Lecture notes to the 1-st year master course

Particle Physics 1

Nikhef - Autumn 2011

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Contents

Lecture 0

Introduction

The particle physics master course will be given in the autumn semester of 2011 and contains two parts: Particle Physics 1 (PP1) and Particle Physics 2 (PP2). The PP1 course consists of 12 lectures (Monday and Wednesday morning) and mainly follows the material as discussed in the books of Halzen and Martin and Griffiths.

These notes are my personal notes made in preparation of the lectures. They can be used by the students but should not be distributed. The original material is found in the books used to prepare the lectures (see below).

The contents of particle physics 1 is the following:

- Lecture 1: Concepts and History
- Lecture 2 - 5: Electrodynamics of spinless particles
- Lecture 6 - 8: Electrodynamics of spin 1/2 particles
- Lecture 9: The Weak interaction
- Lecture 10 - 12: Electroweak scattering: The Standard Model

Each lecture of 2×45 minutes is followed by a 1 hour problem solving session.

The particle physics 2 course contains the following topics:

- The Higgs Mechanism
- Quantum Chromodynamics

In addition the master offers in the next semester topical courses (not obligatory) on the particle physics subjects: CP Violation, Neutrino Physics and Physics Beyond the Standard Model

Examination

The examination consists of two parts: Homework (weight=1/3) and an Exam (weight=2/3).

Literature

The following literature is used in the preparation of this course (the comments reflect my personal opinion):

Halzen & Martin: "Quarks & Leptons: an Introductory Course in Modern Particle Physics ":
Although it is somewhat out of date (1984), I consider it to be the best book in the field for a master course. It is somewhat of a theoretical nature. It builds on the earlier work of Aitchison (see below). Most of the course follows this book.

Griffiths: "Introduction to Elementary Particle Physics", second, revised ed.
The text is somewhat easier to read than H & M and is more up-to-date (2008) (e.g. neutrino oscillations) but on the other hand has a somewhat less robust treatment in deriving the equations.

Perkins: "Introduction to High Energy Physics", (1987) 3-rd ed., (2000) 4-th ed.
The first three editions were a standard text for all experimental particle physics. It is dated, but gives an excellent description of, in particular, the experiments. The fourth edition is updated with more modern results, while some older material is omitted.

Aitchison: "Relativistic Quantum Mechanics"
(1972) A classical, very good, but old book, often referred to by H & M.

Aitchison & Hey: "Gauge Theories in Particle Physics"
(1982) 2nd edition: An updated version of the book of Aitchison; a bit more theoretical.
(2003) 3rd edition (2 volumes): major rewrite in two volumes; very good but even more theoretical. It includes an introduction to quantum field theory.

Burcham & Jobes: "Nuclear & Particle Physics"
(1995) An extensive text on nuclear physics and particle physics. It contains more (modern) material than H & M. Formula's are explained rather than derived and more text is spent to explain concepts.

Das & Ferbel: "Introduction to Nuclear and Particle Physics"
(2006) A book that is half on experimental techniques and half on theory. It is more suitable for a bachelor level course and does not contain a treatment of scattering theory for particles with spin.

Martin and Shaw: "Particle Physics ", 2-nd ed.
(1997) A textbook that is somewhere inbetween Perkins and Das & Ferbel. In my opinion it has the level inbetween bachelor and master.

Particle Data Group: "Review of Particle Physics"
This book appears every two years in two versions: the book and the booklet. Both of them list all aspects of the known particles and forces. The book also contains concise, but excellent short reviews of theories, experiments, accellerators, analysis techniques, statistics etc. There is also a version on the web: http://pdg.lbl.gov

The Internet:

In particular Wikipedia contains a lot of information. However, one should note that Wikipedia does not contain original articles and they are certainly not reviewed! This means that they cannot be used for formal citations.

In addition, have a look at google books, where (parts of) books are online available.

About Nikhef

Nikhef is the Dutch institute for subatomic physics. Although the name Nikhef is kept, the acronym "Nationaal Instituut voor Kern en Hoge Energie Fysica" is no longer used. The name Nikhef is used to indicate simultaneously two overlapping organisations:

- Nikhef is a national research lab funded by the foundation FOM; the dutch foundation for fundamental research of matter.

- Nikhef is also a collaboration between the Nikhef institute and the particle physics departements of the UvA (A'dam), the VU (A'dam), the UU (Utrecht) and the RU (Nijmegen) contribute. In this collaboration all dutch activities in particle physics are coordinated.

In addition there is a collaboration between Nikhef and the Rijks Universiteit Groningen (the former FOM nuclear physics institute KVI) and there are contacts with the Universities of Twente, Leiden and Eindhoven.
For more information go to the Nikhef web page: http://www.nikhef.nl

The research at Nikhef includes both accelerator based particle physics and astroparticle physics. A strategic plan, describing the research programmes at Nikhef can be found on the web, from: www.nikhef.nl/fileadmin/Doc/Docs & pdf/StrategicPlan.pdf .

The accelerator physics research of Nikhef is currently focusing on the LHC experiments: Alice ("Quark gluon plasma"), Atlas ("Higgs") and LHCb ("CP violation"). Each of these experiments search answers for open issues in particle physics (the state of matter at high temperature, the origin of mass, the mechanism behind missing antimatter) and hope to discover new phenomena (eg supersymmetry, extra dimensions). The LHC started in 2009 and is currently producing data at increasing luminosity. The first results came out at the ICHEP 2010 conference in Paris, while the latest news of this summer on the search for the Higgs boson and "New Physics" have been discussed in the EPS conference in Grenoble and the lepton-photon conference in Mumbai. So far no convincing evidence for the Higgs particle or for New Physics have been observed.

In preparation of these LHC experiments Nikhef is/was also active at other labs: STAR (Brookhaven), D0 (Fermilab) and Babar (SLAC). Previous experiments that ended their activities are: L3 and Delphi at LEP, and Zeus, Hermes and HERA-B at Desy.

A more recent development is the research field of astroparticle physics. It includes Antares & KM3NeT ("cosmic neutrino sources"), Pierre Auger ("high energy cosmic rays"), Virgo & ET ("gravitational waves") and Xenon ("dark matter").

Nikhef houses a theory departement with research on quantum field theory and gravity, string theory, QCD (perturbative and lattice) and B-physics.

Driven by the massive computing challenge of the LHC, Nikhef also has a scientific computing departement: the Physics Data Processing group. They are active in the development of a worldwide computing network to analyze the huge datastreams from the (LHC-) experiments ("The Grid").

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History of Particle Physics

The book of Griffiths starts with a nice historical overview of particle physics in the previous century. Here's a summary:

Atomic Models

1897 *Thomson:* Discovery of Electron. The atom contains electrons as "plums in a pudding".

1911 *Rutherford:* The atom mainly consists of empty space with a hard and heavy, positively charged nucleus.

1913 *Bohr:* First quantum model of the atom in which electrons circled in stable orbits, quatized as: $L = \hbar \cdot n$

1932 *Chadwick:* Discovery of the neutron. The atomic nucleus contains both protons and neutrons. The role of the neutrons is associated with the binding force between the positively charged protons.

The Photon

1900 *Planck:* Description blackbody spectrum with quantized radiation. No interpretation.

1905 *Einstein:* Realization that electromagnetic radiation itself is fundamentally quantized, explaining the photoelectric effect. His theory received scepticism.

1916 *Millikan:* Measurement of the photo electric effect agrees with Einstein's theory.

1923 *Compton:* Scattering of photons on particles confirmed corpuscular character of light: the Compton wavelength.

Mesons

1934 *Yukawa:* Nuclear binding potential described with the exchange of a quantized field: the pi-meson or pion.

1937 *Anderson & Neddermeyer:* Search for the pion in cosmic rays but he finds a weakly interacting particle: the muon. (Rabi: "Who ordered that?")

1947 *Powell:* Finds both the pion and the muon in an analysis of cosmic radiation with photo emulsions.

Anti matter

1927 *Dirac* interprets negative energy solutions of Klein Gordon equation as energy levels of holes in an infinite electron sea: "positron".

1931 *Anderson* observes the positron.

1940-1950 *Feynman* and *Stückelberg* interpret negative energy solutions as the positive energy of the anti-particle: QED.

Neutrino's

1930 *Pauli* and *Fermi* propose neutrino's to be produced in β-decay ($m_\nu = 0$).

1958 *Cowan* and *Reines* observe inverse beta decay.

1962 *Lederman* and *Schwarz* showed that $\nu_e \neq \nu_\mu$. Conservation of lepton number.

Strangeness

1947 *Rochester* and *Butler* observe V^0 events: K^0 meson.

1950 *Anderson* observes V^0 events: Λ baryon.

The Eightfold Way

1961 *Gell-Mann* makes particle multiplets and predicts the Ω^-.

1964 Ω^- particle found.

The Quark Model

1964 *Gell-Mann* and *Zweig* postulate the existance of quarks

1968 Discovery of quarks in electron-proton collisions (SLAC).

1974 Discovery charm quark (J/ψ) in SLAC & Brookhaven.

1977 Discovery bottom quarks (Υ) in Fermilab.

1979 Discovery of the gluon in 3-jet events (Desy).

1995 Discovery of top quark (Fermilab).

Broken Symmetry

1956 *Lee and Yang* postulate parity violation in weak interaction.

1957 *Wu* et. al. observe parity violation in beta decay.

1964 *Christenson, Cronin, Fitch & Turlay* observe CP violation in neutral K meson decays.

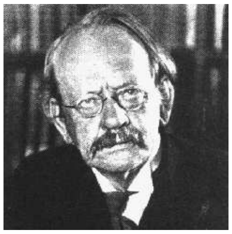
The Standard Model

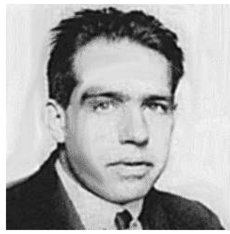
1978 *Glashow*, *Weinberg*, *Salam* formulate Standard Model for electroweak interactions

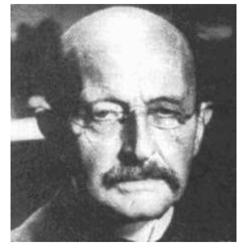
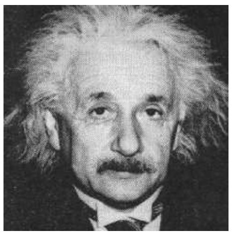
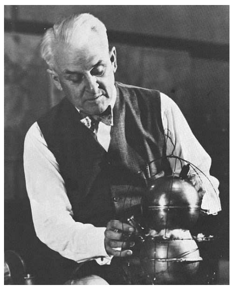
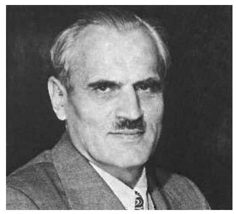
1983 W-boson has been found at CERN.

1984 Z-boson has been found at CERN.

1989-2000 LEP collider has verified Standard Model to high precision.









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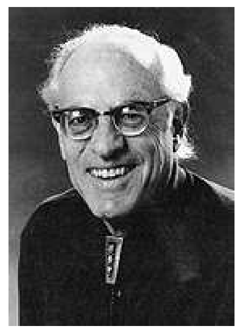
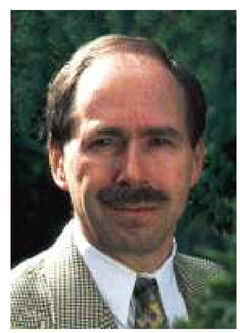











Natural Units

We will often make use of *natural units*. This means that we work in a system where the action is expressed in units of Planck's constant:

$$\hbar \approx 1.055 \times 10^{-34}\text{Js}$$

and velocity is expressed in units of the light speed in vacuum:

$$c = 2.998 \times 10^{8}\text{m/s}.$$

In other words we often use $\hbar = c = 1$.

This implies, however, that the results of calculations must be translated back to measureable quantities in the end. Conversion factors are the following:

quantity	conversion factor	natural unit	normal unit
mass	$1\text{ kg} = 5.61 \times 10^{26}\text{ GeV}$	GeV	GeV/c^2
length	$1\text{ m} = 5.07 \times 10^{15}\text{ GeV}^{-1}$	GeV^{-1}	$\hbar c/\text{ GeV}$
time	$1\text{ s} = 1.52 \times 10^{24}\text{ GeV}^{-1}$	GeV^{-1}	$\hbar/\text{ GeV}$
unit charge	$\text{e} = \sqrt{4\pi\alpha}$	1	$\sqrt{\hbar c}$

Cross sections are expressed in *barn*, which is equal to 10^{-24}cm^2. Energy is expressed in GeV, or 10^9 eV, where 1 eV is the kinetic energy an electron obtains when it is accelerated over a voltage of 1V.

Exercise -1:

Derive the conversion factors for mass, length and time in the table above.

Exercise 0:

The Z-boson particle is a carrier of the weak force. It has a mass of 91.1 GeV. It can be produced experimentally by annihilation of an electron and a positron. The mass of an electron, as well as that of a positron, is 0.511 MeV.

(a) *Can you guess what the Feynman interaction diagram for this process is? Try to draw it.*

(b) *Assume that an electron and a positron are accelerated in opposite directions and collide head-on to produce a Z-boson in the lab frame. Calculate the beam energy required for the electron and the positron in order to produce a Z-boson.*

(c) *Assume that a beam of positron particles is shot on a target containing electrons. Calculate the beam energy required for the positron beam in order to produce Z-bosons.*

(d) *This experiment was carried out in the 1990's. Which method do you think was used? Why?*

Lecture 1

Particles and Forces

Introduction

After Chadwick had discovered the neutron in 1932, the elementary constituents of matter were the *proton* and the *neutron* inside the atomic nucleus, and the *electron* circling around it. With these constituents the atomic elements could be described as well as the chemistry with them. The answer to the question: "What is the world made of?" was indeed rather simple. The force responsible for interactions was the electromagnetic force, which was carried by the *photon*.

There were already some signs that there was more to it:

- Dirac had postulated in 1927 the existence of *anti-matter* as a consequence of his relativistic version of the Schrodinger equation in quantum mechanics. (We will come back to the Dirac theory later on.) The anti-matter partner of the electron, the positron, was actually discovered in 1932 by Anderson (see Fig. 1.1).

- Pauli had postulated the existence of an invisible particle that was produced in nuclear beta decay: the *neutrino*. In a nuclear beta decay process:

  $$N_A \rightarrow N_B + e^-$$

  the energy of the emitted electron is determined by the mass difference of the nuclei N_A and N_B. It was observed that the kinetic energy of the electrons, however, showed a broad mass spectrum (see Fig. 1.2), of which the maximum was equal to the expected kinetic energy. It was as if an additional invisible particle of low mass is produced in the same process: the (anti-) neutrino.

1.1 The Yukawa Potential and the Pi meson

The year 1935 is a turning point in particle physics. Yukawa studied the strong interaction in atomic nuclei and proposed a new particle, a π-meson as the carrier of the nuclear force. His idea was that the nuclear force was carried by a **massive** particle

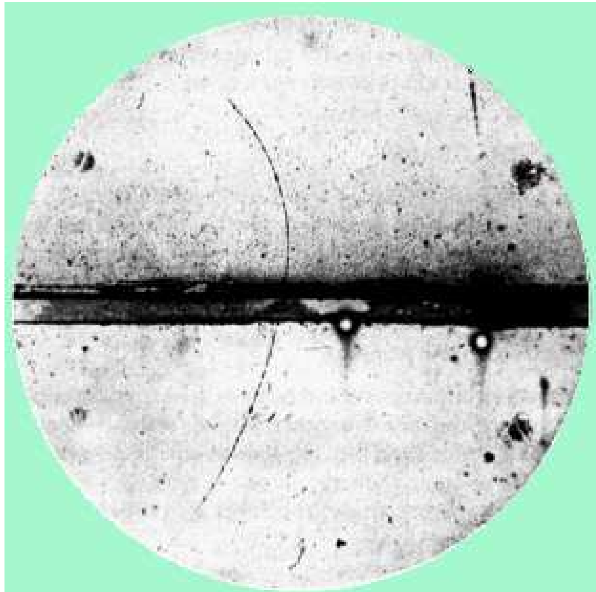


Figure 1.1: The discovery of the positron as reported by Anderson in 1932. Knowing the direction of the B field Anderson deduced that the trace was originating from an anti electron. *Question: how?*

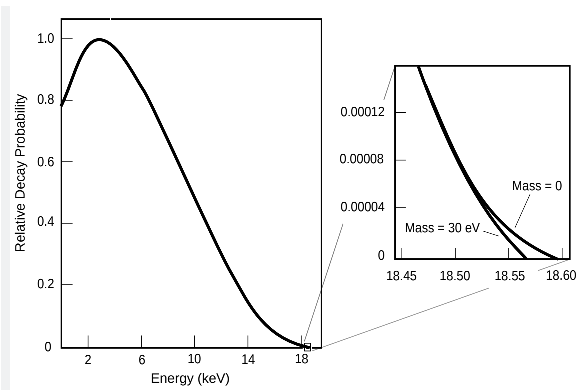


Figure 1. The Beta Decay Spectrum for Molecular Tritium
The plot on the left shows the probability that the emerging electron has a particular energy. If the electron were neutral, the spectrum would peak at higher energy and would be centered roughly on that peak. But because the electron is negatively charged, the positively charged nucleus exerts a drag on it, pulling the peak to a lower energy and generating a lopsided spectrum. A close-up of the endpoint (plot on the right) shows the subtle difference between the expected spectra for a massless neutrino and for a neutrino with a mass of 30 electron volts.

Figure 1.2: The beta spectrum as observed in tritium decay to helium. The endpoint of the spectrum can be used to set a limit of the neutrino mass. *Question: how?*

(in contrast to the massless photon) such that the range of this force is limited to the nuclei.

The qualitative idea is that a virtual particle, the force carrier, can be created for a time $\Delta t < \hbar/2mc^2$. Electromagnetism is transmitted by the massless photon and has an infinite range, the strong force is transmitted by a massive meson and has a limited range, depending on the mass of the meson.

The Yukawa potential (also called the OPEP: One Pion Exchange Potential) is of the form:

$$U(r) = -g^2 \, \frac{e^{-r/R}}{r}$$

where R is called the *range* of the force.

For comparison, the electrostatic potential of a point charge e as seen by a test charge e is given by:

$$V(r) = -e^2 \, \frac{1}{r}$$

The electrostatic potential is obtained in the limit that the range of the force is infinite: $R = \infty$. The constant g is referred to as the *coupling constant* of the interaction.

Exercise 1:

(a) *The wave equation for an electromagnetic potential V is given by:*

$$\Box\, V = 0 \qquad ; \qquad \Box \equiv \partial_\mu \partial^\mu \equiv \frac{\partial^2}{\partial t^2} - \nabla^2$$

which in the static case can be written in the form of Laplace equation:

$$\nabla^2\, V = 0$$

Assuming spherical symmetry, show that this equation leads to the Coulomb potential V(r)
Hint: remember spherical coordinates.

(b) *The wave equation for a massive field is the Klein Gordon equation:*

$$\Box\, U + m^2\, U = 0$$

which, again in the static case can be written in the form:

$$\nabla^2\, U - m^2\, U = 0$$

Show, again assuming spherical symmetry, that Yukawa's potential is a solution of the equation for a massive force carrier. What is the relation between the mass m of the force carrier and the range R of the force?

(c) *Estimate the mass of the π-meson assuming that the range of the nucleon force is* $1.5 \times 10^{-15}\,\mathrm{m} = 1.5\,\mathrm{fm}$.

Yukawa called this particle a *meson* since it is expected to have an intermediate mass between the electron and the nucleon. In 1937 Anderson and Neddermeyer, as well as Street and Stevenson, found that cosmic rays indeed consist of such a middle weight particle. However, in the years after, it became clear that two things were not right:

(1) This particle did not interact strongly, which was very strange for a carrier of the strong force.

(2) Its mass was somewhat too low.

In fact this particle turned out to be the *muon*, the heavier brother of the electron.

In 1947 Powell (as well as Perkins) found the pion to be present in cosmic rays. They took their photographic emulsions to mountain tops to study the contents of cosmic rays (see Fig. 1.3). (In a cosmic ray event a cosmic proton scatters with high energy on an atmospheric nucleon and produces many secondary particles.) Pions produced in the atmosphere decay long before they reach sea level, which is why they were not observed before.

1.2 Strange Particles

After the pion had been identified as Yukawa's strong force carrier and the anti-electron was observed to confirm Dirac's theory, things seemed reasonably under control. The muon was a bit of a mystery. It lead to a famous quote of Isidore Rabi at the conference: "Who ordered *that*?"

But in December 1947 things went all wrong after Rochester and Butler published so-called V^0 events in cloud chamber photographs. What happened was that charged cosmic particles hit a lead target plate and as a result many different types of particles were produced. They were classified as:

baryons: particles whose decay product ultimately includes a proton.

mesons: particles whose decay product ultimately include only leptons or photons.

Why were these events called *strange*? The mystery lies in the fact that certain (neutral) particles were produced (the "V^0's") with a large cross section ($\sim 10^{-27} cm^2$), while they decay according to a process with a small cross section ($\sim 10^{-40} cm^2$). The explanation to this riddle was given by Abraham Pais in 1952 and is called *associated production*. This means that strange particles are always *produced* in pairs by the strong interaction. It was suggested that strange particle carries a *strangeness* quantum number. In the strong interaction one then has the conservation rule $\Delta S = 0$, such that a particle with S=+1 (e.g. a K meson) is simultaneously produced with a particle with S=-1 (e.g. a Λ baryon). These particles then individually *decay* through the weak interaction, which does not conserve strangeness. An example of an associated production event is seen in Fig. 1.4.

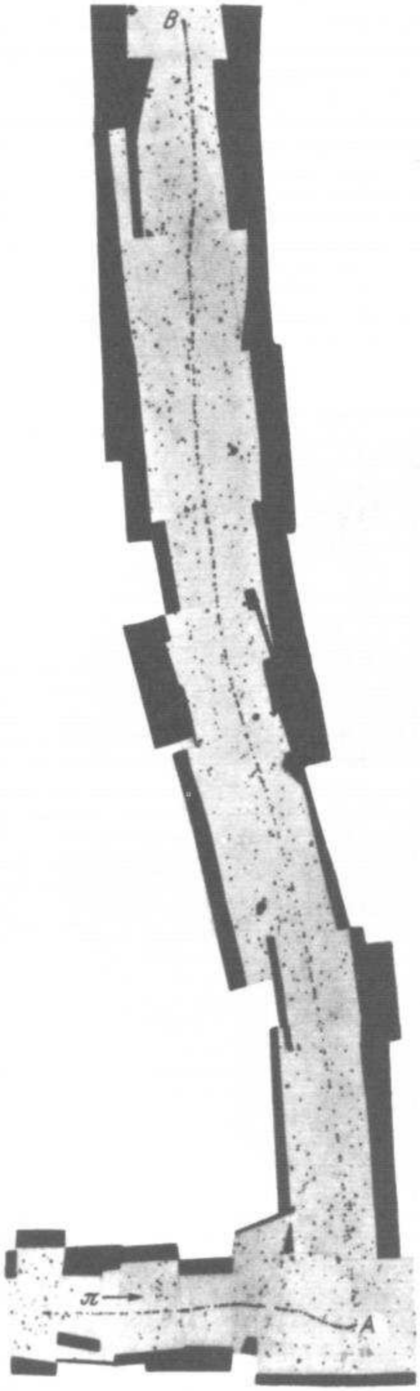


Figure 1.4 One of Powell's earliest pictures showing the track of a pion in a photographic emulsion exposed to cosmic rays at high altitude. The pion (entering from the left) decays into a muon and a neutrino (the latter is electrically neutral, and leaves no track). Reprinted by permission from C. F. Powell, P. H. Fowler, and D. H. Perkins, *The Study of Elementary Particles by the Photographic Method* (New York: Pergamon, 1959). First published in *Nature* **159,** 694 (1947).

Figure 1.3: A pion entering from the left decays into a muon and an invisible neutrino.

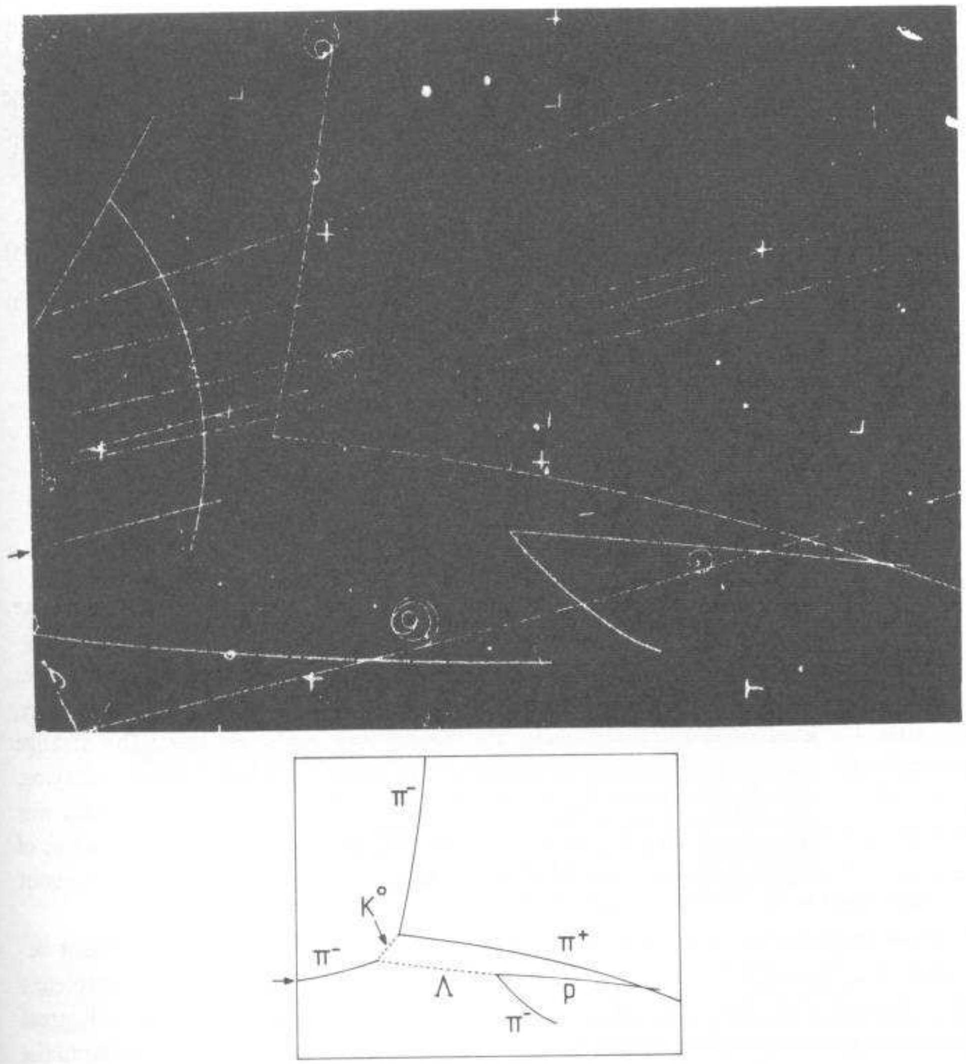


Fig. 2.7 A bubble chamber picture of the associated production reaction $\pi^- + p \to K^0 + \Lambda$. The incoming pion is indicated by the arrow, and the unseen neutrals are detected by their decays $K^0 \to \pi^+ + \pi^-$ and $\Lambda \to \pi^- + p$. This picture was taken in the 10 inch (25 cm) bubble chamber at the Lawrence Berkeley Radiation Laboratory. (Photograph courtesy of the Lawrence Berkeley Radiation Laboratory.)

Figure 1.4: A bubble chamber picture of associated production.

In the years 1950 - 1960 many elementary particles were discovered and one started to speak of the particle zoo. A quote: "The finder of a new particle used to be awarded the Nobel prize, but such a discovery now ought to be punished by a $10.000 fine."

1.3 The Eightfold Way

In the early 60's Murray Gell-Mann (at the same time also Yuvan Ne'eman) observed patterns of symmetry in the discovered mesons and baryons. He plotted the spin 1/2 baryons in a so-called octet (the "eightfold way" after the eighfold way to Nirvana in Buddhism). There is a similarity between Mendeleev's periodic table of elements and the supermultiplets of particles of Gell Mann. Both pointed out a deeper structure of matter. The eightfold way of the lightest baryons and mesons is displayed in Fig. 1.5 and Fig. 1.6. In these graphs the Strangeness quantum number is plotted vertically.

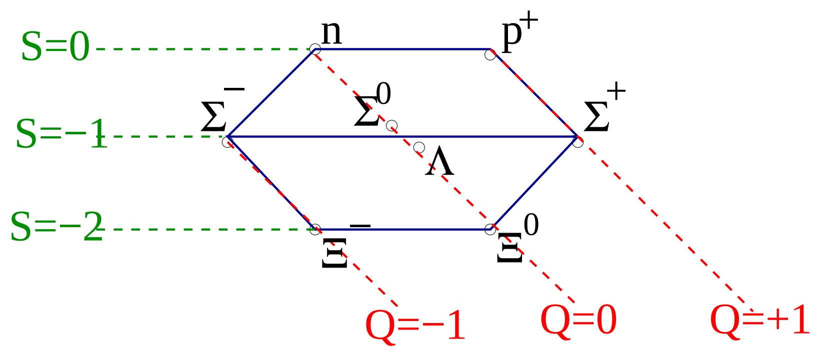


Figure 1.5: Octet of lightest baryons with spin=1/2.

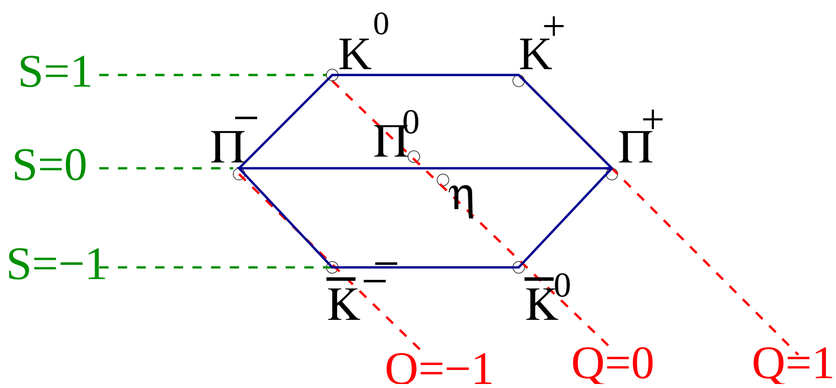


Figure 1.6: Octet with lightest mesons of spin=0

Also heavier hadrons could be given a place in *multiplets*. The baryons with spin=3/2 were seen to form a decuplet, see Fig. 1.7. The particle at the bottom (at S=-3) had not been observed. Not only was it found later on, but also its predicted mass was found to be correct! The discovery of the Ω^- particle is shown in Fig. 1.8.

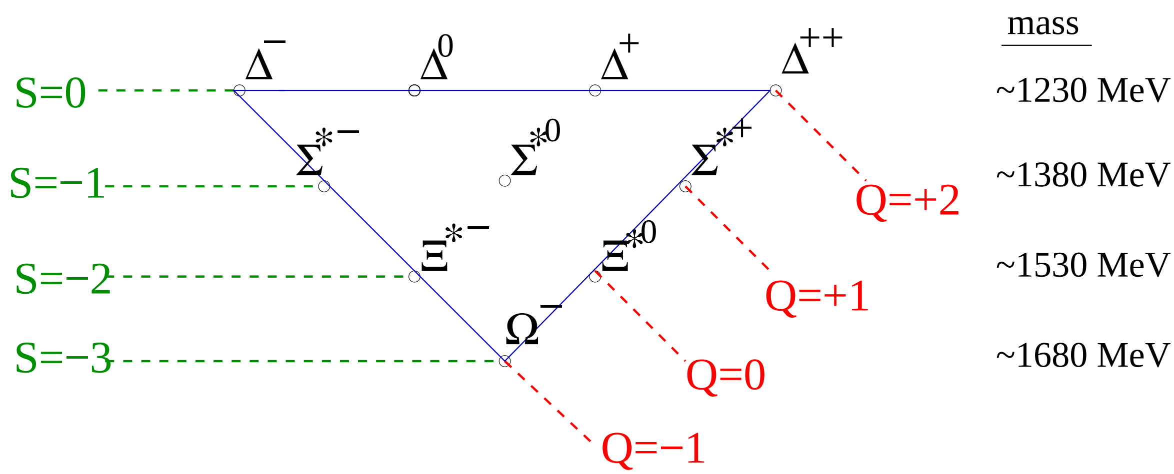


Figure 1.7: Decuplet of baryons with spin=3/2. The Ω^- was not yet observed when this model was introduced. It's mass was predicted.

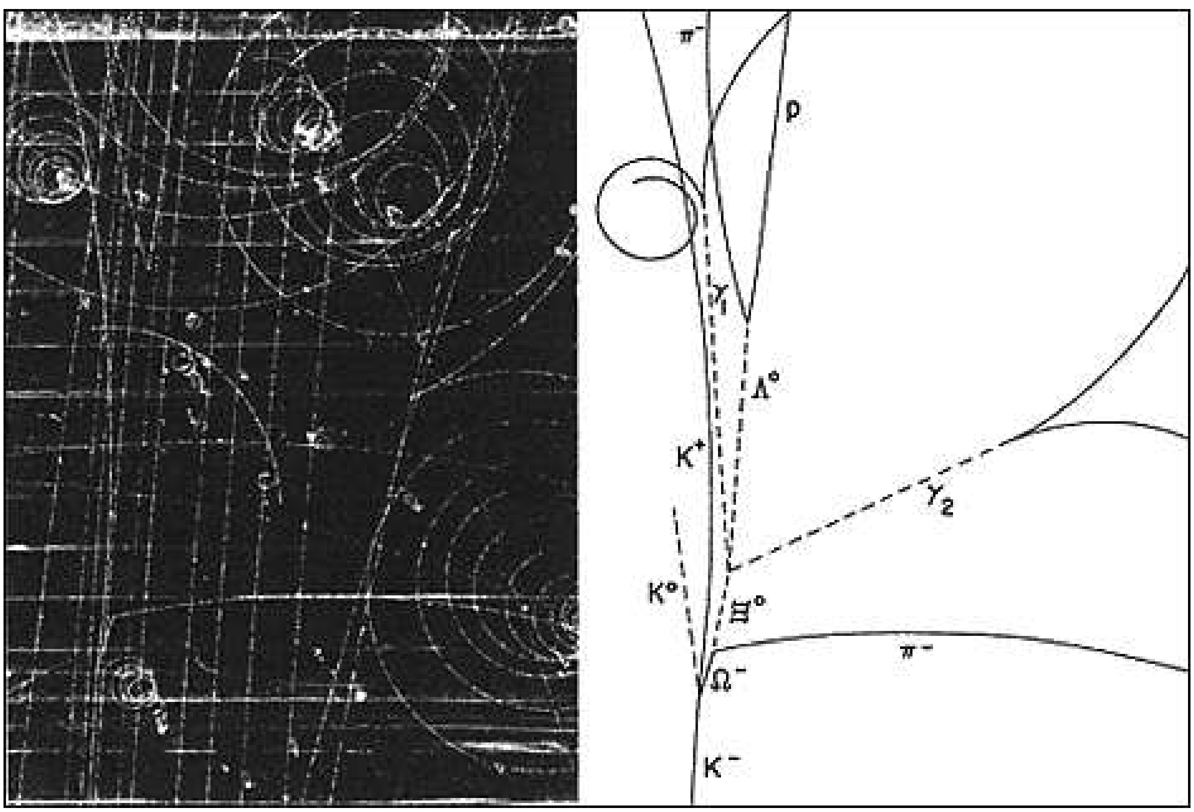


Figure 1.8: Discovery of the omega particle.

1.4 The Quark Model

The observed structure of hadrons in multiplets hinted at an underlying structure. Gell-Mann and Zweig postulated indeed that hadrons consist of more fundamental partons: the quarks. Initially three quarks and their anti-particle were assumed to exist (see Fig. 1.9). A baryon consists of 3 quarks: (q, q, q), while a meson consists of a quark and an antiquark: $(q, \overline{q})$. Mesons can be their own anti-particle, baryons cannot.

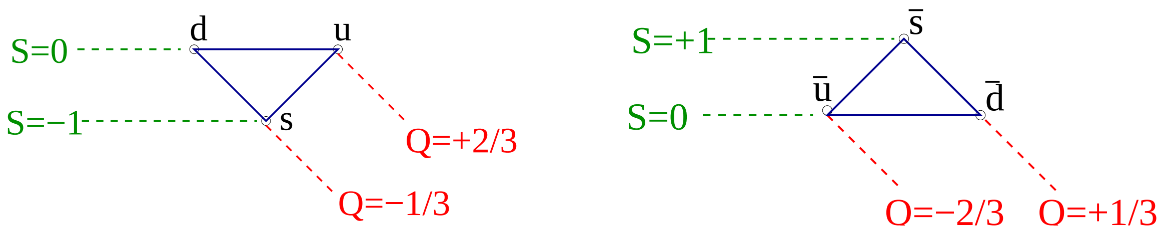


Figure 1.9: The fundamental quarks: u,d,s.

Exercise 2:
Assign the quark contents of the baryon decuplet and the meson octet.

How does this explain that baryons and mesons appear in the form of octets, decuplets, nonets etc.? For example a baryon, consisting of 3 quarks with 3 flavours (u,d,s) could in principle lead to 3x3x3=27 combinations. The answer lies in the fact that the wave function of fermions is subject to a symmetry under exchange of fermions. The total wave function must be anti-symmetric with respect to the interchange of two fermions.

$$\psi\left(baryon\right) = \psi\left(space\right) \cdot \phi\left(spin\right) \cdot \chi\left(flavour\right) \cdot \zeta\left(color\right)$$

These symmetry aspects are reflected in group theory where one encounters expressions as: $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ and $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

For more information on the static quark model read §2.10 and §2.11 in H&M, §5.5 and §5.6 in Griffiths, or chapter 5 in the book of Perkins.

1.4.1 Color

As indicated in the wave function above, a quark has another internal degree of freedom. In addition to electric charge a quark has a different charge, of which there are 3 types. This charge is referred to as the color quantum number, labelled as *r, g, b*. Evidence for the existence of color comes from the ratio of the cross section:

$$R \equiv \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = N_C \sum_i Q_i^2$$

where the sum runs over the quark types that can be produced at the available energy. The plot in Fig. 1.10 shows this ratio, from which the result $N_C = 3$ is obtained.

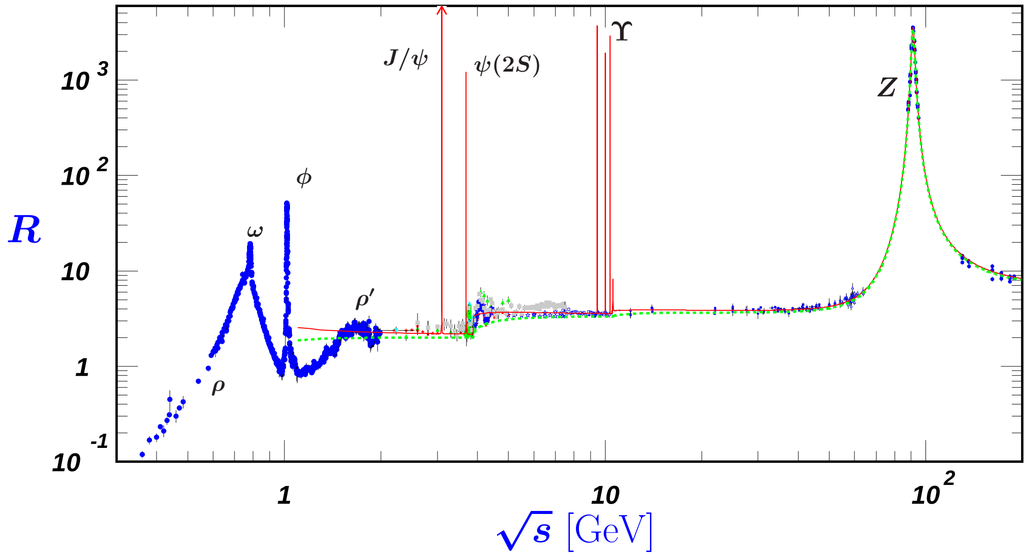


Figure 1.10: The R ratio.

Exercise 3: The Quark Model

(a) *Quarks are fermions with spin 1/2. Show that the spin of a meson (2 quarks) can be either a triplet of spin 1 or a singlet of spin 0.*
Hint: Remember the Clebsch Gordon coefficients in adding quantum numbers.
In group theory this is often represented as the product of two doublets leads to the sum of a triplet and a singlet: $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$ *or, in terms of quantum numbers:* $1/2 \otimes 1/2 = 1 \oplus 0$.

(b) *Show that for baryon spin states we can write:* $1/2 \otimes 1/2 \otimes 1/2 = 3/2 \oplus 1/2 \oplus 1/2$ *or equivalently* $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}$

(c) *Let us restrict ourselves to two quark flavours: u and d. We introduce a new quantum number, called* isospin *in complete analogy with spin, and we refer to the u quark as the isospin +1/2 component and the d quark to the isospin -1/2 component (or u= isospin "up" and d=isospin "down"). What are the possible isospin values for the resulting baryon?*

(d) *The Δ^{++} particle is in the lowest angular momentum state ($L = 0$) and has spin $J_3 = 3/2$ and isospin $I_3 = 3/2$. The overall wavefunction (L⇒space-part, S⇒spin-part, I⇒isospin-part) must be anti-symmetric under exchange of any of the quarks. The symmetry of the space, spin and isospin part has a consequence for the required symmetry of the Color part of the wave function. Write down the color part of the wave-function taking into account that the particle is color neutral.*

(e) *In the case that we include the s quark the flavour part of the wave function becomes:* $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$.
In the case that we include all 6 quarks it becomes: $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6}$. *However, this is not a good symmetry. Why not?*

1.5 The Standard Model

The fundamental constituents of matter and the force carriers in the Standard Model can be represented as follows:

The fundamental particles:

charge	**Quarks**		
$\frac{2}{3}$	u (up) 1.5–4 MeV	c (charm) 1.15–1.35 GeV	t (top) (174.3 ± 5.1) GeV
$-\frac{1}{3}$	d (down) 4–8 MeV	s (strange) 80–130 MeV	b (bottom) 4.1–4.4 GeV
charge	**Leptons**		
0	ν_e (e neutrino) < 3 eV	ν_μ (μ neutrino) < 0.19 MeV	ν_τ (τ neutrino) < 18.2 MeV
-1	e (electron) 0.511 MeV	μ (muon) 106 MeV	τ (tau) 1.78 GeV

The forces, their mediating bosons and their relative strength:

Force	Boson	Relative strength
Strong	g (8 gluons)	$\alpha_s \sim \mathcal{O}(1)$
Electromagnetic	γ (photon)	$\alpha \sim \mathcal{O}(10^{-2})$
Weak	Z^0,$W^\pm$ (weak bosons)	$\alpha_W \sim \mathcal{O}(10^{-6})$

Some definitions:

hadron (greek: strong)	particle that feels the strong interaction
lepton (greek: light, weak)	particle that feels only weak interaction
baryon (greek: heavy)	particle consisting of three quarks
meson (greek: middle)	particle consisting of a quark and an anti-quark
pentaquark	a hypothetical particle consisting of 4 quarks and an anti-quark
fermion	half-integer spin particle
boson	integer spin particle
gauge-boson	force carrier as predicted from local gauge invariance

In the Standard Model forces originate from a mechanism called local gauge invariance, which wil be discussed later on in the course. The strong force (or color force) is mediated by gluons, the weak force by intermediate vector bosons, and the electromagnetic force by photons. The fundamental diagrams are represented below.

