xi$  may be determined by measuring the release energy of an expanding cloud (Bourdel *et al.*, 2004; Stewart *et al.*, 2006). In this case, however, an appreciable temperature dependence was found (Stewart *et al.*, 2006), which makes extrapolations to  $T = 0$  difficult. In particular, at finite temperature, the relation between the density distributions at  $a = 0$  and at  $a = \infty$  involves the complete function  $\xi(\theta)$  because the Fermi temperature continuously decreases as one moves away from the trap center.

On the theoretical side, the ground state properties of a resonantly interacting Fermi gas have been obtained numerically by fixed-node Green function Monte Carlo calculations. They provide quantitative results for the equation of state (Astrakharchik *et al.*, 2004; Carlson *et al.*, 2003) at arbitrary values of  $a$  and in particular at unitarity. The resulting values for  $\xi(0)$  are 0.43 (Carlson *et al.*, 2003) or 0.41 (Astrakharchik *et al.*, 2004). Very recently, the chemical potential and the gap of the unitary Fermi gas at zero temperature have been calculated analytically from an effective field theory using an  $\epsilon = 4 - d$

<sup>20</sup> In a trap, the chemical potential  $\mu_{\text{trap}} \sim R^2$  is reduced by a factor  $\sqrt{\xi(0)}$  and not  $\xi(0)$  as in the homogeneous case, because the density in the trap center is increased by the attractive interaction.

expansion (Nishida and Son, 2006). The possibility of such an expansion is based on an observation made by Nussinov and Nussinov (2004), that a unitary Fermi gas in four dimensions is in fact an ideal Bose gas. Indeed, in  $d = 4$ , a two-particle bound state in a zero range potential only appears at infinitely strong attraction. Thus, already at  $\varepsilon_b = 0^+$ , the resulting dimer size vanishes. At finite density, therefore, one ends up with a non-interacting BEC, similar to the situation as  $a \rightarrow 0^+$  in three dimensions. The expansion about the upper critical dimension  $d = 4$  may be complemented by an expansion around the lower critical dimension, which is two for the present problem (Nishida and Son, 2007). Indeed, for  $d \leq 2$  a bound state at zero binding energy appears for an arbitrary weak attractive interaction, as shown explicitly in section V.A. A unitary Fermi gas in  $d \leq 2$  thus coincides with the non-interacting gas and  $\xi(0) \equiv 1$  for all  $d \leq 2$  (Nussinov and Nussinov, 2004). The  $d - 2$  expansion, however, only captures non-superfluid properties like the equation of state while all effects associated with superfluidity are nonperturbative. Combining these two expansions within a Borel-Padé method, the field-theoretical results for the universal parameter  $\xi(0)$  give values in the range  $\xi(0) = 0.36 - 0.39$  for different Padé-approximants (Nishida and Son, 2007).

*Critical temperature and pseudogap* Within a single-channel description, a zero range interaction  $V(\mathbf{x} - \mathbf{x}') = g_0 \delta(\mathbf{x} - \mathbf{x}')$  between fermions of opposite spin  $\sigma$  gives rise to an interaction Hamiltonian in momentum space

$$\hat{H}' = \frac{g_0}{2V} \sum_{\sigma} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{Q}} c_{\mathbf{k}+\mathbf{Q}, \sigma}^{\dagger} c_{-\mathbf{k}, -\sigma}^{\dagger} c_{-\mathbf{k}', -\sigma} c_{\mathbf{k}'+\mathbf{Q}, \sigma}. \quad (135)$$

Here  $c_{\mathbf{k}, \sigma}^{\dagger}$  are fermion creation operators with momentum  $\mathbf{k}$  and spin  $\sigma$  and  $V$  is the volume of the system. Moreover,  $\mathbf{k} - \mathbf{k}'$  is the momentum transfer due to the interaction and  $\mathbf{Q}$  the conserved total momentum in the two-particle scattering process. The bare coupling strength  $g_0$  is determined by the s-wave scattering length  $a$  after a regularization, in which the delta potential is replaced by the proper pseudopotential with finite strength  $g$  (see below). For attractive interactions  $g < 0$ ,<sup>21</sup> the Hamiltonian (135) was first discussed by Gor'kov and Melik-Barkhudarov (1961). In the weak-coupling regime  $k_F|a| \ll 1$ , where the magnitude of the scattering length is much less than the average interparticle spacing, they showed that a BCS-instability to state

with bound pairs appears at a temperature

$$T_c = \frac{8e^C}{(4e)^{1/3}\pi e^2} T_F \exp(-\pi/(2k_F|a|)) \quad (136)$$

( $C = 0.577$  is Euler's constant). As expected for a weak-coupling BCS-instability, the critical temperature vanishes with an essential singularity. The absence of an energy cutoff in the interaction leaves the Fermi temperature as the characteristic scale. For typical densities and off-resonant scattering lengths in cold gases, the parameter  $k_F|a| \approx 0.02$  is very small, so (136) is applicable in principle. In practice, however, fermionic superfluidity in dilute gases, where  $T_F$  is only of order micro-Kelvin, is unobservable unless  $k_F|a|$  becomes of order one. In fact, the range of accessible coupling strengths on the BCS side of the crossover is limited by the finite level spacing in the trap or, alternatively, by the trap size  $R$ , which must be larger than the size  $\xi_b \approx \hbar v_F/(k_B T_c)$  of a Cooper pair (Tinkham, 1996). Using the local density approximation, the condition  $k_B T_c \gtrsim \hbar \omega$  on the BCS side is equivalent to  $\xi_b \lesssim R$  and implies particle numbers  $N \gtrsim N_{\star} = \exp(3\pi/(2k_F|a|))$ . Since  $N_{\star} = 10^5$  at  $k_F|a| = 0.4$  this shows that with typical values for the particle numbers in a trap, the regime  $k_F|a| \ll 1$  is no longer described by the theory of a locally homogeneous system. Instead, for  $N < N_{\star}$  one reaches a regime which is similar to that of pairing in nuclei, where the resulting energy gap obeys  $\Delta_0 \ll \hbar \omega$ , see e.g. Heiselberg and Mottelson (2002).

In the strong-coupling regime  $k_F|a| \gtrsim 1$  near the unitarity limit, where the critical temperature lies in an accessible range of order  $T_F$  itself, no analytical solution of the problem is available. In particular, the singular nature of the two-particle scattering amplitude  $f(k) = i/k$  right at unitarity rules out any obvious perturbative approach. It is only far out on the BEC side of the problem, where  $k_F a \ll 1$  again provides a small parameter. In this regime, the binding energy  $\varepsilon_b$  is much larger than the Fermi energy  $\varepsilon_F$ . At temperatures  $k_B T \ll \varepsilon_b$  therefore, a purely bosonic description applies for a dilute gas of strongly bound pairs<sup>22</sup> with density  $n/2$  and a repulsive interaction described by the dimer-dimer scattering length of Eq. (130). Its dimensionless coupling constant  $(n/2)^{1/3} a_{dd} = 0.16 k_F a$  is much smaller than one in the regime  $1/k_F a \gtrsim 2$ . Since the dimers eventually approach an ideal Bose gas, with density  $n/2$  and mass  $2M$ , the critical temperature in the BEC limit is obtained by converting the associated ideal BEC condensation temperature into the original Fermi energy. In the homogeneous

<sup>21</sup> Note that the model (135) does not make sense in the regime  $g > 0$ , where it describes *repulsive* fermions. However, with a proper pseudopotential, the two-particle interaction has a bound state for positive scattering length. The Hamiltonian (135) is then understood to describe fermions along this branch and not in their continuum states, where the interaction would be repulsive

<sup>22</sup> Note that the pseudopotential bound state is strongly bound in the BEC limit of the crossover only as far as the scales relevant for the BCS-BEC crossover are concerned, while it is a very weakly bound state on the scale of the actual interatomic potential, being the highest, so-called rovibrational state of a total number  $N_b \gg 1$  of bound states, see section I.A

case this gives  $T_c(a \rightarrow 0) = 0.218 T_F$  while in a trap the numerical factor is 0.518. The fact that  $T_c$  is completely independent of the coupling constant in the BEC limit is simple to understand: On the BCS side, superfluidity is destroyed by fermionic excitations, namely the breakup of pairs. The critical temperature is therefore of the same order as the pairing gap at zero temperature, consistent with the well known BCS relation  $2\Delta_0/k_B T_c = 3.52$ . A relation of this type is characteristic for a situation, in which the transition to superfluidity is driven by the gain in *potential* energy associated with pair formation. In particular, the formation and condensation of fermion pairs occur at the same temperature. By contrast, on the BEC side, the superfluid transition is driven by a gain in *kinetic* energy, associated with the condensation of preformed pairs. The critical temperature is then on the order of the degeneracy temperature of the gas, which is completely unrelated to the pair binding energy.

To lowest order in  $k_F a$  in this regime, the shift Eq. (11) in the critical temperature due to the repulsive interaction between dimers is positive and linear in  $k_F a$ . The critical temperature in the homogeneous case therefore has a maximum as a function of the dimensionless inverse coupling constant  $v = 1/k_F a$ , as found in the earliest calculation of  $T_c$  along the BCS-BEC crossover by Nozières and Schmitt-Rink (1985). More recent calculations of the universal curve  $\theta_c(v)$  (Haussmann *et al.*, 2007) indeed show a maximum around  $v \approx 1$ , which is rather small, however (see Fig. 28). The associated universal ratio  $T_c/T_F = 0.16$  at the unitarity point  $v = 0$  agrees well with the value 0.152(7) obtained from precise Quantum Monte Carlo calculations for the negative-U Hubbard model at low filling by Burovski *et al.* (2006). Considerably larger values 0.23 and 0.25 for the ratio  $T_c/T_F$  at unitarity have been found by Bulgac *et al.* (2006) from auxiliary field Quantum Monte Carlo calculations and by Akkineni *et al.* (2006) from restricted path integral Monte Carlo methods, the latter working directly with the continuum model. In the presence of a trap, the critical temperature has been calculated by Perali *et al.* (2004). In this case, no maximum is found as a function of  $1/k_F a$  because the repulsive interaction between dimers on the BEC side leads to a density reduction in the trap center, which eliminates the  $T_c$ -maximum at fixed density.

The increasing separation between the pair *formation* and the pair *condensation* temperature as  $v$  varies between the BCS- and the BEC-limit implies that in the regime  $-2 \lesssim v \lesssim +2$  near unitarity, there is a substantial range of temperatures above  $T_c$ , where preformed pairs exist, but do not form a superfluid. From recent path integral Monte Carlo calculations, the characteristic temperature  $T^*$  below which strong pair correlations appear has been found to be of order  $T^* \approx 0.7 T_F$  at unitarity (Akkineni *et al.*, 2006), which is at least three times the condensation temperature  $T_c$  at this point. It has been shown by Randeria *et al.* (1992) and Trivedi and Randeria (1995), that the existence of pre-

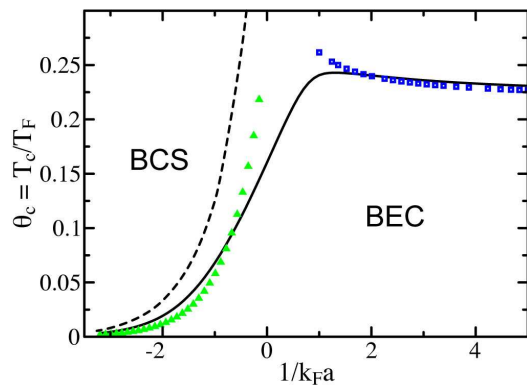


FIG. 28 Critical temperature of the homogeneous gas as a function of the coupling strength. The exact asymptotic results Eq. (136) and Eq. (11) in the BCS and BEC limits are indicated by green triangles and blue squares, respectively. At unitarity  $T_c = 0.16 T_F$ . The dashed line schematically denotes the evolution of  $T^*$ . Reprinted with permission from (Haussmann *et al.*, 2007).

formed pairs in the regime  $T_c < T \lesssim T^*$  leads to a normal state very different from a conventional Fermi liquid. For instance, the spin susceptibility is strongly suppressed due to singlet formation above the superfluid transition temperature<sup>23</sup>. This is caused by the strong attractive interactions near unitarity, which leads to pairs in the superfluid, whose size is of the same order than the interparticle spacing. The temperature range between  $T_c$  and  $T^*$  may be considered as a regime of strong superconducting fluctuations. Such a regime is present also in high-temperature superconductors, where it is called the Nernst region of the pseudogap phase (Lee *et al.*, 2006). Its characteristic temperature  $T^*$  approaches  $T_c$  in the regime of weak coupling (see Fig. 28). It disappears in underdoped cuprates, where  $T_c$  vanishes. Remarkably, in these systems, the temperature below which the spin susceptibility is suppressed, however, becomes *larger* at small doping (Lee *et al.*, 2006). Apart from the different nature of the pairing in both cases (s- versus d-wave), the nature of the pseudogap in the cuprates, which appears in the proximity of a Mott-insulator with antiferromagnetic order, is thus a rather complex set of phenomena, which still lack a proper microscopic understanding (Lee *et al.*, 2006). For a discussion of the relevance and limitations of the analogy between the pseudogap phase in the BCS-BEC crossover to that in high  $T_c$  cuprates see e.g. the reviews by Randeria (1998) and by Chen *et al.* (2005).

