Physics 20 Lecture Notes

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Contents

Chapter 1

Vectors

1.1 The Philosophy of Physics

Everything I'm going to tell you in the next two weeks is a lie.

The reason for this is that physics is an experimental science. Over the course of the Scientific Revolution, it slowly became accepted that if you wanted to know how the Universe works, you had to actually go do an experiment and find out. This may seem obvious to us today, but this way of thinking has important ramifications for the way we do physics, which can be easy to forget when first learning about the subject. At its core, physics is about coming up with models to try to understand how the Universe works, and then making predictions from those models which we can test in an experiment.

Because the ultimate laws of physics are not handed down to us from some higher being, it's left to us to try to figure out what they are. Unfortunately, we are faced with the problem that our experimental data will always be incomplete. There's always a region of the Universe we haven't been able to observe yet, a complicated type of material we haven't been able to build in a lab yet, or some very short distance scale which we haven't been able to probe yet.

As a result of this, any theory of physics comes with an asterisk on it, which tells us that it is only valid *as far as we know so far*. Newtonian mechanics, the subject of Physics 20, was the prevailing theory of how mechanical bodies behaved, up until the early 1900s. After observing the ways in which the Universe appeared to behave, Isaac Newton and other scientists realized that they could assume certain mathematical rules about how physical bodies should behave which fit this data, and allowed them to make further experimental predictions which were subsequently verified. But eventually, once experiments became sensitive enough to make detailed measurements involving the motion of light, Newtonian mechanics was replaced with a more complete theory, called Special

Relativity.

These facts are relevant to this course in two primary ways. First, it puts what you are about to learn into perspective, which is always a useful thing to know before taking any course on any subject. Newtonian mechanics is a model for how the world works, which has been proven to be very successful for describing the behavior of what me might call "everyday" objects, bodies which are average in physical size and move at speeds much slower than the speed of light. In some sense, this theory is "wrong," in that we now have theories which are more accurate, and only approximately reproduce Newtonian mechanics under certain conditions. However, Newtonian mechanics is still an incredibly useful tool. If I were studying, for example, the projectile motion of a rock fired from a catapult, in principle, a theory of the detailed microscopic properties of the spring used to build the catapult would provide me with more information than Newtonian mechanics. But this is usually unnecessary in any practical application - once I measure the speed with which my catapult can launch rocks, then I can make use of this information to study the motion of the rock with a very high level of accuracy, more than enough to be able to get it over the walls of my enemy's castle. However, if I wanted to build a better, more powerful spring, I would eventually need to know more about the chemistry of the materials it's made of.

Secondly, the fact that the practice of physics involves doing experiments and inferring laws from them means that there is an art form to doing physics. Being able to look at experimental data, decide what information is actually relevant to what you need to know, and then building a theory from there takes a lot of training. If you continue to do scientific research as a career, you probably won't be competent at this until you've finished your PhD, and only after years working as a professional researcher will you become skilled at it. But the ability to look at a situation, decide what information is relevant, and answer a question is a skill which you can start honing now, and this is really the most useful thing which you're going to learn in your early physics education. At the beginning of your education, this skill will be mostly developed by accepting some general physical principles, and attempting to make a physical prediction that could be tested in an experiment. As you begin to take lab classes, you'll start to learn how to actually do the experimental part of this process. Ultimately, if you continue on to become a professional researcher, you'll learn how to actually look at the Universe and say something new about it. But whether or not you continue on to become a professional physicist, I think you'll find that developing this ability will be incredibly useful to you in life in general, and this sort of mindset is the one I hope to center this course around.

With that said, let's start learning about some physics.

1.2 Vectors

In Newtonian mechanics, we want to understand how material bodies interact with each other and how this affects their motion through space. In order to be able to make quantitative statements about this, we need to develop a sort of mathematical language for describing motion. Part of this language involves calculus, since we know that calculus lets us talk about how things change in a quantitative way. The other part of this language involves the notion of a vector. A vector is a mathematical object which not only has a magnitude, but also an orientation in space. For example, the velocity of a material object is a vector. It has a magnitude, which we call its speed. This tells us how "quickly" the object is moving, in other words how much distance it travels in some time. It also has a direction - moving to the left is different from moving to the right.

This is in contrast with what we call scalars, which are simply just numbers, with no direction. For example, the temperature of an object is a scalar - temperature doesn't "point" in a direction. However, if we were to put two bodies of different temperature in contact and ask about the heat flow between them, we would be talking about a vector - the heat would flow at some rate, and in some direction (from the hot body to the cold body).

When we write vectors down on paper, there are two common notations. One way is to write them with an arrow over them, such as $\vec{a}$. Another way is to simply write them in bold, such as $\mathbf{a}$. When I want to talk about the magnitude of a vector, I will usually write this as the name of the vector without any special style, just as a. Another notation is to write the magnitude as $|\vec{a}|$.

One type of vector which we'll talk about a lot is the displacement between two objects, which is shown in Figure 1.1. If I have two objects somewhere in space, the displacement from one object to the other is a vector quantity, whose magnitude is the distance between them, and whose orientation points from the first object to the second object. We can use this example to demonstrate the first mathematical operation we can do with vectors, which is vector addition.

Consider the three points in space, A, B, and C. These could represent the location of some particles interacting in some way, or maybe this is just some map of where my house is in relation to the rest of Goleta. In any event, we call the vector pointing from point A to point B $\vec{v}_{AB}$. Likewise, the vector from point B to C is $\vec{v}_{BC}$, and the vector from A to C is $\vec{v}_{AC}$. We define the vector sum of $\vec{v}_{AB}$ and $\vec{v}_{BC}$ to be equal to $\vec{v}_{AC}$,

$$\vec{v}_{AB} + \vec{v}_{BC} = \vec{v}_{AC}. \tag{1.1}$$

This agrees with our intuitive sense of moving through space - if I walk from A to B, and then from B to C, I've effectively walked from A to C. Notice,

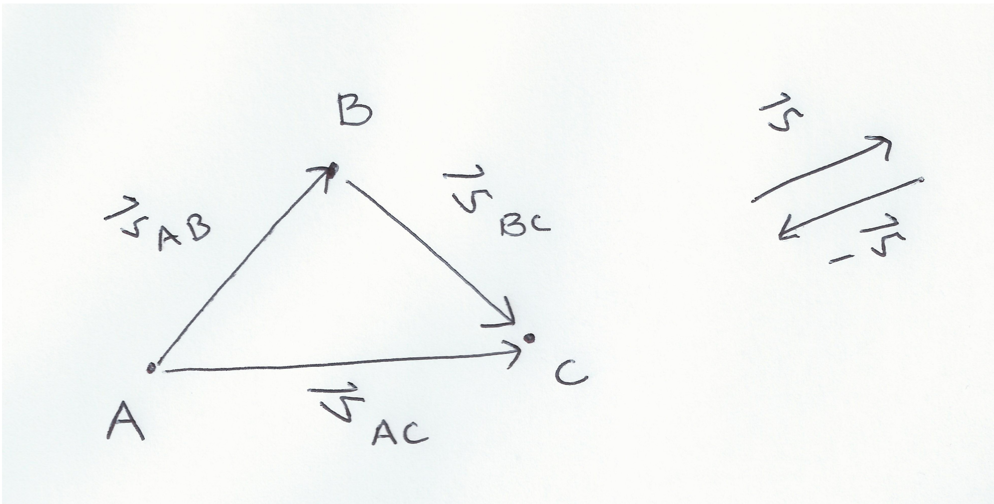


Figure 1.1: The most beautiful vectors ever drawn.

however, that if I had walked these two different routes in straight lines, the total distance I would have walked would NOT be the same. On the homework you'll explore the difference between these two distances.

Notice that graphically, we add two displacement vectors by taking the tail of the second vector and placing it at the tip of the first vector. In the example of the three points in space, the vectors were already lined up in this way. There will be other situations in which we want to add vectors that do not necessarily line up in this fashion. One example of this is when we want to add or subtract velocity vectors, in order to find the relative velocity between two objects. In this situation, we take one of the vectors, and move it so that the tail of one vector lines up with the tip of the other vector, while maintaining the length and orientation of the vector that we moved. This process is known as "parallel transport." An example of this is shown in figure 1.2.

We can also multiply vectors by numbers. When we multiply a vector by a positive number, we simply multiply the magnitude by that number, while leaving the orientation unchanged. When we multiply a vector by a negative number, we multiply its magnitude by the absolute value of this number, while reversing its orientation. As for notation, if I want to write the result of multiplying $\vec{v}$ by a number α, I will denote it as $\alpha\vec{v}$. One important fact is that scalar multiplication distributes over vector addition, which is to say

$$\alpha\left(\vec{a}+\vec{b}\right)=\alpha\vec{a}+\alpha\vec{b}. \tag{1.2}$$

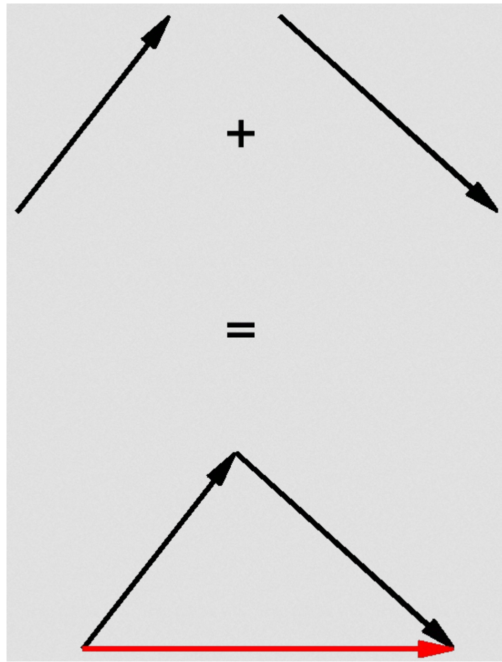


Figure 1.2: Parallel transporting a vector so that it can be added to another one.

Geometrically, the above statement says that I am just multiplying all of the side lengths in the triangle formed by the three vectors by the same amount, in order to get a similar triangle. It is also true that

$$(\alpha + \beta)\,\vec{v} = \alpha\vec{v} + \beta\vec{v}. \tag{1.3}$$

1.3 Coordinate Systems

Now, in principle, we could perform vector manipulations in a completely geometric way, by drawing them with their lengths and orientations, and finding the resulting vector sums. However, it is usually easier to work with something called a coordinate system. A coordinate system consists of a point called the origin, and a choice of vectors, equal to the number of dimensions of space, called axes. This is demonstrated in Figure 1.3. In this case I'm working in two dimensions, which would be appropriate if I were, for example, considering only objects sitting on a flat surface, say on the surface of a table. The origin in some sense is a point of reference which I'll use to describe all other points. The two vectors $\hat{x}$ and $\hat{y}$ are the vectors which define my two coordinate axes. The "hat" on the names of the vectors is a symbol which means these are a special type of vector, whose magnitude is equal to one (in some suitable set of units, say one meter). The two vectors have an angle of 90 degrees between them.

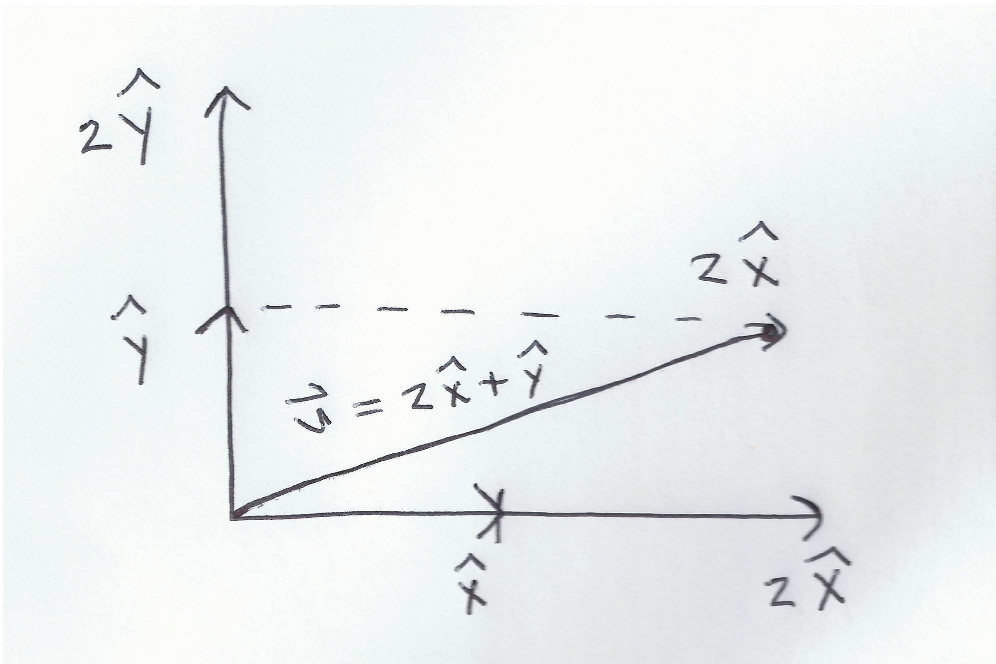


Figure 1.3: Representing a vector in a coordinate system.

The reason that a coordinate system is useful is because it allows me to write any vector as a set of numbers that describe the relation between that vector and my basis. These numbers are called coordinates, and they can be used to perform vector manipulations in a concise way. In Figure 1.3 I've drawn the vector sum of $2\hat{x}$ and $\hat{y}$. Notice that to compute the sum, I moved one of the vectors so that its tail was in contact with the tip of the other vector. The resulting vector can be written as

$$\vec{v} = 2\hat{x} + \hat{y}. \tag{1.4}$$

The fact that $\vec{v}$ is equal to this vector sum allows us to develop an efficient method of working with vectors. The idea is to represent the vector $\vec{v}$ in terms of the two numbers in this vector sum,

$$\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{1.5}$$

Any vector that I can draw can be represented in this way, for some choice of numbers. Imagine I have another vector,

$$\vec{w} = 3\hat{x} + 4\hat{y}. \tag{1.6}$$

Using the rules of vector addition, I can write

$$\vec{v} + \vec{w} = 2\hat{x} + \hat{y} + 3\hat{x} + 4\hat{y} = 5\hat{x} + 5\hat{y}. \tag{1.7}$$

If I write the above statement in my new language of coordinates, I find

$$\vec{v} + \vec{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}. \tag{1.8}$$

Thus, the vector sum, in coordinates, is just the sum of the coordinates of the two vectors. Figure 1.4 shows a graphical representation of why this works. In general, we work with the notation

$$\vec{v} = v_x\hat{x} + v_y\hat{y}. \tag{1.9}$$

The number v_x is called the x coordinate, and v_y is the y coordinate.

We can also represent scalar multiplication this way,

$$5\vec{v} = 5\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}. \tag{1.10}$$

An important fact to always remember is that a vector is a quantity which exists in its own right, without a coordinate system! The displacement between campus and my house has a well defined length and orientation, regardless of how I describe it. Choosing a coordinate system is just a way to make calculations easier. Depending on which coordinate system we pick, we will get different values for the coordinates of a vector. But the vector will always be the same vector.

In order to specify a vector, I can do two things. First, I can just specify its coordinates. This tells me how to form the vector out of the coordinate vectors, similar to what I have written above. Alternatively, I can specify the magnitude of the vector, and its angle with respect to some fixed direction in space. As a matter of convention, we usually choose to specify vectors in terms of the angle they form with the $\hat{x}$ vector (although nothing says that we MUST make this choice). With a little trigonometry, we can see that

$$v_x = v\cos\theta\ ;\ v_y = v\sin\theta \tag{1.11}$$

Because our coordinate vectors are perpendicular, we can use the Pythagorean theorem to show that

$$v = \sqrt{v_x^2 + v_y^2}. \tag{1.12}$$

Figure 1.5 shows how this works.

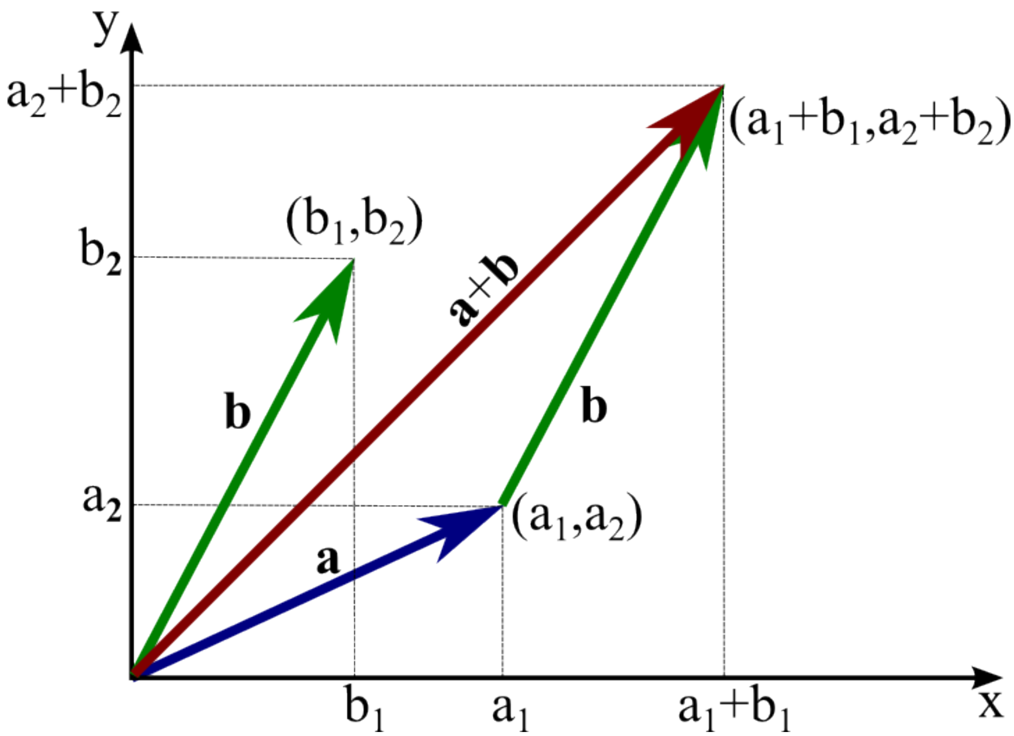


Figure 1.4: A geometric representation of how we can use coordinates to add vectors. Notice the use of parallel transport to add two vectors whose tails are initially both at the origin.

1.4 The Scalar Product

There is another commonly used operation we're going to want to perform with vectors, which goes by many names. It is often called the scalar product, but it is also referred to as the dot product, or inner product. Given two vectors $\vec{a}$ and $\vec{b}$, their scalar product is defined as

$$\vec{a} \cdot \vec{b} = ab \cos \theta \tag{1.13}$$

Geometrically, this product gives us a sort of sense of how much $\vec{a}$ lies along $\vec{b}$, which is shown in Figure 1.6. While not immediately obvious, it will turn out that this mathematical object shows up a lot in physics, and so it is useful to give it a special status, and give it a name.

The scalar product distributes, in the sense that

$$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}. \tag{1.14}$$

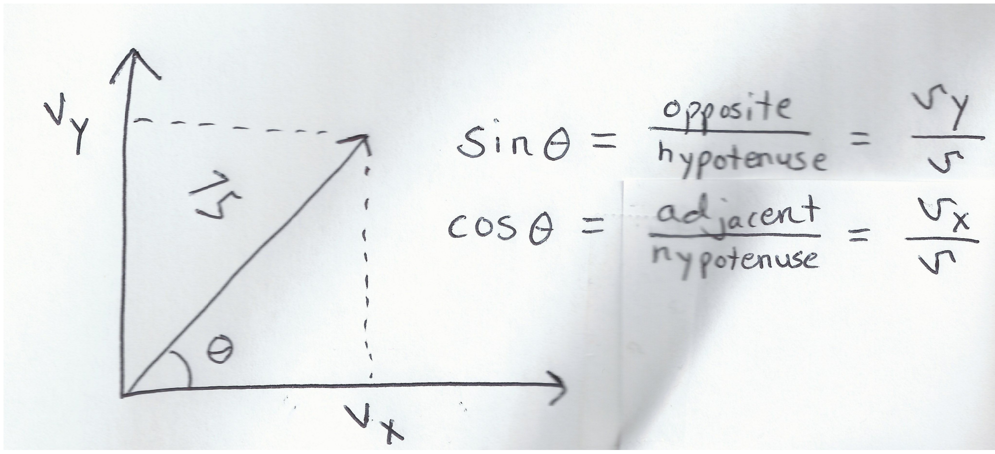


Figure 1.5: The relation between a vector's coordinates and its length and orientation.

This fact can be proven with some knowledge of geometry and trigonometry. For our coordinate vectors, since we know the lengths and orientations, we can see that

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = 1 \; ; \; \hat{x} \cdot \hat{y} = 0. \tag{1.15}$$

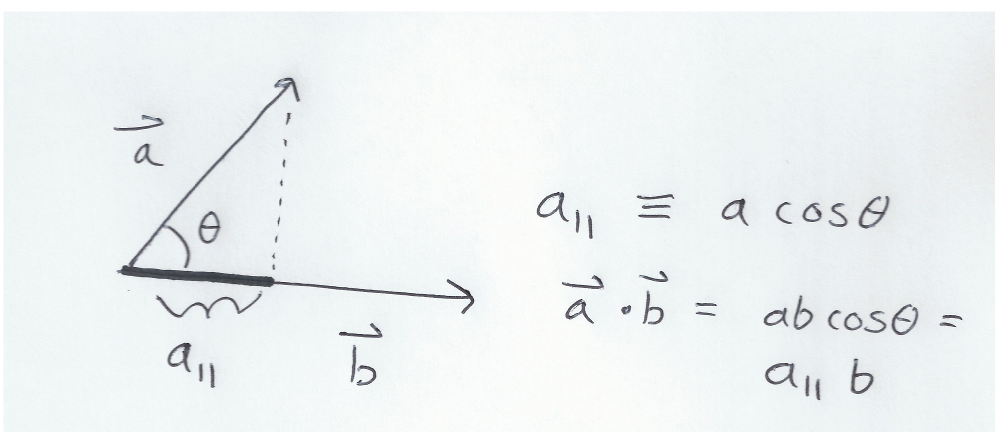


Figure 1.6: A geometric representation of the dot product.

Using the above facts, we see that the dot product of two vectors can easily be written in terms of their components,

$$\vec{a}\cdot\vec{b} = (a_x\hat{x} + a_y\hat{y})\cdot(b_x\hat{x} + b_y\hat{y}) = a_xb_x\hat{x}\cdot\hat{x}+a_xb_y\hat{x}\cdot\hat{y}+a_yb_x\hat{y}\cdot\hat{x}+a_yb_y\hat{y}\cdot\hat{y} = a_xb_x+a_yb_y. \tag{1.16}$$

Notice that the final expression would not be as simple if we had chosen a more complicated set of coordinate vectors. A special case of the above expression tells us that

$$a = |\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}. \tag{1.17}$$

1.5 Three Dimensions

The idea of a vector easily generalizes to three dimensions. We add another coordinate vector, which we often call $\hat{z}$, which is perpendicular to both and $\hat{x}$ and $\hat{y}$. Any vector in three dimensions can be written in the form

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}, \tag{1.18}$$

and the scalar product is written as

$$\vec{v}\cdot\vec{w} = v_xw_x + v_yw_y + v_zw_z. \tag{1.19}$$

The expression for the length of a vector in terms of the dot product with itself is still the same. An example of a three-dimensional coordinate system is shown in Figure 1.7.

Tomorrow we'll learn about how to use some of these ideas to describe the motion of particles.

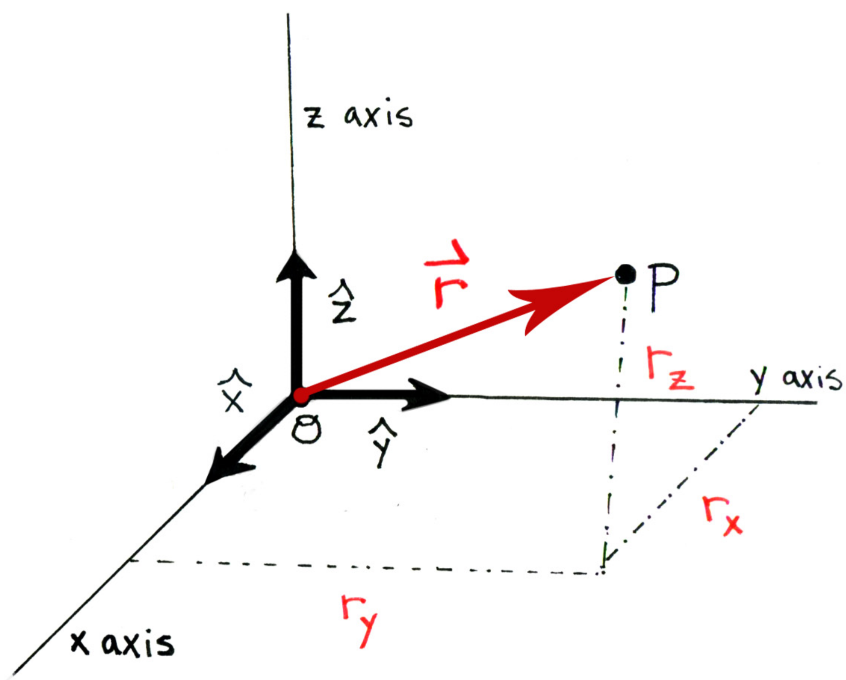


Figure 1.7: Representing a vector in three dimensions. Image credit: Kristen Moore

Chapter 2

Kinematics

2.1 Displacement and Velocity

In order to understand the motion of physical objects, we need to develop the mathematical language to describe it, which is usually referred to as Kinematics. Because we know that we can use vectors to describe objects with magnitude and orientation, we are going to use vectors to describe motion. The idea is to define the position vector of an object as the vector from the origin of some coordinate system to that object, which is demonstrated in Figure 2.1. We usually refer to the position vector of the object as $\vec{r}(t)$, since of course the word position starts with the letter p. The notation emphasizes that, in general, the location of the object will change with time. Often we will simply refer to the position vector of a particle as its position.

If my object is at a given point at some time t_1, and then it is at another point at time t_2, the displacement vector over that time interval is defined as

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1) . \tag{2.1}$$

Remember: this is not the same thing as the total distance traveled between the two points! In general, even if the particle follows some sort of wiggly path in between these two points, the displacement is still defined as the above vector quantity.

Using the displacement, we can define the average velocity over the time interval, which is given by

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}. \tag{2.2}$$

This agrees with our usual notion of velocity as being a notion of distance traveled per time, although it is important to remember that this is a vector

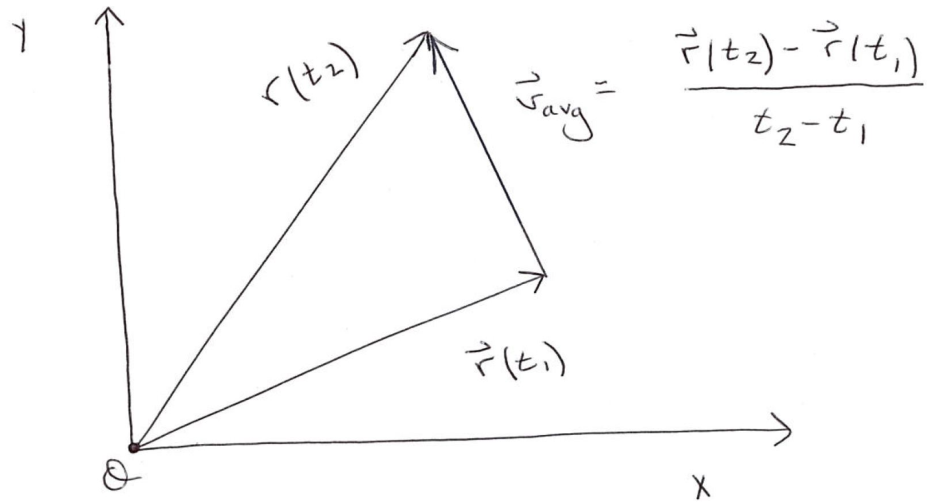


Figure 2.1: The definition of the average velocity over a time interval, defined in terms of the displacement vectors.

quantity. This is shown in Figure 2.1. In the limit that the time interval goes to zero, we recover what is called the instantaneous velocity,

$$\vec{v}\,(t) = \lim_{t_1 \to t_2} \frac{\vec{r}\,(t_2) - \vec{r}\,(t_1)}{t_2 - t_1} \equiv \frac{d\vec{r}}{dt}. \tag{2.3}$$

This defines the (instantaneous) velocity of an object as the derivative of the position with respect to time. This idea is shown in Figure 2.2. If we write out the position and velocity vectors in terms of coordinates, what we find is that

$$\vec{v}\,(t) = \lim_{t_1 \to t_2} \begin{pmatrix} \frac{r_x(t_2) - r_x(t_1)}{t_2 - t_1} \\ \frac{r_y(t_2) - r_y(t_1)}{t_2 - t_1} \\ \frac{r_z(t_2) - r_z(t_1)}{t_2 - t_1} \end{pmatrix} = \begin{pmatrix} \frac{dr_x}{dt} \\ \frac{dr_y}{dt} \\ \frac{dr_z}{dt} \end{pmatrix} \tag{2.4}$$

and so we have

$$v_x\,(t) = \frac{dr_x}{dt}, \tag{2.5}$$

and similarly for the y and z components. The magnitude of the velocity is referred to as the *speed*.

