

QUANTUM FIELD THEORY AND FUNCTIONAL INTEGRALS

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ABSTRACT. These notes were inspired by the course “Quantum Field Theory from a Functional Integral Point of View” given at the University of Zurich in Spring 2017 by Santosh Kandel. We describe Feynman’s path integral approach to quantum mechanics and quantum field theory from a functional integral point of view, where the main focus lies in Euclidean field theory. The notion of Gaussian measure and the construction of the Wiener measure are covered. Moreover, we recall the notion of classical mechanics and the Schrödinger picture of quantum mechanics, where it shows the equivalence to the path integral formalism, by deriving the quantum mechanical propagator out of it. Additionally, we give an introduction to elements of constructive quantum field theory.

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1. INTRODUCTION

We want to give a review of quantum field theory, using perturbative methods with the notion of Feynman path integrals. In classical mechanics we consider an action functional¹

$$S(q) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t))dt,$$

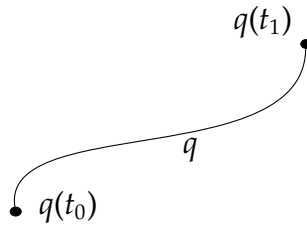


FIGURE 1. The path of least action, i.e. the solution to $\delta S = 0$, between two points $x = q(t_0)$ and $y = q(t_1)$ in space-time.

where $L(q, \dot{q}) = \frac{1}{2}m\|\dot{q}\|^2 - V(q)$ is called the *Lagrangian* function of the paths $q: [t_0, t_1] \rightarrow \mathbb{R}^n$ with some function $V \in C^\infty(\mathbb{R}^n)$ depending on q , called the *potential energy*. We denote by $\text{Path}_{(x,y)}^{[t_0,t_1]}(\mathbb{R}^n)$ the space of all such paths with $q(t_0) = x$ and $q(t_1) = y$. By considering the methods of variational calculus, one can show that the solutions of the equation $\delta S = 0$ for fixed endpoints (i.e. the extremal points of S) give us the classical trajectory of the particle with mass $m \in \mathbb{R}^+$. The equations following from $\delta S = 0$ are called the Euler–Lagrange equations (EL), and they are exactly the equations of motion obtained from Newtonian mechanics. Newton’s equations of motion appear from the law $F = ma(t) = m\ddot{q}(t)$ (read it “force equals mass times acceleration”). To see this, we recall that the *momentum* in physics is given by $p = mv$, where v denotes the velocity of the particle with mass m . Then, by the fact that $v = \dot{q}$, one considers the coordinates $\dot{q} = \frac{p}{m}$ and $\dot{p} = -\nabla V$, where ∇ denotes the gradient operator. The *Hamiltonian* approach considers the space with these coordinates to be the classical phase space (classical space of *states*) given by $T^*\mathbb{R}^n \ni (q, p)$ endowed with a *symplectic form*² given by

$$\omega = \sum_{i=1}^n dq^i dp_i.$$

Moreover, one considers a *total energy function* (or a *Hamiltonian function*) $H(q, p) = \frac{\|p\|^2}{2m} + V$, where V is again a potential energy function. In the physics literature, the first term of H is called the *kinetic energy*. This function is said to be *Hamiltonian* if there is a vector field X_H such that

$$\iota_{X_H}\omega = -dH,$$

¹In the physics literature, it is common to denote the time-derivatives by “dots”, i.e. $\frac{d}{dt}q(t) = \dot{q}(t)$.

²we will not always write \wedge between forms but secretly always mean the exterior product between them, i.e. for two differential forms α, β , we have $\alpha\beta = \alpha \wedge \beta$.

where ι denotes the *contraction map* (also called *interior derivative*). The vector field X_H is called the *Hamiltonian vector field* of H . In the case at hand, since ω is nondegenerate, every function is Hamiltonian and its Hamiltonian vector field is uniquely determined. For H being the total energy function and the canonical symplectic form on the cotangent space, we get the following Hamiltonian vector field: A vector field on $T^*\mathbb{R}^n$ has the form general form $X = X^i \partial_{q^i} + X_i \partial_{p_i}$. Thus, applying the equation for being the Hamiltonian vector field of H we get $-dH = X_i dq^i + X^i dp_i = \iota_X \omega$. Now since $dH = \partial_i V dq^i + \frac{p_i}{m}$, we get the coefficients of the vector field to be $X_i = -\partial_i V$ and $X^i = \frac{p_i}{m}$. Hence, we get the Hamiltonian vector field

$$X_H = -\partial_i V \partial_{q^i} + \frac{p_i}{m} \partial_{p_i}.$$

Naturally, X_H induces a *Hamiltonian flow* $T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$.

An approach of quantization of the above is to associate to $T^*\mathbb{R}^n$ the space of square integrable functions $L^2(\mathbb{R}^n)$ on \mathbb{R}^n . The Hamiltonian flow can then be replaced by a linear map

$$e^{\frac{i}{\hbar} \widehat{H}}: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n),$$

where $\widehat{H} := -\frac{\hbar^2}{2m} \Delta + V$ denotes the *Hamilton operator*, which is the canonical quantization of the classical Hamiltonian function, where $\Delta = \sum_{1 \leq j \leq n} (\partial_{x^j})^2$ denotes the Laplacian. Note that the space of states is now given by a Hilbert space \mathcal{H}_0 and the observables as operators on \mathcal{H}_0 . One can show that the action of this operator can be expressed as an integral of the form

$$(e^{\frac{i}{\hbar} \widehat{H}} \psi)(x) = \int K(x, y) \psi(y) dy,$$

for $\psi \in \mathcal{H}_0$, where K denotes the *integral kernel* for the operator. Feynman showed in [4] that this kernel (quantum mechanical *propagator*) can be seen as a *path integral*, which is given by

$$K(x, y) = \int_{\text{Path}_{(x,y)}^{[t_0, t_1]}(\mathbb{R}^n)} e^{\frac{i}{\hbar} S(q)} \mathcal{D}q.$$

where S denotes the action of the classical system and \mathcal{D} a measure on the path space (see also figure 2).

Since \mathcal{D} is suppose to be a “measure” on an infinite-dimensional space, it is mathematically ill-defined. However, one can still make sense of such an integral in several ways; one of them is by considering its *perturbative expansion* in formal power series with *Feynman diagrams* as coefficients. This procedure is mathematically well-defined. These notes are based on [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

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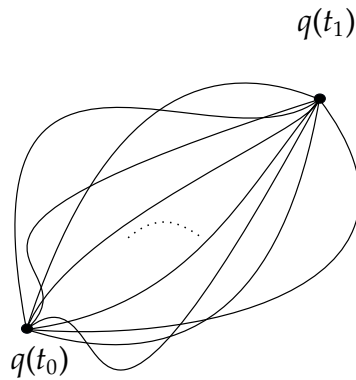


FIGURE 2. Illustration of the fact that all the paths between $x = q(t_0)$ and $y = q(t_1)$ are taken into account.

Part 1. A Brief Recap of Classical Mechanics

2. NEWTONIAN MECHANICS WITH EXAMPLES

Consider a particle of mass m moving in \mathbb{R}^n . The position of a particle $x = (x_1, \dots, x_n)$ is a vector in \mathbb{R}^n . More precisely $x(t) = (x_1(t), \dots, x_n(t))$ is the position of the particle at time t . Let $v(t)$ and $a(t)$ denote the velocity and the acceleration at time t respectively. Then

$$(1) \quad v(t) = \dot{x}(t) = (\dot{x}_1(t), \dots, \dot{x}_n(t)),$$

$$(2) \quad a(t) = \ddot{x}(t) = (\ddot{x}_1(t), \dots, \ddot{x}_n(t)),$$

where $\dot{x}_i(t) = \frac{d}{dt}x_i(t)$ and $\ddot{x}_i(t) = \frac{d}{dt}\dot{x}_i(t) = \frac{d^2}{dt^2}x_i(t)$. We recall Newton's second law of motion:

$$(3) \quad m\ddot{x}(t) = F(x(t), \dot{x}(t)),$$

where F is a force acting on the particle with mass m . Hence, the trajectories of motion are given by solutions of (3). We note that (3) is a system of second order ordinary differential equations and is nonlinear in general³.

Example 2.0.1 (The free particle on \mathbb{R}^n). The force $F = 0$, which implies that (3) becomes $\ddot{x} = 0$, hence the trajectories of motion are given by $x(t) = at + b$ with $a, b \in \mathbb{R}^n$.

Example 2.0.2 (Harmonic oscillator in one dimension ($n = 1$)). The force is given by $F = -Kx$ (Hooke's law), where $K = \omega^2 m$ is the so-called spring constant. Then the equation of motion becomes $m\ddot{x} + Kx = 0$. Hence the trajectories of motion are given by

$$(4) \quad x(t) = a \cos(\omega t) + b \sin(\omega t),$$

with $a, b \in \mathbb{R}$.

Thus, in Newtonian mechanics, we are interested in solving the equation (3). One way to try to solve (3) would be to try to find conserved quantities which may help simplifying the problem.

³Nonlinearity depends on the nature of F

2.1. Conservation of Energy. Assume that the force F depends only on the position and it has the form $F = -\nabla V(x)$, where $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is some function. Such a force F is called a conservative force and V is called the potential energy of F . Since (3) is a second order differential equation, the state space or phase space of (3) is $\mathbb{R}^{2n} = \{(x, v) \mid x, v \in \mathbb{R}^n\}$. Define the total energy function E by

$$(5) \quad E(x, v) = \frac{1}{2}m\|v\|^2 + V(x),$$

where $\|v\|^2 = \langle v, v \rangle$ with the standard inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^n . The main significance of the total energy function is that it is conserved, meaning that its value along any trajectory of motion is constant.

Proposition 2.1.1. Suppose a particle moving on \mathbb{R}^n satisfying Newton's law of the form (3). Then

$$(6) \quad \frac{d}{dt}E(x(t), \dot{x}(t)) = 0,$$

along any trajectory $x(t)$ satisfying (3).

Proof. Along a solution $x(t)$ of (3) we have

$$(7) \quad \begin{aligned} \frac{d}{dt}E(x, v) &= \sum_{i=1}^n \frac{\partial E}{\partial x_i} \dot{x}_i + \sum_{i=1}^n \frac{\partial E}{\partial v_i} \dot{v}_i \\ &= \sum_{i=1}^n \frac{\partial V}{\partial x_i} v_i + m \sum_{i=1}^n v_i \dot{v}_i \\ &= (\nabla V + ma)v \\ &= (-F + ma)v \\ &= 0, \end{aligned}$$

□

Definition 2.1.1 (Constant of motion). Let f be a function on the phase space \mathbb{R}^{2n} . We say f is a constant of motion if $\frac{d}{dt}f = 0$ along $(x(t), \dot{x}(t))$, whenever $x(t)$ is a trajectory of motion.

Remark 2.1.1. Constants of motion are conserved quantities.

By proposition 2.1.1, the total energy is a constant of motion. Next, using an example, we investigate that the conservation of energy helps us to understand the solution of the equation of motion. Let us rewrite (3) in terms of first order equations

$$(8) \quad \begin{aligned} \frac{d}{dt}x_i(t) &= v_i(t), \quad i = 1, 2, \dots, n \\ \frac{d}{dt}v_i(t) &= \frac{1}{m}F_i(x(t)), \quad i = 1, 2, \dots, n \end{aligned}$$

For simplicity, assume $n = 1$. Hence we have

$$(9) \quad \begin{aligned} \frac{d}{dt}x(t) &= v(t), \\ \frac{d}{dt}v(t) &= \frac{1}{m}F(x(t)) \end{aligned}$$

By conservation of energy, we know that $\frac{d}{dt}E(x, v) = 0$ along $(x(t), v(t))$, whenever $(x(t), v(t))$ satisfy (9). Let $E(x(t), v(t)) = E_0$. Then

$$(10) \quad \frac{1}{2}m\dot{x}(t)^2 + V(x(t)) = E_0,$$

and thus

$$(11) \quad \dot{x}(t) = \pm \sqrt{\frac{2(E_0 - V(x(t)))}{m}},$$

which can be solved using separation of variables. From this example, we learned that the conservation of energy helps us simplify the given system of equation in the one dimensional case (previous example), we were able to reduce the second order equation into a first order equation and even solve the equation. A general “mantra” is: the knowledge of conserved quantities helps to simplify the equation of motion.

3. HAMILTONIAN MECHANICS

3.1. The general formulation. Hamiltonian mechanics gives a systematic approach to understand conserved quantities. Consider a particle moving in \mathbb{R}^n . The idea is to think of the total energy as a function of position and momentum rather than a function of position and velocity:

$$(12) \quad H(x, p) = \frac{1}{2m} \sum_{j=1}^n p_j^2 + V(x),$$

where $p_j = m\dot{x}_j$. Now the system of equations (12) can be written as

$$(13) \quad \begin{aligned} \frac{d}{dt}x_i(t) &= \dot{x}_i(t) = \frac{1}{m}p_i = \frac{\partial H}{\partial p_i} \\ \frac{d}{dt}p_i(t) &= m\frac{d}{dt}\dot{x}_i(t) = -\frac{\partial V}{\partial x_i} = -\frac{\partial H}{\partial x_i}. \end{aligned}$$

The equations of (13), i.e.

$$(14) \quad \dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

are called Hamilton’s equations.

3.2. The Poisson bracket. The previous observation implies that in Hamiltonian mechanics we consider the phase space to be

$$\mathbb{R}^{2n} := \{(x, p) \mid x, p \in \mathbb{R}^n\}.$$

It turns out that \mathbb{R}^{2n} has more structures. If f and g are smooth functions on \mathbb{R}^{2n} , one can define the Poisson bracket

$$(15) \quad \{f, g\} := \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial x_j} \frac{\partial f}{\partial p_j} \right).$$

Exercise 3.2.1. Verify that the Poisson bracket satisfies the following properties. Let f, g and h be smooth function on \mathbb{R}^{2n} . Then

- (1) $\{f, g\} = -\{g, f\}$
- (2) $\{f, g + ch\} = \{f, g\} + c\{f, h\}, \quad c \in \mathbb{R}$
- (3) $\{f, gh\} = \{f, g\}h + \{f, h\}g$
- (4) $\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$ (Jacobi identity)

Example 3.2.1. Let p_j and x_j be momentum and position observables as images of the following maps respectively.

$$(16) \quad \begin{aligned} (x, p) &\longmapsto p_j \\ (x, p) &\longmapsto x_j. \end{aligned}$$

Then $\{x_i, x_j\} = 0 = \{p_i, p_j\}$ and $\{x_i, p_j\} = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta.

Next we will see that we can use the Poisson bracket to describe the conserved quantities. For that we need the following proposition.

Proposition 3.2.1. Let $f \in C^\infty(\mathbb{R}^{2n})$. Then

$$(17) \quad \frac{d}{dt}f = \{f, H\}$$

along a solution of Hamilton's equations $\{(x(t), p(t))\} \subset \mathbb{R}^{2n}$.

Proof. Exercise. □

Corollary 3.2.1. Let $f \in C^\infty(\mathbb{R}^{2n})$. Then f is conserved along solutions of Hamilton's equations iff

$$\{f, H\} = 0.$$

Proof. By proposition 3.2.1 $\frac{d}{dt}f = \{f, H\}$ along solutions $(x(t), p(t))$ of Hamilton's equations. By definition, f is conserved if $\frac{d}{dt}f = 0$ iff $\{f, H\} = 0$. □

Remark 3.2.1. Given any $f \in C^\infty(\mathbb{R}^{2n})$, we can define Hamilton's equations by

$$(18) \quad \begin{aligned} \dot{x}_i &= \frac{\partial f}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial f}{\partial x_i} \\ i &= 1, 2, \dots, n \end{aligned}$$

For the next remarks we assume familiarity with basic differential geometry notions such as vector fields, differential forms etc.

Remark 3.2.2. \mathbb{R}^{2n} has a canonical symplectic structure $\omega = \sum_{i=1}^n dp_i \wedge dx_i$. Given $f \in C^\infty(\mathbb{R}^{2n})$ there exists a vector field X_f , called the Hamiltonian vector field of f , defined by

$$(19) \quad \omega(X_f, \cdot) = -df$$

The flow of X_f is given by solutions of (18). In this case, one can check that

$$(20) \quad \{f, g\} = \omega(X_f, X_g).$$

This means that if (N, ω) is a symplectic manifold, then we can define the Poisson bracket of $f, g \in C^\infty(N)$ using (20).

Remark 3.2.3. Let $f \in C^\infty(\mathbb{R}^{2n})$ and X_f be the corresponding Hamiltonian vector field. The flow of X_f (or in other words the solutions of (18)) defines one-parameter diffeomorphisms

$$(21) \quad \begin{aligned} \Phi_{X_f}^t : \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^{2n} \\ (x, p) &\longmapsto \Phi_{X_f}^t(x, p) = (x(t), p(t)), \end{aligned}$$

where $(x(t), p(t))$ satisfy Hamilton's equations with $x(0) = x$ and $p(0) = p$. Then, assuming that the flow is complete, we get

- (1) $\Phi_{X_f}^t$ preserves ω (i.e. $(\Phi_{X_f}^t)^*\omega = \omega$). Such a map is called a symplectomorphism.
- (2) $\Phi_{X_f}^t$ preserves the volume form $v = dx_1 dx_2 \cdots dx_n dp_1 dp_2 \cdots dp_n$ (i.e. $(\Phi_{X_f}^t)^*v = v$). This is known as Liouville's theorem.

Remark 3.2.4. Let $f, g \in C^\infty(\mathbb{R}^{2n})$. Then f is conserved along the solutions of Hamilton's equations of g iff $\{f, g\} = 0$ (This is an instance of Noether's theorem).

4. LAGRANGIAN MECHANICS

There are two important points in this formalism:

- Mechanics on a configuration space.
- Basic theorems are invariant under actions of diffeomorphisms of the configuration space. It is useful to compute conserved quantities.

4.1. Lagrangian system. Let M be a smooth manifold (we will usually consider $M = \mathbb{R}^n$). A Lagrangian system with configuration space M consists of a smooth real valued function $L : TM \times \mathbb{R} \rightarrow \mathbb{R}$, where TM denotes the tangent bundle of M (e.g. if $M = \mathbb{R}^n$, then $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \{(x, v)\}$). L is called the Lagrangian function or simply Lagrangian. Lagrangian mechanics uses special ideas such as the least action principle from calculus of variation. Let $x_0, x_1 \in M$ and $P(M, x_0, x_1) := \{\gamma : [t_0, t_1] \subset \mathbb{R} \rightarrow M \mid \gamma(t_0) = x_0, \gamma(t_1) = x_1\}$, which is the space of parameterized paths joining x_0 to x_1 .

Definition 4.1.1 (Action functional). The action functional $S : P(M, x_0, x_1) \rightarrow \mathbb{R}$ of the Lagrangian system (M, L) is defined by

$$(22) \quad S(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t), t) dt.$$

From now on we take $M = \mathbb{R}^n$. We are interested in understanding “critical points” of S . Let $h : [t_0, t_1] \rightarrow \mathbb{R}^n$ be such that $\gamma + h \in P(\mathbb{R}^n, x_0, x_1)$ and $h(t_0) = h(t_1) = 0$. We think of h as a small variation of $\gamma \in P(\mathbb{R}^n, x_0, x_1)$. Then, if we change $\gamma(t)$ by h , we get

$$(23) \quad S(\gamma + \varepsilon h) = \int_{t_0}^{t_1} L(\gamma(t) + \varepsilon h(t), \dot{\gamma}(t) + \varepsilon \dot{h}(t), t) dt,$$

which needs to be extremal with respect to the parameter ε . Hence

$$(24) \quad \frac{d}{d\varepsilon} S(\gamma + \varepsilon h) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \gamma} h + \frac{\partial L}{\partial \dot{\gamma}} \dot{h} \right) dt = 0.$$

For the second part, we use integration by parts, which gives

$$(25) \quad \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{\gamma}} \dot{h}(t) dt = \underbrace{\frac{\partial L}{\partial \dot{\gamma}} h \Big|_{t_0}^{t_1}}_{=0} - \int_{t_0}^{t_1} \frac{d}{dt} \frac{\partial L}{\partial \dot{\gamma}} h(t) dt.$$

The last term remains and by the product rule we get

$$(26) \quad \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \gamma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\gamma}} \right) h(t) dt = 0.$$

Definition 4.1.2 (Extremal point/Critical point). An extremal (or critical) point of S is some $x \in P(\mathbb{R}^n, x_0, x_1)$ such that

$$(27) \quad \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) h dt = 0$$

along x for all paths h such that $h(t_0) = h(t_1) = 0$.

Theorem 4.1.1. A path $x \in P(\mathbb{R}^n, t_0, t_1)$ is an extremal of S iff along x we have

$$(28) \quad \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$$

The proof for this theorem follows from the following lemma.

Lemma 4.1.1. Let $f : [t_0, t_1] \rightarrow \mathbb{R}^n$ be a continuous path and

$$(29) \quad \int_{t_0}^{t_1} fh dt = 0$$

for all continuous $h : [t_0, t_1] \rightarrow \mathbb{R}^n$ such that $h(t_0) = h(t_1) = 0$. Then $f \equiv 0$ on $[t_0, t_1]$.

Proof. For simplicity assume $n = 1$, i.e. $f : [t_0, t_1] \rightarrow \mathbb{R}$ and $h : [t_0, t_1] \rightarrow \mathbb{R}$. By contradiction assume there is some $t \in [t_0, t_1]$ such that $f(t) > 0$. Then by continuity there is some $\delta > 0$ such that $f > 0$ on $(t - \delta, t + \delta)$. Let h be a continuous function on $[t_0, t_1]$ such that h vanishes outside $(t - \delta, t + \delta)$ but $h > 0$ on $(t - \delta/2, t + \delta/2)$. Then

$$\int_{t_0}^{t_1} fh dt > 0,$$

which is a contradiction. □

Definition 4.1.3 (Euler-Lagrange equations). The equations

$$(30) \quad \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$$

are called the Euler-Lagrange (EL) equations of $S(x)$.

