PHYSICS 150

Michel van Veenendaal

Contents

1
Classical Mechanics

The major foundations for classical mechanics were made by Isaac Newton (1643-1727). Newton was born on Christmas day, the fourth of January 1643. Newton's birth date directly demonstrates the importance of planetary motion and the need of accurate measurements in physics. Obviously, January 4 is not really Christmas day. However, England was still adhering to the Julian calendar introduced by Julius Caesar in 46 BC. In this system, it is assumed that a year lasts 365 and a quarter days (hence, the leap year every fourth year). This is pretty accurate, but unfortunately the real year is actually 11 minutes shorter. This was rectified in 1582 by the introduction by pose Gregory XIII of the Gregorian calendar and followed in most of catholic Europe. Unfortunately, the wisdom of the Gregorian calendar was not directly followed in protestant nations, such as the British Empire (including their American colonies). Although 11 minutes is not a lot, it does add up over the centuries. By the time it was finally introduced in England in 1752, 11 days needed to be corrected (and people complain when loosing an hour when daylight saving time starts). Isaac Newton was born in the small village of Woolsthorpe the only son of a waelthy local farmer, also named Isaac Newton, who died three months before he was born.

In 1661, at the age of seventeen Trinity College at Cambridge persuaded Isaac's mother to have him enter the university. In 1665, he derived a generalized binomial theorem. In 1665, the Great Plague broke out. This was one of the several recurrences of the bubonic plague that occured in Europe since the Black Death in 1348-50 till about 1750. In London, the Great Plague killed 100,000 people. about 20% of the population. As a precaution Cambridge University was closed for a period of eighteen months and Newton had to return to Woolthorpe Manor. Although that might at first seem a big disadvantage, for Newton, being left to think for himself was very fruitful. It was during this 18-month break that he developed calculus, gained

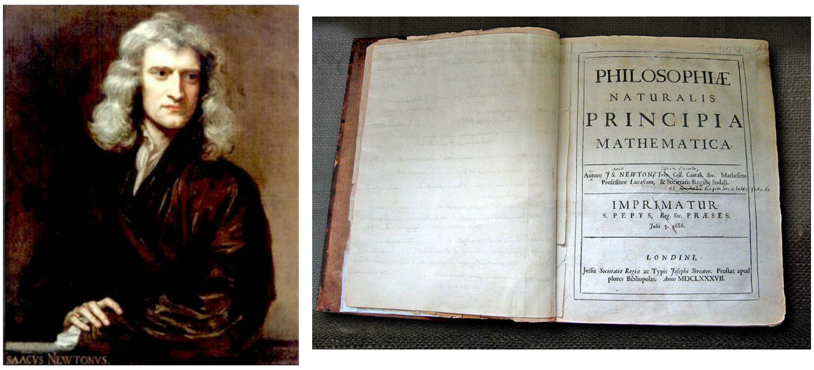

Figure 1.1 Sir Isaac Newton and a copy of his most famous work *PhilosophiæNaturalis Principia Mathematica*

significant insight into the laws of planetary motion, and did major work in optics. His theory of mechanics was published only two decades later in 1687 in *PhilosophiæNaturalis Principia Mathematica* published (*Mathematical Principles of Natural Philosophy*), often simply referred to as the Principia. In it, he laid out three major laws, which we discuss in the coming sections.

1.1 Newton's first law

Newton's first law reads

Lex I: Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

or in plain old English

Law I: Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed,

often abreviated to "a body in motion stays in motion." This phrase is very famous and might seem somewhat obvious. First of all, one might wonder why Newton even wrote it down. This was not really his invention, but already discovered by Galileo Galilei and others. Furthermore, there is no need to write it down, since it is already contained in Newton's second, to which we will come back later. It seems so simple, you might even think you could have thought of that yourself. However, just as many riddles and magic tricks seem so obvious after you are told their solution or how they work, figuring them out yourself is not that easy. To understand the revolutionary nature of the ideas of the scientific revolution in the sixteenth and seventeenth century, it is important to understand the background against which they were developed. You really have to empty your mind and try to imagine that you were living in those days. That you were educated ideas that are not correct, but seem reasonable. Now not only do you have to figure out that what you are being taught is incorrect, but you also have to propose a valid alternative. Newton's theory of motion was that alternative for almost two centuries, when it was realized that it had deficiencies when the speed approaches the speed of light and when looking at length scales of the order of the size of atoms. However, just because Newton's theory is not entirely correct does not make it wrong. It is still used for most applications that do not involve these limits (partially, because calculating things using relativistic quantum mechanics is horribly complicated).

1.1.1 Aristotle's natural philosophy

Whereas the scientific revolution had already started, many universities, including Cambridge, still clung to the ideas of Aristotle (384 BC-322 BC), see Fig. 1.2. Despite being born more than two millenia before Newton, the philosophical framework developed by Aristotle still loomed large in Western civilization. His writings incorporating the works of his teacher Plato and Socrates (Plato's teacher) were the first to create a comprehensive philosophy on subjects in the natural sciences, logics, politics, arts, etc. The adoption by the Roman Catholic church of many of the Aristotelian philosophies made it even harder to develop new ideas on mechanics.

Aristotle was born in 384 BC in the small city of Stagira in the Macedonian region of Greece. At age eighteen, he went to Athens to continue his education at Plato's academy. After spending time in Asia minor, Aristotle was requested by Philips II of Macedon to become the teacher of his son Alexander in 343 BC. Alexander (356 BC 323 BC) established one of the largest empires and became known as Alexander the Great. Aristotle also taught Ptolemy (367 BC 283 BC), a Macedonian general under Alexander the Great, (323 BC 283 BC) and Cassander (350 BC 297 BC) who became kings of Egypt and Macedonia, respectively after the split of the empire following Alexander's death.

Aristotle's works are an encyclopedia of the knowledge in ancient Greece.

Figure 1.2 Aristotle's concepts in natural philosophy dominated western civilization for two millenia. Aristotle was an important teacher who had Alexander the Great as his pupil.

Besides including the current though on a very wide variety of subject, Aristotle also made numerous original contributions. Aristotle's ambitions were therefore much larger than Newton's. His aim was to develop a natural philosophy. It was not is intention to develop a modern scientific theory that could be subjected to experimental verification. Physics or natural philosophy was more seen as a conceptual puzzle. A larger question that Aristotle dealt with, but modern physics not (at least not directly) is, for example, how an apple can grow into an apple tree. This led Aristotle to the concept of actual and potential forms. The apple has the actual form of an apple but the potential form of an apple tree. Related to that is the idea of "nature", which is the tendency to actualize its potential. The nature of an apple is to grow into an apple tree. Obviously, this does not need to happen because other things might prevent this actualization, but left to its own, the apple would become an apple tree. We can apply the same philosophy to motion. It is in a rock's nature to fall to the center of the earth. On the other hand, the nature of smoke directs it to go upward; it is in the nature of water to go back to the ocean. Aristotle believed that every object was made out of the four elements: earth, water, air, and fire. This qualitatively explains why a falling feather falls more slowly than a falling rock. Even though both are solid, the feather has more air inside it suppressing the nature of the earth part to fall to the earth (and helping birds to fly). The rock is mainly earth and therefore wants to be on the ground. Therefore, natural processes are internally goal-directed.

The nature does not explain everything. The reason a rock falls is because of its natural motion. However, a rock that is on the earth's surface is already in its natural place. Therefore, moving a rock sideways or upwards is not a natural motion. Aristotle called this forced motion. Now let us try to come up with an equation that describes this motion and let us only look at horizontal motion. Apparently, we need a force to keep something moving at a particular speed. We have not really defined force, but basically it says something about how hard we need to push to keep something moving. Obviously, this force should depend on the speed. We all know from our own experience that it is more difficult to move at a higher speed. In addition, it should probably depend on the nature of things. Moving in a medium that is close to your own nature is more difficult. It is easy to move in air, but more difficult in water, because air is more different from earth than water. On the other hand when you are stuck in the sand or when your feet are buried in concrete, it is very hard to move since you are already in your natural state and therefore you should remain at rest. Now if we were to put this in equation form, it would look something like

$$F = Rv, \tag{1.1}$$

where F is the force you need to apply to make an object move. The force increases if the speed v becomes larger. In addition there is a factor R that describes a resistance that depends on the difference in nature between the object that you want to move and the medium that it is moving in. For example, when you are stuck in sand, a solid object should be at rest and the resistance is large. On the other, it is easy to move something in air and the force should be less. Therefore, the resistance should be less. Mind you, the Greeks never wrote down an equation like this. But it is close to what they had in mind and modern notation makes life so much easier (although you might disagree), that it is convenient to use.

Great, you might say, so now we are learning physics of ancient Greeks? This equation must surely be incorrect. Actually it is not. This equation is Stokes' equation derived in 1851 by George Stokes, See Fig. 1.3. It describes the drag force on a (spherical) particle in a viscous fluid. The drag force is given by $F_d = Rv$. So the force that needs to be applied to maintain a constant motion is $F = F_d$ or $F = Rv$. Note that there is a different philosophy behind it. The Greeks felt it was the object resisting the motion. If the object is its proper nature, then it will resist a lot against the movement. The modern interpretation is that it is the medium in which the object is moving is resisting the motion. However, both arrive at the same equation (ok, the Greeks never wrote this equation down...). So if the equation is correct, then what is the problem? How can someone that has the right equation be wrong?

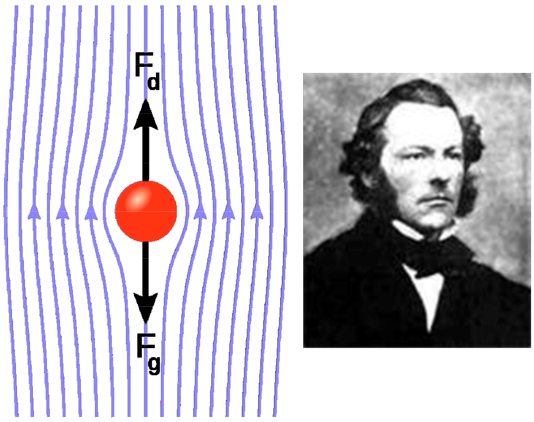

Figure 1.3 Aristotle's laws of motion are under certain circumstances equal to Stokes' law from the nineteenth century. However, in many situations Aristotle's laws are simply incorrect and certainly not fundamental laws of physics.

The problems start when you take a correct equation and start interpreting it as a law which is applicable everywhere. And this is were you should be careful, not only in physics, but in every field of research. Just because it works sometimes does not necessarily make it correct and certainly does not make it a law. So why does it go wrong? Let us rewrite the equation somewhat

$$v = \frac{F}{R}. \tag{1.2}$$

This equation tells you the velocity v that the object will have when you apply a certain force F. Note the R is now in the denominator. R is the resistance that tells you how difficult it is to move something. For example, it is easy to push something forward in air (low R), it becomes more difficult in water (higher R), but when a solid object is buried under the ground, it is in its nature and therefore must resist a lot to the motion. However, let us see if we can decrease the resistance even more. Then it becomes even easier to make an object move at a high speed. But what happens if $R = 0$. Then any push, no matter how small, will make the object move at infinite speed. To the ancient Greeks' credit, they did think about this situation. They concluded that moving at infinite speed did not make any sense. However, they did not conclude that the equation was wrong. They concluded that a vacuum cannot exist. This is known by the statement *Horror vacui* (which is Latin nor Greek) or *Nature abhors a vacuum*. Then again, nobody had seen a vacuum in the first place. Aristotle's view of matter is that there is matter everywhere. The density can flucuate, but never go to zero, so that a vacuum never exists. Everytime that a vacuum might develop, matter flows in to prevent this from happening. The problem of the vacuum however remained problematic till the twentieth century. We will come back to this later.

Before we continue with the development of classical mechanics, one might wonder why a theory that, as we shall see, wrong could persist for close to two millenia. How can you ignore facts and measurements? First of all, measurements are quite often hard to produce. In particular, as was the case in ancient Greece, if you do not have proper measuring equipment such as a clock. And even when you have measurements, then can be hard to interpret. As we saw above, you might arrive at an equation that is correct, but leads to wrong results if you interpret it incorrectly. Furthermore, for the ancient Greeks having a consistent philosophy was more important than a theory that might interpret some messy experiments. This sounds at first surprising, but even nowadays people often dismiss facts when they are in conflict with their philosophy, their prejudices or their religious beliefs. Artistotle's philosophy had a certain consistency which the modern theory of classical mechanics does not directly have. His notion of the nature of objects, giving them an internal goal, explained why a rock falls to the earth, but, in addition, gives a natural explanation why an apple grows into an apple tree. On the other hand, gravitation is downright spooky. What is this force pulling objects? Gravitation? I do not see any gravitation. Furthermore, explaining why an apple grows into an apple tree is horribly complicated in modern science. Also, the idea that Earth was at the center at the universe was favored by Aristotle over other ideas by others that the Earth rotates around the Sun. It just makes a lot more sense that the Earth, and implicitly humans, are at the center.

However, the ideas that inanimate objects, such as a rock, have an internal purpose was not entirely satisfactory either. Already before Newton, René Descartes (1596-1650), see Fig. 1.4, started developing a new type of philosopy that draws clearer distinctions between the laws of physics and the intellectual and spiritual world. Descartes distinguished the mind ("thinking substance"), bodies ("extended substance"), and God ("infinite substance"). The laws of motion of physics only apply to bodies. The existence of the mind was demonstrated by Descartes by the fact that he could doubt his own existence: "Cogito ergo sum" or "I think, therefore I am". Although this

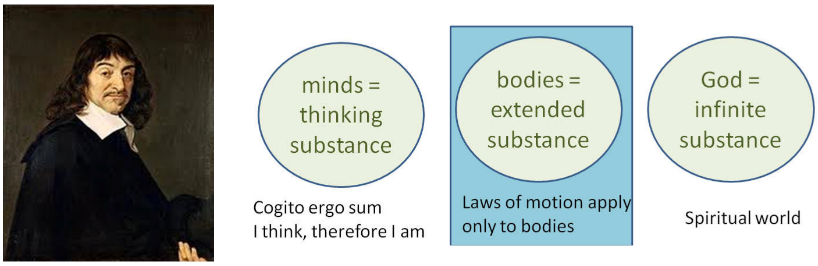

Figure 1.4 René Descartes developed a mechanical philosophy that separated the minds, the bodies, and the spiritual world. Laws of motion only apply to bodies and not to the thinking and spiritual substances.

division is rather nice, it also does pose some problems. Modern people are rather willing to believe that a rock has no internal purpose, but what about a mountain, the Earth, the universe? Or, if we start from very small, what about DNA, cells, a human? Many people think that some of these have an internal purpose. The tremendous growth of scientific thought leaves many with the impression that science is threatening the other two substances. Is life just a bunch of chemical reactions; are humans just DNA soup?

Still, the idea that facts can be ignored seems strange to people living in the twenty first century. You cannot just make stuff up? Let us consider some examples how poor measurements combined with arguments of a more philosophical nature can bring us further from a factual world. Let us take an American who is trying to explaing the world record long jump to a Dutch guy. The world record is 29 ft and 4.5 inches. This is 8.95 meters. However, suppose we are not entirely certain about our units and take roughly 3 feet in a meter. We then end up with $29/3$ m $+4.5\times2.5$ cm$=9.77$ meter. That is already quite a bit off. However, now let us assume that the person who is trying to explain that same distance has no clue how much a meter is. He therefore tries to explain it in a more graphic way by jumping over people lying in a straight line on the ground. This would give roughly a distance of five persons and a head. However, let us just round that off to six persons (we do not want to sell the record holder short) **?**. Now let us take another step. The person explaining this is really impressed by the long jumper. The record has stood since 1991, which is incredible. In addition, the record holder is American, so we definitely do not want that Dutch man to have a bad impression of this record. So let us add another person for good measure. The Dutchman now calculates it back into his units, adding a little bit more along the way since the average Dutch person is taller than

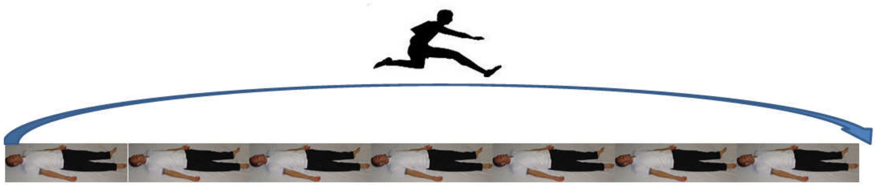

Figure 1.5 Although we are used to accuracy in scientific works, this was not always taken for granted. Often facts and numbers were adjusted to convey other messages.

the average American. We now end up with 12.65 meter or 41 feet and 6.25 inches. Sure, you might say. This is ridiculous. You cannot just add a person. This sounds more like a fish tale than science. We can call it a fish tale. A bit more formally we can call it a literary device to create respect. The story is now no longer scientifically correct. However, in some ways it is still try, because the person holding the world record is still the greatest long jumper.

Okay, apart from fishermen, who would ever do such a thing? Let us look at another example. The oldest known (documented) human being was Jeanne Louise Calment who lived from 21 February 1875 to 4 August 1997, i.e. 122 years and 164 days. She lived in Arles and met the Dutch painter Vincent van Gogh in 1888. However, person living longer have been mentioned. For example, the first emperor of Japan lived from 711 BC to 585 BC to the ripe age of 126 years. However, that pales in comparison with Kay Kavus, a shah of Persia who lived 500 years. You might easily discard this as mythology or legend. However, what about Methuselah who lived 969 years as is described in Genesis 5:1-32 of the Bible. Now for many people this is an area where facts and religious beliefs start to conflict with each other. We can interpret this as a literary device. The Bible is describing events long before the Bible was written. These were the forefathers that required a lot of respect. Since noone had ever met them, it made sense that they lived longer since everything was better in the time of our forefathers. Judging by the example from Japan and Persia this was apparently a common literary device in ancient texts. Some Sumerian kings were even said to have lived more than 10,000 years. However, many people grown up after the scientific revolution are very uncomfortable doing this. If it is not a fact, then it must be untrue. Several explanations to account for this longevity were given. A reason good by that the conditions were ideal before the Flood described in the Bible. Another theory states that the longevity genes were lost in the Flood. Lifespans decreased significantly after Noah (who died at 950).

Of course, some might interject that the Bible must be factually correct because it comes from divine inspiration. Take for example the narrative of the Flood: "God plans to destroy humans because they have gone astray. God warns Noah to build a boat and use pitch Every species of animal [· · ·] are to be saved [· · ·]. The flood comes and destroys all life on the Earth. The waters subside slowly and Noah sends out a dove, [· · ·] and raven. The boat comes to rest on the top of Mt. Ararat." Clearly this implies that a flood must have happened. However, this text is not from the Bible. It comes the flood epic in Gilgamesh with some minor adaptation: "The gods' plan to destroy humands because they have gone astray. The god Ea warns Utnapishtim to build a boat and use pitch. Every species of animal and of craftsmen are to be saved, as well as their family. The flood comes and destroys all life on the Earth. The waters subside slowly and Utnapsihtim sends out a dove, swallow and raven. The boat comes to rest on the top of Mt. Nisir." Now the epic of Gilgamesh is significantly older than the Bible. Note that the Exodus takes place around 1290-1235BC during the reign of Rameses II. It can still be divinely inspired, but clearly the gods (plural!) that the Sumerian were talking about are clearly different from the one in the Bible. Obviously, this still does not mean it is incorrect. People have wondered why there are flood stories. One of the reasons could be that fossils of sea animals are found on mountain tops. Modern science ascribes this to plate tectonics, but to the ancient civilizations the presence of these fossils must have been very confusing.

Related to the genealogy, is the age of the Earth. The age has been estimated from adding the ages of the forefathers to be around 6,000 years. This is certainly different from the age estimated by modern scientist, which is around 4.5 billion years. Unfortunately such a young Earth/universe creates a different set of problems. Since the universe is so big, 6,000 years is an incredibly short time. Our own galaxy is estimated to measure about 100,000 light years across (a lightyear is the distance light travels in a year, therefore looking into the sky is looking into the past). So we would only be able to see a very small portion of our galaxy. New stars should be appearing every night as their light finally reaches Earth. One of the nearest galaxies is the Andromeda galaxy, which is 2.5 million light years away. If the universe was created 6,000 years ago, we would not be able to see it. Other galaxies are estimated to be more than 12 billion light years away. Of course, the universe could have been created 6,000 years ago with all the light rays and all. However, this makes the Creator also a big deceiver, which is not very satisfactory. Other explanations, invoking general relativity involve slowing down time during the creation allowing 14 billion years to pass by in six days.

Now the point here is not to say that the Bible is not correct, but merely to point out that the nice separation between the physical and the spiritual world that Descartes described does not entirely work for many people since it creates a conflict between them. However, Descartes might point out that the Bible deals with the spiritual world and should not be treated as a scientific document. Just because it is not 100% accurate on the scientific facts does not mean it cannot be true on the spiritual one. In addition, one might argue that for the people who wrote the Bible the whole geological and astronomical history was rather irrelevant. What for them was important is how their civilization came about. Ancient civilizations in the fertile crescent between the Eufrates and the Tigris started around 3,500 BC. Looking at it that way, 6,000 years is not a bad estimate at all. However, what is important for us to note is that separating scientific facts from other influences, such as philosophy, religion, and other biases is not easy at all. The ideas that developed in the sixteenth and early seventeenth centuries that the laws of nature independent of an underlying purpose (a mechanical philosophy) were ideas that were far less trivial than they might appear to us. And the notion that nature just goes about its business and that things develop at random without any underlying purpose is still something that is difficult to accept for many people.

While philosophers were busy undoing Aristotle's notion that objects have an internal purpose, others were looking into more detail at the laws of motion. The area where Aristotle's theory really becomes problematic is with projectiles. If a force is needed to maintain movement, then how do we explain the flight of an arrow? After the arrow leaves the bow, there is no longer an obvious force working on the arrow. Several rather convoluted ideas were developed trying to explain how the medium in which the arrow moves (air...) effectively pushes the arrow forward. However, none of them very satisfactory.

Other areas were also problematic. The Sun, the planets, and, in particular, the Moon are obviously massive objects. Yet they do not seem to follow their nature, that is falling back to Earth. Obviously, this was noticed by the Greeks, who offered a couple of solutions. Other ancient civilizations had suggested that the celestial objects were attached to a sphere called the firmament. Outside of that sphere, the Gods lived, breathing aether, the element of the Gods. An alternative solution, doing away with the firmament, is that the celestial objects move around in the aether. The aether is then a special element, where the usual laws of nature do not apply. Therefore, massive objects do not have to follow their nature and fall to Earth. This element is in addition to the usual four classical elements: earth, water, wind, and fire. In more modern terminology, these elements are the different phases of materials: solids, liquids, and gases. In addition, fire can be understood as combustion or chemical reactions transforming one material into another. Note, that fire/combustion connects for example solids with gases, when, for example, smoke escapes burning wood. Aether is therefore the fifth element necessary to keep the sky from falling on our heads. Note that in latin, fifth element is quitessence. These concepts were remarkably similar in many civilizations. For example, in Japanese philosophy, one encounters Chi (earth), Sui (water), Fu (wind), and Ka (fire). Surprisingly, they also recognize a fifth element Ku (usually translated as void or heaven) describing everything that cannot by the regular elements.

Of course, you might think that several centuries later this whole notion was gone. However, this was not the case. It might not have been entirely the same aether as the ancient Greeks were thinking about, but the problem lied in the propagation of light. It was generally assumed that light moved in a way similar to sound. Sound travels through air by making the molecules in the air vibrate. This is the reason why sound changes when the medium changes allowing you to talk like Donald Duck when you inhale helium. The underlying physics is that helium atoms are a lot lighter than the oxygen and nitrogen molecules that make up most of the air. The lighter molecules prefer to vibrate with a higher frequency than the heavier ones, making your voice higher. Light was generally assumed to propagate by making a medium vibrate. However, this medium cannot be air because light travels in outer space, but sound is impossible without air. This medium was known as the aether. It was not until 1887, that Albert Michelson and Edward Morley demonstrated that there is no such thing as an aether. Unfortunately, their experiment is rather complex and we therefore do not discuss it at this stage. Therefore, light, unlike sound, travels by itself. It vibrates on its own and does not require a medium.

The first different thought on intertia were developed in the fourteenth century by the French priest Jean Buridan (c1300-1358) perhaps following ideas by the Persian polymath Avicenna (c980-1037). Buridan's writing appears remarkable close to Newton's thoughts. Consider his words on throwing an object:

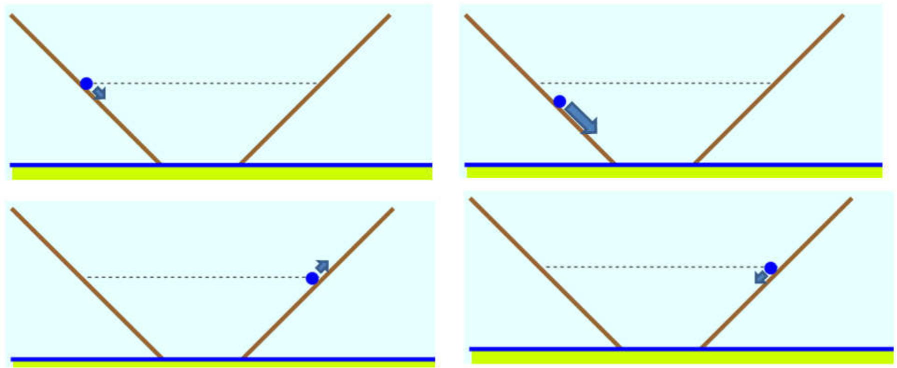

Figure 1.6 Galileo Galilei's experiment on motion: A ball released from a certain height on one leg will (approximately) reach the same height on the opposite leg.

...after leaving the arm of the thrower, the projectile would be moved by an impetus given to it by the thrower and would continue to be moved as long as the impetus remained stronger than the resistance, and would be of infinite duration were it not diminished and corrupted by a contrary force resisting it or by something inclining it to a contrary motion

So it is not the projectile that is loosing its impetus because there is no longer a force working on the projectile, but the projectile is loosing its impetus because there is a force working on it. Therefore, in the absence of an external force the projectile would simply keep moving.

Great strides in the understanding of motion were made by Galileo Galilei (1564-1642). Galilei made a concerted effort to understand nature by means of experiments (although some of these experiments were actually only thought experiments). The key to understanding motion in its "purest" form depends on the elimination of friction. This is easier said than done. One of his experiments involved rolling a ball down one leg and up another one, see Fig. 1.6. When the resistance is small, the ball will approximately reach the same height on the other leg. However, when the angle with the ground of the other leg is decreased, see Fig. 1.7, the ball still reaches the same height, but know it has to travel a greater distance. We can keep doing this until the second leg is lying on the ground. Galileo concluded that in this limit, the ball still wants to go back to the same height, but this implies it keeps travelling till infinity because it never reaches it. Therefore, unless acted upon by an external force, an object remain in motion. This is exactly Newton's first law.