The velocity vector is different from the displacement vector in an important way. The displacement vector is an oriented line from the origin to the location of the particle, and has physical units of length. The velocity vector, however,

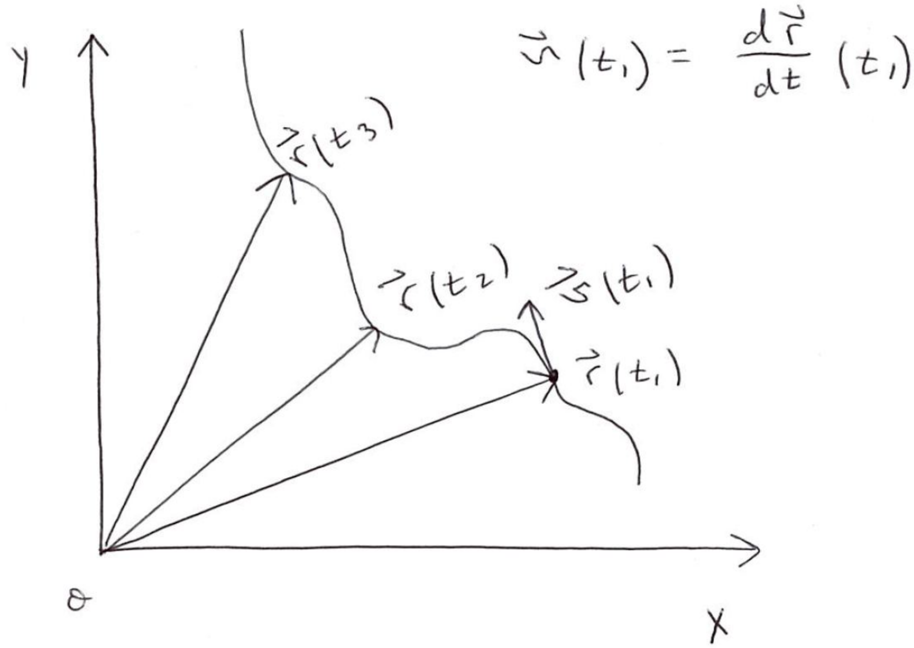


Figure 2.2: The definition of the instantaneous velocity as the time derivative of the displacement. The velocity at the first time is given, while several other position vectors along the motion of the particle are also shown.

has units of length divided by time, and it is NOT a displacement vector that extends in space between two points, even though this is how we've drawn the average velocity. As shown in Figure 2.2, the tail of the instantaneous velocity vector is usually placed at the location of the particle at that time, but it is important to remember that drawing the vector as something that extends in space is really just a way for us to graphically represent the fact that it has a magnitude and direction. In reality, velocity doesn't have a length that extends through space, so in some sense it's not really correct to draw it on the same set of axes as the displacement vectors, although we usually still do this anyways.

Another important fact about the velocity vector is that if we shift the origin of our coordinates, while maintaining the orientation of the axes, the components of the velocity vector don't change, whereas the components of the displacement vector do change. We'll discuss this in more detail when we cover Galilean relativity on day four.

We can find the displacement in terms of the velocity by integrating,

$$\Delta\vec{r} = \int_{t_1}^{t_2} \hat{v}\,(t)\,dt = \left[\int_{t_1}^{t_2} v_x\,(t)\,dt\right]\hat{x} + \left[\int_{t_1}^{t_2} v_y\,(t)\,dt\right]\hat{y} + \left[\int_{t_1}^{t_2} v_z\,(t)\,dt\right]\hat{z}. \quad (2.6)$$

One can check that this is correct using the fundamental theorem of calculus for each component of the position and velocity. The total distance traveled by the object, often referred to as the arc length of its trajectory, is given by the integral of its speed,

$$s = \int_{t_1}^{t_2} |\vec{v}\,(t)\,|dt \quad (2.7)$$

where s of course stands for "arc," since no other important terms I just mentioned start with the letter s.

After defining the velocity, we can also discuss higher order derivatives. The second derivative of the position, or the first derivative of the velocity, is referred to as the acceleration,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}. \quad (2.8)$$

The average acceleration over some time is given by

$$\vec{a}_{\text{avg}} = \frac{\vec{v}\,(t_2) - \vec{v}\,(t_1)}{t_2 - t_1}. \quad (2.9)$$

Again, the change in velocity can be obtained from the integral of the acceleration,

$$\vec{v}\,(t_2) - \vec{v}\,(t_1) = \int_{t_1}^{t_2} \vec{a}\,(t)\,dt \quad (2.10)$$

Usually, acceleration is the highest order derivative that we consider. This is because Newton's laws tell us what the acceleration of an object is in terms of the forces acting on it. However, higher order derivative occasionally come up. The third derivative of position, or first derivative of acceleration, is usually referred to as the "jerk." The fourth derivative is often referred to as the "jounce," or sometimes the "snap," while some sources cite the fifth and sixth derivatives as the "crackle" and "pop," respectively.

It is absolutely imperative to remember that these are all VECTOR quantities! They must be manipulated as such, making sure to correctly add the different components together.

2.2 Motion at Constant Acceleration

While the mathematical language of kinematics that we've developed so far allows us to describe arbitrary motion, it is quite often the case that we are studying motion in which the acceleration is constant. Therefore, it is useful to derive some equations that describe the motion of a particle in this special case.

For simplicity, we'll assume that the motion begins at time zero, and denote the **final** time as simply t, with **intermediate** times (being integrated over) referred to as t'. The notation we'll use for any quantity at time zero will be

$$\vec{v}\,(0) \equiv \vec{v}_0. \tag{2.11}$$

Now, we know that we can find the final velocity according to

$$\vec{v}\,(t) = \vec{v}_0 + \int_0^t \vec{a}\; dt'. \tag{2.12}$$

In terms of components, we have

$$\vec{v}\,(t) = \vec{v}_0 + \left[\int_0^t a_x dt'\right]\hat{x} + \left[\int_0^t a_y dt'\right]\hat{y} + \left[\int_0^t a_z dt'\right]\hat{z}. \tag{2.13}$$

Because all of the components of the acceleration are constant, this simply becomes

$$\vec{v}\,(t) = \vec{v}_0 + a_x t\hat{x} + a_y t\hat{y} + a_z t\hat{z} = \vec{v}_0 + \vec{a}t. \tag{2.14}$$

Remember that the acceleration is constant in the sense that it does not change in time. It is, however, still a vector quantity, with a magnitude and a direction.

Now, similarly we can find the position according to

$$\vec{r}\,(t) = \vec{r}_0 + \int_0^t \vec{v}\,(t')\, dt' = \vec{r}_0 + \int_0^t \left(\vec{v}_0 + \vec{a}t'\right) dt'. \tag{2.15}$$

Because $\vec{v}_0$ and $\vec{a}$ are constant, the result is

$$\vec{r}\,(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \tag{2.16}$$

This result is often referred to as one of the "kinematic equations," but keep in mind that it is applicable **only** in the case that the acceleration is constant throughout the motion. The above expression is absolutely untrue when the acceleration changes with time.

Also, notice that our final result is specified in terms of three quantities that we need to provide: the (constant) acceleration, the initial starting velocity, and the initial starting position. These pieces of information we need to provide about the behavior of the motion at its beginning are often referred to as "initial conditions," or sometimes "boundary values."

In one dimension, the above kinematic equation reads

$$r_x(t) = r_{x0} + v_{x0}t + \frac{1}{2}a_x t^2. \tag{2.17}$$

In higher dimensions, this expression will be true *for each component separately*, since the kinematic equation is a vector equation.

I want to pause for a moment to point out that up until now in this course, we haven't actually done any *physics* yet. Everything we've talked about so far has involved how to develop a good language for describing motion, and is really just *math*. So far I haven't told you anything about what sort of motion actually occurs in nature as a result of some sort of physical principles. I'm going to change that now by introducing the subject of projectile motion.

2.3 Projectile Motion

Projectile motion is the motion that occurs for a material body under the influence of gravity alone. An example of this would be the motion of a bullet after it has been fired from a gun (assuming that we are neglecting the effects of air resistance). For reasons which we will come to understand later on in the course, for objects moving near the surface of the Earth, it is usually a very good approximation to say that gravity causes the objects to move with a constant acceleration, with a value that is completely independent of the body in question. We generally refer to this value as g, and on Earth the numerical value for its magnitude is approximately 9.8 meters per second squared. Its orientation is directed towards the ground.

Since the motion of a body under the influence of gravity is described by a constant acceleration, we can use the equation we derived in the previous section to describe its motion. Let's imagine that we have a gun which fires a bullet with some initial velocity, held at some initial starting point. This is shown in Figure 2.3. What we want to do is find the position of the bullet as a function of time, after it is fired.

In order to solve this problem efficiently, we're going to want to set up a coordinate system. The common convention is to set up a coordinate system whose x axis is parallel with the ground, and whose y axis points vertically upwards away from the ground. The origin is generally placed so that the x coordinate of the starting point of the bullet is zero - in other words, it is aligned horizontally with the nozzle of the gun. As for the vertical location of the origin, we have two choices that both make sense. We can either take the origin to be at the location of the nozzle of the gun, or we can choose the origin so that it is on the ground. In this case, I'm going to orient my coordinates so that the origin is on the ground, but any other choice would be fine, so long as I am consistent with my choice. This is shown in Figure 2.3.

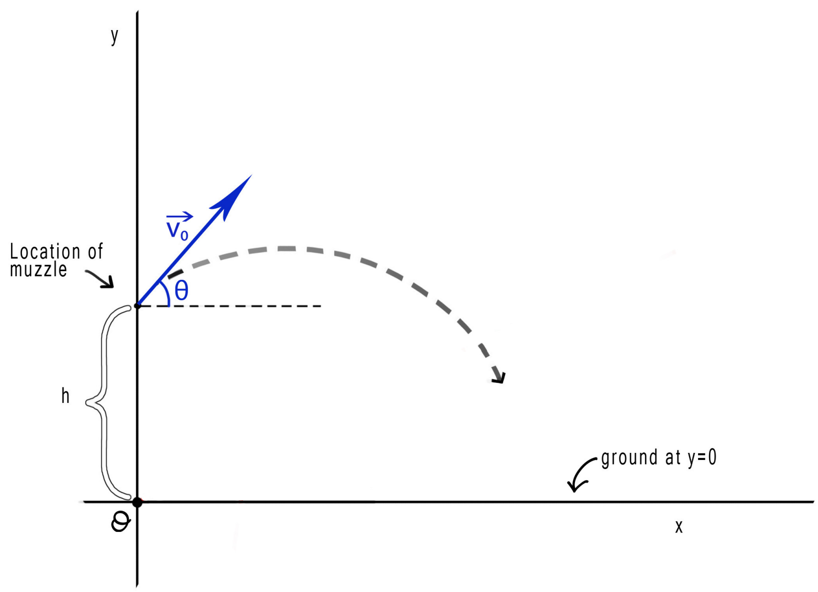


Figure 2.3: The set up of our projectile motion problem, complete with coordinate axes. The dotted line is a path we suspect the particle might take. Image credit: Kristen Moore

Now, the equation that we derived in the previous case will give me what I want to know (the location of the bullet as a function of time), so long as I provide it with three things: the initial location of the bullet, the initial velocity of the bullet, and the (constant) value for the acceleration. In our coordinates, we can see that the initial location of the bullet is specified by

$$\vec{r}_0 = \begin{pmatrix} 0 \\ h \end{pmatrix}, \tag{2.18}$$

where h is the height of the nozzle of the gun off of the ground. As for the initial velocity, we can either specify this in terms of components, or the initial magnitude and angle. Usually, it is more physically intuitive to specify the magnitude and angle, so we write our initial velocity as

$$\vec{v}_0 = \begin{pmatrix} v_0 \cos\theta \\ v_0 \sin\theta \end{pmatrix}, \tag{2.19}$$

where v_0 is the initial magnitude of the velocity of the bullet (its speed), and θ is the angle the initial velocity makes with the horizontal axis (the ground).

As for the acceleration, we are told that it will have a magnitude equal to g, pointed towards the ground. Therefore, if we take g to be the positive magnitude of the acceleration, the vector form of the acceleration will be

$$\vec{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}. \tag{2.20}$$

Notice that the y component is negative! This is a result of the fact that we oriented $\hat{y}$ to be pointing up away from the ground, whereas the acceleration points down towards the ground.

With this information clarified, I can now make use of my kinematic equation. The x component of the motion is described by the equation

$$r_x(t) = r_{x0} + v_{x0}t + \frac{1}{2}a_x t^2, \tag{2.21}$$

or,

$$r_x(t) = v_0 \cos\theta t. \tag{2.22}$$

The y coordinate of the motion is described by

$$r_y(t) = r_{y0} + v_{y0}t + \frac{1}{2}a_y t^2, \tag{2.23}$$

or,

$$r_y(t) = h + v_0 \sin\theta t - \frac{1}{2}gt^2. \tag{2.24}$$

Combining these together, we can write the position vector of the bullet as a function of time as

$$\vec{r}(t) = \begin{pmatrix} v_0 \cos\theta t \\ h + v_0 \sin\theta t - \frac{1}{2}gt^2 \end{pmatrix} \tag{2.25}$$

So now we have an expression for the position of the bullet as a function of time. However, we know that this expression has an obvious limit to its validity. Eventually, one of two things will happen. One possibility is that the bullet will shortly fall back to the ground, at which point it will no longer be under the influence of gravity, and projectile motion will no longer be an adequate description of its trajectory. The second possibility is that we have an unimaginably powerful, rocket-powered rifle, and we've fired the bullet with such a large initial speed that it is able to travel very far from the surface of the Earth, and the approximation of constant acceleration is no longer valid. If the speed is large enough, it is possible for the bullet to completely escape the Earth's gravity, and never return.

Assuming that the first situation is more accurate, one thing we would like to do is figure out the time at which the bullet hits the ground, and then proceeds to do something else which is not correctly described by projectile motion (bounces off the ground, gets stuck in the dirt, or whatever). Now, in order to figure this out, we need to understand how to take a physical constraint (the bullet hits the ground) and turn it into an appropriate mathematical statement, which will somehow allow us to get to the ultimate answer we want.

Now, if we think about it, the statement that the bullet has hit the ground, in our vector language, is really the statement that the y coordinate has become zero. Thus, the mathematical statement of our condition is

$$r_y\left(t\right) = h + v_0 \sin\theta t - \frac{1}{2} g t^2 = 0. \tag{2.26}$$

What we have now is an expression which can be solved to find the possible values of t which satisfy this condition. This is really just a math problem, and we can use the quadratic formula to see that the possible solutions to this problem are

$$t = \frac{-v_0 \sin\theta \pm \sqrt{v_0^2 \sin^2\theta + 2gh}}{-g}. \tag{2.27}$$

Of course, the quadratic formula has two possible solutions. How could this be the case? Presumably, there is ONE time when the bullet hits the ground, not two. Well, we need to apply some more physical reasoning. Certainly, the time must be in the future, and so it should be a positive number. This rules out one of the possible answers, and so we see that the time that the bullet hits the ground must be

$$t_g = \frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh}}{g}. \tag{2.28}$$

An easy thing to immediately notice about this expression is that it is inversely proportional to g, so that when the acceleration due to gravity is larger, it takes a shorter amount of time for the bullet to hit the ground. This makes intuitive sense - if the acceleration due to gravity is not very strong, we expect it to not influence the motion of the bullet as much, and so it will take longer for the motion of the bullet to deviate from its initial velocity, which is pointing up and away from the ground.

We can also ask how high the bullet will go before starting to fall back down. We know physically that the bullet will travel up, come to a stop, and then turn around. Therefore, the maximum height must be attained when the vertical component of the velocity goes to zero. The y component of velocity as a function of time can be found by differentiating the y coordinate, and we find

$$v_y\left(t\right) = v_0 \sin\theta - gt. \tag{2.29}$$

If we set this equal to zero, the time at which the maximum height is obtained is

$$t_{\max} = \frac{v_0 \sin\theta}{g}. \tag{2.30}$$

The maximum height is found by evaluating the y coordinate at this time,

$$y_{\max} = r_y\left(\frac{v_0 \sin\theta}{g}\right) = h + v_0 \sin\theta \frac{v_0 \sin\theta}{g} - \frac{1}{2} g\left(\frac{v_0 \sin\theta}{g}\right)^2, \tag{2.31}$$

or

$$y_{\max} = h + \frac{v_0^2 \sin^2\theta}{2g}. \tag{2.32}$$

Notice that the maximum height obtained depends on the starting height, but the time it takes to get there does not. Also notice that in order to find a maximum value, we had to take a derivative. Does this sound reminiscent of anything you remember covering in your calculus courses?

Also, notice that in the case that $h = 0$, which is the case that the bullet is fired from the ground, the amount of time it takes to reach the maximum height is half the amount of time it takes to reach the ground. This says that **the amount of time it takes the bullet to rise to its maximum height is the same as the amount of time it takes to fall back down to the original starting height**, which is a nice feature of projectile motion to remember.

One last question we might want to know the answer to is how far the bullet travels horizontally before hitting the ground - that is, what is the x coordinate of the bullet at the time when the bullet hits the ground. In order to figure this out, we simply evaluate the expression for the x coordinate at the time we found previously,

$$x_{\max} = r_x\left(t = \frac{v_0\sin\theta + \sqrt{v_0^2\sin^2\theta + 2gh}}{g}\right) \tag{2.33}$$
$$= \frac{v_0\cos\theta\left(v_0\sin\theta + \sqrt{v_0^2\sin^2\theta + 2gh}\right)}{g}.$$

There's another nice feature of projectile motion, which is that it's relatively easy to write down a relation between the x and y coordinates of the motion at any given time. Because the x coordinate increases linearly with time, there is a one to one relationship between the x coordinate of the bullet, and the time that has elapsed. If we invert this relationship, we find

$$t = \frac{r_x}{v_0\cos\theta}. \tag{2.34}$$

If I specify a value for the x coordinate, the above expression will tell me how much time has elapsed when the bullet has that x coordinate. Using this expression, I can write the y coordinate as a function of x coordinate, to find

$$r_y\left(r_x\right) = h + v_0\sin\theta\left(\frac{r_x}{v_0\cos\theta}\right) - \frac{1}{2}g\left(\frac{r_x}{v_0\cos\theta}\right)^2, \tag{2.35}$$

or

$$r_y\left(r_x\right) = h + \tan\theta r_x - \frac{g}{2v_0^2\cos^2\theta}r_x^2. \tag{2.36}$$

Thus, if we plot the y coordinate against the x coordinate, which indicates the bullet's trajectory, we will find a parabola, which is a generic feature of projectile motion. Specifically, this means that in Figure 2.3, the shape of the dotted line is parabolic.

So far we've talked about how to describe the motion of particles using vectors and kinematics, and studied one physical case where we were told that the acceleration was constant. Tomorrow we'll talk about Newton's laws, and we'll try to start understanding why objects move in the various ways that they do.

Chapter 3

Newton's Laws

3.1 Forces and Newton's Laws

Now that we have a way of describing the motion of bodies, we want to start developing a model for *why* they move in the ways they do. Newtonian mechanics is a model which tells us how to compute the acceleration of a body in terms of something called the *force* acting on it, an idea which we will clarify in today's lecture.

The rules which tell us how objects behave under the influence of a force are generally referred to as Newton's Laws. Unfortunately, this is very bad terminology. The use of the word "law" implies that we somehow know these rules to be absolute fact, and that they hold under all circumstances. Of course, we know today this is not true - special relativity and quantum mechanics provide a more accurate description of physics. Even before we knew this, it was still true that Newtonian mechanics was a theory of physics, just like every other model we have ever created to describe the universe. In common language, the word "theory" is often used as a pejorative, in an attempt to discredit some idea as being unfounded or untested. However, in scientific language, any model of the universe is a theory of physics, even if it has been experimentally tested and confirmed to be accurate in a variety of situations.

What then, exactly, is a force? Intuitively, a force on an object is a sort of "push" or "pull" that results in the object experiencing some sort of change in its motion. Ultimately, all of the forces that act on a body are a result of interactions it experiences with other bodies, and so the idea of forces is really just a succinct mathematical way to describe how bodies interact with each other. There are various ways that bodies can interact with each other, and thus various types of forces that can act on objects. Objects can exert gravitational forces on each other, and electrically charged particles experience forces from electric fields. We have a variety of models and theories which tell us exactly what these forces should be in order to correctly describe the motion of bodies in these situations. In general, there can be several different forces which act on an object.

In particular, the force acting on a body is a vector quantity. It has a magnitude (roughly speaking, how much we're pushing on the body), along with a direction (roughly speaking, where we are trying to push it). When more than one type of force acts on a body, we say that the *net force* acting on it is the *vector sum* of all of the individual forces. In order to study the motion of bodies in a quantitative way, we need to develop a precise set of rules which tell us, given a certain physical situation, exactly what forces are present on a body, and how they affect its motion. As we mentioned above, in Newtonian Mechanics, these rules are referred to as Newton's Laws, and there are three of them.

The first of Newton's laws says that if a body is under the influence of zero **net** force, then its acceleration is zero. In other words, if a body interacts with other bodies in such a way that the **net** force acting on it is zero, its motion is unperturbed, and it has a constant velocity.

Newton's second law generalizes this statement. It says that the **net** force acting on a body is equal to the mass of that body, times its acceleration,

$$\vec{F} = m\vec{a}. \tag{3.1}$$

Notice that when the force is zero, this reduces to Newton's first law. The mass of an object is a positive, scalar number which tells us how much a body resists changes in its motion. It is a somewhat abstract quantity, and is different from the weight of an object. The weight of an object is the force that object experiences due to the effects of gravity. If I take a ball on Earth and move it to the Moon, where the effects of gravity are weaker, then the weight of the ball will be reduced. But its *mass* will stay the same. If I were to take a magnetic ball out into empty space far from any other bodies, where any gravitational effects are negligible, and study its reaction to a magnetic force, I would be learning something about its mass.

Newton's second law isn't very useful unless I start telling you something about what types of forces can act on an object and how they behave. But before I start giving examples of forces, Newton's third law tells us that there are certain conditions that any valid force must obey. In particular, whenever two bodies interact, the *magnitude* of the force exerted on each body is the same, while the *directions* of the forces are opposite. This is often stated by saying that bodies exert equal and opposite forces on each other. For example, two bodies with mass will be attracted to each other through gravity, and the force they exert on each other will be the same in magnitude. The direction of the force on one body is such that the force points towards the other body, so that the two forces are opposite in direction. This is sketched in Figure 3.1. You'll explore Newton's law of gravitation more in the homework.

To clarify the difference between the two forces in a force pair, we often develop a subscript notation. If I have two bodies which I will call A and B, then the force from body A acting on body B is written $\vec{F}_{\text{A on B}}$. Newton's third law then reads

$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}. \tag{3.2}$$

3.2 Some Examples of Forces

As I mentioned previously, the gravitational interaction between two bodies is one way two bodies can exert forces on each other. For small bodies moving near the surface of the Earth, as we mentioned yesterday, it is usually a good approximation to say that the acceleration of a body due to gravity is constant, and we call this constant acceleration $\vec{g}$. As a result, the force acting on the body due to gravity is

$$\vec{F}_g = m\vec{g}, \tag{3.3}$$

where m is the mass of the body. In principle, the small body ALSO exerts a force on the Earth. You'll explore this more in the second homework.

Another way that two bodies can interact is through electromagnetic forces. When I take a cup of coffee and sit it down on a table, I know that despite the force of gravity pulling the cup down, the cup sits still, and so its acceleration must be zero. This must mean that some other force is acting on the cup, in order to result in a net force of zero on the cup. The other force of course comes from the interaction of the cup with the table. In principle, the reason that the cup and table do not simply pass straight through each other is related to chemistry. At a microscopic level, the electrons in the atoms of the cup and the table are electrically charged, and repel each other through the force of electricity.

If we had a knowledge of how these electrons interacted with each other, and a lot of patience, in principle, we could calculate the force that the table and cup exert on each other. However, if we just want to model the interaction of the cup and the table on a larger scale, we don't really need to do this. Because we know the cup is not accelerating, then we know that the net force on the cup must be zero. Therefore, the net effect of the interaction of the cup and the table must be such that the force from the table on the cup must be the negative of the force acting from the Earth on the cup, so that the two forces

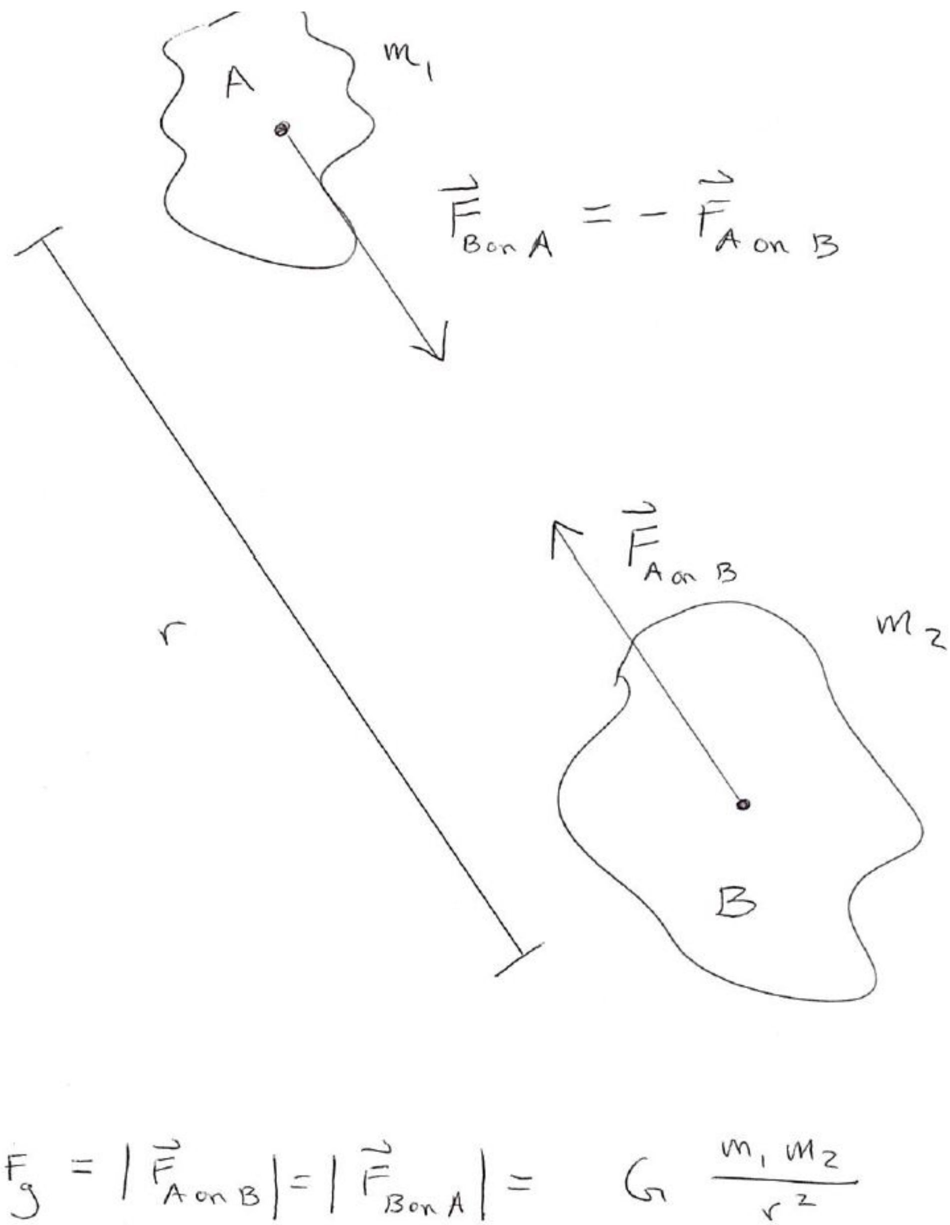


Figure 3.1: Two massive bodies will exert a gravitational force on each other, and it will obey Newton's third law. Newton's law of gravitation, which you'll work with in the homework, is shown at the bottom.

add to zero. Thus, we say

$$\vec{N} = -\vec{F}_g, \tag{3.4}$$

and we call the force between the cup and table the *normal* force.