*Extended BCS description of the crossover* A simple approximation, which covers the complete range of coupling strengths analytically, is obtained by assuming that, at

<sup>23</sup> For a proposal to measure the spin susceptibility in trapped Fermi gases see Recati *et al.* (2006).

least for the ground state, only zero momentum pairs are relevant. In the subspace of states with only zero momentum pairs, all contributions in Eq. (135) with  $\mathbf{Q} \neq 0$  vanish. The resulting Hamiltonian

$$\hat{H}'_{BCS} = \frac{g_0}{2V} \sum_{\sigma} \sum_{k, k'} c_{k, \sigma}^{\dagger} c_{-k, -\sigma}^{\dagger} c_{-k', -\sigma} c_{k', \sigma} \quad (137)$$

thus involves only *two* momentum sums. Eq. (137) is in fact just the reduced BCS-Hamiltonian, which is a standard model Hamiltonian to describe the phenomenon of superconductivity. It is usually solved by a variational Ansatz

$$\Psi_{BCS}(1, 2, \dots, N) = \hat{\mathcal{A}}[\phi(1, 2)\phi(3, 4) \cdots \phi(N-1, N)] \quad (138)$$

in which an identical two-particle state  $\phi(1, 2)$  is assumed for each pair. Here the arguments  $1 = (\mathbf{x}_1, \sigma_1)$  etc. denote position and spin,  $\hat{\mathcal{A}}$  is the operator which antisymmetrizes the many-body wavefunction and we have assumed an even number of fermions for simplicity. The wave function (138) is a simple example of a so-called Pfaffian state (see section VII.C) with  $(N-1)!!$  terms, which is just the square root of the determinant of the completely antisymmetric  $N \times N$  matrix  $\phi(i, j)$ . In second quantization, it can be written in the form  $|\Psi_{BCS}(1, 2, \dots, N)\rangle = (\hat{b}_0^{\dagger})^{N/2}|0\rangle$  of a Gross-Pitaevskii like state. The operator  $\hat{b}_0^{\dagger} = \sum_k \phi_k c_{k, \uparrow}^{\dagger} c_{-k, \downarrow}^{\dagger}$  creates a pair with zero total momentum, with  $\phi_k = V^{-1/2} \int \phi(\mathbf{x}) \exp -i\mathbf{k}\mathbf{x}$  the Fourier transform of the spatial part of the two particle wave function  $\phi(1, 2)$  in Eq. (138). It is important to note, however, that  $\hat{b}_0^{\dagger}$  is *not* a Bose operator. It develops this character only in the limit, where the two-particle wave function  $\phi(1, 2)$  has a size much smaller than the interparticle spacing (see below). To avoid the difficult task of working with a fixed particle number, it is standard practice to use a coherent state

$$|\text{BCS}\rangle = C_{BCS} \exp(\alpha \hat{b}_0^{\dagger}) |0\rangle = \prod_k (u_k + v_k c_{k, \uparrow}^{\dagger} c_{-k, \downarrow}^{\dagger}) |0\rangle. \quad (139)$$

Since  $\langle \hat{b}_0^{\dagger} \hat{b}_0 \rangle = |\sum_k \phi_k u_k v_k|^2 = |\alpha|^2 = N/2$  by the number equation (see below), this state is characterized by a macroscopic occupation of a single state, which is a bound fermion pair with zero total momentum. The amplitudes  $u_k, v_k$  are connected to the two-particle wavefunction via  $v_k/u_k = \alpha \phi_k$ . Since  $u_k^2$  or  $v_k^2$  are the probabilities of a pair  $\mathbf{k} \uparrow, -\mathbf{k} \downarrow$  being empty or occupied, they obey the normalization  $u_k^2 + v_k^2 = 1$ . The overall normalization constant  $C_{BCS} = \exp -\frac{1}{2} \sum_k \ln(1 + |\alpha \phi_k|^2) \rightarrow \exp -|\alpha|^2/2$  approaches the standard result of a coherent state of bosons in the strong coupling limit, where  $|\alpha \phi_k|^2 \approx v_k^2 \ll 1$  for all  $k$ . In this limit,  $\hat{b}_0^{\dagger}$  is indeed a Bose operator and the wave function Eq. (138) is that of an ideal BEC of dimers. In fact, antisymmetrization becomes irrelevant in the limit where the occupation  $v_k^2$  of

all fermion states is much less than one<sup>24</sup>. The BCS wave function has the gap  $\Delta$  as a single variational parameter, which appears in the fermion momentum distribution

$$v_k^2 = \frac{1}{2} \left( 1 - \frac{\varepsilon_k - \mu}{\sqrt{(\varepsilon_k - \mu)^2 + \Delta^2}} \right). \quad (140)$$

With increasing strength of the attractive interaction, this evolves continuously from a slightly smeared Fermi distribution to a rather broad distribution  $v_k^2 \rightarrow \Delta^2/4(\varepsilon_k - \mu)^2 \sim (1 + (k\xi_b)^2)^{-2}$  in the BEC limit, where the chemical potential is large and negative (see below). Its width  $\xi_b^{-1}$  increases as the pair size  $\xi_b = \hbar/\sqrt{2M|\mu|}$  approaches zero. Experimentally the fermionic momentum distribution near the Feshbach resonance has been determined from time-of-flight measurements by Regal *et al.* (2005). Accounting for the additional smearing due to the trap, the results are in good agreement with Monte-Carlo calculations of the momentum distribution for the model (135) (Astrakharchik *et al.*, 2005b). An analysis of the distribution at finite temperature allows to determine the decrease of the average fermionic excitation gap with temperature (Chen *et al.*, 2006b).

Within the extended BCS description, the magnitude of  $\Delta$  is determined by the standard gap equation

$$-\frac{1}{g_0} = \frac{1}{2V} \sum_k \frac{1}{\sqrt{(\varepsilon_k - \mu)^2 + \Delta^2}} \quad (141)$$

where  $E_k = \sqrt{(\varepsilon_k - \mu)^2 + \Delta^2}$  is the BCS quasiparticle energy. In conventional superconductors the momentum sum in Eq. (141) is restricted to a thin shell around the Fermi energy and the solution  $\Delta \sim \exp -1/|g_0|N(0)$  for  $g_0 < 0$  depends only on the density of states per spin  $N(0)$  in the normal state right *at* the Fermi energy. In cold gases, however, there is no such cutoff as long as  $\varepsilon_F \ll \varepsilon^*$ . Moreover, the true dimensionless coupling constant  $N(0)|g| = 2k_F|a|/\pi$  is far from small, approaching infinity at the Feshbach resonance. The pairing interaction thus affects fermions deep in the Fermi sea and eventually completely melts the Fermi sphere. Within the pseudopotential approximation the apparent divergence in Eq. (141) can be regularized by the formal replacement  $1/g_0 \rightarrow 1/g - 1/2V \sum_k (1/\varepsilon_k)$ . Physically, this amounts to integrating out the high energy contributions in Eq. (141) where the spectrum is unaffected by the pairing. A general procedure for doing this, including the case of strong pairing in nonzero angular momentum states, has been given by Randeria *et al.* (1990). Converting the sum over  $k$  to an integral over the free particle density of states, the renormalized gap equation at

<sup>24</sup> Note that the wavefunction (138) still contains  $(N-1)!!$  terms even in the BEC limit. In practice, however, only a single term is relevant, unless one is probing correlations between fermions in different pairs.

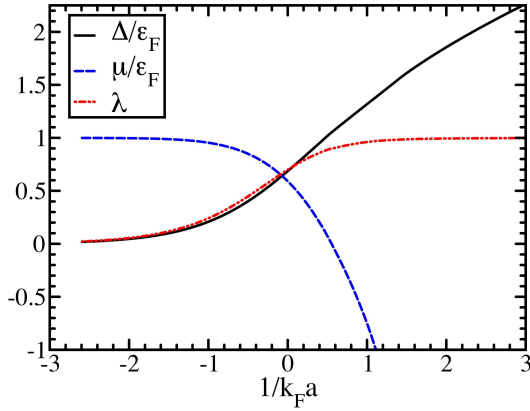


FIG. 29 Solution of the gap and number equations (142) and (143) for the reduced BCS Hamiltonian Eq. (137). The dimensionless gap parameter, chemical potential and condensate fraction (144) of the ground state are shown as a function of the dimensionless interaction parameter  $1/k_F a$ .

zero temperature can then be written in the form

$$\frac{1}{k_F a} = (\tilde{\mu}^2 + \tilde{\Delta}^2)^{1/4} P_{1/2}(x). \quad (142)$$

Here  $\tilde{\mu} = \mu/\varepsilon_F$  and  $\tilde{\Delta} = \Delta/\varepsilon_F$  are the dimensionless chemical potential and gap respectively, while  $P_{1/2}(x)$  is a Legendre function of the first kind. The parameter  $x = -\mu/(\mu^2 + \Delta^2)^{1/2}$  varies between  $-1$  in the BCS- and  $+1$  in the BEC-limit because the fermion chemical potential continuously drops from  $\mu = \varepsilon_F$  in weak-coupling to  $\mu \rightarrow -\varepsilon_b/2$  for strongly bound pairs and  $|\mu| \gg \Delta$  in both limits. Physically the behavior of the chemical potential in the BEC-limit can be understood by noting that the energy gained by adding *two* fermions is just the molecular binding energy. The detailed evolution of  $\mu$  as a function of the dimensionless coupling strength  $1/k_F a$  follows from the equation  $N = 2 \sum_k v_k^2$  for the average particle number. In dimensionless form this gives

$$\frac{4}{\pi} = \tilde{\mu}(\tilde{\mu}^2 + \tilde{\Delta}^2)^{1/4} P_{1/2}(x) + (\tilde{\mu}^2 + \tilde{\Delta}^2)^{3/4} P_{-1/2}(x). \quad (143)$$

The equations (142, 143), originally discussed by Eagles (1969) and Leggett (1980), determine the gap, the chemical potential and related quantities like the condensate fraction (Ortiz and Dukelsky, 2005)

$$\lambda_{\text{BCS}} = \frac{n_0}{n} |_{\text{BCS}} = \frac{3\pi}{16} \frac{\tilde{\Delta}^2}{\text{Im}(\tilde{\mu} + i\tilde{\Delta})^{1/2}} \quad (144)$$

for arbitrary coupling (see Figure 29).

They provide a simple approximation for the crossover between weak-coupling  $1/k_F a \rightarrow -\infty$  and the BEC-limit  $1/k_F a \rightarrow \infty$  within the variational Ansatz Eq. (139) for the ground state wave function. In fact, as realized long ago by Richardson and Gaudin, the results are *exact* for the reduced BCS Hamiltonian Eq. (137) and not just of

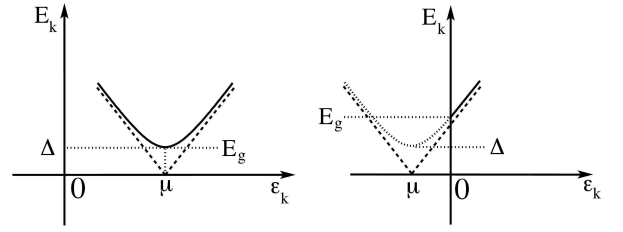


FIG. 30 Change in the fermionic excitation spectrum of the extended BCS description as the chemical potential changes from positive to negative values.

a *variational* nature as usually presented in textbooks<sup>25</sup>. Both the gap and the condensate fraction increase continuously with coupling strength, while the chemical potential becomes negative for  $1/k_F a > 0.55$ . The values  $\xi_{\text{BCS}}(0) = 0.59$ ,  $\tilde{\Delta}_{\text{BCS}} = 0.69$  and  $\lambda_{\text{BCS}} = 0.70$  for the chemical potential, the gap and the condensate fraction at unitarity differ, however, considerably from the corresponding results  $\xi(0) \approx 0.4$  and  $\tilde{\Delta} \approx \lambda \approx 0.5$  obtained by both numerical and field-theoretic methods for the physically relevant model (135). For strong coupling, the gap increases like  $\tilde{\Delta} = 4(3\pi k_F a)^{-1/2}$ . Apparently, this is much smaller than the two-particle binding energy. To explain why  $2\Delta$  differs from the energy  $\varepsilon_b$  of a strongly bound dimer even in the BEC limit, it is necessary to determine the minimum value of the energy  $E_k = \sqrt{(\varepsilon_k - \mu)^2 + \Delta^2}$  for single fermion excitations. For negative chemical potentials, this minimum is not at  $\Delta$  as in the usual situation  $\mu > 0$ , but at  $\sqrt{\Delta^2 + \mu^2}$  (see Fig. 30). Since  $|\mu| \gg \Delta$  in the BEC limit, the minimum energy for a single fermionic excitation is therefore  $|\mu| = \varepsilon_b/2$  and not  $\Delta$  (Randeria *et al.*, 1990).

The size of the pairs  $\xi_b$  continuously shrinks from an exponentially large value  $k_F \xi_b \simeq \varepsilon_F/\Delta$  in the BCS-limit to essentially zero  $\xi_b \simeq a$  in the BEC-limit of tightly bound pairs. For weak coupling, the size of the pair coincides with the coherence length  $\xi$ . This is no longer the case on the BEC-side of the crossover, however. Indeed, as shown by Pistoiesi and Strinati (1996) and Engelbrecht *et al.* (1997), the coherence length reaches a minimum value on the order of the interparticle spacing around the unitary limit and then increases slowly to approach the value  $\xi = (4\pi n a_{dd})^{-1/2}$  of a weakly interacting Bose gas of dimers with density  $n/2$ . This minimum is closely related to a maximum in the critical velocity around the unitary point. Indeed, as discussed in section IV.D, the critical momentum for superfluid flow within a mean-field description is simply  $k_c \approx 1/\xi$ . More precisely, the critical velocity  $v_c$  on the

<sup>25</sup> For a review see Dukelsky *et al.* (2004). It is interesting to note that although the BCS wave function (139) gives the exact thermodynamics of the model, its number projected form does not seem to be exact beyond the trivial weak coupling or BEC limit, see Ortiz and Dukelsky (2005).