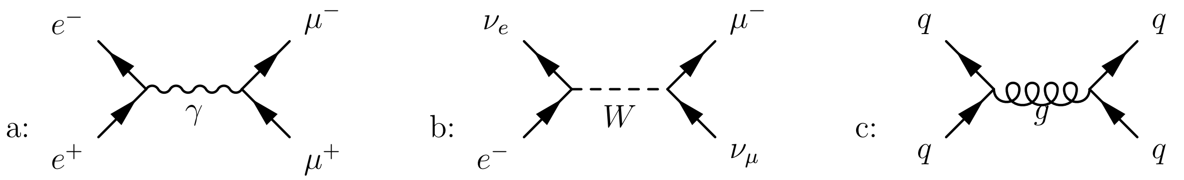


Figure 1.11: Feynman diagrams of fundamental lowest order perturbation theory processes in a: electromagnetic, b: weak and c: strong interaction.

There is an important difference between the electromagnetic force on one hand, and the weak and strong force on the other hand. The photon does not carry charge and, therefore, does not interact with itself. The gluons, however, carry color and do interact amongst each other. Also, the weak vector bosons carry weak isospin and undergo a self coupling.

The strength of an interaction is determined by the coupling constant as well as the mass of the vector boson. Contrary to its name the couplings are not constant, but vary as a function of energy. At a momentum transfer of 10^{15} GeV the couplings of electromagnetic, weak and strong interaction all have the same value. In the quest of unification it is often assumed that the three forces unify to a grand unification force at this energy.

Due to the self coupling of the force carriers the running of the coupling constants of the weak and strong interaction are opposite to that of electromagnetism. Electromagnetism becomes weaker at low momentum (i.e. at large distance), the weak and the strong force become stronger at low momentum or large distance. The strong interaction coupling even diverges at momenta less than a few 100 MeV (the perturbative QCD description breaks down). This leads to confinement: the existence of colored objects (i.e. objects with net strong charge) is forbidden.

Finally, the Standard Model includes a, not yet observed, scalar Higgs boson, which provides mass to the vector bosons and fermions in the Brout-Englert-Higgs mechanism.

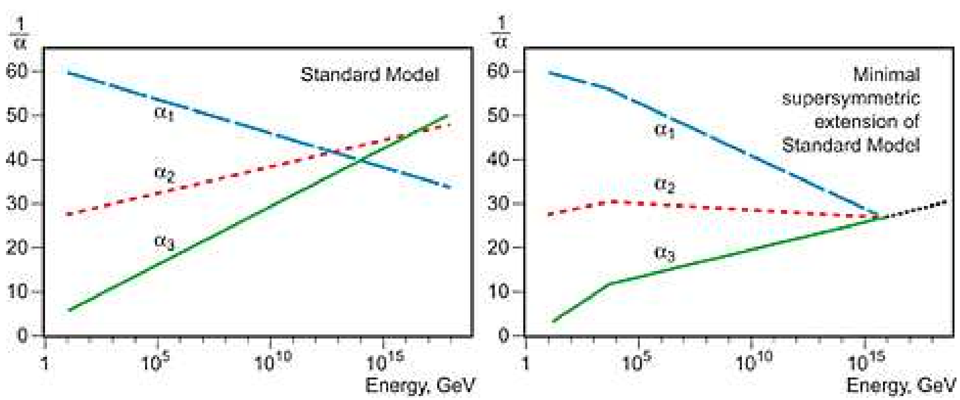


Figure 1.12: Running of the coupling constants and possible unification point. On the left: Standard Model. On the right: Supersymmetric Standard Model.

Open Questions

- Does the Higgs in fact exist?
- Why are the masses of the particles what they are?
- Why are there 3 generations of fermions?
- Are quarks and leptons truly fundamental?
- Why is the charge of the electron *exactly* opposite to that of the proton. Or: why is the total charge of leptons and quarks exactly equal to 0?
- Is a neutrino its own anti-particle?
- Can all forces be described in a single theory (unification)?
- Why is there no anti matter in the universe?
- What is the source of dark matter?
- What is the source of dark energy?

Lecture 2

Wave Equations and Anti Particles

Introduction

In the course we develop a quantum mechanical framework to describe electromagnetic scattering, in short Quantum Electrodynamics (QED). The way we build it up is that we first derive a framework for non-relativistic scattering of spinless particles, which we then extend to the relativistic case. Also we will start with the wave equations for particles without spin, and address the spin 1/2 particles later on in the lectures ("the Dirac equation').

What is a spinless particle? There are two ways that you can think of it: either as charged mesons (e.g. pions or kaons) for which the strong interaction has been "switched off" or for electrons or muons for which the fact that they are spin-1/2 particles is ignored. In short: it not a very realistic case.

2.1 Non Relativistic Wave Equations

If we start with the non relativistic relation between kinetic energy and momentum

$$E = \frac{\vec{p}^2}{2m}$$

and make the quantum mechanical substitution:

$$E \rightarrow i\frac{\partial}{\partial t} \qquad \text{and} \qquad \vec{p} \rightarrow -i\vec{\nabla}$$

then we end up with Schrödinger's equation:

$$\boxed{i\frac{\partial}{\partial t}\,\psi = \frac{-1}{2m}\,\nabla^2\psi}$$

In electrodynamics we have the continuity equation ("Gauss law") which relates a current to a change of charge:

$$\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$$

where j = the current density and ρ = the charge density.

This is a rather general law stat can be stated in words as: "The change of charge in a given volume equals the current through the surrounding surface."

Can we make use of the continuity equation in quantum mechnics? Let us multiply the Schrödinger equation from the left by ψ^* and do the same for the complex conjugates:

$$\begin{aligned}
\psi^*\, i\frac{\partial\psi}{\partial t} &= \psi^* \left(\frac{-1}{2m}\right) \nabla^2\psi \\
\psi\, -i\frac{\partial\psi^*}{\partial t} &= \psi \left(\frac{-1}{2m}\right) \nabla^2\psi^* \\
\hline
\frac{\partial}{\partial t}\underbrace{(\psi^*\,\psi)}_{\rho} &= -\vec{\nabla}\cdot\underbrace{\left[\frac{i}{2m}\left(\psi\,\vec{\nabla}\psi^* - \psi^*\,\vec{\nabla}\psi\right)\right]}_{\vec{j}}
\end{aligned} \quad -$$

In the result we can recognize again the continuity equation if we interpret the density and current as indicated.

Example: Consider a solution to the Schrödinger equation for a free particle:

$$\psi = N\, e^{i(\vec{p}\vec{x}-Et)} \qquad \text{(show it is a solution)}$$

then:

$$\begin{aligned}
\rho &= \psi^*\,\psi = |N|^2 \\
\vec{j} &= \frac{i}{2m}\left(\psi\,\vec{\nabla}\psi^* - \psi^*\,\vec{\nabla}\psi\right) = \frac{|N|^2}{m}\,\vec{p}
\end{aligned}$$

Exercise 4:

Derive the expressions for ρ and $\vec{j}$ in the above example explicitly starting from the Schrödinger equation and its complex conjugate.

2.2 Relativistic Wave Equations

If we start with the relativistic equation:

$$E^2 = \vec{p}^2 + m^2$$

and again make the substitution:

$$E \to i\frac{\partial}{\partial t} \qquad \text{and} \qquad \vec{p} \to -i\vec{\nabla}$$

then we end up with the Klein Gordon equation for a wavefunction ϕ:

$$\boxed{-\frac{\partial^2}{\partial t^2}\phi = -\nabla^2\phi + m^2\,\phi}$$

or in 4-vector notation:

$$\left(\Box + m^2\right)\ \phi(x) = 0$$
$$\text{or}\ :\quad \left(\partial_\mu \partial^\mu + m^2\right)\ \phi(x) = 0$$

A solution is again provided by plane waves:

$$\phi(x) = N\ e^{-ip_\mu x^\mu} \qquad\qquad \text{with eigenvalues}\ \ E^2 = \vec{p}^2 + m^2$$

In the same way as before we can define a current density by multiplying the K.G. equation for ϕ from the left with ϕ^* and doing the same to the complex conjugate equation:

$$\begin{aligned}
-i\phi^* \left(-\frac{\partial^2 \phi}{\partial t^2}\right) &= -i\phi^* \left(-\nabla^2 \phi + m^2\ \phi\right) \\
i\phi \left(-\frac{\partial^2 \phi^*}{\partial t^2}\right) &= i\phi \left(-\nabla^2 \phi^* + m^2 \phi^*\right) \\
\hline
\frac{\partial}{\partial t} \underbrace{i\left(\phi^*\ \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right)}_{\rho} &= \vec{\nabla} \cdot \underbrace{\left[i\ \left(\phi^*\ \vec{\nabla}\phi - \phi\ \vec{\nabla}\phi^*\right)\right]}_{-\vec{j}} \qquad +
\end{aligned}$$

where we can recognize again the continuity equation. In 4-vector notation it becomes:

$$\begin{aligned}
j^\mu &= \left(\rho, \vec{j}\right) = i\left[\phi^*\left(\partial^\mu \phi\right) - \left(\partial^\mu \phi^*\right)\phi\right] \\
\partial_\mu j^\mu &= 0
\end{aligned}$$

Let us substitute the plane wave solutions $\phi = N\ e^{-ipx}$ then:

$$\begin{aligned}
\rho &= 2\ |N|^2\ E \\
\vec{j} &= 2\ |N|^2\ \vec{p} \\
\text{or}\ :\ \rightarrow\ j^\mu &= 2\ |N|^2\ p^\mu
\end{aligned}$$

Exercise 5:
Derive the expressions for ρ and $\vec{j}$ explicitly starting from the Klein Gordon equation.

But now we really have an interpretation problem! There are **two** solutions: $E = \pm\sqrt{\vec{p}^2 + m^2}$. The solution with $E < 0$ is difficult to interpret as it means $\rho < 0$.

Exercise 6:
The relativitic energy-momentum relation can be written as:

$$E = \sqrt{\vec{p}^2 + m^2} \tag{2.1}$$

This is linear in $E = \partial/\partial t$, but we don't know what to do with the square root of the momentum operator. However, for small $\vec{p}$ we can expand the expression in powers of $\vec{p}$. Do this up and including to order $\vec{p}^2$ and write down the resulting wave equation. Determine the probability density and the current density.

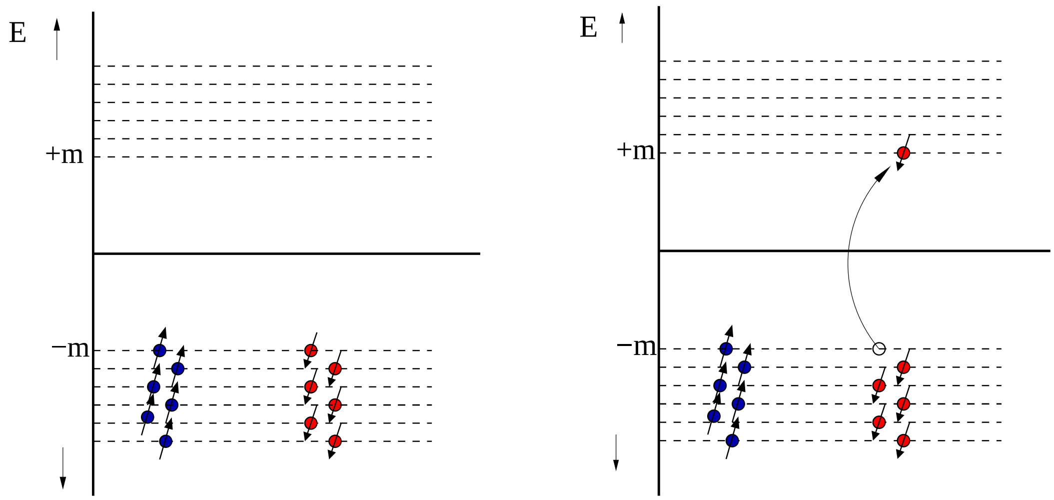


Figure 2.1: Dirac's interpretation of negative energy solutions: "holes"

2.3 Interpretation of negative energy solutions

2.3.1 Dirac's interpretation

In 1927 Dirac offered a new interpretation of the negative energy states. He introduced a new wave equation which in fact was *linear* in time and space, which will be discussed later on in the course. It turned out to automatically describe particles with spin 1/2. At this point in the course we consider spinless particles. Stated otherwise: the wave function ψ or ϕ is a scalar quantity as there is no individual spin "up" or spin "down" component.

According to the Pauli exclusion principle, Dirac knew that there can not be two identical particles in the same quantum state. Dirac's picture of the vacuum and of a particle are schematically represented in Fig. 2.1.

The plot shows all the avaliable energy levels of an electron. It's lowest absolute energy level is given by $|E| = m$. Dirac imagined the vacuum to contain an infinite number of states with negative energy which are all occupied. Since an electron is a spin-1/2 particle each state can only contain one spin "up" electron and one spin-"down" electron. All the negative energy levels are filled. Such a vacuum ("sea") is not detectable since the electrons in it cannot interact, i.e. go to another state.

If energy is added to the system, an electron can be kicked out of the sea. It now gets a positive energy with $E > m$. This means this electron becomes visible as it can now interact. At the same time a "hole" in the sea has appeared. This hole can be interpreted as a positive charge at that position. Dirac's hope was that he could describe the proton in such a way.

2.3.2 Pauli-Weisskopf Interpretation

Pauli and Weiskopf proposed a simpler scheme in 1934 in which they re-interpreted the opposite sign solutions of the Klein Gordon equation as the opposite charges:

ρ = electric charge density

$\vec{j}$ = electric current density

and the $-$ and $+$ solutions indicate the electron and positron. The positron then had of course the mass as the electron. The positron was discovered in 1931 by Anderson.

2.3.3 Feynman-Stückelberg Interpretation

The current density for a particle with charge $-e$ and momentum $(E, \vec{p})$ is:

$$j^\mu(-e) = -2e\ |N|^2\ p^\mu = -2e\ |N|^2\ (E, \vec{p})$$

The current density for a particle with charge $+e$ and momentum $(E, \vec{p})$ is:

$$j^\mu(+e) = +2e\ |N|^2\ p^\mu = -2e\ |N|^2\ (-E, -\vec{p})$$

This means that the positive energy solution for a positron **is** the negative energy solution for an electron.

Note that indeed the wave function $Ne^{ip_\mu x^\mu} = Ne^{ip_\mu x^\mu}$ is invariant under: $p^\mu \to -p^\mu$ and $x_\mu \to -x_\mu$. So the wave functions that describe particles also describe anti-particles. The negative energy solutions give particles travelling backwards in time. They are the same as the positive energy solutions of anti-particles travelling forward in time. This is indicated in Fig. 2.2.

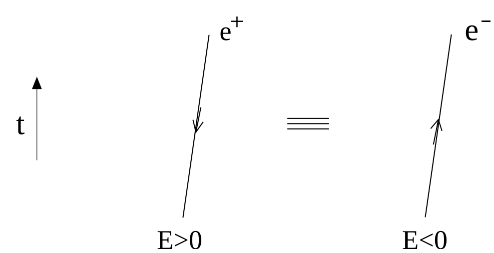


Figure 2.2: A positron travelling forward in time **is** an electron travelling backwards in time.

As a consequence of the Feynman-Stückelberg interpretation the process of an absorption of a positron with energy $-E$ is the same as the emission of an electron with energy E (see Fig.2.3). In the calculations with Feynman diagrams we have made the convention that all scattering processes are calculated in terms of *particles* and **not** antiparticles. As an example, the process of an incoming positron scattering off a potential will be calculated as that of an electron travelling back in time (see Fig. 2.4).

Let us consider the scattering of an electron in a potential. The probability of a process is calculated in perturbation theory in terms of basic scattering processes (i.e.

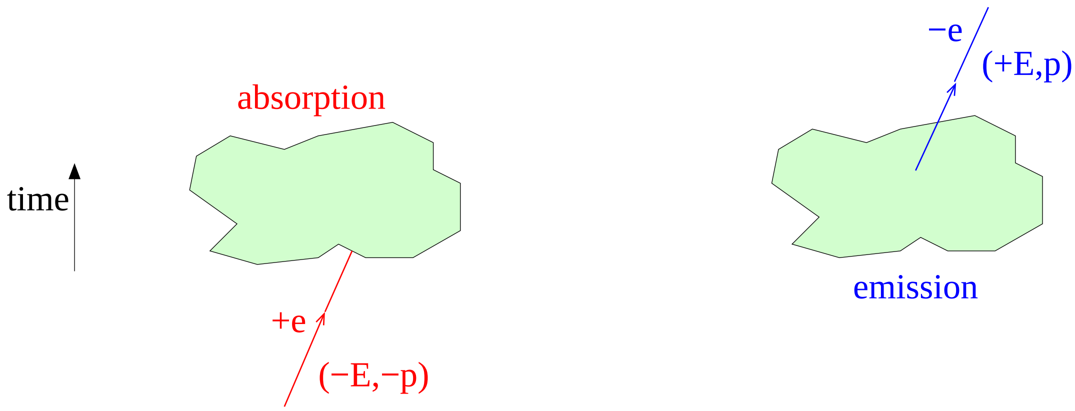


Figure 2.3: There is no difference between the process of an absorption of a positron with $p^\mu = (-E, -\vec{p})$ and the emission of an electron with $p^\mu = (e, \vec{p})$.

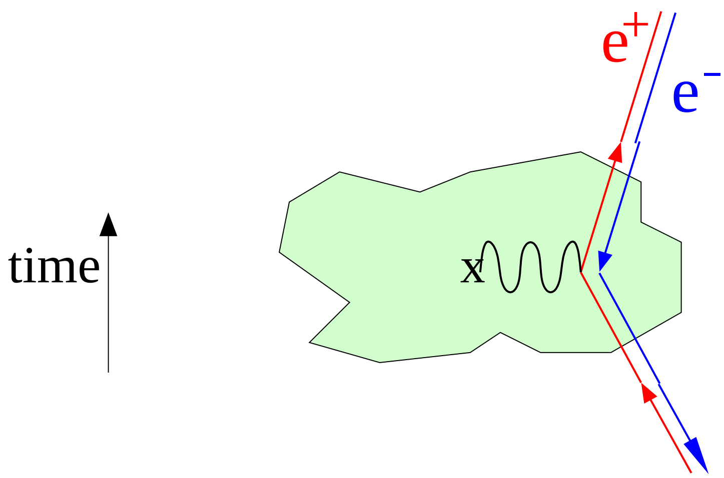


Figure 2.4: In terms of the charge current density $j^\mu_{+(E,\vec{p})}(+e) \equiv j^\mu_{-(E,\vec{p})}(-e)$

Feynman diagrams). In Fig. 2.5 the first and second order scattering of the electron is illustrated. To first order a single photon carries the interaction between the electron and the potential. When the calculation is extended to second order the electron interacts twice with the field. It is interesting to note that this scattering can occur in two time orderings as indicated in the figure. Note that the observable path of the electron before and after the scattering process is identical in the two processes. Because of our anti-particle interpretation, the second picture is also possible. It can be viewed in two ways:

- The electron scatters at time t_2 runs back in time and scatters at t_1.
- First at time t_1 "spontaneously" an e^-e^+ pair is created from the vacuum. Later-on, at time t_2, the produced positron annihilates with the incoming electron, while the produced electron emerges from the scattering process.

In quantum mechanics both time ordered processes (the left and the right picture) must be included in the calculation of the cross section. We realize that the vacuum has become a complex environment since particle pairs can spontaneously emerge from it and dissolve into it!

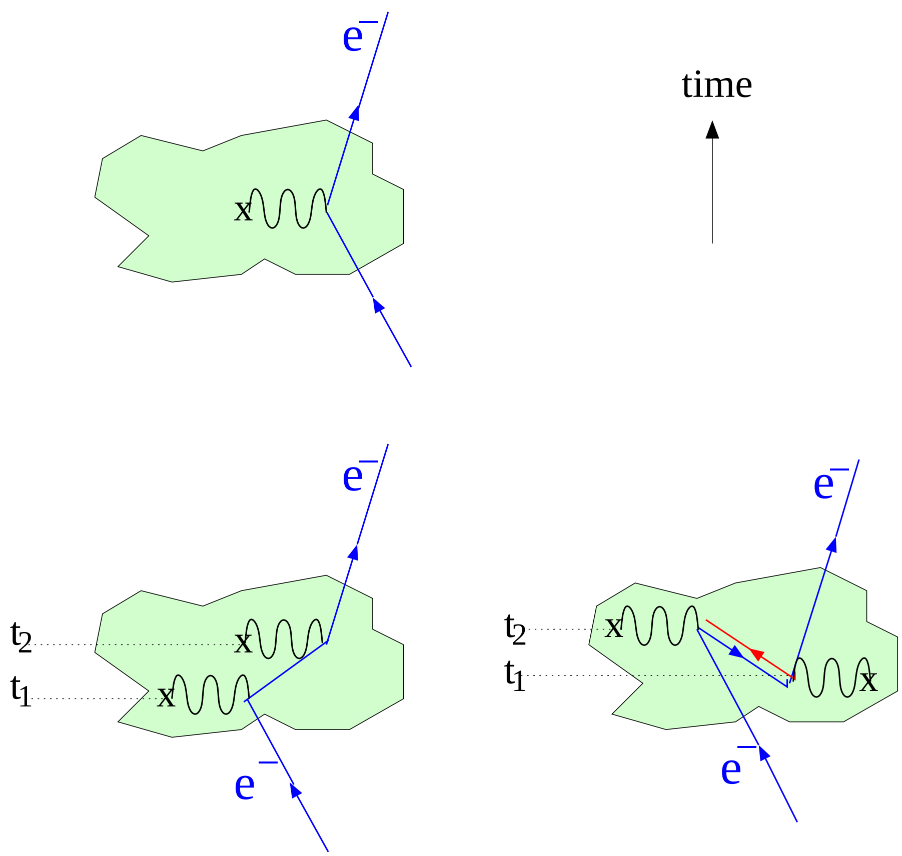
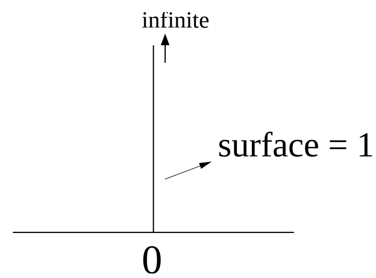


Figure 2.5: First and second order scattering.

2.4 The Dirac Deltafunction



The definition of the Dirac delta function is:

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases}$$

in such a way that:

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

In that case one has: $f(x)\,\delta(x) = f(0)\,\delta(x)$ for any function f. Therefore:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = \int_{-\infty}^{\infty} f(0)\,\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0)$$

Exercise 7:

The consequences of the definition of the Dirac Delta function are the following:

(a) *Prove that:*

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

(b) *Prove that:*

$$\delta\left(g\left(x\right)\right) = \sum_{i=1}^{n} \frac{1}{\left|g'\left(x_i\right)\right|}\,\delta\left(x - x_i\right)$$

where the sum i runs over the 0-points of $g(x)$, i.e.:$g(x_i) = 0$.
Hint: make a Taylor expansion of g around the 0-points.

Exercise 8

Characteristics of the Dirac delta function:

(a) *Calculate* $\int_0^3 \ln(1+x)\,\delta(\pi - x)\,dx$

(b) *Calculate* $\int_0^3 (2x^2 + 7x + 4)\,\delta(x-1)\,dx$

(c) *Calculate* $\int_0^3 \ln(x^3)\,\delta(x/e - 1)\,dx$

(d) *Simplify* $\delta\left(\sqrt{(5x-1)} - x - 1\right)$

(e) *Simplify* $\delta(\sin x)$ *and draw the function*

Lecture 3

The Electromagnetic Field

3.1 Maxwell Equations

As we eventually want to calculate processes in QED, let us look at the electromagnetic field and the photon. The Maxwell equations in vacuum are:

(1)	$\vec{\nabla} \cdot \vec{E} = \rho$	Gauss law
(2)	$\vec{\nabla} \cdot \vec{B} = 0$	No magnetic poles
(3)	$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$	Faraday′s law of induction
(4)	$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$	Relate B field to a current

From the first and the fourth equation we can indeed derive the continuity equation:

$$\boxed{\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}}$$

In scattering with particles we want to work relativistic, so it would be suitable if we could formulate Maxwell equations in a covariant way; i.e. in a manifestly Lorentz invariant way.

To do this we introduce a mathematical tool: the potential $A^\mu = \left(V, \vec{A}\right)$. We note at this point that the fields $\vec{E}, \vec{B}$ are physical, while the potential is *not*. Remember that the following identities are valid for any vector field $\vec{A}$ and scalar field V:

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\nabla} V\right) &= 0 \qquad \text{(rotation of gradient is 0)} \\ \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) &= 0 \qquad \text{(divergence of a rotation is 0)} \end{aligned}$$

We *choose* the potential in such a way that two Maxwell equations are automatically fullfilled:

1. $\vec{B} = \vec{\nabla} \times \vec{A}$
   Then, automatically it follows that: $\vec{\nabla} \cdot \vec{B} = 0$.

2. $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$
   Then, automatically it follows that: $\vec{\nabla} \times \vec{E} = -\frac{\partial(\vec{\nabla}\times\vec{A})}{\partial t} - 0 = -\frac{\partial \vec{B}}{\partial t}$.

So, by a suitable defition of how the potential A^μ is related to the physical fields, automatically Maxwell equations (2) and (3) are fullfilled.

<u>**Exercise 9:**</u>

(a) *Show that Maxwell's equations can be written as:*

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$$

Hint: Derive the expressions for ρ *and* $\vec{j}$ *explicitly and note that* $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = -\nabla^2 \vec{A} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right)$

(b) *It can be made even more compact by introducing the tensor:* $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. *Show that with this definition Maxwell's equations reduce to:*

$$\partial_\mu F^{\mu\nu} = j^\nu$$

Intermezzo: 4-vector notation

Assume that we have a contravariant vector:

$$A^\mu = \left(A^0, A^1, A^2, A^3\right) = \left(A^0, \vec{A}\right)$$

then the covariant vector is obtained as:

$$A_\nu = (A_0, A_1, A_2, A_3) = g_{\mu\nu} A^\mu = \left(A^0, -A^1, -A^2 - A^3\right) = \left(A^0, -\vec{A}\right)$$

since we use the metric sensor:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

There is one exception to this: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. For the derivative 4-vector we then find:

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \qquad\qquad \partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

which is opposite to the contravariant and covariant behaviour of a usual 4-vector A^μ defined above.

3.2 Gauge Invariance

Since we have introduced the potential A^μ as a mathematical tool rather than as a physical field we can choose **any** A^μ potential as long as the $\vec{E}$ and $\vec{B}$ fields don't change. After re-examining the equations that define A we realize that there is a freedom to make so-caled gauge transformations which do not affect the physical fields $\vec{E}$ and $\vec{B}$:

$$\begin{aligned} A^\mu \rightarrow A'^\mu &= A^\mu + \partial^\mu \lambda && \text{or} \\ A_\mu \rightarrow A'_\mu &= A_\mu + \partial_\mu \lambda && \text{for any scalar field } \lambda \end{aligned}$$

In terms of the Voltage V and vectors potential $\vec{A}$ we have:

$$\begin{aligned} V' &= V + \frac{\partial \lambda}{\partial t} \\ \vec{A}' &= \vec{A} - \vec{\nabla} \lambda \end{aligned}$$

<u>**Exercise 10:**</u>
Show explicitly that in such gauge transformations the $\vec{E}$ and $\vec{B}$ fields do not change:

$$\begin{aligned} \vec{B}' &= \vec{\nabla} \times \vec{A}' = ... = \vec{B} \\ \vec{E}' &= -\frac{\partial \vec{A}'}{\partial t} - \vec{\nabla} V' = ... = \vec{E} \end{aligned}$$

The laws of physics are gauge invariant. This implies that we can choose any gauge to calculate physics quantities. It is most elegant if we can perform all calculations in a way that is manifestly gauge invariant. However, sometimes we choose a particular gauge in order to make the expressions in calculations simpler.

A gauge choice that is often made is called the <u>*Lorentz condition*</u>, in which we choose A^μ according to:

$$\boxed{\partial_\mu A^\mu = 0}$$

<u>**Exercise 11:**</u>
Show that it is always possible to define a A^μ field according to the Lorentz gauge. To do this assume that for a given A^μ field one has: $\partial_\mu A^\mu \neq 0$. Give then the equation for the gauge field λ by which that A^μ field must be transformed to obtain the Lorentz gauge.

In the Lorentz gauge the Maxwell equations simplify further:

$$\begin{aligned} \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu &= j^\nu && \text{now becomes :} \\ \partial_\mu \partial^\mu A^\nu &= j^\nu \end{aligned}$$

However, A^μ still has some freedom since we have fixed: $\partial_\mu (\partial^\mu \lambda)$, but we have not yet fixed $\partial^\mu \lambda$! In other words a gauge transformation of the form:

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \lambda \qquad \text{with :} \quad \Box\lambda = \partial_\mu \partial^\mu \lambda = 0$$

is still allowed within the Lorentz gauge $\partial_\mu A^\mu = 0$. However, we can in addition impose the *Coulomb condition*:

$$\boxed{A^0 = 0 \qquad \text{or equivalently :} \qquad \vec{\nabla} \cdot \vec{A} = 0}$$

At the same time we realize, however, that this is not elegant as we give the "0-th component" or "time-component" of the 4-vector a special treatment. Therefore the choice of this gauge is not Lorentz invariant. This means that one has to chose a different gauge condition if one goes from one reference frame to a different reference frame. This is allowed since the choice of the gauge is irrelevant for the physics observables, but it sometimes considered "not elegant".

3.3 The photon

Let us turn to the wave function of the photon. We start with Maxwell's equation and consider the case in vacuum:

$$\Box A^\mu = j^\mu \qquad \rightarrow \quad \text{vacuum :} \ \ j^\mu = 0 \quad \rightarrow \qquad \Box A^\mu = 0$$

Immediately we recognize in each component the Klein Gordon equation of a quantum mechanical particle with mass $m = 0$: $(\Box + m^2)\,\phi(x) = 0$ (see previous Lecture). This particle is the photon.

The plane wave solutions of the massless K.-G. equation are:

$$A^\mu (x) = N \varepsilon^\mu (\vec{p}) \ e^{-ip_\nu x^\nu} \qquad \text{with :} \quad p^2 = p_\mu p^\mu = 0$$

We are describing a *vector* field A^μ since the field has a Lorentz index μ. The vector $\varepsilon^\mu(\vec{p})$ is the *polarization* vector: it has 4 components. Does this mean that the photon has 4 independent polarizations (degrees of freedom)?

Let us take a look at the gauge conditions and we see that there are some restrictions:

- Lorentz condition:

  $$\partial_\mu A^\mu = 0 \quad \Rightarrow \quad p_\mu \, \varepsilon^\mu = 0$$

  This reduces the number of independent components to three. For the gauge field this implies $\Box\lambda = 0$ and we see that we can choose the gauge field as:

  $$\begin{aligned} \lambda &= iae^{-ip_\nu x^\nu} \\ \partial^\mu \lambda &= ap^\mu e^{-ip_\nu x^\nu} \end{aligned}$$

where a is a constant. Thus the gauge transformation looks like

$$A^\mu \to A'^\mu = N\left(\varepsilon^\mu e^{-ip_\nu x^\nu} + ap^\mu e^{-ip_\nu x^u}\right)$$

or, in terms of the polarization vector:

$$\varepsilon^\mu \to \varepsilon'^\mu = \varepsilon^\mu + ap^\mu$$

Therefore, different polarization vectors which differ by a multiple of p^μ describe the same physical photon.

- Coulomb condition:

  We choose the zero-th component of the gauge field such that: $\varepsilon^0 = 0$. Then the Lorentz condition reduces to:

$$\begin{cases} A^0 = 0 \\ \vec{\nabla}\cdot\vec{A} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \varepsilon^0 = 0 \\ \vec{\varepsilon}\cdot\vec{p} = 0 \end{cases}$$

So, instead of 4 degrees of freedom (ε^μ) we now only have 2 independent polarization vectors which are perpendicular to the three-momentum of the photon. If the photon travels along the z-axis the polarization degrees of freedom can be:

- *transverse polarizations:*

$$\vec{\varepsilon}_1 = (1,0,0) \qquad \vec{\varepsilon}_2 = (0,1,0)$$

- *circular polarizations:*

$$\vec{\varepsilon}_+ = \frac{-\vec{\varepsilon}_1 - i\vec{\varepsilon}_2}{\sqrt{2}} \qquad \vec{\varepsilon}_- = \frac{+\vec{\varepsilon}_1 - i\vec{\varepsilon}_2}{\sqrt{2}}$$

Exercise 12

Show that the circular polarization vectors ε_+ and ε_- transform under a rotation of angle ϕ around the z-axis as:

$$\begin{aligned} \vec{\varepsilon}_+ \to \vec{\varepsilon}\,'_+ &= e^{-i\phi}\vec{\varepsilon}_+ \\ \vec{\varepsilon}_- \to \vec{\varepsilon}\,'_- &= e^{i\phi}\vec{\varepsilon}_- \\ \text{or } \vec{\varepsilon}\,'_i &= e^{-im\phi}\,\vec{\varepsilon}_i \end{aligned}$$

Hence $\vec{\varepsilon}_+$ and $\vec{\varepsilon}_-$ describe a photon of helicity +1 and -1 respectively.

Since the photon is a spin-1 particle we would expect $m_z = -1, 0, +1$. How about helicity 0? The transversality equation $\vec{\varepsilon}\cdot\vec{p} = 0$ arises due to the fact that the photon is massless. For massive vector fields (or virtual photon fields!) this component is allowed: $\vec{\varepsilon}//\vec{p}$.

3.4 The Bohm Aharanov Effect

Later on in the course we will see that the presence of a vector field $\vec{A}$ affects the phase of a wave function of the particle. The phase factor is affected by the presence of the field in the following way:

$$\psi' = e^{i\frac{q}{\hbar}\alpha(\vec{r},t)}\psi$$

where q is the charge of the particle, $\hbar$ is Plancks constant, and α is given by:

$$\alpha(\vec{r},t) = \int_r d\vec{r}' \cdot A(\vec{r}',t)$$

Let us now go back to the famous two-slit experiment of Feynman in which he considers the interference between two possible electron trajectories. From quantum mechanics we know that the intensity at a detection plate positioned behind the two slits shows an interference pattern depending on the relative phases of the wave functions ψ_1 and ψ_2 that travel different paths. For a beautiful description of this see chapter 1 of the “Feynman Lectures on Physics” volume 3 (“2-slit experiment”) and pages 15-8 to 15-14 in volume 2 (“Bohm-Aharanov”). The idea is schematically depicted in Fig. 3.1.

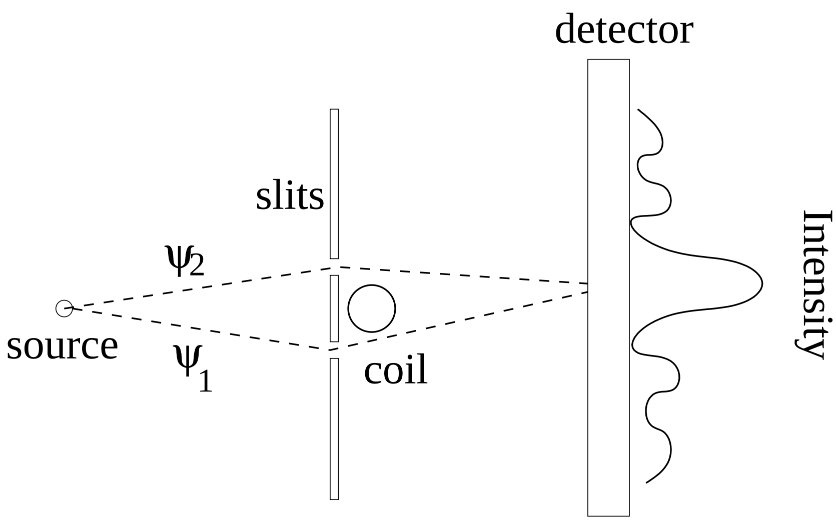


Figure 3.1: The schematical setup of an experiment that investigates the effect of the presence of an A field on the phase factor of the electron wave functions.

In case a field $\vec{A}$ is present the phases of the wave functions are affected, such that the wave function on the detector is:

$$\psi = \psi_1\, e^{iq\alpha_1(\vec{r},t)} + \psi_2\, e^{iq\alpha_2(\vec{r},t)} = \left(\psi_1\, e^{iq(\alpha_1-\alpha_2)} + \psi_2\right)\, e^{iq\alpha_2}$$

We note that the interference between the two amplitudes depends on the relative phase:

$$\begin{aligned}\alpha_1 - \alpha_2 &= \int_{r_1} d\vec{r}_1' A_1 - \int_{r_2} d\vec{r}_2' A_2 = \oint d\vec{r}' \cdot \vec{A}(\vec{r}',t) \\ &= \int_S \vec{\nabla} \times \vec{A}(\vec{r}',t) \cdot d\vec{S} = \int_S \vec{B} \cdot d\vec{S} = \Phi\end{aligned}$$

where we have used Stokes theorem to relate the integral around a closed loop to the magnetic flux through the surface. In this way the presence of a magnetic field can affect, (i.e. *shift*) the interference pattern on the screen.

Let us now consider the case that a very long and thin solenoid is positioned in the setup of the two-slit experiment. Inside the solenoid the B-field is homogeneous and outside it is $B = 0$ (or sufficiently small), see Fig. 3.2. However, from electrodynamics we recall the $\vec{A}$ field is **not** zero outside the coil. There is a lot of $\vec{A}$ circulation around the thin coil. The electrons in the experiment pass through this $\vec{A}$ field which quantum mechanically affects the phase of their wave function and therefor also the interference pattern on the detector. On the other hand, there is no B field in the region, so classically there is no effect. Experimentally it has been verified (in a technically difficult experiment) that the interference pattern will indeed shift.

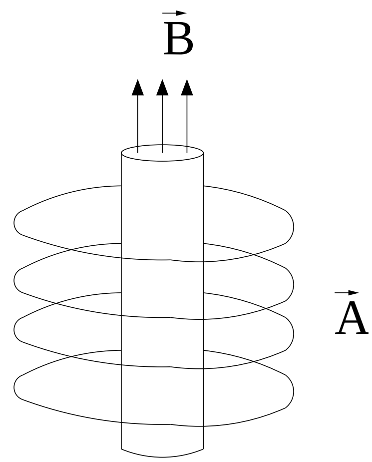


Figure 3.2: Magnetic field and vector potential of a long solenoid.

Discussion:
We have introduced the vector potential as a mathematical tool to write Maxwells equations in a Lorentz covariant form. In this formulation we noticed that the A-field has some arbitraryness due to gauge invariance. Quantummechanically we observe, however, that the A field is *not* just a mathematical tool, but gives a more fundamental description of “forces”. The aspect of gauge invariance seems an unwanted (*“not nice”*) aspect now, but later on it will turn out to be a fundamental concept in our description of interactions.

Exercise 13 *The delta function*

(a) *Show that*

$$\frac{d^3p}{(2\pi)^3 2E} \tag{3.1}$$

is Lorentz invariant ($d^3p = dp_x dp_y dp_z$)*. Do this by showing that*

$$\int M(p)\; 2d^4p\; \delta(p^2 - m^2)\; \theta(p^0) = \int M(\vec{p}) \frac{d^3p}{E}. \tag{3.2}$$

The 4-vector p *is* (E, p_x, p_y, p_z)*, and* $M(p)$ *is a Lorentzinvariant function of* p *and* $\theta(p^0)$ *is the Heavyside function.*

(b) *The delta-function can have many forms. One of them is:*

$$\delta(x) = \lim_{\alpha\to\infty} \frac{1}{\pi} \frac{\sin^2 \alpha x}{\alpha x^2} \tag{3.3}$$

Make this plausible by sketching the function $\sin^2(\alpha x)/(\pi\alpha x^2)$ *for two relevant values of* α*.*

(c) *Show that another (important!) representation of the Dirac delta function is given by*

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx}\, dk$$

To do this use the definition of Fourier transforms:

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(k)\, e^{ikx}\, dk \\ g(k) &= \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx \end{aligned}$$

Lecture 4

Perturbation Theory and Fermi's Golden Rule

4.1 Non Relativistic Perturbation Theory

Let us start to examine a scattering process: $A + B \rightarrow C + D$. As an example we take in mind the case where two electrons scatter in an electromagnetic potential A^μ as schematically depicted in Fig. 4.1

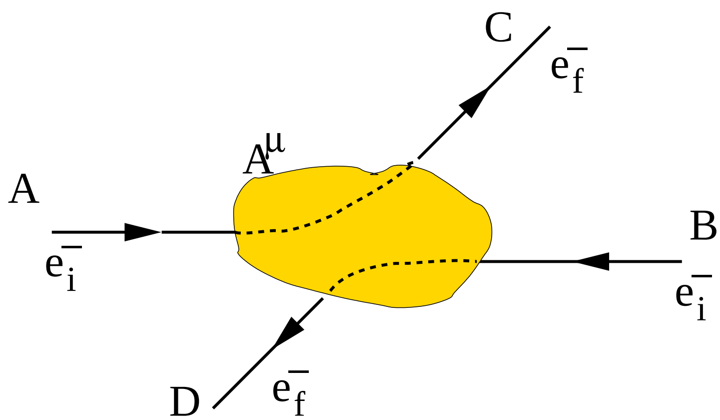


Figure 4.1: Scattering of two electrons in a electromagnetic potential.

