Christoph Schiller

MOTION MOUNTAIN
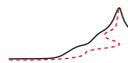
THE ADVENTURE OF PHYSICS – VOL.IV

THE QUANTUM OF CHANGE

Christoph Schiller

Motion Mountain

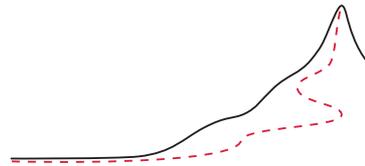The Adventure of Physics
Volume IV

The Quantum of Change

To Britta, Esther and Justus Aaron

τῷ ἐμοὶ δαίμονι

Die Menschen stärken, die Sachen klären.
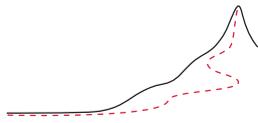
PREFACE

This book series is for anybody who is curious about motion in nature. How do things, people, animals, images and empty space move? The answer leads to many adventures, and this volume presents those due to the discovery that there is a smallest possible change value in nature. This smallest change value, the quantum of action, leads to what is called *quantum physics*. In the structure of modern physics, shown in Figure 1, quantum physics covers four of eight points. The present volume introduces the foundations of quantum theory, deduces the structure of atoms and explains the appearance of probabilities, wave functions and colours.

The present introduction to quantum physics arose from a threefold aim I have pursued since 1990: to present the basics of quantum motion in a way that is simple, up to date and captivating.

In order to be *simple*, the text focuses on concepts, while keeping mathematics to the necessary minimum. Understanding the concepts of physics is given precedence over using formulae in calculations. The whole text is within the reach of an undergraduate.

In order to be *up to date*, the text is enriched by the many gems – both theoretical and empirical – that are scattered throughout the scientific literature.

In order to be *captivating*, the text tries to startle the reader as much as possible. Reading a book on general physics should be like going to a magic show. We watch, we are astonished, we do not believe our eyes, we think, and finally we understand the trick. When we look at nature, we often have the same experience. Indeed, every page presents at least one surprise or provocation for the reader to think about. Numerous interesting challenges are proposed.

The motto of the text, *die Menschen stärken, die Sachen klären*, a famous statement on pedagogy, translates as: 'To fortify people, to clarify things.' Clarifying things – and adhering only to the truth – requires courage, as changing the habits of thought produces fear, often hidden by anger. But by overcoming our fears we grow in strength. And we experience intense and beautiful emotions. All great adventures in life allow this, and exploring motion is one of them. Enjoy it.

Christoph Schiller

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* 'First move, then teach.' In modern languages, the mentioned type of *moving* (the heart) is called *motivating*; both terms go back to the same Latin root.

Complete, unified description of motion
Adventures: describing precisely *all motion*, understanding
the origin of colours, space -time and particles, enjoying
extreme thinking, calculating masses and couplings,
catching a further, tiny glimpse of bliss (vol. VI).

PHYSICS:
Describing motion with precision,
i.e., using the least action principle.

An arrow indicates an
increase in precision by
adding a **motion limit.**

General relativity
Adventures: the
night sky, measu-
ring curved and
wobbling space,
exploring black
holes and the
*universe, space
and time* (vol. II).

Quantum theory
with classical gravity
Adventures: bouncing
neutrons, under-
standing tree
growth (vol. V).

Quantum field theory
(the 'standard model')
Adventures: building
accelerators, under-
standing quarks, stars,
bombs and the basis of
life, matter & radiation
(vol. V).

Classical gravity
Adventures:
climbing, skiing,
space travel,
the wonders of
astronomy and
geology (vol. I).

Special relativity
Adventures: light,
magnetism, length
contraction, time
dilation and
$E_0 = mc^2$
(vol. II).

Quantum theory
Adventures: biology,
birth, love, death,
chemistry, evolution,
enjoying colours, art,
paradoxes, medicine
and high-tech business
(vol. IV and vol. V).

c
limits
fast
motion

G
limits
uniform
motion

h, e, k
limit
tiny
motion

Galilean physics, heat and electricity
The world of *everyday motion*: human scale, slow and weak.
Adventures: sport, music, sailing, cooking, describing
beauty and understanding its origin (vol. I);
using electricity, light and computers,
understanding the brain and people (vol. III).

FIGURE 1 A complete map of physics, the science of motion, as first proposed by Matvei Bronshtein
(b. 1907 Vinnytsia, d. 1938 Leningrad). The Bronshtein cube starts at the bottom with everyday motion,
and shows the connections to the fields of modern physics. Each connection increases the precision of
the description and is due to a limit to motion that is taken into account. The limits are given for
uniform motion by the gravitational constant G, for fast motion by the speed of light c, and for tiny
motion by the Planck constant h, the elementary charge e and the Boltzmann constant k.

Using this book

Marginal notes refer to bibliographic references, to other pages or to challenge solutions. In the colour edition, marginal notes, pointers to footnotes and links to websites are typeset in green. Over time, links on the internet tend to disappear. Most links can be recovered via www.archive.org, which keeps a copy of old internet pages. In the free pdf edition of this book, available at www.motionmountain.net, all green pointers and links are clickable. The pdf edition also contains all films; they can be watched directly in Adobe Reader.

Solutions and hints for *challenges* are given in the appendix. Challenges are classified as easy (e), standard student level (s), difficult (d) and research level (r). Challenges for which no solution has yet been included in the book are marked (ny).

Advice for learners

Learning allows us to discover what kind of person we can be. Learning widens knowledge, improves intelligence and provides a sense of achievement. Therefore, learning from a book, especially one about nature, should be efficient and enjoyable. Avoid bad learning methods like the plague! Do not use a marker, a pen or a pencil to highlight or underline text on paper. It is a waste of time, provides false comfort and makes the text unreadable. And do not learn from a screen. In particular, never, ever, learn from the internet, from videos, from games or from a smartphone. Most of the internet, almost all videos and all games are poisons and drugs for the brain. Smartphones are dispensers of drugs that make people addicted and prevent learning. Nobody putting marks on paper or looking at a screen is learning efficiently or is enjoying doing so.

In my experience as a pupil and teacher, one learning method never failed to transform unsuccessful pupils into successful ones: if you read a text for study, summarize every section you read, *in your own words and images, aloud*. If you are unable to do so, read the section again. Repeat this until you can clearly summarize what you read in your own words and images, aloud. And *enjoy* the telling aloud! You can do this alone or with friends, in a room or while walking. If you do this with everything you read, you will reduce your learning and reading time significantly; you will enjoy learning from good texts much more and hate bad texts much less. Masters of the method can use it even while listening to a lecture, in a low voice, thus avoiding to ever take notes.

Advice for teachers

A teacher likes pupils and likes to lead them into exploring the field he or she chose. His or her enthusiasm is the key to job satisfaction. If you are a teacher, before the start of a lesson, picture, feel and tell yourself how you enjoy the topic of the lesson; then picture, feel and tell yourself how you will lead each of your pupils into enjoying that topic as much as you do. Do this exercise consciously, every day. You will minimize trouble in your class and maximize your teaching success.

This book is not written with exams in mind; it is written to make teachers and students *understand* and *enjoy* physics, the science of motion.

Feedback

The latest pdf edition of this text is and will remain free to download from the internet. I would be delighted to receive an email from you at fb@motionmountain.net, especially on the following issues:

— What was unclear and should be improved?
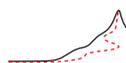— What story, topic, riddle, picture or film did you miss?

Also help on the specific points listed on the www.motionmountain.net/help.html web page is welcome. All feedback will be used to improve the next edition. You are welcome to send feedback by mail or by sending in a pdf with added yellow notes, to provide illustrations or photographs, or to contribute to the errata wiki on the website. If you would like to translate a chapter of the book in your language, please let me know.
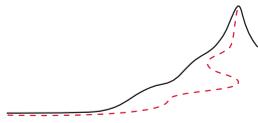
On behalf of all readers, thank you in advance for your input. For a particularly useful contribution you will be mentioned – if you want – in the acknowledgements, receive a reward, or both.

Support

Your donation to the charitable, tax-exempt non-profit organisation that produces, translates and publishes this book series is welcome. For details, see the web page www.motionmountain.net/donation.html. The German tax office checks the proper use of your donation. If you want, your name will be included in the sponsor list. Thank you in advance for your help, on behalf of all readers across the world.

The paper edition of this book is available, either in colour or in black and white, from www.amazon.com, in English and in certain other languages. And now, enjoy the reading.

CONTENTS

The Quantum of Change

In our quest to understand how things move,
we discover that there is a smallest change value in nature,
implying that motion is fuzzy,
that boxes are never tight,
that matter is composed of elementary units,
and that light and interactions are streams of particles.
The smallest change value explains why antimatter exists,
why particles are unlike gloves,
why copying machines do not exist,
why probabilities are reasonable,
and how all colours in nature are formed.
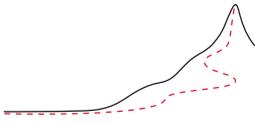
MINIMUM ACTION – QUANTUM THEORY FOR POETS

> " Natura [in operationibus suis] non facit saltus.**
>
> 15th century "

Climbing Motion Mountain up to this point, we completed three legs. We came across Galileo's mechanics (the description of motion for kids), then continued with Einstein's relativity (the description of motion for science-fiction enthusiasts), and finally explored Maxwell's electrodynamics (the description of motion for business people). These three classical descriptions of motion are impressive, beautiful and useful. However, they have a small problem: they are wrong. The reason is simple: none of them describes *life*.

Whenever we observe a flower or a butterfly, such as those of Figure 2, we enjoy the bright colours, the motion, the wild smell, the soft and delicate shape or the fine details of their symmetries. However, we know:

> ▷ Classical physics cannot explain any characteristic length or time scale observed in nature.

Now, flowers and animals – but also many non-living systems – have characteristic sizes, size ranges and proportions; and they have characteristic rhythms. And indeed, classical physics cannot explain their origin, because

> ▷ The classical constants of nature – the gravitational constant G, the ideal gas constant R, the speed of light c, the vacuum permittivity ε_0 and the vacuum permeability μ_0 – do not allow defining length or time units: They cannot be combined to yield a length or time value. And they cannot be be used to build a meter bar.

In fact, the classical constants do not even allow us to measure speed or force values, even though these measurements are fractions of c and c^4/G; because in order to measure fractions, we need to define fractions first; however, defining fractions also requires length or time scales and units, which classical physics does not allow.

Without measurements, there are also no emotions! Indeed, our emotions are triggered by our senses. And all the impressions and all the information that our senses

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Ref. 1   ** 'Nature [in its workings] makes no jumps.'

FIGURE 2 Examples of quantum machines (© Linda de Volder).

provide us are – among others – measurements. Since classical physics does not provide measurement scales, we know:

▷ Classical physics does not allow understanding senses or emotions.

The reason for all these limitations is the following connection:

▷ Classical physics alone cannot be used to build any measurement device.

Every sense contains measurement devices. And every measurement device, like any pattern or rhythm, needs an internal scale, or, more generally, an internal measurement unit. Because classical physics does not provide any scale, classical physics does not explain how measurement devices work, not how senses work, and not how emotions appear.

To understand emotions and life, we need to go *beyond* classical physics. Take any example of a pleasant situation,* such as a beautiful evening sky, a waterfall, a happy child or a caress. Classical physics is not able to explain any aspect of the situation: First, the colours and their origin remain mysterious. Secondly, all shapes, sizes and proportions remain mysterious. Thirdly, the timing and the duration of the involved processes can-

Challenge 2 s

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* The photograph on page 14 shows a female glow worm, *Lampyris noctiluca*, as commonly found in the United Kingdom (© John Tyler, www.johntyler.co.uk/gwfacts.htm).

not be understood. Fourthly, all the sensations and emotions produced by the situation remain mysterious. To understand and explain these aspects, we need *quantum theory*. In fact, we will find out that both *life* and *every type of pleasure* are examples of quantum motion. Emotions are quantum processes.

In the early days of physics, the impossibility to describe life and pleasure was not seen as a shortcoming, because neither senses nor material properties nor scales were thought to be related to motion. And pleasure was not considered a serious subject of investigation for a respectable researcher anyway. Today, the situation is different. In our

adventure we have learned that our senses of time, hearing, touch, smell and sight are primarily *detectors of motion*. Without motion, there would be no senses. Furthermore, all detectors are made of matter. During the exploration on electromagnetism we began to understand that all properties of matter are due to motions of charged constituents. Density, stiffness, colour and all other material properties result from the electromag-

netic behaviour of the Lego bricks of matter: namely, the molecules, the atoms and the electrons. Thus, the properties of matter are also consequences of motion. Moreover,

we saw that these tiny constituents are *not* correctly described by classical electrodynamics. We even found that light itself does not behave classically. Therefore the inability of classical physics to describe matter, light and the senses is indeed due to its intrinsic limitations.

In fact, every failure of classical physics can be traced back to a single, fundamental

discovery made in 1899 by Max Planck:[*]

▷ In nature, action values smaller than $\hbar = 1.06 \cdot 10^{-34}$ Js are not observed.

All attempts to observe physical action values smaller than this fail.[**] In other words, in nature – as in a good cinema film – there is always some action. The existence of a smallest action value – the so-called *quantum principle* – is in complete contrast with clas-

sical physics. (Why?) Despite this contrast, the quantum principle has passed an enorm-

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[*] Max Planck (1858–1947), professor of physics in Berlin, was a central figure in thermostatics and modern physics. He discovered and named the *Boltzmann constant k* and the *quantum of action h*, often called Planck's constant. His introduction of the quantum hypothesis gave birth to quantum theory. He also made the works of Einstein known in the physical community, and later organized a job for him in Berlin. He received the Nobel Prize for physics in 1918. He was an important figure in the German scientific establishment; he also was one of the very few who had the courage to tell Adolf Hitler *face to face* that it was a bad idea to fire Jewish professors. (He got an outburst of anger as answer.) Famously modest, with many tragedies in his personal life, he was esteemed by everybody who knew him.

[**] In fact, this story is a slight simplification: the constant originally introduced by Planck was the (unreduced) constant $h = 2\pi\hbar$. The factor 2π leading to the final quantum principle was added somewhat later, by other researchers.

This somewhat unconventional, but didactically useful, approach to quantum theory is due to Niels Bohr.

Nowadays, it is hardly ever encountered in the literature, despite its simplicity.

Niels Bohr (b. 1885 Copenhagen, d. 1962 Copenhagen) was one of the great figures of modern physics. A daring thinker and a polite man, he made Copenhagen University into the new centre of development of quantum theory, overshadowing Göttingen. He developed the description of the atom in terms of quantum theory, for which he received the 1922 Nobel Prize in Physics. He had to flee Denmark in 1943 after the German invasion, because of his Jewish background, but returned there after the war, continuing to attract the best physicists across the world.
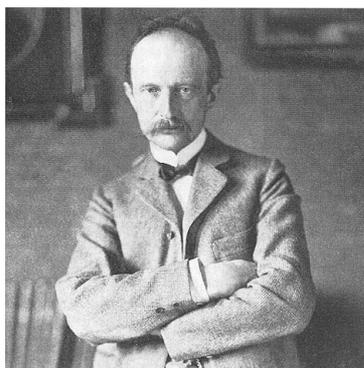
FIGURE 3 Max Planck (1858–1947)

FIGURE 4 Niels Bohr
(1885–1962)

ous number of experimental tests, many of which we will encounter in this part of our mountain ascent. Above all, the quantum principle has never failed even a single test. The fundamental constant $\hbar$, which is pronounced 'aitch-bar', is called the *quantum of action*, or alternatively *Planck's constant*. Planck discovered the quantum principle when studying the properties of incandescent light, i.e., of light emanating from hot bodies. But the quantum principle also applies to motion of matter, and even, as we will see later, to motion of empty space, such as gravitational waves.

The quantum principle states that no experiment can measure an action smaller than $\hbar$. For a long time, Einstein tried to devise experiments to overcome this limit. But he failed in all his attempts: nature does not allow it, as Bohr showed again and again. The same occurred to many other researchers.

We recall that in physics – as in the theatre – action is a measure for the *change* occurring in a system. The quantum principle can thus rephrased as

> ▷ In nature, a change smaller than $\hbar = 1.06 \cdot 10^{-34}$ Js cannot be observed.

Therefore, a smallest action implies that *there is a smallest change value in nature*. If we compare two observations, there will always be change between them. Thus the quantum of action would perhaps be better named the *quantum of change*.

Can a minimum change really exist in nature? To accept the idea, we need to explore three points, detailed in Table 1. We need to show that a smaller change is *never observed* in nature, show that smaller change values *can never* be observed, and finally, show that *all consequences* of this smallest change, however weird they may be, apply to nature. In fact, this exploration constitutes all of quantum physics. Therefore, these checks are all we do in the remaining of this part of our adventure. But before we explore some of the experiments that confirm the existence of a smallest change, we directly present some of its more surprising consequences.

TABLE 1 How to convince yourself and others that there is a smallest action, or smallest change ℏ in nature. Compare this table with the two tables in volume II, that about maximum speed on page 26, and that about maximum force on page 109.

STATEMENT	TEST
The smallest action value ℏ is observer-invariant.	Check all observations.
Local change or action values < ℏ are not observed.	Check all observations.
Local change or action values < ℏ cannot be produced.	Check all attempts.
Local change or action values < ℏ cannot even be imagined.	Solve all paradoxes.
The smallest local change or action value ℏ is a principle of nature.	Deduce quantum theory from it.
	Show that all consequences, however weird, are confirmed by observation.

The effects of the quantum of action on rest

Since action is a measure of change, a minimum observable action means that two successive observations of the same system always differ by at least ℏ. In every system, there is always *something* happening. As a consequence we find:

> ▷ In nature *there is no rest*.

Everything moves, all the time, at least a little bit. *Natura facit saltus.** True, these jumps are tiny, as ℏ is too small to be observable by any of our senses. Nevertheless, rest can be observed only macroscopically, and only as a long-time or many-particle average. For example, the quantum of action implies that in a mountain – an archetypal 'system at rest' – all the atoms and electrons are continually buzzing around. In short,

> ▷ There is motion *inside matter*.

Since there is a minimum action for all observers, and since there is no rest, we deduce:

> ▷ In nature *there is no perfectly straight or perfectly uniform motion*.

Forget all you have learnt so far: Inertial motion is an approximation! An object can move in straight, uniform motion only approximately, and only when observed over long distances or long times. We will see later that the more massive the object is, the better

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* 'Nature makes jumps.'

the approximation is. (Can you confirm this?) So *macroscopic* observers can still talk about space-time symmetries; and *special* relativity can thus be reconciled with quantum theory.

Also free fall, or motion along a geodesic, exists only as a long-time average. So *general* relativity, which is based on the existence of freely-falling observers, cannot be correct when actions of the order of $\hbar$ are involved. Indeed, the reconciliation of the quantum principle with general relativity – and thus with curved space – is a big challenge. (The solution is simple only for weak, everyday fields.) The issues involved are so mind-shattering that they form a separate, final, part of this adventure. We thus explore situations without gravity first.

THE CONSEQUENCES OF THE QUANTUM OF ACTION FOR OBJECTS

Have you ever wondered why leaves are green? You probably know that they are green because they absorb blue (short-wavelength) and red (long-wavelength) light, while allowing green (medium-wavelength) light to be reflected. How can a system filter out the small and the large, and let the middle pass through? To do so, leaves must somehow *measure* the frequency. But we have seen that classical physics does not allow measurement of time (or length) intervals, as any measurement requires a measurement unit, and classical physics does not allow such units to be defined. On the other hand, it takes only a few lines to confirm that with the help of the quantum of action $\hbar$ (and the Boltzmann constant k, both of which Planck discovered), fundamental units for *all* measurable quantities can be defined, including time and therefore frequency. (Can you find a combination of the speed of light c, the gravitational constant G and the quantum of action $\hbar$ that gives a time? It will only take a few minutes.)

In short, measurements are only possible at all because of the existence of the quantum of action.

> ▷ All measurements *are quantum effects*.

When Planck saw that the quantum of action allowed defining all units in nature, he was as happy as a child; he knew straight away that he had made a fundamental discovery, even though (in 1899) quantum theory did not yet exist. He even told his seven-year-old son Erwin about it, while walking with him through the woods around Berlin. Planck explained to his son that he had made a discovery as important as universal gravity. Indeed, Planck knew that he had found the key to understanding many of the effects that were then unexplained.

> ▷ In nature, *all times and all frequencies are due to the quantum of action*.

All processes that take time are quantum processes. If you prefer, *waiting* is a quantum effect! In particular, without the quantum of action, oscillations and waves could not exist:

> ▷ Every colour is a quantum effect.

But this is not all. Planck also realized that the quantum of action allows us to understand the *size* of all things.

▷ Every size is a quantum effect.

Can you find the combination of c, G and $\hbar$ that yields a length? With the quantum of action, it was finally possible to determine the maximum size of mountains, of trees and of humans. Planck knew that the quantum of action confirmed what Galileo had already deduced long before him: that sizes are due to fundamental, smallest scales in nature.

Max Planck also understood that the quantum of action $\hbar$ was the *last missing* constant of nature. With $\hbar$, it becomes possible to define a *natural* unit for *every* observable property in nature. Together, c, G and $\hbar$ allow to define units that are independent of culture or civilization – even extraterrestrials would understand them.[*] In short, $\hbar$ allows understanding *all* observables. Therefore, with $\hbar$ it is possible to draw the diagram shown in Figure 1 that encompasses *all* motion in nature, and thus *all* of physics.

In our environment, the size of all objects is related and due to the size of atoms. In turn, the size of atoms is a direct consequence of the quantum of action. Can you derive an approximation for the size of atoms, knowing that it is given by the motion of electrons of mass m_e and charge e, constrained by the quantum of action? This connection, a simple formula, was discovered in 1910 by Arthur Erich Haas, 15 years before quantum theory was formulated.

▷ Atom sizes are quantum effects.

At the time, Haas was widely ridiculed.[**] Nowadays, his formula for the size of atoms is found in all textbooks, including this one. In determining the size of atoms, the quantum of action has another important consequence:

▷ *Gulliver's travels* are impossible.

There are no tiny people and no giant ones. Classically, nothing speaks against the idea; but the quantum of action prevents it. Can you supply the detailed argument?

But if rest does not exist, how can *shapes* exist? Any shape of everyday life, including that of a flower, is the result of body parts remaining *at rest* with respect to each other. Now, all shapes result from interactions between the constituents of matter, as shown most clearly in the shapes of molecules. But how can a molecule, such as the water molecule H_2O, shown in Figure 5, have a shape? In fact, a molecule does not have a *fixed* shape, but its shape fluctuates, as would be expected from the quantum of action. Despite the fluctuations, every molecule does have an *average* shape, because different angles and distances correspond to different energies. Again, these average length and angle val-

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[*] In fact, it is also possible to define all measurement units in terms of the speed of light c, the gravitational constant G and the electron charge e. Why is this not fully satisfactory?

[**] Before the discovery of $\hbar$, the only simple length scale for the electron was the combination $e^2/(4\pi\varepsilon_0 m_e c^2) \approx 3\,\text{fm}$; this is ten thousand times smaller than an atom. We stress that any length scale containing e is a quantum effect, and not a classical length scale, because e is the quantum of electric charge.
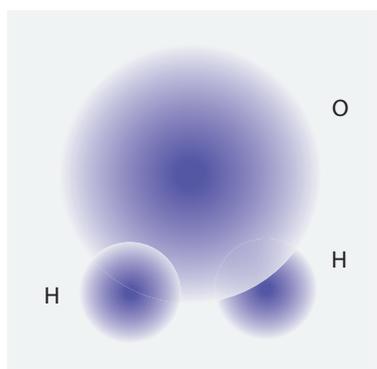
FIGURE 5 An artist's impression of a water molecule made of two hydrogen (H) and one oxygen (O) atom.

FIGURE 6 Max Born (1882–1970)

ues only exist because the quantum of action yields fundamental length scales in nature. Without the quantum of action, there would be *no* shapes in nature.

▷ All shapes are quantum effects.

All shapes in everyday life are due to molecular shapes, or to their generalizations.

The *mass* of an object is also a consequence of the quantum of action, as we will see later on. Since all material properties – such as density, colour, stiffness or polarizability – are defined as combinations of length, time and mass units, we find:

▷ All material properties arise from the quantum of action.

In short, the quantum of action determines the size, shape, colour, mass, and all other properties of objects, from stones to whipped cream.

WHY 'QUANTUM'?

Quantum effects surround us on all sides. However, since the quantum of action is so small, its effects on *motion* appear mostly, but not exclusively, in *microscopic* systems. The study of such systems was called *quantum mechanics* by Max Born, one of the major

TABLE 2 Some small systems in motion and the observed action values for their changes.

SYSTEM AND CHANGE	ACTION	MOTION
Light		
Smallest amount of light absorbed by a coloured surface	$1\,\hbar$	quantum
Smallest impact when light reflects from mirror	$2\,\hbar$	quantum
Smallest consciously visible amount of light	$c.\,5\,\hbar$	quantum
Smallest amount of light absorbed in flower petal	$1\,\hbar$	quantum
Blackening of photographic film	$c.\,3\,\hbar$	quantum
Photographic flash	$c.\,10^{17}\,\hbar$	classical
Electricity		
Electron ejected from atom or molecule	$c.\,1\text{–}2\,\hbar$	quantum
Electron extracted from metal	$c.\,1\text{–}2\,\hbar$	quantum
Electron motion inside microprocessor	$c.\,2\text{–}6\,\hbar$	quantum
Signal transport in nerves, from one molecule to the next	$c.\,5\,\hbar$	quantum
Current flow in lightning bolt	$c.\,10^{38}\,\hbar$	classical
Materials		
Tearing apart two neighbouring iron atoms	$c.\,1\text{–}2\,\hbar$	quantum
Breaking a steel bar	$c.\,10^{35}\,\hbar$	classical
Basic process in superconductivity	$1\,\hbar$	quantum
Basic process in transistors	$1\,\hbar$	quantum
Basic magnetization process	$1\,\hbar$	quantum
Chemistry		
Atom collision in liquid at room temperature	$1\,\hbar$	quantum
Shape oscillation of water molecule	$c.\,1-5\,\hbar$	quantum
Shape change of molecule, e.g. in chemical reaction	$c.\,1-5\,\hbar$	quantum
Single chemical reaction curling a hair	$c.\,2-6\,\hbar$	quantum
Tearing apart two mozzarella molecules	$c.\,300\,\hbar$	quantum
Smelling one molecule	$c.\,10\,\hbar$	quantum
Burning fuel in a cylinder in an average car engine explosion	$c.\,10^{37}\,\hbar$	classical
Life		
Air molecule hitting eardrum	$c.\,2\,\hbar$	quantum
Smallest sound signal detectable by the ear	Challenge 10 ny	
Single DNA duplication step during cell division	$c.\,100\,\hbar$	quantum
Ovule fertilization	$c.\,10^{14}\,\hbar$	classical
Smallest step in molecular motor	$c.\,5\,\hbar$	quantum
Sperm motion by one cell length	$c.\,10^{15}\,\hbar$	classical
Cell division	$c.\,10^{19}\,\hbar$	classical
Fruit fly's wing beat	$c.\,10^{24}\,\hbar$	classical
Person walking one body length	$c.\,2\cdot10^{36}\,\hbar$	classical
Nuclei and stars		
Nuclear fusion reaction in star	$c.\,1-5\,\hbar$	quantum
Explosion of gamma-ray burster	$c.\,10^{80}\,\hbar$	classical

contributors to the field.* Later, the term *quantum theory* became more popular.

Quantum theory arises from the existence of *smallest* measurable values in nature, generalizing the idea that Galileo had in the seventeenth century. As discussed in detail earlier on, it was Galileo's insistence on 'piccolissimi quanti' – smallest quanta – of matter that got him into trouble. We will soon discover that the idea of a smallest change is necessary for a precise and accurate description of matter and of nature as a whole. Therefore Born adopted Galileo's term for the new branch of physics and called it 'Quantentheorie' or 'theory of quanta'. The English language adopted the Latin singular 'quantum' instead of the plural used in most other languages.

Note that the term 'quantum' does *not* imply that all measurement values are *multiples* of a smallest one: this is so only in a few cases.

Quantum theory is the description of microscopic motion. Quantum theory is *necessary* whenever a process produces an action value of the order of the quantum of action. Table 2 shows that all processes on atomic and molecular scales, including biological and chemical processes, are quantum processes. So are processes of light emission and absorption. These phenomena can *only* be described with quantum theory.

Table 2 also shows that the term 'microscopic' has a different meaning for a physicist and for a biologist. For a biologist, a system is 'microscopic' if it requires a microscope for its observation. For a physicist, a system is *microscopic* if its characteristic action is of the order of the quantum of action. In other words, for a physicist a system is usually microscopic if it is *not even* visible in a (light) microscope. To increase the confusion, some quantum physicists nowadays call their own class of microscopic systems 'mesoscopic', while others call their systems 'nanoscopic'. Both terms were introduced only to attract attention and funding: they are conceptually useless.

THE EFFECT OF THE QUANTUM OF ACTION ON MOTION

There is another way to characterize the difference between a microscopic, or quantum, system and a macroscopic, or classical, one. A smallest action implies that the difference between the action values S of two successive observations of the same system, a time Δt apart, cannot vanish. We have

$$|S(t + \Delta t) - S(t)| = |(E \pm \Delta E)(t + \Delta t) - Et| = |E\Delta t \pm t\Delta E \pm \Delta E\Delta t| \geqslant \frac{\hbar}{2} \, . \quad (1)$$

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* Max Born (b. 1882 Breslau, d. 1970 Göttingen) first studied mathematics, then turned to physics. A professor at Göttingen University, he made the city one of the world centres of physics. He developed quantum mechanics with his assistants Werner Heisenberg and Pascual Jordan, and then applied it to scattering, solid-state physics, optics and liquids. He was the first to understand that the wave function, or state function, describes a probability amplitude. Later, Born and Emil Wolf wrote what is still the main textbook on optics. Many of Born's books were classics and read all over the world.

Born attracted to Göttingen the most brilliant talents of the time, receiving as visitors Hund, Pauli, Nordheim, Oppenheimer, Goeppert-Mayer, Condon, Pauling, Fock, Frenkel, Tamm, Dirac, Mott, Klein, Heitler, London, von Neumann, Teller, Wigner, and dozens of others. Being Jewish, Born lost his job in 1933, when criminals took over the German government. He emigrated, and became professor in Edinburgh, where he stayed for 20 years. Physics at Göttingen never recovered from this loss. For his elucidation of the meaning of the wave function he received the 1954 Nobel Prize in Physics.

FIGURE 7 Werner Heisenberg (1901–1976)

The factor 1/2 arises because a smallest action $\hbar$ automatically implies an action indeterminacy of half its value. Now the values of the energy E and time t – but not of (the positive) ΔE or Δt – can be set to zero if we choose a suitable observer. Thus, the existence of a quantum of action implies that in any system the evolution is constrained by

$$\Delta E \Delta t \geqslant \frac{\hbar}{2} \, , \tag{2}$$

where E is the energy of the system and t is its age, so that ΔE is the change of energy and Δt is the time between two successive observations.

Challenge 11 e   By a similar reasoning, we find that for any physical system the position and momentum are constrained by

$$\Delta x \Delta p \geqslant \frac{\hbar}{2} \, , \tag{3}$$

where Δx is the indeterminacy in position and Δp is the indeterminacy in momentum. These two famous relations were called *indeterminacy relations* by their discoverer, Werner Heisenberg.* In English they are often called 'uncertainty relations'; however, this term is incorrect. The quantities are not uncertain, but *undetermined*. Because of the quantum of action, system observables have *no* definite value. There is no way to ascribe

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* It is often said that the indeterminacy relation for energy and time has a different weight from that for momentum and position. This is a wrong idea, propagated by the older generation of physicists, which has survived through many textbooks for over 70 years. Just forget it. It is essential to remember that all four quantities appearing in the inequalities describe the *internal* properties of the system. In particular, t is a time variable deduced from changes observed *inside* the system, and not the time coordinate measured by an outside clock; similarly, the position x is not the external space coordinate, but the position characterizing the system.

Ref. 7   Werner Heisenberg (1901–1976) was an important theoretical physicist and an excellent table-tennis and tennis player. In 1925, as a young man, he developed, with some help from Max Born and Pascual Jordan, the first version of quantum theory; from it he deduced the indeterminacy relations. For these achievements he received the Nobel Prize in Physics in 1932. He also worked on nuclear physics and on turbulence. During the Second World War, he worked on the nuclear-fission programme. After the war, he published several successful books on philosophical questions in physics, slowly turned into a crank, and tried unsuccessfully – with some half-hearted help from Wolfgang Pauli – to find a unified description of nature based on quantum theory, the 'world formula'.

a precise value to momentum, position, or any other observable of a quantum system. We will use the term 'indeterminacy relation' throughout. The habit to call the relation a 'principle' is even more mistaken.

Any system whose indeterminacy is of the order of $\hbar$ is a quantum system; if the indeterminacy product is much larger, the system is classical, and then classical physics is sufficient for its description. So even though classical physics assumes that there are *no* measurement indeterminacies in nature, a system is *classical* only if its indeterminacies are *large* compared to the minimum possible ones!

In other terms, quantum theory is necessary whenever we try to measure some quantity as precisely as possible. In fact, every measurement is itself a quantum process. And the indeterminacy relation implies that measurement precision is limited. The quantum of action shows that

> ▷ *Motion cannot be observed to infinite precision.*

In other words, the microscopic world is *fuzzy*. This fact has many important consequences and many strange ones. For example, if motion cannot be observed with infinite precision, the very concept of motion needs to be handled with great care, as it cannot be applied in certain situations. In a sense, the rest of our quest is just an exploration of the implications of this result.

In fact, as long as space-time is *flat*, it turns out that we *can* retain the concept of motion to describe observations, provided we remain aware of the limitations implied by the quantum principle.

The surprises of the quantum of action

The quantum of action $\hbar$ implies a fuzziness of all motion. This fuzziness also implies the existence of short-time deviations from energy, momentum and angular-momentum conservation in microscopic systems. For general assurance it must be stressed that for long observation times – surely for all times longer than a microsecond – conservation holds. But in the first part of our adventure, we realized that any type of non-conservation implies the existence of *surprises* in nature. Well, here are some of them.

Since precisely uniform motion does not exist, a system moving in one dimension only – such as the hand of a clock – always has the possibility of moving a bit in the opposite direction, thus leading to incorrect readings. Indeed, quantum theory predicts that clocks have essential limitations:

> ▷ Perfect clocks do not exist.

The deep implications of this statement will become clear step by step.

It is also impossible to avoid that an object makes small displacement sideways. In fact, quantum theory implies that, strictly speaking,

> ▷ Neither uniform nor one-dimensional motion exists.

Also this statement harbours many additional surprises.

Quantum limitations apply also to metre rules. It is impossible to ensure that the rule is completely at rest with respect to the object being measured. Thus the quantum of action implies again, on the one hand, that measurements are possible, and on the other hand:

▷ Measurement accuracy is limited.

It also follows from the quantum of action that any inertial or freely-falling observer must be *large*, as only large systems approximate inertial motion.

▷ An observer cannot be microscopic.

If humans were not macroscopic, they could neither observe nor study motion.

Because of the finite accuracy with which microscopic motion can be observed, we discover that

▷ *Faster-than-light motion* is possible in the microscopic domain.

Quantum theory thus predicts *tachyons*, at least over short time intervals. For the same reason,

▷ *Motion backwards in time* is possible over microscopic times and distances.

In short, a quantum of action implies the existence of microscopic time travel. However, this remains impossible in the macroscopic domain, such as everyday life.

But there is more. Imagine a moving car suddenly disappearing for good. In such a situation, neither momentum nor energy would be conserved. The action change for such a disappearance is large compared to $\hbar$, so that its observation would contradict even classical physics – as you may wish to check. However, the quantum of action allows a *microscopic* particle, such as an electron, to disappear for a *short* time, provided it reappears afterwards.

▷ The quantum of action implies that there is no permanence in nature.

The quantum of action also implies:

▷ The vacuum is not empty.

If we look at empty space twice, the two observations being separated by a tiny time interval, some energy will be observed the second time. If the time interval is short enough, the quantum of action will lead to the observation of radiation or matter particles. Indeed, particles can appear anywhere from nowhere, and disappear just afterwards: the action limit requires it. In summary, nature exhibits short-term appearance and disappearance of matter and radiation. In other words, the classical idea of an *empty* vacuum is correct only when the vacuum is observed over a *long* time.

The quantum of action implies that compass needles cannot work. If we look twice in
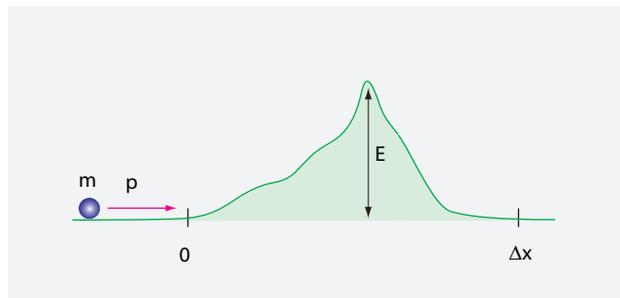
FIGURE 8 Hills are never high enough.

quick succession at a compass needle, or even at a house, we usually observe that it stays oriented in the same direction. But since physical action has the same dimensions as angular momentum, a minimum value for action implies a minimum value for angular momentum. Even a macroscopic object has a minimum value for its rotation. In other words, quantum theory predicts

 ▷ Everything rotates.

An object can be non-rotating only approximately, when observations are separated by long time intervals.

For microscopic systems, the quantum limits on rotation have specific effects. If the rotation angle can be observed – as for molecules – the system behaves like a macroscopic object: its position and orientation are fuzzy. But for a system whose rotation angle cannot be observed, the quantum of action limits the angular momentum to multiples of $\hbar/2$. In particular, all microscopic bound systems – such as molecules, atoms, or nuclei – contain rotational motion and rotating components.

Transformation, life and Democritus

At the beginning of our adventure, we mentioned that the Greeks distinguished three types of changes: transport, growth, and transformation. We also mentioned that Democritus had deduced that all these types of changes – including life and death – were in fact the same, and due to the motion of atoms. The quantum of action makes exactly this point.

First of all, a minimum action implies that cages in zoos are dangerous and banks are not safe. A cage is a feature that needs a lot of energy to overcome. Physically speaking, the wall of a cage is an energy hill, resembling the real hill shown in Figure 8. Imagine that a particle with momentum p approaches one side of the hill, which is assumed to have width Δx.

In everyday life – and thus in classical physics – the particle will never be observed on the other side of the hill if its kinetic energy $p^2/2m$ is *less* than the height E of the hill. But imagine that the missing momentum to overcome the hill, $\Delta p = \sqrt{2mE} - p$, satisfies $\Delta x \Delta p \leqslant \hbar/2$. The particle will have the possibility to overcome the hill, despite
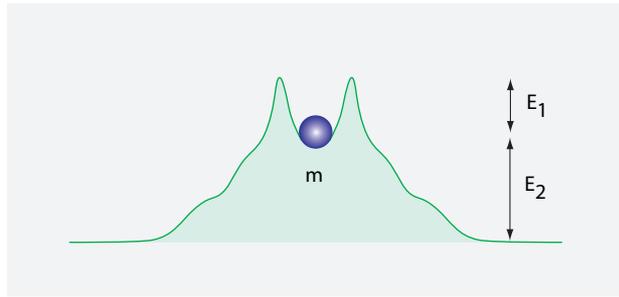
FIGURE 9 Leaving enclosures.

its insufficient energy. The quantum of action thus implies that a hill of width

$$\Delta x \leqslant \frac{\hbar/2}{\sqrt{2mE} - p} \tag{4}$$

is *not* an obstacle to a particle of mass *m*. But this is not all. Since the value of the particle momentum p is itself undetermined, a particle can overcome the hill even if the hill is *wider* than the value (4) – although the broader it is, the lower the probability will be. So any particle can overcome *any* obstacle. This is called the *tunnelling effect*, for obvious reasons. Classically, tunnelling is impossible. In quantum theory, the feat is possible, be-

cause the wave function does not vanish at the location of the hill; sloppily speaking, the wave function is non-zero inside the hill. It thus will be also non-zero behind the hill. As a result, quantum systems can penetrate or 'tunnel' through hills.

In short, the minimum-action principle implies that there are no tight boxes in nature. Thanks to the tunnelling effect,

    ▷ Matter is not impenetrable.

The penetrability of all matter is in contrast to everyday, classical observation. Can you
explain why lion cages work *despite* the quantum of action?

By the way, the quantum of action also implies that a particle with a kinetic energy greater than the energy height of a hill can be *reflected* by the hill. Also this effect is impossible in classical physics.

The minimum-action principle also implies that bookshelves are dangerous. Why? Shelves are obstacles to motion. A book on a shelf is in the same situation as the mass in Figure 9: the mass is surrounded by energy hills hindering its escape to the outer, lower-energy world. But thanks to the tunnelling effect, escape is always possible. The same picture applies to a branch of a tree, a nail in a wall, or anything attached to anything else. Things can never be permanently fixed together. In particular, we will discover that every example of light emission – even radioactivity – results from this effect.

In summary, the quantum of action thus implies that

    ▷ Decay is part of nature.

Note that decay often appears in everyday life, under a different name: *breaking*. In fact,

FIGURE 10 Identical objects with crossing paths.

all breakages require the quantum of action for their description. Obviously, the cause of breaking is often classical, but the *mechanism* of breaking is always quantum. Only 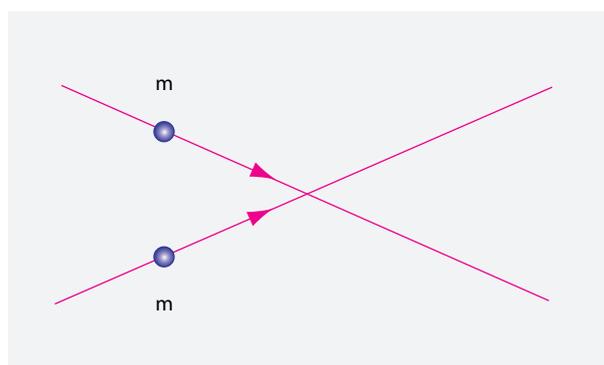objects that obey quantum theory can break. In short, there are no stable excited systems in nature. For the same reason, by the way, no memory can be perfect. (Can you confirm this?)

Taking a more general view, *ageing* and *death* also result from the quantum of action. Death, like ageing, is a composition of breaking processes. When dying, the mechanisms in a living being break. Breaking is a form of decay, and is due to tunnelling. Death is thus a quantum process. Classically, death does not exist. Might this be the reason why so many people believe in immortality or eternal youth?

We will also discover that the quantum of action is the reason for the importance of the action observable in classical physics. In fact, the existence of a *smallest* action is the reason for the *least*-action principle of classical physics.

A minimum action also implies that matter cannot be continuous, but must be composed of smallest entities. Indeed, any flow of a truly continuous material would contradict the quantum principle. Can you give the precise argument? Of course, at this point in our adventure, the non-continuity of matter is no longer a surprise. But the quantum of action implies that even *radiation* cannot be continuous. As Albert Einstein was the first to state clearly, light is made of quantum particles.

Even more generally, the quantum of action implies that in nature

▷ All flows and all waves are made of microscopic particles.

The term 'microscopic' (or 'quantum') is essential, as such particles do *not* behave like little stones. We have already encountered several differences, and we will encounter others shortly. For these reasons, there should be a special name for microscopic particles; but so far all proposals, of which *quanton* is the most popular, have failed to catch on.

The quantum of action has several strange consequences for microscopic particles. Take two such particles with the same mass and composition. Imagine that their paths cross, and that at the crossing they approach each other very closely, as shown in Figure 10. A minimum action implies that in such a situation, if the distance becomes small enough, the two particles can switch roles, without anybody being able to avoid, or notice, it. Thus, in a volume of gas it is *impossible* – thanks to the quantum of action – to
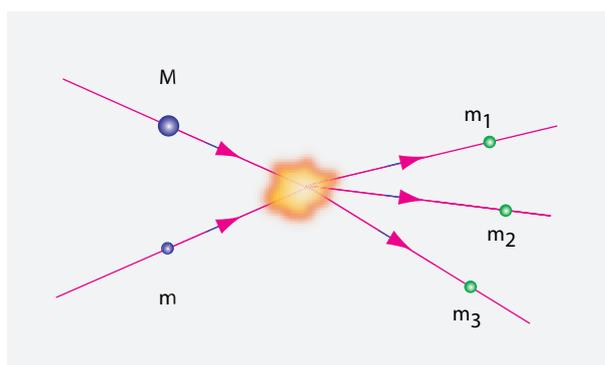
FIGURE 11 Transformation through reaction.

follow particles moving around and to say which particle is which. Can you confirm this deduction, and specify the conditions, using the indeterminacy relations? In summary

> ▷ In nature it is impossible to distinguish between identical particles.

Can you guess what happens in the case of light?

But matter deserves still more attention. Imagine again two particles – even two different ones – approaching each other very closely, as shown in Figure 11. We know that if the approach distance gets small, things get fuzzy. Now, the minimum-action principle makes it possible for something to happen in that small domain as long as resulting outgoing products have the same total linear momentum, angular momentum and energy as the incoming ones. Indeed, ruling out such processes would imply that arbitrarily small actions could be observed, thus eliminating nature's fuzziness, as you may wish to check for yourself. In short,

> ▷ The quantum of action allows transformation of matter.

One also says that the quantum of action allows particle *reactions*. In fact, we will discover that *all* kinds of reactions in nature, including breathing, digestion, and all other chemical and nuclear reactions, are due just to the existence of the quantum of action.

One type of process that is especially dear to us is *growth*. The quantum of action implies that all growth happens in small steps. Indeed,

> ▷ All growth processes in nature are quantum processes.

Above all, as mentioned already, the quantum of action explains life. Only the quantum of action makes reproduction and heredity possible. Birth, sexuality and death are consequences of the quantum of action.

So Democritus was both right and wrong. He was right in deducing fundamental constituents for matter and radiation. He was right in unifying all change in nature – from transport to transformation and growth – as motion of particles. But he was wrong in assuming that the small particles behave like stones. As we will show in the following, the smallest particles behave like *quantons*: they behave randomly, and they behave partly

Challenge 18 s

Challenge 19 s

Challenge 20 e

as waves and partly as particles.

Randomness – a consequence of the quantum of action

What happens if we try to measure a change smaller than the quantum of action? Nature has a simple answer: we get *random results*. If we build an experiment that tries to produce a change or action of the size of a quarter of the quantum of action, the experiment will produce, for example, a change of *one* quantum of action in a quarter of the cases, and *no* change in three quarters of the cases,* thus giving an *average* of one quarter of $\hbar$.

> ▷ Attempts to measure actions below $\hbar$ lead to random results.

If you want to condense quantum physics in one key statement, this is it.

The quantum of action leads to randomness at microscopic level. This connection can be seen also in the following way. Because of the indeterminacy relations, it is impossible to obtain definite values for both the momentum and the position of a particle. Obviously, definite values are also impossible for the individual components of an experimental set-up or an observer. Therefore, initial conditions – both for a system and for an experimental set-up – cannot be exactly duplicated. The quantum of action thus implies that whenever an experiment on a microscopic system is performed twice, the outcomes will (usually) be different. The outcomes could only be the same if both the system and the observer were in exactly the same configuration each time. However, because of the second principle of thermodynamics, and because of the quantum of action, reproducing a configuration is impossible. Therefore,

> ▷ Microscopic systems behave randomly.

Obviously, there will be some *average* outcome; but in all cases, microscopic observations are *probabilistic*. Many find this conclusion of quantum theory the most difficult to swallow. But fact is: the quantum of action implies that the behaviour of quantum systems is strikingly different from that of classical systems. The conclusion is unavoidable:

> ▷ Nature behaves randomly.

Can we observe randomness in everyday life? Yes. Every *window* proves that nature behaves randomly on a microscopic scale. Everybody knows that we can use a train window either to look at the *outside* landscape or, by concentrating on the reflected image, to observe some interesting person *inside* the carriage. In other words, observations like that of Figure 12 show that glass reflects some of the light particles and lets some others pass through. More precisely, glass reflects a random selection of light particles; yet the average proportion is constant. In these properties, partial reflection is similar to the tunnelling effect. Indeed, the partial reflection of photons in glass is a result of the quantum of action. Again, the situation can be *described* by classical physics, but the precise amount of reflection cannot be *explained* without quantum theory. We retain:

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* In this context, 'no change' means 'no change' in the physical variable to be measured; generally speaking, there is always some change, but not necessarily in the variable being measured.
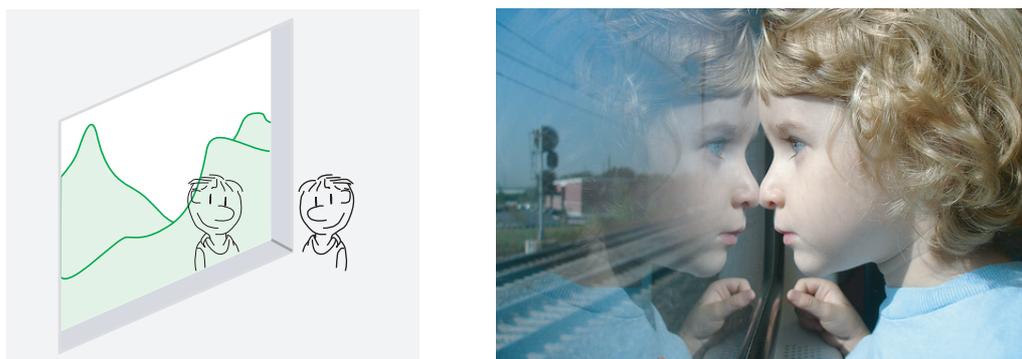
FIGURE 12 A famous quantum effect: how do train windows manage to show two superimposed images? (Photo © Greta Mansour)
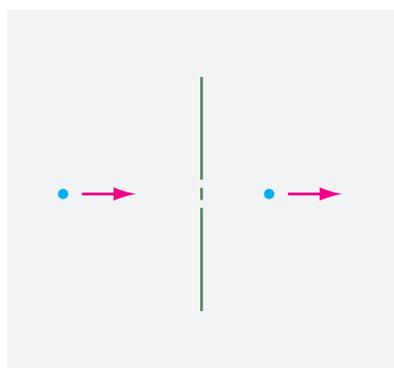
FIGURE 13 A particle and a screen with two nearby slits.

▷ Quantons move randomly.

Without the quantum of action, train journeys would be much more boring.

Waves – a consequence of the quantum of action

The quantum of action implies an important result about the *paths* of particles. If a particle travels from one point to another, there is no way to say which path it has taken in between. Indeed, in order to distinguish between two possible, but slightly different, paths, actions smaller than $\hbar$ would have to be measured reliably. In particular, if a particle is sent through a screen with two sufficiently close slits, as illustrated in Figure 13, it is impossible to say which slit the particle passed through. This impossibility is fundamental.

We already know phenomena of motion for which it is not possible to say with precision how something moves or which path is taken behind two slits: *waves* behave in this way. All waves are subject to the indeterminacy relations

$$\Delta\omega\Delta t \geqslant \frac{1}{2} \quad \text{and} \quad \Delta k\Delta x \geqslant \frac{1}{2} \; . \tag{5}$$

A wave is a type of motion described by a *phase* that changes over space and time. This turns out to hold for all motion. In particular, this holds for matter.

We saw above that quantum systems are subject to

$$\Delta E \Delta t \geqslant \frac{\hbar}{2} \quad \text{and} \quad \Delta p \Delta x \geqslant \frac{\hbar}{2} \; . \tag{6}$$

We are thus led to ascribe a frequency and a wavelength to a quantum system:

$$E = \hbar\omega \quad \text{and} \quad p = \hbar k = \hbar \frac{2\pi}{\lambda} \; . \tag{7}$$

The energy–frequency relation for *light* and the equivalent momentum–wavelength relation were deduced by Max Planck in 1899. In the years from 1905 onwards, Albert Einstein confirmed that the relations are valid for all examples of emission and absorption of light. In 1923 and 1924, Louis de Broglie* predicted that the relation should hold also for all quantum *matter* particles. The experimental confirmation came a few years later. (This is thus another example of a discovery that was made about 20 years too late.) In short, the quantum of action implies:

▷ Matter particles behave like waves.

In particular, the quantum of action implies the existence of *interference* for streams of matter.

Particles – a consequence of the quantum of action

The quantum of action, the smallest change, implies that flows cannot be arbitrarily weak. This applies to *all* flows: in particular, it applies to rivers, solid matter flows, gas flows, light beams, energy flows, entropy flows, momentum flows, angular momentum flows, probability flows, signals of all kind, electrical charge flows, colour charge flows and weak charge flows.

Water flows in rivers, like any other matter flow, cannot be arbitrary small: the quantum of action implies that there is a smallest matter flow in nature. Depending on the situation, the smallest matter flow is a molecule, an atom or a smaller particle. Indeed, the quantum of action is also at the origin of the observation of a smallest charge in electric current. Since all matter can flow, the quantum of action implies:

▷ All matter has particle aspects.

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* Louis de Broglie (b. 1892 Dieppe, d. 1987 Paris), physicist and professor at the Sorbonne. The energy–frequency relation for light had earned Max Planck and Albert Einstein the Nobel Prize in Physics, in 1918 and 1921. De Broglie expanded the relation to predict the wave nature of the electron (and of all other quantum matter particles): this was the essence of his doctoral thesis. The prediction was first confirmed experimentally a few years later, in 1927. For the prediction of the wave nature of matter, de Broglie received the Nobel Prize in Physics in 1929. Being an aristocrat, he did no more research after that. For example, it was Schrödinger who then wrote down the wave equation, even though de Broglie could equally have done so.

In the same way, the quantum of action, the smallest change, implies that light cannot be arbitrarily faint. There is a smallest illumination in nature; it is called a *photon* or a *light quantum*. Now, light is a wave, and the argument can be made for any other wave as well. In short, the quantum of action thus implies:

▷ All waves have particle aspects.

This has been proved for light waves, water waves, X-rays, sound waves, plasma waves, fluid whirls and any other wave type that has ever been observed. There is one exception: gravitational waves have finally been observed in 2016, many decades after their prediction; it is expected that their particle-like aspects, the *gravitons*, also exist, though this might take a long time to prove by experiment.

In summary, the quantum of action states:

▷ If something moves, it is made of quantum particles, or quantons.

Later on we will explore and specify the exact differences between a quantum particle and a small stone or a grain of sand. We will discover that matter quantons move differently, behave differently under rotation, and behave differently under exchange.

Quantum information

In computer science, the smallest unit of change is called a 'bit change'. The existence of a smallest change in nature implies that computer science – or information science – can be used to describe nature, and in particular quantum theory. This analogy has attracted much research in the past decades, and explored many interesting questions: Is unlimited information storage possible? Can information be read out and copied completely? Can information be transmitted while keeping it secret? Can information transmission and storage be performed independently of noise? Can quantum physics be used to make new types of computers? So far, the answer to all these questions is negative; but the hope to change the situation is not dead yet.

The analogy between quantum theory and information science is limited: information science can describe only the 'software' side of devices. For a physicist, the 'hardware' side of nature is central. The hardware of nature enters the description whenever the actual value $\hbar$ of the quantum of action must be introduced.

As we explore the similarities and differences between nature and information science, we will discover that the quantum of action implies that macroscopic physical systems cannot be copied – or 'cloned', as quantum theorists like to say. Nature does not allow copies of macroscopic objects. In other words:

▷ Perfect copying machines do not exist.

The quantum of action makes it impossible to gather and use all information in a way that allows production of a perfect copy.

The exploration of copying machines will remind us again that the precise order in which measurements are performed in an experiment matters. When the order

of measurements can be reversed without affecting the net result, physicists speak of 'commutation'. The quantum of action implies:

    ▷ Physical observables do not commute.

We will also find that the quantum of action implies that systems are not always independent, but can be *entangled*. This term, introduced by Erwin Schrödinger, describes one of the most absurd consequences of quantum theory. Entanglement makes everything in nature connected to everything else. Entanglement produces effects that *seem* (but are not) faster than light.

    ▷ Entanglement produces a (fake) form of non-locality.

Entanglement implies that trustworthy communication cannot exist.

We will also discover that *decoherence* is an ubiquitous process in nature that influences all quantum systems. For example, it allows measurements on the one hand and makes quantum computers impossible on the other.

Curiosities and fun challenges about the quantum of action

Even if we accept that no experiment performed so far contradicts the minimum action, we still have to check that the minimum action does not contradict reason. In particular, the minimum action must also be consistent with all *imagined* experiments. This is not self-evident.

<div align="center">* *</div>

Where is the quantum scale in a pendulum clock?

<div align="center">* *</div>

When electromagnetic fields come into play, the value of the action (usually) depends on the choice of the vector potential, and thus on the choice of gauge. We saw in the part on electrodynamics that a suitable choice of gauge can change the value of the action by adding or subtracting any desired amount. Nevertheless, there is a smallest action in nature. This is possible, because in quantum theory, physical gauge changes cannot add or subtract any amount, but only multiples of *twice* the minimum value. Thus they do not allow us to go below the minimum action.

<div align="center">* *</div>

Adult plants stop growing in the dark. Without light, the reactions necessary for growth cease. Can you show that this is a quantum effect, not explainable by classical physics?

<div align="center">* *</div>

Most quantum processes in everyday life are electromagnetic. Can you show that the quantum of action must also hold for nuclear processes, i.e., for processes that are not electromagnetic?

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Is the quantum of action independent of the observer, even near the speed of light? This question was the reason why Planck contacted the young Einstein, inviting him to Berlin, thus introducing him to the international physics community.

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The quantum of action implies that tiny people, such as *Tom Thumb*, cannot exist. The quantum of action implies that fractals cannot exist in nature. The quantum of action implies that '*Moore's law*' of semiconductor electronics, which states that the number of transistors on a chip doubles every two years, cannot be valid for ever. Why not?

＊＊

Take a horseshoe. The distance between the two ends is not fixed, since otherwise their position and velocity would be known at the same time, contradicting the indeterminacy relation. Of course, this reasoning is also valid for any other solid object. In short, both quantum mechanics and special relativity show that rigid bodies do not exist, albeit for different reasons.

＊＊

Angular momentum has the same dimensions as action. A smallest action implies that there is a smallest angular momentum in nature. How can this be, given that some particles have spin zero, i.e., have no angular momentum?

＊＊

Could we have started the whole discussion of quantum theory by stating that there is a minimum angular momentum instead of a minimum action?

＊＊

Niels Bohr, besides propagating the idea of a minimum action, was also an enthusiast of the so-called *complementarity principle*. This is the idea that certain pairs of observables of a system – such as position and momentum – have linked *precision*: if one observable of the pair is known to high precision, the other observable is necessarily known with low precision. Can you deduce this principle from the minimum action?

The dangers of buying a can of beans

Another way to show the absurd consequences of quantum theory is given by the ultimate product warning, which according to certain well-informed lawyers should be printed on every can of beans and on every product package. It shows in detail how deeply our human condition fools us.

Warning: care should be taken when **looking** at this product:

- It emits heat radiation.
- Bright light has the effect to compress this product.

Warning: care should be taken when **touching** this product:

- Part of it could heat up while another part cools down, causing severe burns.

Warning: care should be taken when **handling** this product:

- This product consists of at least 99.999 999 999 999 % empty space.
- This product contains particles moving with speeds higher than one million kilo-metres per hour.
- Every kilogram of this product contains the same amount of energy as liberated by about one hundred nuclear bombs.*
- In case this product is brought in contact with antimatter, a catastrophic explosion will occur.
- In case this product is rotated, it will emit gravitational radiation.

Warning: care should be taken when **transporting** this product:

- The force needed depends on its velocity, as does its weight.
- This product will emit additional radiation when accelerated.
- This product attracts, with a force that increases with decreasing distance, every other object around, including its purchaser's kids.

Warning: care should be taken when **storing** this product:

- It is impossible to keep this product in a specific place and at rest at the same time.
- Except when stored underground at a depth of several kilometres, over time cosmic radiation will render this product radioactive.
- This product may disintegrate in the next 10^{35} years.
- It could cool down and lift itself into the air.
- This product warps space and time in its vicinity, including the storage container.
- Even if stored in a closed container, this product is influenced and influences all other objects in the universe, including your parents in law.
- This product can disappear from its present location and reappear at any random place in the universe, including your neighbour's garage.

Warning: care should be taken when **travelling away from** this product:

- It will arrive at the expiration date before the purchaser does so.

Warning: care should be taken when **using** this product:

- Any use whatsoever will increase the entropy of the universe.
- The constituents of this product are exactly the same as those of any other object in the universe, including those of rotten fish.

All these statements are correct. The impression of a certain paranoid side to quantum physics is purely coincidental.

Ref. 11 * A standard nuclear warhead has an explosive yield of about 0.2 megatons (implied is the standard explosive trinitrotoluene or TNT), about thirteen times the yield of the Hiroshima bomb, which was 15 kilotonne. A megatonne is defined as 1 Pcal=4.2 PJ, even though TNT delivers about 5 % slightly less energy than this value. In other words, a megaton is the energy content of about 47 g of matter. That is less than a handful for most solids or liquids.

A SUMMARY: QUANTUM PHYSICS, THE LAW AND INDOCTRINATION

The mere existence of a quantum of action, a quantum of change, has many deep consequences: randomness, wave-particle duality, matter transformation, death, and, above all, new thinking habits.

Don't all the deductions from the quantum of action presented so far look wrong, or at least crazy? In fact, if you or your lawyer made some of the statements on quantum physics in court, maybe even under oath, you might end up in prison! However, all the above statements are correct: they are all confirmed by experiment. And there are many more surprises to come. You may have noticed that, in the preceding examples, we have made no explicit reference to electricity, to the nuclear interactions or to gravity. In these domains the surprises are even more astonishing. Observation of antimatter, electric current without resistance, the motion inside muscles, vacuum energy, nuclear reactions in stars, and – maybe one day – the boiling of empty space, will fascinate you as much as they have fascinated, and still fascinate, thousands of researchers.
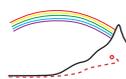
In particular, the consequences of the quantum of action for the early universe are mind-boggling. Just try to explore for yourself its consequences for the big bang. Together, all these topics will lead us a long way towards the aim of our adventure. The consequences of the quantum of action are so strange, so incredible, and so numerous, that quantum physics can rightly be called the description of motion for *crazy* scientists. In a sense, this generalizes our previous definition of quantum physics as the description of motion related to pleasure.

Unfortunately, it is sometimes claimed that 'nobody understands quantum theory'. This is wrong. In fact, it is worse than wrong: it is indoctrination and disinformation. Indoctrination and disinformation are methods that prevent people from making up their own mind and from enjoying life. In reality, the consequences of the quantum of action can be understood and enjoyed by everybody. In order to do so, our first task on our way towards completing our adventure will be to use the quantum of action to study our classical standard of motion: the motion of light.

> " Nie und nirgends hat es Materie ohne
> Bewegung gegeben, oder kann es sie geben.
> Friedrich Engels, *Anti-Dühring*.* "

* 'Never and nowhere has matter existed, nor can it exist, without motion.' Friedrich Engels (1820–1895) was one of the theoreticians of Marxism.
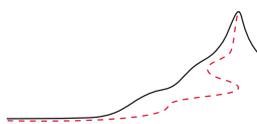
LIGHT – THE STRANGE CONSEQUENCES OF THE QUANTUM OF ACTION

> Alle Wesen leben vom Lichte,
> jedes glückliche Geschöpfe.
> Friedrich Schiller, *Wilhelm Tell*.**

S ince all the colours of materials are quantum effects, it becomes mandatory to tudy the properties of light itself. If a smallest change really exists, then there hould also be a smallest illumination in nature. This conclusion was already drawn in ancient Greece, for example by Epicurus (341–271 BCE), who stated that light is a stream of little particles. The smallest possible illumination would then be that due to a single light particle. Today, the particles are called *light quanta* or *photons*. Incredibly, Epicurus himself could have checked his prediction with an experiment.

HOW DO FAINT LAMPS BEHAVE?

Around 1930, Brumberg and Vavilov found a beautiful way to check the existence of photons using the naked eye and a lamp. Our eyes do not allow us to *consciously* detect single photons, but Brumberg and Vavilov found a way to circumvent this limitation. In fact, the experiment is so simple that it could have been performed many centuries earlier; but nobody had had a sufficiently daring imagination to try it.

Brumberg and Vavilov constructed a mechanical shutter that could be opened for time intervals of 0.1 s. From the other side, in a completely dark room, they illuminated the opening with extremely weak green light: about 200 aW at 505 nm, as shown in Figure 14. At that intensity, whenever the shutter opens, on average about 50 photons can pass. This is just the sensitivity threshold of the eye. To perform the experiment, they repeatedly looked into the open shutter. The result was simple but surprising. Sometimes they observed light, and sometimes they did not. Whether they did or did not was completely random. Brumberg and Vavilov gave the simple explanation that at low lamp powers, because of fluctuations, the number of photons is above the eye threshold half the time, and below it the other half. The fluctuations are random, and so the conscious detection of light is as well. This would *not* happen if light were a *continuous* stream: in that case, the eye would detect light at each and every opening of the shutter. (At higher light intensities, the percentage of non-observations quickly decreases, in accordance with the explanation given.)

In short, a simple experiment proves:

** 'From light all beings live, each fair-created thing.' Friedrich Schiller (b. 1759 Marbach, d. 1805 Weimar), poet, playwright and historian.
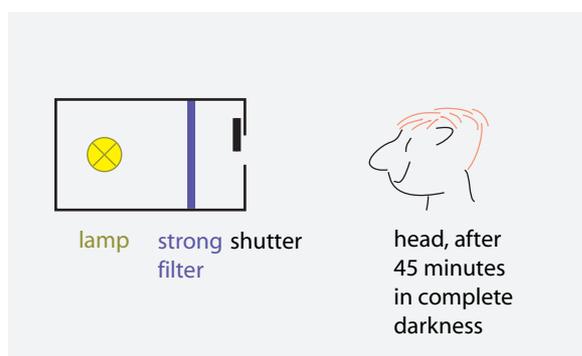
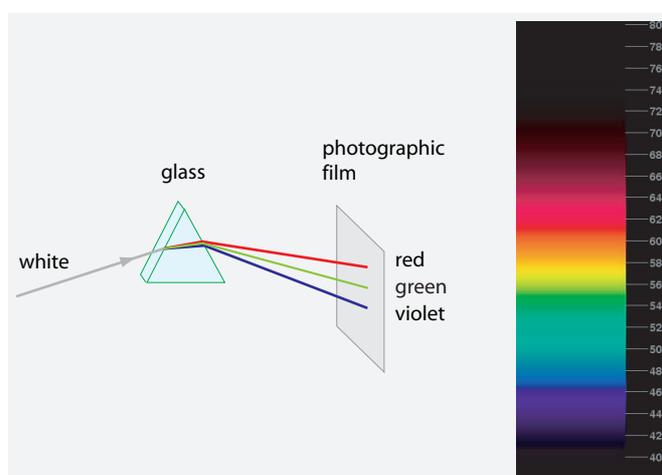FIGURE 14 How to experience single photon effects (see text).

FIGURE 15 How does a white-light spectrum appear at extremely long screen distances? (The short-screen-distance spectrum shown, © Andrew Young, is optimized for CRT display, not for colour printing, as explained on mintaka.sdsu.edu/ GF/explain/optics/rendering. html.)

> ⊳ Light is made of photons.

Nobody knows how the theory of light would have developed if this simple experiment had been performed 100 or even 2500 years earlier.

The reality of photons becomes more convincing if we use devices to help us. A simple way is to start with a screen behind a prism illuminated with white light, as shown in Figure 15. The light is split into colours. As the screen is placed further and further away, the illumination intensity cannot become arbitrarily small, as that would contradict the quantum of action. To check this prediction, we only need some black-and-white photographic film. Film is blackened by daylight of any colour; it becomes dark grey at medium intensities and light grey at lower intensities. Looking at an extremely light grey film under the microscope, we discover that, even under uniform illumination, the grey shade is actually composed of black spots, arranged more or less densely. All these spots have the same size, as shown in Figure 16. This regular size suggests that a photographic film reacts to single photons. Detailed research confirms this conjecture; in the twentieth century, the producers of photographic films have elucidated the underlying atomic mechanism in all its details.
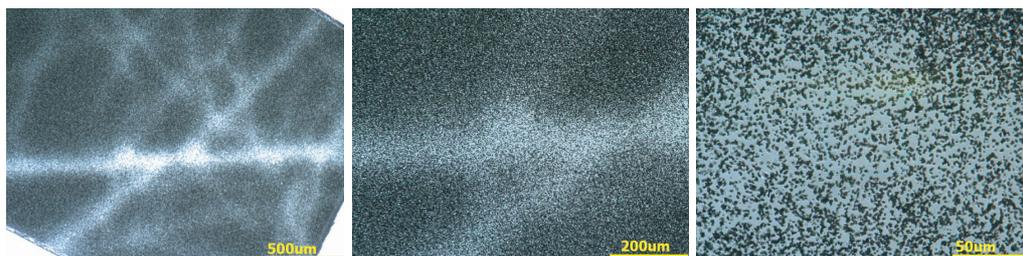
FIGURE 16 Exposed photographic film at increasing magnification (© Rich Evans).

FIGURE 17 Detectors that allow photon counting: photomultiplier tubes (left), an avalanche photodiode (top right, *c*. 1 cm) and a microchannel plate (bottom right, *c*. 10 cm) (© Hamamatsu Photonics).

Single photons can be detected most elegantly with electronic devices. Such devices can be photomultipliers, photodiodes, microchannel plates or rod cells in the eye; a selection is shown in Figure 17. Also these detectors show that low-intensity light does *not* produce a homogeneous colour: on the contrary, low-intensity produces a random pattern of equal spots, even when observing typical wave phenomena such as interference patterns, as shown in Figure 18. Today, recording and counting individual photons is a standard experimental procedure. Photon counters are part of many spectroscopy setups, such as those used to measure tiny concentrations of materials. For example, they are used to detect drugs in human hair.

All experiments thus show the same result: whenever sensitive light detectors are constructed with the aim of 'seeing' as accurately as possible – and thus in environments as
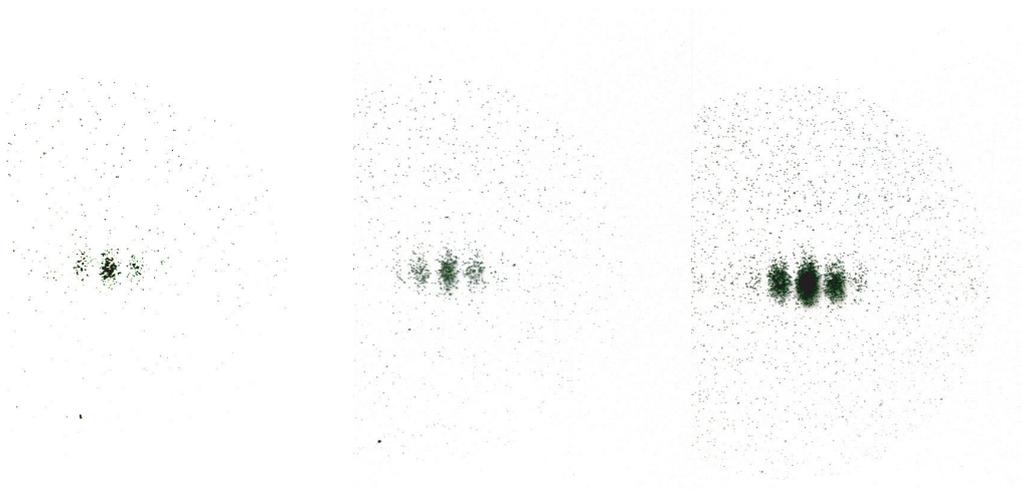
FIGURE 18 Light waves are made of particles: observation of photons – black spots in these negatives – in a low intensity double slit experiment, with exposure times of 1, 2 and 5 s, using an image intensifier (© Delft University of Technology).

light detectors
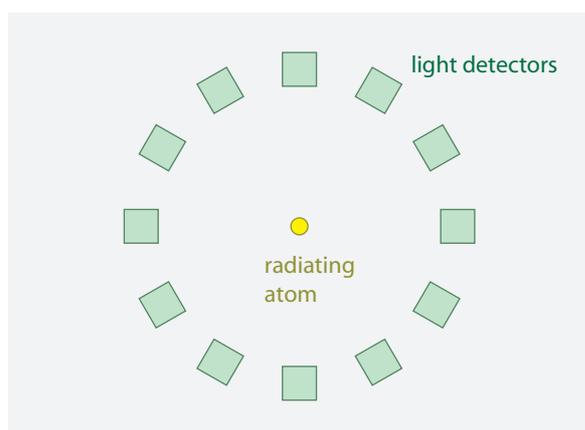
radiating atom

FIGURE 19 An atom radiating *one* photon triggers only *one* detector and recoils in only *one* direction.

dark as possible – one finds that light manifests as a stream of *light quanta*. Nowadays they are usually called *photons*, a term that appeared in 1926. Light of low or high intensity corresponds to a stream with a small or large number of photons.

A particularly interesting example of a low-intensity source of light is a single atom. Atoms are tiny spheres. When atoms radiate light or X-rays, the radiation should be emitted as a spherical wave. But in all experiments – see Figure 19 for a typical set-up – the light emitted by an atom is *never* found to form a spherical wave, in contrast to what we might expect from everyday physics. Whenever a radiating atom is surrounded by many detectors, only a *single* detector is triggered. Only the average over many emissions and detections yields a spherical shape. The experiments shows clearly that partial photons cannot be detected.

All experiments in dim light thus show that the continuum description of light is

incorrect. All such experiments thus prove directly that light is a stream of particles, as Epicurus had proposed in ancient Greece. More precise measurements confirm the role of the quantum of action: every photon leads to the *same* amount of change. All photons of the same frequency blacken a film or trigger a scintillation screen in the same way. In short, the amount of change induced by a single photon is indeed the *smallest* amount of change that light can produce.

If there were no smallest action value, light could be packaged into arbitrarily small amounts. But nature is different. In simple terms: the classical description of light by a *continuous* vector potential $A(t, x)$, or electromagnetic field $F(t, x)$, whose evolution is described by a principle of least action, is *wrong*. Continuous functions do not describe the observed particle effects. A modified description is required. The modification has to be significant only at low light intensities, since at high, everyday intensities the classical Lagrangian describes all experimental observations with sufficient accuracy.*

At which intensities does light cease to behave as a continuous wave? Human eyesight does not allow us to consciously distinguish single photons, although experiments show that the hardware of the eye is in principle able to do so. The faintest stars that can be seen at night produce a light intensity of about $0.6\,\text{nW/m}^2$. Since the pupil of the eye is small, and we are not able to see individual photons, photons must have energies smaller than 100 aJ. Brumberg and Vavilov's experiment yields an upper limit of around 20 aJ.

An exact value for the quantum of action found in light must be deduced from laboratory experiment. Some examples are given in the following.

PHOTONS

In general, all experiments show that a beam of light of frequency f or angular frequency ω, which determines its colour, is accurately described as a stream of photons, each with the same energy E given by

$$E = \hbar\, 2\pi f = \hbar\, \omega \, . \tag{8}$$

This relation was first deduced by Max Planck in 1899. He found that for light, the smallest measurable action is given by the quantum of action $\hbar$. In short, *colour* is a property of photons. A coloured light beam is a hailstorm of corresponding photons.