BEC side of the crossover coincides with the sound velocity according to the Landau criterion. Near unitarity, this velocity reaches  $c = v_F \sqrt{\xi(0)/3} \approx 0.36 v_F$  (see below). On the BCS side, the destruction of superfluidity does not involve the excitation of phonons but is due to pair-breaking. As shown by Combescot *et al.* (2006) and Sensarma *et al.* (2006), the resulting critical velocity exhibits a maximum  $v_c^{max} \approx 0.36 v_F$  around the unitarity point which is close to the value of the local sound velocity there. Using a moving optical lattice near the trap center, this prediction has been verified recently by Miller *et al.* (2007).

*Failure of extended BCS-theory* In the regime of weak-coupling, the gap  $\hat{\Delta}_{BCS} = 8/e^2 \exp -\pi/2k_F|a|$  is exponentially small. Using Eq. (136) for the critical temperature, however, the ratio  $2\Delta/k_B T_c$  differs from the well known BCS-value  $2\pi/e^C = 3.52$  by a factor  $(4e)^{1/3}$ . The reason for this discrepancy is subtle and important from a basic point of view. It has to do with the fact that the reduced BCS-model contains only zero momentum pairs and thus no density fluctuations are possible. By contrast, the original model (135) includes such fluctuations, which are present in any neutral system. The innocent looking BCS assumption of pairs with zero momentum only thus eliminates an important part of the physics. To understand why the reduced BCS-model fails to account for the correct low energy excitations, it is useful to rewrite the interaction Eq. (137) in real space, where

$$\hat{H}'_{BCS} = \frac{g_0}{2V} \sum_{\sigma} \int_{\mathbf{x}} \int_{\mathbf{x}'} \hat{\psi}_{\sigma}^{\dagger}(\mathbf{x}) \hat{\psi}_{-\sigma}^{\dagger}(\mathbf{x}) \hat{\psi}_{-\sigma}(\mathbf{x}') \hat{\psi}_{\sigma}(\mathbf{x}'). \quad (145)$$

The associated non-standard (note the order of  $\mathbf{x}$  and  $\mathbf{x}'$ !) 'interaction'  $V_{BCS}(\mathbf{x} - \mathbf{x}') = g_0/V$  has *infinite* range, but is scaled with  $1/V$  to give a proper thermodynamic limit. It is well known, that models of this type are exactly soluble and that their phase transitions are described by mean-field theory. Moreover, while Eq. (145) is invariant under a *global* gauge transformation  $\psi_{\sigma}(\mathbf{x}) \rightarrow e^{i\phi} \psi_{\sigma}(\mathbf{x})$  (particle number is conserved), this is no longer the case if  $\phi(\mathbf{x})$  varies spatially. In the exactly soluble reduced BCS model therefore, the change in energy associated with a slowly varying phase does not vanish like  $(\nabla\phi(\mathbf{x}))^2$  as it should for a superfluid (see Appendix). The omission of pairs with finite momentum thus eliminates the well known Bogoliubov-Anderson mode  $\omega(q) = cq$  of a neutral superfluid (Anderson, 1958b). A proper description of the crossover problem in a neutral system like cold gases requires to account for both the bosonic as well as the fermionic excitations for an arbitrary strength of the coupling. By contrast, for charged superconductors, the Bogoliubov-Anderson mode is pushed up to a high frequency plasmon mode, which is irrelevant for the description of superconductivity. A reduced BCS model, in which they are omitted anyway, is thus appropriate.

The absence of the collective Bogoliubov-Anderson mode implies that already the leading corrections to the ground state in powers of  $k_F|a|$  are incorrect in a description which is based on just solving the gap- and number-equation for arbitrary coupling. For weak interactions, one misses the Gorkov, Melik-Barkhudarov reduction of the gap. Indeed, as shown by Heiselberg *et al.* (2000), density fluctuations give rise to a screening of the attractive interaction at finite density, changing the dimensionless coupling constant of the two-particle problem to  $g_{\text{eff}} = g + g^2 N(0)(1 + 2 \ln 2)/3 + \dots$ . Since the additional contribution to the two-body scattering amplitude  $g < 0$  is positive, the bare attraction between two fermions is weakened. Due to the nonanalytic dependence of the weak-coupling gap or transition temperature on the coupling constant  $k_F a$ , the renormalization  $g \rightarrow g_{\text{eff}}$  gives rise to a reduction of both the gap and the critical temperature by a finite factor  $(4e)^{-1/3} \approx 0.45$ . The universal ratio  $2\Delta/k_B T_c = 3.52$ , valid in weak coupling, is thus unaffected. It is interesting to note that a naive extrapolation of the Gorkov Melik-Barkhudarov result  $\Delta/\varepsilon_F = (2/e)^{7/3} \exp -\pi/2k_F|a|$  for the zero temperature gap to infinite coupling  $k_F a = \infty$  gives a value  $\Delta(v=0) = 0.49\varepsilon_F$  which is close to the one obtained from Quantum Monte Carlo calculations at unitarity (Carlson *et al.*, 2003). In the BEC limit, the condensate fraction Eq. (144) reaches the trivial limit one of the ideal Bose gas like  $\lambda_{BCS} = 1 - 2\pi n a^3 + \dots$ . The correct behavior, however, is described by the Bogoliubov result Eq. (10) with density  $n/2$  and scattering length  $a \rightarrow a_{dd}$ , i.e. it involves the *square root* of the gas parameter  $n a^3$ . As pointed out by Lamacraft (2006), the presence of pairs with nonzero momentum is also important for measurements of noise correlations in time-of-flight pictures near the BCS-BEC crossover. As discussed in section III.B, such measurements provide information about the truncated density correlation function

$$G_{\uparrow\downarrow}(\mathbf{k}_1, \mathbf{k}_2) = \langle \hat{n}_{\uparrow}(\mathbf{k}_1) \hat{n}_{\downarrow}(\mathbf{k}_2) \rangle - \langle \hat{n}_{\uparrow}(\mathbf{k}_1) \rangle \langle \hat{n}_{\downarrow}(\mathbf{k}_2) \rangle. \quad (146)$$

The momenta  $\mathbf{k}_{1,2} = M\mathbf{x}_{1,2}/\hbar t$  are simply related to the positions  $\mathbf{x}_{1,2}$  at which the correlations are determined after a free flight expansion for time  $t$ . While the formation of bound states shows up as a peak at  $\mathbf{k}_1 = -\mathbf{k}_2$  as observed by Greiner *et al.* (2005a) in a molecular gas above condensation, the presence of pairs with nonzero momenta give rise to an additional contribution proportional to  $1/|\mathbf{k}_1 + \mathbf{k}_2|$ . This reflects the depletion of the condensate due to bosons at finite momentum as found in the Bogoliubov theory of an interacting Bose gas.

*Crossover thermodynamics* As pointed out above, a complete description of the BCS-BEC crossover involves both bosonic and fermionic degrees of freedom. Since the fermionic spectrum has a gap, only the bosonic Bogoliubov-Anderson mode is relevant at low temperatures (Liu, 2006). Similar to the situation in superfluid  $^4\text{He}$ , these sound modes determine the low tempera-

ture thermodynamics along the full BCS-BEC crossover. They give rise to an entropy

$$S(T) = V \frac{2\pi^2}{45} k_B \left( \frac{k_B T}{\hbar c} \right)^3 + \dots \quad (147)$$

which vanishes like  $T^3$  for arbitrary coupling strength. The associated sound velocity  $c$  at zero temperature follows from the ground state pressure or the chemical potential via the thermodynamic relation  $Mc^2 = \partial p / \partial n = n \partial \mu / \partial n$ . It was shown by Engelbrecht *et al.* (1997), that the sound velocity decreases monotonically from  $c = v_F / \sqrt{3}$  to zero on the BEC side. There, to lowest order in  $k_F a$ , it is given by the Gross-Pitaevskii result  $(c/v_F)^2 = k_F a_{dd} / (6\pi)$  for the sound velocity of a dilute gas of dimers with a repulsive interaction obtained from Eq. (130). At unitarity,  $c = v_F \sqrt{\xi(0)/3} \approx 0.36 v_F$  is related to the Fermi velocity by the universal constant  $\xi(0)$ . This has been used in recent measurements of the sound velocity along the axial  $z$  direction of an anisotropic trap by Joseph *et al.* (2007). The velocity  $c_0 = c(z=0)$  measured near the trap center is given by  $c_0^2 = \sqrt{\xi(0)} (v_F^{(0)})^2 / 5$ . Here,  $v_F^{(0)}$  is the Fermi velocity of a non-interacting gas, which is fixed by the associated Fermi energy in the trap by  $\varepsilon_F(N) = M(v_F^{(0)})^2 / 2$ . A factor  $3/5$  arises from averaging over the transverse density profile (Capuzzi *et al.*, 2006). An additional factor  $1/\sqrt{\xi(0)}$  is due to the fact that at unitarity, the attractive interactions increase the local density  $n(0)$  in the trap center compared to that of a non-interacting Fermi gas by a factor  $\xi(0)^{-3/4}$ . The local Fermi velocity  $v_F[n(0)]$  is thus enhanced by  $\xi(0)^{-1/4}$  compared to  $v_F^{(0)}$ . Precise measurements of the ratio  $c_0/v_F^{(0)} = 0.362 \pm 0.006$  (Joseph *et al.*, 2007) at the lowest available temperatures thus give  $\xi(0) = 0.43 \pm 0.03$ , in good agreement with the Quantum Monte Carlo results. In addition, the independence of this ratio on the density has been tested over a wide range, thus confirming the universality at the unitary point.

Numerical results on the crossover problem at finite temperatures have been obtained by Bulgac *et al.* (2006), by Burovski *et al.* (2006) and by Akkineni *et al.* (2006) at the unitarity point. More recently, they have been complemented by analytical methods, using expansions around the upper and lower critical dimension (Nishida, 2007) or in the inverse number  $1/N$  of a strongly attractive Fermi gas with  $2N$  components (Nikolic and Sachdev, 2007). A rather complete picture of the crossover thermodynamics for arbitrary couplings and temperatures in the spin-balanced case was given by Haussmann *et al.* (2007) on the basis of a variational approach to the many-body problem developed by Luttinger-Ward and DeDominicis-Martin. Although the theory does not capture correctly the critical behavior near the superfluid transition, which is a continuous transition of the 3D XY-type for arbitrary coupling, the results obey standard thermodynamic relations and the specific relation  $p = 2u/3$  at unitarity at the level of a few

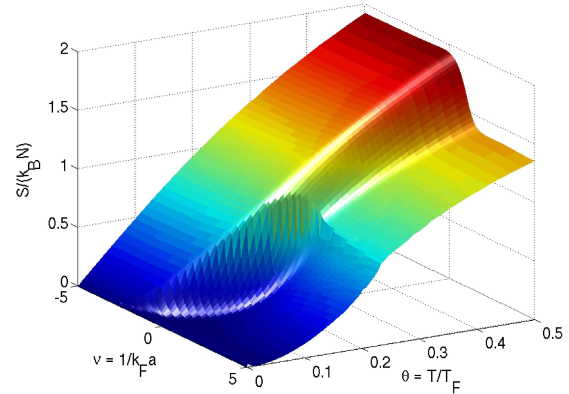


FIG. 31 Entropy per particle in units of  $k_B$  as a function of the dimensionless coupling  $1/k_F a$  and temperature  $\theta = T/T_F$ . The crossover from a Fermi gas with a tiny superfluid regime as  $T \rightarrow 0$  to a Bose gas which is superfluid below  $T \approx 0.2 T_F$  occurs in a narrow range of coupling strengths. An isentropic ramp, starting in the normal phase on the negative  $a$  side at sufficiently low  $T$  leads to a molecular condensate. Reprinted with permission from Haussmann *et al.* (2007).

percent. In addition, the resulting value  $T_c/T_F \approx 0.16$  for the critical temperature at unitarity agrees rather well with the presently most precise numerical results by Burovski *et al.* (2006). As an example, a 3D plot of the entropy per particle is shown in Fig. 31. Apparently, the freezing out of fermionic excitations with increasing coupling  $v$  leads to a strong suppression of the low temperature entropy. An adiabatic ramp across the Feshbach resonance from the BEC to the BCS side is thus associated with a lowering of the temperature, as emphasized by Carr *et al.* (2004). In particular, it is evident from this picture that a molecular condensate can be reached by going isentropically from negative to positive scattering lengths even if the initial state on the fermionic side is above the transition to superfluidity, as was the case in the experiments by Greiner *et al.* (2003). The plot provides a quantitative picture of how an attractive gas of fermions gradually evolves into a repulsive gas of bosons. Apparently, most of the quantitative change happens in the range  $-2 \lesssim v \lesssim +2$ , which is precisely the regime accessible experimentally with cold atoms. Note, that the exact entropy and its first derivative are continuous near the critical temperature. Indeed, the singular contribution to  $S(T)$  is proportional to  $|T - T_c|^{(1-\alpha)}$  and  $\alpha < 0$  for the 3D XY-transition. Moreover, the value  $S(T_c)/N \approx 0.7 k_B$  of the entropy per particle at  $T_c$  at the unitarity point (Haussmann *et al.*, 2007), provides a limit on the entropy of any initial state which is required to reach the superfluid regime  $T < T_c$  near unitarity by an adiabatic process.