Corollary 4.1.1. A path $x \in P(\mathbb{R}^n, x_0, x_1)$ is an extremal of S iff it satisfies the Euler-Lagrange equations.

4.2. Hamilton's least action principle. Recall that we defined the total energy function by

$$E(x, v) = \frac{1}{2}m\|v\|^2 + V(x),$$

where the first term is the kinetic energy and the second the potential energy.

Theorem 4.2.1. Define $L(\gamma(t), \dot{\gamma}(t), t) = \frac{1}{2}m\|\dot{\gamma}(t)\|^2 - V(\gamma(t))$. Then an extremal path $\gamma(t)$ of S solves the system (8).

Proof. Exercise. □

Remark 4.2.1. Even though only an extremal path of S is involved here, it is called Hamilton's least action principle.

Next we briefly investigate how Hamilton's equations and the EL equations are related.

5. THE LEGENDRE TRANSFORM

Let f be a convex function, i.e. $f''(x) > 0$. Let $p \in \mathbb{R}$ and define $g(x) = px - f(x)$. Then $g'(x) = p - f'(x)$. Since f is convex (i.e. f' is increasing), there is a unique x_0 such that $g(x_0) = 0$. We denote this x_0 by $x(p)$. Moreover, $f''(x) > 0$ implies $g''(x) < 0$, and hence g has a maximum at $x(p)$. In this case the Legendre transform of f is defined by

$$(31) \quad \mathcal{L}f(p) = \max_x g(x) = \max_x (px - f(x)).$$

Example 5.0.1. Let $f(x) = x^2$, then $\mathcal{L}f(p) = \frac{1}{4}p^2$.

Example 5.0.2. Let $f(x) = \frac{1}{2}x^2$, then $\mathcal{L}f(p) = \frac{1}{2}p^2$.

More generally, let V be a finite dimensional vector space and V^* be its dual and $f : V \rightarrow \mathbb{R}$ be a function. Then $\mathcal{L}f : V^* \rightarrow \mathbb{R}$ is defined by

$$(32) \quad \mathcal{L}f(p) = \max_{x \in V} (p(x) - f(x)),$$

where $p(x)$ is the pairing between $x \in V$ and $p \in V^*$. If f is convex, then $\mathcal{L}f$ exists.

Exercise 5.0.1. Show that if f is convex, then so is $\mathcal{L}f$. Moreover, show that $\mathcal{L}(\mathcal{L}f) = f$.

Example 5.0.3. Let A be an $n \times n$ positive definite matrix and $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $f(x) = \frac{1}{2}\langle Ax, x \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^n . Then

$$\mathcal{L}f(\omega) = \frac{1}{2}\langle A^{-1}\omega, \omega \rangle.$$

Let us now consider a Lagrangian system (\mathbb{R}^n, L) , i.e. $L : \underbrace{\mathbb{R}^n \times \mathbb{R}^n}_{\ni(x,v,t)} \times \mathbb{R} \rightarrow \mathbb{R}$. Let $H(x, p, t)$

be the Legendre transform of L in v -direction.

Theorem 5.0.1. *The system of EL equations are equivalent to Hamilton's equation with H defined as above.*

Proof. Exercise. □

Part 2. The Schrödinger Picture of Quantum Mechanics

Classical physics is inconsistent at the level of atoms and molecules. For example, the hydrogen atom which is composed of two particles a proton of charge $+e$ and an electron of charge $-e$. If we follow classical mechanics, then the charged electron would radiate energy continuously causing the atom to collapse. But this is not true. We need quantum mechanics to explain the stability of molecules and atoms.⁴

⁴We refer to a standard physics book on quantum mechanics for the motivation leading to postulates of quantum mechanics.

6. POSTULATES OF QUANTUM MECHANICS

6.1. First Postulate. The pure states of a quantum mechanical system are rays in a Hilbert space \mathcal{H} , i.e. one dimensional subspaces of \mathcal{H} . The Hilbert space \mathcal{H} is called the **space of states**. Define

$$P\mathcal{H} := (\mathcal{H} \setminus \{0\})/(\mathbb{C} \setminus \{0\}).$$

Let $\phi, \psi \in \mathcal{H} \setminus \{0\}$. We say $\phi \sim \psi$ iff there is an $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\phi = \alpha\psi$. Then $P\mathcal{H}$ is the set of equivalence classes with respect to this equivalence relation.

Lemma 6.1.1. There is a canonical bijection

$$\{\text{1-dimensional subspaces of } \mathcal{H}\} \longleftrightarrow P\mathcal{H}.$$

Proof. Let L be a one dimensional subspace of \mathcal{H} , and $\phi \in L$ such that $\phi \neq 0$. Define

$$\beta(L) = [\phi], \quad [\phi] \in P\mathcal{H}.$$

Let us check that β is well defined. Let $\psi \in L \setminus \{0\}$. Then there is an $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\psi = \alpha\phi$ (since L is a one dimensional subspace). Thus $[\psi] = [\phi]$. This shows that β is well defined. One can easily check that β is a bijection. \square

Remark 6.1.1. More precisely, the space of pure states is $P\mathcal{H}$.

From now on when we say a state we mean $\psi \in \mathcal{H}$ such that $\|\psi\| = 1$ (these are called **normalized states**). The concept of a state as a ray in \mathcal{H} leads to the probability interpretation in quantum mechanics. This means that if a physical system is in the state ψ , then the probability that it is in the state ϕ is $|\langle\psi, \phi\rangle|^2$. Since we assume $\|\phi\| = 1, \|\psi\| = 1$, clearly $0 \leq |\langle\psi, \phi\rangle|^2 \leq 1$.

6.2. Second Postulate. Quantum mechanical observables are **self adjoint operators** on \mathcal{H} . Let A be an observable. Then the **expectation** of A in the state ψ is defined as

$$(33) \quad \langle A \rangle_\psi = \frac{\langle A\psi, \psi \rangle}{\langle \psi, \psi \rangle}$$

6.3. Third Postulate. The Hamiltonian \widehat{H} is the infinitesimal generator of the unitary group $U(t) = e^{-\frac{i}{\hbar}t\widehat{H}}$. It describes the dynamics of the system. Let ψ be a state. Then time evolution is described by the **Schrödinger equation**

$$(34) \quad i\hbar \frac{d}{dt}\psi(t) = \widehat{H}\psi(t).$$

Using an Ansatz for equation (34), we get a solution of the form $\psi(t) = e^{-\frac{i}{\hbar}t\widehat{H}}\psi(0)$. In the so-called **Heisenberg picture**, the Schrödinger equation takes the form

$$(35) \quad \hbar \frac{d}{dt}A(t) = [i\widehat{H}, A(t)],$$

where A is an observable and $[,]$ is the commutator of operators, defined by $[A, B] = AB - BA$.

Lemma 6.3.1. Let $\phi(t)$ and $\psi(t)$ be solutions of (34), such that $\phi(0) = \phi$ and $\psi(0) = \psi$. Then

$$\langle \phi(t), \psi(t) \rangle = \langle \phi, \psi \rangle, \quad \forall t$$

Proof. We have

$$(36) \quad \phi(t) = e^{-\frac{i}{\hbar}t\widehat{H}}\phi(0),$$

$$(37) \quad \psi(t) = e^{-\frac{i}{\hbar}t\widehat{H}}\psi(0),$$

and since $e^{-\frac{i}{\hbar}t\widehat{H}}$ is a unitary operator, we get the result. \square

6.4. Summary of CM and QM. The following should summarize the differences of classical and quantum mechanics.

	Classical Mechanics	Quantum Mechanics
State space	T^*M (cotangent bundle)	$P\mathcal{H}$, where \mathcal{H} is a Hilbert space
Observables	$C^\infty(T^*M)$ (smooth functions on the cotangent bundle)	Self adjoint operators on \mathcal{H}
Dynamics	Described by Hamilton's equation associated to a Hamiltonian function $H \in C^\infty(T^*M)$	Described by the Schrödinger equation associated to a quantum Hamiltonian operator \widehat{H} : $i\hbar \frac{d}{dt}\psi(t) = \widehat{H}\psi(t)$

Next, we will define basic notations and concepts used to define quantum mechanical systems.⁵

7. ELEMENTS OF FUNCTIONAL ANALYSIS

Let \mathcal{H} be a Hilbert space (we always assume it is separable, i.e. there exists a basis). An operator in \mathcal{H} is a pair $(A, D(A))$ where $D(A)$ is a subspace of \mathcal{H} , called the domain of A , and $A : D(A) \rightarrow \mathcal{H}$ is a linear map. We can always assume that $D(A)$ is dense in \mathcal{H} .

Definition 7.0.1 (Bounded operator). A linear map $A : D(A) \rightarrow \mathcal{H}$ is called bounded if there exists some $\varepsilon > 0$ such that for all $\psi \in D(A)$

$$\|A\psi\| \leq \varepsilon\|\psi\|.$$

Otherwise, we say A is unbounded.

Remark 7.0.1. If A is bounded, A can be always extended to a bounded operator $\widetilde{A} : \mathcal{H} \rightarrow \mathcal{H}$. Hence, when we talk about bounded operator, we always consider $A : \mathcal{H} \rightarrow \mathcal{H}$.

Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. Then there is a unique operator $A^* : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle, \quad \forall \phi, \psi \in \mathcal{H}.$$

⁵This is fairly standard. One can find them in any text book about functional analysis.

Definition 7.0.2 (Adjoint/Self adjoint operator). We call A^* the adjoint of A . Moreover, a bounded operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is called self adjoint if $A^* = A$.

Example 7.0.1. Let $\mathcal{H} = L^2([0, 1])$ and $X : \mathcal{H} \rightarrow \mathcal{H}$, $(Xf)(x) = xf(x)$. Then

$$\|Xf\|^2 = \int_0^1 x^2 |f(x)|^2 dx \leq \int_0^1 |f(x)|^2 dx = \|f\|^2,$$

which implies that $\|Xf\| \leq \|f\|$ and thus X is bounded. Let now $f, g \in L^2([0, 1])$. Then

$$(38) \quad \langle f, Xg \rangle = \int_0^1 \overline{f(x)} xg(x) dx = \int_0^1 \overline{xf(x)} g(x) dx = \langle Xf, g \rangle,$$

and thus $A^* = A$. Hence A is self adjoint.

7.1. Unbounded operators.

Example 7.1.1. Let $\mathcal{H} = L^2(\mathbb{R})$. Let X be the multiplication operator like before and define its domain $D(X) = \{\phi \in L^2(\mathbb{R}) \mid x\phi(x) \in L^2(\mathbb{R})\}$. We claim that

(1) $D(X)$ is dense in $L^2(\mathbb{R})$.

(2) X is unbounded.

Let $\phi \in L^2(\mathbb{R})$. Define $\phi_n = \phi\chi_{[-n,n]}$, where χ denotes the characteristic function. Then it is clear that $x\phi_n \in L^2(\mathbb{R})$ and by the dominated convergence theorem $\phi_n \xrightarrow{n \rightarrow \infty} \phi$, in $L^2(\mathbb{R})$. This proves (1). To see that X is unbounded, consider $\phi_n = \frac{1}{\sqrt{n}}\chi_{[0,n]}$, then $\|\phi_n\| = 1$ for all n , but

$$\|X\phi_n\|^2 = \frac{1}{n} \int_0^1 x^2 dx = \frac{n^2}{3} \xrightarrow{n \rightarrow \infty} \infty.$$

Thus X is unbounded, proving (2).

7.2. Adjoint of an unbounded operator. Let A be an unbounded operator in \mathcal{H} with domain $D(A)$. Define $D(A^*) = \{\phi \in \mathcal{H} \mid \langle \phi, A \cdot \rangle$ is a bounded linear functional on $D(A^*)\}$. Using Riesz's theorem, one can show that if $\phi \in D(A^*)$, then there is a unique $\psi \in \mathcal{H}$ such that

$$\langle \psi, \chi \rangle = \langle \phi, A\chi \rangle, \quad \forall \chi \in D(A).$$

We define $A^*\phi = \psi$.

Definition 7.2.1 (Symmetric operator). Let A be an unbounded operator with $D(A)$. We say A is symmetric if

$$\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle, \quad \forall \phi, \psi \in D(A).$$

Moreover, A is self adjoint if $D(A) = D(A^*)$ and $A^*\phi = A\phi$.

Exercise 7.2.1. Show that if A is symmetric, then $D(A) \subseteq D(A^*)$. Hence, A is self adjoint iff A is symmetric and $D(A) = D(A^*)$.

Exercise 7.2.2. Let $\mathcal{H} = L^2(\mathbb{R})$ and $V : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable map. Define the domain

$$D(V(x)) = \{\phi \in L^2(\mathbb{R}) \mid V(x)\phi(x) \in L^2(\mathbb{R})\}$$

for the operator

$$(39) \quad V(X) : D(V(X)) \longrightarrow L^2(\mathbb{R})$$

$$(40) \quad \phi \longmapsto V(x)\phi$$

Proposition 7.2.1. $V(X)$ is self adjoint.

Proof. We need to check that $D(V(X))$ is dense in $L^2(\mathbb{R})$. $V(X)$ is symmetric and $D(V(X)) = D(V(X)^*)$. It is easy to check that $D(V(X))$ is dense in $L^2(\mathbb{R})$. Since V is a real valued function, $V(X)$ is symmetric as well. We only need to show $D(V(X)^*) \subseteq D(V(X))$. For this let $\phi \in D(V(X)^*)$. We want to show that $V(x)\phi(x) \in L^2(\mathbb{R})$. Since $\phi \in D(V(X)^*)$, we get that $\psi \mapsto \langle \phi, V(X)\psi \rangle$ is a bounded linear functional on $D(V(X))$. In fact, it can be extended to a bounded linear functional on $L^2(\mathbb{R})$ (since $D(V(X))$ is dense). Hence by Riesz's theorem there is a unique $\chi \in L^2(\mathbb{R})$ such that

$$(41) \quad \langle \chi, \psi \rangle = \langle \phi, V(X)\psi \rangle, \quad \forall \psi \in L^2(\mathbb{R}),$$

thus

$$(42) \quad \int_{\mathbb{R}} \overline{\chi(x)}\psi(x)dx = \int_{\mathbb{R}} \overline{\phi(x)}V(x)\psi(x)dx, \quad \forall \psi \in L^2(\mathbb{R}),$$

and hence

$$(43) \quad \int_{\mathbb{R}} \overline{\chi(x)}\psi(x)dx = \int_{\mathbb{R}} \overline{\phi(x)V(x)\psi(x)}dx, \quad \forall \psi \in L^2(\mathbb{R}).$$

which shows that $\chi = V(x)\phi$ a.e., and therefore $V(x)\phi \in L^2(\mathbb{R})$. Hence $\phi \in D(V(X))$. \square

Similarly one can show that the operator P , defined by $P\psi(x) = -i\hbar \frac{d}{dx}\psi(x)$, is a self adjoint operator with domain

$$D(P) = \{\psi \in L^2(\mathbb{R}) \mid k\widehat{\psi}(k) \in L^2(\mathbb{R})\},$$

where

$$\widehat{\psi}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx}\psi(x)dx$$

is the Fourier transform of ψ . Next we mention two technical results without proof.

Theorem 7.2.1 (Spectral theorem/ Functional calculus). *Let A be a self adjoint operator on \mathcal{H} . Let $L(\mathcal{H})$ denote the space of bounded linear operators in \mathcal{H} . Then, there is a unique map*

$$\{\text{Bounded measurable functions on } \mathbb{R}\} \xrightarrow{\widehat{\phi}} L(\mathcal{H}),$$

such that

(1) $\widehat{\phi}$ is linear and $\widehat{\phi}(fg) = \widehat{\phi}(f)\widehat{\phi}(g)$ for all bounded measurable functions f, g on \mathbb{R} .

(2) $\widehat{\phi}(f) = (\widehat{\phi}(f))^*$

(3) $\|\widehat{\phi}(h)\| \leq \|h\|_\infty$

(4) If $h_n \xrightarrow{n \rightarrow \infty} x$ and for all n we have $|h_n(x)| \leq |x|$, then for all $\psi \in D(A)$

$$\widehat{\phi}(h_n)\psi \xrightarrow{n \rightarrow \infty} A\psi.$$

(5) $A\psi = \lambda\psi$ for $\lambda \in \mathbb{C}$.

We can use this theorem to produce bounded operators from a self adjoint operator, e.g. let $f(x) = e^{itx}$. We can see that f is bounded and measurable. Hence we can talk about $f(A) = e^{itA}$ as a bounded linear operator on \mathcal{H} .

Theorem 7.2.2 (Stone's theorem). *Let A be a self adjoint operator on \mathcal{H} . Define $U(t) = e^{itA}$. Then*

(1) $U(t)$ is a unitary operator:

$$\langle U(t)\phi, U(t)\psi \rangle = \langle \phi, \psi \rangle$$

for all $\phi, \psi \in \mathcal{H}$. Moreover, $U(t+s) = U(t) \circ U(s)$.

(2) For $\phi \in \mathcal{H}$ and $t \rightarrow t_0$ we have that $U(t)\phi \rightarrow U(t_0)\phi$ in \mathcal{H} (strong convergence)

(3) The limit $\lim_{t \rightarrow 0} \frac{U(t)\psi - \psi}{t}$ exists in \mathcal{H} for all $\psi \in D(A)$ and

$$\lim_{t \rightarrow 0} \frac{U(t)\psi - \psi}{t} = iA\psi.$$

(formally, this means $\frac{d}{dt}U(t) = iA$)

(4) Let $\psi \in \mathcal{H}$ such that the limit $\lim_{t \rightarrow 0} \frac{U(t)\psi - \psi}{t}$ exists. Then $\psi \in D(A)$.

Moreover, if $U(t)$, for $t \in \mathbb{R}$, is a family of unitary operators such that (1) and (2) hold, then $U(t) = e^{itA}$ for some self adjoint operator A .

Definition 7.2.2 (Strongly continuous one parameter unitary group). A family $U(t)$ satisfying (1) and (2) of theorem 7.2.2 is called strongly continuous one parameter unitary group and A is called the infinitesimal generator.

Definition 7.2.3 (Resolvent). Let A be an operator with domain $D(A)$ and let $\lambda \in \mathbb{C}$. We say that λ is in the resolvent set $\rho(A)$ of A if

(1) $\lambda I - A: D(A) \rightarrow \mathcal{H}$ is bijective,

(2) $(\lambda I - A)^{-1}$ is a bounded operator.

Definition 7.2.4 (Spectrum). The spectrum $\sigma(A)$ of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

One can actually check that if λ is an eigenvalue of A , then $\lambda \in \sigma(A)$. We call the set of eigenvalues of A the point spectrum of A .

Let $\mathcal{A}[0, 1]$ denote the set of absolutely continuous L^2 -functions on $[0, 1]$.

Example 7.2.1. Consider the operator $T = i \frac{d}{dx}$ on $L^2([0, 1])$ with domain $D(T) = \mathcal{A}[0, 1]$. Then $\sigma(T) = \mathbb{C}$ (just a differential equation).

Example 7.2.2. Consider the operator $T = i \frac{d}{dx}$ with domain $D(T) = \{f \in \mathcal{A}[0, 1] \mid f(0) = 0\}$. We claim that $\rho(T) = \mathbb{C}$.

Proof. Let $\lambda \in \mathbb{C}$ and define

$$S_\lambda g(x) := i \int_0^x e^{-i\lambda(x-s)} g(s) ds.$$

One can show that $(T - \lambda I)S_\lambda g = g$ for all $g \in L^2([0, 1])$. Moreover, $S_\lambda(T - \lambda I)g = g$ for all $g \in D(T)$. We need to show that S_λ is bounded. Indeed, we have

$$\begin{aligned} \|S_\lambda g\|_2^2 &= \int_0^1 |S_\lambda g(x)|^2 dx \leq \sup_{x \in [0, 1]} |S_\lambda g(x)|^2 = \sup_{x \in [0, 1]} \left| \int_0^x e^{-i\lambda(x-s)} g(s) ds \right|^2 \\ &\leq \sup_{x \in [0, 1]} \left(\int_0^x |e^{-i\lambda(x-s)} g(s)| ds \right)^2 \leq \left(\sup_{x \in [0, 1]} \left| \int_0^x e^{-i\lambda(x-s)} dx \right|^2 \right) \left(\sup_{x \in [0, 1]} \left| \int_0^x g(s) ds \right|^2 \right) \\ &\leq C(\lambda) \|g\|_2^2. \end{aligned}$$

□

7.3. Quantization of a classical system. We want to talk about quantization of a classical system by considering a “quantization map” between classical and quantum data. Consider a map \mathcal{Q} , which maps a classical system to a quantum system. The classical (path space) space of states (T^*M, ω) , which is a symplectic manifold coming from a cotangent space, is mapped to a Hilbert space \mathcal{H} . Moreover, the space of observables $C^\infty(T^*M)$ is mapped to the space of self adjoint operators. We know that $C^\infty(T^*M)$ is endowed with a Poisson bracket $\{ , \}$, but the question is what its image is under \mathcal{Q} .

Example 7.3.1. Let $T^*M = \mathbb{R}^{2n} = \{(x, p) \mid x, p \in \mathbb{R}^n\}$. Then x_i, p^i represent position and momentum observables and $\{x_i, p^i\} = \delta_{ij}$. Denote by \widehat{x}_i the operator given by multiplication with x_i and by $\widehat{p}^i := -i\hbar \frac{\partial}{\partial x_i}$. Then their commutator bracket is given by $[\widehat{x}_i, \widehat{p}^i] = i\hbar \delta_{ij}$.

The previous example can be generalized such that given $\{f, g\}$ for $f, g \in C^\infty(T^*M)$ it will be mapped by \mathcal{Q} to

$$\frac{1}{i\hbar} [\mathcal{Q}(f), \mathcal{Q}(g)],$$

or, by considering $\frac{df}{dt} = \{f, H\}$, we get the quantum image

$$i\hbar \frac{d}{dt} A(t) = [A(t), \widehat{H}],$$

which is basically the Schrödinger equation for the Heisenberg picture.