The value of *Planck's constant* can be determined from measurements of black bodies or other light sources. All such measurements coincide and yield

$$\hbar = 1.054\,571\,726(47) \cdot 10^{-34}\,\text{Js} \, , \tag{9}$$

a value so small that we can understand why photons go unnoticed by humans. For example, a green photon with a wavelength of 555 nm has an energy of 0.37 aJ. Indeed, in normal light conditions the photons are so numerous that the continuum approximation for the electromagnetic field is highly accurate. In the dark, the insensitivity of the signal processing of the human eye – in particular the slowness of the light receptors – makes
photon counting impossible. However, the eye is not far from the maximum possible
sensitivity. From the numbers given above about dim stars, we can estimate that humans

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* The transition from the classical case to the quantum case used to be called *quantization*. This concept, and the ideas behind it, are only of historical interest today.

are able to see *consciously*, under ideal conditions, flashes of about half a dozen photons; in normal conditions, the numbers are about ten times higher.

Let us explore the other properties of photons. Above all, photons have no measurable (rest) mass and no measurable electric charge. Can you confirm this? In fact, experiments can only provide an upper limit for both quantities. The present experimental upper limit for the (rest) mass of a photon is 10^{-52} kg, and for the charge is $5 \cdot 10^{-30}$ times the electron charge. These limits are so small that we can safely say that both the mass and the charge of the photon vanish.

We know that intense light can *push* objects. Since the energy, the lack of mass and the speed of photons are known, we deduce that the photon momentum is given by

$$p = \frac{E}{c} = \hbar \frac{2\pi}{\lambda} \quad \text{or} \quad \boldsymbol{p} = \hbar \boldsymbol{k} . \tag{10}$$

In other words, if light is made of particles, we should be able to play billiard with them. This is indeed possible, as Arthur Compton showed in a famous experiment in 1923. He directed X-rays, which are high-energy photons, onto graphite, a material in which electrons move almost freely. He found that whenever the electrons in the material are hit by the X-ray photons, the deflected X-rays change colour. His experiment is shown in Figure 20. As expected, the strength of the hit is related to the deflection angle of the photon. From the colour change and the deflection angle, Compton confirmed that the photon momentum indeed satisfies the expression $\boldsymbol{p} = \hbar \boldsymbol{k}$.

All other experiments agree that photons have momentum. For example, when an atom *emits* light, the atom feels a *recoil*. The momentum again turns out to be given by the expression $\boldsymbol{p} = \hbar \boldsymbol{k}$. In short, the quantum of action determines the momentum of the photon.

The value of a photon's momentum respects the indeterminacy relation. Just as it is impossible to measure exactly both the wavelength of a wave and the position of its crest, so it is impossible to measure both the momentum and the position of a photon. Can you confirm this? In other words, the value of the photon momentum is a direct consequence of the quantum of action.

From our study of classical physics, we know that light has a property beyond its colour: light can be *polarized*. That is only a complicated way to say that light can *turn* the objects that it shines on. In other words, light has an angular momentum oriented (mainly) along the axis of propagation. What about photons? Measurements consistently find that each light quantum carries an *angular momentum* given by $L = \hbar$. It is called its *helicity*. The quantity is similar to one found for massive particles: one therefore also speaks of the *spin* of a photon. In short, photons somehow 'turn' – in a direction either parallel or antiparallel to their direction of motion. Again, the magnitude of the photon helicity, or spin, is no surprise; it confirms the classical relation $L = E/\omega$ between energy and angular momentum that we found in the section on classical electrodynamics. Note that, counterintuitively, the angular momentum of a single photon is fixed, and thus independent of its energy. Even the most energetic photons have $L = \hbar$. Of course, the value of the helicity also respects the limit given by the quantum of action. The many consequences of the helicity (spin) value $\hbar$ will become clear in the following.
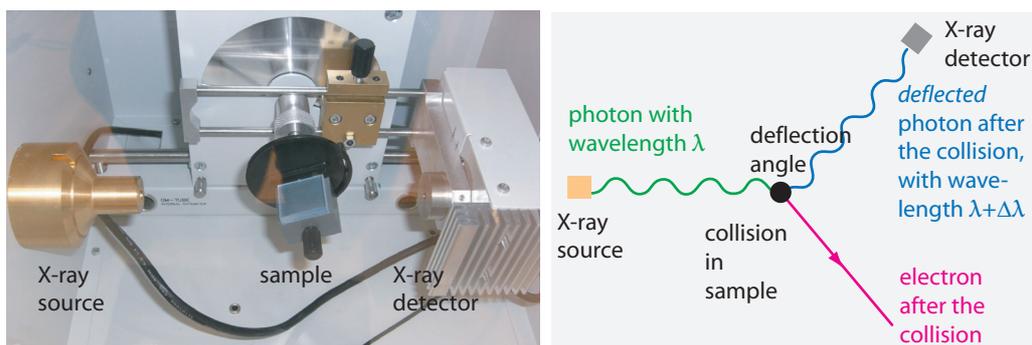
FIGURE 20 A modern version of Compton's experiment fits on a table. The experiment shows that photons have momentum: X-rays – and thus the photons they consist of – change frequency when they hit the electrons in matter in exactly the same way as predicted from colliding particles (© Helene Hoffmann).

WHAT IS LIGHT?

> La lumière est un mouvement luminaire de
> corps lumineux.
>
> Blaise Pascal[*]

In the seventeenth century, Blaise Pascal used the above statement about light to make fun of certain physicists, ridiculing the blatant use of a circular definition. Of course, he was right: in his time, the definition was indeed circular, as no meaning could be given to any of the terms. But whenever physicists study an observation with care, philosophers lose out. All those originally undefined terms now have a definite meaning and the circular definition is resolved. Light is indeed a type of motion; this motion can rightly be called 'luminary' because, in contrast to the motion of material bodies, it has the unique property $v = c$; the luminous bodies, called *light quanta* or *photons*, are characterized, and differentiated from all other particles, by their dispersion relation $E = cp$, their energy $E = \hbar\omega$, their spin $L = \hbar$, the vanishing of all other quantum numbers, and the property of being the quanta of the electromagnetic field.

In short, *light is a stream of photons*. It is indeed a 'luminary movement of luminous bodies'. Photons provide our first example of a general property of the world on small scales: *all waves and all flows in nature are made of quantum particles*. Large numbers of (coherent) quantum particles – or *quantons* – behave as and form waves. We will see shortly that this is the case even for matter. Quantons are the fundamental constituents of *all* waves and *all* flows, without exception. Thus, the everyday continuum description of light is similar in many respects to the description of water as a continuous fluid: photons are the atoms of light, and continuity is an approximation valid for large numbers of particles. Single quantons often behave like classical particles.

Physics books used to discuss at length a so-called *wave–particle duality*. Let us be clear from the start: quantons, or quantum particles, are *neither* classical waves *nor* clas-

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[*] 'Light is the luminary movement of luminous bodies.' Blaise Pascal (b. 1623 Clermont, d. 1662 Paris), important mathematician and physicist up to the age of 26, after which he became a theologian and philosopher.

sical particles. In the microscopic world, quantons are the fundamental objects.

However, there is much that is still unclear. Where, inside matter, do these mono-chromatic photons come from? Even more interestingly, if light is made of quantons, all electromagnetic fields, even static ones, must be made of photons as well. However, in static fields nothing is flowing. How is this apparent contradiction resolved? And what implications does the particle aspect have for these static fields? What is the difference between quantons and classical particles? The properties of photons require more careful study.

The size of photons

First of all, we might ask: what are these photons made of? All experiments so far, per-formed down to the present limit of about 10^{-20} m, give the same answer: 'we can't find anything'. This is consistent with both a vanishing mass and a vanishing size of photons. Indeed, we would intuitively expect a body with a finite size to have a finite mass. Thus, although experiments can give only an upper limit, it is consistent to claim that a photon has *zero size*.

A particle with zero size cannot have any constituents. Thus a photon cannot be di-vided into smaller entities: photons are not composite. For this reason, they are called *elementary* particles. We will soon give some further strong arguments for this result. Challenge 35 s (Can you find one?) Nevertheless, the conclusion is strange. How can a photon have vanishing size, have no constituents, and still be *something*? This is a hard question; the answer will appear only in the last volume of our adventure. At the moment we simply have to accept the situation as it is. We therefore turn to an easier question.

Are photons countable? – Squeezed light

> Also gibt es sie doch.
>
> Max Planck[*]

We saw above that the simplest way to count photons is to distribute them across a large screen and then to absorb them. But this method is not entirely satisfactory, as it destroys the photons. How can we count photons without destroying them?

One way is to reflect photons in a mirror and measure the recoil of the mirror. It seems almost unbelievable, but nowadays this effect is becoming measurable even for small numbers of photons. For example, it has to be taken into account in relation to the Vol. II, page 181 laser mirrors used in gravitational wave detectors, whose position has to be measured with high precision.

Another way of counting photons without destroying them involves the use of special high-quality laser cavities. It is possible to count photons by the effect they have on atoms cleverly placed inside such a cavity.

In other words, light intensity can indeed be measured without absorption. These measurement show an important issue: even the best light beams, from the most sophist-

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[*] 'Thus they do exist after all.' Max Planck, in his later years, said this after standing silently, for a long time, in front of an apparatus that counted single photons by producing a click for each photon it detected. For a large part of his life, Planck was sceptical of the photon concept, even though his own experiments and conclusions were the starting point for its introduction.

icated lasers, *fluctuate* in intensity. There are *no* steady beams. This comes as no surprise: if a light beam did *not* fluctuate, observing it twice would yield a vanishing value for the action. However, there is a minimum action in nature, namely $\hbar$. Thus any beam and any flow in nature *must* fluctuate. But there is more.

A light beam is described, in a cross section, by its intensity and phase. The change – or action – that occurs while a beam propagates is given by the product of intensity and phase. Experiments confirm the obvious deduction: the intensity and phase of a beam behave like the momentum and position of a particle in that they obey an indeterminacy relation. You can deduce it yourself, in the same way as we deduced Heisenberg's relations. Using as characteristic intensity $I = E/\omega$, the beam energy divided by the angular frequency, and calling the phase φ, we get*

$$\Delta I \, \Delta\varphi \geqslant \frac{\hbar}{2} \ . \tag{12}$$

Equivalently, the indeterminacy product for the average photon number $n = I/\hbar = E/\hbar\omega$ and the phase φ obeys:

$$\Delta n \, \Delta\varphi \geqslant \frac{1}{2} \ . \tag{13}$$

For light emitted from an ordinary lamp, so-called *thermal light*, the indeterminacy product on the left-hand side of the above inequality is a large number. Equivalently, the indeterminacy product for the action (12) is a large multiple of the quantum of action.

For laser beams, i.e., beams of *coherent light*,** the indeterminacy product is close to 1/2. An illustration of coherent light is given in Figure 22.

Today it is possible to produce light for which the product of the two indeterminacies in equation (13) is near 1/2, but whose two values *differ* (in the units of the so-called *phasor space* illustrated in Figure 21). Such light is called *non-classical* or *squeezed*. The photon statistics is either hyper- or sub-Poissonian. Such light beams require involved laboratory set-ups for their production and are used in many modern research applications. Non-classical light has to be treated extremely carefully, as the smallest disturbances transforms it back into ordinary coherent (or even thermal light), in which Poisson (or even Bose-Einstein) statistics hold again. A general overview of the main types of light beams is given in Figure 21, together with their intensity and phase behaviour. (Several properties shown in the figure are defined for a single phase space cell only.)

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* A large photon number is assumed in the expression. This is obvious, as $\Delta\varphi$ cannot grow beyond all bounds, more precisely, not beyond 2π. The exact relations are

$$\Delta I \, \Delta \cos\varphi \geqslant \frac{\hbar}{2}|\langle \sin\varphi \rangle|$$

$$\Delta I \, \Delta \sin\varphi \geqslant \frac{\hbar}{2}|\langle \cos\varphi \rangle| \tag{11}$$

where $\langle x \rangle$ denotes the expectation value of the observable x.

** Coherent light is light for which the photon number probability distribution is Poissonian; in particular, the variance is equal to the mean photon number. Coherent light is best described as composed of photons in coherent quantum states. Such a (canonical) *coherent state*, or *Glauber state*, is formally a state with $\Delta\varphi \to 1/n$ and $\Delta n \to n$.
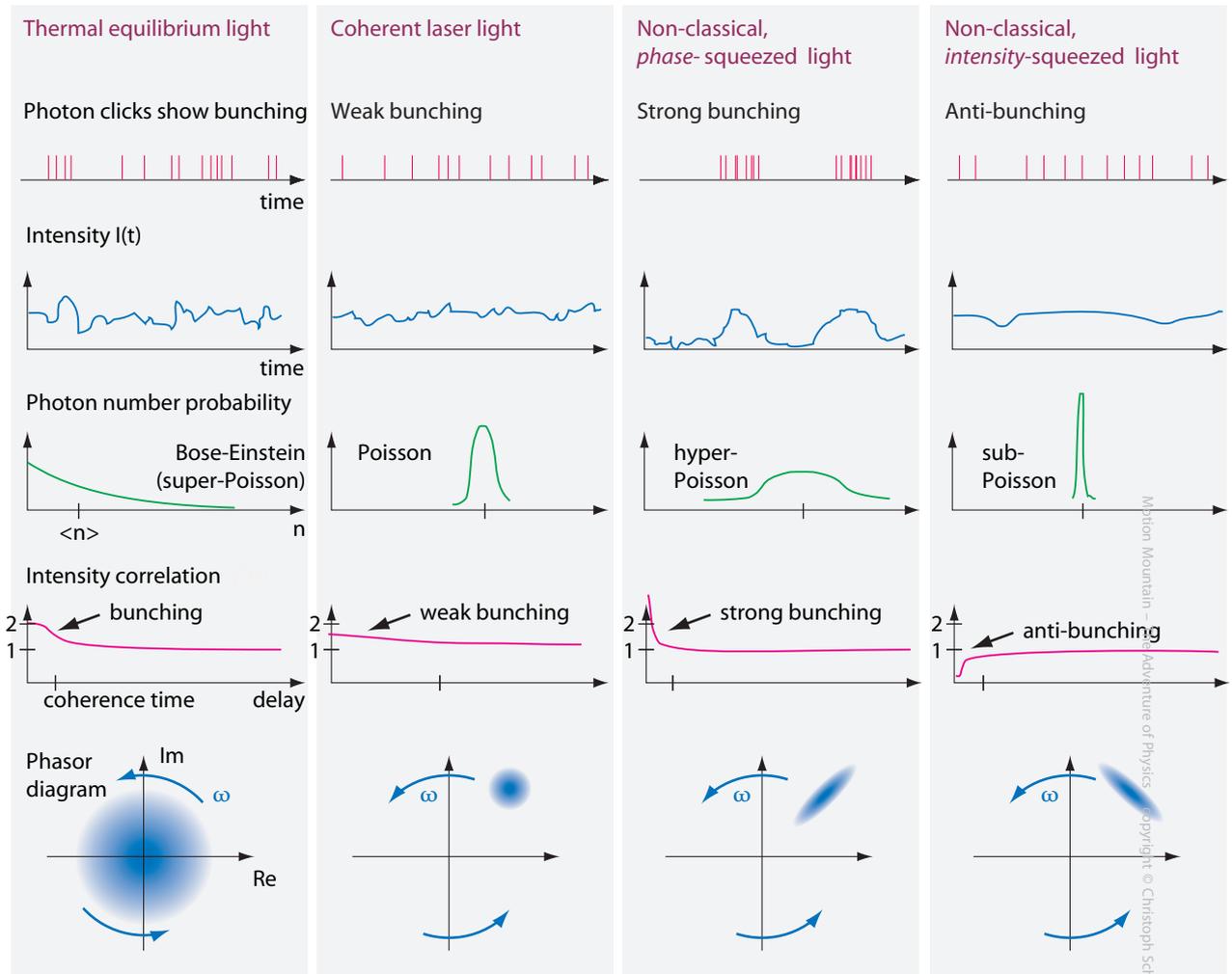
FIGURE 21 Four types of light and their photon properties: thermal light, laser light, and two extreme types of non-classical, squeezed light.

One extreme of non-classical light is *phase-squeezed light*. Since a phase-squeezed light beam has an (almost) determined phase, the photon number in such a beam fluctuates from zero to (almost) infinity. In other words, in order to produce coherent laser light that approximates a pure sine wave as perfectly as possible, we must accept that the photon number is as undetermined as possible. Such a beam has extremely small phase fluctuations that provide high precision in interferometry; the phase noise is as low as possible.

The other extreme of non-classical light is a beam with a given, fixed number of photons, and thus with an extremely high phase indeterminacy. In such an *amplitude-squeezed light beam*, the phase fluctuates erratically.* This sort of squeezed, non-classical

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* The most appropriate quantum states to describe such light are called *number states*, sometimes *Fock states*. These states are stationary, thus eigenstates of the Hamiltonian, and contain a fixed number of photons.
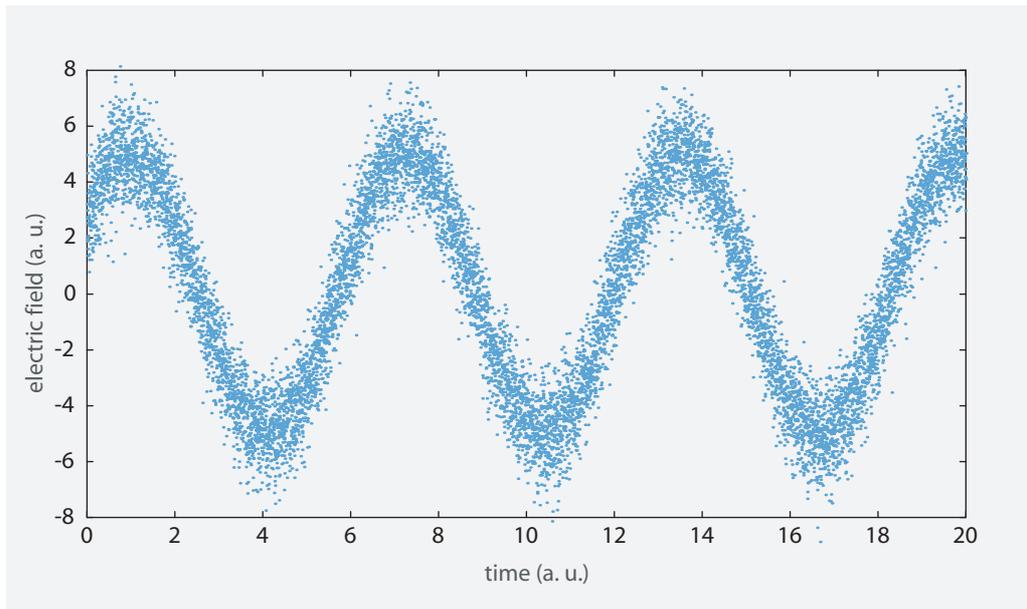
FIGURE 22 A simple way to illustrate the indeterminacy of a light beam's intensity and phase: the measured electric field of a coherent electromagnetic wave with low intensity, consisting of about a dozen photons. The cloudy sine wave corresponds to the phasor diagram at the bottom of the second column in the previous overview. For large number of photons, the relative noise amplitude is negligible. (© Rüdiger Paschotta)

light is ideal for precision intensity measurements as it provides the lowest intensity noise available. This kind of light shows *anti-bunching* of photons. To gain more insight, sketch the graphs corresponding to Figure 22 for phase-squeezed and for amplitude-squeezed light.

In contrast, the *coherent light* that is emitted by laser pointers and other lasers lies between the two extreme types of squeezed light: the phase and photon number indeterminacies are of similar magnitude.

The observations about thermal light, coherent laser light and non-classical light highlight an important property of nature: the number of photons in a light beam is not a well-defined quantity. In general, it is undetermined, and it *fluctuates*. Photons, unlike stones, cannot be counted precisely – as long as they are propagating and not absorbed. In flight, it is only possible to determine an approximate, average photon number, within the limits set by indeterminacy. Is it correct to claim that the number of photons at the beginning of a beam is not necessarily the same as the number at the end of the beam?

The fluctuations in the number of photons are of most importance at optical frequencies. At radio frequencies, the photon number fluctuations are usually negligible, due to the low photon energies and the usually high photon numbers involved. Conversely, at gamma-ray energies, wave effects play little role. For example, we saw that in deep, dark intergalactic space, far from any star, there are about 400 photons per cubic centimetre; they form the cosmic background radiation. This photon density number, like the number of photons in a light beam, also has a measurement indeterminacy. Can you estimate it?
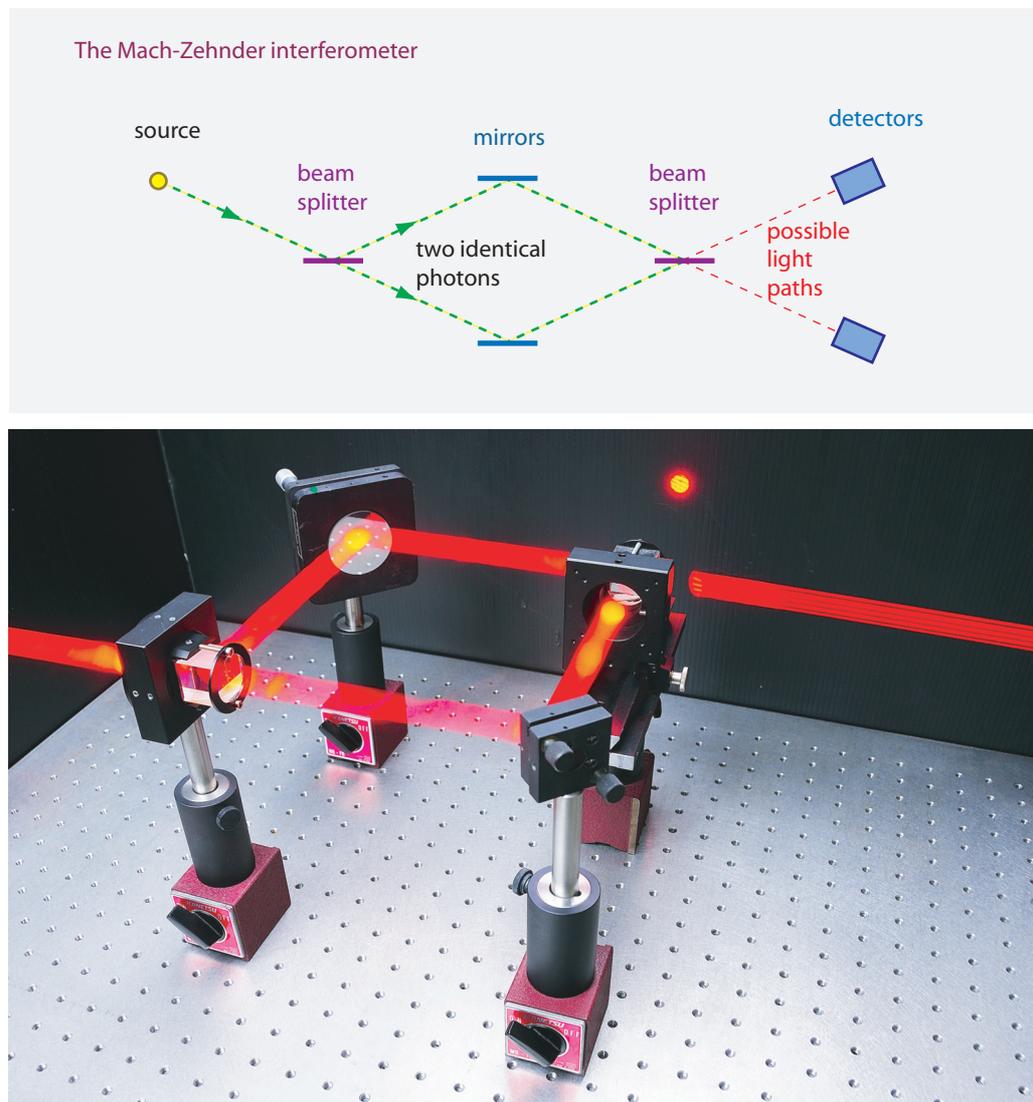
FIGURE 23 The Mach–Zehnder interferometer and a practical realization, about 0.5 m in size (© Félix Dieu and Gaël Osowiecki).

In short, unlike pebbles, *photons are countable, but their number is not fixed.* And this is not the only difference between photons and pebbles.

THE POSITIONS OF PHOTONS

Where is a photon when it moves in a beam of light? Quantum theory gives a simple answer: nowhere in particular. This is proved most spectacularly by experiments with interferometers, such as the basic interferometer shown in Figure 23. Interferometers show that even a beam made of a *single* photon can be split, led along two different paths, and then recombined. The resulting interference shows that the single photon cannot be

said to have taken either of the two paths. If one of the two paths is blocked, the pattern on the screen changes. In other words, somehow the photon must have taken both paths at the same time. Photons cannot be localized: they have no position.*

We come to the conclusion that macroscopic light pulses have paths, but the individual photons in it do not. Photons have neither position nor paths. Only large numbers of photons can have positions and paths, and then only approximately.

The impossibility of localizing photons can be quantified. Interference shows that it is impossible to localize photons in the direction *transverse* to the motion. It might seem less difficult to localize photons *along* the direction of motion, when it is part of a light pulse, but this is a mistake. The quantum of action implies that the indeterminacy in the longitudinal position is given at least by the wavelength of the light. Can you confirm this? It turns out that photons can only be localized within a *coherence length*. In fact, the transversal and the longitudinal coherence length differ in the general case. The longitudinal coherence length (divided by *c*) is also called temporal coherence, or simply, the *coherence time*. It is also indicated in Figure 21. The impossibility of localizing photons is a consequence of the quantum of action. For example, the transverse coherence length is due to the indeterminacy of the transverse momentum; the action values for paths leading to points separated by less than a coherence length differ by less than the quantum of action $\hbar$. Whenever a photon is detected somewhere, e.g., by absorption, a *precise* statement on its direction or its origin *cannot* be made. Sometimes, in special cases, there can be a high probability for a certain direction or source, though.

Lack of localisation means that photons *cannot* be simply visualized as short wave trains. For example, we can increase the coherence length by sending light through a narrow filter. Photons are truly *unlocalizable* entities, specific to the quantum world. Photons are neither little stones nor little wave packets. Conversely, 'light path', 'light pulse position' and 'coherence' are properties of a photon ensemble, and do *not* apply to a single photon.

Whenever photons can *almost* be localized along their direction of motion, as in coherent light, we can ask how photons are lined up, one after the other, in a light beam. Of course, we have just seen that it does not make sense to speak of their *precise* position. But do photons in a perfect beam arrive at almost-regular intervals?

To the shame of physicists, the study of photon correlations was initiated by two astronomers, Robert Hanbury Brown and Richard Twiss, in 1956, and met with several years of disbelief. They varied the transversal distance of the two detectors shown in Figure 24 – from a few to 188 m – and measured the intensity correlations between them. Hanbury Brown and Twiss found that the intensity fluctuations within the volume of coherence are correlated. Thus the photons themselves are correlated. With this experiment, they were able to measure the diameter of numerous distant stars.

Inspired by the success of Hanbury Brown and Twiss, researchers developed a simple method to measure the probability that a second photon in a light beam arrives at a given time after the first one. They simply split the beam, put one detector in the first branch, and varied the position of a second detector in the other branch. The set-up is sketched in Figure 25. Such an experiment is nowadays called a *Hanbury Brown Twiss experiment*.

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* We cannot avoid this conclusion by saying that photons are *split* at the beam splitter: if we place a detector in each arm, we find that they never detect a photon at the same time. Photons cannot be divided.
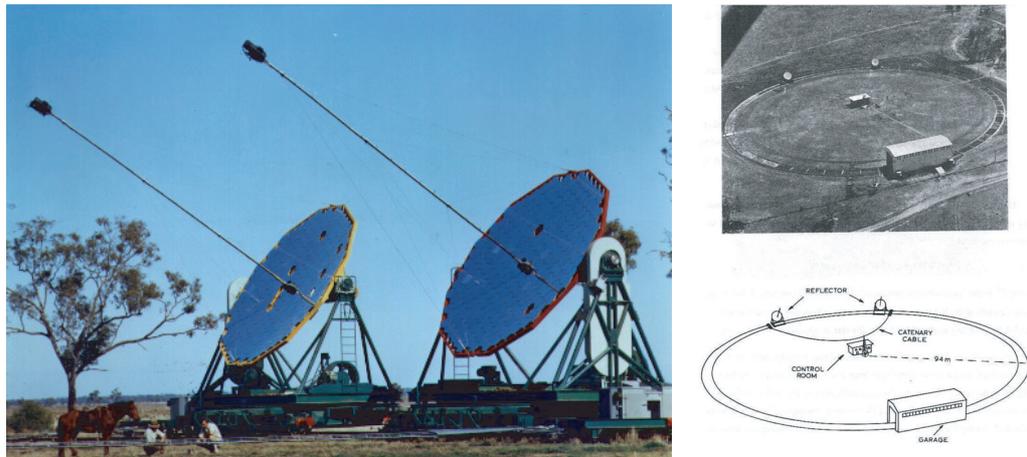
FIGURE 24 The original experimental set-up with which Hanbury Brown and Twiss measured stellar diameters at Narrabri in Australia. The distance between the two light collectors could be changed by moving them on rails. The light detectors are at the end of the poles and each of them, as they wrote, 'collected light as rain in a bucket.' (© John Davis).
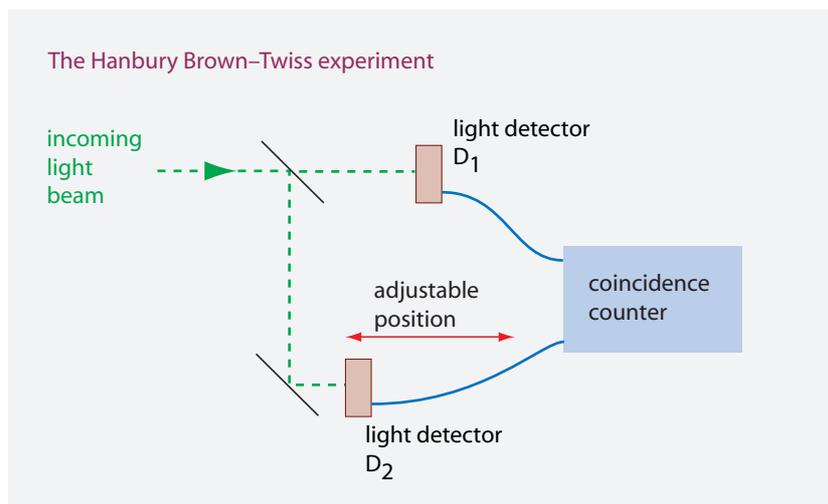
FIGURE 25 How to measure photon statistics with an electronic intensity correlator or coincidence counter, the variation being measured by varying the position of a detector.

One finds that, for coherent light within the volume of coherence, the clicks in the two counters – and thus the photons themselves – are *correlated*. To be more precise, such experiments show that whenever the first photon hits, the second photon is most likely to hit just afterwards. Thus, photons in light beams are *bunched*. Bunching is one of the many results showing that photons are quantons, that they are indeed necessary to describe light, and that they are unlocalizable entities. As we will see below, the result also implies that photons are bosons.

Every light beam has an upper time limit for bunching: the *coherence time*. For times longer than the coherence time, the probability for bunching is low, and independent of the time interval, as shown in Figure 25. The coherence time characterizes every light
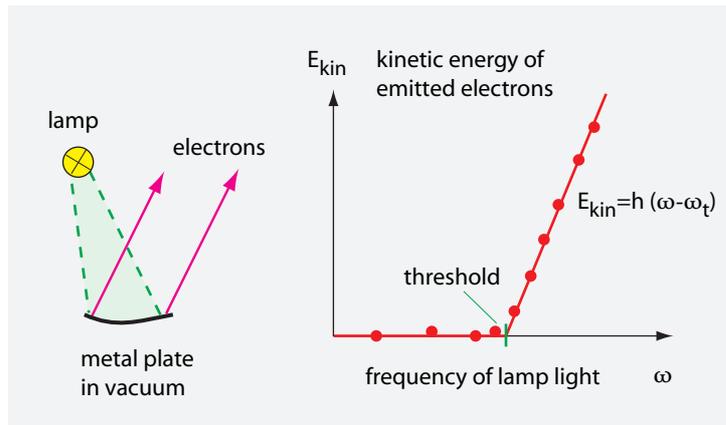
FIGURE 26 The kinetic energy of electrons emitted in the photoelectric effect.

beam. In fact, it is often easier to think in terms of the *coherence length* of a light beam. For thermal light, the coherence length is only a few micrometres: a small multiple of the wavelength. The largest coherence lengths, of over 300 000 km, are obtained with research lasers that have an extremely narrow laser bandwith of just 1 Hz. Interestingly, coherent light is even found in nature: several special stars have been found to emit it.

Although the intensity of a good laser beam is almost constant, the photons do not arrive at regular intervals. Even the best laser light shows bunching, though with different statistics and to a lesser degree than lamp light, as illustrated in Figure 21. Light whose photons arrive regularly, thus exhibiting so-called *(photon) anti-bunching*, is obviously non-classical in the sense defined above; such light can be produced only by special experimental arrangements. Extreme examples of this phenomenon are being investigated at present by several research groups aiming to construct light sources that emit one photon at a time, at regular time intervals, as reliably as possible. In short, we can state that the precise photon statistics in a light beam depends on the mechanism of the light source.

In summary, experiments force us to conclude that light is made of photons, but also that photons cannot be localized in light beams. It makes no sense to talk about the position of a photon in general; the idea makes sense only in some special situations, and then only approximately and as a statistical average.

Are photons necessary?

In light of the results uncovered so far, the answer to the above question is obvious. But the issue is tricky. In textbooks, the *photoelectric effect* is usually cited as the first and most obvious experimental proof of the existence of photons. In 1887, Heinrich Hertz observed that for certain metals, such as lithium or caesium, incident ultraviolet light leads to charging of the metal. Later studies of the effect showed that the light causes emission of electrons, and that the energy of the ejected electrons does not depend on the intensity of the light, but only on the difference between $\hbar$ times its frequency and a material-dependent threshold energy. Figure 26 summarizes the experiment and the measurements.

In classical physics, the photoelectric effect is difficult to explain. But in 1905, Albert Einstein deduced the measurements from the assumption that light is made of photons of energy $E = \hbar\omega$. He imagined that this energy is used partly to take the electron over the threshold, and partly to give it kinetic energy. More photons only lead to more electrons, not to faster ones. In 1921, Einstein received the Nobel Prize for the explanation of the photoelectric effect. But Einstein was a *genius*: he deduced the correct result by a somewhat incorrect reasoning. The (small) mistake was the assumption that a classical, continuous light beam would produce a different effect. In fact, it is easy to see that a classical, continuous electromagnetic field interacting with discrete matter, made of discrete atoms containing discrete electrons, would lead to exactly the same result, as long as the motion of electrons is described by quantum theory. Several researchers confirmed

this early in the twentieth century. The photoelectric effect by itself does *not* imply the existence of photons.

Indeed, many researchers in the past were unconvinced that the photoelectric effect shows the existence of photons. Historically, the most important argument for the *necessity* of light quanta was given by Henri Poincaré. In 1911 and 1912, aged 57 and only a few months before his death, he published two influential papers proving that the radiation law of black bodies – in which the quantum of action had been discovered by Max Planck

– *requires* the existence of photons. He also showed that the amount of radiation emitted by a hot body is *finite* only because of the *quantum* nature of the processes leading to light emission. A description of these processes in terms of classical electrodynamics would lead to (almost) *infinite* amounts of radiated energy. Poincaré's two influential papers convinced most physicists that it was worthwhile to study quantum phenomena in more detail. Poincaré did not know about the action limit $S \geqslant \hbar$; yet his argument is based on the observation that light of a given frequency has a minimum intensity, namely a single photon. Such a one-photon beam may be split into two beams, for example by using a half-silvered mirror. However, taken together, those two beams never contain more than a single photon.

Another interesting experiment that requires photons is the observation of 'molecules

of photons'. In 1995, Jacobson et al. predicted that the de Broglie wavelength of a *packet* of photons could be observed. According to quantum theory, the packet wavelength is given by the wavelength of a single photon divided by the number of photons in the packet. The team argued that the packet wavelength could be observable if such a packet could be split and recombined without destroying the cohesion within it. In 1999, this effect was indeed observed by de Pádua and his research group in Brazil. They used a careful set-up with a nonlinear crystal to create what they call a *biphoton*, and observed its interference properties, finding a reduction in the effective wavelength by the predicted factor of two.

Since then, packages with three and even four entangled photons have been created and observed.

Yet another argument for the necessity of photons is the above-mentioned recoil felt

by atoms emitting light. The recoil measured in these cases is best explained by the emission of a photon in a particular direction. In contrast, classical electrodynamics predicts the emission of a spherical wave, with no preferred direction.

Obviously, the observation of *non-classical light*, also called *squeezed light*, also argues for the existence of photons, as squeezed light proves that photons are indeed an *intrinsic*

aspect of light, necessary even when interactions with matter play no role. The same is
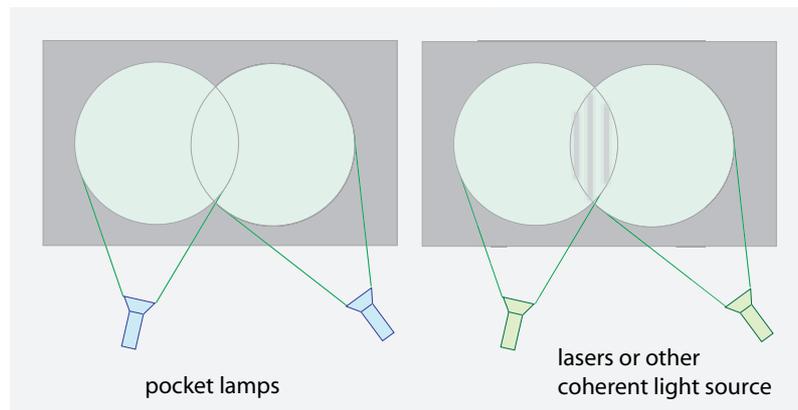
FIGURE 27 Two situations in which light crosses light: different light sources lead to different results.

true for the Hanbury Brown–Twiss effect.

Finally, the spontaneous decay of excited atomic states also requires the existence of photons. This cannot be explained by a continuum description of light.

In summary, the concept of a photon is indeed necessary for a precise description of light; but the details are often subtle, as the properties of photons are unusual and require a change in our habits of thought. To avoid these issues, most textbooks stop discussing photons after coming to the photoelectric effect. This is a pity, as it is only then that things get interesting. Ponder the following. Obviously, all electromagnetic fields are made of photons. At present, photons can be counted for gamma rays, X-rays, ultraviolet light, visible light and infrared light. However, for lower frequencies, such as radio waves, photons have not yet been detected. Can you imagine what would be necessary to count the photons emitted from a radio station? This issue leads directly to the most important question of all:

Interference: how can a wave be made up of particles?

> Die ganzen fünfzig Jahre bewusster Grübelei haben mich der Antwort auf die Frage 'Was sind Lichtquanten?' nicht näher gebracht. Heute glaubt zwar jeder Lump er wisse es, aber er täuscht sich.
>
> Albert Einstein, 1951 *

If a light wave is made of particles, we must be able to explain each and every wave property in terms of photons. The experiments mentioned above already hint that this is possible only because photons are *quantum* particles. Let us take a more detailed look at this connection.

Light can *cross* other light undisturbed, for example when the light beams from two pocket lamps shine through each other. This observation is not hard to explain with

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* 'Fifty years of conscious brooding have not brought me nearer to the answer to the question 'What are light quanta?' Nowadays every bounder thinks he knows it, but he is wrong.' Einstein wrote this a few years before his death in a letter to Michele Besso.

Challenge 40 s

Ref. 28
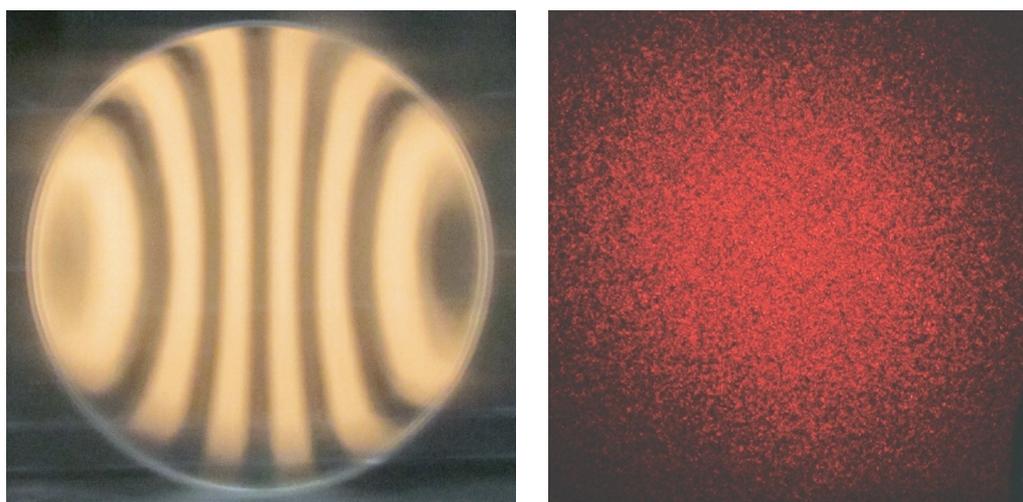
FIGURE 28 Examples of interference patterns that appear when coherent light beams cross: the interference produced by a self-made parabolic telescope mirror of 27 cm diameter, and a speckle laser pattern on a rough surface (© Mel Bartels, Epzcaw).

photons; since photons do not interact with each other, and are point-like, they 'never' hit each other. In fact, there *is* an extremely small positive probability for their interaction, as we will find out later, but this effect is not observable in everyday life.

But if two *coherent* light beams, i.e., two light beams of identical frequency and fixed phase relation cross, we observe alternating bright and dark regions: so-called *interference fringes*. The schematic set-up is shown in Figure 27. Examples of actual interference effects are given in Figure 28 and Figure 29. How do these interference fringes appear?* How can it be that photons are not detected in the dark regions? We already know the only possible answer: the brightness at a given place corresponds to the *probability* that a photon will arrive there. The fringes imply:

▷ Photons behave like moving little *arrows*.

Some further thought leads to the following description:

— The arrow is always perpendicular to the direction of motion.
— The arrow's direction stays fixed in space when the photons move.
— The length of an arrow shrinks with the square of the distance travelled.
— *The probability of a photon arriving somewhere is given by the square of an arrow.*
— *The final arrow is the* sum *of all the arrows arriving there by all possible paths.*
— Photons emitted by single-coloured sources are emitted with arrows of constant length pointing in the direction ωt; in other words, such sources spit out photons with a *rotating* mouth.
— Photons emitted by incoherent sources – e.g., thermal sources, such as pocket lamps – are emitted with arrows of constant length pointing in *random* directions.

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* If lasers are used, fringes can only be observed if the two beams are derived from a single beam by splitting, or if two expensive high-precision lasers are used. (Why?)
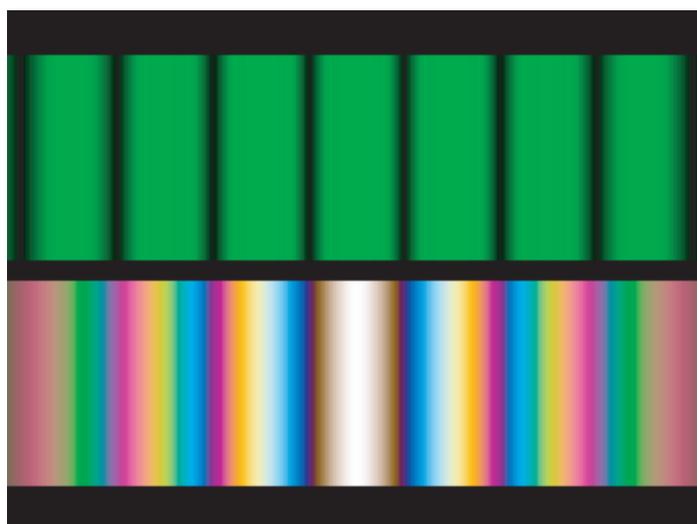
FIGURE 29 Top: calculated interference patterns – and indistinguishable from observed ones under ideal, "textbook" conditions – produced by two parallel narrow slits illuminated with green light and with white light. Bottom: two Gaussian beams interfering at an angle (© Dietrich Zawischa, Rüdiger Paschotta).

With this simple model* we can explain the wave behaviour of light. In particular, we can describe the interference stripes seen in laser experiments, as shown schematically in Figure 30. You can check that in some regions the two arrows travelling through the two slits add up to zero *for all times*. No photons are detected there: those regions are black. In other regions, the arrows always add up to the maximal value. These regions are always bright. Regions in between have intermediate shades. Obviously, in the case of usual pocket lamps, shown in the left-hand diagram of Figure 27, the brightness in the common region also behaves as expected: the averages simply add up.

Obviously, the photon model implies that an interference pattern is built up as the sum of a large number of single-photon hits. Using low-intensity beams, we should therefore be able to see how these little spots slowly build up an interference pattern by accumulating in the bright regions and never hitting the dark regions. This is indeed the case, as

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* The model gives a correct description of light except that it neglects polarization. To add polarization, it is necessary to combine arrows that rotate in both senses around the direction of motion.
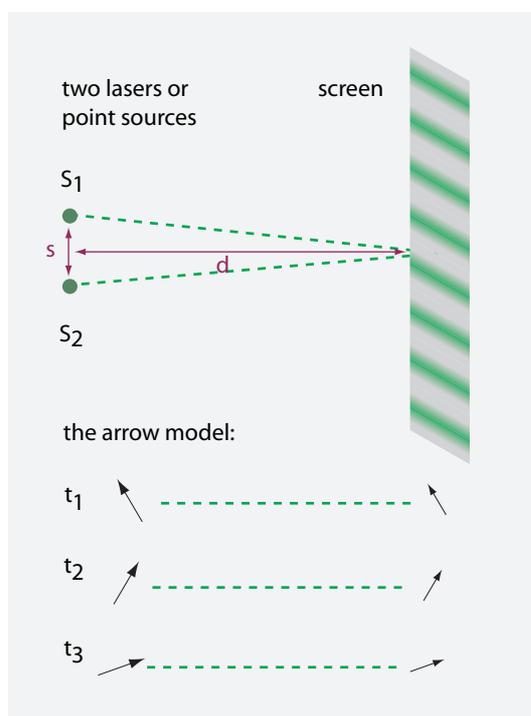
FIGURE 30 Interference and the description of light with arrows (at three instants of time).

we have seen earlier on. All experiments confirm this description.

In other words, interference is the superposition of coherent light fields or, more generally, of coherent electromagnetic fields. Coherent light fields have specific, more regular photon behaviour, than incoherent light fields. We will explore the details of photon statistics in more detail shortly.

In summary, photons are quantum particles. Quantum particles can produce interference patterns – and all other wave effects – when they appear in large numbers, because they are described by an arrow whose length squared gives the probability for its detection.

Interference of a single photon

It is important to point out that interference between two light beams is *not* the result of two different photons cancelling each other out or being added together. Such cancellation would contradict conservation of energy and momentum. Interference is an effect applicable to each photon separately – as shown in the previous section – because each photon is spread out over the whole set-up: each photon takes all possible paths. As Paul Dirac stressed:

▷ Each photon interferes only with itself.

Interference of a photon with itself only occurs because photons are quantons, and not classical particles.
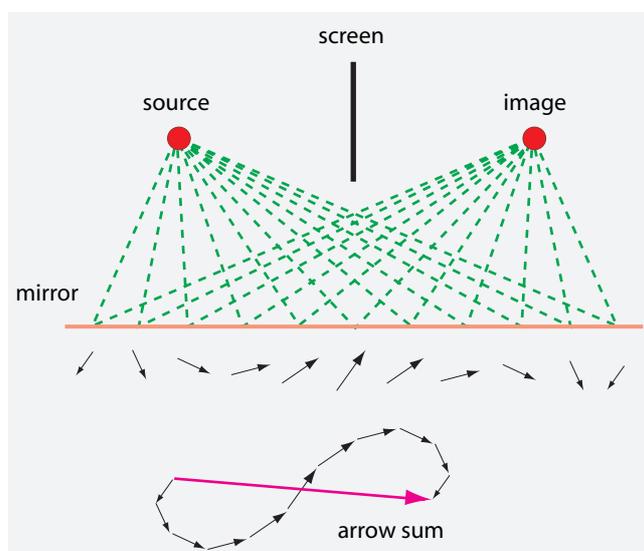
FIGURE 31 Light reflected by a mirror, and the corresponding arrows (at an instant of time).

Dirac's oft-quoted statement leads to a famous paradox: if a photon can interfere only with itself, how can two laser beams from two different lasers interfere with each other? The answer given by quantum physics is simple but strange: in the region where the beams interfere – as mentioned above – it is impossible to say from which source a photon has come. The photons in the crossing region *cannot* be said to come from a specific source. Photons, also in the interference region, are quantons, and they indeed interfere only with themselves.

Another description of the situation is the following:

> ▷ A photon interferes only within its volume of coherence. And in that volume, it is impossible to distinguish photons.

In the coherence volume formed by the longitudinal and transversal coherence length – sometimes also called a *phase space cell* – we cannot completely say that light is a flow of photons, because a flow cannot be defined in it. Despite regular claims to the contrary, Dirac's statement is correct, as we will see below. It is a strange consequence of the quantum of action.

REFLECTION AND DIFFRACTION DEDUCED FROM PHOTON ARROWS

Waves also show *diffraction*. Diffraction is the change of propagation direction of light or any other wave near edges. To understand this phenomenon with photons, let us start with a simple mirror, and study *reflection* first. Photons (like all quantum particles) move from source to detector by *all* possible paths. As Richard Feynman,* who discovered this explanation, liked to stress, the term 'all' has to be taken literally. This is not a big deal in

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* Richard ('Dick') Phillips Feynman (b. 1918 New York City, d. 1988 Los Angeles), physicist, was one of the founders of quantum electrodynamics. He also discovered the 'sum-over-histories' reformulation of quantum theory, made important contributions to the theory of the weak interaction and to
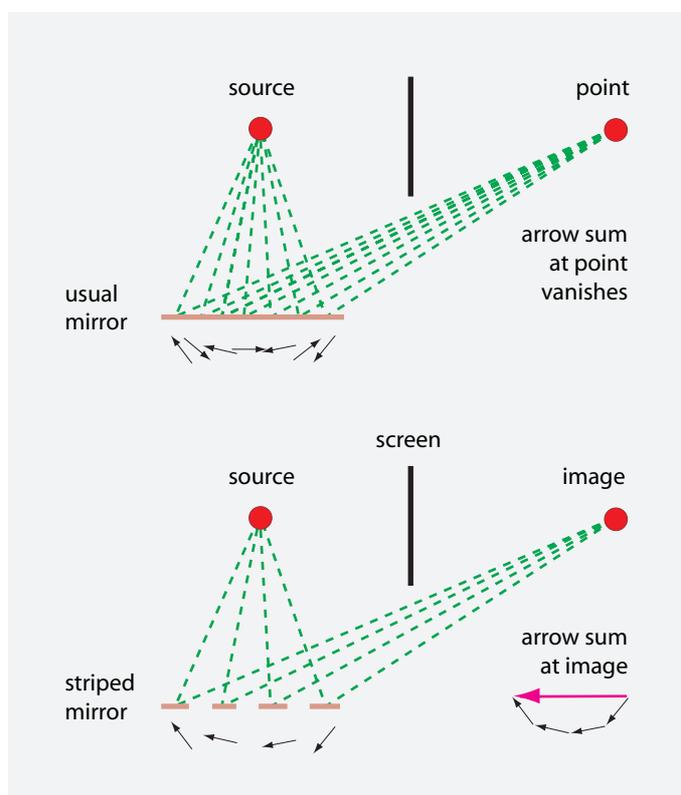
FIGURE 32 The light reflected by a badly-placed mirror and by a grating.

the explanation of interference. But in order to understand a mirror, we have to include all possibilities, however crazy they seem, as shown in Figure 31.

As stated above, a light source emits rotating arrows. To determine the probability that light arrives at a certain location within the image, we have to add up all the arrows arriving at the same time at that location. For each path, the arrow orientation at the image is shown – for convenience only – below the corresponding segment of the mirror. The angle and length of the arriving arrow depends on the path. Note that the sum of all the arrows does not vanish: light does indeed arrive at the image. Moreover, the largest contribution comes from the paths near to the middle. If we were to perform the same calculation for another image location, (almost) no light would get there.

In short, the rule that reflection occurs with the incoming angle equal to the outgoing angle is an approximation, following from the arrow model of light. In fact, a detailed

quantum gravity, and co-authored a famous textbook, the *Feynman Lectures on Physics*, now online at www. feynmanlectures.info. He is one of those theoretical physicists who made his career mainly by performing complex calculations – but he backtracked with age, most successfully in his teachings and physics books, which are all worth reading. He was deeply dedicated to physics and to enlarging knowledge, and was a collector of surprising physical explanations. He helped building the nuclear bomb, wrote papers in top-less bars, avoided to take any professional responsibility, and was famously arrogant and disrespectful of authority. He wrote several popular books on the events of his life. Though he tried to surpass the genius of Wolfgang Pauli throughout his life, he failed in this endeavour. He shared the 1965 Nobel Prize in Physics for his work on quantum electrodynamics.
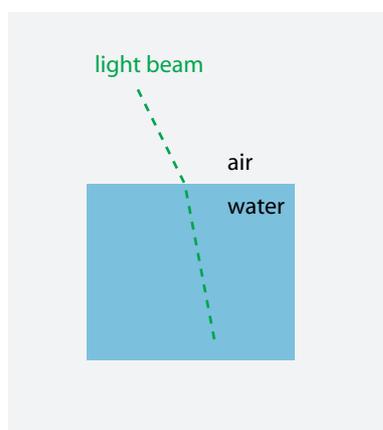
FIGURE 33 If light were made of little stones, they would
move faster in water.

calculation, with more arrows, shows that the approximation is quite precise: the errors
are much smaller than the wavelength of the light.

The proof that light does indeed take all these strange paths is given by a more spe-
cialized mirror. As show in Figure 32, we can repeat the experiment with a mirror that
reflects only along certain stripes. In this case, the stripes have been carefully chosen so
that the corresponding path lengths lead to arrows with a bias in one direction, namely
to the left. The arrow addition now shows that such a specialized mirror – usually called
a *grating* – allows light to be reflected in unusual directions. Indeed, this behaviour is
standard for waves: it is called *diffraction*. In short, the arrow model for photons allows
us to describe this wave property of light, provided that photons follow the 'crazy' prob-
ability scheme. Do not get upset! As was said above, quantum theory is the theory for
crazy people.

You may wish to check that the arrow model, with the approximations it generates
by summing over all possible paths, automatically ensures that the quantum of action is
indeed the smallest action that can be observed.

Challenge 42 s

Refraction and partial reflection from photon arrows

All waves have a *signal velocity*. The signal velocity also depends on the medium in which
they propagate. As a consequence, waves show *refraction* when they move from one me-
dium into another with different signal velocity. Interestingly, the naive particle picture
of photons as little stones would imply that light is faster in materials with high refractive
Challenge 43 e   indices: the so-called *dense* materials. (See Figure 33.) Can you confirm this? However,
experiments show that light in dense materials moves *slowly*. The wave picture has no
Challenge 44 e   difficulty explaining this observation. (Can you confirm this?) Historically, this was one
of the arguments *against* the particle theory of light. In contrast, the arrow model of light
Challenge 45 e   presented above is able to explain refraction properly. It is not difficult: try it.

Waves also *reflect partially* from materials such as glass. This is one of the most dif-
ficult wave properties to explain with photons. But it is one of the few effects that is *not*
explained by a classical wave theory of light. However, it *is* explained by the arrow model,
as we will find out. Partial reflection confirms the first two rules of the arrow model. Par-

tial reflection shows that photons indeed behave *randomly*: some are reflected and other are not, without any selection criterion. The distinction is purely statistical. More about this issue shortly.

FROM PHOTONS TO WAVES

In waves, the fields *oscillate in time and space*. One way to show how waves can be made of particles is to show how to build up a sine wave using a large number of photons. A sine wave is a coherent state of light. The way to build them up was explained in detail by Roy Glauber. In fact, to build a pure sine wave, we need a superposition of a beam with one photon, a beam with two photons, a beam with three photons, and so on. Together, they give a perfect sine wave. As expected, its photon number fluctuates to the highest possible degree.

If we repeat the calculation for non-ideal beams, we find that the indeterminacy relation for energy and time is respected: every emitted beam will possess a certain spectral width. Purely monochromatic light does not exist. Similarly, no system that emits a wave *at random* can produce a monochromatic wave. All experiments confirm these results.

In addition, waves can be *polarized*. So far, we have disregarded this property. In the photon picture, polarization is the result of carefully superposing beams of photons spinning clockwise and anticlockwise. Indeed, we know that linear polarization can be seen as a result of superposing circularly-polarized light of both signs, using the proper phase. What seemed a curiosity in classical optics turns out to be a fundamental justification for quantum theory.

Finally, photons are *indistinguishable*. When two photons of the same colour cross, there is no way to say afterwards which of the two is which. The quantum of action makes this impossible. The indistinguishability of photons has an interesting consequence. It is impossible to say which emitted photon corresponds to which arriving photon. In other words, there is no way to follow the path of a photon, as we are used to following the path of a billiard ball. Photons are indeed indistinguishable. In addition, the experiment by Hanbury Brown and Twiss implies that photons are bosons. We will discover more details about the specific indistinguishability of bosons later on.

In summary, we find that light waves *can* indeed be described as being built of particles. However, this is only correct with the proviso that *photons*

— are not precisely countable – never with a precision better than $\sqrt{N}$,
— are not localizable – never with a precision better than the coherence length,
— have no size, no charge and no (rest) mass,
— show a phase that increases as ωt, i.e., as the product of frequency and time,
— carry spin 1,
— of the same frequency are indistinguishable bosons – within a coherence volume,
— can take any path whatsoever – as long as allowed by the boundary conditions,
— have no discernable origin, and
— have a detection probability given by the square of the sum of amplitudes[*] for all allowed paths leading to the point of detection.

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[*] The amplitude of a photon field, however, cannot and should not be identified with the wave function of any massive spin 1 particle.

In other words, light can be described as made of particles *only* if these particles have special, *quantum* properties. These quantum properties differ from everyday particles and allow photons to behave like waves whenever they are present in large numbers.

CAN LIGHT MOVE FASTER THAN LIGHT? – REAL AND VIRTUAL PHOTONS

In a vacuum, light can move faster than c, as well as slower than c. The quantum principle provides the details. As long as this principle is obeyed, the speed of a short light flash can differ – though only by a tiny amount – from the 'official' value. Can you estimate the allowable difference in arrival time for a light flash coming from the dawn of time?

The arrow description for photons gives the same result. If we take into account the crazy possibility that photons can move with any speed, we find that all speeds very different from c cancel out. The only variation that remains, translated into distances, is the indeterminacy of about one wavelength in the longitudinal direction, which we mentioned above.

In short, light, or real photons, can indeed move faster than light, though only by an amount allowed by the quantum of action. For everyday situations, i.e., for high values of the action, all quantum effects average out, including light and photon velocities different from c.

Not only the position, but also the energy of a single photon can be undefined. For example, certain materials split one photon of energy $\hbar\omega$ into two photons, whose two energies add up to the original one. Quantum mechanics implies that the energy partitioning is known only when the energy of one of the two photons is measured. Only at that very instant is the energy of the second photon known. Before the measurement, both photons have undefined energies. The process of energy fixing takes place *instantaneously*, even if the second photon is far away. We will explain below the background to this and similar strange effects, which *seem* to be faster than light. In fact, despite the appearance, these observations do *not* involve faster-than-light transmission of energy or information.

More bizarre consequences of the quantum of action appear when we study *static* electric fields, such as the field around a charged metal sphere. Obviously, such a field must also be made of photons. How do they move? It turns out that static electric fields are made of *virtual* photons. Virtual photons are photons that do not appear as free particles: they only appear for an extremely short time before they disappear again. In the case of a static electric field, they are *longitudinally* polarized, and do not carry energy away. Virtual photons, like other virtual particles, are 'shadows' of particles that obey

$$\Delta x \Delta p \leqslant \hbar/2 \ . \tag{14}$$

Rather than obeying the usual indeterminacy relation, they obey the opposite relation, which expresses their very brief appearance. Despite their intrinsically short life, and despite the impossibility of detecting them directly, virtual particles have important effects. We will explore them in detail shortly.

In fact, the vector potential A allows *four* polarizations, corresponding to the four coordinates (t, x, y, z). It turns out that for the photons one usually talks about – the free or *real* photons – the polarizations in the t and z directions cancel out, so that one

observes only the x and y polarizations in actual experiments.

For bound or *virtual* photons, the situation is different. All four polarizations are possible. Indeed, the z and t polarizations of virtual photons – which do not appear for real photons, i.e., for free photons – are the ones that can be said to be the building blocks of *static* electric and magnetic fields.

In other words, static electric and magnetic fields are continuous flows of virtual photons. In contrast to real photons, virtual photons can have mass, can have spin directions not pointing along the path of motion, and can have momentum opposite to their direction of motion. Exchange of virtual photons leads to the *attraction* of bodies of different charge. In fact, virtual photons *necessarily* appear in any description of electromagnetic interactions. Later on we will discuss their effects further – including the famous attraction of neutral bodies.

We have seen already early on that virtual photons, for example those that are needed to describe collisions of charges, must be able to move with speeds higher than that of light. This description is required in order to ensure that the speed of light remains a limit in all experiments.

In summary, it might be intriguing to note that virtual photons, in contrast to real photons, are not bound by the speed of light; but it is also fair to say that virtual photons move faster-than-light only in a *formal* sense.

INDETERMINACY OF ELECTRIC FIELDS

We have seen that the quantum of action implies an indeterminacy for light intensity. Since light is an electromagnetic wave, this indeterminacy implies similar, separate limits for electric and magnetic fields at a given point in space. This conclusion was first drawn in 1933 by Bohr and Rosenfeld. They started from the effects of the fields on a test particle of mass m and charge q, which are described by:

$$m\boldsymbol{a} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{b}) \;. \tag{15}$$

Since it is impossible to measure both the momentum and the position of a particle, they deduced an indeterminacy for the electrical field, given by

$$\Delta E = \frac{\hbar}{q\,\Delta x\,t} \;, \tag{16}$$

where t is the measurement time and Δx is the position indeterminacy. Thus every value of an electric field, and similarly of a magnetic field, possesses an indeterminacy. The state of the electromagnetic field behaves like the state of matter in this respect: both follow an indeterminacy relation.

HOW CAN VIRTUAL PHOTON EXCHANGE LEAD TO ATTRACTION?

Exchange of real photons always leads to recoil. But exchange of *virtual* photons can lead either to attraction or repulsion, depending on the signs of the two charges involved. This is worth looking at.

We start with two localized charges of same sign, located both on the x-axis, and want to determine the momentum transferred from the charge on the right side via a virtual photon to the charge on the left side.

For the virtual photon, the important part of its state *in momentum space* is its imaginary part, which, *if emitted by a negative charge,* has a positive peak (delta function shape) at the negative of its momentum value and a negative peak at its positive momentum value.

When the virtual photon hits the other charged particle, on the left, it can push it either to the left or to the right. The probability amplitude for each process is given by the particle charge times the photon momentum value times i times time. Both amplitudes need to be added.

In the case that the second particle has the *same* charge as the first, the effect of the virtual photon absorption in momentum space is to add a wave function that originally was antisymmetric and positively valued on the positive axis, and that is then shifted to the left, to a second wave function which originally was the negative of the first, but is then shifted to the right. The result for this one-photon absorption process is a real-valued, antisymmetric function in momentum space, with positive values for negative momenta, and negative values for positive momenta.

To understand repulsion, we need to add the wave function for this one-photon process to the zero-photon (thus unmodified) function of the second particle, and then square the sum. This unmodified function was positive in the case of same charges. The squaring process of the sum yields a probability distribution in momentum space whose maximum is at a negative momentum value; thus the second particle has been *repelled* from the first.

If the charges had different signs, the maximum of the sum would be at a positive momentum value, and the second particle would be *attracted* to the first. In short, attraction or repulsion is determined by the interference between the wave function for one-photon absorption (more precisely, for odd-photon-number absorption) and the wave function for zero-photon absorption (more precisely, for even-photon-number absorption).

Can two photons interfere?

In 1930, Paul Dirac made the famous statement already mentioned earlier on:

▷ Each photon interferes only with itself. Interference between two different photons never occurs.

Often this statement is misinterpreted as implying that light from two separate photon *sources* cannot interfere. Unfortunately, this false interpretation has spread through a part of the literature. Everybody can check that this statement is incorrect with a radio: signals from two distant radio stations transmitting on the same frequency lead to beats in amplitude, i.e., to *wave interference*. (This should not to be confused with the more common *radio interference*, which usually is simply a superposition of intensities.) Radio transmitters are coherent sources of photons, and any radio receiver shows that signals form two different sources can indeed interfere.

In 1949, interference of fieds emitted from two different photon sources has been

demonstrated also with *microwave* beams. From the nineteen fifties onwards, numerous experiments with two lasers and even with two thermal light sources have shown light interference. For example, in 1963, Magyar and Mandel used two ruby lasers emitting light pulses and a rapid shutter camera to produce spatial interference fringes.

However, all these experimental results with two interfering sources do not contradict the statement by Dirac. Indeed, two photons cannot interfere for several reasons.

— Interference is a result of the space-time propagation of waves; photons appear only when the energy–momentum picture is used, mainly when interaction with matter takes place. The description of space-time propagation and the particle picture are mutually exclusive – this is one aspect of the complementary principle. Why does Dirac seem to mix the two in his statement? Dirac employs the term 'photon' in a very general sense, as quantized state of the electromagnetic field. When two coherent beams are superposed, the quantized entities, the photons, cannot be ascribed to either of the sources. Interference results from superposition of two coherent states, not of two particles.

— Interference is only possible if one *cannot know* where the detected photon comes from. The quantum mechanical description of the field in a situation of interference never allows ascribing photons of the superposed field to one of the sources. In other words, if it is possible to say from which source a detected photon comes from, interference *cannot* be observed.

— Interference between two coherent beams requires a correlated or fixed phase between them, i.e., an undetermined particle number; in other words, interference is possible if and only if the photon number for each of the two beams is unknown. And a beam has an unknown photon number when the number indeterminacy is of similar size as the average number.

The statement of Dirac thus depends on the definition of the term 'photon'. A better choice of words is to say that interference is always between two (indistinguishable) state histories, but never between two quantum particles. Or, as expressed above:

▷ A photon interferes only *within* its volume of coherence, i.e., within its own cell of phase space. Outside, there is no interference. And inside that volume or cell, it is impossible to distinguish photons, states or histories.

The concept of 'photon' remains deep even today. The quantum particle model of coherence and light remains fascinating to this day. Summarizing, we can say: *Two different electromagnetic beams can interfere, but two different photons cannot.*

Curiosities and fun challenges about photons

Can one explain refraction with photons? Newton was not able to do so, but today we can. In refraction by a horizontal surface, as shown in Figure 34, the situation is translationally invariant along the horizontal direction. Therefore, the momentum component along this direction is conserved: $p_1 \sin \alpha_1 = p_2 \sin \alpha_2$. The photon energy $E = E_1 = E_2$ is obviously conserved. The index of refraction n is defined in terms of momentum and
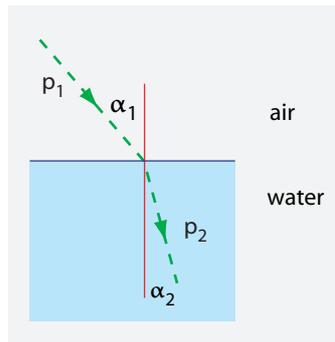
FIGURE 34 Refraction and photons.

energy as

$$n = \frac{cp}{E} \, . \tag{17}$$

Challenge 50 e   The 'law' of refraction follows:

$$\frac{\sin \alpha_1}{\sin \alpha_2} = n \, . \tag{18}$$

The relation is known since the middle ages.

There is an important issue here. In a material, the velocity of a photon $v = \delta E/\delta p$ in a light ray *differs* from the phase velocity $u = E/p$ that enters into the calculation. In summary, inside matter, the concept of photon must be used with extreme care.

∗ ∗

If an electromagnetic wave has amplitude A, the photon density d is

$$d = \frac{A^2}{\hbar \omega} \, . \tag{19}$$

Challenge 51 ny   Can you show this?

∗ ∗

Challenge 52 e   Show that for a laser pulse in vacuum, the coherence volume increases during propagation, whereas the volume occupied in phase space remains constant. Its entropy is constant, as its path is reversible.

∗ ∗

A typical effect of the quantum 'laws' is the yellow colour of the lamps used for street illumination in most cities. They emit pure yellow light of (almost) a single frequency; that is why no other colours can be distinguished in their light. According to classical electrodynamics, harmonics of that light frequency should also be emitted. Experiments show, however, that this is not the case; classical electrodynamics is thus wrong. Is this
Challenge 53 s   argument correct?

FIGURE 35 The blue shades of the sky and the colours of clouds are due to various degrees of Rayleigh, Mie and Tyndall scattering (© Giorgio di Iorio).

* *

How can you check whether a single-photon-triggered bomb is functional without exploding it? This famous puzzle, posed by Avshalom Elitzur and Lev Vaidman, requires interference for its solution. Can you find a way?

* *

What happens to photons that hit an object but are not absorbed or transmitted? Generally speaking, they are scattered. *Scattering* is the name for any process that changes the motion of light (or that of any other wave). The details of the scattering process depend on the object; some scattering processes only change the direction of motion, others also change the frequency. Table 3 gives an overview of processes that scatter light.

All scattering properties depend on the material that produces the deflection of light. Among others, the study of scattering processes explains many colours of transparent materials, as we will see below.

We note that the bending of light due to gravity is not called scattering. Why not?

A summary on light: particle and wave

In summary, light is a stream of light quanta or photons. A single photon is the smallest possible light intensity of a given colour. Photons, like all quantons, are quite different from everyday particles. In fact, we can argue that the only (classical) particle aspects of photons are their quantized energy, momentum and spin. In all other respects, photons are *not* like little stones. Photons move with the speed of light. Photons cannot be localized in light beams. Photons are indistinguishable. Photons are bosons. Photons have no mass, no charge and no size. It is more accurate to say that *photons are calculating devices to precisely describe observations about light.*

TABLE 3 Types of light scattering.

SCATTERING TYPE	SCATTERER	DETAILS	EXAMPLES
Rayleigh scattering	atoms, molecules	elastic, intensity changes as $1/\lambda^4$, scatterers smaller than $\lambda/10$	blue sky, red evening sky, blue cigarette smoke
Mie scattering	transparent objects, droplets	elastic, intensity changes as $1/\lambda^{0.5}$ to $1/\lambda^2$, scatterer size around λ	blue sky, red evenings, blue distant mountains
Geometric scattering	edges	elastic, scatterer size larger than λ	better called *diffraction*, used in interference
Tyndall scattering	non-transparent objects	elastic, angle weakly or not wavelength-dependent	smog, white clouds, fog, white cigarette smoke
Smekal–Raman scattering	excited atoms, molecules	inelastic, light gains energy	used in lidar investigations of the atmosphere
Inverse Raman scattering	atoms, molecules	inelastic, light loses energy	used in material research
Thomson scattering	electrons	elastic	used for electron density determination
Compton scattering	electrons	inelastic, X-ray lose energy	proves particle nature of light (see page 46)
Brillouin scattering	acoustic phonons, density variations in solids/fluids	inelastic, frequency shift of a few GHz	used to study phonons and to diagnose optical fibres
Von Laue or X-ray scattering	crystalline solids	elastic, due to interference at crystal planes	used to determine crystal structures; also called *Bragg diffraction*

The strange properties of photons are the reason why earlier attempts to describe light as a stream of (classical) particles, such as the attempt of Newton, failed miserably, and were rightly ridiculed by other scientists. Indeed, Newton upheld his theory against all experimental evidence – especially with regard to light's wave properties – which is something that a physicist should never do. Only after people had accepted that light is a wave, and then discovered and understood that quantum particles are fundamentally different from classical particles, was the quanton description successful.

The quantum of action implies that *all waves are streams of quantons*. In fact, all waves

are *correlated* streams of quantons. This is true for light, for any other form of radiation, and for all forms of matter waves.

The indeterminacy relations show that even a single quanton can be regarded as a wave; however, whenever it interacts with the rest of the world, it behaves as a particle. In fact, it is *essential* that all waves be made of quantons: if they were not, then interactions would be non-local, and objects could not be localized at all, contrary to experience.

To decide whether the wave or the particle description is more appropriate, we can use the following criterion. Whenever matter and light interact, it is more appropriate to describe electromagnetic radiation as a wave if the wavelength λ satisfies

$$\lambda \gg \frac{\hbar c}{kT} \,, \tag{20}$$

where $k = 1.4 \cdot 10^{-23}$ J/K is Boltzmann's constant and T is the temperature of the particle. If the wavelength is much *smaller* than the quantity on the right-hand side, the particle description is most appropriate. If the two sides are of the same order of magnitude, both descriptions play a role. Can you explain the criterion?
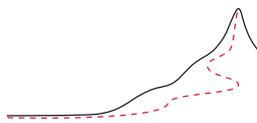
MOTION OF MATTER – BEYOND CLASSICAL PHYSICS

> All great things begin as blasphemies.
>
> George Bernard Shaw

T he existence of a smallest action has numerous important consequences for he motion of matter. We start with a few experimental results that show hat the quantum of action is indeed the smallest measurable action value, also in the case of matter. Then we show that the quantum of action implies the existence of a phase and thus of the wave properties of matter. Finally, from the quantum of action, we deduce for the motion of matter the same description that we already found for light: matter particles behave like rotating arrows.

WINE GLASSES, PENCILS AND ATOMS – NO REST

> Otium cum dignitate.**
>
> Cicero, *De oratore*.

If the quantum of action is the smallest observable change in a physical system, then two observations of the same system must always differ. Thus there cannot be perfect rest in nature. Is that true? Experiments show that this is indeed the case.

A simple consequence of the lack of perfect rest is the impossibility of completely filling a glass of wine. If we call a glass at maximum capacity (including surface tension effects, to make the argument precise) 'full', we immediately see that the situation requires the liquid's surface to be completely at rest. This is never observed. Indeed, a completely quiet surface would admit two successive observations that differ by less than $\hbar$. We could try to reduce all motions by reducing the temperature of the system. To achieve absolute rest we would need to reach absolute zero temperature. Experiments show that this is impossible. (Indeed, this impossibility, the so-called *third 'law' of thermodynamics*, is equivalent to the existence of a minimum action.) All experiments confirm: *There is no rest in nature.* In other words, the quantum of action proves the old truth that a glass of wine is always partially empty *and* partially full.

The absence of microscopic rest, predicted by the quantum of action, is confirmed in many experiments. For example, a pencil standing on its tip *cannot* remain vertical, as shown in Figure 36, even if it is isolated from all disturbances, such as vibrations, air molecules and thermal motion. This – admittedly very academic – conclusion follows

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** 'Rest with dignity.'
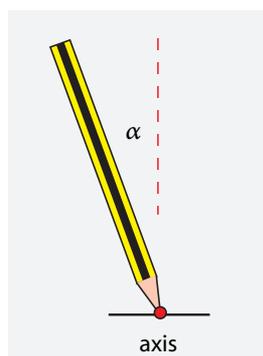
FIGURE 36 A falling pencil.

from the indeterminacy relation. In fact, it is even possible to calculate the time after which a pencil *must* have fallen over. In practice however, pencils fall over much earlier, because in usual conditions, external disturbances are much larger than the effects of the quantum of action.

But the most important consequence of the absence of rest is another. The absence of rest for the electrons inside atoms prevents them from falling into the nuclei, despite their mutual attraction. In other words, the existence and the size of atoms, and thus of all matter, is a direct consequence of the absence of microscopic rest! We will explore this consequence in more detail below. Since we are made of atoms, we can say: we only exist and live because of the quantum of action.