The ingredients to calculate the counting rate for a scattering process: $A+B \rightarrow C+D$ are:

1. The *transition probability* W_{fi} to go from an initial state $''i''$ to a final state $''f''$.

2. The experimental conditions called the "*flux*" factor. It includes both the beam intensity and the target density.

3. The Lorentz invariant *"phase space"* factor Φ (also referred to as dLIPS). It takes care of the fact that experiments usually can not observe individual states but integrate over a number of (allmost identical) states.

The formula for the calculation of a (differential) cross section is:

$$d\sigma = \frac{W_{fi}}{\text{Flux}}\,\Phi$$

Note that the "real" physics, (i.e. the Feynman diagrams) is included in the transition probability W_{fi}. The flux and the phase space factors are the necessary "bookkeeping" needed to compare the physics theory with a realistic experiment. (The calculation of the phase space can in fact be rather involved.)

4.1.1 The Transition Probability

In order to calculate the transition probability we use the framework of non-relativistic perturbation theory. In the end we will see how we can use the result in a Lorentz covariant way and apply it to relativistic scattering.

Consider the scattering of a particle in a potential as depicted in Fig. 4.2 Assume that *before* the interaction takes place, as well as *after*, the system is described by the non-relativistic Schrödinger equation:

$$i\,\frac{\partial\psi}{\partial t} = H_0\,\psi$$

where H_0 is the unperturbed Hamiltonian, which does not have a time dependence. Solutions of this equation can be written in the as:

$$\psi_m = \phi_m(\vec{x})\,e^{-iE_m t}$$

with eigenvalues E_m.
The ϕ_m form a complete set orthogonal eigenfunctions of: $H_0\phi_m = E_m\phi_m$, so:

$$\int \phi_m^*(\vec{x})\,\phi_n(\vec{x})\,d^3x = \delta_{mn}$$

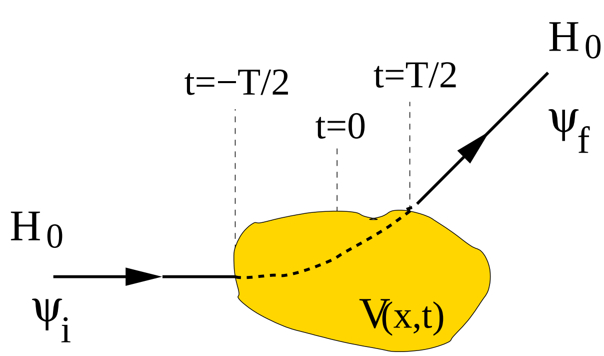


Figure 4.2: Scattering of a particle in a potential.

Assume that at $t = 0$ a perturbation occurs such that the system is described by:

$$i\frac{\partial \psi}{\partial t} = \left(H_0 + V\left(\vec{x}, t\right)\right)\, \psi \tag{4.1}$$

The solutions ψ can generally be written as:

$$\psi = \sum_{n=0}^{\infty} a_n(t)\, \phi_n(\vec{x})\, e^{-iE_n t} \tag{4.2}$$

where $a_n(t)$ is the coëfficient to find the system in state "n".

To determine these coëfficients $a_n(t)$ substitute 4.2 in 4.1:

$$\begin{aligned} i\sum_{n=0}^{\infty} \frac{da_n(t)}{dt}\, \phi_n(\vec{x})\, e^{-iE_n t} + i\sum_{n=0}^{\infty} (-i)\, E_n\, a_n(t)\, \phi_n(\vec{x})\, e^{-iE_n t} \quad &= \\ \sum_{n=0}^{\infty} E_n\, a_n(t)\, \phi_n(\vec{x})\, e^{-iE_n t} + \sum_{n=0}^{\infty} V(\vec{x}, t)\, a_n(t)\, \phi_n(\vec{x})\, e^{-iE_n t} \end{aligned}$$

and the two terms proportional to E_n cancel.

Multiply the resulting equation from the left with: $\psi_f^* = \phi_f^*(\vec{x})\, e^{iE_f t}$ and integrate over volume d^3x to obtain:

$$\begin{aligned} i\sum_{n=0}^{\infty} \frac{da_n(t)}{dt} \underbrace{\int d^3x\, \phi_f^*(\vec{x})\, \phi_n(\vec{x})}_{\delta_{fn}}\, e^{-i\left(E_n - E_f\right)t} \quad &= \\ \sum_{n=0}^{\infty} a_n(t) \int d^3x\, \phi_f^*(\vec{x})\, V(\vec{x}, t)\, \phi_n(\vec{x})\, e^{-i\left(E_n - E_f\right)t} \end{aligned}$$

Next we use the orthonormality relation:

$$\int d^3x\, \phi_m^*(\vec{x})\, \phi_n(\vec{x}) = \delta_{mn}$$

so that we find:

$$\frac{da_f(t)}{dt} = -i\sum_{n=0}^{\infty} a_n(t) \int d^3x\, \phi_f^*(\vec{x})\, V(\vec{x}, t)\, \phi_n(\vec{x})\, e^{-i\left(E_n - E_f\right)t}$$

We will assume two simplifications:

- We prepare the incoming wave in a single state: The incoming wave is: $\psi_i = \phi_i(\vec{x})\, e^{-iE_i t}$. In other words: $a_i(-\infty) = 1$ and $a_n(-\infty) = 0$ for $(n \neq i)$.
- We will assume that the inital condition is true during the time that the perturbation happens! This implies that we work with a *weak* interaction. In fact this is the lowest order in perturbation theory in which we replace $\sum_{n=0}^{\infty}$ by just one term: $n = i$. It means that $a_f(t) << 1$ is assumed at all times.

Then we get:

$$\frac{da_f(t)}{dt} = -i \int d^3x\, \phi_f^*(\vec{x})\, V(\vec{x},t)\, \phi_i(\vec{x})\, e^{-i\left(E_i - E_f\right)t}$$

Our aim is to determine $a_f(t)$:

$$a_f(t') = \int_{-T/2}^{t'} \frac{da_f(t)}{dt}\, dt = -i \int_{-T/2}^{t'} dt \int d^3x \left[\phi_f(\vec{x})\, e^{-iE_f t}\right]^* V(\vec{x},t) \left[\phi_i(\vec{x})\, e^{-iE_i t}\right]$$

We define the *transition amplitude* T_{fi} as the amplitude to go from state i final state f at the end of the interaction:

$$T_{fi} \equiv a_f(T/2) = -i \int_{-T/2}^{T/2} dt \int d^3x\, \phi_f^*(\vec{x},t)\, V(\vec{x},t)\, \phi_i(\vec{x},t)$$

Finally we take the limit: $T \to \infty$. Then we can write the expression in 4-vector notation:

$$\boxed{T_{fi} = -i \int d^4x\, \phi_f^*(x)\, V(x)\, \phi_i(x)}$$

Note:
The expression for T_{fi} has a manifest Lorentzinvariant form. It is true for each Lorentz frame. Although we started with Schrödinger's equation (i.e. non-relativistic) we will always use it: also for relativistic frames.

1-st and 2-nd order perturbation

What is the meaning of the initial conditions: $a_i(t) = 1, a_n(t) = 0$? It implies that the potential can only make **one** quantum perturbation from the initial state i to the final state f. For example the perturbation: $i \to n \to f$ is not included in this approximation (it is a 2^{nd} order perturbation).

If we want to improve the calculation to second order in perturbation theory we replace the approximation $a_n(t) = 0$ by the first order result:

$$\begin{aligned} \frac{da_f(t)}{dt} &= -i\, V_{fi}\, e^{i\left(E_f - E_i\right)t} \\ &+ (-i)^2 \left[\sum_{n \neq i} V_{ni} \int_{-T/2}^{t} dt'\, e^{i(E_n - E_i)t'}\right] V_{fn}\, e^{i\left(E_f - E_n\right)t} \end{aligned}$$

where we have assumed that the perturbation is time independent and introduced the notation:

$$V_{fi} \equiv \int d^3x \phi_f^*(\vec{x})\, V(\vec{x})\, \phi_i(\vec{x})$$

See the book of Halzen and Martin how to work out the second order calculation. A graphical illustration of the first and second order perturbation is given in Fig. 4.3.

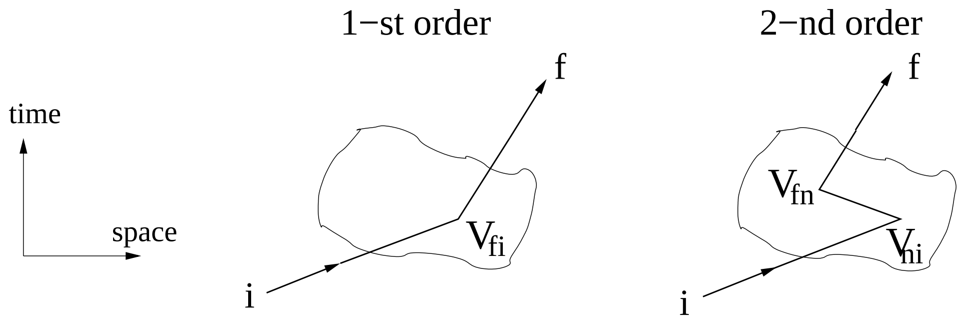


Figure 4.3: First and Second order approximation in scattering.

Can we interpret $|T_{fi}|^2$ as the probability that a particle has scattered from state i to state f? Consider the case where the perturbation is time **in**dependent. Then:

$$T_{fi} = -i\, V_{fi} \int_{-\infty}^{\infty} dt\; e^{i\left(E_f - E_i\right)t} = -2\pi i\, V_{fi}\, \delta\left(E_f - E_i\right)$$

The δ-function expresses energy conservation in $i \to f$. From the uncertainty princple it can then be inferred that the transition between two exactly defined energy states E_i and E_f must be infinitely seperated in time. Therefore the quantity $|T_{fi}|^2$ is not a meaningfull quantity. We define instead the transition probability per unit time as:

$$W_{fi} = \lim_{T\to\infty} \frac{|T_{fi}|^2}{T}$$

The calculation of the transition probability is non-trivial as it involves the square of a δ-function. A proper treatment is rather lengthy[1] and involves the use of wave packets. Instead we will apply a "trick". If we assume that the interaction occurs during a time period T from $t = -T/2$ until $t = +T/2$ we can write:

$$\begin{aligned} |W_{fi}| &= \lim_{T\to\infty} \frac{1}{T}\, |V_{fi}|^2 \int_{-\infty}^{\infty} dt\; e^{i\left(E_f - E_i\right)t} \cdot \int_{-T/2}^{T/2} dt'\; e^{i\left(E_f - E_i\right)t'} \\ &= |V_{fi}|^2\; 2\pi\delta\left(E_f - E_i\right) \cdot \lim_{T\to\infty} \frac{1}{T} \underbrace{\int_{-T/2}^{T/2} dt'}_{T} \end{aligned}$$

The δ-function in the first integral implies that there is only contribution for E_f equal to E_i in the second integral.

Then we note that the arbitrary chosen time period T drops out of the formula such that the transition probability per unit time becomes:

$$W_{fi} = \lim_{T\to\infty} \frac{|T_{fi}|^2}{T} = 2\pi\; |V_{fi}|^2\; \delta\left(E_f - E_i\right)$$

This is the transition probability for a given initial state into a specific final state.

[1] see e.g. the book by K.Gottfried, "Quantum Mechanics" (1966), Volume 1, sections 12, 56.

In particle physics experiments we typically have:

- Well prepared initial states
- An integral over final states that are reached: $\rho(E_f)dE_f$.

Finally we arrive at Fermi's Golden rule:

$$\boxed{\begin{aligned} W_{fi} &= 2\pi \int dE_f \, \rho\,(E_f)\; |V_{fi}|^2 \; \delta\,(E_f - E_i) \\ &= 2\pi \; |V_{fi}|^2 \; \rho\,(E_i) \end{aligned}}$$

<u>**Exercise 14**</u>
Assume that there is a constant perturbation potential between $t = -T/2$ and $t = T/2$.
(a) Write down the expression for T_{fi} at time $T/2$ and do the integral over t.
(b) Write down the expression for W_{fi}. Show that this expression corresponds to the one in the lecture in the limit that $T \to \infty$.
(c) Assume that density for final states $\rho(E_f)$ is a constant and perform the integral over all final states dE_f. Compare it to the expression of Fermi's Golden rule.
Hint: $\int_{-\infty}^{+\infty} \frac{sin^2 x}{x^2}\, dx = \pi$

4.1.2 Normalisation of the Wave Function

Let us assume that we are working with solutions of the Klein-Gordon equation:

$$\phi = N\, e^{-ipx}$$

We normalise the wave function in a given volume V to 1:

$$\int_V \phi^* \, \phi \, dV = 1 \qquad \Rightarrow \qquad N = \frac{1}{\sqrt{V}}$$

The probability density for a Klein Gordon wave is given by (see Lecture 2):

$$\rho = 2\; |N|^2 \; E \qquad \Rightarrow \qquad \rho = \frac{2E}{V}$$

In words: in a given volume V there are $2E$ particles. The fact that ρ is proportional to E is needed to compensate for the Lorentz contraction of the volume element d^3x such that $\rho\, d^3x$ remains constant. The volume V is arbitrary and in the end it must drop out of any calculation of a scattering process.

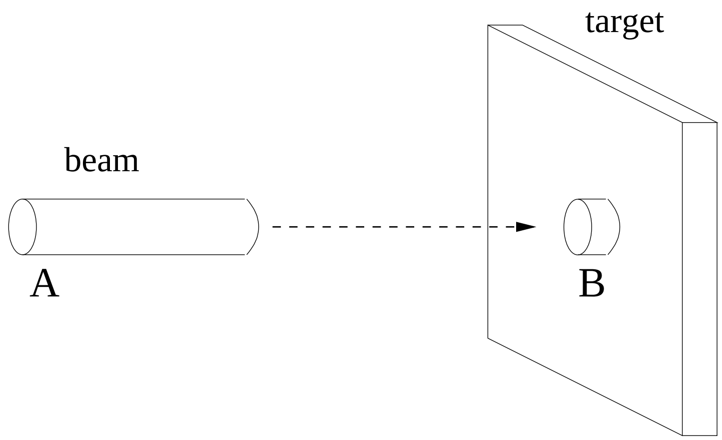


Figure 4.4: A beam incident on a target.

4.1.3 The Flux Factor

The flux factor or the initial flux corresponds to the amount of particles that pass each other per unit area and per unit time. This is easiest to consider in the lab frame. Consider the case that a beam of particles (A) is shot on a target (B), see Fig. 4.4

The number of beam particles that pass through unit area per unit time is given by $|\vec{v}_A|\, n_A$. The number of target particles per unit volume is n_B. The density of particles n is given by $n = \rho = \frac{2E}{V}$ such that:

$$\text{Flux} = |\vec{v}_A|\, n_a\, n_b = |\vec{v}_A|\, \frac{2E_A}{V}\, \frac{2E_B}{V}$$

Exercise 15
In order to provide a general, Lorentz invariant expression for the flux factor replace $\vec{v}_A$ by $\vec{v}_A - \vec{v}_B$ and show using: $\vec{v}_A = \vec{P}_A/E_A$ and $\vec{v}_B = \vec{P}_B/E_B$, that:

$$\boxed{\text{Flux} = 4\,\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}\;/\;V^2}$$

4.1.4 The Phase Space Factor

How many quantum states can be put into a given volume V? Assume the volume is rectangular with sides L_x, L_y, L_z. A particle with momentum p has a "size" given by: $\lambda = 2\pi/p$. Using periodic boundary conditions to ensure no net particle flow out of the volume we see that the number of states with a momentum between $\vec{p} = (0,0,0)$ and $\vec{p} = (p_x, p_y, p_z)$ is

$$N = n_x\, n_y\, n_z = \frac{L_x}{\lambda_x}\, \frac{L_y}{\lambda_y}\, \frac{L_z}{\lambda_z} = \frac{L_x p_x}{2\pi}\, \frac{L_y p_y}{2\pi}\, \frac{L_z p_z}{2\pi} = \frac{V}{(2\pi)^3}\, p_x\, p_y\, p_z$$

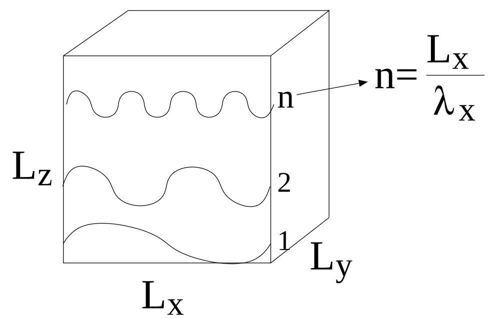


Figure 4.5: Schematic calculation of the number of states in a box of volume V.

An alternative view is given by Burcham & Jobes on page 305. The number of final states is given by the total size of the available phase space for the final state divided by the volume of the elementary cell: h^3 (within an elementary cell states cannot be distinguished):

$$N = \frac{1}{h^3}\int dx\, dy\, dz\, dp_x\, dp_y\, dp_z = \frac{V}{(2\pi)^3\,\hbar^3}\int dp_x\, dp_y\, dp_z = \frac{V}{(2\pi)^3}\, p_x\, p_y\, p_z$$

As a consequence, the number of states with momentum between $\vec{p}$ and $\vec{p}+d\vec{p}$ (i.e. between (p_x, p_y, p_z) and $(p_x+dp_x, p_y+dp_y, p_z+dp_z)$) is:

$$dN = \frac{V}{(2\pi)^3}\, dp_x\, dp_y\, dp_z$$

The wave functions were normalized according to $\int_V \rho dV = 2E$, therefore the number of states per particle is:

$$\#states/particle = \frac{V}{(2\pi)^3}\,\frac{d^3p}{2E}$$

For a process in the form $A + B \to C + D + E + ...$ with N final state particles the Lorentz invariant phase space factor is:

$$d\Phi = \mathrm{dLIPS} = \prod_{i=1}^{N} \frac{V}{(2\pi)^3}\,\frac{d^3p_i}{2E_i}$$

4.1.5 Summary

Finally we arrive at the formula to calculate a cross section for the process

$$A_i + B_i \to C_f + D_f + \ldots$$

$$\boxed{\begin{aligned} d\sigma_{fi} &= \frac{1}{\text{flux}}\, W_{fi}\, d\Phi \\ W_{fi} &= \lim_{T\to\infty} \frac{|T_{fi}|^2}{T} \\ T_{fi} &= -i \int d^4x\, \psi_f^*(x)\, V(x)\, \psi_i(x) \\ d\Phi &= \prod_{i=1}^{N} \frac{V}{(2\pi)^3} \frac{d^3\vec{p}_i}{2E_i} \\ \text{flux} &= 4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}\, /\, V^2 \end{aligned}}$$

Exercise 16
Show that the cross section does not depend on the arbitrary volume V.

Exercise 17
Why is the phase space factor indeed Lorentz invariant? (Hint: Just refer to a previous exercise.)

4.2 Extension to Relativistic Scattering

The transition amplitude of the scattering process $A + B \to C + D$, for incoming and outgoing plane waves $\phi = Ne^{-ipx}$ takes the form:

$$T_{fi} = -i\, N_A N_B N_C N_D\, (2\pi)^4\, \delta(p_A + p_B - p_C - p_D)\, \mathcal{M}$$

where $\mathcal{M}$ is the so-called *Matrix element* and the delta function takes care of the energy and momentum conservation in the process.

To find the transition probability we square this expression:

$$\begin{aligned} |T_{fi}|^2 &= |N_A N_B N_C N_D|^2\ |\mathcal{M}|^2 \int d^4x\, e^{-i(p_A+p_B-p_C-p_D)x} \times \int d^4x'\, e^{-i(p_A+p_B-p_C-p_D)x'} \\ &= |N_A N_B N_C N_D|^2\ |\mathcal{M}|^2\, (2\pi)^4\, \delta^4(p_A + p_B - p_C - p_D) \times \lim_{T,V\to\infty} \int_{TV} d^4x \\ &= |N_A N_B N_C N_D|^2\ |\mathcal{M}|^2\, (2\pi)^4\, \delta^4(p_A + p_B - p_C - p_D) \times \lim_{T,V\to\infty} TV \end{aligned}$$

This gives for the transition probability per unit time and volume:

$$\begin{aligned} W_{fi} &= \lim_{T,V\to\infty} \frac{|T_{fi}|^2}{TV} \\ &= |N_A N_B N_C N_D|^2\ |\mathcal{M}|^2\ (2\pi)^4\, \delta\left(p_A + p_B - p_C - p_D\right) \end{aligned}$$

Indeed we see that the delta funtion provides conservation of energy and momentum.

The cross section is again given by[2]:

$$d\sigma = \frac{W_{fi}}{\text{Flux}}\Phi_2$$

The phase space factor is:

$$\Phi_2 = \frac{V\, d^3p_C}{(2\pi)^3 \cdot 2E_C}\ \frac{V\, d^3p_D}{(2\pi)^3 \cdot 2E_D}$$

and the Flux factor is:

$$\text{Flux} = 4\sqrt{\left(p_A \cdot p_B\right)^2 - m_A^2\, m_B^2}\ /\ V^2$$

Taking it all together with $N = 1\sqrt{V}$:

$$d\sigma = \frac{1}{V^4}\, |\mathcal{M}|^2\, (2\pi)^4 \delta^4\left(p_A + p_B - p_C - p_D\right) \cdot \frac{V^2}{4\sqrt{\left(p_A \cdot p_B\right)^2 - m_A^2 m_B^2}} \cdot \frac{V\, d^3p_C}{(2\pi)^3\ 2E_C} \frac{V\, d^3p_D}{(2\pi)^3\ 2E_D}$$

In this formula the arbitrary volume factors V cancel again.

[2] Usually we will write this as:

$$d\sigma = \frac{|\mathcal{M}|^2}{\text{Flux}}\, d\Phi$$

and absorb the delta function in the phase space factor.

We finally have for the cross section of $A+B \to C+D$:

$$d\sigma = \frac{(2\pi)^4\ \delta^4\left(p_A+p_B-p_C-p_D\right)}{4\sqrt{\left(p_A\cdot p_B\right)^2-m_A^2 m_B^2}} \cdot |\mathcal{M}|^2 \cdot \frac{d^3p_C}{(2\pi)^3\ 2E_C}\ \frac{d^3p_D}{(2\pi)^3\ 2E_D}$$

Similarly the formula for decay $A \to C+D$ is:

$$d\Gamma = \frac{(2\pi)^4\ \delta^4\left(p_A-p_C-p_D\right)}{2E_A} \cdot |\mathcal{M}|^2 \cdot \frac{d^3p_C}{(2\pi)^3\ 2E_C}\ \frac{d^3p_D}{(2\pi)^3\ 2E_D}$$

Exercise 18 *(See also H&M Ex. 4.2)*
Calculate the two particle phase space in the interaction $A+B \to C+D$.

(a) *Start with the expression:*

$$\Phi_2 = \int (2\pi)^4\ \delta^4\left(p_A+p_B-p_C-p_D\right)\ \frac{d^3\vec{p_C}}{(2\pi)^3\ 2E_C}\ \frac{d^3\vec{p_D}}{(2\pi)^3\ 2E_D}$$

Do the integral over d^3p_D using the δ function and show that we can write:

$$\Phi_2 = \int \frac{1}{(2\pi)^2}\ \frac{p_f^2\ dp_f\ d\Omega}{4E_C E_D}\ \delta\left(E_A+E_B-E_C-E_D\right)$$

where we have made use spherical coordinates (i.e.: $d^3p_C = |p_C|^2 dp_C\ d\Omega$) and of $p_f \equiv |p_C|$.

(b) *In the C.M. system we can write: $\sqrt{s} \equiv W = E_A+E_B$. Show that the expression becomes (hint: calculate dW/dp_f):*

$$\Phi_2 = \int \frac{1}{(2\pi)^2}\ \frac{p_f}{4}\left(\frac{1}{E_C+E_D}\right)\ dW\ d\Omega\ \delta\left(W-E_C-E_D\right)$$

So that we finally get:

$$\Phi_2 = \frac{1}{4\pi^2}\ \frac{p_f}{4\sqrt{s}}\ d\Omega$$

(c) *Show that the flux factor in the center of mass is:*

$$F = 4p_i\sqrt{s}$$

and hence that the differential cross section for a $2 \to 2$ process in the center of mass frame is given by:

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{1}{64\pi^2 s}\ \frac{p_f}{p_i}\ |\mathcal{M}|^2$$

For the decay rate $A \to B+C$ one finds ($4p_i\sqrt{s} \to 2E_A = 2m_A$):

$$\left.\frac{d\Gamma}{d\Omega}\right|_{cm} = \frac{1}{32\pi^2 m_A^2}\ p_f\ |\mathcal{M}|^2$$

Lecture 5

Electromagnetic Scattering of Spinless Particles

Introduction

In this lecture we discuss electromagnetic scattering of spinles particles. First we describe an example of a charged particle scattering in an external electric field. Second we derive the cross section for two particles that scatter in each-others field. We end the lecture with a prescription how to treat antiparticles.

In classical mechanics the equations of motion can be derived using the variational principle of Hamilton which states that the action integral I should be stationary under arbitrary variations of the generalized coordinates $q_i, \dot{q}_i$: $\delta I = 0$, where:

$$I = \int_{t0}^{t1} \mathcal{L}(q_i, \dot{q}_i) dt \quad \text{with} \quad \mathcal{L}(q_i, \dot{q}_i) = T - V$$

This leads to the Euler Lagrange equations of motion (see Appendix A):

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \qquad .$$

These may also be written in the form

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} \quad \text{with} \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad ,$$

the generalized (or canonical) momentum.

5.1 Electrodynamics

How do we introduce electrodynamics in the wave equation of a system? The Hamiltonian of a free particle is:

$$H \, \psi = \frac{\vec{p}^2}{2m} \, \psi$$

In the presence of an electromagnetic field the equation of movement is:

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

The Hamiltonian that leads to the desired equation of motion is (see e.g. Jackson):

$$H\ \psi = \left[\frac{1}{2m}\left(\vec{p} - q\vec{A}(\vec{r},t)\right)^2 + q\Phi(\vec{r},t)\right]\ \psi$$

This means that we replace the kinematic energy and momentum by the canonical energy and momentum: $E \rightarrow E - q\Phi$ and $\vec{p} \rightarrow \vec{p} - q\vec{A}$. In 4-vec notation:

$$p^\mu \longrightarrow p^\mu - qA^\mu$$

This is called minimal substitution contains the essential physics of electrodynamics.

<u>Exercise 19</u>

The Lagrangian for a charged particle moving in a electromagnetic field is:

$$\mathcal{L} = \frac{1}{2}m\vec{v}^2 + q\vec{v}\cdot\vec{A}(\vec{r},t) - q\Phi(\vec{r},t)$$

(a) *Show that for a uniform magnetic field, we may take:*

$$V = 0, \qquad \vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$$

If we choose the z-axis in the direction of $\vec{B}$ we have in cylindrical coordinates (r, ϕ, z):

$$V = 0, \quad A_r = 0, \quad A_\phi = \frac{1}{2}Br, \quad A_z = 0$$

Hint: In cylindrical coördinates the cross product is defined as:

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}, \frac{\partial A_z}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\left[\frac{\partial\left(rA_\phi\right)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]\right)$$

(b) *Write down the Lagrangian in cylindrical coördinates*

(c) *Write out the Lagrangian equations:*

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) = \frac{\partial \mathcal{L}}{\partial q_\alpha}$$

in the cylindrical coördinates.

(d) *Show that the equation of motion in terms of the coordinate $\dot{\phi}$ yields (assume r=constant):*

$$\dot{\phi} = 0 \quad \text{or} \quad \dot{\phi} = -\frac{qB}{m}$$

i.e. that it is in agreement with the law:

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

In quantum mechanics we make the replacement $p^\mu \to i\partial^\mu$, such that we have now:

$$\boxed{\partial^\mu \to \partial^\mu + iqA^\mu}$$

This is the heart of quantum electrodynamics. As we will see later in the lectures this substitution is mandatory in order make the theory quantum electrodynamics locally gauge invariant! (This was exactly the substitution in the example of the Bohm-Aharanov effect where $\vec{p} \to \vec{p} - q\vec{A}$ in the phase of the wave function.)

Start with the free particle Klein-Gordon equation:

$$\left(\partial_\mu \partial^\mu + m^2\right)\,\phi = 0$$

and substitute $\partial^\mu \to \partial^\mu - ieA^\mu$ for a particle with charge $-e$:

$$(\partial_\mu - ieA_\mu)\,(\partial^\mu - ieA^\mu)\;\phi + m^2\phi = 0$$

which is of the form:

$$\left(\partial_\mu \partial^\mu + m^2 + V(x)\right)\,\phi = 0$$

from which we derive for the perturbation potential:

$$V(x) = -ie\,(\partial_\mu A^\mu + A_\mu \partial^\mu) - e^2 A^2$$

Since e^2 is small ($\alpha = e^2/4\pi = 1/137$) we can neglect the second order term: $e^2A^2 \approx 0$.

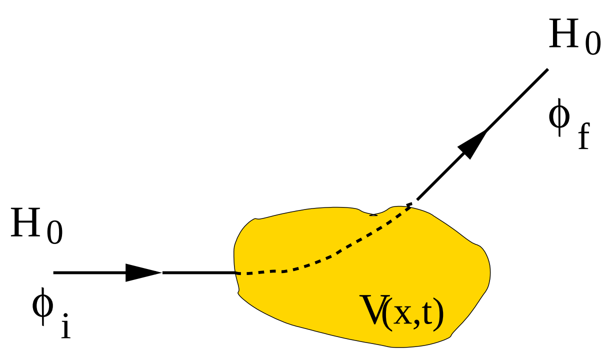


Figure 5.1: Scattering potential

From the previous lecture we take the general expression for the transition amplitude:

$$\begin{aligned} T_{fi} &= -i\int d^4x\;\phi_f^*(x)\,V(x)\,\phi_i(x) \\ &= -i\int d^4x\;\phi_f^*(x)\,(-ie)\;(A_\mu\partial^\mu + \partial_\mu A^\mu)\;\phi_i(x) \end{aligned}$$

Use now partial integration to calculate:

$$\int d^4x\;\phi_f^*\,\partial_\mu\,(A^\mu\,\phi_i) = \underbrace{\left[\phi_f^*\,A^\mu\,\phi_i\right]_{-\infty}^{\infty}}_{=0} - \int \partial_\mu\left(\phi_f^*\right)\,A^\mu\,\phi_i\,d^4x$$

note that at $t = -\infty$ and at $t = +\infty$: $A^\mu = 0$.

We then get:

$$T_{fi} = -i \int -ie \underbrace{\left[\phi_f^*(x)\left(\partial_\mu \phi_i(x)\right) - \left(\partial_\mu \phi_f^*(x)\right)\phi_i(x)\right]}_{j_\mu^{fi}} A^\mu \, d^4x$$

We had the definition of a Klein-Gordon current density:

$$j_\mu = -ie \left[\phi^* \left(\partial_\mu \phi\right) - \left(\partial_\mu \phi^*\right)\phi\right]$$

In complete analogy we now define the "transition current density" to go from initial state i to final state f:

$$j_\mu^{fi} = -ie \left[\phi_f^* \left(\partial_\mu \phi_i\right) - \left(\partial_\mu \phi_f^*\right)\phi_i\right]$$

so that we arrive at:

$$\boxed{T_{fi} = -i \int j_\mu^{fi} \, A^\mu \, d^4x}$$

This is the expression for the transition amplitude for going from free particle solution i to free particle solution f in the presence of a perturbation caused by an electromagnetic field.

If we substitute the free particle solutions of the unperturbed Klein-Gordon equation in initial and final states we find for the transition current of spinless particles:

$$\boxed{\begin{array}{rcl} \phi_i & = & N_i \, e^{-ip_i x} \quad ; \quad \phi_f^* = N_f^* e^{ip_f x} \\ j_\mu^{fi} & = & -e N_i N_f^* \left(p_\mu^i + p_\mu^f\right) \, e^{i\left(p_f - p_i\right)x} \end{array}}$$

Verify that the conservation law $\partial^\mu j_\mu^{fi} = 0$ holds. From this equation it can be derived that the charge is conserved in the interaction.

5.2 Scattering in an External Field

Consider the case that the external field is a static field of a point charge Z located in the origin:

$$A_\mu = \left(V, \vec{A}\right) = \left(V, \vec{0}\right) \qquad \text{with} \qquad V(x) = \frac{Ze}{4\pi|\vec{x}|}$$

The transition amplitude is:

$$\begin{aligned} T_{fi} &= -i \int j_{fi}^\mu \, A_\mu \, d^4x \\ &= -i \int (-e) \, N_i N_f^* \, \left(p_i^\mu + p_f^\mu\right) \, A_\mu \, e^{i\left(p_f - p_i\right)x} \, d^4x \end{aligned}$$

Insert that $A_\mu = \left(V, \vec{0}\right)$ and thus: $p^\mu A_\mu = E\,V$:

$$T_{fi} = i \int eN_i N_f^* \ (E_i + E_f) \ V(x)\, e^{i\left(p_f - p_i\right)x}\, d^4x$$

Split the integral in a part over time and in a part over space and note that $V(\vec{x})$ is not time dependent. Use also again: $\int e^{i\left(E_f - E_i\right)t} dt = 2\pi\, \delta\left(E_f - E_i\right)$ to find that:

$$T_{fi} = ieN_i N_f^* \ (E_i + E_f) \ 2\pi\, \delta\left(E_f - E_i\right) \int \frac{Ze}{4\pi|\vec{x}|}\, e^{-i\left(\vec{p}_f - \vec{p}_i\right)\vec{x}}\, d^3x$$

Now we make use of the Fourier transform:

$$\frac{1}{|\vec{q}|^2} = \int d^3x\, e^{i\vec{q}\vec{x}}\, \frac{1}{4\pi|\vec{x}|}$$

Using this with $\vec{q} \equiv (\vec{p}_f - \vec{p}_i)$ we obtain:

$$T_{fi} = ieN_i N_f^* \ (E_i + E_f) \ 2\pi\, \delta\left(E_f - E_i\right) \ \frac{Ze}{|\vec{p}_f - \vec{p}_i|^2}$$

The next step is to calculate the transition probability:

$$\begin{aligned} W_{fi} &= \lim_{T\to\infty} \frac{|T_{fi}|^2}{T} \\ &= \lim_{T\to\infty} \frac{1}{T} \left|N_i N_f^*\right| \left[2\pi\, \delta\left(E_f - E_i\right)\right]^2 \left(\frac{Ze^2\left(E_i + E_f\right)}{|\vec{p}_f - \vec{p}_i|^2}\right)^2 \end{aligned}$$

We apply again our "trick" (or calculate the integral explicitly and let $T \to \infty$):

$$\begin{aligned} \lim_{T\to\infty} \left[2\pi\, \delta\left(E_f - E_i\right)\right]^2 &= 2\pi\, \delta\left(E_f - E_i\right) \cdot \lim_{T\to\infty} \int_{-T/2}^{T/2} dt\, e^{i\left(E_f - E_i\right)t} \\ &= 2\pi\, \delta\left(E_f - E_i\right) \cdot \lim_{T\to\infty} \underbrace{\int_{-T/2}^{T/2} e^{i0t} dt}_{T} \\ &= \lim_{T\to\infty} 2\pi\, \delta\left(E_f - E_i\right) \cdot T \end{aligned}$$

Putting this back into W_{fi} we obtain:

$$W_{fi} = \lim_{T\to\infty} \frac{1}{T} \cdot T \ |N_i N_f|^2 \ 2\pi\, \delta\left(E_f - E_i\right) \left(\frac{Ze^2\left(E_i + E_f\right)}{|\vec{p}_f - \vec{p}_i|^2}\right)^2$$

The cross section is given by[1]:

$$d\sigma = \frac{W_{fi}}{\text{Flux}}\, \text{dLips}$$

[1] Note that $E = m_0\gamma$ and $\vec{p} = m_0\gamma\vec{v}$ so that $\vec{v} = \vec{p}/E$.

$$\begin{aligned}
&\text{with :} \\
\text{Flux} &= \vec{v}\,\frac{2E_i}{V} = \frac{\vec{p}_i}{E_i}\frac{2E_i}{V} = \frac{2\vec{p}_i}{V} \\
\text{dLips} &= \frac{V}{(2\pi)^3}\frac{d^3p_f}{2E_f} \\
\text{Normalization : } N &= \frac{1}{\sqrt{V}} \qquad\qquad \rightarrow \qquad \int_V \phi^*\phi dV = 1
\end{aligned}$$

In addition, from energy and momentum conservation we write $E = E_i = E_f$ and $p = |\vec{p}_f| = |\vec{p}_i|$

Putting everything together:

$$d\sigma = \frac{1}{V^2}\,2\pi\,\delta\left(E_f - E_i\right)\cdot\left(\frac{Ze^2\left(E_i + E_f\right)}{|\vec{p}_f - \vec{p}_i|^2}\right)^2\cdot\frac{V}{2\,|\vec{p}_i|}\,\frac{V}{(2\pi)^3}\,\frac{d^3p_f}{2E_f}$$

Note that the arbitrary volume V drops from the expression!
Use now $d^3p_f = p_f^2\,dp_f\,d\Omega$ and $|p_f| = |p_i| = p$ to get:

$$\begin{aligned}
d\sigma &= \frac{1}{(2\pi)^2}\delta\left(E_f - E_i\right)\left(\frac{Ze^2\left(E_i + E_f\right)}{|\vec{p}_f - \vec{p}_i|^2}\right)^2\frac{p_f^2\,dp_f\,d\Omega}{2|\vec{p}_i|\,2E_f} \\
&= \frac{1}{(2\pi)^2}\delta\left(E_f - E_i\right)\left(\frac{Ze^2\left(E_i + E_f\right)}{\underbrace{2p^2\left(1 - \cos\theta\right)}_{4p^2\sin^2\theta/2}}\right)^2\frac{p\,dp\,d\Omega}{4E}
\end{aligned}$$

now, since $E^2 = m^2 + \vec{p}^2$, use $p\,dp = E\,dE$ such that:

$$\frac{p\,dp\,d\Omega}{4E}\,\delta\left(E_f - E_i\right) = \frac{dE\,\delta\left(E_f - E_i\right)\,d\Omega}{4} = \frac{d\Omega}{4}$$

We arrive at the expression for the differential cross section:

$$d\sigma = \left(\frac{Ze^2E}{4\pi p^2\sin^2\theta/2}\right)^2\,d\Omega$$

or:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{Z^2E^2e^4}{16\pi^2p^4\sin^4\theta/2} = \frac{Z^2E^2\alpha^2}{p^4\sin^4\theta/2}}$$

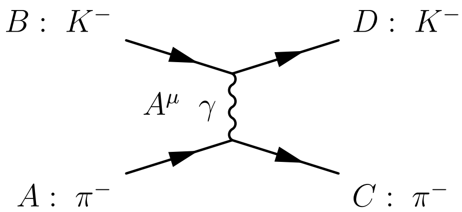
In the classical (i.e. non-relativistic) limit we can take $E \rightarrow m$ and $E_{kin} = \frac{p^2}{2m}$ such that:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{Z^2m^2\alpha^2}{4m^2E_{kin}^2\sin^4\theta/2} = \frac{Z^2\alpha^2}{4E_{kin}^2\sin^4\theta/2}}$$

the well known Rutherford scattering formula.

5.3 Spinless $\pi - K$ Scattering

Let us proceed to the case of QED scattering of a π^- particle on a K^- particle. We ignore the fact that pions and kaons also are subject to the strong interaction (e.g. we could consider scattering at large distances).



We know from the previous calculation how a particle scatters in an external field. In this case the field is not external as the particles scatter in each others field. How do we deal with it?

Ansatz:
Consider first only the pion. It scatters in the field of the kaon. How do we find the field generated by the kaon? This field is again caused by the transition current j^μ_{BD} of the scattering kaon. The field is then found by solving Maxwell's equations for this current (adopting the Lorentz gauge condition):

$$\partial_\nu \partial^\nu A^\mu = j^\mu_{BD} = -eN_B N_D^* \ (p^\mu_B + p^\mu_D) \ e^{i(p_D - p_B)x}$$

(see the previous section.)
Since $\partial_\nu \partial^\nu e^{-iqx} = -q^2 \, e^{-iqx}$ we can verify that

$$A^\mu = \frac{e}{q^2} N_B N_D^* \ (p^\mu_B + p^\mu_D) \ e^{i(p_D - p_B)x} = -\frac{1}{q^2} \, j^\mu_{BD} \quad ,$$

where we have used that $q = (p_D - p_B) = -(p_C - p_A)$ is the 4-vector momentum that is transmitted from the pion particle to the kaon particle via the A^μ field, i.e. the photon.

In this case the transition amplitude becomes:

$$T_{fi} = -i \int j^\mu_{AC} \, A_\mu \, d^4x = -i \int j^\mu_{AC} \, \frac{-1}{q^2} j^{BD}_\mu \, d^4x = -i \int j^\mu_{AC} \, \frac{-g_{\mu\nu}}{q^2} \, j^\nu_{BD} \, d^4x$$

Note:

1. The expression is symmetric in the two currents. It does not matter whether we scatter the pion in the field of the kaon or the kaon in the field of the pion.

2. There is only scattering if $q^2 \neq 0$. This is interesting as for a "normal" photon one has $q^2 = m^2 = 0$. It implies that we deal with *virtual* photons; i.e. photons that are "off mass-shell".