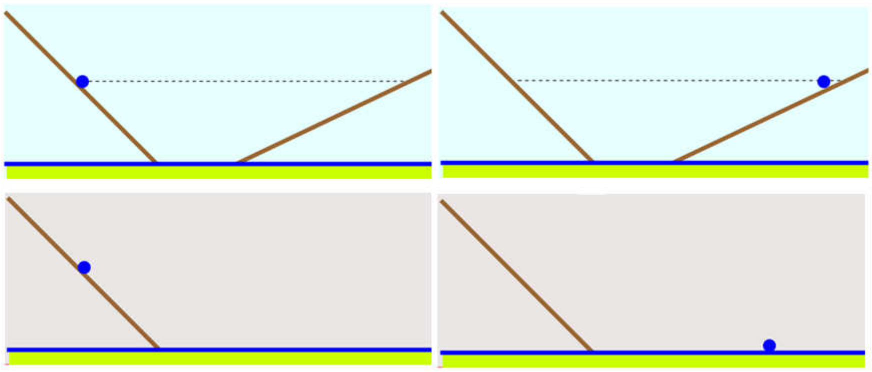

Figure 1.7 When one of the legs is lowered, the ball will still reach the same height, but now it has to travel a larger distance. Therefore, when the angle goes to zero, the ball will go on forever since it still wants to reach the same height, but it will never get there.

1.1.2 Constant velocity

From the velocity v , we can calculate the distance x travelled in time t from

$$x = vt. \tag{1.3}$$

For example,

$$v = 2 \text{ m/s} \quad \text{and} \quad t = 10 \text{ s} \quad \Rightarrow \quad x = vt = 2 \times 10 = 20 \text{ m}. \tag{1.4}$$

For $t = 0$ s, Equation (1.24) always has the object at $x = 0$. However, we do not always want that. A more general equation is

$$x = vt + x_0, \tag{1.5}$$

where the object is at $x = x_0$ at time $t = 0$. We can now solve some more complicated problems.

Suppose an object is at $x = 10$ km at $t = 0$. Where will the object be at $t = 4$ h later given an average velocity of 40 km/h.

Answer:

$$x = vt + x_0 = 40t + 10 \quad \text{km} \quad \Rightarrow \tag{1.6}$$

$$\text{at} \quad t = 4 \quad \text{h, we have} \quad x = 40 \times 4 + 10 \quad \text{km} = 170 \quad \text{km} \tag{1.7}$$

Using this equation, we can also tackle more interesting problems.

Two cars start driving towards each other. One car starts in a city 100 km away and drives 40 km/h. The other car drives at a speed of 60 km/h. If the cars drive towards each other when and where do they meet?

Answer:
We have two equations of motion (one for each car)

$$x_1 = v_1 t + x_{10} = 60t \ \ \text{km} \tag{1.8}$$

$$x_2 = v_2 t + x_{20} = -40t + 100 \ \ \text{km}. \tag{1.9}$$

The minus sign is very important. Not only have the cars different speeds, they are also driving in opposite directions. So far the motion of the two cars are separate problems. However, to find out when they meet, we have to combine the two equations. At the time when they meet (t_{meet}), they are in the same place, so their positions have to be equal.

$$x_1 = x_2 \quad \Rightarrow \quad 60t_{\text{meet}} = -40t_{\text{meet}} + 100 \ \ \text{km}. \tag{1.10}$$

We can rewrite this as

$$100t_{\text{meet}} = 100 \quad \Rightarrow \quad t_{\text{meet}} = 1 \ \ \text{h}. \tag{1.11}$$

Where are the cars at the time of the meeting? We can find this by inserting t_{meet} in one of the equations of motion

$$x_1 = 60t_{\text{meet}} = 60 \times 1 = 60 \ \ \text{km}. \tag{1.12}$$

Obviously, we find the same value if we insert it in the second equation of motion

$$x_2 = -40t_{\text{meet}} + 100 = -40 \times 1 + 100 = 60 \ \ \text{km}, \tag{1.13}$$

which is a good way to check your result.

Now what would have happened if we had messed up the sign? In that case, we solve a different problem. The cars are now driving in the same direction, but one car has a 100 km head start. We can also solve this problem

$$\begin{aligned} 60t_{\text{meet}} = 40t_{\text{meet}} + 100 \ \ \text{km} \quad &\Rightarrow \quad 20t_{\text{meet}} = 100 \\ &\Rightarrow \quad t_{\text{meet}} = 5 \ \ \text{h}. \end{aligned} \tag{1.14}$$

They now meet at

$$x = 60t_{\text{meet}} = 300 \ \ \text{km}. \tag{1.15}$$

Note that the meeting point now does not lie between the initial positions of the cars.

We can mess up the problems even more. What happens if we interchange the velocities? Now the equations of motion are

$$x_1 = v_1 t + x_{10} = 40t \ \text{ km} \tag{1.16}$$
$$x_2 = v_2 t + x_{20} = 60t + 100 \ \text{ km}, \tag{1.17}$$

where we take the cars driving in the same direction. When do they meet now?

$$x_1 = x_2 \quad \Rightarrow \quad 40t_{\text{meet}} = 60t_{\text{meet}} + 100 \ \text{ km}. \tag{1.18}$$

The solution is now

$$-20t_{\text{meet}} = 100 \quad \Rightarrow \quad t_{\text{meet}} = -5 \ \text{ h}. \tag{1.19}$$

This looks strange. How can they meet at a negative time? The answer is it cannot if the cars starting driving at $t = 0$. The car driving 40 km/h never overtakes the one driving at 60 km/h if that car has a head start. However, the answer $t_{\text{meet}} = -5$ h does have physical meaning. Suppose the cars did not start driving at $t = 0$, but were already driving for several hours then they were apparently at the same point five hours earlier. Therefore, the solution is the answer to the question:

If at time $t = 0$ two cars are 100 km apart. The car driving 60 km/h is ahead of the car driving 40 km/h. If the cars started at the same point, when did they start driving?
The answer is then 5 hours earlier.

1.2 Newton's second law

1.2.1 Constant acceleration

Newton's first law deals with objects in the absence of a force and states that they keep moving at a constant speed. Newton's second law describes what happens to the speed if an external force is applied. Before we can do this, we need to look at speed in more detail. Speed is the distance travelled divided by the time it takes, or in equation form

$$v = \frac{\Delta x}{\Delta t}, \tag{1.20}$$

where Δ indicates that we are talking about a difference. So $\Delta x = x_{\text{end}} - x_{\text{begin}}$ is the difference between the final and initial position. We can therefore also write it as

$$v = \frac{x_{\text{end}} - x_{\text{begin}}}{t_{\text{end}} - t_{\text{begin}}}, \tag{1.21}$$

Likewise $\Delta t = t_{\text{end}} - t_{\text{begin}}$ is the change in time needed to go from the begin point to the end point. For example, if you have traveled 50 miles at a constant speed for 1 hour, then the speed is 50 miles/h (say: miles per hour). Generally, in science we prefer to use the metric system. In that case, for example, running 1 kilometer for 10 min gives a speed of 1 km divided by 1/6 hour or 6 km/h. The most typical unit of expressing speed is meters per second or m/s.

Now we can also turn this the other way around

$$\Delta x = v\Delta t \tag{1.22}$$

or the distance travelled is equal to the velocity times the time travelled. Likewise, given the distance travelled and the speed, the time that it takes is given by

$$\Delta t = \frac{\Delta x}{v}. \tag{1.23}$$

Velocity is very close to speed, but in addition to the size the velocity also indicates the direction. Therefore cars driving in opposite directions on a road at the same speed do not have the same velocity since they drive in different directions. In three dimensions, velocity is a vector whose magnitude is equal to the speed. We will come back to this later.

Let us look at this by having an object move at a constant velocity. The position is then given by

$$x = vt, \tag{1.24}$$

where we take the position where the object is a $t = 0$ as the point with $x = 0$. Now let us check that Eq. (1.21) indeed gives us the velocity

$$v = \frac{x_{\text{end}} - x_{\text{begin}}}{t_{\text{end}} - t_{\text{begin}}} = \frac{v(t + \Delta t) - vt}{t + \Delta t - t} = \frac{v\Delta t}{\Delta t} = v, \tag{1.25}$$

which indeed does what it is supposed to do. Therefore, we can obtain the velocity from the change in position as a function of time. So position, time, and velocity are intimately related with each.

However, let us first make our lives a little bit more difficult and consider the situation where the speed is not constant. How do we determine the speed? Well, consider you are sitting in an accelerating car. However, the speedometer is changing all the time. Every time you look at it the speed is different. What can you do? It would be nice if we could freeze time, because then the speedometer would stand still and we would know exactly what the speed would be. However, Equation (1.20) shows that you can also determine the speed by dividing the change in distance by the change in time. However, no we are running into problems. If we freeze time again, then everything is standing still. So we are stuck with a paradox. First, we find that when we freeze time the velocity is constant and then we find that nothing is moving. To add to our difficulties, the time is not changing, so $\Delta t = 0$, which means we are dividing by zero which would give infinity.

These problems involving things that are infinitesimally small and problems involving infinity already bothered the ancient Greeks. Zeno of Elea (490-430BC) had a whole array of paradoxes related to them. As Aristotle put it "That which is in locomotion must arrive at the half-way stage before it arrives at the goal." Let us consider an example. Suppose we want to cross 1 meter at 1 meter per second. First, we need to reach the 1/2 meter point. This takes 1/2 second. Then we need reach the halfway point between 1/2 and 1. That takes 1/4 second. Reaching the next half-way point takes 1/8 second and so on. However, in the end there is an infinite number of half way points to reach. This requires an infinite number of tasks, which is impossible to accomplish and therefore the end is never reached. We can express the total time as

$$\Delta t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots . \qquad (1.26)$$

However, this paradox is not really a paradox at all. Let us multiply Δt by 1/2

$$\frac{1}{2}\Delta t = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots . \qquad (1.27)$$

However, the right-hand side is equal to $\Delta t - \frac{1}{2}$. Therefore,

$$\frac{1}{2}\Delta t = \Delta t - \frac{1}{2} \quad \Rightarrow \quad \frac{1}{2}\Delta t = \frac{1}{2} \quad \Rightarrow \quad \Delta t = 1 \ \text{ s.} \qquad (1.28)$$

This is of course exactly equal to the time it takes to cross 1 m at 1 m/s. Therefore, freezing time is apparently not a good idea and leads to weird results. Now let us consider what happens if we do not freeze the time, but only look a very small time later. Then the speedometer has barely changed, so the speed still looks constant, but the object has moved (albeit by a very small amount).

This dependence is known as parabolic. More generally, we can write

$$x = \frac{1}{2}at^2, \qquad (1.29)$$

where the presence of the $\frac{1}{2}$ will become apparent later. We also use the letter a for the constant and not v. We also return to that later. So at times $t = 0, 1, 2, 3, 4$, the object is at $\frac{1}{2}a, 2a, \frac{9}{2}a, 8a$, respectively. Now let us look at a particular time t and little bit later at $t + \Delta t$. We can calculate the velocity

$$v = \frac{x_{\mathrm{end}} - x_{\mathrm{begin}}}{t_{\mathrm{end}} - t_{\mathrm{begin}}} = \frac{\frac{1}{2}a(t+\Delta t)^2 - \frac{1}{2}at^2}{(t+\Delta t) - t}. \tag{1.30}$$

Now we can write

$$(t+\Delta t)^2 = (t+\Delta t)(t+\Delta t) = t^2 + 2t\Delta t + (\Delta t)^2. \tag{1.31}$$

Now we have to realize that when Δt is small, its square $(\Delta t)^2$ is even smaller. Let us take an example. If $\Delta t = 0.001$, then $(\Delta t)^2 = 0.000001$. The difference in size only increases when Δt becomes smaller. So at some point we can simply neglect the $(\Delta t)^2$ term and write

$$(t+\Delta t)^2 \cong t^2 + 2t\Delta t, \tag{1.32}$$

where the $\cong$ stands for approximately equal. Putting that back into the equation gives

$$v \cong \frac{\frac{1}{2}a(t^2 + 2t\Delta t) - \frac{1}{2}at^2}{\Delta t} = \frac{at\Delta t}{\Delta t} = at. \tag{1.33}$$

Therefore we have

$$x = \frac{1}{2}at^2 \qquad \Leftrightarrow \qquad v = at \tag{1.34}$$

Therefore, we can calculate the instantaneous velocity from the difference in position in a very small time interval and divide this by the time interval.

Now we notice that when the position changes as t^2, the speed is no longer constant but increases in time.

$$\frac{\Delta v}{\Delta t} = \frac{a\Delta t}{\Delta t} = a. \tag{1.35}$$

The quantity a is known as the acceleration, which in this case is a constant. Since $a = \Delta v/\Delta t$, its unit is m/s/s=m/s^2.

Example.– Given an acceleration of 5 m/s^2, what are the velocity and position at $t = 10$ s?

Using Eq. (1.34), we find

$$x = \frac{1}{2}at^2 = \frac{1}{2}5 \times 10^2 = 250 \text{ m}, \qquad v = at = 5 \times 10 = 50 \text{ m/s}. \tag{1.36}$$

Example.– If the acceleration is 4 $\mathrm{m/s^2}$, what is the velocity at $x = 50$ m?

This problem splits into two parts. First, we need to find the time that the object is a $x = 50$ m. This can be done via

$$x = \frac{1}{2}at^2 = \frac{1}{2}4 \times t^2 = 50 \quad \Rightarrow \quad t^2 = 25 \quad t = 5 \ \ \mathrm{s}. \tag{1.37}$$

Secondly, we need to find the velocity at that time:

$$v = at = 4 \times 5 = 20 \ \ \mathrm{m/s}. \tag{1.38}$$

1.2.2 Newton's second law: $F = ma$

We have now find that the motion of an object is described by several different components: the position x, the velocity v, and the acceleration a. But what happens now when we apply a force? What component changes. We saw earlier that the ancient Greeks thought that it requires a force to change the position. This means that the force is proportional to the velocity (which is the change in position with respect to time). Therefore, in order to maintain a constant velocity, a constant force F is needed. This proportionality can be written as $F = Rv$, where R describes the "nature" of the object. This leads to the intuitive conclusion that it requires a larger force to maintain a constant velocity. It also means that if the force is removed the object will go back to its natural state, i.e. rest. However, this seems logical to many people, it is not correct.

First let us try to find out what quantity a force connects to. The concept of "nature" is not really well-defined from a physics point of view. There is no simple experiment that tells me what the value of R is in the equation $F = Rv$. However, there is a well-defined property of a material and that is its mass. Relative masses can be easily determined by using a balance. Although accurate measurements are not that easy, it is not too difficult to find that different masses react differently under a different force. The astronaut in Fig. 1.8 applies (approximately) the same force on each of the balls while he is blowing. It is not really necessary to do this in space, but it is rather nice to have the balls floating. Clearly, the ball with the greatest mass obtains a much smaller final velocity than the wooden ball and the ping-pong ball. It was therefore proposed that we should not use the concept of nature, but mass.

Secondly, the first law of Newton already states that in the absence of a

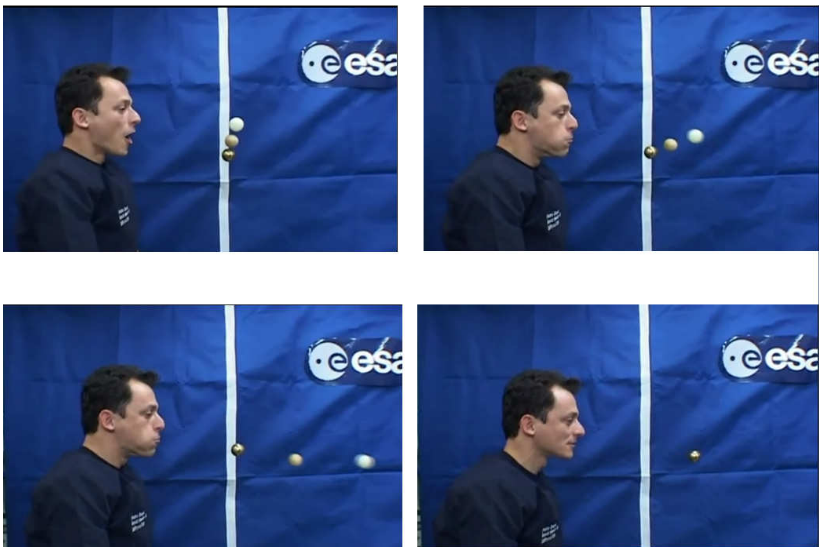

Figure 1.8 The inertia of an object depends on its mass. While the astronaut is blowing, he applies an (approximately) constant force on each of the balls. However, then do not accelerate in the same way and obtain different finite velocities.

force, an object keeps moving at the same velocity. Therefore, no force in needed to maintain a constant velocity. However, a force is needed to change the velocity. In the previous Section, we saw that the change in velocity is the acceleration. The correct law is therefore

$$F = ma. \tag{1.39}$$

The unit of force is

$$\mathrm{kg} \times \frac{\mathrm{m}}{\mathrm{s}^2} \equiv \mathrm{N}, \tag{1.40}$$

where N stands for Newton, since kilogram meter per second squared is such a mouth full. We see that the acceleration can be written as

$$a = \frac{F}{m}. \tag{1.41}$$

Therefore, if the mass increases, the acceleration decreases. This directly explains why the metal ball (with the greater mass) accelerates more slowly than the wooden and plastic ball. If the astronaut is blowing with the same force F for a time Δt, then the final velocity v_{final} is given by

$$v_{\mathrm{final}} = a\Delta t = \frac{F\Delta t}{m}. \tag{1.42}$$

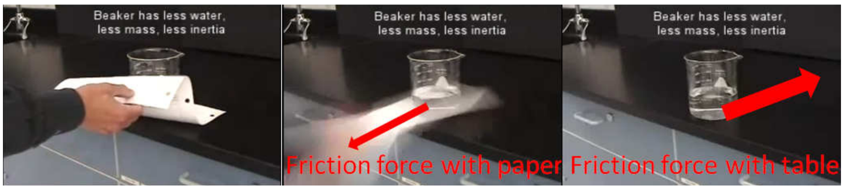

Figure 1.9 A beaker is resting on a peace of paper. The paper is suddenly pulled away. There is a friction force between the paper and the beaker. However, due to the large mass of the beaker, this force gives the beaker only a small acceleration. Before the beaker can significantly displace itself, it lands on the table and quickly decelerates.

The balls keep moving at the speed v_{final} after the astronaut stops blowing (they will slow down because of air resistance. However, removing the air from the experiment makes it very difficult to blow and has other annoying side effects for the astronaut).

Mass is directly related to the concept of intertia. Now we can look at it in two ways. First, an object at rest tends to stay at rest. Second, an object in motion tends to stay in motion. From a physics point of view these are the same. Both state that an objects tends to keep its original speed, except in the former the speed in zero, whereas in the latter the speed is finite. However, for our daily experience these two can often seem different. Let us first look at the object at rest. Figure 1.9 shows a typical staple in a magician's repertoire, although in the end it is basic physics. A piece of paper is under a beaker. The paper is suddenly pulled away. The beaker wants to follow, but the friction force between the smooth glass surface of the beaker and the paper is too small to give a significant acceleration to the beaker. By the time the paper is gone, the beaker has moved only a little and has a small velocity. However, the friction with the table brings the beaker quickly to a stop. For this "trick" to work, it is important that the friction between the beaker and the paper is small. Using sand paper or taping the paper to the beaker will have obviously disastrous results. It is also important that the beaker is pulled away quickly. When the paper is pulled slowly, the friction force can give the beaker sufficient acceleration to make it follow the paper (for the experts: the friction force is approximately proportional to the weight of the beaker. However, if the beaker does not move the friction force is actually somewhat larger than for the case where the paper starts sliding under the beaker).

The second example involves a car and a passenger (dummy) moving at a constant velocity. The velocity of the car is suddenly changed by the

Figure 1.10 A car and its passenger are moving at the same velocity. The car suddenly decelerates. The friction force between the seat and the passenger is insufficient to significantly the velocity of the dummy.

impact of the car with a wall. The friction force of the car seat on the driver is insufficient to slow down the velocity of the dummy. The dummy keeps moving forward at almost the same speed, before the steering wheel and wind shield cause a sudden deceleration of the dummy. This is clear reminder to wear a seat belt at all times!

We like to end this Section by reminding you that although the equation $F = ma$ looks relatively simple, it is far from simple to invent it. The great leap forward by Galileo, Newton, and others became possible because they placed themselves in ideal conditions. Before them, all the efforts were doomed to fail because one tried to incorporate friction and other kinds of resistance into a fundamental law. Deriving such an elegant expression became possible when these forces were removed from the consideration. However, this is almost like placing yourself in outer space. A great example of thinking outside the box.

Example.– If the mass if 10 kg and the force applied on it is 2 N, then what is the acceration?

$$F = ma \quad \Rightarrow \quad 2 = 10a \quad \Rightarrow \quad a = \frac{2}{10} = \frac{1}{5} = 0.2 \ \frac{\mathrm{m}}{\mathrm{s}^2}. \tag{1.43}$$

Example.– If the velocity after 40 seconds is 80 m/s and the mass is 2 kg, what was the applied force?

Step one is to calculate the acceleration

$$v = at \quad \Rightarrow \quad 80 = 40a \quad \Rightarrow \quad a = 2 \ \frac{\mathrm{m}}{\mathrm{s}^2}. \tag{1.44}$$

From the acceleration, we can obtain the applied force

$$F = ma = 2 \times 2 = 4 \ \ \mathrm{N}. \tag{1.45}$$

1.3 Gravity

The prototypical example of a situation where the velocity changes is under the influence of gravity. Let us go back to Galileo. Galileo devised some clever experiments to study the motion of object in gravitation. Although most people know him as the person who dropped objects from the leaning tower of pisa, there are some problems with this story. First of all, it is not at all clear that Galileo Galilei actually threw balls of the leaning tower of Pisa. Such experiments might have been performed by Girolamo Borro in 1575. It is certain the Simon Stevin (1548-1620) threw objects of different mass from the Nieuwe Kerk (new church) in Delft, even before Galileo supposedly performed the same experiment. However, it appears that Galileo did a similar thought experiment. The surprising result was that both objects hit the ground at the same time. This is in complete contradiction with Aristotle's philosophy that states that heavier objects should fall faster because it is in their nature. Of course Aristotle might still say that the tendency of the lighter ball to fall to earth is comparable to the heavier ball. It clearly does not work for a piece of paper or a feather. This is indeed more difficult to show. However, in a vacuum (however, good pumps were not available in those days), feather and a ball do indeed fall in the same way. This is clearly in disagreement with Aristotle. Then again, it also appears to contradict the law $F = ma$, since doesn't that say that heavier objects have more intertia and therefore should be accelerated more slowly in a gravitational pull?

Galileo decided to look into this more carefully. Although the leaning-tower story is about as indestructable as the apple that never fell on Newton's head, the experiment itself is not ideal to study gravitation. Balls fall very fast and, in addition, air resistance quickly leads to an almost constant velocity. Galileo wanted more control. He therefore studied rolling balls, but

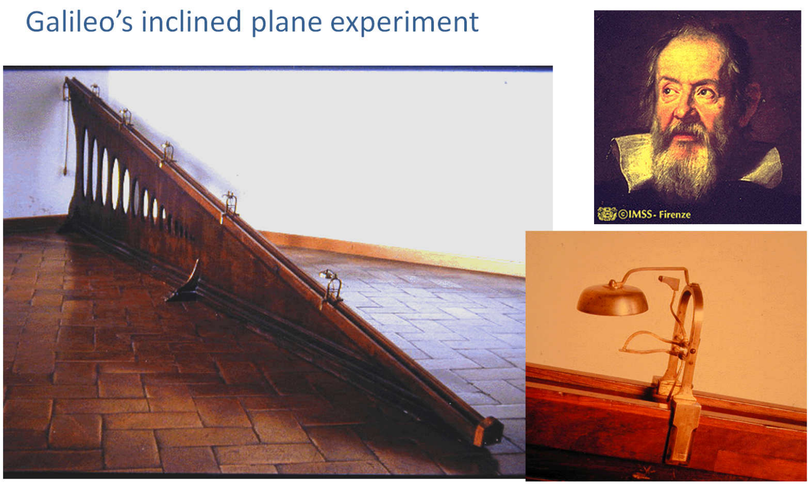

Figure 1.11 Galileo's inclined plane experiment. Galileo determined the distances travelled in each time interval by attaching bells to the slope.

on an inclined plane, see Fig. 1.11. Less dramatic, but definitely more controllable. The force (gravity) that moves the ball forward becomes less if the angle with the ground becomes smaller. Still, the balls roll pretty fast and it is not easy to figure out at what time the ball passes a certain point. Again, Galileo thought of a clever trick. He attached bells to the inclined plane which would sound every time the ball passed it. By rearranging the bells until the the rhythm was exactly right. He then measured the distances between the bells and found them to be in proportion 1, 3, 5, 7, 9,. . . , or, measured from the beginning 0, 1, 4, 9, 16, 25, etc. The ratios were always the same, independent of the angle of the inclined plane. This behavior can be described by t^2, where t is the time. Therefore, Galileo found that gravity gives a constant acceleration, regardless of the mass. The conventional letter to denote this acceleration is g. The gravitational pull is therefore mg. Insterting this in Newton's law gives

$$F = ma \quad \Rightarrow \quad mg = ma \quad \Rightarrow \quad a = g. \tag{1.46}$$

The quantity mg is known as the weight. Since it is directly proportional to mass, people often confuse the two. The problem lies in the gravitational acceleration g. Although g does not differ too much over the Earth's surface (its average being 9.806 m/s^2), it is not a constant. There are several reasons for the deviations from the average value. First, of all the Earth is not a perfect sphere, but is somewhat bulged at the equator as a result of the spinning of the Earth. Since the acceleration decreases when you are

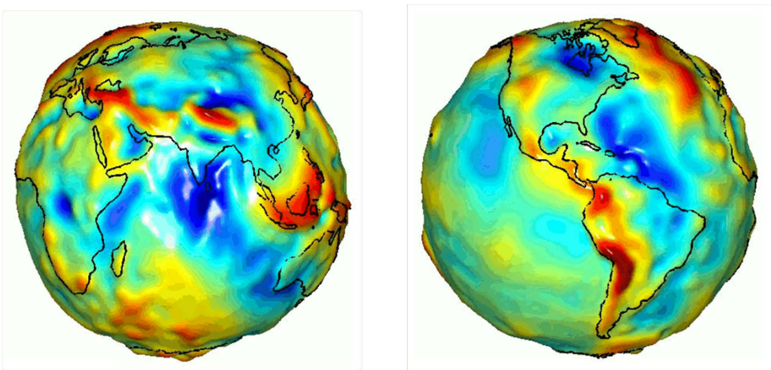

Figure 1.12 The devitations from Earth's gravity, relative to an idealized smooth Earth, the earth ellipsoid. Larger and smaller values of g are indicated in read and blue, respectively.

further away from the Earth's center, g is smaller at the equator. Additionally, the Earth is spinning around its axis, which also somewhat reduces the effective g at the equator. The values of the gravitational acceleration are $g = 9.780$ m/s^2 at the Equator to about 9.832 m/s^2 at the poles. So an object weighs 0.5% more at the poles than at the equator. However, note that the mass has not changed. In addition, there are other factors that can change the gravitational constant, such as the topography (the presence of mountains and valleys) and the geology (the local density of the materials in the Earth). Such variations are shown in Fig. 1.12. Therefore, going to the top of Mount Everest at 8,848 meters altitude causes a weight decrease of 0.28% (then again, the climb to the top probably produces a greater weight loss than gravity). However, these variations are all pretty minor. Larger effects occur when travelling to different planets. The gravitational acceleration at the Moon is only 1.62 m/s^2. Therefore, you weigh about 6 times less, allowing you to jump around very easily even while wearing very heavy spacesuits. On the other hand, you would weigh about 28 times more on the surface of the Sun. However, the ambient temperature of 6000 K (about 10,000 degrees Fahrenheit) will incinerate you, before you can even test how this will affect your jumping skills. However, be careful when comparing this with astronauts that are weightless when in orbit around the Earth. Do not make the mistate into thinking that they have gone away far enough from Earth that they no longer experience the gravitational pull of the Earth. In a typical orbit of a spaceship around the Earth (at an alitude of about 400 km), gravity is still 90% of what it is at the Earth's surface. We will come back to this later.