No infinite measurement precision

Not only does the quantum of action prevent the existence of rest; the quantum of action also prevents the observation or measurement of rest. In order to check whether an object is at rest, we need to observe its position with high precision. Because of the wave properties of light, we need a high-energy photon: only a high-energy photon has a small wavelength and thus allows a precise position measurement. As a result of this high energy, however, the object is disturbed. Worse, the disturbance itself is not precisely measurable; so there is no way to determine the original position even by taking the disturbance into account. In short, perfect rest cannot be observed – even if it existed.

Indeed, all experiments in which systems have been observed with highest precision confirm that perfect rest does not exist. The absence of rest has been confirmed for electrons, neutrons, protons, ions, atoms, molecules, atomic condensates and crystals. The absence of rest has been even confirmed for objects with a mass of about a tonne, as used in certain gravitational wave detectors. No object is ever at rest.

The same argument on measurement limitations also shows that no measurement, of any observable, can ever be performed to infinite precision. This is another of the far-reaching consequences of the quantum of action.

Cool gas

The quantum of action implies that rest is impossible in nature. In fact, even at extremely low temperatures, all particles inside matter are in motion. This fundamental lack of rest

is said to be due to the so-called *zero-point fluctuations*. A good example is provided by the recent measurements of Bose–Einstein condensates. They are trapped gases, with a small number of atoms (between ten and a few million), cooled to extremely low temperatures (around 1 nK). The traps allow to keep the atoms suspended in mid-vacuum. These cool and trapped gases can be observed with high precision. Using elaborate experimental techniques, Bose–Einstein condensates can be put into states for which $\Delta p \Delta x$ is almost exactly equal to $\hbar/2$ – though never lower than this value. These experiments confirm directly that there is no observable rest, but a fundamental *fuzziness* in nature. And the fuzziness is described by the quantum of action.

This leads to an interesting puzzle. In a normal object, the distance between the atoms is much larger than their de Broglie wavelength. (Can you confirm this?) But today it is possible to cool objects to extremely low temperatures. At sufficiently low temperatures, less than 1 nK, the wavelength of the atoms may be larger than their separation. Can you imagine what happens in such cases?

Flows and the quantization of matter

> " Die Bewegung ist die Daseinsform der Materie.
> Friedrich Engels, *Anti-Dühring*.*

Not only does the quantum of action make rest impossible, it also makes impossible any situation that does not change in time. The most important examples of (apparently) stationary situations are *flows*. The quantum of action implies that *no* flow can be stationary. More precisely, a smallest action implies that no flow can be continuous. *All flows fluctuate*. In nature, all flows are made of smallest entities: all flows are made of quantum particles. We saw above that this is valid for light; it also applies to matter flows. Two simple kinds of flow from our everyday experience directly confirm this consequence from the quantum of action: flows of fluids and flows of electricity.

Fluid flows and quantons

The flow of matter also exhibits smallest units. We mentioned early on in our adventure that a consequence of the particulate structure of liquids is that oil or any other smooth liquid produces *noise* when it flows through even the smoothest of pipes. We mentioned that the noise we hear in our ears in situations of absolute silence – for example, in a snowy and windless landscape in the mountains or in an anechoic chamber – is partly due to the granularity of blood flow in the veins. All experiments confirm that all flows of matter produce vibrations. This is a consequence of the quantum of action, and of the resulting granularity of matter. In fact, the quantum of action can be determined from noise measurements in fluids.

Knocking tables and quantized conductivity

If electrical current were a continuous flow, it would be possible to observe action values as small as desired. The simplest counter-example was discovered in 1996, by José Costa-Krämer and his colleagues. They put two metal wires on top of each other on a kitchen

* 'Motion is matter's way of being.'
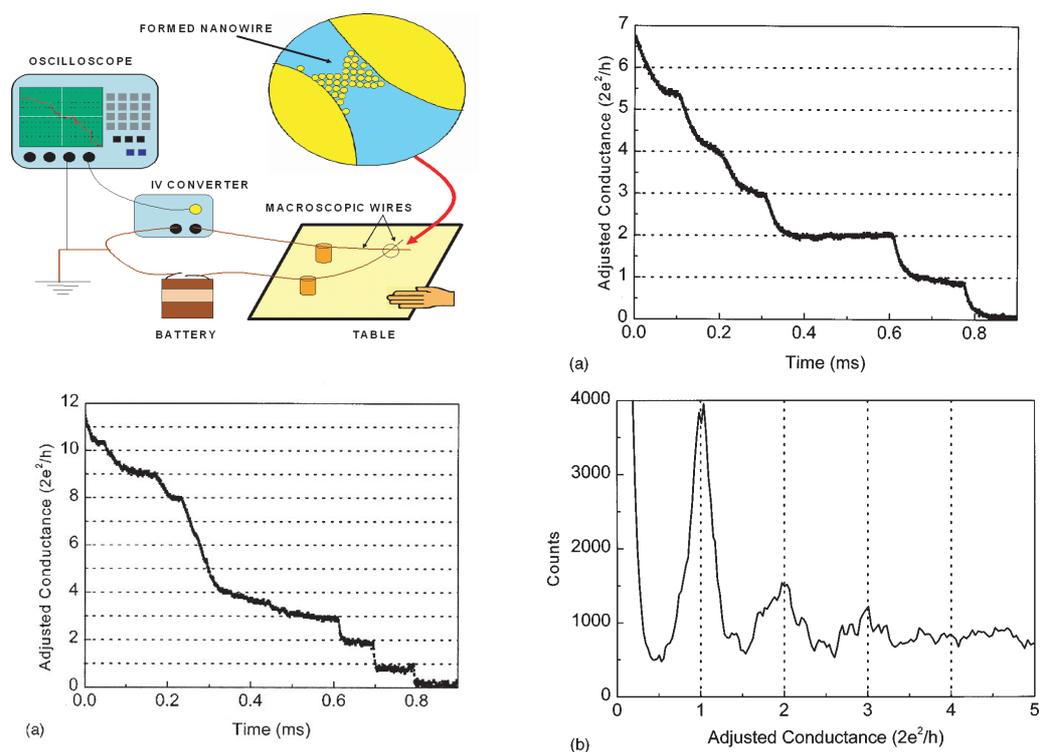
FIGURE 37 Steps in the flow of electricity in metal wire crossings: the set-up, the nanowires at the basis of the effect, and three measurement results (© José Costa-Krämer, AAPT from Ref. 39).

table and attached a battery, a current-voltage converter – or simply a resistor – and a storage oscilloscope to them. Then they measured the electrical current while *knocking* on the table. That is all.

Knocking the table breaks the contact between the two wires. In the last millisecond before the wires detach, the conductivity and thus the electrical current diminishes in regular steps of about 7 μA, as can easily be seen on the oscilloscope. Figure 37 shows such a measurement. This simple experiment could have beaten, if it had been performed a few years earlier, a number of other, enormously expensive experiments which discovered this same quantization at costs of several million euro each, using complex set-ups at extremely low temperatures.

In fact, the quantization of conductivity appears in any electrical contact with a small cross-section. In such situations the quantum of action implies that the conductivity can only be a multiple of $2e^2/\hbar \approx (12\,906\,\Omega)^{-1}$. Can you confirm this result? Note that electrical conductivity can be as small as required; only the *quantized* electrical conductivity has the minimum value of $2e^2/\hbar$.

Challenge 60 e

Many more elaborate experiments confirm the observation of conductance steps. They force us to conclude that there is a *smallest electric charge* in nature. This smallest charge has the same value as the charge of an electron. Indeed, electrons turn out to be part of every atom, in a construction to be explained shortly. In metals, a large number of electrons can move freely: that is why metals conduct electricity so well and work as
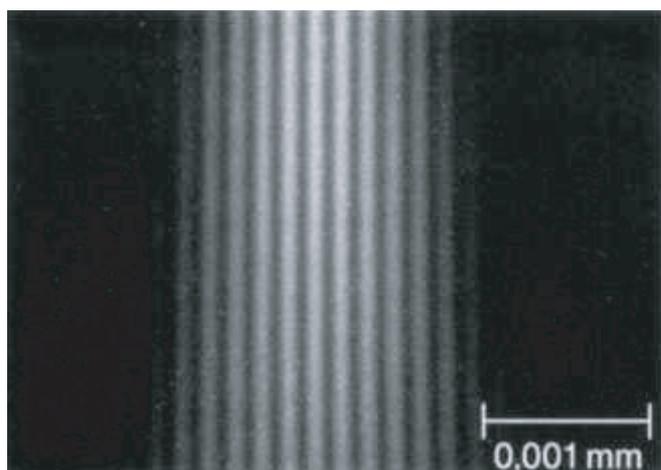
FIGURE 38 Electrons beams diffract and interfere at multiple slits (© Claus Jönsson).

mirrors.

In short, matter and electricity flow in smallest units. Depending on the flowing material, the smallest flowing units of matter may be 'molecules', 'atoms', 'ions', or 'electrons'. All of them are *quantum* particles, or *quantons*. In short, the quantum of action implies that matter is made of quantons. Matter quantons share some properties with ordinary stones, but also differ from them in many ways. A stone has position and momentum, mass and acceleration, size, shape, structure, orientation and angular momentum, and colour. We now explore each of these properties for quantons, and see how they are related to the quantum of action.

Matter quantons and their motion – matter waves

Ref. 40   In 1923 and 1924, the influential physicist Louis de Broglie pondered the consequences of the quantum of action for matter particles. He knew that in the case of light, the quantum of action connects wave behaviour to particle behaviour. He reasoned that the same should apply to matter. It dawned to him that streams of matter particles with the same momentum should behave as waves, just as streams of light quanta do. He thus predicted that like for light, coherent matter flows should have a wavelength λ and angular frequency ω given by

$$\lambda = \frac{2\pi\,\hbar}{p} \quad \text{and} \quad \omega = \frac{E}{\hbar}\,, \tag{21}$$

where p and E are the momentum and the energy, respectively, of the single particles. Equivalently, we can write the relations as

$$\boldsymbol{p} = \hbar\boldsymbol{k} \quad \text{and} \quad E = \hbar\omega\,. \tag{22}$$

All these relations state that matter quantons also behave as waves. For everyday objects, the predicted wavelength is unmeasurably small – though not for microscopic particles.

Soon after de Broglie's prediction, experiments began to confirm it. Matter streams
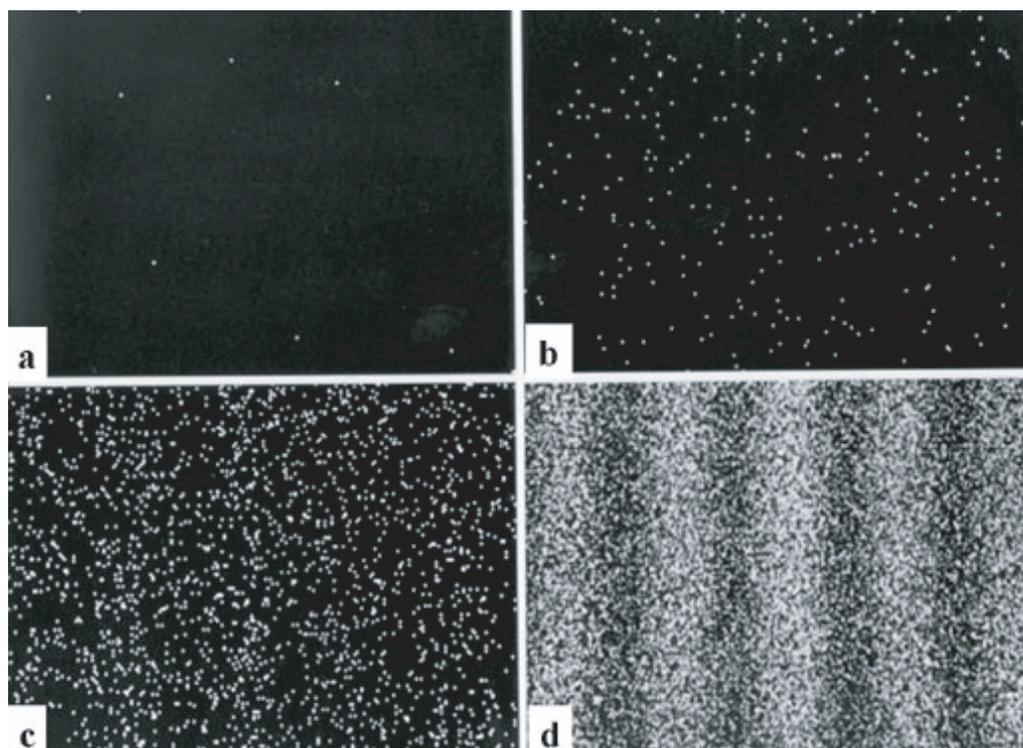
FIGURE 39 Formation over time of the interference pattern of electrons, here in a low-intensity double-slit experiment: (a) 8 electrons, (b) 270 electrons, (c) 2000 electrons, (d) 6000 electrons, after 20 minutes of exposure. The last image corresponds to the situation shown in the previous figure. (© Tonomura Akira/Hitachi).

were observed to diffract, refract and interfere; and all observations matched the values predicted by de Broglie. Because of the smallness of the wavelength of quantons, careful experiments are needed to detect these effects. But one by one, all experimental confirmations of the wave properties of light were repeated for matter beams. For example, just as light is diffracted when it passes around an edge or through a slit, matter is also diffracted in these situations. This is true even for electrons, the simplest particles of everyday matter, as shown in Figure 38. In fact, the experiment with electrons is quite difficult. It was first performed by Claus Jönsson in Tübingen in 1961; in the year 2002 it was voted the most beautiful experiment in all of physics. Many years after Jönsson, the experiment was repeated with a modified electron microscope, as shown in Figure 39.

Inspired by light interferometers, researchers began to build *matter* interferometers. Matter interferometers have been used in many beautiful experiments, as we will find out. Today, matter interferometers work with beams of electrons, nucleons, nuclei, atoms, or even large molecules. Just as observations of light interference prove the wave character of light, so the interference patterns observed with matter beams prove the wave character of matter. They also confirm the value of $\hbar$.

Like light, matter is made of particles; like light, matter behaves as a wave when large numbers of particles with the same momentum are involved. But although beams of large

molecules behave as waves, everyday objects – such as cars on a motorway – do not. There are several reasons for this. First, for cars on a motorway the relevant wavelength is extremely small. Secondly, the speeds of the cars vary too much. Thirdly, cars can be counted. In summary, streams of cars with the same speed cannot be made coherent.

If matter behaves like a wave, we can draw a strange conclusion. For any wave, the position and the wavelength cannot both be sharply defined simultaneously: the indeterminacies of the wave number $k = 2\pi/\lambda$ and of the position X obey the relation

$$\Delta k \Delta X \geq \frac{1}{2} \; . \tag{23}$$

Similarly, for every wave the angular frequency $\omega = 2\pi f$ and the instant T of its peak amplitude cannot both be sharply defined. Their indeterminacies are related by

$$\Delta \omega \Delta T \geq \frac{1}{2} \; . \tag{24}$$

Using de Broglie's wave properties of matter (22), we get

$$\Delta p \Delta X \geqslant \frac{\hbar}{2} \qquad \text{and} \qquad \Delta E \Delta T \geqslant \frac{\hbar}{2} \; . \tag{25}$$

These famous relations are called *Heisenberg's indeterminacy relations*. They were discovered by Werner Heisenberg in 1925. They are valid for all quantum particles, be they matter or radiation. The indeterminacy relations state that there is no way to simultaneously ascribe a precise momentum and position to a quantum system, nor to simultaneously ascribe a precise energy and age. The more accurately one quantity is known, the less accurately the other is.[*] As a result, matter quantons – rather like stones – can *always* be localized, but always only approximately. On the other hand, we saw that photons often cannot be localized.

Both indeterminacy relations have been checked experimentally in great detail. All experiments confirm them. In fact, every experiment proving that matter behaves like a wave is a confirmation of the indeterminacy relation – and vice versa.

When two variables are linked by indeterminacy relations, one says that they are *complementary* to each other. Niels Bohr systematically explored all possible such pairs. You can also do that for yourself. Bohr was deeply fascinated by the existence of a complementarity principle, and he later extended it in philosophical directions. In a well-known scene, somebody asked him what was the quantity complementary to precision. He answered: 'clarity'.

We remark that the usual, *real*, matter quantons always move more slowly than light. Due to the inherent fuzziness of quantum motion, it should not come to a surprise that exceptions exist. Indeed, in some extremely special cases, the quantum of action allows the existence of particles that move faster than light – so-called *virtual* particles – which we will meet later on.

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[*] A policeman stopped the car being driven by Werner Heisenberg. 'Do you know how fast you were driving?' 'No, but I know exactly where I was!'

In summary, the quantum of action means that matter quantons do not behave like point-like stones, but as waves. In particular, like for waves, the values of position and momentum cannot both be exactly defined for quantons. The values are fuzzy – position and momentum are undetermined. The more precisely one of the two is known, the less precisely the other is known.

MASS AND ACCELERATION OF QUANTONS

Matter quantons, like stones, have mass. Indeed, hits by single electrons, atoms or molecules can be detected, if sensitive measurement set-ups are used. Quantons can also be slowed down or accelerated. We have already explored some of these experiments in the section on electrodynamics. However, quantons differ from pebbles. Using the time–energy indeterminacy relation, you can deduce that

$$a \leqslant \frac{2mc^3}{\hbar} \ . \tag{26}$$

Thus there is a *maximum acceleration* for quantons.[*] Indeed, no particle has ever been observed with a higher acceleration than this value. In fact, no particle has ever been observed with an acceleration anywhere *near* this value. The quantum of action thus prevents rest but also limits acceleration.

WHY ARE ATOMS NOT FLAT? WHY DO SHAPES EXIST?

The quantum of action determines all sizes in nature. In particular, it determines all *shapes*. Let us start to explore this topic.

Experiments show that all *composed* quantons, such as atoms or molecules, have structures of finite size and often with complex shape. The size and the shape of every composed quanton are due to the motion of their constituents. The motion of the constituents is due to the quantum of action; but how do they move?

In 1901, Jean Perrin and independently, in 1904, Nagaoka Hantaro proposed that atoms are small 'solar systems'. In 1913, Niels Bohr used this idea, combining it with the quantum of action, and found that he could predict the size and the colour of hydrogen atoms, two properties that had not until then been understood. We will perform the calculations below. Even Bohr knew that the calculations were not completely understood, because they seemed to assume that hydrogen atoms were flat, like the solar system is. But first of all, atoms are observed to be spherical. Secondly, a flat shape would contradict the quantum of action. Indeed, the quantum of action implies that the motion of quantum constituents is fuzzy. Therefore, all composed quantons, such as atoms or molecules, must be made of *clouds* of constituents.

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* We note that this acceleration limit is different from the acceleration limit due to general relativity:

$$a \leqslant \frac{c^4}{4Gm} \ . \tag{27}$$

In particular, the quantum limit (26) applies to microscopic particles, whereas the general-relativistic limit applies to macroscopic systems. Can you confirm that in each domain the relevant limit is the smaller of the two?
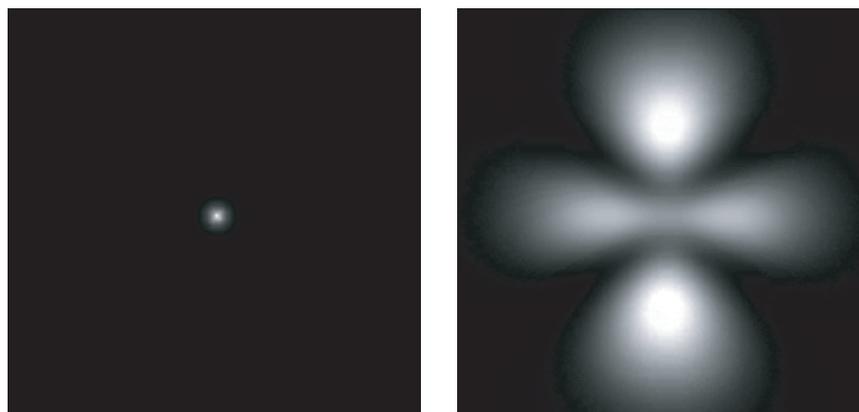
FIGURE 40 Probability clouds: a hydrogen atom in its spherical ground state (left) and in a non-spherical excited state (right) as seen by an observer travelling around it (QuickTime film produced with Dean Dauger's software package 'Atom in a Box', available at daugerresearch.com).

In short, the quantum of action predicts:

▷ Atoms are spherical clouds.

Experiment and theory confirm that the shape of any atom is due to the cloud, or probability distribution, of its lightest components, the electrons. The quantum of action thus states that atoms or molecules are not hard balls, as Democritus or Dalton believed, but that they are clouds. *Matter is made of clouds.*

Atomic electron clouds are not infinitely hard, but can to a certain degree interpenetrate and be deformed. The region where this deformation occurs is called a chemical *bond*. Bonds lead to molecules. Molecules, being composed of atoms, are composed of (deformed) spherical clouds. Bonds also lead to liquids, solids, flowers and people. A detailed exploration confirms that all shapes, from the simplest molecules to the shape of people, are due to the interactions between electrons and nuclei of the constituent atoms. Nowadays, molecular shapes can be calculated to high precision. Small molecules, like water, have shapes that are fairly rigid, though endowed with a certain degree of elasticity. Large molecules, such as polymers or peptides, have flexible shapes. These shape changes are essential for their effects inside cells and thus for our survival. A large body of biophysical and biochemical research is exploring molecular shape effects.

In summary, the quantum of action implies that shapes exist – and that they fluctuate. For example, if a long molecule is held fixed at its two ends, it cannot remain at rest in between. Such experiments are easy to perform nowadays, for example with DNA. In fact, all experiments confirm that the quantum of action prevents rest, produces sizes and shapes, and enables chemistry and life.

In nature, all sizes and shapes are due to the quantum of action. Now, every macroscopic object and every quantum object with a non-spherical shape is able to rotate. We therefore explore what the quantum of action can say about rotation.
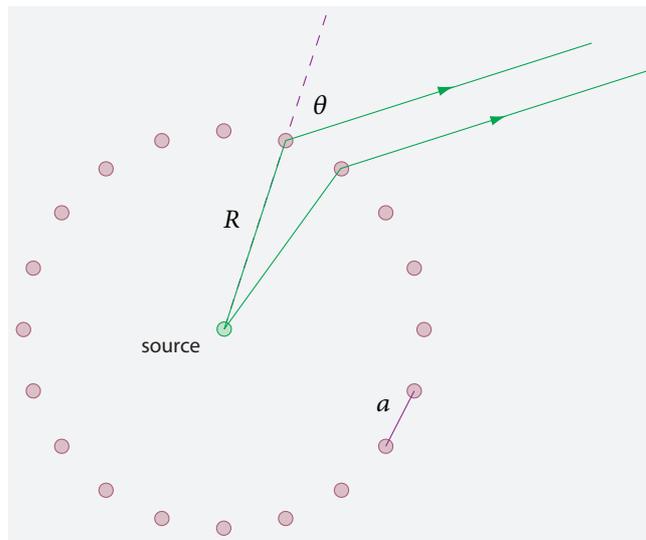
FIGURE 41 The quantization of angular momentum.

ROTATION, QUANTIZATION OF ANGULAR MOMENTUM, AND THE LACK OF NORTH POLES

> " Tristo è quel discepolo che non avanza il suo
> maestro.
>   Leonardo da Vinci[*] "

In everyday life, rotation is a frequent type of motion. Wheels are all around us. It turns out that the quantum of action has important consequences for rotational motion. First of all, we note that action and angular momentum have the same physical dimension: both are measured in Js or Nms. It only takes a little thought to show that if matter or radiation has a momentum and wavelength related by the quantum of action, then angular momentum is fixed in multiples of the quantum of action. This beautiful argument is due to Dicke and Wittke.

Imagine a circular fence, made of N vertical steel bars spaced apart at a distance $a = 2\pi R/N$, as shown in Figure 41. At the centre of the fence, imagine a source of matter or radiation that can emit particles towards the fence in any chosen direction. The linear momentum of such a particle is $p = \hbar k = 2\pi\hbar/\lambda$. At the fence slits, the wave will interfere. Outside the fence, the direction of the motion of the particle is determined by the condition of constructive interference. In other words, the angle θ, describing the direction of motion outside the fence, is given by $a \sin\theta = M\lambda$, where M is an integer. Through the deflection due to the interference process, the fence receives a linear momentum $p \sin\theta$, or an angular momentum $L = pR \sin\theta$. Combining all these expressions, we find that the angular momentum transferred to the fence is

$$L = NM\hbar \ . \tag{28}$$

Ref. 46

Challenge 65 e

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[*] 'Sad is that disciple who does not surpass his master.' This statement from one of his notebooks, the Codice Forster III, is sculpted in large letters in the chemistry aula of the University of Rome *La Sapienza*.

In other words, the angular momentum of the fence is an integer multiple of $\hbar$. Fences can only have integer intrinsic angular momenta (in units of $\hbar$). The generalization of the argument to all bodies is also correct. (Of course, this latter statement is only a hint, not a proof.)

> ▷ The measured intrinsic angular momentum of bodies is always a multiple of $\hbar$.

Quantum theory thus states that every object's angular momentum increases in steps. Angular momentum is quantized. This result is confirmed by all experiments.

But rotation has more interesting aspects. Thanks to the quantum of action, just as linear momentum is usually fuzzy, so is angular momentum. There is an indeterminacy relation for angular momentum L. The complementary variable is the phase angle φ of the rotation. The indeterminacy relation can be expressed in several ways. The simplest approximation – and thus *not* the exact expression – is

$$\Delta L \, \Delta \varphi \geqslant \frac{\hbar}{2} \; . \tag{29}$$

This is obviously an approximation: the relation is only valid for large angular momenta. In any case, the expression tells us that rotation behaves similarly to translation. The expression cannot be valid for small angular momentum values, as $\Delta \varphi$ by definition cannot grow beyond 2π. In particular, angular-momentum eigenstates have $\Delta L = 0$.*

The indeterminacy of angular momentum appears for all macroscopic bodies. We can say that the indeterminacy appears for all cases when the angular phase of the system can be measured.

The quantization and indeterminacy of angular momentum have important consequences. Classically speaking, the poles of the Earth are the places that do not move when observed by a non-rotating observer. Therefore, at those places matter would have a defined position and a defined momentum. However, the quantum of action forbids this. There cannot be a North Pole on Earth. More precisely, the idea of a *fixed* rotational axis is an approximation, not valid in general. This applies in particular to rotating quantum particles.

Rotation of quantons

The effects of the quantum of action on the rotation of microscopic particles, such as atoms, molecules or nuclei, are especially interesting. We note again that action and angular momentum have the same units. The precision with which angular momentum

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* An *exact* formulation of the indeterminacy relation for angular momentum is

$$\Delta L \, \Delta \varphi \geqslant \frac{\hbar}{2} |1 - 2\pi P(\pi)| \; , \tag{30}$$

where $P(\pi)$ is the normalized probability that the angular position has the value π. For an angular-momentum eigenstate, one has $\Delta \varphi = \pi/\sqrt{3}$ and $P(\pi) = 1/2\pi$. This exact expression has been tested and confirmed by experiments.

can be measured depends on the precision of the rotation angle. But if a *microscopic* particle rotates, this rotation might be unobservable: a situation in fundamental contrast with the case of *macroscopic* objects. Experiments indeed confirm that many microscopic particles have unobservable rotation angles. For example, in many (but not all) cases, an atomic nucleus rotated by half a turn cannot be distinguished from the unrotated nucleus.

If a microscopic particle has a *smallest* unobservable rotation angle, the quantum of action implies that the angular momentum of that particle *cannot* be zero. It must always be rotating. Therefore we need to check, for each particle, what its smallest unobservable angle of rotation is. Physicists have checked all particles in nature in experiments, and found smallest unobservable angles (depending on the particle type) of 0, 4π, 2π, $4\pi/3$, π, $4\pi/5$, $2\pi/3$ etc.

Let us take an example. Certain nuclei have a smallest unobservable rotation angle of *half* a turn. This is the case for a prolate nucleus (one that looks like a rugby ball) turning around its short axis, such as a ^{23}Na nucleus. In this case, both the largest observable rotation angle and the indeterminacy are thus a *quarter* turn. Since the change, or action, produced by a rotation is the number of turns multiplied by the angular momentum, we find that the angular momentum of this nucleus is $2 \cdot \hbar$.

As a general result, we deduce from the minimum angle values that the angular momentum of a microscopic particle can be 0, $\hbar/2$, $\hbar$, $3\hbar/2$, $2\hbar$, $5\hbar/2$, $3\hbar$ etc. In other words, the intrinsic angular momentum of a particle, usually called its *spin*, is an integer multiple of $\hbar/2$. Spin describes how a particle behaves under rotations.

How can a particle rotate? At this point, we do not yet know how to *picture* the rotation. But we can *feel* it – just as we showed that light is made of rotating entities: all matter, including electrons, can be *polarized*. This is shown clearly by the famous Stern–Gerlach experiment.

Silver, Stern and Gerlach – polarization of quantons

After a year of hard work, in 1922, Otto Stern and Walther Gerlach* completed a beautiful experiment to investigate the polarization of matter quantons. They knew that inhomogeneous magnetic fields act as polarizers for rotating charges. Rotating charges are present in every atom. Therefore they let a beam of silver atoms, extracted from an oven by evaporation, pass an inhomogeneous magnetic field. They found that the beam splits into *two separate* beams, as shown in Figure 42. No atoms leave the magnetic field region in intermediate directions. This is in full contrast to what would be expected from classical physics.

Ref. 50

The splitting into *two* beams is an intrinsic property of silver atoms; today we know that it is due to their spin. Silver atoms have spin $\hbar/2$, and depending on their orientation in space, they are deflected either in the direction of the field inhomogeneity or against it. The splitting of the beam is a pure quantum effect: there are no intermediate options. Indeed, the Stern–Gerlach experiment provides one of the clearest demonstrations that classical physics does not work well in the microscopic domain. In 1922, the

---

* Otto Stern (1888–1969) and Walther Gerlach (1889–1979) worked together at the University of Frankfurt. For his subsequent measurement of the anomalous magnetic moment of the proton, Stern received the Nobel Prize in Physics in 1943, after he had to flee National Socialism.
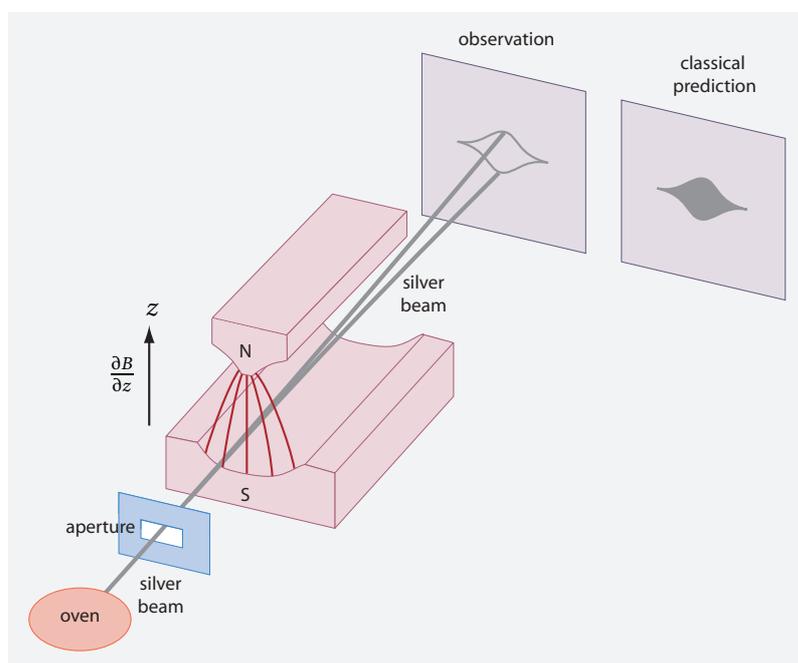
FIGURE 42 The Stern–Gerlach experiment.

result seemed so strange that it was studied in great detail all over the world.

When one of the two beams – say the 'up' beam – is passed through a second set-up, all the atoms end up in the 'up' beam. The other possible exit, the 'down' beam, remains unused in this case. In other words, the up and down beams, in contrast to the original beam, cannot be split further. This is not surprising.

But if the second set-up is rotated by $\pi/2$ with respect to the first, again two beams – 'right' and 'left' – are formed, and it does not matter whether the incoming beam is directly from the oven or from the 'up' part of the beam. A partially-rotated set-up yields a partial, uneven split. The proportions of the two final beams depend on the angle of rotation of the second set-up.

We note directly that if we split the beam from the oven first vertically and then horizontally, we get a different result from splitting the beam in the opposite order. (You can check this yourself.) Splitting processes do not commute. When the order of two operations makes a difference to the net result, physicists call them *non-commutative*. Since all measurements are also physical processes, we deduce that, in general, measurements and processes in quantum systems are non-commutative.

Beam splitting is direction-dependent. Matter beams behave almost in the same way as polarized light beams. Indeed, the inhomogeneous magnetic field acts on matter somewhat like a polarizer acts on light. The up and down beams, taken together, define a polarization direction. Indeed, the polarization direction can be rotated, with the help of a *homogeneous* magnetic field. And a rotated beam in a unrotated magnet behaves like an unrotated beam in a rotated magnet.

In summary, matter quantons can be polarized. We can picture polarization as the orientation of an internal rotation axis of the massive quanton. To be consistent, the
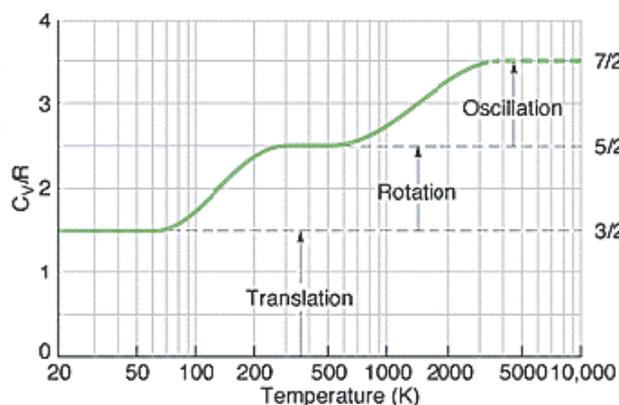
FIGURE 43 An idealized graph of the heat capacity of hydrogen over temperature (© Peter Eyland).

rotation axis must be imagined to precess around the direction of polarization. Thus, massive quantum particles resemble photons also in their polarizability.

CURIOSITIES AND FUN CHALLENGES ABOUT QUANTUM MATTER

> It is possible to walk while reading, but not to read while walking.
>
> Serge Pahaut

The quantum of action implies that there are *no fractals* in nature. Everything is made of particles. And particles are clouds. Quantum theory requires that all shapes in nature be 'fuzzy' *clouds*.

* *

Can atoms rotate? Can an atom that falls on the floor roll under the table? Can atoms be put into high-speed rotation? The answer is 'no' to all these questions, because angular momentum is quantized; moreover, atoms are not solid objects, but clouds. The macroscopic case of an object turning more and more slowly until it stops does not exist in the microscopic world. The quantum of action does not allow it.

Ref. 51

* *

Light is refracted when it enters dense matter. Do matter waves behave similarly? Yes, they do. In 1995, David Pritchard showed this for sodium waves entering a gas of helium and xenon.

Ref. 52

* *

Many quantum effects yield curves that show steps. An important example is the molar heat of hydrogen H_2 gas, shown in Figure 43. In creasing the temperature from 20 to 8 000 K, the molar heat is shows two steps, first from $3R/2$ to $5R/2$, and then to $7R/2$. Can you explain the reason?

* *

Most examples of quantum motion given so far are due to electromagnetic effects. Can

Challenge 67 s

you argue that the quantum of action must also apply to nuclear motion, and in particular, to the nuclear interactions?

* *

There are many other formulations of the indeterminacy principle. An interesting one is due to de Sabbata and Sivaram, who explained in 1992 that the following intriguing relation between temperature and time also holds:

$$\Delta T \Delta t \geqslant \hbar/k. \tag{31}$$

Ref. 53

Here, k is the Boltzmann constant. All experimental tests so far have confirmed the result.

* *

Challenge 68 e

Here is a trick question: what is the moment of inertia of an electron? Why?

FIRST SUMMARY ON THE MOTION OF QUANTUM PARTICLES

In summary, the 'digital' beam splitting seen in the Stern–Gerlach experiment and the wave properties of matter force us to rethink our description of motion. They show that microscopic matter motion follows from the quantum of action, the smallest observable action value. In special relativity, the existence of a maximum speed forced us to introduce the concept of space-time, and then to refine our description of motion. In general relativity, the maximum force obliged us to introduce the concepts of horizon and curvature, and then again to refine our description of motion. At the present point, the existence of the quantum of action and the wave behaviour of matter force us to take two similar steps: we first introduce the concept of a wave function, and then we refine our description of matter motion.
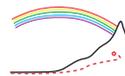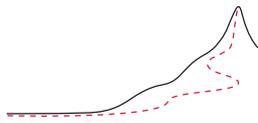
THE QUANTUM DESCRIPTION OF MATTER AND ITS MOTION

> " Die Quanten sind doch eine hoffnungslose
> Schweinerei!** Max Born

I n everyday life and in classical physics, we say that a system *has* a position, that
t *is* oriented in a certain direction, that it *has* an axis of rotation, and that
t *is* in a state with specific momentum. In classical physics, we can talk in this
way because the *state* – the situation a system 'is' in and the properties a system 'has'
– coincide with the *results of measurement*. They coincide because measurements can
always be imagined to have a negligible effect on the system.

However, because of the existence of a smallest action, the interaction necessary to
perform a measurement on a system *cannot* be made arbitrarily small. Therefore, the
quantum of action makes it *impossible* for us to continue saying that a system *has* mo-
mentum, *has* position or *has* an axis of rotation. The quantum of action forces us to use
the idea of the rotating arrow and to introduce the concept of *wave function* or *state
function*. Let us see why and how.

States and measurements – the wave function

The Stern–Gerlach experiment shows that the measured values of spin *orientation* are
not intrinsic, but result from the measurement process (in this case, from the interaction
with the applied inhomogeneous field). This is in contrast to the spin *magnitude*, which
is intrinsic and independent of state and measurement. In short, the quantum of action
forces us to distinguish carefully three concepts:

— the state of the system;
— the operation of measurement;
— the result or outcome of the measurement.

In contrast to the classical, everyday case, the *state* of a quantum system – the proper-
ties a system 'has' – is *not* described by the outcomes of measurements. The simplest
illustration of this difference is the system made of a single particle in the Stern–Gerlach
experiment. The experiment shows that a spin measurement on a general (oven) particle
state sometimes gives 'up' (say +1), and sometimes gives 'down' (say −1). So a general
atom, in an oven state, has no intrinsic orientation. Only *after* the measurement, an atom
is either in an 'up' state or in a 'down' state.

** 'Those quanta are a hopeless dirty mess!'

It is also found that feeding 'up' states into a second measurement apparatus gives only 'up' states: thus certain special states, called *eigenstates*, do remain unaffected by measurement.

Finally, the Stern–Gerlach experiment and its variations show that states can be rotated by applied fields: atom states have a direction or orientation in space. The experiments also show that the states rotate as the atoms move through space.

The experimental observations can be described in a straightforward way. Since measurements are operations that take a state as input and produce an output state and a measurement result, we can say:

▷ *States* are described by rotating arrows, or rotating vectors.
▷ *Measurements* of observables are operations on the state vectors.
▷ *Measurement results* are *real numbers*; and like in classical physics, they usually depend on the observer.

In particular, we have distinguished two quantities that are not distinguished in classical physics: states and measurement results. Given this distinction, quantum theory follows quite simply, as we shall see.

Given that the quantum of action is not vanishingly small, any measurement of an observable quantity is an interaction with a system and thus a transformation of its state. Therefore, quantum physics describes physical observables as *operators*, or equivalently, as transformations. The Stern–Gerlach experiment shows this clearly: the interaction with the field influences the atoms: some in one way, and some in another way. In fact, all experiments show:

▷ Mathematically, states are *complex vectors*, or rotating arrows, in an abstract space. This space of all possible states or arrows is a *Hilbert space*.
▷ Mathematically, measurements are linear transformations, more precisely, they are described by self-adjoint, or *Hermitean*, *operators* (or matrices).
▷ Mathematically, *changes of viewpoint* are described by *unitary operators* (or matrices) that act on states, or arrows, and on measurement operators.

Quantum-mechanical experiments also show that a measurement of an observable can only give a result that is an *eigenvalue* of the corresponding transformation. The resulting states after the measurement, those exceptional states that are *not* influenced when the corresponding variable is measured, are the *eigenvectors*. In short, every expert on motion must know what an eigenvalue and an eigenvector is.

For any linear transformation T, those special vectors ψ that are transformed into multiples of themselves,

$$T\psi = \lambda\psi \tag{32}$$

are called *eigenvectors* (or *eigenstates*), and the multiplication factor λ is called the associated *eigenvalue*. Experiments show:

▷ The state of the system *after* a measurement is given by the eigenvector corresponding to the measured eigenvalue.
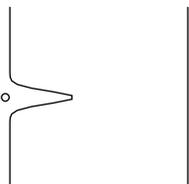
In the Stern–Gerlach experiment, the eigenstates are the 'up' and the 'down' states. In general, the eigenstates are those states that do not change when the corresponding variable is measured. Eigenvalues of Hermitean operators are always real, so that consistency is ensured: all measurement results are real numbers.

In summary, the quantum of action obliges us to distinguish between three concepts that are mixed together in classical physics: the *state* of a system, a *measurement* on the system, and the *measurement result*. The quantum of action forces us to change the vocabulary with which we describe nature, and obliges to use more differentiated concepts. Now follows the main step: the description of *motion* with these concepts. This is what is usually called 'quantum theory'.

Visualizing the wave function: rotating arrows and probability clouds

We just described the state of a quanton with an arrow. In fact, this is an approximation for localized quantons. More precisely,

> ▷ The state of a quantum particle is described by a spatial distribution of arrows, a so-called *wave function*.

To develop a visual image of the wave function, we first imagine a quantum particle that is localized as much as possible. In this case, the wave function for a free quanton can be described simply by a single rotating arrow.

Experiments show that when a *localized* quanton travels through space, the attached arrow rotates. If the particle is non-relativistic and if spin can be neglected, the rotation takes place in a plane perpendicular to the direction of motion. The end of the arrow then traces a *helix* around the direction of motion. In this case, the state at a given time is described by the angle of the arrow. This angle is the *quantum phase*. The quantum phase is responsible for the wave properties of matter, as we will see. The wavelength and the frequency of the helix are determined by the momentum and the kinetic energy of the particle.

If the particle is *not localized* – but still non-relativistic and still with negligible spin effects – the state, or the wave function, defines a rotating arrow *at each point in space*. The rotation still takes place in a plane perpendicular to the direction of motion. But now we have a distribution of arrows that all trace helices parallel to the direction of motion. At each point in space and time, the state has a quantum phase and a length of the arrow. The arrow lengths decrease towards spatial infinity.

Figure 44 shows an example of evolution of a wave function for non-relativistic particles with negligible spin effects. The direction of the arrow at each point is shown by the colour at the specific point. The length of the arrow is shown by the brightness of the colour. For non-relativistic particles with negligible spin effects, the wave function $\psi(t, x)$ is thus described by a length and a phase: it is a *complex* number at each point in space. The phase is essential for interference and many other wave effects. What measurable property does the amplitude, the length of the local arrow, describe? The answer was given by the famous physicist Max Born:
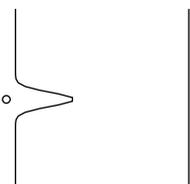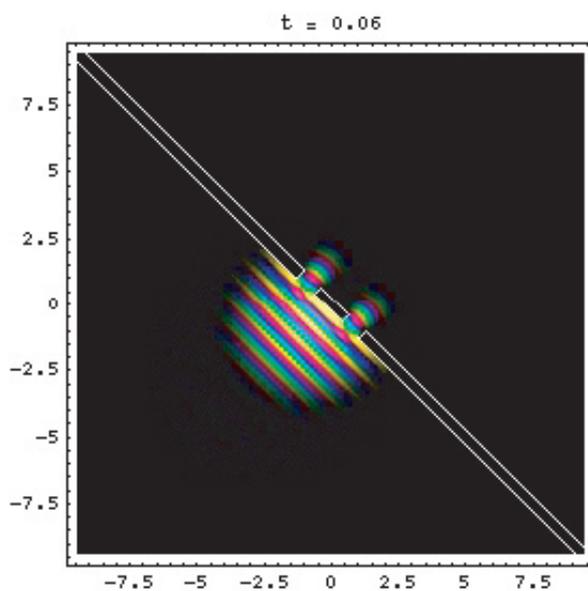
FIGURE 44 The motion of a wave function, the quantum state, through a double slit, showing both the particle and the wave properties of matter. The density of the state, related to the arrow length, is displayed by brightness, and the local phase is encoded in the colour. (QuickTime film © Bernd Thaller)

> ▷ The amplitude of the wave function is a *probability amplitude*. The square of the amplitude, i.e., the quantity $|\psi(t, x)|^2$, gives the probability to find the particle at the place x at time t.

In other terms, a wave function is a combination of two ideas: on the one hand, a wave function is a *cloud*; on the other hand, at each point of the cloud one has to imagine an arrow. Over time, the arrows rotate and the cloud changes shape.

> ▷ A wave function is a cloud of rotating arrows.

Describing the state of a matter particle with a cloud of rotating arrows is the essential step to picture the wave properties of matter.

We can clarify the situation further.

> ▷ In every process in which the phase of the wave function is not important, the cloud image of the wave function is sufficient and correct.

For example, the motion of atoms of molecules in gases or liquids can be imagined as the motion of cloudy objects. It needs to be stressed that the clouds in question are quite hard: it takes a lot of energy to deform atomic clouds. The hardness of a typical crystal is directly related to the hardness of the atomic clouds that are found inside. Atoms are extremely stiff, or hard clouds.

On the other hand,

> ▷ In every process in which the phase of the wave function does play a role,
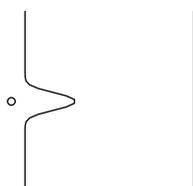
the cloud image of the wave function needs to be expanded with rotating arrows at each point.

This is the case for interference processes of quantons, but also for the precise description of chemical bonds.

Teachers often discuss the best way to explain wave functions. Some teachers prefer to use the cloud model only, others prefer not to use any visualization at all. Both approaches are possible; but the most useful and helpful approach is to imagine the state or wave function of a *non-relativistic* quantum particle as an arrow at every point in space. The rotation frequency of the set of arrows is the *kinetic energy* of the particle; the wavelength of the arrow motion – the period of the helical curve that the tip of the arrows – or of the average arrow – traces during motion – is the *momentum* of the quantum particle.

An arrow at each point in space is a (mathematical) *field*. The field is concentrated in the region where the particle is located, and the amplitude of the field is related to the probability to find the particle. Therefore the state field, the wave function or state function, is an *arrow cloud*. It is usually called with the greek letter ψ.

Note that even though the wave function can be seen as defining an arrow at every point in space, the wave function as a whole can also be described as one, single vector, this time in a Hilbert space. For free particles, i.e., particles that are not subject to external forces, the Hilbert space is infinite dimensional! Nevertheless, it is not hard to calculate in such spaces. The scalar product of two wave functions is the spatial integral of the product of the complex conjugate of the first function and the (unconjugated) second function. With this definition, all vector concepts (unit vectors, null vectors, basis vectors, etc.) can be meaningfully applied to wave functions.

In summary, for non-relativistic particles without spin effects, *the state or wave function of a quantum particle is a cloud, or a distributed wave, of rotating arrows.* This aspect of a quantum cloud is unusual. Since a quantum cloud is made of little arrows, every point of the cloud is described by a local density and a local orientation. This latter property does not occur in any cloud of everyday life.

For many decades it was tacitly assumed that a wave function ψ cannot be visualized more simply than with a cloud of rotating arrows. Only the last years have shown that there are other visualization for such quantum clouds; one possible visualization is presented in the last volume of this series.

The state evolution – the Schrödinger equation

The description of the state of a non-relativistic quanton with negligible spin effects as a rotating cloud *completely* determines how the wave function evolves in time. Indeed, for such quantum particles the evolution follows from the total energy, the sum of kinetic and potential energy $T + V$, and the properties of matter waves:

▷ The local rate of change of the state arrow(s) $\psi(x)$, or simply ψ, is produced
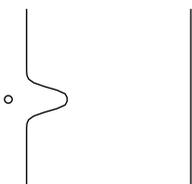
by the local total energy, or Hamiltonian, $H = T + V$:

$$i\hbar \frac{\partial}{\partial t}\psi = H\psi \; . \qquad (33)$$

This famous equation is *Schrödinger's equation of motion.*[*] This *evolution equation* applies to all quantum systems and is one of the high points of modern physics.

In fact, Erwin Schrödinger had found his equation in two different ways. In his first
paper, he deduced it from a variational principle. In his second paper, he deduced the evolution equation directly, by asking a simple question: how does the state evolve? He knew that the state of a quanton behaves both like a wave and like a particle. A wave is described by a field, which he denoted $\psi(t, x)$. If the state ψ behaves like a wave, then the corresponding wave function must be an amplitude W multiplied by a phase factor $e^{ikx-\omega t}$. The state can thus be written as

$$\psi(t, x) = W(t, x)\, e^{ikx-\omega t} \; . \qquad (34)$$

The amplitude W is the *length* of the local arrow; the phase is the *orientation* of the local arrow. Equivalently, the amplitude is the local density of the cloud, and the phase is the local orientation of the cloud.

We know that the quantum wave must also behave like a particle of mass m. In particular, the non-relativistic relation between energy and momentum $E = \boldsymbol{p}^2/2m + V(\boldsymbol{x})$ – where $V(\boldsymbol{x})$ is the potential at position $\boldsymbol{x}$ – must be fulfilled for these waves. The two
de Broglie relations (22) for matter wavelength and matter frequency then imply

$$i\hbar \frac{\partial\psi}{\partial t} = H\psi = \frac{-\hbar^2}{2m}\nabla^2\psi + V(\boldsymbol{x})\psi \; . \qquad (35)$$

This is the complete form of Schrödinger's wave equation. ∇^2 is the Laplace operator, essentially the second derivative over space. It states how the arrow wave, the wave function ψ associated to a particle, evolves over time. In 1926, this *wave equation* for the complex field ψ became instantly famous when Schrödinger used it, by inserting the potential felt by an electron near a proton, to calculate the energy levels of the hydrogen atom. In a hydrogen atom, light is emitted by the single electron inside that atom; therefore a precise description of the motion of the electron in a hydrogen atom allows us to describe the light frequencies it can emit. (We will perform the calculation and the comparison with
experiment below.) First of all, the Schrödinger equation explained that only *discrete* colours are emitted by hydrogen. In addition, the frequencies of the emitted light were found to be in agreement with the prediction of the equation to five decimal places. Finally, the

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[*] Erwin Schrödinger (b. 1887 Vienna, d. 1961 Vienna) was famous for being a *physicien bohémien*, always living in a household with two women. In 1925 he discovered the equation that brought him international fame, and the Nobel Prize in Physics in 1933. He was also the first to show that the radiation discovered by Victor Hess in Vienna was indeed coming from the cosmos. He left Germany, and then again Austria, out of dislike for National Socialism, and was a professor in Dublin for many years. There he published his famous and influential book *What is life?*. In it, he came close to predicting the then-unknown nucleic acid DNA from theoretical insight alone.

FIGURE 45 Erwin Schrödinger (1887–1961)

size of atoms was predicted correctly. These were important results, especially if we keep in mind that classical physics cannot even explain the existence of atoms, let alone their light emission! In contrast, quantum physics explains all properties of atoms and their colours to high precision. In other words, the discovery of the quantum of action led the description of the motion of matter to a new high point.

In fact, the exact description of matter quantons is only found when both *spin effects* and the *relativistic* energy–momentum relation are taken into account. We do this below. No deviations between the full relativistic calculations and experiments have ever been found. And even today, predictions and measurements of atomic spectra remain the most precise and accurate in the whole study of nature: in the cases that experimental precision allows it, the calculated values agree with experiments to 13 decimal places.

Self-interference of quantons

Waves interfere. All experiments, including the examples shown in Figure 38 and Figure 39, confirm that all quantum particles, and in particular all matter quantons, show interference. Interference is a direct consequence of the Schrödinger equation, as the film of Figure 44 shows. The film illustrates the solution of the Schrödinger equation for a quantum particle moving through a double slit. The film visualizes how a double slit induces diffraction and interference for a matter particle.

It turns out that the Schrödinger equation completely reproduces and explains the observations of matter interference: also the interference of matter quantons is due to the evolution of clouds of rotating arrows. And like in all interference phenomena, the local intensity of the interference pattern turns out to be proportional to the square $|W|^2$ of the local wave amplitude. And the local wave amplitude results from the phase of the interfering wave trains. The analogy with light interefence is complete; even the formulae are the same.

We note that even though the wave function is spread out over the whole detection screen just before it hits the screen, it nevertheless yields only a localized spot on the screen. This effect, the so-called *collapse of the wave function*, is explored in detail below.
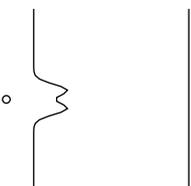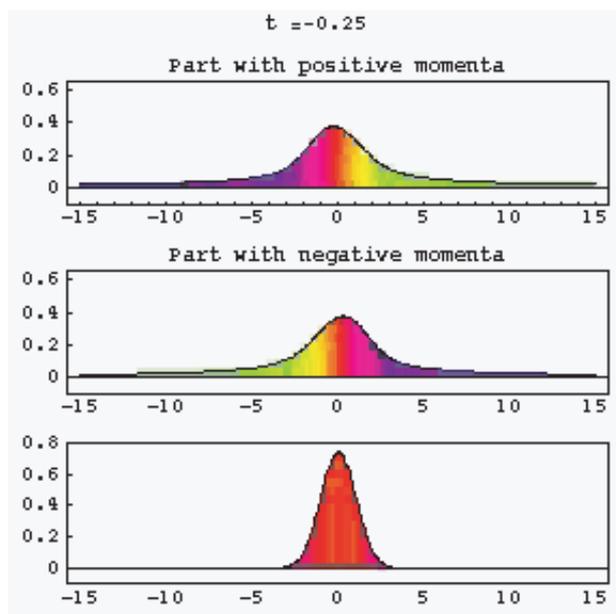
FIGURE 46 The evolution of a wave function (lowest curve) with zero momentum, and the motion of its parts with positive and negative momenta. Local phase is encoded in the colour. (QuickTime film © Bernd Thaller)

The speed of quantons

Let us delve a little into the details of the description given by the Schrödinger equation (35). The equation expresses a simple connection: the classical speed of a matter particle is the *group velocity* of the wave function ψ. Seen from far away, the wave function thus moves like a *classical* particle would.

But we know from classical physics that the group velocity is not always well defined: in cases where the group dissolves into several peaks, the concept of group velocity is not of much use. These are also the cases in which quantum motion is very different from classical motion, as we will soon discover. But for well-behaved cases, such as free or almost free particles, we find that the wave function moves in the same way as a classical particle does.

The Schrödinger equation makes another point: velocity and position of matter are not independent variables, and cannot be chosen at will. The initial condition of a system is given by the initial value of the wave function alone. No derivatives have to be (or can be) specified. Indeed, experiments confirm that quantum systems are described by a *first-order* evolution equation, in stark contrast to classical systems. The reason for this contrast is the quantum of action and the limit it poses on the possible state variables of a particle.

Dispersion of quantons

For free quantum particles, the Schrödinger's evolution equation implies *dispersion*, as illustrated in Figure 46. Imagine a wave function that is localized around a given starting position. Such a wave function describes a quantum system at rest. When time passes,
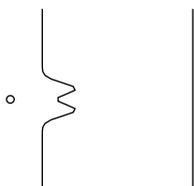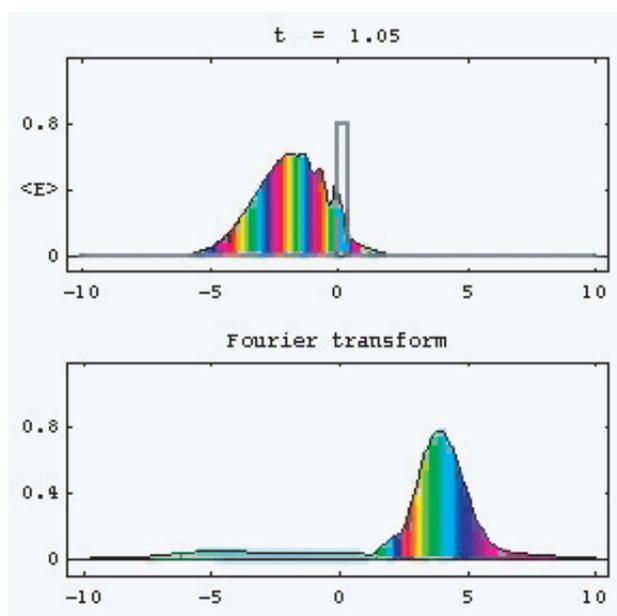
FIGURE 47 The tunnelling of a wave function through a potential hill (the rectangular column): most of the wave function is reflected, and part of the wave function passes to the other side. Local phase is encoded in the colour. (QuickTime film © Bernd Thaller)

this wave function will *spread out* in space. Indeed, Schrödinger's evolution equation is similar, mathematically, to a diffusion equation. In the same way that a drop of ink in water spreads out, also the state of a localized quantum particle will spread out in space. True, the most probable position stays unchanged, but the probability to find the particle at large distances from the starting position increases over time. For quantum particles, this spreading effect is indeed observed by all experiments. The spread is a consequence of the wave aspect of matter, and thus of the quantum of action $\hbar$. It occurs for quantons at rest and therefore also for quantons in motion. For macroscopic objects, the spreading effect is not observed, however: cars rarely move away from parking spaces. Indeed, quantum theory predicts that for macroscopic systems, the effect of spreading is negligibly small. Can you show why?

Challenge 70 e

Challenge 71 ny

In summary, the wave aspect of matter leads to the spreading of wave functions. Wave functions show dispersion.

Tunnelling and limits on memory – damping of quantons

'Common sense' says that a slow ball cannot roll over a high hill. More precisely, classical physics says that if the kinetic energy T is smaller than the potential energy V that the ball would have at the top of the hill, then the ball cannot reach the top of the hill. In contrast, according to quantum theory, there is a non-vanishing probability of passing the hill for *any* energy of the ball.

In quantum theory, hills and obstacles are described by potential barriers, and objects by wave functions. Any initial wave function will spread *beyond* any potential barrier of finite height and width. The wave function will also be non-vanishing *at* the location of the barrier. In short, any object can overcome any hill or barrier, as shown in Figure 48.
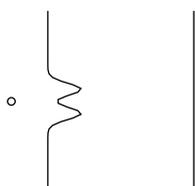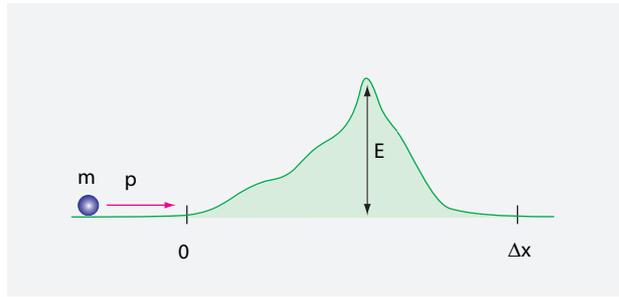
FIGURE 48 Climbing a hill.

This effect is called the *tunnelling effect*. It is in complete contrast to everyday experience – and to classical mechanics.

The tunnelling effect results from a new aspect contained in the quantum description of hills: in nature, any obstacle can be overcome with a *finite* effort. No obstacle is infinitely difficult to surmount. Indeed, only for a potential of infinite height would the wave function vanish and fail to spread to the other side. But such potentials exist only as approximations; in nature potentials are always of finite value.

How large is the tunnelling effect? Calculation shows that the transmission probability P is given approximately by

$$P \approx \frac{16T(V-T)}{V^2} \, e^{-\frac{2w}{\hbar}\sqrt{2m(V-T)}} \tag{36}$$

where w is the width of the hill, V its height, and m and T the mass and the kinetic energy of the particle. For a system of large number of particles, the probability is (at most) the product of the probabilities for the different particles.

Let us take the case of a car in a garage, and assume that the car is made of 10^{28} atoms at room temperature. A typical garage wall has a thickness of 0.1 m and a potential height of $V = 1\,\text{keV} = 160\,\text{aJ}$ for the passage of an atom. We get that the probability of finding the car outside the garage is

$$P \approx \left(10^{-\left(10^{12}\right)}\right)^{\left(10^{28}\right)} \approx 10^{-\left(10^{40}\right)} . \tag{37}$$

The smallness of this value (just try to write it down, to be convinced) is the reason why it is never taken into account by the police when a car is reported missing. (Actually, the probability is even considerably smaller. Can you name at least one effect that has been forgotten in this simple calculation?)

Obviously, tunnelling can be important only for small systems, made of a few particles, and for thin barriers, with a thickness of the order of $\hbar/\sqrt{2m(V-T)}$. For example, tunnelling of single atoms is observed in solids at high temperature, but is not important in daily life. For electrons, the effect is more pronounced: the barrier width w

Challenge 72 ny

Challenge 73 e

Challenge 74 s

Farady cage
with high
electric
potential

screen with
intereference
pattern that
depends on
potential

beam splitter

charged matter beam
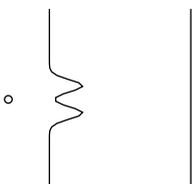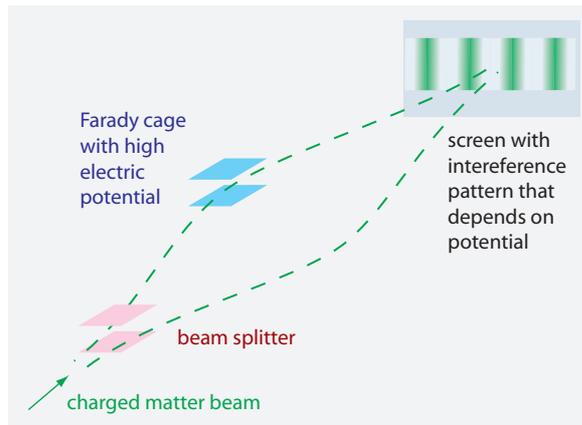
FIGURE 49 A localized electric potential
in an interferometer leads to a shift of the
interference pattern.

for an appreciable tunnelling effect is

$$w \approx \frac{0.5\,\text{nm} \sqrt{\text{aJ}}}{\sqrt{V - T}} \; . \tag{38}$$

At room temperature, the kinetic energy T is of the order of 6 zJ; increasing the temperature obviously increases the tunnelling. As a result, electrons tunnel quite easily through barriers that are a few atoms in width. Indeed, every TV tube uses tunnelling at high temperature to generate the electron beam producing the picture. The necessary heating is the reason why in the past, television tubes took some time to switch on.

The tunnelling of electrons also limits the physical size of computer memories. Memory chips cannot be made arbitrary small. Silicon integrated circuits with one terabyte of random-access memory (RAM) will probably never exist. Can you imagine why? In fact, tunnelling limits the working of any type of memory, including that of our brain. Indeed, if we were much hotter than 37°C, we could not remember anything!

Since light is made of particles, it can also tunnel through potential barriers. The best – or highest – potential barriers for light are mirrors; mirrors have barrier heights of the order of one attojoule. Tunnelling implies that light *can* be detected behind any mirror. These so-called *evanescent waves* have indeed been detected; they are used in various high-precision experiments and devices.

THE QUANTUM PHASE

We have seen that the *amplitude* of the wave function, the probability amplitude, shows the same effects as any wave: dispersion and damping. We now return to the *phase* of the wave function and explore it in more detail.

Whereas the amplitude of a wave function is easy to picture – just think of the (square root of the) density of a real cloud – the phase takes more effort. As mentioned, states or wave functions are *clouds with a local phase*: they are clouds of rotating arrows, i.e., clouds of objects that rotate and can be rotated. In case of an everyday water cloud, a local rotation of droplets has no effect on the cloud. In contrast, in quantum theory, the local rotation of the cloud, thus the local change of its phase, does have a measurable
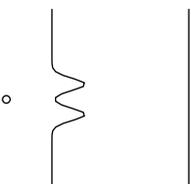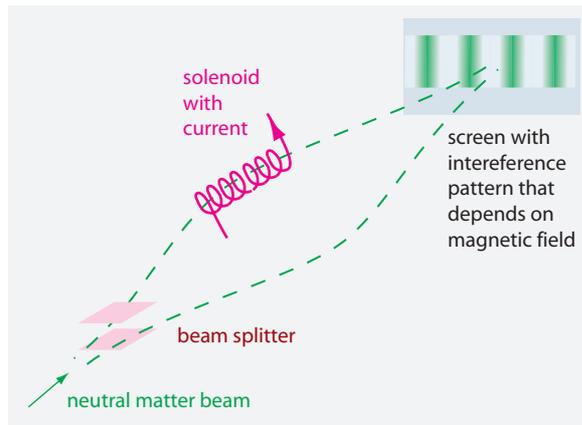
FIGURE 50 Magnetic fields change the phase of a spinning particle.

effect. Let us explore this point.

The phase of free matter waves behaves like the phase of photons: it evolves with time, and thus increases along the path of a moving particle. The phase can be pictured by a small rotating arrow. The angular velocity with which the phase rotates is given by the famous relation $\omega = E/\hbar$. In short,

> ▷ We can picture the wave function of a free quantum particle as a moving cloud of arrows; the arrows rotate with constant frequency while the cloud disperses at the same time.

Above all, the phase is that aspect of the wave function that leads to interference effects. When two partial wave functions are separated and recombined after a relative phase change, the phase change will determine the interference pattern. This is the origin of the electron beam interference observations shown in Figure 38. Without the quantum phase, there would be no extinction and no interference.

The phase of a wave function can be influenced in many ways. The simplest way is the use of electric fields. If the wave function of a *charged* particle is split, and one part is led through a region with an electric field, a phase change will result. The arrangement is shown in Figure 49. A periodic change of the electric potential should yield a periodic shift of the interference pattern. This is indeed observed.

Another simple case of phase manipulation is shown in Figure 50: also a magnetic field changes the phase of a spinning neutral particle – if it contains charges – and thus influences the interference behaviour.

A famous experiment shows the importance of the phase in an even more surprising way: the *Aharonov–Bohm effect*. The effect is famous for two reasons: it is counterintuitive and it was predicted before it was observed. Look at the set-up shown in Figure 51. A matter wave of charged particles is split into two by a cylinder – positioned at a right angle to the matter's path – and the matter wave recombines behind it. Inside the cylinder there is a magnetic field; outside, there is none. (A simple way to realize such a cylinder is a long solenoid.) Quantum physics predicts that an interference pattern will be observed, and that the position of the stripes will depend on the value of the mag-
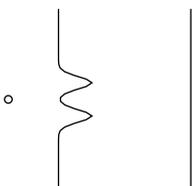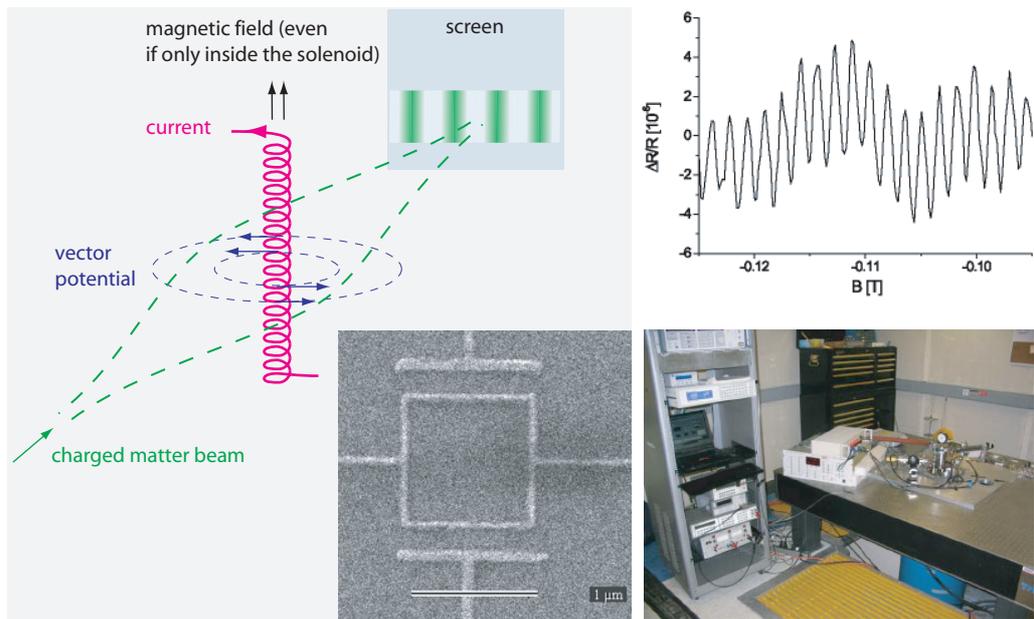
FIGURE 51 The Aharonov–Bohm effect: the influence of the magnetic vector potential on interference (left) and a measurement confirmation (right), using a microscopic sample that transports electrons in thin metal wires (© Doru Cuturela).
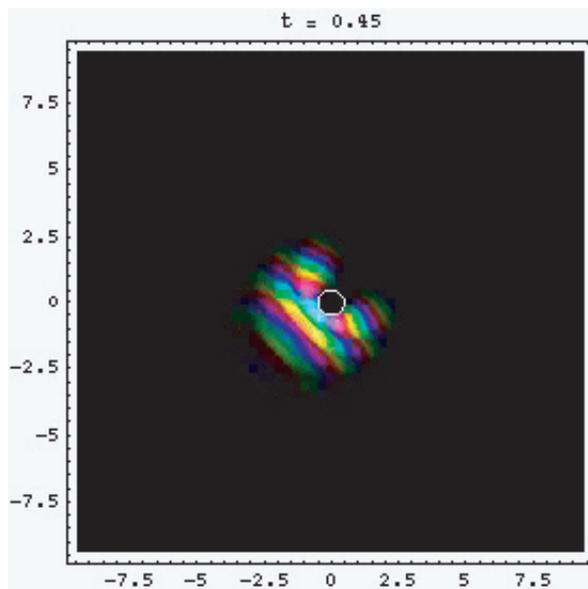
FIGURE 52 The motion of a wave function around a solenoid showing the Aharonov–Bohm effect. The density of the state is displayed by brightness, and the local phase is encoded in the colour. (QuickTime film © Bernd Thaller)

netic field. This happens even though the wave never enters the region with the field! The surprising effect has been observed in countless experiments.

The reason for the Aharonov–Bohm effect is simple: for a charged particle, the phase
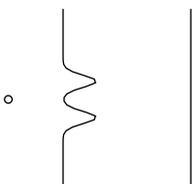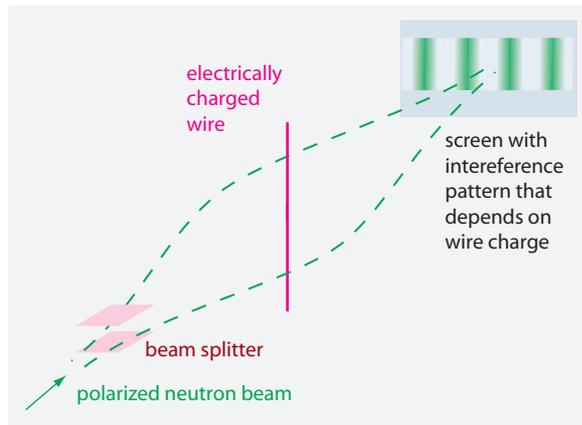
FIGURE 53 The Aharonov–Casher effect: the influence of charge on the phase leads to interference even for interfering neutrons.

of a wave function is determined by the vector potential **A**, not by the magnetic field **B**. The vector potential around a solenoid does not vanish – as we know from the section on electrodynamics – but circulates around the solenoid. This circulation distinguishes the two sides of the solenoid and leads to a phase shift – one that indeed depends on the magnetic field value – and thus produces interference, even though the particle never interacts with the magnetic field itself.

A further example for phase manipulation is the so-called *Aharonov–Casher effect*, which even occurs for neutral particles, as long as they have a magnetic moment, such as neutrons have. The phase of a *polarized* neutron will be influenced by an electric field, so that the arrangement shown in Figure 53 will show an interference pattern that depends on the applied electric potential.

Another case of phase manipulation will be presented later on: also gravitational fields can be used to rotate wave functions. Even the acceleration due to rotational motion can do so. In fact, it has been possible to measure the rotation of the Earth by observing the change of neutron beam interference patterns.

Another important class of experiments that manipulate the phase of wave functions are possible with macroscopic quantum phenomena. In superconductivity and in superfluidity, the phase of the wave function is regularly manipulated with magnetic and electric fields. This possibility has many important technical applications. For example, the so-called *Josephson effect* is used to measure electric potential differences by measuring the frequency of emitted radio waves, and so-called *superconducting quantum interference devices*, or SQIDs, are used to measure tiny magnetic fields.

We note that all these experiments confirm that the *absolute* phase of a wave function *cannot* be measured. However, *relative phases* – phase differences or phase changes – *can* be measured. Can you confirm this?

All the phase shift effects just presented have been observed in numerous experiments. The phase is an essential aspect of the wave function: the phase leads to interference and is the main reason for calling it *wave* function in the first place. Like in any wave, the phase evolves over time and it can be influenced by various external influences. Above all, the experiments show that a localized quantum particle – thus when the spread of the wave function can be neglected – is best imagined as a rotating arrow;
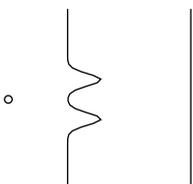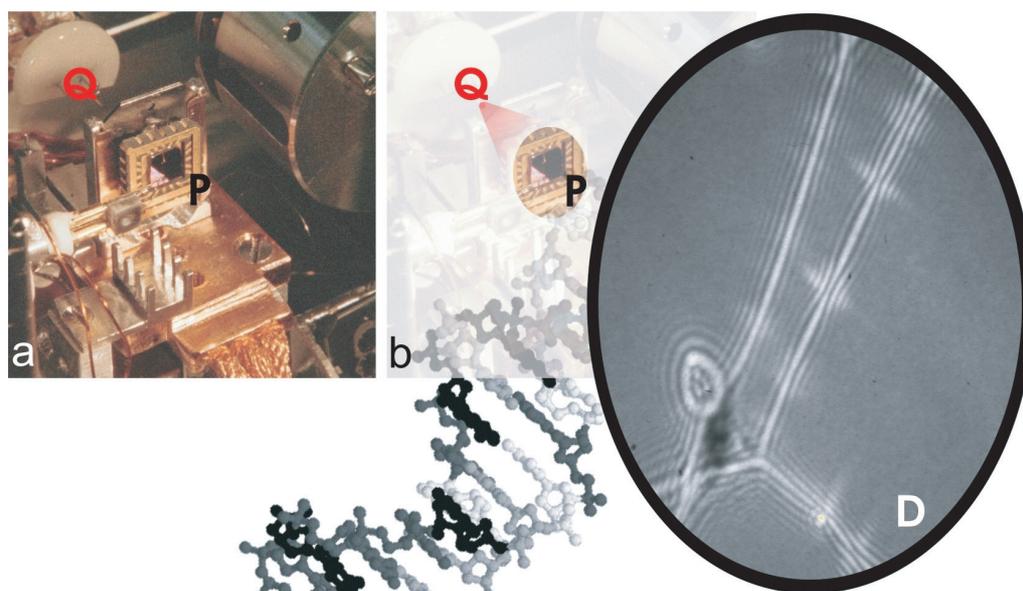
FIGURE 54 An electron hologram of DNA molecules (© Hans-Werner Fink/Wiley VCH).

in contrast, whenever the spread cannot be neglected, the wave function is best imagined as a wave of arrows rotating at each point in space.