### C. Experiments near the unitarity limit

*Thermodynamic properties* The thermodynamics of a strongly interacting Fermi gas near unitarity has been studied experimentally by the Duke group (Kinast *et al.*, 2005). It has been found, that the spatial profiles of both the trapped and released gas are rather close to a Thomas-Fermi profile with a size parameter determined from hydrodynamic scaling. This is consistent with the fact that the equation of state of the unitary gas is identical with that of a non-interacting gas up to a scale factor. Strictly speaking, however, this is true only at zero temperature. At finite temperature, the function  $\xi(\theta)$  defined in Eq. (133) is clearly different from that of an ideal Fermi gas. Assuming a Thomas-Fermi profile, the effective temperature of a near unitary gas can be inferred from fitting the observed cloud profiles with a Thomas-Fermi distribution at finite temperature. The temperature dependent internal energy  $E(T)$  and specific heat of the gas could then be determined by adding a well defined energy to the gas through a process, where the cloud is released abruptly and recaptured after a varying expansion time (Kinast *et al.*, 2005). The resulting  $E(T)$  curve essentially follows that of a non-interacting gas above a characteristic temperature  $\tilde{T}_c \approx 0.27 T_F^{(0)}$ . The data below  $\tilde{T}_c$  follow a power law  $E(T) - E_0 \sim T^a$  with an exponent  $a \simeq 3.73$ , which is not far from the exact low  $T$  result  $E(T) - E_0 = V \cdot (\pi^2/30)k_B T (k_B T / \hbar c)^3$  of a uniform superfluid in the phonon-dominated regime  $k_B T \ll M c^2$ . Quantitatively, the data are well described by a finite temperature generalization of the extended BCS description of the crossover (Chen *et al.*, 2005). In particular, the characteristic temperature  $\tilde{T}_c$  is close to the theoretically expected transition temperature  $T_c^{trap} \approx 0.25 T_F^{(0)}$  to the superfluid state. Indeed, within the local density approximation (LDA), the superfluid transition in a trap occurs when the local Fermi temperature  $T_F[n(0)]$  in the trap center reaches the critical value  $T_c$  of the homogeneous gas at density  $n(0)$ . At the unitarity point, the latter is obtained from the universal ratio  $T_c/T_F \approx 0.16$ . As noted above, the attractive interactions increase the local density in the trap center compared to the non-interacting gas with (bare) Fermi energy  $\varepsilon_F(N) = \hbar\bar{\omega} \cdot (3N)^{1/3} = k_B T_F^{(0)}$ . Thus  $T_F[n(0)]$  is enhanced by a factor  $1/\sqrt{\xi}$ . Neglecting for simplicity the variation of the chemical potential over the size of the trap due to the local variation of  $T_F$ , this translates into a critical temperature  $T_c^{trap}/T_F^{(0)} = 0.16 \cdot \xi^{-1/2}(T_c) \approx 0.25$ , where  $\xi(T_c) \approx 0.39$  is again a universal number at unitarity (Haussmann *et al.*, 2007).

Experimental results on thermodynamic properties which do not rely on the difficult issue of a proper temperature calibration are possible by using the virial theorem Eq. (134). It allows to measure the energy of the strongly interacting gas directly from its density profile. Within LDA, the contributions to  $\langle \hat{H}_{trap} \rangle$  from the three spatial directions are identical, even in an anisotropic trap.

The total energy per particle thus follows directly from the average mean square radius  $E/N = 3M\omega_z^2 \langle z^2 \rangle$  along (say) the axial direction. The predicted linear increase of  $\langle z^2 \rangle$  at unitarity with the energy input has been verified experimentally by Thomas *et al.* (2005). More generally, since the internal energy per particle is equal to  $f(\theta) - \theta f'(\theta)$  in units of the bare Fermi energy, the universal function  $f(\theta)$  is in principle accessible by measuring the density profile. A possible way to do thermometry for such measurements has recently been developed by Luo *et al.* (2007). They have measured the dependence  $S(E)$  of the entropy on energy by determining the energy from the mean square radii and the entropy by adiabatically ramping the magnetic field far into the BCS regime. There, the gas is essentially an ideal Fermi gas in a trap and its entropy may be inferred again from  $\langle z^2 \rangle$ . At low energies,  $S(E) \sim (E - E_0)^b$  is fit to a power law with an exponent  $b \approx 0.59$  (note that in the phonon dominated regime of a uniform gas one expects  $b = 3/4$  exactly at *arbitrary* coupling). The associated ground state energy has been determined by Hu *et al.* (2007a) from a fit to the theory of Nozières and Schmitt-Rink (1985), generalized to the symmetry broken state (see (Engelbrecht *et al.*, 1997)). The resulting value  $(E_0/N\varepsilon_F)_{trap} = 0.48 \pm 0.03$  implies  $\xi(0) = 0.40 \pm 0.03$ , because the total energy in the trap is just twice of the result  $(E_0/N\varepsilon_F) = f(0) = 3\xi(0)/5$  obtained in a uniform system. At higher energies, a different power law for  $S(E)$  is found. The crossover energy is used to provide a characteristic temperature scale  $\tilde{T}_c/T_F^{(0)} = 0.29 \pm 0.03$ , which agrees well with the critical temperature inferred from the earlier measurements.

*Condensate fraction* The standard signature of BEC in ultracold gases is the appearance of a bimodal density distribution below the condensation temperature. Fitting the density profile from the absorption image to a superposition of a Thomas Fermi profile and a Gaussian background from the thermal atoms allows a rather precise measurement of the condensate fraction  $n_0/n$ . In the case of fermionic superfluidity this does not work because pairs break in time-of-flight. A way out is the rapid transfer technique (Regal *et al.*, 2004a; Zwierlein *et al.*, 2004) in which the fragile pairs on the BCS side of the crossover are preserved by sweeping the magnetic field towards the BEC side of the resonance, transforming them to stable molecules. In an adiabatic situation, each fermionic pair is thereby transformed into a tightly bound dimer and a time-of-flight analysis of the molecular condensate allows to infer the momentum distribution of the original pairs on the BCS side of the crossover. The resulting absorption images are shown in Fig. 32.

In practice, the process is non-adiabatic and essentially projects the initial many-body state onto that of a molecular condensate. A theoretical analysis of the condensate fraction extracted from the rapid transfer technique has been given by Perali *et al.* (2005) and by Altman and Vishwanath (2005). The fraction of



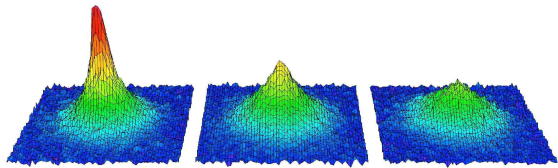


FIG. 32 Time-of-flight images showing a fermionic condensate after projection onto a molecular gas. The images show condensates at different detunings  $\Delta B = 0.12, 0.25,$  and  $0.55$  G (left to right) from the Feshbach resonance (BCS side), starting with initial temperatures  $T/T_F = 0.07$ . Reprinted with permission from Regal *et al.* (2004a).

molecules depends on the sweep rate and, in a strongly non-adiabatic situation, provides information about pair correlations in the initial state even in the absence of a condensate (Altman and Vishwanath, 2005). Experimentally, the observed condensate fractions in  $^{40}\text{K}$  were at most around 0.14 (Regal *et al.*, 2004a), much smaller than the expected equilibrium values  $n_0/n \approx 0.5$  at unitarity. A possible origin of this discrepancy may be the rather short lifetime of  $^{40}\text{K}$  dimers near resonance on the order of 100 ms. Yet, the qualitative features of the phase diagram agree with an analysis based on an equilibrium theory (Chen *et al.*, 2006a). For  $^6\text{Li}$ , in turn, the condensate fractions determined via the rapid transfer technique turned out to be much larger, with a maximum value 0.8 at  $B \approx 820$  G, on the BEC side of the resonance (Zwierlein *et al.*, 2004) (see Fig. 33). Apparently, in this case, there were no problems with the lifetime, however the experimental definition of  $n_0/n$  via the ratio of the particle numbers in the central peak to the total number overestimates the true condensate fraction. This is probably due to the fact that the central peak contains an appreciable contribution from particles which are removed from the condensate by the strong interactions even at zero temperature. The observation (Zwierlein *et al.*, 2004) that the condensate fraction decreases to zero on the BEC side (see Fig. 33) instead of approaching one is probably caused by fast vibrational relaxation into deeply bound states further away from the resonance.

*Collective modes* Collective modes in a harmonic trap have been a major tool to study cold gases in the BEC regime, where the dynamics is determined by superfluid hydrodynamics. For a mixture of fermions with an adjustable attractive interaction, the corresponding eigenfrequencies have been worked out by Stringari (2004) and by Heiselberg (2004). For a highly elongated trap, where the axial trap frequency  $\omega_z$  is much smaller than the radial confinement frequency  $\omega_\perp$ , two important eigenmodes are the axial and radial compression modes, with

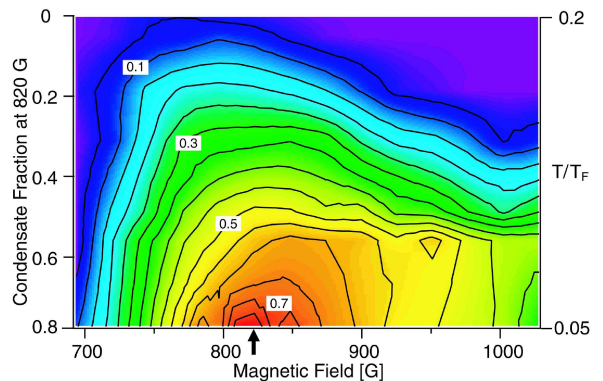


FIG. 33 Condensate fraction as a function of magnetic field and temperature. The highest condensate fractions and onset temperatures are obtained on the BEC side, close to the resonance at  $B_0 = 834$  G. As a measure of temperature, the condensate fraction at 820 G (see arrow) is used as the vertical axis. Reprinted with permission from Zwierlein *et al.* (2004).

respective frequencies (Heiselberg, 2004; Stringari, 2004)

$$\omega_B = \omega_z \sqrt{3 - \frac{1}{\gamma + 1}} \quad \text{and} \quad \omega_r = \omega_\perp \sqrt{2(\gamma + 1)}. \quad (148)$$

They are completely determined by the effective polytropic index  $\gamma = d \ln p / d \ln n - 1$  and thus give information about the equation of state  $p(n)$  along the crossover. Since  $\gamma = 2/3$  exactly for the unitary Fermi gas, one obtains universal numbers for these frequencies precisely at the Feshbach resonance, as pointed out by Stringari (2004). In particular the radial compression mode frequency  $\omega_r$  is equal to  $\omega_r = \sqrt{10/3} \omega_\perp$  in the BCS-limit and at unitarity, while it approaches  $\omega_r = 2 \omega_\perp$  in the BEC-limit, where  $\gamma = 1$ . The fact that the chemical potential Eq. (9) of a dilute repulsive Bose gas is larger than its mean-field limit  $\mu_{\text{Bose}} \sim n$ , implies  $\gamma > 1$ , i.e. the BEC-limit is reached from *above*. The expected non-monotonic behavior of  $\omega_r$  as a function of  $1/k_F a$  has recently been observed by Altmeyer *et al.* (2007). Remarkably, it provides the first quantitative test of the Lee-Huang-Yang correction to the chemical potential Eq. (9) of a dilute repulsive Bose gas (of dimers) (see Fig. 34).

The issue of damping of the collective modes in the BCS-BEC crossover has raised a number of questions, which - remarkably - are connected with recent developments in QCD and field theory. As pointed out by Gelman *et al.* (2004), not only the thermodynamics but also dynamical properties like the kinetic coefficients are described by universal scaling functions at the unitarity point. An example is the shear viscosity  $\eta$  which determines the damping of sound and collective oscillations in trapped gases. At unitarity, its dependence on density  $n$  and temperature  $T$  is fixed by dimensional arguments to be  $\eta = \hbar n \alpha(T/\mu)$ , where  $\mu$  is the chemical potential and  $\alpha(x)$  a dimensionless universal function. At zero temperature, in particular,  $\eta(T = 0) = \alpha_\eta \hbar n$  is linear in the density, which defines a 'quantum viscosity' coefficient  $\alpha_\eta$ .

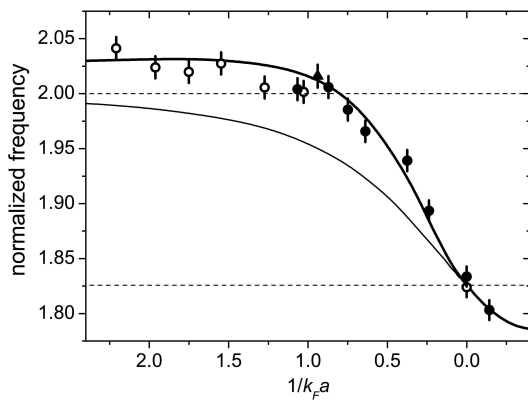


FIG. 34 Normalized frequency  $\omega_r/\omega_\perp$  of the radial compression mode as a function of dimensionless interaction parameter  $1/k_F a$  on the BEC side of the crossover. The result at unitarity agrees well with the predicted value  $\sqrt{10/3} = 1.826$ . The BEC limit  $\omega_r/\omega_\perp = 2$  is approached from *above* due to the positive correction to the ground state chemical potential beyond mean field, predicted by Lee, Huang and Yang. The monotonic line is the result of an extended BCS mean field theory. Reprinted with permission from Altmeyer *et al.* (2007).