Example.– If an object hits the ground with a velocity of 20 m/s, from what height was it dropped?

The time can be expressed in terms of the gravitational acceleration and the velocity as

$$v = gt \quad \Rightarrow \quad t = \frac{v}{g}. \tag{1.47}$$

The height is then given by

$$h = \frac{1}{2}gt^2 = \frac{1}{2}g\left(\frac{v}{g}\right)^2 = \frac{1}{2}\frac{v^2}{g} = \frac{1}{2}\frac{20^2}{9.8} \cong 20.4 \quad \text{m}. \tag{1.48}$$

1.3.1 General formulas

In the equations used in the previous Section, $x = v = 0$ at time $t = 0$. Obviously, this does not have to be the case. For example, an object can already have a finite velocity at $t = 0$. One can obtain more general equations that reflect this fact. The more general equations are where we have taken the y coordinate since many problems deal with gravity and the acceleration is in y axis. Note that for $t = 0$, we have $y(0) = y_0$ and $v_y(0) = v_0$.

$$\begin{aligned} y(t) &= \frac{1}{2}at^2 + v_0 t + y_0 \\ v_y(t) &= at + v_0, \end{aligned} \tag{1.49}$$

Let us give an example to see how this works:

Example.– Let us consider an object thrown upwards with an initial velocity of $v_0 = 5$ m/s a a height of 10 m. For simplicity, let us take the gravitational acceleration $g = -10$ m/s^2.

We can insert the initial conditions into Eq. (1.49), giving

$$y(t) = -5t^2 + 5t + 10 \tag{1.50}$$

$$v_y(t) = -10t + 5 \tag{1.51}$$

These functions are plotted in Figs. 1.13 and 1.14. When writing down these equations, it is very important that you get the signs correct. The gravitional acceleration is downwards and therefore gets a minus sign. The object is thrown upwards and therefore gets a plus sign. We took $y_0 = 10$ m. This

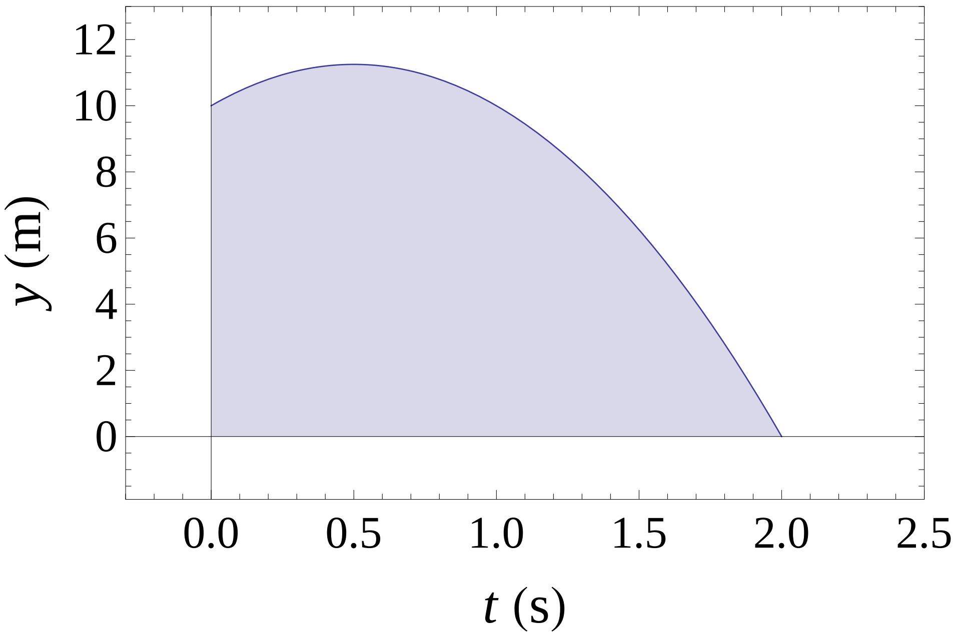

Figure 1.13 The vertical position of an object thrown upwards with a velocity $v_0 = 5$ m/s from a height of 10 m.

means that we have taken the ground as $y = 0$. Note that this is a choice that we make. We could have take the point where we throw the object upwards as $y_0 = 0$. Then the ground is at $y = -10$ m. However, it is probably less confusing to be consistent and take the ground always at $y = 0$. Now that we have the equations of motion down, we can ask ourselves some important questions:

(a) When is the object back at the same height of 10 m?

For this, we need to know when the height is equal to 10 m, or

$$y(t) = 10 \qquad \Rightarrow \qquad -5t^2 + 5t + 10 = 10 \tag{1.52}$$

This equation is easily solved giving

$$-5t^2 + 5t = 0 \qquad \Rightarrow \qquad -5t(t-1) = 0 \qquad \Rightarrow \qquad t = 0, 1 \;\; \text{s.} \tag{1.53}$$

We find two solutions. The first, $t = 0$ seconds is the time when we threw the object into the air. The object then goes up after which the graviational acceleration pulls it down again. The second time, $t = 1$ s, is the time when the object passes the point where it was thrown upwards. We can

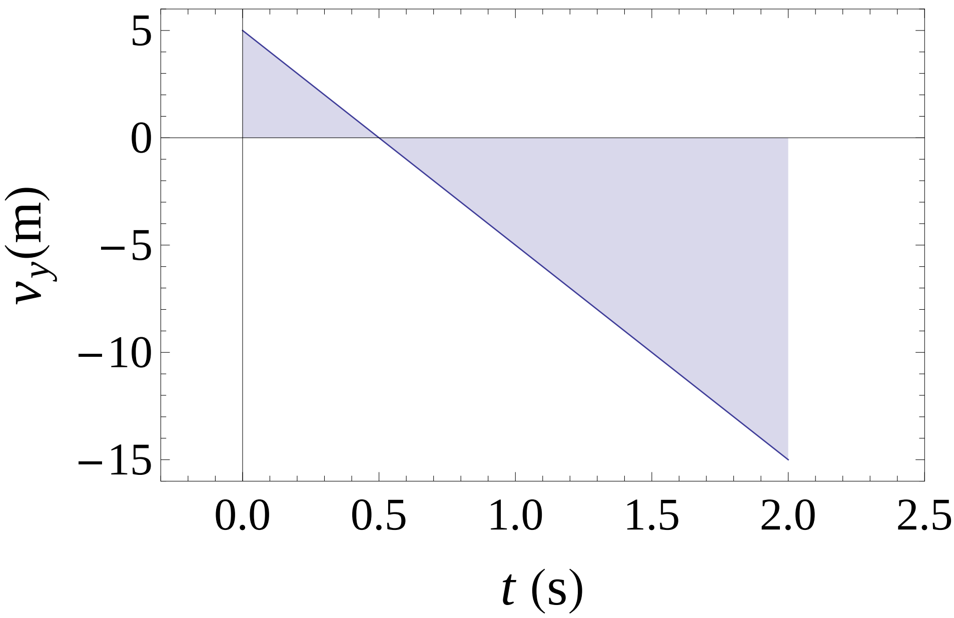

Figure 1.14 The velocity of an object thrown upwards with a velocity $v_0 =$ 5 m/s from a height of 10 m.

doublecheck this by inserting it back into the equation of motion, $y(1) = -5 \times 1^2 + 5 \times 1 + 10 = 10$ m. We can also calculate the velocity which at $t = 1$ s is

$$v_y(1) = -10 \times +5 = -5 \ \ \text{m/s}, \tag{1.54}$$

using Eq. (1.51). Note that this is exactly the initial speed, but the sign has changed. First, the object was thrown upwards (positive sign) and,at $t = 1$ s, it is going down (negative sign).

(b) When does the object hit the ground?

Since we take the ground at $y = 0$, we have to solve the equation

$$-5t^2 + 5t + 10 = 0. \tag{1.55}$$

This is a quadratic equation of the form

$$at^2 + bt + c = 0. \tag{1.56}$$

The roots of this quadratic equation are given by the well-known formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.57}$$

Inserting that $a = -5$, $b = 5$, and $c = 10$ gives

$$t = \frac{-5 \pm \sqrt{5^2 - 4(-5)10}}{2 \times (-5)} = \frac{-5 \pm \sqrt{25 + 200}}{-10} = \frac{-5 \pm \sqrt{225}}{-10} \tag{1.58}$$

$$= \frac{-5 \pm 15}{-10} = \begin{cases} \frac{10}{-10} = -1 \\ \frac{-20}{-10} = 2 \end{cases} \tag{1.59}$$

This is somewhat confusing. We obtain the right answer namely $t = 2$ s, which we can also see by checking the plot in Fig. 1.13. However, we also find that $t = -1$ s works. This almost seems like going backwards in time. What does this all mean? The equations of motion in (1.49) give the trajectory of the object. However, they contain nothing of the physical stuff like when it is being thrown, when it hits the ground, etc. This is something we all put in there ourselves. Equation (1.49) is the motion of an object that starts at $t = -\infty$ s (yes, indeed that is minus infinity, before the creation of the universe) at a postion of $y(-\infty) = -\infty$ m, whereever that may be.... It will also keep on going forever. Therefore, it is crucial that we put these physical bounds into the equation. However, the time $t = -1$ s is not entirely unphysical. At time $t = 0$ s, we had said that the object *isthrown* upwards with a velocity of 5 m/s at a height of 10 m. This is not really contained in the equation. The only things that the equations say is that the object *is* at 10 m height and *has* a velocity of 5 m/s. It does not say anything about being thrown at time $t = 0$. It only describes the situation at $t = 0$, not how the object obtained this height and this velocity. The solution that $y = 0$ at $t = -1$ seconds tells us that we could have obtained the same situation at $t = 0$ by throwing the object upwards 1 second earlier with a velocity of

$$v_y(-1) = -10 \times (-1) + 5 = 15 \quad \text{m/s}. \tag{1.60}$$

This situation is shown in Fig. 1.15. The next question we can ask is

(c) When is the object at its maximum and what height does it reach?

This is more complicated and we have several ways of calculating it. First, we can look at the solution from (a). For Fig. 1.15, we see that the parabola is symmetric around its maximum. We also know that it passes the same height at $t = 0$ and $t = 1$ seconds. The maximum is therefore in the center

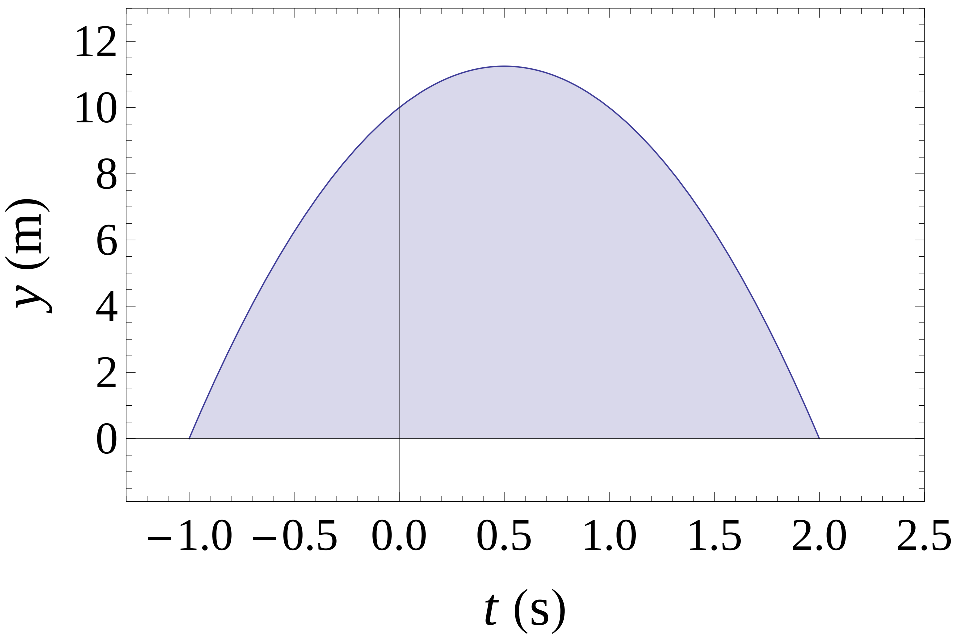

Figure 1.15 An alternative way of looking at the same problem. The same trajectory is obtained by throwing the object upwards with a velocity of 15 m/s at a time $t = -1$ s.

of those two times, or

$$t_{\max} = \frac{0+1}{2} = \frac{1}{2} = 0.5 \quad \text{s.} \tag{1.61}$$

The height can now be found by inserting the time into the equation of motion in (1.50). giving

$$y(\frac{1}{2}) = -5\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) + 10 = -\frac{5}{4} + \frac{5}{2} + 10 = \frac{5}{4} + 10 = 11\frac{1}{4} \quad \text{m.}$$

We could have found the same result by taking the looking at the times when it was on the ground $t = -1$ and 2 seconds, which gives again $t_{\max} = (-1+2)/2 = 1/2$ s.

There is however a different way of looking at the problem. Let us ask ourselves first, what defines the maximum in the equation of motion? Before the object reaches the maximum, it is going upwards ($v_y > 0$). After reaching the maximum, it is going down ($v_y < 0$). Therefore, at some point the velocity v_y changes from positive to negative. There is only one number that can be positive and negative at the same time and that is zero. Therefore,

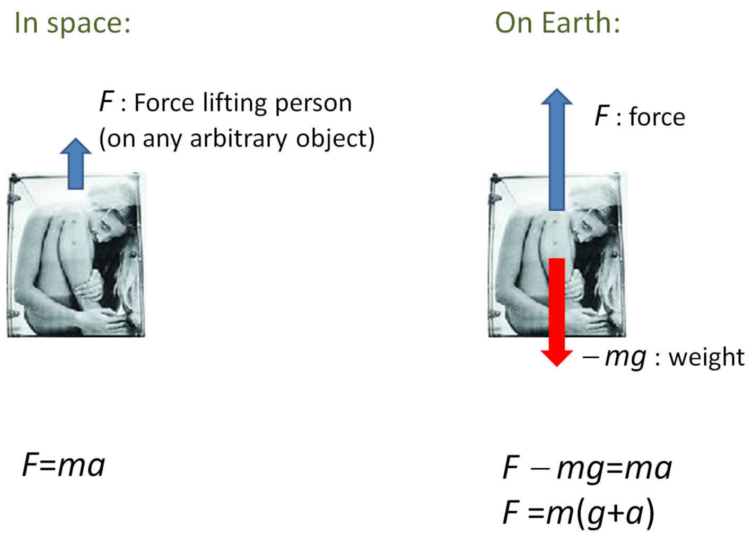

Figure 1.16 Comparison between lifting an object in space and on Earth. In space, the effect of gravity is absent and a much smaller force is needed to give the same acceleration as on Earth.

we want to find the time for which $v_y = 0$. This is given by

$$v_y(-1) = -10t + 5 = 0 \quad \Rightarrow \quad t = \frac{5}{10} = \frac{1}{2} \text{ s.} \tag{1.62}$$

This is the same time as we found earlier.

1.3.2 Weight and mass

Different instruments measure different things. The correct way to measure mass is to use a balance. A balance compares different weights, but since the gravitational acceleration is the same on both sides of the scale you are effectively measuring mass. However, most scales actually measure weight because the work with springs or other sensors that are sensitive to the weight. They display shows the result in pounds or kilograms, the unit of mass. However, when you would measure your weight on the Moon the result will be incorrect.

1.3.3 Gravity and acceleration

Although g is an acceleration, one should not confuse it with the acceleration in $F = ma$. Let us compare lifting an object in space and on Earth, see Fig.

1.16. In space, one only has to overcome the inertia of an object in order to give it a certain acceleration a. The force F is then given by

$$F = ma, \tag{1.63}$$

where m is the mass of the object. Note that if you only give the object a very small acceleration, little force is needed. Also, once you set it in motion, there is no need to apply any additional force since once the object is in motion, it will remain in motion. You need to apply an additional force in the opposite direction to stop the object.

On Earth, one also has to overcome gravity. The force is then given by

$$F - mg = ma \quad \Rightarrow \quad F = m(g + a) \tag{1.64}$$

We have to take care of the signs here. The force is working in the opposite direction as gravity and one needs a minus sign to account for this. One needs a larger force in order to obtain the same acceleration as in space. The other way of writing the equation is

$$a = \frac{F}{m} - g. \tag{1.65}$$

Depending on the size of the force, the acceleration can be positive or negative. If $F > mg$, where mg is the weight of the object, then the object moves upwards. However, if $F < mg$ the object falls back to the ground. But even when there is no acceleration, you still need a force of $F = mg$ to prevent the object from falling.

In a spaceship orbiting Earth, objects and people are weightless. At first, you might think that this is a result of the fact that the spaceship is not close to Earth and therefore the gravitational acceleration is close to zero. However, is that the case? We know that spaceships and spacestations that orbit the Earth are closer than the Moon. Now, the Moon is held into orbit by the graviational pull of the Earth. In fact, the spacestation orbits that Earth at an altitude of about 400 kilometers. This is significantly closer than the Moon, which is 400,000 km away from the Earth. Given that the radius of the Earth is 6,400 km, the gravitational acceleration is smaller by a factor $(6,400/6,800)^2 \cong 0.89$, which is only 11% smaller than at the Earth's surface. Athough it is a great way to loose some weight (but not mass!), it certainly does not the weightlessness.

Let us return to the question: what is weight? Weight is what you measure when you stand on a scale. The scale measures the gravitational pull that the Earth exerts on you. This force $F = -mg$ is the weight (not that the negative sign indicates that gravity is pulling you down). However, although gravity

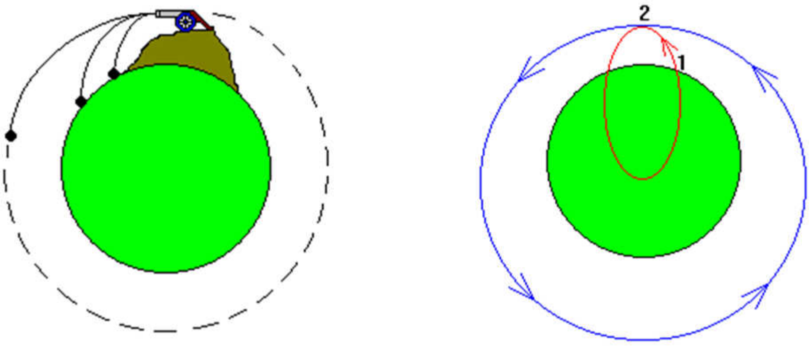

Figure 1.17 Generally, cannonball shot from a cannon generally hits the Earth's surface after a while, see the left side of the Figure. However, for a certain velocity the cannonball falls towards the Earth, but it will never reach it and keeps going around the Earth (in the absence of air resistance). The right side shows that also for other conditions, the canon ball would "circle" around the center of earth, if it wasn't for the annoying Earth's surface.

is pulling you towards the center of the Earth, you are not accelerating when you are standing on a scale. This is because the scale is preventing you from falling. This force is exactly equal to your weight, so

$$F_{\text{gravity}} + F_{\text{scale on you}} = -mg + F_{\text{scale on you}} = 0 = ma \quad \Rightarrow \quad a = 0 \tag{1.66}$$

so the acceleration is zero. Now the force that the scale exerts on you is exactly equal, but opposite, to the force that you exert on the scale

$$F_{\text{you on scale}} = -F_{\text{scale on you}} = -mg. \tag{1.67}$$

Now let us put the scale into an accelerating elevator. Note that the elevator should be accelerating not just simply moving at a constant velocity. We can now ask again the question: what does the scale indicate?

$$F_{\text{gravity}} + F_{\text{scale on you}} = -mg + F_{\text{scale on you}} = -ma. \tag{1.68}$$

In the case that $a > 0$, the elevator will be accelerating downwards. The gravitational pull has not changed, but we want to know what the scale indicates. We can solve the above equation

$$F_{\text{scale on you}} = m(g - a). \tag{1.69}$$

So the scale feels an effective weight of

$$F_{\text{you on scale}} = -m(g - a). \tag{1.70}$$

For the very special case that $g = a$ (basically the elevator is just falling without any resistance), the scale indicates that your weight is zero. You are weightless.

However, a spaceship is not an elevator. A dropping elevator eventually hits the ground. A spaceship or spacestation does not crash into the Earth's surface and still the people and objects in there are weightless. The clue lies in the observation that it is not only falling down, but also moving forward. Figure 1.17 shows the situation for a cannonball. Generally, the velocity is too small and the cannonball will hit the Earth's surface. However, for a certain velocity, the cannonball is still falling, but it simply never reaches the ground and it will continue to orbit around the Earth (in the absence of air resistance). Obviously, we cannot do this experiment too close to the Earth's surface since the Earth is not entirely spherical and the cannonball will hit the ground somewhere. So, we have to shoot it from a higher vantage point. We still need to shoot the cannon ball at about 28,000 km/hour. You can save yourself 1670 km/hour if you shoot the cannonball in the direction of the Earth's rotation. These are very high velocities considering that a bullet leaves an M16 rifle at about 3,500 km/h. This is in fact common practice with rockets which are usually launched eastward. However, in principle, the same thing would happen under different conditions, see the right side of Fig. 1.17. If the Earth's surface was not there (say, all the Earth's mass was concentrated in a very tiny ball), then the cannonball would describe an ellipse around the center of the Earth. Note, that it would not describe an ellipse if the cannonball could magically travel through the Earth. The reason for that is the dependence of the gravitational acceleration inside the Earth. If the cannonball is at a certain distance from the center of the Earth, all the matter which is at a greated distance to the center of the Earth does not contribute to the gravitational acceleration.

It is also possible to recreate free-fall conditions in a plane. At first, you might think that just dropping out of the sky would do the trick. However, this does not quite work because of the air resistance (drag) that prevents a good free fall. However, with a place one can correct the imperfect free fall. By making the plane describe a parabola in the sky, one can obtain roughly 25 seconds of weightlessness. This is followed by 25 seconds of feeling twice the gravitational acceleration.

Besides the obvious effect of weightlessness, other things are also different in zero gravity. For example, flames burn differently, see Fig. 1.18. At Earth, convection plays an important in determining the shape of the flame.Om Earth, convection plays an important role. Cold air is denser and therefore heavier and will sink. Hot air is less dense and will rise causing the narrowing

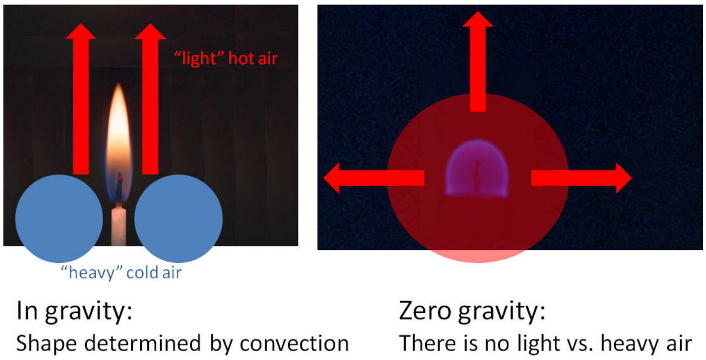

Figure 1.18 Flames burn differently in the presence (left) and absence (right) of gravity. With gravity, the convection of the air gives the flames its familiar shape. In zero gravity, there is no such thing as "heavy" and "light" air since everything is weightless. The heat radiates out uniformly giving the flame a spherical shape.

of the flame. Without gravity, the air is weightless and there is nothing like "light" and "heavy" air. The hot air will move away from the flame uniformly. This causes the to become more spherical.

Also liquids behave differently in zero gravity. Without gravitational acceleration, the liquid does not splash on the floor, but floats in the air, much like a bubble on Earth. Although the surface tension keeping the liquid together is not strong, in the absence of gravity it is the force dominating the behavior of the liquid. This makes it possible to perform some cool shock wave experiments, see Fig. 1.19

1.3.4 Gravity: the mysterious force

Let us consider the expression for the gravitational force in more detail $F = mg$. This expression for the force solves our problem from a mathe-

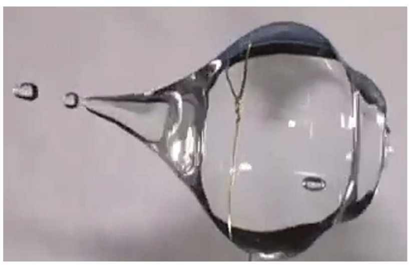

Figure 1.19 A shock wave propagating through a ball of weightless water.