CAN TWO ELECTRON BEAMS INTERFERE? ARE THERE COHERENT ELECTRON BEAMS?

Ref. 59 Do coherent electron sources exist? The question is tricky. Results in the literature, such as the one illustrated in Figure 54, state that is possible to make holograms with electron beams.* However, when one asks these authors about the meaning of coherence, they answer that electron coherence is only transversal, not longitudinal. Transversal coherence is determined by the possible size of wavefronts with a given phase. The upper limit of this size is given by the interactions such a state has with its environment. All this behaviour is as expected for actual coherence.

However, the concept of 'transversal coherence' is a misnomer. The ability to interfere with oneself, as implies in the term 'transversal coherence' is not the correct definition of coherence. Transversal coherence, be it for photons or for matter particles, only expresses the smallness of the particle source. Both small lamps (and lasers) can show interference when the beam is split and recombined with identical path length; this is not a proof of coherence of the light field. A similar reasoning shows that monochromaticity is not a proof for coherence either.

A state is called *coherent* if it possesses a well-defined phase throughout a given domain of space or time. The size of the spatial region or of the time interval defines the degree of coherence. This definition yields coherence lengths of the order of the source size for small 'incoherent' sources. Even for a small coherence length, the size of an interference pattern or the distance d between its maxima can be much larger than the

Ref. 60 * In 2002, the first holograms have been produced that made use of neutron beams.
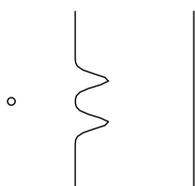
coherence length l or the source size s. In short, a large size (or a persistent duration in time) of an interference pattern alone is *not* a proof of coherence.

Let us recall the situation for light. A light source is coherent if it produces an approximate sine wave over a certain length or time. Due to the indeterminacy relation, in any coherent beam of light, the photon number is undetermined. The same requirement applies to coherent electron beams: an undetermined electron number is needed for coherence. That is impossible, as electrons carry a conserved charge. Coherent electron beams do not exist.

In summary, even though an electron can interfere with itself, and even though it is possible to produce interference between two light sources, interference between two electron sources is impossible. Indeed, nobody has every managed to produce interference between two electron sources. There is no conventional concept of coherence for electron beams.

The least action principle in quantum physics

In nature, motion happens in a way that minimizes change. Indeed, in classical phys- ics, the principle of least action – or principle of cosmic lazyness – states: in nature, the motion of a particle happens along that particular path – out of all possible paths with the same end points – for which the action is minimal. This principle of cosmic laziness or cosmic efficiency was stated mathematically by saying that in nature, the *variation* δS of the action is zero. Action or change minimization explains all classical evolution equations. We now transfer this idea to the quantum domain.

For quantum systems, we need to redefine both the concept of action and the concept of variation: first of all, we have to find a description of action that is based on operators; secondly, we need to define the action variation without paths, as the concept of 'path' does not exist for quantum systems; thirdly, since there is a smallest action in nature, a vanishing variation is not a clearly defined concept, and we must overcome this hurdle. There are two main ways to achieve this goal: to describe the motion of quantum systems as a superposition of all possible paths, or to describe action with the help of wave functions. Both approaches are equivalent.

In the first approach, the *path integral formulation*, the motion of a quantum particle is described as a democratic superposition of motions along all possible paths. (We called it the 'arrow model' above.) For each path, the evolution of the arrow is determined, and at the end point, the arrows from all paths are added. The action for each path is the number of turns that the arrow performs along the path. The result from this exercise is that the path for which the arrow makes the smallest number of turns is usually (but not always!) the most probable path. A more precise investigation shows that classical, macroscopic systems always follow only the path of smallest action, whereas quantum systems follow all paths.

In the second approach to quantum physics, action is defined with the help of wave functions. In classical physics, we defined the action (or change) as the integral of the Lagrangian between the initial and final points in time, and the Lagrangian itself as the difference between kinetic and potential energy. In quantum physics, the simplest definition is the *quantum action* defined by Julian Schwinger. Let us call the initial and final
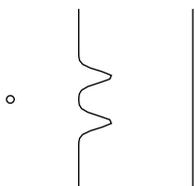
states of the system ψ_i and ψ_f. The action S between these two states is defined as

$$S = \langle \psi_i \,|\, \int L \,dt \,|\, \psi_f \rangle \,, \tag{39}$$

where L is the Lagrangian (operator). The angle brackets represent the 'multiplication' of states and operators as defined in quantum theory.[*] In simple words, also in quantum theory, action – i.e., the change occurring in a system – is the integral of the Lagrangian. The Lagrangian operator L is defined in the same way as in classical physics: the Lagrangian $L = T - V$ is the difference between the kinetic energy T and the potential energy V operators. The only difference is that, in quantum theory, the momentum and position *variables* of classical physics are replaced by the corresponding *operators* of quantum physics.[**]

To transfer the concept of action variation δS to the quantum domain, Julian Schwinger introduced the straightforward expression

$$\delta S = \langle \psi_i \,|\, \delta \int L \,dt \,|\, \psi_f \rangle \,. \tag{40}$$

The concept of path is not needed in this expression, as the variation of the action is based on varying wave functions instead of varying particle paths.

The last classical requirement to be transferred to the quantum domain is that, because nature is lazy, the variation of the action must vanish. However, in the quantum domain, the variation of the action cannot be zero, as the smallest observable action is the quantum of action. As Julian Schwinger discovered, there is only one possible way to express the required minimality of action:

$$\delta S = \langle \psi_i \,|\, \delta \int L \,dt \,|\, \psi_f \rangle = -i\hbar \,\delta \langle \psi_i | \psi_f \rangle \,. \tag{41}$$

This so-called *quantum action principle* describes all motion in the quantum domain. Classically, the right-hand side is zero – since $\hbar$ is taken to be zero – and we then recover the minimum-action principle $\delta S = 0$ of classical physics. But in quantum theory, whenever we try to achieve small variations, we encounter the quantum of action and changes of (relative) phase. This is expressed by the right-hand side of the expression. The right side is the reason that the evolution equations for the wave function – Schrödinger's equation for the spinless non-relativistic case, or Dirac's equation for the spin 1/2 relativistic case – are valid in nature.

In other words, all quantum motion – i.e., the quantum evolution of a state ψ or $|\psi\rangle$ – happens in such a way that the action variation is the same as $-i$ times the quantum of action $\hbar$ times the variation of the scalar product between initial and final states. In

* We skip the details of notation and mathematics here; in the simplest description, states are wave functions, operators act on these functions, and the product of two different brackets is the integral of the function product over space.
** More precisely, there is also a condition governing the ordering of operators in a mixed product, so that the non-commutativity of operators is taken into account. We do not explore this issue here.
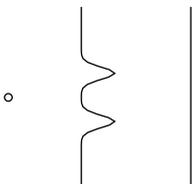
simple terms, in the actual motion, the intermediate states are fixed by the requirement that they must lead from the initial state to the final state with the smallest number of effective turns of the state phase. The factor $-i$ expresses the dependence of the action on the rotation of the wave function.

In summary, the least action principle is also valid in quantum physics, provided one takes into account that action values below $\hbar$ cannot be found in experiments. The least action principle governs the evolution of wave function. The least action principle thus explains the colour of all things, all other material science, all chemistry and all biology, as we will see in the following.

The motion of quantons with spin

> Everything turns.
>
> Anonymous

What is the origin of the quantum phase? Classical physics helps to answer the question. Like everyday objects, also quantons can rotate around an axis: we speak of particle *spin*. But if quantum particles can spin, they should possess angular momentum. And indeed, experiments confirm this deduction.

In particular, electrons have spin. The full details of electron spin were deduced from experiments by two Dutch students, George Uhlenbeck and Samuel Goudsmit, in 1925. They had the guts to publish what Ralph Kronig had also suspected: that electrons rotate around an axis with a projected component of the angular momentum given by $\hbar/2$. In fact, this value – often called spin 1/2 for short – is valid for *all* elementary *matter* particles. (In contrast, all known elementary *radiation* particles have spin values of $\hbar$, or spin 1 for short.)

If a spinning particle has angular momentum, it must be possible to rearrange the axis by applying a torque, to observe precession, to transfer the spin in collisions, etc. All these effects are indeed observed; for example, the Stern–Gerlach experiment already allows all these observations. The only difference between particle spin and classical angular momentum is that particle spin is quantized, as we deduced above.

In other words, the *spin L* of a quantum particle has all the properties of a rotation around an axis. As a consequence, spinning *charged* quantum particles act as small dipole magnets, with the magnet oriented along the axis of rotation. The observed strength of the dipole magnet, the *magnetic moment*, is proportional to the spin and to the conversion factor $-e/2m_e$, as expected from classical physics. Therefore, the natural unit for the magnetic moment of the electron is the quantity $\mu_B = e\hbar/2m_e$; it is called *Bohr's magneton*. It turns out that the magnetic moment $\boldsymbol{\mu}$ of quantons behaves differently from that of classical particles. The quantum effects of spin are described by the so-called *g-factor*, which is a pure number:

$$\boldsymbol{\mu} = g \frac{-e}{2m_e} \boldsymbol{L} = -g\mu_B \frac{\boldsymbol{L}}{\hbar} , \quad \text{with} \quad \mu_B = \frac{e\hbar}{2m_e} . \tag{42}$$

From the observed optical spectra, Uhlenbeck and Goudsmit deduced a *g*-factor of 2 for the electron. Classically, one expects a value $g = 1$. The experimental value $g = 2$ was
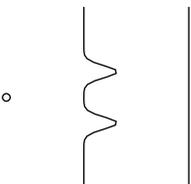
explained by Llewellyn Thomas as a relativistic effect a few months after its experimental discovery.

By 2004, experimental techniques had become so sensitive that the magnetic effect of a single electron spin attached to an impurity (in an otherwise non-magnetic material) could be detected. Researchers now hope to improve these so-called 'magnetic-resonance-force microscopes' until they reach atomic resolution.

In 1927, Wolfgang Pauli* discovered how to include spin 1/2 in a quantum-mechanical description: instead of a state function described by a single complex number, a state function with *two* complex components is needed. The reason for this expansion is simple. In general, the little rotating arrow that describes a quantum state does *not* rotate around a *fixed* axis, as is assumed by the Schrödinger equation; the *axis of rotation* has also to be specified at each position in space. This implies that two additional parameters are required at each space point, bringing the total number of parameters to four real numbers, or, equivalently, two complex numbers. Nowadays, Pauli's equation for quantum mechanics with spin is mainly of conceptual interest, because – like that of Schrödinger – it does not comply with special relativity.

In summary, the non-relativistic description of a quanton with spin implies the use of wave functions that specify *two* complex numbers at each point in space and time. The additional complex number describe the local rotation plane of the spin. The idea of including the local rotation plane was also used by Dirac when he introduced the relativistic description of the electron, and the idea is also used in all other wave equations for particles with spin.

RELATIVISTIC WAVE EQUATIONS

In 1899, Max Planck had discovered the quantum of action. In 1905, Albert Einstein published the theory of special relativity, which was based on the idea that the speed of light c is independent of the speed of the observer. The first question Planck asked himself was whether the value of the quantum of action would be independent of the speed of the observer. It was his interest in this question that led him to invite Einstein to Berlin. With this invitation, he made the patent-office clerk famous in the world of physics.

Experiments show that the quantum of action is indeed independent of the speed of the observer. All observers find the same minimum value. To include special relativity into quantum theory, we therefore need to find the correct quantum Hamiltonian H, i.e., the correct energy operator.

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* Wolfgang Ernst Pauli (b. 1900 Vienna, d. 1958 Zürich), at the age of 21, wrote one of the best texts on special and general relativity. He was the first to calculate the energy levels of hydrogen using quantum theory, discovered the exclusion principle, incorporated spin into quantum theory, elucidated the relation between spin and statistics, proved the CPT theorem, and predicted the neutrino. He was admired for his intelligence, and feared for his biting criticisms, which led to his nickname, 'conscience of physics'. Despite this trait, he helped many people in their research, such as Heisenberg with quantum theory, without claiming any credit for himself. He was seen by many, including Einstein, as the greatest and sharpest mind of twentieth-century physics. He was also famous for the 'Pauli effect', i.e., his ability to trigger disasters in laboratories, machines and his surroundings by his mere presence. As we will see shortly, one can argue that Pauli actually received the Nobel Prize in Physics in 1945 – officially 'for the discovery of the exclusion principle' – for finally settling the question of how many angels can dance on the tip of a pin.
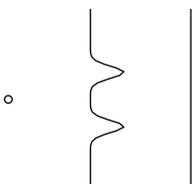
For a free relativistic particle, the classical Hamiltonian function – that is, the energy of the particle – is given by

$$H = \pm\sqrt{c^4 m^2 + c^2 \boldsymbol{p}^2} \quad \text{with} \quad \boldsymbol{p} = \gamma m \boldsymbol{v} \; . \tag{43}$$

Thus we can ask: what is the corresponding Hamilton operator for the quantum world? The simplest answer was given, in 1949 by T.D. Newton and E.P. Wigner, and in 1950, by L.L. Foldy and S.A. Wouthuysen. The operator is almost the same one:

$$H = \beta\sqrt{c^4 m^2 + c^2 \boldsymbol{p}^2} \quad \text{with} \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \; . \tag{44}$$

The signs appearing in the matrix operator β distinguish, as we will see, between particles and antiparticles. The numbers +1 and −1 appear twice, to take care of the two possible spin directions for each case.

With this relativistic Hamiltonian operator for spin 1/2 particles – and with all others – the wave function is described by *four* complex numbers, *two* for particles and *two* for antiparticles. Why? We saw above that a quantum particle with spin requires two complex components for its state; this followed from the requirement to specify, at each point in space, the length of the arrow, its phase, and its plane of rotation. Earlier on we also found that relativity automatically introduces antimatter. (We will explore the issue in more detail below.) Both matter and antimatter are thus part of any relativistic description of quantum effects. The wave function for a particle has vanishing antiparticle components, and vice versa. In total, the wave function for relativistic spin 1/2 particle has thus *four* complex components.

The Hamilton operator yields the velocity operator $\boldsymbol{v}$ through the same relation that is valid in classical physics:

$$\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \beta\frac{\boldsymbol{p}}{\sqrt{c^4 m^2 + c^2 \boldsymbol{p}^2}} \; . \tag{45}$$

This velocity operator shows a continuum of eigenvalues, from minus to plus the speed of light. The velocity $\boldsymbol{v}$ is a constant of motion, as are the momentum $\boldsymbol{p}$ and the energy

$$E = \sqrt{c^4 m^2 + c^2 \boldsymbol{p}^2} \; . \tag{46}$$

Also the orbital angular momentum $\boldsymbol{L}$ is defined as in classical physics, through

$$\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p} \; . \tag{47}$$

The orbital angular momentum $\boldsymbol{L}$ and the spin $\boldsymbol{\sigma}$ are separate constants of motion. A particle (or antiparticle) with positive (or negative) angular momentum component has
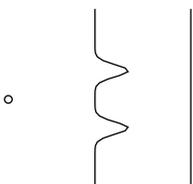
a wave function with only one non-vanishing component; the other three components vanish.

But alas, the representation of relativistic motion named after Foldy and Wouthuysen is not the simplest when it comes to take electromagnetic interactions into account. The simple identity between the classical and quantum-mechanical descriptions is lost when electromagnetism is included. We will solve this problem below, when we explore Dirac's evolution equation for relativistic wave functions.

Bound motion, or composite vs. elementary quantons

When is an object composite, and not elementary? Whenever it contains internal, or bound motion. When is this the case? Quantum theory gives several pragmatic answers.

The first criterion for compositeness is somewhat strange: an object is composite when its gyromagnetic ratio is different from the one predicted by quantum electrodynamics. The *gyromagnetic ratio* γ – not to be confused with the relativistic dilation factor – is defined as the ratio between the magnetic moment M and the angular momentum L:

$$\boldsymbol{M} = \gamma \boldsymbol{L} \ . \tag{48}$$

The gyromagnetic ratio γ is measured in units of $\mathrm{s^{-1}T^{-1}}$, i.e., C/kg, and determines the energy levels of magnetic spinning particles in magnetic fields; it will reappear later in the context of magnetic resonance imaging. All candidates for elementary particles have spin 1/2. The gyromagnetic ratio for spin-1/2 particles of magnetic moment M and mass m can be written as

$$\gamma = \frac{M}{\hbar/2} = g\frac{e}{2m} \ . \tag{49}$$

The criterion for being elementary can thus be reduced to a condition on the value of the dimensionless number g, the so-called *g-factor*. (The expression $e\hbar/2m$ is often called the *magneton* of the particle.) If the g-factor *differs* from the value predicted by quantum electrodynamics for point particles – about 2.0 – the object is *composite*. For example, a $^4\mathrm{He}^+$ helium ion has spin 1/2 and a g value of $14.7 \cdot 10^3$. Indeed, the radius of the helium ion is $3 \cdot 10^{-11}$ m, obviously a finite value, and the ion is a composite entity. For the proton, one measures a g-factor of about 5.6. Indeed, experiments yield a finite proton radius of about 0.9 fm and show that it contains several constituents.

The neutron, which has a magnetic moment despite being electrically neutral, must therefore be composite. Indeed, its radius is approximately the same as that of the proton. Similarly, molecules, mountains, stars and people must be composite. According to this first criterion, the only elementary particles are *leptons* (i.e., electrons, muons, tauons and neutrinos), *quarks*, and *intermediate bosons* (i.e., photons, W-bosons, Z-bosons and gluons). More details on these particles will be revealed in the chapters on the nucleus.

Another simple criterion for compositeness has just been mentioned: *any object with a measurable size is composite*. This criterion yields the same list of elementary particles as the first. Indeed, the two criteria are related. The simplest model for composite structures
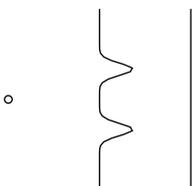
predicts that the g-factor obeys

$$g - 2 = \frac{R}{\lambda_C} \qquad (50)$$

where R is the radius and $\lambda_C = h/mc$ is the Compton wavelength of the system. This expression is surprisingly precise for helium-4 ions, helium-3, tritium ions and protons, as you may wish to check. The tables in Appendix B in the next volume make the same point. In short, the second criterion for compositeness is equivalent to the first.

A third criterion for compositeness is more general: *any object larger than its Compton length is composite*. The argument is simple. An object is composite if one can detect *internal* motion, i.e., motion of some components. Now the action of any part with mass m_{part} moving inside a composed system of size r obeys

$$S_{part} < 2\pi\, r\, m_{part}\, c < \pi\, r\, m\, c \qquad (51)$$

where m is the mass of the *composite* object. On the other hand, following the principle of quantum theory, this action, to be observable, must be larger than $\hbar/2$. Inserting this condition, we find that for any composite object*

$$r > \frac{\hbar}{2\pi\, m\, c}\;. \qquad (52)$$

The right-hand side differs only by a factor $4\pi^2$ from the so-called *Compton (wave)length*

$$\lambda = \frac{h}{m\, c} \qquad (53)$$

of an object. Thus any object *larger* than its own Compton wavelength is composite; and any object *smaller* than the right-hand side of expression (52) is elementary. Again, only leptons, quarks and intermediate bosons passed the test. (For the Higgs boson discovered in 2012, the test has yet to be performed, but it is expected to comply as well.) All other objects are composite. In short, this third criterion produces the same list as the previous ones. Can you explain why?

A fourth criterion for compositeness is regularly cited by Steven Weinberg: a particle is elementary if it appears in the Lagrangian of the standard model of particle physics, i.e., in the description of the fundamental building blocks of nature. Can you show that this criterion follows from the previous ones?

Interestingly, we are not yet finished with this topic. Even stranger statements about compositeness will appear when gravity is taken into account. Just be patient: it is worth it.

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* Can you find the missing factor of 2? And is the assumption that the components must always be lighter than the composite a valid one?
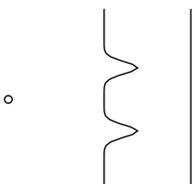
Curiosities and fun challenges about quantum motion of matter

> " Die meisten Physiker sind sehr naiv, sie glauben
> immer noch an wirkliche Wellen oder
> Teilchen.*
> "
>
> Anton Zeilinger

Take the sharpest knife edge or needle tip you can think of: the quantum of action implies that their boundaries are not sharp, but fuzzy, like the boundaries of clouds. Take the hardest or most solid object you can think of, such as diamond or a block of tungsten: the quantum of action implies that its surface is somewhat soft. All experiments confirm these statements. *Nothing in nature is really sharp or really solid.* Quantum physics thus disagrees with several ideas of the ancient Greek atomists.

∗ ∗

Do hydrogen atoms exist? Most types of atom have been imaged with microscopes, photographed under illumination, levitated one by one, and even moved with needles, one by one, as the picture on page 344 in volume I shows. Researchers have even moved single atoms by using laser beams to push them. However, not a single one of these experiments has measured or imaged hydrogen atoms. Is that a reason to doubt the existence of hydrogen atoms? Taking this not-so-serious discussion seriously can be a lot of fun.

∗ ∗

Is the wave function 'real'? More precisely, is the wave function really a cloud? Some physicists still doubt this. This dying group of physicists, often born around the middle of the twentieth century, have heard so often – incorrectly and usually from questionable authorities – that a wave function has no reality that they stopped asking and answering the simplest questions. To dispel their doubts, ask them whether they have a non-zero height or whether they think that atoms are round. If they agree, they have admitted that wave functions have some sort of reality. All everyday objects are made of elementary particles that are so unmeasurably small that we can call them point-like. Therefore, the size, surface area and volume of all everyday objects are exclusively due to wave functions. Every length, area and volume is a proof that wave functions have some sort of reality.

∗ ∗

Two observables can commute for two different reasons: either they are very *similar* – such as the coordinates x and x^2 – or they are very *different* – such as the coordinate x and the momentum p_y. Can you give an explanation for this?

∗ ∗

Space and time translations commute. Why then do the momentum operator and the Hamiltonian not commute in general?

∗ ∗

Ref. 68

Challenge 82 s

Challenge 83 e

Challenge 84 d

Challenge 85 ny

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* 'Most physicists are very naive; they still believe in real waves or real particles.' Anton Zeilinger, physicist at the University of Vienna, is well-known for his experiments on quantum mechanics.
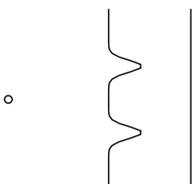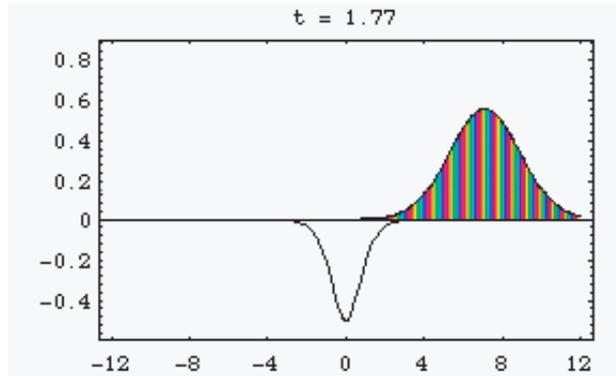
FIGURE 55 A special potential well that does *not* disturb a wave function. Colour indicates phase. (QuickTime film © Bernd Thaller)

There exist special potentials that have *no* influence on a wave function. Figure 55 shows an example. This potential has reflection coefficient zero for all energies; the scattered wave has no reflected part. The mathematical reason is fascinating. The potential well has the shape of a soliton of the Korteweg–de Vries equation; this equation is related to the Schrödinger equation.

∗ ∗

Any bound system in a non-relativistic state with no angular momentum obeys the relation

$$\langle r^2 \rangle \, \langle T \rangle \;\; \geqslant \; \frac{9\hbar^2}{8m} \;, \tag{54}$$

where m is the reduced mass and T the kinetic energy of the components, and r is the size of the system. Can you deduce this result, and check it for the ground state of hydrogen?

∗ ∗

In high school, it often makes sense to visualize electron wave functions as a special type of fluid-like matter, called *electronium*, that has a negative charge density. In this visualization, an atom is a positive nucleus surrounded by an electronium cloud. Deforming the electronium cloud around a nucleus requires energy; this happens when a photon of the correct frequency is absorbed, for example. When atoms of the right kind approach each other, the electronium clouds often form stable bridges – chemical bonds.

∗ ∗

Quantum theory allows for many unusual bound states. Usually we think of bound states as states of low energy. But there are situations in which bound states arise due to external forcing with oscillating potentials. We encountered such a situation in classical physics: the vertically driven, upside-down pendulum that remain vertical despite being unstable. Similar situations also occur in quantum physics. Examples are Paul traps, the helium atom, negative ions, Trojan electrons and particle accelerators.

Ref. 69

Challenge 86 s

Vol. I, page 319

Ref. 70

∗ ∗
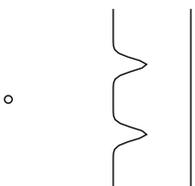
One often reads that the universe might have been born from a quantum fluctuation.

Can you explain why this statement make no sense?

A SUMMARY ON MOTION OF MATTER QUANTONS

In summary, the motion of massive quantons, i.e., of quantum matter particles, can be described in two ways:

— At high magnification, quantum matter particles are described by wave functions that move like *advancing, rotating and precessing clouds of arrows.* The local cloud orientation, or local phase, follows a wobbling motion. The square of the wave function, i.e., the density of the cloud, is the probability for finding the particle at a given spot.
— Seen from far away, at low magnification, a moving massive quantum particle behaves as a single advancing, rotating and precessing arrow. The details of the rotation and precession of the arrow depend on the energy and momentum of the particle and the potential it is subjected to. The arrow is a probability amplitude: the squared length of the arrow is the probability to observe the particle. If a particle can get from a starting point to a final point in several ways, the probability amplitudes for each way add up.

The single rotating arrow results from a cloud average. The single arrow combines particle and wave properties. A full rotation of the arrow corresponds to the quantum of action $\hbar$. This central feature implies that a non-relativistic particle whose spin can be neglected follows the Schrödinger equation, and that a relativistic electron follows the Dirac equation. The Dirac equation agrees with all known experiments. In particular, the Dirac equation describes all of materials science, chemistry and biology, as we will find out.

To continue with the greatest efficiency on our path across quantum physics, we explore three important topics: the indistinguishability of particles of the same kind, the spin of quantum particles, and the meaning of probabilities.
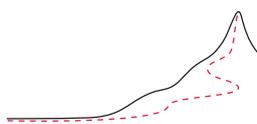
PERMUTATION OF PARTICLES – ARE PARTICLES LIKE GLOVES?

Why are we able to distinguish twins from each other? Why can we distinguish hat looks alike, such as a copy from an original? Most of us are convinced that henever we compare an original with a copy, we can find a difference. This conviction turns out to be correct also in the quantum domain, but the conclusion is not straightforward.

Think about any method that allows you to distinguish objects: you will find that it runs into trouble for point-like particles. Therefore, in the quantum domain something must change about our ability to distinguish particles and objects.

We could argue that differences between an original object and a copy can always be made to disappear: it should be sufficient to use the same number and type of atoms. In fact, the quantum of action shows that this is not sufficient, even though all atoms of the same type are indeed indistinguishable copies of each other! In the following we explore the most important consequences on motion of the indistinguishability of atoms and of the distinguishability of macroscopic objects.

DISTINGUISHING MACROSCOPIC OBJECTS

A number of important properties of objects are highlighted by studying a combinatorial puzzle: the *glove problem*. It asks:

> How many surgical gloves (for the right hand) are necessary if m doctors need to operate w patients in a hygienic way, so that nobody gets in contact with the body fluids of anybody else?

The same problem also appears in other settings. For example, it also applies to computers, interfaces and computer viruses or to condoms, men and women – and is then called the *condom problem*. To be clear, the optimal number of gloves is *not* the product mw. In fact, the problem has three subcases.

— The simple case $m = w = 2$ already provides the most important ideas needed. Are you able to find the optimal solution and procedure?

— In the case $w = 1$ and m odd, the solution is $(m + 1)/2$ gloves. The corresponding expression $(w+1)/2$ holds for the case $m = 1$ and w odd. This is the optimal solution, as you can easily check yourself.

— A solution with a simple procedure for all other cases is given by $\lceil 2w/3+m/2 \rceil$ gloves, where $\lceil x \rceil$ means the smallest integer greater than or equal to x. For example, for two

doctors and three patients this gives only three gloves. (However, this formula does not always give the optimal solution; better values exist in certain subcases.)

Enjoy working on the puzzle. You will find that three basic properties of gloves determine the solution. First, gloves have two sides, an interior and an exterior one, that can be distinguished from each other. Secondly, gloves turned inside out exchange left and right and can thus be distingusihed from gloves that are not reversed. Thirdly, gloves can be distinguished from each other.

Now we come back to our original aim: Do the three basic properties of gloves also apply to quantum particles? We will explore the issue of double-sidedness of quantum particles in the last part of our mountain ascent. The question whether particles can be turned inside out will be of importance for their description and their motion. We will also explore the difference between right- and left-handed particles, though in the next part of our adventure. In the present chapter we concentrate on the third issue, namely whether objects and particles can always be distinguished from copies. We will find that *elementary* particles do not behave like gloves – but in a much more surprising manner.

In everyday life, distinction of macroscopic objects can be achieved in two ways. On the one hand, we are able to distinguish objects – or people – from each other because they differ in their *intrinsic properties*, such as their mass, colour, size or shape. On the other hand, we are able to distinguish objects even if they have the same intrinsic properties. Any game of billiard shows us that by following the path of each ball, we can distinguish it from the other balls. In short, we can distinguish objects with identical properties also using their *state*.

The state of a billiard ball is given by its position, its linear and its angular momentum. We are able to distinguish two identical billiard balls because the measurement error for the position of each ball is much smaller than the size of the ball itself. The different states of two billiard balls allow us to track each ball. However, in the microscopic domain, this is not possible! Let us take two atoms of the same type. Two such atoms have exactly the same intrinsic properties. To distinguish them in collisions, we would need to keep track of their motion. But due to the quantum of action and the ensuing indeterminacy relation, we have no chance to achieve this. In fact, a simple experiment from the nineteenth century showed that even nature itself is not able to do it! This profound result was discovered studying systems which incorporate a large number of colliding atoms of the same type: *gases*.

Distinguishing atoms

What is the entropy of a gas? The calculation of the entropy S of a simple gas, made of N simple particles[*] of mass m moving in a volume V, gives

$$\frac{S}{kN} = \ln\left[\frac{V}{\Lambda^3}\right] + \frac{3}{2} + \frac{\ln\alpha}{N} \,. \tag{55}$$

Here, k is the Boltzmann constant, ln the natural logarithm, T the temperature, and $\Lambda = \sqrt{2\pi\hbar^2/mkT}$ is the thermal wavelength (approximately the de Broglie wavelength of the

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[*] Particles are *simple* if they are fully described by their momentum and position; atoms are simple particles. Molecules are not simple, as they are describe also by their orientation.
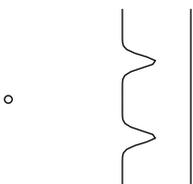
FIGURE 56 Willard Gibbs (1839–1903)

Challenge 92 e

particles making up the gas). In this result, the pure number α is equal to 1 if the particles are distinguishable like billiard balls, and equal to $1/N!$ if they are not distinguishable at all. Measuring the entropy of a simple gas thus allows us to determine α and therefore to test experimentally whether particles are distinguishable.

Challenge 93 e

It turns out that only the second case, $\alpha = 1/N!$, describes nature. We can easily check this without even performing the measurement: only in the second case does the entropy of two volumes of identical gas *add* up.* The result, often called *Gibbs' paradox*,** thus

Ref. 73

proves that the microscopic components of matter are *indistinguishable*: in a system of quantum particles – be they electrons, protons, atoms or small molecules – there is no way to say which particle is which.

Indistinguishability of particles is thus an experimental property of nature. It holds without exception. For example, when radioactivity was discovered, people thought that it contradicted the indistinguishability of atoms, because decay seems to single out certain atoms compared to others. But quantum theory then showed that this is not the case and that even atoms and molecules are indistinguishable.

Since $\hbar$ appears in the expression for the entropy, indistinguishability is a quantum effect. Indeed, indistinguishability plays no role if quantum effects are negligible, as is the case for billiard balls. Nevertheless, indistinguishability is important in everyday life. We will find out that the properties of everyday matter – plasma, gases, liquids and solids – would be completely different without indistinguishability. For example, we will discover that without it, knifes and swords would not cut. In addition, the soil would not carry us; we would fall right through it. To illuminate the issue in more detail, we explore the following question.

Challenge 94 d

* Indeed, the entropy values observed by experiment, for a monoatomic gas, are given by the so-called Sackur–Tetrode formula

$$\frac{S}{kN} = \ln\left[\frac{V}{N\Lambda^3}\right] + \frac{5}{2} \tag{56}$$

which follows when $\alpha = 1/N!$ is inserted above. It was deduced independently by the German physicist Otto Sackur (1880–1914) and the Dutch physicist Hugo Tetrode (1895–1931). Note that the essential parameter is the ratio between V/N, the classical volume per particle, and Λ^3, the de Broglie volume of a quantum particle.

** Josiah Willard Gibbs (1839–1903), US-American physicist who was, with Maxwell and Planck, one of the three founders of statistical mechanics and thermodynamics; he introduced the concept of *ensemble* and the term *thermodynamic phase*.
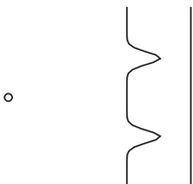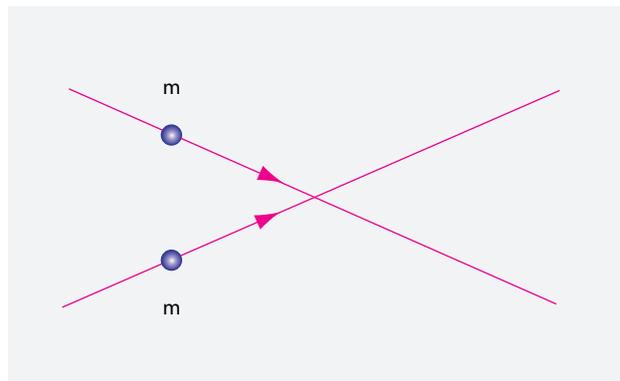
FIGURE 57 Identical objects with crossing paths.

WHY DOES INDISTINGUISHABILITY APPEAR IN NATURE?

Take two quantum particles with the same mass, the same composition and the same shape, such as two atoms of the same kind. Imagine that their paths cross, and that they approach each other to small distances at the crossing, as shown in Figure 57. In a gas, both a collision of atoms or a near miss are examples. Now, all experiments ever performed show that at small distances it is impossible to say whether the two quantons have switched roles or not.

> ▷ It is *impossible* in a gas to follow quantum particles moving around and to determine which one is which. Tracking colliding quantons is impossible.

The impossibility to distinguish nearby particles is a direct consequence of the quantum of action $\hbar$. For a path that brings two approaching particles very close to each other, a role switch requires only a small amount of change, i.e., only a small (physical) action. However, we know that there is a smallest observable action in nature. Keeping track of each quantum particle at small distances would require action values *smaller* than the quantum of action. The existence of the quantum of action thus makes it impossible to keep track of quantum particles when they come too near to each other. Any description of systems with several quantons must thus take into account that after a close encounter, it is impossible to say which quanton is which.

If we remember that quantum theory describes quantons as clouds, the indistinguishability appears even more natural. Whenever two clouds meet and depart again, it is impossible to say which cloud is which. On the other hand, if two particles are kept distant enough, one does have an effective distinguishability; indistinguishability thus appears only when the particles come close.

In short, indistinguishability is a natural, unavoidable consequence of the existence of a smallest action value in nature. This result leads us straight away to the next question:

CAN QUANTUM PARTICLES BE COUNTED?

In everyday life, we can count objects because we can distinguish them. Since quantum particles cannot always be distinguished, we need some care in determining how to count
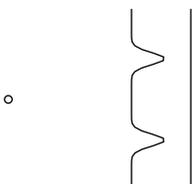
them. The first step in counting particles is the definition of what is meant by a situation without any particle at all. This seems an easy thing to do, but later on we will encounter situations where already this step runs into difficulties. In any case, the first step of counting is thus the *specification of the vacuum*. Any counting method requires that the situation without particles is clearly separated from situations with particles.

The second step necessary for counting is the specification of an observable useful for determining quantum particle number. The easiest way is to choose one of those conserved quantum numbers that add up under composition, such as electric charge. Counting itself is then performed by measuring the total charge and dividing by the unit charge.

In everyday life, the weight or mass is commonly used as observable. However, it cannot be used generally in the quantum domain, except for simple cases. For a large number of particles, the interaction energy will introduce errors. For very large particle numbers, the gravitational binding energy will do so as well. But above all, for transient phenomena, unstable particles or short measurement times, mass measurements reach their limits. In short, even though counting stable atoms through mass measurements works in everyday life, the method is not applicable in general; especially at high particle energies, it cannot be applied.

Counting with the help of conserved quantum numbers has several advantages. First of all, it works also for transient phenomena, unstable particles or short measurement times. Secondly, it is not important whether the particles are distinguishable or not; counting always works. Thirdly, virtual particles are not counted. This is a welcome state of affairs, as we will see, because for virtual particles, i.e., particles for which $E^2 \neq p^2c^2 + m^2c^4$, there is *no way* to define a particle number anyway. Using a conserved quantity is indeed the best particle counting method possible.

The side effect of counting with the help of quantum numbers is that antiparticles count negatively! Also this consequence is a result of the quantum of action. We saw above that the quantum of action implies that even in vacuum, particle–antiparticle pairs are observed at sufficiently high energies. As a result, an antiparticle must count as minus one particle. In other words, any way of counting quantum particles can produce an error due to this effect. In everyday life this limitation plays no role, as there is no antimatter around us. The issue does play a role at higher energies, however. It turns out that there is no general way to count the exact number of particles and antiparticles separately; only the sum can be defined. In short, quantum theory shows that particle counting is never perfect.

In summary, nature does provide a way to count quantum particles even if they cannot be distinguished, though only for everyday, low energy conditions; due to the quantum of action, antiparticles count negatively. Antiparticles thus provide a limit to the counting of particles at high energies, when the mass–energy equivalence becomes important.

WHAT IS PERMUTATION SYMMETRY?

Since quantum particles are countable but indistinguishable, there exists a symmetry of nature for systems composed of several identical quantons. *Permutation symmetry*, also called *exchange symmetry*, is the property of nature that observations are unchanged under exchange of identical particles. Permutation symmetry forms one of the four pil-

lars of quantum theory, together with space-time symmetry, gauge symmetry and the not yet encountered renormalization symmetry. Permutation symmetry is a property of *composed* systems, i.e., of systems made of many (identical) subsystems. Only for such systems does indistinguishability play a role.
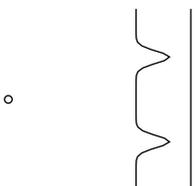
In other words, 'indistinguishable' is not the same as 'identical'. Two quantum particles of the same type are not the *same*; they are more like exact copies of each other. On the other hand, everyday life experience shows us that two copies can always be distinguished under close inspection, so that the term 'copy' is not fully appropriate either.

> ▷ Quantons, quantum particles, are countable and completely indistinguishable.* Quantum particles are *perfect copies* of each other.

Being perfect copies, not even nature can distinguish particles; as a result, permutation symmetry appears.

In the next chapter, we will discover that permutation is partial rotation. Permutation symmetry thus is a symmetry under partial rotations. Can you find out why?

INDISTINGUISHABILITY AND WAVE FUNCTION SYMMETRY

The indistinguishability of quantum particles leads to important conclusions about the description of their state of motion. This happens because it is impossible to formulate a description of motion that includes indistinguishability right from the start. (Are you

able to confirm this?) We need to describe a n-particle state with a state $\Psi_{1\ldots i\ldots j\ldots n}$ which assumes that distinction is possible, as expressed by the ordered indices in the notation, and we introduce the indistinguishability afterwards.

Indistinguishability, or permutation symmetry, means that the exchange of any two quantum particles results in the same physical observations.** Now, two quantum states have the same physical properties if they differ at most by a phase factor; indistinguishability thus requires

$$\Psi_{1\ldots i\ldots j\ldots n} = e^{i\alpha}\, \Psi_{1\ldots j\ldots i\ldots n} \tag{57}$$

for some unknown angle α. Applying this expression twice, by exchanging the same couple of indices again, allows us to conclude that $e^{2i\alpha} = 1$. This implies that

$$\Psi_{1\ldots i\ldots j\ldots n} = \pm\, \Psi_{1\ldots j\ldots i\ldots n}\,, \tag{58}$$

in other words, a wave function is either *symmetric* or *antisymmetric* under exchange of indices. (We can also say that the eigenvalue for the exchange operator is either $+1$ or -1.)

> ▷ Quantum theory thus predicts that quantum particles can be indistinguish-

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* The word 'indistinguishable' is so long that many physicists sloppily speak of 'identical' particles nevertheless. Take care.
** We therefore have the same situation that we encountered already several times: *an overspecification of the mathematical description*, here the explicit ordering of the indices, *implies a symmetry of this description*, which in our case is a symmetry under exchange of indices, i.e., under exchange of particles.
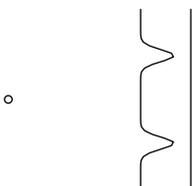
able in one of two distinct ways.*

▷ Particles corresponding to *symmetric* wave functions – those which transform under particle exchange with a '+' in equation (58) – are called** *bosons*.

▷ Particles corresponding to *antisymmetric* wave functions – those which transform under particle exchange with a '−' in equation (58) – are called*** *fermions*.

Experiments show that the exchange behaviour depends on the *type* of particle. Photons are found to be bosons. On the other hand, electrons, protons and neutrons are found to be fermions. Also about half of the atoms are found to behave as bosons (at moderate energies), the other half are fermions. To determine they type of atom, we need to take into account the spin of the electron and that of the nucleus.

In fact, a composite of an *even* number of fermions (at moderate energies) – or of any number of bosons (at any energy) – turns out to be a boson; a composite of an *odd* number of fermions is (always) a fermion. For example, ^{4}He is a boson, ^{3}He a fermion. Also the natural isotopes ^{23}Na, ^{41}K, ^{85}Rb, ^{87}Rb and ^{133}Cs are bosons, because they have odd numbers of electrons and of nucleons; in contrast, ^{40}K and ^{134}Cs are fermions (and, in this case, also radioactive).

The behaviour of photons

A simple experiment, shown in Figure 58, allows observing an important aspect of photon behaviour. Take a source that emits two indistinguishable photons, i.e., two photons of identical frequency and polarization, at the same time. The photon pair is therefore in an entangled state. In the laboratory, such a source can be realized with a down-converter, a material that converts a photon of frequency $2f$ into two photons of frequency f. The two entangled photons, after having travelled exactly the same distance, are made to enter the two sides of an ideal beam splitter (for example, a half-silvered mirror). Two detectors are located at the two exits of the beam splitter. Experiments show that both photons are always detected together on the *same* side, and never separately on

Ref. 75

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* This conclusion applies to three-dimensional space. In two dimensions there are more possibilities. Such possibilities have been and partly still are topic of research.
** 'Bosons' are named after the physicist Satyenra Nath Bose (b. 1894 Calcutta, d. 1974 Calcutta) who first described the statistical properties of photons. The work was later expanded by Albert Einstein, so that one speaks of Bose–Einstein statistics.
*** The term 'fermion' is derived from the name of the physicist and Nobel Prize winner Enrico Fermi (b. 1901 Rome, d. 1954 Chicago) famous for his all-encompassing genius in theoretical and experimental physics. He mainly worked on nuclear and elementary particle physics, on spin and on statistics. For his experimental work he was called 'quantum engineer'. He is also famous for his lectures, which are still published in his own hand-writing, and his brilliant approach to physical problems. Nevertheless, his highly deserved Nobel Prize was one of the few cases in which the prize was given for a discovery which turned out to be incorrect. He left Italy because of the bad treatment his Jewish wife was suffering and emigrated to the USA. Fermi worked on the Manhattan project that built the first atomic bombs. After the Second World War, he organized one of the best physics department in the world, at the University of Chicago, where he was admired by everybody who worked with him.
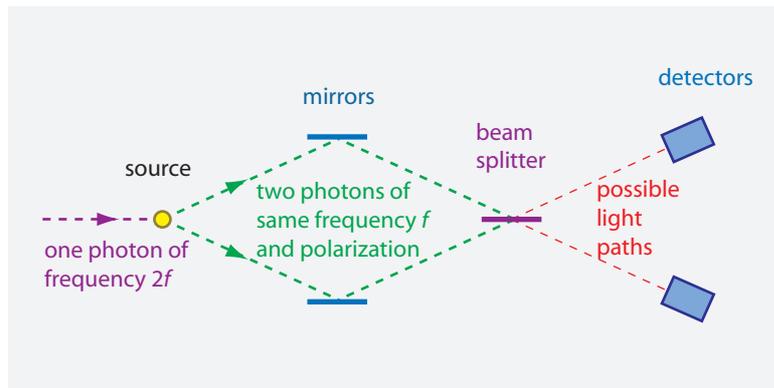
Ref. 74
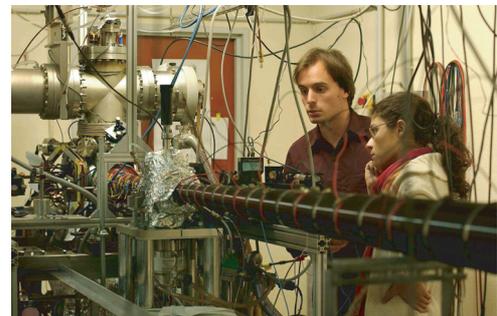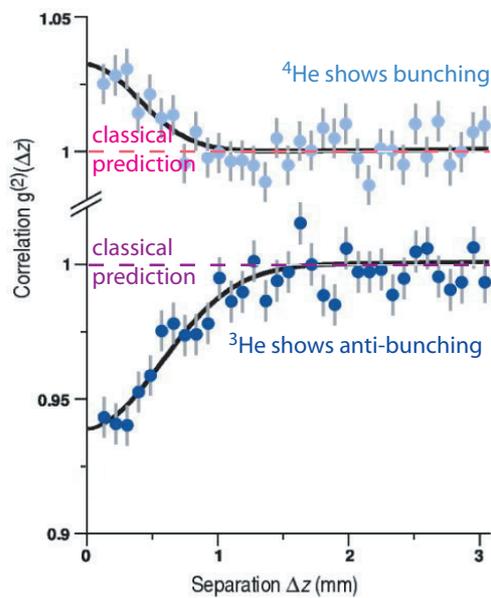
FIGURE 59 Bunching and antibunching of ^{3}He and ^{4}He helium!bunching atoms: the measurement result, the detector and the experiment (from atomoptic.iota.u-psud.fr/research/helium/helium.html, photo © Denis Boiron, Jerome Chatin).

opposite sides. This happens because the two options where one of the photons is transmitted and the other reflected interfere destructively. (The discussion mentioned above applies also here: despite two photons being involved, also in this case, when investigating the details, only one photon interferes with itself.)

The experiment shows that photons are *bosons*. Indeed, in the same experiment, fermions behave in exactly the opposite way; two fermions are always detected separately on *opposite* sides, never together on the same side.

Page 59

Ref. 76
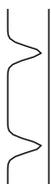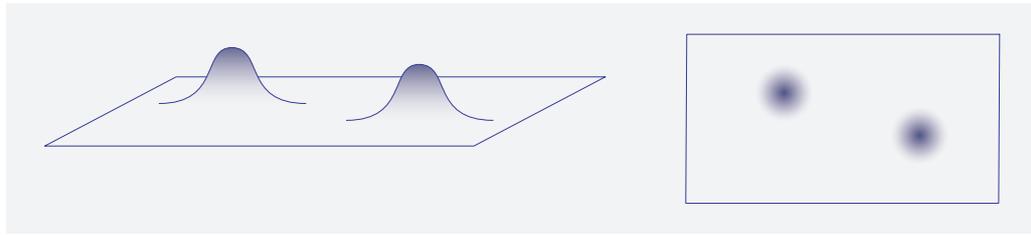
FIGURE 60 Picturing particles as localized excitations (left) or clouds (right).

Bunching and antibunching

Another way to test the exchange character of a particle is the Hanbury Brown–Twiss experiment described earlier on. First of all, this beautiful experiment shows that quantum particles behave differently than classical particles. In addition, compared to classical particles, fermions show antibunching – because of Pauli's exclusion principle – and bosons show bunching. Hanbury Brown and Twiss performed the experiment with photons, which are bosons.

In 2005, a French–Dutch research collaboration performed the experiment with atoms. By using an extremely cold helium gas at 500 nK and a clever detector principle, they were able to measure the correlation curves typical for the effect. The results, shown in Figure 59, confirm that ^{3}He is a fermion and ^{4}He is a boson, as predicted from the composition rule of quantum particles.

The energy dependence of permutation symmetry

If experiments force us to conclude that nobody, not even nature, can distinguish between two particles of the same type, we deduce that they do not form two separate entities, but some sort of unity. Our naive, classical sense of particle as a separate entity from the rest of the world is thus an incorrect description of the phenomenon of 'particle'. Indeed, no experiment can track particles with identical intrinsic properties in such a way that they can be distinguished with certainty. This impossibility has been checked experimentally with all elementary particles, with nuclei, with atoms and with numerous molecules.

How does this fit with everyday life, i.e., with classical physics? Photons do not worry us much here. Let us focus the discussion on matter particles. We know to be able to distinguish electrons by pointing to different wires in which they flow; also, we can distinguish our fridge, with its electrons and atoms, from that of our neighbour. While the quantum of action makes distinction impossible, everyday life does allow it.

The simplest explanation for both observations is to imagine a microscopic particle, especially an elementary one, as a *bulge*, i.e., as a localized excitation of the vacuum, or as a tiny cloud. Figure 60 shows two such bulges and two clouds representing particles. It is evident that if particles are too near to each other, it makes no sense to distinguish them; we cannot say any more which is which.

The bulge image shows that either for large distances or for high potential walls separating them, distinction of identical particles does become possible. In such situations, measurements allowing us to track particles independently do exist – as we know from

everyday life. In other words, we can specify a limit energy at which permutation symmetry of objects or particles separated by a distance d becomes important. It is given by

$$E = \frac{c\,\hbar}{d}\ .\tag{59}$$

Are you able to confirm the expression? For example, at everyday temperatures we *can* distinguish atoms inside a solid from each other, since the energy so calculated is much higher than the thermal energy of atoms. To have fun, you might want to determine at what energy two truly identical human twins become indistinguishable. Estimating at what energies the statistical character of trees or fridges will become apparent is then straightforward.

To sum up, in daily life we are able to distinguish objects and thus people for two reasons: because they are made of *many* parts, and because we live in a *low energy* environment. The bulge image of particles purveys the idea that distinguishability exists for objects in everyday life but not for particles in the microscopic domain.

The energy issue immediately adds a new aspect to the discussion. How can we describe fermions and bosons in the presence of virtual particles and of antiparticles?

Indistinguishability in quantum field theory

Quantum field theory, as we will see in the next volume, simply puts the cloudy bulge idea of Figure 60 into mathematical language. A situation without any bulge is called *vacuum state*. Quantum field theory describes all particles of a given type as *excitations* of a single fundamental field. Particles are indistinguishable because each particle is an excitation of the same basic substrate and each excitation has the same properties. A situation with one particle is then described by a vacuum state acted upon by a *creation operator*. Adding a second particle is described by adding a second creation operator, and subtracting a particle by adding a *annihilation operator*; the latter turns out to be the adjoint of the former.

Quantum field theory studies how creation and annihilation operators must behave to describe observations.* It arrives at the following conclusions:

— Field operators for particles with half-integer spin are *fermions* and imply (local) anticommutation.
— Field operators for particles with with integer spin are *bosons* and imply (local) commutation.
— For all field operators, the commutator, respectively anticommutator, taken at two points with space-like separations, vanishes.

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* Whenever the relation

$$[b, b^\dagger] = bb^\dagger - b^\dagger b = 1\tag{60}$$

holds between the creation operator $b^\dagger$ and the annihilation operator b, the operators describe a *boson*. The dagger can thus be seen as describing the operation of adjoining; a double dagger is equivalent to no dagger. If the operators for particle creation and annihilation anticommute

$$\{d, d^\dagger\} = dd^\dagger + d^\dagger d = 1\tag{61}$$

they describe a *fermion*. The so defined bracket is called the *anticommutator bracket*.

— Antiparticles of fermions are fermions, and antiparticles of bosons are bosons.
— Virtual particles behave under exchange like their real counterparts.

These connections are at the basis of quantum field theory. They describe how quantons behave under permutation.

But why are quantum particles identical? Why are all electrons identical? Lead by experiment, quantum field theory describes electrons as identical excitations of the vacuum, and as such as identical by construction. Of course, this answer is not really satisfying. We will find a better one only in the final part of our mountain ascent.

How accurately is permutation symmetry verified?

Are electrons *perfect* fermions? In 1990, a simple but effective experiment testing their fermion behaviour was carried out by Ramberg and Snow. They sent an electric current of 30 A through a copper wire for one month and looked for X-ray emission. They did not find any. They concluded that electrons are always in an antisymmetric state, with a symmetric component of less than

$$2 \cdot 10^{-26} \tag{62}$$

of the total state. In short, electrons are always in an antisymmetric state: they are fermions.

The reasoning behind this elegant experiment is the following. If electrons would not always be fermions, every now and then an electron could fall into the lowest energy level of a copper atom, leading to X-ray emission. The lack of such X-rays implies that electrons are fermions to a very high accuracy. X-rays could be emitted only if they were bosons, at least part of the time. Indeed, two electrons, being fermions, cannot be in the same quantum state: this restriction is called the *Pauli exclusion principle*. It applies to all fermions and is the topic of the next chapter.

Copies, clones and gloves

Can classical systems be indistinguishable? They can: large molecules are examples – provided they are made of exactly the same isotopes. Can *large* classical systems, made of a mole or more particles be indistinguishable? This simple question effectively asks whether a *perfect* copy, or (physical) *clone*, of a physical system is possible.

It could be argued that any factory for mass-produced goods, such as one producing shirt buttons or paper clips, shows that copies are possible. But the appearance is deceiving. Seen under a microscope, there is usually some difference. Is this always the case? In 1982, the Dutch physicist Dennis Dieks and independently, the US-American physicists Wootters and Zurek, published simple proofs that quantum systems cannot be copied. This is the famous *no-cloning theorem*.

A *copying machine* is a machine that takes an original, reads out its properties and produces a copy, leaving the original unchanged. This definition seems straightforward. However, we know that if we extract information from an original, we have to interact with it. As a result, the system will change at least by the quantum of action. We thus expect that due to quantum theory, copies and originals can never be identical.*

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* This seems to provide a solution against banknote forgeries. In fact, Stephen Wiesner proposed to use

Quantum theory indeed shows that copying machines are impossible. A copying machine is described by an operator that maps the state of an original system to the state of the copy. In other words, a copying machine is linear. This linearity leads to a problem. Simply stated, if a copying machine were able to copy originals either in state $|A\rangle$ or in state $|B\rangle$, it could not work if the state of the original were a superposition $|A\rangle + |B\rangle$. Let us see why.

A copy machine is a device described by an operator U that changes the starting state $|s\rangle_c$ of the copy in the following way:

— If the original is in state $|A\rangle$, a copier acts on the copy $|s\rangle_c$ as

$$U|A\rangle|s\rangle_c = |A\rangle|A\rangle_c \ . \tag{63}$$

— If the original is in state $|B\rangle$, a copier acts on the copy $|s\rangle_c$ as

$$U|B\rangle|s\rangle_c = |B\rangle|B\rangle_c \ . \tag{64}$$

As a result of these two requirements, an original in the state $|A + B\rangle$ is treated by the copier as

$$U|A + B\rangle|s\rangle_c = |A\rangle|A\rangle_c + |B\rangle|B\rangle_c \ . \tag{65}$$

This is in contrast to what we want, which would be

$$U_{\text{wanted}}|A + B\rangle|s\rangle_c = (|A\rangle + |B\rangle)(|A\rangle_c + |B\rangle_c) \ . \tag{66}$$

In other words, a copy machine cannot copy a state completely.* This is the so-called *no-cloning theorem*.

The impossibility of copying is implicit in quantum theory. If we were able to clone systems, we could measure a variable of a system and a second variable on its copy. We would be thus able to beat the indeterminacy relation in both copies. This is impossible. In short, copies are always imperfect.

The lack of quantum mechanical copying machines is disappointing. Such science fiction machines could be fed with two different inputs, such as a lion and a goat, and produce a superposition: a chimaera. Quantum theory shows that all these imaginary beings or situations cannot be realized.

Other researchers then explored how near to perfection a copy can be, especially in the case of classical systems. To make a long story short, these investigations show that also the copying or cloning of macroscopic systems is impossible. In simple words, *copying machines do not exist*. Copies can always be distinguished from originals if observations

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quantum theory already in 1970; he imagined to use polarizations of stored single photons as bits of serial numbers. Can you explain why this cannot work?

* The no-cloning theorem puts severe limitations on quantum computers, as computations often need copies of intermediate results. The theorem also shows that faster-than-light communication is impossible in EPR experiments. In compensation, *quantum cryptography* becomes possible – at least in the laboratory. Indeed, the no-cloning theorem shows that nobody can copy a quantum message without being noticed. The specific ways to use this result in cryptography are the 1984 Bennett–Brassard protocol and the 1991 Ekert protocol.

are made with sufficient care. In particular, this is the case for biological clones; biological clones are identical twins born following separate pregnancies. They differ in their finger prints, iris scans, physical and emotional memories, brain structures, and in many other aspects. (Can you specify a few more?) In short, biological clones, like identical twins, are not copies of each other.

Challenge 100 s

In summary, everyday life objects such as photocopies, billiard balls or twins are always distinguishable. There are two reasons: first, quantum effects play no role in everyday life, so that there is no danger of unobservable exchange; secondly, perfect clones of classical systems do not exist anyway, so that there always are tiny differences between any two objects, even if they look identical at first sight. Gloves, being classical systems, can thus always be distinguished.

Summary

As a consequence of the quantum of action $\hbar$, quantum particles are *indistinguishable*. This happens in one of two ways: they are either bosons or fermions. Not even nature is able to distinguish between identical quantum particles.

Despite the indistinguishability of quantons, the state of a physical system cannot be copied to a second system with the same particle content. Therefore, perfect clones do not exist in nature.
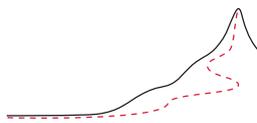
ROTATIONS AND STATISTICS
– VISUALIZING SPIN

S pin is the observation that matter beams can be *polarized*: rays can be rotated. Spin thus describes how particles behave under rotations. Particles are thus not imply point-like: quantum particles can rotate around an axis. This proper rotation is called *spin*; like macroscopic rotation, spin is described by an angular momentum.

In the following, we recall that the spin of quantons is quantized in units of $\hbar/2$. Then we show a deep result: the value of spin determines whether a quantum particle, and any general quantum system, is a boson or a fermion. And we will show that spin is the rotation of quantons.

QUANTUM PARTICLES AND SYMMETRY

The general background for the appearance of spin was clarified by Eugene Wigner in 1939.** He started by recapitulating that any quantum particle, if *elementary*, must behave like an *irreducible representation* of the set of all viewpoint changes. This set of viewpoint changes forms the symmetry group of flat space-time, the so-called *inhomogeneous Lorentz group*. Why?

We have seen in the chapter on symmetry, in the first volume of this adventure, that the symmetry of any composite system leads to certain requirements for the components of the system. If the components do not follow these requirements, they cannot build a symmetric composite.

We know from everyday life and precision experiments that all physical systems are symmetric under translation in time and space, under rotation in space, under boosts, and – in many cases – under mirror reflection, matter–antimatter exchange and motion reversal. We know these symmetries from everyday life; for example, the usefulness of what we call 'experience' in everyday life is simply a consequence of time translation symmetry. The set of all these common symmetries, more precisely, of all these symmetry transformations, is called the *inhomogeneous Lorentz group*.

These *symmetries*, i.e., these changes of viewpoints, lead to certain requirements for the components of physical systems, i.e., for the elementary quantum particles. In mathematical language, the requirement is expressed by saying that *elementary particles* must be *irreducible representations* of the symmetry group.

** Eugene Wigner (b. 1902 Budapest, d. 1995 Princeton), theoretical physicist, received the Nobel Prize in Physics in 1963. He wrote over 500 papers, many about various aspects of symmetry in nature. He was also famous for being the most polite physicist in the world.

Every textbook on quantum theory carries out this reasoning in systematic detail. Starting with the Lorentz group, one obtains a list of all possible irreducible representations. In other words, on eobtains a list of all possible ways that elementary particles can behave. * Cataloguing the possibilities, one finds first of all that every elementary particle is described by *four-momentum* – no news so far – by an internal angular momentum, the *spin*, and by a set of *parities*.

— Four-momentum results from the translation symmetry of nature. The momentum value describes how a particle behaves under translation, i.e., under position and time shift of viewpoints. The magnitude of four-momentum is an invariant property, given by the mass, whereas its orientation in space-time is free.

— Spin results from the rotation symmetry of nature. The spin value describes how an object behaves under rotations in three dimensions, i.e., under orientation change of viewpoints.** The magnitude of spin is an invariant property, and its orientation has various possibilities with respect to the direction of motion. In particular, the spin of massive quantum particles behaves differently from that of massless quantum particles.

For *massive* quantum particles, the inhomogeneous Lorentz group implies that the invariant magnitude of spin is $\sqrt{J(J+1)}\,\hbar$, often written, by oversimplification, as J. It is thus customary to say and write 'spin J' instead of the cumbersome 'spin $\sqrt{J(J+1)}\,\hbar$'. Since the value of the quantum number J specifies the magnitude of the angular momentum, it gives the representation under rotations of a given particle type. The exploration shows that the spin quantum number J can be any multiple of 1/2, i.e., it can take the values 0, 1/2, 1, 3/2, 2, 5/2, etc. As summarized in Table 4, experiments show that electrons, protons and neutrons have spin 1/2, the W and Z particles spin 1 and helium atoms spin 0. In addition, the representation of spin J is $2J + 1$ dimensional, meaning that the spatial orientation of the spin has $2J + 1$ possible values. For electrons, with $J = 1/2$, there are thus two possibilities; they are usually called 'up' and 'down'. Spin thus only takes *discrete* values. This is in contrast with linear momentum, whose representations are infinite dimensional and whose possible values form a *continuous* range.

Also *massless* quantum particles are characterized by the value of their spin. It can take the same values as in the massive case. For example, photons and gluons have spin 1. For massless particles, the representations are one-dimensional, so that massless particles are completely described by their *helicity*, defined as the projection of the spin onto the direction of motion. Massless particles can have positive or negative helicity, often also called right-handed and left-handed polarization. There is no other freedom for the orientation of spin in the massless case.

— To complete the list of particle properties, the remaining, discrete symmetries of the inhomogeneous Lorentz group must be included. Since *motion inversion*, *spatial parity* and *charge inversion* are parities, each elementary particle has to be described by three additional numbers, called T, P and C, each of which can only take the values

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* To be of physical relevance for quantum theory, representations have to be *unitary*. The full list of *irreducible* and *unitary* representations of viewpoint changes thus provides the range of possibilities for any particle that wants to be *elementary*.
** The group of physical rotations is also called SO(3), since mathematically it is described by the group of **Special Orthogonal** 3 by 3 matrices.

TABLE 4 Particle spin as representation of the rotation group.

SPIN [$\hbar$]	SYSTEM unchanged after rotation by	MASSIVE EXAMPLES elementary	composite	MASSLESS EXAMPLES elementary
0	any angle	Higgs boson	mesons, nuclei, atoms	none [a]
1/2	2 turns	e, μ, τ, q, ν_e, ν_μ, ν_τ	nuclei, atoms, molecules, radicals	none, as neutrinos have a tiny mass
1	1 turn	W, Z	mesons, nuclei, atoms, molecules, toasters	photon γ, gluon g
3/2	2/3 turn	none [a]	baryons, nuclei, atoms	none [a]
2	1/2 turn	none	nuclei	'graviton' [b]
5/2	2/5 turn	none	nuclei	none
3	1/3 turn	none	nuclei [c]	none
etc.[c]	etc.[c]	etc.[c]	etc.[c]	none possible

a. Supersymmetry, a symmetry conjectured in the twentieth century, predicts elementary particles in these and other boxes.
b. The graviton has not yet been observed.
c. Nuclei exist with spins values up to at least 101/2 and 51 (in units of $\hbar$). Ref. 83

+1 or −1. Being parities, these numbers must be *multiplied* to yield the value for a composed system.

In short, the symmetries nature lead to the classification of all elementary quantum particles by their mass, their momentum, their spin and their P, C and T parities.

TYPES OF QUANTUM PARTICLES

The spin values observed for all quantum particles in nature are given in Table 4. The parities and all known intrinsic properties of the elementary particles are given in Table 5. Spin and parities together are called *quantum numbers*. All other intrinsic properties of quantons are related to interactions, such as mass, electric charge or isospin, and we will explore them in the next volume.

TABLE 5 Elementary particle properties.

PARTICLE	MASS m^a	LIFETIME τ OR ENERGY WIDTH,[b] MAIN DECAY MODES	ISOSPIN I, SPIN J,[c] PARITY P, CHARGE PARITY C	CHARGE, ISOSPIN, STRANGE-NESS,[c] CHARM, BEAUTY,[d] TOPNESS: $QISCBT$	LEPTON & BARYON[e] NUM-BERS $L\,B$
Elementary radiation (bosons)					
photon γ	0 ($<10^{-53}$ kg)	stable	$I(J^{PC}) =$ $0, 1(1^{--})$	000000	0, 0
$W^{\pm}$	80.398(25) GeV$/c^2$	2.124(41) GeV 67.60(27) % hadrons, 32.12(36) % $l^+\nu$	$J = 1$	$\pm$100000	0, 0
Z	91.1876(21) GeV$/c^2$	2.65(2) $\cdot$ 10^{-25} s or 2.4952(23) GeV$/c^2$ 69.91(6) % hadrons, 10.0974(69) % l^+l^-	$J = 1$	000000	0, 0
gluon	0	stable	$I(J^P) = 0(1^-)$	000000	0, 0
Elementary matter (fermions): leptons					
electron e	9.109 382 15(45) $\cdot$ 10^{-31} kg = 81.871 0438(41) pJ$/c^2$ = 0.510 998 910(13) MeV$/c^2$ = 0.000 548 579 909 43(23) u gyromagnetic ratio $\mu_e/\mu_B = -1.001\,159\,652\,1811(7)$	$> 13 \cdot 10^{30}$ s	$J = \frac{1}{2}$	$-100\,000$	1, 0
muon μ	0.188 353 130(11) yg = 105.658 3668(38) MeV$/c^2$ = 0.113 428 9256(29) u gyromagnetic ratio $\mu_\mu/(e\hbar/2m_\mu) = -1.001\,165\,9208(6)$	2.197 03(4) µs 99 % $e^- \bar{\nu}_e \nu_\mu$	$J = \frac{1}{2}$	-100000	1, 0
tau τ	1.776 84(17) GeV$/c^2$	290.6(1.0) fs	$J = \frac{1}{2}$	-100000	1, 0
el. neutrino ν_e	< 2 eV$/c^2$		$J = \frac{1}{2}$		1, 0
muon neutrino ν_μ	< 2 eV$/c^2$		$J = \frac{1}{2}$		1, 0
tau neutrino ν_τ	< 2 eV$/c^2$		$J = \frac{1}{2}$		1, 0
Elementary matter (fermions): quarks[f]					
up u	1.5 to 3.3 MeV$/c^2$	see proton	$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$	$+\frac{2}{3}+\frac{1}{2}0000$	0, $\frac{1}{3}$
down d	3.5 to 6 MeV$/c^2$	see proton	$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$	$-\frac{1}{3}-\frac{1}{2}0000$	0, $\frac{1}{3}$
strange s	70 to 130 MeV$/c^2$		$I(J^P) = 0(\frac{1}{2}^+)$	$-\frac{1}{3}0-1000$	0, $\frac{1}{3}$
charm c	1.27(11) GeV$/c^2$		$I(J^P) = 0(\frac{1}{2}^+)$	$+\frac{2}{3}00+100$	0, $\frac{1}{3}$

TABLE 5 (Continued) Elementary particle properties.

Particle	Mass m [a]	Lifetime τ or energy width, [b] main decay modes	Isospin I, spin J, [c] parity P, charge parity C	Charge, isospin, strange-ness, [c] charm, beauty, [d] topness: QISCBT	Lepton & baryon [e] num-bers $L\,B$
bottom b	4.20(17) GeV/c^2	$\tau = 1.33(11)$ ps	$I(J^P) = 0(\frac{1}{2}^+)$	$-\frac{1}{3}000-10$	$0, \frac{1}{3}$
top t	171.2(2.1) GeV/c^2		$I(J^P) = 0(\frac{1}{2}^+)$	$+\frac{2}{3}0000+1$	$0, \frac{1}{3}$

Observed elementary boson

Higgs boson	126 GeV/c^2		$J = 0$		

Notes:

a. See also the table of SI prefixes on page 206. About the eV/c^2 mass unit, see page 210.

b. The *energy width* Γ of a particle is related to its lifetime τ by the indeterminacy relation $\Gamma\tau = \hbar$. There is a difference between the *half-life* $t_{1/2}$ and the lifetime τ of a particle: they are related by $t_{1/2} = \tau \ln 2$, where $\ln 2 \approx 0.693\,147\,18$; the half-life is thus shorter than the lifetime. The unified *atomic mass unit* u is defined as 1/12 of the mass of a carbon 12 atom at rest and in its ground state. One has 1 u = $\frac{1}{12}m(^{12}\text{C})$ = 1.660 5402(10) yg.

c. To keep the table short, its header does not explicitly mention *colour*, the – confusingly named – charge of the strong interactions. It has to be added to the list of basic object properties. Quantum numbers containing the word 'parity' are multiplicative; all others are additive. Parity P and charge parity C are written as + or −. Time parity T (not to be confused with topness T), better called motion inversion parity, is equal to CP in all known particles. The isospin I (or I_Z) appears twice in the table; it is defined only for up and down quarks and their composites, such as the proton and the neutron. In the literature one also sees references to the so-called G-parity, defined as $G = (-1)^{IC}$.

The table header also does not mention the *weak charge* of the particles. The details on weak charge g, or, more precisely, on the *weak isospin*, a quantum number assigned to all left-handed fermions (and right-handed anti-fermions), but to no right-handed fermion (and no left-handed antifermion), are given in the section on the weak interactions.

d. 'Beauty' is now commonly called *bottomness*; similarly, 'truth' is now commonly called *topness*. The signs of the quantum numbers S, I, C, B, T can be defined in different ways. In the standard assignment shown here, the sign of each of the non-vanishing quantum numbers is given by the sign of the charge of the corresponding quark.

e. If supersymmetry existed, R-parity would have to be added to this column. R-parity is a multiplicative quantum number related to the lepton number L, the baryon number B and the spin J through the definition $R = (-1)^{3B+L+2J}$. All particles from the standard model are R-even, whereas their conjectured supersymmetric partner particles would be R-odd. However, supersymmetry is now known to be in contrast with experiment.

f. For the precise definition and meaning of quark masses, see page 233 in volume V.
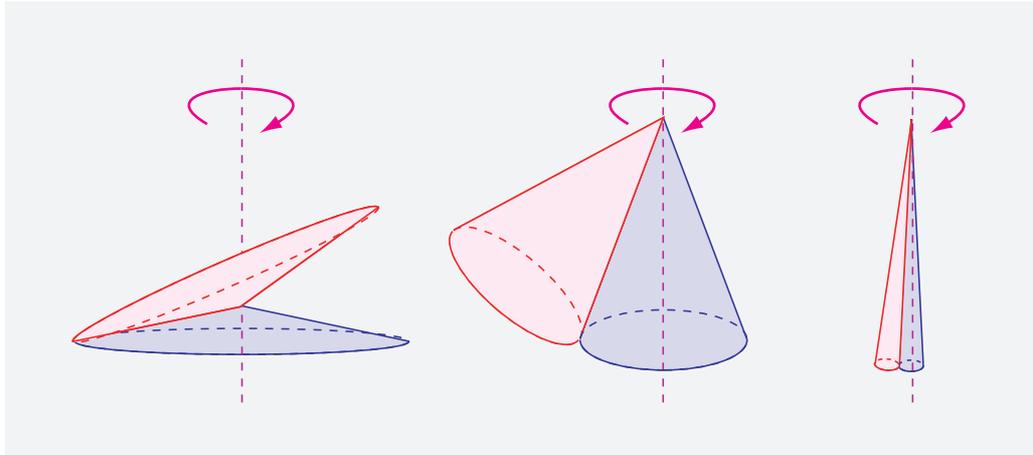
FIGURE 61 Illustrating an argument showing why rotations by 4π are equivalent to no rotation at all (see text).

SPIN 1/2 AND TETHERED OBJECTS

A central result of quantum theory is that spin 1/2 is a possibility in nature, even though this value does not appear in everyday life. For a system to have spin 1/2 means that for such a system only a rotation by *two turns* is equivalent to *none* at all, while one by *one turn* is not. No *simple* systems with this property exist in everyday life, but such systems do exist in microscopic systems: electrons, neutrinos, silver atoms and molecular radicals all have spin 1/2. Table 4 gives a more extensive list.

The mathematician Hermann Weyl used a simple image to explain that the rotation by two turns is equivalent to zero turns, whereas one turns differs. Take two cones, touching each other at their tips as well as along a line, as shown in Figure 61. Hold one cone and roll the other around it. When the rolling cone, after a full turn around the other cone, i.e., around the vertical axis, has come back to the original position, it has rotated by some

angle. If the cones are wide, as shown on the left, the final rotation angle is small. The limit of extremely wide cones gives no rotation at all. If the cones are very thin, like needles, the moving cone has rotated by (almost) 720 degrees; this situation is like a coin rolling around a second coin of the same size, both lying on a table. The rolling coins rotates by two turns, thus by 720 degrees. Also in this case, the final rotation angle is small. The result for 0 degrees and for 720 degrees is the same. If we imagine the cone angle to vary continuously, this visualization shows that a 0 degree rotation can be continuously

changed into a 720 degree rotation. In contrast, a 360 degree rotation *cannot* be 'undone' in this way.