Writing out the expression we find:

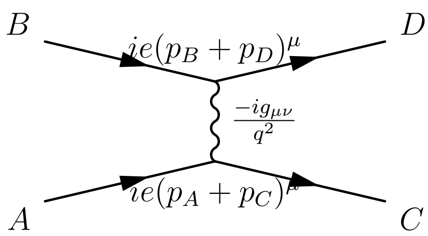
$$T_{fi} = -ie^2 \int (N_A N_C^*) \ (p^\mu_A + p^\mu_C) \ e^{i(p_C - p_A)x} \cdot \frac{-1}{q^2} \cdot (N_B N_D^*) \ (p^\mu_B + p^\mu_D) \ e^{i(p_D - p_B)x} \, d^4x$$

Next, do the integrals over x in order to obtain the energy-momentum conservation δ-functions:

$$T_{fi} = -ie^2 \left(N_A N_C^*\right) \left(p_A^\mu + p_C^\mu\right) \frac{-1}{q^2} \left(N_B N_D^*\right) \left(p_\mu^B + p_\mu^D\right) (2\pi)^4 \, \delta^4 \left(p_A + p_B - p_C - p_D\right)$$

Usually this is written in terms of the *matrix element* $\mathcal{M}$ as:

$$\begin{aligned} T_{fi} &= -i\, N_A N_B N_C^* N_D^* \; (2\pi)^4 \, \delta^4 \left(p_A + p_B - p_C - p_D\right) \cdot \mathcal{M} \\ \text{with :} \quad -i\mathcal{M} &= \underbrace{ie \left(p_A + p_C\right)^\mu}_{\text{vertex factor}} \cdot \underbrace{\frac{-ig_{\mu\nu}}{q^2}}_{\text{propagator}} \cdot \underbrace{ie \left(p_B + p_D\right)^\nu}_{\text{vertex factor}} \end{aligned}$$



The matrix element $\mathcal{M}$ contains:

a vertex factor: for each vertex we introduce the factor: iep^μ, where:
- · e is the intrinsic coupling strength of the particle to the e.m. field.
- · p^μ is the sum of the 4-momenta before and after the scattering (remember the particle/anti-particle convention).

a propagator: for each internal line (photon) we introduce a factor $\frac{-ig_{\mu\nu}}{q^2}$, where:
- · q is the 4-momentum of the exchanged photon quantum.

Using Fermi's golden rule we can proceed to calculate the relativistic transition probability:

$$W_{fi} = \lim_{T,V\to\infty} \frac{\left|T_{fi}\right|^2}{TV} = \lim_{T,V\to\infty} \frac{1}{TV} \; \left|N_A N_B N_C^* N_D^*\right|^2 \; \left|\mathcal{M}\right|^2 \; \left|(2\pi)^4 \delta^4 \left(p_A + p_B - p_C - p_D\right)\right|^2$$

Again we use the "trick" :

$$\delta(p) = \lim_{T,V\to\infty} \frac{1}{(2\pi)^4} \int_{-T/2}^{+T/2} dt \int_{-V/2}^{+V/2} d^3x \, e^{ipx}$$

such that

$$\lim_{T,V\to\infty} \frac{1}{TV} \, |\delta^4(p)|^2 = \frac{1}{TV} \, TV \, \delta(p)$$

We get for the transition amplitude:

$$W_{fi} = \left|N_A N_B N_C N_D\right|^2 \; \left|\mathcal{M}\right|^2 \; (2\pi)^4 \;\; \delta^4 \left(p_A + p_B - p_C - p_D\right)$$

For the scattering process: $A + B \rightarrow C + D$ the cross section is obtained from:

$$\begin{aligned} d\sigma &= \frac{W_{fi}}{\text{Flux}} \text{dLips} \\ \text{Flux} &= 4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}/V^2 \\ \text{dLips} &= \frac{V}{(2\pi)^3} \frac{d^3 p_C}{2E_C} \frac{V}{(2\pi)^3} \frac{d^3 p_D}{2E_D} \end{aligned}$$

The volume V cancels again and we obtain:

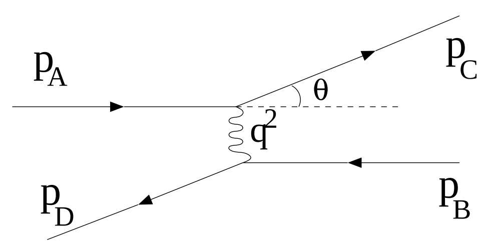
$$d\sigma = \frac{(2\pi)^4 \delta^4 (p_A + p_B - p_C - p_D)}{4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}} |\mathcal{M}|^2 \frac{d^3 p_C}{(2\pi)^3 2E_C} \frac{d^3 p_D}{(2\pi)^3 2E_D}$$

which leads to the differential cross section for $2 \rightarrow 2$ electromagnetic scattering is (see exercise 18):

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{s} \left| \frac{\vec{p}_f}{\vec{p}_i} \right| |\mathcal{M}|^2$$

We will work it out for the relativistic case that: $E = p$, i.e. $m \approx 0$.

$$\begin{aligned} p_A^\mu &= (p, p, 0, 0) \\ p_B^\mu &= (p, -p, 0, 0) \\ p_C^\mu &= (p, p\cos\theta, p\sin\theta, 0) \\ p_D^\mu &= (p, -p\cos\theta, -p\sin\theta, 0) \\ q^\mu &= (p_D - p_B)^\mu = (0, p(1-\cos\theta), -p\sin\theta, 0) \end{aligned}$$



We calculate the matrix element and the differential cross section using:

$$\begin{aligned} (p_A + p_C)^\mu &= (2p, p(1+\cos\theta), p\sin\theta, 0) \\ (p_B + p_D)^\mu &= (2p, -p(1+\cos\theta), -p\sin\theta, 0) \end{aligned}$$

to get:

$$\begin{aligned} (p_A + p_C)^\mu \; g_{\mu\nu} \; (p_B + p_D)^\nu &= p^2 (6 + 2\cos\theta) \\ q^2 &= -2p^2 (1 - \cos\theta) \end{aligned}$$

We then find for the matrix element:

$$\begin{aligned} -i\mathcal{M} &= ie \, (p_A + p_C)^\mu \, \frac{-ig_{\mu\nu}}{q^2} \, ie \, (p_B + p_D)^\nu \\ \mathcal{M} &= e^2 \frac{p^2 (6 + 2\cos\theta)}{2p^2 (1 - \cos\theta)} = e^2 \left(\frac{3 + \cos\theta}{1 - \cos\theta} \right) \end{aligned}$$

and then:

$$|\mathcal{M}|^2 = \left(e^2\right)^2 \left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2$$

Finally we obtain from:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\,\frac{1}{s}\,\frac{p}{p}\,|\mathcal{M}|^2$$

the cross section ($\alpha = e^2/4\pi$):

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\,\frac{1}{s}\,\left(e^2\right)^2 \left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2 = \frac{\alpha^2}{4s}\left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2}$$

This is the QED cross section for spinless scattering.

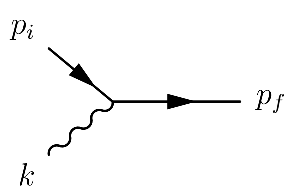
5.4 Particles and Anti-Particles

We have seen that the negative energy state of a particle can be interpreted as the positive energy state of its anti-particle. How does this effect energy conservation that we encounter in the δ-functions? We have seen that the Matrix element has the form of:

$$\mathcal{M} \propto \int \phi_f^*(x)\,V(x)\,\phi_i(x)\,dx$$

Let us examine four cases:

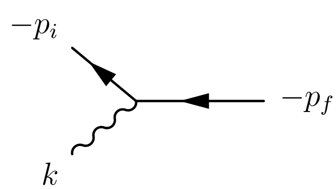
- Scattering of an electron and a photon:



$$\begin{aligned}\mathcal{M} &\propto \int \left(e^{-ip_f x}\right)^* e^{-ikx}\,e^{-ip_i x}\,dx \\ &= \int e^{-i\left(p_i+k-p_f\right)x}\,dx \\ &= (2\pi)^4\,\delta\left(E_i+\omega-E_f\right)\,\delta^3\left(\vec{p_i}+\vec{k}-\vec{p_f}\right)\end{aligned}$$

$\Rightarrow$ Energy and momentum conservation are guaranteed by the δ-function.

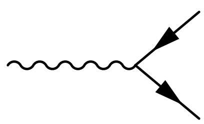
- Scattering of a positron and a photon:



Replace the anti-particles always by particles by reversing ($E, \vec{p} \rightarrow -E, -\vec{p}$) such that now: incoming state $= -p_f$, outgoing state $= -p_i$:

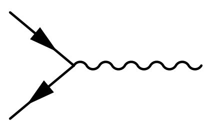
$$\begin{aligned}\mathcal{M} &\propto \int \left(e^{-i(-p_i)x}\right)^* e^{-ikx}\,e^{-i(-p_f)x}\,dx \\ &= \int e^{-i\left(p_i-p_f+k\right)x}\,dx \\ &= (2\pi)^4\,\delta\left(E_i+\omega-E_f\right)\,\delta^3\left(\vec{p_i}+\vec{k}-\vec{p_f}\right)\end{aligned}$$

- Electron positron pair production:



$$\begin{aligned}\mathcal{M} &\propto \int \left(e^{-ip_- x}\right)^* e^{-i(-p_+ + k)x}\, dx \\ &= \int e^{-i(k-p_+-p_-)x}\, dx \\ &= (2\pi)^4\ \delta\left(k - p_- - p_+\right)\end{aligned}$$

- Electron positron annihilation:



$$\begin{aligned}\mathcal{M} &\propto \int \left(e^{-i(k-p_+)x}\right)^* e^{-i(p_-)x}\, dx \\ &= \int e^{-i(p_- + p_+ - k)x} \\ &= (2\pi)^4\ \delta\left(p_- + p_+ - k\right)\end{aligned}$$

<u>**Exercise 20**</u>

Decay rate of $\pi^0 \rightarrow \gamma\gamma$:

(a) *Write down the expression for the total decay rate Γ for the decay: $A \rightarrow C + D$*

(b) *Assume that particle A is a π^0 particle with a mass of 140 MeV and that particles C and D are photons. Draw the Feynman diagram for this decay*

   (i) *assuming the pion is a $u\bar{u}$ state.*

   (ii) *assuming the pion is a $d\bar{d}$ state.*

(c) *For the Matrix element we have: $\mathcal{M} \sim f_\pi\, e^2$, where for the decay constant we insert $f_\pi = m_\pi$.*

   (i) *Where does the factor e^2 come from?*

   (ii) *What do you think is the meaning of the factor f_π? Describe it qualitatively.*

(d) *The π^0 is actually a $u\bar{u} + d\bar{d}$ wave with 3 colour degrees of freedom.*

   (i) *Give the expression for the decay rate.*

   (ii) *Calculate the decay rate expressed in GeV.*

   (iii) *Convert the rate into seconds using the conversion table of the introduction lecture.*

   (iv) *How does the value compare to the Particle Data Group (PDG) value?*

Lecture 6

The Dirac Equation

Introduction

It is sometimes said that Schrödinger had first discovered the Klein-Gordon equation before the equation carrying his own name, but that he had rejected it because it was quadratic in $\frac{\partial}{\partial t}$. In Lecture 2 we have seen how indeed the Klein-Gordon equation leads to the interpretation of negative probabilities: $\rho = 2|N|^2 E$, where the energy can be: $E = \pm\sqrt{\vec{p}^2 + m^2}$.

To avoid this problem Dirac in 1928 tried to make a relativistic correct equation that was linear in $\frac{\partial}{\partial t}$. He wanted to combine the merits of a linear combination (no negative probabilites) with the relativistic correctness of the K.G. equation. Since he wanted the equation to be linear in $\frac{\partial}{\partial t}$, Lorentz covariance requires it to be also linear in $\vec{\nabla}$.

What Dirac found, to his own great surprise, was an equation that describes particles with spin $\frac{1}{2}$, i.e. the fundamental fermions. At the same time he predicted the existence of anti-matter. This was not taken serious untill 1932, when Anderson found the anti-electron: the positron.

6.1 Dirac Equation

Write the Hamiltonian in a general form[1]:

$$H\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\ \psi \tag{6.1}$$

with coëfficients $\alpha_1, \alpha_2, \alpha_3, \beta$. These must be chosen such that after squaring one finds:

$$H^2\psi = \left(\vec{p}^2 + m^2\right)\ \psi$$

Let us try eq 6.1 and see what happens:

$$\begin{aligned} H^2\psi &= (\alpha_i p_i + \beta m)^2\ \psi \qquad \text{with : } \ i = 1,2,3 \\ &= \left(\underbrace{\alpha_i^2}_{=1}\ p_i^2 + \underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{=0 \quad i>j}\ p_i p_j + \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{=0} p_i m + \underbrace{\beta^2}_{=1} m^2 \right)\ \psi \end{aligned}$$

[1] Here $\vec{\alpha} \cdot \vec{p} = \alpha_x p_x + \alpha_y p_y + \alpha_z p_z$

So we have the following requirements:

- $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1$
- $\alpha_1, \alpha_2, \alpha_3, \beta$ anti-commute with each other.

Note that Dirac discovered this just a few years after the beginning of the formulation of quantum mechanics and commuting operators. He was highly interested in the mathematical behaviour of the operators.

Immediatly we conclude that $\vec{\alpha}, \beta$ cannot be ordinary numbers, but that they must be matrices. They now operate on a wave function which has become a column vector (called a *spinor*). This was not believed when Dirac first published his theory.

The lowest dimensional matrices that have the desired behaviour are 4×4 matrices (see the book of Aitchison (1972) chapter 8; section 1). The choice of the $(\vec{\alpha}, \beta)$ is however *not* unique. Here we choose the Dirac-Pauli representations:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad ; \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that the physics is independent of the representation. It only depends on the anti-commuting behaviour of the operators. Another representation is the Weyl representation:

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \qquad ; \qquad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Exercise 21

(a) *Write a general Hermitian* 2×2 *matrix in the form* $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ *where* a *and* c *are real. Write then* $b = s + it$ *and show that the matrix can be written as:* $\{(a+c)/2\}\, I + s\sigma_1 - t\sigma_2 + \{(a-c)/2\}\, \sigma_3$
How can we conclude that $\vec{\alpha}$ *and* β *cannot be* 2×2 *matrices?*

(b) *Show that the* $\vec{\alpha}$ *and* β *matrices in both the Dirac-Pauli as well as in the Weyl representation have the required anti-commutation behaviour.*

One can show using the fact that the energy must be real (see Aitchison) that the α_i and β matrices are Hermitian:

$$\alpha_i^\dagger = \alpha_i \qquad ; \qquad \beta^\dagger = \beta$$

6.2 Covariant form of the Dirac Equation

We had

$$H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi$$

Now we replace: $H \rightarrow i\frac{\partial}{\partial t}$, $\vec{p} \rightarrow -i\vec{\nabla}$ to find:

$$i\frac{\partial}{\partial t}\psi = \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\ \psi$$

Multiply this equation from the left side by β (note that $\beta^2 = 1$):

$$\begin{aligned} i\beta\frac{\partial}{\partial t}\psi &= -i\beta\vec{\alpha}\cdot\vec{\nabla}\psi + m\,\psi \\ i\beta\frac{\partial}{\partial t}\psi &+ i\beta\vec{\alpha}\cdot\vec{\nabla}\psi - m\psi = 0 \\ \left(i\beta\frac{\partial}{\partial t}\psi\right. &+ \left. i\beta\alpha_1\frac{\partial}{\partial x} + i\beta\alpha_2\frac{\partial}{\partial y} + i\beta\alpha_3\frac{\partial}{\partial z}\right)\ \psi - m\psi = 0 \end{aligned}$$

in which we see a nice symmetric structure arising. We write the equation in a covariant notation:

$$\boxed{\begin{aligned} (i\gamma^\mu\partial_\mu - m)\ \psi &= 0 \\ \text{with}:\ \gamma^\mu = (\beta, \beta\vec{\alpha}) &\equiv \text{Dirac } \gamma\text{-matrices} \end{aligned}}$$

In fact the Dirac eq. are really 4 coupled differential equations:

$$\text{for each } j=1,2,3,4 \quad : \quad \sum_{k=1}^{4}\left[\sum_{\mu=0}^{3} i\left(\gamma^\mu\right)_{jk}\ \partial_\mu - m\delta_{jk}\right]\ (\psi_k) = 0$$

$$\text{or} \quad : \quad \left[i \underbrace{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}_{\gamma^\mu} \cdot\partial_\mu - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot m \right] \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

or even more specific:

$$\left[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\frac{i\partial}{\partial t} + \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\frac{i\partial}{\partial x} + \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\frac{i\partial}{\partial y} + \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\frac{i\partial}{\partial z} - \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} m\right] \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Take note of the use of the Dirac (or spinor) indices ($j, k = 1, 2, 3, 4$) simultaneously with the Lorentz indices ($\mu = 0, 1, 2, 3$).

On the other hand, there is an alternative and very short notation: an electron is described by:

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0 \quad \Rightarrow \quad (i\ \partial\!\!\!/ - m)\,\psi = 0$$

while the equation:

$$i\,\partial\!\!\!/\,\psi = 0$$

contains everything you want to know about a neutrino (assuming $m = 0$).

6.3 The Dirac Algebra

From the definitions of $\vec{\alpha}$ and β we can derive the following relation:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Thus:

$$\left(\gamma^0\right)^2 = \mathbb{1} \quad ; \quad \left(\gamma^1\right)^2 = \left(\gamma^2\right)^2 = \left(\gamma^2\right)^2 = -\mathbb{1}$$

Also we have the Hermitean conjugates:

$$\begin{aligned} \gamma^{0\dagger} &= \gamma^0 \quad ; \quad \beta^\dagger = \beta \\ \gamma^{i\dagger} &= \left(\beta\alpha^i\right)^\dagger = \alpha^{i\dagger}\beta^\dagger = \alpha^i\beta = -\gamma^i \end{aligned}$$

Then the relation $\left\{\gamma^k, \gamma^0\right\} = 0$ implies:

$$\begin{aligned} \gamma^k\gamma^0 &= -\gamma^0\gamma^k = \gamma^0\gamma^{k\dagger} \\ \text{thus}: \ \gamma^0\gamma^k\gamma^0 &= \gamma^{02}\gamma^{k\dagger} = \gamma^{k\dagger} \end{aligned}$$

In general:

$$\boxed{\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0}$$

In words this means that we can undo a hermitean conjugate $\gamma^{\mu\dagger}\gamma^0$ by moving a γ^0 "through it": $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$

Furthermore we can define:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

with the characteristics:

$$\gamma^{5\dagger} = \gamma^5 \qquad \left(\gamma^5\right)^2 = \mathbb{1} \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0$$

6.4 Current Density

Similarly to the case of the Schrödinger and the Klein-Gordon equations we can derive a continuity equation to determine the current density j^μ: Write the Dirac equation as:

$$i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^k\frac{\partial\psi}{\partial x^k} - m\psi = 0 \qquad\qquad k = 1, 2, 3$$

We work now with matrices, so instead of complex conjugates we use Hermitean conjugates:

$$-i\frac{\partial\psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial\psi^\dagger}{\partial x^k}\left(-\gamma^k\right) - m\psi^\dagger = 0$$

But now we have a problem! The additional $-$ sign in $\left(-\gamma^k\right)$ disturbs the Lorentz invariant form of the equation. This means we cannot use this equation.
We can restore Lorentz covariance by multiplying the equation from the right by γ^0. Or, in other words, we can define the *adjoint spinor* as: $\overline{\psi} = \psi^\dagger\gamma^0$.

$$\text{Dirac spinor}: \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \qquad \text{Adjoint Dirac spinor}: \left(\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3, \overline{\psi}_4\right)$$

The adjoint Dirac equation the becomes:

$$-i\frac{\partial\overline{\psi}}{\partial t}\gamma^0 - i\frac{\partial\overline{\psi}}{\partial x^k}\gamma^k - m\overline{\psi} = 0 \qquad k = 1, 2, 3$$

Now we multiply the Dirac equation from the left by $\overline{\psi}$ and we mulitply the adjoint Dirac equation from the right by ψ:

$$\begin{array}{rcl} \left(i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi}\right)\psi & = & 0 \\ \overline{\psi}\left(i\partial_\mu\gamma^\mu\psi - m\psi\right) & = & 0 \\ \hline & + & \_\_ \\ \overline{\psi}\left(\partial_\mu\gamma^\mu\psi\right) + \left(\partial_\mu\overline{\psi}\gamma^\mu\right)\psi & = & 0 \end{array}$$

We recognize again the continuity equation:

$$\partial_\mu j^\mu = 0 \qquad \text{with}: \quad j^\mu = \left(\overline{\psi}\gamma^\mu\psi\right)$$

6.4.1 Dirac Interpretation

Consider

$$j^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = \sum_{i=1}^{4}|\psi_i|^2 > 0$$

Therefore the probability density is always greater then 0! This is the historical motivation of Dirac's work.

However, we had seen in the Pauli-Weiskopf interpretation that $j^\mu = \left(\rho, \vec{j}\right)$ was the *charge current density.* In that case:

$$j^\mu = -e\overline{\psi}\gamma^\mu\psi$$

is the electric 4-vector current density (just as we used it before). In the Feynman-Stückelberg interpretation the particle solution with negative energy **is** the antiparticle solution with positive energy.

Note:
In the case of Klein-Gordon waves, the current of an antiparticle ($j^\mu = 2|N|^2 p^\mu$) gets a minus sign w.r.t. the current of the particle, due to reversal of 4-momentum. In order to keep this convention an additional, ad-hoc, $-$ sign is required for the current of a spin-1/2 antiparticle (e.g. positron). This additional $-$ sign between particles and antiparticles is only required for fermionic currents and not for bosonic currents. It is related to the spin-statistics connection: bosonic wavefunctions are symmetric, and fermionic wavefunctions are anti-symmetric. In field theory[2] the extra minus sign is related to the resulting fact that bosonic field operators follow *commutation* relations, while fermionic field operators follow *anti-commutation* relations. This was realized first by W.Pauli in 1940. In conclusion: fermionic anti-particle currents get an ad-hoc additional $-$ sign to maintain the Feynman-Stückelberg interpretation!

If we use the ansatz: $\psi = u(p)e^{-ipx}$ for the spinor ψ then we get for the interaction current density 4-vector:

$$\begin{aligned} j^\mu_{fi} &= -e u_f^\dagger \gamma^0 \gamma^\mu u_i e^{i(p_f - p_i)x} \\ &= -e \overline{u}_f \gamma^\mu u_i e^{-iqx} \end{aligned}$$

$$j^\mu_{fi} = -e \left(\quad \overline{u}_f \quad \right) \left(\quad \gamma^\mu \quad \right) \left(u_i \right) \cdot e^{-iqx}$$

Exercise 22: Traces and products of γ matrices

For the γ matrices we have:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu}$$

Use this relation to show that:

(a) $\not a \not b + \not b \not a = 2\,(a \cdot b)$

(b) i) $\gamma_\mu \gamma^\mu = 4$

ii) $\gamma_\mu \not a \gamma^\mu = -2 \not a$

iii) $\gamma_\mu \not a \not b \gamma^\mu = 4\,(a \cdot b)$

iv) $\gamma_\mu \not a \not b \not c \gamma^\mu = -2 \not c \not b \not a$

(c) i) $\mathrm{Tr}\; \mathbb{1} = 4$

ii) Tr *(odd number of γ_μ's)* $= 0$

iii) $\mathrm{Tr}\,(\not a \not b) = 4\,(a \cdot b)$

iv) $\mathrm{Tr}\,(\not a \not b \not c \not d) = 4\;[\,(a\cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)\,]$

(d) i) $\mathrm{Tr}\,\gamma^5 = \mathrm{Tr}\, i\,\gamma^0\gamma^1\gamma^2\gamma^3 = 0$

ii) $\mathrm{Tr}\,\gamma^5 \not a \not b = 0$

iii) *Optional excercise for "die-hards":* $\mathrm{Tr}\,\gamma^5 \not a \not b \not c \not d = -4\,i\,\varepsilon_{\alpha\beta\gamma\delta}\, a^\alpha b^\beta c^\gamma d^\delta$ *where $\varepsilon_{\alpha\beta\gamma\delta} = +1\,(-1)$ for an even (odd) permutation of 0,1,2,3; and 0 if two indices are the same.*

[2] See Aitchison & Hey, 3rd edition §7.2

Lecture 7

Solutions of the Dirac Equation

7.1 Solutions for plane waves with $\vec{p} = 0$

We look for free particle solutions of:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\ \psi = 0}$$

A *quick* way to get wave solutions with $\vec{p} = 0$ is to realize that this implies $-i\vec{\nabla}\ \psi = 0$, or that the wavefunction ψ has no explicit space dependence. In that case the Dirac equation $(i\gamma^\mu \partial_\mu - m)\ \psi = 0$ reduces to $i\gamma^0 \frac{\partial \psi}{\partial t} = m\psi$, or written in the Dirac-Pauli representation:

$$\begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_A}{\partial t} \\ \frac{\partial \psi_B}{\partial t} \end{pmatrix} = -i\, m \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad \Rightarrow \quad \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} e^{-imt}\, \psi_A(0) \\ e^{+imt}\, \psi_B(0) \end{pmatrix}$$

where the solution is given immediately. Note that ψ_A represents a two-component spinor with positive energy and ψ_B a two-component spinor with negative energy. In the following, however, we will follow the standard textbook method to derive the Dirac solutions.

Exercise 23
Each of the four components of the Dirac equation satisfies the Klein Gordon equation: $(\partial_\mu \partial^\mu + m^2)\, \psi_i = 0$.
Show this explicitly by operating on the Dirac equation from the left with: $\gamma^\nu \partial_\nu$.
Hint: Use the anticommutation relation of the γ-matrices.

Ansatz:
This suggests to try the plane wave solutions:

$$\psi(x) = u(p)\, e^{-ipx}$$

Since $\psi(x)$ is a 4-component spinor, also $u(p)$ is a 4-component spinor. After substitution in the Dirac equation we find what is called the Dirac equation in the momentum representation:

$$\boxed{\begin{array}{rcl} (\gamma^\mu p_\mu - m)\ u(p) & = & 0 \\ \text{or}: \quad (\not{p} - m)\ u(p) & = & 0 \end{array}}$$

Remember that the Dirac equation is a linear set of equations (use here the Pauli-Dirac representation):

$$\left[\begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} p^i - \begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix} m \right] \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

In fact we can recognize two coupled equations:

$$\left\{ \begin{array}{rcl} (\vec{\sigma} \cdot \vec{p})\ u_B & = & (E - m)\, u_A \\ (\vec{\sigma} \cdot \vec{p})\ u_A & = & (E + m)\, u_B \end{array} \right.$$

where u_A and u_B are now each two component spinors.

Let us first look at solutions for a particle at rest: $\vec{p} = 0$:

$$\left\{ \begin{array}{rcl} (\vec{\sigma} \cdot \vec{p})\ u_B & = & (E - m)\, u_A \\ (\vec{\sigma} \cdot \vec{p})\ u_A & = & (E + m)\, u_B \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{rcl} E\, u_A & = & m\, u_A \\ E\, u_B & = & -m\, u_B \end{array} \right.$$

For these equations there are 4 independent solutions, the eigenvectors:

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad u^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad u^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with eigenvalues: $E = m,\ m,\ -m,\ -m$, respectively.
$u^{(1)}$, $u^{(2)}$ are the positive energy solutions of e^-.
$u^{(3)}$, $u^{(4)}$ are the negative energy solutions of e^- and thus the positive energy solutions of e^+.

We define the antiparticle solutions as follows:

$$\begin{array}{rcl} v^{(1)}(p) & = & u^{(4)}(-p) \\ v^{(2)}(p) & = & -u^{(3)}(-p) \end{array}$$

The $-$sign in $u^{(3)}$ is chosen such that the charge conjugation transformation (see later) implies $u^{(1)} \rightarrow v^{(1)}$ and $u^{(2)} \rightarrow v^{(2)}$.

7.2 Solutions for moving particles $\vec{p} \neq 0$

Again look at:

$$\begin{cases} (\vec{\sigma} \cdot \vec{p})\ u_B &= (E-m)\, u_A \\ (\vec{\sigma} \cdot \vec{p})\ u_A &= (E+m)\, u_B \end{cases}$$

Choose now for the two $E > 0$ solutions:

$$u_A^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad u_A^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then it follows:

$$\begin{aligned} u_B^{(1)} &= \frac{\vec{\sigma} \cdot \vec{p}}{E+m}\, u_A^{(1)} = \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ u_B^{(2)} &= \frac{\vec{\sigma} \cdot \vec{p}}{E+m}\, u_A^{(2)} = \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

So, the two independent solutions are:

$$u^{(1)} = \begin{pmatrix} u_A^{(1)} \\ u_B^{(1)} \end{pmatrix} \quad , \quad u^{(2)} = \begin{pmatrix} u_A^{(2)} \\ u_B^{(2)} \end{pmatrix}$$

Analogously: choose for the two $E < 0$ solutions:

$$u_B^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad u_B^{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then it follows:

$$\begin{aligned} u_A^{(3)} &= \frac{\vec{\sigma} \cdot \vec{p}}{E-m}\, u_B^{(3)} = -\frac{\vec{\sigma} \cdot \vec{p}}{|E|+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ u_A^{(4)} &= \frac{\vec{\sigma} \cdot \vec{p}}{E-m}\, u_B^{(4)} = -\frac{\vec{\sigma} \cdot \vec{p}}{|E|+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

So, the two independent solutions are now:

$$u^{(3)} = \begin{pmatrix} u_A^{(3)} \\ u_B^{(3)} \end{pmatrix} \quad , \quad u^{(4)} = \begin{pmatrix} u_A^{(4)} \\ u_B^{(4)} \end{pmatrix}$$

To gain insight, let us write them out in more detail.
Use the explicit representation:

$$\vec{\sigma} \cdot \vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z$$

we find:

$$(\vec{\sigma} \cdot \vec{p})\ u_A^{(1)} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

and similar for $u_A^{(2)}$, $u_B^{(3)}$, $u_B^{(4)}$.

Then we find the solutions:

$$\text{electron spinors :} \quad u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad , \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$\text{positron spinors :} \quad v^{(1)} = N \begin{pmatrix} \frac{p_x-ip_y}{|E|+m} \\ \frac{-p_z}{|E|+m} \\ 0 \\ 1 \end{pmatrix} \quad , \quad v^{(2)} = N \begin{pmatrix} \frac{-p_z}{|E|+m} \\ \frac{-(p_x+ip_y)}{|E|+m} \\ -1 \\ 0 \end{pmatrix}$$

and we can verify that the $u^{(1)}$ - $u^{(4)}$ solutions are indeed orthogonal.

Exercise 24

Show explicitly that the Dirac equations describes relativistic particles. To do this substitute the expression:

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A \qquad \text{into} \qquad u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m} u_B$$

Hint: Work out the product $(\vec{\sigma}\cdot\vec{p})^2$ *in components.*

7.3 Particles and Anti-particles

The spinors $u(p)$ of matter waves are solutions of the Dirac equation:

$$(\not{p} - m)\ u(p) = 0 \qquad \Rightarrow \text{solutions with } p^0 = E > 0$$

For the antiparticles (the solutions $v(p)$) we have substituted $v(p) = u(-p)$. Remember that we interpret an antiparticle as a particle travelling back in time. Let us make the same substitution in the Dirac equation (for negative p^0 !):

$$(-\not{p} - m)\ u(-p) = 0 \qquad \Rightarrow \text{replaced } p \rightarrow -p$$

Then we find for solutions with the new p^0 (=E>0) the Dirac equation for anti-particles:

$$\boxed{(\not{p} + m)\ v(p) = 0}$$

7.3.1 The Charge Conjugation Operation

The Dirac equation for a particle in an electromagnentic field is obtained by substituting $\partial_\mu \rightarrow \partial_\mu + iqA_\mu$ in the free Dirac equation. For an electron ($q = -e$) this leads to:

$$[\gamma^\mu (i\partial_\mu + eA_\mu) - m]\,\psi = 0 \quad .$$

Similarly, there must be a Dirac equation describing the positron ($q = +e$):

$$[\gamma^\mu (i\partial_\mu - eA_\mu) - m]\,\psi_C = 0 \quad ,$$

where the positron wave function ψ_C is obtained by a one-to-one correspondence with the electron wave function ψ. Let us assume that the positron wave function can be obtained using a charge conjugation matrix C, which operates as follows:

$$\psi_C = C\,\overline{\psi}^T = C\gamma^0\psi^* \quad .$$

We note that $\overline{\psi}$ is the "row-wise" solution of the adjoint Dirac equation (while $\psi^\dagger$ is not! - see previous lecture) and $\overline{\psi}^T$ is the associated column vector (like ψ).
Let us take the complex conjugate of the electron equation:

$$[-\gamma^{\mu *} (i\partial_\mu - eA_\mu) - m]\,\psi^* = 0$$

Assume that there is a matrix $(C\gamma^0)$, such that:

$$-(C\gamma^0)\gamma^{\mu *} = \gamma^\mu (C\gamma^0)$$

then we can use the complex conjugated electron equation to show that:

$$\begin{aligned} (C\gamma^0)\;[-\gamma^{\mu *} (i\partial_\mu - eA_\mu) - m]\,\psi^* &= 0 \\ [\gamma^\mu (i\partial_\mu - eA_\mu) - m]\,C\gamma^0\psi^* &= 0 \end{aligned}$$

and that we indeed obtain the positron equation if $\psi_C = C\gamma^0\psi^*$.
A possible choice of the matrix $(C\gamma^0)$ can be shown to be:

$$C\gamma^0 = i\gamma^2 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \quad .$$

7.4 Normalisation of the Wave Function

We choose again (similar to the Klein-Gordon case) a normalisation of the wave function such that there are $2E$ particles in a unit volume:

$$\int_V \rho dV = \int \overline{\psi}\gamma^0\psi\, dV = \int \psi^\dagger\gamma^0\gamma^0\psi dV = \int \psi^\dagger\psi dV$$

Substitute now the plane wave solution: $\psi = u(p)\, e^{-ipx}$:

$$\int \rho dV = \int u^\dagger(p)\, e^{ipx}\, u(p)\, e^{-ipx}\, dV = u^\dagger(p) u(p) \int_V dV$$

Choose the unit volume and normalise to $2E$:

$$\int_V dV = 1 \qquad ; \qquad u^\dagger(p) u(p) = 2E$$

where for u we must substitute: $u^1(p), u^2(p), v^1(p), v^2(p)$. Using orthogonality of the solutions we get the relations:

$$\begin{aligned} u^{(r)\dagger}\, u^{(s)} &= 2E\, \delta_{rs} \qquad r,s = 1,2 \\ v^{(r)\dagger}\, v^{(s)} &= 2E\, \delta_{rs} \qquad r,s = 1,2 \end{aligned}$$

Explicit calculation gives:

$$\begin{aligned} u^{(1)\dagger}\, u^{(1)} &= N^2 \left(1, 0, \frac{p_z}{E+m}, \frac{p_x - ip_y}{E+m}\right) \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} = 2E \\ \ldots\ldots &\Rightarrow \frac{N^2}{(E+m)^2} \left((E+m)^2 + \underbrace{p_x^2 + p_y^2 + p_z^2}_{E^2-m^2} \right) = 2E \\ \ldots\ldots &\Rightarrow \frac{N^2}{(E+m)^2} (2E\,(E+m)) = 2E \\ &\Rightarrow N = \sqrt{E+m} \end{aligned}$$

Analogously for $u^{(2)}$, $v^{(1)}$, $v^{(2)}$.

7.5 The Completeness Relation

Let's look again at the Hermitian conjugate Dirac equation for the adjoint spinors $\overline{u}$, $\overline{v}$:

$$\begin{aligned} \text{Dirac}: &\qquad (\not{p} - m)\, u = 0 \\ \text{Look at}: &\qquad [(\gamma^\mu p_\mu - m)\ u = 0]^\dagger \qquad \Rightarrow \qquad u^\dagger \left(\gamma^{\mu\dagger} p_\mu - m\right) = 0 \end{aligned}$$

Multiply this from the right side by γ^0:

$$u^\dagger \gamma^{\mu\dagger} \gamma^0 p_\mu - u^\dagger \gamma^0 m = 0$$

Use now: $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$ to find:

$$\begin{aligned} \underbrace{u^\dagger \gamma^0}_{\overline{u}} \gamma^\mu \underbrace{\gamma^0 \gamma^0}_{1} p_\mu - \underbrace{u^\dagger \gamma^0}_{\overline{u}} m &= 0 \\ \text{then}: \qquad \overline{u} \gamma^\mu p_\mu - \overline{u} m &= 0 \end{aligned}$$

The conjugate Dirac equation is therefore:

$$\boxed{\overline{u}\ (\not p\ - m) = 0}$$

Also in exactly the same way:

$$(\not p\ + m)\ v = 0 \qquad \Rightarrow \qquad \overline{v}\ (\not p\ + m) = 0$$

We can now (see exercise 25) derive the *completeness relations*:

$$\boxed{\begin{aligned} \sum_{s=1,2} u^{(s)}(p)\,\overline{u}^{(s)}(p) &= (\not p\ + m) \\ \sum_{s=1,2} v^{(s)}(p)\,\overline{v}^{(s)}(p) &= (\not p\ - m) \end{aligned}}$$

Note: $u\,\overline{u}$ is **not** an inproduct but we have here 4x4 matrix relations:

$$\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \cdot (....) = \begin{pmatrix} & \gamma^\mu & \end{pmatrix} \cdot p_\mu + \begin{pmatrix} & \mathbb{1} & \end{pmatrix} \cdot m$$

These relations will be used later on in the calculation of the Feynman diagrams. (**Note**: $\sum_{s=3,4} u^{(s)}(p)\,\overline{u}^{(s)}(p) = \sum_{s=1,2} v^{(s)}(-p)\,\overline{v}^{(s)}(-p) = -(\not p\ + m)$)

Exercise 25: *(See also H&M p.110-111 and Griffiths p. 242)*
The spinors u, v, $\bar{u}$ and $\bar{v}$ are solutions of respectively:

$$\begin{aligned} (\not p - m)\,u &= 0 \\ (\not p + m)\,v &= 0 \\ \bar{u}\,(\not p - m) &= 0 \\ \bar{v}\,(\not p + m) &= 0 \end{aligned}$$

(a) *Use the orthogonality relations:*

$$\begin{aligned} u^{(r)\dagger}\,u^{(s)} &= 2E\,\delta_{rs} \\ v^{(r)\dagger}\,v^{(s)} &= 2E\,\delta_{rs} \end{aligned}$$

to show that:

$$\begin{aligned} \bar{u}^{(s)}\,u^{(s)} &= 2m \\ \bar{v}^{(s)}\,v^{(s)} &= -2m \end{aligned}$$

(b) *Show that:* $(\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2$

(c) *Derive the completeness relations:*

$$\begin{aligned} \sum_{s=1,2} u^{(s)}(p)\,\bar{u}^{(s)}(p) &= \not p + m \\ \sum_{s=1,2} v^{(s)}(p)\,\bar{v}^{(s)}(p) &= \not p - m \end{aligned}$$

7.6 Helicity

The Dirac spinors for a given momentum p have a two-fold degeneracy. This implies that there must be an additional observable that commutes with H and p and the eigenvalues of which distinguish between the degenerate states.
Could the extra quantum number be spin? So, eg.: $u^{(1)}$ = spin "up", and $u^{(2)}$ =spin "down"?
No! Because spin does not commute with H (see exercise 26).

Exercise 26: *(Exercise 7.8 Griffiths, see also Exercise 5.4 of H & M)*
The purpose of this problem is to demonstrate that particles described by the Dirac equation carry "intrinsic" angular momentum ($\vec{S}$) in addition to their orbital angular momentum ($\vec{L}$). We will see that $\vec{L}$ and $\vec{S}$ are not conserved individually but that their sum is.

(a) *Compare the Dirac equation*

$$(\gamma^\mu p_\mu - m)\,\psi = 0\ ,$$

with Schrödinger's equation

$$H\psi = E\psi\ ,$$

and derive an expression for the Hamiltonian H from this (see previous lecture).

(b) *The orbital angular momentum is $\vec{L} = \vec{r} \times \vec{p}$. Show that $[p_i, x_j] = -i\delta_{ij}$ and use this to show that $\vec{L}$ does not commute with H:*

$$\left[H, \vec{L}\right] = -i\gamma^0\,(\vec{\gamma} \times \vec{p})\ .$$

(c) *Show that $\vec{S}$, given by:*

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

also does not commute with H:

$$\left[H, \vec{S}\right] = i\gamma^0\,(\vec{\gamma} \times \vec{p})\ .$$

We see from (b) and (c) that the sum of the commutators is equal to 0, and therefore $\vec{J} = \vec{L} + \vec{S}$ is conserved.

The fact that spin is not a good quantum number can also be realised upon inspection of the solutions u:

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$

So solutions can have: $p_x \neq 0$ & $p_y \neq 0$ & $p_z \neq 0$.

The spin operator is defined as:

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

If it commutes, the states should be eigenstates of the spin operator and we expect:

$$\vec{\Sigma}\, u^{(1)} = s\, u^{(1)} \quad ?$$

This is not possible as can be seen by requiring the equation:

$$\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} p_z / (E+m) \\ (p_x + ip_y) / (E+m) \end{pmatrix} \end{pmatrix} \stackrel{?}{=} s \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} p_z / (E+m) \\ (p_x + ip_y) / (E+m) \end{pmatrix} \end{pmatrix}$$

to be true for any p_x, p_y, p_z.
However, it can be made to work if we define the *helicity* λ as:

$$\lambda = \frac{1}{2}\vec{\Sigma} \cdot \hat{p} \equiv \frac{1}{2} \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix}$$

We could interpret the helicity as the "spin component in the direction of movement". (Or: we choose $p_x = p_y = 0$ and consider only σ_z in the equation above). In this case the orbital angular momentum is zero by definition and and $J = S$ is conserved.