It's important to remember that we didn't compute this normal force using any sort of deeper principle, or any theory of how the cup and table should interact. We simply imposed the physical condition that the cup isn't accelerating, along with the knowledge that gravity is acting on the cup, to *infer* what the force between the cup and table must be. Because we've defined the normal force in terms of how one might go about measuring it, we sometimes say we've given the normal force an *operational definition*. The normal force is always defined to be *perpendicular* to the surface over which two bodies meet. In general, there can also be forces as a result of contact between two bodies which point *parallel* to the direction of the surface. We'll deal with these later in the lecture, when we discuss friction.

3.3 Free-Body Diagrams

In general, a body can have many different forces acting on it, and as we've seen, we need to add these forces together as a vector sum, in order to find the net force acting on the body. In order to do this, it's convenient to work with something called a *free-body diagram*. To see how this works, let's set up a coordinate system, where the origin is centered on the location of the coffee cup I just mentioned. We're going to make the approximation that the cup is a point-like object, which, despite sounding somewhat silly, turns out to be a surprisingly reasonable assumption, provided that the shape and overall structure of the body doesn't change much (later in the course we'll understand why this approximation works so well when we discuss the notion of center of mass). If I wanted to study how a physical body was squeezed or compressed due to a force, then this approximation wouldn't be so useful, but for our current purposes it will suffice. In particular, we will choose a set of coordinate axes where the x direction is aligned along the surface of the table, and the y direction points upwards away from the surface of the table. This is shown in Figure 3.2.

The idea is to use this free-body diagram to indicate the forces acting on the body in question. The way we do this is by drawing the forces acting on the body as vectors, with the tails of the vectors located at the origin. The forces are drawn in the direction that they point in space. However, while the vectors have an orientation, and a magnitude, they do **not** have units of length, and they are **not** displacement vectors - they don't actually extend physically out into space. Force has units of *Newtons*, or kilogram-meters per second squared.

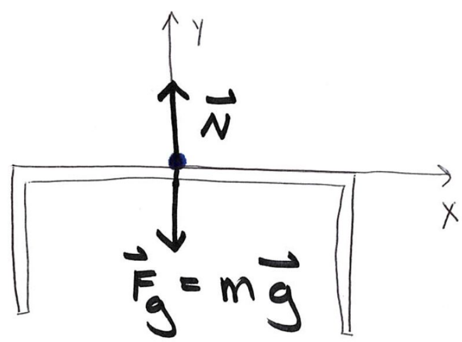


Figure 3.2: A free-body diagram for a cup sitting still on a table.

In our example, the force of gravity acts on the cup and points down, while the normal force points upwards. We've drawn the two vectors with the same length, to indicate that their magnitudes are the same.

We can also consider a more complicated example. Imagine that I had tied a string to the coffee cup, and I begin to pull on the string at some angle with respect to the surface of the table. Let's assume that I know how much force this exerts on the cup, and I call this force $\vec{F}_s$. My free-body diagram now looks like Figure 3.3 (I've dispensed with the table for the sake of clarity). This sort of force is often referred to as a tension force.

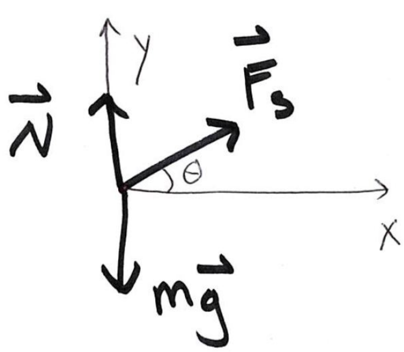


Figure 3.3: A free-body diagram for a cup, sitting on a table, being pulled by a string.

I now want to compute the net force on the cup, assuming that these are the three forces acting on it. It is usually useful to do this component by component.

I'll start by computing the x component of the net force, which means I need to add the x components of each of the three forces. Now, we know that gravity points downwards, so its x component is zero. As far as the x component of the tension is concerned, I can use my usual trigonometry relation to write

$$F_{sx} = F_s \cos\theta, \tag{3.5}$$

assuming that the direction and magnitude of this force are two quantities which I'm given. For the moment, we'll still assume that the table only exerts an upwards normal force on the cup, so that there is no friction to consider. Thus, the total force in the x direction is

$$F_x = F_s \cos\theta. \tag{3.6}$$

From Newton's second law, this tells us that the acceleration in the x direction is

$$a_x = \frac{F_s \cos\theta}{m}, \tag{3.7}$$

where m is the mass of the coffee cup.

As for the y components of the net force, we must add the y components of the weight, the normal force, and the tension. The result is

$$F_y = -mg + N + F_s \sin\theta, \tag{3.8}$$

where I've used the specific form of the gravitational force. Now, I do **not** know a priori what the value of the normal force is. But, if I happen to be moving the string in such a way that there is no *vertical* acceleration of the coffee cup, then the net force in the y direction must be zero. If I set the above expression equal to zero, then I know that

$$N = mg - F_s \sin\theta. \tag{3.9}$$

In the case that there is no string, or the case that the string is completely horizontal, the second term on the right is zero, and we recover the previous situation. Notice that pulling up on the string at an angle reduces the normal force between the cup and table (can you see why this makes intuitive sense?).

3.4 Friction

We know from everyday experience that if I were to take my coffee cup and slide it horizontally along the table, and then let go, eventually the coffee cup would come to a stop. Because the velocity of the cup has changed over time, we know that the cup is accelerating, and so it must have a force acting on it. Of course, this is the result of friction between the cup and the table. Again, the reason for this force is a result of complicated chemical reactions between the cup and the table at the atomic level. In principle, if we knew the detailed laws of how these materials interact, we could figure out what this force is.

However, experience has shown that it is usually possible to model the friction between two objects in a very simple form, and that it comes in two types. The first type of friction is called kinetic friction, and it occurs between two objects that are in contact and moving against each other. Empirical evidence suggests that the magnitude of the force due to kinetic friction can be written as

$$f_k = \mu_k N, \tag{3.10}$$

where N is the magnitude of the normal force between the two objects, and μ_k is a parameter called the *coefficient of kinetic friction.* This parameter is something we can measure from experiment, and is taken to be a property of the two bodies. For example, if I was sliding a coffee cup across a wooden table, I would look up the coefficient of kinetic friction between ceramic and wood. Experience has shown that, to a good approximation, this parameter does not depend very much on how quickly the objects are moving past each other, and so we will take it to be a constant. The direction of the force is along the surface of contact, and for each body it points *opposite to the direction of motion.*

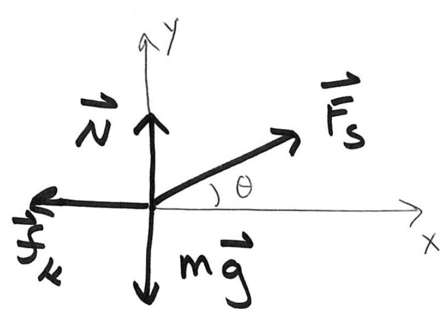


Figure 3.4: A free-body diagram for a cup sitting on a table being pulled by a string, and subject to a kinetic friction force.

If I revisit my previous free-body diagram and include the effects of kinetic friction while the coffee cup is being pulled by the string, it would look like Figure 3.4. Because friction acts along the surface of contact, it only has an x component. Therefore, the net force in the x direction is now

$$F_x = -\mu_k N + F_s \cos\theta. \tag{3.11}$$

The magnitude of the normal force is still determined by the condition that the coffee cup does not accelerate vertically, and if we use the value we found previously, we can write

$$F_x = -\mu_k mg + \mu_k F_s \sin\theta + F_s \cos\theta. \tag{3.12}$$

If we pull on the string so that the motion along the table is at constant velocity, then all of the components of the force must be zero, including the x component above. Thus, if we set the above expression to zero, we can write

$$F_s = \frac{\mu_k mg}{\mu_k \sin\theta + \cos\theta}, \tag{3.13}$$

which tells us, for a given angle, how large the magnitude of the tension must be in order to move the block at constant velocity.

Notice that the above expression for the required tension force has a very nontrivial dependence on the angle. We might initially expect that if we want to pull the coffee cup horizontally with the smallest required force, we should pull the string horizontally. But this is not true, because there are two competing effects at play here. While it is true that lifting the string projects a smaller component of the force along the x direction, it also helps reduce the normal force, since the vertical component of the tension now helps compensate the effects of gravity. If we minimize the above expression as a function of angle, we can figure out the ideal angle to pull the string at. The result turns out to be

$$\tan\theta_{\text{ideal}} = \mu_k. \tag{3.14}$$

The above resuls are valid while the two objects are moving against each other. We also know from experience that two objects placed in contact with each other will generally experience friction as a result of their contact, even when they are not moving. This is the reason that a cup placed on a slanted table does not slide down if the slant is not too steep, even though gravity is pulling it down. The force between the table and the cup that is preventing any motion in this case is called static friction.

Static friction is similar to the normal force in the sense that we usually figure out what it is by imposing some physical constraint. For example, if a cup is not sliding across a slanted table, it must be because there exists a static friction force helping to balance the force of gravity. However, we know from experience that if we tip the table enough, the coffee cup will eventually begin to slide. Thus, there is a **maximum** amount of resisting force that friction can provide. Experience has shown that we can find out what this maximum value is in terms of the normal force, such that

$$f_s \leq \mu_s N, \tag{3.15}$$

where μ_s is called the *coefficient of static friction.*

An example of this is shown in Figure 3.5, where we've drawn a rectangular block on an inclined plane at some angle. We might ask ourselves, how much can we tip the ramp before the block begins to slide? Well, this is a matter of determining, as a function of the angle, what static friction force would be required to hold the block in place. Once the angle becomes large enough that this hypothetical force exceeds the maximum allowed frictional force, we know the block will begin to slide.

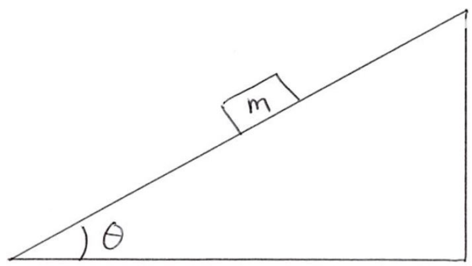


Figure 3.5: Behold the block on a ramp, in all its glory.

To answer this question, we set up a free body diagram for the block, as shown in Figure 3.6. We've chosen to align the x direction pointing downwards along the ramp, and the y direction perpendicular to the ramp, pointing out. We've indicated the presence of a potential normal and static friction force, which together describe the interaction of the block with the ramp, as well as the force due to gravity. Using trigonometry, we can see that the components of the gravitational force are

$$F_{gx} = F_g \sin\theta = mg\sin\theta \; ; \; F_{gy} = -F_g \cos\theta = -mg\cos\theta, \tag{3.16}$$

where m is the mass of the block.

Sometimes working out the components of the gravitational force in situations like this can be frustrating, since the location of where the angle is defined (the corner of the ramp) is not aligned with the origin of our free-body diagram (the location of the block). A good way to check that you have the right expression is to make sure that the forces reduce to a sensible limit when the angle goes to zero. In this case the ramp is flat, the above expressions would tell us that there is no component of the force along the x direction. This implies that there is no component of the gravitational force pointing along the surface of the ramp, as should be the case when the ramp is flat.

Now, the net force in the y direction will be due to the normal force and

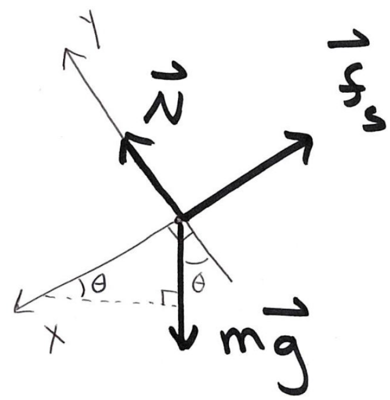


Figure 3.6: A free-body diagram for a block on a ramp, subject to gravity and static friction.

component of gravity along that direction,

$$F_y = N - mg\cos\theta. \tag{3.17}$$

If we assume that the block stays sitting on the surface of the ramp, it will not be accelerating perpendicular to it, which means that the net force in the y direction must be zero, and so

$$N = mg\cos\theta. \tag{3.18}$$

When the angle goes to zero, the cosine term is equal to one, and we recover the usual result for a flat table.

As for the x direction, the two relevant forces are static friction, and gravity. Adding these two forces, we find

$$F_x = mg\sin\theta - f_s. \tag{3.19}$$

Notice that I've oriented the static friction force so that it will point opposite to the force of gravity, in order to compensate its effects. If I want the block to not slide at all, then the acceleration along the x direction should be zero, which means the net force along the x direction should be zero. Equating the above expression to zero, I find

$$mg\sin\theta = f_s. \tag{3.20}$$

Now, we know that the **maximum** frictional force we can sustain is given by

$$f_s \leq \mu_s N = \mu_s mg\cos\theta. \tag{3.21}$$

This means that we must have

$$mg\sin\theta \leq \mu_s mg\cos\theta, \tag{3.22}$$

or,

$$\sin\theta \leq \mu_s\cos\theta. \tag{3.23}$$

Since $\cos\theta$ is always positive over the range of angles we are considering, we can divide it over the inequality, and we ultimately find that we must have

$$\tan\theta \leq \mu_s \tag{3.24}$$

in order for the block not to slide.

In the homework you'll explore some more problems that involve working with forces and free-body diagrams. Tomorrow we'll discuss the ideas of Galilean Relativity, Inertial Reference Frames, and the Equivalence Principle, and how we can use them to help us solve physics problems.

Chapter 4

Galilean Relativity

4.1 Changing Coordinate Systems

In previous lectures, I've told you that when I do physics problems, it doesn't matter what choice of coordinate system I make. Today I'm going to explore this statement in a little more detail.

Let's imagine that I have two bodies in space, interacting gravitationally, shown in Figure 4.1. The force they experience depends on the distance between them, and their two masses. I've included a choice of coordinates, and indicated the position vectors of each body.

Now, what if I wanted to make another choice of coordinate system? I've indicated in Figure 4.2 the same physical system, but with another coordinate system chosen. This new coordinate system is drawn in blue, and I've labeled the coordinate axes with primes. Now, because the position vector of an object is defined with respect to the origin of some coordinate system, it becomes clear that with a new set of coordinates, the position vectors of my bodies will change. This is also shown.

Now, the question I want to ask is, how are the position vectors in the two coordinate systems related to each other? To answer this question, I've also drawn another vector, $\vec{d}$, which is the displacement vector from the origin of the first coordinate system to the origin of the second coordinate system. Using the laws of vector addition, it becomes clear that

$$\vec{r}_1\,' = \vec{r}_1 - \vec{d}, \tag{4.1}$$

and likewise for the position of the second body. Alternatively, I could rearrange this to write

$$\vec{d} + \vec{r}_1\,' = \vec{r}_1. \tag{4.2}$$

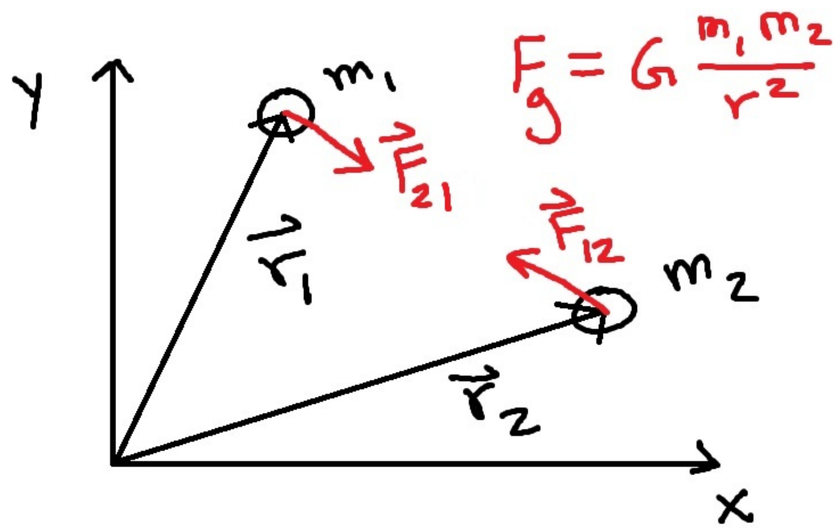


Figure 4.1: Two massive bodies will exert a gravitational force on each other, according to Newton's law of gravitation. Apologies for MS Paint.

This vector equation just says that if I start at the origin of the original coordinate system, move to the origin of the new coordinate system, and then move along the vector $\vec{r}_1\,'$, I'll end up at the location of the first body. This is what I should expect, since the net result of moving from the origin of the first coordinate system to the first body is described by the displacement vector $\vec{r}_1$.

Now, despite the fact that I can use two different coordinate systems to describe my physical problem, the forces experienced by the bodies should be the same regardless. *The forces are vector quantities which act on the bodies, and have an existence in their own right, independent of a choice of coordinate system.* This is made explicit by the formula for the force. It depends on the **physical distance** between the two bodies, and their masses, which are all quantities that have nothing to do with my choice of reference frame. So the forces are always the same, and so are the masses.

The fact that this is true means that for either of the bodies, the expression $\vec{F}/m$, or the force on that body divided by its mass, is always the same, no matter what choice of coordinates I pick, so that

$$\vec{F}/m = \vec{F}\,'/m. \tag{4.3}$$

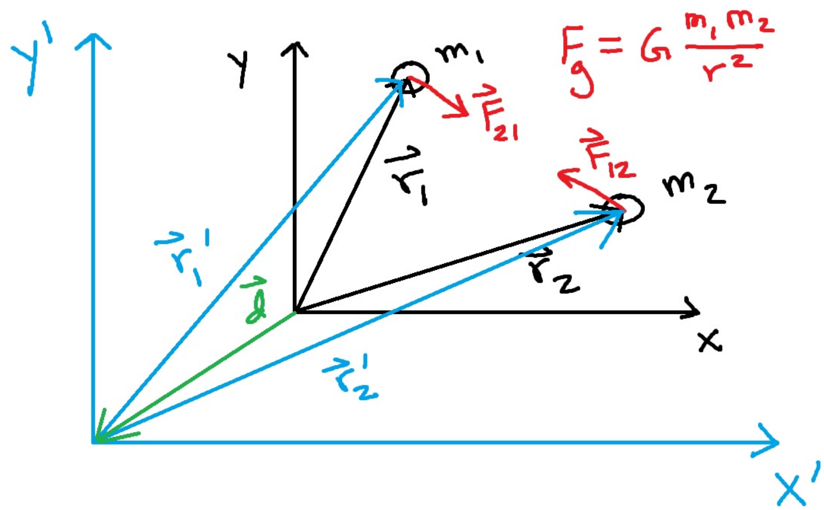


Figure 4.2: A second choice of coordinate system will, in general, lead to different position vectors.

Now, I told you that the way that we actually determine the motion of a body is by using Newton's second law,

$$\vec{F}/m = \vec{a}. \tag{4.4}$$

If it is indeed true that I can use any choice of coordinate system to do a physics problem, then *Newton's law should be true no matter which coordinate system I choose.* If it is, and the quantity $\vec{F}/m$ is the same regardless of the choice of coordinates, then it better be true that the acceleration is also the same in both coordinate systems.

However, the acceleration is something which is defined in terms of the position vector. Since the position vector DOES depend on the choice of coordinate system, I had better be more careful in making sure that the acceleration vector is actually the same. In the original coordinate system, I have

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}. \tag{4.5}$$

Now, in the new coordinate system, I have

$$\vec{a}\,' = \frac{d^2\vec{r}\,'}{dt^2} = \frac{d^2}{dt^2}\left(\vec{r} - \vec{d}\right). \tag{4.6}$$

However, the vector $\vec{d}$ is just a constant displacement vector, and so its time derivative is zero. Therefore, we find

$$\vec{a}\,' = \frac{d^2\vec{r}}{dt^2} = \vec{a}. \tag{4.7}$$

So everything checks out.

4.2 Galilean Relativity

But now let's imagine I do something a little different. Imagine I were to pick a new frame whose origin was not sitting still with respect to the old frame, but instead moving at a constant velocity with respect to the old frame. That is, imagine I had

$$\vec{d}\,(t) = \vec{d}_0 + t\vec{v}_{no}, \tag{4.8}$$

where $\vec{v}_{no}$ and $\vec{d}_0$ are some constant vectors. Notice that the vector $\vec{v}_{no}$ is *the velocity of the origin of the new frame with respect to the old frame*. We can see this because $\vec{d}$ is nothing other than the *position vector* of the origin of the new frame with respect to the old frame. If I therefore want to compute the velocity of the origin of the new frame, I need to compute the time derivative of $\vec{d}$,

$$\frac{d}{dt}\vec{d} = \frac{d}{dt}\left(\vec{d}_0 + \vec{v}_{no}t\right) = \vec{v}_{no}, \tag{4.9}$$

as claimed. Because this vector tells me the velocity of the origin of the new coordinate system with respect to the origin of the old coordinate system, we often just say that this is "the velocity of the new frame with respect to the old frame." The subscript "*no*" stands for "new with respect to old."

I now want to understand how this might change the physics of my problem. Again, so long as $t \neq 0$, the position vectors with respect to one frame will be different from the position vectors with respect to another frame. But, I can again show that the accelerations will stay the same, and so Newton's laws will still be valid. If I compute the acceleration in the new frame this time, I find

$$\vec{a}\,' = \frac{d^2\vec{r}\,'}{dt^2} = \frac{d^2}{dt^2}\left(\vec{r} - \vec{d}\right) = \frac{d^2}{dt^2}\left(\vec{r} - \vec{d}_0 - t\vec{v}_{no}\right). \tag{4.10}$$

However, the second derivative with respect to time will kill off the second and third terms, and so again

$$\vec{a}\,' = \frac{d^2\vec{r}}{dt^2} = \vec{a}. \tag{4.11}$$

Thus, if Newton's laws hold in the first frame, they also hold in the new frame. This result is called the principle of *Galilean relativity*, and the change of coordinate system we have performed is called a *Galilean transformation.*

This result tells us that there is really no way to prefer one of these frames over the other. Newton's laws, which we believe to be the "laws of physics," hold the same way in both frames. So there is really no way we can say which one is the "correct frame," or which origin is the one sitting still.

However, while the accelerations are the same, the velocities will be different. We can see this by considering the velocity of body number one in the new frame. What we find is that

$$\vec{v}_1{}' = \frac{d\vec{r}_1{}'}{dt} = \frac{d}{dt}\left(\vec{r}_1 - \vec{d}_0 - t\vec{v}_{no}\right) = \vec{v}_1 - \vec{v}_{no}. \tag{4.12}$$

We see that the velocity is shifted by the velocity of the new frame with respect to the old one. To make my notation a little more explicit, I can write this as

$$\vec{v}_{1n} = \vec{v}_{1o} - \vec{v}_{no}, \tag{4.13}$$

where the subscripts would be read as "body one with respect to new frame," "body one with respect to old frame," and "new frame with respect to old frame."

Now, velocity transformations can be a little annoying, because it's easy to accidentally get the subscripts backwards, or be dyslexic about which frame is which. There is a notational method I find very helpful to make sure you have the transformation correct. If I take the above equation and rearrange it, I have

$$\vec{v}_{1o} = \vec{v}_{1n} + \vec{v}_{no}. \tag{4.14}$$

Notice that on the right side, the letter "n" shows up twice, once on the outside of the two letters, and once on the inside. If I imagine these two copies of "n" "canceling", then they leave me with simply "1o,"

$$1n + no \rightarrow 1o. \tag{4.15}$$

The result is the correct arrangement of subscripts on the left side.

I can also generalize this formula a little bit. Imagine that I have three objects or points in space, A, B, and C, which are moving with respect to each other. If I associate the origin of a coordinate system with two of them, and then consider the velocity of the third object, what I have is that

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}. \tag{4.16}$$

This equation says that ***the velocity of object A as observed from the location of C is the same as the vector sum of the velocity of object A with respect to the location of object B, plus the velocity of object B with respect to the location of object C***. I find this form of the velocity addition law to be the easiest one to remember, because I simply imagine "canceling out" the two Bs.

Another important property to remember is that for any two objects A and B,

$$\vec{v}_{AB} = -\vec{v}_{BA}. \tag{4.17}$$

Intuitively, this says that if I think you're moving to the left, you instead think that I'm moving to the right. I can verify this claim pretty easily from looking at Figure 4.2. If $\vec{d}$ is the position vector of the new frame with respect to the old frame, then certainly, $-\vec{d}$ must be the position vector of the old frame, as viewed by the new frame, since $-\vec{d}$ points in the *opposite* direction, from the origin of the new frame to the origin of the old frame. When we take a time derivative to get the velocity, the minus sign carries through.

These ideas are actually probably familiar to you already, although you may not have thought about them in this way. To give an example of these ideas, imagine I'm standing on the Earth and I fire a gun horizontally. I've drawn this in Figure 4.3. I've also set up a coordinate system centered around myself, indicated in blue, with the bullet moving along the x direction, with velocity $\vec{v}_b$ with respect to my coordinates. For the sake of simplicity, I'll ignore the effects of gravity in this example (I'll assume that I'm only worried about time periods short enough that the bullet hasn't started to fall noticeably).

I've also shown a car which I imagine happens to be driving by me as I fire the gun. Let's say that with respect to me, the car isn't moving in the y direction, but it is moving along the x direction with some velocity $\vec{v}_c$ with respect to me. Now, I've also drawn some coordinates centered around the car, labeled in red. If the driver of the car measures the velocity of the bullet, then we can use the velocity transformation rule to see that they will find

$$\vec{v}_b\,' = \vec{v}_b - \vec{v}_c. \tag{4.18}$$

If we want to be really careful that this is right, we can use the letter B for bullet, C for car, and G for ground (where I'm standing). Then the above equation says

$$\vec{v}_{BC} = \vec{v}_{BG} - \vec{v}_{CG} = \vec{v}_{BG} + \vec{v}_{GC}, \tag{4.19}$$

which obeys the rule for the subscripts that I mentioned previously.