There are systems in everyday life that behave like spin 1/2, but they are not simple: all such systems are *tethered*. The most well-known system is the belt. Figure 62 and Figure 63 show that a rotation by 4π of a belt buckle is equivalent to no rotation at all: this is easily achieved by moving the belt around. You may want to repeat the process by your-

self, using a real belt or a strip of paper, in order to get a feeling for it. The untangling process is often called the *belt trick*, but also *scissor trick*, *plate trick*, *string trick*, *Philippine wine dance* or *Balinese candle dance*. It is sometimes incorrectly attributed to Dirac,
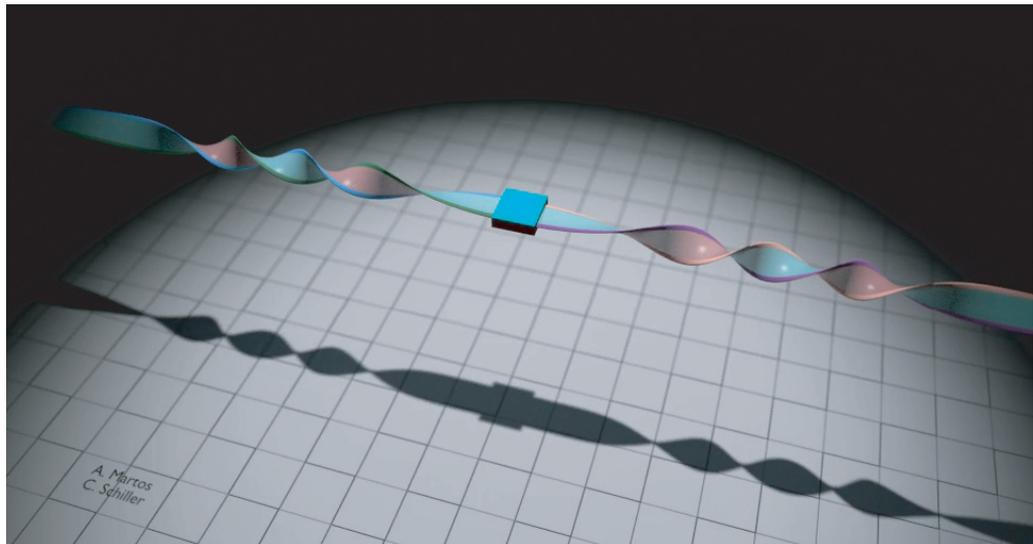
FIGURE 62 Assume that the belt cannot be observed, but the square object can, and that it represents a particle. The animation then shows that such a particle (the square object) can return to the starting position after rotation by 4π (and not after 2π). Such a 'belted' particle thus fulfils the defining property of a *spin 1/2 particle*: rotating it by 4π is equivalent to no rotation at all. The belt thus represents the spinor wave function; for example, a 2π rotation leads to a twist; this means a change of the sign of the wave function. A 4π rotation has no influence on the wave function. You can repeat the trick at home, with a paper strip. The equivalence is shown here with two attached belts, but the trick works with any positive number of belts! (QuickTime film © Antonio Martos)
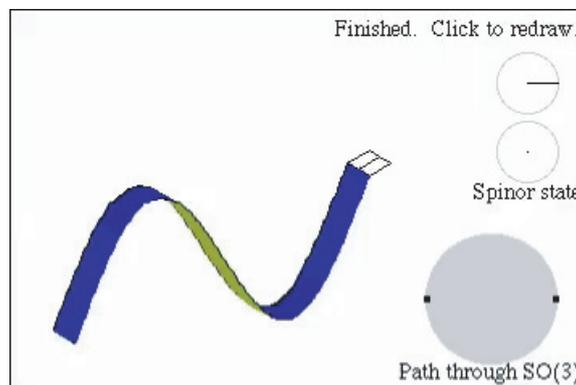
FIGURE 63 The belt trick with a simple belt: a double rotation of the belt buckle is equivalent to no rotation. (QuickTime film © Greg Egan)

because he used it extensively in his lectures.

The human body has such a belt built in: the *arm*. Just take your hand, put an object on it for clarity, such as a cup, and turn the hand and object by 2π by twisting the arm. After a second rotation the whole system will be untangled again, as shown in Figure 64. The trick is even more impressive when *many* arms are used. You can put your two hands (if you chose the correct starting position) under the cup or you can take a friend or two
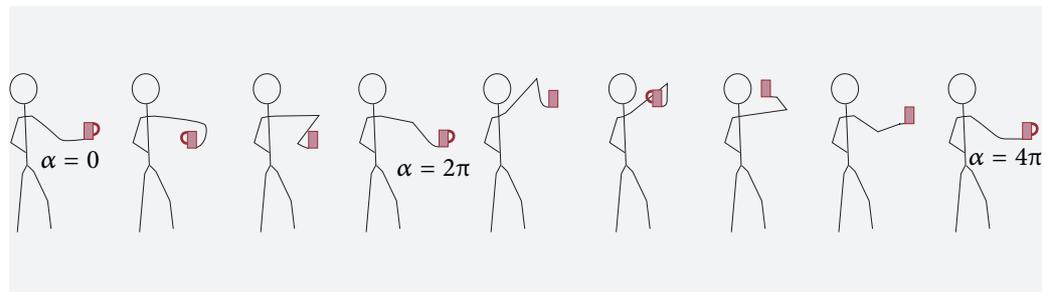
FIGURE 64 The human arm as spin 1/2 model.
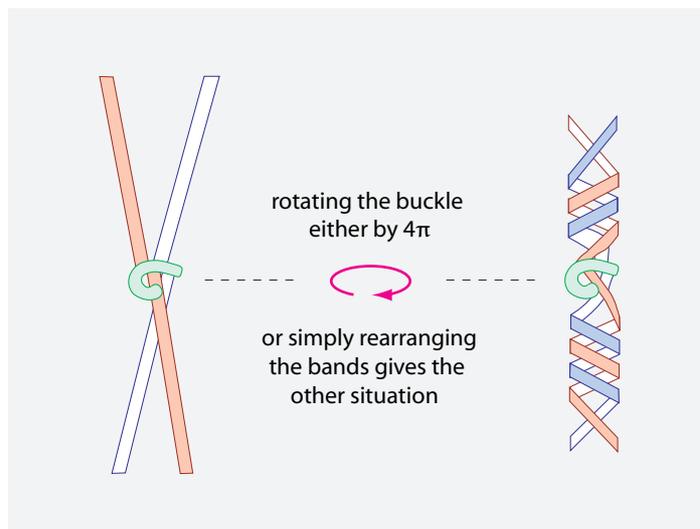
FIGURE 65 The generalized belt trick, modelling the rotation behaviour of a spin 1/2 particle: independently of the number of bands or tubes or strings attached, the two situations can be transformed into each other, either by rotating the central object by 4π or by keeping the central object fixed and moving the bands around it.

who each keep a hand attached to the cup together with you. The belt trick can still be performed, and the whole system untangles after two full turns.

This leads us to the most general way to show the connection between tethering and spin 1/2. Just glue any number of threads, belts or tubes, say half a metre long, to some object, as shown in Figure 65. (With many such tails, is not appropriate any more to call it a belt buckle.) Each band is supposed to go to spatial infinity and be attached there. Instead of being attached at spatial infinity, we can also imagine the belts attached to a distant, fixed object, like the arms are attached to a human body. If the object, which represents the particle, is rotated by 2π, twists appear in its tails. If the object is rotated by an additional turn, to a total of 4π, all twists and tangles can be made to disappear, without moving or turning the object. You really have to experience this in order to believe it. And the process really works with *any* number of bands glued to the object. The website www.evl.uic.edu/hypercomplex/html/dirac.html provides a animation showing this process with four attached belts.

In short, all these animations show that belt buckles, and in fact all (sufficiently) tethered systems, return to their original state only after rotations by 4π, and *not* after rotations by 2π only. *Tethered objects behave like spin 1/2 particles.* In fact, tethered ob-
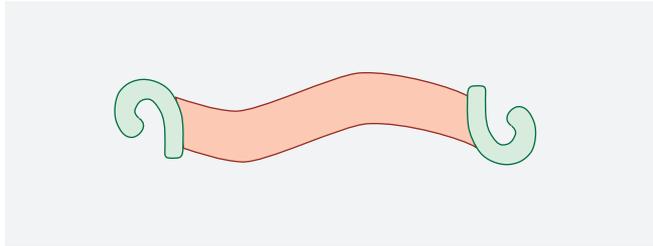
FIGURE 66 Two belt buckles connected by a belt, one way of visualizing two spin 1/2 particles.

jects, such as belt buckles, are the *only* systems that reproduce spin 1/2 properties. In the last part of our adventure we will discover the deep underlying reason for the equivalence between spin 1/2 particles and tethered systems.

Exploring the symmetries of wave functions, quantum theory shows that rotations *require* the existence of spin for all quantum particles. An investigation of the wave function shows that wave functions of elementary matter particles behave under rotation like tethered objects. For example, a wave function whose tethered equivalent is tangled acquires a negative sign.

In summary, quantum theory implies the existence of the slightly counter-intuitive spin 1/2 value. In particular, it appears for elementary matter particles.

The extension of the belt trick

But why do experiments show that all fermions have half-integer spin and that all bosons have integer spin? In particular, why do electrons obey the Pauli exclusion principle? At first sight, it is not clear what the spin value has to do with the statistical properties of a particle. In fact, there are several ways to show that rotations and statistics are connected.

The first proof, due to Wolfgang Pauli, used the details of quantum field theory and was so complicated that its essential ingredients were hidden. It took several decades to convince

everybody that a further observation about belts was the central part of the proof.

Starting from the bulge model of quantum particles shown in Figure 60, we can imagine a tube connecting two particles, similar to a belt connecting two belt buckles, as shown in Figure 66. The buckles represent the particles. The tube keeps track of their relative orientation. If one particle/buckle is rotated by 2π along any axis, a twist is inserted into the belt. As just shown, if the same buckle is rotated by another 2π, bringing the total to 4π, the ensuing double twist can easily be undone without moving or rotating the buckles.

Now we look again at Figure 66. If we take the two buckles and simply *swap* their positions, a twist is introduced into the belt. If we swap them again, the twist will disappear. In short, two connected belt buckles return to their original state only after a double exchange, and not after a single exchange.

In other words, if we take each buckle to represent a particle and a twist to mean a factor −1, the belt exactly describes the phase behaviour of spin 1/2 wave functions, both under *rotation* and under *exchange*. In particular, we see that rotation and exchange behaviour are related.

Similarly, also the belt trick itself can be extended to exchange. Take two buckles that
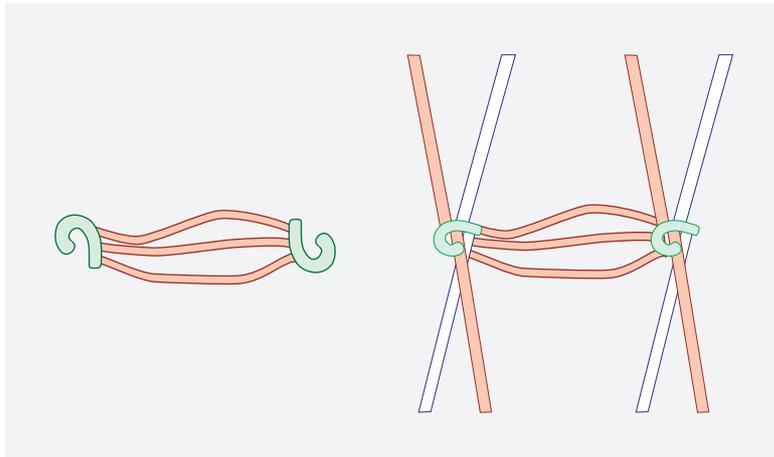
FIGURE 67 Extended belt models for two spin 1/2 particles.
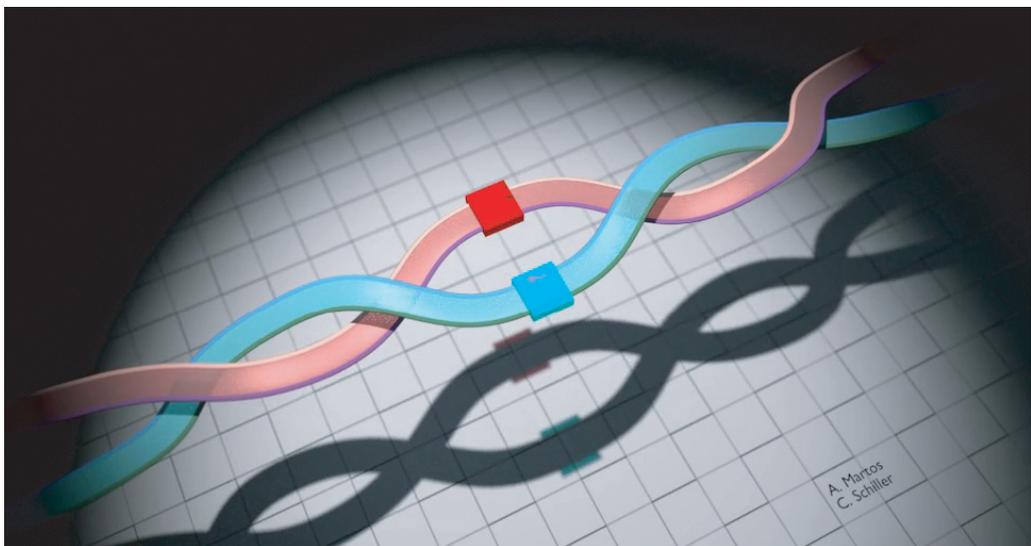
FIGURE 68 Assume that the belts cannot be observed, but the square objects can, and that they represent particles. We know from above that belted buckles behave as spin 1/2 particles. The animation shows that two such particles return to the original situation if they are switched in position twice (but not once). Such particles thus fulfil the defining property of *fermions*. (For the opposite case, that of bosons, a simple exchange would lead to the identical situation.) You can repeat the trick at home using paper strips. The equivalence is shown here with two belts per particle, but the trick works with any positive number of belts attached to each buckle. This animation is the essential part of the proof that spin 1/2 particles are fermions. This is called the *spin–statistics theorem*. (QuickTime film © Antonio Martos)

are connected with *many* bands or threads, like in Figure 67 or in Figure 68. The band can connect the particles, or go to spatial infinity, or both. An exchange of the two buckles produces quite a messy tangle. But almost incredibly, in all cases, a second exchange leads back to the original situation, if the belts are properly rearranged. You might want to test

yourself that the behaviour is also valid if additional particles are involved, as long as you always exchange the same two particles twice.

We conclude that tethered objects behave like fermions under exchange. These observations together form the spin–statistics theorem for spin 1/2 particles: *spin and exchange behaviour are related*. Indeed, these almost 'experimental' arguments can be put into exact mathematical language by studying the behaviour of the configuration space of particles. These investigations result in the following statements:

> ▷ Objects of spin 1/2 are fermions.*

> ▷ Exchange and rotation of spin 1/2 particles are similar processes.

In short, objects that behave like spin 1/2 particles under rotations also behave like fermions under exchange. And vice versa. The exchange behaviour of particles determines their statistical properties; the rotation behaviour determines their spin. By extending the belt trick to several buckles, each with several belts, we thus visualized the spin–statistics theorem for fermions.

Note that all these arguments require three dimensions of space, because there are no tangles (or knots) in fewer or more dimensions.** And indeed, spin exists only in three spatial dimensions.

The belt trick leads to interesting puzzles. We saw that a spin 1/2 object can be modelled by imagining that a belt leading to spatial infinity is attached to it. If we want to model the spin behaviour with attached one-dimensional *strings* instead of *bands*, what is the minimum number of strings we need? More difficult is the following puzzle: Can the belt trick be performed if the buckle is glued into a mattress, thus with the mattress acting like 'infinitely many' belts?

Angels, Pauli's exclusion principle and the hardness of matter

Why are we able to knock on a door? Why can stones not fly through tree trunks? How does the mountain we are walking on carry us? Why can't we walk across walls? In classical physics, we avoided this issue, by taking solidity as a defining property of matter. But we cannot do so any more: we have seen that matter consists mainly of low density electron clouds. The quantum of action thus forces us to *explain* the quantum of matter.

The explanation of the impenetrability of matter is so important that it led to a Nobel prize in physics. The interpenetration of bodies is made impossible by *Pauli's exclusion principle* among the electrons inside atoms. Pauli's exclusion principle states:

> ▷ Two fermions cannot occupy the same quantum state.

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* A mathematical observable behaving like a spin 1/2 particle is neither a vector nor a tensor, as you may want to check. An additional concept is necessary; such an observable is called a *spinor*. We will introduce it in detail later on.

** Of course, knots and tangles do exist in higher dimensions. Instead of considering knotted one-dimensional lines, one can consider knotted planes or knotted higher-dimensional hyperplanes. For example, deformable planes can be knotted in four dimensions and deformable 3-spaces in five dimensions. However, the effective dimensions that produce the knot are always three.

All experiments known confirm the statement.

Why do electrons and other fermions obey Pauli's exclusion principle? The answer can be given with a beautifully simple argument. We know that exchanging two fermions produces a minus sign in the total wave function. Imagine these two fermions being, as a classical physicist would say, located at the same spot, or as a quantum physicist would say, in the same state. If that could be possible, an exchange would change nothing in the system. But an exchange of fermions must produce a minus sign for the total state. Both possibilities – no change at all as well as a minus sign – cannot be realized at the same time. There is only one way out: two fermions must avoid to ever be in the same state. This is Pauli's exclusion principle.

The exclusion principle is the reason that two pieces of matter in everyday life cannot penetrate each other, but have to *repel* each other. For example, take a bell. A bell would not work if the colliding pieces that produce the sound would interpenetrate. But in any example of two interpenetrating pieces, the electrons from different atoms would have to be at the same spot: they would have to be in the same states. This is impossible. Pauli's exclusion principle forbids interpenetration of matter. Bells only work because of the exclusion principle.

Why don't we fall through the floor, even though gravity pulls us down, but remain standing on its surface? Again, the reason is Pauli's exclusion principle. Why does the floor itself not fall? It does not fall, because the matter of the Earth cannot interpenetrate and the atoms cannot made to approach each other than a certain minimal distance. In other words, Pauli's exclusion principle implies that atomic matter cannot be compressed indefinitely. At a certain stage an effective *Pauli pressure* appears, so that a compression limit ensues. For this reason for example, planets made of atomic matter – or neutron stars made of neutrons, which also have spin 1/2 and thus also obey the exclusion principle – do not collapse under their own gravity.

The exclusion principle is the reason that atoms are *extended* electron clouds and that different atoms have different sizes. In fact, the exclusion principle forces the electrons in atoms to form *shells*. When electrons are added around a nucleus and when one shell is filled, a new shell is started. This is the origin of the periodic systems of the elements.

The size of any atom is the size of its last shell. Without the exclusion principle, atoms would be as small as a hydrogen atom. In fact, most atoms are considerably larger. The same argument applies to nuclei: their size is given by the last nucleon shell. Without the exclusion principle, nuclei would be as small as a single proton. In fact, they are usually about 100 000 times larger.

The exclusion principle also settles an old question: How many *angels* can dance on the top of a pin? (Note that angels, if at all, must be made of fermions, as you might want to deduce from the information known about them, and that the top of a pin is a single point in space.) Both theory and experiment confirm the answer already given by
Thomas Aquinas in the Middle Ages: Only *one* angel! The fermion exclusion principle could also be called 'angel exclusion principle'. To stay in the topic, the principle also shows that *ghosts* cannot be objects, as ghosts are supposed to be able to traverse walls.

Let us sum up. Simplifying somewhat, the exclusion principle keeps things around us in shape. Without the exclusion principle, there would be no three-dimensional objects. Only the exclusion principle fixes the diameter of atomic clouds, keeps these clouds from merging, and holds them apart. This repulsion is the origin for the size of soap, planets

and neutron stars. All shapes of solids and fluids are a direct consequence of the exclusion principle. In other words, when we knock on a table or on a door, we prove experimentally that these objects and our hands are made of fermions.

So far, we have only considered fermions of spin 1/2. We will not talk much about particles with odd spin of higher value, such as 3/2 or 5/2. Such particles can all be seen as being composed of spin 1/2 entities. Can you confirm this?

We did not talk about lower spins than 1/2 either. A famous theorem states that a spin value between 0 and 1/2 is impossible in three dimensions. Smaller spins are impossible because the largest rotation angle that can be distinguished and measured in three dimensions is 4π. There is no way to measure a larger angle; the quantum of action makes this impossible. Thus there cannot be any spin value between 0 and 1/2 in nature.

Is spin a rotation about an axis?

The spin of a particle behaves experimentally like an intrinsic angular momentum, adds up like angular momentum, is conserved as part of angular momentum, is described like angular momentum and has a name synonymous with angular momentum. Despite all this, for many decades a strange and false myth was spread in many physics courses and textbooks around the world: "Spin 1/2, despite its name, is *not* a rotation about an axis." It is time to finish with this example of incorrect thinking.

Electrons do have spin 1/2 and are charged. Electrons and all other charged particles with spin 1/2 do have a magnetic moment.* A magnetic moment is expected for any rotating charge. In other words, spin 1/2 does behave like rotation. However, assuming that a particle consists of a *continuous* charge distribution in rotational motion gives the wrong value for the magnetic moment. In the early days of the twentieth century, when physicists were still thinking in classical terms, they concluded that charged spin 1/2 particles thus cannot be rotating. This myth has survived through many textbooks. The correct deduction, however, is that the assumption of continuous charge distribution is wrong. Indeed, charge is quantized; nobody expects that elementary charge is continuously spread over space, as that would contradict its quantization.

The other reason for the false myth is rotation itself. The myth is based on classical thinking and maintains that any rotating object must have *integer* spin. Since half integer spin is not possible in classical physics, it is argued that such spin is not due to rotation. But let us recall what rotation is. Both the belt trick for spin 1/2 as well as the integer spin case remind us: a *rotation* of one body around another is a fraction or a multiple of an exchange. What we call a rotating body in everyday life is a body continuously exchanging the positions of its parts – and vice versa.

> ▷ Rotation and exchange are the same process.

Now, we just found that spin is exchange behaviour. Since rotation is exchange and spin is exchange, it follows that

> ▷ Spin *is* rotation.

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* This magnetic moment can easily be measured in an experiment; however, not one of the Stern–Gerlach type. Why not?

Challenge 112 e

Ref. 82

Challenge 113 ny
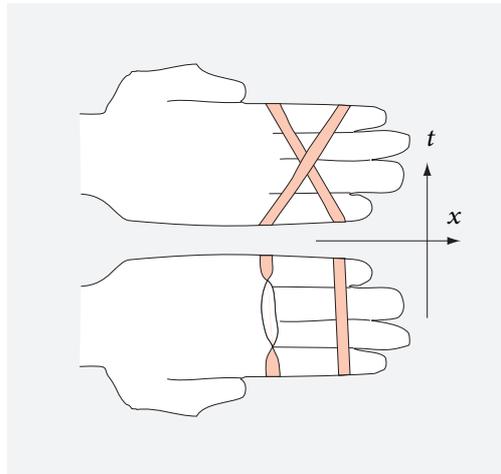
FIGURE 69 Equivalence of exchange and rotation in space-time.

Since we deduced spin, like Wigner, from rotation invariance, this conclusion is not a surprise. In addition, the belt model of a spin 1/2 particle tells us that such a particle can rotate continuously without any hindrance. Also the magnetic moment then gets its correct value. In short, we are allowed to maintain that spin is rotation about an axis, without any contradiction to observations, even for spin 1/2.

In summary, the belt model shows that also spin 1/2 is rotation, as long as we assume that only the buckle can be observed, not the belt(s), and that elementary charge is not continuously distributed in space.*

Since permutation properties and spin properties of fermions are so well described by the belt model, we could be led to the conclusion that these properties might really be consequence of such belt-like connections between particles and the outside world. Maybe for some reason we only observe the belt buckles, not the belts themselves. In the final part of this walk we will discover whether this idea is correct.

ROTATION REQUIRES ANTIPARTICLES

The connection between rotation and antiparticles may be the most astonishing conclusion from the experiments showing the existence of spin. So far, we have seen that rotation requires the existence of spin, that spin appears when relativity is introduced

into quantum theory, and that relativity requires antimatter. Taking these three statements together, the conclusion of the title is not surprising any more: rotation requires antiparticles. Interestingly, there is a simple argument making the same point with the belt model, if it is extended from space alone to full *space-time*.

To learn how to think in space-time, let us take a particle and reduce it to two short tails, so that the particle is a short line segment. When moving in a 2+1 dimensional

---

* Obviously, the exact structure of the electron still remains *unclear* at this point. Any angular momentum S is given classically by $S = \Theta\omega$; however, neither the moment of inertia Θ, connected to the rotation radius and electron mass, nor the angular velocity ω are known at this point. We have to wait quite a while, until the final part of our adventure, to find out more.
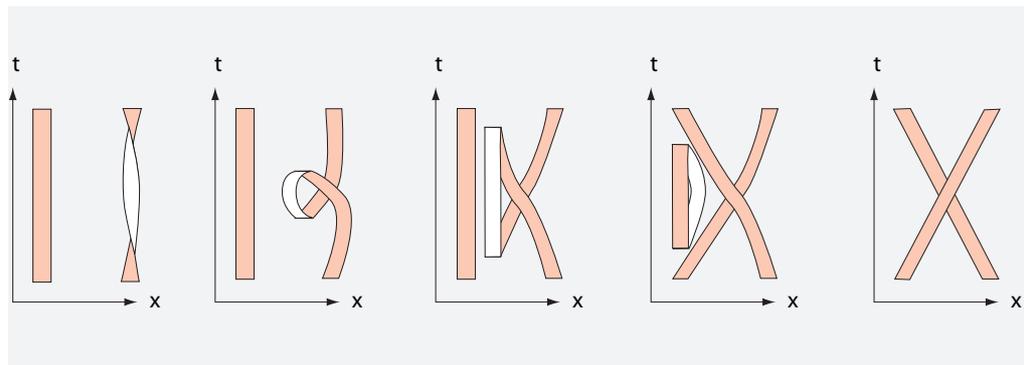
FIGURE 70 Belts in space-time: rotation and antiparticles.

space-time, the particle is described by a ribbon. Playing around with ribbons in space-time, instead of belts in space, provides many interesting conclusions. For example, Figure 69 shows that wrapping a rubber ribbon around the fingers can show, again, that a rotation of a body by 2π in presence of a second one is the same as exchanging the positions of the two bodies.* Both sides of the hand transform the same initial condition, at one edge of the hand, to the same final condition at the other edge. We have thus successfully extended a known result from space to space-time: rotation and exchange are equivalent.

If you think that Figure 69 is not a satisfying explanation, you are right. A more satisfying explanation must include a smooth sequence of steps realizing the equivalence between rotation and exchange. This is shown in Figure 70. We assume that each particle is described by a segment; in the figure, the two segments lie horizontally. The leftmost diagram shows two particles: one at rest and one being rotated by 2π. The deformation of the ribbons shows that this process is equivalent to the exchange in position of two particles, which is shown in the rightmost diagram.

But the essential point is made by the intermediate diagrams. We note that the sequence showing the equivalence between rotation and exchange requires the use of a loop. But such a loop in space-time describes the appearance of a particle–antiparticle pair! In other words, without antiparticles, the equivalence of rotation and exchange would not hold. In short, rotation in space-time requires the existence of antiparticles.

WHY IS FENCING WITH LASER BEAMS IMPOSSIBLE?

When a sword is approaching dangerously, we can stop it with a second sword. Many old films use such scenes. When a laser beam is approaching, it is impossible to fend it off with a second beam, despite all science fiction films showing so. Banging two laser beams against each other is impossible. The above explanation of the spin–statistics theorem shows why.

The electrons in the swords are fermions and obey the Pauli exclusion principle. Fermions make matter impenetrable. On the other hand, the photons in laser beams are

---

* Obviously, the full argument would need to check the full spin 1/2 model of Figure 65 in four-dimensional space-time. But doing this is not an easy task; there is no good visualization yet.

J = 0   J = 1/2   J = 1
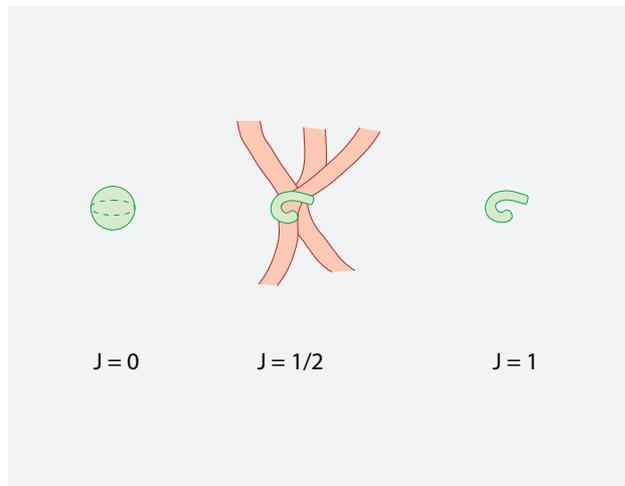
FIGURE 71 Some visualizations of
spin representations.

bosons. Two bosons *can* be in the same state; bosons allow interpenetration. Matter is
impenetrable because at the fundamental level *it is composed of fermions.* Radiation is
composed of bosons; light beams can cross each other. The distinction between fermi-
ons and bosons thus explains why objects can be touched while images cannot. In the
first part of our mountain ascent we started by noting this difference; now we know its
origin.

Spin, statistics and composition

Under rotations, integer spin particles behave differently from half-integer particles. In-
teger spin particles do not show the strange sign changes under rotations by 2π. In
the belt imagery, integer spin particles need no attached strings. In particular, a spin
0 particle obviously corresponds to a sphere. Models for other important spin values are
shown in Figure 71. Exploring their properties in the same way as above, we arrive at the
full *spin–statistics theorem*:

> ▷ Exchange and rotation of objects are similar processes.
>
> ▷ Objects of half-integer spin are fermions. They obey the Pauli exclusion
> principle.
>
> ▷ Objects of integer spin are bosons.

You might prove by yourself that this suffices to show the following rule:

> ▷ Composites of bosons, as well as composites of an even number of fermi-
> ons (at low energy), are bosons; composites of an uneven number of fermi-
> ons are fermions.*

* This rule implies that spin 1 and higher can also be achieved *with* tails; can you find such a representation?
   Note that composite fermions can be bosons only up to that energy at which the composition breaks
down. Otherwise, by packing fermions into bosons, we could have fermions in the same state.

These connections express basic characteristics of the three-dimensional world in which we live. To which class of particles do tennis balls, people, trees, mountains and all other macroscopic objects belong?

THE SIZE AND DENSITY OF MATTER

The three spatial dimensions have many consequences for physical systems. We know that all matter is made of fermions, such as electrons, protons and neutrons. The exclusion principle has an interesting consequence for systems made of N *identical* fermions; such systems obey the following expression for momentum p and size l:

$$\Delta p \, \Delta l \gtrsim N^{1/3} \hbar \,. \tag{67}$$

Can you derive it? This *extended indeterminacy relation* provides a simple way to estimate the spatial size of matter systems. In particular, the extended indeterminacy relation implies that the average energy per quanton increases with quanton density. Can you show this?

The extended indeterminacy relation implies that matter systems whose extension is due to electrons – thus all condensed matter systems – essentially have similar matter and energy densities. The extended indeterminacy relation also implies that nuclei, which are composed of protons and neutrons, all have essentially the same matter density.

For bosons, the components of radiation, there is no extended indeterminacy relation, as the number of components N in a particular quantum state does not have any effect or limits. The indeterminacy relation thus does not limit the power density of laser light; and indeed, the power density of laser beams varies much more than the matter density of solids.

The indeterminacy relation highlights a further difference between matter and radiation. As we saw above, a system of N identical bosons, such as a laser beam, obeys an indeterminacy between the number and the phase which is easily derived from the energy–time indeterminacy relation. The number–phase relation can be written, approximately, as

$$\Delta N \, \Delta \varphi \gtrsim 1 \,. \tag{68}$$

It is important in the use of lasers in precision experiments. The relation limits how close a system can get to a pure sine wave; indeed for a pure sine wave, the indeterminacy product would be zero.

For fermions, where the maximum number in the same state is 1, the number–phase uncertainty relation reduces to a total uncertainty on the phase. In other words, we find – again – that we cannot have fermion beams that behave as waves. There are no classical fermion waves, no coherent fermion beams, in nature.

A SUMMARY ON SPIN AND INDISTINGUISHABILITY

The quantum of action $\hbar$ implies that physical systems are made of two types of indistinguishable quantum particles: *bosons* and *fermions*. The two possible exchange behaviours are related to the particle spin value, because exchange is related to rotation. The connec-

tion between spin and rotation implies that antiparticles exist. It also implies that spin is intrinsically a three-dimensional phenomenon.

Experiments show that radiation is made of elementary particles that behave as bosons. Bosons have integer spin. Two or more bosons, such as two photons, *can share* the same state. This sharing makes laser light possible.

Experiments show that matter is made of elementary particles that behave as fermions. Fermions have half-integer spin. They obey Pauli's exclusion principle: two fermions *cannot be* in the same state. The exclusion principle between electrons explains the structure and (partly) the size of atoms, as well as the chemical behaviour of atoms, as we will find out later on. Together with the electrostatic repulsion of electrons, the exclusion principle explains the incompressibility of matter and its lack of impenetrability.

Fermions make matter 'hard', bosons allow light beams to cross.

Limits and open questions of quantum statistics

The topic of quantum particle statistics remains a research field in theoretical and experimental physics. In particular, researchers have searched and still are searching for generalizations of the possible exchange behaviours of particles.

In two spatial dimensions, the effect of a particle exchange on the wave function is a continuous phase, in contrast to three dimensions, where the result is a sign. Two-dimensional quantum objects are therefore called *anyons* because they can have 'any' spin. Anyons appear as quasi-particles in various experiments in solid state physics, because the set-up is often effectively two-dimensional. The *fractional quantum Hall effect*, perhaps the most interesting discovery of modern experimental physics, has pushed any-
ons onto the stage of modern research.

Other theorists generalized the concept of fermions in other ways, introducing par-
afermions, parabosons, plektons and other hypothetical concepts. Oscar Greenberg has spent most of his professional life on this issue. His conclusion is:

▷ In 3 + 1 space-time dimensions, only fermions and bosons exist.

Can you show that this result implies that the ghosts appearing in Scottish tales do not
exist?

From a different viewpoint, the belt model of spin 1/2 invites to study the behaviour of braids, open links and knots. (In mathematics, braids and open links are made of strands extending to infinity.) This fascinating part of mathematical physics has become important with in modern unified theories, which all state that particles, especially at high energies, are not point-like, but extended entities. The quest is to understand what happens to permutation symmetry in a unified theory of nature. A glimpse of the difficulties appears already above: how can Figures 60, 65 and 70 be reconciled and combined? We will
settle this issue in the final part of our mountain ascent.
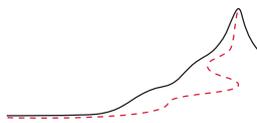
SUPERPOSITIONS AND PROBABILITIES – QUANTUM THEORY WITHOUT IDEOLOGY

> " The fact that an adequate philosophical presentation has been so long delayed is no doubt caused by the fact that Niels Bohr brainwashed a whole generation of theorists into thinking that the job was done fifty years ago. "
>
> Murray Gell-Mann

Why is this famous physical issue arousing such strong emotions? In particular, who is brainwashed, Gell-Mann, the discoverer of the quarks, or most of the world's physicists working on quantum theory who follow Niels Bohr's opinion? In the twentieth century, quantum mechanics has thrown many in disarray. We have a simple aim: we want to *understand* quantum theory. Quantum mechanics is unfamiliar for two reasons: it allows *superpositions* and it leads to *probabilities*. In this chapter we explore and clarify these two topics – until we understand quantum theory.

Probabilities appear whenever an aspect of a microscopic system is measured. The quantum of action, the smallest change value found in nature, leads to the appearance of probabilities in measurements.

Superpositions appear because the quantum of action radically changed the two most basic concepts of classical physics: state and system. The *state* is not defined and described any more by the specific values taken by position and momentum, but by the specific wave function 'taken' by the position and momentum operators.** In addition, in classical physics a *system* was described and defined as a set of permanent aspects of nature; permanence was defined as negligible interaction with the environment. Quantum mechanics shows that these definitions have to be modified.

Clarifying the origin of superpositions and probabilities, as well as the concepts of system and state, will help us to avoid getting lost on our way to the top of Motion Mountain. Indeed, quite a number of researchers have lost their way since quantum theory appeared, including important physicists like Murray Gell-Mann and Steven Weinberg.

** It is equivalent, and often conceptually clearer, to say that the state is described by a complete set of commuting operators. In fact, the discussion of states is somewhat simplified in the Heisenberg picture. However, here we study the issue in the Schrödinger picture only, i.e., using wave functions.

Every such `artistic impression' is wrong.

(Why?)

FIGURE 72 An artist's impression of a macroscopic superposition is impossible – because such superpositions are not found in our environment.

WHY ARE PEOPLE EITHER DEAD OR ALIVE?

The evolution equation of quantum mechanics is *linear* in the wave function; the linearity reflects the existence of superpositions. Superpositions imply that we can imagine and try to construct systems where the state ψ is a superposition of two radically distinct situations, such as those of a dead and of a living cat. This famous fictional animal is called *Schrödinger's cat* after the originator of the example. Is it possible to produce it? And how would it evolve in time? We can ask the same two questions in other situations. For example, can we produce a superposition of a state where a car is inside a closed garage with a state where the car is outside? What happens then?

Macroscopic superpositions are strange. Such situations are not observed in everyday life, and only very rarely in the laboratory. The reason for this rareness is an important aspect of what is often called the 'interpretation' of quantum mechanics. In fact, such strange situations *are* possible, and the superposition of macroscopically distinct states has actually been observed in a few cases, though not for cats, people or cars. To get an idea of the constraints, let us specify the situation in more detail.*

MACROSCOPIC SUPERPOSITIONS, COHERENCE AND INCOHERENCE

The object of discussion are linear superpositions of the type $\psi = a\psi_a + b\psi_b$, where ψ_a and ψ_b are macroscopically distinct states of the system under discussion, and where a and b are some complex coefficients. States are called *macroscopically distinct* when each state corresponds to a different macroscopic situation, i.e., when the two states can be distinguished using the concepts or measurement methods of classical physics. In particular, this means that the physical action necessary to transform one state into the other must be much larger than $\hbar$. For example, two different positions of a body composed of a large number of atoms are macroscopically distinct. The state of a cat that is living and of the same cat when it is dead also differ by many quanta of action.

A 'strange' situation is thus a superposition of macroscopically distinct states. Let us work out the essence of such macroscopic superpositions more clearly. Given two macroscopically distinct states ψ_a and ψ_b, any superposition of the type $\psi = a\psi_a + b\psi_b$ is called a *pure state*. Since the states ψ_a and ψ_b can interfere, one also talks about a *(phase) coherent superposition*. In the case of a superposition of macroscopically distinct states,

* Most what can be said about this topic has been said by three important researchers: Niels Bohr, one of the fathers of quantum physics, John von Neumann, who in the nineteen-thirties stressed the differences between evolution and decoherence, and by Heinz Dieter Zeh, who in the nineteen-seventies stressed the importance of baths and the environment in the decoherence process.

the scalar product $\psi_a^\dagger \psi_b$ is obviously vanishing. In case of a coherent superposition, the coefficient product $a^* b$ is different from zero. This fact can also be expressed with the help of the *density matrix* ρ of the system, defined as $\rho = \psi \otimes \psi^\dagger$. In the present case it is given by

$$
\begin{aligned}
\rho_{\text{pure}} = \psi \otimes \psi^\dagger &= |a|^2 \psi_a \otimes \psi_a^\dagger + |b|^2 \psi_b \otimes \psi_b^\dagger + a\,b^* \psi_a \otimes \psi_b^\dagger + a^*\,b\,\psi_b \otimes \psi_a^\dagger \\
&= (\psi_a, \psi_b) \begin{pmatrix} |a|^2 & a\,b^* \\ a^*\,b & |b|^2 \end{pmatrix} \begin{pmatrix} \psi_a^\dagger \\ \psi_b^\dagger \end{pmatrix} .
\end{aligned}
\tag{69}
$$

We can then say that whenever the system is in a pure, or coherent state, then its density matrix, or *density functional*, contains off-diagonal terms of the same order of magnitude as the diagonal ones.[*] Such a density matrix corresponds to the above-mentioned strange situations that we never observe in daily life. We will shortly understand why.

We now have a look at the opposite situation, a density matrix for macroscopic distinct states with *vanishing* off-diagonal elements. For two states, the example

$$
\begin{aligned}
\rho_{\text{mixed}} &= |a|^2 \psi_a \otimes \psi_a^\dagger + |b|^2 \psi_b \otimes \psi_b^\dagger \\
&= (\psi_a, \psi_b) \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \begin{pmatrix} \psi_a^\dagger \\ \psi_b^\dagger \end{pmatrix}
\end{aligned}
\tag{71}
$$

describes a system which possesses *no* phase coherence at all. (Here, $\otimes$ denotes the non-commutative dyadic product or tensor product which produces a tensor or matrix starting from two vectors.) Such a diagonal density matrix cannot be that of a pure state; the density matrix describes a system which is in the state ψ_a with probability $|a|^2$ and which is in the state ψ_b with probability $|b|^2$. Such a system is said to be in a *mixed state*, because its state is *not known*, or equivalently, is in a *(phase) incoherent superposition*: interference effects cannot be observed in such a situation. A system described by a mixed state is always *either* in the state ψ_a *or* in the state ψ_b. In other words, a diagonal density matrix for macroscopically distinct states is not in contrast, but in agreement with everyday experience.

In the picture of density matrices, the non-diagonal elements contain the difference between normal, i.e., incoherent, and unusual or strange, i.e., coherent, superpositions.

The experimental situation is clear: for macroscopically distinct states, only diagonal density matrices are observed in everyday life. Almost all systems in a coherent macroscopic superposition somehow lose their off-diagonal matrix elements. How does this process of *decoherence* – also called *disentanglement* in certain settings – take place? The density matrix itself shows the way.

---

[*] Using the density matrix, we can rewrite the evolution equation of a quantum system:

$$
\dot\psi = -iH\psi \quad \text{becomes} \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H, \rho] .
\tag{70}
$$

Both are completely equivalent. (The new expression is sometimes also called the *von Neumann equation*.) We won't actually do any calculations here. The expressions are given so that you recognize them when you encounter them elsewhere.

DECOHERENCE IS DUE TO BATHS

In thermodynamics, the density matrix ρ for a large system is used for the definition of its entropy S – and of all its other thermodynamic quantities. These studies show that

$$S = -k \, \mathrm{tr} \, (\rho \ln \rho) \tag{72}$$

where tr denotes the *trace*, i.e., the sum of all diagonal elements, and k is the Boltzmann constant. The expression is thus the quantum mechanical definition of entropy.

We now remind ourselves that a physical system with a large and constant entropy is called a *bath*. In simple physical terms, a bath is a system to which we can ascribe a temperature. More precisely,

> A *(physical) bath* – also called a *(thermodynamic) reservoir* – is any large system for which the concept of *equilibrium* can be applied.

Experiments show that in practice, this is equivalent to the condition that a bath consists of many interacting subsystems. For this reason, all macroscopic quantities describing the state of a bath show small, irregular *fluctuations*, a property that will be of central importance shortly.

An everyday bath is also a physical bath: indeed, a thermodynamic bath is similar to an extremely large warm water bath, one for which the temperature does not change even if we add some cold or warm water to it. The physical concept of bath, or reservoir, is thus an abstraction and a generalization of the everyday concept of bath. Other examples of physical baths are: an intense magnetic field, a large amount of gas, or a large solid. (The meanings of 'intense' and 'large' of course depend on the system under study.)

The definition (72) of entropy tells us that the loss of off-diagonal elements corresponds to an increase in entropy. In addition, any increase in entropy of a reversible system, such as the quantum mechanical system in question, is due to an interaction with a bath.

In short, *decoherence is due to interaction with a bath*. In addition, decoherence is a process that increases entropy: decoherence is irreversible. We will now show that baths are everywhere, that decoherence thus takes place everywhere and all the time, and that therefore, macroscopic superpositions are (almost) never observed.

HOW BATHS LEAD TO DECOHERENCE – SCATTERING

Where is the bath interacting with a typical system? The bath must be outside the system we are talking about, i.e., in its *environment*. Indeed, we know experimentally that a typical environment is large and characterized by a temperature. Some examples are listed in Table 6. In short,

> ▷ Any environment is a bath.

We can even go further: for every experimental situation, there is a bath *interacting* with the system under study. Indeed, every system which can be observed is not isolated, as it obviously interacts at least with the observer; and every observer by definition contains

TABLE 6 Common and less common baths with their main properties.

BATH TYPE	TEMPERATURE	WAVELENGTH	PARTICLE FLUX	CROSS SECTION (ATOM)	HIT time $1/\sigma\varphi$ FOR	
	T	λ_{eff}	φ	σ	ATOM[a]	BALL[a]
matter baths						
solid, liquid	300 K	10 pm	10^{31} /m^{2}s	10^{-19} m^2	10^{-12} s	10^{-25} s
air	300 K	10 pm	10^{28} /m^{2}s	10^{-19} m^2	10^{-9} s	10^{-22} s
laboratory vacuum	50 mK	10 μm	10^{18} /m^{2}s	10^{-19} m^2	10 s	10^{-12} s
photon baths						
sunlight	5800 K	900 nm	10^{23} /m^{2}s		10^{-4} s	10^{-17} s
'darkness'	300 K	20 μm	10^{21} /m^{2}s		10^{-2} s	10^{-15} s
cosmic microwaves	2.7 K	2 mm	10^{17} /m^{2}s		10^{2} s	10^{-11} s
terrestrial radio waves						
Casimir effect					very large	
Unruh radiation of Earth	40 zK				very large	
nuclear radiation baths						
radioactivity		10 fm	1 /m^{2}s	10^{-25} m^2	10^{25} s	10^{12} s
cosmic radiation	>1000 K	10 fm	10^{-2} /m^{2}s	10^{-25} m^2	10^{27} s	10^{14} s
solar neutrinos	≈ 10 MK	10 fm	10^{11} /m^{2}s	10^{-47} m^2	10^{36} s	10^{15} s
cosmic neutrinos	2.0 K	3 mm	10^{17} /m^{2}s	10^{-62} m^2	10^{45} s	10^{24} s
gravitational baths						
gravitational radiation	$5 \cdot 10^{31}$ K	10^{-35} m			very large	

a. Values are rough estimates. The macroscopic ball is assumed to have a 1 mm size.

a bath, as we will show in more detail shortly. Usually however, the most important baths we have to take into consideration are the atmosphere around a system, the radiation or electromagnetic fields interacting with the system, or, if the system itself is large enough to have a temperature, those degrees of freedom of the system which are not involved in the superposition under investigation.

Since every physical system is in contact with a bath, every density matrix of a macroscopic superposition will lose its diagonal elements eventually. At first sight, this direction of thought is not convincing. The interactions of a system with its environment can be made extremely small by using clever experimental set-ups; that would imply that the time for decoherence can be made extremely large. Thus we need to check how much time a superposition of states needs to decohere. It turns out that there are two standard ways to estimate the *decoherence time*: either by modelling the bath as large number of colliding particles, or by modelling it as a continuous field.

If the bath is described as a set of particles randomly hitting the microscopic system, it is best characterized by the effective wavelength λ_{eff} of the particles and by the average interval t_{hit} between two hits. A straightforward calculation shows that the decoherence

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time t_d is in any case smaller than this time interval, so that

$$t_d \leqslant t_{\text{hit}} = \frac{1}{\varphi\sigma} \ , \tag{73}$$

where φ is the flux of particles and σ the cross-section for the hit.* Typical values are given in Table 6. We easily note that for macroscopic objects, decoherence times are extremely short. (We also note that nuclear and gravitational effects lead to large decoherence times and thus can be neglected.) *Scattering leads to fast decoherence of macroscopic systems.* However, for atoms or smaller systems, the situation is different, as expected. *Microscopic systems can show long decoherence times.*

We note that the quantum of action $\hbar$ appears in the expression for the decoherence time, as it appears in the area σ. Decoherence is a quantum process.

How baths lead to decoherence – relaxation

A second method to estimate the decoherence time is also common. Any interaction of a system with a bath is described by a relaxation time t_r. The term *relaxation* designates any process which leads to the return to the equilibrium state. The terms *damping* and *friction* are also used. In the present case, the relaxation time describes the return to equilibrium of the combination bath and system. Relaxation is an example of an irreversible evolution. A process is called *irreversible* if the reversed process, in which every component moves in opposite direction, is of very low probability.** For example, it is usual that a glass of wine poured into a bowl of water colours the whole water; it is very rarely observed that the wine and the water separate again, since the probability of all water and wine molecules to change directions together at the same time is rather low, a state of affairs making the happiness of wine producers and the despair of wine consumers.

Now let us simplify the description of the bath. We approximate it by a single, unspecified, scalar field which interacts with the quantum system. Due to the continuity of space, such a field has an infinity of degrees of freedom. They are taken to model the many degrees of freedom of the bath. The field is assumed to be in an initial state where its degrees of freedom are excited in a way described by a temperature T. The interaction of the system with the bath, which is at the origin of the relaxation process, can be described by the repeated transfer of small amounts of energy E_{hit} until the relaxation

---

* The decoherence time is derived by studying the evolution of the density matrix $\rho(x, x')$ of objects localized at two points x and x'. One finds that the off-diagonal elements follow $\rho(x, x', t) = \rho(x, x', 0)e^{-\Lambda t(x-x')^2}$, where the localization rate Λ is given by

$$\Lambda = k^2 \varphi \sigma_{\text{eff}} \tag{74}$$

where k is the wave number, φ the flux and σ_{eff} the cross-section of the collisions, i.e., usually the size of the macroscopic object.

One also finds the surprising result that a system hit by a particle of energy E_{hit} *collapses* the density matrix roughly down to the de Broglie (or thermal de Broglie) wavelength of the hitting particle. Both results together give the formula above.

** Beware of other definitions which try to make something deeper out of the concept of irreversibility, such as claims that 'irreversible' means that the reversed process is *not at all* possible. Many so-called 'contradictions' between the irreversibility of processes and the reversibility of evolution equations are due to this mistaken interpretation of the term 'irreversible'.

Ref. 98

Ref. 99

process is completed.

The objects of interest in this discussion, like the mentioned cat, person or car, are described by a mass m. Their main characteristic is the maximum energy E_r which can be transferred from the system to the environment. This energy describes the interactions between system and environment. The superpositions of macroscopic states we are interested in are solutions of the Hamiltonian evolution of these systems.

Ref. 100   The initial coherence of the superposition, so disturbingly in contrast with our everyday experience, disappears exponentially within a *decoherence time* t_d given by[*]

$$t_d = t_r \frac{E_{\text{hit}}}{E_r} \frac{e^{E_{\text{hit}}/kT} - 1}{e^{E_{\text{hit}}/kT} + 1} \tag{77}$$

where k is again the *Boltzmann constant* and like above, E_r is the maximum energy which can be transferred from the system to the environment. Note that one always has $t_d \leqslant t_r$. After the decoherence time t_d is elapsed, the system has evolved from the coherent to the incoherent superposition of states, or, in other words, the density matrix has lost its off-diagonal terms. One also says that the phase coherence of this system has been destroyed. Thus, after a time t_d, the system is found either in the state ψ_a or in the state ψ_b, respectively with the probability $|a|^2$ or $|b|^2$, and not any more in a coherent superposition which is so much in contradiction with our daily experience. Which final state is selected depends on the precise state of the bath, whose details were eliminated from the calculation by taking an *average* over the states of its microscopic constituents.

The important result is that for all *macroscopic* objects, the decoherence time t_d is extremely small. In order to see this more clearly, we can study a special simplified case. A macroscopic object of mass m, like the mentioned cat or car, is assumed to be at the same time in two locations separated by a distance l, i.e., in a superposition of the two corresponding states. We further assume that the superposition is due to the object moving as a quantum mechanical oscillator with frequency ω between the two locations; this is the simplest possible system that shows superpositions of an object located in two different positions. The energy of the object is then given by $E_r = m\omega^2 l^2$, and the smallest transfer energy $E_{\text{hit}} = \hbar\omega$ is the difference between the oscillator levels. In a macroscopic situation, this last energy is much smaller than kT, so that from the preceding expression we

Ref. 102   get

$$t_d = t_r \frac{E_{\text{hit}}^2}{2E_r \, kT} = t_r \frac{\hbar^2}{2mkTl^2} = t_r \frac{\lambda_T^2}{l^2} \tag{78}$$

---

[*] This result is derived as in the above case. A system interacting with a bath always has an evolution given

Ref. 101   by the general form

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] - \frac{1}{2t_o}\sum_j [V_j\rho, V_j^\dagger] + [V_j, \rho V_j^\dagger] \,, \tag{75}$$

where ρ is the density matrix, H the Hamiltonian, V the interaction, and t_o the characteristic time of the

Challenge 126 ny   interaction. Are you able to see why? Solving this equation, one finds for the elements far from the diagonal $\rho(t) = \rho_0 e^{-t/t_0}$. In other words, they disappear with a characteristic time t_o. In most situations one has a relation of the form

$$t_0 = t_r \frac{E_{\text{hit}}}{E_r} = t_{\text{hit}} \tag{76}$$

or some variations of it, as in the example above.

in which the frequency ω has disappeared. The quantity $\lambda_T = \hbar/\sqrt{2mkT}$ is called the *thermal de Broglie wavelength* of a particle.

We note again that the quantum of action $\hbar$ appears in the expression for the decoherence time. Decoherence is a quantum process.

It is straightforward to see that for practically all macroscopic objects the typical decoherence time t_d is extremely short. For example, setting $m = 1$ g, $l = 1$ mm and $T = 300$ K we get $t_d/t_r = 1.3 \cdot 10^{-39}$. Even if the interaction between the system and the environment would be so weak that the system would have as relaxation time the age of the universe, which is about $4 \cdot 10^{17}$ s, the time t_d would still be shorter than $5 \cdot 10^{-22}$ s, which is over a million times faster than the oscillation time of a beam of light (about 2 fs for green light). For Schrödinger's cat, the decoherence time would be even shorter. These times are so short that we cannot even hope to *prepare* the initial coherent superposition, let alone to observe its decay or to measure its lifetime.

For microscopic systems however, the situation is different. For example, for an electron in a solid cooled to liquid helium temperature we have $m = 9.1 \cdot 10^{-31}$ kg, and typically $l = 1$ nm and $T = 4$ K; we then get $t_d \approx t_r$ and therefore the system can stay in a coherent superposition until it is relaxed, which confirms that for this case coherent effects can indeed be observed if the system is kept isolated. A typical example is the be-
Ref. 103   haviour of electrons in superconducting materials. We will mention a few more below.

In 1996 the first actual measurement of decoherence times was published by the Paris
Ref. 104   team led by Serge Haroche. It confirmed the relation between the decoherence time and the relaxation time, thus showing that the two processes have to be distinguished at microscopic scale. In the meantime, many other experiments confirmed the decoherence
Ref. 105   process with its evolution equation, both for small and large values of t_d/t_r. A particularly
Ref. 106   beautiful experiment has been performed in 2004, where the disappearance of two-slit interference for C_{70} molecules was observed when a bath interacts with them.

Summary on decoherence, life and death

Our exploration showed that *decoherence results from coupling to a bath in the environment.* Decoherence is a quantum statistical effect, i.e., a thermodynamic effect. Decoherence follows from quantum theory, is an irreversible process, and thus occurs automatically. Above all, decoherence is a process that has been observed in experiments.

The estimates of decoherence times in everyday life told us that both the preparation and the survival of superpositions of macroscopically different states is made *impossible* by the interaction with any bath found in the environment. This is the case even if the usual measure of this interaction, given by the friction of the motion of the system, is very small. Even if a macroscopic system is subject to an extremely low friction, leading to a very long relaxation time, its decoherence time is still vanishingly short. Only carefully designed microscopic systems in expensive laboratory set-ups can reach substantial decoherence times.

Our everyday environment is full of baths. Therefore,

> ▷ Coherent superpositions of macroscopically distinct states never appear in everyday life, due to the rapid decoherence times induced by baths in the environment.

Cars cannot be in and out of a garage at the same time. We cannot be dead and alive at the same time. An illustration of a macroscopic superposition – see Figure 122 – is impossible. In agreement with the explanation, coherent superpositions of macroscopic states appear in some special laboratory situations.

What is a system? What is an object?

In classical physics, a system is a part of nature that can be isolated from its environment. However, quantum mechanics tells us that isolated systems do not exist, since interactions cannot be made vanishingly small. The contradiction can be solved with the results above: they allow us to define the concept of system with more accuracy.

> ▷ A *system* is any part of nature that interacts *incoherently* with its environment.

This implies:

> ▷ An *object* is a part of nature interacting with its environment only through baths.

In particular, we get:

> ▷ A system is called *microscopic* or *quantum mechanical* and can described by a wave function ψ whenever
> — it is almost isolated, with $t_{evol} = \hbar/\Delta E < t_r$, and
> — it is in *incoherent* interaction with its environment.

In short, a microscopic or quantum mechanical system can be described by a wave function only if it interacts incoherently and weakly with its environment. (For such a system, the energy indeterminacy ΔE is larger than the relaxation energy.) In contrast, a bath is never isolated in the sense just given, because the evolution time of a bath, the time scale during which its properties change, is always much larger than its relaxation time. Since all macroscopic bodies are in contact with baths – or even contain one – they cannot be described by a wave function. In particular, it is impossible to describe any measuring apparatus with the help of a wave function.

We thus conclude:

> ▷ A *macroscopic system* is a system with a decoherence time much shorter than any other evolution time of its constituents.

Obviously, macroscopic systems also interact incoherently with their environment. Thus cats, cars and television news speakers are all macroscopic systems.

Entanglement

One possibility is left over by the two definitions of system: what happens in the situation in which the interactions with the environment are *coherent*? We will encounter some ex-

amples shortly. Following the definitions, they are neither microscopic nor macroscopic systems.

> ▷ A 'system' in which the interaction with its environment is coherent is called *entangled*.

Such 'systems' are not described by a wave function, and strictly speaking, they are *not* systems. In these situations, when the interaction is coherent, one speaks of *entanglement*. For example, one says that a particle or set of particles is said to be *entangled* with its environment.

Entangled, i.e., coherently interacting systems can be divided, but must be disentangled when doing so. The act of division leads to detached entities; detached entities interact incoherently. Quantum theory shows that nature is not made of detached entities, but that it is made of *detachable* entities. In quantum theory, the criterion of detachment is the incoherence of interaction. Coherent superpositions imply the surprising consequence that there are systems which, even though they look being made of detached parts, are not. Entanglement poses a limit to detachment. All surprising properties of quantum mechanics, such as Schrödinger's cat, are consequences of the classical prejudice that a system made of two or more parts can obviously be detached into two subsystems without disturbance. But coherent superpositions, or entangled systems, do not allow detachment without disturbance. Whenever we assume to be able to detach entangled systems, we get strange or incorrect conclusions, such as apparent faster-than-light propagation, or, as one says today, non-local behaviour. Let us have a look at a few typical examples.

Entangled situations are observed in many experiments. For example, when an electron and a positron annihilate into two photons, the polarisations of these two photons are entangled, as measured already in 1949. Also when an excited atom decays in steps, emitting two photons, the photon polarisations are entangled, as was first shown in 1966 with the help of calcium atoms. Similarly, when an unstable molecule in a singlet state, i.e., in a spin 0 state, decays or splits into debris, the spins of the debris are entangled, as observed in the 1970s. Also the spontaneous parametric down-conversion of photons produces entanglement. In a non-linear optical material, an incoming photon is converted into two outgoing photons whose added energies correspond to the energy of the incoming photon. In this case, the two outgoing photons are entangled both in their polarisation and in their direction. In 2001, the spins of two extremely cold caesium gas samples, with millions of atoms each and located a few millimetres apart, have been entangled. Also position entanglement has been regularly observed, for example for closely spaced ions inside ion traps.

Is quantum theory non-local? A bit about the Einstein–Podolsky–Rosen paradox

> “ [Mr. Duffy] lived a little distance away from his body ...
>
> James Joyce, *A Painful Case* ”
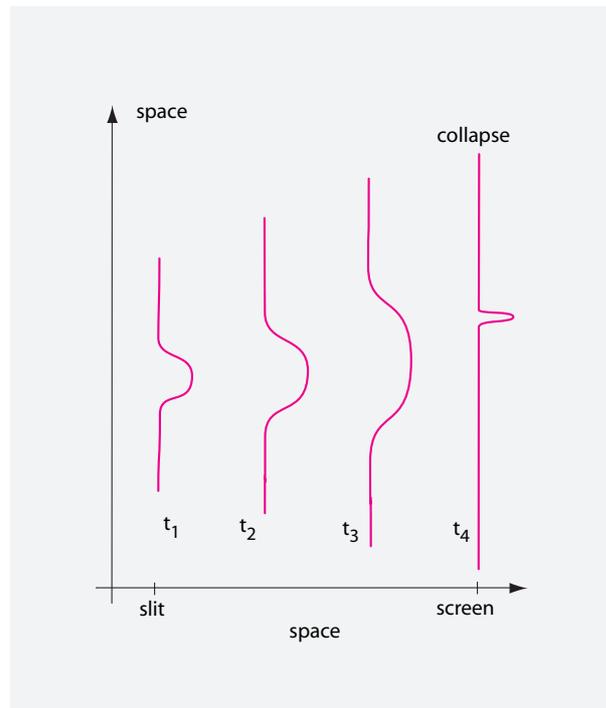
FIGURE 73 Quantum mechanical motion: an electron wave function (actually its module squared) from the moment it passes a slit until it hits a screen.

It is often suggested, incorrectly, that wave function collapse or quantum theory are non-local.* The issue needs clarification.

We start by imagining an electron hitting a screen after passing a slit. Following the description just deduced, the collapse process proceeds schematically as depicted in Figure 73. An animation that includes another example of a collapse process – inspired by Bohm's thought experiment – can be seen in the lower left corners on these pages, starting at page 115. The collapse process has a surprising side: due to the shortness of the decoherence time, during this (and any other) wave function collapse the maximum of the wave function usually changes position faster than light. Is this reasonable?

A situation is called *acausal* or *non-local* if energy is transported faster than light. Using Figure 73 you can determine the energy velocity involved, using the results on signal propagation. The result is a value smaller than c. A wave function whose maximum moves faster than light does *not automatically* imply that energy moves faster than light.

In other words, quantum theory contains speeds greater than light, but no *energy* speeds greater than light. In classical electrodynamics, the same happens with the scalar and the vector potentials if the Coulomb gauge is used. We have also encountered speeds faster than that of light in the motion of shadows and scissors, and in many other observations. Any physicist now has two choices: he can be straight, and say that there is no non-locality in nature; or he can be less straight, and claim there is. In the latter

* This continues a topic that we know already: we have explored a different type of non-locality, in general relativity, earlier on.
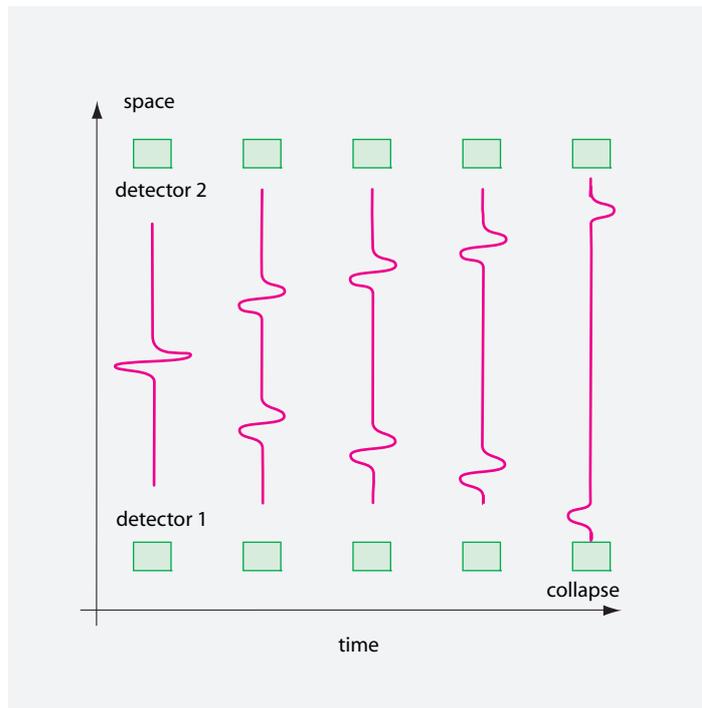
FIGURE 74 Bohm's thought experiment.

case, he has to claim that even classical physics is non-local. However, nobody dares to claim this. In fact, there is a danger in this more provoking usage of the term 'non-local': a small percentage of those who claim that the world is non-local after a while start to believe that there really is faster-than-light energy transport in nature. These people become prisoners of their muddled thinking. On the other hands, muddled thinking helps to get more easily into newspapers. In short, even though the definition of non-locality is not unanimous, here we stick to the stricter one, and define non-locality as *energy* transport faster than light.

An often cited thought experiment that shows the pitfalls of non-locality was proposed by Bohm* in the discussion around the so-called Einstein–Podolsky–Rosen paradox. In the famous EPR paper the three authors tried to find a contradiction between quantum mechanics and common sense. Bohm translated their rather confused paper into a clear thought experiment that is shown schematically in Figure 74. When two particles in a spin 0 state move apart, measuring one particle's spin orientation implies an *immediate* collapse also of the other particle's spin, namely in the exactly opposite direction. This happens instantaneously over the whole separation distance; no speed limit is obeyed. In other words, entanglement seems to lead to faster-than-light communication.

However, in Bohm's experiment, no energy is transported faster than light. No non-locality is present, despite numerous claims of the contrary by certain authors. The two

* David Joseph Bohm (1917–1992), was an influential physicist. He codiscovered the Aharonov–Bohm effect and spent a large part of his later life investigating the connections between quantum physics and philosophy.

entangled electrons belong to one system: assuming that they are separate only because the wave function has two distant maxima is a conceptual mistake. In fact, no signal can be transmitted with this method; the decoherence is a case of prediction which looks like a signal without being one. Bohm's experiment, like any other EPR-like experiment, does not allow communication faster than light. We already discussed such cases in the section on electrodynamics.

Bohm's experiment has actually been performed. The first and most famous realization was due, in 1982, by Alain Aspect; he used photons instead of electrons. Like all latter tests, it has fully confirmed quantum mechanics.

In fact, experiments such as the one by Aspect confirm that it is impossible to treat either of the two particles as a system by itself; it is impossible to ascribe any physical property, such as a spin orientation, to either of them alone. (The Heisenberg picture would express this restriction even more clearly.) Only the two electrons together form a physical system, because only the pair interacts incoherently with the environment.

The mentioned two examples of apparent non-locality can be dismissed with the remark that since obviously no energy flux faster than light is involved, no problems with causality appear. Therefore the following example is more interesting. Take two identical atoms, one in an excited state, one in the ground state, and call l the distance that separates them. Common sense tells that if the first atom returns to its ground state emitting a photon, the second atom can be excited only after a time $t = l/c$ has been elapsed, i.e., after the photon has travelled to the second atom.

Surprisingly, this conclusion is wrong. The atom in its ground state has a non-zero probability to be excited at the same moment in which the first is de-excited. This has been shown most simply by Gerhard Hegerfeldt. The result has also been confirmed experimentally.

More careful studies show that the result depends on the type of superposition of the two atoms at the beginning: coherent or incoherent. For incoherent superpositions, the intuitive result is correct; the counter-intuitive result appears only for coherent superpositions. Again, a careful discussion shows that no real non-locality of energy is involved.

In summary, faster-than-light speeds in wave function collapse do not contradict the limit on energy speed of special relativity. Collapse speeds are phase velocities. In nature, phase velocities are unlimited; unlimited phase velocities never imply energy transport faster than light. In addition, we recover the result that physical systems are only clearly defined if they interact incoherently with their environment.

Curiosities and fun challenges about superpositions

Some people wrongly state that atoms in a superposition of two states centred at different positions can be photographed. (This lie is even used by some sects to attract believers.) Why is this not true?

* *

In a few cases, the superposition of different macroscopic states can actually be observed by lowering the temperature to sufficiently small values and by carefully choosing suitably small masses or distances. Two well-known examples of coherent superpositions are those observed in gravitational wave detectors and in Josephson junctions. In the

first case, one observes a mass as heavy as 1000 kg in a superposition of states located at different points in space: the distance between them is of the order of 10^{-17} m. In the second case, in superconducting rings, superpositions of a state in which a macroscopic current of the order of 1 pA flows in clockwise direction with one where it flows in counter-clockwise direction have been produced.

* *

Superpositions of magnetization in up and down direction at the same time have been observed for several materials.

* *

Since the 1990s, the sport of finding and playing with new systems in coherent mac-
roscopic superpositions has taken off across the world. The challenges lie in the clean experiments necessary. Experiments with single atoms in superpositions of states are
among the most popular ones.

* *

In 1997, coherent atom waves were extracted from a cloud of sodium atoms.

* *

Macroscopic objects usually are in incoherent states. This is the same situation as for light. The world is full of 'macroscopic', i.e., incoherent light: daylight, and all light from lamps, from fire and from glow-worms is incoherent. Only very special and carefully constructed sources, such as lasers or small point sources, emit coherent light. Only these sources allow studying interference effects. In fact, the terms 'coherent' and 'incoherent' originated in optics, since for light the difference between the two, namely the capacity to interfere, had been observed centuries before the case of matter.

Coherence and incoherence of light and of matter manifest themselves differently, because matter can stay at rest but light cannot and because matter is made of fermions,
but light is made of bosons. Coherence can be observed easily in systems composed of bosons, such as light, sound in solids, or electron pairs in superconductors. Coherence is less easily observed in systems of fermions, such as systems of atoms with their electron clouds. However, in both cases a decoherence time can be defined. In both cases coherence in many particle systems is best observed if all particles are in the same state (superconductivity, laser light) and in both cases the transition from coherent to incoherent is due to the interaction with a bath. A beam is thus incoherent if its particles arrive randomly in time and in frequency. In everyday life, the rarity of observation of coherent matter superpositions has the same origin as the rarity of observation of coherent light.

* *

We will discuss the relation between the environment and the *decay* of unstable systems
later on. The phenomenon is completely described by decoherence.

* *

Can you find a method to measure the *degree* of entanglement? Can you do so for a system made of many particles?

* *

The study of entanglement leads to a simple conclusion: *teleportation contradicts correlation*. Can you confirm the statement?

* *

Are ghost images in TV sets, often due to spurious reflections, examples of interference?

* *

What happens when two monochromatic electrons overlap?

* *

Some people say that quantum theory could be used for quantum computing, by using coherent superpositions of wave functions. Can you give a general reason that makes this aim very difficult – even though not impossible – even without knowing how such a quantum computer might work, or what the so-called *qubits* might be?

Why do probabilities and wave function collapse appear in measurements?

Measurements in quantum mechanics are puzzling also because they lead to statements in which *probabilities* appear. For example, we speak about the probability of finding an electron at a certain distance from the nucleus of an atom. Statements like this belong to the general type 'when the observable A is measured, the probability to find the outcome a is p.' In the following we will show that the probabilities in such statements are inevitable for any measurement, because, as we will show, (1) any measurement and any observation is a special case of decoherence or disentanglement process and (2) all decoherence processes imply the quantum of action. (Historically, the process of measurement was studied *before* the more general process of decoherence. That explains in part why the topic is so confused in many peoples' minds.)

What is a measurement? As already mentioned earlier on, a measurement is any interaction which produces a record or a memory. (Any effect of everyday life is a record; but this is not true in general. Can you give some examples of effects that are records and some effects which are not?) Measurements can be performed by machines; when they are performed by people, they are called *observations*. In quantum theory, the process of measurement is not as straightforward as in classical physics. This is seen most strikingly when a quantum system, such as a single electron, is first made to pass a diffraction slit, or better – in order to make its wave aspect become apparent – a double slit and then is made to hit a photographic plate, in order to make also its particle aspect appear. Experiment shows that the blackened dot, the spot where the electron has hit the screen, cannot be determined in advance. (The same is true for photons or any other particle.) However, for large numbers of electrons, the spatial distribution of the black dots, the so-called *diffraction pattern*, can be calculated in advance with high precision.

The outcome of experiments on microscopic systems thus forces us to use probabilities for the description of microsystems. We find that the probability distribution $p(\boldsymbol{x})$ of the spots on the photographic plate can be calculated from the wave function ψ of the electron at the screen surface and is given by $p(\boldsymbol{x}) = |\psi^{\dagger}(\boldsymbol{x})\psi(\boldsymbol{x})|^2$. This is in fact a special
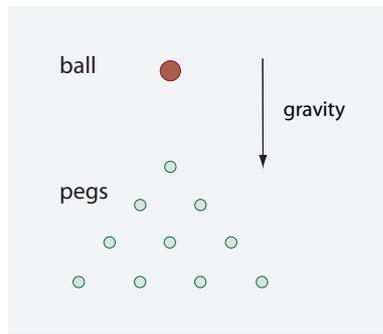
FIGURE 75 A system showing probabilistic behaviour: ball falling through an array of pegs.

case of the general *first property of quantum measurements*:

> ▷ The measurement of an observable A for a system in a state ψ gives as result one of the eigenvalues a_n, and the probability P_n to get the result a_n is given by

$$P_n = |\varphi_n^\dagger \psi|^2 \,, \tag{79}$$

> where φ_n is the eigenfunction of the operator A corresponding to the eigenvalue a_n.*

Experiments also show a *second property of quantum measurements:*

> ▷ After a measurement, the observed quantum system is in the state φ_n corresponding to the measured eigenvalue a_n. One also says that during the measurement, the wave function has *collapsed* from ψ to φ_n.

Ref. 119

These two experimental properties can also be generalized to the more general cases with degenerate and continuous eigenvalues.

Obviously, the experimental results on the measurement process require an explanation. At first sight, the sort of probabilities encountered in quantum theory are different from the probabilities we encounter in everyday life. Take roulette, dice, the system shown in Figure 75, pachinko machines or the direction in which a pencil on its tip falls: all have been measured experimentally to be random (assuming no cheating by the designer or operators) to a high degree of accuracy. These everyday systems do not puzzle us. We unconsciously assume that the random outcome is due to the small, but uncontrollable variations of the starting conditions or the environment every time the experiment is repeated.**

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* All linear transformations transform some special vectors, called *eigenvectors* (from the German word *eigen* meaning 'self') into multiples of themselves. In other words, if T is a transformation, e a vector, and

$$T(e) = \lambda e \tag{80}$$

where λ is a scalar, then the vector e is called an *eigenvector* of T, and λ is associated *eigenvalue*. The set of all eigenvalues of a transformation T is called the *spectrum* of T.
** To get a feeling for the limitations of these unconscious assumptions, you may want to read the already

But microscopic systems seem to be different. The two properties of quantum measurements just mentioned express what physicists observe in every experiment, even if the initial conditions are taken to be *exactly* the same every time. But why then is the position for a single electron, or most other observables of quantum systems, not predictable? In other words, what happens during the collapse of the wave function? How long does the collapse take? In the beginning of quantum theory, there was the perception that the observed unpredictability is due to the lack of information about the state of the particle. This lead many to search for so-called 'hidden variables'. All these attempts were doomed to fail, however. It took some time for the scientific community to realize that the unpredictability is *not* due to the lack of information about the state of the particle, which is indeed described *completely* by the state vector ψ.

In order to uncover the origin of probabilities, let us recall the nature of a measurement, or better, of a general observation.

▷ Any observation is the production of a record.

The record can be a visual or auditive memory in our brain, or a written record on paper, or a tape recording, or any such type of object. As explained in the previous volume, an object is a record if it cannot have arisen or disappeared by chance. To avoid the influence of chance, all records have to be protected as much as possible from the *external* world; e.g. one typically puts archives in earthquake safe buildings with fire protection, keeps documents in a safe, avoids brain injury as much as possible, etc.

On top of this, records have to be protected from their *internal* fluctuations. These internal fluctuations are due to the many components any recording device is made of. If the fluctuations were too large, they would make it impossible to distinguish between the possible contents of a memory. Now, fluctuations decrease with increasing size of a system, typically with the square root of the size. For example, if a hand writing is too small, it is difficult to read if the paper gets brittle; if the magnetic tracks on tapes are too small, they demagnetize and lose the stored information. In other words, a record is rendered stable against internal fluctuations by making it of sufficient size. Every record thus consists of many components and shows small fluctuations.

The importance of size can be expressed in another way: every system with memory, i.e., every system capable of producing a record, contains a *bath*. In summary, the statement that any observation is the production of a record can be expressed more precisely as:

▷ Any observation of a system is the result of an interaction between that system and a bath in the recording apparatus.