One can verify that indeed λ commutes with the Hamiltonian $H = \vec{\alpha} \cdot \vec{p} + \beta m$:

$$\left[H, \vec{\Sigma} \cdot \hat{p}\right] = ... = 0$$

Choose $\vec{p} = ((0, 0, p)$. For the spin component in the direction of movement we have the eigenvalues:

$$\begin{aligned} \frac{1}{2}(\vec{\sigma} \cdot \hat{p})\ u_A &= \frac{1}{2}\sigma_3 u_A = \pm\frac{1}{2}u_A \\ \frac{1}{2}(\vec{\sigma} \cdot \hat{p})\ u_B &= \frac{1}{2}\sigma_3 u_B = \pm\frac{1}{2}u_B \end{aligned}$$

Positive helicity = spin and momentum parallel
Negative helicity = spin and momentum anti-parallel

Exercise 27: *(Exercise 5.5 of H & M)*

(a) *Use the equations*

$$(\vec{\sigma} \cdot \vec{p})\, u_A = (E + m)\, u_B \tag{7.1}$$

to show that, for a non-relativistic electron with velocity β*,* u_B *is een factor* $\frac{1}{2}\beta$ *smaller then* u_A*. In a non-relativistic description* ψ_A *and* ψ_B *are often called respectively the "large" and "small" components of the electron wavefunction.*

(b) *Show that the Dirac equation for an electron with charge* $-e$ *in the non-relativistic limit in an electromagnetic field* $A^\mu = (A^0, \mathbf{A})$ *reduces to the Schrödinger-Pauli equation*

$$\left(\frac{1}{2m}\left(\vec{p} + e\vec{A}\right)^2 + \frac{e}{2m}\vec{\sigma} \cdot B - eA^0\right) \psi_A - E_{NR}\psi_A \,, \tag{7.2}$$

where the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$*, and the non-relativistic energy* $E_{NR} = E - m$*. Assume that* $|eA^0| << m$*.*
Do this by substituting $p^\mu + eA^\mu$ *for* p^μ *in eq 7.1 and solve the equations for* ψ_A*. Use:*

$$\vec{p} \times \vec{A} + \vec{A} \times \vec{p} = -i\vec{\nabla} \times \vec{A} \,,$$

where $\vec{p} = -i\vec{\nabla}$*.*
The term with eA^0 *in 7.2 is a constant potential energy that is of no further importance. The term with* $\vec{B}$ *arises due to the fact that* $\vec{p}$ *and* $\vec{A}$ *don't commute. In this term we recognise the magnetic field:*

$$-\vec{\mu} \cdot \vec{B} = -g\frac{e}{2m}\vec{S} \cdot \vec{B} \,.$$

Here g *is the* gyromagnetic ratio, *i.e. the ratio between the magnetic moment of a particle and its spin. Classicaly we have* $g = 1$*, but according to the Direc equation* $(\vec{S} = \frac{1}{2}\vec{\sigma})$ *one finds* $g = 2$*. The current value of* $(g - 2)/2$ *is according to the Particle Data Book*

$$(g - 2)/2 = 0.001159652193 \pm 0.000000000010$$

This number, and its precision, make QED the most accurate theory in physics. The deviation from $g = 2$ *is caused by high order corrections in perturbation theory.*

Lecture 8

Spin 1/2 Electrodynamics

8.1 Feynman Rules for Fermion Scattering

With the spinor solutions of the Dirac equation we finally have the tools to calculate cross section for fermions (spin-1/2 particles). Analogously to the case of spin 0 particles (K.G.-waves) we determine the solutions of the Dirac equations in the presence of a perturbation potential. So we work with the free spin-1/2 solutions $\psi = u(p)\, e^{-ipx}$ that satisfy the free Dirac equation: $(\gamma_\mu p^\mu - m)\,\psi = 0$.

In order to introduce an electromagnetic perturbation we make again the substitution for a particle with $q = -e$: $p^\mu \to p^\mu + eA^\mu$. The Dirac equation for an electron then becomes:

$$(\gamma_\mu p^\mu - m)\,\psi + e\gamma_\mu A^\mu \psi = 0 \tag{8.1}$$

Again, we would like to have a kind of Schrödinger equation, ie. an equation of the type:

$$(H_0 + V)\,\psi = E\psi$$

In order to get to this form, we multiply eq 8.1 from the left by γ^0:

$$\begin{aligned}
\to \quad & \left(\gamma^0\gamma_\mu p^\mu - \gamma^0 m\right)\psi + e\gamma^0\gamma_\mu A^\mu\psi = 0 \\
\to \quad & \left(E - \gamma^0\gamma^k p^k - \gamma^0 m\right)\psi = -e\gamma^0\gamma_\mu A^\mu\psi \\
\to \quad & E\psi = \underbrace{\left(\gamma^0\gamma^k p^k + \gamma^0 m\right)}_{H_0 = \vec{\alpha}\cdot\vec{p} + \beta m}\psi - \underbrace{e\gamma^0\gamma_\mu A^\mu}_{V}\psi
\end{aligned}$$

For such a theory we can write, in analogy to spinless scattering:

$$T_{fi} = -i \int \psi_f^\dagger(x)\, V(x)\, \psi_i(x)\, d^4x$$

Note, that the difference with the case of the KG solutions in spinless scattering is that we had:

$$T_{fi} = -i \int \psi_f^*(x)\, V(x)\, \psi_i(x)\, d^4x$$

where we now have Hermite conjugates instead of complex conjugates.

We substitute for the potential: $V(x) = -e\gamma^0\gamma_\mu A^\mu$ to obtain the expression:

$$\begin{aligned} T_{fi} &= -i\int \psi_f^\dagger(x)\ \left(-e\gamma^0\gamma_\mu A^\mu(x)\right)\ \psi_i(x) d^4x \\ &= -i\int \overline{\psi}_f(x)\ (-e)\,\gamma_\mu\psi_i(x)A^\mu(x)\ d^4x \end{aligned}$$

For the current density we had in a previous lecture the expression:

$$j^\mu = -e\overline{\psi}\gamma^\mu\psi$$

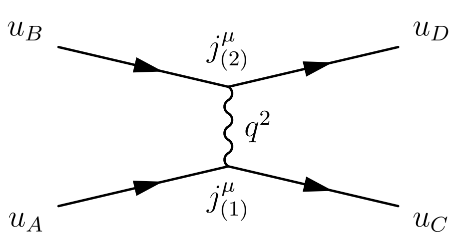
So we find, in complete analogy to the spinless particle case:

$$\begin{aligned} T_{fi} &= -i\int j_\mu^{fi}\ A^\mu\ d^4x \\ \text{with}\quad j_\mu^{fi} &= -e\,\overline{\psi}_f\,\gamma_\mu\,\psi_i \\ &= -e\overline{u}_f\,\gamma_\mu\,u_i\,e^{i(p_f-p_i)x} \end{aligned}$$

and j_μ^{fi} can be interpreted as the electromagnetic transition current between state i and state f.

Remember that:
$$j_\mu^{fi} = \begin{pmatrix} \ \overline{u}_f\ \end{pmatrix}\begin{pmatrix} & & \\ & \gamma_\mu & \\ & & \end{pmatrix}\begin{pmatrix} \\ u_i \\ \\ \end{pmatrix} = \left(j^{fi}\right)_\mu$$

Similar to the spinless case we will use the A^μ solutions of the Maxwell equations to determine the Feynman rules for scattering of particle with spin. Consider again the case in which particle 1 scatters in the field of particle 2: ie. we consider the interaction: $A + B \to C + D$:



We had from Maxwell:

$$\Box A^\mu = j^\mu_{(2)}$$

to which the solution was:

$$A^\mu = -\frac{1}{q^2}j^\mu_{(2)}$$

The transition amplitude is then again:

$$T_{fi} = -i\int j_\mu^{(1)}\ \frac{-1}{q^2}\ j^\mu_{(2)}\ d^4x = -i\int j^\mu_{(1)}\ \frac{-g_{\mu\nu}}{q^2}\ j^\nu_{(2)}\ d^4x$$

which is symmetric in terms of particle (1) and (2). We insert the explicit expression for the current:

$$j^\mu_{fi} = -e\overline{u}_f\gamma^\mu u_i\,e^{i(p_f-p_i)x}$$

to obtain:

$$T_{fi} = -i \int -e\overline{u}_C \gamma^\mu u_A \, e^{i(p_C - p_A)x} \cdot \frac{-g_{\mu\nu}}{q^2} \cdot -e\overline{u}_D \gamma^\nu u_B \, e^{i(p_D - p_B)x} \, d^4x$$

So that we arrive at the "Feynman Rules":

$$\begin{aligned} T_{fi} &= -i\,(2\pi)^4 \; \delta^4 \left(p_D + p_C - p_B - p_A\right) \cdot \mathcal{M} \\ -i\mathcal{M} &= \underbrace{ie\left(\overline{u}_C \gamma^\mu u_A\right)}_{\text{vertex}} \cdot \underbrace{\frac{-ig_{\mu\nu}}{q^2}}_{\text{propagator}} \cdot \underbrace{ie\left(\overline{u}_D \gamma^\nu u_B\right)}_{\text{vertex}} \end{aligned}$$

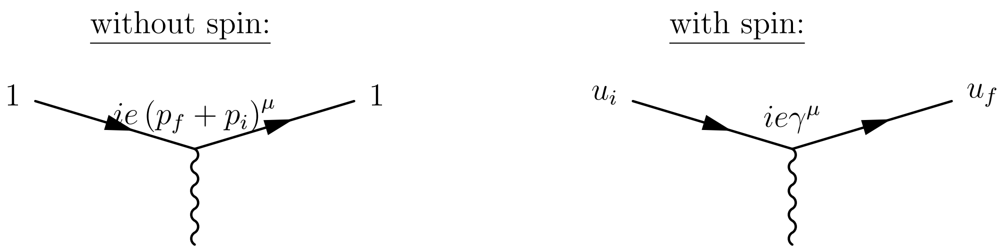


Figure 8.1: Vertex factors for *left:* spinless particles, *right:* spin 1/2 particles.

Exercise 28:

A spinless electron can interact with A^μ only via its charge; the coupling is proportional to $(p_f + p_i)^\mu$. An electron with spin, on the other hand, can also interact with the magnetic field via its magnetic moment. This coupling involves the factor $i\sigma^{\mu\nu}\,(p_f - p_i)$. The relation between the Dirac current and the Klein-Gordon current can be studied as follows:

(a) *Define the antisymmetric $\sigma^{\mu\nu}$ tensor as:*

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$$

Show that the Gordon decomposition of the Dirac current can be made:

$$\overline{u}_f \gamma^\mu u_i = \frac{1}{2m}\overline{u}_f \left[(p_f + p_i)^\mu + i\sigma^{\mu\nu}\,(p_f - p_i)_\nu\right] u_i$$

Hint: Start with the term proportional to $\sigma^{\mu\nu}$ and use: $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ and use the Dirac equations: $\gamma^\nu p_{i\nu} u_i = m u_i$ and $\overline{u}_f \gamma^\nu p_{f\nu} = m\overline{u}_f$.

(b) *(optional) Make exercise 6.2 on page 119 of H& M which shows that the Gordon decomposition in the non-relativistic limit leads to an electric and a magnetic interaction. (Compare also to exercise 27.)*

8.2 Electron - Muon Scattering

We proceed to use the Feynman rules to calculate the cross section of the process: $e^-\mu^- \to e^-\mu^-$. We want to calculate the *unpolarized cross section:*

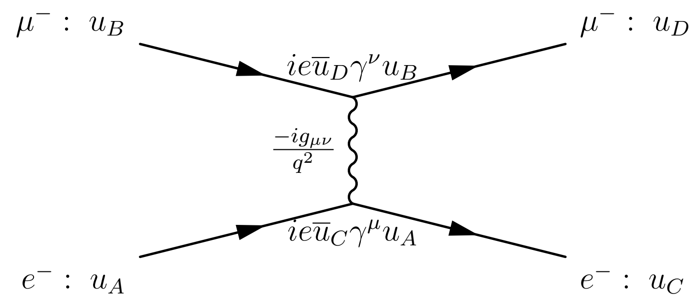
- The incoming particles are not polarized. This implies that we average over spins in the initial state.
- The polarization of the final state particles is not measured. This implies that we sum over the spins in the final state.

The spin summation and averaging means that we replace the matrix element by:

$$|\mathcal{M}|^2 \to \overline{|\mathcal{M}|^2} = \frac{1}{(2s_A+1)(2s_B+1)} \sum_{Spin} |\mathcal{M}|^2$$

where $2s_A+1$ is the number of spin states of particle A and $2s_B+1$ for particle B. So the product $(2s_A+1)(2s_B+1)$ is the number of spin states in the initial state.

$-i\mathcal{M} =$



We have to take the square of the diagram and sum over all spin states. For a given spin state:

$$\begin{aligned}
-i\mathcal{M} &= -e^2\,\overline{u}_C\gamma^\mu u_A\,\frac{-i}{q^2}\,\overline{u}_D\gamma_\mu u_B \\
|\mathcal{M}|^2 &= e^4\left[(\overline{u}_C\gamma^\mu u_A)\,\frac{1}{q^2}\,(\overline{u}_D\gamma_\mu u_B)\right]\left[(\overline{u}_C\gamma^\nu u_A)\,\frac{1}{q^2}\,(\overline{u}_D\gamma_\nu u_B)\right]^* \\
&= \frac{e^4}{q^4}\,L^{\mu\nu}_{\text{electron}}\,L^{\text{muon}}_{\mu\nu}
\end{aligned}$$

Intermezzo:
If

$$\mathcal{M} = A^\mu B_\mu$$

then

$$\begin{aligned}
|\mathcal{M}|^2 &= [A^\mu B_\mu]\,[A^\nu B_\nu]^* \\
&= (A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3)\,(A_0^*B_0^* - A_1^*B_1^* - A_2^*B_2^* - A_3^*B_3^*) \\
&= |A_0|^2|B_0|^2 - A_0A_1^*B_0B_1^* - A_0A_2^*B_0B_2^* - A_0A_3^*B_0B_3^*
\end{aligned}$$

$$\begin{aligned}
&-A_1A_0^*B_1B_0^* + |A_1|^2|B_1|^2 + A_1A_2^*B_1B_2^* + A_1A_3^*B_1B_3^* \\
&-A_2A_0^*B_2B_0^* + A_2A_1^*B_2B_1^* + |A_2|^2|B_2|^2 + A_2A_3^*B_2B_3^* \\
&-A_3A_0^*B_3B_0^* + A_3A_1^*B_3B_1^* + A_3A_2^*B_3B_2^* + |A_3|^2|B_3|^2 \\
= \;& \alpha^{\mu\nu}\beta_{\mu\nu} \\
&\text{with}: \; \alpha^{\mu\nu} = A^\mu A^{\nu *} \\
&\qquad\quad\;\; \beta_{\mu\nu} = B_\mu B_\nu{}^*
\end{aligned}$$

Next we proceed to take into account the spin. We have:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} = \frac{1}{(2s_A+1)}\frac{1}{(2s_B+1)}\sum_{\text{Spin}}|\mathcal{M}|^2 &= \frac{1}{4}\frac{e^4}{q^4}\,L^{\mu\nu}_{\text{electron}}\,L^{\text{muon}}_{\mu\nu} \\
\text{with}: \quad L^{\mu\nu}_{\text{electron}} &= \sum_{e-\text{spin}}[\overline{u}_C\gamma^\mu u_A]\;[\overline{u}_C\gamma^\nu u_A]^* \\
L^{\mu\nu}_{\text{muon}} &= \sum_{\mu-\text{spin}}[\overline{u}_D\gamma^\mu u_B]\;[\overline{u}_D\gamma^\nu u_B]^*
\end{aligned}$$

$L^{\mu\nu}$ is called the lepton tensor.

We have now split the sum over all spinstates in a sum over electron spins and a sum over muon spins. So, for each vertex there is a tensor $L^{\mu\nu}$ which has the form:

$$L^{\mu\nu} = \underbrace{\left[\begin{pmatrix} \;\overline{u}\; \end{pmatrix}\begin{pmatrix} & & \\ & \gamma^\mu & \\ & & \end{pmatrix}\begin{pmatrix} \\ u \\ \\ \end{pmatrix}\right]}_{\text{a \underline{number}}}\;\underbrace{\left[\begin{pmatrix} \;\overline{u}\; \end{pmatrix}\begin{pmatrix} & & \\ & \gamma^\nu & \\ & & \end{pmatrix}\begin{pmatrix} \\ u \\ \\ \end{pmatrix}\right]^*}_{\text{a \underline{number}}}$$

These numbers are called: *bilinear covariants*. Their general form is $\overline{\psi}\,(4\times 4)\,\psi$ and they have specific properties under Lorentz transformations (see Halzen & Martin section 5.6 or Griffiths section 7.3 for characteristics). They will also appear in the weak interaction later on, but there they will have a different form then the pure vector form: $\overline{\psi}\gamma^\mu\psi$.

Note: To do the spin summation is rather tedious. The rest of the lecture is just calculations in order to do this!

Since we work with numbers complex conjugation is the same as hermitean conjugation:

$$\begin{aligned}
[\overline{u}_C\gamma^\nu u_A]^* &= [\overline{u}_C\gamma^\nu u_A]^\dagger \\
\text{while} \quad [\overline{u}_C\gamma^\nu u_A]^\dagger &= \left[u_C^\dagger\gamma^0\gamma^\nu u_A\right]^\dagger = \left[u_A^\dagger\gamma^{\nu\dagger}\gamma^0 u_C\right] \\
&= \left[\overline{u}_A\gamma^0\gamma^{\nu\dagger}\gamma^0 u_C\right] = [\overline{u}_A\gamma^\nu u_C]
\end{aligned}$$

$\Rightarrow$ Complex conjugation just reverses the order in the product!

Using this aspect we then write for the lepton tensor:

$$L_{\rm e}^{\mu\nu} = \sum_{e\text{ spin}} (\overline{u}_C \gamma^\mu u_A) \cdot (\overline{u}_A \gamma^\nu u_C)$$

Next we write out the tensor explicitly in all components and we sum over all incoming spin states s and over all outgoing spins s':

$$L_{\rm e}^{\mu\nu} = \sum_{s'} \sum_{s} \overline{u}_{C\alpha}^{(s')} \, \gamma_{\alpha\beta}^{\mu} \, u_{A\beta}^{(s)} \cdot \overline{u}_{A\gamma}^{(s)} \, \gamma_{\gamma\delta}^{\nu} \, u_{C\delta}^{(s')} \cdot$$

where α, β, γ, δ are the individual matrix element indices that take care of the matrix multiplication.

At this point we apply <u>Casimir's Tric:</u>
Get the factor $u_{C\delta}^{(s')}$ all the way up in front such that it falls outside the summation over s. Why can we do this?
Because we have written out all terms of the matrix multiplication in indices; i.e. in *numbers*. The behaviour of the matrix multiplication is still valid because of the sum rules of the indices!

So, now we have:

$$L_{\rm e}^{\mu\nu} = \underbrace{\sum_{s'} u_{C\delta}^{(s')} \, \overline{u}_{C\alpha}^{(s')}}_{(\not{p}_C + m)_{\delta\alpha}} \, \gamma_{\alpha\beta}^{\mu} \cdot \underbrace{\sum_{s} u_{A\beta}^{(s)} \, \overline{u}_{A\gamma}^{(s)}}_{(\not{p}_A + m)_{\beta\gamma}} \, \gamma_{\gamma\delta}^{\nu}$$

and we can use the completeness relations (see previous lecture)[1]:

$$\sum_s u^{(s)} \, \overline{u}^{(s)} = \not{p} + m$$

(Remember that these are 4×4 relations which are valid for each component.)
So we use the completeness relations in order to do the sums over the spins!

The result is:

$$L_{\rm e}^{\mu\nu} = (\not{p}_C + m)_{\delta\alpha} \, \gamma_{\alpha\beta}^{\mu} \, (\not{p}_A + m)_{\beta\gamma} \, \gamma_{\gamma\delta}^{\nu}$$

Here is the next trick: look at the indices α, β, γ, δ; they are components of 4×4 matrices. Perform the sum over the indices α, β, γ and say that the result is: A. Then we find that $L_{\rm e}^{\mu\nu} \propto A_{\delta\delta}$ and we have to do the remaining sum over δ, which means that we take the <u>trace</u> of the matrix. In other words, the fact that we sum over all indices means:

$$L_{\rm e}^{\mu\nu} = \mathrm{Tr}\left[(\not{p}_C + m) \, \gamma^\mu \, (\not{p}_A + m) \, \gamma^\nu\right]$$

[1] for anti-fermions this gives an overall "−" sign in the tensor: $L_{\rm e}^{\mu\nu} \to -L_{\rm e}^{\mu\nu}$ for each particle → anti-particle.

Where are we at this point? We look at the reaction $e^-\mu^- \to e^-\mu^-$ and we have:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{(2s_A+1)(2s_B+1)} \sum_{Spin} |\mathcal{M}|^2 \\
&= \frac{1}{(2s_A+1)(2s_B+1)} \frac{e^4}{q^4} L_{\rm e}^{\mu\nu} L^{\rm m}_{\mu\nu} \\
\text{with :} \quad & L_{\rm e}^{\mu\nu} = {\rm Tr}\left[(\not p_C + m)\ \gamma^\mu\ (\not p_A + m)\ \gamma^\nu\right] \\
& L^{\rm m}_{\mu\nu} = {\rm Tr}\left[(\not p_D + m)\ \gamma_\mu\ (\not p_B + m)\ \gamma_\nu\right]
\end{aligned}$$

In order to evaluate these expressions we make use of trace identities.

<u>Intermezzo: Trace theorems</u>

- In general:
  - ${\rm Tr}\,(A+B) = {\rm Tr}(A) + {\rm Tr}(B)$
  - ${\rm Tr}\,(ABC) = {\rm Tr}\,(CAB) = {\rm Tr}\,(BCA)$
- For γ-matrices: from the definition: $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ it follows:
  - Tr (odd number of γ_μ's = 0). $\Rightarrow$ only 0 on the diagonal.
  - ${\rm Tr}\,(\gamma^\mu\gamma^\nu) = 4\,g^{\mu\nu}$. $\Rightarrow$ note that this is a matrix of traces!
  - ${\rm Tr}\,(\not a\ \not b) = 4\,a\cdot b$
  - ${\rm Tr}\,(\not a\ \not b\ \not c\ \not d) = 4\ [\,(a\cdot b)\,(c\cdot d) - (a\cdot c)\,(b\cdot d) + (a\cdot d)\,(b\cdot c)\,]$

We are calculating:

$$\begin{aligned}
L_{\rm e}^{\mu\nu} &= {\rm Tr}\left[(\not p_C + m)\ \gamma^\mu\ (\not p_A + m)\ \gamma^\nu\right] \quad \text{.....write it out.....} \\
&= \underbrace{{\rm Tr}\left[\not p_C\,\gamma^\mu\ \not p_A\,\gamma^\nu\right]}_{\text{case 2}} + \underbrace{{\rm Tr}\left[m\gamma^\mu m\gamma^\nu\right]}_{\text{case 1}} + \underbrace{{\rm Tr}\left[\not p_C\,\gamma^\mu m\gamma^\nu\right]}_{3\gamma's\Rightarrow 0} + \underbrace{{\rm Tr}\left[m\gamma^\mu\ \not p_A\,\gamma^\nu\right]}_{3\gamma's\Rightarrow 0}
\end{aligned}$$

<u>Case 1:</u> ${\rm Tr}\left[m\gamma^\mu m\gamma^\nu\right] = m^2{\rm Tr}\left[\gamma^\mu\gamma^\nu\right] = 4m^2g^{\mu\nu}$

<u>Case 2:</u> ${\rm Tr}\left[\not p_C\,\gamma^\mu\ \not p_A\,\gamma^\nu\right] = ?$
Use the rule for ${\rm Tr}\,(\not a\ \not b\ \not c\ \not d)$ with $a = p_C$ and $c = p_A$, but what are b and d?
$\Rightarrow b$ must be chosen such that $\gamma_\alpha \cdot b = \gamma^\mu$. $\quad \Rightarrow b = g^{\alpha\mu}$.
$\Rightarrow d$ must be chosen such that $\gamma_\beta \cdot d = \gamma^\nu$. $\quad \Rightarrow d = g^{\beta\nu}$.
Therefore (note that μ and ν are Lorentz-indices while the trace theorem works in Dirac space!):

$$\begin{aligned}
& {\rm Tr}\left[\not p_C\,\gamma^\mu\ \not p_A\,\gamma^\nu\right] \\
&= 4\left[\left(p_{C\alpha}g^{\alpha\mu}\right)\left(p_{A\beta}g^{\beta\nu}\right) - \left(p_C^\alpha p_{A\alpha}\right)\left(g^{\alpha\mu}g^{\beta\nu}g_{\alpha\beta}\right) + \left(p_{C\alpha}g^{\alpha\nu}\right)\left(p_{A\beta}g^{\beta\mu}\right)\right] \\
&= 4\left[p_C^\mu p_A^\nu + p_C^\nu p_A^\mu - (p_C\cdot p_A)\,g^{\mu\nu}\right]
\end{aligned}$$

Finally we find for the tensors:

$$\begin{aligned} L_{\mathrm{e}}^{\mu\nu} &= 4\left[p_C^{\mu}p_A^{\nu}+p_C^{\nu}p_A^{\mu}-\left(p_C\cdot p_A-m_e^2\right)\, g^{\mu\nu}\right] \\ L^{\mathrm{m}}_{\mu\nu} &= 4\left[p_{D\mu}p_{B\nu}+p_{D\nu}p_{B\mu}-\left(p_D\cdot p_B-m_m^2\right)\, g_{\mu\nu}\right] \end{aligned}$$

To recapitulate, the matrix element for $e^-\mu^-\to e^-\mu^-$:

$$\overline{|\mathcal{M}|^2}=\frac{1}{(2s_A+1)\,(2s_B+1)}\cdot\frac{e^4}{q^4}\cdot L_{\mathrm{e}}^{\mu\nu}\,L^{\mathrm{m}}_{\mu\nu}$$

and will just fill in the results of the tensors we just calculated:

$$\begin{aligned} L_{\mathrm{e}}^{\mu\nu}\,L^{\mathrm{m}}_{\mu\nu} &= 4\left[p_C^{\mu}p_A^{\nu}+p_C^{\nu}p_A^{\mu}-\left(p_C\cdot p_A-m_e^2\right)\, g^{\mu\nu}\right]\cdot 4\left[p_{D\mu}p_{B\nu}+p_{D\nu}p_{B\mu}-\left(p_D\cdot p_B-m_m^2\right)\, g_{\mu\nu}\right] \\ &= 16\,\cdot \\ &\quad (p_C\cdot p_D)\,(p_A\cdot p_B)+(p_C\cdot p_B)\,(p_A\cdot p_D)-(p_C\cdot p_A)\,(p_D\cdot p_B)+(p_C\cdot p_A)\,m_m^2 \\ &\quad (p_C\cdot p_B)\,(p_A\cdot p_D)+(p_C\cdot p_D)\,(p_A\cdot p_B)-(p_C\cdot p_A)\,(p_D\cdot p_B)+(p_C\cdot p_A)\,m_m^2 \\ &\quad -(p_C\cdot p_A)\,(p_D\cdot p_B)-(p_C\cdot p_A)\,(p_D\cdot p_B)+(p_C\cdot p_A)\,(p_D\cdot p_B)\cdot 4-(p_C\cdot p_A)\,m_m^2\cdot 4 \\ &\quad +m_e^2\,(p_D\cdot p_B)+m_e^2\,(p_D\cdot p_B)-4m_e^2\,(p_D\cdot p_B)+4m_e^2m_m^2 \\ &= 32\cdot\left[(p_A\cdot p_B)\,(p_C\cdot p_D)+(p_A\cdot p_D)\,(p_C\cdot p_B)-m_e^2\,(p_D\cdot p_B)-m_m^2\,(p_A\cdot p_C)+2m_e^2m_m^2\right] \end{aligned}$$

We then obtain:

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{e^4}{q^4}\cdot L_{\mathrm{e}}^{\mu\nu}\,L^{\mathrm{m}}_{\mu\nu} \\ &= 8\frac{e^4}{q^4}\left[(p_C\cdot p_D)\,(p_A\cdot p_B)+(p_C\cdot p_B)\,(p_A\cdot p_D)-m_e^2\,(p_D\cdot p_B)-m_m^2\,(p_A\cdot p_C)+2m_e^2m_m^2\right] \end{aligned}$$

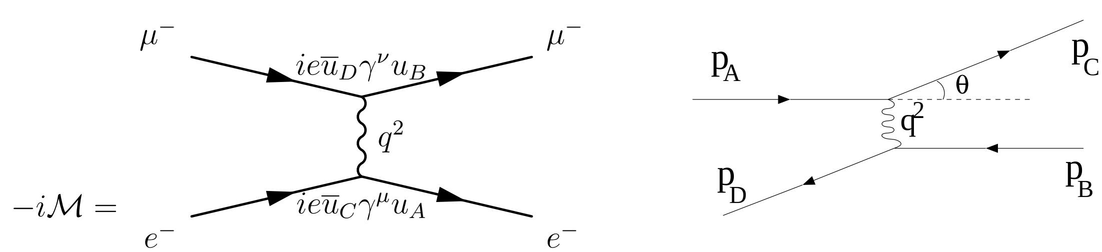


Figure 8.2: $e^-\mu^-\to e^-\mu^-$ scattering. left: the Feynman diagram. right: the scattering process.

Let us consider the ultrarelativistic limit; ie. we ignore the masses of the particles with respect to their momentum. Also we use the Mandelstam variables:

$$\begin{aligned} s &\equiv (p_A+p_B)^2=p_A^2+p_B^2+2\,(p_A\cdot p_B) && \simeq 2\,(p_A\cdot p_B) \\ t &\equiv (p_D-p_B)^2\equiv q^2 && \simeq -2\,(p_D\cdot p_B) \\ u &\equiv (p_A-p_D)^2 && \simeq -2\,(p_A\cdot p_D) \end{aligned}$$

In addition we have the following relations following from energy and momentum conservation ($p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu$):

$$\begin{aligned} (p_A + p_B)^2 &= (p_C + p_D)^2 \\ (p_D - p_B)^2 &= (p_C - p_A)^2 \\ (p_A - p_D)^2 &= (p_B - p_C)^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} p_A \cdot p_B &= p_C \cdot p_D \\ p_D \cdot p_B &= p_C \cdot p_A \\ p_A \cdot p_D &= p_B \cdot p_C \end{aligned}$$

such that:

$$\begin{aligned} (p_A \cdot p_B)\ (p_C \cdot p_D) &= \frac{1}{2}s\,\frac{1}{2}s = \frac{1}{4}s^2 \\ (p_A \cdot p_D)\ (p_C \cdot p_B) &= \left(-\frac{1}{2}u\right)\left(-\frac{1}{2}u\right) = \frac{1}{4}u^2 \\ q^4 = (p_D - p_B)^4 &= t^2 \end{aligned}$$

Then the ultrarelativistic limit gives us:

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{t^2}\left(\frac{1}{4}s^2 + \frac{1}{4}u^2\right) = 2e^4\left(\frac{s^2+u^2}{t^2}\right)$$

We define the particle momenta now according to Fig. 8.2:

Take now: $p_A = (p, p, 0, 0)$ $\quad p_C = (p, p\cos\theta, p\sin\theta, 0)$

$p_B = (p, -p, 0, 0)$ $\quad p_D = (p, -p\cos\theta, -p\sin\theta, 0)$

We get the for the Mandelstam variables:

$$s = 4p^2 \qquad t = -2p^2(1-\cos\theta) \qquad u = -2p^2(1+\cos\theta)$$

and we finally obtain the differential cross section:

$$\begin{aligned} \left.\frac{d\sigma}{d\Omega}\right|_{c.m.} &= \frac{1}{64\pi^2}\cdot\frac{1}{s}\cdot\overline{|\mathcal{M}|^2} \\ &= \frac{\alpha^2}{2s}\cdot\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2} \\ \text{with} \qquad & \alpha = \frac{e^2}{4\pi} \end{aligned}$$

8.3 Crossing: the process $e^+e^- \to \mu^+\mu^-$

We will use the "crossing" principle to obtain $\overline{|\mathcal{M}|^2}_{(e^+e^-\to\mu^+\mu^-)}$ from the result of $\overline{|\mathcal{M}|^2}_{(e^-\mu^-\to e^-\mu^-)}$

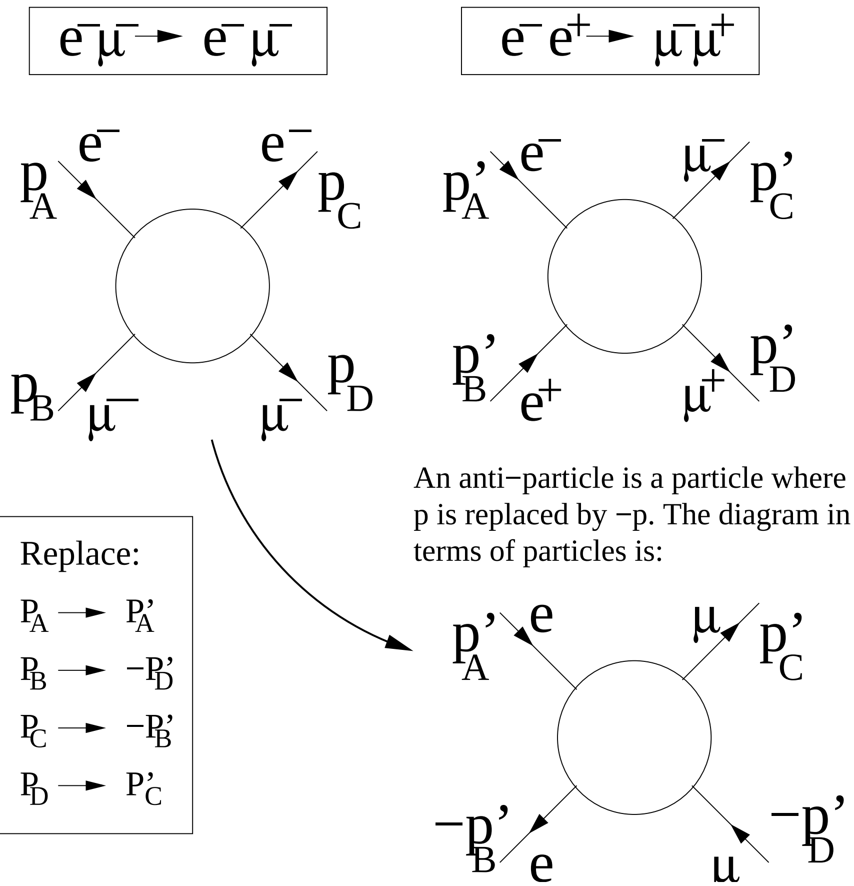


Figure 8.3: The principle of crossing. Use the anti-particle interpretation of a particle with the 4-momentum reversed in order to related the Matrix element of the "crossed" reaction to the original one.

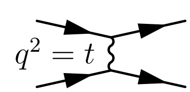
So we replace in the previously obtained result:

$$\begin{aligned}
s &= 2\,(p_A\cdot p_B) && \to && -2\,(p'_A\cdot p'_D) = u'\\
t &= -2\,(p_A\cdot p_C) && \to && 2\,(p'_A\cdot p'_B) = s'\\
u &= -2\,(p_A\cdot p_D) && \to && -2\,(p'_A\cdot p'_C) = t'
\end{aligned}$$

such that we have[2]:

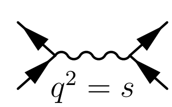
$$\overline{|\mathcal{M}|}^2_{e^-\mu^-\to e^-\mu^-} = 2e^4\frac{s^2+u^2}{t^2}$$

"t-channel":

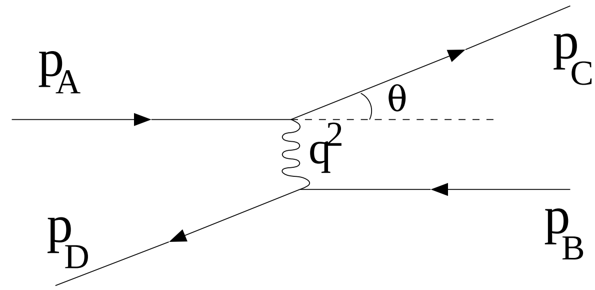


$$\overline{|\mathcal{M}|}^2_{e^-e^+\to\mu^-\mu^+} = 2e^4\frac{u'^2+t'^2}{s'^2}$$

"s-channel":



[2]We ignored two times the "−" sign introduced by replacing fermions by antifermions!



Again we go to the center of mass:

$$\begin{aligned} p_A &= (p, p, 0, 0,) \\ p_B &= (p, -p, 0, 0,) \\ p_C &= (p, p\cos\theta, p\sin\theta, 0,) \\ p_D &= (p, -p\cos\theta, -p\sin\theta, 0,) \end{aligned}$$

We calculate the Mandelstam variables:

$$\begin{aligned} s &= 2\,(p_A \cdot p_B) = 4p^2 \\ t &= -2\,(p_A \cdot p_C) = -2p^2\,(1-\cos\theta) \\ u &= -2\,(p_A \cdot p_D) = -2p^2\,(1+\cos\theta) \end{aligned}$$

We immediately get for the matrix element:

$$\overline{|\mathcal{M}|}^2 = 2e^4\,\frac{t^2+u^2}{s^2} = e^4\,\left(1+\cos^2\theta\right)$$

This means that we obtain for the cross section:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\,\left(1+\cos^2\theta\right)}$$

To calculate the total cross section for the process we integrate over the azimuthal angle ϕ and the polar angle θ:

$$\boxed{\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4}{3}\pi\frac{\alpha^2}{s}}$$

Exercise 29:
Can you easily obtain the cross section of the process $e^+e^- \to e^+e^-$ from the result of $e^+e^- \to \mu^+\mu^-$? If **yes**: *give the result, if* **no**: *why not?*

Exercise 30: *The process $e^+e^- \to \pi^+\pi^-$*
We consider scattering of spin 1/2 electrons with spin-0 pions. We assume point-particles; i.e. we forget that the pions have a substructure consisting of quarks. Also we only consider electromagnetic interaction and we assume that the particle masses can be neglected.

(a) *Consider the process of electron - pion scattering: $e^-\pi^- \to e^-\pi^-$. Give the matrix element $\mathcal{M}$ for this process.*

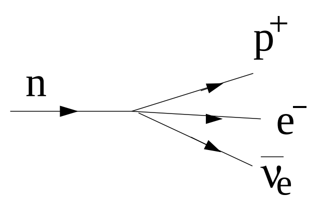
(b) *Use the principle of crossing to find the matrix element for $e^+e^- \to \pi^+\pi^-$ (Note: watch out for ad-hoc $-$-sign: see footnotes previous pages for antiparticles!)*

(c) *Determine the differential cross section $d\sigma/d\Omega$ in the center-of-mass of the e^+e^--system*

Lecture 9

The Weak Interaction

In 1896 Henri Becquerel observed that Uranium affected photographic plates. He was studying the effect of fluorescence, which he thought was caused by the X-rays that were discovered by Wilhelm Röntgen. To test his hypothesis he wanted to observe that this fluorescence radiation also affected photographic plates. He discovered by accident that the Uranium salt he used also affected the photographic plate when they were **not** exposed to sunlight. Thus he discovered natural radioactivity.
We know now that the weak interaction in nature is based on the decay: $n \rightarrow p + e^- + \overline{\nu}_e$ and has a lifetime of $\tau = 886s$.



Compare the lifetimes of the following decays:

	weak :	$\pi^- \rightarrow \mu^- \overline{\nu}_\mu$	$\tau = 2.6 \cdot 10^{-8}$ sec
		$\mu^- \rightarrow e^- \overline{\nu}_e \nu_\mu$	$\tau = 2.2 \cdot 10^{-6}$ sec
with :	e.m. :	$\pi^0 \rightarrow \gamma\gamma$	$\tau = 8.4 \cdot 10^{-17}$ sec
	strong :	$\rho \rightarrow \pi\pi$	$\tau = 4.4 \cdot 10^{-23}$ sec ($\Gamma = 150$ MeV)

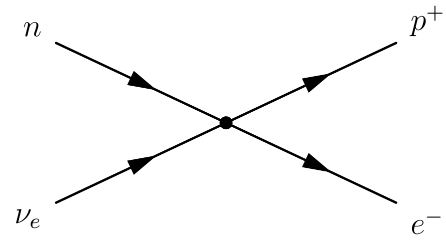
and realise that the lifetime of a process is inversely proportional to the strength of the interaction. Note in addition that:

1. All fermions "feel" the weak interaction. However, when present the electromagnetic and strong interactions dominate.

2. Neutrino's feel only the weak interaction. This is the reason why they are so hard to detect.

9.1 The 4-point interaction

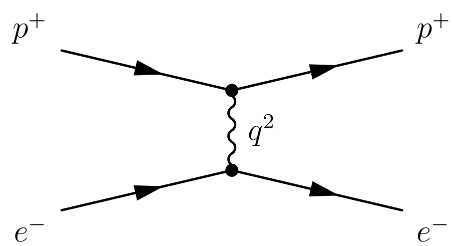
Based on the model of electromagnetic interactions Fermi invented in 1932 the so-called *4-point interaction model*, introducing the Fermi constant as the strength of the interaction: $G_F \approx 1.166 \cdot 10^{-5}\text{GeV}^{-2}$



The "Feynman diagram" of the 4-point interaction "neutrino scattering on a neutron" has the following matrix element:

$$\mathcal{M} = G_F \ (\overline{u}_p \gamma^\mu u_n) \ (\overline{u}_e \gamma_\mu u_\nu)$$

This is to be compared to the electromagnetic diagram for electron proton scattering:



Here the matrix element was:

$$\mathcal{M} = \frac{4\pi\alpha}{q^2} \ (\overline{u}_p \gamma^\mu u_p) \ (\overline{u}_e \gamma_\mu u_e)$$

1. $e^2 = 4\pi\alpha$ is replaced by G_F
2. $1/q^2$ is removed

We take note of the following facts of the weak interaction:

1. The hadronic current j^h_μ has $\Delta Q = 1$, the leptonic current has $\Delta Q = -1$. We refer to this as: *charged currents*, since there is a net charge transferred from the hadron current to the lepton current. We will see later that neutral weak currents turn out to exist as well.
2. There is a coupling constant G_F, which now plays a similar rôle as α in QED.
3. There is no propagator; ie. a "4-point interaction".
4. The currents have what is called a "vector character" similar as in QED. This means that the currents are of the form $\overline{\psi}\gamma^\mu\psi$.

The vector character of the interaction was in fact just a guess that turned out successful to describe many aspects of β-decay. There was no reason for this choice apart from similarity of QED. In QED the reason that the interaction has a vector behaviour is the fact that the force mediator, the foton, is a spin-1, or vector particle.

In the most general case the matrix element of the 4-point interaction can be written as:

$$\mathcal{M} = G_F \left(\overline{\psi}_p \ (4 \times 4) \ \psi_n\right) \left(\overline{\psi}_e \ (4 \times 4) \ \psi_\nu\right)$$

where (4×4) are combinations of γ-matrices. Lorentz invariance of the interaction puts restrictions on the form of the bilinear covariants of any possible interaction.