Definition 7.3.1 (Quantization). A quantization of a classical system $(\mathbb{R}^{2n}, \omega)$ is an argument of a quantum Hilber space \mathcal{H} together with a linear map

$$\mathcal{Q}: C^\infty(\mathbb{R}^{2n}) \rightarrow \{\text{self adjoint operators on } \mathcal{H}\}$$

such that the following hold:

(q1) \mathcal{Q} is linear,

(q2) $\mathcal{Q}(1) = \text{id}_{\mathcal{H}}$,

(q3) $\mathcal{Q}(x_i) = \widehat{x}_i$, $\mathcal{Q}(p_i) = \widehat{p}_i$,

(q4) $[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar\mathcal{Q}(\{f, g\})$,

(q5) $\mathcal{Q}(\phi \circ f) = \phi(\mathcal{Q}(f))$ for any map $\phi: \mathbb{R} \rightarrow \mathbb{R}$.

Remark 7.3.1. The problem is that (q1) – (q5) are inconsistent. Even (q1), (q3), and (q5) are inconsistent.

Example 7.3.2. Consider $n = 1$. We want to know what the image of the classical observable x^2p^2 is under \mathcal{Q} , i.e. $\mathcal{Q}(x^2p^2)$. We write

$$x^2p^2 = \frac{(x^2 + p^2)^2 - x^4 - p^4}{2}.$$

Then we use (q3) and (q5) to get the quantum observables

$$\frac{(\widehat{x}^2 + \widehat{p}^2)^2 - \widehat{x}^4 - \widehat{p}^4}{2} = \frac{\widehat{p}^2\widehat{x}^2 + \widehat{x}^2\widehat{p}^2}{2}.$$

On the other hand we have

$$xp = \frac{(x + p)^2 - x^2 - p^2}{2} \implies \mathcal{Q}(x^2p^2) = \mathcal{Q}((xp)^2) = \left(\frac{(\widehat{x}^2 + \widehat{p}^2)^2 - \widehat{x}^4 - \widehat{p}^4}{2} \right)^2,$$

which implies

$$\mathcal{Q}(x^2p^2) = \left(\frac{\widehat{p}^2\widehat{x}^2 + \widehat{x}^2\widehat{p}^2}{2} \right)^2,$$

which is in general not what we get before.

The question here is: what are general approaches to a solution? Even (q1), (q2), (q4) and (q5) are not consistent. We can have two different solutions:

- Keep (q1), (q2), (q3), (q4) and choose an appropriate domain for \mathcal{Q} ,
- Keep (q1), (q2), (q3) and demand (q4) holds asymptotically, i.e.

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar\mathcal{Q}(\{f, g\}) + O(\hbar^2).$$

We have two different approaches:

- (1) (Canonical quantization) Here we quantize the observables x_i, p_i as the image of \mathcal{Q} , i.e. $x_i \mapsto \widehat{x}_i$ and $p_i \mapsto \widehat{p}_i$. Moreover, $f(x, p) \mapsto f(\widehat{x}, \widehat{p})$ and the question will be what to do for $x_i p_j$? More precisely, there is an ordering problem. We need to know how to define $\mathcal{Q}(x_i^2 p_j^2)$.
- (2) (Wick ordering quantization) Consider $z = x + i\alpha p$ and $\bar{z} = x - i\alpha p$. Then write $f(x, p)$ as $f(z, \bar{z})$, e.g.

$$f(z, \bar{z}) = \sum_{ij} a_{ij} z_i^{r_i} \bar{z}_j^{r_j},$$

and with $\widehat{z} = \widehat{x} + i\alpha \widehat{p}$, $\widehat{\bar{z}} = \widehat{x} - i\alpha \widehat{p}$ we get

$$\mathcal{Q}_{Wick}(f) = f(\widehat{z}, \widehat{\bar{z}}) = \sum_{ij} (\widehat{z}_j^{r_j})^* \widehat{z}_i^{r_i}.$$

Example 7.3.3. Consider $n = 1$. Then, by writing $x = \frac{1}{2}(z + \bar{z})$, we get

$$\begin{aligned} \mathcal{Q}_{Wick}(x^2) &= \mathcal{Q}_{Wick}\left(\frac{1}{4}z^2 + 2z\bar{z} + \bar{z}^2\right) \\ &= \frac{1}{4}\left((\widehat{x} + i\alpha\widehat{p})^2 + 2(\widehat{x} + i\alpha\widehat{p})(\widehat{x} - i\alpha\widehat{p}) + (\widehat{x} + i\alpha\widehat{p})^2\right) \\ &= \frac{1}{4}\left(\widehat{x}^2 - \alpha^2\widehat{p}^2 + i\alpha(\widehat{x}\widehat{p} + \widehat{p}\widehat{x}) + 2(\widehat{x}^2 + \alpha^2\widehat{p}^2 + i\alpha[\widehat{x}, \widehat{p}]) + \widehat{x}^2 - \alpha^2\widehat{p}^2 - i\alpha(\widehat{x}\widehat{p} + \widehat{p}\widehat{x})\right) \\ &= \frac{1}{4}(4\widehat{x}^2 + 2i\alpha[\widehat{x}, \widehat{p}]) = \widehat{x}^2 - \frac{1}{2}\hbar\alpha I, \end{aligned}$$

where I is the identity operator.

- (3) (Weyl Quantization) Consider $n = 1$. We define $\mathcal{Q}_{Weyl}(x, p) := \frac{\widehat{x}p + p\widehat{x}}{2}$. E.g. $\mathcal{Q}_{Weyl}(x^2 p) = \mathcal{Q}_{Weyl}(x p x) = \frac{\widehat{x}^2 p + \widehat{x}p\widehat{x} + p\widehat{x}^2}{3!}$. More generally,

$$\mathcal{Q}_{Weyl}(x_i p_j) = \frac{1}{(n+m)!} \sum_{\sigma \in S_{n+m}} \widehat{x}_{\sigma(1)} \cdots \widehat{x}_{\sigma(n)} \widehat{p}_{\sigma(n+1)} \cdots \widehat{p}_{\sigma(n+m)}.$$

Exercise 7.3.1. Let g be any polynomial in x and p . Then

$$\mathcal{Q}_{Weyl}(x \cdot g) = \mathcal{Q}_{Weyl}(x) \mathcal{Q}_{Weyl}(g) - \frac{i\hbar}{2} \mathcal{Q}_{Weyl}\left(\frac{\partial g}{\partial p}\right) = \mathcal{Q}_{Weyl}(g) \mathcal{Q}_{Weyl}(x) - \frac{i\hbar}{2} \mathcal{Q}_{Weyl}\left(\frac{\partial g}{\partial p}\right)$$

$$\mathcal{Q}_{Weyl}(p \cdot g) = \mathcal{Q}_{Weyl}(p) \mathcal{Q}_{Weyl}(g) + \frac{i\hbar}{2} \mathcal{Q}_{Weyl}\left(\frac{\partial g}{\partial x}\right) = \mathcal{Q}_{Weyl}(g) \mathcal{Q}_{Weyl}(p) - \frac{i\hbar}{2} \mathcal{Q}_{Weyl}\left(\frac{\partial g}{\partial x}\right)$$

Proposition 7.3.1. Let f be a polynomial in x and p of degree at most 2 and g be any polynomial. Then

$$[\mathcal{Q}_{Weyl}(f), \mathcal{Q}_{Weyl}(g)] = i\hbar \mathcal{Q}_{Weyl}(\{f, g\}).$$

Proof. Let $f = g$. Then $\{x, g\} = \frac{\partial g}{\partial p}$. Using exercise 7.3.1 we get

$$[\mathcal{Q}_{Weyl}(x), \mathcal{Q}_{Weyl}(g)] = \frac{i\hbar}{2} \mathcal{Q}_{Weyl}\left(\frac{\partial g}{\partial p}\right) + \frac{i\hbar}{2} \mathcal{Q}_{Weyl}\left(\frac{\partial g}{\partial p}\right) = i\hbar \mathcal{Q}\left(\frac{\partial g}{\partial p}\right).$$

□

Remark 7.3.2. This is not possible for arbitrary polynomials f and g , because of the NO-GO theorem of Gronewald.

7.4. More on self adjoint operators.

Theorem 7.4.1. *Let A be a self adjoint operator. Then $\sigma(A) \subseteq \mathbb{R}$.*

Proof. Assume A is bounded. Let $\lambda = a + ib$ with $b \neq 0$. We claim that $\lambda \in \rho(A)$. Let $\psi \in \mathcal{H}$. Moreover, define $T := (A - \lambda I)$. Then

$$\begin{aligned} \langle (A - \lambda I)\psi, (A - \lambda I)\psi \rangle &= \langle (A - aI)\psi - ib\psi, (A - aI)\psi - ib\psi \rangle \\ &= \|T\psi\|^2 - \langle ib\psi, T\psi \rangle - \langle T\psi, ib\psi \rangle + b^2\|\psi\|^2 \\ &= \|T\psi\|^2 + b^2\|\psi\|^2 \\ &> b^2\|\psi\|^2. \end{aligned}$$

Hence $\langle (A - \lambda I)^*(A - \lambda I)\psi, \psi \rangle > b^2\|\psi\|^2$ and thus $(A - \lambda I)^*(A - \lambda I)$ is a positive operator. Moreover, we can show that $(A - \lambda I)^{-1}$ is bounded. □

Remark 7.4.1. There are also plenty examples for unbounded operators.

7.5. Eigenvalues of single Harmonic Oscillator. Let $\mathcal{H} = L^2(\mathbb{R})$ and consider the Hamiltonian $H(x, p) = \frac{1}{2m}p^2 + \frac{kx^2}{2}$ with $k = m\omega^2$. Then going to the corresponding operator formulation, we have $\widehat{p} = i\hbar \frac{d}{dx}$ and \widehat{x} is just multiplication by x . Then the Hamilton operator is given by

$$\widehat{H} = \frac{1}{2m}\widehat{p}^2 + \frac{k\widehat{x}^2}{2} = \frac{1}{2m}(\widehat{p}^2 + (m\omega\widehat{x})^2).$$

We will only do formal computations (i.e. we forget about the domains). Define $a = \frac{m\omega\widehat{x} + i\widehat{p}}{\sqrt{2\hbar m\omega}}$ and $a^* = \frac{m\omega\widehat{x} - i\widehat{p}}{\sqrt{2\hbar m\omega}}$.

Lemma 7.5.1. We have

$$\widehat{H} = \hbar\omega \left(a^*a + \frac{1}{2}I \right).$$

Lemma 7.5.2. The following hold:

- (1) $[a, a^*] = I$,
- (2) $[a, a^*a] = a$,
- (3) $[a^*, a^*a] = -a^*$

Proof. Exercise □

Proposition 7.5.1. Assume that ψ is an eigenvector of a^*a with eigenvalue λ . Then

$$(44) \quad a^*a(a\psi) = (\lambda - 1)a\psi,$$

$$(45) \quad a^*a(a^*\psi) = (\lambda + 1)a^*\psi.$$

Remark 7.5.1. The consequence of this proposition is that either $a\psi$ is an eigenvector or $a\psi = 0$. We know that $a^*a \geq 0$ so all eigenvalues are non-negative. Hence, if ψ is an eigenvector with eigenvalue λ , then there is some number N such that $a^N\psi \neq 0$ but $a^{N+1}\psi = 0$.

Define $\psi_0 = a^N\psi$. Then $a^*a\psi_0 = 0$ and thus ψ_0 is an eigenvector of zero eigenvalue.

Proposition 7.5.2. Let ψ_0 be such that $\|\psi_0\| = 1$ and $a\psi_0 = 0$. Then, $\psi_n := (a^*)^n\psi_0$, for $n \geq 0$, satisfies the following:

$$(i) \quad a^*\psi_n = \psi_{n+1},$$

$$(ii) \quad (a^*a)\psi_n = n\psi_n,$$

$$(iii) \quad \langle \psi_n, \psi_m \rangle = n!\delta_{mn},$$

$$(iv) \quad a\psi_{n+1} = (n+1)\psi_n$$

Remark 7.5.2. Our goal is to find some $\psi_0 \in L^2(\mathbb{R})$ such that $a\psi_0 = 0$ and $\|\psi_0\| = 1$.

Define $\tilde{x} = \frac{x}{\sqrt{\frac{\hbar}{m\omega}}}$, then $\frac{d}{d\tilde{x}} = \sqrt{\frac{\hbar}{m\omega}} \frac{d}{dx}$. Thus

$$a = \frac{1}{\sqrt{2}} \left(\tilde{x} + \frac{d}{d\tilde{x}} \right), \quad a^* = \frac{1}{\sqrt{2}} \left(\tilde{x} - \frac{d}{d\tilde{x}} \right).$$

We want to solve the equation $a\psi_0 = 0$. This is equivalent to $\frac{d\psi_0}{d\tilde{x}} + \tilde{x}\psi_0 = 0$, which implies that

$$\psi_0(x) = \sqrt{\frac{2m\omega}{\hbar}} e^{-\frac{m\omega}{2\hbar}x^2} \in \mathcal{S}(\mathbb{R}).$$

Here $\mathcal{S}(\mathbb{R})$ represents the space of Schwartz functions on \mathbb{R} (see Subsection 8.2)

Proposition 7.5.3. For $H_n(\tilde{x})$ satisfying $H_0(\tilde{x}) = 1$ and $H_{n+1}(\tilde{x}) = \frac{1}{\sqrt{2}} \left(2\tilde{x}H_n(\tilde{x}) - \frac{dH_n(\tilde{x})}{d\tilde{x}} \right)$ we have

$$\psi_n(\tilde{x}) = H_n(\tilde{x})\psi_0(\tilde{x}).$$

Remark 7.5.3. One can check that the family $\{\psi_n\}$ forms an orthogonal basis of $L^2(\mathbb{R})$.

We want to ask the following question: Is $\{\hbar\omega(n + \frac{1}{2})$ for $n = 0, 1, 2, \dots\}$ the full spectrum of \widehat{H} ? The answer is yes, but the proof is not straight forward.

7.6. **Weyl Quantization on \mathbb{R}^{2n} .** Let f be a sufficiently nice function, e.g. $f \in \mathcal{S}(\mathbb{R}^{2n})$. We define $\mathcal{Q}_{Weyl}(f)$ as an operator on $L^2(\mathbb{R}^n)$ by

$$\mathcal{Q}_{Weyl}(f) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \widehat{f}(a, b) \underbrace{e^{i(a\widehat{x} + b\widehat{p})}}_{U(a, b)} da db,$$

where \widehat{f} denotes the Fourier transform of f . We can compute $U(a, b)$ by using the BCH formula: $e^{A+B} = e^{[A, B]/2} e^A e^B$ if $[[A, B], B] = [A, [A, B]]$. Formally, we get

$$U(a, b) = e^{-\frac{1}{2}[ia\widehat{x}, ib\widehat{p}]} e^{ia\widehat{x}} e^{ib\widehat{p}} = e^{\frac{i\hbar}{2}ab} e^{ia\widehat{x}} e^{ib\widehat{p}}.$$

Exercise 7.6.1. Show $(e^{ib\widehat{p}}\psi)(x) = \psi(x + \hbar b)$.

Using the exercise, we get $U(a, b)\psi(x) = e^{i\hbar ab} e^{ia\widehat{x}}\psi(x + \hbar b)$. There are some nice properties for the Weyl quantization:

- If $f \in \mathcal{S}(\mathbb{R}^{2n})$, then $\mathcal{Q}_{Weyl}(f)$ is a bounded operator on $L^2(\mathbb{R}^n)$. In fact, it is a Hilbert-Schmidt operator.
- The map $\mathcal{Q}_{Weyl}: \mathcal{S}(\mathbb{R}^{2n}) \rightarrow L^2(\mathbb{R}^n)$ is one-to-one.
- Let $f, g \in \mathcal{S}(\mathbb{R}^{2n})$. Then $[\mathcal{Q}_{Weyl}(f), \mathcal{Q}_{Weyl}(g)] = i\hbar \mathcal{Q}_{Weyl}(\{f, g\}) + O(\hbar^2)$.

8. SOLVING SCHRÖDINGER EQUATIONS, FOURIER TRANSFORM AND PROPAGATOR

Recall that in the Hamiltonian formalism of classical mechanics the dynamics (time evolution) was generated by Hamilton's equations associated to a Hamiltonian function $H \in C^\infty(T^*M)$. In quantum mechanics, it is postulated that time evolution is described by the Schrödinger equation associated to the quantum Hamiltonian \widehat{H} : Given $\psi \in \mathcal{H}$ we consider

$$(46) \quad \begin{cases} i\hbar \frac{d}{dt} \psi(t) &= \widehat{H} \psi(t) \\ \psi(0) &= \psi \end{cases}$$

Before we discuss how to solve the Schrödinger equation (SE), let us briefly mention some features of the equation.

- (1) The SE is a linear equation: If $\psi_1(t)$ and $\psi_2(t)$ solve the SE with $\psi_1(0) = \psi_1$ and $\psi_2(0) = \psi_2$, then $\alpha\psi_1(t) + \beta\psi_2(t)$ solve the SE with

$$\alpha\psi_1(0) + \beta\psi_2(0) = \alpha\psi_1 + \beta\psi_2.$$

Remark 8.0.1. The linear SE can easily be generalized to a nonlinear equation but we do not discuss that here.

- (2) The SE is deterministic in the sense that given $\psi \in \mathcal{H}$, there is a canonical way to produce $\psi(t)$ (we will make this precise later).
- (3) **Unitarity:** $\|\psi(t)\|^2 = \|\psi\|^2$ for all t (compare this with conservation of energy in classical mechanics).

8.1. Solving the Schrödinger equation. We start with a simple situation, namely we assume that $\{\lambda_j\}_{j \in I}$ are eigenvalues of \widehat{H} and $\{\phi_{\lambda_j}\}$ form an orthonormal basis of \mathcal{H} , where ϕ_{λ_j} is an eigenvector associated to the eigenvalue λ_j , i.e. the equation $\widehat{H}\phi_{\lambda_j} = \lambda_j\phi_{\lambda_j}$ holds. We want to solve

$$(47) \quad \begin{cases} i\hbar\phi_{\lambda_j}(t) &= \widehat{H}\phi_{\lambda_j}(t) \\ \phi_{\lambda_j}(0) &= \phi_{\lambda_j} \end{cases}$$

We want to formulate the idea for solving this equation. Look for solutions of the form

$$\phi_{\lambda_j}(t) = f(t)\phi_{\lambda_j}.$$

From (47) it follows that

$$(48) \quad \begin{cases} i\hbar f'(t)\phi_{\lambda_j} &= \lambda_j f(t)\phi_{\lambda_j} \\ f(0) &= 1 \end{cases}$$

Clearly we can take $f(t) = e^{-\frac{i}{\hbar}\lambda_j t}$, and we see that $\phi_{\lambda_j}(t) = e^{-\frac{i}{\hbar}t\lambda_j}\phi_{\lambda_j}$ solves (47). Note that we can write

$$(49) \quad \phi_{\lambda_j}(t) = e^{-\frac{i}{\hbar}t\widehat{H}}\phi_{\lambda_j}.$$

Now equation (49) together with the linearity of the SE suggests that “formally” for all $\psi \in \mathcal{H}$,

$$(50) \quad \psi(t) = e^{-\frac{i}{\hbar}t\widehat{H}}\psi$$

solves the SE (46). In fact, if $\psi \in D(\widehat{H})$, then using Stone’s theorem it can be deduced that $\psi(t) \in D(\widehat{H})$ for all t , and in this case $\psi(t)$ defined as in (50) indeed solves the SE (46). Hence (50) can be interpreted as a canonical time evolution of $\psi \in \mathcal{H}$. This is what is usually referred as the deterministic feature of the SE.

Remark 8.1.1. To define $\psi(t) = e^{-\frac{i}{\hbar}t\widehat{H}}\psi$ we do not need the assumption that it has an eigenbasis. We only need \widehat{H} to be self adjoint.

Definition 8.1.1 (Propagator). The operator $U(t) = e^{-\frac{i}{\hbar}t\widehat{H}}$ is called the (quantum mechanical) propagator.

Lemma 8.1.1. If $\{\phi_{\lambda_j}\}$ is an eigenbasis with ϕ_{λ_j} being eigenvectors associated to the eigenvalues λ_j then

$$(51) \quad U(t) = \sum_{j=1}^n e^{-\frac{i}{\hbar}t\lambda_j}\phi_{\lambda_j}^* \otimes \phi_{\lambda_j},$$

where $\phi_{\lambda_j}^* \in \mathcal{H}^*$ is the dual of ϕ_{λ_j} .

Proof. Let $\psi \in \mathcal{H}$. Then we can write it as a linear combination $\psi = \sum_{k=1}^n c_k \phi_{\lambda_k}$. We know that

$$(52) \quad U(t)\psi = \sum_{k=1}^n c_k U(t)\phi_{\lambda_k} = \sum_{k=1}^n c_k e^{-\frac{i}{\hbar}t\lambda_k} \phi_{\lambda_k}.$$

On the other hand

$$(53) \quad \left(\sum_{j=1}^n e^{-\frac{i}{\hbar}t\lambda_j} \phi_{\lambda_j}^* \otimes \phi_{\lambda_j} \right) \psi = \sum_{k,j=1}^n c_k e^{-\frac{i}{\hbar}t\lambda_k} \phi_{\lambda_j} \underbrace{\phi_{\lambda_j}^*(\phi_{\lambda_k})}_{=\delta_{jk}} \phi_{\lambda_j} = \sum_{k=1}^n c_k e^{-\frac{i}{\hbar}t\lambda_k} \phi_{\lambda_k}.$$

Thus for all ψ we get

$$U(t)\psi = \left(\sum_{j=1}^n e^{-\frac{i}{\hbar}t\lambda_j} \phi_{\lambda_j}^* \otimes \phi_{\lambda_j} \right) \psi.$$

□

Let us give a short summary of the discussion so far.