By the way, since baths imply friction, we can also say: memory needs friction. In addition, any observation measuring a physical quantity uses an interaction *depending* on that same quantity. With these seemingly trivial remarks, we can describe in more detail

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mentioned story of those physicists who built a machine that could predict the outcome of a roulette ball from the initial velocity imparted by the croupier.

the quantum
mechanical
system

apparatus, e.g. eye, ear,
or machine, with memory,
i.e. coupled to a bath

H

H $_{\text{int}}$

t $_{\text{r}}$

describes
its possible
states

is determined
by the type
of measurement

describes its
friction, e.g.
due to heat flow
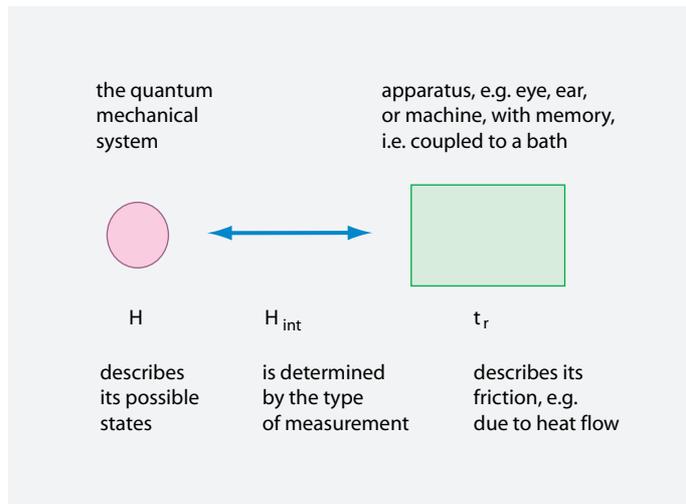
the process of observation, or, as it is usually called in the quantum theory, the measurement process.

Any measurement *apparatus*, or *detector*, is characterized by two main aspects, shown in Figure 76: the interaction it has with the microscopic system, and the bath it contains to produce the record. Any description of the measurement process thus is the description of the evolution of the microscopic system *and* the detector; therefore one needs the Hamiltonian for the particle, the interaction Hamiltonian, and the bath properties (such as the relaxation time t_{r}). The interaction specifies what is measured and the bath realizes the memory.

We know that only classical thermodynamic systems can be irreversible; quantum systems are not. We therefore conclude: a measurement system *must* be described classically: otherwise it would have no memory and would not be a measurement system: it would not produce a record! Memory is a classical effect. (More precisely, memory is an effect that only appears in the classical limit.) Nevertheless, let us see what happens if we describe the measurement system quantum mechanically.

Let us call A the observable which is measured in the experiment and its eigenfunctions φ_n. We describe the quantum mechanical system under observation – often a particle – by a state ψ. The full state of the system can always be written as

$$\psi = \psi_p\,\psi_{\text{other}} = \sum_n c_n\varphi_n\,\psi_{\text{other}}\;.\tag{81}$$

Here, ψ_p is the aspect of the (particle or system) state that we want to measure, and ψ_{other} represents all other degrees of freedom, i.e., those not described – *spanned*, in mathematical language – by the operator A corresponding to the observable we want to measure. The numbers $c_n = |\varphi_n^\dagger\psi_p|$ give the expansion of the state ψ_p, which is taken to be normalized, in terms of the basis φ_n. For example, in a typical position measurement, the functions φ_n would be the position eigenfunctions and ψ_{other} would contain the information about the momentum, the spin and all other properties of the particle.

How does the system–detector interaction look like? Let us call the state of the apparatus before the measurement χ_{start}. The measurement apparatus itself, by definition, is a device which, when it is hit by a particle in the state $\varphi_n \psi_{\text{other}}$, changes from the state χ_{start} to the state χ_n. One then says that the apparatus has *measured* the eigenvalue a_n corresponding to the eigenfunction φ_n of the operator A. The index n is thus the record of the measurement; it is called the *pointer* index or variable. This index tells us in which state the microscopic system was before the interaction. The important point, taken from our previous discussion, is that the states χ_n, being records, are macroscopically distinct, precisely in the sense of the previous section. Otherwise they would not be records, and the interaction with the detector would not be a measurement.

Of course, during measurement, the apparatus sensitive to φ_n changes the part ψ_{other} of the particle state to some other situation $\psi_{\text{other},n}$, which depends on the measurement and on the apparatus; we do not need to specify it in the following discussion.* But let us have an intermediate check of our reasoning. Do apparatuses as described here exist? Yes, they do. For example, any photographic plate is a detector for the position of ionizing particles. A plate, and in general any apparatus measuring position, does this by changing its momentum in a way depending on the measured position: the electron on a photographic plate is stopped. In this case, χ_{start} is a white plate, φ_n would be a particle localized at spot n, χ_n is the function describing a plate blackened at spot n and $\psi_{\text{other},n}$ describes the momentum and spin of the particle after it has hit the photographic plate at the spot n.

Now we are ready to look at the measurement process itself. For the moment, let us disregard the bath in the detector, and let us just describe it with a state as well, which we call χ_{start}. In the time before the interaction between the particle and the detector, the combined system (including the detector) was in the initial state ψ_i given simply by

$$\psi_i = \psi_p \chi_{\text{start}} = \sum_n c_n \varphi_n \psi_{\text{other}} \chi_{\text{start}} \, , \tag{84}$$

where ψ_p is the (particle or system) state. After the interaction, using the just mentioned, experimentally known characteristics of the apparatus, the combined state ψ_a is

$$\psi_a = \sum_n c_n \varphi_n \psi_{\text{other},n} \chi_n \, . \tag{85}$$

This evolution from ψ_i to ψ_a follows from the evolution equation applied to the particle–detector combination. Now, the combined state ψ_a is a superposition of macroscopically

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* How does the interaction look like mathematically? From the description we just gave, we specified the final state for every initial state. Since the two density matrices are related by

$$\rho_{\text{f}} = T \rho_{\text{i}} T^{\dagger} \tag{82}$$

we can deduce the Hamiltonian from the matrix T. Are you able to see how?

Challenge 135 ny

By the way, one can say in general that an apparatus measuring an observable A has a system interaction Hamiltonian depending on the pointer variable A, and for which one has

$$[H + H_{\text{int}}, A] = 0 \, . \tag{83}$$

distinct states: it is a superposition of distinct macroscopic states of the detector. In our example ψ_a could correspond to a superposition of one state where a spot on the left upper corner is blackened on an otherwise white plate with another state where a spot on the right lower corner of the otherwise white plate is blackened. Such a situation is never observed. Let us see why.

The density matrix ρ_a of the combined state ψ_a after the measurement given by

$$\rho_a = \psi_a \otimes \psi_a^{\dagger} = \sum_{n,m} c_n c_m^* (\varphi_n \psi_{\text{other},n} \chi_n) \otimes (\varphi_m \psi_{\text{other},m} \chi_m)^{\dagger} \ , \tag{86}$$

contains large non-diagonal terms, i.e., terms for $n \neq m$, whose numerical coefficients are different from zero. Now let us take the bath back in. From the previous section we know the effect of a bath on such a macroscopic superposition. We found that a density matrix such as ρ_a decoheres extremely rapidly. We assume here that the decoherence time is negligibly small.* After decoherence, the off-diagonal terms vanish, and only the final, diagonal density matrix ρ_{f}, given by

$$\rho_{\text{f}} = \sum_n |c_n|^2 (\varphi_n \psi_{\text{other},n} \chi_n) \otimes (\varphi_n \psi_{\text{other},n} \chi_n)^{\dagger} \tag{87}$$

remains and has experimental relevance. As explained above, such a density matrix describes a mixed state, and the numbers $P_n = |c_n|^2 = |\varphi_n^{\dagger} \psi_p|^2$ give the probability of measuring the value a_n and of finding the particle in the state $\varphi_n \psi_{\text{other,n}}$ as well as the detector in the state χ_n. But this is precisely what the two properties of quantum measurements state.

We therefore find that describing a measurement as an evolution of a quantum system interacting with a macroscopic detector, itself containing a bath, we can *deduce* the two properties of quantum measurements, probabilistic outcomes and the collapse of the wave function, from the quantum mechanical evolution equation. The decoherence time t_{d} of the previous section becomes the time of collapse for the case of a measurement; in addition we find

$$t_{\text{collapse}} = t_{\text{d}} < t_{\text{r}} \ . \tag{88}$$

In other words, the collapse time is always smaller than the relaxation time of the bath. We thus have a formula for the time the wave function takes to collapse. All experimental measurements of the time of collapse have confirmed this result.

Ref. 104

WHY IS $\hbar$ NECESSARY FOR PROBABILITIES?

At first sight, one could argue that the two properties of quantum measurements do not contain $\hbar$, and thus are not consequences of quantum theory. However, this argument is incorrect.

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* Note however, that an *exactly* vanishing decoherence time, which would mean a *strictly* infinite number of degrees of freedom of the bath or the environment, is in contradiction with the evolution equation, and in particular with unitarity, locality and causality. It is essential in the whole argument not to confuse the logical consequences of a extremely small decoherence time with those of an exactly vanishing decoherence time.

Decoherence is a quantum process, because $\hbar$ appears in the expression of the decoherence time. Since the collapse of the wave function is based on decoherence, it is a quantum process as well. Also probabilities are due to the quantum of action.

In addition, we have seen that the concept of wave function appears only because the quantum of action $\hbar$ is not zero. Wave functions, their collapse and probabilities are due to the quantum of change $\hbar$.

These results recall a statement made earlier on: probabilities appear whenever an experiment attempts to detect changes, i.e., action values, smaller than $\hbar$. Most puzzles

around measurement are due to such attempts. However, nature does not allow such measurements; in every such attempt, probabilities appear.

Hidden variables

A large number of people are not satisfied with the explanation of probabilities in the quantum world. They long for more mystery in quantum theory. They do not like the idea that probabilities are due to baths and to the quantum of action. The most famous prejudice such people cultivate is the idea that the probabilities are due to some hidden aspect of nature which is still unknown to humans. Such imagined, unknown aspects are called *hidden variables*.

The beautiful thing about quantum mechanics is that it allows both conceptual and experimental tests on whether such hidden variables exist – without the need of knowing them. Obviously, hidden variables controlling the evolution of microscopic system would contradict the statement that action values below $\hbar$ cannot be detected. The smallest observable action value is the reason for the random behaviour of microscopic systems. The smallest action thus *excludes* hidden variables. But let us add some more detailed arguments.

Historically, the first, somewhat abstract argument against hidden variables was given by John von Neumann.* An additional no-go theorem for hidden variables was pub-

lished by Kochen and Specker in 1967, and independently by John Bell in 1969. The theorem states:

> ▷ Non-contextual hidden variables are impossible, if the Hilbert space has a dimension equal or larger than three.

The theorem is about *non-contextual* variables, i.e., about hidden variables *inside* the quantum mechanical system. The Kochen–Specker theorem thus states that there is no non-contextual hidden variables model, because mathematics forbids it. This result essentially eliminates all possibilities for hidden variables, because usual quantum mechanical systems have Hilbert space dimensions larger than three.

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* János Neumann (b. 1903 Budapest, d. 1957 Washington DC) influential mathematician. One of the greatest and clearest scientific minds of the twentieth century, he settled many issues, especially in applied mathematics and quantum theory, that others still struggle with today. He then worked on the atomic and the hydrogen bomb, on ballistic missiles, and on general defence problems. For the bomb research, he strongly influenced the building of the earliest electronic computers, extending the ideas of Konrad Zuse. At the end of his life, he wanted to change the weather with nuclear bombs. He died of a cancer that was due to his exposure to nuclear radiation when watching bomb tests.

We cannot avoid noting that there are no restricting theorems about *contextual* hidden variables, i.e., variables in the environment and in particular, in the baths contained in it. Indeed, their necessity was shown above!

Also common sense eliminates hidden variables, without any recourse to mathematics, with a simple argument. If a quantum mechanical system had internal hidden variables, the measurement apparatus would have zillions of them.[*] And this would mean that it could not work as a measurement system.

Despite all arguments, researchers have always been looking for experimental tests on hidden variables. Most tests are based on the famed *Bell's inequality*, a beautifully simple relation published by John Bell[**] in the 1960s.

Can we distinguish quantum theory and locally realistic theories that use hidden variables? Bell's starting idea is to do so by measuring the polarizations of two correlated photons. Quantum theory says that the polarization of the photons is fixed only at the time it is measured, whereas local realistic models – the most straightforward type of hidden variable models – claim that the polarization is fixed already in advance by a hidden variable. As Bell found out, experiments can be used to decide which alternative is correct.

Imagine that the polarization is measured at two distant points A and B. Each observer can measure 1 or -1 in each of his favourite direction. Let each observer choose two directions, 1 and 2, and call their results a_1, a_2, b_1 and b_2. Since the measurement results all are either 1 or -1, the value of the specific expression $(a_1 + a_2)b_1 + (a_2 - a_1)b_2$ has always the value ± 2.

Imagine that you repeat the experiment many times, assuming that the hidden variables appear statistically. You then can deduce (a special case of) Bell's inequality for two *hidden variables*; it predicts that

$$|(a_1 b_1) + (a_2 b_1) + (a_2 b_2) - (a_1 b_2)| \leqslant 2 . \tag{89}$$

Here, the expressions in brackets are the averages of the measurement products over a large number of samples. This hidden variable prediction holds independently of the directions of the involved polarizers.

On the other hand, for the case that the polarizers 1 and 2 at position A and the corresponding ones at position B are chosen with angles of $\pi/4$, *quantum theory* predicts that

$$|(a_1 b_1) + (a_2 b_1) + (a_2 b_2) - (a_1 b_2)| = 2\sqrt{2} > 2 . \tag{90}$$

This prediction is in complete contradiction with the hidden variable prediction.

Now, all experimental checks of Bell's inequality have confirmed standard quantum mechanics and falsified hidden variables. There are no exceptions.

Another measurable contradiction between quantum theory and locally realistic theories has been predicted by Greenberger, Horn and Zeilinger in systems with three en-
tangled particles. Again, quantum theory has been confirmed in all experiments.

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[*] Which leads to the definition: one zillion is 10^{23}.
[**] John Stewart Bell (1928–1990), theoretical physicist who worked mainly on the foundations of quantum theory.

In summary, *hidden variables do not exist.* Of course, this is not really surprising. The search for hidden variables is based on a misunderstanding of quantum mechanics or on personal desires on how the world should be, instead of taking it as it is: there is a smallest measurable action value, $\hbar$, in nature.

Summary on probabilities and determinism

> « Geometrica demonstramus quia facimus; si
> physica demonstrare possemus, faceremus.
>   Giambattista Vico* »

We draw a number of conclusions which we need for the rest of our mountain ascent. Note that these conclusions, even though in agreement with all experiments, are not yet shared by all physicists! The whole topic is a problem for people who prefer ideology to facts.

In everyday life, probabilities often do not appear or are not noted. Quantum theory shows:

▷ Probabilities appear whenever a process tries to distinguish between situations that differ by about one quantum of action $\hbar$.
▷ The precise mechanism for the appearance of probabilities is due to the involved baths.

In short: probabilities appear whenever an experiment tries to distinguish between *close situations*. In more detail:

— Probabilities do not appear in measurements because the state of the quantum system is unknown or fuzzy, but because the detailed state of an interacting bath in the environment is unknown. Quantum mechanical probabilities are of statistical origin and are due to baths in the environment or in the measurement apparatus, in combination with the quantum of action $\hbar$. The probabilities are due to the large number of degrees of freedom contained in the bath. These large numbers make the outcome of experiments – especially those whose possible outcomes differ by about $\hbar$ – unpredictable. If the state of the involved bath were known, the outcome of an experiment could be predicted. The probabilities of quantum theory are due to the quantum of action and are 'thermodynamic' in origin.

In other words, there are *no fundamental* probabilities in nature. All probabilities in nature are due to decoherence; in particular, all probabilities are due to the statistics of the many particles – some of which may even be virtual – that are part of the baths in the environment. Modifying well-known words by Albert Einstein, we can agree on the following: 'nature does not play dice.' Therefore we called ψ the *wave function* – instead of 'probability amplitude', as is often done. An even better term would be *state function*.

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* 'We are able to demonstrate geometrical matters because we make them; if we could prove physical matters we would be able to make them.' Giovanni Battista Vico (b. 1668 Napoli, d. 1744 Napoli) important philosopher and thinker. In this famous statement he points out a fundamental distinction between mathematics and physics.

— Every observation in everyday life is a special case of decoherence. What is usually called the 'collapse of the wave function' is a decoherence process due to the interaction with the baths present in the environment or in the measuring apparatus. Because humans are warm-blooded and have memory, humans themselves are measurement apparatuses. The fact that our body temperature is 37°C is thus the reason that we see only a single world, and no superpositions. (Actually, there are many additional reasons; can you name a few?)

— Every measurement is complete when the microscopic system has interacted with the bath in the measuring apparatus. Quantum theory as a description of nature does not require detectors; the evolution equation describes all examples of motion. However, *measurements* do require the existence of detectors. A detector, or measurement apparatus, is a machine that records observations. Therefore, it has to include a bath, i.e., has to be a classical, macroscopic object. In this context one speaks also of a *classical apparatus*. This necessity of the measurement apparatus to be classical had been already stressed in the very early stages of quantum theory.

— All measurements, being decoherence processes that involve interactions with baths, are irreversible processes and increase entropy.

— Every measurement, like every example of decoherence, is a special case of quantum mechanical evolution, namely the evolution for the combination of a quantum system, a macroscopic detector and a bath. Since the evolution equation is relativistically invariant, no causality problems appear in measurements; neither do locality problems or logical problems appear.

— Since both the evolution equation and the measurement process do not involve quantities other than space-time, Hamiltonians, baths and wave-functions, no other quantity plays a role in measurement. In particular, no human observer nor any consciousness is involved or necessary. Every measurement is complete when the microscopic system has interacted with the bath in the apparatus. The decoherence inherent in every measurement takes place even if nobody is looking. This trivial consequence is in agreement with the observations of everyday life, for example with the fact that the Moon is orbiting the Earth even if nobody looks at it.* Similarly, a tree falling in the middle of a forest makes noise even if nobody listens. Decoherence is independent of human observation, of the human mind and of human existence.

— In every measurement the quantum system interacts with the detector. Since there is a minimum value for the magnitude of action, every *observation influences the observed*. Therefore every measurement *disturbs* the quantum system. Any precise description of observations must also include the description of this disturbance. In the present section the disturbance was modelled by the change of the state of the system from ψ_{other} to $\psi_{\text{other,n}}$. Without such a change of state, without a disturbance of the quantum system, a measurement is impossible.

— Since the complete measurement process is described by quantum mechanics, unitarity is and remains the basic property of evolution. There are no non-unitary processes in quantum mechanics.

— The description of the collapse of the wave function as a decoherence process is an

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* The opposite view is sometimes falsely attributed to Niels Bohr. The Moon is obviously in contact with many radiation baths. Can you list a few?

explanation exactly in the sense in which the term 'explanation' was defined earlier on; it describes the relation between an observation and all the other aspects of reality, in this case the bath in the detector or the environment. The collapse of the wave function has been measured, calculated and *explained*. The collapse is not a question of 'interpretation', i.e., of opinion, as unfortunately often is suggested.*

— It is not useful to speculate whether the evolution for a *single* quantum measurement could be determined if the state of the environment around the system were known. Measurements need baths. But a bath is, to an excellent approximation, irreversible and thus cannot be described by a wave function, which behaves reversibly.**

In short:

▷ Quantum mechanics is deterministic.
▷ Baths are probabilistic.
▷ Baths are probabilistic because of the quantum of action.

In summary, there is *no* irrationality in quantum theory. Whoever uses quantum theory as argument for superstitions, irrational behaviour, new age beliefs or ideologies is guilty of disinformation. The statement by Gell-Mann at the beginning of this chapter is such an example. Another is the following well-known, but incorrect statement by Richard Feynman:

... nobody understands quantum mechanics.

Nobel Prizes obviously do not prevent views distorted by ideology. The correct statement is:

▷ The quantum of action and decoherence are the key to understanding quantum theory.

In fact, $\hbar$ and decoherence allow clarifying many other issues. We explore a few interesting ones.

WHAT IS THE DIFFERENCE BETWEEN SPACE AND TIME?

Space and time differ. Objects are localized in space but not in time. Why is this the case? In nature, most bath–system interactions are mediated by a potential. All potentials are by definition position dependent. Therefore, every potential, being a function of the position x, commutes with the position observable (and thus with the interaction Hamiltonian). The decoherence induced by baths – except if special care is taken – thus first of all destroys the non-diagonal elements for every superposition of states centred

* This implies that the so-called 'many worlds' interpretation is wishful thinking. The conclusion is confirmed when studying the details of this religious approach. It is a belief system, not based on facts.
** This very strong type of determinism will be very much challenged in the last part of this text, in which it will be shown that time is not a fundamental concept, and therefore that the debate around determinism looses most of its interest.

at different locations. In short, *objects are localized because they interact with baths via potentials.*

For the same reason, objects also have only one spatial orientation at a time. If the system–bath interaction is spin-dependent, the bath leads to 'localization' in the spin variable. This occurs for all microscopic systems interacting with magnets. As a result, macroscopic superpositions of magnetization are almost never observed. Since electrons, protons and neutrons have a magnetic moment and a spin, this conclusion can even be extended: everyday objects are never seen in superpositions of different rotation states because their interactions with baths are spin-dependent.

As a counter-example, most systems are not localized in time, but on the contrary exist for very long times, because practically all system–bath interactions do *not* commute with time. In fact, this is the way a bath is defined to begin with. In short, *objects are permanent because they interact with baths.*

Are you able to find an interaction which is momentum-dependent instead of position-dependent? What is the consequence for macroscopic systems?

In other words, in contrast to general relativity, quantum theory produces a distinction between space and time. In fact, we can *define* position as the observable that commutes with interaction Hamiltonians. This distinction between space and time is due to the properties of matter and its interactions. We could not have deduced this distinction in general relativity.

Are we good observers?

Are humans classical apparatuses? Yes, they are. Even though several prominent physicists claim that free will and probabilities are related, a detailed investigation shows that this in not the case. Our senses are classical machines because they obey their definition: human senses record observations by interaction with a bath. Our brain is also a classical apparatus: the neurons are embedded in baths. Quantum probabilities do *not* play a determining role in the brain.

Any observing entity, be it a machine or a human being, needs a bath and a memory to record its observations. This means that observers have to be made of matter; an observer cannot be made of radiation. Our description of nature is thus severely biased: we describe it from the standpoint of matter. That is a bit like describing the stars by putting the Earth at the centre of the universe: we always put matter at the centre of our description. Can we eliminate this basic anthropomorphism? We will find out as we continue our adventure.

What relates information theory, cryptology and quantum theory?

Physics means talking about observations of nature. Like any observation, also measurements produce information. It is thus possible to translate much (but not all) of quantum theory into the language of information theory. In particular, the existence of a smallest change value in nature implies that the information about a physical system can never be complete, that information transport has its limits and that information can never be fully trusted. The details of these studies form a fascinating way to look at the microscopic world.

The analogy between quantum theory and information theory becomes even more interesting when the statements are translated into the language of cryptology. Cryptology is the science of transmitting hidden messages that only the intended receiver can decrypt. In our modern times of constant surveillance, cryptology is an important tool to protect personal freedom.*

The quantum of action implies that messages can be sent in an (almost) safe way. Listening to a message is a measurement process. Since there is a smallest action $\hbar$, we can detect whether somebody has tried to listen to a message that we sent. To avoid a man-in-the-middle attack – somebody who pretends to be the receiver and then sends a copy of the message to the real, intended receiver – we can use *entangled* systems as signals or messages to transmit the information. If the entanglement is destroyed, somebody has listened to the message. Usually, quantum cryptologists use communication systems based on entangled photons.

The major issue of *quantum cryptology*, a large modern research field, is the key distribution problem. All secure communication is based on a secret key that is used to decrypt the message. Even if the communication channel is of the highest security – like entangled photons – one still has to find a way to send the communication partner the secret key necessary for the decryption of the messages. Finding such methods is the main aspect of quantum cryptology. However, close investigation shows that all key exchange methods are limited in their security.

In short, due to the quantum of action, nature provides limits on the possibility of sending encrypted messages. The statement of these limits is (almost) equivalent to the statement that change in nature is limited by the quantum of action.

Is the universe a computer?

The quantum of action provides a limit to secure information exchange. This connection allows us to brush aside several incorrect statements often found in the media. Stating that 'the universe is information' or that 'the universe is a computer' is as reasonable as saying that the universe is an observation or a chewing-gum dispenser. Any expert of motion should beware of these and similarly fishy statements; people who use them either deceive themselves or try to deceive others.

Does the universe have a wave function? And initial conditions?

The wave function of the universe is frequently invoked in discussions about quantum theory. Various conclusions are deduced from this idea, for example on the irreversibility of time, on the importance of initial conditions, on changes required to quantum theory and much more. Are these arguments correct?

The first thing to clarify is the meaning of 'universe'. As explained already, the term can have two meanings: either the collection of all matter and radiation, or this collection *plus* all of space-time. Let us also recall the meaning of 'wave function': it describes the

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* Cryptology consists of the field of *cryptography*, the art of coding messages, and the field of *cryptoanalysis*, the art of deciphering encrypted messages. For a good introduction to cryptology, see the text by Albrecht Beutelspacher, Jörg Schwenk & Klaus-Dieter Wolfenstätter, *Moderne Verfahren der Kryptographie*, Vieweg 1995.

state of a system. The state distinguishes two otherwise identical systems; for example, position and velocity distinguish two otherwise identical ivory balls on a billiard table. Alternatively and equivalently, the state describes changes in time.

Does the universe have a state? If we take the wider meaning of universe, it does not.

Talking about the state of the universe is a contradiction: by definition, the concept of state, defined as the non-permanent aspects of an object, is applicable only to *parts* of the universe.

We then can take the narrower sense of 'universe' – the sum of all matter and radiation only – and ask the question again. To determine the state of all matter and radiation, we need a possibility to measure it: we need an environment. But the environment of matter and radiation is space-time only; initial conditions cannot be determined since we need measurements to do this, and thus an apparatus. An apparatus is a material system with a bath attached to it; however, there is no such system outside the universe.

In short, quantum theory does not allow for measurements of the universe.

▷ The universe has no state.

Beware of anybody who claims to know something about the wave function of the universe. Just ask him Wheeler's question: If you know the wave function of the universe, why aren't you rich?

Despite this conclusion, several famous physicists have proposed evolution equations for the wave function of the universe. (The best-known is, ironically, the Wheeler–
DeWitt equation.) It seems a silly point, but not one prediction of these equations has been compared to experiment; the arguments just given even make this impossible in principle. Exploring such equations, so interesting it may seem at first sight, must therefore be avoided if we want to complete our adventure and avoid getting lost in false beliefs.

There are many additional twists to this story. One is that space-time itself, even without matter, might be a bath. This speculation will be shown to be correct in the last volume of this adventure. The result seems to allow speaking of the wave function of the universe. But then again, it turns out that time is *undefined* at the scales where space-time is an effective bath; this again implies that the concept of state is not applicable there.

A lack of 'state' for the universe is a strong statement. It also implies a lack of initial conditions! The arguments are precisely the same. This is a tough result. We are so used to think that the universe has initial conditions that we never question the term. (Even in this text the mistake might appear every now and then.) But there are no initial conditions for the universe.

We can retain as summary, valid even in the light of the latest research: The universe is *not* a system, has *no* wave function and *no* initial conditions – independently of what is meant by 'universe'.
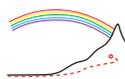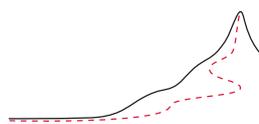
COLOURS AND OTHER INTERACTIONS BETWEEN LIGHT AND MATTER

> " Rem tene; verba sequentur.** "
>
> Cato

Stones and all other objects have colours. Why? In other words, what is the specific way in which charged quantum particles that are found inside stones and inside all other objects interact with electromagnetic fields? In this chapter, we first give an overview of the various ways that colours in nature result from the quantum of action, i.e., from the interaction between matter quantons and photons. Then we explore the simplest such system: we show how the quantum of action leads to the various colours produced by hydrogen atoms. After this, we discover that the interaction between matter and radiation leads to other surprising effects, especially when special relativity is taken into account.

THE CAUSES OF COLOUR

Quantum theory explains all colours in nature. Indeed, all the colours that we observe are due to electrically *charged* particles. More precisely, colours are due to the interactions of charged particles with photons. All colours are thus quantum effects.

So far, we have explored the motion of quantons that are described by mass only. Now we study the motion of particles that are electrically charged. The charged particles at the basis of most colours are electrons and nuclei, including their composites, from ions, atoms and molecules to fluids and solids. Many colour issues are still topic of research. For example, until recently it was unclear why exactly asphalt is black. The exact structure of the chemical compounds, the *asphaltenes*, that produce the very dark brown colour was unknown. Only recent research has settled this question. In fact, the development of new colourants and colour effects is an important part of modern industry.

An overview of the specific mechanisms that generate colour is given in the following table. The table includes all colours that appear in everyday life. (Can you find one that is missing?)

** 'Know the subject and the words will follow.' Marcus Porcius Cato, (234–149 BCE) or Cato the elder, Roman politician famous for his speeches and his integrity.

TABLE 7 Causes of colour.

COLOUR TYPE	EXAMPLE	DETAILS

Class I: Colours due to simple excitations
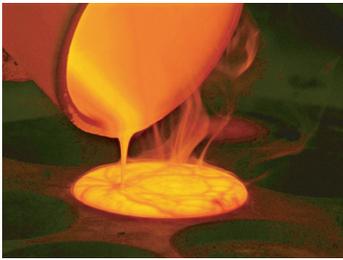
1. Incandescence and free charge radiation

| Carbon arc lamp, hot steel, lightbulb wire, most stars, magma, lava, hot melts | Colours are due to continuous spectrum emitted by all hot matter; colour sequence, given by Wien's rule, is black, red, orange, yellow, white, blue-white (molten lead and silver © Graela) |

| Wood fire, candle | Wood and wax flames are yellow due to incandescence if carbon-rich and oxygen-poor |
| White fireworks, flashlamp, sparklers | Due to metals burning to oxide at high temperature, such as magnesium, zinc, iron, aluminium or zirconium (sparkler © Sarah Domingos) |

| Nuclear reactors, synchroton light sources, free electron lasers | Due to fast free charges: Vavilov–Čerenkov radiation is due to speed of particle larger than the speed of light in matter, Bremsstrahlung is due to the deceleration of charged particles (nuclear reactor core under water, courtesy NASA) |
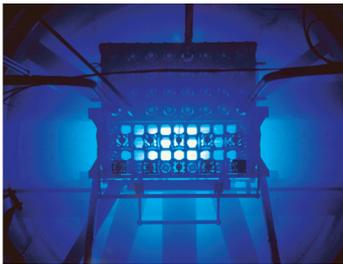
2. Atomic gas excitations

| Red neon lamp, blue argon lamp, UV mercury lamp, yellow sodium street lamps, most gas lasers, metal vapour lasers, some fluorescence | Colours are due to transitions between atomic energy levels (gas discharges © Pslawinski) |

| Aurora, triboluminescence in scotch tape, crystalloluminescence in strontium bromate | In air, blue and red colours are due to atomic and molecular energy levels of nitrogen, whereas green, yellow, orange colours are due to oxygen (aurora © Jan Curtis) |

| Lightning, arcs, sparks, coloured fireworks, most coloured flames, some electroluminescence | Colour lines are due to energy levels of highly excited atoms (flames of K, Cu, Cs, B, Ca © Philip Evans) |

TABLE 7 Causes of colour (continued).

COLOUR TYPE	EXAMPLE	DETAILS
	3. Vibrations and rotations of molecules	
	Bluish water, blue ice when clear, violet iodine, red-brown bromine, yellow-green chlorine, red flames from CN or blue-green flames from CH, some gas lasers, blue ozone leading to blue and grey evening sky	Colours are due to quantized levels of rotation and vibrations in molecules (blue iceberg © Marc Shandro)

Class II: Colours due to ligand field effects

	4. Transition metal compounds	
	Green malachite $Cu_2CO_3(OH)_2$, blue cobalt oxide, blue azurite $Cu_3(CO_3)_2(OH)_2$, red to brown hematite Fe_2O_3, green MnO, white $Mn(OH)_2$, brown manganite, chrome green Cr_2O_3, green praesodymium, pink europium and yellow samarium compounds, piezochromic and thermochromic $Cr_2O_3 - Al_2O_3$ UV and electron phosphors, scintillation, some fluorescence, some lasers	Colours are due to electronic states of the ions; phosphors are used in cathodes tubes for TV/computer displays and on fluorescent lamp tubes (green malachite on yellow kasolite, a uranium mineral, picture width 5 mm, found in Kolwezi, Zaire/Congo, © Stephan Wolfsried, television shadow mask photo © Planemad)
	5. Transition metal impurities	
	Ruby, emerald, alexandrite, perovskites, corresponding lasers	Electronic states of transition metal ions are excited by light and thus absorb specific wavelengths (ruby on calcite from Mogok, Myanmar, picture width 3 cm, © Rob Lavinsky)

TABLE 7 Causes of colour (continued).

Colour type	Example	Details

Class III: Colours due to molecular orbitals

6. Organic compounds

Red haemoglobin in blood, blue blood haemocyanin, green chlorophyll in plants, yellow or orange carotenes in carrots, flowers and yellow autumn leaves, red or purple anthocyanins in berries, flowers and red autumn leaves, blue indigo, red lycopene in tomatoes, red meat from iron-containing myoglobin, brown glucosamine in crust of baked food, brown tannins, black eumelanin in human skin, hair and eye, iron-rich variation pheomelanin in redheads, black melanin also in cut apples and bananas as well as in movable sacks in chameleons, brown-black asphalt, some fluorescence, chemiluminescence, phosphorescence, halochromism, electrochromism and thermochromism, dye lasers

Colours are due to conjugated π-bonds, i.e. to alternating single and double bonds in molecules; floral pigments are almost all anthocyanins, betalains or carotenes; used in colourants for foods and cosmetics, in textile dyes, in electrochromic displays, in inks for colour printers, in photosensitizers (narcissus © Thomas Lüthi, blood on finger © Ian Humes, berries © Nathan Wall, hair courtesy dusdin)

Glow-worms, some bacteria and fungi, most deep-sea fish, octopi, jellyfish, and other deep-sea animals

Bioluminescence is due to excited molecules, generally called *luciferines* (angler fish, length 4.5 cm, © Steve Haddock)

TABLE 7 Causes of colour (continued).

COLOUR TYPE	EXAMPLE	DETAILS

7. Inorganic charge transfer

Blue sapphire, blue lapis lazuli, green amazonite, brown-black magnetite Fe_3O_4 and most other iron minerals (colouring basalt black, beer bottles brown, quartz sand yellow, and many other rocks with brown or red tones), black graphite, purple permanganate, orange potassium dichromate, yellow molybdates, red hematite Fe_2O_3, some fluorescence

Light induces change of position of an electron from one atom to another; for example, in blue sapphire the transition is between Ti and Fe impurities; many paint pigments use charge transfer colours; fluorescent analytical reagents are used in molecular medicine and biology (magnetite found in Laach, Germany, picture width 10 mm, © Stephan Wolfsried, sand desert Evelien Willemsen)

Class IV: Colours due to energy band effects

8. Metallic bands

Gold (green in transmission), pyrite, iron, brass, alloys, silver, copper, ruby glass

Colours in reflection and in transmission are due to transitions of electrons between overlapping bands (saxophone © Selmer)

9. Pure semiconductor bands

Silicon, GaAs, black galena PbS, red cinnabar HgS, cadmium yellow CdS, black CdSe, red CdS_xSe_{1-x}, white ZnO, orange vermillion HgS, colourless diamond, black to gold piezochromic SmS

Colours are due to electron transitions between separate bands; colour series is black, red, orange, yellow, white/colourless; some used as pigments (zinc oxide courtesy Walkerma)

10. Doped semiconductor bands

Blue, yellow, green and black diamond; LEDs; semiconductor lasers; solar cells; ZnS and $Zn_xCd_{1-x}S$ based and other phosphors

Colours are due to transitions between dopants and semiconductor bands (e.g. blue diamond: boron acceptors, black diamond: nitrogen donors) (quantum dots © Andrey Rogach)

TABLE 7 Causes of colour (continued).

Colour type	Example	Details
	11. Colour centres	
	Amethyst, smoky quartz, fluorite, green diamonds, blue, yellow and brown topaz, brown salt, purple colour of irradiated glass containing Mn^{2+}, lyoluminescence, some fluorescence, F-centre lasers	Colours are due to colour centres, i.e. to electrons or to holes bound at crystal vacancies; colour centres are usually created by radiation (amethyst © Rob Lavinsky)
	Some light-dependent sunglasses	The photochromic colouring is due to colour centres formed by the UV light of the Sun

Class V: Colours due to physical and geometrical optics

	12. Dispersive refraction and polarization	
	Cut diamond, cut zirconia, halos and sun dogs formed by ice crystals in the air	Spectral decomposition (sparkle or 'fire' of gemstones) is due to dispersion in crystals (zirconia photo © Gregory Phillips)
	Rainbow	Colours of primary and secondary bow are due to dispersion in water droplets
	Green flash	dispersion in the atmosphere shifts the sun colours
	13. Scattering	
	Blue sky, blue colouring of distant mountains, red sunset; colour intensification by pollution; blue quartz	Blue light is scattered more than red light by Rayleigh scattering, when scatterers (molecules, dust) are smaller than the wavelength of light (Tokyo sunset © Altus Plunkett, blue quartz © David Lynch)

TABLE 7 Causes of colour (continued).

Colour type	Example	Details
	White colour of hair, milk, beer foam, clouds, fog, cigarette smoke coming out of lungs, snow, whipped cream, shampoo, stars in gemstones	The white colour is due to wavelength-independent Mie scattering, i.e. scattering at particles larger than the wavelength of light (snow man © Andreas Kostner)
	Blue human skin colour in cold weather, blue and green eyes in humans, blue monkey skin, blue turkey necks, most blue fish, blue reptiles, blue cigarette smoke	Tyndall blue colours are due to scattering on small particles in front of a dark background (blue poison frog *Dendrobates azureus* © Lee Hancock)
	Ruby glass	The red colour of Murano glass is due to scattering by tiny colloidal gold particles included in the glass in combination with the metallic band structure of gold (ruby glass © murano-glass-shop.it)
	Nonlinearities, Raman effect, potassium dihydrogen phosphate (KDP)	Frequency-shifting scattering, second harmonic generation and other nonlinearities of certain materials change the colour of light impinging with high intensities (800 nm to 400 nm frequency doubling ring laser © Jeff Sherman)
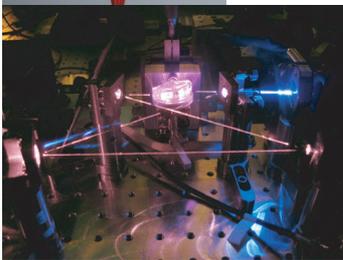
14. Interference (without diffraction)

	Nacre, oil films, soap bubbles, coatings on camera lenses, eyes of cats in the dark, wings of flies and dragonflies, fish scales, some snakes, pearls, tempering colours of steel	Thin film interference produces a standard colour sequence that allows precise thickness determination (abalone shell © Anne Elliot)
	Polarization colours of thin layers of birefringent crystals or thicker layers of stressed polymers	Colours are due to interference, as shown by the dependence on layer thickness (photoelasticity courtesy Nevit Dilmen)
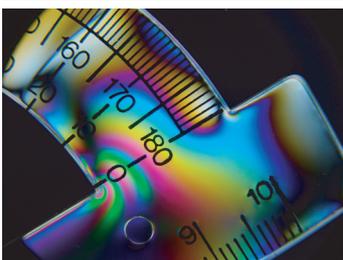
TABLE 7 Causes of colour (continued).

Colour type	Example	Details
	Supernumerary rainbows (see page 102 in volume III)	Due to interference, as shown by the dependence on drop size
	Iridescent beetles, butterflies and bird feathers, iridescent colours on banknotes and on cars	Due to scattering at small structures or at nanoparticles, as shown by the angular dependence of the colour (mallard duck © Simon Griffith)

15. Diffraction (with interference)

Colour type	Example	Details
	Opal	Colours are due to the tiny spheres included in the water inside the opal; colours can change if the opal dries out (polished Brazilian opal © Opalsnopals)
	Aureole, glory, corona	Colours are due to diffraction at the tiny mist droplets (aeroplane condensation cloud iridescence © Franz Kerschbaum)
	Diffraction gratings, CDs, vinyl records, some beetles and snakes	Colours are due to diffraction and interference at tiny, regular pits (CD illuminated by flashlamp © Alfons Reichert)
	Photonic crystals	A modern research topic
	Cholesteric liquid crystals, certain beetles	Colours are due to diffraction and interference in internal material layers (liquid crystal colours © Ingo Dierking)
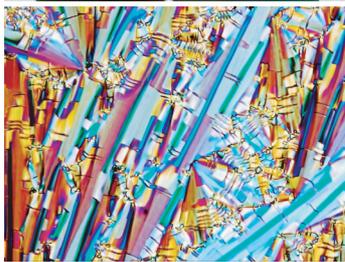
Class VI: Colours due to eye limitations

Colour type	Example	Details
Fechner colours, as on lite.bu.edu/vision/applets/Color/Benham/Benham.html	Benham's wheel or top	Colours are due to different speed response of different photoreceptors

TABLE 7 Causes of colour (continued).

COLOUR TYPE	EXAMPLE	DETAILS
Internal colour production when eyes are stimulated	Phosphenes	Occur through pressure (rubbing, sneeze), or with electric or magnetic fields
Polarization colours	Haidinger's brush	See page 113 in volume III
Colour illusions, as on www.psy.ritsumei.ac.jp/~akitaoka/color9e.html	Appearing and disappearing colours	Effects are due to combinations of brain processing and eye limitations
False colour output of eye, as described on page 196 in volume III	Red light can be seen as green	Observable with adaptive optics, if red light is focused on a green-sensitive cone
Colour-blind or 'daltonic' person, see page 209 in volume III, with reduced colour spectrum	Protan, deutan or tritan	Each type limits colour perception in a different way

Colours fascinate. Fascination always also means business; indeed, a large part of the chemical industry is dedicated to synthesizing colourants for paints, inks, clothes, food and cosmetics. Also evolution uses the fascination of colours for its own business, namely propagating life. The specialists in this domain are the flowering plants. The chemistry of colour production in plants is extremely involved and at least as interesting as the production of colours in factories. Practically all flower colourants, from white, yellow, orange, red to blue, are from three chemical classes: the *carotenoids*, the *anthocyanins* (flavonoids) and the *betalains*. These colourants are stored in petals inside dedicated containers, the *vacuoles*. There are many good review articles providing the details.

Ref. 131

Even though colours are common in plants and animals, most higher animals do not produce many colourants themselves. For example, humans produce only one colourant: *melanin*. (*Hemoglobin*, which colours blood red, is not a dedicated colourant, but transports the oxygen from the lungs through the body. Also the pink *myoglobin* in the muscles is not a dedicated colourant.) Many higher animals, such as birds, need to *eat* the colourants that are so characteristic for their appearance. The yellow colour of legs of pigeons is an example. It has been shown that the connection between colour and nutrition is regularly used by potential mates to judge from the body colours whether a proposing partner is sufficiently healthy, and thus sufficiently attractive.

Ref. 132

Above all, the previous table distinguished six main classes among the causes of colours. The study of the first class, the colours of incandescence, led Max Planck to discover the quantum of action. In the meantime, research has confirmed that in each class, all colours are due to the quantum of action $\hbar$. The relation between the quantum of action and the material properties of atoms, molecules, liquids and solids are so well known that colourants can now be designed on the computer.

In summary, an exploration of the causes of colours found in nature confirms that all colours are due to quantum effects. We show this by exploring the simplest example: the colours of atomic gas excitations.
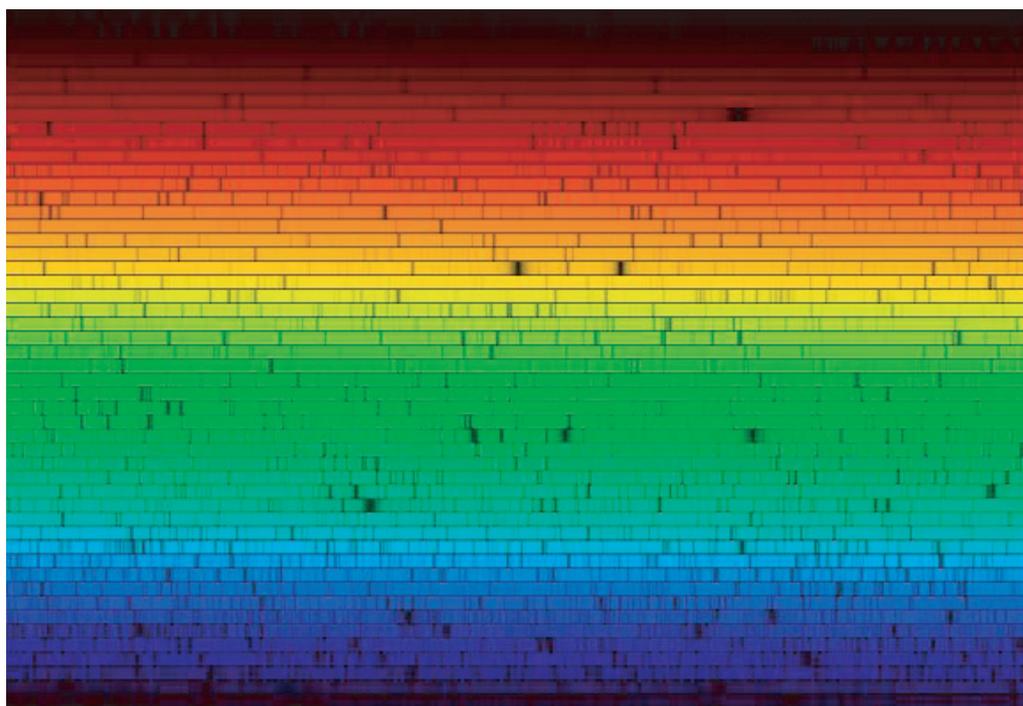
FIGURE 77 The spectrum of daylight: a *stacked* image of an extended rainbow, showing its Fraunhofer lines (© Nigel Sharp, NOAO, FTS, NSO, KPNO, AURA, NSF).

USING THE RAINBOW TO DETERMINE WHAT STARS ARE MADE OF

Near the beginning of the eighteenth century, Bavarian instrument-maker Joseph Fraunhofer* and the English physicist William Wollaston noted that the rainbow *lacks* certain colours. These colours appear as black lines when the rainbow is spread out in sufficient breadth. Figure 77 shows the lines in detail; they are called *Fraunhofer lines* today. In 1860, Gustav Kirchhoff and Robert Bunsen showed that the colours *missing* in the rainbow were exactly those colours that certain elements *emit* when heated. In this way they managed to show that sodium, calcium, barium, nickel, magnesium, zinc, copper and iron are present in the Sun. Looking at the rainbow thus tells us what the Sun is made of.

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* Joseph Fraunhofer (b. 1787 Straubing, d. 1826 Munich), having been orphaned at the age of 11, learned lens-polishing. He taught himself optics from books. He entered an optical company at the age of 19, ensuring the success of the business by producing the best available lenses, telescopes, micrometers, optical gratings and optical systems of his time. He invented the spectroscope and the heliometer. He discovered and counted 476 lines in the spectrum of the Sun; these lines are now named after him. (Today, Fraunhofer lines are still used as measurement standards: the second and the metre are defined in terms of them.) Physicists from all over the world would buy their equipment from him, visit him, and ask for copies of his publications. Even after his death, his instruments remained unsurpassed for generations. With his telescopes, in 1837 Bessel was able to make the first measurement of parallax of a star, and in 1846 Johann Gottfried Galle discovered Neptune. Fraunhofer became a professor in 1819. He died young, from the consequences of the years spent working with lead and glass powder.
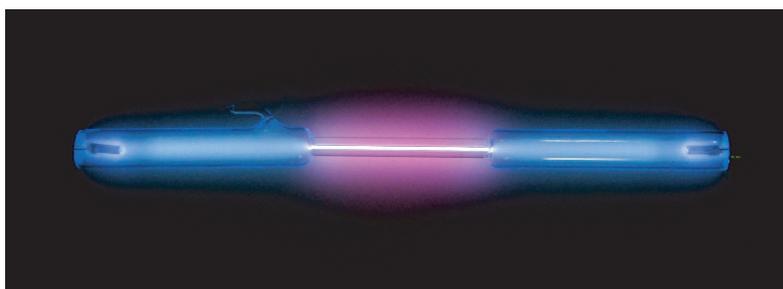
Of the 476 Fraunhofer lines that Kirchhoff and Bunsen observed, 13 did not correspond to any known element. In 1868, Jules Janssen and Joseph Lockyer independently predicted that these unknown lines were from an unknown element. The element was eventually found on Earth, in an uranium mineral called cleveite, in 1895. The new element was called *helium*, from the Greek word ἥλιος 'helios' – Sun.

In 1925, using an equation developed by Saha and Langmuir, the young physicist Cecilia Payne (b. 1900 Wendover, England, d. 1979 Cambridge, Massachusetts) taught the world how to deduce the mass percentage of each element from the light spectrum of a star. She did so in her brilliant PhD thesis. Above all, she found that hydrogen and helium were the two most abundant elements in the Sun, in stars, and thus in the whole universe. This went completely against the ideas of the time, but is now common knowledge. Payne had completed the study of physics in Cambridge, UK, but had not received a degree there because she was a woman. So she left for the United States, where the situation was somewhat better, and where she worked on her PhD thesis; eventually, she became professor at Harvard University, and later head of its astronomy department. Above all, Payne became an important role model for many female scientists.

Despite being the second most common element in the universe, helium is rare on Earth because it is a light noble gas that does not form chemical compounds. Helium atoms on Earth thus rise in the atmosphere and finally escape into space.

Understanding the colour lines produced by each element had started to become interesting already before the discovery of helium; but afterwards the interest increased further, thanks to the increasing number of applications of colour knowledge in chemistry, physics, technology, crystallography, biology and lasers. Colours are big business, as the fashion industry, the media and the advertising business show.

In summary, colours are specific mixtures of light frequencies. Light is an electromagnetic wave and is emitted by moving charges. For a physicist, colours thus result from the interaction of charged matter with the electromagnetic field. Now, *sharp* colour lines cannot be explained by classical electrodynamics. We need quantum theory to explain them.

WHAT DETERMINES THE COLOURS OF ATOMS?

The simplest colours to study are the sharp colour lines emitted or absorbed by single atoms. Single atoms are found in gases. The simplest atom to study is that of hydrogen. As shown in Figure 78, hot hydrogen gas emits light. The light consists of a handful of

sharp spectral lines that are shown on the left of Figure 79. Already in 1885, the Swiss schoolteacher Johann Balmer (1828–1898) had discovered that the wavelengths of visible hydrogen lines obey the formula:

$$\frac{1}{\lambda_m} = R\left(\frac{1}{4} - \frac{1}{m^2}\right) \quad \text{for} \quad m = 3, 4, 5, \dots . \tag{91}$$

Careful measurements, which included the hydrogen's spectral lines in the infrared and in the ultraviolet, allowed Johannes Rydberg (1854–1919) to generalize this formula to:

$$\frac{1}{\lambda_{mn}} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right) , \tag{92}$$

where n and $m > n$ are positive integers, and the so-called *Rydberg constant R* has the value $10.97\,\mu m^{-1}$; easier to remember, the inverse value is $1/R = 91.16$ nm. All the colour lines emitted by hydrogen satisfy this simple formula. Classical physics cannot explain this result at all. Thus, quantum theory has a clearly defined challenge here: to explain the formula and the value of R.

Incidentally, the transition λ_{21} for hydrogen is called the *Lyman-alpha line*. Its wavelength, 121.6 nm, lies in the ultraviolet. It is easily observed with telescopes, since most of the visible stars consist of excited hydrogen. The Lyman-alpha line is routinely used to determine the speed of distant stars or galaxies, since the *Doppler effect* changes the wavelength when the speed is large. The record in 2004 was a galaxy with a Lyman-alpha line shifted to 1337 nm. Can you calculate the speed with which it is moving away from the Earth?

From the start, it was clear that the colours of hydrogen are due to the motion of its electron. (Why?) The first way to deduce Balmer's formula from the quantum of action was found by Niels Bohr in 1903. Bohr understood that in contrast to planets circling the Sun, the electron moving around the proton has only a discrete number of possible motion states: the angular momentum of the electron is quantized. Assuming that the angular momentum of the electron is an integer multiple of $\hbar$ directly yields Balmer's formula and explains the numerical value of the Rydberg constant R. This calculation is so famous that it is found in many secondary school books. The result also strengthened Bohr's decision to dedicate his life to the exploration of the structure of the atom.

Twenty years time later, in 1926, Erwin Schrödinger solved his equation of motion for an electron moving in the electrostatic potential $V(r) = e^2/4\pi\varepsilon_0 r$ of a point-like proton. By doing so, Schrödinger reproduced Bohr's result, deduced Balmer's formula and became famous in the world of physics. However, this important calculation is long and complex. It can be simplified.

In order to understand hydrogen colours, it is not necessary to solve an equation of motion for the electron; it is sufficient to compare the energies of the initial and final states of the electron. This can be done most easily by noting that a specific form of the action must be a multiple of $\hbar/2$. This approach, a generalization of Bohr's explanation, was developed by Einstein, Brillouin and Keller, and is now named EBK quantization. It

Hydrogen: spectral lines and
energy levels
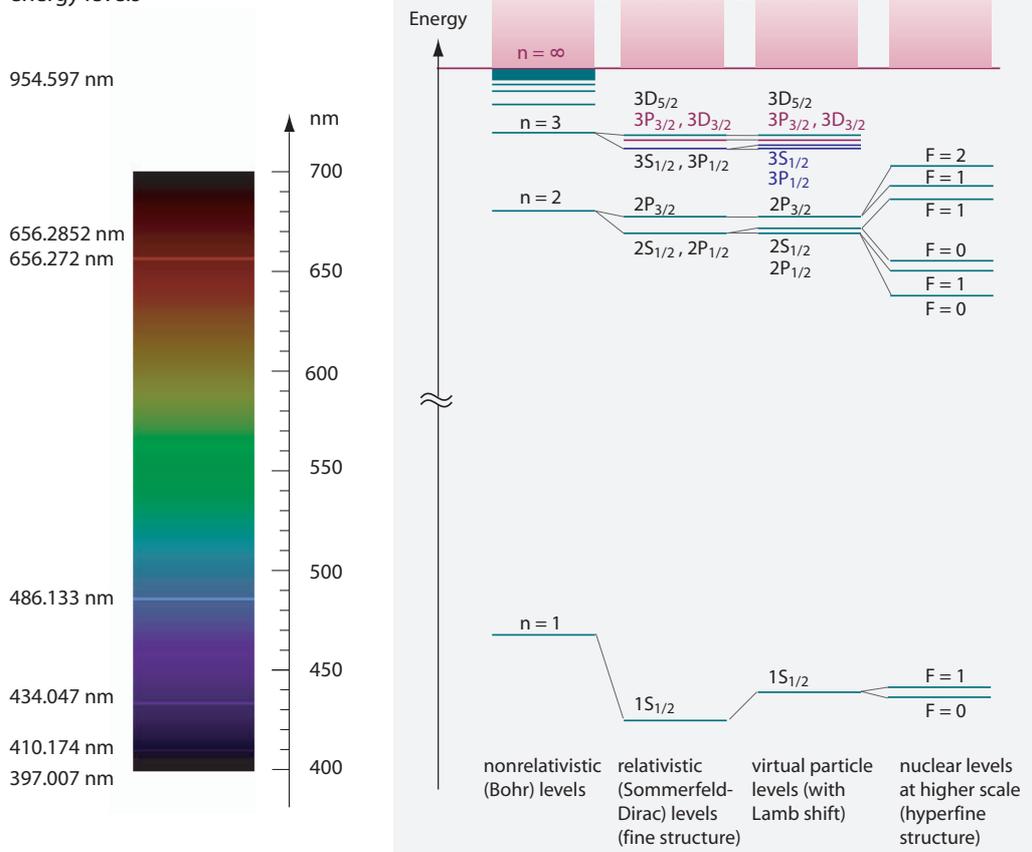
FIGURE 79 Atomic hydrogen: the visible spectrum of hydrogen (NASA) and its calculated energy levels, in four approximations of increasing precision. Can you associate the visible lines to the correct level transitions?

relies on the fact that the action S of any quantum system obeys

$$S = \frac{1}{2\pi} \oint dq_i p_i = \left(n_i + \frac{\mu_i}{4} \right) \hbar \tag{93}$$

for every coordinate q_i and its conjugate momentum p_i. The expression reflects the similarity between angular momentum and action. Here, n_i can be zero or any positive integer, and μ_i is the so-called *Maslov index*, an even integer, which in the case of atoms has the value 2 for the radial and azimuthal coordinates r and θ, and 0 for the rotation angle φ. The integral is to be taken along a full orbit. In simple words, *the action S is a half-integer multiple of the quantum of action.* This result can be used to calculate the energy levels of periodic quantum systems. Let us do so for hydrogen atoms.

Any rotational motion in a spherical potential $V(r)$ is characterized by a constant energy E and constant angular momenta L and L_z. Therefore the conjugate momenta

for the coordinates r, θ and φ are

$$p_r = \sqrt{2m(E - V(r)) - \frac{L^2}{r^2}}$$

$$p_\theta = \sqrt{L^2 - \frac{L_z^2}{\sin^2 \theta}}$$

$$p_\varphi = L_z \ . \tag{94}$$

Using these expressions in equation (93) and setting $n = n_r + n_\theta + n_\varphi + 1$, we get* the result

$$E_n = -\frac{1}{n^2}\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} = -\frac{Rhc}{n^2} = -\frac{c^2 m\alpha^2}{2n^2} \approx -\frac{2.19\,\text{aJ}}{n^2} \approx -\frac{13.6\,\text{eV}}{n^2} \ . \tag{97}$$

These *energy levels* E_n, the non-relativistic Bohr levels, are shown in Figure 79. Using the idea that a hydrogen atom emits a single photon when its electron changes from state E_n to E_m, we get exactly the formula deduced by Balmer and Rydberg from observations! The match between observation and calculation is about four digits. For (almost) the first time ever, a material property, the colour of hydrogen atoms, had been explained from a fundamental principle of nature. Key to this explanation was the quantum of action $\hbar$. (This whole discussion assumes that the electrons in hydrogen atoms that emit light are in eigenstates. Can you argue why this is the case?)

In short, the quantum of action implies that only certain specific energy values for an electron are allowed inside an atom. The lowest energy level, for $n = 1$, is called the *ground state*. Its energy value 2.19 aJ is the *ionization energy* of hydrogen; if that energy is added to the ground state, the electron is no longer bound to the nucleus. The ionization energy thus plays the same role for electrons around atoms as does the *escape velocity*, or better, the escape *energy*, for satellites or rockets shot from planets.

In the same way that the quantum of action determines the colours of the hydrogen atom, it determines the colours of all other atoms. All Fraunhofer lines, whether observed in the infrared, visible or ultraviolet, are due to the quantum of action. In fact, every colour in nature is due to a mixture of colour lines, so that all colours, also those of solids and liquids, are determined by the quantum of action.
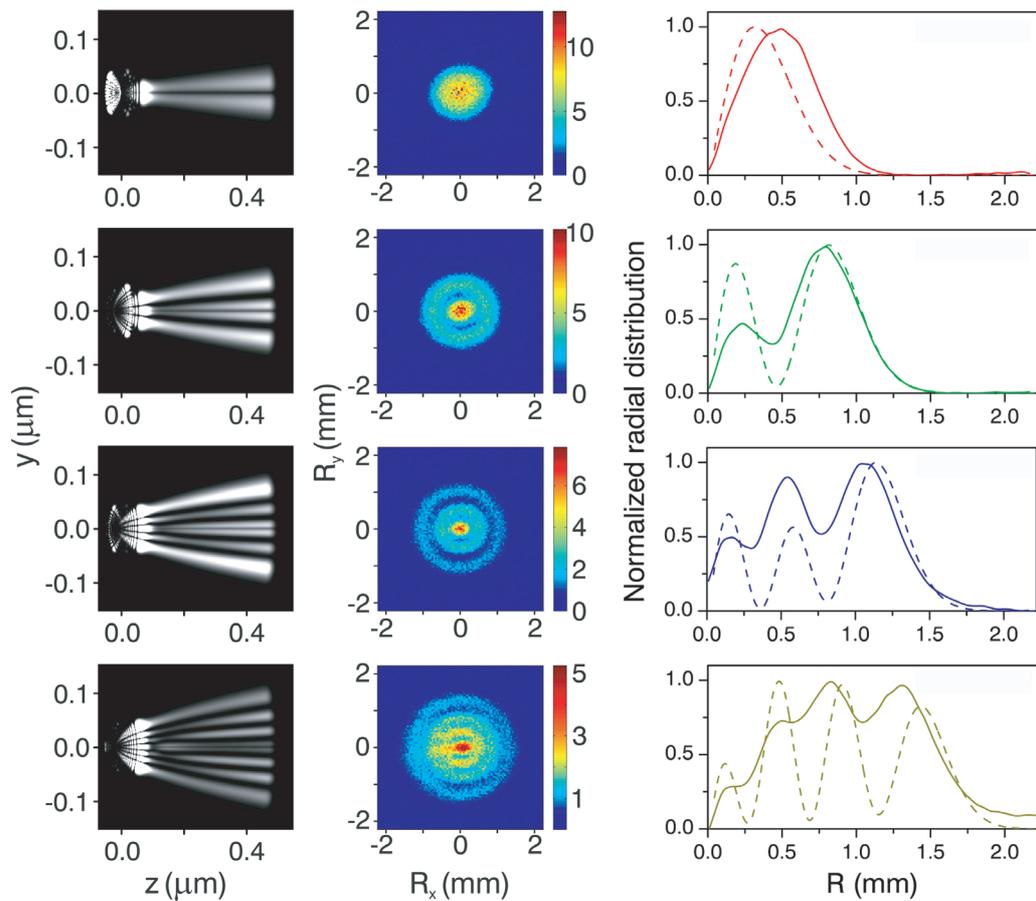
FIGURE 80 The figure shows the calculated and the measured nodal structure of the hydrogen atom in a weak external electric field, magnified by an electrostatic lens. The patterns are two-dimensional interference shadows of the wave functions. Left column: how the wave function is projected from the atoms to the macroscopic screen; central column: the measured nodal structure; right column: comparison of the measured (solid) and calculated (dashed) electron densities. (© Aneta Stodolna/APS, from Ref. 138).

THE SHAPE OF ATOMS

Free atoms are spherical. Atoms in external fields are deformed. Whatever the situation, the shape of atoms is due to the shape of the wave function. The simplest case is the

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* The calculation is straightforward. After insertion of $V(r) = e/4\pi\varepsilon_0 r$ into equation (94) one needs to perform the (tricky) integration. Using the general result

$$\frac{1}{2\pi} \oint \frac{dz}{z} \sqrt{Az^2 + 2Bz - C} = -\sqrt{C} + \frac{B}{\sqrt{-A}} \tag{95}$$

one gets

$$\left(n_r + \frac{1}{2}\right)\hbar + L = n\hbar = \frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{m}{-2E}} \ . \tag{96}$$

This leads to the energy formula (97).

hydrogen atom. Its wave functions – more precisely, the eigenfunctions for the first few energy levels – are illustrated on the right hand side of Figure 81. These functions had been calculated by Erwin Schrödinger already in 1926 and are found in all textbooks. We do not perform the calculation here, and just show the results.

The square of the wave function is the probability density of the electron. This density quickly decreases with increasing distance from the nucleus. Like for a real cloud, the density is never zero, even at large distances. We could thus argue that all atoms have infinite size; in practice however, chemical bonds or the arrangement of atoms in solids show that it is much more appropriate to imagine atoms as clouds of finite size.

Surprisingly, the first measurement of the wave function of an atom dates only from the year 2013; it was performed with a clever photoionization technique by Aneta Stodolna and her team. The beautiful experimental result is shown in Figure 80. The figures confirm that wave functions, in contrast to probability densities, have nodes, i.e. lines – or better, surfaces – where their value is zero.

In summary, all experiments confirm that the electron in the hydrogen atom forms wave functions in exactly the way that is predicted by quantum theory. In particular, the shape of atoms is found to agree with the calculation from quantum mechanics.

The size of atoms

The calculation of the hydrogen energy levels also yields the effective radius of the electron orbits. It is given by

$$ r_n = n^2 \frac{\hbar^2 4\pi\varepsilon_0}{m_e e^2} = \frac{\hbar}{m_e c \alpha} = n^2 a_0 \approx n^2 \, 52.918\,937 \text{ pm} \text{ , with } n = 1, 2, 3, \ldots \quad (98) $$

We again see that, in contrast to classical physics, quantum theory allows only certain specific orbits around the nucleus. (For more details about the fine-structure constant α, see below.) To be more precise, these radii are the average sizes of the electron clouds surrounding the nucleus. The smallest orbital radius value, 53 pm for $n = 1$, is called the *Bohr radius*, and is denoted by a_0.

In a gas of hydrogen atoms, most atoms are in the ground state described by $r_1 = a_0$ and E_1. On the other hand, quantum theory implies that a hydrogen atom excited to the level $n = 500$ is about 12 μm in size: larger than many bacteria! Such blown-up atoms, usually called *Rydberg atoms*, have indeed been observed in the laboratory, although they are extremely sensitive to perturbations.

In short, the quantum of action determines *the size of atoms*. The result thus confirms the prediction by Arthur Erich Haas from 1910. In other words

▷ The quantum of action determines the size of all things.

In 1915, Arnold Sommerfeld understood that the analogy of electron motion with orbital gravitational motion could be continued in two ways. First of all, electrons can move, on average, on *ellipses* instead of circles. The quantization of angular momentum then implies that only selected eccentricities are possible. The higher the angular momentum, the larger the number of possibilities: the first few are shown in Figure 81. The
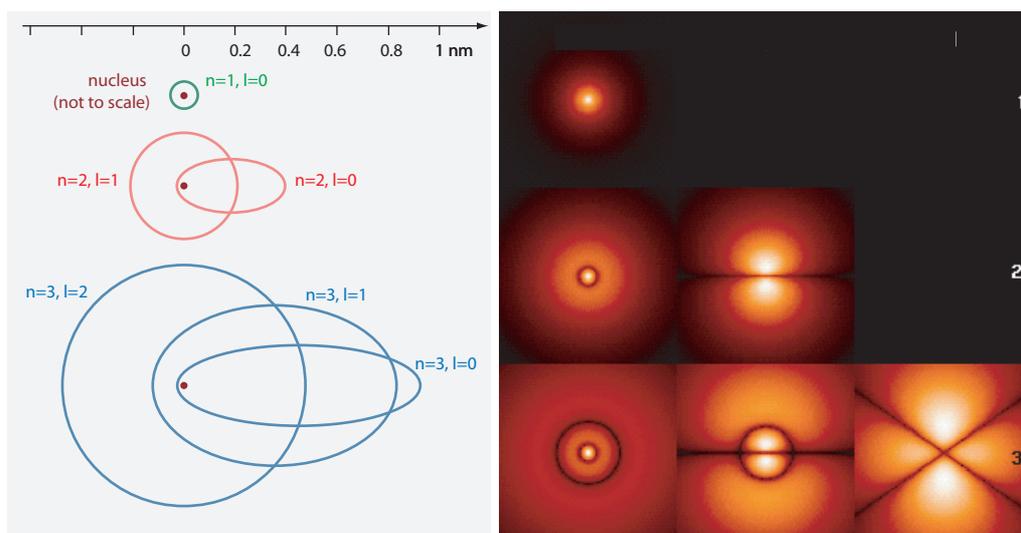
FIGURE 81 The imagined, but not existing and thus false electron orbits of the Bohr–Sommerfeld model of the hydrogen atom (left) and the correct description, using the probability density of the electron in the various states (right) (© Wikimedia).

highest eccentricity corresponds to the minimum value $l = 0$ of the so-called azimuthal quantum number, whereas the case $l = n - 1$ correspond to circular orbits. Furthermore, the ellipses can have different orientations in space.

Ref. 137

The second point Sommerfeld noted was that the speeds of the electron in hydrogen are – somewhat – relativistic: the speed values are not negligible compared to the speed of light. Indeed, the orbital frequency of electrons in hydrogen is

$$f_n = \frac{1}{n^3} \frac{e^4 m_{\mathrm{e}}}{4\varepsilon_0^2 h^3} = \frac{1}{n^3} \frac{m_e c^2 \alpha^2}{h} \approx \frac{6.7 \,\mathrm{PHz}}{n^3} \tag{99}$$

and the electron speed is

$$v_n = \frac{1}{n} \frac{e^2}{4\pi\varepsilon_0 \hbar} = \frac{\alpha c}{n} \approx \frac{2.2 \,\mathrm{Mm/s}}{n} \approx \frac{0.007 \, c}{n} \,. \tag{100}$$

As expected, the further the electron's orbit is from the nucleus, the more slowly it moves. This result can also be checked by experiment: exchanging the electron for a muon allows us to measure the time dilation of its lifetime. Measurements are in excellent agreement with the calculations.

Ref. 136

In short, Sommerfeld noted that Bohr's calculation did not take into account relativistic effects. And indeed, high-precision measurements show slight differences between the Bohr's non-relativistical energy levels and the measured ones. The calculation must be improved.

Relativistic hydrogen

Measuring atomic energy levels is possible with a much higher precision than measuring wave functions. In particular, energy level measurements allow to observe relativistic effects.

Also in the relativistic case, the EBK action has to be a multiple of $\hbar/2$. From the relativistic expression for the kinetic energy of the electron in a hydrogen atom

$$E + c^2 m = \sqrt{p^2 c^2 + m^2 c^4} - \frac{e^2}{4\pi\varepsilon_0 r} \tag{101}$$

we get the expression

$$p_r^2 = 2mE\left(1 + \frac{E}{2c^2 m}\right) + \frac{2me^2}{4\pi\varepsilon_0 r}\left(1 + \frac{E}{c^2 m}\right) . \tag{102}$$

We now introduce, for convenience, the so-called *fine-structure constant*, as $\alpha = e^2/(4\pi\varepsilon_0 \hbar c) = \sqrt{4\pi\hbar R/mc} \approx 1/137.036$. ($\alpha$ is a dimensionless constant; $R = 10.97\ \mu\text{m}^{-1}$ is the Rydberg constant.) The radial EBK action then implies that

$$E_{nl} + c^2 m = \frac{c^2 m}{\sqrt{1 + \frac{\alpha^2}{\left(n - l - \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - \alpha^2}\right)^2}}} . \tag{103}$$

This result, first found by Arnold Sommerfeld in 1915, is correct for point-like, i.e., non-rotating electrons. In reality, the electron has spin 1/2; the correct relativistic energy levels thus appear when we set $l = j \pm 1/2$ in the above formula. The result can be approximated by

$$E_{nj} = -\frac{R}{n^2}\left(1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4}\right) + ...\right) . \tag{104}$$

It reproduces the hydrogen spectrum to an extremely high accuracy. If we compare the result with the non-relativistic one, we note that each non-relativistic level n is split in n different levels. This splitting is illustrated in Figure 79. In precision experiments, the splitting of the lines of the hydrogen spectrum is visible as the so-called *fine structure*. The magnitude of the fine structure depends on α, a fundamental constant of nature. Since Arnold Sommerfeld discovered the importance of this fundamental constant in this context, the name he chose, the *fine-structure constant*, has been taken over across the world. The fine-structure constant describes the *strength* of the electromagnetic interaction; the fine-structure constant is the electromagnetic coupling constant.

Modern high-precision experiments show additional effects that modify the colours of atomic hydrogen. They are also illustrated in Figure 79. Virtual-particle effects and the coupling of the proton spin give additional corrections. But that is still not all: isotope effects, Doppler shifts and level shifts due to environmental electric or magnetic fields

FIGURE 82 Paul Dirac (1902–1984)

also influence the hydrogen spectrum. The final effect on the hydrogen spectrum, the famous Lamb shift, will be a topic later on.

Relativistic wave equations – again

> The equation was more intelligent than I was.
> Paul Dirac about his equation, repeating
> a statement made by Heinrich Hertz.

What is the evolution equation for the wave function in the case that relativity, spin and interactions with the electromagnetic field are taken into account? We could try to generalize the representation of relativistic motion given by Foldy and Wouthuysen to the case of particles with electromagnetic interactions. Unfortunately, this is not a simple matter. The simple identity between the classical and quantum-mechanical descriptions is lost if electromagnetism is included.

Charged quantum particles are best described by another, equivalent representation of the Hamiltonian, which was discovered much earlier, in 1926, by the British physicist Paul Dirac.* Dirac found a neat trick to take the square root appearing in the relativistic energy operator. In Dirac's representation, the Hamilton operator is given by

$$H_{\text{Dirac}} = \beta m + \boldsymbol{\alpha} \cdot p \ . \tag{105}$$

The quantities β and the three components $(\alpha_1, \alpha_2, \alpha_3) = \boldsymbol{\alpha}$ turn out to be complex 4×4 matrices.

In Dirac's representation, the position operator x is not the position of a particle, but has additional terms; its velocity operator has only the eigenvalues plus or minus the velocity of light; the velocity operator is not simply related to the momentum operator;

---

* Paul Adrien Maurice Dirac (b. 1902 Bristol, d. 1984 Tallahassee), bilingual physicist, studied electrotechnics in Bristol, then went to Cambridge, where he later became a professor, holding the chair that Newton had once held. In the years from 1925 to 1933 he published a stream of papers, of which several were worth a Nobel Prize; he received it in 1933. Dirac unified special relativity and quantum theory, predicted antimatter, worked on spin and statistics, predicted magnetic monopoles, speculated on the law of large numbers, and more besides. His introversion, friendliness and shyness, and his deep insights into nature, combined with a dedication to beauty in theoretical physics, made him a legend all over the world during his lifetime. For the latter half of his life he tried, unsuccessfully, to find an alternative to quantum electrodynamics, of which he was the founder, as he was repelled by the problems of infinities. He died in Florida, where he lived and worked after his retirement from Cambridge.

t = 12.75
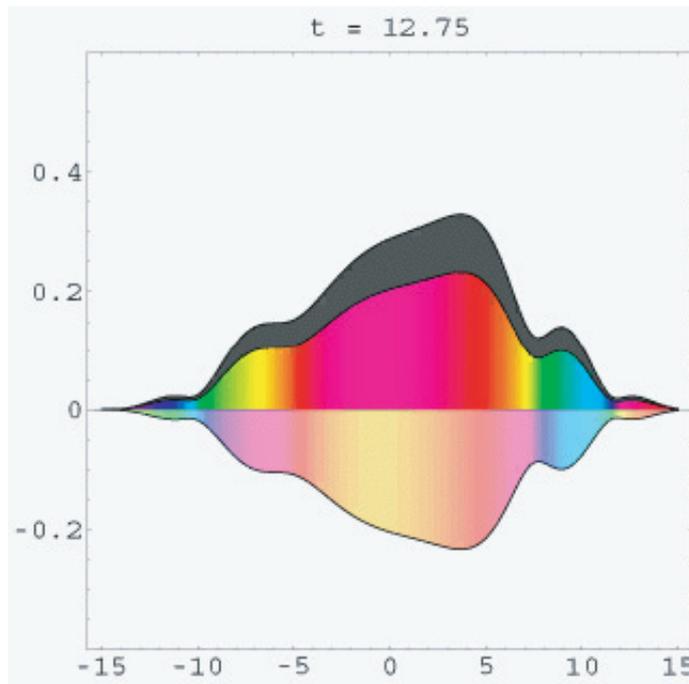
FIGURE 83 The famous
Zitterbewegung: the
superposition of positive and
negative energy states leads to
an oscillation around a mean
vale. Colour indicates phase;
two coloured curves are shown,
as the Dirac equation in one
dimension has only two
components (not four); the grey
curve is the probability density.
(QuickTime film © Bernd Thaller)

the equation of motion contains the famous 'Zitterbewegung' term; orbital angular momentum and spin are not separate constants of motion.