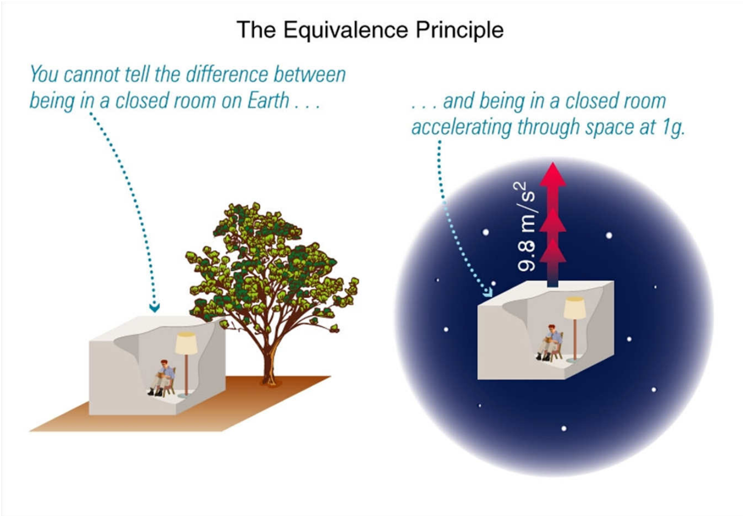

Figure 1.20 Einstein's thought experiment of gravity. Einstein concluded that for a person inside a room without windows, there is no way of knowing whether he is at the Earth's surface experiencing g due to the Earth's graviational field or being accelerated in free space with an acceleration equal to g.

matical point of view, meaning that it gives the right result for the type of experiments that Galileo performed. However, the more you look at it and think about it the more confusing and mysterious it becomes. We will not be able to provide every answer here, but asking the right questions is often a crucial part in solving a problem. First, we have not solved in any way the problem about the nature of the force. What is this mysterious force that just acts through nowhere? Are there any strings somewhere pulling us back to Earth? Can this force act through vacuum or does it need some substance to mediate the interaction? Second, the gravitational force is proportional to the mass. So not only does it act in some invisible fashion, in some way the Earth is able to determine the mass of an object and apply a force on it that is directly proportional to it. How on Earth does the Earth do that? Thirdly, why does the gravitational force work on the mass, which is exactly the same quantity as the one that appears in m times a, i.e. the inertia. We will talk later about different forces, but none of them works on mass. They all work on different quantities (such as charge for the Coulomb interaction). Why doesn't gravity work on a different quantity, say *nonsense* (abbreviated by n)? So one object could have a mass (inertia) of 5 kg, but have a *nonsense* of 15 kg, whereas another object could have the same mass of 5 kg, but a *nonsense* of only 10 kg. Then the gravitational forces given by $n \times g$ would be different and the accelerations would be 3 and 2 m/s^2, respectively. Now this might seem like nonsense, but people have actually looked into this and found the *nonsense* and the mass to be equivalent up to many decimal places. This may sound absolutely silly to you, but it is this line of thinking that led Einstein to his theory of general relativity. Einstein wondered why there is such a strange similarity between acceleration a and gravity g and he concluded that they are equivalent. Einstein formulated the equivalence principle. He imagined someone sitting in a room with no windows. On Earth, he this person would experience the gravitational acceleration of 9.8 m/s^2 of the Earth. However, Einstein concluded that this experience would be exactly the same if he was sitting in the same room in free space (without any gravity) being accelerated with 9.8 m/s^2. Confusing? It certainly is. How can standing still on the earth give exactly the same experience as being accelerated in free space? However, this equivalence explains why the inertia in ma and the mass in the gravitational force mg is exactly the same. This still does not answer the first couple of questions, but this is not really the right place to talk about distortions in the space-time continuum. However, it does show that asking the right questions is about just as important as actually solving the questions.

1.4 The solar system

Ancient civilizations had an entirely different idea of the universe than the modern one. The view is reflected in the old testament. It was believed that the Earth was flat and resting on columns (the pillars of the Earth). There was water below the Earth and also above the Earth. The Earth was protected from the upper waters by the firmament. The Sun, Moon, and starts were also attached to the firmament. Precipitation was the result of the opening of flood gates in the firmament.

At the start of the renaissance in the sixteenth century, it was still generally assumed that the Earth is a the center of our solar system. However, although this followed Aristotle's view of our solar systems, some of the ancient Greeks had a surprisingly well developed insights into our solar systems. Let us try to developed a view of our solar system in a number of steps.

STEP 1: How big is the earth?

The first thing that you need to know is the radius of the earth. Although, you might stick to the point of view that the Earth is flat, it is not difficult

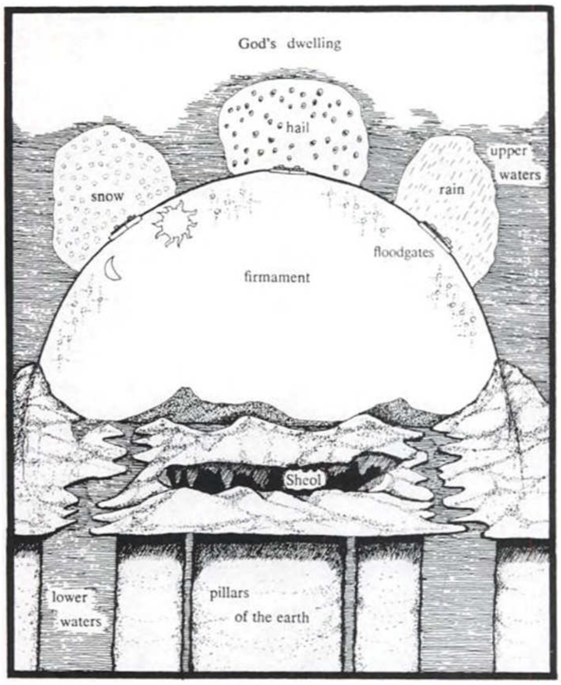

Figure 1.21 The view of the universe for ancient civilizations. The upper and lower waters were separated during the creation. The Earth was flat and resting on pillars. The Sun, Moon and stars are attached to the firmament. The firmament also contain floodgates.

to establish that the Earth has a curvature. For example, of a boat at the horizon you first see the mast and only later the hull. The radius of the Earth was first determined by the Greek Eratosthenes (279-194 B.C.). He knew that on noon at the longest day of the year the sun was almost at its zenith in Syene. Therefore, no shadows were cast. However, Eratosthenes was in Alexandria and there a shadow was cast atnoon on the summer solstice. Eratosthenes took a large obelisk and measured its shadow. Since he knew the height of the obelisk, he could determine that the Sun appeared at an angle of 7.2 degrees south of the zenith. This actually determines the curvature of the earth between Syene and Alexandria. What was left was to determine the distance between Syene and Alexandria. To this end (astronomers apparently had some influence in those days), he ordered some soldiers to walk from Alexandria to Syene to determine the distance. It turn out to be 5000 stadia, roughly 750 km. The circumference of the Earth can then be determined

$$\text{circumference} = 750 \times \frac{360^\circ}{7.2^\circ} = 37,500 \text{ km}, \tag{1.71}$$

which is close enough to the real value of 40,000 km. The diameter is then obtained by dividing the circumference by π giving 12,700 km. Since the Earth is spinning around its axis in 24 hours, this means that a person at the equator moves at a velocity of about 1600 km/h due to the rotation of the Earth.

An alternative example is by watching the sunset which does not involve giving soldiers marching orders. Suppose you are lying down, watching the Sun set. You start your stopwatch just after the Sun goes down. Then you stand up and watch the Sun set again. Suppose your height is h =1.7 m and the time elapsed is t =11.1 s. What is the radius of the Earth?
Solution: The idea is that if the Sun is setting the line connecting your eyes to the Sun is a tangent to the Earth's surface, see Fig. 1.22. Then you raise yourself by h and draw a new tangent. This is drawn with great exaggeration see Fig. 1.22. This forms a the red lines in the Figure form a triangle with 90° angle. We can therefore use Pythagoras theorem

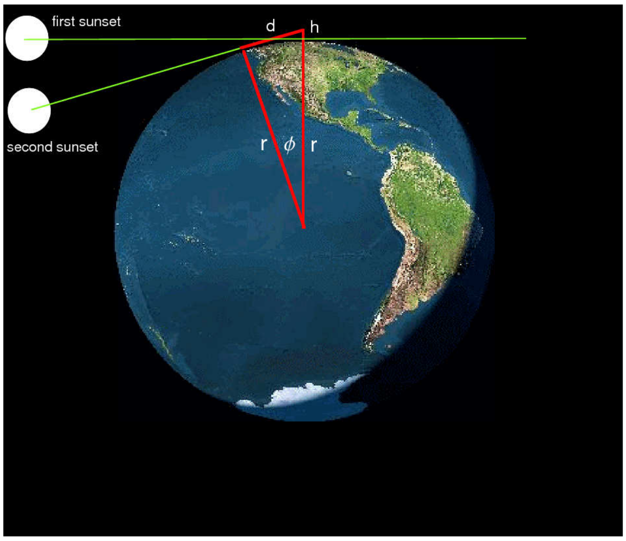

Figure 1.22 Schematic diagram to calculate the size of the Earth by measuring to different sunsets. Aperson lying down and one standing see a different sunset by increasing the distance to the center of the Earth by h. Note that h is greatly exaggerated.

$$d^2 + r^2 = (r+h)^2 \qquad \Rightarrow \qquad d^2 = 2rh + h^2 \cong 2rh. \tag{1.72}$$

We can do the last approximation since $h \ll r$. Now, $d = r\tan\phi$, where ϕ is given in Fig. 1.22 and we can also write

$$r^2 \tan^2\phi = 2rh \qquad \Rightarrow \qquad r = \frac{2h}{\tan^2\phi} \tag{1.73}$$

We have to determine ϕ. However, we now that 11 s has passed and that the Earth turns 360° in $24 \times 3600 = 86400$ s. The angle is then given by

$$\phi = \frac{11.1}{86400} \times 360^\circ = 0.04625^\circ. \tag{1.74}$$

This gives for the radius

$$r = \frac{2 \times 1.7}{\tan^2 0.04625} = 5.22 \times 10^6 \text{ m}, \tag{1.75}$$

or about 5,000 km giving a diameter of roughly 10,000 km. Again not exact, but a reasonable estimate nevertheless.

STEP 2: What is the size of the Moon?

This idea is again thought to originate from Eratosthenes. However, it was first carried out by Aristarchus of Samos (310-230 B.C.). The idea is to determine the relative sizes of the Earth and the Moon. Since we know the size of the Earth, calculating the size of the Moon, should then be a piece of cake. The trick was to make use of a lunar eclips, when the Moon is passing through the shadow of the Earth. We want to compare the time it takes the Moon to travel to the shadow of the Earth (which is roughly equal to the size of the Earth if the Sun is sufficiently far away, so that its rays can be considered parallel) to the time it takes for the moon to traverse its own diameter. This can be done by measuring the difference between the time that a moon covers a bright star and the time that the star reappears. Doing this, Aristarchus found that the Moon's diameter is about 3/8 that of the Earth or 4700 km (it should be closer to 1/4). Okay, slightly off the real diameter of 3476 km, but we are interested in the right order of magnitude.

STEP 3: What is the distance between the Earth and the Moon?

Now that we have the absolute diameter of the Moon, we can determine the distance by measuring the angular diameter on the sky. The angle occupied

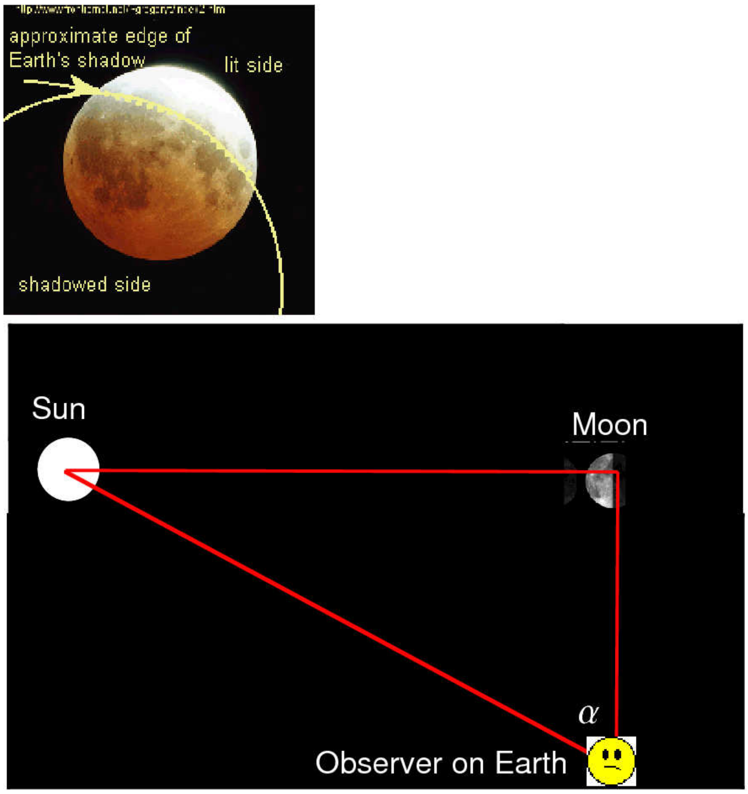

Figure 1.23 The top part shows the Moon moving through the shadow of the Earth. The lower part shows how to determine the distance to the Sun when the Moon is in its third quarter.

by the full moon is about 0.5°. or $\frac{0.5^\circ 2\pi}{360^\circ} = 0.0087$ arcradians.

$$\text{distance}_{\text{Earth-Moon}} = \frac{3476}{0.0087} \cong 400,000 \text{ km}. \tag{1.76}$$

This distance between the Earth and the Moon varies between 363,300-405,500 km, since the orbit of the Moon around the Earth is elliptical and not spherical.
Now that we determined the distance to the Moon, we might even try something more ambitious: Can we determine the distance to the moon as well?

STEP 4: What is the distance between the Earth and the Sun?

This is a bit trickier to determine than the distance to the moon. Again, our good old friend Aristarchus found a method to determine the distance.

The trick is to find a right angle. When we have a half Moon, the Sun-Moon-Earth angle is 90°. Since we know the distance from the Earth to the Moon, we can determine the distance Earth-Sun, when we know the angle α to the Sun, see Fig. 1.23. This is quite a difficult determination. There is the unhealthy aspect of looking directly at the Sun, but in addition there is the complication that the angle is almost 90°. In fact, it is 89.853°. The calculation of the distance would then be

$$\text{distance}_{\text{Earth}-\text{Sun}} = \frac{\text{distance}_{\text{Earth}-\text{Moon}}}{\cos\alpha} = \frac{400,000}{\cos 89.85} \cong 150,000,000 \text{ km}.$$

Aristarchus was off by a factor 20. Still, it means that he ended up with a number in the millions of kilometers. Note that this distance to the Sun means that we are flying through space at the relaxed speed of 100,000 km/h. While we are at it, we can also calculate the size of the Sun. Since the Earth rotates around the Sun in 365 days, the Earth moves at a speed of

$$v = \frac{2\pi 150,000,000}{365 \times 24} = 108,000 \quad \text{km/h}. \tag{1.77}$$

For comparison, the fastest speed achieved on land is a rocket sled that managed to go 10,385 km/h. NASA's X-43, an unmanned experimental hypersonic aircraft, the fastest one on record, flew at 12,144 km/h.

STEP 5: What is the size of the Sun?

Just as for the Moon, we can again make use of the angle that the Sun occupies in space. From solar eclipses, we know that the angular diameter of the Sun is more or less that of the Moon, i.e. 0.5°. Using the distance from the Earth to the Sun, we can obtain a diameter of

$$\text{diameter} = 150 \times 10^6 \text{ km } \times \frac{0.5^\circ}{360^\circ} \times 2\pi \cong 1.3 \times 10^6 \text{ km}, \tag{1.78}$$

which is several times larger than the size of the Moon's orbit around Earth.

If this was all known more than two centuries before Christ, why did this knowledge get lost for many centuries, only to be rediscovered after the renaissance? Well, it did not really get lost, but it was largely ignored. One of the reasons is that people prefer ignore things that they do not find appealing. The idea of people walking upside down on the other side of the Earth is just hard to believe. Also, wherever we are on Earth, the surface looks more or less flat. A second factor is that people like to place themselves at the center of the universe. If the Sun is that far away, it is more logical that the Earth circles around it in 365 days, as opposed to the Sun circling around the Earth in 24 h at roughly 4 million miles per hour. However, that makes the Earth less important than the Sun. So it is easier to ignore the fact that the Sun is that far away. Of course, we may find that all very silly, but still people like to stick to comfortable ideas and deny or ignore things that makes them feel less special or makes them appear small and irrelevant. The idea that the Earth rotates around the Sun is now reasonable well accepted, but how many philosophies and religions have as foundation that humans are absolutely unique and that the universe is just a backdrop to our existence?

It is the idea that humans are central in the creation that defeated the heliocentric model (the idea that the planets rotate around the Sun and not the Earth. The latter is known as the geocentric model). The geocentric model with the Earth at the center was refined by Ptolomy (90-168 AD). Ptolomy was a Greek living in Egypt, around the time that the Roman Empire was at its greatest extent. The basis of the model was that the universe was made up of celestial spheres. The Sun was on a sphere about 1210 Earth radii. This is off by about a factor 20 (which is not too bad...). The outer sphere was at 20,000 Earth radii, which coincidentally is very close to the distance between the Earth and the Sun. However, it is only about 0.000015 light years (the distance light travels in a year) or about 7 light minutes (light takes a little over 8 minutes to travel from the Sun to the Earth at a speed of 300,000 km/s). The distance to the nearest star (Proxima Centauri) is 4.2 light years, which is significantly further away. The Greeks however noticed the the planets do not follow simple movements in the sky, but often seemed to move backwards (retrograde motion), see Fig. 1.24. This was solved by introducing epicycles, additional circular motions that the planets made in their orbits. Ptolomy's work on the motion of the planets was published in his influential book Almagest, around 150 AD.

1.4.1 Science between the Greeks and the Renaissance

The geocentric view held sway until the fifteenth century. Again we seem to be hopping between the ancient Greeks and the renaissance. Did really nothing happen in between? The Roman had a giant empire, but their interests were not in fundamental science and philosophy, which they essentially took from the Greeks. The Romans were great engineers who constructed building such as the colosseum (72-80 AD) and the Pantheon (126). The Pantheon is still the building with the largest dome made out of non reinforced concrete. Some might say the the achievements of the ancient Egyptians, such as the great pyramids were even greater. However, the pyramids while fantastic

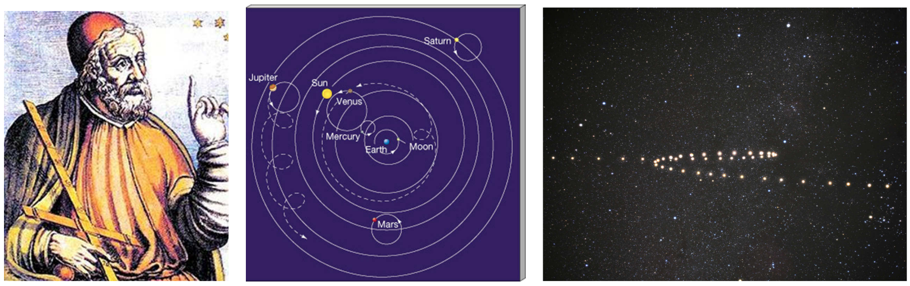

Figure 1.24 Ptolomy perfected the geocentric model by introducing epicycles, little additional circles that the planets made in their orbits to explains the peculiar motion that some of the planets made in the sky.

achievements are less difficult from an engineering point of view than, say, the pantheon. You can see this for yourself. Try to build a pyramid or a building ressembling the pantheon out of sand on the beach. And they have to be the same, not only the outside, but also the inside. This is easy for the pyramids which are not really buildings, but giant mounds (although they have some relatively small interior spaces in there that you can no doubt reproduce in a sand castle. However, the Pantheon, see Fig. 1.25 has an interior space that exactly fits a sphere with a diameter that is slightly over 43 meters. Now try to reproduce that in a sand castle.

However, size is obviously more impressive. In fact, some people are so impressed by the great pyramids in Egypt that they claim that they were built by aliens. First, you might say that it is kind of strange that there are also pyramids that are less impressive than the Great Pyramids. The answer to that is that those pyramids were not built by aliens only the really impressive ones. Second, you might wonder why aliens would come to Earth to teach us how to stack stones into a perfect pyramids. It seems like such a wasteful excersize. Wouldn't you need something else if you were an alien? And it the aliens really wanted to teach us something, why not teach the ancient Egyptians, who were living in the bronze age, how to make stainless steel, which they could then use to make nice stainless steel frame buildings? Furthermore, the largest pyramid on Earth is not in Egypt, but actually in North Korea (of all places): the 330 m high Ryugyong Hotel in Pyongyang. Compare this to the Great Pyramid of Giza/Khufu/Cheops, which is only 146.5 meters high. In addition, the Ryugyong Hotel is a functional building (if the North Korean finally manage to finish it). Again compare this with the pyramids which are built for a single guest that has no intention of checking out (at least not voluntarily). So apparently, the North Koreans

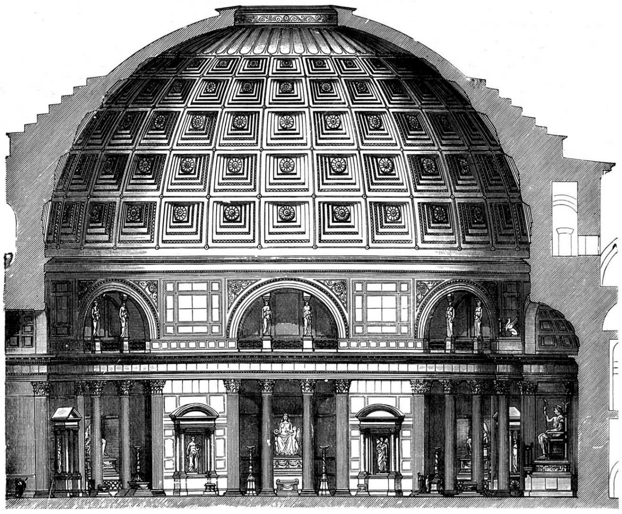

Figure 1.25 One of the great architectural feats by the Romans: the Pantheon (126 AD).

are out-building the aliens civilizations that are familiar with interstellar space travel. This does not make much sense.
However, people believing in aliens have more up their sleeve. Look at the geat pyramid of Giza. Take four times its base and divide it by the height and you get

$$\frac{4 \times \text{Base}}{\text{Height}} = \frac{4 \times 115.2}{146.5} = 3.14 \cong \pi, \tag{1.79}$$

which is rather surprising since the ancient Egyptians were not supposed to known about π, hence aliens. We can try to view this as a scientific hypothesis. Now what is the sceptical scientist to do here? Now you might try to argue that π has something to do with the area or the circumference of a circle and that the division of base and height seems more like a tangent which is not directly related to π, but you already feel that that argument is not going to fly. 3.14, that can hardly be a coincidence, right? Does the alien believer beat the sceptical scientist here?

Let us first have a look how Egyptians measured things. Their standard measure was the Egyptian Royal cubit. The cubit was subdivided into 7 palms, which were again subdivided into 4 fingers, making a total of 28 digits. Now the Egyptian architects had to convey to the engineers how to construct the pyramid. Now let us fix the height at 28 fingers. If we now

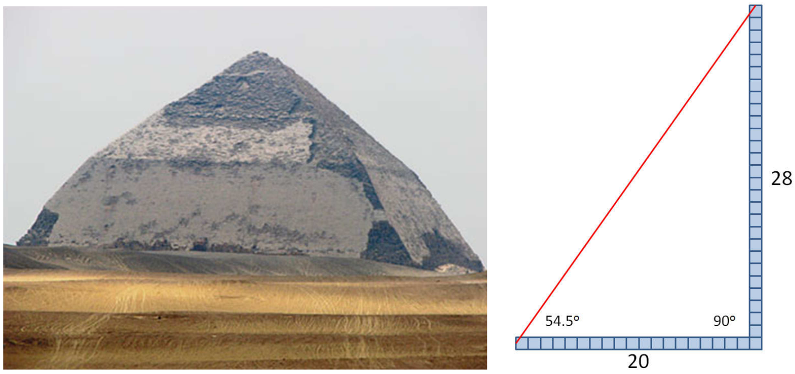

Figure 1.26 The stepped pyramid (2600 BC) and a hypothesis on how the ancients Egyptians might have worked with angles.

take half the base also at 28 fingers, then the angle is

$$\theta = \arctan\frac{28}{28} = \arctan 1 = 45^\circ. \tag{1.80}$$

Now this might not be entirely daring enough, so let us try something steeper by reducing half the base to, say, 20?

$$\theta = \arctan\frac{28}{20} = \arctan 1.4 = 54.5^\circ. \tag{1.81}$$

All right construction starts, but at some point the pyramid starts to crumble at the base and the whole thing is succumbing under its own weight. Disaster! Now what to do. The easiest thing to do is to continue building at a somewhat less daring angle. How about 30? All right, this gives an angle of 43.0°. Now let us compare this with the measured angles of the stepped pyramid that was built for Sneferu: 54.8 and 43.4. That's not too far off.

Accident? Let us continue with Sneferu's son Khufu or Cheops. Obviously, not wanting to repeat the disaster of his father, he decides to start with a somewhat more conservative angle. A half-base of 22 gives an angle of $\arctan\frac{28}{22} = 51.84^\circ$. This is scarily close to the measured angle of 51.82° for the Great Pyramid of Giza/Khufu/Cheops (2560 BC). The next pharao is Khafra or Chephren. Apparently, he does not want to be outdone by his father and decides to go for a steeper pyramid. Taking half-base of 21 gives an angle of $\arctan\frac{28}{21} = 53.13^\circ$. The measured angle of the other Great Pyramid is 53.1°.

Now what about π? Well, let us calculate the ratio of the base and the heights of the Great Pyramid of Gizeh

$$\frac{4 \times \text{Base}}{\text{Height}} = \frac{4 \times 22}{28} = \frac{22}{7} = 3.1428. \quad (1.82)$$

Now this is not the same as $\pi = 3.1415$, but the first three digits are the same. Interestingly, 22/7 was the upper bound for π found by Archimedes (287-212 BC) more than two millenia later, but maybe Archimedes was an alien...

Now what does this proof? Not directly anything, since I have no proof that the argument about the determination of the angles is correct. You are still free to believe that the pyramids are built by aliens, although it is not entirely obvious why you want to stick to *that* hypothesis. What does this teach us about science? A number of things. First, do not get fooled by coincidences. Just because something happens to be close, does not make it the same. Second, keep thinking! Just because you found some interesting hypothesis, does not directly make it a correct one. And thirdly, failing to understand something does not prove something else. Just because you do not understand where the 3.14 comes from is not a direct proof that aliens built the pyramids.