- The operator $U(t) = e^{-\frac{i}{\hbar}t\widehat{H}}$ can be used to describe time evolution of states in a canonical way.
- If \widehat{H} has a eigenbasis $\{\phi_{\lambda_j}\}$, corresponding to the eigenvalues λ_j , then $U(t)$ can be described explicitly as

$$U(t) = \sum_{j=1}^n e^{-\frac{i}{\hbar}t\widehat{H}} \phi_{\lambda_j}^* \otimes \phi_{\lambda_j}.$$

8.2. The Schrödinger equation for the free particle moving on \mathbb{R} . Recall that we have $\mathcal{H} = L^2(\mathbb{R})$ and $\widehat{H} = \frac{1}{2m}\widehat{p}^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$. Hence the SE (46) becomes

$$(54) \quad \begin{cases} i\hbar \frac{\partial}{\partial t} \psi(x, t) &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) \\ \psi(x, 0) &= \psi(x) \end{cases}$$

Here we will discuss how to solve (54) with Fourier transform. We will also try to find an explicit representation of $U(t)$.

8.2.1. Digression on Fourier Transform. We will briefly recall the definition and properties of the Fourier transform. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions on \mathbb{R}^n . Recall that $f \in \mathcal{S}(\mathbb{R}^n)$ roughly means that $f \in C^\infty(\mathbb{R}^n)$ and f and all its derivatives approach to zero as $|x| \rightarrow \infty$ faster than any polynomial function approaches to infinity. Now let $f \in \mathcal{S}(\mathbb{R}^n)$. The Fourier transform $\mathcal{F}(f)$, or simply \widehat{f} , of f is defined by

$$(55) \quad \widehat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\langle k, x \rangle} f(x) dx,$$

where $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ again denotes the standard inner product on \mathbb{R}^n . We want to list some properties of the Fourier transform without proofs:

(i) If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$.

(ii) Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$(56) \quad \frac{\partial \widehat{f}}{\partial x_j} = ik_j \widehat{f},$$

$$(57) \quad x_j \widehat{f} = i \frac{\partial \widehat{f}}{\partial k_j}$$

(iii) Let $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$(58) \quad \mathcal{F}^{-1}(\widehat{f})(x) = f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\langle k, x \rangle} \widehat{f}(k) dk.$$

This is called the inverse Fourier transform.

(iv) Let $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$(59) \quad \int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\widehat{f}(k)|^2 dx.$$

This is called Plancherel's formula.

(v)

Theorem 8.2.1 (Combined inversion and Plancherel formula). *The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ can be extended to a unique bounded map $\mathcal{F} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$. This map can be computed as*

$$(60) \quad \mathcal{F}(f)(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \lim_{A \rightarrow \infty} \int_{|x| \leq A} e^{-i\langle k, x \rangle} f(x) dx.$$

Moreover, the inverse Fourier transform $\mathcal{F}^{-1} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is unitary and

$$(61) \quad \mathcal{F}^{-1}(f)(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \lim_{A \rightarrow \infty} \int_{|x| \leq A} e^{i\langle k, x \rangle} \widehat{f}(k) dk.$$

Remark 8.2.1. If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then

$$\mathcal{F}(f)(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\langle k, x \rangle} f(x) dx,$$

because in this case

$$\lim_{A \rightarrow \infty} \int_{|x| \leq A} e^{-i\langle k, x \rangle} f(x) dx = \int_{\mathbb{R}^n} e^{-i\langle k, x \rangle} f(x) dx$$

by dominated convergence.

(vi) Let f and g be two measurable functions. Then the convolution $f * g$ of f and g is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy,$$

where we assume that the right hand side exists. Suppose $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g).$$

8.3. Solving the Schrödinger equation with Fourier Transform. First, we look for solutions of the form $\psi(x, t) = e^{i(kx - \omega(k)t)}$. From (54), it is clear that $\psi(x, t)$ is a solution iff $\omega(k) = \frac{\hbar k^2}{2m}$. Hence,

$$(62) \quad \psi(x, t) = e^{ikx - i\frac{\hbar k^2}{2m}t}$$

is a solution. However, note that, such $\psi(x, t) \notin L^2(\mathbb{R}^n)$. Therefore, $\psi(x, t)$ is not the solution we are looking for. Here, the idea is to use $\psi(x, t)$ to produce a sensible solution of (54)

Proposition 8.3.1. Let $\psi_0 \in \mathcal{S}(\mathbb{R})$ and let $\widehat{\psi}_0$ be its Fourier transform. Define

$$(63) \quad \psi(x, t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \widehat{\psi}_0(k) e^{i(kx - \omega(k)t)} dk.$$

Then $\psi(x, t)$ is a solution of (54) with $\psi(x, 0) = \psi_0(x)$.

Proof. Since $\widehat{\psi}_0(k) \in \mathcal{S}(\mathbb{R})$, we can check that the derivatives with respect to x and t can be interchanged with the integral sign in the definition of $\psi(x, t)$. Since $e^{i(kx - \omega(k)t)}$ solves the SE, we can easily check that $\psi(x, t)$ solves (54). Moreover,

$$\psi(x, 0) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ikx} \widehat{\psi}_0(k) dk = \psi_0(x),$$

where the last equality holds because of the inverse Fourier transform. □

Corollary 8.3.1. Let ψ_0 be as in proposition 8.3.1. Let $\widehat{\psi}(k, t)$ be the Fourier transform of $\psi(x, t)$ with respect to x . Then

$$\widehat{\psi}(k, t) = \widehat{\psi}_0(k) e^{-i\omega(k)t}.$$

Proof. From proposition 8.3.1 we know

$$\psi(x, t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ikx} (e^{-i\omega(k)t} \widehat{\psi}_0(k)) dk.$$

Thus, the claim follows. □

Form property (vi) of Fourier transforms, formally we get

$$e^{-i\omega(k)t} \widehat{\psi}_0(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \mathcal{F}(K_t * \psi_0),$$

where $\mathcal{F}(K_t) = e^{-i\omega(k)t}$, i.e. $K_t = \mathcal{F}^{-1}(e^{-i\omega(k)t}) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ikx} e^{-i\omega(k)t} dk$. Again, a “formal computation” shows that

$$K_t(x) = \sqrt{\frac{m}{i2\pi\hbar t}} e^{\frac{imx^2}{2\hbar t}}.$$

The computation of $K_t(x)$ is “formal” because $e^{-i\omega(k)t} \notin L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and thus we do not know how to take the inverse Fourier transform of it. Hence, we need a way to make sense of an integral of the form

$$(64) \quad \int_{\mathbb{R}} e^{ikx} e^{-i\omega(k)t} dk.$$

Integrals of the form (64) are called Fresnel Integrals.

8.3.1. *Digression on Fresnel Integrals.* Let Q be a real, symmetric $n \times n$ -matrix with $\det(Q) \neq 0$. An integral of the form

$$\int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Qx, x \rangle} dx$$

is called a Fresnel integral, and is defined as

$$\int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Qx, x \rangle} dx := \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\varepsilon\langle x, x \rangle} e^{\frac{i}{2}\langle Qx, x \rangle} dx.$$

As a matter of fact we have

$$\int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Qx, x \rangle} dx = e^{\frac{\pi i}{4} \text{sign}(Q)} \frac{1}{\left| \det\left(\frac{Q}{2\pi}\right) \right|^{\frac{1}{2}}},$$

where $\text{sign}(Q) = \#\text{positive eigenvalues} - \#\text{negative eigenvalues}$. More generally, for $\omega \in \mathbb{R}^n$, we have

$$(65) \quad \begin{aligned} \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Qx, x \rangle} e^{\langle \omega, x \rangle} dx &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Qx, x \rangle - \frac{1}{2}\varepsilon\langle x, x \rangle} e^{\langle \omega, x \rangle} dx \\ &= \frac{e^{\frac{\pi i}{4} \text{sign}(Q)}}{\left| \det\left(\frac{Q}{2\pi}\right) \right|^{\frac{1}{2}}} e^{\frac{i}{2}\langle Q^{-1}\omega, \omega \rangle}. \end{aligned}$$

We use this general result, to compute

$$(66) \quad \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-i\omega(k)t} e^{ikx} dk.$$

Now, using (65), it can be easily checked that

$$K_t(x) = \sqrt{\frac{m}{2\pi i \hbar t}} e^{\frac{imx^2}{2\hbar t}}.$$

Remark 8.3.1. There is also another way to define (66) (see [7]).

Now we make our previous formal discussion mathematically.

Proposition 8.3.2. Suppose $\psi_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and define

$$\psi(x, t) = \mathcal{F}^{-1} \left(\widehat{\psi}_0(k) e^{-\frac{\hbar k^2 t}{2m}} \right).$$

Then $\psi(x, t) = K_t * \psi_0$, where $K_t(x) = \sqrt{\frac{m}{2\pi i \hbar t}} e^{\frac{i m x^2}{2\hbar}}$.

Proof. We will only briefly sketch the proof. The idea here is to show that

$$(67) \quad \mathcal{F}(K_t * \psi_0) = \widehat{\psi}_0(k) e^{-\frac{i \hbar k^2 t}{2m}}.$$

We can not talk about $\mathcal{F}(K_t)$ as $K_t \notin L^2(\mathbb{R})$. However, we can consider $K_t \chi_{[-n, n]}$ and its Fourier transform. Observe that

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \mathcal{F}(K_t \chi_{[-n, n]} * \psi_0) = \mathcal{F}(K_t \chi_{[-n, n]}) \mathcal{F}(\psi_0).$$

It can be shown that $K_t \chi_{[-n, n]} * \psi_0 \xrightarrow{n \rightarrow \infty} K_t * \psi$ in $L^2(\mathbb{R})$ and

$$\mathcal{F}(K_t \chi_{[-n, n]}) \mathcal{F}(\psi_0) \xrightarrow{n \rightarrow \infty} \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{i \hbar k^2 t}{2m}} \widehat{\psi}_0$$

in $L^2(\mathbb{R})$. These two observations imply that (67) holds and hence

$$K_t * \psi_0 = \mathcal{F}^{-1} \left(\widehat{\psi}_0(k) e^{-\frac{\hbar k^2 t}{2m}} \right).$$

□

8.3.2. *Summary of the discussion.* We have shown that if $\psi_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$e^{-\frac{i}{\hbar} t \widehat{H}} \psi_0 = (\mathcal{F}^{-1} \circ m \circ \mathcal{F}) \psi_0,$$

where $(mf)(k) = e^{-\frac{i \hbar k^2 t}{2m}} f(k)$. This means we have shown that the following diagram is commutative.

$$\begin{array}{ccc} L^1(\mathbb{R}) \cap L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \\ \downarrow e^{-\frac{i}{\hbar} t \widehat{H}} & & \downarrow m \\ L^2(\mathbb{R}) & \xleftarrow{\mathcal{F}^{-1}} & L^2(\mathbb{R}) \end{array}$$

Moreover, we have shown that $(\mathcal{F}^{-1} \circ m \circ \mathcal{F}) \psi_0 = K_t * \psi_0$. Finally, combining these results, we conclude that

$$\left(e^{-\frac{i}{\hbar} t \widehat{H}} \psi_0 \right) (x) = \sqrt{\frac{m}{2\pi i \hbar t}} \int_{\mathbb{R}} e^{\frac{i m (x-y)^2}{2\hbar}} \psi_0(y) dy,$$

i.e. the integral kernel of $e^{-\frac{i}{\hbar} t \widehat{H}}$ is $K_t(x - y) = \sqrt{\frac{m}{2\pi i \hbar t}} e^{\frac{i m (x-y)^2}{2\hbar}}$.

Remark 8.3.2. One can check that $K_t(x)$ satisfies the SE and $\lim_{t \rightarrow 0} K_t(x) = \delta(x)$ in distributional sense.

Definition 8.3.1 (Fundamental solution). $K_t(x)$ is called the fundamental solution of the SE.

Remark 8.3.3. One can easily extend the discussion above for the free particle in \mathbb{R}^n .

Part 3. The Path Integral Approach to Quantum Mechanics

We saw that Hamilton's approach to classical mechanics inspired an axiomatic approach to quantum mechanics. Hence, it is natural to ask whether there is a "Lagrangian formulation" of quantum mechanics. Dirac, who viewed Lagrangian mechanics more fundamental, took first steps towards a Lagrangian formulation of quantum mechanics. Feynman advanced it further, which gave rise to the path integral formulation of quantum theory. Dirac suggested that the quantum mechanical propagator $K(t, x, y)$ may be represented by

$$(68) \quad \int_{\gamma \in P(t, x, y)} e^{\frac{i}{\hbar} S(\gamma)} \mathcal{D}\gamma,$$

where $P(t, x, y)$ is the space of paths $\gamma : [0, t] \rightarrow \mathbb{R}$ joining x to y . Since $P(t, x, y)$ is an infinite dimensional manifold, it is not clear what the integral (68) means.

9. FEYNMAN'S FORMULATION OF THE PATH INTEGRAL

Feynman's idea was to define (68) as a limit of integrals over finite dimensional manifolds, which roughly goes as follows. Let $P_n(t, x, y)$ be the space of piecewise linear paths joining x to y , which consists of n line segments $\ell_{x, x_1}, \ell_{x_1, x_2}, \dots, \ell_{x_{n-1}, y}$. Clearly, to define $\gamma \in P_n(t, x, y)$, we need to specify (x_1, \dots, x_{n-1}) . This means that we can identify $P_n(t, x, y)$ with \mathbb{R}^{n-1} . Hence, we can define

$$(69) \quad \int_{\gamma \in P(t, x, y)} e^{\frac{i}{\hbar} S(\gamma)} \mathcal{D}\gamma := \lim_{n \rightarrow \infty} A(n, t) \int_{\gamma \in P_n(t, x, y)} e^{\frac{i}{\hbar} S(\gamma)} dx_1 \cdots dx_{n-1},$$

where $A(n, t)$ is some constant depending on n and t .

9.1. Free Propagator for the free particle on \mathbb{R} . We have already shown that

$$(70) \quad K(t, x, y) = \sqrt{\frac{m}{2\pi i \hbar t}} e^{\frac{i}{\hbar} \frac{m}{2t} (x-y)^2}.$$

Let us now give a path integral derivation of $K(t, x, y)$. Let $0 = t_0 < \dots < t_n = t$ with $t_i - t_{i-1} = \frac{t}{n} =: \Delta t$. Moreover, let $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and let γ be the piecewise linear path joining x to y such that $\gamma(t_i) = x_i$ and the line segment joining x_{i-1} to x_i is given by

$$\gamma(s) = \frac{1}{\Delta t} ((t_i - s)x_{i-1} + (s - t_{i-1})x_i), \quad s \in [t_{i-1}, t_i], \quad i = 1, 2, \dots, n$$

Then

$$(71) \quad S(\gamma) = \frac{1}{2} m \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(x_i - x_{i-1})^2}{(\Delta t)^2} ds = \frac{1}{2} m \sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{\Delta t}$$

and thus

$$(72) \quad A(n, t) \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} S(\gamma)} dx_1 \cdots dx_{n-1} = A(n, t) \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} \frac{m}{2} \sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{\Delta t}} dx_1 \cdots dx_{n-1}$$

Define $f_i = \sqrt{\frac{m}{2\hbar\Delta t}} x_i$. Then by change of variables, this integral will be

$$(73) \quad \begin{aligned} A(n, t) \left(\frac{2\hbar\Delta t}{m} \right)^{\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} e^{i \sum_{i=1}^n (f_i - f_{i-1})^2} df_1 \cdots df_{n-1} &= A(n, t) \left(\frac{2\hbar\Delta t}{m} \right)^{\frac{n-1}{2}} \frac{(\pi i)^{\frac{n-1}{2}}}{\sqrt{n}} e^{\frac{i}{\hbar} (f_n - f_1)^2} \\ &= A(n, t) \left(\frac{2\hbar\Delta t}{m} \right)^{\frac{n-1}{2}} \frac{(\pi i)^{\frac{n-1}{2}}}{\sqrt{n}} e^{\frac{i}{\hbar} \frac{m}{2n\Delta t} (x_n - x_1)^2} \\ &= A(n, t) \left(\frac{2\pi i \hbar \Delta t}{m} \right)^{\frac{n-1}{2}} \left(\frac{m}{2n\pi i \hbar \Delta t} \right)^{\frac{1}{2}} e^{\frac{i}{\hbar} \frac{m}{2n\Delta t} (x_n - x_1)^2} \\ &= A(n, t) \left(\frac{2\pi i \hbar \Delta t}{m} \right)^{\frac{n-1}{2}} \left(\frac{m}{2n\pi i \hbar \Delta t} \right)^{\frac{1}{2}} e^{\frac{i}{\hbar} \frac{m}{2t} (y-x)^2} \end{aligned}$$

Define $A(n, t) := \left(\frac{m}{2\pi i \hbar t} \right)^{\frac{n}{2}}$, then

$$(74) \quad \begin{aligned} \int_{\gamma \in P(t, x, y)} e^{\frac{i}{\hbar} S(\gamma)} \mathcal{D}\gamma &= \lim_{n \rightarrow \infty} A(n, t) \int_{\mathbb{R}^{n-1}} e^{\frac{i}{\hbar} S(\gamma)} dx_1 \cdots dx_{n-1} \\ &= \left(\frac{m}{2\pi i \hbar t} \right)^{\frac{1}{2}} e^{\frac{i}{\hbar} \frac{m}{2t} (x-y)^2} \\ &= K(t, x, y). \end{aligned}$$

Next we show how to derive the path integral representation of the propagator associated with a Hamiltonian of the form $\widehat{H}_0 + V(\widehat{x})$, where $\widehat{H}_0 = \frac{1}{2m} \widehat{p}^2$ is the free Hamiltonian. Let us recall the Kato-Lie-Trotter product formula. Let A and B be self adjoint operators on a Hilbert space \mathcal{H} with domains $D(A)$ and $D(B)$ respectively. Assume that $A + B$ is densely defined and essentially self adjoint on $D(A) \cap D(B)$. Then

$$(75) \quad \lim_{n \rightarrow \infty} \left(e^{\frac{i}{\hbar} t A} e^{\frac{i}{\hbar} t B} \right)^n = e^{it(A+B)}$$

in the strong operator topology (i.e. $A_n \rightarrow A$ iff $\|A_n \psi - A \psi\| \xrightarrow{n \rightarrow \infty} 0$ for all $\psi \in \mathcal{H}$). We assume that $V(\widehat{x})$ is sufficiently nice so that the assumption of the Kato-Lie-Trotter product formula is satisfied. Then for all $\psi \in L^2(\mathbb{R})$, we have

$$(76) \quad e^{-\frac{i}{\hbar} t (\widehat{H}_0 + V(\widehat{x}))} \psi = \lim_{n \rightarrow \infty} \left(e^{-\frac{i}{\hbar} \frac{t}{n} \widehat{H}_0} e^{-\frac{i}{\hbar} \frac{t}{n} V(\widehat{x})} \right)^n \psi$$

Let us compute the right hand side of (76). Recall that

$$(77) \quad \left(e^{-\frac{i}{\hbar} \frac{t}{n} \widehat{H}_0} \psi \right) (x_1) = \sqrt{\frac{m}{2\pi i \hbar \frac{t}{n}}} \int_{\mathbb{R}} e^{\frac{i}{\hbar} \frac{m}{2\frac{t}{n}} (x_1 - x_0)^2} dx_0$$

and

$$(78) \quad \left(e^{-\frac{i}{\hbar} \frac{t}{n} V(\bar{x})} \psi \right) (x) = e^{-\frac{i}{\hbar} \frac{t}{n} V(x)} \psi(x).$$

Using these two relations, we compute

$$(79) \quad \left(\left(e^{-\frac{i}{\hbar} \frac{t}{n} \widehat{H}_0} e^{-\frac{i}{\hbar} \frac{t}{n} V(\bar{x})} \right) \psi \right) (x_1) = \sqrt{\frac{m}{2\pi i \hbar \frac{t}{n}}} \int_{\mathbb{R}} e^{\frac{i}{\hbar} \frac{m}{2\hbar} (x_1 - x_0)^2} e^{-\frac{i}{\hbar} \frac{t}{n} V(x_0)} \psi(x_0) dx_0.$$

Repeatedly applying the process we get

$$(80) \quad \begin{aligned} \left(\left(e^{-\frac{i}{\hbar} \frac{t}{n} \widehat{H}_0} e^{-\frac{i}{\hbar} \frac{t}{n} V(\bar{x})} \right)^n \psi \right) (x_n) &= \left(\frac{m}{2\pi i \hbar \frac{t}{n}} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} \frac{m}{2\hbar} \sum_{k=1}^n (x_k - x_{k-1})^2 - \frac{i}{\hbar} \frac{t}{n} \sum_{k=1}^n V(x_{k-1})} \psi(x_0) dx_0 dx_1 \cdots dx_{n-1} \\ &= \left(\frac{m}{2\pi i \hbar \frac{t}{n}} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} \sum_{k=1}^n \frac{t}{n} \left\{ \frac{m}{2} \left(\frac{x_k - x_{k-1}}{\frac{t}{n}} \right)^2 - V(x_{k-1}) \right\}} \psi(x) dx dx_1 \cdots dx_{n-1} \end{aligned}$$

$$(81) \quad = \int_{\mathbb{R}} \left\{ \left(\frac{m}{2\pi i \hbar \frac{t}{n}} \right)^{\frac{n}{2}} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{\hbar} \sum_{k=1}^n \frac{t}{n} \left\{ \frac{m}{2} \left(\frac{x_k - x_{k-1}}{\frac{t}{n}} \right)^2 - V(x_{k-1}) \right\}} dx_1 \cdots dx_{n-1} \right\} \psi(x_0) dx_0$$