So why use this horrible Hamiltonian? Because only the Dirac Hamiltonian can easily be used for charged particles. Indeed, it is transformed to the Hamiltonian coupled to the electromagnetic field by the so-called *minimal coupling*, i.e., by the substitution

$$ \boldsymbol{p} \rightarrow \boldsymbol{p} - q\boldsymbol{A} \,, \tag{106} $$

that treats electromagnetic momentum like particle momentum. With this prescription, Dirac's Hamiltonian describes the motion of charged particles interacting with an electromagnetic field $\boldsymbol{A}$. The minimal coupling substitution is not possible in the Foldy–Wouthuysen Hamiltonian. In the Dirac representation, particles are pure, point-like, structureless electric charges; in the Foldy–Wouthuysen representation they acquire a charge radius and a magnetic-moment interaction. (We will come to the reasons below, in the section on QED.)

In more detail, the simplest description of an electron (or any other elementary, stable,

electrically-charged particle of spin 1/2) is given by the action S and Lagrangian

$$S = \int \mathcal{L}_{\text{QED}} d^4 x \quad \text{where} \tag{107}$$

$$\mathcal{L}_{\text{QED}} = \overline{\psi} \left(i\hbar c \slashed{D} - c^2 m \right) \psi - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \quad \text{and}$$

$$\slashed{D}_\mu = \gamma^\mu (\partial_\mu - ie A_\mu)$$

The first, matter term in the Lagrangian leads to the Dirac equation: it describes how elementary, charged, spin 1/2 particles are moved by electromagnetic fields. The second, radiation term leads to Maxwell's equations, and describes how electromagnetic fields are moved by the charged particle wave function. Together with a few calculating tricks, these equations describe what is usually called *quantum electrodynamics*, or QED for short.

As far as is known today, the relativistic description of the motion of charged matter and electromagnetic fields given the QED Lagrangian (107) is *perfect*: no differences between theory and experiment have ever been found, despite intensive searches and despite a high reward for anybody who would find one. All known predictions fully agree with all measurements. In the most spectacular cases, the correspondence between theory and measurement extends to more than thirteen digits. But even more interesting than the precision of QED are certain of its features that are missing in classical electrodynamics. Let's have a quick tour.

Getting a first feeling for the Dirac equation

The QED Lagrangian implies that the wave function of a charged particle in a potential follows the Dirac equation:

$$i\hbar\gamma^\mu (\partial_\mu - ie A_\mu)\psi = mc\psi \ . \tag{108}$$

The many indices should not make us forget that this equation simply states that the eigenvalue of the energy–momentum operator is the rest mass (times the speed of light c). In other words, the equation states that the wave ψ moves with a phase velocity c.

The wave function ψ has four complex components. Two describe the motion of particles, and two the motion of antiparticles. Each type of particle needs two complex components, because the equation describes spin and particle density. Spin is a rotation, and a rotation requires three real parameters. Spin and density thus require four real parameters; they can be combined into two complex numbers, both for particles and for antiparticles.

Challenge 152 e Each of the four components of the wave function of a relativistic spinning particle follows the relativistic Schrödinger–Klein–Gordon equation. This means that the relativistic energy–momentum relation is followed by each component separately.

Challenge 153 e The relativistic wave function ψ has the important property that a rotation by 2π changes its sign. Only a rotation by 4π leaves the wave function unchanged. This is the typical behaviour of spin 1/2 particles. For this reason, the four-component wave function of a spin 1/2 particle is called a *spinor*.
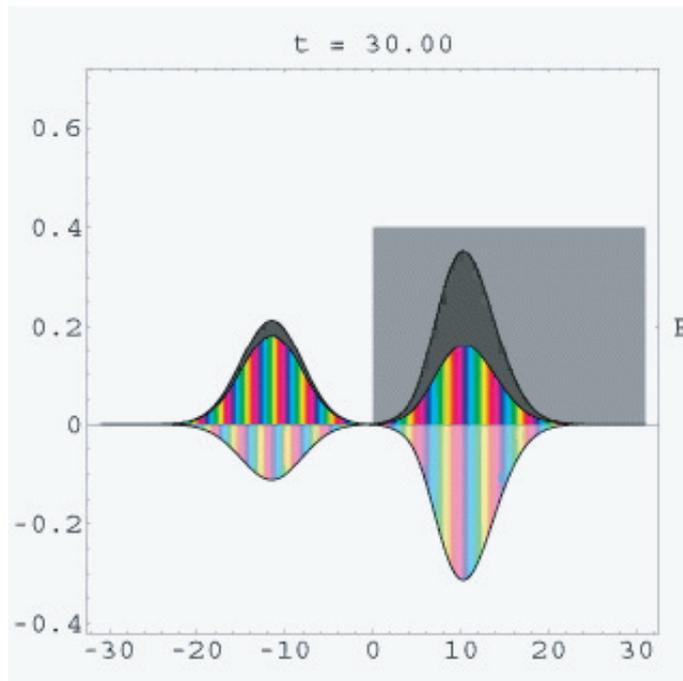
FIGURE 84 Klein's paradox: the motion of a relativistic wave function that encounters a very steep potential. Part of the wave function is transmitted; this part is antimatter, as the larger lower component shows. (QuickTime film © Bernd Thaller)

ANTIMATTER

'Antimatter' is now a household term. Interestingly, the concept appeared *before* there was any experimental evidence for it. The relativistic expression for the energy E of an electron with charge e in the field of a charge Q is

$$E + \frac{Qe}{4\pi\varepsilon_0 r} = \sqrt{m^2 c^4 + p^2 c^2} \ . \tag{109}$$

This expression also allows solutions with *negative* energy and opposite charge $-e$, if the negative root is used. Quantum theory shows that this is a general property, and these solutions correspond to what is called *antimatter*.

Indeed, the antimatter companion of the electron was predicted in the 1920s by Paul Dirac from his equation (108), which is based on the above relativistic energy relation (109). Unaware of this prediction, Carl Anderson discovered the antielectron in 1932, and called it the *positron*. (The correct name would have been 'positon', without the 'r'. This correct form is used in the French language.) Anderson was studying cosmic rays, and noticed that some 'electrons' were turning the wrong way in the magnetic field he had applied to his apparatus. He checked his apparatus thoroughly, and finally deduced that he had found a particle with the same mass as the electron but with positive electric charge.

The existence of positrons has many strange implications. Already in 1928, before their discovery, the Swedish theorist Oskar Klein had pointed out that Dirac's equation for

electrons makes a strange prediction: when an electron hits a sufficiently steep potential wall, the reflection coefficient is larger than unity. Such a wall will reflect *more* than is thrown at it. In addition, a large part of the wave function is transmitted through the wall. In 1935, after the discovery of the positron, Werner Heisenberg and Hans Euler explained the paradox. They found that the Dirac equation predicts that whenever an electric field exceeds the critical value of

$$E_c = \frac{m_e c^2}{e \lambda_e} = \frac{m_e^2 c^3}{e \hbar} = 1.3 \, \text{EV/m} \, , \tag{110}$$

the vacuum will spontaneously generate electron–positron pairs, which are then separated by the field. As a result, the original field is reduced. This so-called *vacuum polarization* is the reason for the reflection coefficient greater than unity found by Klein. Indeed, steep potentials correspond to high electric fields.

Vacuum polarization shows that, in contrast to everyday life, the *number* of particles is not a constant in the microscopic domain. Only the *difference* between particle number and antiparticle number turns out to be conserved. Vacuum polarization thus limits our possibility to count particles in nature!

Vacuum polarization is a weak effect. It has been only observed in particle collisions of high energy. In those case, the effect even increases the fine-structure constant! Later on we will describe truly gigantic examples of vacuum polarization that are postulated around charged black holes.

Of course, the generation of electron–positron pairs is not a *creation* out of nothing, but a *transformation* of energy into matter. Such processes are part of every relativistic description of nature. Unfortunately, physicists have a habit of calling this transformation 'pair creation', thus confusing the issue somewhat. The transformation is described by quantum field theory, which we will explore in the next volume.

Virtual particles

Despite what was said so far, action values smaller than the smallest action value do have a role to play. We have already encountered one example: in a collision between two electrons, there is an exchange of virtual photons. We learned that the exchanged virtual photon cannot be observed. Indeed, the action S for this exchange obeys

$$S \leqslant \hbar \, . \tag{111}$$

In short, *virtual* particles appear only as mediators in interactions. They cannot be observed. Virtual particles, in contrast to ordinary, real particles, do not obey the relation $E^2 - p^2 c^2 = m^2 c^4$. For example, the kinetic energy can be negative. Indeed, virtual particles are the opposite of 'free' or real particles. They may be observed in a vacuum if the measurement time is very short. They are intrinsically short-lived.

Virtual photons are the cause for electrostatic potentials, for magnetic fields, for the Casimir effect, for spontaneous emission, for the van der Waals force, and for the Lamb shift in atoms. A more detailed treatment shows that in every situation with virtual photons there are also, with even lower probability, virtual electrons and virtual

positrons.

Massive virtual particles are essential for vacuum polarization, for the limit in the number of the elements, for black-hole radiation and for Unruh radiation. Massive virtual particles also play a role in the strong interaction, where they hold the nucleons together in nuclei, and in weak nuclear interaction, where they explain why beta decay happens and why the Sun shines.

In particular, virtual particle–antiparticle pairs of matter and virtual radiation particles together form what we call the *vacuum*. In addition, virtual radiation particles form what are usually called static fields. Virtual particles are needed for a full description of all interactions. In particular, virtual particles are responsible for every decay process.

Curiosities and fun challenges about colour and atoms

Where is the sea bluest? Sea water, like fresh water, is blue because it absorbs red and green light. The absorption is due to a vibrational band of the water molecule that is due to a combination of symmetric and asymmetric molecular stretches. The absorption is weak, but noticeable. At 700 nm (red), the $1/e$ absorption length of water is 1 m.

Ref. 141

Sea water can also be of bright colour if the sea floor reflects light. In addition, sea water can be green, if it contains small particles that scatter or absorb blue light. Most often, these particles are soil or plankton. (Satellites can determine plankton content from the 'greenness' of the sea.) Thus the sea is especially blue if it is deep, quiet and cold; in that case, the ground is distant, soil is not mixed into the water, and the plankton content is low. The Sargasso Sea is 5 km deep, quiet and cold for most of the year. It is often called the *bluest* of the Earth's waters.

Lakes can also be blue if they contain small mineral particles. The particles scatter light and lead to a blue colour for reasons similar to the blue colour of the sky. Such blue lakes are found in many places on Earth.

* *

On modern high-precision measurements of the hydrogen spectra, listen to the undisputed master of the field: enjoy the 2012 talk by Theodor Hänsch, who has devoted a large part of his life to the topic, at www.mediatheque.lindau-nobel.org.

* *

The hydrogen atom bears many fascinating aspects. In 2015, Friedmann and Hagen showed that a well-known formula for $\pi = 3.14159265...$ can be extracted from the colour spectrum. Quantum mechanics and colours are beautiful subjects indeed.

Ref. 142

* *

If atoms contain orbiting electrons, the rotation of the Earth, via the Coriolis acceleration, should have an effect on their motion, and thus on the colour of atoms. This beautiful prediction is due to Mark Silverman; the effect is so small, however, that it has not yet been measured.

Ref. 136

* *

Light is diffracted by material gratings. Can matter be diffracted by light gratings? Surprisingly, it actually can, as predicted by Dirac and Kapitza in 1937. This was accomplished for the first time in 1986, using atoms. For free electrons, the feat is more difficult; the clearest confirmation came in 2001, when new laser technology was used to perform a beautiful measurement of the typical diffraction maxima for electrons diffracted by a light grating.

<div style="text-align:center">* *</div>

Light is totally reflected when it is directed to a dense material at a large enough angle so that it cannot enter the material. A group of Russian physicists have shown that if the dense material is excited, the intensity of the totally-reflected beam can be *amplified*. It is unclear whether this will ever lead to applications.

<div style="text-align:center">* *</div>

The ways people handle *single* atoms with electromagnetic fields provide many beautiful examples of modern applied technologies. Nowadays it is possible to levitate, to trap, to excite, to photograph, to deexcite and to move single atoms just by shining light onto them. In 1997, the Nobel Prize in Physics has been awarded to the originators of the field, Steven Chu, Claude Cohen-Tannoudji and William Philips.

<div style="text-align:center">* *</div>

Given two mirrors and a few photons, it is possible to capture an atom and keep it floating between the two mirrors. This feat, one of several ways to isolate single atoms, is now standard practice in laboratories. Can you imagine how it is done?

<div style="text-align:center">* *</div>

An example of modern research is the study of *hollow* atoms, i.e., atoms missing a number of inner electrons. They have been discovered in 1990 by J.P. Briand and his group. They appear when a completely ionized atom, i.e., one without any electrons, is brought in contact with a metal. The acquired electrons then orbit on the outside, leaving the inner shells empty, in stark contrast with usual atoms. Such hollow atoms can also be formed by intense laser irradiation.

<div style="text-align:center">* *</div>

Relativistic quantum effects can be seen with the unaided eye. The two most important ones concern gold and mercury. The yellow colour of gold – which has atomic number 79 – is due to the transition energy between 5d and 6s electrons, which absorbs blue light. Without relativistic effects, this transition would lie in the ultraviolet, similar to the transition between 4d and 5s electrons for silver, and gold would be colourless. The yellow colour of gold is thus a relativistic effect.

Mercury – which has atomic number 80 – has a filled 6s shell. Due to the same relativistic effects that appear in gold, these shells are contracted and do not like to form bonds. For this reason, mercury is still liquid a room temperature, in contrast to all other metals. Relativity is thus the reason that mercury is liquid, and that thermometers work.

<div style="text-align:center">* *</div>

Is phosphorus phosphorescent?

* *

It is possible to detect the passage of a single photon through an apparatus without ab-
sorbing it. How would you do this?

Material properties

Like the size of hydrogen atoms, also the size of all other atoms is fixed by the quantum
of action. Indeed, the quantum of action determines to a large degree the interactions
among electrons. By doing so, the quantum of change determines all the interactions
between atoms in everyday matter; therefore it determines all other material properties.
The elasticity, the plasticity, the brittleness, the magnetic and electric properties of ma-
terials are equally fixed by the quantum of action. Only $\hbar$ makes electronics possible! We
will study some examples of material properties in the next volume. Various details of
the general connection between $\hbar$ and material properties are still a subject of research,
though none is in contradiction with the quantum of action. Material research is among
the most important fields of modern science, and most advances in our standard of living
result from it. We will explore some aspects in the next volume.

In summary, materials science has confirmed that quantum physics is also the correct
description of all materials; quantum physics has confirmed that all material properties
of everyday life are of electromagnetic origin; and quantum physics has confirmed that
all material properties of everyday life are due to interactions that involve electrons.

A tough challenge: the strength of electromagnetism

The great physicist Wolfgang Pauli used to say that after his death, the first thing he
would ask god would be to explain Sommerfeld's fine-structure constant. (Others used
to comment that after god will have explained it to him, he will think a little, and then
snap: 'Wrong!')

The *fine-structure constant*, introduced by Arnold Sommerfeld, is the dimensionless
constant of nature whose value is measured to be

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137.035\,999\,679(94)} \approx 0.007\,297\,352\,5376(50)\ . \tag{112}$$

This number first appeared in explanations of the fine structure of atomic colour spectra;
hence its strange name. Sommerfeld was the first to understand its general importance.
It is central to quantum electrodynamics for several reasons.

First of all, the fine-structure constant describes the *strength* of electromagnetism.
The number α results from the interaction of two electric charges e. Writing Coulomb's
relation for the force F between two electrons as

$$F = \alpha\frac{\hbar c}{r^2} \tag{113}$$

it becomes clear that the fine-structure constant describes the strength of electromagnet-

ism. A higher value for the fine-structure constant α would mean a stronger attraction or repulsion between charged bodies. Thus the value of α determines the sizes of atoms, and indeed of all things, as well as all colours in nature.

Secondly, it is only because the fine-structure constant α is so small that we are able to talk about particles at all. Indeed, only because the fine-structure constant is much smaller than 1 it is possible to distinguish particles from each other. If the number α were near to or larger than 1, particles would interact so strongly that it would not be possible to observe them separately or to talk about particles at all.

This leads on to the third reason for the importance of the fine-structure constant. Since it is a dimensionless number, it implies some yet-unknown mechanism that fixes its value. Uncovering this mechanism is one of the challenges remaining in our adventure. As long as the mechanism remains unknown – as was the case in 2016 – we do not understand the colour and size of a single thing around us!

Small changes in the strength of electromagnetic attraction between electrons and protons would have numerous important consequences. Can you describe what would happen to the size of people, to the colour of objects, to the colour of the Sun, or to the workings of computers, if the strength were to double? And what if it were to gradually drop to half its usual value?

Since the 1920s, explaining the value of α has been seen as one of the toughest challenges facing modern physics. That is the reason for Pauli's fantasy. In 1946, during his Nobel Prize lecture, he repeated the statement that a theory that does not determine this number cannot be complete. Since that time, physicists seem to have fallen into two classes: those who did not dare to take on the challenge, and those who had no clue. This fascinating story still awaits us.

The problem of the fine-structure constant is so deep that it leads many astray. For example, it is sometimes claimed that it is impossible to change physical units in such a way that $\hbar$, c and e are all equal to 1 at the same time, because to do so would change the number $\alpha = 1/137.036...(1)$. Can you explain why the argument is wrong?

A summary on colours and materials

In summary, the quantum of action $\hbar$ – together with the interaction between electromagnetic fields and the electrons inside atoms, molecules, liquids and solids – determines the size, the shape, the colour and the material properties of all things around us. The quantum of action determines mechanical properties such as hardness or elasticity, magnetic properties, thermal properties such as heat capacity or heat of condensation, optical properties such as transparency, and electrical properties such as metallic shine. In addition, the quantum of action determines all chemical and biological aspects of matter. This connection is the topic of the next volume.

The strength of the electromagnetic interaction is described by the fine-structure constant $\alpha \approx 1/137.036$. Its value is yet unexplained.

Challenge 157 s

Ref. 149

Challenge 158 s

QUANTUM PHYSICS IN A NUTSHELL
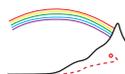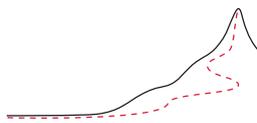
Compared to classical physics, quantum theory is definitely more complex. The basic idea however, is simple: in nature there is a smallest change, or a smallest action with the value $\hbar$ = 1.1 · 10^{-34} Js. More precisely, all of quantum theory can be resumed in one sentence:

▷ In nature, actions or changes smaller than $\hbar$ = 1.054 571 800(13) · 10^{-34} Js
are not observed.

This smallest action value, the *quantum of action*, leads to all the strange observations made in the microscopic domain, such as the wave behaviour of matter, indeterminacy relations, decoherence, randomness in measurements, indistinguishability, quantization of angular momentum, tunnelling, pair creation, decay and particle reactions.

The essence of quantum theory is thus the lack of infinitely small change. The mathematics of quantum theory is abstract and involved, though. Was this part of our walk worth the effort? It was: the results are profound, and the accuracy of the description is *complete*. We first give an overview of these results and then turn to the questions that are still left open.

Physical results of quantum theory

The existence of a smallest action value in nature leads directly to the main lesson we learned about motion in the quantum part of our adventure:

▷ If it moves, it is made of quantons, or quantum particles.

This statement applies to every physical system, thus to all objects and to all images, i.e., to all matter and radiation. Moving stuff is made of *quantons*. Stones, water waves, light, sound waves, earthquakes, gelatine and everything else we can interact with is made of quantum particles.

In our exploration of relativity we discovered that also horizons and the vacuum can move. If all moving entities are made of quantum particles, what does this imply for horizons and empty space? We can argue that no fundamental problems are expected for horizons, because one way to describe horizons is as an extreme state of matter. But the details for the quantum aspects of vacuum are not simple; they will be the topic of the last part of this adventure.

Earlier in our adventure we asked: what are matter, radiation and interactions? Now we know: they all are composites of elementary quantum particles. In particular, interactions are exchanges of elementary quantum particles.

An *elementary quantum particle* is a countable entity that is smaller than its own Compton wavelength. All elementary particles are described by energy–momentum, mass, spin, C, P and T parity. However, as we will see in the next volume, this is not yet the complete list of particle properties. About the *intrinsic* properties of quantum particles, i.e., those that do not depend on the observer, quantum theory makes a simple statement:

> ▷ In nature, all intrinsic properties of quantons, or quantum particles – with the exception of mass – such as spin, electric charge, strong charge, parities etc., appear as integer multiples of a basic unit. Since all physical systems are made of quantons, in composed systems all intrinsic properties – with the exception of mass* – either add or multiply.

In summary, all moving entities are made of quantum particles described by *discrete* intrinsic properties. To see how deep this result is, you can apply it to all those moving entities for which it is usually forgotten, such as ghosts, spirits, angels, nymphs, daemons, devils, gods, goddesses and souls. You can check yourself what happens when their particle nature is taken into account.

> “ *Deorum injuriae diis curae.*** ”
>
> Tiberius, as reported by Tacitus.

RESULTS ON THE MOTION OF QUANTUM PARTICLES

Quantons, or quantum particles, differ from everyday particles: quantum particles *interfere*: they behave like a mixture of particles and waves. This property follows directly from the existence of $\hbar$, the smallest possible action in nature. From the existence of $\hbar$, quantum theory deduces all its statements about quantum particle motion. We summarize the main ones.

There is *no rest* in nature. All objects obey the indeterminacy relation, which states that the indeterminacies in position x and momentum p follow

$$\Delta x \Delta p \geqslant \hbar/2 \quad \text{with} \quad \hbar = 1.1 \cdot 10^{-34} \,\text{Js} \tag{114}$$

and making rest an impossibility. The state of quantum particles is defined by the same observables as in classical physics, with the difference that observables do not commute. Classical physics appears in the limit that the Planck constant $\hbar$ can effectively be set to zero.

Quantum theory introduces a *probabilistic element* into motion. Probabilities result from the quantum of action through the interactions with the baths that are part of the

---

* More precisely, together with mass, also mixing angles are not quantized. These properties are defined in the next volume.
** 'Offenses of gods are care of the gods.'

environment of every physical system. Equivalently, probabilities result in every experiment that tries to induce a change that is smaller than the quantum of action.

Quantum particles *behave like waves*. The associated de Broglie wavelength λ is given by the momentum p through

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p} \tag{115}$$

both in the case of matter and of radiation. This relation is the origin of the wave behaviour of light and matter. The light particles are called photons; their observation is now standard practice. Quantum theory states that particle waves, like all waves, interfere, refract, disperse, dampen, can be dampened and can be polarized. This applies to photons, electrons, atoms and molecules. All waves being made of quantum particles, all waves can be seen, touched and moved. Light for example, can be 'seen' in photon-photon scattering in vacuum at high energies, can be 'touched' using the Compton effect, and can be 'moved' by gravitational bending. Matter particles, such as molecules or atoms, can be seen in electron microscopes and can be touched and moved with atomic force microscopes. The interference and diffraction of wave particles is observed daily in the electron microscope.

Matter waves can be imagined as clouds that *rotate locally*. In the limit of negligible cloud size, quantum particles can be imagined as rotating little arrows. Equivalently, quantons have a *phase*.

Particles *cannot be enclosed forever*. Even though matter is impenetrable, quantum theory shows that tight boxes or insurmountable obstacles do not exist. Enclosure is never forever. Waiting long enough always allows us to overcome any boundary, since there is a finite probability to overcome any obstacle. This process is called *tunnelling* when seen from the spatial point of view and is called *decay* when seen from the temporal point of view. Tunnelling explains the working of television tubes as well as radioactive decay.

All particles and all particle beams *can be rotated*. Particles possess an intrinsic angular momentum called *spin*, specifying their behaviour under rotations. Bosons have integer spin, fermions have half integer spin. An even number of bound fermions or any number of bound bosons yield a composite boson; an odd number of bound fermions yield a low-energy fermion. Solids are impenetrable because of the fermion character of its electrons in the atoms.

Identical particles are *indistinguishable*. Radiation is made of indistinguishable particles called bosons, matter of fermions. Under exchange of two fermions at space-like separations, the wave function changes sign, whereas for two bosons the wave function remains unchanged. All other properties of quantum particles are the same as for classical particles, namely countability, interaction, mass, charge, angular momentum, energy, momentum, position, as well as impenetrability for matter and penetrability for radiation. Perfect copying machines do not exist.

In collisions, particles *interact locally*, through the exchange of other particles. When matter particles collide, they interact through the exchange of virtual bosons, i.e., off-shell bosons. Motion change is thus due to particle exchange. Exchange bosons of even spin mediate only attractive interactions. Exchange bosons of odd spin mediate repulsive interactions as well.

The properties of collisions imply the non-conservation of particle number. In collisions, particles can appear – i.e., can be 'created' – or disappear – i.e., can be 'annihilated'. This is valid both for bosons and for fermions.

The properties of collisions imply the existence of *antiparticles*, which are regularly observed in experiments. Elementary fermions, in contrast to many elementary bosons, differ from their antiparticles; they can be created and annihilated only in pairs. Elementary fermions have non-vanishing mass and move slower than light.

Particles can *decay* and be *transformed*. Detailed investigations show that collisions imply the non-conservation of particle type. In collisions, selected particles can change their intrinsic properties. This observation will be detailed in the next volume. Equivalently, the quantum of action implies that things *break* and living beings *die*.

Images, made of radiation, are described by the same observables as matter: position, phase, speed, mass, momentum etc. – though their values and relations differ. Images can only be localized with a precision of the wavelength λ of the radiation producing them.

The appearance of Planck's constant $\hbar$ implies that *length scales* and *time scales* exist in nature. Quantum theory introduces a fundamental jitter in every example of motion. Thus the infinitely small is eliminated. In this way, lower limits to structural dimensions and to many other measurable quantities appear. In particular, quantum theory shows that it is impossible that on the electrons in an atom small creatures live in the same way that humans live on the Earth circling the Sun. Quantum theory shows the impossibility of Lilliput.

Clocks and metre bars have *finite precision*, due to the existence of a smallest action and due to their interactions with baths. On the other hand, all measurement apparatuses must contain baths, since otherwise they would not be able to record results.

Quantum effects leave no room for cold fusion, astrology, teleportation, telekinesis, supernatural phenomena, creation out of nothing, multiple universes, or faster than light phenomena – the EPR paradox notwithstanding.

ACHIEVEMENTS IN ACCURACY AND PRECISION

Apart from the conceptual changes, quantum theory improved the accuracy of predictions from the few – if any – digits common in classical mechanics to the full number of digits – sometimes thirteen – that can be measured today. The limited precision is usually *not* given by the inaccuracy of theory, it is given by the measurement accuracy. In other words, the agreement is only limited by the amount of money the experimenter is willing to spend. Table 8 shows this in more detail.

TABLE 8 Selected comparisons between classical physics, quantum theory and experiment.

OBSERVABLE	CLASSICAL PREDICTION	PREDICTION OF QUANTUM THEORY[a]	MEASUREMENT	COST ESTIMATE
Simple motion of bodies				
Indeterminacy	0	$\Delta x \Delta p \geqslant \hbar/2$	$(1 \pm 10^{-2})\,\hbar/2$	10 k€

OBSERVABLE	CLASSICAL PREDICTION	PREDICTION OF QUANTUM THEORY[a]	MEASUREMENT	COST ESTIMATE
Matter wavelength	none	$\lambda p = 2\pi\hbar$	$(1 \pm 10^{-2})\,\hbar$	10 k€
Compton wavelength	none	$\lambda_c = h/m_e c$	$(1 \pm 10^{-3})\,\lambda$	20 k€
Pair creation rate	0	σE	agrees	100 k€
Radiative decay time in hydrogen	none	$\tau \sim 1/n^3$	(1 ± 10^{-2})	5 k€
Smallest angular momentum	0	$\hbar/2$	$(1 \pm 10^{-6})\,\hbar/2$	10 k€
Casimir effect/pressure	0	$p = (\pi^2 \hbar c)/(240 r^4)$	(1 ± 10^{-3})	30 k€
Colours of objects				
Spectrum of hot objects	diverges	$\lambda_{max} = hc/(4.956\,kT)$	$(1 \pm 10^{-4})\,\Delta\lambda$	10 k€
Lamb shift	none	$\Delta\lambda = 1057.86(1)$ MHz	$(1 \pm 10^{-6})\,\Delta\lambda$	50 k€
Rydberg constant	none	$R_\infty = m_e c \alpha^2/2h$	$(1 \pm 10^{-9})\,R_\infty$	50 k€
Stefan–Boltzmann constant	none	$\sigma = \pi^2 k^4/60\hbar^3 c^2$	$(1 \pm 3 \cdot 10^{-8})\,\sigma$	20 k€
Wien's displacement constant	none	$b = \lambda_{max} T$	$(1 \pm 10^{-5})\,b$	20 k€
Refractive index of water	none	1.34	within a few %	1 k€
Photon-photon scattering	0	from QED: finite	agrees	50 M€
Electron gyromagnetic ratio	1 or 2	2.002 319 304 365(7)	2.002 319 304 361 53(53)	30 M€
Muon anomalous magnetic moment	0	$11\,659\,1827(63) \cdot 10^{-11}$	$11\,659\,2080(60) \cdot 10^{-11}$	100 M€
Composite matter properties				
Atom lifetime	$\approx 1\,\mu s$	∞	$> 10^{20}$ a	1 €
Muonium hyperfine splitting	none	4 463 302 542(620) Hz	4 463 302 765(53) Hz	1 M€
Molecular size and shape	none	from QED	within 10^{-3}	20 k€

a. All these predictions are calculated from the basic physical constants given in Appendix A.

We notice that the predicted values *do not differ* from the measured ones. If we remember that classical physics does not allow us to calculate any of the measured values, we get an idea of the progress quantum physics has achieved. This advance in understanding is due to the introduction of the quantum of action $\hbar$. Equivalently, we can state: no description of nature without the quantum of action is complete.

In summary, quantum theory is precise and accurate. In the microscopic domain quantum theory is in *perfect* correspondence with nature; despite prospects of fame and riches, despite the largest number of researchers ever, no contradiction between observation and theory has been found yet. On the other hand, explaining the measured value

of the fine-structure constant, $\alpha = 1/137.035\,999\,074(44)$, remains an open problem of the electromagnetic interaction.

Is quantum theory magic?

Studying nature is like experiencing magic. Nature often looks different from what it is. During magic we are fooled – but only if we forget our own limitations. Once we start to see ourselves as part of the game, we start to understand the tricks. That is the fun of magic. The same happens in quantum motion.

∗ ∗

Nature seems irreversible, even though it isn't. We never remember the future. We are fooled because we are macroscopic.

∗ ∗

Nature seems decoherent, even though it isn't. We are fooled again because we are macroscopic.

∗ ∗

There are no clocks in nature. We are fooled by those of everyday life because we are surrounded by a huge number of particles.

∗ ∗

Motion often seems to disappear, even though it is eternal. We are fooled again, because our senses cannot experience the microscopic domain.

∗ ∗

Objects seem distinguishable, even though the statistical properties of their components show that they are not. We are fooled because we live at low energies.

∗ ∗

Matter seems continuous, even though it isn't. We are fooled because of the limitations of our senses.

∗ ∗

Motion seems deterministic in the classical sense, even though it is random. We are fooled again because we are macroscopic.

∗ ∗

In short, our human condition permanently fools us. The answer to the title question is: classical physics is like magic, and the tricks are uncovered by quantum theory. That is its main attraction.

Quantum theory is exact, but can do more

We can summarize this part of our adventure with a simple statement:

▷ Quantum physics is the description of matter and radiation *without* the concept of infinitely small.

All change in nature, in fact, everything is described by *finite* quantities, and above all, by the smallest change possible in nature, the quantum of action $\hbar$ = $1.054\,571\,800(13) \cdot 10^{-34}$ Js.

All experiments, without exception, show that the quantum of action $\hbar$ is the smallest observable change. The description of nature with the quantum of action is thus *exact* and *final*. The smallest measurable action $\hbar$, like the maximum energy speed c, is a fundamental property of nature. One could also call both of them *fundamental truths*.

Since quantum theory follows logically and completely from the smallest measurable action $\hbar$, the simplest way – and the only way – to disprove quantum theory is to find an observation that contradicts the smallest change value $\hbar$. Try it!

Even though we have deduced a fundamental property of nature, if we turn back to the

start of our exploration of quantum theory, we cannot hide a certain disappointment. We know that classical physics cannot explain life. Searching for the details of microscopic motion, we encountered so many interesting aspects that we have *not yet* achieved the explanation of life. For example, we know what determines the speed of electrons in atoms, but we do not know what determines the running speed of an athlete. In fact, we have not even discussed the properties of any solid or liquid, let alone those of more complex structures like living beings.

In other terms, after this introduction into quantum theory, we must still connect quantum processes to our everyday world. Therefore, the topic of the next volume will be the exploration of the motion of and inside living things – and of the motion inside all kind of matter, from solids to stars, using the quantum of action as a foundation. After that, we will explore the motion of empty space.
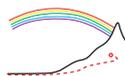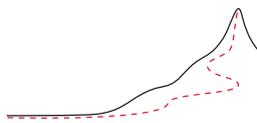
UNITS, MEASUREMENTS AND CONSTANTS

Measurements are comparisons with standards. Standards are based on *units*. Many different systems of units have been used throughout the world. Most of these standards confer power to the organization in charge of them. Such power can be misused; this is the case today, for example in the computer industry, and was so in the distant past. The solution is the same in both cases: organize an independent and global standard. For measurement units, this happened in the eighteenth century: in order to avoid misuse by authoritarian institutions, to eliminate problems with differing, changing and irreproducible standards, and – this is not a joke – to simplify tax collection and to make it more just, a group of scientists, politicians and economists agreed on a set of units. It is called the *Système International d'Unités*, abbreviated *SI*, and is defined by an international treaty, the 'Convention du Mètre'. The units are maintained by an international organization, the 'Conférence Générale des Poids et Mesures', and its daughter organizations, the 'Commission Internationale des Poids et Mesures' and the 'Bureau International des Poids et Mesures' (BIPM). All originated in the times just before the French revolution.

SI units

All SI units are built from seven *base units*. Their simplest definitions, translated from French into English, are the following ones, together with the dates of their formulation and a few comments:

■ 'The *second* is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.' (1967) The 2019 definition is equivalent, but much less clear.*

■ 'The *metre* is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.' (1983) The 2019 definition is equivalent, but much less clear.*

■ 'The *kilogram*, symbol kg, is the SI unit of mass. It is defined by taking the fixed numerical value of the Planck constant h to be $6.626\,070\,15 \cdot 10^{-34}$ when expressed in the unit J · s, which is equal to $kg \cdot m^2 \cdot s^{-1}$.' (2019)*

■ 'The *ampere*, symbol A, is the SI unit of electric current. It is defined by taking the fixed numerical value of the elementary charge e to be $1.602\,176\,634 \cdot 10^{-19}$ when expressed in the unit C, which is equal to A · s.' (2019)* This definition is equivalent to: One ampere is $6.241\,509\,074... \cdot 10^{18}$ elementary charges per second.

■ 'The *kelvin*, symbol K, is the SI unit of thermodynamic temperature. It is defined by

taking the fixed numerical value of the Boltzmann constant k to be $1.380649 \cdot 10^{-23}$ when expressed in the unit $J \cdot K^{-1}$.' (2019)*

▪ 'The mole, symbol mol, is the SI unit of amount of substance. One mole contains exactly $6.02214076 \cdot 10^{23}$ elementary entities.' (2019)*

▪ 'The *candela* is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency $540 \cdot 10^{12}$ hertz and has a radiant intensity in that direction of (1/683) watt per steradian.' (1979) The 2019 definition is equivalent, but much less clear.*

We note that both time and length units are defined as certain properties of a standard example of motion, namely light. In other words, also the Conférence Générale des Poids et Mesures makes the point that the observation of motion is a *prerequisite* for the definition and construction of time and space. *Motion is the fundament of every observation and of all measurement.* By the way, the use of light in the definitions had been proposed already in 1827 by Jacques Babinet.**

From these basic units, all other units are defined by multiplication and division. Thus, all SI units have the following properties:

▪ SI units form a system with *state-of-the-art precision*: all units are defined with a precision that is higher than the precision of commonly used measurements. Moreover, the precision of the definitions is regularly being improved. The present relative uncertainty of the definition of the second is around 10^{-14}, for the metre about 10^{-10}, for the kilogram about 10^{-9}, for the ampere 10^{-7}, for the mole less than 10^{-6}, for the kelvin 10^{-6} and for the candela 10^{-3}.

▪ SI units form an *absolute* system: all units are defined in such a way that they can be reproduced in every suitably equipped laboratory, independently, and with high precision. This avoids as much as possible any error or misuse by the standard-setting organization. In fact, the SI units are as now as near as possible to Planck's natural units, which are presented below. In practice, the SI is now an international standard defining the numerical values of the seven constants $\Delta\nu_{Cs}$, c, $\hbar$, e, k, N_A and K_{cd}. After over 200 years of discussions, the CGPM has little left to do.

▪ SI units form a *practical* system: the base units are quantities of everyday magnitude. Frequently used units have standard names and abbreviations. The complete list includes the seven base units just given, the supplementary units, the derived units and the admitted units.

The *supplementary* SI units are two: the unit for (plane) angle, defined as the ratio of arc length to radius, is the *radian* (rad). For solid angle, defined as the ratio of the subtended area to the square of the radius, the unit is the *steradian* (sr).

The *derived* units with special names, in their official English spelling, i.e., without capital letters and accents, are:

Ref. 152

NAME	ABBREVIATION	NAME	ABBREVIATION
hertz	$Hz = 1/s$	newton	$N = kg\,m/s^2$
pascal	$Pa = N/m^2 = kg/m\,s^2$	joule	$J = Nm = kg\,m^2/s^2$
watt	$W = kg\,m^2/s^3$	coulomb	$C = As$
volt	$V = kg\,m^2/As^3$	farad	$F = As/V = A^2s^4/kg\,m^2$
ohm	$\Omega = V/A = kg\,m^2/A^2s^3$	siemens	$S = 1/\Omega$
weber	$Wb = Vs = kg\,m^2/As^2$	tesla	$T = Wb/m^2 = kg/As^2 = kg/Cs$
henry	$H = Vs/A = kg\,m^2/A^2s^2$	degree Celsius	°C (see definition of kelvin)
lumen	$lm = cd\,sr$	lux	$lx = lm/m^2 = cd\,sr/m^2$
becquerel	$Bq = 1/s$	gray	$Gy = J/kg = m^2/s^2$
sievert	$Sv = J/kg = m^2/s^2$	katal	$kat = mol/s$

We note that in all definitions of units, the kilogram only appears to the powers of 1, 0 and −1. Can you try to formulate the reason?

The *admitted* non-SI units are *minute*, *hour*, *day* (for time), *degree* $1° = \pi/180$ rad, *minute* $1' = \pi/10\,800$ rad, *second* $1'' = \pi/648\,000$ rad (for angles), *litre*, and *tonne*. All other units are to be avoided.

All SI units are made more practical by the introduction of standard names and abbreviations for the powers of ten, the so-called *prefixes*:*

POWER	NAME		POWER	NAME		POWER	NAME		POWER	NAME	
10^1	deca	da	10^{-1}	deci	d	10^{18}	Exa	E	10^{-18}	atto	a
10^2	hecto	h	10^{-2}	centi	c	10^{21}	Zetta	Z	10^{-21}	zepto	z
10^3	kilo	k	10^{-3}	milli	m	10^{24}	Yotta	Y	10^{-24}	yocto	y
10^6	Mega	M	10^{-6}	micro	μ	unofficial:			Ref. 153		
10^9	Giga	G	10^{-9}	nano	n	10^{27}	Xenta	X	10^{-27}	xenno	x
10^{12}	Tera	T	10^{-12}	pico	p	10^{30}	Wekta	W	10^{-30}	weko	w
10^{15}	Peta	P	10^{-15}	femto	f	10^{33}	Vendekta	V	10^{-33}	vendeko	v
						10^{36}	Udekta	U	10^{-36}	udeko	u

▪ SI units form a *complete* system: they cover in a systematic way the full set of observables of physics. Moreover, they fix the units of measurement for all other sciences as well.

* Some of these names are invented (yocto to sound similar to Latin *octo* 'eight', zepto to sound similar to Latin *septem*, yotta and zetta to resemble them, exa and peta to sound like the Greek words ἑξάκις and πεντάκις for 'six times' and 'five times', the unofficial ones to sound similar to the Greek words for nine, ten, eleven and twelve); some are from Danish/Norwegian (atto from *atten* 'eighteen', femto from *femten* 'fifteen'); some are from Latin (from *mille* 'thousand', from *centum* 'hundred', from *decem* 'ten', from *nanus* 'dwarf'); some are from Italian (from *piccolo* 'small'); some are Greek (micro is from μικρός 'small', deca/deka from δέκα 'ten', hecto from ἑκατόν 'hundred', kilo from χίλιοι 'thousand', mega from μέγας 'large', giga from γίγας 'giant', tera from τέρας 'monster').

Translate: I was caught in such a traffic jam that I needed a microcentury for a picoparsec and that my car's fuel consumption was two tenths of a square millimetre.

> ■ SI units form a *universal* system: they can be used in trade, in industry, in commerce, at home, in education and in research. They could even be used by extraterrestrial civilizations, if they existed.
> ■ SI units form a *self-consistent* system: the product or quotient of two SI units is also an SI unit. This means that in principle, the same abbreviation, e.g. 'SI', could be used for every unit.

The SI units are not the only possible set that could fulfil all these requirements, but they are the only existing system that does so.[*]

The meaning of measurement

Every measurement is a comparison with a standard. Therefore, any measurement requires *matter* to realize the standard (even for a speed standard), and *radiation* to achieve the comparison. The concept of measurement thus assumes that matter and radiation exist and can be clearly separated from each other.

Every measurement is a comparison. Measuring thus implies that space and time exist, and that they differ from each other.

Every measurement produces a measurement result. Therefore, every measurement implies the *storage* of the result. The process of measurement thus implies that the situation before and after the measurement can be distinguished. In other terms, every measurement is an *irreversible* process.

Every measurement is a process. Thus every measurement takes a certain amount of time and a certain amount of space.

All these properties of measurements are simple but important. Beware of anybody who denies them.

Planck's natural units

Since the exact form of many equations depends on the system of units used, theoretical physicists often use unit systems optimized for producing simple equations. The chosen units and the values of the constants of nature are related. In microscopic physics, the system of *Planck's natural units* is frequently used. They are defined by setting $c = 1$, $\hbar = 1$, $G = 1$, $k = 1$, $\varepsilon_0 = 1/4\pi$ and $\mu_0 = 4\pi$. Planck units are thus defined from combinations of fundamental constants; those corresponding to the fundamental SI units are given in Table 10.[**] The table is also useful for converting equations written in natural units back to SI units: just substitute every quantity X by X/X_{Pl}.

---

[*] Apart from international units, there are also *provincial* units. Most provincial units still in use are of Roman origin. The mile comes from *milia passum*, which used to be one thousand (double) strides of about 1480 mm each; today a nautical mile, once defined as minute of arc on the Earth's surface, is defined as exactly 1852 m. The inch comes from *uncia*/*onzia* (a twelfth – now of a foot). The pound (from *pondere* 'to weigh') is used as a translation of *libra* – balance – which is the origin of its abbreviation lb. Even the habit of counting in dozens instead of tens is Roman in origin. These and all other similarly funny units – like the system in which all units start with 'f', and which uses furlong/fortnight as its unit of velocity – are now officially defined as multiples of SI units.

[**] The natural units x_{Pl} given here are those commonly used today, i.e., those defined using the constant $\hbar$, and not, as Planck originally did, by using the constant $h = 2\pi\hbar$. The electromagnetic units can also be defined with other factors than $4\pi\varepsilon_0$ in the expressions: for example, using $4\pi\varepsilon_0\alpha$, with the *fine-structure constant* α, gives $q_{Pl} = e$. For the explanation of the numbers between brackets, see below.

TABLE 10 Planck's (uncorrected) natural units.

NAME			DEFINITION		VALUE
Basic units					
the Planck length	l_{Pl}	$=$	$\sqrt{\hbar G/c^3}$	$=$	$1.616\,0(12)\cdot10^{-35}$ m
the Planck time	t_{Pl}	$=$	$\sqrt{\hbar G/c^5}$	$=$	$5.390\,6(40)\cdot10^{-44}$ s
the Planck mass	m_{Pl}	$=$	$\sqrt{\hbar c/G}$	$=$	$21.767(16)\,\mu$g
the Planck current	I_{Pl}	$=$	$\sqrt{4\pi\varepsilon_0 c^6/G}$	$=$	$3.479\,3(22)\cdot10^{25}$ A
the Planck temperature	T_{Pl}	$=$	$\sqrt{\hbar c^5/Gk^2}$	$=$	$1.417\,1(91)\cdot10^{32}$ K
Trivial units					
the Planck velocity	v_{Pl}	$=$	c	$=$	0.3 Gm/s
the Planck angular momentum	L_{Pl}	$=$	$\hbar$	$=$	$1.1\cdot10^{-34}$ Js
the Planck action	S_{aPl}	$=$	$\hbar$	$=$	$1.1\cdot10^{-34}$ Js
the Planck entropy	S_{ePl}	$=$	k	$=$	13.8 yJ/K
Composed units					
the Planck mass density	ρ_{Pl}	$=$	$c^5/G^2\hbar$	$=$	$5.2\cdot10^{96}$ kg/m^3
the Planck energy	E_{Pl}	$=$	$\sqrt{\hbar c^5/G}$	$=$	2.0 GJ $= 1.2\cdot10^{28}$ eV
the Planck momentum	p_{Pl}	$=$	$\sqrt{\hbar c^3/G}$	$=$	6.5 Ns
the Planck power	P_{Pl}	$=$	c^5/G	$=$	$3.6\cdot10^{52}$ W
the Planck force	F_{Pl}	$=$	c^4/G	$=$	$1.2\cdot10^{44}$ N
the Planck pressure	p_{Pl}	$=$	$c^7/G\hbar$	$=$	$4.6\cdot10^{113}$ Pa
the Planck acceleration	a_{Pl}	$=$	$\sqrt{c^7/\hbar G}$	$=$	$5.6\cdot10^{51}$ m/s^2
the Planck frequency	f_{Pl}	$=$	$\sqrt{c^5/\hbar G}$	$=$	$1.9\cdot10^{43}$ Hz
the Planck electric charge	q_{Pl}	$=$	$\sqrt{4\pi\varepsilon_0 c\hbar}$	$=$	1.9 aC $= 11.7$ e
the Planck voltage	U_{Pl}	$=$	$\sqrt{c^4/4\pi\varepsilon_0 G}$	$=$	$1.0\cdot10^{27}$ V
the Planck resistance	R_{Pl}	$=$	$1/4\pi\varepsilon_0 c$	$=$	$30.0\,\Omega$
the Planck capacitance	C_{Pl}	$=$	$4\pi\varepsilon_0\sqrt{\hbar G/c^3}$	$=$	$1.8\cdot10^{-45}$ F
the Planck inductance	L_{Pl}	$=$	$(1/4\pi\varepsilon_0)\sqrt{\hbar G/c^7}$	$=$	$1.6\cdot10^{-42}$ H
the Planck electric field	E_{Pl}	$=$	$\sqrt{c^7/4\pi\varepsilon_0\hbar G^2}$	$=$	$6.5\cdot10^{61}$ V/m
the Planck magnetic flux density	B_{Pl}	$=$	$\sqrt{c^5/4\pi\varepsilon_0\hbar G^2}$	$=$	$2.2\cdot10^{53}$ T

The natural units are important for another reason: whenever a quantity is sloppily called 'infinitely small (or large)', the correct expression is 'as small (or as large) as the corresponding corrected Planck unit'. As explained throughout the text, and especially in the final part, this substitution is possible because almost all Planck units provide, within a correction factor of order 1, the extremal value for the corresponding observable – some an upper and some a lower limit. Unfortunately, these correction factors are not yet widely known. The exact extremal value for each observable in nature is obtained

when G is substituted by $4G$ and $4\pi\varepsilon_0$ by $4\pi\varepsilon_0\alpha$ in all Planck quantities. These extremal values, or *corrected Planck units*, are the *true natural units*. To exceed the extremal values

is possible only for some extensive quantities. (Can you find out which ones?)

OTHER UNIT SYSTEMS

A central aim of research in high-energy physics is the calculation of the strengths of all interactions; therefore it is not practical to set the gravitational constant G to unity, as in the Planck system of units. For this reason, high-energy physicists often only set $c = \hbar = k = 1$ and $\mu_0 = 1/\varepsilon_0 = 4\pi$,* leaving only the gravitational constant G in the equations.

In this system, only one fundamental unit exists, but its choice is free. Often a standard length is chosen as the fundamental unit, length being the archetype of a measured quantity. The most important physical observables are then related by

$$
\begin{aligned}
1/[l^2] &= [E]^2 = [F] = [B] = [E_{\text{electric}}]\,, \\
1/[l] &= [E] = [m] = [p] = [a] = [f] = [I] = [U] = [T]\,, \\
1 &= [v] = [q] = [e] = [R] = [S_{\text{action}}] = [S_{\text{entropy}}] = \hbar = c = k = [\alpha]\,, \quad (116) \\
[l] &= 1/[E] = [t] = [C] = [L] \quad \text{and} \\
[l]^2 &= 1/[E]^2 = [G] = [P]
\end{aligned}
$$

where we write $[x]$ for the unit of quantity x. Using the same unit for time, capacitance and inductance is not to everybody's taste, however, and therefore electricians do not use this system.**

Often, in order to get an impression of the energies needed to observe an effect under study, a standard energy is chosen as fundamental unit. In particle physics the most common energy unit is the *electron volt* eV, defined as the kinetic energy acquired by an electron when accelerated by an electrical potential difference of 1 volt ('proton volt' would be a better name). Therefore one has $1\,\text{eV} = 1.6 \cdot 10^{-19}$ J, or roughly

$$
1\,\text{eV} \approx \tfrac{1}{6}\,\text{aJ} \tag{117}
$$

which is easily remembered. The simplification $c = \hbar = 1$ yields $G = 6.9 \cdot 10^{-57}\,\text{eV}^{-2}$ and allows one to use the unit eV also for mass, momentum, temperature, frequency, time

and length, with the respective correspondences $1\,\text{eV} \equiv 1.8 \cdot 10^{-36}\,\text{kg} \equiv 5.4 \cdot 10^{-28}\,\text{Ns} \equiv 242\,\text{THz} \equiv 11.6\,\text{kK}$ and $1\,\text{eV}^{-1} \equiv 4.1\,\text{fs} \equiv 1.2\,\mu\text{m}$.

---

* Other definitions for the proportionality constants in electrodynamics lead to the Gaussian unit system often used in theoretical calculations, the Heaviside–Lorentz unit system, the electrostatic unit system, and

the electromagnetic unit system, among others.

** In the list, l is length, E energy, F force, E_{electric} the electric and B the magnetic field, m mass, p momentum, a acceleration, f frequency, I electric current, U voltage, T temperature, v speed, q charge, R resistance, P power, G the gravitational constant.

The web page www.chemie.fu-berlin.de/chemistry/general/units_en.html provides a tool to convert various units into each other.

Researchers in general relativity often use another system, in which the *Schwarzschild radius* $r_{\text{S}} = 2Gm/c^2$ is used to measure masses, by setting $c = G = 1$. In this case, mass and length have the *same* dimension, and $\hbar$ has the dimension of an area.

To get some feeling for the unit eV, the following relations are useful. Room temper-ature, usually taken as 20°C or 293 K, corresponds to a kinetic energy per particle of 0.025 eV or 4.0 zJ. The highest particle energy measured so far belongs to a cosmic ray with an energy of $3 \cdot 10^{20}$ eV or 48 J. Down here on the Earth, an accelerator able to pro-duce an energy of about 105 GeV or 17 nJ for electrons and antielectrons has been built, and one able to produce an energy of 14 TeV or 2.2 μJ for protons will be finished soon. Both are owned by CERN in Geneva and have a circumference of 27 km.

The lowest temperature measured up to now is 280 pK, in a system of rhodium nuclei held inside a special cooling system. The interior of that cryostat may even be the coolest point in the whole universe. The kinetic energy per particle correspond-ing to that temperature is also the smallest ever measured: it corresponds to 24 feV or 3.8 vJ $= 3.8 \cdot 10^{-33}$ J. For isolated particles, the record seems to be for neutrons: kinetic energies as low as 10^{-7} eV have been achieved, corresponding to de Broglie wavelengths of 60 nm.

Curiosities and fun challenges about units

The Planck length is roughly the de Broglie wavelength $\lambda_B = h/mv$ of a man walking comfortably ($m = 80$ kg, $v = 0.5$ m/s); this motion is therefore aptly called the 'Planck stroll.'

* *

The Planck mass is equal to the mass of about 10^{19} protons. This is roughly the mass of a human embryo at about ten days of age.

* *

The most precisely measured quantities in nature are the frequencies of certain milli-second pulsars, the frequency of certain narrow atomic transitions, and the Rydberg constant of *atomic* hydrogen, which can all be measured as precisely as the second is defined. The caesium transition that defines the second has a finite linewidth that limits the achievable precision: the limit is about 14 digits.

* *

The most precise clock ever built, using microwaves, had a stability of 10^{-16} during a running time of 500 s. For longer time periods, the record in 1997 was about 10^{-15}; but values around 10^{-17} seem within technological reach. The precision of clocks is limited for short measuring times by noise, and for long measuring times by drifts, i.e., by sys-tematic effects. The region of highest stability depends on the clock type; it usually lies between 1 ms for optical clocks and 5000 s for masers. Pulsars are the only type of clock for which this region is not known yet; it certainly lies at more than 20 years, the time elapsed at the time of writing since their discovery.

* *

The shortest times measured are the lifetimes of certain 'elementary' particles. In par-ticular, the lifetime of certain D mesons have been measured at less than 10^{-23} s. Such times are measured using a bubble chamber, where the track is photographed. Can you

estimate how long the track is? (This is a trick question – if your length cannot be observed with an optical microscope, you have made a mistake in your calculation.)

* *

The longest times encountered in nature are the lifetimes of certain radioisotopes, over 10^{15} years, and the lower limit of certain proton decays, over 10^{32} years. These times are thus much larger than the age of the universe, estimated to be fourteen thousand million

years.

* *

Variations of quantities are often much easier to measure than their values. For example, in gravitational wave detectors, the sensitivity achieved in 1992 was $\Delta l / l = 3 \cdot 10^{-19}$ for

lengths of the order of 1 m. In other words, for a block of about a cubic metre of metal it is possible to measure length changes about 3000 times smaller than a proton radius. These set-ups are now being superseded by ring interferometers. Ring interferometers measuring frequency differences of 10^{-21} have already been built; and they are still being

improved.

Precision and accuracy of measurements

Measurements are the basis of physics. Every measurement has an *error*. Errors are due to lack of precision or to lack of accuracy. *Precision* means how well a result is reproduced when the measurement is repeated; *accuracy* is the degree to which a measurement corresponds to the actual value.

Lack of precision is due to accidental or *random errors*; they are best measured by the *standard deviation*, usually abbreviated σ; it is defined through

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \; , \tag{118}$$

where $\bar{x}$ is the average of the measurements x_i. (Can you imagine why $n - 1$ is used in

the formula instead of n?)

For most experiments, the distribution of measurement values tends towards a normal distribution, also called *Gaussian distribution*, whenever the number of measurements is increased. The distribution, shown in Figure 85, is described by the expression

$$N(x) \approx e^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \; . \tag{119}$$

The square σ^2 of the standard deviation is also called the *variance*. For a Gaussian distri-

bution of measurement values, 2.35σ is the full width at half maximum.

Lack of accuracy is due to *systematic errors*; usually these can only be estimated. This estimate is often added to the random errors to produce a *total experimental error*, some-

times also called *total uncertainty*. The *relative* error or uncertainty is the ratio between the error and the measured value.

For example, a professional measurement will give a result such as 0.312(6) m. The
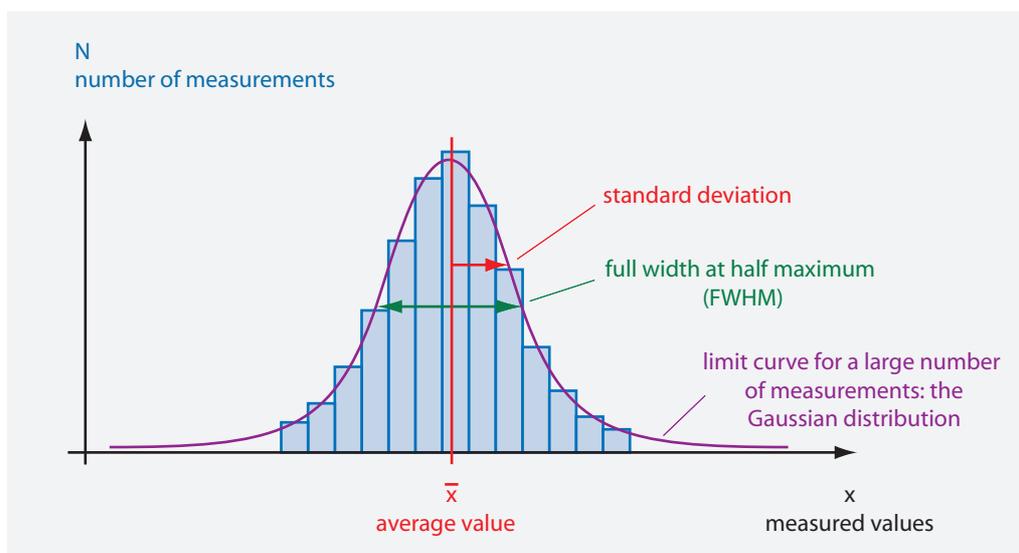
FIGURE 85 A precision experiment and its measurement distribution. The precision is high if the width of the distribution is narrow; the accuracy is high if the centre of the distribution agrees with the actual value.

number between the parentheses is the standard deviation σ, in units of the last digits. As above, a Gaussian distribution for the measurement results is assumed. Therefore, a value of 0.312(6) m implies that the actual value is expected to lie

Challenge 170 e

— within 1σ with 68.3 % probability, thus in this example within 0.312 ± 0.006 m;
— within 2σ with 95.4 % probability, thus in this example within 0.312 ± 0.012 m;
— within 3σ with 99.73 % probability, thus in this example within 0.312 ± 0.018 m;
— within 4σ with 99.9937 % probability, thus in this example within 0.312 ± 0.024 m;
— within 5σ with 99.999 943 % probability, thus in this example within 0.312 ± 0.030 m;
— within 6σ with 99.999 999 80 % probability, thus within 0.312 ± 0.036 m;
— within 7σ with 99.999 999 999 74 % probability, thus within 0.312 ± 0.041 m.

Challenge 171 s

(Do the latter numbers make sense?)

Note that standard deviations have one digit; you must be a world expert to use two, and a fool to use more. If no standard deviation is given, a (1) is assumed. As a result, among professionals, 1 km and 1000 m are *not* the same length!

What happens to the errors when two measured values A and B are added or subtracted? If the all measurements are independent – or uncorrelated – the standard deviation of the sum *and* that of difference is given by $\sigma = \sqrt{\sigma_A^2 + \sigma_B^2}$. For both the product or ratio of two measured and uncorrelated values C and D, the result is $\rho = \sqrt{\rho_C^2 + \rho_D^2}$, where the ρ terms are the *relative* standard deviations.

Challenge 172 s

Assume you measure that an object moves 1 m in 3 s: what is the measured speed value?

LIMITS TO PRECISION

What are the limits to accuracy and precision? There is no way, even in principle, to measure a length x to a *precision* higher than about 61 digits, because in nature, the ratio between the largest and the smallest measurable length is $\Delta x/x > l_{\text{Pl}}/d_{\text{horizon}} = 10^{-61}$.

(Is this ratio valid also for force or for volume?) In the final volume of our text, studies of clocks and metre bars strengthen this theoretical limit.

But it is not difficult to deduce more stringent practical limits. No imaginable machine can measure quantities with a higher precision than measuring the diameter of the Earth within the smallest length ever measured, about 10^{-19} m; that is about 26 digits of precision. Using a more realistic limit of a 1000 m sized machine implies a limit of 22 digits. If, as predicted above, time measurements really achieve 17 digits of precision, then they are nearing the practical limit, because apart from size, there is an additional practical restriction: cost. Indeed, an additional digit in measurement precision often means an additional digit in equipment cost.

PHYSICAL CONSTANTS

In physics, general observations are deduced from more fundamental ones. As a consequence, many measurements can be deduced from more fundamental ones. The most fundamental measurements are those of the physical constants.

The following tables give the world's best values of the most important physical constants and particle properties – in SI units and in a few other common units – as published in the standard references. The values are the world averages of the best measurements made up to the present. As usual, experimental errors, including both random and estimated systematic errors, are expressed by giving the standard deviation in the last digits. In fact, behind each of the numbers in the following tables there is a long

story which is worth telling, but for which there is not enough room here.

In principle, *all* quantitative properties of matter can be calculated with quantum theory – more precisely, equations of the standard model of particle – and a set of *basic* physical constants that are given in the next table. For example, the colour, density and elastic properties of any material can be predicted, in principle, in this way.

TABLE 11 Basic physical constants.

QUANTITY	SYMBOL	VALUE IN SI UNITS	UNCERT.[a]
Constants that define the SI measurement units			
Vacuum speed of light [c]	c	299 792 458 m/s	0
Original Planck constant [c]	h	$6.626\,070\,15 \cdot 10^{-34}$ Js	0
Reduced Planck constant, quantum of action	$\hbar$	$1.054\,571\,817\ldots \cdot 10^{-34}$ Js	0
Positron charge [c]	e	0.160 217 6634 aC	0
Boltzmann constant [c]	k	$1.380\,649 \cdot 10^{-23}$ J/K	0
Avogadro's number	N_{A}	$6.022\,140\,76 \cdot 10^{23}$ 1/mol	0
Constant that *should* define the SI measurement units			

TABLE 11 (Continued) Basic physical constants.

QUANTITY	SYMBOL	VALUE IN SI UNITS	UNCERT.[a]
Gravitational constant	G	$6.674\,30(15) \cdot 10^{-11}\,\mathrm{Nm^2/kg^2}$	$2.2 \cdot 10^{-5}$
Other fundamental constants			
Number of space-time dimensions		$3 + 1$	0 [b]
Fine-structure constant [d] or	$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$	$1/137.035\,999\,084(21)$	$1.5 \cdot 10^{-10}$
e.m. coupling constant	$= g_{em}(m_e^2 c^2)$	$= 0.007\,297\,352\,5693(11)$	$1.5 \cdot 10^{-10}$
Fermi coupling constant [d] or	$G_F/(\hbar c)^3$	$1.166\,3787(6) \cdot 10^{-5}\,\mathrm{GeV^{-2}}$	$5.1 \cdot 10^{-7}$
weak coupling constant	$\alpha_w(M_Z) = g_w^2/4\pi$	$1/30.1(3)$	$1 \cdot 10^{-2}$
Strong coupling constant [d]	$\alpha_s(M_Z) = g_s^2/4\pi$	$0.1179(10)$	$8.5 \cdot 10^{-3}$
Weak mixing angle	$\sin^2\theta_W(\overline{MS})$	$0.231\,22(4)$	$1.7 \cdot 10^{-4}$
	$\sin^2\theta_W$ (on shell)	$0.222\,90(30)$	$1.3 \cdot 10^{-3}$
	$= 1 - (m_W/m_Z)^2$		
CKM quark mixing matrix	$|V|$	$\begin{pmatrix} 0.97383(24) & 0.2272(10) & 0.00396(9) \\ 0.2271(10) & 0.97296(24) & 0.04221(80) \\ 0.00814(64) & 0.04161(78) & 0.999100(34) \end{pmatrix}$	
Jarlskog invariant	J	$3.08(18) \cdot 10^{-5}$	
PMNS neutrino mixing m.	$|P|$	$\begin{pmatrix} 0.82(2) & 0.55(4) & 0.150(7) \\ 0.37(13) & 0.57(11) & 0.71(7) \\ 0.41(13) & 0.59(10) & 0.69(7) \end{pmatrix}$	
Electron mass	m_e	$9.109\,383\,7015(28) \cdot 10^{-31}\,\mathrm{kg}$	$3.0 \cdot 10^{-10}$
		$5.485\,799\,090\,65(16) \cdot 10^{-4}\,\mathrm{u}$	$2.9 \cdot 10^{-11}$
		$0.510\,998\,950\,00(15)\,\mathrm{MeV}$	$3.0 \cdot 10^{-10}$
Muon mass	m_μ	$1.883\,531\,627(42) \cdot 10^{-28}\,\mathrm{kg}$	$2.2 \cdot 10^{-8}$
		$105.658\,3755(23)\,\mathrm{MeV}$	$2.2 \cdot 10^{-8}$
Tau mass	m_τ	$1.776\,82(12)\,\mathrm{GeV}/c^2$	$6.8 \cdot 10^{-5}$
El. neutrino mass	m_{ν_e}	$< 2\,\mathrm{eV}/c^2$	
Muon neutrino mass	m_{ν_μ}	$< 2\,\mathrm{eV}/c^2$	
Tau neutrino mass	m_{ν_τ}	$< 2\,\mathrm{eV}/c^2$	
Up quark mass	u	$21.6(+0.49/-0.26)\,\mathrm{MeV}/c^2$	
Down quark mass	d	$4.67(+0.48/-0.17)\,\mathrm{MeV}/c^2$	
Strange quark mass	s	$93(+11/-5)\,\mathrm{MeV}/c^2$	
Charm quark mass	c	$1.27(2)\,\mathrm{GeV}/c^2$	
Bottom quark mass	b	$4.18(3)\,\mathrm{GeV}/c^2$	
Top quark mass	t	$172.9(0.4)\,\mathrm{GeV}/c^2$	
Photon mass	γ	$< 2 \cdot 10^{-54}\,\mathrm{kg}$	
W boson mass	$W^\pm$	$80.379(12)\,\mathrm{GeV}/c^2$	
Z boson mass	Z^0	$91.1876(21)\,\mathrm{GeV}/c^2$	
Higgs mass	H	$125.10(14)\,\mathrm{GeV}/c^2$	
Gluon mass	$g_{1...8}$	$c.\ 0\,\mathrm{MeV}/c^2$	

a. Uncertainty: standard deviation of measurement errors.
b. Measured from to 10^{-19} m to 10^{26} m.
c. Defining constant.
d. All coupling constants depend on the 4-momentum transfer, as explained in the section on
renormalization. *Fine-structure constant* is the traditional name for the electromagnetic coupling constant g_{em} in the case of a 4-momentum transfer of $Q^2 = m_e^2 c^2$, which is the smallest one possible. At higher momentum transfers it has larger values, e.g., $g_{em}(Q^2 = M_W^2 c^2) \approx 1/128$. In contrast, the strong coupling constant has lover values at higher momentum transfers; e.g., $\alpha_s(34 \text{ GeV}) = 0.14(2)$.

Why do all these basic constants have the values they have? For any basic constant *with a dimension*, such as the quantum of action $\hbar$, the numerical value has only historical meaning. It is $1.054 \cdot 10^{-34}$ Js because of the SI definition of the joule and the second. The question why the value of a *dimensional* constant is not larger or smaller therefore always requires one to understand the origin of some *dimensionless* number giving the ratio between the constant and the corresponding *natural unit* that is defined with c, G,
k, N_A and $\hbar$. Details and values for the natural units are given in the dedicated section.

In other words, understanding the sizes of atoms, people, trees and stars, the duration of molecular and atomic processes, or the mass of nuclei and mountains, implies understanding the ratios between these values and the corresponding natural units. The key to understanding nature is thus the understanding of all measurement ratios, and thus of all dimensionless constants. This quest, including the understanding of the fine-structure constant α itself, is completed only in the final volume of our adventure.

The basic constants yield the following useful high-precision observations.

TABLE 12 Derived physical constants.

QUANTITY	SYMBOL	VALUE IN SI UNITS	UNCERT.
Vacuum permeability	μ_0	1.256 637 062 12(19) µH/m	$1.5 \cdot 10^{-10}$
Vacuum permittivity	$\varepsilon_0 = 1/\mu_0 c^2$	8.854 187 8128(13) pF/m	$1.5 \cdot 10^{-10}$
Vacuum impedance	$Z_0 = \sqrt{\mu_0/\varepsilon_0}$	376.730 313 668(57) Ω	$1.5 \cdot 10^{-10}$
Loschmidt's number at 273.15 K and 101 325 Pa	N_L	2.686 780 111... $\cdot 10^{25}$ 1/m^3	0
Faraday's constant	$F = N_A e$	96 485.332 12... C/mol	0
Universal gas constant	$R = N_A k$	8.314 462 618... J/(mol K)	0
Molar volume of an ideal gas at 273.15 K and 101 325 Pa	$V = RT/p$	22.413 969 54... l/mol	0
Rydberg constant ^{a}	$R_\infty = m_e c \alpha^2/2h$	10 973 731.568 160(21) m^{-1}	$1.9 \cdot 10^{-12}$
Conductance quantum	$G_0 = 2e^2/h$	77.480 917 29... µS	0
Magnetic flux quantum	$\varphi_0 = h/2e$	2.067 833 848... fWb	0
Josephson frequency ratio	$2e/h$	483.597 8484... THz/V	0
Von Klitzing constant	$h/e^2 = \mu_0 c/2\alpha$	25 812.807 45... Ω	0
Bohr magneton	$\mu_B = e\hbar/2m_e$	9.274 010 0783(28) yJ/T	$3.0 \cdot 10^{-10}$
Classical electron radius	$r_e = e^2/4\pi\varepsilon_0 m_e c^2$	2.817 940 3262(13) fm	$4.5 \cdot 10^{-10}$
Compton wavelength of the electron	$\lambda_C = h/m_e c$	2.426 310 238 67(73) pm	$3.0 \cdot 10^{-10}$
	$\lambdabar_c = \hbar/m_e c = r_e/\alpha$	0.386 159 267 96(12) pm	$3.0 \cdot 10^{-10}$

TABLE 12 (Continued) Derived physical constants.

QUANTITY	SYMBOL	VALUE IN SI UNITS	UNCERT.
Bohr radius [a]	$a_\infty = r_e/\alpha^2$	52.917 721 0903(80) pm	$1.5 \cdot 10^{-10}$
Quantum of circulation	$h/2m_e$	3.636 947 5516(11) cm^2/s	$3.0 \cdot 10^{-10}$
Specific positron charge	e/m_e	175.882 001 076(55) GC/kg	$3.0 \cdot 10^{-10}$
Cyclotron frequency of the electron	$f_c/B = e/2\pi m_e$	27.992 489 872(9) GHz/T	$3.0 \cdot 10^{-10}$
Electron magnetic moment	μ_e	$-9.284\,764\,7043(28)$ yJ/T	$3.0 \cdot 10^{-10}$
	μ_e/μ_B	$-1.001\,159\,652\,181\,28(18)$	$1.7 \cdot 10^{-13}$
	μ_e/μ_N	$-1\,838.281\,971\,88(11) \cdot 10^3$	$6.0 \cdot 10^{-11}$
Electron g-factor	g_e	$-2.002\,319\,304\,362\,56(35)$	$1.7 \cdot 10^{-13}$
Muon–electron mass ratio	m_μ/m_e	206.768 2830(46)	$2.2 \cdot 10^{-8}$
Muon magnetic moment	μ_μ	$-4.490\,448\,30(10) \cdot 10^{-26}$ J/T	$2.2 \cdot 10^{-8}$
Muon g-factor	g_μ	$-2.002\,331\,8418(13)$	$6.3 \cdot 10^{-10}$
Atomic mass unit	$1\,u = m_{12C}/12$	1.660 539 066 60(50) $\cdot 10^{-27}$ kg	$3.0 \cdot 10^{-10}$
Proton mass	m_p	1.672 621 923 69(51) $\cdot 10^{-27}$ kg	$3.1 \cdot 10^{-10}$
		1.007 276 466 621(53) u	$5.3 \cdot 10^{-11}$
		938.272 088 16(29) MeV	$3.1 \cdot 10^{-10}$
Proton–electron mass ratio	m_p/m_e	1 836.152 673 43(11)	$6.0 \cdot 10^{-11}$
Specific proton charge	e/m_p	9.578 833 1560(29) $\cdot 10^7$ C/kg	$3.1 \cdot 10^{-10}$
Proton Compton wavelength	$\lambda_{C,p} = h/m_pc$	1.321 409 855 39(40) fm	$3.1 \cdot 10^{-10}$
Nuclear magneton	$\mu_N = e\hbar/2m_p$	5.050 783 7461(15) $\cdot 10^{-27}$ J/T	$3.1 \cdot 10^{-10}$
Proton magnetic moment	μ_p	1.410 606 797 36(60) $\cdot 10^{-26}$ J/T	$4.2 \cdot 10^{-10}$
	μ_p/μ_B	1.521 032 202 30(46) $\cdot 10^{-3}$	$3.0 \cdot 10^{-10}$
	μ_p/μ_N	2.792 847 344 63(82)	$2.9 \cdot 10^{-10}$
Proton gyromagnetic ratio	$\gamma_p = 2\mu_p/h$	42.577 478 518(18) MHz/T	$4.2 \cdot 10^{-10}$
Proton g factor	g_p	5.585 694 6893(16)	$2.9 \cdot 10^{-10}$
Neutron mass	m_n	1.674 927 498 04(95) $\cdot 10^{-27}$ kg	$5.7 \cdot 10^{-10}$
		1.008 664 915 95(43) u	$4.8 \cdot 10^{-10}$
		939.565 420 52(54) MeV	$5.7 \cdot 10^{-10}$
Neutron–electron mass ratio	m_n/m_e	1 838.683 661 73(89)	$4.8 \cdot 10^{-10}$
Neutron–proton mass ratio	m_n/m_p	1.001 378 419 31(49)	$4.9 \cdot 10^{-10}$
Neutron Compton wavelength	$\lambda_{C,n} = h/m_nc$	1.319 590 905 81(75) fm	$5.7 \cdot 10^{-10}$
Neutron magnetic moment	μ_n	$-0.966\,236\,51(23) \cdot 10^{-26}$ J/T	$2.4 \cdot 10^{-7}$
	μ_n/μ_B	$-1.041\,875\,63(25) \cdot 10^{-3}$	$2.4 \cdot 10^{-7}$
	μ_n/μ_N	$-1.913\,042\,73(45)$	$2.4 \cdot 10^{-7}$
Stefan–Boltzmann constant	$\sigma = \pi^2 k^4/60\hbar^3 c^2$	56.703 744 19... nW/m^{2}K^4	0
Wien's displacement constant	$b = \lambda_{max}T$	2.897 771 955... mmK	0
		58.789 257 57... GHz/K	0
Electron volt	eV	0.160 217 6634... aJ	0
Bits to entropy conversion const.	$k \ln 2$	10^{23} bit = 0.956 994... J/K	0

TABLE 12 (Continued) Derived physical constants.

QUANTITY	SYMBOL	VALUE IN SI UNITS	UNCERT.
TNT energy content		3.7 to 4.0 MJ/kg	$4 \cdot 10^{-2}$

a. For infinite mass of the nucleus.

Some useful properties of our local environment are given in the following table.

TABLE 13 Astronomical constants.