For any possible theory (or "force") the bilinear covariants can be of the following type:

	current	# components	# γ-matrices	spin
Scalar	$\overline{\psi}\psi$	1	0	0
Vector	$\overline{\psi}\gamma^\mu\psi$	4	1	1
Tensor	$\overline{\psi}\sigma^{\mu\nu}\psi$	6	2	2
Axial vector	$\overline{\psi}\gamma^\mu\gamma^5\psi$	4	3	1
Pseudo scalar	$\overline{\psi}\gamma^5\psi$	1	4	0

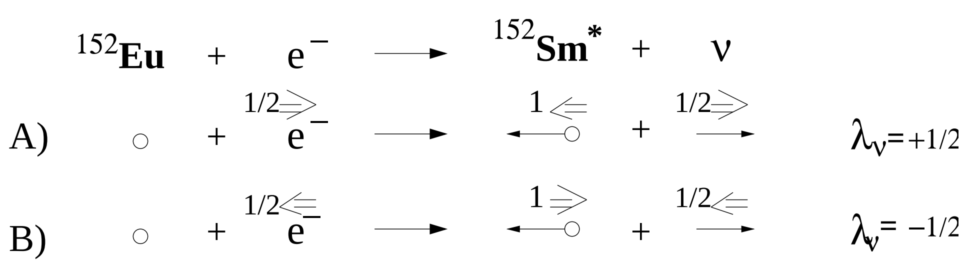
Table 9.1: Possible forms of the bilinear covariants. $\sigma^{\mu\nu} \equiv \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$. Note that the total number of components is 16.

In the most general case the 4-point weak interaction can be written as:

$$\mathcal{M} = G_F \sum_{i,j}^{S,P,V,A,T} C_{ij}\,(\overline{u}_p\, O_i\, u_n)\ (\overline{u}_e\, O_j\, u_\nu)$$

where O_i, O_j are operators of the form S, V, T, A, P.
It can be shown with Dirac theory (see eg. Perkins: "Introduction to High Energy Physics", 3rd edition, appendix D) that:
S, P, T interactions in $n \to pe\overline{\nu}_e$ imply: helicity e = helicity $\overline{\nu}_e$,
V, A, interactions in $n \to pe\overline{\nu}_e$ imply: helicity e = -helicity $\overline{\nu}_e$.

In 1958 Goldhaber et. al. measured experimentally that the weak interaction is of the type: V, A, (ie. it is _not_ S, P, T). See Perkins ed 3, §7.5 for a full description of the experiment. The basic idea is the following.
Consider the electron capture reaction: $^{152}\text{Eu} + e^- \to ^{152}\text{Sm}^*(\text{J} = 1) + \nu$



By studying the consecutive decay $^{152}\text{Sm}^* \to ^{152}\text{Sm} + \gamma$ it was observed that only case B actually occurred. In other words: neutrino's have helicity -1/2. From this it was concluded that in the weak interaction only the V, A currents are involved and not S, P, T!

9.1.1 Lorentz covariance and Parity

Let us consider a Lorentz transformation: $x'^{\nu} = \Lambda^{\nu}_{\mu} x^{\nu}$. The Dirac equation in each of the two frames is then, respectively:

$$\begin{aligned} i\gamma^{\mu}\frac{\partial\psi(x)}{\partial x^{\mu}} - m\psi(x) &= 0 \\ i\gamma^{\nu}\frac{\partial\psi'(x')}{\partial x'^{\nu}} - m\psi'(x') &= 0 \end{aligned}$$

For the wave function there must exist a relation with an operator S, such that:

$$\psi'(x') = S\psi(x)$$

Since the Dirac spinor is of the form $\psi(x) = u(p)\, e^{-ipx}$, S is independent of x and only acts on the spinor u. The Dirac equation after the Lorentz transformation becomes:

$$i\gamma^{\nu}\frac{\partial S\left(\psi(x)\right)}{\partial x'^{\nu}} - mS\left(\psi(x)\right) = 0$$

and if we act on this equation by S^{-1} from the left:

$$iS^{-1}\gamma^{\nu}\frac{S\left(\partial\psi(x)\right)}{\partial x'^{\nu}} - mS^{-1}S\psi(x) = 0$$

This equation is consistent with the orginal Dirac equation if the relation
$S^{-1}\gamma^{\nu}S = \Lambda^{\nu}_{\mu}\gamma^{\mu}$ holds and we used that $\partial/\partial x^{\mu} = \Lambda^{\nu}_{\mu}\,\partial/\partial x'^{\nu}$.

Let us now take a look at the parity operator which inverts space: ie. $t \to t\,;\, \vec{r} \to -\vec{r}$. The parity Lorentz transformation is:

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Which is the "Dirac" operator that gives: $\psi'(x') = S\psi(x)$?
The easiest way is to find it is to use the relation: $S_p^{-1}\gamma^{\mu}S_p = \Lambda^{\mu}_{\nu}\gamma^{\nu} = (\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3)$, or, more explicitly, to find the matrix S_p for which:

$$\begin{gathered} S_p^{-1}\gamma^0 S_p = \gamma^0 \\ S_p^{-1}\gamma^k S_p = -\gamma^k \end{gathered}$$

which has the solution $S_p = \gamma^0$.

Alternatively, we can get the parity operator from the Dirac equation. Assume that the wave function $\psi\left(\vec{r}, t\right)$ is a solution of the Dirac equation:

$$\left(\gamma_0\frac{\partial}{\partial t} + \gamma_k\frac{\partial}{\partial x^k} - m\right)\psi\left(\vec{r}, t\right) = 0$$

then, after a parity transformation we find:

$$\left(\gamma_0 \frac{\partial}{\partial t} - \gamma_k \frac{\partial}{\partial x^k} - m\right) \psi\left(-\vec{r}, t\right) = 0$$

So, $\psi\left(-\vec{r}, t\right)$ is **not** a solution of the Dirac equation due to the additional - sign! Multiply the Dirac equation of the parity transformed spinor from the left by γ^0, to find:

$$\begin{aligned} & \gamma_0 \left(\gamma_0 \frac{\partial}{\partial t} - \gamma_k \frac{\partial}{\partial x^k} - m\right) \psi\left(-\vec{r}, t\right) = 0 \\ \Rightarrow \quad & \left(\gamma_0 \frac{\partial}{\partial t}\gamma_0 + \gamma_k \frac{\partial}{\partial x^k}\gamma_0 - m\gamma_0\right) \psi\left(-\vec{r}, t\right) = 0 \\ \Rightarrow \quad & \left(\gamma_0 \frac{\partial}{\partial t} + \gamma_k \frac{\partial}{\partial x^k} - m\right) \gamma_0 \psi\left(-\vec{r}, t\right) = 0 \end{aligned}$$

We conclude that if $\psi\left(\vec{r}, t\right)$ is a solution of the Dirac equation, then $\gamma_0 \psi\left(-\vec{r}, t\right)$ is also a solution (in the mirror world).
In other words: under the parity operation ($S = \gamma^0$): $\psi\left(\vec{r}, t\right) \rightarrow \gamma_0 \psi\left(-\vec{r}, t\right)$.

An interesting consequence can be derived from the explicit representation of the γ^0 matrix:

$$\gamma^0 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}$$

from which it is seen that the parity operator has an opposite sign for the positive and negative solutions. In other words: fermions and anti-fermions have opposite parity.

What does this imply for the currents in the interactions? Under the Parity operator we get:

$$\begin{array}{llcllcll} S: & \overline{\psi}\psi & \rightarrow & \overline{\psi}\gamma^0\gamma^0\psi & = & \overline{\psi}\psi & \text{Scalar} \\ P: & \overline{\psi}\gamma^5\psi & \rightarrow & \overline{\psi}\gamma^0\gamma^5\gamma^0\psi & = & -\overline{\psi}\gamma^5\psi & \text{Pseudo Scalar} \\ V: & \overline{\psi}\gamma^\mu\psi & \rightarrow & \overline{\psi}\gamma^0\gamma^\mu\gamma^0\psi & = & \left\{ \begin{array}{c} \overline{\psi}\gamma^0\psi \\ -\overline{\psi}\gamma^k\psi \end{array} \right. & \text{Vector} \\ A: & \overline{\psi}\gamma^\mu\gamma^5\psi & \rightarrow & \overline{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi & = & \left\{ \begin{array}{c} -\overline{\psi}\gamma^0\psi \\ \overline{\psi}\gamma^k\psi \end{array} \right. & \text{Axial Vector.} \end{array}$$

We had concluded earlier that the weak matrix element in neutron decay is of the form:

$$\mathcal{M} = G_F \sum_{i,j}^{V,A} C_{ij}\ \left(\overline{u}_p O_i u_p\right)\ \left(\overline{u}_e O_j u_\nu\right)$$

<u>But:</u> if there is a contribution from vector <u>as well as</u> from axial vector then we must have parity violation!

9.2 The $V - A$ interaction

It turns out that the only change that is needed in the pure vector coupling of Fermi is:

$$(\overline{u}_e \, \gamma^\mu \, u_\nu) \rightarrow \left(\overline{u}_e \, \gamma^\mu \, \frac{1}{2}\left(1 - \gamma^5\right) \, u_\nu\right)$$

This is the famous $V - A$ interaction where the vector coupling and the axial vector coupling are equally strong present. The consequence is that there is **maximal violation of parity** in the weak interaction.

Exercise 31: Helicity vs Chirality

(a) Write out the **chirality operator** γ^5 *in the Dirac-Pauli representation.*

(b) The helicity operator is defined as $\lambda = \vec{\sigma} \cdot \hat{p}$. Show that helicity operator and the chirality operator have the same effect on a spinor solution, i.e.

$$\gamma^5 \, \psi = \gamma^5 \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)} \end{pmatrix} \approx \lambda \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)} \end{pmatrix} = \lambda \, \psi$$

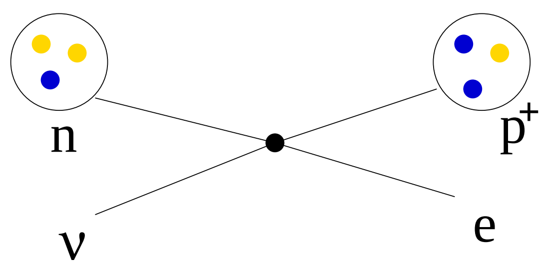
in the ultrarelativistic limit that $E >> m$.

(c) Explain why the weak interaction is called **left-handed**.

For neutron decays there is a complication to test the $V - A$ structure since the neutron and the proton are not point particles. The observed matrix element for neutron decays is:

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \left(\overline{u}_p \gamma^\mu \left(C_V - C_A \gamma^5\right) u_n\right) \left(\overline{u}_e \gamma_\mu \left(1 - \gamma^5\right) u_\nu\right)$$

It has the follwing values for the vector and axial vector couplings:
$C_V = 1.000 \pm 0.003$, $C_A = 1.260 \pm 0.002$
However, the fundamental weak interaction between the quarks and the leptons are pure $V - A$.

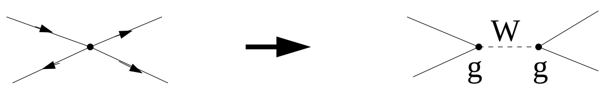


9.3 The Propagator of the weak interaction

The Fermi theory has a 4-point interaction: there is no propagator involved to transmit the interaction from the lepton current to the hadron current. However, we know now that forces are carried by bosons:

- the electromagnetic interaction is carried by the massless photon which gives rise to a $\rightarrow \frac{1}{q^2}$ propagator
- the weak interaction is carried by the massive W, Z bosons, for which we have the propagators: $\frac{1}{M_W^2 - q^2}$ and $\frac{1}{M_Z^2 - q^2}$.

Let us consider an interaction at low energy; ie. the case that $M_W^2 >> q^2$. In that case the propagator reduces to $\frac{1}{M_W^2}$.



strength: $\sim \frac{G_F}{\sqrt{2}}$ $\sim \frac{g^2}{8M_W^2}$

We interpret the coupling constant g of the weak interaction exactly like e in QED.

How "weak" is the weak interaction? In QED we have: $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$
In the weak interaction it turns out: $\alpha_w = \frac{g^2}{4\pi} = \frac{1}{29}$
The interaction is weak because the mass M_W is high! The intrinsic coupling constant is not small in comparison to QED. As a consequence it will turn out that at high energies: $q^2 \sim M_W^2$ the weak interaction is comparable in strength to the electromagnetic interaction.

9.4 Muon Decay

Similar to the process $e^+e^- \rightarrow \mu^+\mu^-$ in QED, the muon decay process $\mu^- \rightarrow e^-\overline{\nu}_e\nu_\mu$ is the standard example of a weak interaction process.

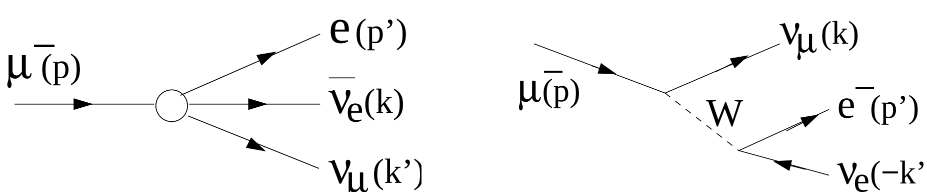


Figure 9.1: Muon decay: *left:* Labelling of the momenta, *right:* Feynman diagram. Note that for the spinor of the outgoing antiparticle we use: $u_{\nu_e}(-k') = v_{\nu_e}(k')$.

Using the Feynman rules we can write for the matrix element:

$$\mathcal{M} = \frac{g}{\sqrt{2}} \left(\underbrace{\overline{u}(k)}_{\text{outgoing } \mu_\nu} \gamma^\mu \frac{1}{2}\left(1-\gamma^5\right) \underbrace{u(p)}_{\text{incoming } \mu} \right) \underbrace{\frac{1}{M_W^2}}_{\text{propagator}} \frac{g}{\sqrt{2}} \left(\underbrace{\overline{u}(p')}_{\text{outgoing } e} \gamma_\mu \frac{1}{2}\left(1-\gamma^5\right) \underbrace{v(k')}_{\text{outgoing } \overline{\nu}_e} \right)$$

Next we square the matrix element and sum over the spin states, exactly similar to the case of $e^+e^- \to \mu^+\mu^-$. Then we use again the tric of Casimir as well as the completeness relations to convert the sum over spins into a trace. The result is:

$$\overline{|\mathcal{M}|}^2 = \frac{1}{2}\sum_{\text{Spin}} |\mathcal{M}|^2 = \frac{1}{2}\left(\frac{g^2}{8M_W^2}\right)^2 \cdot \text{Tr}\left\{\gamma^\mu\left(1-\gamma^5\right)(\not{p}' + m_e)\gamma^\nu\left(1-\gamma^5\right)\not{k}'\right\}$$
$$\cdot \text{Tr}\left\{\gamma_\mu\left(1-\gamma^5\right)\not{k}\,\gamma_\nu\left(1-\gamma^5\right)(\not{p} + m_\mu)\right\}$$

Now we use some more trace theorems (see below) and also $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$ to find the result:

$$\boxed{\overline{|\mathcal{M}|}^2 = 64\, G_F^2\, (k\cdot p')\,(k'\cdot p)}$$

Intermezzo: Trace theorems used (see also Halzen & Martin p 261):

$$\text{Tr}\left(\gamma^\mu\,\not{a}\,\gamma^\nu\,\not{b}\right)\cdot\text{Tr}\left(\gamma_\mu\,\not{c}\,\gamma_\nu\,\not{d}\right) = 32\,\left[(a\cdot c)\,(b\cdot d) + (a\cdot d)\,(b\cdot c)\right]$$
$$\text{Tr}\left(\gamma^\mu\,\not{a}\,\gamma^\nu\gamma^5\,\not{b}\right)\cdot\text{Tr}\left(\gamma_\mu\,\not{c}\,\gamma_\nu\gamma^5\,\not{d}\right) = 32\,\left[(a\cdot c)\,(b\cdot d) - (a\cdot d)\,(b\cdot c)\right]$$
$$\text{Tr}\left(\gamma^\mu\left(1-\gamma^5\right)\not{a}\,\gamma^\nu\left(1-\gamma^5\right)\not{b}\right)\cdot\text{Tr}\left(\gamma_\mu\left(1-\gamma^5\right)\not{c}\,\gamma_\nu\left(1-\gamma^5\right)\not{d}\right) = 256\,(a\cdot c)\,(b\cdot d)$$

The decay width we can find by applying Fermi's golden rule:

$$\begin{aligned}
d\Gamma &= \frac{1}{2E}\overline{|\mathcal{M}|}^2\, dQ \\
\text{where : } dQ &= \frac{d^3p'}{(2\pi)^3\, 2E'}\cdot\frac{d^3k}{(2\pi)^3\, 2\omega}\cdot\frac{d^3k'}{(2\pi)^3\, 2\omega'}\cdot(2\pi)^4\,\delta^4\left(p-p'-k'-k\right) \\
\text{with :} E &= \text{muon energy} \\
E' &= \text{electron energy} \\
\omega' &= \text{electron neutrino energy} \\
\omega &= \text{muon neutrino energy}
\end{aligned}$$

First we evaluate the expression for the matrix element. We have the relation $(p = (m_\mu, 0, 0, 0))$:

$$p = p' + k + k' \qquad\qquad \text{so :} \qquad (k + p') = (p - k')$$

We can also see the following relations to hold:

$$\begin{aligned}
(k+p')^2 &= \underbrace{k^2}_{=0} + \underbrace{p'^2}_{m_e^2\approx 0} + 2\,(k\cdot p') \\
(p-k')^2 &= \underbrace{p^2}_{m_\mu^2\equiv m^2} + \underbrace{k'^2}_{=0} - 2\,\underbrace{(p\cdot k')}_{m\omega'}
\end{aligned}$$

Therefore we have the relation: $2\,(k \cdot p') = m^2 - 2m\omega'$, which we use to rewrite the matrix element as:

$$\overline{|\mathcal{M}|}^2 = 64\, G_F^2\, (k \cdot p')\, (k' \cdot p) = 32\, G_F^2\, \left(m^2 - 2m\omega'\right)\, m\omega'$$

We had the expression for the decay time:

$$d\Gamma = \frac{1}{2E}\, \overline{|\mathcal{M}|}^2\, dQ = \frac{16 G_F^2}{m}\, \left((m^2 - 2m\omega'\right) m\omega'\, dQ$$

(E is replaced by m since the decaying muon is in rest). For the total decay width we must integrate over the phase space:

$$\Gamma = \int \frac{1}{2E}\, \overline{|\mathcal{M}|}^2\, dQ = \frac{16 G_F^2}{m} \int \left((m^2 - 2m\omega'\right) m\omega'\, dQ$$

We note that the integrand only depends on the neutrino energy ω'. So, let us first perform the integral in dQ over the <u>other</u> energies and momenta:

$$\begin{aligned}\int_{\text{other}} dQ &= \frac{1}{8\,(2\pi)^5} \int \delta\,(m - E' - \omega' - \omega)\; \delta^3\left(\vec{p}' + \vec{k}' + \vec{k}\right)\, \frac{d^3\vec{p}'}{E'}\, \frac{d^3\vec{k}'}{\omega'}\, \frac{d^3\vec{k}}{\omega} \\ &= \frac{1}{8\,(2\pi)^5} \int \delta\,(m - E' - \omega' - \omega)\, \frac{d^3\vec{p}'\, d^3\vec{k}'}{E'\omega'\omega}\end{aligned}$$

since the δ-function gives 1 for the integral over $\vec{k}$.

We also have the relation:

$$\omega = |k| = |\vec{p}' + \vec{k}'| = \sqrt{E'^2 + \omega'^2 + 2E'\omega' \cos\theta}$$

where θ is the angle between the electron and the electron neutrino. We choose the z-axis along $\vec{k}'$, the direction of the electron neutrino. From the equation for ω we derive:

$$d\omega = \frac{-2E'\omega' \sin\theta}{2\underbrace{\sqrt{E'^2 + \omega'^2 + 2E'\omega' \cos\theta}}_{\omega}}\, d\theta \qquad \Leftrightarrow \qquad d\theta = \frac{-\omega\, d\omega}{E'\omega' \sin\theta}$$

Next we integrate over $d^3\vec{p}' = E'^2 \sin\theta\, dE'\, d\theta\, d\phi$ with $d\theta$ as above:

$$\begin{aligned}dQ &= \frac{1}{8\,(2\pi)^5} \int \delta\,(m - E' - \omega' - \omega)\, \frac{E'^2 \sin\theta}{E'}\, dE'\, d\theta\, d\phi\, \frac{d^3\vec{k}'}{\omega'}\, \frac{1}{\omega} \\ &= \frac{1}{8\,(2\pi)^5}\, 2\pi \int \delta\,(m - E' - \omega' - \omega)\; dE'\, d\omega\, \frac{d^3\vec{k}'}{\omega'^2}\end{aligned}$$

(using the relation: $E' \sin\theta\, d\theta = -\frac{\omega}{\omega'}\, d\omega$).

Since we integrate over ω, the δ-function will cancel:

$$dQ = \frac{1}{8\,(2\pi)^4} \int dE'\, \frac{d^3\vec{k}'}{\omega'^2}$$

such that the full expression for Γ becomes:

$$\Gamma = \frac{2G_F^2}{(2\pi)^4} \int \left(m^2 - 2m\omega'\right) \omega' \, dE' \, \frac{d^3\vec{k}'}{\omega'^2}$$

Next we do the integral over k' as far as possible with:

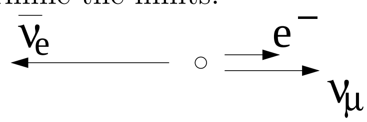
$$\int d^3\vec{k}' = \int \omega'^2 \, \sin\theta' \, d\omega' \, d\theta' \, d\phi' = 4\pi \int \omega'^2 \, d\omega$$

so that we get:

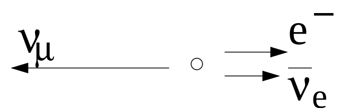
$$\Gamma = \frac{G_F^2 m}{(2\pi)^3} \int (m - 2\omega') \; \omega' \, d\omega' \, dE'$$

Before we do the integral over ω' we have to determine the limits:

- maximum electron neutrino energy: $\omega' = \frac{1}{2}m$



- minimum electron neutrino energy: $\omega' = \frac{1}{2}m - E'$



Therefore we obtain the spectrum:

$$\frac{d\Gamma}{dE'} = \frac{G_F^2 m}{(2\pi)^3} \int_{\frac{1}{2}m - E'}^{\frac{1}{2}m} (m - 2\omega') \; \omega' \, d\omega' = \frac{G_F^2 m^2}{12\pi^3} \, E'^2 \left(3 - 4\frac{E'}{m}\right)$$

which can be measured experimentally.

Finally we obtain for the decay of the muon:

$$\boxed{\Gamma \equiv \frac{1}{\tau} = \frac{G_F^2 \, m^5}{192 \, \pi^3}}$$

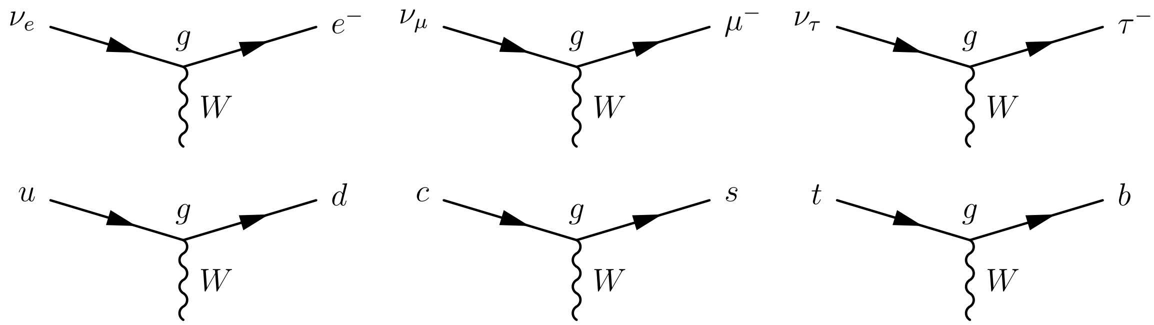
A measurement of the muon lifetime: $\tau = 2.19703 \pm 0.00004 \, \mu s$ determines the Fermi coupling constant: $G_F = (1.16639 \pm 0.00002) \cdot 10^{-5} \mathrm{GeV}^{-2}$. This is the standard method to determine G_F or $\frac{g^2}{M_W^2}$.

9.5 Quark mixing

In muon decay we studied the weak interaction acting between leptons: electron, muon, electron-neutrino and muon-neutrino. We have seen in the process of neutron decay that the weak interaction also operates between the quarks. All fundamental fermions are susceptible to the weak interaction. Both the leptons and quarks are usually ordered in a representation of three generations:

$$\underline{\text{Leptons :}} \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \qquad \underline{\text{Quarks :}} \quad \begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}$$

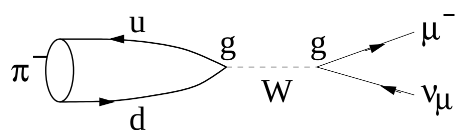
In a first assumption the charged current weak interaction works inside the generation doublets:



To test the validity of this model for quarks let us look at the examples of quark diagrams of pion decay and kaon decay:

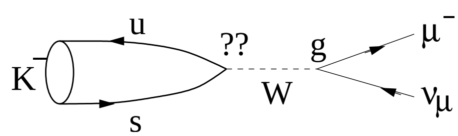
1. pion decay

   $\pi^- \rightarrow \mu^- \overline{\nu}_\mu$
   $\Gamma_{\pi^-} \propto \frac{g^4}{M_W^4} \propto G_F^2$



2. kaon decay

   $K^- \rightarrow \mu^- \overline{\nu}_\mu$
   This decay does occur!



9.5.1 Cabibbo - GIM mechanism

We have to modify the model by the replacements:

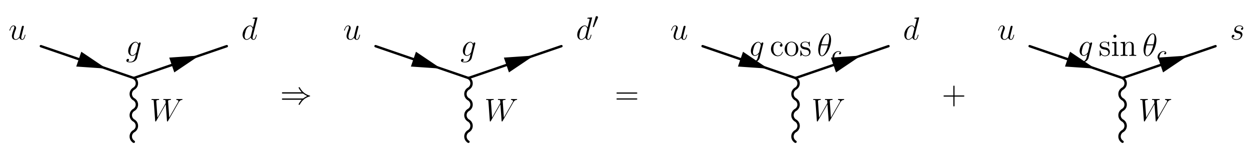
$$\begin{aligned} d &\rightarrow d' = d\ \cos\theta_c + s\ \sin\theta_c \\ s &\rightarrow s' = -d\ \sin\theta_c + s\ \cos\theta_c \end{aligned}$$

or, in matrix representation:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

where θ_c is the Cabibbo mixing angle.

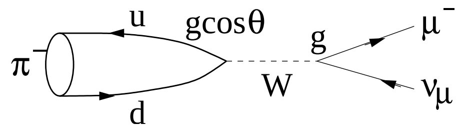
In terms of the diagrams the replacement implies:



Both the u, d coupling and the u, s coupling exist. In this case the diagrams of pion decay and kaon decay are modified:

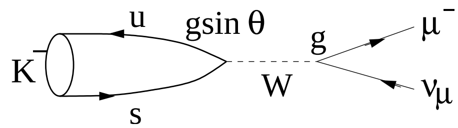
1. Pion decay

   $\pi^- \longrightarrow \mu^- \overline{\nu}_\mu$
   $\Gamma_{\pi^-} \propto G_F^2 \cos^2\theta_c$



2. Kaon decay

   $K^- \longrightarrow \mu^- \overline{\nu}_\mu$
   $\Gamma_{K^-} \propto G_F^2 \sin^2\theta_c$



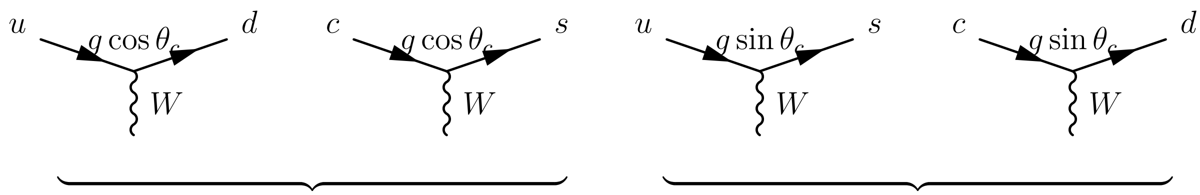
In order to check this we can compare the decay rate of the two reactions. A proper calculation gives:

$$\frac{\Gamma\left(K^-\right)}{\Gamma\left(\pi^-\right)} \approx \tan^2\theta_c \cdot \left(\frac{m_\pi}{m_K}\right)^3 \left(\frac{m_K^2 - m_\mu^2}{m_\pi^2 - m_\mu^2}\right)^2$$

As a result the Cabibbo mixing angle is observed to be:

$$\boxed{\theta_C = 12.8^o}$$

The couplings for the first two generations are:



Cabibbo "favoured" decay   Cabibbo "suppressed" decay

Formulated in a different way:

- The flavour eigenstates u, d, s, c are the mass eigenstates. They are the solution of the total Hamiltonian describing quarks; ie. mainly strong interactions.

- The states $\begin{pmatrix} u \\ d' \end{pmatrix}$, $\begin{pmatrix} c \\ s' \end{pmatrix}$ are the eigenstates of the weak interaction Hamiltonian, which affects the decay of the particles.

The relation between the mass eigenstates and the interaction eigenstates is a rotation matrix:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

with the Cabibbo angle as the mixing angle of the generations.

9.5.2 The Cabibbo - Kobayashi - Maskawa (CKM) matrix

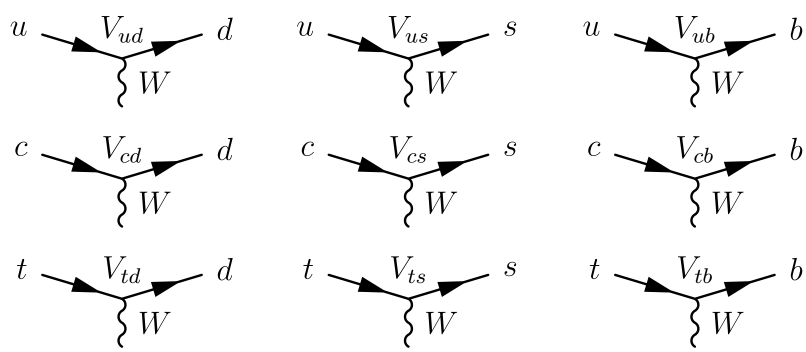
We extend the picture of the previous section to include all three generations. This means that we now make the replacement:

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} u \\ d' \end{pmatrix} \begin{pmatrix} c \\ s' \end{pmatrix} \begin{pmatrix} t \\ b' \end{pmatrix}$$

with in the most general way can be written as:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \underbrace{\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}}_{\text{CKM}-\text{matrix}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

The "g" couplings involved are:



It should be noted that the matrix is not uniquely defined since the phases of the quark wavefunctions are not fixed. The standard representation of this unitary 3×3 matrix contains three mixing angles between the quark generations θ_{12}, θ_{13}, θ_{23}, and one complex phase δ:

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}s_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$.

In the Wolfenstein parametrization this matrix is:

$$V_{CKM} \approx \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

It can be easy seen to includes 4 parameters:

3 real parameters : λ, A, ρ
1 imaginary parameter : $i\eta$

This imaginary parameter is the source of CP violation in the Standard Model. It means that it defines the difference between interactions involving matter and those that involve anti-matter.

We further note that, in case neutrino particles have mass, a similar mixing matrix also exists in the lepton sector. The Pontecorvo-Maki-Nakagawa-Sakata matrix U_{PMNS} is then defined as follows:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \underbrace{\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}}_{PMNS-matrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

In a completely similar way this matrix relates the mass eigenstates of the leptons (ν_1, ν_2, ν_3) to the weak interaction eigenstates (ν_e, ν_μ, ν_τ). There is an interesting open question whether neutrino's are their own anti-particles ("Majorana" neutrino's) or not ("Dirac" neutrino's). In case neutrinos are of the Dirac type, the U_{PMNS} matrix has one complex phase, similar to the quark mixing matrix. Alternatively, if neutrinos are Majorana particles, the U_{PMNS} matrix includes three complex phases.

It is currently not clear whether the explanation for a matter dominated universe lies in quark flavour physics ("baryogenesis") or in lepton flavour physics ("leptogenesis") and whether it requires physics beyond the Standard Model. It is however interesting to note that there exist 3 generations of particles!

Exercise 32: **Pion Decay**

Usually at this point the student is asked to calculate pion decay, which requires again quite some calculations. The ambitious student is encourage to try and do it (using some help from the literature). However, the exercise below requires little or no calculation but instead insight in the formalism.

(a) *Draw the Feynman diagram for the decay of a pion to a muon and an anti-neutrino:* $\pi^- \longrightarrow \mu^- \overline{\nu}_\mu$.

Due to the fact that the quarks in the pion are not free particles we cannot just apply the Dirac formalism for free particle waves. However, we know that the interaction is transmitted by a W^- *and therefore the coupling must be of the type:* V *or* A. *(Also, the matrix element must be a Lorentz scalar.) It turns out the decay amplitude has the form:*

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \left(q^\mu f_\pi\right) \left(\overline{u}(p)\gamma_\mu \left(1-\gamma^5\right) v(k)\right)$$

where p^μ *and* k^μ *are the 4-momenta of the muon and the neutrino respectively, and* q *is the 4-momentum carried by the* W *boson.* f_π *is called the decay constant.*

(b) *Can the pion also decay to an electron and an electron-neutrino? Write down the Matrix element for this decay.*
Would you expect the decay width of the decay to electrons to be larger, smaller, or similar to the decay width to the muon and muon-neutrino?
Base your argument on the available phase space in each of the two cases.

The decay width to a muon and muon-neutrino is found to be:

$$\Gamma = \frac{G_F^2}{8\pi} f_\pi^2 \, m_\pi \, m_\mu^2 \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2}\right)^2$$

The measured lifetime of the pion is $\tau_\pi = 2.6 \cdot 10^{-8} s$ *which means that* $f_\pi \approx m_\pi$. *An interesting observation is to compare the decay width to the muon and to the electron:*

$$\frac{\Gamma(\pi^- \to e^- \overline{\nu}_e)}{\Gamma(\pi^- \to \mu^- \overline{\nu}_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \approx 1.2 \cdot 10^{-4} \quad !!$$

(c) *Can you give a reason why the decay rate into an electron and an electron-neutrino is strongly suppressed in comparison to the decay to a muon and a muon-neutrino. Consider the spin of the pion, the handedness of the W coupling and the helicity of the leptons involved.*

Lecture 10

Local Gauge Invariance

In the next three lectures the Standard Model of electroweak interactions will be introduced. We will do this via the principle of gauge invariance. The idea of gauge invariance forms now such a firm basis of the description of forces that I feel it is suitable to be discussed in these lectures. As these lectures are not part of a theoretical master course we will follow a - hopefully - intuitive approach. Certainly we will try to focus, as we did before, on the concepts rather then on formal derivations.
A good book on this topic is:
Chris Quigg, "Gauge Theories of the Strong, Weak, and Electromagnetic Interactions", in the series of "Frontiers in Physics", Benjamin Cummings.

10.1 Introduction

The reason why we chose the Lagrangian approach in field theory is that it is particularly suitable to discuss symmetry or invariance principles and conservation laws that they are related to. Symmetry principles play a fundamental role in particle physics. In general one can distinguish[1] in general 4 groups of symmetries. There is a theorem stating that a symmetry is always related to a quantity that is fundamentally unobservable. Some of these unobservables are mentioned below:

- permutation symmetries: Bose Einstein statistics for integer spin particles and Fermi Dirac statistics for half integer spin particles. The unobservable is the identity of a particle.

- continuous space-time symmetries: translation, rotation, acceleration, etc. The related unobservables are respectively: absolute position in space, absolute direction and the equivalence between gravity and accelleration.

- discrete symmetries: space inversion, time inversion, charge inversion. The unobservables are absolute left/right handedness, the direction of time and an absolute definition of the sign of charge. A famous example in this respect is to try and

[1] T.D. Lee: "Particle Physics and Introduction to Field Theory"

make an absolute definition of matter and anti-matter. Is this possible? This question will be addressed in the particle physics II course.

- unitary symmetries or internal symmetries: gauge invariances. These are the symmetries discussed in these lectures. As an example of an unobservable quantity we can mention the absolute phase of a quantum mechanical wave function.

The relation between symmetries and conservation laws is expressed in a fundamental theorem by Emmy Noether: each continuous symmetry transformation under which the Lagrangian is invariant in form leads to a conservation law. Invariances under *external* operations as time and space translation lead to conservation of energy and momentum, and invariance under rotation to conservation of angular momentum Invariances under *internal* operations, like the rotation of the complex phase of wave functions lead to conserved currents, or more specific, conservation of charge.

We believe that the fundamental elementary interactions of the quarks and leptons can be understood as consequences of gauge symmetry priciples. The idea of local gauge invariant theory will be discussed in the first lecture and will be further applied in the unified electroweak theory in the second lecture. In the third lecture we will calculate the electroweak process $e^+e^- \to \gamma, Z \to \mu^+\mu^-$, using the techniques we developed before.

10.2 Lagrangian

In classical mechanics the Lagrangian may be regarded as the fundamental object, leading to the equations of motions of objects. From the Lagrangian, one can construct "the action" and follow Hamilton's principle of least action to find the physical path:

$$\delta S = \delta \int_{t_1}^{t_2} dt \; L\left(q, \dot{q}\right) = 0$$

where $q, \dot{q}$ are the generalized coordinate and velocity.

Exercise 33:
Prove that satisfaction of Hamilton's principle is guaranteed by the Euler Lagrange equations:

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

The classical theory does not treat space and time symmetrically as the Lagrangian might depend on the *parameter* t. This causes a problem if we want to make a relativisticaly covariant theory.

In a field theory the Lagrangian in terms of generalized coordinates is replaced $L(q, \dot{q})$ by a Lagrangian density in terms of fields $\phi\left(x\right)$ and their gradients:

$$\mathcal{L}\left(\phi\left(x\right), \partial_\mu \phi\left(x\right)\right) \quad \text{where} \quad L \equiv \int d^3x \; \mathcal{L}\left(\phi, \partial_\mu \phi\right)$$

The fields may be regarded as a separate generalized coordinate at each value of its argument: the space-time coordinate x. In fact, the field theory is the limit of a system of n degrees of freedom where n tends to infinity.

In this case the principle of least action becomes:

$$\delta \int_{t_1}^{t_2} d^4x \, \mathcal{L}\left(\phi, \partial_\mu \phi\right) = 0$$

where t_1, t_2 are the endpoints of the path.

This is guaranteed by the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi\left(x\right)} = \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\left(x\right)\right)}$$

which in turn lead to the equation of motion for the fields.

Note: If the Lagrangian is a Lorentz scalar, then the theory is automatically relativistic covariant.

What we will do next is to try and construct the Lagrangian for electromagnetic and weak interaction based on the idea of gauge invariance (or gauge symmetries).

Exercise 34: Lagrangians versus equations of motion

(a) *Show that the Euler Lagrange equations of the Lagrangian*

$$\mathcal{L} = \mathcal{L}_{KG}^{free} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2$$

of a real scalar field ϕ leads to the Klein-Gordon equation.

For a complex scalar field one can show that the Lagrangian becomes:

$$\mathcal{L} = \left|\partial^\mu \phi\right|^2 - m^2 \left|\phi\right|^2$$

(b) *Show that the Euler Lagrange equations of the Lagrangian*

$$\mathcal{L} = \mathcal{L}_{Dirac}^{free} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi$$

leads to the Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) = 0$$

and its adjoint. To do this, consider ψ and $\bar{\psi}$ as independent fields.

(c) *Show that the Lagrangian*

$$\mathcal{L} = \mathcal{L}_{EM} = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - j^\mu A_\mu = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu$$

leads to the Maxwell equations:

$$\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = j^\nu$$

Hence the current is conserved ($\partial_\nu j^\nu = 0$), since $F^{\mu\nu}$ is antisymmetric.

10.3 Where does the name "gauge theory" come from?

The idea of gauge invariance as a dynamical principle is due to Hermann Weyl. He called it *"eichinvarianz"* ("gauge" = "calibration"). Hermann Weyl[2] was trying to find a geometrical basis for both gravitation and electromagnetism. Although his effort was unsuccesfull the terminology survived. His idea is summarized here.

Consider a change in a function $f(x)$ between point x_μ and point $x_\mu + dx_\mu$. If the space has a uniform scale we expect simply:

$$f(x+dx) = f(x) + \partial^\mu f(x) dx_\mu$$

But if in addition the scale, or the unit of measure, for f changes by a factor $(1 + S^\mu dx_\mu)$ between x and $x + dx$, then the value of f becomes:

$$\begin{aligned} f(x+dx) &= \left(f(x) + \partial^\mu f(x) dx_\mu\right)\,(1 + S^\nu dx_\nu) \\ &= f(x) + \left(\partial^\mu f(x) + f(x) S^\mu\right) dx_\mu + O(dx)^2 \end{aligned}$$

So, to first order, the increment is:

$$\Delta f = \left(\partial^\mu + S^\mu\right) f\ dx_\mu$$

In other words Weyl introduced a modified differential operator by the replacement: $\partial^\mu \rightarrow \partial^\mu + S^\mu$.

One can see this in analogy in electrodynamics in the replacement of the momentum by the canonical momentum parameter: $p^\mu \rightarrow p^\mu - qA^\mu$ in the Lagrangian, or in Quantum Mechanics: $\partial^\mu \rightarrow \partial^\mu + iqA^\mu$, as was discussed in the earlier lectures. In this case the "scale" is $S^\mu = iqA^\mu$. If we now require that the laws of physics are invariant under a change:

$$(1 + S^\mu dx_\mu) \rightarrow (1 + iqA^\mu dx_\mu) \approx \exp\left(iqA^\mu dx_\mu\right)$$

then we see that the change of scale gets the form of a change of a phase. When he later on studied the invariance under phase transformations, he kept using the terminology of "gauge invariance".

10.4 Phase Invariance in Quantum Mechanics

The expectation value of a quantum mechanical *observable* is typically of the form:

$$\langle O \rangle = \int \psi^* O \psi$$

If we now make the replacement $\psi(x) \rightarrow e^{i\alpha}\psi(x)$ the expectation value of the observable remains the same. We say that we cannot measure the absolute phase of the wave

[2] H. Weyl, *Z. Phys.* **56**, 330 (1929)

function. (We can only measure *relative* phases between wavefunctions in interference experiments, see eg. the CP violation observables.)