Then we get

$$(82) \quad \begin{aligned} \lim_{n \rightarrow \infty} \left(\left(e^{-\frac{i}{\hbar} \frac{t}{n} \widehat{H}_0} e^{-\frac{i}{\hbar} \frac{t}{n} V(\bar{x})} \right) \psi \right) (x) &= \\ &= \int_{\mathbb{R}} \left\{ \lim_{n \rightarrow \infty} \left(\frac{m}{2\pi i \hbar \frac{t}{n}} \right)^{\frac{n}{2}} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{\hbar} \sum_{k=1}^n \frac{t}{n} \left\{ \frac{m}{2} \left(\frac{x_k - x_{k-1}}{\frac{t}{n}} \right)^2 - V(x_{k-1}) \right\}} dx_1 \cdots dx_{n-1} \right\} \psi(x_0) dx_0 = \\ &= \int_{\mathbb{R}} K(t, x, x_0) \psi(x_0) dx_0, \end{aligned}$$

where

$$K(t, x, x_0) = \lim_{n \rightarrow \infty} \left(\frac{m}{2\pi i \hbar \frac{t}{n}} \right)^{\frac{n}{2}} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{\hbar} \sum_{k=1}^n \frac{t}{n} \left\{ \frac{m}{2} \left(\frac{x_k - x_{k-1}}{\frac{t}{n}} \right)^2 - V(x_{k-1}) \right\}} dx_1 \cdots dx_{n-1}.$$

Moreover, observe that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{t}{n} \left\{ \frac{m}{2} \left(\frac{x_k - x_{k-1}}{\frac{t}{n}} \right)^2 - V(x_{k-1}) \right\}$$

can be interpreted as

$$\int_0^t \left(\frac{m}{2} \|\dot{\gamma}(s)\|^2 - V(\gamma(s)) \right) ds.$$

This means that

$$(83) \quad \int_{\gamma \in P(t, x, y)} e^{\frac{i}{\hbar} S(\gamma)} \mathcal{D}\gamma := \lim_{n \rightarrow \infty} \left(\frac{m}{2\pi i \hbar \frac{t}{n}} \right)^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{\hbar} S(\gamma)} dx_1 \cdots dx_{n-1} = K(t, x, y).$$

10. CONSTRUCTION OF THE WIENER MEASURE

We saw that Feynman defined the path integral $\int_{\gamma \in P(t,x,y)} e^{\frac{i}{\hbar} S(\gamma)} \mathcal{D}\gamma$ as a limit of integrals over finite dimensional manifolds. Now we plan to investigate whether or not it is possible to define a probability measure on $P(t, x, y)$, which is of the form

$$\frac{e^{\frac{i}{\hbar} S(\gamma)} \mathcal{D}\gamma}{Z},$$

where Z is some quantity for normalization of the measure. A short answer to this question is no. However, if we replace i by -1 (i.e. Wick rotate) then it is possible to construct a measure of the desired form on a suitable $P(t, x, y)$. This was done by Wiener in 1923 for the case $V(x) = 0$ and it is known as Wiener measure. From now on we assume $V(x) = 0$ and $S(\gamma) = \frac{1}{2} \int_0^t \|\dot{\gamma}(s)\|^2 ds$. The basic ideas are the following:

- Interpret

$$(84) \quad A(n, t) \int_{\substack{E \subseteq \mathbb{R}^{n-1} \\ \text{measurable}}} e^{-S(\gamma)} dx_1 \cdots dx_{n-1}$$

as a measure of a certain "measurable" subset of $P(t, x, y)$.

- Instead of taking the limit $n \rightarrow \infty$, try to extend this "measure" defined by (84) to a measure on $P(t, x, y)$.

Essentially, the idea comes from Molecular-kinetic theory. Einstein showed that, if $\rho(x, t)$ is the probability density for finding the Brownian particle at location x and at time t , then it satisfies the diffusion equation

$$(85) \quad \frac{\partial}{\partial t} \rho(x, t) = D \frac{\partial^2}{\partial x^2} \rho(x, t),$$

where D is the diffusion constant. This immediatly implies

$$\rho(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}},$$

if we insist that $\lim_{t \rightarrow 0} \rho(x, t) = \delta_0(x)$, where δ_0 is the Dirac delta function. This implies that for any measurable set $E \subseteq \mathbb{R}$, the probability of finding the Brownian particle in E at time t is given by

$$(86) \quad \frac{1}{\sqrt{4\pi Dt}} \int_E e^{-\frac{x^2}{4Dt}} dx.$$

From now on we take $2D = 1$. more generally, $\frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{(x-y)^2}{2(t_2-t_1)}}$ is the probability density of finding the particle at y at time $t = t_2$ if it was at x at time $t = t_1$. This means, given $0 = t_0 < t_1 < \cdots < t_n \leq t$ and $E = \prod_{i=1}^n (\alpha_i, \beta_i]$, we can observe that

$$(87) \quad A(n, t) \int_E e^{-\frac{1}{2} \sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}} dx_1 \cdots dx_n$$

can be interpreted as the probability of finding the Brownian particle in $(\alpha_i, \beta_i]$ at time $t = t_i$. Hence it should not be surprising to interpret (84) as a “measure” of a suitable subset of $P(t, x, y)$. Let us try to make this precise and construct the Wiener measure. First, we need some notations and definitions.

- We write

$$C_0([0, 1]) = \{x : [0, 1] \rightarrow \mathbb{R} \mid x \text{ is continuous at } x(0)\},$$

which are paths starting at 0. Recall that $C_0([0, 1])$ is a Banach space with the norm

$$\|x\| = \sup_{t \in [0, 1]} |x(t)|.$$

Hence, it is a topological space. Let $\mathcal{B}(C_0([0, 1]))$ denote the Borel σ -algebra of $C_0([0, 1])$ with respect to the topology induced by the norm $\|\cdot\|$.

- Fix $t \in [0, 1]$, define $\text{ev}_t : C_0([0, 1]) \rightarrow \mathbb{R}$, $\text{ev}_t(x) = x(t)$. It is easy to check that ev_t is continuous and hence it is Borel measurable. More generally, given $t_1, \dots, t_n \in [0, 1]$, define

$$\begin{aligned} P(t_1, \dots, t_n) : C_0([0, 1]) &\longrightarrow \mathbb{R}^n \\ x &\longmapsto P(t_1, \dots, t_n)(x) = (x(t_1), \dots, x(t_n)), \end{aligned}$$

i.e. $P(t_1, \dots, t_n) = (\text{ev}_{t_1}, \dots, \text{ev}_{t_n})$, thus $P(t_1, \dots, t_n)$ is continuous and hence Borel measurable.

- Given $t_1, \dots, t_n \in [0, 1]$ and $(\alpha_1, \beta_1] \times \dots \times (\alpha_n, \beta_n] = \prod_{i=1}^n (\alpha_i, \beta_i] \subseteq \mathbb{R}^n$, define

$$I\left(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i]\right) = P(t_1, \dots, t_n)^{-1}\left(\prod_{i=1}^n (\alpha_i, \beta_i]\right) = \left\{x \in C_0([0, 1]) \mid (x(t_1), \dots, x(t_n)) \in \prod_{i=1}^n (\alpha_i, \beta_i]\right\}.$$

Observe that $I(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i])$ is Borel measurable. Also note that

$$(88) \quad I\left(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i]\right) = \bigcap_{i=1}^n \text{ev}_{t_i}^{-1}((\alpha_i, \beta_i]).$$

From (88) it is clear that we can always assume $t_1 \leq t_2 < \dots < t_{n-1} \leq t_n$.

Exercise 10.0.1. Let $t_1, \dots, t_n \in [0, 1]$ and $t_1 < t_2 < \dots < t_n$. Moreover, let $t_{k-1} < s < t_k$. Check that

$$I\left(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i]\right) = I\left(t_1, \dots, t_{k-1}, s, t_k, \dots, t_n, \prod_{i=1}^{k-1} (\alpha_i, \beta_i] \times \mathbb{R} \times \prod_{i=k}^n (\alpha_i, \beta_i]\right).$$

Hint: use that $I = \bigcap_{i=1}^n \text{ev}_{t_i}^{-1}((\alpha_i, \beta_i])$.

Let \mathcal{J} be the collection of all $I(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i])$, where $n \in \mathbb{N}$ (note that we always include zero in \mathbb{N}) and $\alpha_i \leq \beta_i$ with $\alpha_i, \beta_i \in \mathbb{R} \cup \{\infty\}$ or all i .

Exercise 10.0.2. Check that \mathcal{J} is a semialgebra, i.e.

- (1) $\emptyset, C_0([0, 1]) \in \mathcal{J}$
- (2) If $I, J \in \mathcal{J}$, then $I \cap J \in \mathcal{J}$.
- (3) If $I \in \mathcal{J}$, then $C_0([0, 1]) \setminus I$ is a finite disjoint union of elements in \mathcal{J} .

Solution. We have:

- (1) $\emptyset = \text{ev}_1^{-1}((1, 1])$, and thus $\emptyset \in \mathcal{J}$.
- (2) Let $I = \bigcap_{i=1}^n \text{ev}_{t_i}^{-1}((\alpha_i, \beta_i])$ and $J = \bigcap_{j=1}^m \text{ev}_{s_j}^{-1}((\gamma_j, \delta_j])$. Then

$$I \cap J = \bigcap_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \left(\text{ev}_{t_i}^{-1}((\alpha_i, \beta_i]) \cap \text{ev}_{s_j}^{-1}((\gamma_j, \delta_j]) \right).$$

Note that $\text{ev}_{t_i}^{-1}((\alpha_i, \beta_i]) \cap \text{ev}_{s_j}^{-1}((\gamma_j, \delta_j])$ is of the form $\text{ev}_t^{-1}((a, b])$.

- (3) Let $I \in \mathcal{J}$ with $I = \text{ev}_t^{-1}((\alpha, \beta])$. Then

$$C_0([0, 1]) \setminus I = \text{ev}_t^{-1}((-\alpha, \alpha]) \cup \text{ev}_t^{-1}((\beta, \alpha]) \in \mathcal{J}.$$

We leave the general case as an exercise. □

Theorem 10.0.1 (Wiener). *There is unique probability measure μ on $\mathcal{B}(C_0([0, 1]))$, such that*

$$(89) \quad \mu \left(I \left(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i] \right) \right) = \frac{\int_{\prod_{i=1}^n (\alpha_i, \beta_i]} e^{-\frac{1}{2} \sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}} dx_1 \cdots dx_n}{\sqrt{(2\pi)^n t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})}}$$

Let us give a small overview of the proof strategy:

- First, we will define $\mu(I)$ for $I \in \mathcal{J}$ by (89).
- Then we will use the Caratheodory extension construction.

Given $I(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i])$, define $\mu(I)$ as in (89). First we show that μ is well defined i.e. if $t_{k-1} < s < t_k$, then

$$(90) \quad \mu \left(I \left(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i] \right) \right) = \mu \left(I \left(t_1, \dots, t_{k-1}, s, t_k, \dots, t_n, \prod_{i=1}^{k-1} (\alpha_i, \beta_i] \times \mathbb{R} \times \prod_{i=k}^n (\alpha_i, \beta_i] \right) \right)$$

To verify (90), we need the following lemma:

Lemma 10.0.1 (Kolmogorov-Chapman equation). Define $K(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$. Then

$$(91) \quad \int_{\mathbb{R}} K(t_1, x, y) K(t_2, y, z) dy = K(t_1 + t_2, x, z).$$

In other words

$$(92) \quad \frac{1}{\sqrt{(2\pi)^2 t_1 t_2}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2t_1}} e^{-\frac{(y-z)^2}{2t_2}} dy = \frac{1}{\sqrt{2\pi(t_1 + t_2)}} e^{-\frac{(x-z)^2}{2(t_1+t_2)}}.$$

Proof of Theorem 10.0.1. Note that

$$(93) \quad \mu \left(I \left(t_1, \dots, t_{k-1}, s, t_k, \dots, t_n, \prod_{i=1}^{k-1} (\alpha_i, \beta_i] \times \mathbb{R} \times \prod_{i=k}^n (\alpha_i, \beta_i] \right) \right) = \\ = \frac{\int_{\prod_{i=1}^{k-1} (\alpha_i, \beta_i] \times \mathbb{R} \times \prod_{i=k}^n (\alpha_i, \beta_i]} e^{-\frac{1}{2} \left\{ \sum_{i=1}^{k-2} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}} + \sum_{i=k+1}^n \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}} + \frac{(y - x_{k-1})^2}{s - t_{k-1}} + \frac{(x_k - y)^2}{t_k - s} \right\}} dx_1 \cdots dx_{k-1} dy dx_k \cdots dx_n}{\sqrt{(2\pi)^{n+1} t_1 (t_2 - t_1) \cdots (t_{k-1} - t_{k-2}) (s - t_{k-1}) (t_k - s) \cdots (t_n - t_{n-1})}}.$$

Using Lemma 10.0.1, we see that

$$(94) \quad \frac{\int_{\prod_{i=1}^n (\alpha_i, \beta_i]} e^{-\frac{1}{2} \sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}} dx_1 \cdots dx_n}{\sqrt{(2\pi)^{n+1} t_1 (t_2 - t_1) \cdots (t_k - t_{k-1}) \cdots (t_n - t_{n-1})}} = \mu \left(I \left(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i] \right) \right).$$

□

Exercise 10.0.3. Check that if $I, J \in \mathcal{J}$ and $I \cap J = \emptyset$, $I \cup J \in \mathcal{J}$, then

$$\mu(I \cup J) = \mu(I) + \mu(J),$$

i.e. μ is finitely additive. Hint: Use

$$I = I \left(t_1, \dots, t_n, \prod_{i=1}^n (\alpha_i, \beta_i] \right) \\ J = J \left(s_1, \dots, s_m, \prod_{j=1}^m (\gamma_j, \delta_j] \right).$$

A fact of the construction is that μ is countably additive on \mathcal{J} . Now, by the Caratheodory extension construction, μ induces a unique measure on $\sigma(\mathcal{J})$, the σ -algebra generated by \mathcal{J} . We will denote this “measure” again by μ . To prove theorem 10.0.1, we will show that $\sigma(\mathcal{J}) = \mathcal{B}(C_0([0, 1]))$, which is the content of the following proposition.

Proposition 10.0.1.

$$\sigma(\mathcal{J}) = \mathcal{B}(C_0([0, 1])).$$

Proof. We already know that $\mathcal{J} \subset \mathcal{B}(C_0([0, 1]))$. Hence $\sigma(\mathcal{J}) \subset \mathcal{B}(C_0([0, 1]))$. To show the converse, it suffices to show that for any $\delta > 0$,

$$\overline{B_\delta(x_0)} := \{x \in C_0([0, 1]) \mid \|x - x_0\| \leq \delta\} \subset \sigma(\mathcal{J}).$$

Fix $\delta > 0$ and $x_0 \in C_0([0, 1])$. Our goal will be to show that

$$\overline{B_\delta(x_0)} = \bigcap_{N=1}^{\infty} K_N,$$

where $K_N \in \sigma(\mathcal{J})$. Note that for fixed $t \in [0, 1]$ we have

$$(95) \quad \overline{B_\delta(x_0)} \subset \{x \in C_0([0, 1]) \mid |x(t) - x_0(t)| \leq \delta\}.$$

Let $\{t_k\}_{k=1}^{\infty}$ be a dense subset of $[0, 1]$ and define

$$K_N = \{x \in C_0([0, 1]) \mid |x(t_j) - x_0(t_j)| \leq \delta \text{ for } j = 1, 2, \dots, N\}.$$

Then by (95), $\overline{B_\delta(x_0)} \subset \bigcap_{N=1}^{\infty} K_N$. To show the reverse inclusion, we will show that

$$x \notin \overline{B_\delta(x_0)} \implies x \notin \bigcap_{N=1}^{\infty} K_N.$$

Assume that $x \notin \overline{B_\delta(x_0)}$. Then there is an $s \in [0, 1]$ such that

$$|x(s) - x_0(s)| \geq \delta + \delta_1$$

for some $\delta_1 > 0$. Now, choose a subsequence $\{t_{k_j}\}$ of $\{t_k\}$ such that $t_{k_j} \rightarrow s$ (this can be done since $\{t_k\}$ is dense). Since x and x_0 are both continuous, we get

$$(96) \quad x(t_{k_j}) \longrightarrow x(s),$$

$$(97) \quad x_0(t_{k_j}) \longrightarrow x_0(s).$$

Thus for large j we get

$$|x(t_{k_j}) - x_0(t_{k_j})| \geq \delta + \frac{\delta_1}{2},$$

and thus $x \notin \bigcap_{N=1}^{\infty} K_N$. Hence, we were able to construct a measure on $\mathcal{B}(C_0([0, 1]))$. To complete the proof of theorem 10.0.1, we check that μ is a probability measure. Indeed, we have

$$\mu(C_0([0, 1])) = \mu(\text{ev}_1^{-1}(\mathbb{R})) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = 1.$$

This completes the proof of theorem 10.0.1. □

Next we will compute the Wiener measure of the set

$$A_{s,t}^{a,b} = \{x \in C_0([0, 1]) \mid a \leq x(t) - x(s) \leq b\},$$

where $a, b \in \mathbb{R}$ with $a \leq b$, and $s, t \in [0, 1]$ with $0 \leq s < t$. Note that $A_{s,t}^{a,b} = P(s, t)^{-1}(E)$, where $E = \{(x, y) \in \mathbb{R}^2 \mid a \leq x - y \leq b\}$. Hence

$$\begin{aligned}
 \mu(A_{s,t}^{a,b}) &= \frac{1}{\sqrt{(2\pi)^2 s(t-s)}} \iint_E e^{-\frac{y^2}{2s} - \frac{(x-y)^2}{2(t-s)}} dx dy \\
 (98) \qquad &= \frac{1}{\sqrt{(2\pi)^2 s(t-s)}} \int_{\mathbb{R}} \left(\int_a^b e^{-\frac{u^2}{2(t-s)}} du \right) e^{-\frac{y^2}{2s}} dy \\
 &= \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{u^2}{2(t-s)}} du
 \end{aligned}$$

Let us make a short input on pushforward of a measure. Let $(X, \sigma(X), \mu)$ be a measure space, $(Y, \sigma(Y))$ a measurable space and $f : X \rightarrow Y$ a measurable map. Then we can define a measure $f_*\mu$ on $(Y, \sigma(Y))$, which is defined as

$$f_*\mu(P) = \mu(f^{-1}(P)), \quad P \in \sigma(Y).$$

This measure $f_*\mu$ is called the pushforward measure of μ along f . It is easy to check that for any integrable function $\alpha : Y \rightarrow \mathbb{R}$ we have

$$\int_Y \alpha(y) d(f_*\mu(y)) = \int_X (f^*\alpha)(x) d\mu(x),$$

where $(f^*\alpha)(x) = \alpha(f(x))$. Define the map $\alpha_{s,t} : C_0([0, 1]) \rightarrow \mathbb{R}$ by $\alpha_{s,t}(x) = x(t) - x(s)$. Then (98) implies that $(\alpha_{s,t})_*\mu$, where μ is the Wiener measure on $C_0([0, 1])$, is given by

$$(\alpha_{s,t})_*\mu([a, b]) = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{x^2}{2(t-s)}} dx.$$

Thus $(\alpha_{s,t})_*\mu$ is the Gaussian measure on \mathbb{R} , which is centered and it has variance $(t-s)$. As a corollary of this discussion we get

Corollary 10.0.1. The following hold.

(1)

$$\int_{C_0([0,1])} (x(t) - x(s)) d\mu(x) = 0,$$

(2)

$$\int_{C_0([0,1])} (x(t) - x(s))^2 d\mu(x) = t - s.$$

Exercise 10.0.4. Show that

$$\int_{C_0([0,1])} x(s)x(t) d\mu(x) = \min_{s,t \in [0,1]} \{s, t\}.$$

Hint: Assume that $s < t$ and show that

$$\frac{1}{\sqrt{2\pi s(t-s)}} \iint_{\mathbb{R}^2} xye^{-\frac{y^2}{2s}} e^{-\frac{(x-y)^2}{2(t-s)}} dx dy = s.$$

Exercise 10.0.5. Compute

(1)

$$\int_{C_0([0,1])} \left(\int_0^1 x(t) dt \right) d\mu(x),$$

(2)

$$\int_{C_0([0,1])} \left(\int_0^1 x(t)^2 dt \right) d\mu(x).$$

Hint: Use Fubini.

10.1. Towards nowhere differentiability of Brownian Paths. Let $h > 0$ and $0 < \alpha \leq 1$. Define

•

$$C_h^\alpha(s, t) = \{x \in C_0([0, 1]) \mid |x(t) - x(s)| \leq h|t - s|^\alpha\},$$

•

$$C_h^\alpha(t) = \bigcap_{s \in [0,1]} C_h^\alpha(s, t),$$

•

$$C_h^\alpha = \bigcap_{t \in [0,1]} C_h^\alpha(t).$$

One can check that $C_h^\alpha(s, t)$ is closed in $C_0([0, 1])$ and thus $C_h^\alpha(s, t)$, $C_h^\alpha(t)$ and C_h^α are Borel measurable.

Lemma 10.1.1.

$$(99) \quad \mu(C_h^\alpha(s, t)) \leq \sqrt{\frac{2}{\pi}} h|t - s|.$$

Proof. We can write

$$C_h^\alpha(s, t) = \{x \in C_0([0, 1]) \mid -h|t - s|^\alpha \leq x(t) - x(s) \leq h|t - s|^\alpha\} =: A_{s,t}^{-h|t-s|^\alpha, h|t-s|^\alpha}.$$

Assume that $s < t$. Then by (98) we have

$$(100) \quad \begin{aligned} \mu(C_h^\alpha(s, t)) &= \frac{1}{\sqrt{2\pi}} \int_{-h|t-s|^\alpha}^{h|t-s|^\alpha} e^{-\frac{u^2}{2(t-s)}} du \\ &= \frac{1}{\sqrt{2\pi}} \int_{-h|t-s|^{\alpha-\frac{1}{2}}}^{h|t-s|^{\alpha-\frac{1}{2}}} e^{-\frac{u^2}{2}} du \\ &\leq \sqrt{\frac{2}{\pi}} h|t - s|^{\alpha-\frac{1}{2}}, \end{aligned}$$

where we used that $e^{-\frac{u^2}{2}} \leq 1$.