QUANTITY	SYMBOL	VALUE
Tropical year 1900 [a]	a	31 556 925.9747 s
Tropical year 1994	a	31 556 925.2 s
Mean sidereal day	d	$23^h 56' 4.090\,53''$
Average distance Earth–Sun [b]		149 597 870.691(30) km
Astronomical unit [b]	AU	149 597 870 691 m
Light year, based on Julian year [b]	al	9.460 730 472 5808 Pm
Parsec	pc	30.856 775 806 Pm = 3.261 634 al
Earth's mass	$M_\oplus$	$5.973(1) \cdot 10^{24}$ kg
Geocentric gravitational constant	GM	$3.986\,004\,418(8) \cdot 10^{14} \, \mathrm{m^3/s^2}$
Earth's gravitational length	$l_\oplus = 2GM/c^2$	8.870 056 078(16) mm
Earth's equatorial radius [c]	$R_{\oplus\mathrm{eq}}$	6378.1366(1) km
Earth's polar radius [c]	$R_{\oplus\mathrm{p}}$	6356.752(1) km
Equator–pole distance [c]		10 001.966 km (average)
Earth's flattening [c]	$e_\oplus$	1/298.25642(1)
Earth's av. density	$\rho_\oplus$	$5.5 \, \mathrm{Mg/m^3}$
Earth's age	$T_\oplus$	4.50(4) Ga = 142(2) Ps
Earth's normal gravity	g	$9.806\,65 \, \mathrm{m/s^2}$
Earth's standard atmospher. pressure	p_0	101 325 Pa
Moon's radius	$R_{\mathbb{C}\,v}$	1738 km in direction of Earth
Moon's radius	$R_{\mathbb{C}\,h}$	1737.4 km in other two directions
Moon's mass	$M_{\mathbb{C}}$	$7.35 \cdot 10^{22}$ kg
Moon's mean distance [d]	$d_{\mathbb{C}}$	384 401 km
Moon's distance at perigee [d]		typically 363 Mm, historical minimum 359 861 km
Moon's distance at apogee [d]		typically 404 Mm, historical maximum 406 720 km
Moon's angular size [e]		average 0.5181° = 31.08′, minimum 0.49°, maximum 0.55°
Moon's average density	$\rho_{\mathbb{C}}$	$3.3 \, \mathrm{Mg/m^3}$
Moon's surface gravity	$g_{\mathbb{C}}$	$1.62 \, \mathrm{m/s^2}$
Moon's atmospheric pressure	$p_{\mathbb{C}}$	from 10^{-10} Pa (night) to 10^{-7} Pa (day)
Jupiter's mass	$M_{\jupiter}$	$1.90 \cdot 10^{27}$ kg

TABLE 13 (Continued) Astronomical constants.

QUANTITY	SYMBOL	VALUE
Jupiter's radius, equatorial	$R_{\jupiter}$	71.398 Mm
Jupiter's radius, polar	$R_{\jupiter}$	67.1(1) Mm
Jupiter's average distance from Sun	$D_{\jupiter}$	778 412 020 km
Jupiter's surface gravity	$g_{\jupiter}$	24.9 m/s^2
Jupiter's atmospheric pressure	$p_{\jupiter}$	from 20 kPa to 200 kPa
Sun's mass	$M_{\odot}$	1.988 43(3) · 10^{30} kg
Sun's gravitational length	$2GM_{\odot}/c^2$	2.953 250 08(5) km
Heliocentric gravitational constant	$GM_{\odot}$	132.712 440 018(8) · 10^{18} m^3/s^2
Sun's luminosity	$L_{\odot}$	384.6 YW
Solar equatorial radius	$R_{\odot}$	695.98(7) Mm
Sun's angular size		0.53° average; minimum on fourth of July (aphelion) 1888 ″, maximum on fourth of January (perihelion) 1952 ″
Sun's average density	$\rho_{\odot}$	1.4 Mg/m^3
Sun's average distance	AU	149 597 870.691(30) km
Sun's age	$T_{\odot}$	4.6 Ga
Solar velocity around centre of galaxy	$v_{\odot g}$	220(20) km/s
Solar velocity against cosmic background	$v_{\odot b}$	370.6(5) km/s
Sun's surface gravity	$g_{\odot}$	274 m/s^2
Sun's lower photospheric pressure	$p_{\odot}$	15 kPa
Distance to Milky Way's centre		8.0(5) kpc = 26.1(1.6) kal
Milky Way's age		13.6 Ga
Milky Way's size		c. 10^{21} m or 100 kal
Milky Way's mass		10^{12} solar masses, c. 2 · 10^{42} kg
Most distant galaxy cluster known	SXDF-XCLJ 0218-0510	9.6 · 10^9 al

a. Defining constant, from vernal equinox to vernal equinox; it was once used to define the second. (Remember: π seconds is about a nanocentury.) The value for 1990 is about 0.7 s less, corresponding to a slowdown of roughly 0.2 ms/a. (Watch out: why?) There is even an empirical formula for the change of the length of the year over time.

b. The truly amazing precision in the average distance Earth–Sun of only 30 m results from time averages of signals sent from Viking orbiters and Mars landers taken over a period of over twenty years. Note that the International Astronomical Union distinguishes the average distance Earth–Sun from the *astronomical unit* itself; the latter is defined as a fixed and exact length. Also the *light year* is a unit defined as an exact number by the IAU. For more details, see www.iau.org/public/measuring.

c. The shape of the Earth is described most precisely with the World Geodetic System. The last edition dates from 1984. For an extensive presentation of its background and its details, see the

www.wgs84.com website. The International Geodesic Union refined the data in 2000. The radii and the flattening given here are those for the 'mean tide system'. They differ from those of the 'zero tide system' and other systems by about 0.7 m. The details constitute a science in itself.

d. Measured centre to centre. To find the precise position of the Moon at a given date, see the www.fourmilab.ch/earthview/moon_ap_per.html page. For the planets, see the page www.fourmilab.ch/solar/solar.html and the other pages on the same site.

e. Angles are defined as follows: 1 degree = $1° = \pi/180$ rad, 1 (first) minute = $1' = 1°/60$, 1 second (minute) = $1'' = 1'/60$. The ancient units 'third minute' and 'fourth minute', each 1/60th of the preceding, are not in use any more. ('Minute' originally means 'very small', as it still does in modern English.)

Some properties of nature at large are listed in the following table. (If you want a challenge, can you determine whether any property of the universe itself is listed?)

TABLE 14 Cosmological constants.

QUANTITY	SYMBOL	VALUE
Cosmological constant	Λ	$c.\ 1 \cdot 10^{-52}$ m^{-2}
Age of the universe [a]	t_0	$4.333(53) \cdot 10^{17}$ s $= 13.8(0.1) \cdot 10^9$ a
(determined from space-time, via expansion, using general relativity)		
Age of the universe [a]	t_0	over $3.5(4) \cdot 10^{17}$ s $= 11.5(1.5) \cdot 10^9$ a
(determined from matter, via galaxies and stars, using quantum theory)		
Hubble parameter [a]	H_0	$2.3(2) \cdot 10^{-18}$ s$^{-1} = 0.73(4) \cdot 10^{-10}$ a^{-1}
		$= h_0 \cdot 100$ km/s Mpc $= h_0 \cdot 1.0227 \cdot 10^{-10}$ a^{-1}
Reduced Hubble parameter [a]	h_0	$0.71(4)$
Deceleration parameter [a]	$q_0 = -(\ddot{a}/a)_0/H_0^2$	$-0.66(10)$
Universe's horizon distance [a]	$d_0 = 3ct_0$	$40.0(6) \cdot 10^{26}$ m $= 13.0(2)$ Gpc
Universe's topology		trivial up to 10^{26} m
Number of space dimensions		3, for distances up to 10^{26} m
Critical density	$\rho_c = 3H_0^2/8\pi G$	$h_0^2 \cdot 1.878\,82(24) \cdot 10^{-26}$ kg/m^3
of the universe		$= 0.95(12) \cdot 10^{-26}$ kg/m^3
(Total) density parameter [a]	$\Omega_0 = \rho_0/\rho_c$	$1.02(2)$
Baryon density parameter [a]	$\Omega_{B0} = \rho_{B0}/\rho_c$	$0.044(4)$
Cold dark matter density parameter [a]	$\Omega_{CDM0} = \rho_{CDM0}/\rho_c$	$0.23(4)$
Neutrino density parameter [a]	$\Omega_{\nu 0} = \rho_{\nu 0}/\rho_c$	0.001 to 0.05
Dark energy density parameter [a]	$\Omega_{X0} = \rho_{X0}/\rho_c$	$0.73(4)$
Dark energy state parameter	$w = p_X/\rho_X$	$-1.0(2)$
Baryon mass	m_b	$1.67 \cdot 10^{-27}$ kg
Baryon number density		$0.25(1)$ /m^3
Luminous matter density		$3.8(2) \cdot 10^{-28}$ kg/m^3
Stars in the universe	n_s	$10^{22\pm1}$
Baryons in the universe	n_b	$10^{81\pm1}$
Microwave background temperature [b]	T_0	$2.725(1)$ K
Photons in the universe	n_γ	10^{89}
Photon energy density	$\rho_\gamma = \pi^2 k^4/15T_0^4$	$4.6 \cdot 10^{-31}$ kg/m^3

TABLE 14 (Continued) Cosmological constants.

QUANTITY	SYMBOL	VALUE
Photon number density		410.89 /cm^3 or $400 \text{ /cm}^3 (T_0/2.7 \text{ K})^3$
Density perturbation amplitude	$\sqrt{S}$	$5.6(1.5) \cdot 10^{-6}$
Gravity wave amplitude	$\sqrt{T}$	$< 0.71 \sqrt{S}$
Mass fluctuations on 8 Mpc	σ_8	$0.84(4)$
Scalar index	n	$0.93(3)$
Running of scalar index	$dn/d \ln k$	$-0.03(2)$
Planck length	$l_{Pl} = \sqrt{\hbar G/c^3}$	$1.62 \cdot 10^{-35}$ m
Planck time	$t_{Pl} = \sqrt{\hbar G/c^5}$	$5.39 \cdot 10^{-44}$ s
Planck mass	$m_{Pl} = \sqrt{\hbar c/G}$	21.8 µg
Instants in history [a]	t_0/t_{Pl}	$8.7(2.8) \cdot 10^{60}$
Space-time points inside the horizon [a]	$N_0 = (R_0/l_{Pl})^3 \cdot$ (t_0/t_{Pl})	$10^{244\pm1}$
Mass inside horizon	M	$10^{54\pm1}$ kg

a. The index 0 indicates present-day values.
b. The radiation originated when the universe was 380 000 years old and had a temperature of about 3000 K; the fluctuations ΔT_0 which led to galaxy formation are today about $16 \pm 4 \,\mu\text{K} =$

$6(2) \cdot 10^{-6} \, T_0$.

USEFUL NUMBERS

π	$3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 37510_5$
e	$2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47093\ 69995_9$
γ	$0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ 10421\ 59335\ 93992_3$
$\ln 2$	$0.69314\ 71805\ 59945\ 30941\ 72321\ 21458\ 17656\ 80755\ 00134\ 36025_5$
$\ln 10$	$2.30258\ 50929\ 94045\ 68401\ 79914\ 54684\ 36420\ 76011\ 01488\ 62877_2$
$\sqrt{10}$	$3.16227\ 76601\ 68379\ 33199\ 88935\ 44432\ 71853\ 37195\ 55139\ 32521_6$

If the number π is *normal*, i.e., if all digits and digit combinations in its decimal expansion appear with the same limiting frequency, then every text ever written or yet to be written, as well as every word ever spoken or yet to be spoken, can be found coded in its sequence. The property of normality has not yet been proven, although it is suspected to hold. Does this mean that all wisdom is encoded in the simple circle? No. The property is nothing special: it also applies to the number 0.12345678910111213141516171819202021... and many others. Can you specify a few examples?

By the way, in the graph of the exponential function e^x, the point $(0, 1)$ is the only point with two rational coordinates. If you imagine painting in blue all points on the plane with two rational coordinates, the plane would look quite bluish. Nevertheless, the graph goes through only one of these points and manages to avoid all the others.
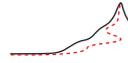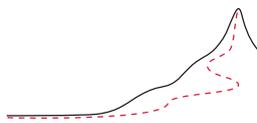
NUMBERS AND VECTOR SPACES

> " A mathematician is a machine that transforms
> coffee into theorems. "
> Paul Erdős (b. 1913 Budapest, d. 1996 Warsaw)

Mathematical concepts can all be expressed in terms of 'sets' and 'relations.' Many fundamental concepts were presented in the last chapter. Why does athematics, given this simple basis, grow into a passion for certain people? How can sets and relations become the center of a person's life? The mathematical appendices present a few more advanced concepts as simply and vividly as possible, for all those who want to understand and to smell the passion for mathematics.

Unfortunately, the passion for mathematics is not easy to spot, because like many other professions, also mathematicians *hide* their passions. In mathematics, this is done through formalism and apparent detachment from intuition. Good mathematical teaching however, puts intuition at the beginning. In this appendix we shall introduce the simplest *algebraic* structures. The appendix in the next volume will present some more involved algebraic structures and then the most important *topological* structures; the third basic type of mathematical structures, *order* structures, are not so important in physics – with one exception: the definition of the real numbers contains an order structure.

Mathematicians are concerned not only with the exploration of concepts, but also with their *classification*. Whenever a new mathematical concept is introduced, mathematicians try to classify all the possible cases and types. This has been achieved most spectacularly for the different types of numbers, for finite simple groups and for many types of spaces and manifolds.

Numbers as mathematical structures

> " A person who can solve $x^2 - 92y^2 = 1$ in less
> than a year is a mathematician. "
> Brahmagupta (b. 598 Sindh, d. 668) (implied:
> solve in *integers*)

Children know: numbers are entities that can be added and multiplied. Mathematicians are more discerning. Any mathematical system with the same basic properties as the *natural* numbers is called a *semi-ring*. Any mathematical system with the same basic properties as the *integers* is called a *ring*. (The terms are due to David Hilbert. Both structures can also be finite rather than infinite.) More precisely, a *ring* $(R, +, \cdot)$ is a set R of ele-

ments with two binary operations, called *addition* and *multiplication*, usually written +
and · (the latter may simply be understood, thus without explicit notation), for which the
following properties hold for all elements $a, b, c \in R$:

— R is a commutative group with respect to addition, i.e.
    $a + b \in R, a + b = b + a, a + 0 = a, a + (-a) = a - a = 0$ and $a + (b + c) = (a + b) + c$;
— R is closed under multiplication, i.e., $ab \in R$;
— multiplication is associative, i.e., $a(bc) = (ab)c$;
— distributivity holds, i.e., $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Many authors add the axiom

— a multiplicative unit exists, i.e., $1a = a1 = a$.

Defining properties such as these are called *axioms*. We stress that axioms are *not* basic
beliefs, as is often stated or implied; axioms are the basic properties used in the definition
of a concept: in this case, of a ring. With the last axiom, one also speaks of a *unital ring*.
   A *semi-ring* is a set satisfying all the axioms of a ring, except that the existence of
neutral and negative elements for addition is replaced by the weaker requirement that if
$a + c = b + c$ then $a = b$. Sloppily, a semi-ring is a ring 'without' negative elements.
   To incorporate division and define the rational numbers, we need another concept. A
number field or *field* K is a ring with

— a multiplicative identity 1, such that all elements a obey $1a = a$;
— at least one element different from zero; and most importantly
— a (multiplicative) inverse a^{-1} for every element $a \neq 0$.

A ring or field is said to be *commutative* if the multiplication is commutative. A non-
commutative field is also called a *skew field*. Fields can be finite or infinite. (A field or a
ring is characterized by its *characteristic p*. This is the smallest number of times one has
to add 1 to itself to give zero. If there is no such number the characteristic is set to 0. p is
always a prime number or zero.) All finite fields are commutative. In a field, all equations
of the type $cx = b$ and $xc = b$ ($c \neq 0$) have solutions for x; there is a unique solution
if $b \neq 0$. To sum up sloppily by focusing on the most important property, a field is a set
of elements for which, together with addition, subtraction and multiplication, a *division*
(by non-zero elements) is also defined. The *rational numbers* are the simplest field that
incorporates the integers.
   The system of the *real numbers* is the minimal extension of the rationals which is
complete and totally ordered.* Can you show that $\sqrt{2}$ is a real, but not a rational number?

---

* A set is mathematically *complete* if physicists call it continuous. More precisely, a set of numbers is *complete*
if every non-empty subset that is bounded above has a least upper bound.
   A set is *totally ordered* if there exists a binary relation ⩽ between pairs of elements such that for all
elements a and b
— if $a ⩽ b$ and $b ⩽ c$, then $a ⩽ c$;
— if $a ⩽ b$ and $b ⩽ a$, then $a = b$;
— $a ⩽ b$ or $b ⩽ a$ holds.
In summary, a set is totally ordered if there is a binary relation that allows saying about any two elements
which one is the predecessor of the other in a consistent way. This is the fundamental – and also the only –
order structure used in physics.
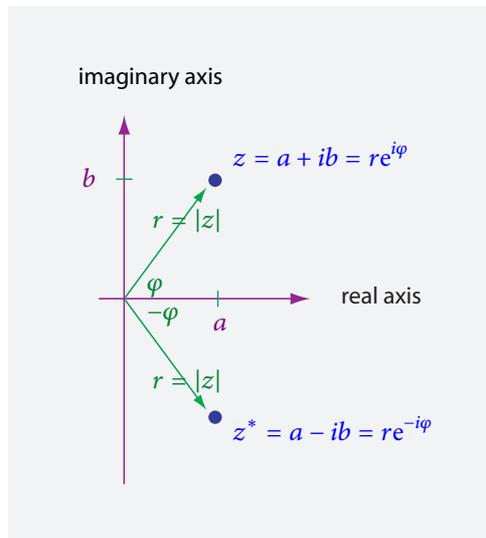
FIGURE 86 Complex numbers are points in the two-dimensional plane; a complex number z and its conjugate z^* can be described in cartesian form or in polar form.

In classical physics and quantum theory, it is always stressed that measurement results are and must be real numbers. But are all real numbers possible measurement results? In
other words, are all measurement results just a subset of the reals?

However, the concept of 'number' is not limited to these examples. It can be gen-
eralized in several ways. The simplest generalization is achieved by extending the real numbers to manifolds of more than one dimension.

Complex numbers

In nature, complex numbers are a useful way to describe in compact form systems and situations that contain a *phase*. Complex numbers are thus useful to describe waves of any kind.

Complex numbers form a two-dimensional manifold. A complex number is defined, in its cartesian form, by $z = a + ib$, where a and b are real numbers, and i is a new symbol, the so-called *imaginary unit*. Under multiplication, the generators of the complex numbers, 1 and i, obey

$$
\begin{array}{c|cc}
\cdot & 1 & i \\
\hline
1 & 1 & i \\
i & i & -1
\end{array}
\tag{120}
$$

often summarized as $i = +\sqrt{-1}$. In a complex number $z = a + ib$, a is called the *real part*, and b the *complex part*. They are illustrated in Figure 86.

The *complex conjugate* z^*, also written $\bar{z}$, of a complex number $z = a + ib$ is defined as $z^* = a - ib$. The *absolute value* $|z|$ of a complex number is defined as $|z| = \sqrt{zz^*} = \sqrt{z^*z} = \sqrt{a^2 + b^2}$. It defines a *norm* on the vector space of the complex numbers. From $|wz| = |w|\,|z|$ follows the *two-squares theorem*

$$
(a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1 b_1 - a_2 b_2)^2 + (a_1 b_2 + a_2 b_1)^2
\tag{121}
$$
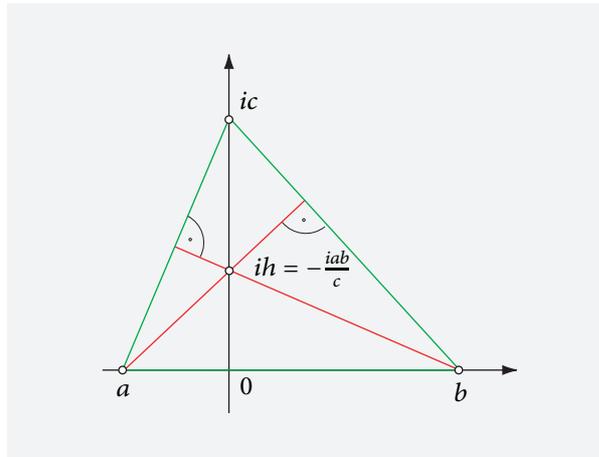
FIGURE 87 A property of triangles
easily provable with complex numbers.

valid for all real numbers a_i, b_i. It was already known, in its version for integers, to Diophantus of Alexandria in the third century CE.

Complex numbers can also be written as ordered pairs (a, A) of real numbers, with their addition defined as $(a, A) + (b, B) = (a + b, A + B)$ and their multiplication defined as $(a, A) \cdot (b, B) = (ab - AB, aB + bA)$. This notation allows us to identify the complex numbers with the *points* on a plane or, if we prefer, to *arrows* in a plane. Translating the definition of multiplication into geometrical language allows us to rapidly prove certain geometrical theorems, such as the one of Figure 87.

Complex numbers $a + ib$ can also be represented as 2×2 matrices

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{with} \quad a, b \in \mathbb{R} \,. \tag{122}$$

Matrix addition and multiplication then correspond to complex addition and multiplication. In this way, complex numbers can be represented by a special type of real matrix. What is $|z|$ in matrix language?

The set $\mathbb{C}$ of complex numbers with addition and multiplication as defined above forms both a commutative two-dimensional field and a vector space over $\mathbb{R}$. In the field of complex numbers, quadratic equations $az^2 + bz + c = 0$ for an unknown z always have *two* solutions (for $a \neq 0$ and counting multiplicity).

Complex numbers can be used to describe the points of a plane. A rotation around the origin can be described by multiplication by a complex number of unit length. Other two-dimensional quantities can also be described with complex numbers. Electrical engineers use complex numbers to describe quantities with phases, such as alternating currents or electrical fields in space.

Writing complex numbers of unit length as $\cos\theta + i\sin\theta$ is a useful method for remembering angle addition formulae. Since one has $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$, one can easily deduce formulae such as $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. By the way, the unit complex numbers form the Lie group SO(2)=U(1).

Every complex number can be written as

$$z = re^{i\varphi} \ .\tag{123}$$

This polar form of writing complex numbers is the reason for introducing them in the first place. The angle φ is called the *phase*; the real number $r = |z|$ is called the *absolute value* or the *modulus* or the *magnitude*. When used to describe oscillations or waves, it makes sense to call r the *amplitude*. The complex exponential function is periodic in $2\pi i$; in other words, we have

$$e^1 = e^{1+2\pi i} \ ,\tag{124}$$

which shows the property we expect from a phase angle.

If one uses the last equation twice, one may write

$$e^1 = e^{1+2\pi i} = (e^{1+2\pi i})^{1+2\pi i} = e^{(1+2\pi i)(1+2\pi i)} = e^{1-4\pi^2+4\pi i} = e^{1-4\pi^2} \ .\tag{125}$$

Oops, that would imply $\pi = 0$! What is wrong here?

Complex numbers can also be used to describe Euclidean plane geometry. Rotations, translations and other isometries, but also reflections, glide reflections and scaling are easily described by simple operations on the complex numbers that describe the coordinate of points.

By the way, there are exactly as many complex numbers as there are real numbers.
Can you show this?

> “ Love is complex: it has real and imaginary parts. ”
>
> Anonymous

Quaternions

The positions of the points on a line can be described by real numbers. Complex numbers can be used to describe the positions of the points of a plane. It is natural to try to generalize the idea of a number to higher-dimensional spaces. However, it turns out that no useful number system can be defined for *three*-dimensional space. A new number system, the *quaternions*, can be constructed which corresponds the points of *four*-dimensional space, but only if the commutativity of multiplication is sacrificed. No useful number system can be defined for dimensions other than 1, 2 and 4.

The quaternions were discovered by several mathematicians in the nineteenth century, among them Hamilton,[*] who studied them for much of his life. In fact, Maxwell's theory of electrodynamics was formulated in terms of quaternions before three-
dimensional vectors were used.
Under multiplication, the quaternions $\mathbb{H}$ form a 4-dimensional algebra over the reals

---

[*] William Rowan Hamilton (b. 1805 Dublin, d. 1865 Dunsink), child prodigy and famous mathematician, named the quaternions after an expression from the Vulgate (Acts. 12: 4).

with a basis $1, i, j, k$ satisfying

$$
\begin{array}{c|cccc}
\cdot & 1 & i & j & k \\
\hline
1 & 1 & i & j & k \\
i & i & -1 & k & -j \\
j & j & -k & -1 & i \\
k & k & j & -i & -1
\end{array}
\tag{126}
$$

These relations are also often written $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. The quaternions $1, i, j, k$ are also called *basic units* or *generators*. The lack of symmetry across the diagonal of the table shows the non-commutativity of quaternionic multiplication. With the quaternions, the idea of a non-commutative product appeared for the first time in mathematics. However, the multiplication of quaternions *is* associative. As a consequence of non-commutativity, polynomial equations in quaternions have many more solutions than in complex numbers: just search for all solutions of the

equation $X^2 + 1 = 0$ to convince yourself of it.

Every quaternion X can be written in the form

$$
X = x_0 + x_1 i + x_2 j + x_3 k = x_0 + \boldsymbol{v} = (x_0, x_1, x_2, x_3) = (x_0, \boldsymbol{v}) ,
\tag{127}
$$

where x_0 is called the *scalar* part and $\boldsymbol{v}$ the *vector* part. The multiplication is thus defined as $(x, \boldsymbol{v})(y, \boldsymbol{w}) = (xy - \boldsymbol{v} \cdot \boldsymbol{w}, x\boldsymbol{w} + y\boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{w})$. The multiplication of two general quaternions can be written as

$$
\begin{aligned}
(a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2) = (&a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 , \ a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2, \\
&a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2, a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2) .
\end{aligned}
\tag{128}
$$

The conjugate quaternion $\overline{X}$ is defined as $\overline{X} = x_0 - \boldsymbol{v}$, so that $\overline{XY} = \overline{Y}\,\overline{X}$. The *norm* $|X|$ of a quaternion X is defined as $|X|^2 = X\overline{X} = \overline{X}X = x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0^2 + \boldsymbol{v}^2$. The norm is multiplicative, i.e., $|XY| = |X|\,|Y|$.

Unlike complex numbers, every quaternion is related to its complex conjugate by

$$
\overline{X} = -\tfrac{1}{2}(X + iXi + jXj + kXk) .
\tag{129}
$$

No relation of this type exists for complex numbers. In the language of physics, a complex number and its conjugate are independent variables; for quaternions, this is not the case. As a result, functions of quaternions are less useful in physics than functions of complex variables.

The relation $|XY| = |X|\,|Y|$ implies the *four-squares theorem*

$$
\begin{aligned}
(a_1^2 &+ a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) \\
&= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4)^2 + (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)^2 \\
&\quad + (a_1 b_3 + a_3 b_1 + a_4 b_2 - a_2 b_4)^2 + (a_1 b_4 + a_4 b_1 + a_2 b_3 - a_3 b_2)^2
\end{aligned}
\tag{130}
$$
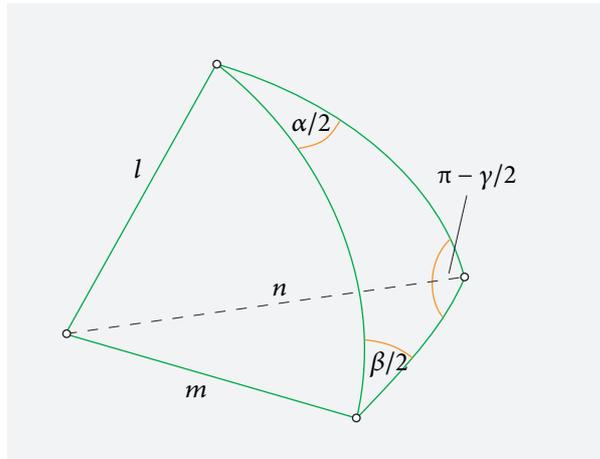
FIGURE 88 Combinations of rotations.

valid for all real numbers a_i and b_i, and thus also for any set of eight integers. It was discovered in 1748 by Leonhard Euler (1707–1783) when trying to prove that each integer is the sum of four squares. (The latter fact was proved only in 1770, by Joseph Lagrange.)

Hamilton thought that a quaternion with zero scalar part, which he simply called a *vector* (a term which he invented), could be identified with an ordinary three-dimensional translation vector; but this is wrong. Such a quaternion is now called a *pure*, or *homogeneous*, or *imaginary* quaternion. The product of two pure quaternions $V = (0, \boldsymbol{v})$ and $W = (0, \boldsymbol{w})$ is given by $VW = (-\boldsymbol{v} \cdot \boldsymbol{w}, \boldsymbol{v} \times \boldsymbol{w})$, where · denotes the scalar product and × denotes the vector product. Note that any quaternion can be written as the ratio of two pure quaternions.

In reality, a pure quaternion $(0, \boldsymbol{v})$ does not behave like a translation vector under coordinate transformations; in fact, a pure quaternion represents a *rotation* by the angle

π or 180° around the axis defined by the direction $\boldsymbol{v} = (v_x, v_y, v_z)$.

It turns out that in three-dimensional space, a *general* rotation about the origin can be described by a *unit* quaternion Q, also called a *normed* quaternion, for which $|Q| = 1$. Such a quaternion can be written as $(\cos\theta/2, \boldsymbol{n}\sin\theta/2)$, where $\boldsymbol{n} = (n_x, n_y, n_z)$ is the normed vector describing the direction of the rotation axis and θ is the rotation angle. Such a unit quaternion $Q = (\cos\theta/2, \boldsymbol{n}\sin\theta/2)$ rotates a pure quaternion $V = (0, \boldsymbol{v})$ into another pure quaternion $W = (0, \boldsymbol{w})$ given by

$$W = QVQ^* . \tag{131}$$

Thus, if we use pure quaternions such as V or W to describe positions, we can use unit quaternions to describe rotations and to calculate coordinate changes. The concatenation of two rotations is then given by the product of the corresponding unit quaternions. Indeed, a rotation by an angle α about the axis $\boldsymbol{l}$ followed by a rotation by an angle β about the axis $\boldsymbol{m}$ gives a rotation by an angle γ about the axis $\boldsymbol{n}$, with the values determined by

$$(\cos\gamma/2, \sin\gamma/2\boldsymbol{n}) = (\cos\beta/2, \sin\beta/2\boldsymbol{m})(\cos\alpha/2, \sin\alpha/2\boldsymbol{l}) . \tag{132}$$
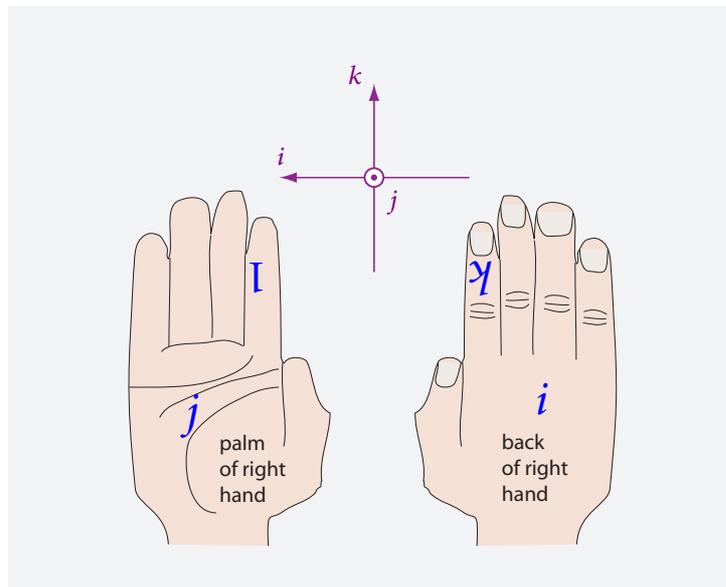
FIGURE 89 The top and back of the right hand, and the quaternions.

One way to show the result graphically is given in Figure 88. By drawing a triangle on a unit sphere, and taking care to remember the factor 1/2 in the angles, the combination of two rotations can be simply determined.

The interpretation of quaternions as rotations is also illustrated, in a somewhat different way, in the motion of any hand. To see this, take a green marker and write the letters 1, i, j and k on your hand as shown in Figure 89. Defining the three possible 180° rotation axes as shown in the figure and taking concatenation as multiplication, the motion of the right hand follows the same 'laws' as those of pure unit quaternions. (One needs to distinguish $+i$ and $-i$, and the same for the other units, by the sense of the arm twist. And the result of a multiplication is that letter that can be read by a person facing you.) You can show that $i^2 = j^2 = k^2 = -1$, that $i^4 = 1$, and conform all other quaternion relations. The model also shows that the rotation angle of the arm is half the rotation angle of the corresponding quaternion. In other words, quaternions can be used to describe the belt trick, if the multiplication VW of two quaternions is taken to mean that rotation V is performed *after* rotation W. Quaternions, like human hands, thus behave like a spin 1/2 particle. Quaternions and spinors are isomorphic.

The reason for the half-angle behaviour of rotations can be specified more precisely using mathematical language. The rotations in three dimensions around a point form the 'special orthogonal group' in three dimensions, which is called SO(3). But the motions of a hand attached to a shoulder via an arm form a different group, isomorphic to the Lie group SU(2). The difference is due to the appearance of half angles in the parametrization of rotations; indeed, the above parametrizations imply that a rotation by 2π corresponds to a multiplication by -1. Only in the twentieth century was it realized that there exist fundamental physical observables that behaves like hands attached to arms: they are called *spinors*. More on spinors can be found in the section on permutation symmetry, where belts are used as an analogy as well as arms. In short, the group SU(2) formed by

the unit quaternions is the *double cover* of the rotation group SO(3).

The simple representation of rotations and positions with quaternions is used by computer programmes in robotics, in astronomy and in flight simulation. In the software used to create three-dimensional images and animations, visualization software, quaternions are often used to calculate the path taken by repeatedly reflected light rays and thus give surfaces a realistic appearance.

The algebra of the quaternions is the only associative, non-commutative, finite-dimensional normed algebra with an identity over the field of real numbers. Quaternions form a non-commutative field, i.e., a skew field, in which the inverse of a quaternion X is $\overline{X}/|X|$. We can therefore define division of quaternions (while being careful to distinguish XY^{-1} and $Y^{-1}X$). Therefore quaternions are said to form a *division algebra*. In fact, the quaternions $\mathbb{H}$, the complex numbers $\mathbb{C}$ and the reals $\mathbb{R}$ are the only three finite-dimensional associative division algebras. In other words, the skew-field of quaternions is the only finite-dimensional real associative non-commutative algebra without divisors of zero. The *centre* of the quaternions, i.e., the set of quaternions that commute with all other quaternions, is just the set of real numbers.

Quaternions can be represented as matrices of the form

$$\begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix} \quad \text{with} \quad A, B \in \mathbb{C} \quad \text{thus} \quad A = a + ib, B = c + id \;, \tag{133}$$

or, alternatively, as

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \quad \text{with} \quad a, b, c, d \in \mathbb{R} \;, \tag{134}$$

where the quaternion X then is given as $X = A + Bj = a + ib + jc + kd$. Matrix addition and multiplication then corresponds to quaternionic addition and multiplication.

The generators of the quaternions can be realized as

$$1 : \sigma_0 \quad , \quad i : -i\sigma_1 \quad , \quad j : -i\sigma_2 \quad , \quad k : -i\sigma_3 \tag{135}$$

where the σ_n are the Pauli spin matrices.[*]

---

[*] The *Pauli spin matrices* are the complex Hermitean matrices

$$\sigma_0 = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{136}$$

all of whose eigenvalues are ± 1; they satisfy the relations $[\sigma_i, \sigma_k]_+ = 2\,\delta_{ik}$ and $[\sigma_i, \sigma_k] = 2i\,\varepsilon_{ikl}\,\sigma_l$. The linear combinations $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$ are also frequently used. By the way, another possible representation of the quaternions is $i : i\sigma_3, j : i\sigma_2, k : i\sigma_1$.

Real 4×4 representations are not unique, as the alternative representation

$$
\begin{pmatrix}
a & b & -d & -c \\
-b & a & -c & d \\
d & c & a & b \\
c & -d & -b & a
\end{pmatrix}
\tag{137}
$$

shows. No representation of quaternions by 3×3 matrices is possible.

These matrices contain real and complex elements, which pose no special problems. In contrast, when matrices with quaternionic elements are constructed, care has to be taken, because quaternionic multiplication is not commutative, so that simple relations such as $\mathrm{tr} AB = \mathrm{tr} BA$ are *not* generally valid.

What can we learn from quaternions about the description of nature? First of all, we see that binary rotations are similar to positions, and thus to translations: all are represented by 3-vectors. Are rotations the basic operations of nature? Is it possible that translations are only 'shadows' of rotations? The connection between translations and rotations is investigated in the last volume of our mountain ascent.

When Maxwell wrote down his equations of electrodynamics, he used quaternion notation. (The now usual 3-vector notation was introduced later by Hertz and Heaviside.) The equations can be written in various ways using quaternions. The simplest is achieved when one keeps a distinction between $\sqrt{-1}$ and the units i, j, k of the quaternions. One then can write all of electrodynamics in a single equation:

$$
\mathrm{d}F = -\frac{Q}{\varepsilon_0}
\tag{138}
$$

where F is the generalized electromagnetic field and Q the generalized charge. These are defined by

$$
\begin{aligned}
F &= E + \sqrt{-1}\,cB \\
E &= iE_x + jE_y + kE_z \\
B &= iB_x + jB_y + kB_z \\
\mathrm{d} &= \delta + \sqrt{-1}\,\partial_t/c \\
\delta &= i\partial_x + j\partial_y + k\partial_z \\
Q &= \rho + \sqrt{-1}\,J/c
\end{aligned}
\tag{139}
$$

where the fields E and B and the charge distributions ρ and J have the usual meanings. The content of equation (138) for the electromagnetic field is exactly the same as the usual formulation.

Despite their charm and their four-dimensionality, quaternions do not seem to be useful for the reformulation of special relativity; the main reason for this is the sign in the expression for their norm. Therefore, relativity and space-time are usually described using real numbers. And even if quaternions were useful, they would not provide additional insights into physics or into nature.

Octonions

In the same way that quaternions are constructed from complex numbers, *octonions* can be constructed from quaternions. They were first investigated by Arthur Cayley (1821–1895). Under multiplication, *octonions* (or *octaves*) are the elements of an eight-dimensional algebra over the reals with the generators $1, i_n$ with $n = 1 \ldots 7$ satisfying

$$
\begin{array}{c|cccccccc}
\cdot & 1 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\
\hline
1 & 1 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\
i_1 & i_1 & -1 & i_3 & -i_2 & i_5 & -i_4 & i_7 & -i_6 \\
i_2 & i_2 & -i_3 & -1 & i_1 & -i_6 & i_7 & i_4 & -i_5 \\
i_3 & i_3 & i_2 & -i_1 & -1 & i_7 & i_6 & -i_5 & -i_4 \\
i_4 & i_4 & -i_5 & i_6 & -i_7 & -1 & i_1 & -i_2 & i_3 \\
i_5 & i_5 & i_4 & -i_7 & -i_6 & -i_1 & -1 & i_3 & i_2 \\
i_6 & i_6 & -i_7 & -i_4 & i_5 & i_2 & -i_3 & -1 & i_1 \\
i_7 & i_7 & i_6 & i_5 & i_4 & -i_3 & -i_2 & -i_1 & -1 \\
\end{array}
\qquad (140)
$$

In fact, 479 other, equivalent multiplication tables are also possible. This algebra is called the *Cayley algebra*; it has an identity and a unique division. The algebra is non-commutative, and also non-associative. It is, however, *alternative*, meaning that for all elements x and y, one has $x(xy) = x^2 y$ and $(xy)y = xy^2$: a property somewhat weaker than associativity. It is the only 8-dimensional real alternative algebra without zero divisors. Because it is not associative, the set $\mathbb{O}$ of all octonions does not form a field, nor even a ring, so that the old designation of 'Cayley numbers' has been abandoned. The octonions are the most general hypercomplex 'numbers' whose norm is multiplicative. Its generators obey $(i_n i_m)i_l = \pm i_n(i_m i_l)$, where the minus sign, which shows the non-associativity, is valid for combinations of indices that are not quaternionic, such as 1-2-4.

Octonions can be represented as matrices of the form

$$
\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \text{ where } A, B \in \mathbb{H}, \quad \text{or as real } 8 \times 8 \text{ matrices.} \qquad (141)
$$

Matrix multiplication then gives the same result as octonionic multiplication.

The relation $|wz| = |w|\,|z|$ allows one to deduce the impressive *eight-squares theorem*

$$
\begin{aligned}
(a_1^2 + a_2^2 &+ a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2 + b_8^2) \\
&= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 - a_5 b_5 - a_6 b_6 - a_7 b_7 - a_8 b_8)^2 \\
&+ (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3 + a_5 b_6 - a_6 b_5 + a_7 b_8 - a_8 b_7)^2 \\
&+ (a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 - a_5 b_7 + a_6 b_8 + a_7 b_5 - a_8 b_6)^2 \\
&+ (a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1 + a_5 b_8 + a_6 b_7 - a_7 b_6 - a_8 b_5)^2 \\
&+ (a_1 b_5 - a_2 b_6 + a_3 b_7 - a_4 b_8 + a_5 b_1 + a_6 b_2 - a_7 b_3 + a_8 b_4)^2 \\
&+ (a_1 b_6 + a_2 b_5 - a_3 b_8 - a_4 b_7 - a_5 b_2 + a_6 b_1 + a_7 b_4 + a_8 b_3)^2 \\
&+ (a_1 b_7 - a_2 b_8 - a_3 b_5 + a_4 b_6 + a_5 b_3 - a_6 b_4 + a_7 b_1 + a_8 b_2)^2 \\
&+ (a_1 b_8 + a_2 b_7 + a_3 b_6 + a_4 b_5 - a_5 b_4 - a_6 b_3 - a_7 b_2 + a_8 b_1)^2
\end{aligned}
\tag{142}
$$

valid for all real numbers a_i and b_i and thus in particular also for all integers. (There are many variations of this expression, with different possible sign combinations.) The theorem was discovered in 1818 by Carl Ferdinand Degen (1766–1825), and then rediscovered in 1844 by John Graves and in 1845 by Arthur Cayley. There is no generalization to higher numbers of squares, a fact proved by Adolf Hurwitz (1859–1919) in 1898.

The octonions can be used to show that a vector product can be defined in more than three dimensions. A *vector product* or *cross product* is an operation $\times$ satisfying

$$
\begin{aligned}
u \times v &= -v \times u && \text{anticommutativity} \\
(u \times v)\, w &= u\,(v \times w) && \text{exchange rule.}
\end{aligned}
\tag{143}
$$

Using the definition

$$
X \times Y = \frac{1}{2}(XY - YX) \,,
\tag{144}
$$

the cross products of imaginary quaternions, i.e., of quaternions of the type $(0, \boldsymbol{u})$, are again imaginary, and correspond to the usual, three-dimensional vector product, thus fulfilling (143). Interestingly, it is possible to use definition (144) for *octonions* as well. In that case, the product of imaginary octonions is also imaginary, and (143) is again satisfied. In fact, this is the only other non-trivial example of a vector product.

In summary: *A vector product exists only in three and in seven dimensions.* Many scholars have conjectured that this relation is connected with a possible ten-dimensionality of nature; however, these speculations have not met with any success.

The symmetries of the forces in nature lead to a well-known question. The unit complex numbers from the Lie group U(1) and the unit quaternions the Lie group SU(2). Do the unit octonions form the Lie group SU(3)?

OTHER TYPES OF NUMBERS

The process of constructing new systems of hypercomplex 'numbers' or real algebras by 'doubling' a given one can be continued ad infinitum. However, octonions, *sedenions* and

all the following doublings are neither rings nor fields, but only non-associative algebras with unity. Other finite-dimensional algebras with unit element over the reals, once called hypercomplex 'numbers', can also be defined: they include the so-called 'dual numbers', 'double numbers', 'Clifford–Lifshitz numbers' etc. They play no role in physics.

Mathematicians have also defined number fields which have 'one and a bit' dimensions, such as algebraic number fields. There is also a generalization of the concept of integers to the complex domain: the *Gaussian integers*, defined as $n + im$, where n and m are ordinary integers. Gauss even defined what are now known as *Gaussian primes*. (Can you find out how?) They are not used in the description of nature, but are important in number theory, the exploration of the properties of integers.

Physicists used to call quantum-mechanical operators 'q-numbers.' But this term has now fallen out of fashion.

Another way in which the natural numbers can be extended is to include numbers larger than infinite. The most important such classes of *transfinite number* are the *ordinals*, the *cardinals* and the *surreals*. The ordinals are essentially an extension of the integers beyond infinity, whereas the surreals are a continuous extension of the reals, also beyond infinity. Loosely speaking, among the transfinites, the ordinals have a similar role as the integers have among the reals; the surreals fill in all the gaps between the ordinals, like the reals do for integers. Interestingly, many series that diverge in $\mathbb{R}$ converge in the surreals. Can you find one example?

The surreals include infinitely small numbers, as do the numbers of *nonstandard analysis*, also called *hyperreals*. In both number systems, in contrast to real numbers, the numbers 1 and 0.999 999... (where an infinite, but hyperfinite string of nines is implied) do not coincide, but are separated by infinitely many other numbers. We explored surreals earlier on. Nonstandard numbers can be used to define the infinitesimals used in integration and differentiation, even at secondary school level.

FROM VECTOR SPACES TO HILBERT SPACES

Vector spaces, also called *linear spaces*, are mathematical generalizations of certain aspects of the intuitive three-dimensional space. A set of elements any two of which can be added together and any one of which can be multiplied by a number is called a vector space, if the result is again in the set and the usual rules of calculation hold.

More precisely, a *vector space* over a number field K is a set of elements, called *vectors*, for which a vector addition and a *scalar multiplication* is defined, such that for all vectors a, b, c and for all numbers s and r from K one has

$$
\begin{aligned}
(a + b) + c = a + (b + c) = a + b + c \quad & \textit{associativity of vector addition} \\
n + a = a \quad & \textit{existence of null vector} \\
(-a) + a = n \quad & \textit{existence of negative vector} \\
1a = a \quad & \textit{regularity of scalar multiplication} \\
(s + r)(a + b) = sa + sb + ra + rb \quad & \textit{complete distributivity of scalar multiplication}
\end{aligned}
\tag{145}
$$

If the field K, whose elements are called *scalars* in this context, is taken to be the real (or

complex, or quaternionic) numbers, one speaks of a real (or complex, or quaternionic) vector space. Vector spaces are also called *linear vector spaces* or simply *linear spaces*.

The complex numbers, the set of all real functions defined on the real line, the set of all polynomials, the set of matrices with a given number of rows and columns, all form vector spaces. In mathematics, a vector is thus a more general concept than in physics. (What is the simplest possible mathematical vector space?)

In physics, the term 'vector' is reserved for elements of a more specialized type of vector space, namely normed inner product spaces. To define these, we first need the concept of a metric space.

A *metric space* is a set with a metric, i.e., a way to define distances between elements. A real function $d(a,b)$ between elements is called a metric if

$$
\begin{aligned}
d(a,b) &\geqslant 0 & \textit{positivity of metric} \\
d(a,b) + d(b,c) &\geqslant d(a,c) & \textit{triangle inequality} \\
d(a,b) = 0 \quad \text{if and only if} \quad a &= b & \textit{regularity of metric}
\end{aligned}
\tag{146}
$$

A non-trivial example is the following. We define a special distance d between cities. If the two cities lie on a line going through Paris, we use the usual distance. In all other cases, we define the distance d by the shortest distance from one to the other travelling via Paris. This strange method defines a metric between all cities in France, the so-called *French railroad distance*.

A *normed* vector space is a linear space with a norm, or 'length', associated to each a vector. A *norm* is a non-negative number $\|a\|$ defined for each vector a with the properties

$$
\begin{aligned}
\|ra\| &= |r| \, \|a\| & \textit{linearity of norm} \\
\|a + b\| &\leqslant \|a\| + \|b\| & \textit{triangle inequality} \\
\|a\| = 0 \quad \text{only if} \quad a &= 0 & \textit{regularity}
\end{aligned}
\tag{147}
$$

Usually there are many ways to define a norm for a given vector space. Note that a norm can always be used to define a metric by setting

$$
d(a,b) = \|a - b\|
\tag{148}
$$

so that all normed spaces are also metric spaces. This is the *natural* distance definition (in contrast to unnatural ones like that between French cities given above).

The norm is often defined with the help of an inner product. Indeed, the most special class of linear spaces are the *inner product spaces*. These are vector spaces with an *inner product*, also called *scalar product* · (not to be confused with the scalar multiplication!)

Challenge 197 s

Challenge 198 s

Challenge 199 ny

which associates a number to each pair of vectors. An inner product space over $\mathbb{R}$ satisfies

$$
\begin{aligned}
a \cdot b &= b \cdot a & &\textit{commutativity of scalar product} \\
(ra) \cdot (sb) &= rs(a \cdot b) & &\textit{bilinearity of scalar product} \\
(a + b) \cdot c &= a \cdot c + b \cdot c & &\textit{left distributivity of scalar product} \\
a \cdot (b + c) &= a \cdot b + a \cdot c & &\textit{right distributivity of scalar product} \\
a \cdot a &\geqslant 0 & &\textit{positivity of scalar product} \\
a \cdot a = 0 \quad \text{if and only if} \quad a &= 0 & &\textit{regularity of scalar product}
\end{aligned}
\tag{149}
$$

for all vectors a, b, c and all scalars r, s. A *real* inner product space of finite dimension is also called a *Euclidean* vector space. The set of all velocities, the set of all positions, or the set of all possible momenta form such spaces.

An inner product space over $\mathbb{C}$ satisfies*

$$
\begin{aligned}
a \cdot b &= \overline{b \cdot a} = \overline{b} \cdot \overline{a} & &\textit{Hermitean property} \\
(ra) \cdot (sb) &= r\overline{s}(a \cdot b) & &\textit{sesquilinearity of scalar product} \\
(a + b) \cdot c &= a \cdot c + b \cdot c & &\textit{left distributivity of scalar product} \\
a \cdot (b + c) &= a \cdot b + a \cdot c & &\textit{right distributivity of scalar product} \\
a \cdot a &\geqslant 0 & &\textit{positivity of scalar product} \\
a \cdot a = 0 \quad \text{if and only if} \quad a &= 0 & &\textit{regularity of scalar product}
\end{aligned}
\tag{150}
$$

for all vectors a, b, c and all scalars r, s. A *complex* inner product space (of finite dimension) is also called a *unitary* or *Hermitean* vector space. If the inner product space is *complete*, it is called, especially in the infinite-dimensional complex case, a *Hilbert space*. The space of all possible states of a quantum system forms a Hilbert space.

All inner product spaces are also metric spaces, and thus normed spaces, if the metric is defined by

$$
d(a,b) = \sqrt{(a - b) \cdot (a - b)} \; .
\tag{151}
$$

Only in the context of an inner product spaces we can speak about angles (or phase differences) between vectors, as we are used to in physics. Of course, like in normed spaces, inner product spaces also allows us to speak about the length of vectors and to define a *basis*, the mathematical concept necessary to define a coordinate system. Which vector spaces or inner product spaces are of importance in physics?

The *dimension* of a vector space is the number of linearly independent basis vectors. Can you define these terms precisely?

A *Hilbert space* is a real or complex inner product space that is also a *complete* metric space. In other terms, in a Hilbert space, distances vary continuously and behave as naively expected. Hilbert spaces usually, but not always, have an infinite number of dimensions.

---

* Two inequivalent forms of the sesquilinearity axiom exist. The other is $(ra) \cdot (sb) = \overline{r}s(a \cdot b)$. The term *sesquilinear* is derived from Latin and means for 'one-and-a-half-linear'.

The definition of Hilbert spaces and vector spaces assume continuous sets to start with. If nature would not be continuous, could one still use the concepts?

Challenge 202 s

Mathematical curiosities and fun challenges

Mathematics provides many counter-intuitive results. Reading a good book on the topic, such as BERNARD R. GELBAUM & JOHN M. H. OLMSTED, *Theorems and Counter-examples in Mathematics*, Springer, 1993, can help you sharpen your mind and make you savour the beauty of mathematics even more.

∗∗

It is possible to draw a curve that meets all points in a square or all points in a cube. This is shown, for example, in the text HANS SAGAN, *Space Filling Curves*, Springer Verlag, 1994. As a result, the distinction between one, two and three dimensions is blurred in pure mathematics. In physics however, dimensions are clearly and well-defined; every object in nature has three dimensions.

Challenge 203 e

∗∗

Challenge 204 ny

Show that two operators A and B obey

$$
\begin{aligned}
\mathrm{e}^A \mathrm{e}^B = \exp(A + B &+ \frac{1}{2}[A, B] \\
&+ \frac{1}{12}[[A, B], B] - \frac{1}{12}[[A, B], A] \\
&- \frac{1}{48}[B, [A, [A, B]]] - \frac{1}{48}[A, [B, [A, B]]] \\
&+ \ldots)
\end{aligned}
\tag{152}
$$

for most operators A and B. This result is often called the *Baker–Campbell–Hausdorff formula* or the *BCH formula*.
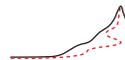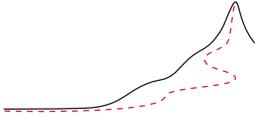
CHALLENGE HINTS AND SOLUTIONS

> *Never make a calculation before you know the answer.*
>
> John Wheeler's motto

Challenge 1, page 10: Do not hesitate to be demanding and strict. The next edition of the text will benefit from it.

Challenge 2, page 16: Classical physics fails in explaining any material property, such as colour or softness. Material properties result from nature's interactions; they are inevitably quantum. Explanations of material properties require, without exception, the use of particles and their quantum properties.

Challenge 3, page 17: Classical physics allows any observable to change *smoothly* with time. In classical physics, there is no minimum value for any observable physical quantity.

Challenge 4, page 20: The higher the mass, the smaller the motion fuzziness induced by the quantum of action, because action is mass times speed times distance: For a large mass, the speed and distance variations are small.

Challenge 5, page 20: The simplest time is $\sqrt{G\hbar/c^5}$. The numerical factor is obviously not fixed; it is changed later on. Using $4G$ instead of G the time becomes the shortest time measurable in nature.

Challenge 7, page 21: The electron charge is special to the electromagnetic interactions; it does not take into account the nuclear interactions or gravity. It is unclear why the length defined with the elementary charge e should be of importance for neutral systems or for the vacuum. On the other hand, the quantum of action $\hbar$ is valid for *all* interactions and *all* observations.

In addition, we can argue that the two options to define a fundamental length – with the quantum of action and with the quantum of charge – are not too different: the electron charge is related to the quantum of action by $e = \sqrt{4\pi\varepsilon_0 \alpha c\hbar}$. The two length scales defined by the two options differ only by a factor near 11.7. In fact, both scales are *quantum* scales.

Challenge 8, page 21: On purely dimensional grounds, the radius of an atom must be

$$r \approx \frac{\hbar^2 4\pi\varepsilon_0}{m_e e^2} \,, \tag{153}$$

which is about 53 nm. Indeed, this guess is excellent: it is just the Bohr radius.

Challenge 9, page 21: Due to the quantum of action, atoms in all people, be they giants or dwarfs, have the same size. This implies that giants cannot exist, as was shown already by Galileo. The argument is based on the given strength of materials; and a same strength everywhere is equivalent to the same properties of atoms everywhere. That dwarfs cannot exist is due to a similar reason; nature is not able to make people smaller than usual (even in the womb they differ markedly from adults) as this would require smaller atoms.

Challenge 12, page 27: A disappearance of a mass m in a time Δt is an action change $c^2 m \Delta t$. That is much larger than $\hbar$ for all objects of everyday life.

Challenge 14, page 29: Tunnelling of a lion would imply action values S of the order of $S = 100 \, \mathrm{kg m^2/s} \gg \hbar$. This cannot happen spontaneously.

Challenge 15, page 30: Every memory, be it human memory or an electronic computer memory, must avoid decay. And decay can only be avoided through high walls and low tunnelling rates.

Challenge 16, page 30: Yes! Many beliefs and myths – from lottery to ghosts – are due to the neglect of quantum effects.

Challenge 17, page 30: Perfectly continuous flow is in contrast to the fuzziness of motion induced by the quantum of action.

Challenge 18, page 31: The impossibility of following two particles along their path appears when their mutual distance d is smaller than their position indeterminacy due to their relative momentum p, thus when $d < \hbar/p$. Check the numbers with electrons, atoms, molecules, bacteria, people and galaxies.

Challenge 19, page 31: Also photons are indistinguishable. See page 63.

Challenge 21, page 36: In the material that forms the escapement mechanism.

Challenge 22, page 36: Growth is not proportional to light intensity or to light frequency, but shows both intensity and frequency *thresholds*. These are quantum effects.

Challenge 23, page 36: All effects mentioned above, such as tunnelling, interference, decay, transformation, non-emptiness of the vacuum, indeterminacy and randomness, are also observed in the nuclear domain.

Challenge 24, page 37: This is not evident from what was said so far, but it turns out to be correct. In fact, there is no other option, as you will see when you try to find one.

Challenge 25, page 37: Tom Thumb is supposedly as smart as a normal human. But a brain cannot be scaled down. Fractals contradict the existence of Planck's length, and Moore's law contradicts the existence of atoms.

Challenge 26, page 37: The total angular momentum counts, including the orbital angular momentum. The orbital angular momentum L is given, using the radius and the linear momentum, $L = r \times p$. The total angular momentum is a multiple of $\hbar$.

Challenge 27, page 37: Yes, we could have!

Challenge 28, page 37: That is just the indeterminacy relation. Bohr expanded this idea to all sort of other pairs of concepts, more in the philosophical domain, such as clarity and precision of explanations: both cannot be high at the same time.

Challenge 29, page 39: The big bang cannot have been an event, for example.

Challenge 32, page 45: Charged photons would be deflected by electric of magnetic fields; in particular, they would not cross undisturbed. This is not observed. Massive photons would be deflected by masses, such as the Sun, much more than is observed.

Challenge 34, page 45: To measure momentum, we need a spatially extended measurement device; to measure position, we need a localized measurement device.

Challenge 35, page 47: Photons are elementary because they realize the minimum action, because they cannot decay, because they cannot be deformed or split, because they have no mass, no electric charge and no other quantum number, and because they appear in the Lagrangian of quantum electrodynamics.

Challenge 36, page 50: The measured electric fields and photon distribution are shown in the famous graphs reproduced in Figure 90.
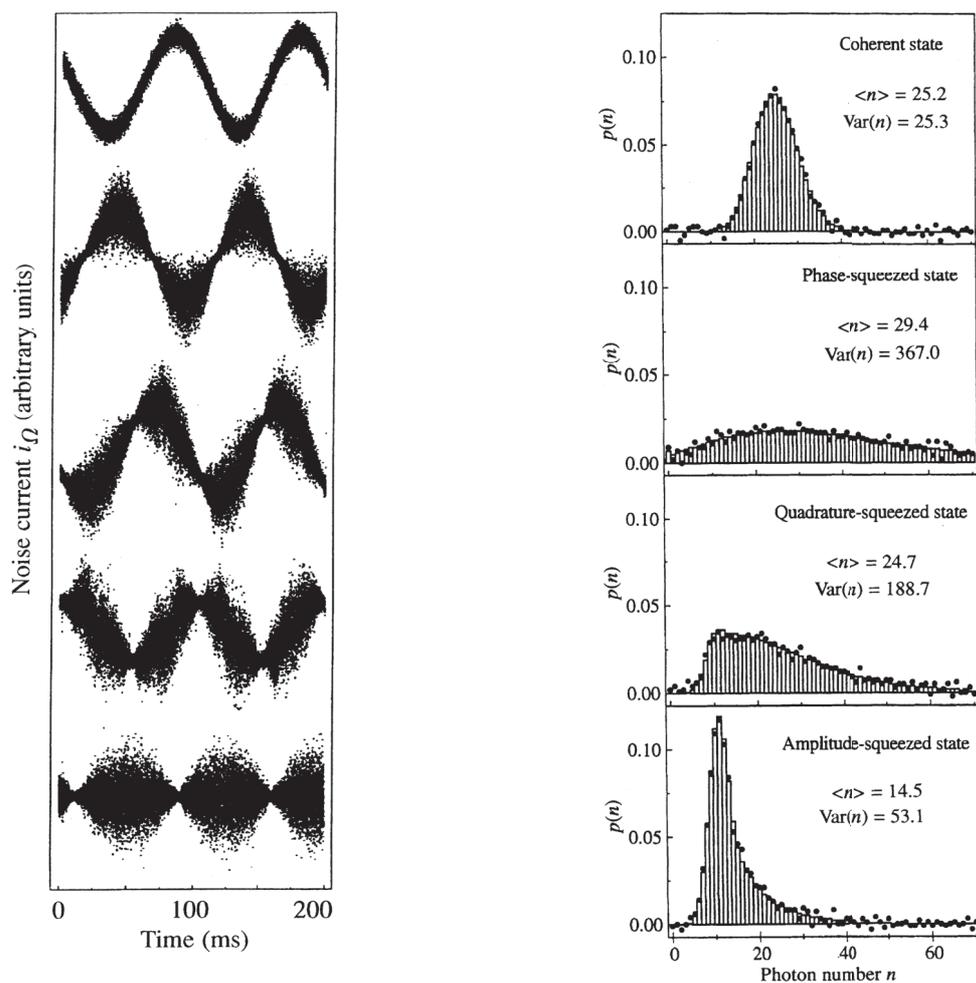
FIGURE 90 Left, from top to bottom: the electric field and its fuzziness measured for a coherent state, for a squeezed vacuum state, for a phase-squeezed state, for a mixed, quadrature-squeezed state and for an amplitude-squeezed state, all with a small number of photons. Right: the corresponding photon number distributions for the uppermost four states. (© G. Breitenbach/Macmillan, from Ref. 19)

Challenge 38, page 50: This is an unclearly posed problem. The radiation is thermal, but the photon number depends on the volume under discussion.

Challenge 40, page 56: Radio photons can be counted using optical pumping experiments in which atomic states are split by a small, 'radio-wavelength' amount, with the help of magnetic fields. Also caesium clocks detect radio photons with optical means. The Josephson effect and magnetic resonance imaging are additional detection methods for radio photons.

Challenge 41, page 57: To be observable to the eye, the interference fringes need to be visible for around 0.1 s. That implies a maximum frequency difference between the two beams of around 10 Hz. This is achievable only if either a single beam is split into two or if the two beams come from high-precision, stabilized lasers.

Challenge 42, page 62: Implicit in the arrow model is the idea that one quantum particle is described by one arrow.

Challenge 48, page 64: Despite a huge number of attempts and the promise of eternal fame, this is the sober conclusion.

Challenge 53, page 68: Yes, the argument is correct. In fact, more detailed discussions show that classical electrodynamics is in contradiction with *all* colours observed in nature.

Challenge 57, page 73: The calculation is not easy, but not too difficult either. For an initial orientation *close* to the vertical, the fall time T turns out to be

$$T = \frac{1}{2\pi} T_0 \ln \frac{8}{\alpha} \tag{154}$$

where α is the starting angle, and a fall through angle π is assumed. Here T_0 is the oscillation time of the pencil for small angles. (Can you determine it?) The indeterminacy relation for the tip of the pencil yields a *minimum* starting angle, because the momentum indeterminacy cannot be made arbitrarily large. You should be able to provide an upper limit. Once this angle is known, you can calculate the maximum time.

Challenge 58, page 74: Use the temperature to calculate the average kinetic energy, and thus the average speed of atoms.

Challenge 59, page 74: At such low temperatures, the atoms cannot be fully distinguished; they form a state of matter with peculiar properties, called a *condensate*. The condensate is not at rest either; but due to its large mass, its fluctuations are greatly reduced, compared to those of a single atom.

Challenge 61, page 78: Only variables whose product has the same units as physical action – Js – can be complementary to each other.

Challenge 62, page 79: Use $\Delta E < E$ and $a\,\Delta t < c$.

Challenge 67, page 86: The quantum of action does not apply only to measurements, it applies to motion itself, and in particular, to all motion. Also effects of the nuclear forces, of nuclear particles and of nuclear radiation particles must comply to the limit. And experiments show that they indeed do. In fact, if they did not, the quantum of action in electrodynamic situations could be circumvented, as you can check.

Challenge 74, page 96: Outside the garage, all atoms need to form the same solid structure again.

Challenge 75, page 97: Terabyte chips would need to have small memory cells. Small cells imply thin barriers. Thin barriers imply high probabilities for tunnelling. Tunnelling implies lack of memory.

Challenge 81, page 108: If a particle were not elementary, its components would be bound by an interaction. But there are no known interactions outside those of the standard model.

Challenge 82, page 109: The difficulties to see hydrogen atoms are due to their small size and their small number of electrons. As a result, hydrogen atoms produce only weak contrasts in X-ray images. For the same reasons it is difficult to image them using electrons; the Bohr radius of hydrogen is only slightly larger than the electron Compton wavelength.

For the first time, in 2008, a research team claimed to have imaged hydrogen atoms adsorbed on graphene with the help of a transmission electron microscope. For details, see J. C. MEYER, C. O. GRIT, M. F. CROMMLE & A. ZETTI, *Imaging and dynamics of light atoms and molecules on graphene*, Nature 454, pp. 319–322, 2008. However, it seems that the report has not been confirmed by another group yet.

More hydrogen images have appeared in recent years. You may search for olympicene on the
internet, for example. For another recent result about hydrogen imaging, see above.

Challenge 84, page 109: This is not easy! Can you use the concept of action to show that there indeed is a fundamental difference between very similar and very different operators?

Challenge 86, page 110: $r = 86\,\text{pm}$, thus $T = 12\,\text{eV}$. This compares to the actual value of $13.6\,\text{eV}$. The trick for the derivation of the formula is to use $\langle \psi \,|\, r_x^2 \,|\, \psi \rangle = \frac{1}{3}\langle \psi \,|\, \boldsymbol{rr} \,|\, \psi \rangle$, a relation valid for states with no orbital angular momentum. It is valid for all coordinates and also for the three momentum observables, as long as the system is non-relativistic.

Challenge 87, page 111: A quantum fluctuation would require the universe to exist already. Such statements, regularly found in the press, are utter nonsense.

Challenge 88, page 112: Point particles cannot be marked; nearby point particles cannot be distinguished, due to the quantum of action.

Challenge 89, page 112: The solution is *two* gloves. In the original setting, if two men and two women want to make love without danger, in theory they need only *two* condoms.

Challenge 94, page 114: The Sackur–Tetrode formula is best deduced in the following way. We start with an ideal monoatomic gas of volume V, with N particles, and total energy U. In phase space, *state sum Z* is given by

$$Z = \frac{V^N}{N!}\frac{1}{\Lambda^{3N}}\;. \tag{155}$$

We use Stirling's approximation $N! \approx N^N/e^N$, and the definition of the entropy as $S = \partial(kT\ln Z)/\partial T$. Inserting the definition of Λ, this gives the Sackur–Tetrode equation.

Challenge 96, page 117: To write anything about two particles on paper, we need to distinguish them, even if the distinction is arbitrary.

Challenge 99, page 123: The idea, also called *quantum money*, is not compatible with the size and lifetime requirements of actual banknotes.

Challenge 100, page 124: Twins differ in the way their intestines are folded, in the lines of their hands and other skin folds. Sometimes, but not always, features like black points on the skin are mirror inverted on the two twins.

Challenge 109, page 135: Three.

Challenge 110, page 135: Not for a mattress. This is not easy to picture.

Challenge 111, page 136: Angels can be distinguished by name, can talk and can sing; thus they are made of a large number of fermions. In fact, many angels are human sized, so that they do not even fit on the tip of a pin.

Challenge 117, page 140: A boson can be represented by an object glued to one infinitesimally thin thread whose two tails reach spatial infinity.

Challenge 118, page 141: Trees, like all macroscopic objects, have a spin value that depends on their angular momentum. Being classical objects whose phase can be observed, the spin value is uncertain. It makes no sense to ask whether trees or other macroscopic objects are bosons or fermions, as they are not quantons.

Challenge 121, page 142: Ghosts, like angels, can be distinguished by name, can talk and can be seen; thus they contain fermions. However, they can pass through walls and they are transparent; thus they cannot be made of fermions, but must be images, made of bosons. That is a contradiction.

Challenge 122, page 144: Macroscopic superpositions cannot be drawn, because observation implies interaction with a bath, which destroys macroscopic superposition.

Challenge 124, page 146: The loss of non-diagonal elements leads to an increase in the diagonal elements, and thus of entropy.

Challenge 127, page 153: The energy speed is given by the advancement of the outer two tails; that speed is never larger than the speed of light.

Challenge 128, page 155: No, as taking a photo implies an interaction with a bath, which would destroy the superposition. In more detail, a photograph requires illumination; illumination is a macroscopic electromagnetic field; a macroscopic field is a bath; a bath implies decoherence; decoherence destroys superpositions.

Challenge 131, page 157: It depends. They can be due to interference or to intensity sums. In the case of radio the effect is clearer. If at a particular frequency the signals changes periodically from one station to another, one has a genuine interference effect.

Challenge 132, page 157: They interfere. But this is a trick question; what is a monochromatic electron? Does it occur in the laboratory?

Challenge 133, page 157: Such a computer requires clear phase relations between components; such phase relations are extremely sensitive to outside disturbances. At present, they do not hold longer than a hundred microseconds, whereas long computer programs require minutes and hours to run.

Challenge 134, page 157: A record is an effect of a process that must be *hard to reverse or undo*. The traces of a broken egg are easy to clean on a large glass plate, but hard in the wool of a sheep. Broken teeth, torn clothes, or scratches on large surfaces are good records. Forensic scientists know many additional examples.

Challenge 138, page 166: Any other bath also does the trick, such as the atmosphere, sound vibrations, electromagnetic fields, etc.

Challenge 139, page 166: The Moon is in contact with baths like the solar wind, falling meteorites, the electromagnetic background radiation of the deep universe, the neutrino flux from the Sun, cosmic radiation, etc.

Challenge 140, page 168: Spatially periodic potentials have the property. Decoherence then leads to momentum diagonalization.

Challenge 142, page 171: If so, let the author know.

Challenge 143, page 182: The red shift value is $z = 9.9995$. From the formula for the longitudinal Doppler shift we get $v/c = ((z + 1)^2 - 1)/((z + 1)^2 + 1)$; this yields 0.984 in the present case. The galaxy thus moves away from Earth with 98.4 % of the speed of light.

Challenge 149, page 184: Hydrogen atoms are in eigenstates for the reasons explained in the chapter on superpositions and probabilities: in a gas, atoms are part of a bath, and thus almost always in energy eigenstates.

Challenge 154, page 195: If several light beams are focused in the space between the mirrors, and if the light beam frequency is properly tuned with respect to the absorption frequencies of the atoms, atoms will experience a restoring force whenever they move away from the focus region. By shining light beams to the focus region from 6 directions, atoms are trapped. The technique of laser cooling is now widely used in research laboratories.

Challenge 155, page 196: No, despite its name, phosphorus is not phosphorescent, but chemoluminescent.

Challenge 157, page 197: This is a trick question. A change in α requires a change in $c, \hbar, e$ or ε_0. None of these changes is possible or observable, as all our measurement apparatus are based on these units. Speculations about change of α, despite their frequency in the press and in scientific journals, are idle talk.

Challenge 158, page 197: A change of physical units such that $\hbar = c = e = 1$ would change the value of ε_0 in such a way that $4\pi\varepsilon_0 = 1/\alpha \approx 137.036$.

Challenge 161, page 207: Mass is a measure of the amount of energy. The 'square of mass' makes no sense.

Challenge 165, page 210: Planck limits can be exceeded for extensive observables for which many particle systems can exceed single particle limits, such as mass, momentum, energy or electrical resistance.

Challenge 167, page 212: Do not forget the relativistic time dilation.

Challenge 168, page 212: The formula with $n-1$ is a better fit. Why?

Challenge 171, page 213: No! They are much too precise to make sense. They are only given as an illustration for the behaviour of the Gaussian distribution. Real measurement distributions are not Gaussian to the precision implied in these numbers.

Challenge 172, page 213: About 0.3 m/s. It is *not* 0.33 m/s, it is *not* 0.333 m/s and it is *not* any longer strings of threes.

Challenge 174, page 219: The slowdown goes *quadratically* with time, because every new slowdown adds to the old one!

Challenge 175, page 220: No, only properties of parts of the universe are listed. The universe itself has no properties, as shown in the last volume.

Challenge 176, page 221: The double of that number, the number made of the sequence of all even numbers, etc.

Challenge 179, page 225: We will find out in the last volume that all measurement values have upper and lower bounds. We will also find out that two physical measurement results cannot differ just from, say, the 300th decimal place onwards. So indeed, all measurement results are real numbers, but not vice versa. It needs to be stressed that for quantum theory, for relativity and also for Galilean physics this restriction has no consequences whatsoever.

Challenge 181, page 226: $|z|^2$ is the *determinant* of the matrix $z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Challenge 186, page 227: Use Cantor's diagonal argument, as in challenge 274.

Challenge 187, page 228: Any quaternion $X = ai+bj+ck$ with $a^2+b^2+c^2 = 1$ solves the equation $X^2 + 1 = 0$; the purely imaginary solutions $+i$ and $-i$ are thus augmented by a continuous sphere of solutions in quaternion space.

Challenge 190, page 230: Any rotation by an angle 2π is described by -1. Only a rotation by 4π is described by $+1$; quaternions indeed describe spinors.

Challenge 192, page 232: Just check the result component by component. See also the mentioned reference.

Challenge 194, page 234: No. Because the unit octonions are not associative, they do not form a group at all. Despite its superficial appeal, this line of reasoning has not led to any insight into the nature of the fundamental interactions.

Challenge 195, page 235: For a Gaussian integer $n + im$ to be prime, the integer $n^2 + m^2$ must be prime, and in addition, a condition on n mod 3 must be satisfied; which one and why?

Challenge 197, page 236: The set that contains only the zero vector.

Challenge 198, page 236: The metric is regular, positive definite and obeys the triangle inequality.

Challenge 200, page 237: Essentially only the vector spaces listed in the appendix (or in the book).

Challenge 201, page 237: If you cannot, blame your math teacher at secondary school, and then look up the definitions. It is not a difficult topic.

Challenge 202, page 238: Spaces could exist approximately, as averages of non-continuous structures. This idea is explored in modern research; an example is given in the last volume of this series.
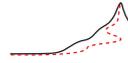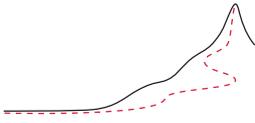
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> Samuel Johnson

> **"** As soon as you write, no time to read remains. **"**
> Anonymous

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3 Bohr explained the indivisibilty of the quantum of action in his famous Como lecture. See N. Bohr, *Atomtheorie und Naturbeschreibung*, Springer, 1931. On page 16 he writes: 'No more is it likely that the fundamental concepts of the classical theories will ever become superfluous for the description of physical experience. The recognition of the indivisibility of the quantum of action, and the determination of its magnitude, not only depend on an analysis of measurements based on classical concepts, but it continues to be the application of these concepts alone that makes it possible to relate the symbolism of the quantum theory to the data of experience.' He also writes: '...the fundamental postulate of the indivisibility of the quantum of action is itself, from the classical point of view, an irrational element which inevitably requires us to forgo a causal mode of description and which, because of the coupling between phenomena and their observation, forces us to adopt a new mode of description designated as complementary in the sense that any given application of classical concepts precludes the simultaneous use of other classical concepts which in a different connection are equally necessary for the elucidation of the phenomena ...' and '...the finite magnitude of the quantum of action prevents altogether a sharp distinction being made between a phenomenon and the agency by which it is observed, a distinction which underlies the customary concept of observation and, therefore, forms the basis of the classical ideas of motion.' Other statements about the indivisibility of the quantum of action can be found in N. Bohr, *Atomic Physics and Human Knowledge*, Science Editions, 1961. See also Max Jammer, *The Philosophy of Quantum Mechanics*, Wiley, first edition, 1974, pp. 90–91. Cited on page 17.

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