But, are we allowed to choose a different phase convention on, say, the moon and on earth, for a wave function $\psi(x)$? In other words, we want to introduce the concept of *local* gauge invariance. This means that the physics observable stays invariant under the replacement:

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)$$

The problem that we face is that the Lagrangian density $\mathcal{L}\left(\psi(x), \partial_\mu\psi(x)\right)$ depends on both on the fields $\psi(x)$ and on the derivatives $\partial_\mu\psi(x)$. The derivative term yields:

$$\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = e^{i\alpha(x)}\ \left(\partial_\mu\psi(x) + i\partial_\mu\alpha(x)\psi(x)\right)$$

The second term spoils the fact that the transformation is simply an overall (unobservable) phase factor. It spoils the phase invariance of the theory. But, if we replace the derivative ∂_μ by the gauge-covariant derivative:

$$\partial_\mu \to D_\mu \equiv \partial_\mu + iqA_\mu$$

and we require that the field A_μ at the same time transforms as:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{q}\partial_\mu\alpha(x)$$

then we see that we get an overall phase factor for the covariant derivative term:

$$\begin{aligned} D_\mu\psi(x) \to D_\mu\psi'(x) &= e^{i\alpha(x)}\left(\partial_\mu\psi(x) + i\partial_\mu\alpha(x)\psi(x) + iqA_\mu(x)\psi(x) - iq\frac{1}{q}\partial_\mu\alpha(x)\psi(x)\right) \\ &= e^{i\alpha(x)}D_\mu\psi(x) \end{aligned}$$

As a consequence, quantities like $\psi^* D_\mu\psi$ will now be invariant under local gauge transformations.

10.5 Phase invariance for a Dirac Particle

We are going to replace in the Dirac Lagrangian:

$$\partial_\mu \to D_\mu \equiv \partial_\mu + iqA_\mu(x)$$

What happens to the Lagrangian?

$$\begin{aligned} \mathcal{L} &= \bar\psi\left(i\gamma^\mu D_\mu - m\right)\ \psi \\ &= \bar\psi\left(i\gamma^\mu\partial_\mu - m\right)\psi - qA_\mu\bar\psi\gamma^\mu\psi \\ &= \mathcal{L}_{free} - \mathcal{L}_{int} \end{aligned}$$

with:

$$\mathcal{L}_{int} = J^\mu A_\mu \qquad \text{and} \qquad J^\mu = q\bar\psi\gamma^\mu\psi$$

which is the familiar current we discussed in previous lectures.

Exercise 35: **Gauge invariance**

(a) (i) *Consider the Lagrangian for a complex scalar field:*

$$\mathcal{L} = |\partial^\mu \phi|^2 - m^2 |\phi|^2 \ .$$

Make a transformation of these fields:

$$\phi(x) \to e^{iq\alpha}\phi(x) \qquad ; \qquad \phi^*(x) \to e^{-iq\alpha}\phi^*(x) \ .$$

Show that the Lagrangian does not change.

(ii) *Do the same for the Dirac Lagrangian while considering the simultaneous transformations:*

$$\psi(x) \to e^{iq\alpha}\psi(x) \qquad ; \qquad \bar{\psi}(x) \to e^{-iq\alpha}\bar{\psi}(x)$$

(iii) *Noether's Theorem: consider an infinitesimal transformation:* $\psi \to \psi' = e^{i\alpha}\psi \approx (1 + i\alpha)\psi$. *Show that the requirement of invariance of the Dirac Lagrangian* $(\delta\mathcal{L}(\psi, \partial_\mu\psi, \bar{\psi}, \partial_\mu\bar{\psi}) = 0)$ *leads to the conservation of charge:* $\partial_\mu j^\mu = 0$, *with:*

$$j^\mu = \frac{ie}{2}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\psi - \bar{\psi}\frac{\mathcal{L}}{\partial\left(\partial_\mu\bar{\psi}\right)}\right) = -e\bar{\psi}\gamma^\mu\psi$$

(b) (i) *Start with the Lagrange density for a complex Klein-Gordon field*

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi$$

and show that a **local** *field transformation:*

$$\phi(x) \to e^{iq\alpha(x)}\phi(x) \qquad ; \qquad \phi^*(x) \to e^{-iq\alpha(x)}\phi^*(x)$$

does **not** *leave the Lagrangian invariant.*

(ii) *Replace now in the Lagrangian:* $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ *and show that the Lagrangian now* **does** *remain invariant, provided that the additional field transforms with the gauge transformation as:*

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu\alpha(x) \ .$$

(c) (i) *Start with the Lagrange density for a Dirac field*

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

and show that a **local** *field transformation:*

$$\psi(x) \to e^{iq\alpha(x)}\psi(x) \qquad ; \qquad \bar{\psi}(x) \to e^{-iq\alpha(x)}\bar{\psi}(x)$$

also does **not** *leave the Lagrangian invariant.*

(ii) Again make the replacement: $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ *where again the gauge field transforms as:*

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x) \ .$$

and show that the physics now **does** *remain invariant.*

In fact, the full QED Lagrangian includes also the so-called kinetic term of the field (the free fotons):

$$\mathcal{L}_{QED} = \mathcal{L}_{free} - J^\mu A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$, where the A fields are given by solutions of the Maxwell equations (see lecture 3):

$$\partial_\mu F^{\mu\nu} = J^\nu \quad .$$

10.6 Interpretation

What does it all mean?
We started from a free field Lagrangian which describes Dirac particles. Then we required that the fields have a U(1) symmetry which couples to the charge q. In other words: the physics does not change if we multiply by a unitary phase factor:

$$\psi(x) \to \psi'(x) = e^{iq\alpha(x)} \psi(x)$$

However, in order to obtain this symmetry we *must* then introduce a gauge field, the photon, which *couples* to the charge q:

$$D_\mu = \partial_\mu + iqA_\mu(x)$$

and which transforms simultaneously as:

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x)$$

the familiar gauge invariance of the electromagnetic field (see Lecture 3: $\alpha \Rightarrow \lambda$)!

This symmetry is called local gauge invariance under U(1) transformations. While ensuring the gauge invariance we have obtained the QED Lagrangian that describes the interactions between electrons and photons!
Note:
If the photon would have a mass, the corresponding term in the Lagrangian would be:

$$\mathcal{L}_\gamma = \frac{1}{2} m^2 A^\mu A_\mu$$

This term obviously violates local gauge invariance, since:

$$A^\mu A_\mu \to (A^\mu - \partial^\mu \alpha)(A_\mu - \partial_\mu \alpha) \neq A^\mu A_\mu$$

Conclusion: the photon must be massless. Later on, in the PPII course, it will be discussed how masses of vector bosons can be generated in the Higgs mechanism.

10.7 Yang Mills Theories

The concept of *non abelian* gauge theories is introduced here in a somewhat historical context as this helps to also understand the origin of the term weak iso-spin and the relation to (strong-) isospin.

Let us look at an example of the isospin system, i.e. the proton and the neutron. Let us also for the moment forget about the electric charge (we switch off electromagnetism and look only at the dominating strong interaction) and write the free Lagrangian for nucleons as:

$$\mathcal{L} = \bar{p}\ (i\gamma^\mu \partial_\mu - m)\ p + \bar{n}\ (i\gamma^\mu \partial_\mu - m)\ n$$

or, in terms of a composite spinor $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$:

$$\mathcal{L} = \bar{\psi}\ (i\gamma^\mu\, I\, \partial_\mu - I\, m)\ \psi \quad \text{with} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If we now, instead of a phase factor as in QED, make a *global* rotation in isospin space:

$$\psi \to \psi' = \exp\left(i\frac{\vec{\tau}\cdot\vec{\alpha}}{2}\right)\psi$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the usual Pauli Matrices [3] and $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is an arbitrary three vector. We have introduced a SU(2) phase transformation of special unitary 2x2 transformations (i.e. unitary 2x2 transformations with det=+1).

What does it mean? We state that, if we forget about their electric charge, the proton and neutron are indistinguishable, similar to the case of two wavefunctions with a different phase). It is just convention which one we call the *proton* and which one the *neutron*. The Lagrangian does not change under such a *global* SU(2) phase rotation. Imposing this requirement on the Lagrangian leads (again Noether's theorem) to the conserved current (use infinitesimal transformation: $\psi \to \psi' = (1 + \frac{i}{2}\vec{\tau}\cdot\vec{\alpha})\psi$):

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\delta\left(\partial_\mu\psi\right) + \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta\bar{\psi} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\bar{\psi}\right)}\delta\left(\partial_\mu\bar{\psi}\right) \\
&= \frac{\delta\mathcal{L}}{\partial\psi}\,\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}\psi + \frac{\delta\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\,\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}\ \left(\partial_\mu\psi\right) + \ldots \\
&= \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\right)\,\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}\,\psi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\,\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}\left(\partial_\mu\psi\right) \\
&= \partial_\mu\,\vec{\alpha}\cdot\left(\frac{i}{2}\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\vec{\tau}\psi\right)
\end{aligned}$$

where the Euler Lagrange relation has been used to eliminate $\partial\mathcal{L}/\partial\psi$. The equation can be written in the form of the continuity equation with corresponding conserved current:

$$\partial_\mu \vec{j}^\mu = 0 \qquad \text{with} \qquad \vec{j}^\mu = \bar{\psi}\gamma^\mu\frac{\vec{\tau}}{2}\psi\ .$$

[3] a representation is: $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$

However, as this is a global gauge transformation, it implies that once we make a definition at given point in space-time, this convention must be respected anywhere in space-time. This restriction seemed unnatural to Yang and Mills in a local field theory.

Can we also make a *local* SU(2) gauge transformation theory? So, let us try to define a theory where we chose the isospin direction differently for any space-time point.

To simplify the notation we define the gauge transformation as follows:

$$\begin{aligned} \psi(x) \to \psi'(x) &= G(x)\psi(x) \\ \text{with } G(x) &= \exp\left(\frac{i}{2}\,\vec{\tau}\cdot\vec{\alpha}(x)\right) \end{aligned}$$

But we have again, as in the case of QED, the problem with the transformation of the derivative:

$$\partial_\mu \psi(x) \to G\,(\partial_\mu \psi) + (\partial_\mu\, G)\;\psi$$

(just write it out yourself).

So, also here, we must introduce a new gauge field to keep the Lagrangian invariant:

$$\mathcal{L} = \bar{\psi}\;(i\gamma^\mu\, D_\mu - Im)\;\psi \quad \text{with} \quad \psi = \begin{pmatrix} p \\ n \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where we introduce the new covariant derivative:

$$I\partial_\mu \to D_\mu = I\partial_\mu + igB_\mu$$

where g is a new coupling constant that replaces the charge e in electromagnetism. The object B_μ is now a (2x2) matrix:

$$B_\mu = \frac{1}{2}\,\vec{\tau}\cdot\vec{b}_\mu = \frac{1}{2}t^a b^a_\mu = \frac{1}{2}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$$

$\vec{b}_\mu = (b_1, b_2, b_3)$ are now three gauge fields. We need now 3 fields rather then 1, one for each of the generators of the symmetry group of SU(2): τ_1, τ_2, τ_3.

We want get again a behaviour:

$$D_\mu\psi \to D'_\mu\psi' = G\,(D_\mu\psi)$$

because in that case the Lagrangian $\bar{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi$ is invariant for local gauge transformations. If we write out the covariant derivative term we get:

$$\begin{aligned} D'_\mu\psi' &= \left(\partial_\mu + igB'_\mu\right)\,\psi' \\ &= G\,(\partial_\mu\psi) + (\partial_\mu G)\,\psi + igB'_\mu\,(G\psi) \end{aligned}$$

If we compare this to the desired result:

$$\begin{aligned} D'_\mu\psi' &= G\,(\partial_\mu\psi + igB_\mu)\,\psi \\ &= G\,(\partial_\mu\psi) + igG\,(B_\mu\psi) \end{aligned}$$

then we see that the desired behaviour is obtained if the gauge field transforms simultaneously as:

$$igB'_\mu \left(G\psi\right) = igG \left(B_\mu \psi\right) - \left(\partial_\mu G\right) \psi$$

which must then be true for all values of the nucleon field ψ. Multiplying this operator equation from the right by G^{-1} we get:

$$B'_\mu = GB_\mu G^{-1} + \frac{i}{g} \left(\partial_\mu G\right) G^{-1}$$

Although this looks rather complicated we can again try to interpret this by comparing to the case of electromagnetism, where $G_{em} = e^{iq\alpha(x)}$.
Then:

$$\begin{aligned} A'_\mu &= G_{em} A_\mu G_{em}^{-1} + \frac{i}{q} \left(\partial_\mu G_{em}\right) G_{em}^{-1} \\ &= A_\mu - \partial_\mu \alpha \end{aligned}$$

which is exactly what we had before.

Exercise 36: *(not required)*
Consider an infinitesimal gauge transformation:

$$G = 1 + \frac{i}{2} \vec{\tau} \cdot \vec{\alpha} \qquad\qquad |\alpha_i| << 1$$

Use the general transformation rule for B'_μ *and use* $B_\mu = \frac{1}{2}\vec{\tau} \cdot \vec{b}_\mu$ *to demonstrate that the fields transform as:*

$$\vec{b}'_\mu = \vec{b}_\mu - \vec{\alpha} \times \vec{b}_\mu - \frac{1}{g} \partial_\mu \vec{\alpha}$$

(use: the Pauli-matrix identity: $(\vec{\tau} \cdot \vec{a})(\vec{\tau} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\tau} \cdot (\vec{a} \times \vec{b})$*).*

So for isospin symmetry the b^a_μ fields transform as an isospin rotation and a gradient term. The gradient term was already present in QED. The rotation term is new. It arises due to the non-commutativity of the 2x2 isospin rotations. If we write out the gauge field transformation formula in components:

$$b'^l_\mu = b^l_\mu - \epsilon_{jkl}\, \alpha^j\, b^k - \frac{1}{g} \partial_\mu \alpha^l$$

we can see that there is a coupling between the different components of the field. This is called self-coupling of the field. The effect of this becomes clear if one also considers the kinetic term of the isospin gauge field (analogous to the QED case):

$$\mathcal{L}_{SU(2)} = \bar{\psi}\, \left(i\gamma^\mu D_\mu - m\right)\, \psi - \frac{1}{4} \vec{F}_{\mu\nu} \vec{F}^{\mu\nu}$$

Introducing the field strength tensor:

$$F_{\mu\nu} = \frac{1}{2} \vec{F}_{\mu\nu} \cdot \vec{\tau} = \frac{1}{2} F^a_{\mu\nu} \tau^a$$

the Lagrangian is usually written as (using the Pauli identity $tr(\tau^a\tau^b) = 2\delta^{ab}$):

$$\mathcal{L}_{SU(2)} = \bar{\psi}\ (i\gamma^\mu D_\mu - m)\ \psi - \frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

with individual components of the field strength tensor:

$$F^l_{\mu\nu} = \partial_\nu b^l_\mu - \partial_\mu b^l_\nu + g\ \epsilon_{jkl}\ b^j_\mu\ b^k_\nu$$

The consequence of the last term is that the Lagrangian term $F_{\mu\nu}F^{\mu\nu}$ contains contributions with 2, 3 and 4 factors of the b-field. These couplings are respectively referred to as bilinear, trilinear and quadrilinear couplings. In QED there's only the bilinear photon propagator term. In the isospin theory there are self interections by a 3-gauge boson vertex and a 4 gauge boson vertex.

10.7.1 What have we done?

We modified the Lagrangian describing isospin 1/2 doublets $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$:

$$\mathcal{L}^{free}_{SU(2)} = \bar{\psi}\ (i\gamma^\mu\partial_\mu - m)\ \psi$$

We made the replacement $\partial_\mu \to D_\mu = \partial_\mu + igB_\mu$ with $B_\mu = \frac{1}{2}\vec{\tau}\cdot\vec{b}_\mu$, to obtain:

$$\begin{aligned}
\mathcal{L}_{SU(2)} &= \bar{\psi}\ (i\gamma^\mu D_\mu - m)\ \psi \\
&= \mathcal{L}^{free}_{SU(2)} - \frac{g}{2}\,\vec{b}_\mu\cdot\bar{\psi}\gamma^\mu\vec{\tau}\psi \\
&= \mathcal{L}^{free}_{SU(2)} - \mathcal{L}^{interaction}_{SU(2)} \\
&= \mathcal{L}^{free}_{SU(2)} - \vec{b}_\mu\cdot\vec{J}^\mu
\end{aligned}$$

where $\vec{J}^\mu = \frac{g}{2}\bar{\psi}\gamma^\mu\vec{\tau}\psi$ is the isospin current.

Let us compare it once more to the case of QED:

$$\mathcal{L}_{U(1)} = \mathcal{L}^{free}_{U(1)} - A_\mu\cdot J^\mu$$

with the electromagnetic current $J^\mu = q\bar{\psi}\gamma^\mu\psi$

We have neglected here the kinetic terms of the fields:

$$\mathcal{L}_{SU(2)} = \bar{\psi}\ (i\gamma^\mu D_\mu - m)\ \psi - \frac{1}{2}trF_{\mu\nu}F^{\mu\nu}$$

which contains self-coupling terms of the fields.

10.7.2 Assessment

We see a symmetry in the $\begin{pmatrix} p \\ n \end{pmatrix}$ system: the isospin rotations.

- If we require local gauge invariance of such transformations we need to introduce $\vec{b}_\mu$ gauge fields.

- But what are they? $\vec{b}_\mu$ must be three massless vector bosons that couple to the proton and neutron. It cannot be the π^-, π^0, π^+ since they are pseudoscalar particles rather then vector bosons. It turns out this theory does not describe the strong interactions. We know now that the strong force is mediated by massless gluons. In fact gluons have 3 colour degrees of freedom, such that they can be described by 3x3 unitary gauge transformations (SU(3)), for which there are 8 generators, listed here:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

  The strong interaction will be discussed later on in the particle physics course. Next lecture we will instead look at the weak interaction and introduce the concept of weak iso-spin.

- Also, we have started to say that the symmetry in the p, n system is only present if we neglect electromagnetic interactions, since obviously from the charge we can absolutely define the proton and the neutron state in the doublet. In such a case where the symmetry is only approximate, we speak of a *broken symmetry* rather then of an *exact symmetry*.

Lecture 11

Electroweak Theory

In the previous lecture we have seen how imposing a local gauge symmetry requires a modification of the free Lagrangian such that a theory with interactions is obtained. We studied:

- local $U(1)$ gauge invariance:

$$\overline{\psi}\ (i\gamma^\mu D_\mu - m)\ \psi = \overline{\psi}\ (i\gamma^\mu \partial_\mu - m)\ - \underbrace{q\overline{\psi}\gamma^\mu\psi}_{J^\mu}\ A_\mu$$

- local $SU(2)$ gauge invariance:

$$\overline{\psi}\ (i\gamma^\mu D_\mu - m)\ \psi = \overline{\psi}\ (i\gamma^\mu \partial_\mu - m)\ - \underbrace{\frac{g}{2}\overline{\psi}\gamma^\mu\vec{\tau}\psi}_{\vec{J}^\mu}\ \vec{b}_\mu$$

For the $U(1)$ symmetry we can identify the A_μ field as the photon and the Feynman rules for QED, as we discussed them in previous lectures, follow automatically. For the $SU(2)$ case we hoped that we could describe the strong nuclear interactions, but this failed.

Let us now, instead of the strong isospin doublet $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$ introduce the following doublets:

$$\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \text{and} \quad \psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

and we speak instead of "weak isospin" doublets. Note that the fermion fields have an L index (for "left-handed"). These left handed states are defined as:

$$\nu_L = \frac{1}{2}(1-\gamma_5)\ \nu \qquad\qquad u_L = \frac{1}{2}(1-\gamma_5)\ u$$
$$e_L = \frac{1}{2}(1-\gamma_5)\ e \qquad\qquad d_L = \frac{1}{2}(1-\gamma_5)\ d$$

with the familiar projection operators:

$$\psi_L = \frac{1}{2}(1-\gamma_5)\ \psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1+\gamma_5)\ \psi$$

(Remember: for massless particles" $\psi_L = \psi_{-\text{helicity}}$ and $\psi_R = \psi_{+\text{helicity}}$.)

The origin of the weak interaction lies in the fact that we now impose a local gauge symmetry in weak isospin rotations of left handed fermion fields. This means that if we "switch off" charge we cannot distinguish between a ν_L and a e_L or a u_L and a d_L state. The fact that we only impose this on left handed states implies that the weak interaction is completely left-right asymmetric. (Intuitively this is very difficult to accept: why would there be a symmetry for the left-handed states only?!). This is called *maximal violation of parity.*

It will turn out that the three vector fields (b_1, b_2, b_3 from the previous lecture) can later be associated with the carriers of the weak interaction, the W^+, W^-, Z bosons. However, these bosons are not massless. An explicit mass term ($\mathcal{L}_M = K b_\mu b^\mu$) would in fact break the gauge invariance of the theory. Their masses can be generated in a mechanism that is called spontaneous symmetry breaking and involves a new hypothetical particle: the Higgs boson. The main idea of the symmetry breaking mechanism is that the Lagrangian retains the full gauge symmetry, but that the ground state, i.e. the vacuum, is no longer at a symmetric position. The realization of the vacuum selects a preferred direction in isospin space, and thus breaks the symmetry. Future lectures will discuss this aspect in more detail.

To construct the weak $SU(2)_L$ theory we start again with the free Dirac Lagrangian and we impose $SU(2)$ symmetry (but now on the weak isospin doublets):

$$\mathcal{L}_{free} = \overline{\psi_L}\ (i\gamma^\mu \partial_\mu - m)\ \psi_L$$

Again we introduce the covariant derivative:

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + igB_\mu \quad \text{with} \quad B_\mu = \frac{1}{2}\,\vec{\tau}\cdot\vec{b}_\mu$$

then:

$$\mathcal{L}_{free} \rightarrow \mathcal{L}_{free} - \vec{b}_\mu \cdot J^\mu_{weak}$$

with the weak current:

$$J^\mu_{weak} = \frac{g}{2}\overline{\psi_L}\gamma^\mu\vec{\tau}\psi_L$$

This is just a copy from what we have seen in the strong isospin example.

The model for the weak interactions now contains 3 massless gauge bosons (b^1, b^2, b^3). However, in nature we have seen that the weak interaction is propagated by 3 massive bosons W^+, W^-, Z^0.

From the Higgs mechanism it turns out that the physical fields associated with b^1_μ and b^2_μ are the charged W bosons:

$$W^\pm_\mu \equiv \frac{b^1_\mu \mp i b^2_\mu}{\sqrt{2}}$$

11.1 The Charged Current

We will use the definition of the W-fields to re-write the first two terms in the Lagrangian of the weak current:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{free} + \mathcal{L}_{weak}^{int} \\ \text{with} \quad \mathcal{L}_{weak}^{int} &= -\vec{b}_\mu \cdot \vec{J}_{weak}^\mu = -b_\mu^1 J^{1\mu} - b_\mu^2 J^{2\mu} - b_\mu^3 J^{3\mu} \end{aligned}$$

The charged current terms are:

$$\mathcal{L}_{CC} = -b_\mu^1 J^{1\mu} - b_\mu^2 J^{2\mu}$$

with:

$$J^{1\mu} = \frac{g}{2} \overline{\psi_L}\, \gamma^\mu\, \tau_1\, \psi_L \quad ; \quad J^{2\mu} = \frac{g}{2} \overline{\psi_L}\, \gamma^\mu\, \tau_2\, \psi_L$$

Exercise 37:
Show that the re-definition $W_\mu^\pm = \frac{b_\mu^1 \mp i b_\mu^2}{\sqrt{2}}$ *leads to:*

$$\begin{aligned} \mathcal{L}_{CC} &= -W_\mu^+ J^{+\mu} - W_\mu^- J^{-\mu} \\ \textit{with:} \quad J^{+\mu} &= \frac{g}{\sqrt{2}} \overline{\psi_L}\, \gamma^\mu\, \tau^+\, \psi_L \quad ; \quad J^{-\mu} = \frac{g}{\sqrt{2}} \overline{\psi_L}\, \gamma^\mu\, \tau^-\, \psi_L \\ \textit{and with:} \quad \tau^+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

So, for the physical fields W^+ and W^- the leptonic currents are:

$$J^{+\mu} = \frac{g}{\sqrt{2}} \overline{\nu_L}\, \gamma^\mu\, e_L \quad ; \quad J^{-\mu} = \frac{g}{\sqrt{2}} \overline{e_L}\, \gamma^\mu\, \nu_L$$

or written out with the left-handed projection operators:

$$J^{+\mu} = \frac{g}{\sqrt{2}} \overline{\nu}\, \frac{1}{2} \left(1 + \gamma^5\right)\, \gamma^\mu\, \frac{1}{2} \left(1 - \gamma^5\right)\, e \quad .$$

Note that we have the identity:

$$\begin{aligned} \left(1 + \gamma^5\right) \gamma^\mu \left(1 - \gamma^5\right) &= \gamma^\mu + \gamma^5\gamma^\mu - \gamma^\mu\gamma^5 - \gamma^5\gamma^\mu\gamma^5 \\ &= \gamma^\mu - 2\gamma^\mu\gamma^5 + \left(\gamma^5\right)^2 \gamma^\mu \\ &= 2\gamma^\mu \left(1 - \gamma^5\right) \end{aligned}$$

such that we get for the leptonic charge raising current (W^+):

$$\boxed{J^{+\mu} = \tfrac{g}{2\sqrt{2}}\, \overline{\nu}\, \gamma^\mu \left(1 - \gamma^5\right) e}$$

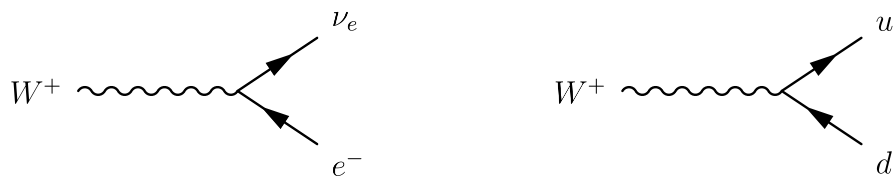
and for the leptonic charge lowering current (W^-):

$$\boxed{J^{-\mu} = \frac{g}{2\sqrt{2}}\, \overline{e}\, \gamma^\mu \,(1-\gamma^5)\, \nu} \quad .$$

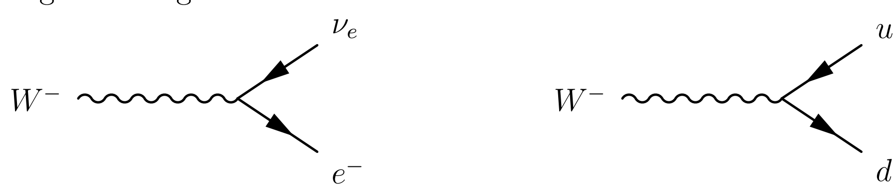
Remembering that a vector interaction has an operator γ^μ in the current and an axial vector interaction a term $\gamma^\mu\gamma^5$, we recognize in the charged weak interaction the famous "V-A" interaction.

The same is true for the quark-currents and we can recognize the following currents in the weak interaction:

Charge raising:



Charge lowering:



11.2 The Neutral Current

11.2.1 Empirical Appraoch

The Lagrangian for weak and electromagnetic interactions is:

$$\begin{aligned}
\mathcal{L}_{EW} &= \mathcal{L}_{free} - \mathcal{L}_{weak} - \mathcal{L}_{EM} \\
\mathcal{L}_{weak} &= W_\mu^+ J^{+\mu} + W_\mu^- J^{-\mu} + b_\mu^3 J_3^\mu \\
\mathcal{L}_{EM} &= a_\mu J_{EM}^\mu
\end{aligned}$$

Let us again look at the interactions for leptons ν, e, then:

$$\begin{aligned}
J_3^\mu &= \frac{g}{2}\, \overline{\psi_L}\, \gamma^\mu\, \tau^3\, \psi_L = \frac{g}{2}\, \overline{\nu_L}\, \gamma^\mu\, \nu_L - \frac{g}{2}\, \overline{e_L}\, \gamma^\mu\, e_L \qquad \left(\text{we used} : \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\
J_{EM}^\mu &= q\, \overline{e}\, \gamma^\mu\, e = q\,(\overline{e_L}\, \gamma^\mu\, e_L) + q\,(\overline{e_R}\, \gamma^\mu\, e_R)
\end{aligned}$$

Exercise 38:
Show explicitly that:

$$\overline{\psi}\, \gamma^\mu\, \psi = \overline{\psi_L}\, \gamma^\mu\, \psi_L + \overline{\psi_R}\, \gamma^\mu\, \psi_R$$

making use of $\psi = \psi_L + \psi_R$ *and the projection operators* $\frac{1}{2}\,(1-\gamma_5)$ *and* $\frac{1}{2}\,(1+\gamma_5)$

Experiments have shown that in contrast to the charged weak interaction, the neutral weak current associated to the Z-boson is *not* purely left-handed, but:

$$J^{\mu f}_{NC} = \frac{g}{2}\,\overline{\psi}^f\,\gamma^\mu\,\left(C^f_V - C^f_A\gamma^5\right)\,\psi^f$$

where C^f_V and C^f_A are no longer equal to 1, but they are constants that express the relative strength of the vector and axial vector components of the interaction. Their value depends on the type of fermion f, as we will see below.

Taking again the leptons $\psi = \begin{pmatrix}\nu\\ e\end{pmatrix}$ we get:

$$J^\mu_{NC} = \frac{g}{2}\,\overline{\nu}\,\gamma^\mu\,\left(C^\nu_V - C^\nu_A\gamma^5\right)\,\nu + \frac{g}{2}\,\overline{e}\,\gamma^\mu\,\left(C^e_V - C^e_A\gamma^5\right)\,e$$

At this point we introduce the left-handed and right-handed couplings:

$$\begin{aligned} C_R &\equiv C_V - C_A \qquad & C_V &= \frac{1}{2}\left(C_R + C_L\right) \\ C_L &\equiv C_V + C_A \qquad & C_A &= \frac{1}{2}\left(C_L - C_R\right) \end{aligned}$$

then:

$$\left(C_V - C_A\gamma^5\right) = \underbrace{C_V - C_A}_{C_R}\left(\frac{1+\gamma^5}{2}\right) + \underbrace{C_V + C_A}_{C_L}\left(\frac{1-\gamma^5}{2}\right)$$

so that we can write:

$$\overline{\psi}\,\gamma^\mu\,\left(C_V - C_A\gamma^5\right)\,\psi = \left(\overline{\psi_L + \psi_R}\right)\,\gamma^\mu\,\left(C_R\psi_R + C_L\psi_L\right) = C_R\,\overline{\psi_R}\gamma^\mu\psi_R + C_L\,\overline{\psi_L}\gamma^\mu\psi_L \quad .$$

For neutrino's we have $C^\nu_L = 1$ and $C^\nu_R = 0$. So, for leptons the observed neutral current can be written as:

$$J^\mu_{NC} = \frac{g}{2}\,\left(\overline{\nu_L}\,\gamma^\mu\,\nu_L\right) + \frac{g}{2}\,\left(C^e_L\,\overline{e_L}\,\gamma^\mu\,e_L\right) + \frac{g}{2}\,\left(C^e_R\,\overline{e_R}\,\gamma^\mu\,e_R\right)$$

We had for the electromagnetic current:

$$J^\mu_{EM} = \qquad q\,\left(\overline{e_L}\,\gamma^\mu\,e_L\right) + q\,\left(\overline{e_R}\,\gamma^\mu\,e_R\right)$$

and for the SU(2) current:

$$J^\mu_3 = \frac{g}{2}\,\left(\overline{\nu_L}\,\gamma^\mu\,\nu_L\right) - \frac{g}{2}\,\left(\overline{e_L}\,\gamma^\mu\,e_L\right)$$

We now insert that J^μ_3 is in fact a linear combination of J^μ_{NC} and J^μ_{EM}:

$$J^\mu_3 = a\cdot J^\mu_{NC} + b\cdot J^\mu_{EM}$$

- look at the ν_L terms: $a = 1$
- look at the e_R terms: $\frac{g}{2}C_R^e + q \cdot b = 0$ $\quad \Rightarrow C_R^e = -\frac{2qb}{g}$
- look at e_L terms : $\frac{g}{2}C_L^e + q \cdot b = -\frac{g}{2}$ $\quad \Rightarrow C_L^e = -1 - \frac{2qb}{g}$

Therefore:

$$C_V = \frac{1}{2}\ (C_R + C_L) \qquad \Rightarrow \qquad C_V^e = -\frac{1}{2} - \frac{2q}{g}b$$

$$C_A = \frac{1}{2}\ (C_L - C_R) \qquad \Rightarrow \qquad C_A^e = -\frac{1}{2}$$

The vector coupling now contains a constant b which gives the ratio in which the $SU(2)$ current $(\frac{g}{2})$ and the electromagnetic current (q) are related. The constant b is a constant of nature and is written as $b = \sin^2\theta$: where θ represents the *weak mixing angle*.
We will study this more carefully below.

11.2.2 Hypercharge vs Charge

Again, we write down the electroweak Lagrangian, but this time we pose a different U(1) symmetry (see H&M[1], Chapter 13):

$$\boxed{\mathcal{L}_{EW} = \mathcal{L}_{free} - g\ \vec{J}^{\mu}_{SU(2)} \cdot \vec{b}_\mu - \tfrac{g'}{2} J^\mu_Y a_\mu}$$

where Y is the so-called *hypercharge* quantum number.
The $U(1)$ gauge invariance in now imposed on the quantity hypercharge rather the charge, and it has a coupling strength $g'/2$.

As before we have the physical charged currents:

$$W^\pm_\mu = \frac{b^1_\mu \mp i b^2_\mu}{\sqrt{2}} \quad .$$

For the neutral currents we say that the physical fields are the following linear combinations:

$$\begin{aligned} A_\mu &= a_\mu \cos\theta_w + b^3_\mu \sin\theta_w \qquad &\text{(massless)} \\ Z_\mu &= -a_\mu \sin\theta_w + b^3_\mu \cos\theta_w \qquad &\text{(massive)} \end{aligned}$$

and the origin of the name *weak mixing angle* for θ_w becomes clear.

We can now write the terms for b^3_μ and a_μ in the Lagrangian:

$$\begin{aligned} -gJ^\mu_3 b^3_\mu - \frac{g'}{2} J^\mu_Y a_\mu &= -\left(g\sin\theta_w J^\mu_3 + g'\cos\theta_w \frac{J^\mu_Y}{2}\right) A_\mu \\ &\quad - \left(g\cos\theta_w J^\mu_3 - g'\sin\theta_w \frac{J^\mu_Y}{2}\right) Z_\mu \\ &\equiv -qJ^\mu_{EM} A_\mu - g_Z J^\mu_{NC} Z_\mu \end{aligned}$$

[1]Halzen and Martin, Quarks & Leptons: "An Introductory Course in Modern Particle Physics"

The weak hypercharge is introuced in complete analogy with the strong hypercharge, for which we have the famous Gellmann - Nishijima relation: $Q = I_3 + \frac{1}{2} Y_S$. In the electroweak theory we use: $Q = T_3 + \frac{1}{2} Y$ which means:

$$\boxed{J^\mu_{EM} = J^\mu_3 + \tfrac{1}{2} J^\mu_Y}$$

then, indeed, for the A_μ field we have:

$$-g \sin\theta_w \left(J^\mu_3 + \frac{g' \cos\theta_w}{g \sin\theta_w} \cdot \frac{1}{2} J^\mu_Y \right) = -e J^\mu_{EM} \quad ,$$

provided the following relation holds:

$$g \sin\theta_w = g' \cos\theta_w = e \quad .$$

The weak mixing angle is defined as the ratio of the coupling constants of the $SU(2)_L$ group and the $U(1)_Y$ group:

$$\tan\theta_w = \frac{g'}{g} \quad .$$

For the Z-currents we then find:

$$\begin{aligned} & - \left(g \cos\theta_w J^\mu_3 - \frac{g'}{2} \sin\theta_w \cdot 2 \left(J^\mu_{EM} - J^\mu_3 \right) \right) Z_\mu \\ = & \ \ldots \\ = & -\frac{e}{\cos\theta_w \sin\theta_w} \left(J^\mu_3 - \sin^2\theta_w J^\mu_{EM} \right) Z_\mu \end{aligned}$$

So we see that:

$$\boxed{J^\mu_{NC} = J^\mu_3 - \sin^2\theta_w \, J^\mu_{EM}}$$

which is in agreement with what we had obtained earlier:

$$J^\mu_3 = a \cdot J^\mu_{NC} + b \cdot J^\mu_{EM} \quad \text{with} \quad a = 1 \ \text{ and } \ b = \sin^2\theta_w$$

11.2.3 Assessment

We introduce a symmetry group $SU(2)_L \otimes U(1)_Y$ and describe electroweak interactions with:

$$-\left(g \, \vec{J}^\mu_L \cdot \vec{b}_\mu + \frac{g'}{2} J^\mu_Y \cdot a_\mu \right)$$

The coupling constants g and g' are free parameters (we can also take e and $\sin^2\theta_w$). The electromagnetic and neutral weak currents are then given by:

$$\begin{aligned} J^\mu_{EM} &= J^\mu_3 + \frac{1}{2} J^\mu_Y \\ J^\mu_{NC} &= J^\mu_3 - \sin^2\theta_w J^\mu_{EM} = \cos^2\theta_w J^\mu_3 - \sin^2\theta_w \frac{J^\mu_Y}{2} \end{aligned}$$

and the interaction term in the Lagrangian becomes:

$$-\left(eJ^{\mu}_{EM}\cdot A_{\mu}+\frac{e}{\cos\theta_w\sin\theta_w}J^{\mu}_{NC}\cdot Z_{\mu}\right)$$

in terms of the physical fields A_μ and Z_μ.

11.3 The Mass of the W and Z bosons

In the electroweak model as introduced here, the gauge fields must be massless, since explicit mass terms ($\sim\phi_\mu\phi^\mu$) are not gauge invariant. In the Standard Model the mass of all particles are generated in the mechanism of spontaneous symmetry breaking, introducing the Higgs particle (see later lectures.) Here we just give an empirical argument to predict the mass of the W and Z particles.

1. Mass terms are of the following form:

$$M_\phi^2=\langle\phi\,|H|\,\phi\rangle\qquad\text{for any field }\phi$$

2. From the comparison with the Fermi 4-point interaction we find:

$$\frac{G_F}{\sqrt{2}}=\frac{g^2}{8M_W^2}\qquad\Rightarrow\qquad M_W^2=\frac{\sqrt{2}g^2}{8G_F}=\frac{\sqrt{2}}{8G_F}\,\frac{e^2}{\sin^2\theta}$$

Thus, we get the following predictions:

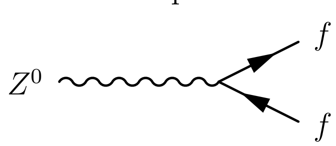
$$\begin{aligned}M_W&=\sqrt{\frac{\sqrt{2}}{8G_F}}\,\frac{e}{\sin\theta_w}=81\text{ GeV}\\M_Z&=M_W\;(g_z/g)=M_W/\cos\theta=91\text{ GeV}\end{aligned}$$

11.4 The Coupling Constants for $Z\to f\overline{f}$

For the neutral Z-current interaction we have for the interaction in general:

$$\begin{aligned}-ig_Z\,J^{\mu}_{NC}\,Z_\mu&=-i\frac{g}{\cos\theta_w}\left(J^\mu_3-\sin^2\theta_w J^\mu_{EM}\right)Z_\mu\\&=-i\frac{g}{\cos\theta_w}\,\overline{\psi}_f\gamma^\mu\underbrace{\left[\frac{1}{2}\left(1-\gamma^5\right)T_3-\sin^2\theta_w Q\right]}_{\frac{1}{2}\left(C_V^f-C_A^f\gamma^5\right)}\psi_f\cdot Z_\mu\end{aligned}$$

which we can represent with the following vertex and Feynman rule:



$$-i\,\frac{g}{\cos\theta_w}\,\gamma^\mu\,\frac{1}{2}\left(C_V^f-C_A^f\gamma^5\right)$$

with:

$$\begin{aligned} C_L^f &= T_3^f - Q^f \sin^2\theta_w \\ C_R^f &= -Q^f \sin^2\theta_w \end{aligned}$$

$$\Rightarrow \begin{aligned} C_V^f &= T_3^f - 2Q^f \sin^2\theta_w \\ C_A^f &= T_3^f \end{aligned}$$

fermion	T_3	Q	Y	C_A^f	C_V^f
ν_e ν_μ ν_τ	$+\frac{1}{2}$	0	-1	$\frac{1}{2}$	$\frac{1}{2}$
e μ τ	$-\frac{1}{2}$	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2} + 2\sin^2\theta_w$
u c t	$+\frac{1}{2}$	$+\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$
d s b	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_w$

Table 11.1: The neutral current vector and axial vector couplings for each of the fermions in the Standard Model.

Exercise 39:

What do you think is the difference between an exact and a broken symmetry?
Can you make a (wild) guess what spontaneous symmetry breaking means?
Which symmetry is involved in the gauge theories below? Which of these gauge symmetries are exact? Why/Why not?

(a) U1(Q) symmetry

(b) SU2(u-d-flavour) symmetry

(c) SU3(u-d-s-flavour) symmetry

(d) SU6(u-d-s-c-b-t) symmetry

(e) SU3(colour) symmetry

(f) SU2(weak-isospin) symmetry

(f) SU5(Grand unified) symmetry

(g) SUSY

Lecture 12

The Process $e^-e^+ \to \mu^-\mu^+$

12.1 The Cross Section of $e^-e^+ \to \mu^-\mu^+$

Equipped with the Feynman rules of the electroweak theory we proceed to calculate the cross section of the electroweak process: $e^-e^+ \to \gamma, Z \to \mu^-\mu^+$. We assume the following kinematics:

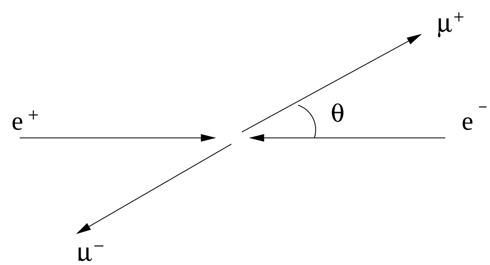


Figure 12.1: Kinematics of the process $e^-e^+ \to \mu^-\mu^+$.