Now, if I notice that the car is passing by me at the same velocity that I fired the bullet, we have

$$\vec{v}_b\,' = \vec{v}_b - \vec{v}_b = 0, \tag{4.20}$$

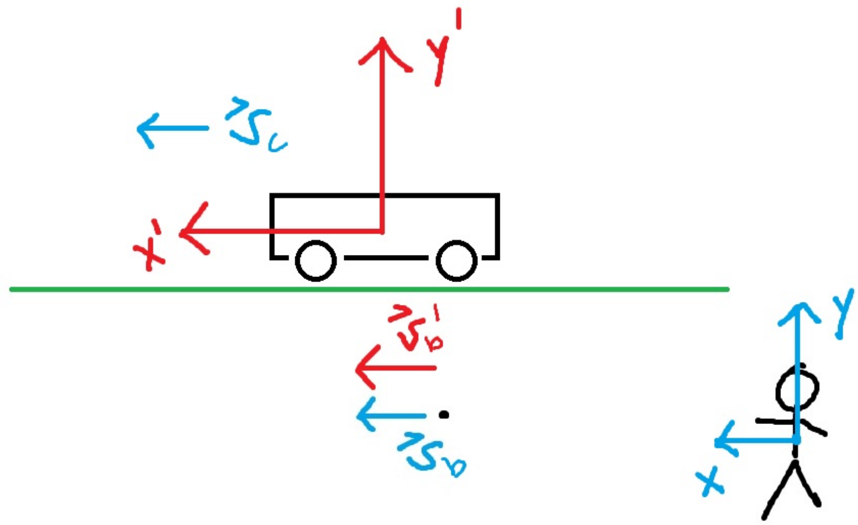


Figure 4.3: A demonstration of Galilean relativity, using the example of a bullet traveling through the air, as observed by the gunman and a driver passing by.

which is to say that according to the driver of the car, the bullet is sitting still. Of course, this makes sense. If the car drives by me at the same velocity as the bullet, the driver of the car should see the bullet *travelling next to him*, sitting still with respect to his coordinates.

Now, it is tempting to say that I am the one who is "really sitting still," and that the car is actually moving. This is because I am on the ground, and the surface of the Earth is familiar to us, for obvious reasons. However, we know in reality that the Earth is actually moving through space, and doesn't really represent any sort of special location. In fact, there is really no difference between the car's choice of coordinates and mine. The driver of the car and I would both say that the bullet is moving at constant velocity, and so it is not accelerating (at least not in the x direction). We also both agree that aside from gravity (which would only affect the y component anyways), there is no force acting on the bullet (at least not in the x direction), since it is in free fall and not in contact with anything. Thus, we both agree that the force and the acceleration are zero, and so we both agree that Newton's second law is obeyed.

4.3 Inertial Reference Frames

Lastly, what if I consider two reference frames whose relative velocities are not constant? Let's go back to our example in Figure 4.1, and imagine that $\vec{d}$ is now given by

$$\vec{d} = \vec{d}_0 + \vec{v}_{no}t + \frac{1}{2}\vec{g}t^2. \tag{4.21}$$

That is to say, the second frame is moving with respect to the first in a way that is quadratic in time, not linear - it is accelerating. If we now go to compute the acceleration of the first body, according to the new frame, we find

$$\vec{a}\,' = \frac{d^2\vec{r}\,'}{dt^2} = \frac{d^2}{dt^2}\left(\vec{r} - \vec{d}_0 - \vec{v}_{no}t - \frac{1}{2}\vec{g}t^2\right) = \vec{a} - \vec{g}, \tag{4.22}$$

and so the acceleration is NOT the same. However, we know that the forces **are** still the same in both frames. Therefore, *it cannot be true that Newton's second law is the same in both frames.* If it is true in one, then it cannot be true in the other.

The same idea can be examined in the case of the car and the bullet. If the car were accelerating with respect to me, then to the driver, *it would appear that the bullet had a net acceleration backwards, despite there not being any forces acting on it in the x direction.*

Assuming that Newton's law is correctly obeyed in the first frame, this frame is know as an *inertial reference frame.* The definition of an inertial reference frame is one in which Newton's laws are obeyed. We have seen that if I have an inertial frame, any other frame which is moving at *constant velocity* with respect to this frame will also obey Newton's laws, and so it is also an inertial reference frame. By starting with one inertial frame, we can find all other inertial frames by considering all of the frames moving with respect to it at constant velocity. However, any frames which are accelerating with respect to this frame will not obey Newton's laws, and we call them non-inertial reference frames.

When I first learned this fact, it seemed somewhat unsatisfying to me. The result we've found is that while the universe doesn't seem to "know" what speed you're moving at, it does seem to be able to make a distinction between which frames are accelerating, and which are not. Why should the first derivative of position be totally unimportant, while the second derivative is absolutely crucial?

An even more unsettling example involves what happens if you stand with your arms outstretched and spin around. Experience tells us that if I stand still, I feel nothing special. But if I start spinning around in a circle, I feel a tension in my arms. This is because my hands are now moving in a circle, and so there is a force required to keep them accelerating, which I feel in my arms. By the way, this effect has nothing to do with whether or not you are standing on the Earth - I could travel far out into intergalactic space, far away from any other bodies, and this would still be the case. But this simple fact actually reveals something profound about the way the universe works: there is a way to distinguish between a rotating and a non-rotating frame. I can *know* if I am "really" spinning or not. How can it be that if I go out into empty space, there's no way to tell what velocity I'm moving at, not even in principle, yet I *can* tell for sure whether or not I am really accelerating or rotating?

If we don't like these ideas, there are two possible ways we can think about getting around them. One idea might be to generalize Newton's laws. Maybe there is a more general law, which takes into account all possible types of frames, and the form of this equation always stays true, no matter what. The other idea is that perhaps if empty space seems to care about acceleration and rotation, perhaps it is actually not so empty as I thought. But it would have to be filled with something pretty weird, something that cares about acceleration, but not velocity.

It turns out that actually both of these ideas can be combined together to help us find a resolution to this problem, but it involves making very radical changes to the way we think about the universe. The resulting theory of physics is known as General Relativity, and I'll talk about it briefly at the very end of the course. But, in the remaining time today, I want to show you a useful tool about accelerated frames which will not only help you do physics problems, but is actually, in disguise, the first step towards solving this puzzle.

4.4 The Equivalence Principle

Let's imagine a situation where I am in a very simple looking rocket traveling through empty space, shown in Figure 4.4. I've drawn a set of coordinate axes, which I know to be an inertial frame. In this frame, I happen to know that the rocket is about to start accelerating upwards, with an acceleration which we will call $\vec{g}$. I'm standing in the rocket, and I'm holding a ball. I've aligned the axes so that at time zero, the floor of the rocket is at $y = 0$, and the ball is at $y = h$. The rocket starts accelerating right at time zero. This type of reference frame, aligned with an accelerating object in such a way that it is momentarily at rest with respect to it, is often called an *instantaneous rest frame* for that object at that time.

Now, at time zero, I let go of the ball, and I ask what happens. Because the ball is now in empty space with no forces acting on it, its acceleration will

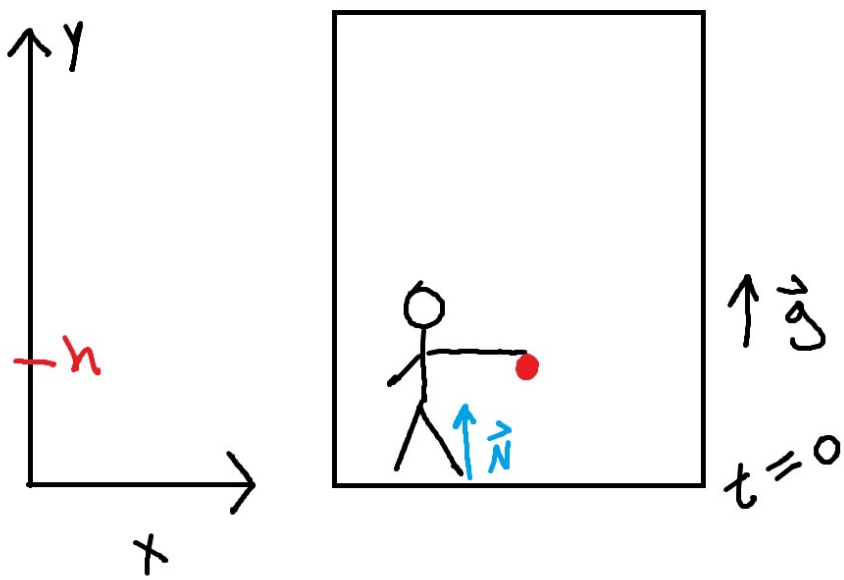


Figure 4.4: Dropping a ball in a rocket... IN SPACE!

be zero. Because it initially had zero velocity, it will continue to not move, and so the ball will stay at $y = h$. The floor of the rocket, however, will accelerate from zero, and so the y coordinate of the floor will be given by

$$y_f = \frac{1}{2}gt^2. \tag{4.23}$$

Eventually, the floor of the rocket will move upwards far enough to hit the ball (which is sitting still). This time occurs when

$$y_f = h \Rightarrow t = \sqrt{\frac{2h}{g}}. \tag{4.24}$$

In general, before the floor hits, the distance between the ball and the floor will be

$$d = y_b - y_f = h - \frac{1}{2}gt^2. \tag{4.25}$$

However, we know this equation looks very familiar. This is exactly the same expression for the height above the ground when I drop a ball on Earth. In fact, let's pretend that I don't know I'm in a rocket accelerating, and imagine my perspective from inside the rocket. As far as I know, I let go of a ball, and the distance between the ball decreased according to the above equation. Of course I'm now convinced I must be standing on the Earth, because clearly this ball is falling under the influence of a uniform gravitational field!

Not only would the motion of the ball look the same, but the sensation I would feel while standing would also be the same. If I were standing in a laboratory on Earth, I would feel a contact force with the floor. I would say that because I am not accelerating, there must be a normal force that the ground exerts on me to compensate gravity, whose magnitude is mg, where m is my mass. But someone watching me accelerate in the rocket would say I have it all wrong - I AM accelerating, at a magnitude of g. The rocket is providing a normal force to *cause* that acceleration, and it is the ONLY force acting on me. This is also shown in the figure.

This is a nice result, and you'll see when doing the homework that it can be incredibly useful in solving some problems. *I can always take an accelerating reference frame, and pretend that I am not actually accelerating, but instead I am subject to a gravitational field.* I can also do the reverse, and remove a gravitational field by assuming that I am in a laboratory in empty space, and that laboratory is accelerating.

Now, this certainly all makes sense in terms of the Newtonian mechanics we've been studying, although you might wonder if it's just a nice mathematical trick. Of course, I could also try seeing what happens to other objects in the rocket. I could shine a laser pen around inside, and the light beam would move along some path. If I were to go out on a limb and try to extend my previous ideas, my conclusion would be that the behavior of light in an accelerating reference frame is the same as its behavior in a gravitational field. In Figure 4.5, we've shown what it might look like if our rocket had a source of light on one side, and a beam of light passed across the length of the rocket. Now, the path of the light beam would appear to bend downwards, thus leading to the conclusion that light is affected by a gravitational field. But of course this is a silly idea, and it seems as though our astronaut in the rocket now has a way to determine, once and for all, that he is actually accelerating, and NOT in a uniform gravitational field.

Amazingly, it turns out that this equivalence of uniform gravity and acceleration is, in fact, always true. Believe it or not, light is affected by gravity - a light beam passing by a massive object will be deflected by it (although not by very much), and our astronaut cannot tell that he is accelerating. You might wonder how something like a massless wave of light could possibly be affected by gravity. There is a way to explain this phenomenon, but again, it requires a dramatically different way of thinking about the universe, which we'll discuss

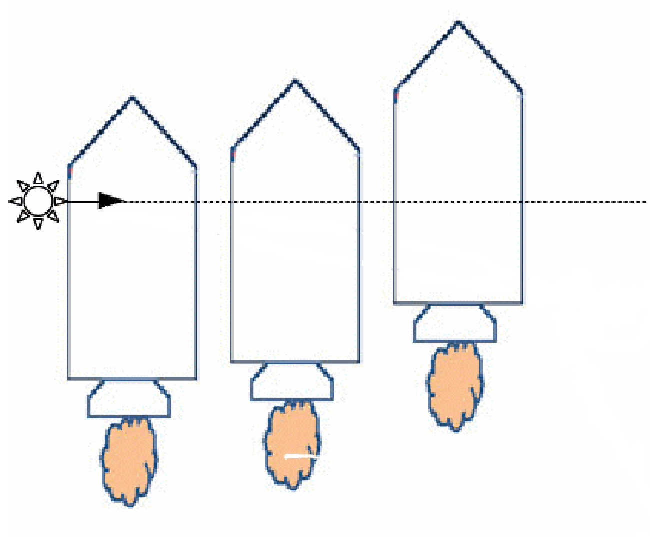


Figure 4.5: A light beam passing through a rocket.

briefly at the end of the course.

That concludes our lightning review of basic mechanics in terms of Newton's laws. Tomorrow, we'll start talking about work and energy, a set of useful ideas which will further help us solve a variety of physics problems.

Chapter 5

Work and Kinetic Energy

5.1 Work

Today I'm going to start introducing the concepts of work and energy, which will be helpful for solving problems in a lot of situations where applying Newton's laws would be cumbersome.

Let's imagine I have a block which I'm pushing across the floor, shown in Figure 5.1. If I'm moving the block at constant velocity, then I know that I have to apply a force to compensate the effects of kinetic friction,

$$\vec{F} = -\vec{f_k} = \mu_k N\hat{x} = \mu_k mg\hat{x}, \tag{5.1}$$

where I'm assuming I'm moving the block in the positive x direction. N is the magnitude of the normal force, m is the mass of the block, and the coefficient of kinetic friction is μ_k.

I know that in order to perform this task, there is some sense in which I need to put some "effort" into compensating the effects of friction. In order to quantify this statement, let's assume I move the block a total distance d. Then, if I'm applying a force along the direction of the motion, a simple definition to quantify this amount of effort might be

$$W = Fd, \tag{5.2}$$

which we say is the amount of *work* that I have done on the block. This definition takes into account how much force I am applying, and for how long of a distance I do that. Intuitively, I would expect that increasing both of these things would correspond to me putting more overall effort into moving the block.

Now, we can also say that because the floor exerts a frictional force on the block, it also does work on the block. In this case, however, the frictional force

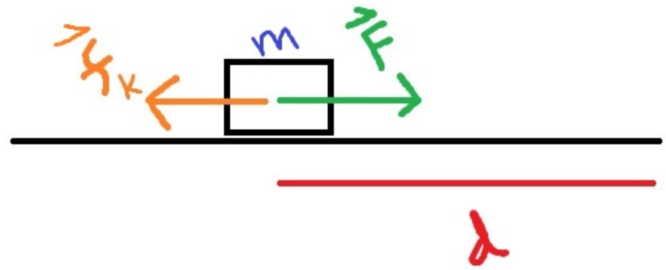


Figure 5.1: Doing work on a block by pushing it across the floor.

opposes the motion of the block - its the reason we need to do anything to make the block move - so perhaps we want to take this into account as well. In this case, we say that the floor does *negative work* on the block, and we write

$$W_f = -f_k d = -Fd, \tag{5.3}$$

where I've used the fact that the magnitudes of the two forces are the same, and I've also included a subscript to clarify that this is the work the floor is doing.

If I add these two together, I find that I get zero, since

$$W_t = W_m + W_f = Fd - Fd = 0, \tag{5.4}$$

where the subscripts stand for "total," "me," and "floor." *Notice that zero net force along the direction of motion implies zero net work.*

I can generalize this definition to include forces pointing in arbitrary directions, which I've shown in Figure 5.2. In this situation, we want to get a sense of how much the force I'm applying goes into moving the block, if I imagine, say, that I am pulling on the block with a string at some angle. In some sense, it seems like this should involve the component of the force along the direction of the block. If the displacement vector for the block's net motion is

$$\vec{d} = d\hat{x}, \tag{5.5}$$

then I define the work to be

$$W_m = \vec{F} \cdot \vec{d} = Fd\cos\theta = F_x d. \tag{5.6}$$

While it may not be obvious that this is the best possible definition, it will become clear in a moment why it is useful.

Now, I can imagine moving the block along a sequence of paths, as shown in Figure 5.3. If the block experiences a net force $\vec{F}_1$ while moving along a

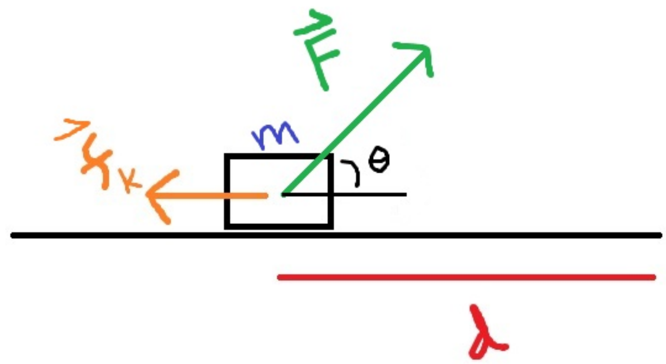


Figure 5.2: Work done by a force which is not along the direction of motion.

displacement $\vec{d}_1$, and a net force $\vec{F}_2$ while moving along a displacement $\vec{d}_2$, we define the total work to be

$$W_t = \vec{F}_1 \cdot \vec{d}_1 + \vec{F}_2 \cdot \vec{d}_2. \tag{5.7}$$

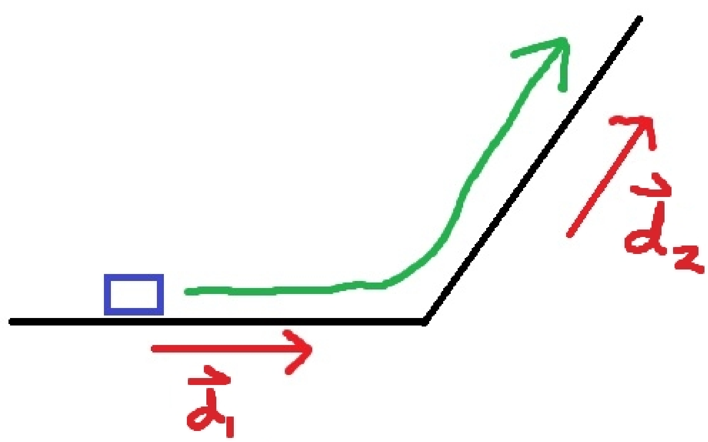


Figure 5.3: The addition of the work done along two paths.

In general, we can imagine that the object moves along an arbitrary path, feeling a force which depends on where it is located. This is shown in Figure 5.4. The notation on the forces emphasizes that the force can depend on where the object is, and the notation on the position of the object emphasizes that this changes with time. Now, over short enough distances, we know a curve will look approximately straight. We can imagine that at time t_1, the object moves a tiny, infinitesimal distance which we call $d\vec{r}$. Over this small section of path, the path is roughly straight, and so we define the infinitesimal work to be

$$dW = \vec{F}\left(\vec{r}\left(t_1\right)\right) \cdot d\vec{r}. \tag{5.8}$$

We write the total work over the path from $\vec{r}\left(t_1\right)$ to $\vec{r}\left(t_2\right)$ as

$$W = \int_{\vec{r}(t_1)}^{\vec{r}(t_2)} \vec{F}\left(\vec{r}\left(t\right)\right) \cdot d\vec{r}\left(t\right). \tag{5.9}$$

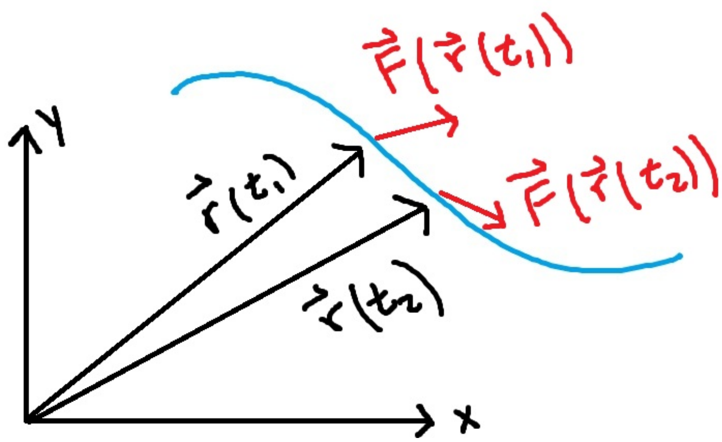


Figure 5.4: The work done along an arbitrary path.

The above object is called a line integral. It tells us to sum up all of the infinitesimal contributions to the work along the path. In order to make sense out of what it means to integrate with respect to a vector, and how to actually compute the above object, we notice that the infinitesimal displacement at a given time is

$$d\vec{r}\left(t\right) = \vec{v}\left(t\right) dt, \tag{5.10}$$

or the velocity at that instant times the time difference. Thus, we can write our integral as

$$W = \int_{t_1}^{t_2} \left[\vec{F}\left(\vec{r}\left(t\right)\right) \cdot \vec{v}\left(t\right)\right] dt, \tag{5.11}$$

which is now just a regular time integral which I know how to compute, assuming I know the path as a function of time, and also what the force is at each point.

While it certainly doesn't look like it from the above expression, the amount of work done over a path is actually independent of how quickly the object moves, so long as the forces only depend on where the particle is in space. While it is beyond the level of this class to prove that, it is a useful fact which can make some calculations easier - if I want to compute the work done while going around in a circle, then so long as the forces doing work on the object only depend on where the particle is located on the circle, I can assume the particle moves in uniform cicular motion when computing the above quantity.

5.2 Kinetic Energy and the Work-Energy Principle

Now that I've defined this object called work, let's put it to use. From Newton's second law, we know that we can write the total force on the particle as

$$\vec{F}\left(\vec{r}\left(t\right)\right) = m\vec{a}\left(t\right), \tag{5.12}$$

where I've assumed that the force, and thus acceleration, may change in time. Using this, I can write the formula for the work as

$$W = m\int_{t_1}^{t_2} \vec{a}\left(t\right)\cdot\vec{v}\left(t\right)dt. \tag{5.13}$$

Now, there is a useful formula for taking the time derivative of a dot product of two vectors. If I have two vectors $\vec{p}(t)$ and $\vec{q}(t)$, then their dot product is

$$\vec{p}\left(t\right)\cdot\vec{q}\left(t\right) = p_x\left(t\right)q_x\left(t\right) + p_y\left(t\right)q_y\left(t\right). \tag{5.14}$$

If I take a time derivative of this, and use the product rule for taking derivatives, I find

$$\frac{d}{dt}\left(\vec{p}\left(t\right)\cdot\vec{q}\left(t\right)\right) = \frac{dp_x}{dt}q_x + p_x\frac{dq_x}{dt} + \frac{dp_y}{dt}q_y + p_y\frac{dq_y}{dt}. \tag{5.15}$$

If I further rearrange this, I can write it as

$$\frac{d}{dt}\left(\vec{p}\left(t\right)\cdot\vec{q}\left(t\right)\right) = \frac{d\vec{p}}{dt}\cdot\vec{q} + \frac{d\vec{q}}{dt}\cdot\vec{p}. \tag{5.16}$$

So there is a "product rule" for dot products as well. However, make sure to realize that I carefully checked this by using the definition of the dot product! I didn't just assume that it was also true for dot products just because it's true for multiplying regular numbers.

With this formula, notice that I can write

$$\frac{d}{dt}\left(v^2\left(t\right)\right) = \frac{d}{dt}\left(\vec{v}\left(t\right)\cdot\vec{v}\left(t\right)\right) = \frac{d\vec{v}}{dt}\cdot\vec{v} + \frac{d\vec{v}}{dt}\cdot\vec{v} = 2\frac{d\vec{v}}{dt}\cdot\vec{v} = 2\vec{a}\cdot\vec{v}. \tag{5.17}$$

Using this, I can write my expression for the work as

$$W = \frac{1}{2}m\int_{t_1}^{t_2}\frac{d}{dt}\left(v^2\left(t\right)\right)dt. \tag{5.18}$$

Since this is just the integral of a derivative, we finally see that

$$W = \frac{1}{2}mv^2\left(t_2\right) - \frac{1}{2}mv^2\left(t_1\right). \tag{5.19}$$

The quantity $K = \frac{1}{2}mv^2$, which depends on the speed of the body, is known as the *kinetic energy* of the body. The above equation tells us that the amount of work done on an object as it travels between two points is the same as the difference between the kinetic energies that that body has at those two points. This is known as the work-energy principle.

So far, it seems like all I have done is define two new quantities - work and kinetic energy. But these ideas are actually very useful in solving problems. To see how this is the case, let's revisit the subject of a block sliding on a table, shown in Figure 5.5. Let's say that I give the block a shove, so that I start it off with some initial velocity $\vec{v}_0$, which we'll take to be entirely along the surface of the table. As the block moves, we know it will be subject to a frictional force, given by

$$\vec{f}_k = -\mu_k N\hat{x} = -\mu_k mg\hat{x}, \tag{5.20}$$

where N is the magnitude of the normal force, m is the mass of the block, and μ_k is the coefficient of kinetic friction. I've taken the positive x direction to be along the direction of the initial velocity, so that the frictional force is negative (I've avoided drawing the gravitational and normal forces on the diagram for the sake of clarity, but they are there of course, acting solely in the vertical direction). Because I give the block a shove and let go, once the block is moving there is no applied force from me.

Now, what I want to know is, how far does the block travel before coming to rest? Well, there are two ways we can answer this question. First, we can solve for the position of the block as a function of time, and then solve for the time that the velocity comes to zero. The x component of the force acting on the block is

$$F_x = -\mu_k mg, \tag{5.21}$$

and so the x component of acceleration is

$$a_x = -\mu_k g. \tag{5.22}$$

Now, if we integrate this equation once, we find

$$v_x\left(t\right) = v_x\left(0\right) + \int_0^t a_x dt' = v_0 - \mu_k gt. \tag{5.23}$$

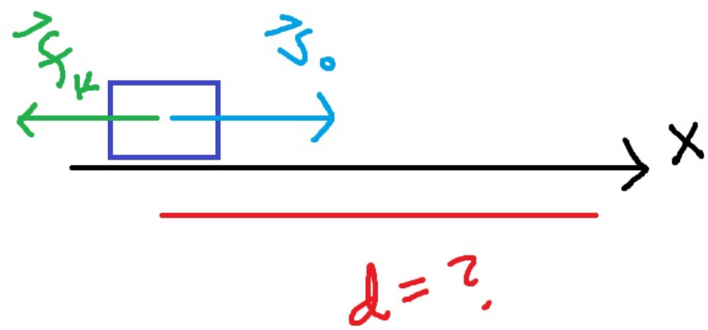


Figure 5.5: A block being brought to a stop by friction.

Integrating again, we see that the position of the block as a function of time is given by

$$x(t) = v_0 t - \frac{1}{2}\mu_k g t^2, \tag{5.24}$$

assuming that we take the starting position to be at $x = 0$.

Now, solving for the condition that the block's velocity is zero, we have

$$v_0 - \mu_k g t = 0 \Rightarrow t_d = \frac{v_0}{\mu_k g}. \tag{5.25}$$

If we use this in the expression for the position, we find

$$d = x(t_d) = v_0 \left(\frac{v_0}{\mu_k g}\right) - \frac{1}{2}\mu_k g \left(\frac{v_0}{\mu_k g}\right)^2 = \frac{1}{2}\frac{v_0^2}{\mu_k g}. \tag{5.26}$$

However, there is a second way to do this problem, which is to make use of the work-energy principle. We know that the change in kinetic energy of the block will be the same as the work done on it during its motion. The net force on the block is due to friction, and so if the block moves a distance d, the work done on it is

$$W = \vec{f}_k \cdot \vec{d} = -f_k d = -\mu_k m g d. \tag{5.27}$$

Make sure to remember that the work is negative! Now, the change in kinetic energy is equal to the final kinetic energy, minus the initial kinetic energy. This is equal to

$$\Delta K = \frac{1}{2}m v_d^2 - \frac{1}{2}m v_0^2 = -\frac{1}{2}m v_0^2, \tag{5.28}$$

since the final velocity is zero. Thus, we have

$$W = \Delta K \Rightarrow -\mu_k mgd = -\frac{1}{2}mv_0^2 \Rightarrow d = \frac{1}{2}\frac{v_0^2}{\mu_k g}. \tag{5.29}$$

This is the same result, but with much less effort! The moral of the story is this: if you only care about what happens at the beginning and end of an object's motion, and don't need to know the full trajectory as a function of time, the work-energy principle can save you a lot of time. Never do more work than you have to!