From a simple fluctuation-dissipation type argument in the *normal* phase, a lower bound  $\alpha_\eta \geq 1/6\pi$  was derived by Gelman *et al.* (2004). Assuming that a hydrodynamic description applies, the effective shear viscosity can be inferred from the damping rate of the collective modes. In particular, the damping of the axial mode near the unitarity limit in the experiments by Bartenstein *et al.* (2004) gives  $\alpha_\eta \approx 0.3$  at the lowest attainable temperatures. Based on these results, it has been speculated that ultracold atoms near a Feshbach resonance are a nearly perfect liquid (Gelman *et al.*, 2004). A proper definition of viscosity coefficients like  $\eta$ , however, requires hydrodynamic damping. At zero temperature, this is usually not the case. It is valid at  $T = 0$  in one dimension, where exact results on  $\alpha_\eta$  have been obtained for arbitrary coupling along the BCS-BEC crossover (Punk and Zwerger, 2006).

At finite temperature, damping is typically associated with the presence of thermally excited quasiparticles, which also give rise to a nonzero entropy density  $s(T)$ . It has been shown by Kovtun *et al.* (2005) that in a rather special class of relativistic field theories, which are dual to some string theory, the ratio  $\eta(T)/s(T) = \hbar/(4\pi k_B)$  has a universal value. For more general models, this value is conjectured to provide a lower bound on  $\eta/s$  for a vanishing chemical potential, i.e. effectively in the zero density limit. Since the ratio does not involve the velocity of light, the string theory bound on  $\eta/s$  may apply also to non-relativistic systems like the unitary Fermi gas, which lack an intrinsic scale beyond temperature and density. From Eq. (147), both entropy density and shear viscosity should therefore vanish like  $T^3$  in the low temperature, superfluid regime. Recent measurements of this

ratio, using the damping of the radial breathing mode in a strongly anisotropic trap have been performed by Turlapov *et al.* (2007) as a function of the energy of the gas. The results indicate a very low viscosity of the unitary gas in both the superfluid and normal regime, and a ratio  $\eta/s$  which is indeed of order  $\hbar/k_B$ . Exact results on *bulk* rather than shear viscosities have been obtained by Son (2007). He noted that the unitary Fermi gas exhibits a conformal symmetry which constrains the phenomenological coefficients in the dissipative part of the stress tensor. In particular, the bulk viscosity vanishes identically in the normal state and thus no entropy is generated in a uniform expansion. In the superfluid phase, which is quite generally characterized by the shear plus three different bulk viscosities (Forster, 1975), this result implies that two of the bulk viscosity coefficients vanish at unitarity, while one of them may still be finite.

*Rf-spectroscopy* A microscopic signature of pairing between fermions is provided by rf-spectroscopy of the gap on the BCS-side of the crossover, where no bound states exist in the absence of a Fermi sea. This was first suggested by Törmä and Zoller (2000). Following earlier work by Regal and Jin (2003) and Gupta *et al.* (2003) in the non-superfluid regime, such an experiment has been performed by Chin *et al.* (2004). An rf-field with frequency  $\omega_L$  is used to drive transitions between one of the hyperfine states  $|2\rangle = |\downarrow\rangle$  which is involved in the pairing and an empty hyperfine state  $|3\rangle$  which lies above it by an energy  $\hbar\omega_{23}$  due to the magnetic field splitting of the bare atom hyperfine levels. In the absence of any interactions, the spectrum exhibits a sharp peak at  $\omega_L = \omega_{23}$ . The presence of attractive interactions between the two lowest hyperfine states  $|1\rangle$  and  $|2\rangle$  will lead to an upward shift of this resonance. In a molecular picture, which is valid far on the BEC-side of the crossover, this shift is expected to coincide with the two-particle binding energy  $\varepsilon_b$ . Indeed, it was shown by Chin and Julienne (2005), that both scattering lengths and molecular binding energies may be extracted from rf-spectra of weakly bound molecules. For a quantitative analysis, however, it is important even at the level of single molecules, to properly account for the nonvanishing interaction  $a_{13,23} \neq 0$  of atoms in state  $|3\rangle$  with those in the initial states  $|1\rangle$  and  $|2\rangle$  forming the molecule (Gupta *et al.*, 2003). In the experiments of Chin *et al.* (2004), the rf-spectrum exhibits a dominant free atom peak centered at  $\omega_L = \omega_{23}$  for temperatures  $T \approx T_F$ . At low temperatures,  $T \lesssim 0.2T_F$  where the gas is superfluid, an additional peak is observed, which is shifted with respect to the free transition. As shown in Figure 35, the shift essentially follows the two-particle binding energy on the BEC-side of the crossover but stays finite on the BCS-side  $a < 0$ . In particular, the size of the shift near unitarity increases with the Fermi energy because the formation of bound pairs is a many-body effect. A theoretical analysis of these observations, which is based on an extended BCS description of pairing gener-

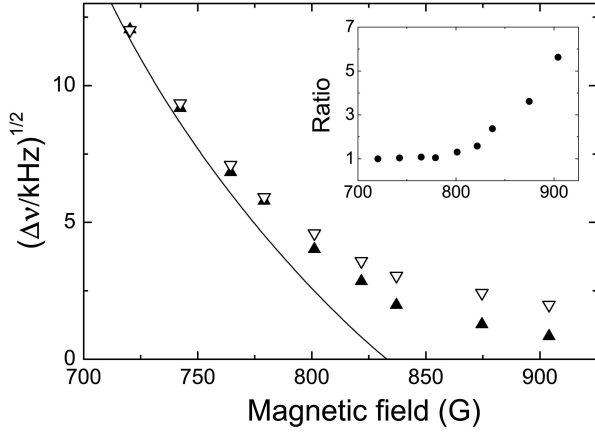


FIG. 35 Effective pairing gap in  ${}^6\text{Li}$  from rf-spectroscopy as a function of magnetic field. The solid line is the two-particle binding energy, which vanishes at the Feshbach resonance at  $B_0 = 834\text{ G}$  coming from the BEC side. The open and closed symbols are for Fermi temperatures  $T_F = 3.6\ \mu\text{K}$  and  $T_F = 1.2\ \mu\text{K}$  respectively. The ratio of the effective pairing gaps strongly depends on  $T_F$  on the BCS side (inset). Reprinted with permission from Chin *et al.* (2004).

alized to finite temperature within a T-matrix formalism, has been given by Kinnunen *et al.* (2004) and He *et al.* (2005); Ohashi and Griffin (2005). Including the necessary average over the inhomogeneous gap parameter  $\Delta(\mathbf{x})$  in a harmonic trap, reasonable agreement with the experimentally observed spectra has been obtained. An important point to realize is that the strong attractive interactions near unitarity lead to an effective 'pairing gap' already in the normal state above  $T_c$ . The RF-shift is therefore not a direct measure of the superfluid order parameter (He *et al.*, 2005; Kinnunen *et al.*, 2004).

The models discussed above rely on the assumption that interactions with the final state  $|3\rangle$  are negligible. When this is not the case, the results can change significantly. For instance, when the interaction constants  $g_{\sigma\sigma'}$  between the states  $|\sigma = 1, 2, 3\rangle$  which are involved are identical, neither mean field ('clock') shifts nor the effects of pairing show up in the long wavelength rf-spectrum (Yu and Baym, 2006). A theory for the average frequency shift of the rf-spectra of homogeneous gases at zero temperature, which takes into account both many-body correlations and the interactions with the final state  $|3\rangle$  has recently been given by Baym *et al.* (2007) and Punk and Zwerger (2007). Near  $T = 0$ , where only a single peak is observed in the rf-spectrum, its position can be determined from a sum rule approach. Introducing the detuning  $\omega = \omega_L - \omega_{23}$  of the RF field from the bare  $|2\rangle - |3\rangle$  transition, the average clock shift

$$\hbar\bar{\omega} = \left( \frac{1}{g_{12}} - \frac{1}{g_{13}} \right) \frac{1}{n_2} \frac{\partial(-u)}{\partial g_{12}^{-1}} \quad (149)$$

can be expressed in terms of the derivative of the ground state energy density  $u$  with respect to the inverse of the renormalized coupling constant  $g_{12} = 4\pi\hbar^2 a_{12}/M$

(Baym *et al.*, 2007). The expression is finite for all coupling strengths  $g_{12}$  and evolves smoothly from the BCS to the BEC-limit. In particular, the average clock shift vanishes, if  $g_{12} = g_{13}$  (Zwierlein *et al.*, 2003a). This is directly connected with the result mentioned above, because the interaction between states  $|2\rangle$  and  $|3\rangle$  drops out for the average shift  $\bar{\omega}$ . For negligible populations of the state  $|3\rangle$ , the derivative  $\partial(-u)/\partial g_{12}^{-1} = \hbar^4 C/M^2$  can be expressed in terms of the constant  $C$  which characterizes the asymptotic behavior  $\lim n_{\mathbf{k}} = C/k^4$  of the momentum distribution at large momenta of the crossover problem in the  $|1\rangle - |2\rangle$  channel (Punk and Zwerger, 2007). Within an extended BCS-description of the ground state wavefunction, the constant  $\hbar^4 C_{\text{BCS}}/M^2 \equiv \Delta^2$  is precisely the square of the gap parameter. In the BCS-limit, Eq. (149) thus reproduces the weak coupling result obtained by Yu and Baym (2006). In the BEC-limit, where the BCS ground state becomes exact again, the asymptotic behavior  $\Delta_{\text{BEC}} = 4\varepsilon_F/\sqrt{3\pi k_F a_{12}}$  gives  $\hbar\bar{\omega} = 2\varepsilon_b(1 - a_{12}/a_{13})$ , with  $\varepsilon_b = \hbar^2/Ma_{12}^2$  the two-particle binding energy. This agrees precisely with the first moment of the spectrum of a bound-free transition in the molecular limit (Chin and Julienne, 2005). The dependence on  $k_F$  drops out, as it must. The most interesting regime is that around the unitarity point  $1/g_{12} = 0$ , where the average rf-shift is simply given by  $\bar{\omega} = -0.46 v_F/a_{13}$ . The prefactor is obtained from a numerical evaluation of the derivative in Eq. (149) (Baym *et al.*, 2007) or - equivalently - the constant  $C$  in the momentum distribution at the unitarity point (Punk and Zwerger, 2007) (note that the dependence  $\bar{\omega} \sim -v_F/a_{13}$  at unitarity also holds in an extended BCS description, however the numerical factor 0.56 differs from the exact value). Compared with locally resolved rf-spectra, which were measured recently by Shin *et al.* (2007), the predicted average shift in the case of  ${}^6\text{Li}$  is almost twice the observed peak position. This is probably due to the fact that  $\bar{\omega}$  has a considerable contribution from the higher frequency part of the spectrum. An unexpected prediction of Eq. (149) is the *linear* dependence of the shift on the Fermi wavevector  $k_F$  at unitarity. Experimentally, the spatial resolution necessary to distinguish this from the naive scaling proportional to  $\varepsilon_F$  has not yet been achieved (Shin *et al.*, 2007).

**Vortices** While the appearance of a pairing gap in the rf-spectrum is a strong indication for superfluidity, it is not a conclusive proof. Indeed, pairing effects appear in the normal state above  $T_c$  as a precursor to superfluidity or may be present even at zero temperature in unbalanced Fermi gases above the Clogston-Chandrasekhar limit, as observed very recently by Schunck *et al.* (2007). A crucial step, which verifies the existence of a paired fermionic superfluid was thus the observation of perfect triangular vortex lattices in rotating Fermi gases near unitarity by Zwierlein *et al.* (2005) (see Fig. 36). Vortex lattices require conservation of vorticity, which is a consequence of

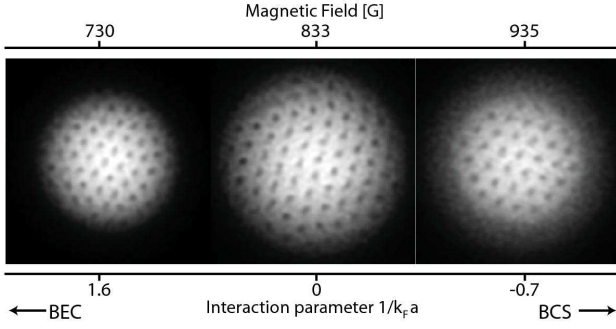


FIG. 36 Vortex lattice in a rotating gas of  ${}^6\text{Li}$  precisely at the Feshbach resonance and on the BEC and BCS side. Reprinted with permission from Zwierlein *et al.* (2005).

superfluid hydrodynamics. The regularity of the lattice shows that all vortices have the same vorticity. Since it is a superfluid of fermionic *pairs*, the expected circulation per vortex is  $h/2M$ . This is indeed the value found from equating the total circulation at a given stirring frequency with the number of vortices and the transverse area of the cloud.

## IX. PERSPECTIVES

From the examples discussed in this review, it is evident that cold atoms provide a novel tool to study the physics of strong correlations in a widely tunable range and in unprecedentedly clean systems. Basic models in many-body physics like the Hubbard-model with on-site interaction or the Haldane pseudopotentials for physics in the lowest Landau level, which were originally introduced in a condensed matter context as an idealized description of strong correlation effects in real materials can now be applied on a quantitative level. In the following, a brief outline of possible future directions is given.