□

Corollary 10.1.1. If $\frac{1}{2} < \alpha \leq 1$, then $\mu(C_h^\alpha(t)) = 0$ and hence $\mu(C_h^\alpha) = 0$.

Proof. Let $\{t_k\} \subseteq [0, 1]$ such that $t_k \rightarrow t$ (for $k \rightarrow \infty$). Now $C_h^\alpha(t) \subseteq C_h^\alpha(t, t_k)$. Thus

$$\mu(C_h^\alpha(t)) \leq \mu(C_h^\alpha(t, t_k)) \leq \sqrt{\frac{2}{\pi}} h |t - t_k|^{\alpha - \frac{1}{2}} \xrightarrow{k \rightarrow \infty} 0.$$

□

Proposition 10.1.1. Let $\frac{1}{2} < \alpha \leq 1$. Then

$$\mu(\{x \in C_0([0, 1]) \mid x \text{ is Hölder continuous of exponent } \alpha\}) = 0.$$

Proof. It is clear since $\{x \in C_0([0, 1]) \mid x \text{ is Hölder continuous of exponent } \alpha\} \subseteq \bigcup_{h=1}^{\infty} C_h^\alpha$. □

Corollary 10.1.2.

$$\mu(\{x \in C_0([0, 1]) \mid x \text{ is differentiable}\}) = 0.$$

Proof. This follows since

$$\{x \in C_0([0, 1]) \mid x \text{ is differentiable}\} \subseteq \{x \in C_0([0, 1]) \mid x \text{ is Hölder continuous of exponent } 1\}.$$

□

The following lemma will play an important role when we discuss nowhere differentiability of Brownian paths.

Lemma 10.1.2. Let $t \in [0, 1]$. Then $\mu(D_t) = 0$, where $D_t = \{x \in C_0([0, 1]) \mid \dot{x}(t) \text{ exists}\}$.

Proof. We can easily check that $D_t \subseteq \bigcup_{h=1}^{\infty} C_h^\bullet(t)$. Moreover, we already know that $\mu(C_h^\bullet(t)) = 0$ and thus $\mu(D_t) = 0$. □

Lemma 10.1.3. Define F on $C_0([0, 1]) \times [0, 1]$ by

$$F(x, t) = \begin{cases} 1, & \text{if } \dot{x}(t) \text{ exists} \\ 0, & \text{otherwise} \end{cases}$$

Then F is measurable on $C_0([0, 1]) \times [0, 1]$.

Proof. We will show that the set $G = \{(x, t) \mid F(x, t) = 1\}$ has measure 0 with respect to $\mu \times m$, where m is the Lebesgue measure on $[0, 1]$. First we observe that $G \subseteq G^*$, where

$$G^* = \{(x, t) \in C_0([0, 1]) \times [0, 1] \mid \lim_{n \rightarrow \infty} f_n(x, t) \text{ exists}\},$$

with $f_n(x, t) = \lim_{n \rightarrow \infty} \frac{x(t - \frac{1}{n}) - x(t)}{\frac{1}{n}}$. Moreover, G^* is measurable (since it is the set where a sequence of measurable functions have a limit). Note that

$$(\mu^* \times m)(G) \leq (\mu^* \times m)(G^*) = (\mu \times m)(G^*),$$

where μ^* is the outer measure associated with the premeasure μ in the construction of the Wiener measure. Now

$$(\mu \times m)(G^*) = \int_0^1 \mu(G_t^*) dt,$$

where $G_t^* = \{x \in C_0([0, 1]) \mid \lim_{n \rightarrow \infty} f_n(x, t) \text{ exists}\}$. If we can show that $\mu(G_t^*) = 0$, then we see that $(\mu \times m)(G^*) = 0 \Rightarrow (\mu^* \times m)(G) = 0 \Rightarrow G$ is measurable. To see that $\mu(G_t^*) = 0$, one can show that

$$G_t^* \subseteq \bigcup_{h=1}^{\infty} \bigcap_{n=1}^{\infty} C_h^\bullet\left(t, t + \frac{1}{n}\right),$$

and use the fact that $\lim_{n \rightarrow \infty} \mu(C_h^\bullet\left(t, t + \frac{1}{n}\right)) = 0$. \square

Theorem 10.1.1 (Nowhere differentiable Brownian paths). *With probability 1, paths $x \in C_0([0, 1])$ are differentiable at most on a subset of Lebesgue measure 0 of $[0, 1]$. (In other words: with probability 1, paths $x \in C_0([0, 1])$ are “nowhere” differentiable.)*

Proof. Let F be defined as in lemma 10.1.3. Note that

$$(101) \quad \int_{C_0([0,1]) \times [0,1]} F(x, t) d\mu(x) dt = \int_0^1 \left(\int_{C_0([0,1])} F(x, t) d\mu(x) \right) dt = \int_0^1 \mu(D_t) dt = 0,$$

by lemma 10.1.2, since for fixed t , $F(x, t) = \chi_{D_t}$. Thus we get

$$\int_{C_0([0,1])} \left(\int_0^1 F(x, t) dt \right) d\mu(x) = 0,$$

whence $\int_0^1 F(x, t) dt = 0$ for almost all $x \in C_0([0, 1])$. For such x , $F(x, t) = 0$ for almost all $t \in [0, 1]$ and thus $\dot{x}(t)$ does not exist for almost all $t \in [0, 1]$. \square

We will state now the following facts without proof.

- (Fact 1) $\mu(\{x \in C_0([0, 1]) \mid x \text{ is Hölder continuous of exponent } \alpha\}) = 1$ for $0 \leq \alpha \leq \frac{1}{2}$ (see [10]).
- (Fact 2) $\mu(\{x \in C_0([0, 1]) \mid x \text{ is Hölder continuous of exponent } \frac{1}{2}\}) = 0$ (see [13, 11]).

Remark 10.1.1. More generally, we can talk about the Wiener measure μ_x on $C_x([a, b]) = \{\omega : [a, b] \rightarrow \mathbb{R} \mid \omega \text{ is continuous and } \omega(a) = x\}$.

Remark 10.1.2. Moreover, there is the Wiener measure μ_x^y on

$$C_x^y([a, b]) = \{\omega : [a, b] \rightarrow \mathbb{R} \mid \omega \text{ is continuous and } \omega(a) = x, \omega(b) = y\}.$$

This measure is the unique measure on $\mathcal{B}(C_x^y([0, 1]))$ such that for all $t_1, \dots, t_n \in (a, b)$

$$\mu_x^y(I(t_1, \dots, t_n), E) = \int_E K_{b-t_n}(y, x_n) K_{t_n-t_{n-1}}(x_n, x_{n-1}) \cdots K_{t_1-a}(x_1, x) dx_1 \cdots dx_n,$$

where $E \subseteq \mathbb{R}^n$ is a measurable set and $K_t(u, v) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(u-v)^2}{2t}}$.

Definition 10.1.1 (Conditional Wiener measure). The Wiener measure μ_x^y is called a conditional Wiener measure.

Remark 10.1.3. μ_x^y is not a probability measure. In fact,

$$\mu_x^y(C_x^y([a, b])) = \frac{1}{\sqrt{2\pi(b-a)}} e^{-\frac{(x-y)^2}{t(b-a)}}.$$

Remark 10.1.4. μ_x^y is called conditional Wiener measure because μ_x and μ_x^y fit in the general framework (see [10, 11]) of a conditional measure

$$\mu_x = \int_{\mathbb{R}} \mu_x^y dy.$$

10.2. The Feynman-Kac Formula. The goal of this subsection is to prove the following theorem.

Theorem 10.2.1 (Feynman-Kac). *Let V be a continuous function on \mathbb{R} , which is bounded from below. Let $\widehat{H}_0 = -\frac{1}{2} \frac{d^2}{dx^2} = \frac{1}{2} \Delta$ and $\widehat{H} = \widehat{H}_0 + V$. Moreover, assume that \widehat{H} is essentially self adjoint. Then for all $\psi \in L^2(\mathbb{R})$*

$$(102) \quad (e^{-t\widehat{H}}\psi)(x_0) = \int_{C_{x_0}([0,t])} \psi(x(t)) e^{-\int_0^t V(x(s)) ds} d\mu_{x_0}(x).$$

The main technical tool, which we are going to use here, is the Trotter product formula, given in the following way. Let A and B be self adjoint operators bounded from below on \mathcal{H} . Assume that $\widehat{H} = A + B$ is essentially self adjoint in $D(A) \cap D(B)$. Denote the unique self adjoint extension of \widehat{H} by \widehat{H} again. Then for all $\phi \in \mathcal{H}$ and for all $t \geq 0$

$$e^{-t\widehat{H}}\phi = \lim_{n \rightarrow \infty} \left((e^{-\frac{t}{n}A} e^{-\frac{t}{n}B})^n \phi \right).$$

Proof of Theorem 10.2.1. We have

$$(103) \quad \left((e^{-\frac{t}{n}\widehat{H}_0} e^{-\frac{t}{n}V}) \psi \right)(x_0) = \int_{\mathbb{R}} K_{\frac{t}{n}}(x_n, x_0) e^{-\frac{t}{n}V(x_1)} \psi(x_1) dx_1.$$

Taking the square of the operator we get

$$(104) \quad \left((e^{-\frac{t}{n}\widehat{H}_0} e^{-\frac{t}{n}V})^2 \psi \right)(x_0) = \iint_{\mathbb{R}^2} K_{\frac{t}{n}}(x_2, x_1) K_{\frac{t}{n}}(x_1, x_n) e^{-\frac{t}{n}(V(x_2)+V(x_1))} \psi(x_1) dx_1 dx_2.$$

Taking the n -th power of the operator, we get

$$(105) \quad \left((e^{-\frac{t}{n}\widehat{H}_0} e^{-\frac{t}{n}V})^n \psi \right)(x_0) = \int_{\mathbb{R}^n} K_{\frac{t}{n}}(x_n, x_{n-1}) \cdots K_{\frac{t}{n}}(x_1, x_0) e^{-\frac{t}{n} \sum_{j=1}^n V(x_j)} \psi(x_n) dx_1 \cdots dx_n,$$

where $x_j = x\left(\frac{jt}{n}\right)$ and thus $x_n = x(t)$. Then (105) is equal to

$$\int_{C_{x_0}([0,t])} \psi(x(t)) e^{-\frac{t}{n} \sum_{j=1}^n V(x(\frac{jt}{n}))} d\mu_{x_0},$$

and thus

$$(106) \quad (e^{-t\widehat{H}}\psi)(x_0) = \lim_{n \rightarrow \infty} \int_{C_{x_0}([0,t])} \psi(x(t)) e^{-\frac{t}{n} \sum_{j=1}^n V(x(\frac{jt}{n}))} d\mu_{x_0}.$$

Since

$$\lim_{n \rightarrow \infty} e^{-\frac{t}{n} \sum_{j=1}^n V(x(\frac{jt}{n}))} = e^{-\int_0^1 V(x(s)) ds},$$

it is enough to justify that limit and integral are interchangeable in (106). This can be justified by using the assumption that V is bounded from below and by Lebesgue's dominated convergence theorem. Details are left to the reader. \square

Remark 10.2.1. The Feynman-Kac formula holds (see [13]) for some general V .

Remark 10.2.2. There is a Feynman-Kac formula with respect to μ_x^y on $C_x^y([0, t])$ as well (see [5]). It simply says that the integral kernel of $e^{-t\widehat{H}}$ is given by

$$K_t(x, y, \widehat{H}) = \int_{C_x^y([0,t])} e^{-\int_0^1 V(x(s)) ds} d\mu_x^y.$$

11. GAUSSIAN MEASURES

11.1. Gaussian measures on \mathbb{R} .

Definition 11.1.1 (Gaussian measure I). A Borel probability measure μ on \mathbb{R} is called Gaussian if it is either the Dirac measure δ_a at $a \in \mathbb{R}$, or it is of the form

$$(107) \quad d\mu(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-a)^2}{2\sigma}},$$

where $a \in \mathbb{R}$, and $\sigma > 0$. The parameters a and σ are called mean and variance of μ respectively.

Remark 11.1.1. If μ is given by (107), we say μ is nondegenerate Gaussian. Moreover, if $a = 0$, then μ is called a centered Gaussian.

Exercise 11.1.1. Check that

$$(108) \quad a = \int_{\mathbb{R}} x d\mu(x),$$

$$(109) \quad \sigma = \int_{\mathbb{R}} (x - a)^2 d\mu(x).$$

Exercise 11.1.1 justifies the names "mean" and "variance" of the Gaussian measure μ given by (107).

Exercise 11.1.2. Given a Borel measure μ , define $\widetilde{\mu} : \mathbb{R} \rightarrow \mathbb{C}$ by

$$\widetilde{\mu}(y) = \int_{\mathbb{R}} e^{iyx} d\mu(x).$$

Check that $\widetilde{\mu}(y) = e^{iay - \frac{1}{2}\sigma y^2}$, if μ is given by (107).

Definition 11.1.2 (Characteristic Functional I). The map $\widetilde{\mu}$ defined as in exercise 11.1.2 is called the charactersitic functional (or Fourier transform) of μ .

Exercise 11.1.3. Let μ be a Borel measure on \mathbb{R} . Show that μ is Gaussian iff

$$(110) \quad \widetilde{\mu}(y) = e^{iay - \frac{1}{2}\sigma y^2}$$

for some $a \in \mathbb{R}$ and $\sigma > 0$.

11.2. Gaussian measures on finite dimensional vector spaces.

Definition 11.2.1 (Gaussian measure II). A Borel probability measure μ on \mathbb{R}^n is called Gaussian, if for all linear maps $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$, the pushforward measure $\alpha_*\mu$ is Gaussian on \mathbb{R} .

This definition is abstract and we will later give a more “working” definition of a Gaussian measure.

Remark 11.2.1. From now on we will identify $(\mathbb{R}^n)^*$ with \mathbb{R}^n , using the standard metric on \mathbb{R}^n , i.e. a linear map $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ will be considered a vector $\alpha \in \mathbb{R}^n$.

Definition 11.2.2 (Characteristic Functional II). Given a finite Borel measure μ on \mathbb{R}^n , define $\widehat{\mu} : \mathbb{R} \rightarrow \mathbb{C}$ by

$$\widehat{\mu}(y) = \int_{\mathbb{R}^n} e^{i\langle y, x \rangle} d\mu(x).$$

$\widehat{\mu}$ is called the Characteristic functional (or Fourier transform) of μ .

Proposition 11.2.1. A Borel measure μ on \mathbb{R}^n is Gaussian iff

$$(111) \quad \widehat{\mu}(y) = e^{-i\langle y, a \rangle - \frac{1}{2}\langle Ky, y \rangle},$$

where $a \in \mathbb{R}^n$ and K is a positive definite symmetric $n \times n$ -matrix. In this case, when μ is nondegenerate, then

$$d\mu(x) = \frac{1}{\det\left(\frac{K}{2\pi}\right)^{\frac{1}{2}}} e^{-\frac{1}{2}\langle K^{-1}(x-a), K^{-1}(x-a) \rangle} dx.$$

Proof. Given a Borel measure μ on \mathbb{R}^n and a linear map $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$, we get

$$(112) \quad \begin{aligned} \widehat{\alpha_*\mu}(t) &= \int_{\mathbb{R}} e^{its} d(\alpha_*\mu)(s) = \int_{\mathbb{R}^n} e^{it\alpha(x)} d\mu(x) \\ &= \int_{\mathbb{R}^n} e^{i\langle t\alpha, x \rangle} d\mu(x) \\ &= \widehat{\mu}(t\alpha), \end{aligned}$$

where we have used in the second equality that

$$\int_X (F^*\alpha)(x) dx = \int_Y f(y) d(\alpha_*\mu)(y).$$

Assume that $\widehat{\mu}$ has the form (110). Then

$$(113) \quad \widehat{\alpha_*\mu}(t) = \widehat{\mu}(t\alpha) = e^{i\langle t\alpha, a \rangle - \frac{1}{2}\langle K(t\alpha), t\alpha \rangle} = e^{it\langle \alpha, a \rangle - \frac{1}{2}t^2\langle K\alpha, \alpha \rangle}.$$

By exercise 11.1.2, $\widehat{\alpha_*\mu}$ is Gaussian on \mathbb{R} . Conversely, assume that $\alpha_*\mu$ is Gaussian on \mathbb{R} for all linear maps $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$. By exercise 11.1.2 we get

$$\widehat{\alpha_*\mu}(t) = e^{ita(\alpha) - \frac{1}{2}\sigma(\alpha)t^2}.$$

Moreover, by exercise 11.1.1, we get

$$(114) \quad a(\alpha) = \int_{\mathbb{R}} t d(\alpha_*\mu)(t),$$

$$(115) \quad \sigma(\alpha) = \int_{\mathbb{R}} (t - a(\alpha))^2 d(\alpha_*\mu)(t).$$

We can check that the application $\alpha \mapsto a(\alpha)$ defines a linear map $\mathbb{R}^n \rightarrow \mathbb{R}$. and hence it can be identified with $a \in \mathbb{R}^n$ as $a(\alpha) = \langle a, \alpha \rangle$. Moreover, the application $\alpha \mapsto \sigma(\alpha)$ defines a quadratic form on \mathbb{R}^n . Hence, there is a symmetric $n \times n$ -matrix K such that $\sigma(\alpha) = \langle K\alpha, \alpha \rangle$. Thus, $\sigma(\alpha) > 0$ for all $\alpha \in \mathbb{R}^n$ implies that K is a positive matrix. The last part of the proof is left as an exercise⁶. \square

Hence, we saw that this abstract definition of a Gaussian measure on \mathbb{R}^n is equivalent to the usual notion of Gaussian measure.

Exercise 11.2.1. Let μ be a Gaussian measure on \mathbb{R}^n of the form

$$d\mu(x) = \det\left(\frac{K}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\langle K(x-a), (x-a) \rangle} dx.$$

Check that

$$a = \int_{\mathbb{R}^n} x d\mu(x) = \left(\int_{\mathbb{R}} x_1 d\mu(x_1), \dots, \int_{\mathbb{R}} x_n d\mu(x_n) \right),$$

and

$$K_{ij}^{-1} = \int_{\mathbb{R}^n} (x_i - a_i)(x_j - a_j) d\mu(x).$$

Definition 11.2.3 (Covariance operator). The vector $a \in \mathbb{R}^n$ is called the mean of the Gaussian measure and the matrix K^{-1} is called the covariance operator of μ . When the mean of a Gaussian measure is 0, then it is called a centered Gaussian (see remark 11.1.1).

Proposition 11.2.2. Let μ be a centered Gaussian measure on \mathbb{R}^n of the form

$$d\mu(x) = \det\left(\frac{K}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\langle Kx, x \rangle} dx.$$

Then

⁶It essentially follows from the one dimensional case and diagonalization of K .

(1) For all $\lambda \in \mathbb{C}^n$,

$$\int_{\mathbb{R}^n} e^{\langle \lambda, x \rangle} d\mu(x) = e^{\frac{1}{2} \langle K^{-1} \lambda, \lambda \rangle}.$$

(2)

$$\int_{\mathbb{R}^n} f(x - \sqrt{t}y) d\mu(y) = \left(e^{\frac{t}{2} L^\mu} f \right) (x),$$

where $L^\mu = \sum_{i=1}^n K_{ij}^{-1} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j}$. Moreover,

$$\int_{\mathbb{R}^n} f(y) d\mu(y) = \left(e^{\frac{1}{2} L^\mu} f \right) (0).$$

(3)

$$\int_{\mathbb{R}^n} p(x) d\mu(x) = p(D_\lambda) e^{-\frac{1}{2} \langle A^{-1} \lambda, \lambda \rangle} \Big|_{\lambda=0},$$

where $p(D_\lambda)$ is a polynomial in derivatives in λ_i -directions $\frac{\partial}{\partial \lambda_i}$ corresponding to the polynomial map $p(x)$, i.e. if $p(x) = x_1 x_2$, then $p(D_\lambda) = \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2}$.

Proof. We prove each point separately:

(1) We have

$$\int_{\mathbb{R}^n} e^{\langle \lambda, x \rangle} d\mu(x) = \det \left(\frac{K}{2\pi} \right)^{\frac{1}{2}} \int_{\mathbb{R}^n} e^{\langle \lambda, x \rangle} e^{-\frac{1}{2} \langle Kx, x \rangle} dx = \frac{\det \left(\frac{K}{2\pi} \right)^{\frac{1}{2}}}{\det \left(\frac{K}{2\pi} \right)^{\frac{1}{2}}} e^{\frac{1}{2} \langle K^{-1} \lambda, \lambda \rangle} = e^{\frac{1}{2} \langle K^{-1} \lambda, \lambda \rangle}.$$

(2) It is sufficient to check that $f(x)$ is of the form $e^{\langle \lambda, x \rangle}$ with $\lambda \in \mathbb{C}^n$ as these function are dense. For $f(x) = e^{\langle \lambda, x \rangle}$ we get

$$f(x - \sqrt{t}y) = e^{\langle \lambda, x \rangle} e^{\langle -\sqrt{t}\lambda, y \rangle},$$

and thus

$$\int_{\mathbb{R}^n} f(x - \sqrt{t}y) d\mu(y) = e^{\langle \lambda, x \rangle} \int_{\mathbb{R}^n} e^{\langle -\sqrt{t}\lambda, y \rangle} d\mu(y) = e^{\langle \lambda, x \rangle} e^{\frac{t}{2} \langle K^{-1} \lambda, \lambda \rangle}.$$

On the other hand

$$L^\mu \left(e^{\langle \lambda, x \rangle} \right) = \langle K^{-1} \lambda, \lambda \rangle e^{\langle \lambda, x \rangle}$$

which means

$$e^{\frac{t}{2} L^\mu} \left(e^{\langle \lambda, x \rangle} \right) = e^{\frac{t}{2} \langle K^{-1} \lambda, \lambda \rangle} e^{\langle \lambda, x \rangle}.$$

Thus we have

$$\int_{\mathbb{R}^n} f(x - \sqrt{t}y) d\mu(y) = \left(e^{\frac{t}{2} L^\mu} f \right) (x),$$

when $f(x) = e^{\langle \lambda, x \rangle}$. The second part can be verified in a similar way.