Now, we can also use the work-energy principle to do problems that would be much more irritating, or possibly intractable, depending on how much math we know. Let's redo this problem, but consider a more complicated situation. Let's say I've taken the table and sanded it down in some strange way, so that the coefficient of friction is not uniform over the table. As a specific example, let's say that

$$\mu_k = Ax^2, \tag{5.30}$$

where A is some constant number. That is, the coefficient is a function of position. If I were to revisit my expression for the acceleration, I would find that

$$a_x = -Agx^2. \tag{5.31}$$

Writing the acceleration in terms of the position, we find

$$\frac{d^2x}{dt^2} = -Agx^2. \tag{5.32}$$

This equation is a differential equation which contains second order derivatives. Not only have you not learned how to solve this yet, it turns out that the solution for the position as a function of time is somewhat gross looking, and difficult to work with.

However, the work-energy principle lets us get to the answer we really want, without the intermediate step of solving for the trajectory. Again, we know that the change in kinetic energy will be equal to the work done on the block. The change in kinetic energy is still the same as before, since the mass, along with starting and ending velocities, are the same. The only thing which is different now is the total work done. But, I know how to compute this - it's just the integral of the force over the distance traveled. So if my block travels a distance d, the work done is

$$W = \int \vec{f}_k \cdot d\vec{x} = -\int_0^d Amgx^2 \, dx = -\frac{1}{3}Amgd^3. \tag{5.33}$$

Notice that in one dimension,

$$d\vec{x} = \vec{v}\,(t)\,dt = \frac{dx}{dt}dt = dx. \tag{5.34}$$

In one dimension, I can simply integrate with respect to x, although if I were working in a higher number of dimensions, I would need to invent some fictitious trajectory for my block to follow, and compute what $\vec{v}\,(t)$ is (which you'll explore on the homework). If I now equate the work done with the change in kinetic energy, I find that

$$-\frac{1}{3}Amgd^3 = -\frac{1}{2}mv_0^2 \Rightarrow d = \left[\frac{3}{2}\frac{v_0^2}{Ag}\right]^{1/3}. \tag{5.35}$$

Needless to say, this is much, much easier than using kinematics!

5.3 Power

The last concept I want to introduce today is power. Power is defined to be the rate at which work is done,

$$P = \frac{dW}{dt}. \tag{5.36}$$

Using our general expression for work, we can see that

$$P = \frac{d}{dt}\int \vec{F}\cdot\vec{v}\;dt = \vec{F}\cdot\vec{v}. \tag{5.37}$$

Because the force acting on an object, along with its velocity, can be functions of time, the power, in general, will also be some function of time. It is equal to the rate of change of the kinetic energy, since we can easily calculate

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = m\vec{a}\cdot\vec{v} = \vec{F}\cdot\vec{v} = P. \tag{5.38}$$

Notice that while in many cases, the net force on an object may be zero, it is often still true that some agent is providing power. In the case of the sliding block, if I am pushing the block along to compensate friction, I am doing work on the block, and so I am expending some effort to do this. The rate at which I do this is the power I am providing.

Actually, in an indirect way, I am actually doing work on the floor. Because the floor does negative work on the block, the net work done on the block from the floor and me is zero, and so is the net power supplied to the block. So where does the result of my work go? Well, we know that when I rub two surfaces together, they get hot. So really I am heating the floor! The power I am providing is going into the thermal energy of the floor, although that is a subject more appropriate for a class on thermodynamics, not classical mechanics.

In many situations, power is a much more important quantity than total work. Walking five miles over the course of a day is pretty easy, but running five miles in an hour is a lot harder! Runners have to train their bodies to be able to output power at a much higher rate than the average person.

Tomorrow we'll continue our discussion of energy by introducing the concepts of potential energy and energy conservation.

Chapter 6

Potential Energy

6.1 Potential Energy

Yesterday we introduced the ideas of work and kinetic energy. Today we're going to expand on these ideas, in order to further build our set of physics problem solving tools.

Let's consider a new type of system that I haven't talked about yet. Let's imagine I have a block sitting on the floor, but it's attached to a nearby wall with a spring. This is shown in Figure 6.1. I'll imagine that the floor has a negligible amount of friction, so that I can effectively ignore it (maybe the floor is actually an air hockey table or something). Initially, I've placed the block so that the spring is not stretched or compressed at all, and is not exerting any forces on the block. I've labeled this position as x_0. If I've taken the location of the wall to be $x = 0$, then we say that the spring has an *equilibrium length* of x_0.

Now, Hooke's law tells us that as we start to move the block a little bit, the force exerted by the spring on the block is given by

$$F = -k\left(x_1 - x_0\right), \tag{6.1}$$

where x_1 is the new location of the block, and k is some positive constant which characterizes the spring. This is shown in Figure 6.2. Notice that when x_1 is to the left of the equilibrium position, the sign of the force is positive, as it should be.

Now let's imagine that I compress the spring by some amount

$$\Delta x = x_1 - x_0. \tag{6.2}$$

After yesterday's lecture, I might want to ask how much work I've done on the block-spring system. Well, I know that the force I exert will be opposite to what

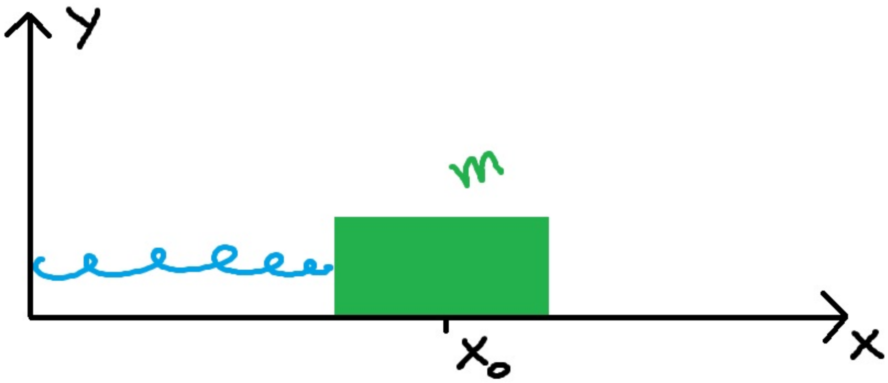


Figure 6.1: A block attached to a spring, sitting at rest.

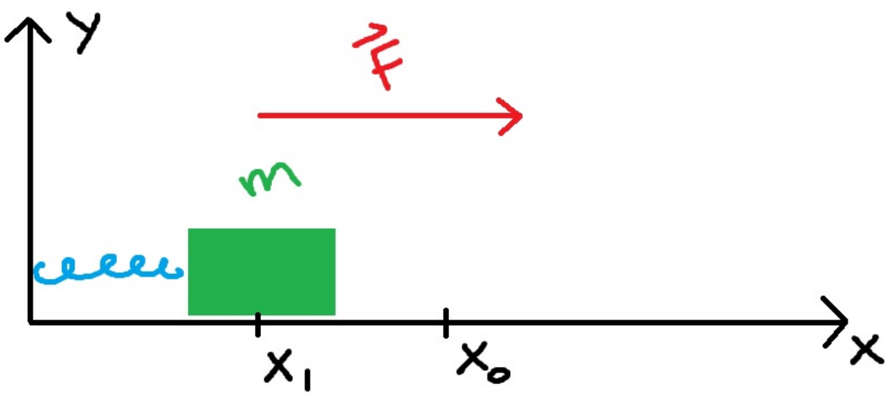


Figure 6.2: Doing work on a block by moving it against the action of a spring.

the spring is exerting on the block,

$$F_m = +k\left(x_1 - x_0\right). \tag{6.3}$$

If I now compute the amount of work that I've done, the result is

$$W_m = \int_{x_0}^{x_1} \vec{F}_m \cdot d\vec{x} = +k \int_{x_0}^{x_1} \left(x - x_0\right) dx = \frac{1}{2} k \left(x_1 - x_0\right)^2 = \frac{1}{2} k \left(\Delta x\right)^2 . \tag{6.4}$$

Notice that my final result doesn't depend on the sign of Δx, and is always positive - this agrees with our intuition that compressing and stretching a string both require some positive effort on my part.

In some sense, while I'm applying a force to the block directly, it might be more natural for me to think of doing work on the spring. The reason I say this is because after the compression has finished, the block is more or less in the same state it was in before I started, and has had zero net work done on it, since it is sitting still before and after the compression, so it's kinetic energy has not changed. The spring, however, has changed noticeably - it is physically smaller as a result of being compressed. We'll elaborate on this idea in a moment.

Now, what happens when I let go of the block? Well, if I were considering the case where I had just done work against friction, nothing. That's because there is only a kinetic friction force opposing the motion of the block while it is moving. Once it stops, and I am not touching it, there is no longer any force doing any work. However, in the case of the spring, this is no longer true. Because the spring is compressed, there will be a force exerted on the block, and once I let go, there is nothing to compensate that force. So the block will start to move back to its equilibrium position.

As it does so, the spring will be doing work on the block. The amount of work done will be

$$W_s = \int_{x_1}^{x_0} \vec{F}_s \cdot d\vec{x} = -k \int_{x_1}^{x_0} (x - x_0)\, dx = \frac{1}{2} k \left(x_1 - x_0\right)^2 = \frac{1}{2} k \left(\Delta x\right)^2 . \quad (6.5)$$

This is the *same* amount of work I did on the spring-block system when I compressed the spring. So what I'm starting to notice is that in some sense, I did work on a system, did something to the spring to change its physical nature, and then, when I let go, the spring restored itself to its original state, while transferring the same amount of work to the block. Furthermore, because the work done on the block is the change in its kinetic energy, I know that after the spring has restored itself to its initial state, it has acquired a nonzero velocity, given by

$$\frac{1}{2} m v^2 = \frac{1}{2} k \left(\Delta x\right)^2 . \quad (6.6)$$

All of this gives me the idea that maybe there is some sort of "stuff" which is being transferred around from place to place in my system, whose amount seems to stay the same. In order to start quantifying this, let me take the expression for the work done during the compression and use it to define a new quantity,

$$U(x) = -\int_{x_0}^{x} F_s\left(x'\right) dx' = k \int_{x_0}^{x} \left(x' - x_0\right) dx' = \frac{1}{2} k \left(x - x_0\right)^2 , \quad (6.7)$$

which I will call the *potential energy* stored in the spring. While I originally considered compression from x_0 to x_1, I'm now considering compression to some arbitrary point x. Notice that it is a function only of the material properties of the spring, described by k, and the location of the end of the spring (where the block is). It is equal to the amount of work I did on the system when I compressed the spring.

Also, notice that the only reason I could define this function in this way is because it was possible for me to write down the force as a function of position. I could not do the same thing for friction, because the frictional force acting on a block is not just something I can write down as a function of position. The force of kinetic friction depends on whether or not the block is moving (it is some constant if it is moving, and zero otherwise), and so it depends on the velocity, not just the location. I describe this by saying that the force from the spring is a *conservative* force, whereas the force from friction is a *nonconservative force*.

I can also use the fundamental theorem of calculus to invert the above relationship, and write the force as

$$F_s\left(x\right) = -\frac{dU}{dx}. \tag{6.8}$$

So alternatively, in a system where I can define a potential energy, I can either specify the force as a function of position, or the potential energy as a function of position, and either one will tell me what the physics is. To see this explicitly, we can take Newton's second law for the spring acting on the block,

$$F_s = ma, \tag{6.9}$$

and use my new definition to write

$$m\ddot{x} = -\frac{dU}{dx}. \tag{6.10}$$

So I've rewritten Newton's laws in terms of this potential energy function.

Now, the observable thing I can measure for the block is its acceleration, and so this tells me that the physically relevant quantity is actually the derivative of the potential energy function, not the potential energy itself. To see this, pretend I defined a second potential energy function,

$$\widetilde{U}\left(x\right) = U\left(x\right) + C, \tag{6.11}$$

where C is some constant. We see that

$$\frac{d\widetilde{U}}{dx} = \frac{d}{dx}\left(U\left(x\right) + C\right) = \frac{dU}{dx}. \tag{6.12}$$

Because the derivative of the potential is what determines the forces and accelerations, then really, either of these definitions is just as good.

The reason for this ambiguity has to do with the fact that I can make different choices for the lower bound of the integral in the definition of the potential energy,

$$U(x) = -\int_{x_0}^{x} F_s(x')\, dx'. \tag{6.13}$$

Let's imagine that my new potential energy is defined to be the same integral, but starting at a different lower bound,

$$\widetilde{U}(x) = -\int_{\widetilde{x}_0}^{x} F_s(x')\, dx'. \tag{6.14}$$

To see the relationship to the old function explicitly, let me break the integral up into two parts,

$$\widetilde{U}(x) = -\int_{\widetilde{x}_0}^{x_0} F_s(x')\, dx' - \int_{x_0}^{x} F_s(x')\, dx'. \tag{6.15}$$

Now, the integral from $\widetilde{x}_0$ to x_0 is

$$-\int_{\widetilde{x}_0}^{x_0} F_s(x')\, dx' = \frac{1}{2}k\left(\widetilde{x}_0 - x_0\right)^2. \tag{6.16}$$

This number depends on k, and also the two possible lower bounds, but it does not depend on x, the actual position of the end of the spring. Therefore, it is just some constant with respect to x. The other integral on the right, of course, is just my original potential energy function, and so I find

$$\widetilde{U}(x) = \frac{1}{2}k\left(\widetilde{x}_0 - x_0\right)^2 + U(x) = U(x) + C. \tag{6.17}$$

This verifies my claim that the ability to shift the potential energy by a constant is a result of the possibility to choose different lower bounds for the integral. Of course, it certainly seems natural to define the lower bound as the position where the spring is in equilibrium, and usually this is what we do, but the important conclusion is that we don't HAVE to, and none of the physics is affected by this choice.

6.2 Conservation of Energy

Now, I want to continue elaborating on this idea that in some sense, there is some sort of "stuff" which is being moved around in this system. In the language

I'm developing, it seems like I did work on the spring, and put potential energy into it. Then, the block lost this potential energy when it relaxed back to equilibrium, but it gave that potential energy to the block in the form of kinetic energy. Motivated by this thinking, let's define the object

$$E = U + K = \frac{1}{2} k \left(x - x_0\right)^2 + \frac{1}{2} m v^2, \tag{6.18}$$

which I will call the *total energy* of the system.

I have a suspicion that this object always stays the same - it is the "stuff" which I am trying to describe. What I would like to show is that it is always the same number, at all points of the motion.

To check this, let's see what happens when we take a time derivative. We have

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2} k \left(x - x_0\right)^2 + \frac{1}{2} m v^2\right] = k \left(x - x_0\right) \frac{dx}{dt} + m v \frac{dv}{dt}, \tag{6.19}$$

where I've used the chain rule in a few places in order to take time derivatives. If I use the definitions of the time derivatives in terms of velocity and acceleration, I have

$$\frac{dE}{dt} = \left[k \left(x - x_0\right) + m a\right] v. \tag{6.20}$$

Using Newton's law, I can write

$$\frac{dE}{dt} = \left[k \left(x - x_0\right) + F\right] v. \tag{6.21}$$

However, I know that the force on the block from the spring is given by

$$F = -k \left(x - x_0\right), \tag{6.22}$$

meaning that the two terms in the brackets cancel each other out, and I find

$$\frac{dE}{dt} = 0. \tag{6.23}$$

So indeed, this total energy function is *constant* in time - it never changes throughout the motion. This principle is called *conservation of energy.*

Notice in particular that I made use of Newton's second law,

$$F = ma \tag{6.24}$$

in order to prove the above statement. So I know that it should hold in any situation in which Newton's second law holds, which is to say that it holds in any inertial reference frame. Now, it is true that according to different reference frames, the block will have different speeds. So in general, we see that the kinetic energy, and hence also the total energy, is a number that depends on which reference frame I am in. But whatever reference frame I use, and whatever value I compute for the total energy, it is always true that that number will stay the same over time.

We say that the total energy is a *conserved* quantity, but it is not an *invariant* quantity. A conserved quantity is a number which stays the same over time in any one frame, but which depends on which frame you are in. An invariant quantity is a number which is the same in all inertial reference frames, for example, the mass of the block. Any particular quantity can be one or the other, or both. The energy is conserved, but not invariant. The mass of the block in this case happens to be conserved and invariant, but that need not be the case. If the block were made of some radioactive material, the atoms in it would decay over time, and its mass would change. But at any given point in time, the mass is the same in all frames.

Of course, energy conservation is an incredibly useful tool. As an example, imagine a situation where instead of slowly compressing the spring, I give it an initial shove, so that it possesses an initial velocity, shown in Figure 6.3. Because the spring is not initially compressed, the initial energy is solely a result of the block's kinetic energy, and so we have

$$E = \frac{1}{2}mv_0^2 \tag{6.25}$$

at the beginning of the motion. Of course, because the total energy is conserved, this will always be the total energy, so that for a general position and general velocity,

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}k\left(x - x_0\right)^2. \tag{6.26}$$

I can use this information to easily answer a lot of questions. For example, if I want to know how much the spring will compress, this corresponds to the block transferring all of its kinetic energy to the spring. This occurs when the block has zero velocity, and so that maximum compression is given by

$$\frac{1}{2}mv_0^2 = \frac{1}{2}k\left(x - x_0\right)^2. \tag{6.27}$$

Rearranging and taking a square root, I find

$$\pm v_0\sqrt{\frac{m}{k}} = x - x_0, \tag{6.28}$$

or,

$$x = x_0 \pm v_0 \sqrt{\frac{m}{k}}. \tag{6.29}$$

So I see that there are two different solutions for the location of the block. This of course makes sense - I know that initially the block will compress, come to a stop, and then start uncompressing. But of course, the block is now traveling in the opposite direction and has some kinetic energy when it comes back to its starting point. Thus, it will keep going, and now it will stretch out the spring, until the spring is extended to some final stopping point. These two points where the block comes to rest are the two points given above.

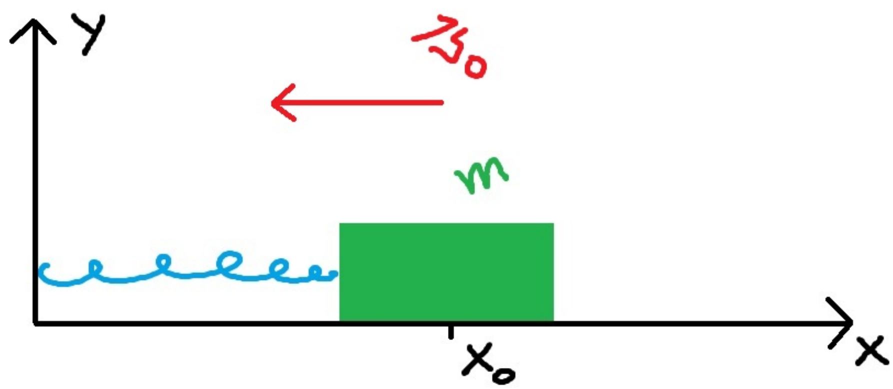


Figure 6.3: Giving a spring an initial velocity.

Keep in mind that if I wanted to find the motion of the block as a function of time, I would need to solve the differential equation

$$m\ddot{x} = -k\left(x - x_0\right). \tag{6.30}$$

While it turns out that this differential equation actually has a nice looking solution, and is not too difficult to do, it still demonstrates the idea that I can learn a lot of information about a particle's trajectory without ever needing to work out the kinematics.

6.3 Potential Energy Diagrams

There's a useful fact about the way that kinetic and potential energies behave, which helps us easily visualize what's going on in our system. Notice that by definition, the kinetic energy is always positive,

$$K = \frac{1}{2}mv^2. \tag{6.31}$$

As a result, we have

$$K \geq 0, \tag{6.32}$$

since the square of the speed can never be negative. This is the reason that the spring compressed to some distance, and then stopped compressing. In order to compress further, it would need to have more potential energy put into it. However, the potential energy must come at the expense of the kinetic energy, since their sum is conserved. Thus, because the potential energy increases as the kinetic energy decreases, and the kinetic energy cannot be less than zero, there is a maximum amount to which the potential energy can increase. This is, of course, the total energy E itself. If we rewrite the above equation in terms of total and potential energy, we have

$$E - U(x) \geq 0 \Rightarrow U(x) \leq E. \tag{6.33}$$

Now, if I think about it, this statement actually tells me something about the motion of the particle. The above statement says that the potential energy can never exceed the total energy. Since the potential energy is a function of position, this means that some regions of space are forbidden to the particle - these are regions in which the potential energy function is more than E. If the particle were to travel to these regions, its potential energy would exceed the total, which is not allowed.

To see how we can visualize these ideas, let's draw a plot of my potential energy function, shown in Figure 6.4. I've plotted the potential energy function, along with a horizontal line, equal to some value E. I'll assume that this is the total energy of my system. The point x_0 represents the equilibrium point, and the points x_i and x_f are the point which satisfy

$$U(x_i) = U(x_f) = E. \tag{6.34}$$

Specifically, this means that, taking x_i to be the smaller value, we have

$$x_i = x_0 - \sqrt{\frac{2E}{k}}, \tag{6.35}$$

as well as

$$x_f = x_0 - \sqrt{\frac{2E}{k}}. \tag{6.36}$$

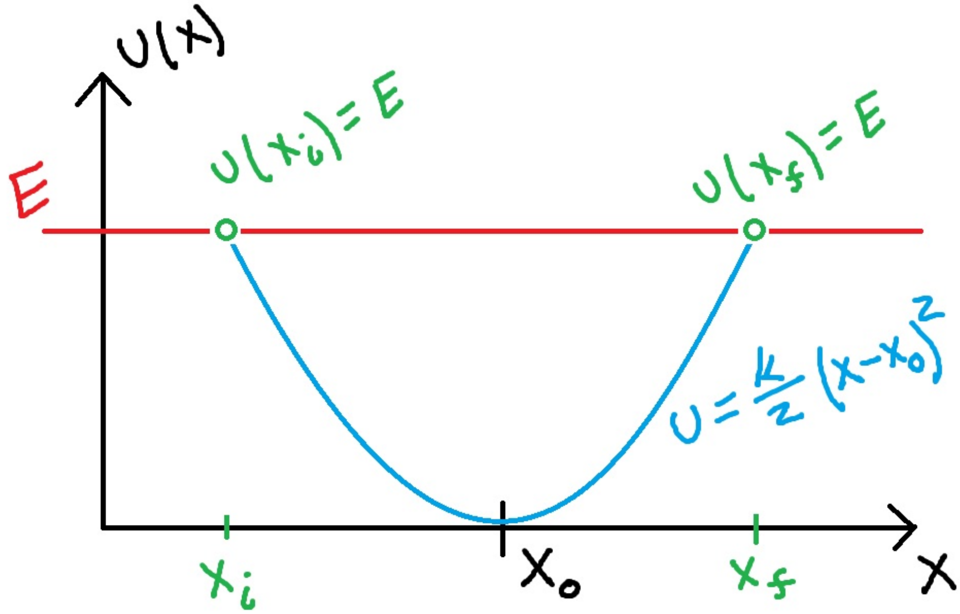


Figure 6.4: Using a potential energy diagram to understand the motion of the block under the influence of the spring force.

From the above considerations, I know the block cannot venture outside of these two points. This is because the potential energy is quadratic, and increases outside of this region. So the block cannot move beyond the region indicated, since otherwise its potential energy would exceed the total energy.

Let's see what else I can say about the motion of the block, just by looking at this potential energy plot. At the bottom of the plot, we have a minimum of the potential energy, and so the first derivative must be zero,

$$\frac{dU}{dx} = 0. \tag{6.37}$$

However, we know this is equal to the force, and so

$$F = 0 \tag{6.38}$$

at a minimum of the potential energy. If I take the block and place it at this point, and then let go, it will sit there, since there is no force acting on it. Thus, we see that places where the potential energy has a minimum correspond to equilibrium points of the system.

Let's also see what happens when I move the block to either side of the minimum. If I place the block slightly to the left, then the force is

$$F = -\frac{dU}{dx} > 0, \tag{6.39}$$

which we can say because the derivative of the plot in this region is negative, since the function is decreasing. This means that the force points in the positive direction, bringing it back to the equilibrium point. If we place the block to the right, then we have

$$F = -\frac{dU}{dx} < 0, \tag{6.40}$$

and the force points to the left, again tending to push it back to the minimum. These ideas help us to develop some intuition about what our potential energy diagram is telling us. Placing the block at the minimum will result in no motion, while displacing it slightly will tend to bring it back towards the equilibrium point.

In general, if I displace my block slightly, then it will continue to experience a force pushing it back to the equilibrium point. Once it reaches this point, the force will begin to oppose its motion. This will continue to occur until the block loses all of its kinetic energy, and the potential energy is again equal to E. The motion then reverses. So we see in general, the block will oscillate back and forth between the two end points of the motion, x_i and x_f.

Because of these behaviors, it is common to make an analogy where we liken this to the rolling of a ball down a "hill." If I were to imagine taking a parabolic surface, shaped like my potential energy plot, and placing a ball on it, when I let go, it would start rolling down towards the bottom of the well, and then roll up the other side, eventually oscillating back and forth. This is a handy tool for thinking about what happens in a potential energy function, but remember that there is a crucial distinction between these two cases. In the case of the ball, I am considering motion throughout two dimensional space, and my axes are x and y, two directions of space. In the case of the potential energy function I am considering, it is really just a one dimensional problem - the block only moves in one dimension, oscillating back and forth between x_i and x_f. This point can cause lots of confusion, if it is not fully appreciated.

We can actually say something even more specific than this. Let's imagine that I take my block and move it to x_i, where it sits still. Once I let go, the block will move to the right. Using the relation between kinetic and potential energy, I can write

$$\frac{1}{2}mv^2 = E - U(x), \tag{6.41}$$

or

$$v = \pm\sqrt{\frac{2}{m}\left(E - U\left(x\right)\right)}. \tag{6.42}$$

Let me consider first the motion from x_i to x_f. Because the block is moving to the right, the velocity is positive, and I have

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}\left(E - U\left(x\right)\right)}. \tag{6.43}$$

If I consider this to be a differential equation involving the position, then after using the method of separation, I find

$$\int_0^{T_1} dt = \int_{x_i}^{x_f} \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U\left(x\right)}}, \tag{6.44}$$

where T_1 is the time it takes to travel from x_i to x_f. Thus,

$$T_1 = \sqrt{\frac{m}{2}} \int_{x_i}^{x_f} \frac{dx}{\sqrt{E - U\left(x\right)}}. \tag{6.45}$$

After the block makes it to the right, it will move back to the left, and since its velocity is now negative, we have

$$T_2 = -\sqrt{\frac{m}{2}} \int_{x_f}^{x_i} \frac{dx}{\sqrt{E - U\left(x\right)}} \tag{6.46}$$

as the time it takes to move from right to left. However, I can get rid of the minus sign by flipping the order of integration, and I get

$$T_2 = \sqrt{\frac{m}{2}} \int_{x_i}^{x_f} \frac{dx}{\sqrt{E - U\left(x\right)}} = T_1. \tag{6.47}$$

So already, I have another useful conclusion - the time it takes to move from left to right is the same as the time to move from right to left. Notice that this doesn't make any assumption about the shape of the potential. But I can actually do better than this. The full period of oscillation between the two points is

$$T = T_1 + T_2 = \sqrt{2m} \int_{x_i}^{x_f} \frac{dx}{\sqrt{E - U\left(x\right)}}. \tag{6.48}$$

The important conclusion here is that I can find the period of oscillation just by integrating a function that depends only on the *shape* of the potential energy function, in between the two end points. I don't need to know anything about the intermediate kinematics of the motion.

As an application of this, let's find the period of an oscillating spring. For a given energy, we know the form of x_i and x_f in terms of the energy E, and so we have

$$T = \sqrt{2m} \int_{x_0 - \sqrt{2E/k}}^{x_0 + \sqrt{2E/k}} \frac{dx}{\sqrt{E - (k/2)(x - x_0)^2}}. \tag{6.49}$$

After doing a little rearranging on this expression, and making some changes of variables in the integral, I can get this expression into the form

$$T = 2\sqrt{\frac{m}{k}} \int_{-1}^{1} \frac{du}{\sqrt{1 - u^2}}. \tag{6.50}$$

Now, the value of this integral turns out to be

$$\int_{-1}^{1} \frac{du}{\sqrt{1 - u^2}} = \pi, \tag{6.51}$$

and thus

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{6.52}$$

This clearly represents an enormously powerful set of tools. I have qualitatively analyzed the motion of the block over its entire range of motion, and even said something exact about its period of oscillation. And I never even had to actually solve for the motion! Notice that in the above expression for the period, there is no dependence on the total energy. This is a unique feature of the harmonic oscillator potential, and is not true for a generic potential.