### A. Quantum magnetism

One of the major challenges with ultracold atoms in optical lattices in the near future lies in the realization and study of configurations, which can serve as tunable model systems for basic problems in quantum magnetism. For two component mixtures of bosonic or fermionic atoms in an optical lattice which are labelled by a pseudo-spin variable  $|\uparrow\rangle, |\downarrow\rangle$  and dominating interactions  $U \gg J$  between them, second order perturbation theory in the kinetic energy term allows one to map the corresponding (Bose-)Hubbard Hamiltonian to an anisotropic Heisenberg model (XXZ model) with effective spin-spin interactions between atoms on neighboring lattice sites:

$$\sum_{\mathbf{R}, \mathbf{R}'} [J_{ex}^z S_{\mathbf{R}}^z S_{\mathbf{R}'}^z \pm J_{ex}^\perp (S_{\mathbf{R}}^x S_{\mathbf{R}'}^x + S_{\mathbf{R}}^y S_{\mathbf{R}'}^y)] \quad (150)$$

where the  $+(-)$  sign holds for the case of fermions or bosons, respectively (Duan *et al.*, 2003; Kuklov and Svistunov, 2003). By tuning the exchange terms  $J_{ex}^z = 2(J_\uparrow^2 + J_\downarrow^2)/U_{\uparrow\downarrow} - 4J_\uparrow^2/U_{\uparrow\uparrow} - 4J_\downarrow^2/U_{\downarrow\downarrow}$  and  $J_{ex}^\perp = 4J_\uparrow J_\downarrow/U_{\uparrow\downarrow}$  via spin-dependent tunneling amplitudes or spin-dependent interactions, one can change between various quantum phases. One intriguing possibility, pointed out by Kuklov and Svistunov (2003) is the realization of supercounterflow, i.e. superfluidity in the relative motion of the two components, while the system is a Mott insulator, as far as the total density is concerned. Another possibility is the formation of topological quantum phases, arising for different exchange coupling along different lattice directions Kitaev (2006), which can be realized with optical lattices Duan *et al.* (2003). The most natural case occurs, however, for equal tunneling amplitudes and onsite interactions, which yields an isotropic Heisenberg type spin-hamiltonian:

$$H = J_{ex} \sum_{\mathbf{R}, \mathbf{R}'} \mathbf{S}_{\mathbf{R}} \cdot \mathbf{S}_{\mathbf{R}'}. \quad (151)$$

Here the superexchange coupling  $J_{ex} = \pm 4J^2/U_{\uparrow\downarrow}$  has a positive (negative) sign for the case of fermions (bosons) and thus favors antiferromagnetically (ferromagnetically) ordered phases. In the case of fermions, it can easily be understood why the antiferromagnetically ordered phase is preferred: an initial spin-triplet state cannot lower its energy via second order hopping processes, as the two spins pointing in the same direction can never be placed on a single lattice site. The spin-singlet term, however, is not subject to this restriction and can lower its energy via a second order exchange hopping process (Auerbach, 2006). Recently, the tunnelling dynamics based on second order hopping events has been observed in an array of tightly confining double wells (Fölling *et al.*, 2007). Remarkably, the observed second order coupling strengths  $\sim J^2/hU$  can be almost on the order of one kHz and thus an order of magnitude larger than the direct magnetic dipole interaction between two alkali atoms on neighboring lattice sites.

The observation of such magnetically ordered quantum phases requires to reach very low temperatures  $k_B T < 4J^2/U$  (for  $J/U \ll 1$ ) in the experiment. Werner *et al.* (2005) have shown that such temperatures could be within reach experimentally, as a Pomeranchuk type cooling effect during the loading of the atoms in the lattice assists in a cooling of the atoms. In particular, it turns out that for an initial temperature of a homogeneous two-component Fermi gas below  $T/T_F \simeq 0.08$ , an antiferromagnetically ordered phase could be reached at unit filling, when the optical lattice potential is ramped up adiabatically. A more robust implementation of spin Hamiltonians might be reached via ground state polar molecules, for which much larger spin-spin coupling strengths have been recently found to be present (Micheli *et al.*, 2006). Such nearest neighbor spin interactions would be mediated via a long range electric

dipole-dipole interaction between the molecules. Several experimental groups are currently pursuing the goal of creating such ground state polar molecules out of ultra-cold atoms via Feshbach sweeps and subsequent photoassociation or by directly slowing and sympathetically cooling stable polar molecules. Noise correlation or Bragg spectroscopy could allow one to uniquely identify such antiferromagnetically ordered phases.

From a condensed matter point of view, one of the most challenging problems is the realization and study of the fermionic repulsive Hubbard model in an optical lattice with adjustable interactions and filling fraction, in particular in 2D. As discussed by Hofstetter *et al.* (2002) such a system would constitute a cold atom version of one of the most intensely studied models in condensed matter physics, allowing to access unconventional normal and *d*-wave superconducting phases as found in the high temperature superconductors (Lee *et al.*, 2006). Of course, realization of these models with cold atoms would not solve the latter problem, however it would be an extremely valuable tool to test some of the still open issues in this field. In this context, specific proposals have been made for realizing so-called resonating valence bond states by adiabatically transforming spin patterns in optical superlattices (Trebst *et al.*, 2006). More complex spin-liquid states might be created by enforcing 'frustration' in antiferromagnetically ordered phases through triangular or Kagome type lattices (Santos *et al.*, 2004). For a detailed review of these models, see Lewenstein *et al.* (2007).

Quantum impurity problems, which have played an important role in the study of magnetism, may be realized in the cold atom context by confining single atoms in a tight optical trap or in a deep optical lattice. For hardcore bosons or fermions, the effective pseudo-spin one half associated with the possible local occupation numbers  $n = 0, 1$  can be coupled to a reservoir of either a BEC or a degenerate Fermi gas, using Raman transitions (Recati *et al.*, 2005).

## B. Disorder

It was noted by Anderson (1958a) that waves in a medium with a static ('quenched') randomness may become localized due to constructive interference between multiply reflected waves. Qualitatively, this happens below a 'mobility edge', where the mean free path  $\ell$  (calculated e.g. in a Born approximation treatment of the scattering by the disorder potential) becomes smaller than the wavelength  $\lambda$ . This so-called Ioffe-Regel criterion applies in a 3D situation with short range disorder and in the absence of interactions<sup>26</sup>. In the interact-

ing case, where both localization due to disorder (Anderson) or due to interactions (Mott) are possible, the problem is still not well understood (see e.g. the reviews by Belitz and Kirkpatrick (1994) and Basko *et al.* (2006)). Cold atoms thus provide a novel tool to investigate the localization problem, in particular since the interactions are tunable over a wide range.

For non-interacting bosons, the ground state in the presence of disorder is trivially obtained by putting all the particles in the lowest single-particle level of the random potential. Adding weak repulsive interactions, a finite number of localized states in the so-called Lifshitz tails of the single-particle spectrum (Lifshitz *et al.*, 1988) will be occupied. As long as the chemical potential is in this low energy range, these states have negligible spatial overlap. For very low densities, repulsive bosons are therefore expected to form a 'Lifshitz glass' of fragmented, local condensates (Lugan *et al.*, 2007). With increasing densities, these local condensates will be coupled by Josephson tunneling and eventually form a superfluid, where coherence is established over the whole sample. A quantitative analysis of this transition was first given by Giamarchi and Schulz (1988) in the particular case of one dimension. Using a quantum hydrodynamic, Luttinger liquid description, they found that weak interactions tend to suppress the effect of Anderson localization (this effect is also present in the case of a commensurate filling in an optical lattice, where also a Mott-insulating phase appears, see Rapsch *et al.* (1999)). Specifically, weak disorder does not destroy the superfluid, provided the Luttinger exponent  $K$  introduced in section V.B is larger than  $3/2$ . Within the Lieb-Liniger model with an effective coupling constant  $\gamma$  defined in Eq. (88), this requires  $\gamma \lesssim 8$ . In the opposite regime  $K < 3/2$  of low densities or strong interactions near the Tonks-Girardeau limit, even weak disorder destroys the superfluid. The ground state then lacks long range phase coherence, consistent with the picture of a 'Lifshitz glass' discussed above. Exact results for the momentum distribution and the local density of states have been obtained by DeMartino *et al.* (2005) in the special limit of the Tonks-Girardeau gas, where the problem is equivalent to non-interacting fermions (see section V.B). Interacting bosons in higher dimensions were studied by Fisher *et al.* (1989), using the Bose-Hubbard model Eq. (65). To account for disorder, the on-site energies  $\epsilon_{\mathbf{R}}$ , are assumed to have a random component, with zero average and finite variance

$$\langle (\epsilon_{\mathbf{R}} - \langle \epsilon_{\mathbf{R}} \rangle) (\epsilon_{\mathbf{R}'} - \langle \epsilon_{\mathbf{R}'} \rangle) \rangle = \Delta \cdot \delta_{\mathbf{R}, \mathbf{R}'}, \quad (152)$$

where  $\Delta$  is a measure of the strength of the disorder. It has been shown by Fisher *et al.* (1989), that even at weak disorder  $\Delta < U/2$  a novel so-called Bose glass phase appears, which separates the SF- and MI-states<sup>27</sup>. At

<sup>26</sup> In 1D and 2D, an arbitrary weak disorder leads to localization, i.e. even waves with  $\lambda \ll \ell$  are localized.

<sup>27</sup> The 'Lifshitz glass' mentioned above may be thought of as the

strong disorder  $\Delta > U/2$ , the MI-states are destroyed completely. The Bose glass is characterized by a vanishing superfluid and condensate density. It will thus show no sharp interference peaks in a time-of-flight experiment after release from an optical lattice (see section IV.B). In contrast to the MI-phase, the Bose glass has both a finite compressibility (i.e. there is no shell structure in a trap) and a continuous excitation spectrum.

A concrete proposal for studying localization effects with cold atoms has been made by Damski *et al.* (2003), who suggested to use laser speckle patterns as a means to realize the frozen disorder. Such patterns have been employed in this context by the groups in Florence (Lye *et al.*, 2005) and in Orsay (Clément *et al.*, 2005). They are produced by a laser beam, which is scattered from a ground glass diffuser (Clement *et al.*, 2006). The speckle pattern has a random intensity  $I$ , which is exponentially distributed  $P(I) \sim \exp(-I/\langle I \rangle)$ . The rms intensity fluctuation  $\sigma_I$  is thus equal to the average intensity  $\langle I \rangle$ . The fluctuations in the intensity give rise to a random value of the optical dipole potential in Eq. (31), which is experienced by the atoms. By varying the detuning  $\Delta$ , this random potential (purely repulsive for  $\Delta > 0$ ) can be tuned in the range between zero and around  $h \cdot 4$  kHz (Clement *et al.*, 2006). An important parameter characterizing the speckle pattern is the spatial correlation length  $\sigma_R$ , which is the scale over which  $\langle I(\mathbf{x})I(0) \rangle$  decays to  $\langle I \rangle^2$ . When the optical setup is diffraction limited, the smallest achievable  $\sigma_R$ 's can take values down to one micrometer (Clement *et al.*, 2006), which is comparable to typical healing lengths  $\xi$  in dilute gases. Reaching this limit is important, since smooth disorder potentials with  $\sigma_R \gg \xi$  are not suitable to study Anderson localization in expanding BEC's. Indeed, the typical range of momenta after expansion reaches up to  $k_{\max} \approx 1/\xi$ . For  $\sigma_R \gg \xi$  therefore, the spectral range of the disorder is much smaller than the momenta of the matter waves. Even speckle patterns with long correlation lengths, however, can lead to a strong suppression of the axial expansion of an elongated BEC. This was observed experimentally by Clément *et al.* (2005); Fort *et al.* (2005); Schulte *et al.* (2005). The effect is not due to Anderson localization, however. Instead, it is caused by classical total reflection, because during expansion the density and chemical potential of the gas decrease. Eventually, therefore, the matter waves have energies below the typical disorder potentials and the gas undergoes fragmentation. A suggestion to realize Anderson localization of non-interacting particles in 1D has recently been made by Sanchez-Palencia *et al.* (2007). It is based on a 1D BEC, which after expansion is transformed into a distribution of free matter waves with momenta up to  $1/\xi$ . For short range disorder with  $\sigma_R < \xi$ , these waves will all be localized, even for rather weak disorder.

A rather direct approach to study the interplay between disorder and strong interactions has recently been followed by the group in Florence, using bi-chromatic optical lattices (Fallani *et al.*, 2007). Such lattices provide a pseudo-random potential if the lattice periods are incommensurate. Adding the incommensurate lattice in the superfluid state, a strong suppression of the interference pattern is found. Starting from a MI-phase, the sharp excitation spectrum is smeared out with increasing amplitude of the incommensurate lattice. Both observations are consistent with the presence of a Bose glass phase for a strong incommensurate lattice potential (Fallani *et al.*, 2007). A different method to realize short range disorder in cold gases has been suggested by Gavish and Castin (2005). It employs a two species mixture of atoms in an optical lattice. Due to a finite inter-species scattering length, the component which is free to move around in the lattice experiences an on-site random potential of the type (152), provided the atoms of the different species or spin state are frozen at random sites. The interplay between disorder and interactions is, of course, also an intriguing problem for fermions. A quantitative phase diagram for short range disorder plus repulsive interactions has been determined in this case by Byczuk *et al.* (2005). So far, however, no experiments have been done in this direction.