(3) Left as an exercise.

□

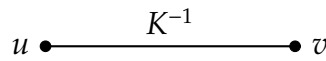
Example 11.2.1. Consider

$$\int_{\mathbb{R}^n} x_i x_j d\mu(x) = K_{ij}^{-1}.$$

More generally,

$$\int_{\mathbb{R}^n} \langle u, x \rangle \langle v, x \rangle d\mu(x) = \langle K^{-1}u, v \rangle.$$

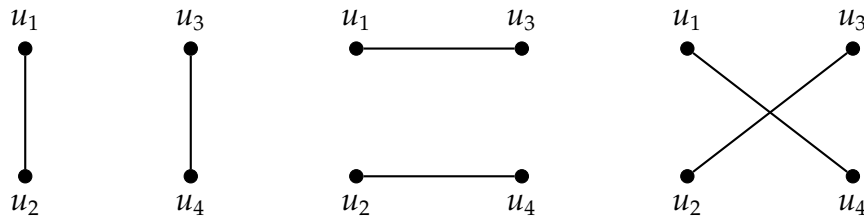
Graphically, it can be represented as



Example 11.2.2. Consider

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^4 \langle u_i, x \rangle \right) d\mu(x) = \langle K^{-1}u_1, u_2 \rangle \langle K^{-1}u_3, u_4 \rangle + \langle K^{-1}u_1, u_3 \rangle \langle K^{-1}u_2, u_4 \rangle + \langle K^{-1}u_1, u_4 \rangle \langle K^{-1}u_2, u_3 \rangle.$$

Graphically, it can be represented as the sum of



where each edge represents K^{-1} .

In general, there is the following theorem.

Theorem 11.2.1 (Wick).

$$\int_{\mathbb{R}^n} \prod_{i=1}^k \langle u_i, x \rangle d\mu(x) = \begin{cases} \sum \langle K^{-1}u_{j_1}, u_{j_2} \rangle \cdots \langle K^{-1}u_{j_{m-1}}, u_{j_m} \rangle, & \text{if } k \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

Exercise 11.2.2. Prove theorem 11.2.1.

11.3. Gaussian measures on real seperable Hilbert spaces. Let \mathcal{H} be a real seperable Hilbert space.

Definition 11.3.1 (Borel measure on \mathcal{H}). A Borel measure μ on \mathcal{H} is a measure defined on $\mathcal{B}(\mathcal{H})$, which is the Borel σ -algebra of \mathcal{H} .

In the previous subsection, we saw that a Gaussian measure on a finite dimensional vector space V is determined by $a \in V$ and a positive symmetric matrix K^{-1} , called the covariance of the Gaussian mean. In this section we will see whether this is the case in the infinite dimensional case as well. Let μ be a Borel measure on \mathcal{H} . Define an operator S_μ on \mathcal{H} by

$$(116) \quad \langle S_\mu(x), y \rangle = \int_{\mathcal{H}} \langle x, z \rangle \langle y, z \rangle d\mu(z)$$

Remark 11.3.1. It may happen that S_μ does not exist.

Let us recall some background material.

- (1) (Trace class operators) Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. We define the squareroot of A by

$$|A| := \sqrt{A^*A},$$

which exists by the spectral theorem. Note that $|A| \geq 0$. Let A be a nonnegative operator on \mathcal{H} . Then

$$\sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle$$

is independent of the choice of an orthonormal basis $\{e_n\}$. In this case, one defines the trace of A as

$$Tr(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle.$$

Definition 11.3.2 (Trace class). The operator A is called trace class if $Tr(|A|) < \infty$.

If A is a trace class operator, then $\sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle$ does not depend on the choice of an orthonormal basis $\{e_n\}$. In this case, we define $Tr(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle$.

- (2) (Bilinear forms/quadratic forms) A bilinear form B with domain $D(B)$ is a bilinear map

$$\begin{aligned} B: D(B) \times D(B) &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto B(x, y), \end{aligned}$$

where $D(B)$ is a dense subspace of \mathcal{H} . Given a bilinear form B on \mathcal{H} , we can define a quadratic form $q(x) = B(x, x)$. A bilinear form B is bounded if there is some $\varepsilon > 0$ such that for all $x, y \in D(B)$

$$|B(x, y)| \leq \varepsilon \|x\| \|y\|.$$

We call B symmetric if $B(x, y) = B(y, x)$ for all x, y . Moreover, B is called positive (definite) if $q(x) \geq 0$ (and $q(x) = 0$ iff $x = 0$) for all x . If B is a bounded, positive and symmetric bilinear form, then there is a bounded linear operator $S_B : \mathcal{H} \rightarrow \mathcal{H}$ such that $B(x, y) = \langle S_B(x), y \rangle$.

This is the end of the background materials. Next, we want to investigate when S_μ exists. We first need some notation. We define

$$\mathcal{T} := \{\text{Trace class, positive, self adjoint operators on } \mathcal{H}\}.$$

Proposition 11.3.1.

$$S_\mu \in \mathcal{T} \iff \int_{\mathcal{H}} \|x\|^2 d\mu(x) < \infty.$$

Proof. Assume $S_\mu \in \mathcal{T}$. Let $\{e_n\}$ be an orthonormal basis of \mathcal{H} . Then

$$\text{Tr}(S_\mu) = \sum_{n=1}^{\infty} \langle S_\mu(e_n), e_n \rangle = \sum_{n=1}^{\infty} \int_{\mathcal{H}} \langle x, e_n \rangle^2 d\mu(x) = \int_{\mathcal{H}} \sum_{n=1}^{\infty} \langle x, e_n \rangle^2 d\mu(x) = \int_{\mathcal{H}} \|x\|^2 d\mu(x),$$

by the monotone convergence theorem. Conversely, assume $\int_{\mathcal{H}} \|x\|^2 d\mu(x) < \infty$. Then, define

$$B(x, y) = \int_{\mathcal{H}} \langle x, z \rangle \langle y, z \rangle d\mu(z).$$

Then

$$|B(x, y)| = \left| \int_{\mathcal{H}} \langle x, z \rangle \langle y, z \rangle d\mu(z) \right| \leq \|x\| \|y\| \int_{\mathcal{H}} \|z\|^2 d\mu(z),$$

and thus B is a bounded bilinear form. Moreover, B is symmetric and positive. Hence, there is a positive self adjoint operator S_μ such that $B(x, y) = \langle S_\mu(x), y \rangle$. Now, we can check

$$\sum_{n=1}^{\infty} \langle S_\mu(e_n), e_n \rangle = \int_{\mathcal{H}} \|x\|^2 d\mu(x) < \infty$$

for any orthonormal basis $\{e_n\}$. Thus $S_\mu \in \mathcal{T}$. □

11.3.1. Characteristic Functionals.

Definition 11.3.3 (Positive definite function). A function $\phi : \mathcal{H} \rightarrow \mathbb{C}$ is called a positive definite if for all $c_1, \dots, c_n \in \mathbb{C}$ and $h_1, \dots, h_n \in \mathcal{H}$ with $n = 1, 2, \dots$ we have

$$(117) \quad \sum_{j,k=1}^n c_k \phi(h_k - h_j) \bar{c}_j \geq 0.$$

Definition 11.3.4 (Characteristic functional III). Let μ be a Borel measure on \mathcal{H} . The characteristic functional (or Fourier transform) $\widehat{\mu}$ of μ is a function $\widehat{\mu} : \mathcal{H} \rightarrow \mathbb{C}$ defined by

$$(118) \quad \widehat{\mu}(y) = \int_{\mathcal{H}} e^{i\langle y, x \rangle} d\mu(x).$$

Remark 11.3.2. It is easy to check that:

- (1) $|\widehat{\mu}(x)| \leq \mu(\mathcal{H})$ for all $x \in \mathcal{H}$.
- (2) If μ is a probability measure, then $\widehat{\mu}(0) = 1$.

(3) If μ is a finite measure, then $\widehat{\mu}$ is uniformly continuous on \mathcal{H} .

Lemma 11.3.1. Let μ be a Borel measure on \mathcal{H} . Then $\widehat{\mu}$ is a positive definite functional on \mathcal{H} .

Proof. Let $h_1, \dots, h_n \in \mathcal{H}$ and $c_1, \dots, c_n \in \mathbb{C}$. Then we get

$$\begin{aligned} \sum_{j,k=1}^n c_j \widehat{\mu}(h_j - h_k) \bar{c}_k &= \int_{\mathcal{H}} \sum_{j,k=1}^n c_j e^{i\langle h_j, x \rangle} e^{-i\langle h_k, x \rangle} \bar{c}_k d\mu(x) \\ &= \int_{\mathcal{H}} \sum_{j,k=1}^n c_j e^{i\langle h_j, x \rangle} \overline{e^{i\langle h_k, x \rangle} c_k} d\mu(x) \\ &= \int_{\mathcal{H}} \left| \sum_{j=1}^n c_j e^{i\langle h_j, x \rangle} \right|^2 d\mu(x) \geq 0. \end{aligned}$$

□

Definition 11.3.5 (Gaussian measure III). A Borel measure μ on \mathcal{H} is called a Gaussian measure on \mathcal{H} if for all $h \in \mathcal{H}$ we get that $(\alpha_h)_* \mu$ is a Gaussian measure on \mathbb{R} , where $\alpha_h : \mathcal{H} \rightarrow \mathbb{R}$ is given by $\alpha_h(x) = \langle h, x \rangle$.

Lemma 11.3.2. Let μ be a Gaussian measure on \mathcal{H} . Then there are functions m and σ on \mathcal{H} such that $\widehat{\mu}(y) = e^{im(y) - \frac{1}{2}\sigma(y)}$.

Proof. Recall that $(\widehat{\alpha_h})_* \widehat{\mu}(t) = \widehat{\mu}(th)$. Since $(\alpha_h)_* \mu$ is a Gaussian measure, we have

$$(\widehat{\alpha_h})_* \widehat{\mu}(t) = e^{im(h)t - \frac{1}{2}t^2\sigma(h)},$$

and thus

$$\widehat{\mu}(h) = (\widehat{\alpha_h})_* \widehat{\mu}(1) = e^{im(h) - \frac{1}{2}\sigma(h)}.$$

□

Exercise 11.3.1. Check that

$$(119) \quad m(y) = \int_{\mathcal{H}} \langle x, y \rangle d\mu(x)$$

$$(120) \quad \sigma(y) = \int_{\mathcal{H}} \langle y, x \rangle^2 d\mu(x)$$

Theorem 11.3.1 (Bochner-Kolmogorov-Milnor-Prokhorov). Let ϕ be a positive definite functional on \mathcal{H} . Then ϕ is a characteristic functional of a Borel probability measure μ on \mathcal{H} if and only if

$$(1) \quad \phi(0) = 1$$

(2) for all $\varepsilon > 0$ there is an $S_\varepsilon \in \mathcal{T}$ such that

$$1 - \operatorname{Re}(\phi(x)) \leq \langle S_\varepsilon x, x \rangle + \varepsilon$$

for all $x \in \mathcal{H}$.

Proof. See [10]. □

Theorem 11.3.2 (Prokhorov). *The following hold:*

(1) Let μ be a Gaussian measure on \mathcal{H} . Then $S_\mu \in \mathcal{T}$.

(2) Let $m \in \mathcal{H}$ and $S \in \mathcal{T}$. Then $\phi(x) = e^{i\langle m, x \rangle - \frac{1}{2}\langle Sx, x \rangle}$ is the characteristic functional of a Gaussian measure.

Proof. We will only consider the centered Gaussian measure, i.e. $m(y) = 0$, i.e. $\widehat{\mu}(x) = e^{-\frac{1}{2}\sigma(x)}$. We want to show that

$$(121) \quad \int_{\mathcal{H}} \|x\|^2 d\mu(x) < \infty.$$

The idea now is to try to find some $S \in \mathcal{T}$ and $C_S > 0$ such that

$$(122) \quad \int_{\mathcal{H}} \langle x, y \rangle^2 d\mu(y) \leq C_S \langle Sx, x \rangle,$$

for all $x \in \mathcal{H}$. Before we discuss a construction of S , let us observe why (122) implies (121). Let $\{e_n\}$ be an orthonormal basis of \mathcal{H} . Then by (122)

$$\sum_{n=1}^{\infty} \int_{\mathcal{H}} \langle e_n, y \rangle^2 d\mu(y) \leq C_S \sum_{n=1}^{\infty} \langle S e_n, e_n \rangle = C_S \text{Tr}(S).$$

This implies that

$$\int_{\mathcal{H}} \|y\|^2 d\mu(y) = \int_{\mathcal{H}} \sum_{n=1}^{\infty} \langle e_n, y \rangle^2 d\mu(y) = \sum_{n=1}^{\infty} \int_{\mathcal{H}} \langle e_n, y \rangle^2 d\mu(y) \leq C_S \text{Tr}(S) < \infty.$$

Hence, our goal will be to construct S such that (122) holds. Since μ is a probability measure, by the previous theorem we get that for all $\varepsilon > 0$ there is some $S_\varepsilon \in \mathcal{T}$ such that

$$(123) \quad 1 - \widehat{\mu}(x) \leq \langle S_\varepsilon x, x \rangle + \varepsilon,$$

for all $x \in \mathcal{H}$. Assume now that $\ker(S_\varepsilon) = \{0\}$. In this case we claim that for all $x \in \mathcal{H} \setminus \{0\}$, we have

$$(124) \quad \int_{\mathcal{H}} \langle x, y \rangle^2 d\mu(x) \leq \frac{4}{\varepsilon} \log\left(\frac{1}{1-2\varepsilon}\right) \langle S_\varepsilon x, x \rangle.$$

Obviously (124) implies (122). To verify (124), we proceed as follows: If $y \in \mathcal{H}$ such that $\langle S_\varepsilon y, y \rangle < \varepsilon$, then from (123) we get $\sigma(y) \leq 2 \log\left(\frac{1}{1-2\varepsilon}\right)$. Given $x \in \mathcal{H} \setminus \{0\}$, take $y = \left(\frac{\varepsilon}{2\langle S_\varepsilon x, x \rangle}\right)^{\frac{1}{2}} x$. Then we can check that $\langle S_\varepsilon y, y \rangle < \varepsilon$ and hence we have

$$(125) \quad \sigma(y) \leq 2 \log\left(\frac{1}{1-2\varepsilon}\right).$$

Note that we use $\ker(S_\varepsilon) = \{0\}$ to define y . Also, we can check that

$$\sigma(y) = \frac{\varepsilon}{2\langle S_\varepsilon x, x \rangle} \sigma(x).$$

Thus for $x \in \mathcal{H} \setminus \{0\}$, we get from (125) that

$$\sigma(x) \leq \frac{4}{\varepsilon} \log\left(\frac{1}{1-2\varepsilon}\right) \langle S_\varepsilon x, x \rangle.$$

Now using $\sigma(x) = \int_{\mathcal{H}} \langle x, y \rangle^2 d\mu(y)$, we verify (124) when $\ker(S_\varepsilon) = \{0\}$. If $\ker(S_\varepsilon) = \{0\}$, then we can construct $S \in \mathcal{T}$ with $S_\varepsilon \leq S$ and $\ker(S) = \{0\}$ as follows: Let $\{\lambda_n\}$ be positive eigenvalues of S_ε and ϕ_n eigenvectors corresponding to λ_n such that $\|\phi_n\| = 1$ and $\phi_n \perp \phi_m$ for $m \neq n$. Moreover, let $\{\psi_j\}$ be an orthonormal basis of $\ker(S_\varepsilon)$. Then $\{\phi_n, \psi_j\}$ form an orthonormal basis of \mathcal{H} . Define the map

$$S : \mathcal{H} \longrightarrow \mathcal{H}$$

$$x \longmapsto S(x) = \sum_n \lambda_n \langle \phi_n, x \rangle \phi_n + \sum_j \frac{1}{j^2} \langle \psi_j, x \rangle \psi_j.$$

Then we can check that $S \in \mathcal{T}$, $\ker(S) = \{0\}$ and thus (123) holds if we replace S_ε by S . Hence repeating the argument above, (124) holds for S . This completes the proof. \square

Now let \mathcal{H} be a separable Hilbert space. Let \mathcal{F} be the set of finite rank projections of \mathcal{H} , i.e. $p \in \mathcal{F}$ iff $p : \mathcal{H} \rightarrow \mathcal{H}$ is a projection and $\dim p(\mathcal{H}) < \infty$. We define the set

$$\mathcal{R} = \{p^{-1}(B) \mid p \in \mathcal{F}, B \subseteq p(\mathcal{H}), B \text{ is Borel measurable}\}.$$

Then it is easy to check that \mathcal{R} is an algebra. However, \mathcal{R} is not a sigma algebra, which can be seen as follows. Let $\overline{B(0,1)}$ be the closed unit ball in \mathcal{H} . When \mathcal{H} is infinite dimensional $\overline{B(0,1)}$ is not a cylinder set, i.e. $\overline{B(0,1)} \notin \mathcal{R}$, as $C \in \mathcal{R}$ implies that C is unbounded. We claim that $\overline{B(0,1)}$ can be written as countable intersections of elements of \mathcal{R} . Let $\{h_n\}$ be a countable dense subset of \mathcal{H} with $h_n \neq 0$ for all n . Moreover, for $N \in \mathbb{N}$, we define the set

$$K_N = \{h \in \mathcal{H} \mid |\langle h, h_n \rangle| \leq \|h_n\|, \forall n = 1, 2, \dots, N\}.$$

Exercise 11.3.2. Show that $K_N \in \mathcal{R}$ for all $N \in \mathbb{N}$.

It is easy to see that $\overline{B(0,1)} \subseteq \bigcap_{N=1}^{\infty} K_N$. Assume that $h \notin \overline{B(0,1)}$. Then there is some $h' \in \mathcal{H} \setminus \{0\}$ such that $\frac{|\langle h, h' \rangle|}{\|h'\|} \geq \delta + 1$ for some $\delta > 0$. Choose then a subsequence $\{h_{n_k}\}$ such that $h_{n_k} \rightarrow h'$. Then $\frac{|\langle h, h_{n_k} \rangle|}{\|h_{n_k}\|} \rightarrow \frac{|\langle h, h' \rangle|}{\|h'\|}$ and thus

$$\frac{|\langle h, h_{n_k} \rangle|}{\|h_{n_k}\|} \geq \delta + 1$$

as $k \rightarrow \infty$. This shows that $h \notin \bigcap_{N=1}^{\infty} K_N$ and hence we have showed that $\bigcap_{N=1}^{\infty} K_N \subseteq \overline{B(0,1)}$. This means $\overline{B(0,1)} = \bigcap_{N=1}^{\infty} K_N$. Next we define a finitely additive measure μ on \mathcal{R} as follows. Let $p \in \mathcal{F}$ and B be a Borel subset of $p(\mathcal{H})$ and $\dim p(\mathcal{H}) = n$. Define

$$\mu(p^{-1}(B)) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_B e^{-\frac{1}{2}\|x\|^2} dx.$$

Exercise 11.3.3. Show that μ is a finitely additive measure on R .

Exercise 11.3.4. Show directly that μ can not be countably additive.

Hence, there is no hope to try to construct the standard Gaussian measure on \mathcal{H} (in the infinite dimensional case the identity operator is not a trace class operator). We ask ourself whether there is a way to make sense of the standard Gaussian measure on \mathcal{H} . The answer is **yes**. There is a way to understand the standard Gaussian measure on \mathcal{H} . The idea is to expand \mathcal{H} so that it supports a countably additive Gaussian measure.

11.4. Standard Gaussian measure on \mathcal{H} . How do we expand \mathcal{H} ? The technical tool we use here is Kolmogorov's theorem. Let us briefly recall this without a proof. For this, let $\{X_i\}_{i \in \mathcal{J}}$ be a family of topological spaces. Assume that for each $I \subseteq \mathcal{J}$ finite, we have a Borel probability measure μ_I on $X_I := \prod_{i \in I} X_i$. Given $J \subseteq I \subseteq \mathcal{J}$, with I finite, let $\pi_{IJ} : X_I \rightarrow X_J$ denote the projection onto the first J coordinates.

Definition 11.4.1 (Compatible family). The family $\{X_I, \mu_I\}_{\substack{I \subseteq \mathcal{J} \\ I \text{ finite}}}$ is said to form a compatible family if for all $J \subseteq I$ we have $(\pi_{IJ})_* \mu_I = \mu_J$.

Theorem 11.4.1 (Kolmogorov). Let $\{X_I, \mu_I\}$ be a compatible family. Then there is a unique probability measure $\mu_{\mathcal{J}}$ on $X_{\mathcal{J}} = \prod_{i \in \mathcal{J}} X_i$ and measurable maps $\pi_I : X_{\mathcal{J}} \rightarrow X_I$ for $I \subseteq \mathcal{J}$ finite such that $(\pi_I)_* \mu_{\mathcal{J}} = \mu_I$.