6.4 Potential Energy in General

Having discussed these ideas extensively for the case of an oscillating spring, it would be nice to generalize them. For any one dimensional system in which the force on the object in question can be written as a function of position alone, we simply define the potential energy as

$$U(x) = -\int_{x_0}^{x} F(x')\, dx'. \tag{6.53}$$

Again, the choice of x_0 is up to me, and does not affect any of the physics.

All of the general claims above will still be true. In particular, we still have

$$F(x) = -\frac{dU}{dx}. \tag{6.54}$$

Also, the total energy, now defined in general to be

$$E = \frac{1}{2}mv^2 + U(x), \tag{6.55}$$

will still be a constant. This is easy to see, since

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{dU}{dt} = mv\frac{dv}{dt} + \frac{dU}{dx}\frac{dx}{dt} = [ma - F]\,v = 0. \tag{6.56}$$

As a result, all of the useful features of potential energy diagrams will still be true, even for arbitrarily funny looking potential energy functions.

Now, for an object moving under an arbitrary force, I might ask the question: "where" is the potential energy stored? In the case of the block oscillating on a spring, the force acting on the block was a result of the spring, and we ascribed the potential energy to the change in the physical nature of the spring. However, it may not always be so clear what the object is that's storing the potential energy. As an example, let's imagine that instead of being attached to the wall by a spring, that my block has a small amount of electric charge on it, labeled by q, and the wall has a small patch of charge where the spring was attached, equal to Q. This is shown in Figure 6.5. Coulomb's law tells me that the force acting on the block is given by

$$F = \frac{1}{4\pi\epsilon_0}\frac{qQ}{x^2}\hat{x}, \tag{6.57}$$

where ϵ_0 is a constant of nature called the *permittivity of free space*. The direction of the force will depend on the signs of q and Q.

Now, again, I can define a potential energy associated with the location of the block,

$$U(x) = -\frac{qQ}{4\pi\epsilon_0}\int_{x_0}^{x}\frac{dx'}{(x')^2} = \frac{qQ}{4\pi\epsilon_0}\frac{1}{x'}\bigg|_{x_0}^{x}, \tag{6.58}$$

where x_0 is some conveniently chosen reference point. If I take this to be infinitely far away, I recover

$$U(x) = \frac{qQ}{4\pi\epsilon_0}\frac{1}{x}. \tag{6.59}$$

Now, the question I naturally ask is, what is this energy, physically? Well, in this case it's not so easy to say. The charge on the wall sits still, and nothing

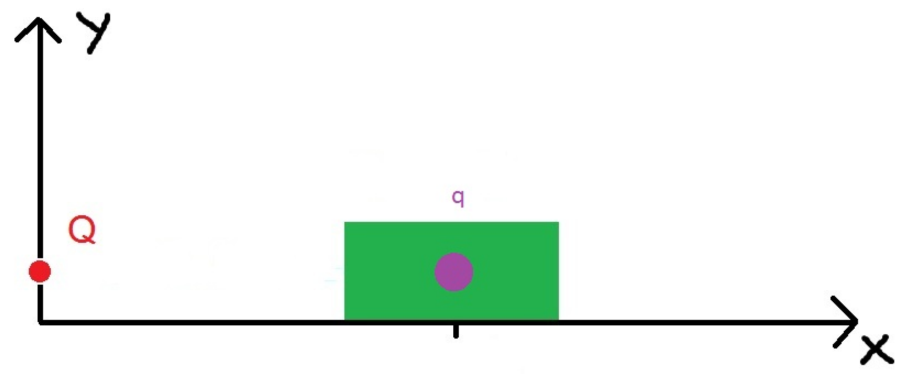


Figure 6.5: In general, it may not be so clear what is "storing" the potential energy in a certain situation.

is obviously different about the system, other than the position of the block. As you will learn in a class on electromagnetism, we typically invent an abstract object known as an "Electric Field," which exists throughout space, and contains energy. Under the right circumstances, this field can change over time, and "move through space," which is what an electromagnetic wave is.

However, the useful feature of potential energy is that I can still *define* a potential energy function mathematically, and leave the physical nature of the energy as a question for another day. So long as I can write the force on the block as a function of its position, then I can define a potential energy, and use all of tips and tricks I've developed here to understand its motion.

6.5 Higher Dimensions and Gravitational Potential

Now, I would like to generalize this to higher dimensions. Let's imagine I want to derive an expression for a gravitational version of potential energy. Analogously to the way that I defined potential energy in one dimension to be the work done on the system while moving along a path, I might do the same in two or three dimensions. This is sketched in Figure 6.6. Since the gravitational force depends only on my location (trivially, since it is just a constant), then I expect I should be able to do this. I therefore propose that

$$U_g\left(\vec{r}\right) = -\oint_C \vec{F} \cdot d\vec{r}. \tag{6.60}$$

My fancy new notation emphasizes that this is a line integral performed over some curve C. I've defined the line integrals to start at the origin of some coordinate system I've set up, but this is only one of many possible choices.

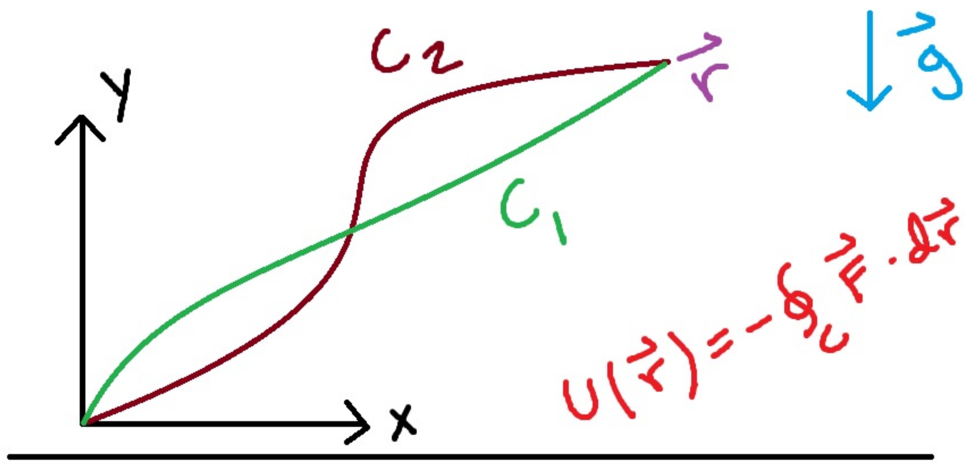


Figure 6.6: Defining potential energy for an arbitrary vector field.

Now, for gravity, I know that I have

$$\vec{F} = -mg\hat{y}. \tag{6.61}$$

If I use this in my definition, and I consider traversing a path which points straight upwards, then I have

$$U_g(\vec{r}) = mg \oint_C \hat{y} \cdot d\vec{r} = mg \int_0^h dy = mgh. \tag{6.62}$$

Therefore, my gravitational potential energy is just a linear function of height (of course, since I can add a constant to this if I want, I can take $h = 0$ to be located wherever I want). Using this expression, I can answer many of the same questions as before. For example, if I fire a bullet into the air with some initial kinetic energy, I can figure out how high it travels by imposing the condition that total energy is conserved, and then solving for the maximum height in a similar fashion. Because the potential energy only depends on height, effectively I again have a one dimensional problem.

6.6 A Subtle Problem

However, there is actually a potential issue here (no pun intended). In one dimension, there was only one way to get between any two points - all you can do is move in a straight line. But in more than one dimension, there are actually a lot of ways I can travel to get to a point. This is also demonstrated in Figure 6.6, where more than one path is drawn. For the case of gravity, my line integral was sufficiently simple to do because the vector field was constant and only pointed in one direction. But for a more general force, I would need to evaluate

$$U(\vec{r}) = -\oint_C \vec{F} \cdot d\vec{r} = -\int \vec{F} \cdot \vec{v}\, dt. \tag{6.63}$$

I'm now faced with an important question: does the value of the line integral depend on which path I take? If it does, then I have a problem - because the value of the line integral will depend on which path I take, I can no longer write a simple potential energy function that just depends on my location in space. Notice that while I told you that a line integral does not depend on how quickly the path is traversed, I didn't make any promises about how it depends on the **path** you take between the two endpoints.

It turns out that sometimes the line integral will depend on the path, and sometimes it won't. For the case of gravity, it turns out that no matter how I perform the line integral, I will always get the same answer, so it is possible to define a gravitational potential energy which only depends on location in space. But sometimes I cannot do this. It would be nice to be able to look at the form of the force as a function of position, and know when this is the case. While it is beyond the scope of this class to prove this, it turns out that the line integral will not depend on the path so long as

$$\vec{\nabla} \times \vec{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} = 0. \tag{6.64}$$

This object is called the curl of the vector field, and if the above equation is true, then it is possible to define a potential energy function. In this case, we often say that the vector field is "curl free." When this works, it turns out that the force is given in terms of the potential energy function as

$$\vec{F} = -\vec{\nabla} U = \left(\frac{\partial F_x}{\partial x} \right) \hat{x} + \left(\frac{\partial F_y}{\partial y} \right) \hat{y} + \left(\frac{\partial F_z}{\partial z} \right) \hat{z}. \tag{6.65}$$

This operation is called the gradient of the function U, and it is a generalization of the derivative to higher dimensions. For those of you not familiar with the funny looking derivative symbols above, they are called partial derivatives. They tell me that I should take a derivative of the object in question with respect to the given coordinate, while pretending that all of the other coordinates are held fixed. So for example,

$$\frac{\partial}{\partial x}\left(x^3 \sin y\right) = 3x^2 \sin y, \tag{6.66}$$

while

$$\frac{\partial}{\partial y}\left(x^3 \sin y\right) = x^3 \cos y. \tag{6.67}$$

For those of you in CCS Physics, you'll end up understanding how to prove these statements once you take the class on vector calculus. While these considerations may seem merely academic, they are actually very important in many areas of physics. For example, it turns out that static electric fields are always curl free and can always be associated with a potential energy function, whereas magnetic fields are never curl free, and can never be associated with a potential energy function.

That concludes our discussion of work and energy, and also the first week of classes. Next week, we'll start off by introducing the concept of momentum.

Chapter 7

Momentum

7.1 Momentum

Last week, we introduced the ideas of potential and kinetic energy, and talked about situations in which their sum, the total energy, was conserved. We found that we could do this when there weren't any nonconservative forces involved in the system, such as friction. Today, I'm going to introduce another concept, momentum, which can be very helpful in solving some problems, especially ones which involve nonconservative forces.

Let's consider a system where two blocks of the same mass slide towards each other on a frictionless table, with equal but opposite velocities. This is shown in Figure 7.1. Let's also imagine we've applied some tape to the sides of the blocks, so that they'll stick together. Once the two blocks hit, we find that as soon as they stick together, they come to a stop, and then their taped-together collective mass sits still on the table.

Clearly, the kinetic energy of the two blocks is not conserved here. The initial kinetic energy of the two blocks was mv^2, while after the collision it is zero. The gravitational potential energy (were this to occur on the Earth) is the same before and after, so the total sum of kinetic plus potential energy is not conserved either. Of course, this is due to the presence of a nonconservative force in our system, which we can not write in terms of a potential energy - the tape causes the two blocks to stick together. The reason for this has to do with the complicated chemical interactions of the tape molecules at short distances, which goes beyond the scope of this course.

However, we do notice something else. The total velocity before and after is conserved. The vector sum of the velocity of the left block and the velocity of the right block always add together to equal zero. Maybe this implies some kind of conservation of total velocity of bodies. However, experience shows that this

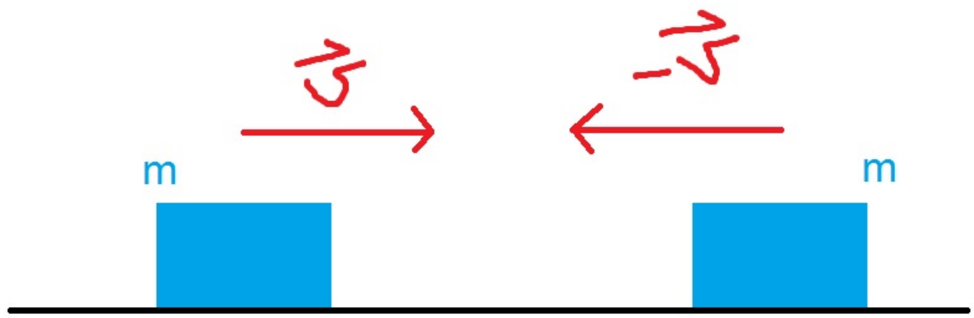


Figure 7.1: Two blocks which violate kinetic energy conservation but which preserve total momentum.

cannot be true in general. We know that if we are driving down the highway and hit a bug, it does not bring our car to a stop. We have the intuitive sense that this is because our car is much more massive. Even though the bug and the car exert the same force on each other (according to Newton's third law), the car keeps moving more or less unaffected, while the same is not true for the bug. With this idea in mind, let's define a simple looking quantity which might capture these two ideas of mass and velocity being important in collisions, by defining

$$\vec{p} = m\vec{v} \tag{7.1}$$

to be the *momentum* of a body.

In the above example, momentum was conserved, since it was zero both before and after the collision. Will this be true for two bodies with arbitrary masses and velocities? It turns out that it will be, which can be shown pretty easily using Newton's second and third law. Newton's second law for a given body says that

$$\vec{F} = m\vec{a}, \tag{7.2}$$

which we can write as

$$\vec{F} = m\frac{d\vec{v}}{dt}. \tag{7.3}$$

Because the mass is constant, we can pull it into the time derivative, and we find

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}. \tag{7.4}$$

Now, for two bodies interacting with each other, which we will call A and B, we know that in general, they must satisfy Newton's third law,

$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}. \tag{7.5}$$

If we define the total momentum of the two bodies to be

$$\vec{p}_T = \vec{p}_A + \vec{p}_B, \tag{7.6}$$

and we assume that the only forces on each body are due to the other one, we see that

$$\frac{d\vec{p}_T}{dt} = \frac{d}{dt}\left(\vec{p}_A + \vec{p}_B\right) = \frac{d\vec{p}_A}{dt} + \frac{d\vec{p}_B}{dt} = \vec{F}_{\text{B on A}} + \vec{F}_{\text{A on B}} = \vec{F}_{\text{B on A}} - \vec{F}_{\text{B on A}} = 0. \tag{7.7}$$

So total momentum is conserved in this situation of two bodies colliding.

7.2 Collisions

The result above - that the momentum of two colliding bodies is conserved - was a nice result which followed simply from Newton's second and third laws. So long as we believe these two laws to be true, then we can use this result in a wide variety of cases, even if we don't fully understand the nature of all of the complicated interactions going on between the two objects. A good example of an application of these ideas is studying collisions between two massive bodies.

In general, we usually speak of three types of collisions between bodies. The first type of collision, elastic scattering, happens when two bodies collide off of each other in such a way that they conserve their kinetic energy. This usually involves the two bodies doing minimal "sticking" to each other, which says that none of their kinetic energy goes into deforming the material properties of the bodies or heating them up in ways that involve complicated microscopic interactions. The second type is called inelastic scattering, which occurs when some of the kinetic energy is lost. This can be due to several mechanics, quite often frictional heating. The extreme case of this is referred to as completely inelastic scattering, and it is when two bodies completely stick together to form one body.

As an example of these ideas, let's consider the case when two blocks stick to each other in more generality. Consider the case shown in Figure 7.2. Let's assume that the velocity of block A is larger than that of block B, so that block A will eventually collide with block B, and then stick to it. Before the collision, the total initial momentum of the system is

$$\vec{p}_i = m_A\vec{v}_A + m_B\vec{v}_B. \tag{7.8}$$

Now, after they stick, the two blocks will move together with some combined velocity, which we will call $\vec{v}_{AB}$, and their total mass will be

$$M = m_A + m_B. \tag{7.9}$$

Thus, their final momentum will be

$$\vec{p}_f = M\vec{v}_{AB} = (m_A + m_B)\,\vec{v}_{AB}. \tag{7.10}$$

Since we know that momentum will be conserved, the final and initial momentum must be the same, and so we can say that

$$m_A\vec{v}_A + m_B\vec{v}_B = (m_A + m_B)\,\vec{v}_{AB}. \tag{7.11}$$

This is an incredibly useful statement, since it lets us find the final velocity of the lump formed out of the two blocks,

$$\vec{v}_{AB} = \frac{m_A\vec{v}_A + m_B\vec{v}_B}{m_A + m_B}. \tag{7.12}$$

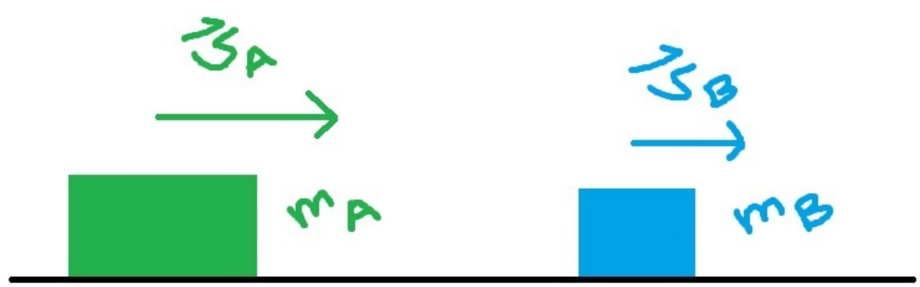


Figure 7.2: Two blocks which will collide and stick together.

In general, we cannot say whether or not a collision will be elastic or inelastic without knowing more detailed information about the interactions between the bodies. Thus, in a totally arbitrary collision we don't know anything about beforehand, we can't apply conservation of kinetic energy, since we have no idea if the collision will actually be elastic. However, so long as two bodies interact with each other on their own, we can always apply momentum conservation, since it follows straight from Newton's second and third laws.

7.3 Momentum Conservation In General

In the above example, we were able to invoke Newton's laws to derive the conservation of momentum for two particles only interacting with themselves. Can we generalize this idea to larger systems? We can, and it will turn out to give us a lot of insight into some of the physics we've been doing up until now.

Let's imagine that I have a collection of particles (N of them to be precise), and they all interact with each other through forces that obey Newton's laws. I'll use the notation $\vec{F}_{ji}$ to indicate the force on particle i from particle j, which we know satisfies

$$\vec{F}_{ji} = -\vec{F}_{ij}. \tag{7.13}$$

I will also write the total momentum of the collection as

$$\vec{p}_T = \sum_{i=1}^{N} \vec{p}_i, \tag{7.14}$$

where $\vec{p}_i$ is the momentum of particle i. Now, the change in total momentum is given by

$$\frac{d\vec{p}_T}{dt} = \sum_{i=1}^{N} \frac{d\vec{p}_i}{dt}. \tag{7.15}$$

Now, from what we saw above, the change in momentum of particle i is given by the force acting on it, which will be

$$\frac{d\vec{p}_i}{dt} = \vec{F}_{\text{ext},i} + \sum_{j=1}^{N} \vec{F}_{ji}, \tag{7.16}$$

where the sum on j is over all of the particles in the system. Notice that we can write it this way with the understanding that

$$\vec{F}_{ii} = 0, \tag{7.17}$$

which is to say that particle i does not exert any forces on itself. The vector $\vec{F}_{\text{ext},i}$ is taken to be the net sum of all forces acting on particle i which come from outside of the collection of N particles. Our expression for the change in momentum tells us

$$\frac{d\vec{p}_T}{dt} = \sum_{i=1}^{N} \left[\vec{F}_{\text{ext},i} + \sum_{j=1}^{N} \vec{F}_{ji} \right] = \vec{F}_{\text{ext}} + \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{F}_{ji}, \tag{7.18}$$

where $\vec{F}_{\text{ext}}$ is defined as the total collection of external forces acting on the collection of N particles from outside of the system.

Now, because of Newton's third law, the forces two bodies exert on each other are equal and opposite. Thus, when we write out the double sum above, the forces will always come in pairs, and they will cancel out. For example, with three particles,

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\vec{F}_{ji} = \sum_{i=1}^{N}\left[\vec{F}_{1i}+\vec{F}_{2i}+\vec{F}_{3i}\right] = \vec{F}_{21}+\vec{F}_{31}+\vec{F}_{12}+\vec{F}_{32}+\vec{F}_{13}+\vec{F}_{23}, \quad (7.19)$$

where I already made use of the fact that the force from a particle on itself is zero. We can rewrite this as

$$\sum_{i=1}^{3}\sum_{j=1}^{N}\vec{F}_{ji} = \vec{F}_{21}+\vec{F}_{31}-\vec{F}_{21}+\vec{F}_{32}-\vec{F}_{31}-\vec{F}_{32} = 0. \quad (7.20)$$

To be more general, we can use Newton's third law to write the double sum as

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{ji} = -\sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{ij}. \quad (7.21)$$

If we change the names of the dummy integration indices on the right, we find

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{ji} = -\sum_{j=1}^{N}\sum_{i=1}^{N}\vec{F}_{ji}. \quad (7.22)$$

Lastly, if we swap the order of summation on the right, we find

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{ji} = -\sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{ji} \Rightarrow \sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{ji} = 0. \quad (7.23)$$

In any event, because the double sum always cancels, we find that

$$\frac{d\vec{p}_T}{dt} = \vec{F}_{\text{ext}}. \quad (7.24)$$

This tells us that the change in total momentum of a collection of particles is equal to the sum of the external forces acting on all of those particles. If the collection of particles is totally isolated, then its total momentum is conserved. Notice that this result was derived solely on the basis of Newton's second and third laws - no other knowledge of the detailed interactions or physics was needed.

7.4 Center of Mass

This realization that the time derivative of total momentum only depends on the external forces helps us to define some new concepts which will allow us to understand some of the physical assumptions we've previously been making. Let's define a new object, defined according to

$$\vec{R} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i, \tag{7.25}$$

and call it the *center of mass*, where M is the total mass. It is an average of the positions of the particles, weighted by their masses.

Let's also define something called the *center of mass velocity*, which is the time derivative of the center of mass

$$\vec{v}_{CM} = \frac{d\vec{R}}{dt} = \frac{1}{M} \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{v}_i = \frac{1}{M} \sum_{i=1}^{N} \vec{p}_i. \tag{7.26}$$

From this result, we see that the center of mass velocity is related to the total mass and total momentum of the system,

$$\vec{p}_T = M\vec{v}_{CM}. \tag{7.27}$$

If we take a second time derivative, we can define the acceleration of the center of mass, and we have

$$\frac{d\vec{p}_T}{dt} = M\vec{a}_{CM}. \tag{7.28}$$

However, from the previous section, we know that the change in total momentum is given by the total external force, and so we have

$$\vec{F}_{\text{ext}} = M\vec{a}_{CM}. \tag{7.29}$$

This result tells us that the total mass times the acceleration of the center of mass is given by the total external force.

This result actually helps us to explain some subtleties of what we've been doing up until now. I've frequently been making reference to "complicated microscopic forces" that occur inside of a body. I've cited this as the reason for why a coffee cup doesn't fall through a table, and also for why friction and sticky collisions don't preserve kinetic energy. But if such a complicated microscopic world makes up all of these material bodies, how can I reliably apply Newton's laws to any sort of large bodies? I keep talking about applying forces to large objects like blocks and carts, without worrying at all about the shape or physical deformations of the body, and how this affects the way I interact with them. Why have I been able to treat my block like a single point object, and talk about its location and acceleration, without specifying details about the orientation of the body?

The answer to these questions is that really, when I talk about these large physical objects obeying Newton's second law, what I'm really saying is that the *center of mass* of these large composite bodies obey Newton's laws, using the total external force, and total mass. It doesn't matter precisely how I handle the block, so long as I know what the net force is that I'm applying over all of the atoms which make up its surface. Furthermore, once I know the total external force, I know this is equal to the total mass, times the acceleration of the center of mass. So really, what I am talking about here is the mass-weighted average of all of the positions in the body. When I take an eraser and throw it across the room, it may spin and rotate and move in a weird way, but the *center of mass* is the location in the body which will move according to the projectile motion formulas we derived. Even if the projectile does something incredibly violent like explode in the middle of flight, if we were to look at the constituent pieces, their center of mass would still move along the usual projectile path. These ideas are demonstrated in Figure 7.3.

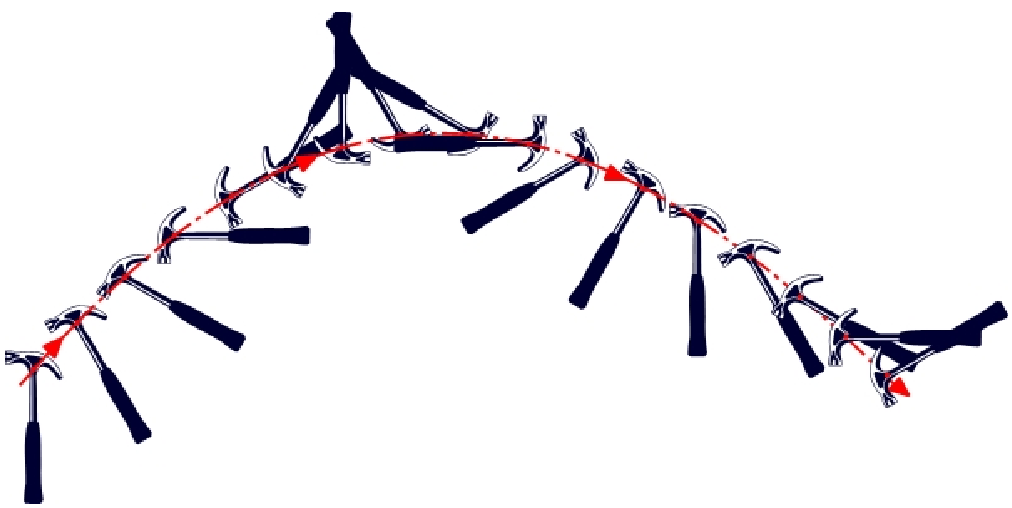


Figure 7.3: A hammer undergoing projectile motion. While it may rotate and generally move in some complicated way, the center of mass continues to follow the path we previously found for projectile motion.

Newton's first law of course is also true, since it is a special case of the second law - if the total external force on the object is zero, then its center of mass will not accelerate. Of course, we know that in many cases, a physical body will still react to forces on them, even when the total is zero. If I squeeze an object inwards so that the total force is zero, I can still change the physical size and shape of the object. But its center of mass will not accelerate.

7.5 Impulse

When the mass of an object is constant, then we've seen that we can write Newton's second law in two forms,

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}. \tag{7.30}$$

However, when the mass changes in time, these two expressions are no longer the same. So the question is, which IS the correct expression, in general? Well, it turns out I've been lying to you somewhat - the second form is actually the one that is correct in full generality. When the mass of an object changes, I can still use the second form, but I cannot use the first form.

In particular, this means that the change in momentum is always equal to the integral of the force over time,

$$\vec{p}(t_2) - \vec{p}(t_1) = \int_{t_1}^{t_2} \vec{F}(t)\, dt. \tag{7.31}$$

We define this object to be something called the *impulse*,

$$\vec{J} \equiv \int_{t_1}^{t_2} \vec{F}(t)\, dt. \tag{7.32}$$

This equation can sometimes be useful in situations in which we know the total amount of force applied as a function of time.