However, not much else remains from the Roman Empire. Pliny the Elder wrote a book on Naturalis Historiæ, which covers just about everything and serves as a model for the modern encyclopedia (although missing the convenient alphabetical organization). The work by Galen (129-200/216) did influential work in medicine. He recognized two different types of blood, but failed to see it as a circulatory system. Instead he thought that venous blood was created in the liver and arterial blood in the heart.

The Roman Empire came down after subsequent invasions by the Visigoths, the Huns, and the Vandals. At the same time, the Empire was split into a western and eastern part. The Eastern Empire, also known as the Byzantine empire was of greater endurance than its western counterpart. It played a great role in transferring knowledge from the islamic empire to the renaissance Italy. Concurring with the decline in the western civilization, the Arab world was on the rise following the teaching of the prophet Mohammed (570-632). One of the innovations that came from the East was the Hindu-Arabic numeral system. The Roman numerals, which are sometimes used, are an inconvenient way to count that hindered the development of mathematics. They consist of letters to represent numbers: M (1000), D (500), C (100), L (50), X (10), V (five), and I (1). If they appear in the right order then they need to be added, thus $VI = V + I = 5 + 1 = 6$. However, if a number appears out of order then it needs to be subtracted from the following larger numeral: $IV = -I + V = -1 + 5 = 4$ (it is not quite known why clocks generally use $IIII$ instead of IV for the number 4). From the number $CIII = C + I + I + I = 100 + 1 + 1 + 1 = 1 - 3$, you can see that Roman did not have the number zero. However, for larger numbers, the system becomes rather cumbersome and intransparent

$$MCMLIV - M + CM + L + IV = 1000 + 900 + 50 + 4 = 1954, \tag{1.83}$$

although practice will undoubtedly help. The Arabic numerals entered Europe around the year 1000.

They also worked on irrational numbers. Rational numbers can be written as fractions, such as 7/8, 113/435, and 90/17. When first comparing numbers such as $1/7 = 0.142857$ and $\pi = 3.141592$, you might not notice too much of a difference (mind you, the decimal system was not yet invented in those days). However, when including more digits you can clearly see a difference. The fraction becomes

$$\frac{1}{7} = 0.142857142857142857142857142857142857142857 1428 \cdots \tag{1.84}$$

It clearly repeats itself after 6 digits. The number π on the other hand never repeats itself

$$\pi = 3.14159265358979323846264338327950288419716939937 51 \cdots \tag{1.85}$$

It just goes on forever. There are other weird things going on. Suppose you make a concerted effort to write down all possible numbers. Let just assume that you think you are done writing down all the numbers between 0 and 1:

$$
\begin{array}{c}
0.\underline{1}3567443567888999441\cdots \\
0.6\underline{7}4333567788889999657\cdots \\
0.64\underline{4}8576532368356678 6\cdots \\
0.887\underline{5}2727776945445657\cdots \\
0.6794\underline{9}687878975655677\cdots \\
0.25343\underline{6}45788657577888\cdots \\
0.347899\underline{8}7975465457577\cdots \\
0.2344364\underline{5}758787897978\cdots
\end{array}
\tag{1.86}
$$

Then some clever guy comes along and takes your list of numbers. He takes the first digit of the first number and adds 1. Now the new number can never be equal to the first number since the first digit is unequal. Then he takes the second number and takes the second digit and adds one, so that the new number is als not equal to the second number, etc. He ends up with the number $0.28560796\cdots$ (where adding 1 to 9 gives 0 again). And, every time you think the list is complete, you can do the same procedure again. So, not only are there an infinite number of irrational numbers, you also cannot count them.

In the islamic world, one was also studying astronomy. Astronomy has always been an important branch of science, essential to calculate seasons and religious days. Abu Ali al-Hasan ibn al-Hasan ibn al-Haytham, better known in the west as Alhazen was one of the better known islamic scientists. One of his works was "Doubts concerning Ptolomy", where he expressed his concerns about the impossibility of the arrangements of the planets suggested by Ptolomy and worries about how the rather abstract mathematical models relates to actual physical motions. However, his intention was not directly to replace Ptolomy's geocentric model. In addition, Alhazen contradicted Aristotle's view that the Milky Way was ignition of emission from the stars occuring in the Earth's atmosphere. The absence of a parallax made this assumption impossible to maintain. Although Alhazen did not take the revolutionary steps taken by renaissance scientists, his doubts on the centuries-old Ptolomaic system was an important step.

While the islamic world was experiencing a golden age, the western world was barely surviving. The early middle ages (500-1000) saw de-urbanization. The only remaining pockets of learning were in monasteries. However, scientific exploration was not a primary concern of the clergy and their major role was the preservation of knowledge from antiquity. A slight brighter period started around the year 800 when Charlemagne united most of western Europe under his rule. This is sometimes known as the Carolingian Renaissance, but that might be somewhat of an overstatement. Having to deal with such a large empire, Charlemagne realized the need of enhancing literacy. For example, not all priests could even read the bible in Latin. However, being able to read is still a far cry from becoming interested in science. However, the Carolingian Empire quickly fell apart due to the splitting of his empire amongst his heirs following Frankish tradition. The western part developed into France, the eastern part developed into modern-day Germany, and the central part, containing Holland, Belgium, Alsace-Lorraine, Switzerland, Burgundy, Savoy, and parts of Northern Italy developed into a region that people would fight over for centuries to come.

Around 1000-1300, Europe enters the High Middle Ages. Despite Renaissance propaganda to portray the Middle Ages as the Dark Ages to offset their own achievements, this period saw significant social change. The population was increasing rapidly to unprecedented levels. The prosperity of this period can still be seen by the magnificent gothic cathedrals built in this period. In addition, there was renewed contact with the Byzantine Empire and the Islamic world, although not always for the right reasons (crusades against the muslims and the reconquista of Spain and Portugal from the Arab empire). These contact led to a rediscovery of the works of Aristotle, Euclid, Ptolomy, Galen, etc. The High Middle Ages also saw a formalization of higher education often a already existing monastic or cathedral schools. Several of these schools were granted charters by the pope. The first was Bologna in 1088, which was at that time in the land owned by the pope (the Papal States). Note that Italy only became unified in 1870 and was agglomeration of different states up to that time. This was followed in the following two centuries by the foundation of universities in Paris (Sorbonne), Salerno, Oxford, Cambridge, Salamanca (Spain), Coimbra (Portugal), Montpellier, Modena, etc. These universities taught what is known as the Studium Generale which consisted of arts (containing everything from philosophy, music, mathematics, natural philosophy) and at least one of the higher faculties: theology, law, or medicine.

There was a renewed interest in scientific exploration. Roger Bacon (1214-1294) was an English friar and philosopher who stressed the need for experimental knowledge. However, our view might have been heavily influenced by the romanticized interpretation of Bacon that occured in the nineteenth century when he was seen as the first modern scientist, far ahead of his time. A more modern view sees Bacon as not too much different from his contemporaries, such as Grosseteste.

However, the High Middle Ages were followed by the Late Middle Ages which saw a significant decline. First, there was a clear climate change occuring around 1300, leading to a significant cooling in Europe. This is also known as the Little Ice Age. This led to the Great Famine of 1315-7, with repeated crop failures. In addition, there were repeated outbreaks of epedemics, such as the plague in 1348-50, known as the Black Death. Finally, there was significant political upheaval such as the fall of Constantinople (modern-day Instanbul and the capital of the Byzantine Empire) to the Ottomans in 1453. Despite all of this there were a number of philosophical developments in the Late Middle Ages. One of them is known as Occam/Ockhams razor, which is traceable to John Duns Scotus (1265-1308) or maybe even Aristotle and Ptolomy. The most familiar expression (although absent in Occam's works) is : "*Entia non sunt multiplicanda praeter necessitatem*"(entities must not be multiplied beyond necessity). We can paraphrase this as "Other things being equal, a simpler explanation is better than a more complex one". Although philosophically nice and practically useful, it is not really scientifically rigorous. Although in science one often finds that simpler solutions are better, this is certainly not always the case.

Other advances of made by the Oxford calculators studying mechanics. The most famous among them are Thomas Bradwardine and his followers William Swineshead, John Dumbleton, and William Heytesbury. They found that "a body moving with constant velocity travels distance and time equal to an accelerated body whose velocity is half the final speed of the accelerated body". Now this is very confusing to read, so let us try to express this into modern terms. The body moving at constant velocity travels at half the final speed of the accelerated body. Let us call the $\frac{1}{2}v$. The distance travelled is then $x = \frac{1}{2}vt$. The accelerated body reaches a final speed of v. Since $v = at$, the acceleration must have been $a = v/t$. The distance travelled by the accelerated body is then

$$x = \frac{1}{2}at^2 = \frac{1}{2}\frac{v}{t}t^2 = \frac{1}{2}vt. \tag{1.87}$$

This is indeed the same as the distance travelled by the body moving at a constant velocity. This result diffused to the continent and was known to philosophers in France, Italy, and other places. This is basically the same as Galileo found with his inclined-plane experiments. So why don't we hear more about this and why do we treat Galileo as a hero and not the Oxford Calculators? Maybe we should. However, although it is a correct statement, it just does not push the envelope far enough. In addition, it lacks the algebraic basis to make it really useful and become the foundation of a real theory of mechanics. So in the end, the whole thing kind of fizzled...

So in just a few pages, we summarized all the accomplishments of 1.5 millenia of scientific thought on the subject of natural philosophy (which includes almost all of the modern-day sciences). We are back at the end of the Middle Ages and the early Renaissance with a flat Earth at the center of the Universe. Well, that is not entirely correct. Most educated people believed that the Earth was actually spherical. This, as is well-known, led to the discovery of America by Christopher Columbus, who mistakenly assumed he had arrived in India. Well, apparently not everything was known about the Earth. However, there was still no simple explanation for the retrograde motion of the planets in the sky.

1.4.2 Geocentric model

This changed in 1543, when Nicolai Copernicus (1473-1543) published *De revolutionibus orbium coelestium*(On the Revolutions of the Celestial Spheres).

The essence of the book is that all the planets rotated around the Sun in circular orbits. Copernicus was born into a wealthy family in Torún in the Kingdom of Poland. He studied at the University of Kraków from 1491-5. He continued his studies at the university of Bologna. Although studying law, his interests were more in the natural sciences and he only obtained his law degree after seven years. He also took a two year break to study medicin at the University of Padua. Apart from several travels, he remained the next forty years of his life in the Prince-Bishopric of Warmia as working secretary and physician to the bishop and as economic administrator of Warmia. He apparently still had time to develop his heliocentric model.

In the Renaissance, there were many advances in a wide variety of fields and the period must have appeared vastly different to the Late Middle Ages that preceded it (at least to the wealthy and educated people). However, the Renaissance has often been romanticized. The contrast with the Middle Ages has been characterized by the phrases *Memento Mori* (Remember you must die) in the Middle Ages and *Carpe Diem* (Seize the Day) in the Renaissance. However, one should not underestimate the power and influence of religion in late fifteenth and sixteenth centuries. From a religious point of view, there were great upheavals with the Protestant Reformation. The Reformation had direct political ramifications with the Habsburg Empire fighting against protestant powers, the religious wars in France, etc. However, in the arts, it was a period of great change. In architecture, the extensive gothic decorations were removed and one was looking for a greater simplicity. In arts, this was found for a large part by a revival of classical ideals in sculpture and in painting. However, one should not entirely write off the Middle Ages as a period where absolutely nothing of interest happened. In particular, Gothic architecture has been an inspiration throughout the next centuries. In addition, the Renaissance was not perfect either. It was for a large part based on an incorrect interpretation of classical ideals and many of these notions survive to this day. Since the paint had disappeared from building and sculpture, it was assumed that it was meant to be monochrome. Since then, the major of the statues have been colorless. In addition, we have even learnt to admire broken statues, such as the Venus de Milo and the Nike of Samothrace. It was a period filled with ideals. The focus turned back on men, but, after centuries of focus on religion, one naturally turned to the phrase that men was created "in the image of God". This certainly sets some pretty high standards for men.

It is in this background that one should see the development of Copernicus' heliocentric view. Certainly, Ptolomy's geocentric model with all those epicycles could not satisfy the idealistic view of the Renaissance man. And if the planets had to turn around the Sun, there is only one possible orbit: the perfect circle. Obviously this view surmounted to openly defying the Roman Catholic church, which, certainly in the days of the Reformation, has serious issues with having their authority chalenged. Giordano Bruno (1548-1600) went even several steps further in his view of the universe: "The universe is then one, infinite, immobile.... It is not capable of comprehension and therefore is endless and limitless, and to that extent infinite and indeterminable, and consequently immobile". An infinite universe: so much for celestial spheres an the presence of heaven beyond the last celestial sphere. In addition, Bruno claimed that the Sun is just another star and all stars are like the Sun. This is another assault on the uniqueness of our solar system and, inherently on us and therefore God Himself. However, it sounds strikingly modern and some have claimed that modern astronomy started with Bruno. Bruno also stated that the laws of motion applied everywhere. This is entirely different from the belief that the universe is finite and divided into a region where the laws of nature applied and the heavens filled with æther, where the laws of heaven applied. For his beliefs, Bruno was burnt at the stake in 1600. Although it has often been stated that Bruno was a martyr of science, dying for his scientific beliefs, the Roman Catholic church had in fact a whole list of acts of heresy against Bruno and the scientific ones were only a minor component.

1.4.3 Galileo

The first person to revolutionize astronomy was Galileo Galilei. He made a wide variety of observations. He observed the remnants of Kepler's supernova. He was certainly not the first to observe them. A supernova was observed in 1054 in the crab constallation by Indian, Arabic, Chinese, and Japanese astronomers (not sure what the Europeans were doing at the time). We now know that a supernova is that has reached the end of its life and that emits more energy in several weeks than the Sun in its entire lifetime. However, Aristotle had posited that supernovas occur in the atmosphere. Galileo studied Kepler's supernova that occured in 1601. It was the explosion of a star 20,000 light years (1.9×10^{17} km) removed from Earth. For a period of several weeks it was brighter than all stars and planets, except Venus. It was visible during the day for three weeks. It was described by Johannes Kepler. This was a good period for supernovas since Tycho Brahes supernova happened in 1572. Galileo wanted to find out if this effect occurred in the Earth's atmosphere. It this was the case, then it was close and it should be possible to observe a parallax. The parallax of an object is

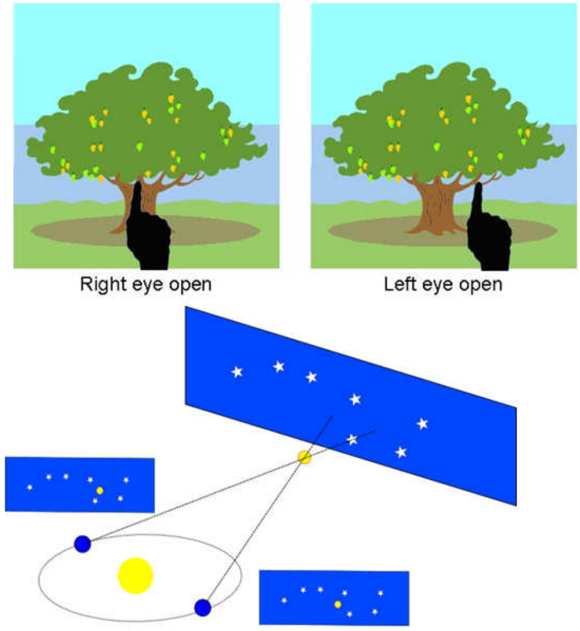

Figure 1.27 The parallax is the movement of an object with respect to the background due to a change in position of the observer. This movement is the same as the change in position of a finger held in front of your eyes when comparing it with either your left or right eye open. To observe the parallax for stars you compare the position at different times of the year.

its motion with respect to the background when the position of the observer is changed. This is the same effect that occurs when you look at your finger with one eye closed. Alternating between your left and right eye, you see the finger move with respect to the background, see Fig. 1.27. This effect is smaller when your finger is further away from your eyes. Galileo was unable to observe any sort of parallax of the supernova. He therefore concluded that the supernova does not occur inside the atmosphere but outside of the Earth. This directly implies that the universe does change, in contradiction with the notions of Aristotle and the church.

Obviously, the distance between your eyes is not sufficient to observe a parallax for a supernova or any star. However, for this you can use the change in position that occurs throughout the year when the Earth moves in its orbit around the Sun. This is a change in the position of the observer of about 300,000,000 km. One of the largest parallaxes of a star is that of 61 Cygni which is a double star system. The motion was first observed by Giuseppi Piazzi in 1804. The first calculation of a distance using the parallax

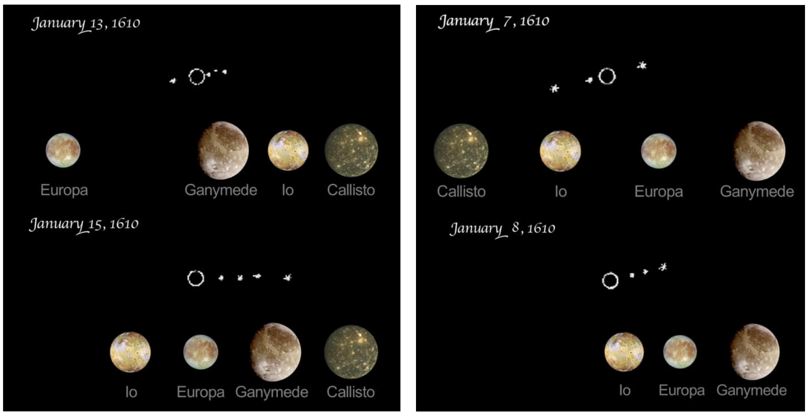

Figure 1.28 Galileo's observations of Jupiter's moons combined with a modern interpretation of the postions of the planet's four largest satellites.

was done by Friedrich Bessel (1784-1846). He determined the distance to 61 Cygna to be 10.4 light years. This is not too far off from modern calculations of 11.4 light years or 1.07×10^{14} km.

A great breakthrough came when the telescope was invented in 1608 by Dutch lens makers such as Hans Lippershey. Galileo heard of this and only a year later he had built his own telescope, which had a magnification of 20 times. He made detailed observations of Jupiter and saw something orbiting the large planet, see Fig. 1.28. He made observations on several days and could only come to the conclusion that Jupiter had several moons of its own. This was another shocking change to the view of the solar system. Galileo's telescope also allowed him to look at celestial objects in more detail. For example, he could clearly see mountains on the Moon. He also observed that the Milky Way was not a nebulous object, but in fact consisted of separate stars.

Galileo also made detailed observations of the phases of Venus. One of the things that he observed was a full phase of Venus. Full phases can only observed when Venus, Earth, and the Sun are in a straight line. In a geocentric system, the Sun is further away than Venus. However, if the three celestial objects are now in a straight line, an observer on Earth is always looking at the dark side of Venus. Therefore, there is no full phase. However, in a heliocentric system there are two options for Venus, Earth, and the Sun to be in a straight line. First, Venus is between the Earth and the Sun, then the dark side of Venus is observed. Or Venus is on the opposite part of its orbit. In that case a full phase is observed. Since Venus is then also at its farthest position from Earth, it should also appear small. This is exactly what Galileo observed. This strengthened Galileo's belief in the heliocentric system. Galileo wrote his beliefs regarding the heliocentric system down in a "Dialogue ceoncerning the two chief world systems" as a series of discussions between Salviati (Copernican and heliocentric view), Simplicio (Ptolomean and geocentric view), and Sagredo, a layman observer. As you might gather from his name, Simplicio was not the smartest of the bunch. This book got Galileo into serious problems with the church who found Galileo "vehemently suspect of heresy". Galileo's sentence of imprisonment was converted into house arrest which he spent in Arcetri from 1633-42. Allegedly, Galileo said "Eppur si muove" (And yet it moves) after his sentence.

1.4.4 Tycho Brahe and Johannes Kepler

Important contributions to the understanding of planetary motion were made by Tycho Brahe and Johannes Kepler. Brahe (1546-1601) was a Danish astronomer who live on Ven, an island between modern-day Denmark and Sweden (it is currently in Sweden). Brahe was from a noble and very rich family. He is also known for loosing part of his nose in sword duel over a mathematical formula with his third cousin Manderup Parsberg. Brahe was known for his very accurate measurements of the motion of the planets. He developed his own "compromise" model of the motion of the planet, which is partially geocentric and partially heliocentric. The Earth is still at the center and the Sun orbit around the Earth. The other planets however orbit around the Sun. Just as with many compromises, this made nobody happy and the model did not get that far. In addition to the planets, Brahe also studied a supernova occured during his lifetime. Brahe died by contracting kidney failure after etiquette prevented him from leaving the table during dinner to go to the bathroom.

However, the person who made good use of Brahe's measurements was his assistant and successor Johannes Kepler (1571-1630). Kepler was born in Weil der Stadt and attended the University of Tübingen. After his studies, he became a mathematics and astronomy teacher in the Austrian city of Graz from 1594 to 1600. There, he wrote his "Mysterium Cosmographicum", which is essentially a defense of the Copernican system. Because of Kepler's refusal to convert to Catholicism, he had to leave his position in Graz. After that, became Brahe's assistant in the Bohemian city of Prague (nowadays the capital of the Czech Republic). However, not too long after Kepler's arrival, Brahe died in 1601. After that he became advisor to Rudolf II, King of Bohemia and Holy Roman Emperor. His employer, although obviously

Catholic, was relatively tolerant to Kepler's Lutheran faith. Although we currently stress Kepler's contributions to astronomy, a major part of his position involved giving astrological advise to Rudolf II.

The detailed measurements done by Brahe and Kepler allowed the latter to perform some serious tests on the geocentric model proposed by Nicolai Copernicus. Although the heliocentric model gave a relatively simple explanation of the retrograde motion of some of the planets in the sky, it was not perfect. Kepler noticed that the assumption of perfect circles for the orbits could not reproduce the actual motion of the planets in the sky. What did this imply? Did that mean that Copernicus model or even the entire geocentric model was wrong? Kepler started to look for different solutions although this goes entirely against the principle of Occam's razor that the simpler and more elegant solution (circles) is preferable to the more complex one (ellipses, ovals, etc.). It also went against the entire Renaissance spirit of harmony and simplicity. Certainly, if God had put the Sun at the center, he would have built the solar system using perfect circles? Why go from the messy Ptolomaic system with epicycles to the perfection of circles only to mess things up again?

Although Kepler's life overlapped with that (Galileo was born before Kepler, but outlived him by 12 years), their spirit was rather different. Although Galileo performed experiments, his mindset was much more of the Renaissance. When Kepler moved away from circular orbits, Galileo did not believe it. He could not believe that God's creation was made up of imperfect shapes. On the other hand, Kepler wasted a lot of time, because he could not identify himself with the Renaissance ideas. Following a circle, the next most obvious shape is an ellipse. However, this was not Kepler's first try. He tried ovals, i.e. egg-shaped orbits, simply because he could not believe that the previous generations had not tried ellipses (still pretty ideal shapes). Coincidentally, similar changes in spirit were also observed in arts and architecture. The period, starting around 1600, is known as the Baroque. Baroque or barroco is the name of an imperfect pearl. The baroque period is known for the shift away from the harmony and classical ideals of the renaissance towards a more dynamics world where emotion played a much larger role. In 1609, Kepler published his *Astronia Nova*. The full title in English is "New Astronomy, Based upon Causes, or Celestial Physics, Treated by Means of Commentaries on the Motions of the Star Mars, from the Observations of Tycho Brahe". The book is over 650 pages and contains the first two of Kepler's laws. The first being:

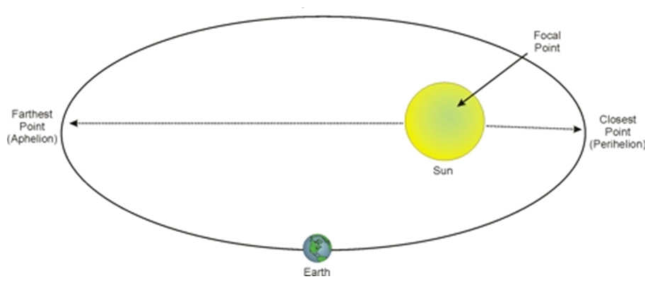

Figure 1.29 Johannes Kepler proposed on accurate measurements by Tycho Brahe that the planet moved in ellipses and not in perfect circles.

The orbit of every planet is an ellipse with the Sun at one of the foci.

This situation is shown in Fig. 1.29. This was a very significant achievement. First of all, gathering the data was a tremendous effort that lasted for years. Second, Kepler's calculations are far from trivial, since they all had to be done by hand. Thirdly, the deviations from circular orbits are relatively small and could easily have been attributed to incorrect observations. Kepler's second is rather more mysterious

a line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

This law is shown in Fig. 1.30, where the wedges indicate the areas that swept out in equal amounts of time. Note that the further the planet is removed from the Sun, the narrower the wedges become. This means that in the same amount of time, the distance travelled becomes smaller. The green arrows in the Figure indicate the velocity. Note that the velocity is always perpendicular to the orbit. This is not a coincidence. In fact, it is the direction of the velocity that determines the orbit. The purple vector is the acceleration. The acceleration always points towards the Sun, because it is due to the gravitational force. It can be split into components parallel and perpendicular to the orbit. The component perpendicular to the orbit makes sure that the planet stays in the orbit. The component parallel to the orbit causes accelerations and decelerations of the planets motion in the orbit. How did Kepler arrive at his second law. He did not have a direct proof of it. A proof was provided only by Newton, 80 years later, and it is rather complicated. Kepler derived his second law using infinitesimals. Basically,

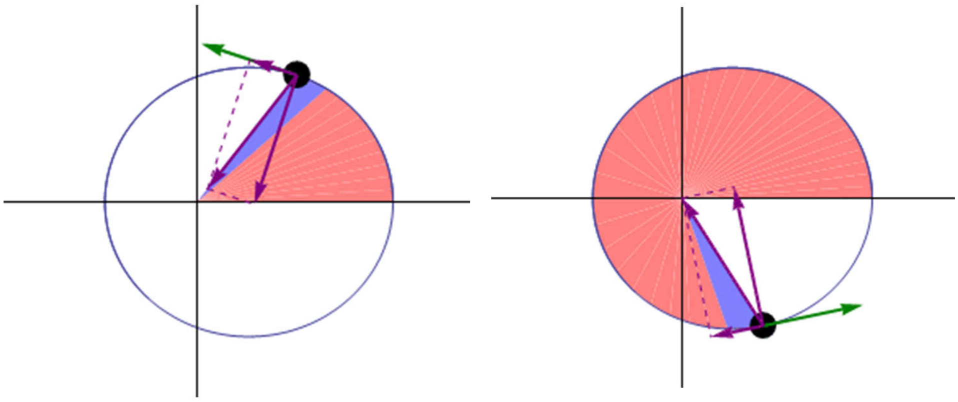

Figure 1.30 The figure illustrates Kepler's second law of planetary motion: a line joining a planet and the Sun sweeps out equal areas during equal intervals of time. The green arrow indicates the velocity of the planet. The purple line is the acceleration which is split into a component along and perpendicular to the orbit.

he determined the areas by dividing the total area into little shapes whose area he knew and then counting the little areas. This might not seem very important, but it is a big step forward.