### C. Nonequilibrium Dynamics

A unique feature of many-body physics with cold atoms is the possibility to modify both the interactions and the external potentials dynamically. In the context of the SF-MI transition, this has been discussed already in section IV.C. Below, we give a brief outline of some of the recent developments in this area.

The basic question about the efficiency of collisions in establishing a new equilibrium from an initial out-of-equilibrium state has been addressed by Kinoshita *et al.* (2006) for Bose gases in one dimension. An array of several thousand 1D tubes created by a strong 2D optical lattice (see section V.B) was subject to a pulsed optical lattice along the axial direction. The zero momentum state is thus depleted and essentially all the atoms are transferred to momenta  $\pm 2\hbar k$ , where  $k$  is the wavevector of the pulsed axial optical lattice. The two wavepackets in each tube separate and then recollide again after a time  $\pi/\omega_0$ , which is half the oscillation period in the harmonic axial trap with frequency  $\omega_0$ . The associated collision energy  $(2\hbar k)^2/M$  was around  $0.45 \hbar\omega_\perp$ , i.e. much smaller than the minimum energy  $2\hbar\omega_\perp$  necessary to excite higher transverse modes. The system thus remains strictly 1D during its time evolution. It was found, that even after several hundred oscillation periods, the initial non-equilibrium momentum distribution was preserved.

This striking observation raises a number of questions. Specifically, is the absence of a broadening of the momentum distribution connected with the integrability of the

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low density or strong disorder limit of the Bose glass.

1D Bose gas? More generally, one can ask, whether and under which conditions, the unitary time-evolution of a non-equilibrium initial state in a strongly interacting but non-integrable quantum system will evolve into a state, in which at least one- and two-particle correlations are stationary and the information about the precise initial conditions is hidden in some - in practice unobservable - high order correlations. The non-equilibrium dynamics for the integrable case of bosons in 1D has recently been addressed by Rigol *et al.* (2007). Taking as a model a Tonks-Girardeau gas on a lattice, they have shown numerically that the momentum distribution at large times is well described by that of an 'equilibrium' state with a density matrix of the form  $\hat{\rho} \sim \exp(-\sum_m \lambda_m \hat{A}_m)$ . Here, the  $\hat{A}_m$  denote the full set of conserved quantities, which are known explicitly for the Tonks-Girardeau gas because it is equivalent to free fermions. The Lagrange multipliers  $\lambda_m$  are fixed by the initial conditions of given expectation values  $\langle \hat{A}_m \rangle$  at  $t = 0$ . This is the standard procedure in statistical physics, where equilibrium is described microscopically as a state of maximum entropy consistent with the given 'macroscopic' data  $\langle \hat{A}_m \rangle$  (Balian, 1991). In particular, an initially double-peaked momentum distribution is preserved in the stationary state described by the maximum entropy density operator  $\hat{\rho}$  (Rigol *et al.*, 2007). The hypothesis, that the absence of momentum relaxation is related to the integrability of the 1D Bose gas, could be tested by going to higher momenta  $k$ , where the 1D scattering amplitude is no longer given by the low energy form of Eq. (85). For such a case, the pseudopotential approximation breaks down and three body collisions or longer ranged interactions can become relevant. For the extreme case of two free 3D colliding BECs, equilibration has indeed been observed to set in after a few collisions (Kinoshita *et al.*, 2006).

In non-integrable systems like the 1D Bose-Hubbard model, where no conserved quantities exist beyond the energy, the standard reasoning of statistical physics gives rise to a micro-canonical density operator, whose equivalent 'temperature' is set by the energy of the initial state<sup>28</sup>. Note that the micro-state  $\exp(-i\hat{H}t/\hbar)|\psi(0)\rangle$ , which evolves from the initial state by the unitary time evolution of a closed system, remains time dependent. Its statistical (von Neumann) entropy vanishes. By contrast, the microcanonical density operator describes a stationary situation, with a nonzero *thermodynamic* entropy. It is determined by the number of energy eigenstates near the exact initial energy which are accessible in an energy range much smaller than the microscopic scale set by the one- or two two-body terms in the Hamiltonian but much larger than the inverse of the recurrence time. For

1D problems, the hypothesis that simple macroscopic observables are eventually well described by such a stationary density operator can be tested quantitatively by using the adaptive time dependent density matrix renormalization group (Daley *et al.*, 2004; White and Feiguin, 2004). In the case of the 1D BHM, an effectively 'thermal' stationary state indeed arises for long time dynamics after a quench from the SF to the MI (Kollath *et al.*, 2007) (see section IV.C). Apparently, however, the description by a stationary 'thermal' density operator is valid only for not too large values of the final repulsion  $U_f$ .

A different aspect of the non-equilibrium dynamics of 1D Bose gases was studied by Hofferberth *et al.* (2007). A single 1D condensate formed in a magnetic micro trap on an atom chip (Folman *et al.*, 2002) is split into two parts by applying rf-potentials (Schumm *et al.*, 2005). The splitting process is done in a phase coherent manner, such that at time  $t = 0$ , the two condensates have a vanishing relative phase. They are kept in a double well potential for a time  $t$  and then released from the trap. As discussed in section III.C, the resulting interference pattern provides information about the statistics of the interference amplitude. In the non-equilibrium situation discussed here, the relevant observable analogous to Eq. (62) is the operator  $\exp(i\hat{\theta}(z, t))$  integrated along the axial  $z$ -direction of the two condensates. Here  $\hat{\theta}(z, t)$  is the time-dependent phase *difference* between the two independently fluctuating condensates. Using the quantum hydrodynamic Hamiltonian Eq. (91), it has been shown by Burkov *et al.* (2007) that the expectation value of this operator decays sub-exponentially for large times  $t \gg \hbar/k_B T$  where the phase fluctuations can be described classically. This behavior is in rather good agreement with experiments (Hofferberth *et al.*, 2007). In particular, it allows to determine the temperature of the 1D gas very precisely.

The dynamics of the superfluid gap parameter in attractive Fermi gases after a sudden change of the coupling constant has been investigated by Barankov and Levitov (2006); Barankov *et al.* (2004) and Yuzbashyan *et al.* (2006) using the exactly integrable BCS Hamiltonian. Such changes in the coupling constant are experimentally feasible by simply changing the magnetic field in a Feshbach resonance. Depending on the initial conditions, different regimes have been found where the gap parameter may oscillate without damping, approaches an 'equilibrium' value different from that associated with the coupling constant after the quench, or decays to zero monotonically if the coupling constant is reduced to very small values.

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<sup>28</sup> In the semiclassical limit, this 'eigenstate thermalization hypothesis' can be derived under relatively weak assumptions, see Srednicki (1994).

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## X. APPENDIX: BEC AND SUPERFLUIDITY

Following the early suggestion of London that superfluidity (SF) in  $^4\text{He}$  has its origin in Bose-Einstein-Condensation (BEC), the relation between both phenomena has been a subject of considerable debate. In the following, we outline their basic definitions and show that SF is the more general phenomenon, which is both necessary and sufficient for the existence of either standard BEC or of quasi-condensates in low dimensional systems. For a detailed discussion of the connections between BEC and superfluidity see also the book by Leggett (2006).

The definition of BEC in an interacting system of bosons is based on the properties of the one-particle density matrix  $\hat{\rho}_1$ , which is conveniently defined by its matrix elements

$$G^{(1)}(\mathbf{x}, \mathbf{x}') = \langle \mathbf{x}' | \hat{\rho}_1 | \mathbf{x} \rangle = \langle \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}') \rangle \quad (153)$$

in position space.<sup>29</sup> As a hermitean operator with  $\text{Tr } \hat{\rho}_1 = \int n(\mathbf{x}) = N$ ,  $\hat{\rho}_1$  has a complete set  $|n\rangle$  of eigenstates with positive eigenvalues  $\lambda_n^{(1)}$  which sum up to  $N$ . As realized by Penrose and Onsager (1956), the criterion for BEC is that there is precisely one eigenvalue  $\lambda_0^{(1)} = N_0$  of order  $N$  while all other eigenvalues are non-extensive<sup>30</sup>. Of course, the separation between extensive and non-extensive eigenvalues  $\lambda_n^{(1)}$  is well defined only in the thermodynamic limit  $N \rightarrow \infty$ . In practice, however, the distinction between the BEC and the normal phase above  $T_c$  is rather sharp even for the typical particle numbers  $N \approx 10^4 - 10^7$  of cold atoms in a trap. This is true in spite of the fact that the non-extensive eigenvalues are still rather large (see below). The macroscopic eigenvalue  $N_0$  determines the number of particles in the condensate. In terms of the single particle eigenfunctions  $\varphi_n(\mathbf{x}) = \langle \mathbf{x} | n \rangle$  associated with the eigenstates  $|n\rangle$  of  $\hat{\rho}_1$ , the existence of a condensate is equivalent to a macroscopic occupation of a single state created by the operator  $\hat{b}_0^\dagger = \int \varphi_0(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x})$ . In a translation invariant situation, the eigenfunctions are plane waves. The eigenvalues  $\lambda_k^{(1)}$

are then just the occupation numbers  $\langle \hat{b}_k^\dagger \hat{b}_k \rangle$  in momentum space. In the thermodynamic limit  $N, V \rightarrow \infty$  at constant density  $n$ , the one particle density matrix

$$\langle \mathbf{x}' | \hat{\rho}_1 | \mathbf{x} \rangle = n_0 + \int_k \tilde{n}(\mathbf{k}) e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \rightarrow n_0 \quad (154)$$

thus approaches a finite value  $n_0 = \lim N_0/V$  as  $r = |\mathbf{x} - \mathbf{x}'| \rightarrow \infty$ . This property is called off-diagonal long range order (ODLRO). Physically, the existence of ODLRO implies that the states in which a boson is removed from an  $N$ -particle system at positions  $\mathbf{x}$  and  $\mathbf{x}'$  have a finite overlap even in the limit when the separation between the two points is taken to infinity<sup>31</sup>. For cold gases, the presence of ODLRO below  $T_c$ , at least over a scale of order  $\mu\text{m}$ , was observed experimentally by measuring the decay of the interference contrast between atomic beams outcoupled from two points at a distance  $r$  from a BEC in a trap (Bloch *et al.*, 2000). The momentum distribution  $\tilde{n}(\mathbf{k})$  of non-condensate particles is singular at small momenta. In the limit  $\mathbf{k} \rightarrow 0$ , it behaves like  $\tilde{n}(\mathbf{k}) \sim c/k$  at zero and  $\tilde{n}(\mathbf{k}) \sim T/k^2$  at finite temperature, respectively (Lifshitz and Pitaevskii, 1980). Since  $k_{min} \sim N^{-1/3}$  in a finite system, the lowest non-extensive eigenvalues of the one particle density matrix are still very large, scaling like  $\lambda_n^{(1)} \sim N^{1/3}$  at zero and  $\lambda_n^{(1)} \sim N^{2/3}$  at finite temperature.

The definition of BEC via the existence of ODLRO (154) is closely related with Feynman's intuitive picture of Bose-Einstein condensation as a transition, below which bosons are involved in existse cycles of infinite size (Feynman, 1953). In fact, a precise connection exists with the superfluid density defined in Eq. (156) below, which can be expressed in terms of the square of the winding number in a Feynman path representation of the equilibrium density matrix (Pollock and Ceperly, 1987). As discussed by Ceperley (1995) and Holzmann and Krauth (1999), the connection between the *condensate* density and infinite cycles is also suggestive in terms of the path integral representation of the one-particle density matrix. It is difficult, however, to put this on a rigorous footing beyond the simple case of an ideal Bose gas (Ueltschi, 2006).

While a microscopic definition of BEC is straightforward at least in principle, the notion of superfluidity is more subtle. On a phenomenological level, the basic properties of superfluids may be explained by introducing a complex order parameter  $\psi(\mathbf{x}) = |\psi(\mathbf{x})| \exp i\phi(\mathbf{x})$ , whose magnitude squared gives the superfluid density  $n_s$ , while the phase  $\phi(\mathbf{x})$  determines the superfluid velocity via  $\mathbf{v}_s = \hbar/M \cdot \nabla\phi(\mathbf{x})$  (Pitaevskii and Stringari, 2003). The latter equation immediately implies that superfluid flow is irrotational and that the circulation  $\Gamma = \oint \mathbf{v}_s ds$

<sup>29</sup> In an inhomogeneous situation, it is sometimes convenient to define a reduced one-particle density matrix  $g^{(1)}$  by  $G^{(1)}(\mathbf{x}, \mathbf{x}') = \sqrt{n(\mathbf{x})n(\mathbf{x}')} g^{(1)}(\mathbf{x}, \mathbf{x}')$ .