7.6 Rocket Motion

There are several important situations in which using momentum considerations is crucial, because we are considering a system whose mass is changing. Consider the case of a rocket powering itself by emitting exhaust. In this situation, as the rocket dumps fuel, its mass is changing. This is indicated in Figure 7.4. With respect to some particular inertial reference frame that we've set up out in space, the velocity of the rocket is v_R, and the velocity of the fuel coming

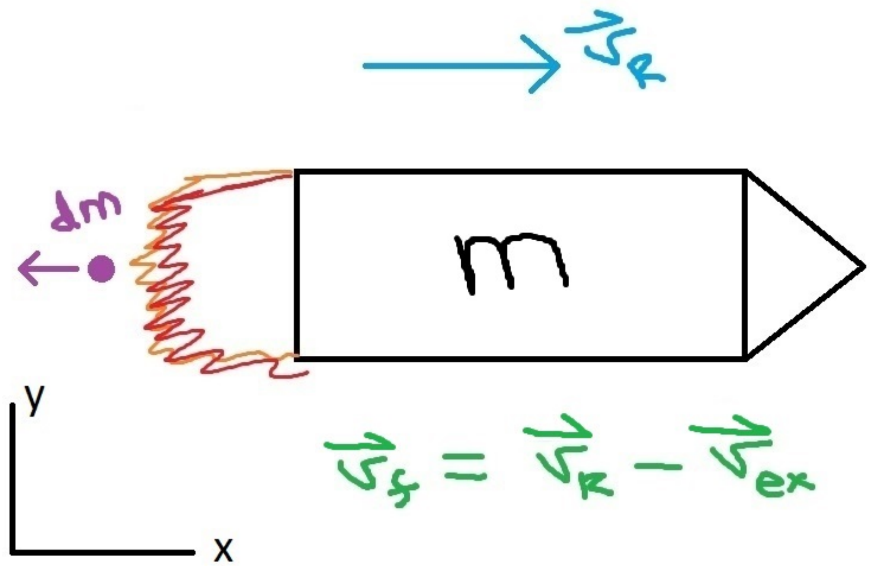


Figure 7.4: A rocket which is accelerating as a result of burning fuel.

out the back, at any particular instant in time, is v_f. Notice that both of these quantities, with respect to the inertial reference frame, will change over time as the rocket accelerates.

Now we want to know - how does burning fuel affect the motion of the rocket? We'll work in one dimension, for simplicity, so that we will often omit the vector symbols on the relevant velocities - we can always do this if the exhaust is emitted straight out the back (keep in mind, however, that the velocities are still signed quantities). Let's assume that at any given moment, the mass of the rocket is some value m. As the rocket expels fuel, the mass of the rocket will change, but we should still be able to apply momentum conservation. We assume that the infinitesimal change in the rocket's mass is dm. If its infinitesimal change in velocity is dv_R, then the rocket's new momentum, after it has emitted the piece of fuel, will be

$$p_R = (m + dm)(v_R + dv_R) . \tag{7.33}$$

Now, consider the infinitesimal piece of fuel that is emitted during this process. The mass of this piece of fuel will be the negative of the change in the rocket mass, and the momentum it carries away with it is given by

$$p_f = -dm \ v_f , \tag{7.34}$$

where v_f is the velocity of the emitted fuel with respect to whatever inertial frame we are using to measure velocities out in space. However, for someone who is on-board the rocket operating its engines, a more natural quantity to consider is the velocity of the fuel with respect to the rocket, since as the rocket operator adjusts the rate of fuel being emitted from the engines, this is the quantity that he can directly adjust. If we use the Galilean velocity transformation formula to write the velocity of the fuel in terms of the velocity of the rocket, emphasizing with our notation that v_f and v_R are with respect to an inertial frame, we have

$$v_f \equiv v_{fI} = v_{fR} + v_{RI} \equiv v_R + v_{fR}. \tag{7.35}$$

It is often convention, however, to work instead with the **exhaust velocity**

$$v_{ex} = -v_{fR}. \tag{7.36}$$

The reason for this is that because fuel is being ejected backwards out of the rocket, v_{fR} will be negative, while v_{ex} will be positive. This is demonstrated in Figure 7.5. Using this velocity transformation, we can write

$$p_f = -dm\left(v_R - v_{ex}\right). \tag{7.37}$$

The total momentum of the rocket and the emitted piece of fuel is thus

$$p_T = (m + dm)\left(v_R + dv\right) - dm\left(v_R - v_{ex}\right). \tag{7.38}$$

If we expand out this expression, we find that

$$p_T = mv_R + mdv + v_{ex}dm + dmdv. \tag{7.39}$$

Now, if we are taking the limit that dm and dv_R both become infinitesimally small, then the last term becomes unimportant compared with the other ones, since it is quadratic in small quantities. Thus, we drop it, and write

$$p_T = mv_R + mdv + v_{ex}dm. \tag{7.40}$$

Now, because there are no other external forces acting on the system, the net change in momentum must be zero. Before the infinitesimal piece of mass was emitted, it was sitting on the rocket, and the momentum of the two was simply mv_R. Thus, we have

$$mv_R = mv_R + mdv + v_{ex}dm, \tag{7.41}$$

or

$$mdv = -v_{ex}dm. \tag{7.42}$$

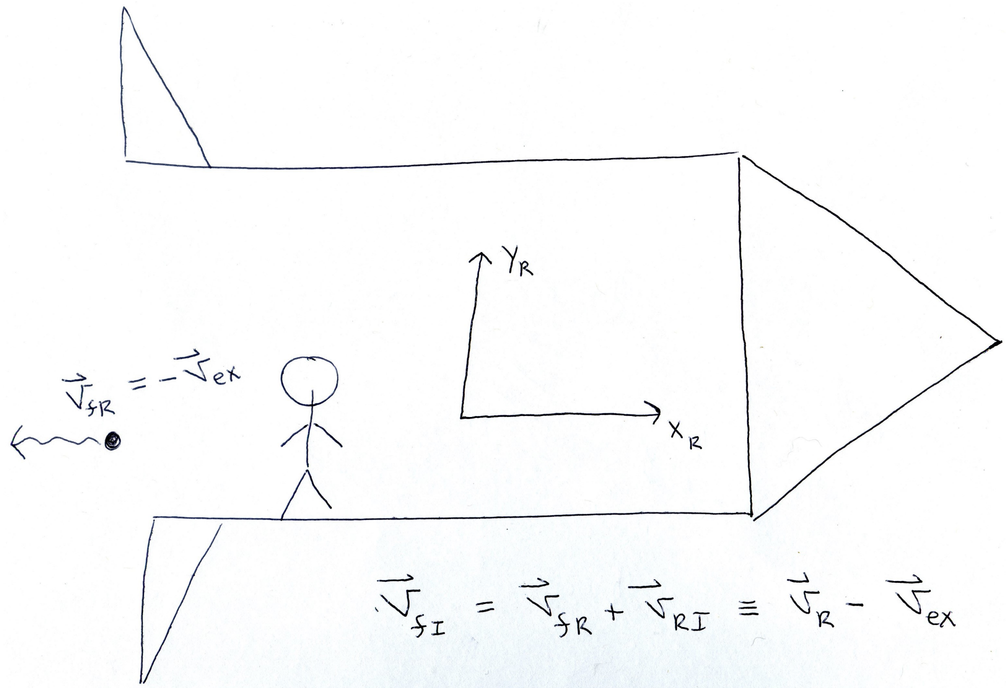


Figure 7.5: The fuel being exhausted from the perspective of an observer on the rocket.

The above expression can be used to find an equation for the velocity as a function of mass. If we rearrange, we find

$$dv = -v_{ex}\frac{dm}{m}. \tag{7.43}$$

If we assume the rocket has some initial mass m_0 and some initial velocity v_0, then we can write

$$\int_{v_0}^{v_R} dv' = -v_{ex}\int_{m_0}^{m} \frac{dm'}{m'}, \tag{7.44}$$

which becomes

$$v_R - v_0 = v_{ex}\ln\left(\frac{m_0}{m}\right). \tag{7.45}$$

So we see that the change in velocity of the rocket depends only on the exhaust speed of the fuel, and the ratio of the original mass to the new mass. Momentum conservation makes this a very easy result to arrive at.

That concludes our overview of the topics which are typically covered in Physics 20. Tomorrow, I'll give you a brief introduction to a more advanced tool in physics, Lagrangian mechanics.

Chapter 8

Advanced Methods: Lagrangian Mechanics

8.1 The Calculus of Variations

So far in this class, we've used a variety of different concepts - Newton's laws, free-body diagrams, work, kinetic energy, potential energy, and momentum. These have all been useful in different situations, depending on the particular physics involved. But really, they all stem from the same basic ideas. Whenever I showed some new result, my derivation always invoked one of Newton's laws. For example, when I showed that the sum of kinetic and potential energy was constant, I invoked Newton's second law,

$$\vec{F} = m\vec{a}, \tag{8.1}$$

in order to show that certain terms were zero. So really, all I've been doing up until now is taking the same information, and repackaging it in different forms. The reason for this is because sometimes different situations are best tackled from different angles, and thinking about Newton's laws in different ways sometimes lets us take a shortcut when it comes to getting to our final answer.

In this spirit, today I'm going to tell you about yet another way to think about Newton's laws. This idea is a little bit different though - at first it might seem like an unusual path to take when thinking about physics, but it will turn out to be one of the most useful ways to tackle physics problems. And, as Richard Feynman himself said, "every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics. He knows that they are all equivalent, and that nobody is ever going to be able to decide which one is right at that level, but he keeps them in his head, hoping that they will give him different ideas for guessing."

What I want to consider first is a math question. Let's imagine I have some function $y(x)$, shown in Figure 8.1. What is the length of this curve connecting the two points? Well, I know that I should be able to write an infinitesimal piece of length of the curve as

$$dl = \sqrt{dx^2 + dy^2}. \tag{8.2}$$

If I do a little rearranging with the infinitesimal pieces, I find

$$dl = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \tag{8.3}$$

With this expression, I can integrate from x_a to x_b, and find the total length

$$l = \int dl = \int_{x_a}^{x_b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \tag{8.4}$$

Now, if I happen to know the form of the function, I can go ahead and easily compute this, which is a common problem in introductory calculus. But what if I ask a totally different question - out of all possible functions, which function *minimizes* the length between the two points?

You might intuitively guess that this is a straight line, and of course this is correct. But as a mathematical question, it is an interesting one to consider - how would I "prove" this? This is a much more subtle question than the usual minimization questions we consider in calculus. In those cases, we have a given function, and we want to know at what point the function is minimized. So what I ask in that situation is, out of all numbers on the real line, which one corresponds to the point where the function is at a minimum? The question I am asking here is much different - I want to know, out of the space of all possible functions that exist, which one minimizes this integral.

The subject of mathematics which addresses this is called the Calculus of Variations. While it may seem overkill to introduce a whole new subject of mathematics to see that the shortest distance between two points is a line, it turns out that there are many other situations in which we want to ask a similar question, and the answer is not nearly so obvious. For example, if I have a rope hanging from a ceiling, what shape will it hang in order to minimize its potential energy? In that case, the potential energy of the rope is the potential energy of all of the little pieces dm under the influence of gravity, and we have

$$U = \int gy\ dm = g\rho \int y\ dl = g\rho \int_{x_a}^{x_b} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \tag{8.5}$$

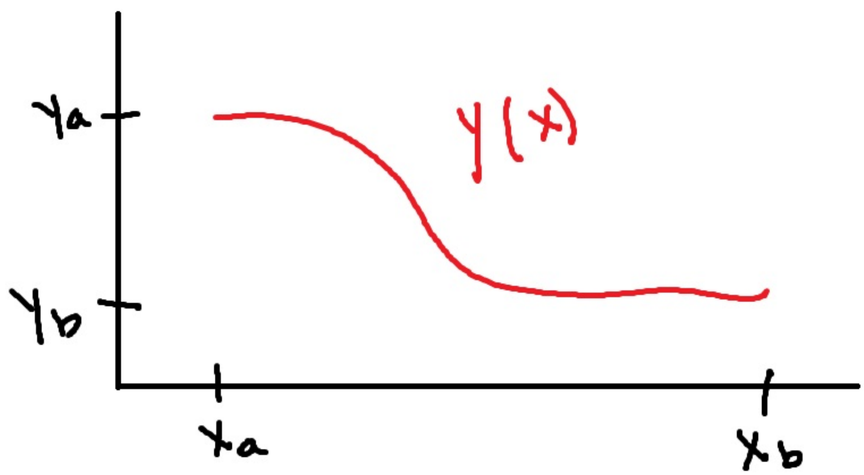


Figure 8.1: A curve connecting two points, whose length we can compute if we perform the appropriate integration.

where ρ is the linear mass density of the rope, y is the height of the rope as a function of x, and x_a and x_b are the endpoints that the rope is tied at. It turns out that in this case the shape of the function is something called a hyperbolic cosine, which isn't exactly as "obvious" as the straight line answer we proposed before.

I can imagine other situations too. What if I have some sort of elastic membrane that I'm using in an engineering application, and I want to know what shape will result in the least amount of tension? Or maybe I need to minimize the amount of time it takes for a certain mechanical process to occur. All of these cases mentioned so far typically involve a situation where I have a quantity which is an integral, and I want to know which function will minimize the value of that integral. It is common to say that what we are minimizing is a *functional* - something which takes a function as input, and returns a number. This is to contrast them with normal functions.

With these facts in mind, I am now going to imagine an arbitrary integral which involves a function $y(x)$, its derivative, and x itself. I write this schematically as

$$I = \int_{x_a}^{x_b} \mathcal{L}\left(x, y, y_x\right) dx, \tag{8.6}$$

where I've defined the notation

$$y_x \equiv \frac{dy}{dx}. \tag{8.7}$$

The quantity $\mathcal{L}$ represents some expression involving the function, which I've yet to specify precisely. In my previous example, I had

$$\mathcal{L}(x, y, y_x) = \sqrt{1 + y_x^2}, \tag{8.8}$$

although in the other cases I mentioned, the form of $\mathcal{L}$ would look different. Given an arbitrary integrand of this form, is there a systematic way to deduce the function which minimizes the integral?

Amazingly enough, it turns out there is a pretty simple looking equation that tells me the answer to this question. Deriving it is somewhat beyond the scope of this course, but it turns out there is a simple differential equation I can use to find the appropriate equation. That differential equation is called the *Euler-Lagrange* equation, and it is

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right). \tag{8.9}$$

The notation here is a little tricky. The derivative on the right with respect to x is a normal derivative. But the partial derivatives acting on $\mathcal{L}$ are instructing me to pretend as though y and y_x were independent, unrelated variables, that $\mathcal{L}$ depends on. So for example, in the case of my curve length problem, I would find

$$\frac{\partial \mathcal{L}}{\partial y_x} = \frac{\partial}{\partial y_x}\left(\sqrt{1 + y_x^2}\right) = \frac{y_x}{\sqrt{1 + y_x^2}}, \tag{8.10}$$

whereas

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{1 + y_x^2}\right) = 0. \tag{8.11}$$

Even though it seems like there should be some sort of "chain rule" relating y and y_x, the funny looking partial derivatives tell me to ignore that, and just think of them as separate things that go into $\mathcal{L}$. We say that the derivative $\partial \mathcal{L}/\partial y$ is zero because $\mathcal{L}$ has no *explicit* dependence on y, although it does have *implicit* dependence, through the appearance of its derivative.

In any event, if we accept the fact that this is the correct way to minimize a functional, we can put it to use. For our example of the length of a curve, the Euler-Lagrange equations tell us that

$$0 = \frac{d}{dx}\left(\frac{y_x}{\sqrt{1 + y_x^2}}\right), \tag{8.12}$$

since my integrand has no explicit dependence on y. Because the x derivative of some expression is zero, this means that that expression must be a constant with respect to x,

$$\frac{y_x}{\sqrt{1+y_x^2}} = C. \tag{8.13}$$

If I square both sides, and do some rearranging, I find

$$y_x = \sqrt{\frac{C^2}{1-C^2}}. \tag{8.14}$$

Because C is some arbitrary constant, this just tells me that the first derivative of the function is equal to some constant value. But of course, this is precisely the mathematical definition of a straight line. The precise value of the constant would be determined by imposing the boundary conditions at each end point - that is, that it actually passes through the required points.

8.2 The Principle of Least Action

After having made this mathematical detour, I want to introduce a new object, called a *Lagrangian*. For a single particle moving under the influence of a potential energy function, we define its Lagrangian to be its kinetic energy, minus its potential energy,

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - U(x), \tag{8.15}$$

where for simplicity I'm just considering one dimension. Furthermore, I will define something called the *action*, which is just the integral of the Lagrangian over time,

$$S = \int_{t_1}^{t_2} \mathcal{L} dt = \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{x}^2 - U(x)\right] dt, \tag{8.16}$$

integrated between some given times t_1 and t_2.

Notice that this object has a similar form as the other integrals I've been considering. In this case, my notation is a little different - t is the variable I'm integrating with respect to, $x(t)$ is a function of that variable, and $\dot{x}$ is its derivative. So, if I wanted to, I could ask - for given times t_1 and t_2, what value of the function $x(t)$, that is the particle's location as a function of time, *minimizes the action*?

To answer this, I can again use the Euler-Lagrange equation. In my new variables, it reads

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right). \tag{8.17}$$

If I inspect my Lagrangian, first of all I see that

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{dU}{dx}, \tag{8.18}$$

since the potential energy is the only place where I have explicit dependence on x. I also see that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}. \tag{8.19}$$

Equating the two sides of the Euler-Lagrange equation, I find

$$m\ddot{x} = -\frac{dU}{dx}, \tag{8.20}$$

which is precisely Newton's equation! So I have discovered an object which, if minimized, results in the correct motion of the particle, assuming I impose the proper boundary conditions at the endpoints. This result tells me that particles under the influence of such a force always move in such a way as to minimize this action functional, a result which is known as *the principle of least action*.

8.3 Lagrangian Mechanics

While this is a very interesting result in its own right, we might wonder how it's useful to us. I've told you that Newton's equation is equivalent to minimizing this thing called the action, but all that tells me is the same differential equation I already knew. How is that helpful?

The reason that this is useful is because it turns out that the Euler-Lagrange is incredibly flexible in terms of how I write down the action, and does not care about what variables I use. Let me explain this with an example, by considering the setup in Figure 8.2. Here, I have a block, with mass m, sitting on a table, attached to a wall with a spring, with spring constant k. The block slides on a frictionless table. Attached to the side of the block is a pendulum which is free to swing from side to side. It has a length L, and the bob has a mass M. I want to derive a set of differential equations which describe the motion of this system.

I could try to tackle this by setting up free-body diagrams for the pendulum and block, and using Newton's laws. The result, not surprisingly, is a complete nightmare. The back reaction of the pendulum on the sliding block makes this a very nontrivial problem. If you don't believe me, try it for yourself - it won't be a fun time.

However, this is a problem which is perfectly suited for Lagrangian mechanics. The principle of least action generalizes to larger systems, and we stipulate

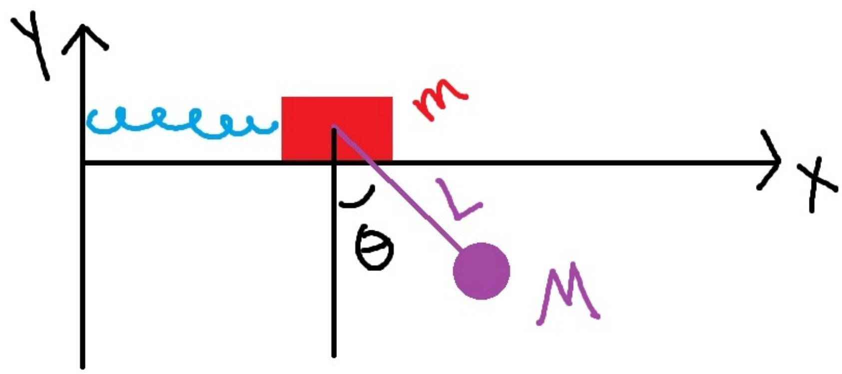


Figure 8.2: The spring-block-pendulum nightmare scenario.

that the Lagrangian is always the kinetic minus the potential energy. This means that we will typically now have several functions which enter into our Lagrangian (the various coordinates of all of the objects), and it turns out that there will be an Euler-Lagrange equation for each function. Let me show how this works by working out the Lagrangian for this system.

There are several sources of potential and kinetic energy in this system. First, we tackle potential energy. The spring has a potential energy of

$$U_s = \frac{1}{2}k\left(x_b - x_0\right)^2, \tag{8.21}$$

where x_b is the x coordinate of the block. The mass on the pendulum has a gravitational potential energy of

$$U_g = Mgy_p, \tag{8.22}$$

where y_p is the y coordinate of the pendulum. We've taken the zero of potential energy to be the height of the table, so that the block's gravitational potential energy is always zero. In addition to the potential energy, we have kinetic energy. The kinetic energy of the block is

$$K_b = \frac{1}{2}m\dot{x}_b^2, \tag{8.23}$$

while the kinetic energy of the pendulum mass is

$$K_p = \frac{1}{2}M\left(\dot{x}_p^2 + \dot{y}_p^2\right). \tag{8.24}$$

Adding these contributions together, we find

$$\mathcal{L} = K - U = \frac{1}{2}m\dot{x}_b^2 + \frac{1}{2}M\left(\dot{x}_p^2 + \dot{y}_p^2\right) - \frac{1}{2}k\left(x_b - x_0\right)^2 - Mgy_p \tag{8.25}$$

for the Lagrangian.

Now, we could go on to solve this problem in terms of the x and y coordinates. But this is actually a bad idea. The reason is because my description of this system right now is *redundant* - I do not really need the x and y coordinates of both the block and the pendulum. If I inspect my system, I notice that I can actually completely describe its orientation in terms of the x coordinate of the block, and the angle θ of the pendulum arm. Because of the physical constraint that the pendulum mass is hooked to the block with a length L, this completely indicates the location of every object in the system. In regular Newtonian mechanics using Newton's laws, this seems somewhat unnatural, since we like to describe our systems in terms of the coordinates of nice inertial reference frames. But as we will see soon, this is no concern at all for Lagrangian mechanics.

With this in mind, let's rewrite the coordinates of the pendulum in terms of its angle. With a little trigonometry, we can write

$$y_p = -L\cos\theta, \tag{8.26}$$

along with

$$x_p = x_b + L\sin\theta. \tag{8.27}$$

Using this, we can write

$$\dot{y}_p = L\dot{\theta}\sin\left(\theta\right), \tag{8.28}$$

along with

$$\dot{x}_p = \dot{x}_b + L\dot{\theta}\cos\left(\theta\right). \tag{8.29}$$

If we plug this back into our Lagrangian, and do some simplifying, we find

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\left(\dot{x}^2 + L^2\dot{\theta}^2 + 2L\dot{x}\dot{\theta}\cos\theta\right) - \frac{1}{2}k\left(x - x_0\right)^2 + MgL\cos\theta, \tag{8.30}$$

where I've dropped the subscript on the block's position, since it's now the only x coordinate we are considering.

The parameters x and θ are called *generalized coordinates.* They are not the components of a position vector in an inertial coordinate system, but they are variables which are functions of time that completely describe the state of my system. It turns out that the calculus of variations does not distinguish between generalized coordinates and regular ones. It says that for each generalized coordinate showing up in the Lagrangian, there is an Euler-Lagrange equation. For my x coordinate, the Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right), \tag{8.31}$$

whereas for my angular coordinate, the equation reads

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right). \tag{8.32}$$

This will give me a pair of differential equations for these two variables.

If I go ahead and take the above derivatives, then after only a few short lines of math, I will ultimately find

$$m\ddot{x} + M\ddot{x} + ML\ddot{\theta}\cos\theta - ML\dot{\theta}^2\sin\theta = -k\left(x - x_0\right) \tag{8.33}$$

for the x equation, along with

$$ML^2\ddot{\theta} + ML\ddot{x}\cos\theta = -MgL\sin\theta \tag{8.34}$$

for the θ equation. Now, unfortunately Lagrangian mechanics doesn't promise that the resulting differential equations will be simple to solve. However, simple differential equations are pretty rare in actual physics, and all told, this one doesn't look too bad. In any event, this is really not a concern to us as physicists. We care about figuring out what the equations of motion are, based on physical principles. Once we've done that, the physics part is over. The rest is just the math problem of how in God's name to solve that system of differential equations. In reality, what we usually do is solve this sort of thing numerically with a computer, which is actually pretty straightforward to do. But the important thing is that we were able to find out the equations of motion at all. To give you a sense of how badly behaved this system is, notice that from the above equations, I can read off the x component of the block's acceleration, which is

$$\ddot{x} = \frac{1}{M+m}\left[-k\left(x - x_0\right) - ML\ddot{\theta}\cos\theta + ML\dot{\theta}^2\sin\theta\right]. \tag{8.35}$$

Imagine trying to derive that form of the acceleration using a free body diagram!

8.4 Noether's Theorem

It most likely goes without saying that the principle of least action is incredibly powerful. The above system would have been a total mess to analyze using forces and free body diagrams, but with only a few short lines of taking derivatives, I derived the correct differential equation for the motion of the system. This alone would make Lagrangian mechanics an indispensable tool.

But the advantages of the Lagrangian formalism go beyond just this - too many really to mention all of them here. But there are a few notable examples. The first one is that it is particularly well-suited to perturbation theory and approximation schemes. In my above example, there is an equilibrium state where the block sits still at the rest position of the spring, while the pendulum hangs motionless under it. If I then nudge the system slightly, it will start to wiggle back and forth, and I might wonder what the frequency of small vibrations in the system is. While I don't have time to go into details here, these small vibrations are called normal modes, and it turns out that if I take my Lagrangian and Taylor expand it around this equilibrium point, it's possible to basically read these frequencies off right from the Lagrangian, if you know how. This is incredibly important in engineering when trying to figure out the resonant frequencies of structures.

The second, and possibly most important advantage of Lagrangian mechanics is that symmetries become very obvious. To explain how this is true, I will give a simple example. Imagine I have a particle moving in two dimensions, and I describe its position in terms of its radius from the origin, and its angle θ from the x axis. On the homework, you will show that the kinetic energy of this particle can be written as

$$K = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2. \tag{8.36}$$

If we assume that the particle is exposed to a potential that only depends on the distance from the origin (which is quite often the case), then the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r), \tag{8.37}$$

where $U(r)$ does not depend on the angle at all.

Now, something I immediately notice about this Lagrangian is that it has no dependence on the angle θ at all. Because the Euler-Lagrange equation for θ is

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right), \tag{8.38}$$

then I immediately see that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0. \tag{8.39}$$

This means that the quantity inside of the parentheses must be a constant which never changes with time. Taking the partial derivative, I see that

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \equiv J, \tag{8.40}$$

which we call the *angular momentum*. When exposed to a potential that only depends on radius, it is a constant in time, and so is conserved. In two dimensions, it is just a scalar number, but in three dimensions it is a vector quantity, which is given by

$$\vec{J} = \vec{r} \times \vec{p}, \tag{8.41}$$

where $\vec{r}$ is the particle's position vector, $\vec{p}$ is its momentum, and the operation between them is known as the *cross product*, an operation between vectors which we haven't discussed in detail here, but comes up frequently in physics.

What has happened here is that a symmetry has immediately led to a conservation law. Because a generalized coordinate was missing from the Lagrangian, the Euler-Lagrange equations immediately told us that the time derivative of some quantity was zero, thus telling us that object is conserved. Clearly, this is a pretty easy way to find conserved quantities, which are always very useful. Another example is a free particle moving alone under the influence of no external forces. In this case, the Lagrangian is simply kinetic energy alone, and we have

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \tag{8.42}$$

Because the Lagrangian does not depend on the position at all, then the Euler-Lagrange equation says

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0, \tag{8.43}$$

and likewise for the other coordinates. But we have

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}, \tag{8.44}$$

which is just the momentum. So the lack of an external force gives rise to momentum conservation.

This idea that symmetries lead to conservation laws is known as *Noether's Theorem*, and it is one of the most important ideas in physics. While it is beyond the scope of this class, it turns out there are actually more conservation laws hiding in these Lagrangians. Seeing them requires a little more mathematical insight, however, since they don't necessarily fall right out of the Euler-Lagrange equations. But the punch line is that Noether's Theorem tells us that *for every symmetry which leaves the action unchanged, there is a corresponding conserved* *quantity.* For example, it is possible to show that because the Lagrangian does not depend explicitly on time, then there is always a conserved quantity associated with this fact. This turns out to be what we have been calling the energy.