When looking at some of the famous scientists in the Renaissance, we are some surprised by the scope of subjects they are working on. The most famous example is Leonardo da Vince who was not only a famous painter, but also left works related to natural philosophy, engineering, and anatomy. However, many of the "modern" scientific disciplines are still rather disconnected. What we nowadays call physics dealt mainly with laws of motion. However, these laws were mainly discussed in philosophical or metaphysical terms, for example, the debate on velocity between the views of Aristotle versus those of Galileo and others. Mathematics was hardly used in physics. Some attemps were made to include mathematics, but it never reached the point of having equations of motion that allowed one to attack any type of problem. In fact, most of the Renaissance physics focused on escaping Aristotle's ideas that had dominated the field for over two millenia. Astronomy focused mainly on observation of the celestial objects, such as the planets and the stars. Very few were trying to make a connection between the laws of motion in physics and the motion of the planets. Again, astronomy in the Renaissance spent most of their time trying to get rid of Ptolomy's geocentric world view. Mathematics on the other hand was a rather separate discipline that was mainly concerned with algebra and geometry. Although

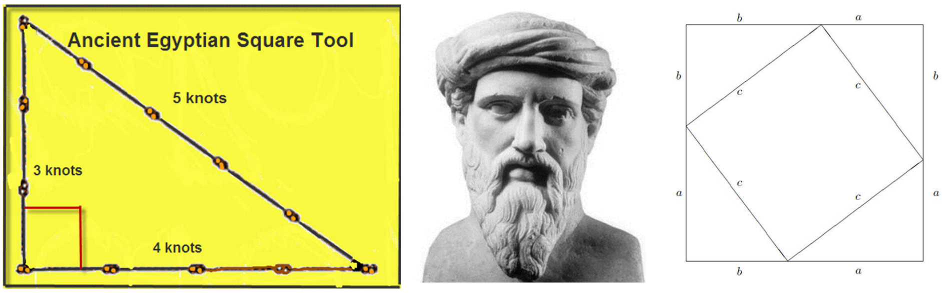

Figure 1.31 The left part shows the Egyptian square tool. A derivation of the rule was given by Pythagoras.

this is somewhat of an oversimplification, it was only during the seventeenth century that clear connection were made between the different disciplines. The most important development in physics was the development of calculus which allowed a mathematics-based physics. The laws of motion expressed in terms of calculus allowed one to approach any possible problem. The initial steps for the development of astronomy were taken by Kepler, who described astronomy as "celestial physics", thereby recognizing that the same physical laws of motion that apply on Earth are also valid for describing the motion of the planets. Granted, his second law of motion is more a phenomenological rule than a derivation based on physical laws.

However, his use of infinitesimals was of significant important and directly linked geometry (the determination of areas, in this case) with laws of motion. However, geometry had a long history, dating back, obviously, to the Greeks. However, the link between geometry and algebra is not directly apparent. So let us look at that in a little more detail.

We have all heard of Pythagoras rule. It is a relation between the sides that applies to triangles with a right angles. Some of these triangles were known to the ancient Egyptian before 2000 BC. One of the best-known examples of Pythagoras rule is the 3-4-5 triangle. The relation between the numbers is given by

$$3^2 + 4^2 = 5^2 \quad \text{or} \quad 9 + 16 = 25. \tag{1.88}$$

By making a string with twelve equidistant knots, one is able to make a square tool, see Fig. 1.31. Other Pythagorean triangles, such as 5-12-13 and 65-72-97 found on the clay tablet Plimpton 322 (1790-1750 BC), where known to the Babylonians. Although, it is not known if they directly related that to triangles. The first real proof came from Pythagoras (570-495 BC). Although the proof is not known, it was generally attributed to him in ancient Greece. There is a great variety of ways to proof the theorem. Let us do one, to get some feeling how to do a mathematical proof. Let us look at the square in Fig. 1.31. The sides are divided into sections a and b. The total area of the square is then

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1.89}$$

However, when connecting the points where the intersections are made, four triangles are formed. The triangles have one right angle, the edges of the square. Figure 1.31 shows that the are of the square can also be written as the area of four triangles and a smaller square with sides of length c

$$4 \times \frac{1}{2}ab + c^2 = 2ab + c^2. \tag{1.90}$$

Since both areas have to be equal, we obtain

$$a^2 + 2ab + b^2 = 2ab + c^2 \qquad \Rightarrow \qquad a^2 + b^2 = c^2. \tag{1.91}$$

This is a good demonstration of the relationship between geometry and algebra. However, let us turn our attention to the calculation of surfaces. One problem that occupied many was the calculation of the circumference and area of a circle. The first good way to calculate these was devised by Archimedes (287-212 BC), probably the greatest physicist and mathematician of hellenistic Greece. Archimedes was born in Syracuse which is on Sicily the island at the tip of the boot of Italy. Archimedes is well-known for discovering the law of buoyancy that Archimedes. Its discovery led Archimedes to run naked through the streets shouting "Eureka" (I found it). He also invented Archimedes' screw and the heat ray devised to set the enemies' ships on fire during the siege of Syracuse by focusing the Sun light on their ships. Tragically, Archimedes was killed after the two-year siege ended and the Romans invaded the city. As the story goes, Archimedes was occupied with his mathematical work and refused to follow orders of a Roman soldier. His last attributed words were: "Do not disturb my circles".

In mathematics circles, on the other hand, his greatest fame is related to the calculation of π. Archimedes realized that the area of a circle must lie between the areas of two polygons. Figure 1.32 shows the situation for two hexagons. Let us take a circle with the radius set to 1. The largest hexagon that fits into the circles touches the circles with its edges. Its area can be calculated by taking the area of 6 triangles, see Fig. 1.32. These triangle can be further split into 2 triangles with one angle equal to $180^\circ/6 = 30^\circ$. Using trigonometry, we are able to derive that the base and height of a triangle with a right angle and a hypotenuse of 1 are $\frac{1}{2}$ and $\frac{1}{2}\sqrt{3}$, respectively. The

Figure 1.32 Archimedes method of determining the area of a circle. Archemides noticed that the area of a circle can be estimated by determining the areas of two different polygons, in this Figure, two hexagons. The area of the circle must lie between these two limits.

area of the triangle is then $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\sqrt{3} = \frac{1}{8}\sqrt{3}$. The total area of the hexagon is then $12 \times \frac{1}{8}\sqrt{3} = \frac{3}{2}\sqrt{3} \cong 2.59$. We can also take a hexagon that is larger than the circle. The smallest of such a hexagon touches the circle with its sides, see Fig. 1.32. The base and the height are now $\frac{1}{3}\sqrt{3}$ and 1, respectively. The area of the triangle is then $\frac{1}{2} \times \frac{1}{3}\sqrt{3} \times 1 = \frac{1}{6}\sqrt{3}$. The area of the hexagon is then $12 \times \frac{1}{6}\sqrt{3} = 2\sqrt{3} \cong 3.46$. Therefore the area of the circle is between 2.59 and 3.46. Now, of course we now that the area of a circle is given by πr^2 (note that, although π is a Greek letter, Archimedes did not use π in this context). Since the radius is one, the area is $\pi \cong 3.14$. Note that although this is a crude estimate, the methodology is more important than the value (one probably could have obtained a better estimate by just measuring the circumference). However, Archimedes managed to extend the procedure up to a 96-gon and was able to show that

$$3\frac{10}{71} = 3.1408 < \pi < 3.1428 = 3\frac{1}{7}. \tag{1.92}$$

Interestingly enough, the upper bound is 22/7 which is the number that looked so surprisingle close to π in the base over height ration of the pyramids. This result is all the more remarkable considering that Archimedes did not have our standard knowledge on trigonometry, the decimal system, let along a calculator! In fact, Archimedes method was the standard method (at least in Europe) used to calculate π for the next 1700 years. Ludolph van Ceulen, the first professor of mathematics at Leiden University in the Netherlands, managed to calculate π up to 35 places using a polygon with 2^{62} sides in 1596. In fact, it was Newton who used series to calculate π, although he did not improve on Van Ceulen's calculation, because in the end this is a boring exercize best left to computers.

Archimedes' method, also known as the method of exhaustion, is in essence the same as Kepler's use of infinitesimals. By dividing an unknown area into many much smaller shapes whose surface is known, one is able to determine the area of the larger surface. We can express this (somewhat sloppily) as an equation

$$\text{total area} = \sum area_{\text{small}}, \tag{1.93}$$

where the summation is indicated Σ, the Greek capital letter S. The summation goes over all the little shapes, whose areas we know, that fit into the total shape.

Kepler still had one more law left in him. In 1612, Rudolf II died and was succeeded by his less tolerant brother Matthias. Kepler spent his remaining years (1612-1630) as a mathematics teacher in Linz, thereby passing by a post at the University of Padua, where he was recommended by Galileo as his successor. However, Kepler preferred to remain in german-speaking Austrian city of Linz. Kepler's third law reads

The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

or in mathematical form

$$\frac{\text{Period}^2}{\text{Axis}^3} = \text{constant}. \tag{1.94}$$

Let us compare this for Earth and Mars. For Earth, the period is, obviously, one year, the semi-major axis can be easily found, giving

$$\frac{(365.25636 \text{ days})^2}{(149,598,261 \text{ km})^3} = 3.9849 \times 10^{-20}. \tag{1.95}$$

For Mars, we find

$$\frac{(686.971 \text{ days})^2}{(227,939,100 \text{ km})^3} = 3.9849 \times 10^{-20}. \tag{1.96}$$

This is particularly impressive. It is therefore not a surprise that Kepler's last book, published in 1619, was call *Harmonices Mundi* or The Harmony of the World.

1.5 The Age of Calculus

It is with Kepler that we see a transition in physics from a world of perfect shapes (circles, spheres, parabolas, etc.) to one dominated by mathematics. This is also where Newton comes in. Although everybody is (at least somewhat) familiar with Newton's three laws, this is not where Newton had the most lasting impact. In particular, since half of Newton's laws are not even his. The importance of Newton lies in the way he approached classical mechanics. He developed a mathematical framework that allowed physicists to attack any problem, not just those of constant acceleration. In fact he, building on earlier work, devised a new way to deal with physical problems. It is known as calculus. Calculus is one of these subjects that many students dread and since this is a non-calculus course, no questions will be asked about them. However, physics revolves around calculus. Calculus is the basis for classical mechanics, electricity and magnetism, thermoquantum mechanics, relativistic quantum mechanics, etc.

However, we already sneaked in some calculus when we were dealing with instantaneous velocities. At some point, we said that the velocity is defined as

$$v = \frac{\Delta x}{\Delta t}, \tag{1.97}$$

i.e. the velocity is the change in position divided by the change in time. We noticed that problems could arise if Δt was very small. But in the end, we could relatively easily work around these problems for a constant acceleration, giving

$$v = \frac{x(t+\Delta t) - x(t)}{t + \Delta t - t} = \frac{\frac{1}{2}a(t+\Delta t)^2 - \frac{1}{2}at^2}{\Delta t}. \tag{1.98}$$

Do not confuse the parentheses. In one case, we write $x(t) = \frac{1}{2}at^2$, where the parentheses indicate that x is a function of t. In the other case $\frac{1}{2}a(t+\Delta t)^2$, we are not talking about a as a function of t, but the parentheses group $(t + \Delta t)$ together. Now if Δt was very small, then $(\Delta t)^2 \ll \Delta t$ and this equation simplifies to

$$v = \frac{\frac{1}{2}a(t^2 + 2t\Delta t) - \frac{1}{2}at^2}{\Delta t} = \frac{\frac{1}{2}a2t\Delta t}{\Delta t} = at. \tag{1.99}$$

In summary, if we know how x depends on the time t, we can figure out how v depends t

$$x = \frac{1}{2}at^2 \quad \Rightarrow \quad v = \frac{\Delta x}{\Delta t} = at. \tag{1.100}$$

Now, if the difference in time is very small, infinitesimally small as it is called, we prefer to use the notation due to Gottfried Wilhelm Leibniz (1646-1716) who invented calculus along with Newton,

$$\Delta \to d \quad \Rightarrow \quad v = \frac{dx}{dt}. \tag{1.101}$$

However, we can do the same procedure for v and obtain the acceleration

$$a = \frac{v(t+\Delta t) - x(t)}{t + \Delta t - t} = \frac{a(t+\Delta t) - at}{\Delta t} = \frac{a\Delta t}{\Delta t} = a. \tag{1.102}$$

Again, if the change is very small, we can write

$$a = \frac{\Delta v}{\Delta t} \quad \to \quad a = \frac{dv}{dt}. \tag{1.103}$$

Therefore, we can write

$$v = \frac{dx(t)}{dt} \quad \text{and} \quad a = \frac{dv(t)}{dt}. \tag{1.104}$$

So, position, velocity, and acceleration are intimately related to each other. Using the expression for the acceleration, we can rewrite Newton's law as

$$F = ma \quad \Rightarrow \quad F = m\frac{dv}{dt} \quad \Rightarrow \quad \frac{dv}{dt} = \frac{F}{m}. \tag{1.105}$$

The last term on the right-hand side, we can say in words as: The change in velocity due to a change in time is given by the force divided by the mass. From this, we can clearly see that the force is changing the velocity and not the position. If there is no net force working on an object ($F = 0$), we find that

$$\frac{dv}{dt} = 0. \tag{1.106}$$

Or, in words, the change in velocity v due to a change in time t is zero. However, this means that only changing the time does not change the velocity. This can only mean that, in the absence of a net force, the velocity is constant or

$$v = \text{constant}. \tag{1.107}$$

However, this is simply Newton's first law. Therefore, Newton's first law is simply a special case of Newton's second law. In fact, it is rather surprising that three centuries later, we still talk about Newton's laws in exactly the same way as Newton.

Now that we have seen that we can go from $x \to v \to a$, is it also possible to go the other way. Earlier on, we talked about the determination of surfaces.

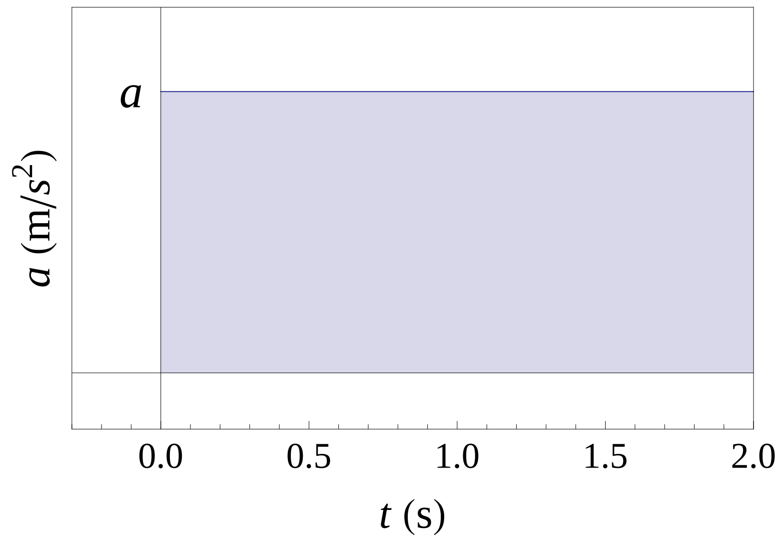

Figure 1.33 The acceleration as a function of time t for the case that the acceleration is equal to the constant a. In the case that the velocity at $t = 0$ equals zero, the final velocity is given by the shaded surface, i.e. $v = at$.

We talked about Kepler's infinitesimals, which we related to earlier work by Archimedes in the determination of π. However, what does determining an area have to do with classical mechanics. Let us look at the expression of v in terms of a constant acceleration a and the time t

$$v = at, \tag{1.108}$$

where we have taken the velocity at $t = 0$ equal to zero. Figure 1.33 shows a plot of the acceleration a as a function of time t. The dependence is very simple since the acceleration does not change as a function of time. The quantity at, which determines the velocity, is simply the area under the straight horizontal line. Let us go one step further and plot $v = at$ as a function of t. This is again a straight line, but now with a slope (the slope is equal to the acceleration a), see Fig. 1.34. We know that the position as a function of time in the case of a constant acceleration is given by

$$x = \frac{1}{2}at^2, \tag{1.109}$$

for the situation where the velocity and position are zero at $t = 0$. Again this equals the area under the line given by the shaded region in Fig. 1.34. The area of the triangle is a little bit more difficult. At a particular time t, the base of the triangle is t. Its height is given by $v = at$. Therefore, the

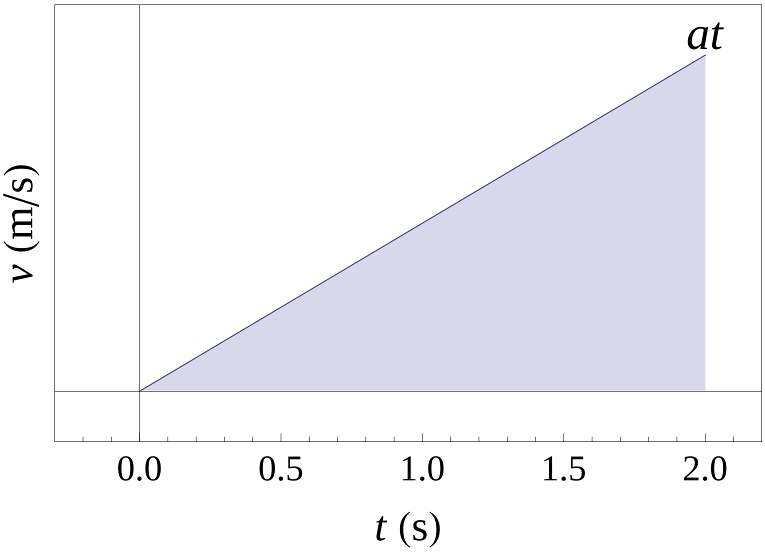

Figure 1.34 The velocity as a function of time t for the case that the acceleration is equal to the constant a. In the case that the position and velocity at $t = 0$ equal zero, the final position is given by the shaded surface, i.e. $x = \frac{1}{2}at^2$.

area is the area of a triangle, which is $\frac{1}{2} \times t \times at = \frac{1}{2}at^2$. This is exactly equal to the expression above. Generally, determining the surface under a curve is more complicated that calculating the area under a triangle. In that case, we have to divide the area into many smaller shapes whose area we know, similarly to Archimedes and Kepler. This is shown in Fig. 1.35. Let us assume that this is the curve for the velocity. The dependence of the velocity as a function of time t is then given by the function $v(t)$. The position at time t is then given by the area under the curve given by $v(t)$ (plus the position at time $t = 0$, which, for convenience, we shall take equal to zero). Somewhat sloppily, we can express this as

$$x = \text{total area} = \sum_{all\ rectangles} \text{area}_{\text{rectangle}}. \tag{1.110}$$

Or, in words, the total area under the curve can be expressed as a sum over all the rectangles that fit under the curve. If we know the expression for $v(t)$ then the area of the rectangles can be easily determined. At a particular time t, the height is given by $v(t)$. The width is given by Δt, where Δt is the time step that we can choose. The smaller we take the time step, the more accurate will be the determination of the area. The area of the rectangle is

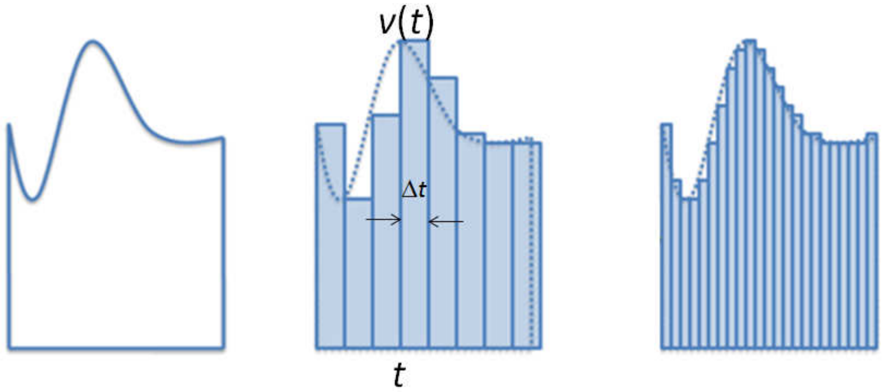

Figure 1.35 This figure shows how the area under a curve can be calculated by dividing the surface into many small rectangles. The smaller the rectangles become, the closer the area of the sum of the rectangles is to the total area under the curve.

then $v(t)\Delta t$. The total area is then

$$x = \sum_t v(t)\Delta t, \tag{1.111}$$

where the summation goes over all the time steps: $0, \Delta t, 2\Delta t, \cdots$. For very small times steps, we use again Leibniz notation and change $\Delta \to d$, but also $\sum \to \int$, where the symbol is an elongated S indicating sum. We then have

$$x = \int v(t)dt \quad \text{and} \quad v = \int a(t)dt. \tag{1.112}$$

This procedure is known as integration.

The bottom line is therefore that differentation and integration allow one to change between position, velocity, and acceleration

$$x(t) \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftarrows}} v(t) \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftarrows}} a(t) \tag{1.113}$$

Therefore, these quantities are intimately related to each other. However, now we understand by Newton's famous book is called *PhilosophiæNaturalis Principia Mathematica* or Mathematical Principles of Natural Philosophy and not "The Three Laws of Mechanics". The three laws are nice, but it is the calculus that is really the meat of the *Principia*.

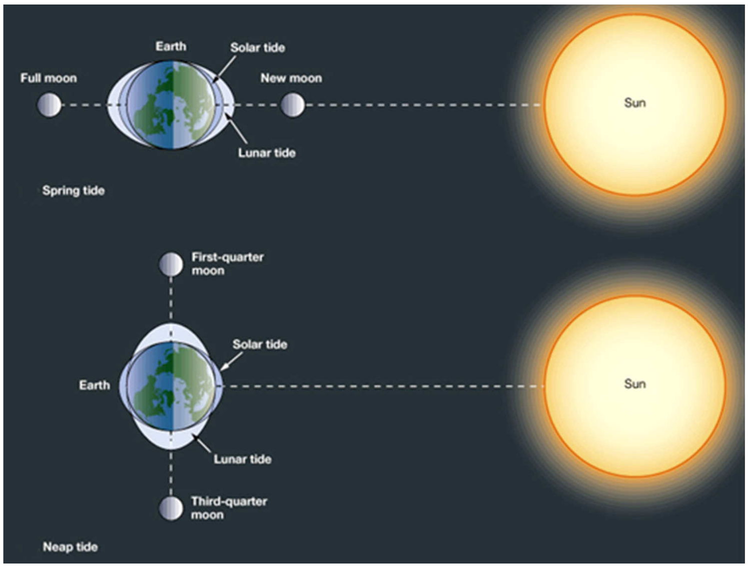

Figure 1.36 The tides are due to the combined gravitational forces of the Moon and the Sun, where the effect of the Moon is about twice that of the Sun. When there is a spring tide, the combined action of the Sun and the Moon gives bigger tides. For first and third quarters Moons, the tides are smaller.

1.6 Newton's third law

1.6.1 Newton's law of Gravitation

The solar system is held together by the gravitational pull of the Sun. We discussed gravity before when talking about constant acceleration on Earth, where we took the force to be equal to mg. However, if we look at the planets then obviously this is a simplification. It is clear that the Sun pulls on the Earth. However, we are all too familiar with the gravitational pull of the Earth. Does the Earth also pull on the Sun? And if so, how much? Does the Sun's pull harder on the Earth? What about the Moon and the Earth.

Most people know that the Earth is feeling the gravity of the Moon, since it is well known that the tides can be related to the gravitational pull of the Earth, see Fig 1.36. In fact, this directly tells you that there must also be a dependence on the distance. If there was no dependence on the distance then the gravitational field of the Moon (and the Sun) would not change when the Earth is rotating around its axis. However, this does not quite explain why there are two tides. At first, you might think that there should be only one tide, because on one side of the Earth the graviational pull is more than on the other side. So when the Earth rotates around its axis,

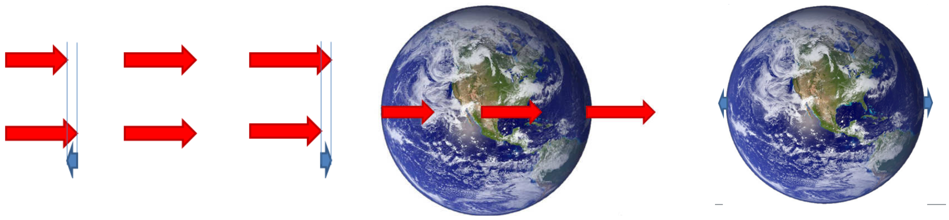

Figure 1.37 The red arrows give an exaggerated picture of the change in gravitational pull by the Moon (or the Sun), which we take to be on the right side. The arrows indicate the values on the left side, center, and right side of the Earth, respectively, see also the Figure of the Earth in the center. However, it is better to look at the differences with respect to the value at the center of the Earth, see the blue arrows. In that case there is an effective for to the left and right on the left and right side of the Earth, respectively.

it would only pass the region with a stronger pull once. However, this is incorrect. It is better to look at the differences of the gravitational pull with respect to the center of the Earth. If we subtract the value at the center of the Earth, then on the side that is closer to the Moon, see Fig. 1.37, there is an effective force towards the Moon. However, on the opposite side, there is an effective force away from the Moon. At a distance from the Moon equal to that at the center of the Earth the effective force is zero. So twice a day, the effective force is pulling the water away from the center of the Earth, therefore there are two tides. The water on the Earth is shaped a bit like jelly that is accelerated in the direction of the Moon. It is elongated along the line connecting the Earth and the Moon. It then creates two blobs, one on each side of the Earth-Moon axis.