There are two Feynman diagrams that contribute to the process:

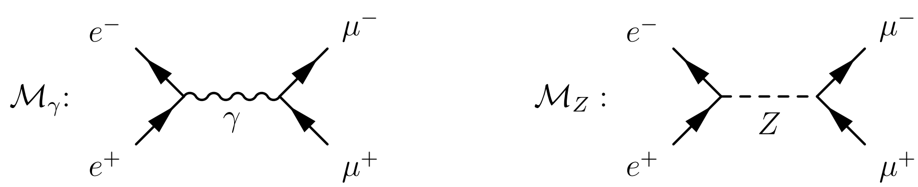


Figure 12.2: Feynman diagrams contributing to $e^-e^+ \to \mu^-\mu^+$

In complete analogy with the calculation of the QED process $e^+e^- \to e^+e^-$ we obtain the cross section using Fermi's Golden rule:

$$d\sigma = \frac{\overline{|\mathcal{M}|}^2}{F}\, dQ$$

With the phase factor dQ flux factor F:

$$dQ \;\; = \;\; \frac{1}{4\pi^2}\,\frac{p_f}{4\sqrt{s}}\, d\Omega$$

$$F = 4p_i\sqrt{s}$$

$$\sigma\left(e^-e^+ \to \mu^-\mu^+\right) = \frac{1}{64\pi^2}\cdot\frac{1}{s}\cdot\overline{|\mathcal{M}|}^2$$

The Matrix element now includes:

$$\mathcal{M}_\gamma = -e^2\left(\overline{\psi}_m\gamma^\mu\psi_m\right)\cdot\frac{g_{\mu\nu}}{q^2}\cdot\left(\overline{\psi}_e\gamma^\nu\psi_e\right)$$

$$\mathcal{M}_Z = -\frac{g^2}{4\cos^2\theta_w}\left[\overline{\psi}_m\gamma^\mu\left(C_V^m - C_A^m\gamma^5\right)\psi_m\right]\cdot\frac{g_{\mu\nu}-q_\mu q_\nu/M_Z^2}{q^2-M_Z^2}\cdot\left[\overline{\psi}_e\gamma^\nu\left(C_V^e - C_A^e\gamma^5\right)\psi_e\right]$$

The propagator for massive vector bosons (Z-boson) is discussed in Halzen & Martin §6.11 and §6.12. The wave equation of a massless spin-1 particle is:

$$\Box^2 A_\mu = 0 \qquad \Rightarrow \qquad i\frac{-g_{\mu\nu}}{q^2}$$

$$\left(\Box^2 + M^2\right)Z_\mu = 0 \qquad \Rightarrow \qquad i\frac{-g_{\mu\nu}+q_\mu q_\nu/M^2}{q^2-M^2}$$

We can simplify the propagator of the Z if we ignore the lepton masses. In practice this means that we work in the limit of high-energy scattering. In that case the Dirac equation becomes:

$$\overline{\psi}_e\ (i\partial_\mu\gamma^\mu - m) = 0 \quad \Rightarrow \quad \overline{\psi}_e\ (\gamma^\mu p_{\mu,e}) = 0$$

Since $p_e = \frac{1}{2}q$ we also have:

$$\frac{1}{2}\overline{\psi}_e\ (\gamma^\mu q_\mu) = 0 \quad \Rightarrow \quad q_\mu\cdot q_\nu/M_z^2 = 0$$

Thus the propagator simplifies:

$$\frac{g_{\mu\nu}-q_\mu q_\nu/M_Z^2}{q^2-M_Z^2} \to \frac{g_{\mu\nu}}{q^2-M_Z^2}$$

Thus we have for the Z-exchange matrix element the expression:

$$\mathcal{M}_Z = \frac{-g^2}{4\cos^2\theta_w}\,\frac{1}{q^2-M_Z^2}\cdot\left[\overline{\psi}_m\,\gamma^\mu\left(C_V^m - C_A^m\gamma^5\right)\,\psi_m\right]\left[\overline{\psi}_e\,\gamma_\mu\left(C_V^e - C_A^e\gamma^5\right)\,\psi_e\right] \quad .$$

To calculate the cross section by summing over $\mathcal{M}_\gamma$ and $\mathcal{M}_Z$ is now straightforward but a rather lengthy procedure: applying Casimir's trick, trace theorems, etc. Let us here try to follow a different approach.

We rewrite the $\mathcal{M}_Z$ matrix element in terms of right-handed and left-handed couplings, using the definitions: $C_R = C_V - C_A$; $C_L = C_V + C_A$. As before we have:

$$\left(C_V - C_A\gamma^5\right) = (C_V - C_A)\cdot\frac{1}{2}\left(1+\gamma^5\right) + (C_V + C_A)\cdot\frac{1}{2}\left(1-\gamma^5\right) \quad .$$

Thus:

$$\left(C_V - C_A\gamma^5\right)\psi = C_R\psi_R + C_L\psi_L \quad .$$

Let us now look back at the QED process:

$$\mathcal{M}_\gamma = \frac{-e^2}{s}\,\left(\overline{\psi}_m\gamma^\mu\psi_m\right)\,\left(\overline{\psi}_e\gamma_\mu\psi_e\right)$$

with (see previous lecture):

$$\begin{aligned}\left(\overline{\psi}_m\gamma^\mu\psi_m\right) &= \left(\overline{\psi_{Lm}}\,\gamma^\mu\,\psi_{Lm}\right) + \left(\overline{\psi_{Rm}}\,\gamma^\mu\,\psi_{Rm}\right)\\ \left(\overline{\psi}_e\gamma_\mu\psi_e\right) &= \left(\overline{\psi_{Le}}\,\gamma_\mu\,\psi_{Le}\right) + \left(\overline{\psi_{Re}}\,\gamma_\mu\,\psi_{Re}\right)\end{aligned}$$

The fact that there are no terms connecting L-handed to R-handed $(\overline{\psi_{Rm}}\,\gamma^\mu\,\psi_{Lm})$ actually implies that we have helicity conservation for high energies (i.e. neglecting $\sim m/E$ terms) at the vertices:

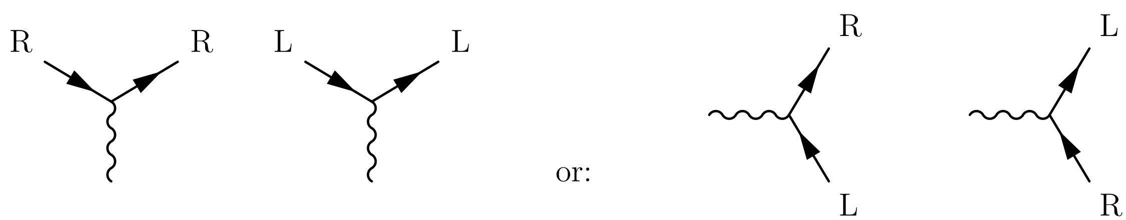


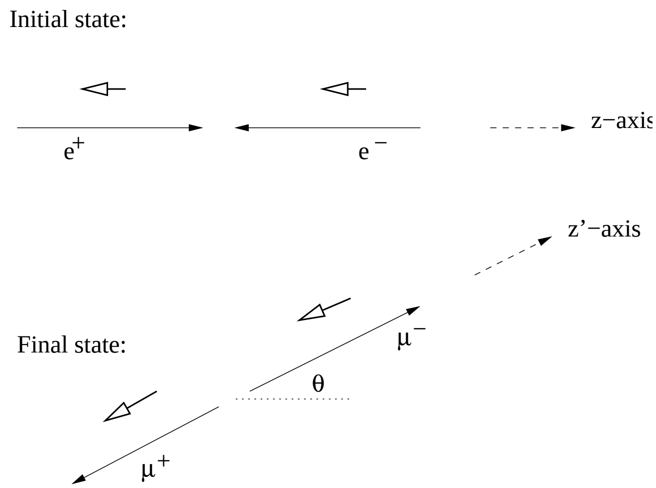
Figure 12.3: Helicity conservation. *left:* A right-handed incoming electron scatters into a right-handed outgoing electron and vice versa in a vector or axial vector interaction . *right:* In the crossed reaction the energy and momentum of one electron is reversed: i.e. in the e^+e^- pair production a right-handed electron and a left-handed positron (or vice versa) are produced. This is the consequence of a spin=1 force carrier. (In all diagrams time increases from left to right.)

As a consequence we can decompose the unpolarized QED scattering process as a sum of 4 cross section contributions (*Note:* $e^+_R \equiv \overline{\psi_{Le}}$ *etc.(!)*)

$$\frac{d\sigma}{d\Omega}^{\text{unpolarized}} = \frac{1}{4}\Big\{\;\frac{d\sigma}{d\Omega}\left(e^-_Le^+_R \to \mu^-_L\mu^+_R\right) + \frac{d\sigma}{d\Omega}\left(e^-_Le^+_R \to \mu^-_R\mu^+_L\right)$$
$$\frac{d\sigma}{d\Omega}\left(e^-_Re^+_L \to \mu^-_L\mu^+_R\right) + \frac{d\sigma}{d\Omega}\left(e^-_Re^+_L \to \mu^-_R\mu^+_L\right)\Big\}$$

where we *average* over the incoming spins and *sum* over the final state spins.

Let us look in more detail at the helicity dependence (H&M §6.6):



In the initial state the e^- and e^+ have opposite helicity (as they produce a spin 1 γ).

The same is true for the final state μ^- and μ^+.

So, in the center of mass frame, scattering proceeds from an initial state with $J_Z = +1$ or -1 along axis $\hat{z}$ into a final state with $J'_Z = +1$ or -1 along axis $\hat{z}'$. Since the interaction proceeds via a photon with spin $J = 1$ the amplitude for scattering over an angle θ is then given by the rotation matrices[1].

$$d^j_{m'm}(\theta) \equiv \left\langle jm' | e^{-i\theta J_y} | jm \right\rangle$$

where the y-axis is perpendicular to the interaction plane.

In the example we have $j = 1$ and $m, m' = \pm 1$

$$\begin{aligned} d^1_{1\,1}(\theta) &= d^1_{-1-1}(\theta) = \frac{1}{2}(1+\cos\theta) \\ d^1_{1-1}(\theta) &= d^1_{-1\,1}(\theta) = \frac{1}{2}(1-\cos\theta) \end{aligned}$$

From this we can see that:

$$\begin{aligned} \frac{d\sigma}{d\Omega}\left(e^-_L e^+_R \to \mu^-_L \mu^+_R\right) &= \frac{\alpha^2}{4s}(1+\cos\theta)^2 = \frac{d\sigma}{d\Omega}\left(e^-_R e^+_L \to \mu^-_R \mu^+_L\right) \\ \frac{d\sigma}{d\Omega}\left(e^-_L e^+_R \to \mu^-_R \mu^+_L\right) &= \frac{\alpha^2}{4s}(1-\cos\theta)^2 = \frac{d\sigma}{d\Omega}\left(e^-_R e^+_L \to \mu^-_L \mu^+_R\right) \end{aligned}$$

Indeed the unpolarised cross section is obtained as the spin-averaged sum over the allowed helicity combinations (*see lecture 8*): $\frac{1}{4} \cdot [(1) + (2) + (3) + (4)] =$

$$\frac{d\sigma}{d\Omega}^{\text{unpol}} = \frac{1}{4}\,\frac{\alpha^2}{4s}\,2\,\left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right] = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \quad .$$

[1]See H&M§2.2:

$$e^{-i\theta J_2}\,|j\;m\rangle = \sum_{m'} d^j_{m\;m'}(\theta)\,|j\;m'\rangle$$

and also appendix H in Burcham & Jobes

Now we go back to the γ, Z scattering. We have the individual contributions of the helicity states, so let us compare the expressions for the matrix-elements $\mathcal{M}_\gamma$ and $\mathcal{M}_Z$:

$$\begin{aligned}\mathcal{M}_\gamma &= -\frac{e^2}{s}\left[\left(\overline{\psi_{Lm}}\,\gamma^\mu\,\psi_{Lm}\right)+\left(\overline{\psi_{Rm}}\,\gamma^\mu\,\psi_{Rm}\right)\right]\cdot\left[\left(\overline{\psi_{Le}}\,\gamma_\mu\,\psi_{Le}\right)+\left(\overline{\psi_{Re}}\,\gamma_\mu\,\psi_{Re}\right)\right]\\ \mathcal{M}_Z &= -\frac{g^2}{4\cos^2\theta_w}\,\frac{1}{s-M_Z^2}\left[C_L^m\left(\overline{\psi_{Lm}}\,\gamma^\mu\,\psi_{Lm}\right)+C_R^m\left(\overline{\psi_{Rm}}\,\gamma^\mu\,\psi_{Rm}\right)\right]\\ &\qquad\cdot\left[C_L^e\left(\overline{\psi_{Le}}\,\gamma_\mu\,\psi_{Le}\right)+C_R^e\left(\overline{\psi_{Re}}\,\gamma_\mu\,\psi_{Re}\right)\right]\end{aligned}$$

At this point we follow the notation of Halzen and Martin and introduce:
$e^-_{L,R}(p) \equiv \psi_{L,R_e}(p)$, $e^+_{L,R}(p) \equiv \psi_{R,L_e}(-p)$, $\mu^-_{L,R}(p) \equiv \psi_{L,R_m}(p)$, $\mu^+_{L,R}(p) \equiv \psi_{R,L_m}(-p)$.
Since the helicity processes do not interfere, we can see (Exercise 40 (a)) that:

$$\begin{aligned}\frac{d\sigma}{d\Omega}_{\gamma,Z}\left(e^-_Le^+_R\to\mu^-_L\mu^+_R\right) &= \frac{\alpha^2}{4s}\,(1+\cos\theta)^2\cdot|1+r\,C_L^mC_L^e|^2\\ \frac{d\sigma}{d\Omega}_{\gamma,Z}\left(e^-_Le^+_R\to\mu^-_R\mu^+_L\right) &= \frac{\alpha^2}{4s}\,(1-\cos\theta)^2\cdot|1+r\,C_R^mC_L^e|^2\end{aligned}$$

with:

$$r=\frac{g^2}{e^2}\,\frac{1}{4\cos^2\theta_w}\,\frac{s}{s-M_z^2}=\frac{\sqrt{2}G_FM_Z^2}{e^2}\,\frac{s}{s-M_Z^2}\quad.$$

where we used that:

$$\frac{G_F}{\sqrt{2}}=\frac{g^2}{8M_W^2}=\frac{g^2}{8M_Z^2\cos^2\theta_w}\quad.$$

Similar expressions hold for the other two helicity configurations.

We note that there is a strange behaviour in the expression of the cross section of the Z-propagator. When $\sqrt{s}\to M_Z$ the cross section becomes ∞. In reality this does not happen (that would be unitarity violation) due to the fact that the Z-particle itself decays and has an intrinsic decay width Γ_Z. This means that the cross section has a Breit Wigner resonance shape. We are not going to derive it, but refer to the literature: e.g. Perkins[2].

Alternatively, a simple argument followed by H&M §2.10 goes as follows: The wave function for a non-stable massive particle state is:

$$\begin{aligned}|\psi(t)|^2 &= |\psi(0)|^2\,e^{-\Gamma t} &&\text{with }\Gamma\text{ the lifetime.}\\ \psi(t) &\sim e^{-iMt}\,e^{-\frac{\Gamma}{2}t} &&\text{with }M\text{ the mass.}\end{aligned}$$

[2]Perkins: Introduction to high energy Physics 3^{rd} ed. §4.8.

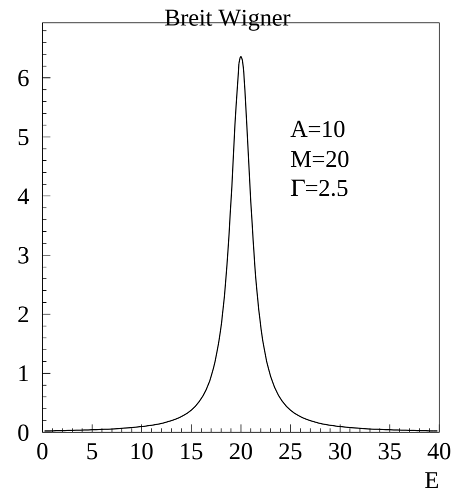
As function of the energy of the e^+e^- pair the state is described by the Fourier transform:

$$\chi(E) = \int \psi(t) e^{iEt} dt \sim \frac{1}{E - M + (i\Gamma/2)} \quad .$$

Such that experimentally we would observe:

$$|\chi(E)|^2 = \frac{A}{(E-M)^2 + (\Gamma/2)^2} \quad ,$$

the so-called Breit-Wigner resonance shape.



In the propagator for the z-boson we replace:

$$\frac{1}{s - M_Z^2} \to \frac{1}{s - \left(M_Z - i\frac{\Gamma_Z}{2}\right)^2} = \frac{1}{s - \left(M_Z^2 - \frac{\Gamma_Z^2}{4}\right) + iM_Z\Gamma_Z}$$

We observe two changes:

1. The maximum of the distribution shifts from $M_Z^2 \to M_Z^2 - \frac{\Gamma_Z^2}{4}$.
2. The expression will be finite because of the term $\propto M_Z\Gamma_Z$

For our expressions in the process $e^-e^+ \to \gamma, Z \to \mu^-\mu^+$ it means that we only replace:

$$r = \frac{\sqrt{2}G_F M_Z^2}{e^2} \cdot \frac{s}{s - M_Z^2} \quad \text{by} \quad r = \frac{\sqrt{2}G_F M_Z^2}{e^2} \cdot \frac{s}{s - \left(M_Z - i\frac{\Gamma_Z}{2}\right)^2}$$

The total unpolarized cross section finally becomes the average over the four L, R helicity combinations. Inserting "lepton universality" $C_L^e = C_L^\mu$; $C_R^e = C_R^\mu$ and therefore also: $C_V^e = C_V^\mu$; $C_A^e = C_A^\mu$, the expression becomes (by writing it out):

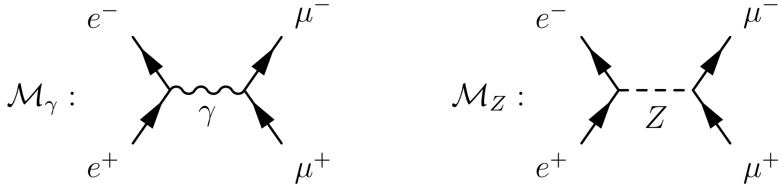
$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{4s}\left[A_0\left(1+\cos^2\theta\right) + A_1\,(\cos\theta)\right] \\ \text{with} \quad & A_0 = 1 + 2\,Re(r)\,C_V^2 + |r|^2\left(C_V^2 + C_A^2\right)^2 \\ & A_1 = 4\,Re(r)\,C_A^2 + 8|r|^2 C_V^2 C_A^2 \end{aligned}$$

In the Standard Model we have: $C_A = -\frac{1}{2}$ and $C_V = -\frac{1}{2} + 2\sin^2\theta$.

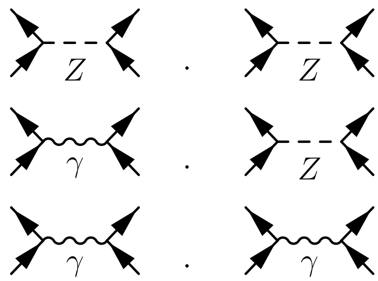
The general expression for $e^-e^+ \to \gamma, Z \to \mu^-\mu^+$ is (assuming seperate couplings for initial and final state):

$$\begin{aligned} A_0 &= 1 + 2Re(r)\,C_V^e C_V^f + |r|^2\left({C_V^e}^2 + {C_A^e}^2\right)\left({C_V^f}^2 + {C_A^f}^2\right) \\ A_1 &= 4Re(r)\,C_A^e C_A^f + 8|r|^2\,C_V^e C_V^f C_A^e C_A^f \end{aligned}$$

To summarize, on the *amplitude level* there are two diagrams that contribute:



Introducing the following notation:



$$\frac{d\sigma}{d\Omega}[Z,Z] = [Z] \cdot [Z] \propto |r|^2$$

$$\frac{d\sigma}{d\Omega}[\gamma Z] = [\gamma] \cdot [Z] \propto Re\,(r)$$

$$\frac{d\sigma}{d\Omega}[\gamma,\gamma] = [\gamma] \cdot [\gamma] \propto 1$$

Explicitly, the expression is:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}[\gamma,\gamma] + \frac{d\sigma}{d\Omega}[Z,Z] + \frac{d\sigma}{d\Omega}[\gamma,Z]$$

with
$$\frac{d\sigma}{d\Omega}[\gamma,\gamma] = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$

$$\frac{d\sigma}{d\Omega}[Z,Z] = \frac{\alpha^2}{4s}|r|^2\left[\left(C_V^{e\,2}+C_A^{e\,2}\right)\left(C_V^{f\,2}+C_A^{f\,2}\right)\left(1+\cos^2\theta\right)+8C_V^eC_V^fC_A^eC_A^f\cos\theta\right]$$

$$\frac{d\sigma}{d\Omega}[\gamma,Z] = \frac{\alpha^2}{4s}Re|r|\left[C_V^eC_V^f\left(1+\cos^2\theta\right)+2C_A^eC_A^f\cos\theta\right]$$

Let us take a look at the cross section close to the peak of the distribution:

$$r \propto \frac{s}{s-\left(M_z - i\frac{\Gamma_Z}{2}\right)^2} = \frac{s}{s-\left(M_z^2-\frac{\Gamma_Z^2}{4}\right)+i\mathcal{M}_Z\Gamma_Z}$$

The peak is located at $s_0 = M_Z^2 - \frac{\Gamma_Z^2}{4}$.
In Exercise 40 (b) we show that:

$$Re(r) = \left(1-\frac{s_0}{s}\right)|r|^2 \quad \text{with} \quad |r|^2 = \frac{s^2}{\left(s-\left(M_Z^2-\frac{\Gamma_Z^2}{4}\right)\right)^2+M_Z^2\Gamma_Z^2}$$

This shows that the interference term is 0 at the peak.

In that case (i.e. at the peak) we have for the cross section terms:

$$A_0 = 1+|r|^2\left(C_V^{e\,2}+C_A^{e\,2}\right)\left(C_V^{f\,2}+C_A^{f\,2}\right)$$

$$A_1 = 8|r|^2\left(C_V^e\,C_A^e\,C_V^f\,C_A^f\right)$$

The total cross section (integrated over $d\Omega$) is then:

$$\sigma(s) = \frac{G_F^2 M_Z^4}{\left(s - \left(M_Z^2 - \frac{\Gamma_Z^2}{4}\right)\right)^2 + M_Z^2\Gamma_Z^2} \cdot \frac{s}{6\pi} \left(C_V^{e\,2} + C_A^{e\,2}\right) \left(C_V^{f\,2} + C_A^{f\,2}\right) \quad .$$

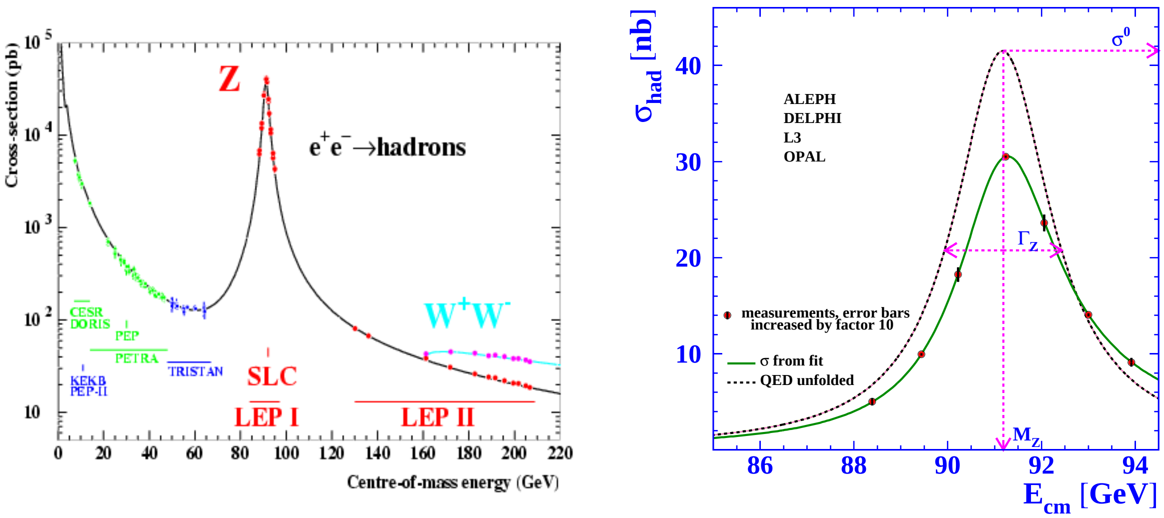
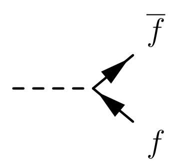


Figure 12.4: *left:* The Z-lineshape as a function of $\sqrt{s}$. *right:* The Lineshape parameters for the lowest order calculations and including higher order corrections.

12.2 Decay Widths

We can also calculate the decay width:

$$\Gamma\left(Z \to f\overline{f}\right)$$



which is according Fermi's golden rule:

$$\begin{aligned} \Gamma\left(Z \to f\overline{f}\right) &= \frac{1}{16\pi}\,\frac{1}{M_Z}\,\left|\overline{\mathcal{M}}\right|^2 \\ &= \frac{g^2}{48\pi}\,\frac{M_z}{\cos^2\theta_w}\,\left(C_V^{f\,2} + C_A^{f\,2}\right) \\ &= \frac{G_F}{6\sqrt{2}}\,\frac{M_Z^3}{\pi}\,\left(C_V^{f\,2} + C_A^{f\,2}\right) \end{aligned}$$

Using this expression for $\Gamma_e \equiv \Gamma(Z \to e^+e^-)$ and $\Gamma_f \equiv \Gamma(Z \to f\overline{f})$ we can re-write:

$$\sigma(s) = \frac{12\pi}{M_Z^2} \cdot \frac{s}{\left(s - \left(M_Z^2 - \frac{\Gamma_Z^2}{4}\right)\right)^2 + M_Z^2\Gamma_Z^2} \cdot \Gamma_e\Gamma_f \quad .$$

Close to the peak we then find:

$$\sigma_{peak} \approx \frac{12\pi}{M_Z^2}\frac{\Gamma_e\Gamma_f}{\Gamma_Z^2} = \frac{12\pi}{M_Z^2}\, BR(Z \to ee) \cdot BR(Z \to ff)$$

Let us now finally consider the case when $f = q$ (a quark). Due to the fact that quarks can be produced in 3 color-states the decay width is:

$$\Gamma(Z \to \overline{q}q) = \frac{G_F}{6\sqrt{2}}\frac{M_Z^3}{\pi}\left({C_V^f}^2 + {C_A^f}^2\right) \cdot N_C$$

with the colorfactor $N_C = 3$. The ratio between the hadronic and leptonic width: $R_l = \Gamma_{had}/\Gamma_{lep}$ can be defined. This ratio can be used to test the consistency of the standard model by comparing the calculated value with the observed one.

12.3 Forward Backward Asymmetry

The forward-backward asymmetry can be defined using the polar angle distribution. At the peak and ignoring the pure photon exchange:

$$\frac{d\sigma}{d\cos\theta} \propto 1 + \cos^2\theta + \frac{8}{3}A_{FB}\cos\theta$$

This defines the forward-backward asymmetry with:

$$A_{FB}^{0,f} = \frac{3}{4}A_e A_f \quad \text{where} \quad A_f = \frac{2C_V^f C_A^f}{C_V^2 + C_A^2}$$

The precise measurements of the forward-backward asymmetry can be used to determine the couplings C_V and C_A.

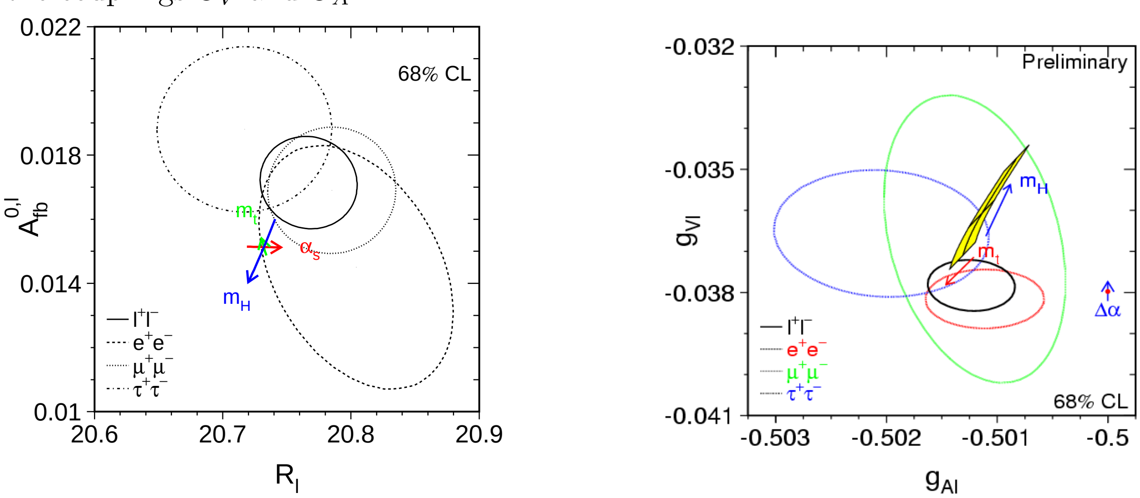


Figure 12.5: *left:* Test of lepton-universality. The leptonic A_{fb} vs. R_l. The contours show the measurements while the arrows show the dependency on Standard Model parameters. *right:* Determination of the vector and axial vector couplings.

12.4 The Number of Light Neutrino Generations

Since the total decay width of the Z must be equal to the sum of all partial widths the following relation holds:

$$\Gamma_Z = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + 3\Gamma_{uu} + 3\Gamma_{dd} + 3\Gamma_{ss} + 3\Gamma_{cc} + 3\Gamma_{bb} + N_\nu \cdot \Gamma_{\nu\nu}$$

From a scan of the Z-cross section as function of the center of mass energy we find:

$$\begin{array}{lcllll}
\Gamma_Z & \approx & 2490 \text{ MeV} & & \\
\Gamma_{ee} & \approx & \Gamma_{\mu\mu} \approx \Gamma_{\tau\tau} = 84 \text{ MeV} & C_V \approx 0 & C_A = -\frac{1}{2} \\
\Gamma_{\nu\nu} & = & 167 \text{ MeV} & C_V = \frac{1}{2} & C_A = \frac{1}{2} \\
\Gamma_{uu} & \approx & \Gamma_{cc} = 276 \text{ MeV} & C_V \approx 0.19 & C_A = \frac{1}{2} \\
\Gamma_{dd} & \approx & \Gamma_{ss} \approx \Gamma_{bb} = 360 \text{ MeV} & C_V \approx -0.35 & C_A = -\frac{1}{2}
\end{array}$$

(Of course $\Gamma_{tt} = 0$ since the top quark is heavier than the Z.)

$$N_\nu = \frac{\Gamma_Z - 3\Gamma_l - \Gamma_{had}}{\Gamma_{\nu\nu}} = 2.984 \pm 0.008 \quad .$$

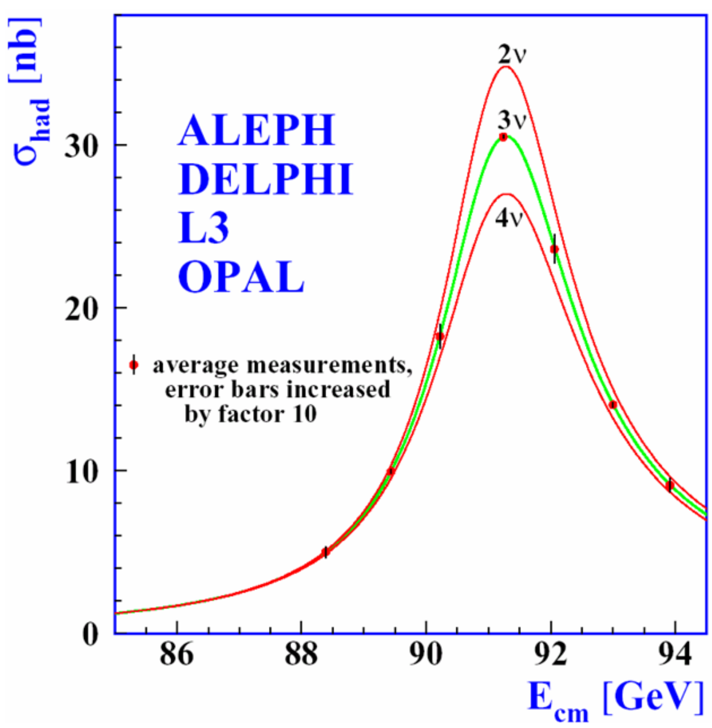


Figure 12.6: The Z-lineshape for resp. $N_\nu = 2,\ 3,\ 4$.

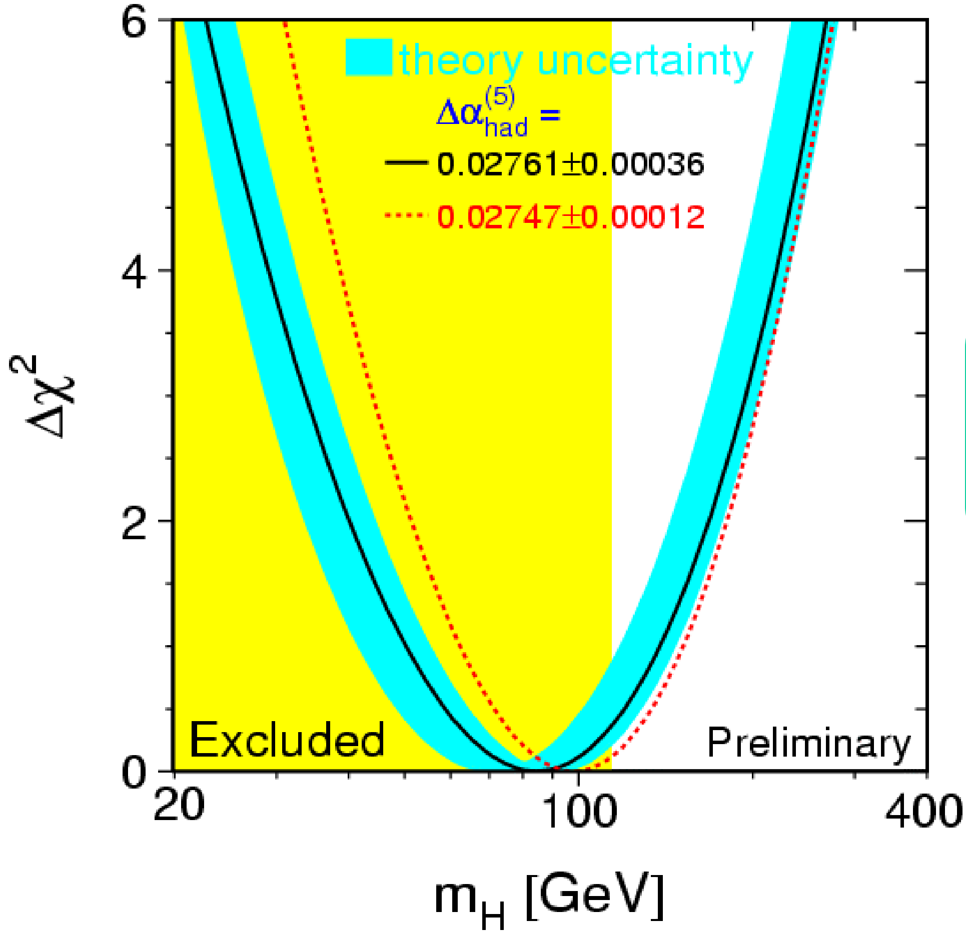


Figure 12.7: Standard Model fit of the predicted value of the Higgs boson.

Exercise 40:

(a) *Show how the unpolarised cross section formula for the process $e^+e^- \to Z, \gamma \to \mu^+\mu^-$ can be obtained from the expression of the helicity cross sections in the lecture:*

$$\frac{d\sigma}{d\Omega}\left(e^-_{L/R}e^+_{R/L} \to \mu^-_{L/R}\mu^+_{R/L}\right) = \frac{\alpha^2}{4s}(1 \pm \cos\theta)^2 \left|1 + r\, C^e_{L/R}C^\mu_{L/R}\right|^2$$

(b) *Show, using the expression of r from the lecture, that close to the peak of the Z-lineshape the expression*

$$Re(r) = \left(1 - \frac{s_0}{s}\right)\, |r|^2$$

with $s_0 = M_z^2 - \Gamma_z^2/4$ holds.

(c) *Show also that at the peak:*

$$\sigma_{peak} \approx \frac{12\pi}{M_z^2}\frac{\Gamma_e\Gamma_\mu}{\Gamma_Z^2}$$

(d) *Calculate the relative contribution of the Z-exchange and the γ exchange to the cross section at the Z peak.*
Use $\sin^2\theta_W = 0.23$, $M_z = 91\, GeV$ and $\Gamma_Z = 2.5\, GeV$.

(e) *The actual line shape of the Z-boson is not a pure Breit Wigner, but it is asymmetrical: at the high $\sqrt{s}$ side of the peak the cross section is higher then expected from the formula derived in the lectures.*
Can you think of a reason why this would be the case?

(f) *The number of light neutrino generations is determined from the "invisible width" of the Z-boson as follows:*

$$N_\nu = \frac{\Gamma_Z - 3\Gamma_l - \Gamma_{had}}{\Gamma_\nu}$$

Can you think of another way to determine the decay rate of $Z \to \nu\bar{\nu}$ directly?
Do you think this method is more precise or less precise?

Appendix A

Variational Calculus

This appendix is a short reminder of variational calculus leading to the Euler Lagrange equations of motion. Let us assume that we have a cartesian coordinate system with coordinates x and y, and consider the distance between an initial position (x_0, y_0) and a final position (x_1, y_1). We ask the simple question: "What is the shortest path between the two points in this space?"

Assume that the path of the particle can be represented as $y = f(x) = y(x)$. So $y(x_0) = y_0$ and $y(x_1) = y_1$.

Consider now the distance dl of two infinitesimal close points:

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1 + y'^2}dx$$

with $y' = dy/dx$.

The total length from (x_0, y_0) to (x_1, y_1) is:

$$l = \int_{x_0}^{x_1} dl = \int_{x_0}^{x_1} \sqrt{1 + y'^2}dx \quad .$$

The problem is to find the function $y(x)$ for which the l is minimal. The variational principle states that for the shortest path this integral should be stationary for possible different paths; i.e. for different functions of $y(x)$.

To find the solution we shall look at a more general case. Assume that the path length is given by the integral:

$$I = \int_{x_0}^{x_1} f(y, y')dx \quad .$$

In the above example we have $f(y, y') = f(y') = \sqrt{1 + y'^2}$.

According to the variational principle the physics path is obtained via $\delta I = 0$. First we consider the infinitesimal change

$$\delta f = \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial y'}\,\delta y'$$

where $\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$.
So we find:

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)$$

and the variation of the integral is:

$$\delta I = \int_{x_0}^{x_1}\left[\underbrace{\frac{\partial f}{\partial y}\delta y}_{(1)} + \underbrace{\frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)}_{(2)}\right] dx \quad .$$

The 2-nd term can be integrated in parts:

$$(2) = -\int_{x_0}^{x_1}\frac{d}{dx}\frac{\partial f}{\partial y'}\,\delta y\, dx + \underbrace{\left[\frac{\partial f}{\partial y'}\,\delta y\right]_{x_0}^{x_1}}_{=0}$$

The second term is zero due to the boundary conditions (the initial and final point are fixed: $\delta y = 0$.)
Therefor a the stationary path requires:

$$\delta I = \int_{x_0}^{x_1}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y(x)\, dx = 0 \quad .$$

This is obtained when the integrand is 0, or:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

For the straight line example above we had $f(y') = \sqrt{1 + y'^2}$, such that $\partial f/\partial y = 0$ and $\partial f/\partial y' = y'/\sqrt{1 + y'^2}$. So the variational principle states that:

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

or that y' is a constant ($dy/dx = a$) and the solution is therefore: $y = ax + b$.

In mechanics involving conservative forces we apply the stationary action principle to the Lagrangian function ($\mathcal{L}$), which depends on the generalized coordinates $(q_i, \dot{q}_i)$:

$$\mathcal{L}(q_i, \dot{q}_i) = T - V$$

such that we write for the equation of motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \quad .$$

Hamilton's principle states that the action integral

$$I = \int_{t_0}^{t_1}\mathcal{L}(q_i, \dot{q}_i)\; dt$$

is stationary: $\delta I = 0$.

Appendix B

Some Properties of Dirac Matrices α_i and β

This appendix lists some properties of the operators α_i and β in the Dirac Hamiltonian:

$$E\psi = i\frac{\partial}{\partial t}\psi = \left(-i\vec{\alpha} \cdot \vec{\nabla} + \beta m\right)\psi$$

1. α_i and β are Hermitean.
   They have real eigenvalues because the operators E and $\vec{p}$ are Hermitean. (Think of a plane wave equation: $\psi = Ne^{-ip_\mu x^\mu}$.)

2. $Tr(\alpha_i) = Tr(\beta) = 0$.
   Since $\alpha_i\beta = -\beta\alpha_i$, we have also: $\alpha_i\beta^2 = -\beta\alpha_i\beta$. Since $\beta^2 = 1$, this implies: $\alpha_i = -\beta\alpha_i\beta$ and therefore $Tr(\alpha_i) = -Tr(\beta\alpha_i\beta) = -Tr(\alpha_i\beta^2) = -Tr(\alpha_i)$, where we used that $Tr(A \cdot B) = Tr(B \cdot A)$.

3. The eigenvalues of α_i and β are ± 1.
   To find the eigenvalues bring α_i, β to diagonal form and since $(\alpha_i)^2 = 1$, the square of the diagonal elements are 1. Therefore the eigenvalues are ± 1. The same is true for β.

4. The dimension of α_i and β matrices is even.
   The $Tr(\alpha_i) = 0$. Make α_i diagonal with a unitary rotation: $U\alpha_i U^{-1}$. Then, using again $Tr(AB) = Tr(BA)$, we find: $Tr(U\alpha_i U^{-1}) = Tr(\alpha_i U^{-1}U) = Tr(\alpha_i)$. Since $U\alpha_i U^{-1}$ has only $+1$ and -1 on the diagonal (see 3.) we have: $Tr(U\alpha_i U^{-1}) = j(+1) + (n-j)(-1) = 0$. Therefore $j = n - j$ or $n = 2j$. In other words: n is even.