Speaking of energy conservation, this leads to an interesting question - what about nonconservative forces, which cannot be written in terms of potential energies? It turns out that Lagrangian mechanics can handle situations like this too, but the terms that we need to add to the Lagrangian will usually depend explicitly on time. Because of this, we no longer have *time translation symmetry*, and there will not be a conserved quantity we can identify as energy. There are some special exceptions to this, where there is a related quantity which will be constant, but is not what we usually think of as an energy.

Should energy be conserved in nature in general? Up until now I've been talking about nonconservative forces, but really these typically arise from microscopic atomic interactions that actually do conserve energy - they tend to increase the random thermal motion of the atoms, which doesn't manifest itself as motion of the macroscopic center of mass. It turns out that in modern theories of physics, energy conservation is actually a somewhat subtle issue. General Relativity tells us that the distinction between space and time is not so clear cut, and so this introduces issues with what we mean by the conserved quantity associated with time symmetry. It turns out there is a very important conservation law in General Relativity, but the object assoicated with it is not necessarily what we are familiar with as an energy. However, this is usually only a practical matter in situations where the effect of gravity is very strong. In everyday physics we do in laboratories on the Earth, the effects related to this issue are barely detectable.

8.5 New Physics

It turns out that many of the ideas we've been talking about, such as forces and Newton's laws, are no longer used in modern theories of physics. A variety of different quantities and ideas make up modern physics theories, such as the Standard Model of Particle Physics, and General Relativity. But the one principle which has survived in all of our physical theories to date is the principle of least action. All of these theories are written in terms of an action with some set of "generalized coordinates." Some of these coordinates are very abstract objects - in the Standard Model, they are quantum fields - strange entities which extend through space and give rise to what we think of as particles. In general relativity, it is the curvature - a function which tells us how much spacetime is distorted by the presence of matter. Typically, we just think of the object in these theories as “the Lagrangian,” and not necessarily as a difference of kinetic and potential energies, since the notion of kinetic vs. potential energy does not necessarily survive in these models. But in all of these cases, an action functional which is being minimized is at the heart of the theory.

In fact, this is the current way in which most new theories are first written down. Because Lagrangian mechanics makes it so easy to see conservation laws and symmetries, and to write down the equation of motion, most new theories are formulated by trying to think of a new Lagrangian which obeys all of the symmetries of nature we think should exist. It turns out that symmetry, in combination with a few other more abstract principles, actually greatly restrict the types of terms which are allowed in Lagrangians, and so this is actually a very useful exercise.

There also exists a related idea, Hamiltonian mechanics, which can be very useful as well. Unfortunately I don’t have time to talk about it in this course, but it is very similar to Lagrangian mechanics. The Hamiltonian is a new function which is derived from the Lagrangian, and leads to a different, yet completely equivalent, set of differential equations for a system. These equations sometimes have nice features, and are more useful in some situations.

In any event, that wraps up all of the material involving classical mechanics, and all of the material which will show up on any of the homeworks or final exam. Tomorrow is going to be a special topics class dedicated to what I think is one of the most amazing ideas in modern physics, Special Relativity.

Appendix A

Tips for Solving Physics Problems

The following is some advice for solving physics problems from previous SIMS instructor Sebastian Fischetti.

In high school, the way I learned physics was very algorithmic: our teacher would teach us a new concept ("Here's what a free body diagram is"), and then he would give us lots of the same problem to repeat over and over again, just using different numbers ("If forces A, B, and C are acting on a block, draw a free body diagram for the block and figure out the total force on the block"). Mostly, it was mindless drilling.

I suspect that many of you may have learned physics in a similar way, and I want to emphasize that that is *not* how physics works at all. Physics is really about creativity - if you encounter a problem you've never seen before, can you think creatively about ways to solve it? Can you apply concepts and ideas you already know to new systems? This is what makes physics hard (and fun!).

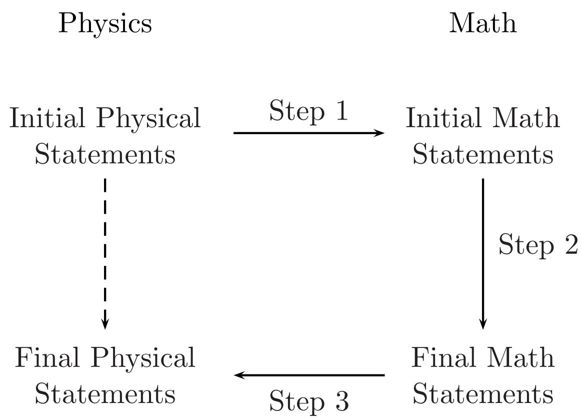
To break it down, a physical problem usually goes something like this: given some information about a physical system (e.g. a baseball is being thrown with some initial velocity; a beam of light is about to enter a piece of glass; the universe is created in some explosion of spacetime), make some new statements about the physical behavior of the system (e.g. when and where the baseball lands; what happens to the light when it enters the glass; what the distribution of matter in the universe looks like later). Going directly from the setup to a conclusion is often difficult, so we use mathematics as an intermediate step. So, solving a physical problem consists of three steps:

1. Convert the information about the physical system into some mathematical statements

2. Use the rules of mathematics to work the mathematical statements into some mathematical conclusions

3. Convert the mathematical conclusions back into physical statements

We can think of the picture as something like this:



The dashed line is what we're really trying to do, but the three solid lines are what the tool of mathematics allows us to do. The important thing to bear in mind is that you're only doing *physics* when you're on the physics side of the diagram; when you're on the math side, you're doing *math*. Thus, Steps 1 and 3 are the steps that require you to do physics, while Step 2 is just doing math. Unfortunately, since Step 2 is usually the easiest, students often learn in high school (and sometimes continue to believe in college) that physics is all about Step 2, and they never learn that Steps 1 and 3 are even important (or even exist!). My goal in this course, and your goal in your future physics courses, is to really focus on Steps 1 and 3, and Step 2 should be given a lower priority.

(By the way, in your introductory physics courses, you may hear of the I SEE method of problem solving, which stands for **I**dentify, **S**et up, **E**xecute, and **E**valuate. Roughly speaking, the Identify and Set up steps are like Step 1, the Execute step is like Step 2, and the Evaluate step is like Step 3)

With that said, I'd like to give you some tips and tricks that I think will be useful in dealing with these three steps.

Step 1

In the first step of the diagram I drew, your goal is to convert some physical statements into mathematical ones. This process requires understanding what physical principles are involved, which will guide you in getting the relevant equations you want. My tips for this step are:

- THINK ABOUT THE PHYSICS OF THE SYSTEM, think about what physical principles you know and if they're relevant, and if they can give you any useful information. Don't just "jump right into the math" if you don't know what you're doing first!

- TALK TO OTHER PEOPLE. Everyone has a different way of thinking about things, and sharing ideas and knowledge can be incredibly useful. Study groups are great for this, and since you already know everyone in the SIMS program, finding people to work with shouldn't be too difficult at all. Of course, your instructors and TAs will also be available for questions and discussion.

- USE YOUR WORDS. I've found that many students think that since physics is mathy, their homeworks should basically be full of equations and calculations with no words. This is bad in two ways: first, physics is about *thinking*, not about math, and you need to use your words to explain your thought process. Second, there's a common saying "the only way to know you really understand something is if you can explain it to someone else." If you write your homeworks as if you're explaining how to do the problem to someone else, you'll find that your own understanding of the problem will increase dramatically. When I start posting homework solutions, note that I often have much more explanatory text than I do equations; you should try to do something similar on your own homeworks.

Step 2

I don't have much to say about this step. This is the "plug and chug" part of the problem, where you take the formulas you obtained in Step 1 and just use algebra/calculus/linear algebra/complex analysis/geometry/vector calculus/differential equations/etc. to solve the equations. There can be a lot of interesting stuff in this step, but it's all math, so I'd ask your math intructors about that.

Step 3

This is the most often skipped step, which I think is very unfortunate, because it's also the most important. The goal of a physical problem is to understand how the universe works, and this is the step that gives you insight into physical systems. The idea here is to think about what your final mathematical answer is telling you about the physics of the problem. My tips are:

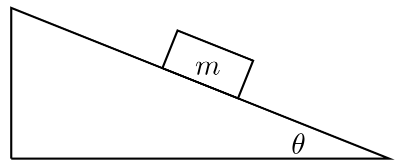
- FORCE YOURSELF TO DO THIS STEP. Many physics problems don't explicitly ask you to think about your final answer, but it's very good to get into the habit of doing it whether you're asked to or not. When you get to a final answer, ask yourself: does this answer make sense? Is there some way I can test it? What is it telling me about the problem? Again, this goes back to what I said above about using your words to discuss the answer.

- CHECK LIMITING CASES. In this course, we're almost exclusively going to stick to variable quantities. The reason variables are so much better than numbers is that you can tune the variables to whatever you want, which makes it easy to check your answer in certain simple cases (for example, if you're working on an inclined plane at angle θ, it's easy to check what happens when the plane becomes flat by setting $\theta = 0$, or when the plane becomes vertical by setting $\theta = \pi/2$).

- USE DIMENSIONAL ANALYSIS. Physics quantities have units, and it's important for units to match! A very, very easy check you can perform on your final answer is to see if all the units agree (I tend to take a *lot* of points off from students who don't bother to check their units, so be warned!).

- Above all, USE AND DEVELOP PHYSICAL INTUITION. If you're working on, say, a classical mechanics problem, you should often have physical intuition for whether or not the answer makes sense (if your answer says that a block with no forces on it should spontaneously start sliding across a flat table, something is clearly wrong). Your intuition is a good check. As you get to more advanced physics concepts, though, sometimes your intuition fails: in special relativity, we have no intuition for how things behave when they move at close to the speed of light, and in quantum mechanics, we have no intuition for how subatomic particles behave. In these cases, you can use your results to physics problems *develop new intuition* about physics. Special relativity blew my mind when I first learned abotu it; now I've used it so often, it seems obvious to me that if I'm in a spaceship moving at close to the speed of light, I should see the stars in front of me turn bluer and scrunch together in front of my ship.

That may have been a lot to take in, so here's a very simple example showing how I might use the steps myself. Consider the problem of finding the acceleration of a block of mass m down a frictionless inclined plane at angle θ from the horizontal, as shown:



Let's break this problem into the three steps:

1. First, we need to take the physical situation above and come up with some quantitative statements about it (i.e. we need to move from the physics side to the math side). We would do this with a free body diagram and by applying Newton's laws: the fact that the block isn't accelerating in the direction perpendicular to the surface of the incline tells us that $mg\cos\theta - N = 0$, where N is the normal force on the block. Newton's second law in the direction parallel to the surface of the incline tells us that $mg\sin\theta = ma$, where a is the acceleration of the block. At this point, we have two equations for two unknown quantities a and N, and we can proceed to the next step.

2. This is the "plug and chug" step; in our case, we just rearrange the equation for the acceleration into $a = g\sin\theta$. So, here's the answer we were looking for. Are we done yet? No! We still need to *think* about it in the last step!

3. Now, we need to take our mathematical answer $a = g\sin\theta$ and think about what it's telling us physically, and consider whether or not it makes sense based on our intuition. The first thing we can do is check if the units work: g has units of acceleration, $\sin\theta$ is dimensionless, so $g\sin\theta$ has units of acceleration. a also has unit of acceleration, we get acceleration = acceleration, as we should. Next, we should consider some limiting cases. Intuitively, it might not be obvious what the block should do on an arbitrary incline, but what if we make the incline flat? A flat incline means that $\theta = 0$, in which case we get $a = 0$. So if the block is on a flat surface, it doesn't accelerate, exactly as we would expect! We can also consider the case of the incline being vertical, i.e. $\theta = \pi/2$. In this case, we'd expect the block to just fall freely under the influence of gravity. Indeed, if we plug $\theta = \pi/2$ into our solution, we get $a = g$, which just means the block is just in freefall. So this, too, agrees with our expectation.

The thing I want to highlight is that most students would stop at the end of Step 2 above when they get to the answer $a = g\sin\theta$, but you'll notice that there's a lot of physical discussion and understanding in Step 3. This was a silly example just to highlight the main ideas, but as you work through more complex problems, try to always focus on that third step, and you'll find your physical intuition should improve significantly.

Appendix B

Taylor's Theorem

One of the most basic questions of introductory calculus is the following: If I have a general function $f(x)$, what is the tangent line at some point x_0? Well, I know that I can find this by computing the derivative, since that's the slope of the function at that point. We know from our calculus courses that the tangent line is given by

$$y\left(x\right) = f'\left(x_0\right)\left(x - x_0\right) + f\left(x_0\right), \tag{B.1}$$

where $y\left(x\right)$ is the equation of the tangent line.

Now, if I'm looking at the function zoomed in very close to this point, then I know that it should look roughly straight, and the tangent line will be a good approximation to the function. But what if I want to do just a little bit better with my approximation? I don't want to work with the full function (maybe it's really complicated looking), but I do want a slightly better approximation than the tangent line. Taylor's theorem is a result which tells us how we can do this. In essence, what it says is that "almost" any function can be expanded in terms of an expansion in powers of $(x - x_0)$, which looks like

$$f\left(x\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(x_0\right)}{n!}\left(x - x_0\right)^n, \tag{B.2}$$

where $f^{(n)}\left(x_0\right)$ is the n^{th} derivative of $f(x)$, evaluated at x_0. This expansion is called a Taylor Series. Figure B.1 shows an example of using a Taylor Series to make a better and better approximation to the sine function.

If we only want to make an approximation one step better than the tangent line, we can write

$$f\left(x\right) \approx f\left(x_0\right) + f'\left(x_0\right)\left(x - x_0\right) + \frac{1}{2} f''\left(x_0\right)\left(x - x_0\right)^2. \tag{B.3}$$

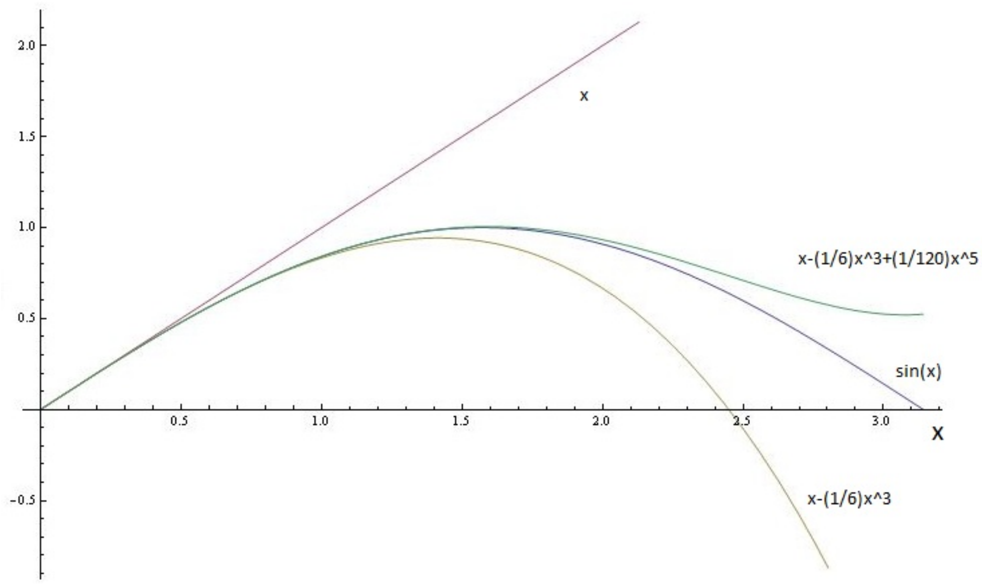


Figure B.1: Approximating the sine function, using higher and higher order Taylor series. Notice that while all of the curves look pretty close to each other for small values of x, the fifth-order expansion is clearly the best approximation for larger values of x. Notice that there are no second or fourth order expansions - because sine is an odd function, all of its even order terms disappear.

Generally, for any given point x, the approximation will get better as we include more and more terms.

As an example of how this can be useful, imagine I want to solve the equation

$$\cos(x) = \lambda x, \tag{B.4}$$

for some number λ. In other words, I have a straight line with slope λ that passes through the origin, and the cosine function, and I want to see where their plots intersect. This is shown in Figure B.2. In general, there is no easy way to do this in closed form.

But let's imagine that I know that λ is really huge, so that the straight line is very steep. Then I know that the point where the two lines should intersect is very close to $x = 0$, because the steeper the line, the more it bends in towards the vertical axis. Then because the intersection will occur near $x = 0$, it seems plausible that we should be able to approximate the value of the cosine function by using a series expansion around zero. To lowest order, we can just replace

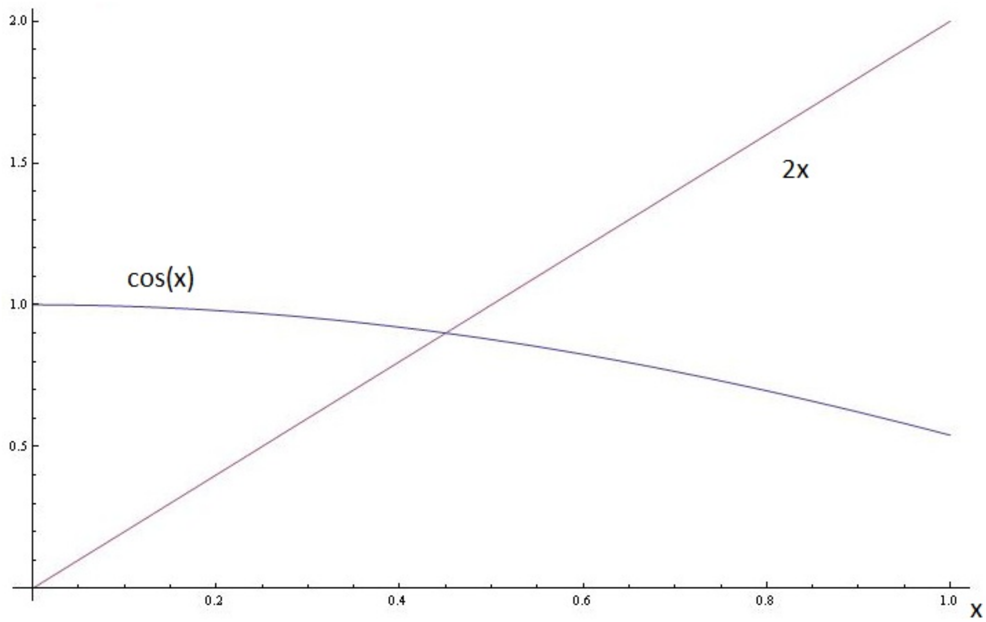


Figure B.2: Finding the intersection of two functions with the help of a Taylor series expansion. The number 2 is not really that large, although it is a good value for making a nice looking plot.

the cosine function with its value at zero, which is simply one, and we find

$$1 = \lambda x, \tag{B.5}$$

or,

$$x = \frac{1}{\lambda}. \tag{B.6}$$

Even for the value $\lambda = 2$, which isn't even that "big," we see that the result we get is 0.5, which, from looking at the plot, seems to be pretty close to the correct value! Notice that our expression gets smaller as λ gets bigger, as we expected.

But by using the power of a taylor series expansion, I can do even better, without very much work. We know that the first derivative of cosine is sine, which is zero at $x = 0$. So there is no correction at that order in the expansion. But the second derivative of cosine is negative itself, which at zero gives negative one. Thus, to second order we can approximate

$$\cos(x) \approx 1 - \frac{1}{2}x^2. \tag{B.7}$$

Despite the fact that I computed the derivatives in this expansion myself, Taylor Series are so useful that the expansions of many functions are documented to very high orders, so you can usually look them up, without doing any math.

Using this expansion, the equation we are trying to solve is now

$$1 - \frac{1}{2}x^2 = \lambda x, \tag{B.8}$$

or,

$$x^2 + 2\lambda x - 2 = 0. \tag{B.9}$$

Taking the positive solution to the quadratic formula, we find

$$x = -\lambda + \sqrt{\lambda^2 + 2}. \tag{B.10}$$

Now, if I consider the function

$$g\left(y\right) = \sqrt{\lambda^2 + y}, \tag{B.11}$$

then I can ALSO Taylor expand THIS function centered around $y = 0$, to find that

$$g\left(y\right) \approx g\left(0\right) + g'\left(0\right)y + \frac{1}{2}g''\left(0\right)y^2. \tag{B.12}$$

After computing all of the derivatives (or just looking up the expansion on Wikipedia), and using the fact that

$$\sqrt{\lambda^2} = \lambda, \tag{B.13}$$

then I ultimately find

$$g\left(y = 2\right) = \sqrt{\lambda^2 + 2} \approx \lambda + \frac{1}{\lambda} - \frac{1}{2\lambda^3}. \tag{B.14}$$

Notice that I chose to do my expansion centered around a point where I could do the square root easily. Using this in my above expression, I see that

$$x \approx \frac{1}{\lambda} - \frac{1}{2\lambda^3}. \tag{B.15}$$

This gives me a nice looking expression for the intersection point in terms of λ. It turns out that even for $\lambda = 2$, my expression gives 0.4375, while a much more sophisticated computational method says that the exact answer, up to four digits, is 0.4502, which is not too far from my value! For $\lambda = 10$, we would have to go out to six decimal places before seeing a difference between my answer and a computer's answer. If we imagine that x is some distance in meters which I'm measuring with a ruler, it's unlikely I'd be able to measure six decimal places!

While it is true that I could have used a computer to get this result for any one specific value of λ, the nice thing about my result is that it gives me a feel for what the intersection point looks like as a function of λ, which can be good for gaining intuition in some situations.

This is an extremely useful tool in physics, because it allows us to make approximations that get better and better until we no longer care about the difference in accuracy. Typically in physics, we are interested in doing experiments, and so we always work with data that has imperfect resolution. In fact, in many cases, we take the philosophy that our job is to just experimentally measure the coefficients in a Taylor series!

As an example, imagine that I want to know how the freezing point of water changes when I add a small amount of salt. I write $T_f(\mu)$ as the freezing temperature, as a function of the salt concentration μ. If I assume that there is some series expansion, I can write the first few terms as

$$T_f(\mu) \approx T_0 + T_1\mu + T_2\mu^2 + ... \tag{B.16}$$

If the salt concentration is very small, then I expect that whatever the true functional dependence on μ is, keeping the first few terms in the series expansion should be reasonable. So, what I do is go out and do an experiment, record a bunch of data, and then I will attempt to fit the data to a quadratic curve. The fit parameters will tell me what the coefficients are. As far as I'm concerned, if the third order term were so small that I could never tell the difference with my thermometer, there's no way for me to know what the third order term is, so I might as well just stop here!

While this may seem like the easy way out, this is actually how a lot of modern physics is usually done! Several theories of physics that we have today are described by equations which could have more terms added to them, but we don't do it because we don't have the ability to measure them yet anyways! With all of this power that the Taylor Series gives us, it's often said that the most important physicist was actually a mathematician, and his name was Brook Taylor.

Appendix C

Differential Equations

There are many instances in physics in which we have a mathematical object we do not know the value of, but which we can solve for by performing some mathematical manipulations. For example, often times we need to solve an algebraic equation for an unknown variable x,

$$2x + 1 = 5 \;\Rightarrow\; 2x = 4 \;\Rightarrow\; x = 4/2 = 2 \tag{C.1}$$

The quantity x is some object which satisfies a mathematical constraint defined by the above equation. By performing basic algebraic manipulations, I can arrive at the value of x.

Sometimes the constraint imposed on a mathematical object is more complicated, and can involve calculus. For example, I could have the equation

$$y'(x) = \frac{dy}{dx} = x. \tag{C.2}$$

The variable y is some function which depends on x, and the above equation puts a constraint on its derivative. To solve this equation, I remember that the integral of a derivative gives the original function back (up to an additive constant), and so I can perform an indefinite integral on both sides with respect to x, thus finding

$$\int \frac{dy}{dx}\, dx = \int x\, dx \;\Rightarrow\; y(x) = \frac{1}{2}x^2 + C. \tag{C.3}$$

Unlike the algebraic equation, this equation has many solutions, since the additive constant C can be any number I like. For example,

$$y(x) = \frac{1}{2}x^2 + 5 \;\Rightarrow\; \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}x^2 + 5\right) = x. \tag{C.4}$$

But what if the equation that places a constraint on the function $y(x)$ involves the function itself? For example, what if we had the equation

$$y' = \frac{dy}{dx} = y + x \tag{C.5}$$

This sort of equation that relates a function to its derivative in some way is known as a differential equation, and they will show up over and over again in your physics classes. If we attempt to proceed as before and integrate both sides with respect to x, we find

$$\int \frac{dy}{dx}\ dx = \int (y + x)\ dx \ \Rightarrow \ y = \int y\ dx + \frac{1}{2}x^2 = \ ??? \tag{C.6}$$

Because I don't know what the function $y(x)$ is, I can't perform the integration on the right. How do we go about finding all of the allowed solutions to this equation?

In general, solving differential equations can be a very complicated subject. Even solving a simple looking differential equation, like the one above, requires some techniques that are too complicated to discuss here, given the amount of time we have in this course (although it would be one of the first techniques you would learn in a course dedicated to differential equations, known as the technique of integrating factors). But even this differential equation is far from the hardest one to solve. We could have higher order derivatives appearing, for example,

$$y'' + y' = y + x. \tag{C.7}$$

We could also have situations in which there are two dependent variables, $y(x)$ and $z(x)$, whose derivatives depend on each other, for example

$$y' = z + x\ ;\ z' = y + x. \tag{C.8}$$

We can even have some functions that have multiple independent variables, $y(x, t)$, with an equation like

$$\frac{d}{dt} y(x, t) = \frac{d}{dx} y(x, t). \tag{C.9}$$

The first two of these you would learn how to solve in a sophomore-level course on differential equations, while the third one you would learn how to solve in a junior or senior-level course on a subject known as partial differential equations. Ultimately, the vast majority of differential equations are too complicated to solve by hand, and require numerical approximation techniques that are performed on a computer.

However, there is a special class of differential equations that are easy to solve, known as separable differential equations. This is a differential equation that takes the form

$$y'(x) = f(x)\, g(y(x)). \tag{C.10}$$

Notice that the left side of the equation involves only the first derivative of y. The right side of the equation involves a function of only x, multiplied by a function of only $y(x)$. For example, we might have the equation

$$y'(x) = (x+1)(y(x)+3). \tag{C.11}$$

In this example,

$$f(x) = x+1 \quad ; \quad g(y(x)) = y(x)+3. \tag{C.12}$$

The reason this type of equation is known as separable is because the right side "separates" into two factors, one involving x, and the other involving y.

There is a simple technique for solving separable equations. The first step is to notice that we can divide both sides of our equation by the part that depends on y. In the example above, this means

$$\frac{y'(x)}{(y(x)+3)} = (x+1). \tag{C.13}$$

Now, because y is a function of x, it makes sense to integrate both sides with respect to x,

$$\int \frac{y'(x)}{(y(x)+3)}\ dx = \int (x+1)\ dx = \frac{1}{2}x^2 + x + C. \tag{C.14}$$

Notice that the integral on the right side is easy to perform.

Now, it may seem that we are still stuck on the left side. But, remember one of the tricks from your calculus class, the method of substitution. That method relies on the fact that underneath the integral sign, we can make the replacement

$$y'(x)\, dx = \frac{dy}{dx} dx \ \rightarrow \ dy. \tag{C.15}$$

Whenever the integrand underneath the integral involves a factor of $y'(x)$, we can use this trick to effectively change variables in the integral, going from an integral over x, to an integral over y. Using this, our equation becomes

$$\int \frac{dy}{(y+3)} = \frac{1}{2}x^2 + x + C \ \Rightarrow \ \ln(y+3) = \frac{1}{2}x^2 + x + C. \tag{C.16}$$

We now have an algebraic equation for y in terms of x, which we can rearrange to arrive at a formula for x. The presence of the additive constant C indicates that there will again be many allowed solutions to this equation, one for each possible value of C (why did I omit the additive constant when I performed the integral on the left?). On the homework, you'll get a chance to gain more practice in solving equations like this (which will include doing the necessary algebraic rearrangement to find what y is), along with some practice in understanding how they relate to actual scientific problems, by choosing to perform some of the extra credit problems.