The total tides are a combined effect of the gravitational forces of the Moon and the Sun. The size of the tide depends on the relative positions of the Moon. We know that the effect of the Moon is larger (the tides due to the Sun are 46% of those due to the Moon). You might think at first that this is due to the fact that the gravitational pull from the Moon is stronger. However, this is not the case. The gravity of the Sun is larger. However, since the distance to the Sun is so much larger, the relative variation in gravity on different sides of the Earth is a lot smaller for the Sun than for the Moon. The tides are large when the gravitational forces of the Moon and the Sun work in the same direction. This occurs for full and new moons. For first- and third-quarter moons, the gravitational forces of the Sun and the Moon are at a 90 degree angle and the tides are smaller, see Fig. 1.36. However, if you really want to accurately predict the tides, you also have to take

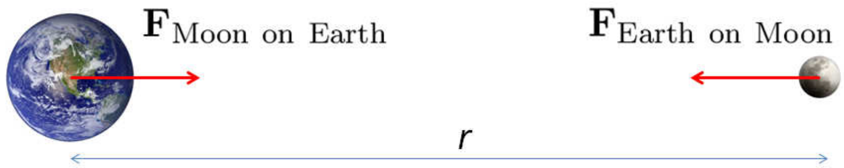

Figure 1.38 The gravitational pull of the Earth on the Moon is equal in magnitude but in opposite direction to the pull of the Moon on the Earth.

into account the tilting of the Earth, local conditions, etc., which turns this problem into a rather nasty one.

However, although we have established that there is a gravitational force of the Moon working on the Earth, we still have not answered the question if the Earth or the Moon is pulling harder. The answer is neither, since both are pulling in exactly the same fashion. We can understand this by looking at the expression of the magnitude of Newton's gravitational force

$$F = G\frac{m_1 m_2}{r^2}. \tag{1.114}$$

We see that the force is determined by the product of the two masses of the objects interacting (say, the Earth and the Moon), divided by the distance squared, and multiplied by a constant G. Therefore, we have two gravitational forces, that of the Earth on the Moon and that of the Moon on the Earth, see Fig. 1.38. They are equal in magnitude, but their direction is opposite

$$\mathbf{F}_{\text{Earth on Moon}} = -\mathbf{F}_{\text{Moon on Earth}}. \tag{1.115}$$

This seems at first counterintuitive. One would expect the object with a larger mass to exert a larger force on the object with the smaller mass. In particular, based on our own experiences, we expect the Earth (large mass) to exert a gravitational pull on us (small mass), and not really the other way around. Apparently, the large mass is confusing, so let us just first look at the gravitational force of two objects with equal masses, see Fig. 1.39. There is no reason why the gravitational forces of the objects on each other should be different

$$\mathbf{F}_{1\text{ on }2} = -\mathbf{F}_{2\text{ on }1}. \tag{1.116}$$

There is a perfect mirror symmetry between the masses 1 and 2 in Fig. 1.39,

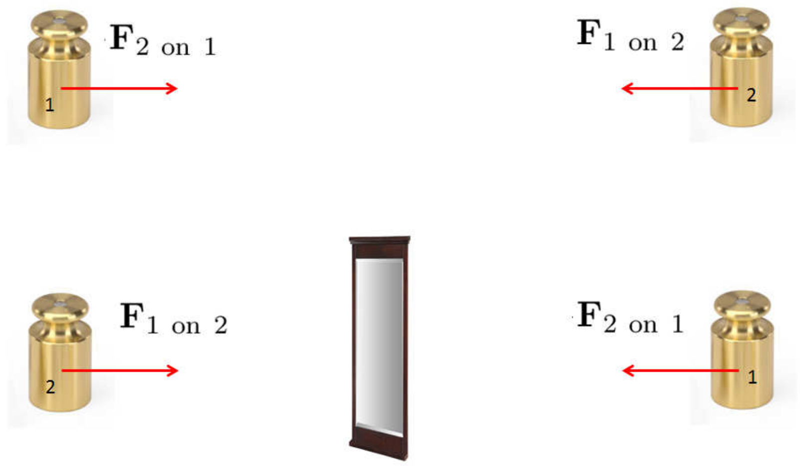

Figure 1.39 One expect the gravity of two equal masses to be the same due to the mirror symmetry of the problem.

so we should also have

$$\mathbf{F}_{2 \text{ on } 1} = -\mathbf{F}_{1 \text{ on } 2} \tag{1.117}$$

(which is obviously the same equation).

Now let us add an additional object of the same mass, see Fig. 1.40. If this object has also the same mass as one, we have, following the above argument,

$$\mathbf{F}_{1 \text{ on } 3} = -\mathbf{F}_{3 \text{ on } 1}. \tag{1.118}$$

We ignore here the gravitational interactions between 2 and 3, which are not relevant for our considerations here. Now let us combine the objects 2 and 3 into one object 23, see Fig. 1.40. The forces are now given by

$$\mathbf{F}_{1 \text{ on } 23} = \mathbf{F}_{1 \text{ on } 2} + \mathbf{F}_{1 \text{ on } 3} = -\mathbf{F}_{23 \text{ on } 1} = -(\mathbf{F}_{2 \text{ on } 1} + \mathbf{F}_{3 \text{ on } 1}) \tag{1.119}$$

So we also find that the forces are equal in magnitude, but in opposite directions if the masses are unequal. We can extend this argument to arbitrary masses. Let us consider masses of $m_1 = 2$ and $m_2 = 3$, see Fig. 1.41. The total force is now proportional to the number of pairs we can make between the different mass units. In the case of 2 and 3, the number of pairs equals 6. In general, the number of pairs is just given by the product of the masses or $m_1 \times m_2$. The mass units are rather arbitrary and we can take them as small as we want and the argument still holds.

The reason why we get confused is because we tend to mix up cause and

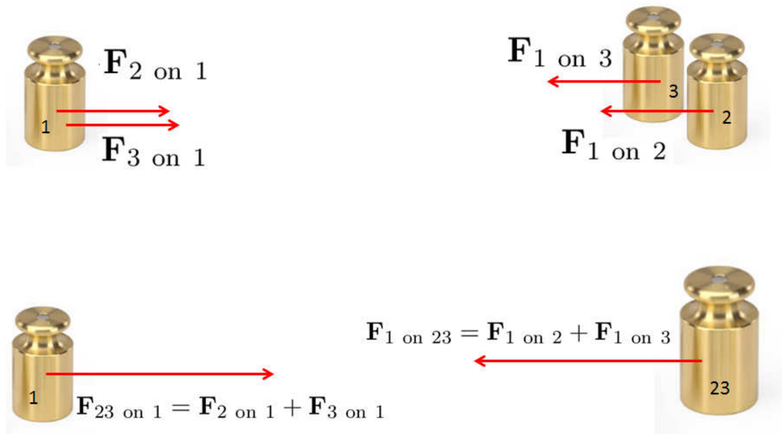

Figure 1.40 In the top part we add an additional mass 3 on the right side. Object 1 exerts a gravitation force on object 3 and object 3 gives an additional pull to object 1. We ignore the interaction between 2 and 3. The bottom part shows the situation if we merge objects 2 and 3.

effect. Although the forces (the cause) is equal in magnitude, that does not mean that the effect (the acceleration) is the same. From the force of the Earth on the Moon, we can calculate the acceleration of the Moon

$$\begin{aligned} F_{\text{Earth on Moon}} &= G\frac{M_{\text{Earth}} m_{\text{Moon}}}{R^2} = m_{\text{Moon}} a_{\text{Moon}} \\ \Rightarrow \quad a_{\text{Moon}} &= G\frac{M_{\text{Earth}}}{R^2} \end{aligned} \tag{1.120}$$

Similarly, we can calculate the acceleration of the Earth

$$\begin{aligned} F_{\text{Moon on Earth}} &= G\frac{M_{\text{Earth}} m_{\text{Moon}}}{R^2} = m_{\text{Earth}} a_{\text{Earth}} \\ \Rightarrow \quad a_{\text{Earth}} &= G\frac{m_{\text{Moon}}}{R^2}. \end{aligned} \tag{1.121}$$

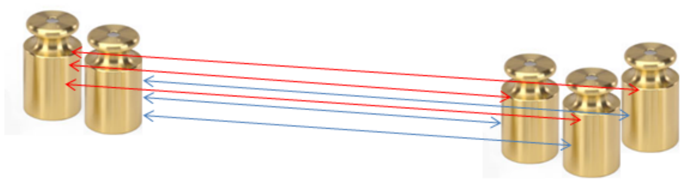

Figure 1.41 The number of force lines between the two masses 1 and 2 is equal to the number of pairs of mass units we can make. That means it is proportional to $m_1 \times m_2$

Note that the acceleration of the Moon is proportional to the mass of the Earth, whereas the acceleration of the Earth is related to the mass of the Moon. This implies that the acceleration of the Moon is larger than the acceleration of the Earth. Still it remains surprising that the gravitational force of you on Earth is equal to that of the Earth on you

$$\mathbf{F}_{\text{Earth on you}} = -\mathbf{F}_{\text{you on Earth}}. \tag{1.122}$$

It suddenly makes you feel a lot stronger. Unfortunately, due to the very large mass of the Earth, it does not really care about your gravitational pull, but the force of the Earth on you is definitely a big deal. In fact, we can calculate it if we use the gravitational constant

$$G = 6.674 \times 10^{-11} \ \text{N m}^2 \ \text{kg}^{-2}. \tag{1.123}$$

We come back to this somewhat later. Inserting the values of the mass and radius of the Earth, we obtain

$$g = a_{\text{you}} = \frac{GM_{\text{Earth}}}{R^2_{\text{Earth}}} = \frac{6.674 \times 10^{-11} \ \times \ 5.97 \times 10^{24}}{6,371,000^2} = 9.81 \text{ m/s}^2,$$

which is the value that we used earlier. On the other hand, the acceleration felt by the Earth is

$$a_{\text{Earth}} = \frac{Gm_{\text{you}}}{R^2_{\text{Earth}}} = \frac{6.674 \times 10^{-11} \ \times \ 80}{6,371,000^2} = 1.31 \times 10^{-22} \text{ m/s}^2$$

if we take a mass of 80 kg. This is why we generally take the Earth as a

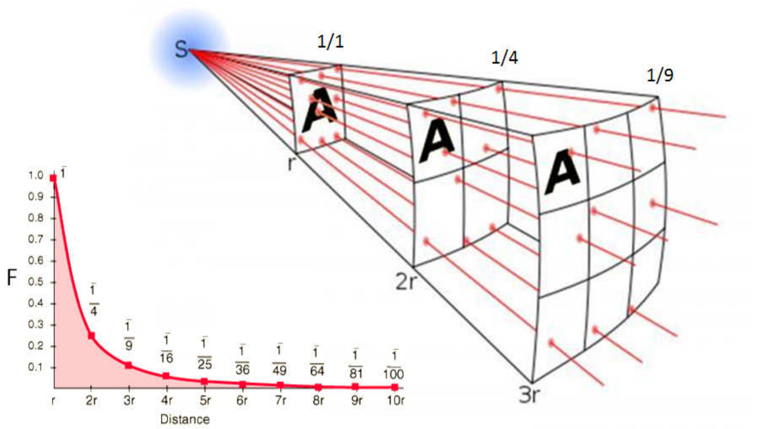

Figure 1.42 The inverse-squared dependence of the force can be understood by considering "force lines" coming from a particular mass. The density of the lines at a particular distance r decreases with the surface area.

"frame of rest" as physicists like to call it. It certainly makes calculating things easier if we simply neglect the effect of small masses on the Earth and simply take the Earth as fixed.

Then we end up with the next part of Newton's gravitational law, the inverse square dependence of r or $F \sim 1/r^2$. This dependence can be understood by looking at Fig. 1.42. Imagine the force of a particular mass is given by force lines. Now force lines do not really exist. In fact, modern theories assume that forces are due to exchange of particles. This works very well for most fundamental forces. Unfortunately, the particles that should give rise to gravity has never been observed. However, for the moment looking at the problem in terms of force lines is sufficient. The density of force lines at a particular distance r is the number of force lines divided by the particular area that we are looking at. If we now increase the distance by a factor, then the area increases by a factor 4, see Fig. 1.42. Equivalently, the area of a sphere, which is given by $4\pi r^2$ increase by a factor 4 if r becomes twice as large. However, the number of force lines remains fixed. Therefore, the density of force lines decreases by a factor four. If we go to a distance $3r$, then the density decreases by a factor 9. Therefore, the force, which is proportional to the density of force lines also decreases by a factor $1/r^2$. This only leaves the factor G in Newton's gravitational force.

1.6.2 Density of the earth

The determination of the factor G was framed in a somewhat different fashion. The gravitational acceleration g can be written as

$$g = G\frac{M_{\rm Earth}}{R^2_{\rm Earth}}. \tag{1.124}$$

If you know the Earth's radius, you can essentially determine the factor GM. This is very important for people on Earth, but it is not the fundamental constant G. However people in the seventeenth and eighteenth century were actually more interested in $M_{\rm Earth}$. The emphasis on fundamental constants only developed later when more were discovered (the speed of light, Boltzmann factor, Planck's constant, etc.). However, the mass of the Earth was something that had their interest. The determination of G was done by Henry Cavendish (1731-1810) using a torsion balance built by John Mitchell (1724-1793). Cavendish was one of the great "amateurs" in science. He was provided for by his noble family being the elder son of Lord Charles Cavendish, son of the second Duke of Devonshire. Cavendish inherited the family fortune later in life. He was a great experimentalist who worked on gravity, electricity, and chemistry. When Maxwell edited unpublished work by Cavendish he realized that many of the results pre-dated the important work and conclusions made by Faraday and Coulomb. He also determined that air consists of 79.167% phlogisticated air and 20.8333% dephlogisticated air. The latter is now known to be oxygen and modern measurements put the percentage at 20.95%. This is impressive indeed. By the way, the phlogiston theory is completely obsolete now. The idea was that flammable materials contained phlogiston. During burning phlogiston is released from materials. The presence of a dephlogisticated gas helps the burning. Unfortunately, this is rather opposite to what happens. Flammable materials are in fact oxygenated when burned, for example, magnesium becomes heavier when burned. However, a lot of materials disappear when burned (they turn into CO_2 and H_2O), so it is understandable that people thought that burning makes the materials lighter. Cavendish is best known for his determination of the gravitational constant. He did this using a torsion experiment, see Fig. 1.43. A rod with two light balls is attached to a torsion rod. When hung between the two large masses the rod will turn. The amount of torsion can then be related to G via Newton's expression for gravity. Henry Cavendish was one of the eccentrics in science. He rarely spoke and his only social events seemed to be the Royal Society Club (a kind of science club). Not that he spoke to there very often. He was incredibly shy of women (does it need to be mentioned that he never married?) and communicated to his servants through notes. He saw his principle heir for only a few minutes each year (who inherited 700,000 pound plus an estate worth about 8,000 pound a year. For comparison, Mr. Darcy, the rich admirer of Elisabeth Bennet in Jane Austen's "Pride and Prejudice" had an income of about 10,000 pound.) Note that the famous Cavendish Laboratory was founded with an endowment by William Cavendish, seventh Duke of Devonshire in 1874.

Figure 1.43

Once Cavendish measured the gravitational constant, he was able to determine the mass of of the earth:

$$G\frac{mM}{r^2} = mg \qquad \Rightarrow \qquad M = \frac{gr^2}{G} = \frac{9.8 \times (6 \times 10^6)^2}{6.67 \times 10^{-11}} = 5.2 \times 10^{24} \ \text{ kg}.$$

Since we know the radius, and therefore the volume, the density is then

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} = \frac{5.2 \times 10^{24}}{\frac{4}{3}\pi(6 \times 10^6)^3} = 5700 \ \text{ kg/m}^3 = 5.7 \ \text{ kg/dm}^3. \quad (1.125)$$

Note that for water the density is 1 kg/dm^3. This number is not entirely a coincidence since 1 dm^3 is a liter and the weight of a liter of water was the definition of a kilogram. The earth is also made out of 34.1% iron (density 7.8), 17.2 % silicon (density 2.3). Also quite a bit of oxygen (28.2%), however this is not in the form of solid oxygen, but as part of compounds. However, the density of the Earth is not uniform. The top layer of several to tens of kilometers is called the crust. Below that is the mantle. However, do not see the Earth as a thin crust upon a fiery ball of burning lava. If you take a piece of mantle, it is just a piece of rock. However, it is still ductile and moves over sufficiently large time scales. The crust moves over the mantle causing what is known as plate tectonics that lead to the formation of mountains in regions where the plats move towards each other. Further deeper into the

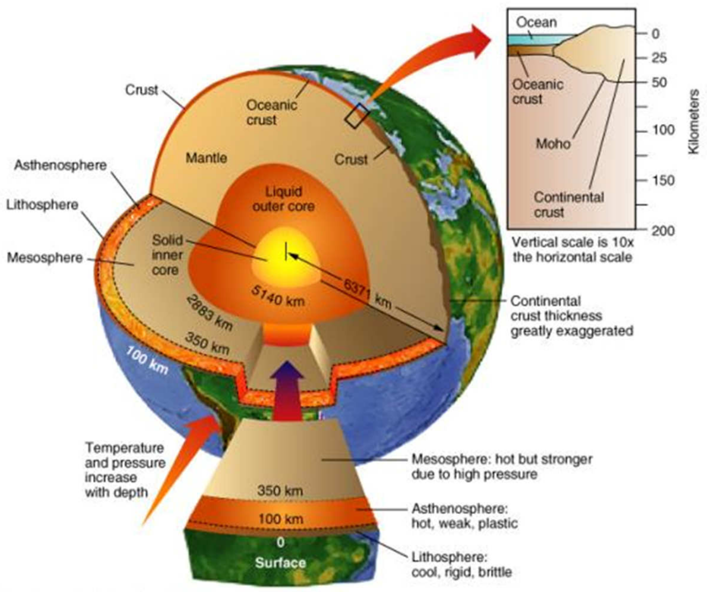

Figure 1.44 The composition of the Earth.

Earth is the core. The core consists of an inner core which is solid and about 70% the size of the Moon. The temperature is about 5700 K, which is about the temperature at the surface of the Sun. The outer core is liquid, a fluid made mainly out of iron an nickel. Although the inner core is liquid, this is not where the lava for volcanoes comes from. The lava occurs at points where the ocean and continental plates meet. Even though the temperature of the mantle is high, it does not melt due to the high pressures. However, in the regions where two plates meet or separate, the pressure is lowered and the rock melts. The rising of molten rock leads to volcanoes.

1.6.3 Action equal minus reaction

Newton's third law reads in Latin:

Lex III: Actioni contrariam semper et qualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse quales et in partes contrarias dirigi.
or in plain English: Law III: To every action there is always an equal and opposite reaction: or the forces of two bodies on each other are always equal and are directed in opposite directions. The first part "To every action there is always an equal and opposite reaction" is the part that everybody always remembers. Unfortunately, it is also the part that is somewhat clumsily formulated, since it implies that there is an object doing the acting and another object doing the reacting. However, as we saw with gravity, there is

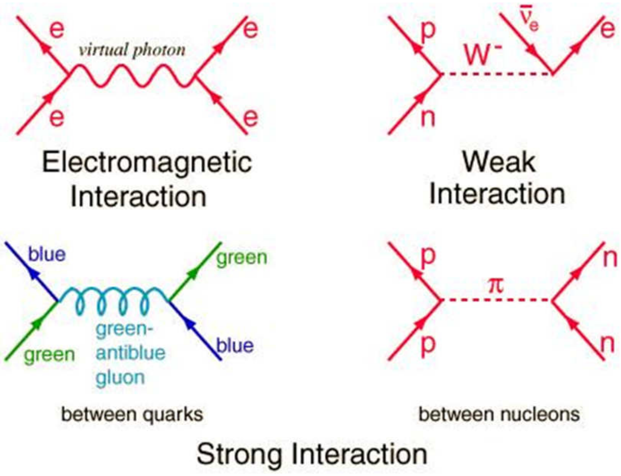

Figure 1.45 Several of the fundamental forces in nature.

no such thing. Forces are interactions between two objects or, when looking at a microscopic level, two particles. Fortunately, the second part is better: "the forces of two bodies on each other are always equal and are directed in opposite directions". This is exactly what we saw when we were looking at gravitation.

Newton's third law always applies. The underlying reason is that all fundamental forces are always interactions between different particles, see Fig. 1.45. For example, the electric force or the Coulomb force is

$$F = f\frac{q_1 q_2}{r^2}, \tag{1.126}$$

where q_1 and q_2 are charges. The other fundamental forces are much more difficult, but the important aspect is that a force is something that occurs between two particles or objects and, just like gravity, one cannot distinguish which one is doing the acting and which one is doing the reacting. However, you might ask, why do we experience that something is acting and something else is reacting? This is just our perception and, in other cases, it is a result of the simplifications that we make. As we saw before, we experience gravity as a force acting on an object. This is because we completely ignore the force of the object acting on the Earth. Even though it has the same magnitude as the force of the Earth on the object, its effect on the Earth is negligible. Now suppose that this was not the case and we had to include the force of the object on Earth. However, if we cannot neglect this force, then we must definitely also include also all the other forces of objects on Earth that are of comparable size or larger. However, if this is the case then the problem becomes completely unsolvable due to the large amount of objects that we need to conclude in our calculations.

Another macroscopic force that satisfies Newton's third law is friction.

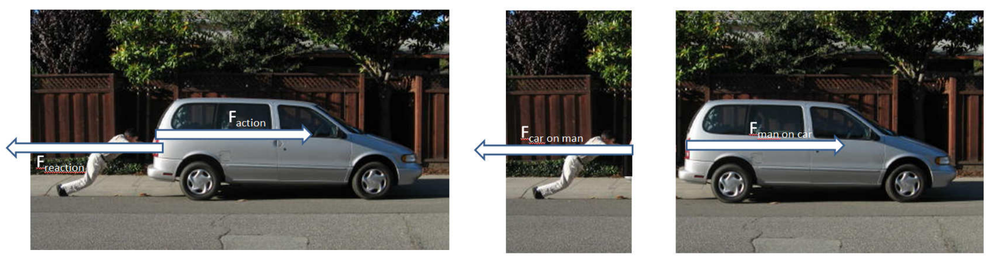

Figure 1.46 The force that the man is exerting on the car is equal but opposite to the force that the car is exerting on the man. However, note that the forces are not working on the same thing.

When looking on a microscopic scale, friction is a horribly complicated force consisting of very many electrostatic interaction between the two surfaces that are experiencing the friction. For example, the friction between a tire and the road depends on the parts of the surface that interact with each via electric interaction of the rubber molecules and the atoms in the road surface. The amount of surface that interacts depends on the weight of the car, the tire pressure, the roughness and material of the road, etc. The more closely you look at this problem, the more surprising it is that quite often you can describe friction simply with a coefficient of friction. However, all these electrostatic interaction satisfy Newton's law and the total friction also satisfies Newton's law.

When considering Newton's third law do not make the following conceptual mistake. If for every action there is an equal but opposite reaction, then the total net force is zero. This is because the forces are acting on different things. For example, see Fig. 1.46. The force exerted by the man on the car equals that of the car on the man. However, they do not cancel since the man only feels the force of the car in this interaction and the car only feels the interaction by the man in this interaction. This does not mean that the car is going to move, since there are more interactions. The car is also interacting with the road. The force of the man on the car is transferred via the wheels to the ground. Due to the interaction with the ground, there is also an interaction of the ground on the car. If the force of the man on the car equals the force of the ground on the car then the car will not move. If on the other hand the force of the man on the car is greater than the force of the ground on the car then the car will move forward. However, the force of the ground on the car can never be greater than the force of the man on the car.

1.6.4 Planets/objects in orbit

Newton's gravitational law can also be used to understand Kepler's laws. We restrict ourselves here to circular orbits, since these are a lot simpler than ellipses. We already noticed earlier that an object or a planet stays in orbit and does not fall into the Earth or the Sun, because it is essentially falling all the time. However, the gravitational force is still acting on the planet/object, so doesn't the velocity change? Actually, the force does change, not in magnitude, but in direction, see Fig. 1.47. The planet/object only stays in a circular orbit if the velocity satisfies a particular condition. At a particular point in the orbit the position vector $\mathbf{r}$ is perpendicular to the velocity vector $\mathbf{v}$, see the arrows in Fig. 1.47. The acceleration due to gravity has to be such that some time later the planet/object is still in the orbit with the velocity perpendicular to the velocity. The change in angle θ between the position vectors has to be exactly the same as the change in angle between the velocity vectors. Therefore the distance travelled $\Delta r = v\Delta t$ divided by the radius or the orbit r, must equal the change in velocity Δv divided by the velocity v due to symmetry

$$\theta = \frac{\Delta r}{r} = \frac{v\Delta t}{r} = \frac{\Delta v}{v} \qquad \Rightarrow \qquad \frac{\Delta v}{\Delta t} = \frac{v^2}{r} \tag{1.127}$$

Remember that for very small changes, we write $\Delta \to d$ and the acceleration is written as

$$a = \frac{dv}{dt}. \tag{1.128}$$

Therefore, the acceleration needed to change the direction is

$$a = \frac{v^2}{r}. \tag{1.129}$$

The acceleration is a result of the gravitational force

$$F = ma \qquad \Rightarrow \qquad G\frac{Mm}{r^2} = m\frac{v^2}{r}. \tag{1.130}$$

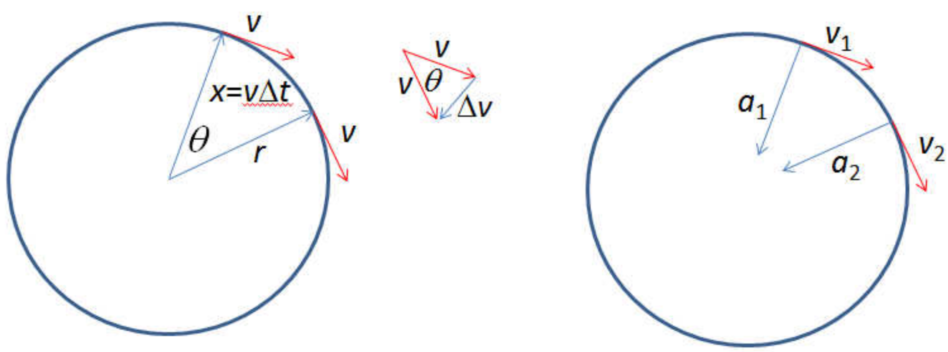

Figure 1.47 The change in velocity due to a central force such as gravitation. Since the force, and hence the acceleration, is perpendicular to the orbit, the magnitude of the velocity does not change, but its direction does.

Now the velocity is given by the circumference $2\pi r$ divided by the period T that it takes the planet to orbit the Sun.

$$G\frac{M}{r^2} = \left(\frac{2\pi r}{T}\right)^2 \frac{1}{r} \qquad \Rightarrow \qquad \frac{T^2}{r^3} = \frac{4\pi^2}{GM} \tag{1.131}$$

The right side is the same for all planets, so we derived Kepler's law (for circular orbits).