<sup>30</sup> In principle it is also possible that more than one eigenvalue is extensive. This leads to so-called fragmented BEC's which may appear e.g. in multicomponent spinor condensates as shown by Ho (1998)

<sup>31</sup> For a discussion of the topological properties of the many-body wave function which are required to give ODLRO see Leggett (1973)



is quantized in an integer number of circulation quanta  $h/M$ . Note, that these conclusions require a finite value  $n_s \neq 0$ , yet are completely independent of its magnitude. In order to connect this phenomenological picture of SF with the microscopic definition of BEC, the most obvious assumption is to identify the order parameter  $\psi(\mathbf{x})$  with the eigenfunction  $\varphi_0(\mathbf{x})$  of  $\hat{\rho}_1$  associated with the single extensive eigenvalue, usually choosing a normalization such that  $\psi(\mathbf{x}) = \sqrt{N_0}\varphi_0(\mathbf{x})$ . Within this assumption, BEC and SF appear as essentially identical phenomena. This simple identification is not valid, however, beyond a Gross-Pitaevskii description or for low dimensional systems. The basic idea of how to define superfluidity in quite general terms goes back to Leggett (1973) and Fisher *et al.* (1973). It is based on considering the sensitivity of the many-body wave function with respect to a change in the boundary condition (bc). Specifically, consider  $N$  bosons in a volume  $V = L^3$  and choose boundary conditions where the many-body wave function

$$\Psi_{\Theta}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_i, \dots, \mathbf{x}_N) \quad (155)$$

is multiplied by a pure phase factor  $e^{i\Theta}$  if  $\mathbf{x}_i \rightarrow \mathbf{x}_i + L\mathbf{e}$  for all  $i = 1 \dots N$  and with  $\mathbf{e}$  a unit vector in one of the directions<sup>32</sup>. The dependence of the many-body energy eigenvalues  $E_n(\Theta)$  on  $\Theta$  leads to a phase dependent equilibrium free energy  $F(\Theta)$ . The difference  $\Delta F(\Theta) = F(\Theta) - F(\Theta = 0)$  is thus a measure for the sensitivity of the many-body system in a thermal equilibrium state to a change in the bc's. Since the eigenstates of a time reversal invariant Hamiltonian can always be chosen real, the energies  $E_n(\Theta)$  and the resulting  $\Delta F(\Theta)$  must be even in  $\Theta$ . For small deviations  $\Theta \ll 1$  from periodic bc's, the expected leading behavior is therefore quadratic. The superfluid density  $n_s(T)$  is then defined by the free energy difference per volume

$$\frac{\Delta F(\Theta)}{V} = \frac{\hbar^2}{2M} n_s(T) \cdot \left(\frac{\Theta}{L}\right)^2 + \dots \quad (156)$$

to leading order in  $\Theta$ . In a superfluid, therefore, a small change in the bc's leads to a change in the free energy per volume which scales like  $\gamma/2 (\Theta/L)^2$ . The associated proportionality constant  $\gamma = \hbar^2 n_s/M$  is called the helicity modulus. Clearly, the definition (156) for superfluidity, which is based only on *equilibrium* properties and also applies to finite systems, is quite different from that for the existence of BEC. Yet, it turns out, that the two phenomena are intimately connected in the sense that a finite superfluid density is both necessary and sufficient for either standard BEC or the existence of quasicondensates in lower dimensions.

Following Leggett (1973), the physical meaning of the phase  $\Theta$  and the associated definition of  $n_s$  can be understood by considering bosons (gas or liquid) in a superfluid state, which are enclosed between two concentric cylinders with almost equal radii  $R$ , co-rotating with an angular frequency  $\mathbf{\Omega} = \Omega \mathbf{e}_z$ <sup>33</sup>. In the rotating frame, the problem is stationary, however it acquires an effective gauge potential  $\mathbf{A} = M\mathbf{\Omega} \wedge \mathbf{x}$ , as shown in Eq. (113). Formally,  $\mathbf{A}$  can be eliminated by a gauge transformation at the expense of a many-body wavefunction, which is no longer single-valued. It changes by a factor  $e^{i\Theta}$  under changes  $\theta_i \rightarrow \theta_i + 2\pi$  of the angular coordinate of each particle  $i$ , precisely as in (155), where  $\Theta = -2\pi MR^2 \Omega / \hbar$  is linear in the angular frequency. The presence of a phase dependent free energy increase  $\Delta F(\Theta)$  of the superfluid in the rotating frame implies, that the equilibrium state in this frame carries a nonzero (kinematic) angular momentum  $L'_z = -\partial \Delta F(\Theta) / \partial \Omega = -(n_s/n) \cdot L_z^{(0)}$ , where  $L_z^{(0)} = I_{cl} \Omega$  is the rigid body angular momentum in the lab frame and  $I_{cl} = NMR^2$  the classical moment of inertia. A fraction  $n_s/n$  of the superfluid thus appears to stay at rest in the lab frame for small angular frequencies  $\Omega \ll \hbar/MR^2$ , where  $\Theta \ll 1$ . As a result, the apparent moment of inertia is smaller than that of classical rigid body like rotation. Superfluidity, as defined by Eq. (156), thus implies the appearance of non-classical rotational inertia (NCRI) (Leggett, 1973). In the context of cold gases, this phenomenon has been observed experimentally by Madison *et al.* (2000). They have shown that a trapped gas in the presence of a small, non-symmetric perturbation remains at zero angular momentum (i.e. no vortex enters), for sufficiently small angular frequencies.

The example of a rotating system shows, that a finite value of  $\Theta$  is associated with a non-vanishing current in the system. Indeed, a finite superfluid density in the sense of Eq. (156) implies the existence of long range current-current correlations, which is the original microscopic definition of  $n_s$  by Hohenberg and Martin (1965). To describe states of a superfluid with finite currents, it is useful to introduce a smoothly varying local phase  $\phi(\mathbf{x})$  on scales much larger than the interparticle spacing, which is connected with the total phase difference between two arbitrary points by  $\Theta = \int ds \cdot \nabla \phi(\mathbf{x})$ . This local phase variable is now precisely the phase of the coarse grained complex order parameter  $\psi(\mathbf{x})$  introduced by Landau. This identification becomes evident by noting that a non-vanishing phase gradient gives rise to a finite superfluid velocity  $\mathbf{v}_s = \hbar/M \cdot \nabla \phi(\mathbf{x})$ . The free energy increase

$$\Delta F(\Theta) = \frac{\gamma(T)}{2} \int d^3x (\nabla \phi(\mathbf{x}))^2 \quad (157)$$

associated with a change in the bc's is therefore just the

<sup>32</sup> For simplicity, we assume isotropy in space. More generally the sensitivity with respect to changes in the bc's may depend on the direction, in which case the superfluid density becomes a tensor.

<sup>33</sup> The walls are assumed to violate perfect cylindrical symmetry, to allow for the transfer of angular momentum to the fluid!

kinetic energy  $M/2 \int n_s(\mathbf{v}_s)^2$  of superfluid flow with velocity  $\mathbf{v}_s$  and density  $n_s$ . An immediate consequence of Eq. (157) is the quite general form of the excitation spectrum of superfluids at low energies. Indeed, considering the fact that phase and particle number are conjugate variables, the operator  $\delta\hat{n}(\mathbf{x})$  for small fluctuations of the density obeys the canonical commutation relation  $[\delta\hat{n}(\mathbf{x}), \hat{\phi}(\mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{x}')$  with the quantized phase operator  $\hat{\phi}(\mathbf{x})$ . For any Bose gas (or liquid) with a finite compressibility  $\kappa = \partial n/\partial\mu \neq 0$  at zero temperature<sup>34</sup>, the energy of small density fluctuations is  $\int (\delta n(\mathbf{x}))^2/2\kappa$ . Combining this with Eq. (157), the effective Hamiltonian for the low lying excitations of an arbitrary superfluid is of the quantum hydrodynamic form

$$\hat{H} = \int d^3x \left[ \frac{\hbar^2 n_s}{2M} (\nabla \hat{\phi}(\mathbf{x}))^2 + \frac{1}{2\kappa} (\delta\hat{n}(\mathbf{x}))^2 \right]. \quad (158)$$

This Hamiltonian describes harmonic phonons with a linear spectrum  $\omega = cq$  and a velocity, which is determined by  $Mc^2 = n_s/\kappa$ . In a translation invariant situation, where  $n_s(T=0) \equiv n$  (see below), this velocity coincides with that of a standard (first sound) compression mode in a gas or liquid. This coincidence is misleading, however, since Eq. (158) describes true elementary excitations and not a hydrodynamic, collision dominated mode. States with a single phonon are thus exact low-lying eigenstates of the strongly interacting many-body system, whose ground state exhibits the property (156). As the Lieb-Liniger solution of a 1D Bose gas shows, this mode exists even in the absence of BEC. It only requires a finite value of the superfluid density, as specified by Eq. (156). A mode of this type will therefore be present in the SF phase of bosons in an optical lattice in *any* dimension and also in the presence of a finite disorder, as long as the superfluid density is non-vanishing.

The connection between SF and BEC now follows from the macroscopic representation (Popov, 1983)

$$\hat{\psi}(\mathbf{x}) = \exp(i\hat{\phi}(\mathbf{x})) \left[ n + \delta\hat{n}(\mathbf{x}) \right]^{1/2} \approx \sqrt{\tilde{n}_0} \exp i\hat{\phi}(\mathbf{x}) \quad (159)$$

of the Bose field in terms of density and phase operators. The parameter  $\tilde{n}_0$  is a quasi-condensate density. Its existence relies on the assumption that there is a broad range  $\xi \ll |\mathbf{x}| \ll \ell_\phi$  of intermediate distances, where the one-particle density matrix is equal to a finite value  $\tilde{n}_0 < n$ ,<sup>35</sup> beyond which only phase fluctuations contribute. Using the harmonic Hamiltonian Eq. (158), the asymptotic

decay of  $G^{(1)}(\mathbf{x}) = \tilde{n}_0 \exp(-\delta\phi^2(\mathbf{x})/2)$  at large separations is thus determined by the mean square fluctuations

$$\delta\phi^2(\mathbf{x}) = \frac{1}{\hbar\kappa} \int \frac{d^d q}{(2\pi)^d} \frac{1 - \cos \mathbf{q}\mathbf{x}}{cq} \coth \frac{\hbar cq}{2k_B T} \quad (160)$$

of the phase difference between points separated by  $\mathbf{x}$ . The representation Eq. (159) and the notion of a quasi-condensate require the existence of a finite superfluid density at least at  $T=0$ , but *not* that of BEC. From Eq. (160), it is straightforward to see, that plain BEC in the sense of a non-vanishing condensate density  $n_0 = \tilde{n}_0 \exp(-\delta\phi^2(\infty)/2)$  at non-zero temperatures only exists in 3D. In two dimensions, the logarithmic divergence of  $\delta\phi^2(\mathbf{x}) \rightarrow 2\eta \ln |\mathbf{x}|$  at large distances leads to an algebraic decay  $g^{(1)}(\mathbf{x}) \sim |\mathbf{x}|^{-\eta}$ , consistent with the Mermin-Wagner-Hohenberg theorem. Since  $\kappa c^2 = n_s(T)/M$ , the exponent  $\eta(T) = (n_s(T)\lambda_T^2)^{-1}$  is related to the exact value of the superfluid density as pointed out in Eq. (100). A similar behavior, due to *quantum* rather than thermal phase fluctuations, applies in 1D at zero temperature, see section V.B.

The arguments above show that SF in the sense of Eq. (156) *plus* the assumption of a finite compressibility  $\kappa$  are sufficient conditions for either plain BEC in 3D or the existence of quasi-condensates in 2D at finite and in 1D at zero temperature<sup>36</sup>. For the reverse question, whether SF is also *necessary* for the existence of BEC, the answer is again yes, in the general sense that BEC is replaced by quasi-condensates in low dimensions. Indeed, for a translation invariant system, Leggett (1998) has shown that the existence of ODLRO in the ground state with an arbitrary small condensate fraction  $n_0/n$  implies perfect superfluidity  $n_s(T=0) \equiv n$  at zero temperature. Note that this includes the case of 2D gases, where only a quasi-condensate survives at non-zero temperature. In 1D, the presence of quasi long range order in  $g^{(1)} \sim |\mathbf{x}|^{-1/2K}$  implies a finite value of  $n_s$  by the relation  $K = \pi\hbar\kappa c$  for bosonic Luttinger liquids. In more general terms, the fact that ODLRO in a BEC implies superfluidity follows from the Nambu-Goldstone theorem. It states that the appearance of (quasi) long range order in the phase implies the existence of an elementary excitation, whose energy vanishes in the limit of zero momentum. As emphasized e.g. by Weinberg (1986), the order parameter phase  $\phi(\mathbf{x})$  introduced above is just the Nambu-Goldstone field associated with the broken  $U(1)$  'gauge' symmetry (this notion has to be treated with care, see e.g. (Wen, 2004)). In the present context, a broken gauge symmetry is just the statement of Eq. (156), that the free energy contains a term quadratic in a phase twist imposed at the boundaries of the system). For systems

<sup>34</sup> This condition rules out the singular case of an ideal Bose gas, where  $\kappa = \infty$  and thus  $c = 0$ . The ideal gas is therefore *not* a SF, even though it has a finite superfluid density which coincides with the condensate density (Fisher *et al.*, 1973).

<sup>35</sup> The short range decay from  $G^{(1)}(\mathbf{0}) = n$  to  $\tilde{n}_0$  is due to fluctuations at scales smaller than  $\xi$  which require a microscopic calculation. The integral (160) is therefore cutoff at  $q_{\max} \approx 1/\xi$ .

<sup>36</sup> In a trap, a quasicondensate in 1D may exist even at finite temperature, provided the cloud size is smaller than the phase coherence length, see section V.B

with a finite compressibility, the dynamics of the Nambu-Goldstone mode is described by the quantum hydrodynamic Hamiltonian Eq. (158). The definition (156) of superfluidity and the resulting generic quantum hydrodynamic Hamiltonian (158) which are connected to long (or quasi long) range order via Eqs. (159, 160), are thus completely general and are not tied e.g. to translation invariant systems. Despite their different definitions based on long range *current* or long range *phase* correlations, the two phenomena SF and BEC (in its generalized sense) are therefore indeed just two sides of the same coin.

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