1.7 Law of conservation of momentum

In an earlier Section, we saw that Newton's first law was a direct consequence of Newton's second law in the absence of a net force

$$\frac{dv}{dt} = \frac{F}{m} = 0 \qquad \Rightarrow \qquad v = \text{constant}. \tag{1.132}$$

Since the change in velocity due to a change in time is zero, the only option is that the change has to be constant. We can do the same when there is more than one object. For example, for two objects in the absence of a new force we have

$$m_1\frac{dv_1}{dt} + m_2\frac{dv_2}{dt} = 0, \tag{1.133}$$

which is just the sum of Eq. 1.132 for objects 1 and 2. However, taking the masses constant as a function of time, we can also rewrite this as

$$\frac{d}{dt}\left(m_1 v_1 + m_2 v_2\right) = 0 \tag{1.134}$$

or the change in the sum of the products of the mass times the velocity due to a change in time is zero. Just as above, this can only be the case if

$$m_1 v_1 + m_2 v_2 = \text{constant}. \tag{1.135}$$

Often the product of the mass times the velocity is defined as the momentum

$$p = mv. \tag{1.136}$$

Using this we can write

$$p_1 + p_2 = \text{constant}, \tag{1.137}$$

which is known as the law of conservation of momentum. This law was first formulated in 1668-70 by John Wallis (1616-1703) after a call from the Royal Society. Wallis is also known for introducing the symbol for infinity: ∞. The absence of a net force does not imply that there are no forces. The objects can still interact with each other:

$$m_1\frac{dv_1}{dt} + m_2\frac{dv_2}{dt} = F_{1 \text{ on } 2} + F_{2 \text{ on } 1} \tag{1.138}$$

However, Newton's third law states that the interactions between two objects are equal but opposite, so

$$F_{1 \text{ on } 2} = -F_{2 \text{ on } 1}, \tag{1.139}$$

which recovers Eq. (1.133) and therefore momentum is still conserved in the presence of internal forces. However, in the presence of an external force, such as gravity, momentum is not a conserved quantity

$$m_1\frac{dv_1}{dt} + m_2\frac{dv_2}{dt} = -m_1 g - m_2 g. \tag{1.140}$$

This does not mean that a typical collision such as billiard balls on a pool table cannot be treated using conservation of momentum. The reason for that is the collisions occur in the plane of the pool table, whereas the gravitional force works in the direction perpendicular to the table. In gravity is cancelled by the normal force of the pool table on the ball. The ball is not accelerating in the vertical direction.

A typical example of conservation of momentum is the recoil of a rifle. This problem is nice since the initial momentum is zero, since both bullet and rifle are at rest. Momentum is created by the explosion of gun powder and the resulting expansion of air. This is a complex process, but since all the forces are internal momentum is still conserved. Now we do make an approximation, since we will only consider the momentum of the bullet and we neglect the momentum carried by the expanding gases through the barrel. Since the initial momentum is zero, conservation of momentum implies for the final momenta

$$M_{\text{rifle}} v_{\text{rifle}} + m_{\text{bullet}} V_{\text{bullet}} = 0. \tag{1.141}$$

The velocity of the rifle is then

$$v_{\text{rifle}} = -\frac{m_{\text{bullet}} V_{\text{bullet}}}{M_{\text{rifle}}}. \tag{1.142}$$

Some typical values for the masses and the velocity of the bullet are

$$m_{\text{bullet}} \cong 10 \text{ grams} \tag{1.143}$$

$$V_{\text{bullet}} \cong 1000 \text{ m/s} \tag{1.144}$$

$$M_{\text{rifle}} \cong 4 \text{ kg} \tag{1.145}$$

This gives a velocity for the rifle of

$$v_{\text{rifle}} = -\frac{m_{\text{bullet}} V_{\text{bullet}}}{M_{\text{rifle}}} = -\frac{0.01 \times 1000}{4} \times 36001000 = 10 \text{ km/h} \quad (1.146)$$

So the larger object obtain a velocity by ejecting a smaller object. This is basically the principle of a rocket. A rocket moves forward by ejecting rocket fuel. Although the rocket fuel is light, it is ejected with a very large velocity. However, our premise of that mass is conserved in the collision is not really valid here, since the rocket is losing mass rapidly due to the expulsion of all the rocket fuel. The equations necessary to solve this problem are significantly more complicated (it's not called rocket science for nothing. . .).

1.7.1 Inelastic Collisions

Another solvable example is that of a completely inelastic collision where one of the objects (let's call that object 2) is initially at rest. After the collision, the two objects form approximately one objects with a combined mass $m_1 + m_2$. Given an initial velocity of object 1 of v_1, conservation of momentum gives

$$m_1 v_1 = (m_1 + m_2) v' \quad \Rightarrow v' = \frac{m_1}{m_1 + m_2} v_1. \quad (1.147)$$

We can now look at some particular limits. If the masses are equal we have

$$m_1 = m_2 = m \qquad \Rightarrow \qquad v' = \frac{m}{m + m} v_1 = \frac{m}{2m} v_1 = \frac{1}{2} v_1, \quad (1.148)$$

so the combined object with a mass of $2m$ continues at only half the initial velocity of object 1 with mass m.

Other typical limits are where one of the masses is significantly larger that the other. For example, if the moving object has a much larger mass then

$$m_1 \gg m_2 \qquad \Rightarrow \qquad v' = \frac{m_1}{m_1 + m_2} v_1 \cong \frac{m_1}{m_1} v_1 = v_1. \quad (1.149)$$

This means that the velocity is hardly changed by the collision. An example is a freight train colliding into a stationary car. One the other hand, if the moving object is very light then

$$m_2 \gg m_1 \qquad \Rightarrow \qquad v' = \frac{m_1}{m_1 + m_2} v_1 \cong \frac{m_1}{m_2} v_1 \cong 0. \quad (1.150)$$

A typical example is driving into a wall.

Another example of an completely inelastic collision is two cars in a head-on collision. Now if the cars have roughly the same masses and velocities (although in opposite directions), then we have

$$mv - mv = 0 = (m+m)v' = 2mv' \qquad \Rightarrow \qquad v' = 0. \tag{1.151}$$

This means that after the collision the combined wreck of the two cars comes to a complete stand still.

1.7.2 Elastic Collisions-Part 1

Conservation of momentum is significantly simplified in the case of completely inelastic collisions. Most problems are more complex. Let us consider the case of a one-dimensional collision. Conservation of momentum then becomes

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'. \tag{1.152}$$

If the masses and initial velocities v_1 and v_2 are known, that still leaves us with two unknowns: v_1' and v_2'. Now this gives a problem. It is possible to solve one equation with one unknown. For example,

$$2x + 3 = 11 \qquad \Rightarrow \qquad x = 4 \tag{1.153}$$

or two equations with two unknowns

$$x + y = 10 \quad \text{and} \quad x - y = 2 \qquad \Rightarrow \qquad x = 6 \quad \text{and} \quad y = 4 \tag{1.154}$$

Sometimes, we can solve three equations with three unknowns,

$$x + y = 10 \quad \text{and} \quad x - y = 2 \quad \text{and} \quad 3x - y = 14 \qquad \Rightarrow \qquad x = 6 \quad \text{and} \quad y = 4$$

But thats only because the third is the sum of the first two. Usually the third equation just ruins it, such as

$$x + y = 10 \quad \text{and} \quad x - y = 2 \quad \text{and} \quad 2x + y = 8,$$

which has no solution. We can increase the number of unknowns and still get a solution

$$x + y + z = 12 \quad \text{and} \quad x - y + z = 4 \quad \text{and} \quad 2x - y - z = 3$$
$$\Rightarrow \qquad x = 5 \quad \text{and} \quad y = 4 \quad \text{and} \quad z = 3$$

But sometimes the third equation does not help:

$$x + y + z = 12 \quad \text{and} \quad x - y + z = 4 \quad \text{and} \quad x + z = 8.$$

This looks like three equations, but in fact they are only two equations since the third is just half the sum of the first two.

However, Equation (1.152) is only one equation with two variables and there is no way to solve that. The situation becomes even worse in two dimensions

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'. \tag{1.155}$$

The vector notation only hides the fact that there are actually two equations, one for each coordinate

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v_{1x}' + m_2 v_{2x}' \tag{1.156}$$

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v_{1y}' + m_2 v_{2y}'. \tag{1.157}$$

However, when the initial conditions are known, there are now two equations with four unknowns. Therefore, we need more information to be able to solve these problems. So far, we have considered collisions that were completely inelastic. After the collision, the objects continue as one object. There is an opposite limit, where the objects are not damaged at all. A typical situation is the collision of billiard ball with each other or bowling balls against the pins. In this case, one can use another condition, namely conservation of energy.

1.8 Conservation of Energy

Before turning our attention back to collisions let us first reconsider the case to objects dropping due to gravity. The equation of motion for an object dropped from a height h is given by

$$y(t) = -\frac{1}{2}gt^2 + h, \tag{1.158}$$

where g is the acceleration due to gravity. The time t when the object hits the ground is given by

$$-\frac{1}{2}gt_{\text{ground}}^2 + h = 0 \qquad \Rightarrow \qquad t_{\text{ground}} = \sqrt{\frac{2h}{g}}. \tag{1.159}$$

The velocity at this time is given by

$$v = -gt_{\text{ground}} = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}, \tag{1.160}$$

where the initial velocity is zero. We can rewrite this somewhat to obtain

$$\frac{1}{2}v^2 = gh \tag{1.161}$$

or when multiplying by the mas m

$$\frac{1}{2}mv^2 = mgh. \tag{1.162}$$

This is a very nice result, since it directly relates the final velocity to the initial height without the need to concern ourselves with detail such as: how long does it take to fall down, the dependence of the position and velocity as a function of time. The result is much more fundamental then is appears at first and is known as conservation of energy. The principle of conservation of energy is very important in physics. On the left-hand side, there is a term related to the velocity. This energy is known as the kinetic energy. We can also express this in terms of momentum $p = mv$

$$E_{\text{kin}} = \frac{1}{2}mv^2 = \frac{p^2}{2m}. \tag{1.163}$$

The term on the right-hand side is related to the gravitational force and the initial height. This quantity is known as the potential energy. We can understand this by looking at the problem of a falling object not in terms of time but in terms of space. Let us take Newton's second law

$$F = ma. \tag{1.164}$$

The following only works in the case of a constant acceleration, but it gives a good idea of the underlying physics. Let us now consider a constant force working on an object in free space or on a frictionless floor for a distance d, see Fig. 1.48. All the force goes into accelerating the object. Applying a force over a distance d gives

$$Fd = mad. \tag{1.165}$$

However, from the equations of motion for constant acceleration, we know that the distance can be written as $d = \frac{1}{2}at^2$, giving

$$Fd = ma\frac{1}{2}at^2 = \frac{1}{2}m(at)^2 = \frac{1}{2}mv^2, \tag{1.166}$$

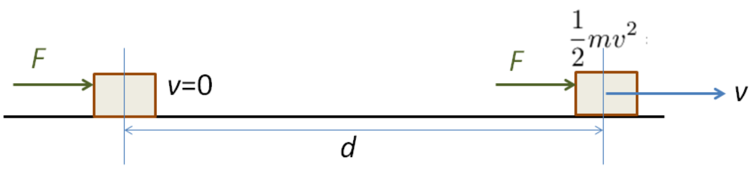

Figure 1.48 By exerting a force upwards in the horizontal direction we are increasing the object's kinetic energy (we are neglecting friction).

since the velocity is given by $v = at$. The quantity on the left-hand side Fd is known as the work. This is the energy that the force puts into the object. Where did that energy go. Well, in this case it went into kinetic energy or energy of motion.

Some of the notion of conservation of energy appeared around the time of Isaac Newton. Gottfried Leibniz identified around 1676-89 the quantity mv^2 with the "living force" (or *vis viva* in Latin). He noticed that this quantity was conserved for systems where the masses do not interact. However, in the eighteenth century quite a few noticed that conservation of momentum alone was insufficient to solve many collision problems. The used the Leibniz's principle as an additional constraint of collisions. In a following Section, we shall deal with these type of collisions, known as elastic collisions. However, the real basis for kinetic energy and its relation to work was laid down by Gaspard-Gustav de Coriolis (1792-1843) and Jean-Victor Poncelet (1788-1867) in the early ninetheenth century.

Now suppose the object is not in free space but is on Earth lying on the ground. We now apply a force upwards that is equal to the gravitational force on the object, i.e. $F = mg$. (Let us ignore the parts where we have to accelerate it to set it in motion and decelerate it when we are at the desired height), see Fig. 1.49.. The work done in that case is equivalent to

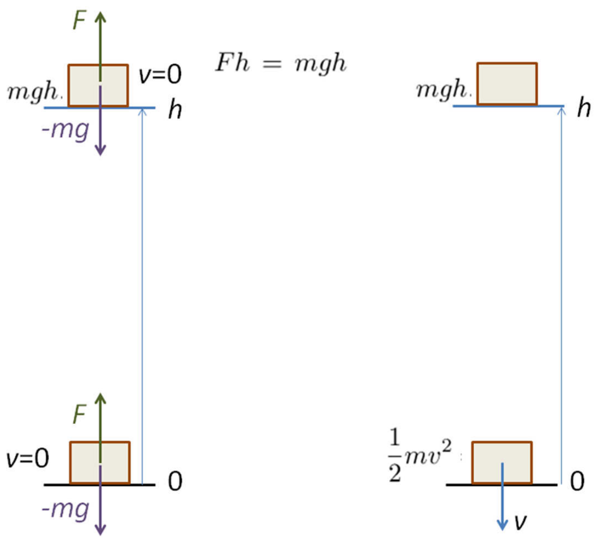

Figure 1.49 By exerting a force upwards in the vertical direction we are increasing the object's potential energy. This energy is converted into kinetic energy when the object is released.

$Fh = mgh$. However, at the end, there object does not have any kinetic energy, since all the work went into overcoming the gravitational force. This does not mean that the object does not have any energy. It has potential energy. It is called potential energy because that energy can be converted into a different energy. As we saw above, it we drop it from that height then, by the time it reaches the ground, it will have a kinetic energy exactly equivalent to the potential energy, or $E_{\text{kin}} = mgh$.

Let us now consider the situation where we do not drop it entirely to the ground but look at a different height h_1. Conservation of energy tells us

$$mgh + \frac{1}{2}mv^2 = \text{constant}. \tag{1.167}$$

Therefore, we can equate the total energy at different heights. If we drop the object from a height h, then we know that the kinetic energy at that height is zero. However, at a intermediate height, we have both potential and kinetic energy. Equating the two gives

$$mgh = mgh_1 + \frac{1}{2}mv_1^2 \quad \Rightarrow \quad \frac{1}{2}mv_1^2 = mg(h - h_1)$$
$$\Rightarrow \quad v_1 = \sqrt{2g(h - h_1)}. \tag{1.168}$$

This shows that the object gains velocity, but less than it would have if it was dropped all the way to the ground since $h - h_1 < h$. However, what happens now if we drop it from height h_1, but also give it the velocity v_1 in the downwards direction. Again, applying conservation of energy and taking into account that there is no potential energy at the ground ($h = 0$), we have

$$\frac{1}{2}mv^2 = mgh_1 + \frac{1}{2}mv_1^2. \tag{1.169}$$

However, using the result from above, we find

$$\frac{1}{2}mv^2 = mgh_1 + mg(h - h_1) = mgh, \tag{1.170}$$

which is exactly equal to the kinetic energy that it would have had if it was dropped from a height h with an initial velocity of zero. However, what is weird is that the kinetic energy does not tell us which way the velocity at height h_1 is. We could have thrown it upwards with a velocity v_1. In that case, it would go up first; come to rest at the maximum of the orbit; then come down again. However, as we saw before, when it is back at h_1, it will again have a velocity of v_1 but now in the downward direction. But what is even stranger is that we could also have thrown it in the horizontal direction with a velocity v_1, see Fig. 1.50. Now let us separate the different directions. We now have a horizontal motion which does not change according to Newton's first law, since there is no net force in the horizontal direction. The kinetic energy related to the horizontal direction is then

$$\frac{1}{2}mv_{\text{horizontal}}^2 = \frac{1}{2}mv_1^2 = mg(h - h_1) \tag{1.171}$$

In the vertical direction, the initial velocity at height h_1 was zero. This is equivalent to dropping an object from height h_1. The kinetic energy for the vertical direction is therefore

$$\frac{1}{2}mv_{\text{vertical}}^2 = mgh_1. \tag{1.172}$$

So the total kinetic energy at the ground is now

$$\begin{aligned}\frac{1}{2}mv^2 &= \frac{1}{2}mv_{\text{horizontal}}^2 + \frac{1}{2}mv_{\text{vertical}}^2 \\ &= mg(h - h_1) + mgh_1 = mgh,\end{aligned} \tag{1.173}$$

which is again the same result! However, the object is now moving with a speed $v_{\text{horizontal}} = \sqrt{2g(h-h_1)}$ in the horizontal direction and $v_{\text{vertical}} = \sqrt{2gh_1}$ in the negative vertical direction.

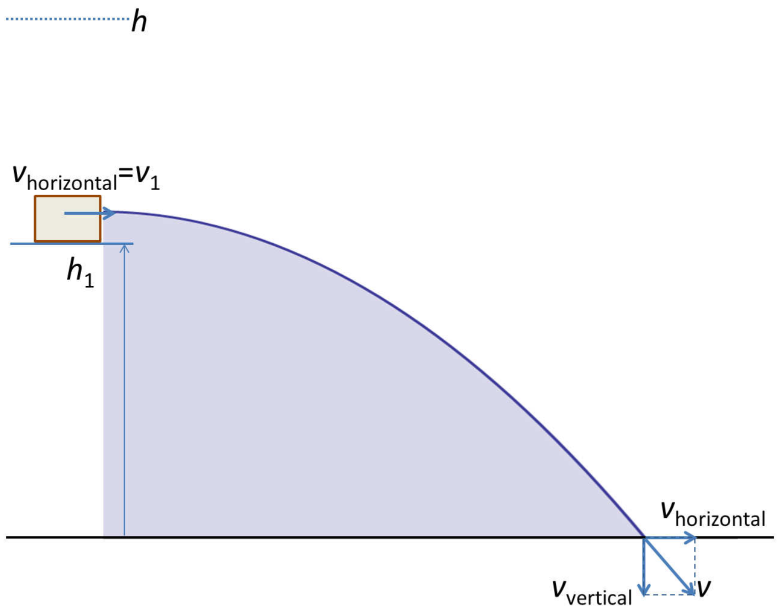

Figure 1.50 Conservation of energy also holds when the object is not thrown in the direction in which the potential energy decreases.

1.8.1 Conservation of Energy and Friction

In the previous Sections, we considered problems in the absence of friction or air resistance. Let us consider again the situation where we push an object in the horizontal plane, but now there is friction between the object and the ground. We push again with the same force F, so the same amount of work Fd is done over a distance d. Let us assume that the initial kinetic energy is much smaller than the work. Let us take the situation where the friction force is equal to the applied force. In that case, there is no acceleration, so the final kinetic energy equals the initial kinetic energy. So we do not gain any kinetic energy. Since the movement is in the horizontal plane, we also are not putting any potential energy into the object. So, we are putting work into the object, but the energy of the object does not change. This seems to be in conflict with the law of conservation of energy. Obviously, this cannot be correct. There are two ways to look at this problem. First, there is a friction force working on the object. This force is working in the opposite direction as the applied force and is therefore equal to $-F_{\text{friction}}d$. Since the object does not accelerate this work is also equal to $-Fd$ and therefore cancel the work by the applied force. We can also say that, although we are putting work into the object, the object is putting work into the ground via the friction forces between the object and the ground. Second, although from a pure mechanical point of view it look like energy is lost. This is why friction is called a nonconservative force, However the energy is not really lost, this energy is simply converted into a different type of energy. The friction between the floor and the object generates heat, which is simply a different type of energy. The friction raises the temperature of the surfaces that are in contact with each other. Therefore, from a thermodynamic point of view (the study of heat and its relation to energy and work), the law of conservation of energy.

Does this mean that energy is always conserved? No, it does not, but energy conservation holds in almost all of our every day experiences. The

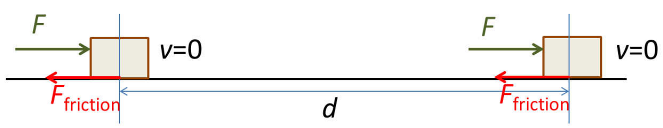

Figure 1.51 When an object experiences friction with the floor while being pushed, it does not gain any kinetic or potential energy when the applied force is equal to the friction force. Instead the work is converted into heat at the contact areas.

concept of energy becomes much more important in other subjects, not only thermodynamics, but also quantum mechanics. In quantum mechanics, that deals with mechanics at an atomic level, one almost never talks about forces, but mainly about kinetic and potential energy. One can also demonstrate that conservation of energy is directly related with the absence of a dependence of the interactions in time (and remember, we arrived at energy because we wanted to get rid of the detailed time dependence of the equations of motion). You might want to point out that the gravitational force changes when an object falls, since its dependence on the height is not constant but changes as $1/r^2$. However, this is not a change in the interaction since gravitation always has a $1/r^2$ potential. Gravitation, even with a more complicated r dependence is still a conservative force. A change in time would be if suddenly the gravitation would, for example, change from a $1/r^2$ potential to a $1/r^3$ potential. Apart from a lot of other disastrous consequences, it would also mean that energy is no longer conserved. When we look at relativity, energy is also not a conserved quantity. Einstein's famous expression $E = mc^2$ implies an equivalence between mass and energy. Mass can be converted into energy and energy into mass. For example, we can create particle and antiparticles out of nothing: out of gamma rays, electrons (particle) and positron (antiparticle) can be created. The gamma rays have no mass, but the electron and positron, even though there each others antiparticles have both a positive mass. They do have an opposite charge. That a lot of energy can be obtained from mass is well known from nuclear explosions. However, this is a totally different topic.

1.8.2 Elastic Collisions-Part 2

In the previous Sections, we discovered another principle, conservation of energy, that can help us solve collision problems. Obviously, this will only help us if the collision indeeds conserves energy. In this case, we are talking about conservation of mechanical energy. This is known as an elastic collision. No energy should be lost in the collision to heating of the objects. In addition, no energy should be lost to create permanent deformations of the objects. The objects are allowed to deform elastically in the same fashion as a spring deforms elastically. A typical example is the collision of two billiard balls which do not deform or heat during the collision. Unfortunately, this still does not allow us to entirely treat the problem of hitting billiard balls since in a real two-dimensional case, we also need to know how the balls hit each other. In addition, there is the problem of spin. So we just consider head-on collisions.

Let us consider an elastic collision in one dimension. We know that momentum is conserved,

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \tag{1.174}$$

Note that if the masses and the two initial velocities are given, there still remain two unknowns v_1' and v_2'. In the case of an elastic collision, we can also use conservation of energy

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2. \tag{1.175}$$

It is convenient to rewrite collect the velocities of each object on one side of the equation

$$\frac{1}{2} m_1 (v_1^2 - v_1'^2) = \frac{1}{2} m_2 (v_2'^2 - v_2^2). \tag{1.176}$$

Using the special product $a^2 - b^2 = (a-b)(a+b)$, we can write

$$m_1 (v_1 - v_1')(v_1 + v_1') = m_2 (v_2' - v_2)(v_2' + v_2). \tag{1.177}$$

We can rearrange the conservation of momentum in the same fashion

$$m_1 (v_1 - v_1') = m_2 (v_2' - v_2). \tag{1.178}$$

Dividing these to equations gives

$$v_1 + v_1' = v_2' + v_2 \quad \Rightarrow \quad v_1 - v_2 = v_2' - v_1' = -(v_1' - v_2'). \tag{1.179}$$

This results shows the magnitude of the relative velocity remains the same, but its sign changes.

Example: Equal masses. For two billiard balls with equal masses the conservation of momentum reduces to

$$m v_1 + m v_2 = m v_1' + m v_2' \quad \Rightarrow \quad v_1 + v_2 = v_1' + v_2' \tag{1.180}$$

In addition, we also have

$$v_1 - v_2 = v_2' - v_1'. \tag{1.181}$$

Adding the two equations gives $v_1 = v_2'$ and subtracting gives $v_2 = v_1'$. So the velocities of 1 and 2 are simply exchanged. A particular case is when one of the balls is at rest $v_2 = 0$. This directly gives $v_1' = 0$ and $v_2' = v_1$. This is often observed when playing pools when the balls have equal masses and in the absence of spin on the balls. It is also well-known from Newton's cradle.

General solution for a collision in one dimension. Two solve this problem, we have two equations: conservation of momentum which we rewrote as

$$m_1(v_1 - v_1') = m_2(v_2' - v_2). \tag{1.182}$$

and the condition for the relative velocities derived above

$$v_1 - v_2 = v_2' - v_1' \quad \Rightarrow \quad v_1' = v_2' - v_1 + v_2. \tag{1.183}$$

Inserting v_1' in the equation for conservation of momentum gives

$$\begin{aligned} m_1[v_1 - (v_2' - v_1 + v_2)] &= m_2(v_2' - v_2) \\ \Rightarrow \quad 2m_1v_1 + (m_2 - m_1)v_2 &= (m_1 + m_2)v_2', \end{aligned} \tag{1.184}$$

giving for v_2',

$$v_2' = \frac{2m_1}{m_1 + m_2}v_1 + \frac{m_2 - m_1}{m_1 + m_2}v_2. \tag{1.185}$$

We can use this to derive v_1',

$$\begin{aligned} v_1' &= v_2' - v_1 + v_2 \\ &= \frac{2m_1}{m_1 + m_2}v_1 + \frac{m_1 - m_2}{m_1 + m_2}v_2 + \frac{m_1 + m_2}{m_1 + m_2}(-v_1 + v_2) \\ &= \frac{m_1 - m_2}{m_1 + m_2}v_1 + \frac{2m_2}{m_1 + m_2}v_2. \end{aligned} \tag{1.186}$$

Apparently, we also could have found this by exchanging 1 and 2. There is clearly some symmetry in the problem.

Example: Target at rest. For the target at rest, $v_2 = 0$, we have

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}v_1 \quad \text{and} \quad v_2' = \frac{2m_1}{m_1 + m_2}v_1. \tag{1.187}$$

Let us consider some limiting cases:

$m_2 \gg m_1$. In this limit, we obtain

$$v_1' \cong -v_1 \quad \text{and} \quad v_2' \cong 0. \tag{1.188}$$

In this limit, the target remains at rest, and the incoming ball simply bounces back. This is comparable to the ball bouncing of a wall.

$m_1 \gg m_2$. In this limit, we obtain

$$v_1' \cong v_1 \quad \text{and} \quad v_2' \cong 2v_1. \tag{1.189}$$

Thus the velocity of m_1 is hardly unchanged, however, mass m_2 takes off