To apply theorem 11.4.1 in our situation, we proceed as follows. Let $\{e_n\}$ be an orthonormal basis of \mathcal{H} . Define a measure μ_n on \mathbb{R}^n by

$$\mu_n(B) = \mu(p_n^{-1}(B)),$$

where $p_n : \mathcal{H} \rightarrow \text{span}\{e_1, \dots, e_n\} \cong \mathbb{R}^n$ is the projection and μ the cylindrical measure defined before. Then it is easy to check that $\{\mathbb{R}^n, \mu_n\}$ form a compatible family of probability measures. Hence, by theorem 11.4.1 there is a probability space $(\Omega, \tilde{\mu})$ and random variables ξ_1, \dots, ξ_n on Ω such that

$$(126) \quad \begin{aligned} \tilde{\mu}(\{\omega \in \Omega \mid (\xi_1(\omega), \dots, \xi_n(\omega)) \in B, B \subseteq \mathbb{R}^n \text{ Borel measurable}\}) \\ = \mu(\{h \in \mathcal{H} \mid (\langle h, e_1 \rangle, \dots, \langle h, e_n \rangle) \in B\}) \text{ (or simply } \mu_n(B)) \end{aligned}$$

Lemma 11.4.1. The $\{\xi_i\}$ are independent and identitcally distributed random variables with mean 0 and variance 1.

Note that using the ξ_i s we can define \mathcal{H} -valued random variables X_n by

$$\begin{aligned} X_n : \Omega &\longrightarrow \mathcal{H} \\ \omega &\longmapsto X_n(\omega) = \sum_{i=1}^n \xi_i(\omega) e_i. \end{aligned}$$

Moreover,

$$(127) \quad (p_n \circ X_n)_* \tilde{\mu} = \mu_n.$$

If $\{X_n\}$ converges in probability (convergence in measure), then it would induce a random variable $X : \Omega \rightarrow \mathcal{H}$ and hence we would get a measure $X_*\tilde{\mu}$ on \mathcal{H} and by construction it would be the standard Gaussian measure on \mathcal{H} . Unfortunately, the bad thing is that the sequence $\{X_n\}$ does not converge in probability. We already know that this is not possible because we have seen that there can not exist a Gaussian measure μ on \mathcal{H} (assuming \mathcal{H} is infinite dimensional) whose characteristic functional is $\widehat{\mu}(x) = e^{-\frac{1}{2}\|x\|^2}$. Let us see directly how $\{X_n\}$ fails to converge in probability. For this it is sufficient to show that $\{X_n\}$ is not Cauchy in probability.

Lemma 11.4.2. $\{X_n\}$ is not Cauchy in probability.

Proof. Let $\varepsilon > 0$ and $n > m$. Then

$$(128) \quad \tilde{\mu}\left(\left\{\omega \in \Omega \mid \left\|\sum_{i=m+1}^n \xi_i(\omega)e_i\right\| > \varepsilon\right\}\right) = \mu_{n-m}\left(\mathbb{R}^{n-m} \setminus \overline{B(0, \varepsilon)}\right)$$

$$(129) \quad = 1 - \mu_{n-m}(\overline{B(0, \varepsilon)})$$

$$(130) \quad \geq 1 - \mu_{n-m}([-\varepsilon, \varepsilon]^{n-m})$$

$$(131) \quad = 1 - (\mu_1([-\varepsilon, \varepsilon]))^{n-m}.$$

Note that $\mu_1([-\varepsilon, \varepsilon]) < 1$ implies that $(\mu_1([-\varepsilon, \varepsilon]))^{n-m} \xrightarrow{n,m \rightarrow \infty} 0$. Here $\mu_1([-\varepsilon, \varepsilon]) = \frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}x^2} dx$. This implies that $\{X_n\}$ is not Cauchy in probability. \square

Hence, our strategy to construct a Banach space containing \mathcal{H} would be the following. First consider a new norm $\|\cdot\|_W$ for which the sequence $\{X_n\}$ is Cauchy in probability. In this case $\{X_n\}$ converges in probability if we consider the Banach space obtained by completing \mathcal{H} with respect to this new norm. This motivates the following definition.

Definition 11.4.2 (Measurable norm). A norm $\|\cdot\|_W$ on \mathcal{H} is said to be measurable if for all $\varepsilon > 0$ there is some $p_0 \in \mathcal{F}$ such that

$$\mu(\{h \in \mathcal{H} \mid \|p_h\|_W > \varepsilon\}) < \varepsilon$$

for all $p \in \mathcal{F}$ such that $p \perp p_0$.

Geometrically it means that $\|\cdot\|_W$ is such that μ is concentrated in a tubular neighborhood of some $p_0 \in \mathcal{F}$. A non-example would be the norm $\|\cdot\|_{\mathcal{H}}$ on \mathcal{H} , which is not measurable.

Theorem 11.4.2 (Gross). *Let $\|\cdot\|_W$ be a measurable norm on \mathcal{H} . Let W be the Banach space obtained by the completion of \mathcal{H} with respect to $\|\cdot\|_W$. Then the sequence $\{X_n\}$ converges in probability in W .*

Theorem 11.4.3 (Gross). *Given a separable real Hilbert space \mathcal{H} , there is a separable Banach space W with a linear continuous dense embedding $\iota : \mathcal{H} \hookrightarrow W$ and a Gaussian measure μ_W on W such that*

$$(132) \quad \begin{aligned} \mu_W(\{w \in W \mid (f_1(w), \dots, f_n(w)) \in B, B \subseteq \mathbb{R}^n \text{ Borel measurable}\}) \\ = \mu(\{h \in \mathcal{H} \mid (\langle h, f_1 \rangle, \dots, \langle h, f_n \rangle) \in B\}) \end{aligned}$$

for all $f_1, \dots, f_n \in W^* \hookrightarrow \mathcal{H}^* \cong \mathcal{H}$. In particular, for all $f \in W^* \subseteq \mathcal{H}$

$$\widehat{\mu}(f) = e^{-\frac{1}{2}\|f\|_{\mathcal{H}}^2}.$$

Here Gaussian measure means that for all $f \in W^*$ we have that $f_*\mu$ is Gaussian on \mathbb{R} .

Remark 11.4.1. If $\|\cdot\|_W$ is a measurable norm on \mathcal{H} , then there is some $c > 0$ such that $\|h\|_W \leq c\|h\|$ for all $h \in \mathcal{H}$ (see [10]). It was expected that $\|\cdot\|_W$ is dominated by $\|\cdot\|$ because we needed a bigger topology on \mathcal{H} to allow convergence of $\{X_n\}$.

Remark 11.4.2. Let A be a positive Hilbert-Schmidt operator on \mathcal{H} . Define a new norm by

$$\|h\|_{W_A} = \|Ah\|.$$

Then $\|\cdot\|_{W_A}$ is a measurable norm on \mathcal{H} (see [10]).

Remark 11.4.3. In the view of remark 11.4.2 we see that there can be many Banach spaces. In other words, we have no uniqueness. However, we do not care.

To elaborate on remark 11.4.3, a slogan here is that \mathcal{H} contains all information about the measure. Our next goal is to make this slogan a little more precise, and this requires some effort. Given a separable Hilbert space \mathcal{H} , we saw that there is a Banach space W , a linear continuous dense embedding $\iota : \mathcal{H} \hookrightarrow W$ and a Gaussian measure μ on W such that $\widehat{\mu}(f) = e^{-\frac{1}{2}\|f\|_{\mathcal{H}}^2}$, where $f \in W^* \subseteq \mathcal{H}^* \cong \mathcal{H}$. Next, we would like to understand whether it is possible to identify \mathcal{H} from a separable Banach space W and a centered Gaussian measure μ . More precisely, given a separable Banach space W and a centered Gaussian measure μ on W , is it possible to find a Hilbert space $\mathcal{H}(\mu)$ together with a linear continuous dense embedding $\iota : \mathcal{H}(\mu) \hookrightarrow W$ such that $\widehat{\mu}(f) = e^{-\frac{1}{2}\|f\|_{\mathcal{H}(\mu)}^2}$. We will start with a separable real Hilbert space \mathcal{H} and a Banach space W and a Gaussian measure μ given by theorem 11.4.3. Then we will try to understand how to recover \mathcal{H} from W and μ . This will give a hint on the construction of $\mathcal{H}(\mu)$ out of W and μ . Let \mathcal{H}, W and μ be as in theorem 11.4.2. Given $f \in W^*$, we have $q_\mu(f) = \int_W f(w)^2 d\mu(w)$. More generally,

$$q_\mu : W^* \times W^* \longrightarrow \mathbb{R}$$

$$(f, g) \longmapsto q_\mu(f, g) = \int_W fg d\mu.$$

Definition 11.4.3 (Covariance of a measure). The map q_μ is called the covariance of μ .

Exercise 11.4.1. Show that $q_\mu(f, g) = \langle f, g \rangle_{\mathcal{H}}$, where $f, g \in W^* \subseteq \mathcal{H}^* \cong \mathcal{H}$.

First, we would like to show that q_μ is a continuous positive definite symmetric bilinear form on W^* . To see this we need a technical tool: Fernique's theorem, which we state without proof.

Theorem 11.4.4 (Fernique). Let W be a separable Banach space and μ be a Gaussian measure on W . Then there is some $\varepsilon = \varepsilon(\mu) > 0$ such that

$$\int_W e^{\varepsilon\|w\|_W^2} d\mu(w) < \infty.$$

Corollary 11.4.1.

$$\int_W \|w\|_W^p d\mu(w) < \infty, \quad \forall p \geq 1.$$

Proposition 11.4.1. q_μ is a continuous bilinear form on W^* .

Proof. We have

$$|q_\mu(f, g)| \leq \int_W |f(w)g(w)| d\mu(w) \leq \|f\|_{W^*} \|g\|_{W^*} \underbrace{\int_W \|w\|_W^2 d\mu(w)}_c.$$

□

Note that $f \in W^*$ implies that $q_\mu(f) < \infty$ and thus $f \in L^2(W, \mu)$. Hence, we have a canonical linear map

$$\begin{aligned} T : W^* &\longrightarrow L^2(W, \mu) \\ f &\longmapsto f. \end{aligned}$$

Lemma 11.4.3. The map T is continuous.

Proof. We have

$$\|T(f)\|_{L^2(W, \mu)} = \int_W f(w)^2 d\mu(w) \leq \|f\|_{W^*}^2 \int_W \|w\|_W^2 d\mu(w).$$

□

Corollary 11.4.2. The norm on W^* induced by q_μ is weaker than $\|\cdot\|_{W^*}$.

Lemma 11.4.4. Let J be the map $J : W^* \rightarrow \mathcal{H}$ given as the composition $W^* \hookrightarrow \mathcal{H}^* \xrightarrow{\sim} \mathcal{H}$. Then $J : (W^*, q_\mu) \rightarrow \mathcal{H}$ is a linear continuous dense isometric embedding.

Proof. It is a direct consequence of the previous corollary. □

Exercise 11.4.2. Given $h \in \mathcal{H}$, define $\alpha_h : W^* \rightarrow \mathbb{R}$ by $\alpha_h(f) = f(h)$. Show that α_h is continuous on W^* with respect to the topology given by q_μ .

Corollary 11.4.3. If $h \in \mathcal{H}$, then $\|h\|_{\mathcal{H}} = \sup_{f \in W^* \setminus \{0\}} \frac{|f(h)|}{\sqrt{q_\mu(f, f)}}$.

Proof. Since $J(W^*)$ is dense in \mathcal{H} , we know that

$$\|h\|_{\mathcal{H}} = \sup_{f \in W^* \setminus \{0\}} \frac{|f(h)|}{\|J(f)\|} = \sup_{f \in W^* \setminus \{0\}} \frac{|f(h)|}{\|f\|_{L^2(W, \mu)}} = \sup_{f \in W^* \setminus \{0\}} \frac{|f(h)|}{\sqrt{q_\mu(f, f)}}.$$

□

Let K be the completion of $T(W^*)$ in $L^2(W, \mu)$. Then we see that J extends to an isometry $J : K \rightarrow \mathcal{H}$.

Exercise 11.4.3. Show that $J : K \rightarrow \mathcal{H}$ is an isomorphism of Hilbert spaces.

Let us summarize what we have seen so far:

(1) We have seen that

$$h \in \mathcal{H} \implies \|h\|_{\mathcal{H}} = \sup_{f \in W^* \setminus \{0\}} \frac{|f(h)|}{\sqrt{q_\mu(f, f)}}.$$

This relation can be thought of as constructing the norm on \mathcal{H} out of W and μ . It will be the key in order to construct \mathcal{H} out of W and μ .

(2) The map $J : K \rightarrow \mathcal{H}$ is an isomorphism of Hilbert spaces. In particular, it is an isomorphism of Banach spaces.

Given a separable Banach space on W and a centered Gaussian measure μ on W , (1) will be used to define a normed space $\mathcal{H}(\mu)$ and (2) will be used to give an inner product on $\mathcal{H}(\mu)$. This way we will be able to construct $\mathcal{H}(\mu)$ out of W and μ .

Definition 11.4.4 ($\mathcal{H}(\mu)$ -norm). Let W be a real separable Banach space and μ a centered Gaussian measure on W . Define a norm $\| \cdot \|_{\mathcal{H}(\mu)}$ by

$$\|w\|_{\mathcal{H}(\mu)} = \sup_{f \in W^* \setminus \{0\}} \frac{|f(w)|}{\sqrt{q_\mu(f, f)}},$$

and $\mathcal{H}(\mu) = \{w \in W \mid \|w\|_{\mathcal{H}(\mu)} < \infty\}$. The space $\mathcal{H}(\mu)$ is called the **Cameron-Martin space**.

Exercise 11.4.4. Show that $\mathcal{H}(\mu)$ is a normed space.

Exercise 11.4.5. Show that $w \in \mathcal{H}(\mu)$ iff $f \mapsto f(w)$ is continuous on W^* if W^* has the topology induced by q_μ .

Proposition 11.4.2. $\mathcal{H}(\mu)$ is a Banach space, i.e. $\| \cdot \|_{\mathcal{H}(\mu)}$ is complete.

Proof. We first show that there is some $c > 0$ such that for all $w \in \mathcal{H}(\mu)$

$$\|w\|_W \leq c \|w\|_{\mathcal{H}(\mu)}.$$

In other words $\iota : (\mathcal{H}(\mu), \| \cdot \|_{\mathcal{H}(\mu)}) \hookrightarrow W$ is continuous. Let $w \in W \setminus \{0\}$. By the Hahn-Banach theorem we can choose $f \in W^*$ such that $\|f\|_{W^*} = 1$ and $f(w) = \|w\|_W$. Moreover, by proposition 11.4.1 we have that $\|f\|_{q_\mu} \leq \tilde{c} \|f\|_{W^*}$ and thus $\|f\|_{q_\mu} \leq \tilde{c}$. Now

$$\|w\|_W = f(w) = |f(w)| = \frac{|f(w)|}{\|f\|_{W^*}} \leq c \frac{|f(w)|}{\|f\|_{q_\mu}} \leq c \|w\|_{\mathcal{H}(\mu)},$$

with $c = \frac{1}{\tilde{c}}$. To show that $\mathcal{H}(\mu)$ is complete, let $\{h_n\}$ be a Cauchy sequence in $\mathcal{H}(\mu)$ with respect to $\| \cdot \|_{\mathcal{H}(\mu)}$. By the previous section it is Cauchy in $(W, \| \cdot \|_W)$. Since W is complete, there is some $h \in W$ such that $h_n \rightarrow h$ in $(W, \| \cdot \|_W)$. We claim that $h \in \mathcal{H}(\mu)$ and $h_n \rightarrow h$ in $\mathcal{H}(\mu)$. Let $\varepsilon > 0$. Choose $m \geq n$ large enough such that

$$f(h_n - h) < \varepsilon \sqrt{q_\mu(f, f)},$$

(use $h_n \rightarrow h$ in W and $f \in W^*$). Therefore

$$\frac{|f(h_n - h)|}{\sqrt{q_\mu(f, f)}} \leq \frac{|f(h_n - h_m)|}{\sqrt{q_\mu(f, f)}} + \frac{|f(h_m - h)|}{\sqrt{q_\mu(f, f)}} \leq \|h_n - h_m\|_{\mathcal{H}(\mu)} + \varepsilon.$$

This shows that $\|h_n - h\|_{\mathcal{H}(\mu)} < \infty$ implies that $h \in \mathcal{H}(\mu)$ and $\|h_n - h\|_{\mathcal{H}(\mu)} \xrightarrow{n \rightarrow \infty} 0$. \square

Even though we haven't been able to show that $\mathcal{H}(\mu)$ is a Banach space, so far, we haven't done anything to show that $\mathcal{H}(\mu) \neq \{0\}$. In order to see this, and that $\mathcal{H}(\mu)$ is a Hilbert space, we will use Bochner integrals.

11.4.1. *Digression on Bochner integrals.* Let $(\Omega, \sigma(\Omega), \mu)$ be a measure space and W a Banach space.

Definition 11.4.5 (Bochner integrable). Let $f : \Omega \rightarrow W$ be a measurable map, where we consider the Borel σ -algebra on W . We say f is Bochner integrable if

$$\int_{\Omega} \|f(\omega)\|_W d\mu(\omega) < \infty.$$

If f is Bochner integrable, it is possible to define $\int_{\Omega} f(\omega) d\mu(\omega) \in W$, i.e. a W -valued integral on Ω , which is called the Bochner integral of f . For us $(\Omega, \sigma(\Omega), \mu)$ will be $(W, \mathcal{B}(W), \mu)$. Let $f \in W^*$, then

$$\int_W \|wf(w)\|_W d\mu(w) \leq \|f\|_{W^*} \int_W \|w\|_W^2 d\mu(w) < \infty,$$

by Fernique's theorem. This implies that, given $f \in W^*$, the map $w \mapsto wf(w)$ is Bochner integrable. Hence $\int_W wf(w) d\mu(w) \in W$. For $f \in W^*$ we define

$$J(f) := \int_W wf(w) d\mu(w) \in W.$$

Exercise 11.4.6. Show that $J(f) \in \mathcal{H}(\mu)$ and $\|J(f)\|_{\mathcal{H}(\mu)} \leq \|f\|_{q_\mu}$.

As a consequence of the exercise we see that $J : W^* \rightarrow \mathcal{H}(\mu)$ is a contraction if W^* is endowed with the norm induced by q_μ , i.e.

$$\|f\|_{q_\mu} = \|f\|_{L^2(W, \mu)}.$$

This shows that $\mathcal{H}(\mu) \neq \{0\}$. Since W^* is dense in K , we have an isometry $J : K \rightarrow \mathcal{H}(\mu)$. Next, we show that J is surjective. Given $h \in \mathcal{H}(\mu)$ such that the map $f \mapsto f(h)$ is continuous on (W^*, q_μ) . We call this map \hat{h} . Note that \hat{h} extends to a continuous linear functional on K . Hence, by Riesz's theorem, \hat{h} can be identified with an element of $\hat{h} \in K$. It is easy to check that $J(\hat{h}) = h$. Thus J is an isomorphism of Banach spaces. Now we give $\mathcal{H}(\mu)$ the Hilbert space structure induced by J . Since K is a separable Hilbert space, so is $\mathcal{H}(\mu)$. We have the following theorem

Theorem 11.4.5 (Gross). *Given a real separable Banach space and a Gaussian measure μ , there exist a Hilbert space $\mathcal{H}(\mu) \subseteq W$ such that $\mathcal{H}(\mu) \hookrightarrow W$ is a linear continuous dense embedding and*

$$\widehat{\mu}(f) = e^{-\frac{1}{2}\|f\|_{\mathcal{H}(\mu)}^2}, \quad \forall f \in W^*.$$

Exercise 11.4.7. Let $\mathcal{H}(\mu)$ be the Cameron-Martin space of (W, μ) . Let $\{e_n\} \subseteq W^*$ be an orthonormal basis of $\mathcal{H}(\mu)$. Show that if $w \in W$, then $w \in \mathcal{H}(\mu)$ iff

$$\sum_{n=1}^{\infty} e_n(w)^2 < \infty,$$

and in this case

$$\|w\|_{\mathcal{H}(\mu)}^2 = \sum_{n=1}^{\infty} e_n(w)^2.$$

Corollary 11.4.4. $\mu(\mathcal{H}(\mu)) = 0$ if $\mathcal{H}(\mu)$ is infinite dimensional.

Proof. Let $\{e_n\} \subseteq W^*$ be as in the previous exercise. Then $\{e_n\}$ is a sequence of independent, identically distributed random variables with mean 0 and variance 1. Hence, by the law of large numbers, we get

$$\sum_{n=1}^{\infty} e_n(w)^2 = \mu \quad a.e.$$

Whereas by exercise 11.4.7 we have

$$\mathcal{H}(\mu) = \left\{ w \in W \mid \sum_{n=1}^{\infty} e_n(w)^2 < \infty \right\}.$$

This implies that $\mu(\mathcal{H}(\mu)) = 0$. □

There is a different way to understand the Cameron-Martin space.

Theorem 11.4.6 (Cameron-Martin). Let $\mathcal{H}(\mu)$ be the Cameron-Martin space of (W, μ) , $h \in W$ and $T_h : W \rightarrow W$ given by $T_h(w) = w - h$. If $h \in \mathcal{H}(\mu)$, then $\mu_h = (T_h)_* \mu$ is absolutely continuous with respect to μ and

$$(133) \quad \frac{d\mu_h}{d\mu} = e^{-\frac{1}{2}\|h\|_{\mathcal{H}(\mu)}^2} e^{\langle h, \cdot \rangle}.$$

Proof. We will compute the Fourier transform of μ_h and $e^{-\frac{1}{2}\|h\|_{\mathcal{H}(\mu)}^2} e^{\langle h, \cdot \rangle} \mu$. Let thus $f \in W^*$. Then we have

$$\widehat{\mu}_h(f) = \int_W e^{if(w)} d\mu_h(w) = \int_W e^{if(w-h)} d\mu(w) = e^{-if(h)} e^{-\frac{1}{2}q_\mu(f,f)}.$$

On the other hand, setting $\tilde{h} = J^{-1}(h)$, we get

$$(134) \quad \begin{aligned} \int_W e^{-\frac{1}{2}\|h\|^2} e^{\tilde{h}(w)} e^{if(w)} d\mu(w) &= e^{-\frac{1}{2}\|h\|^2} \int_W e^{i(f-i\tilde{h})(w)} d\mu(w) \\ &= e^{-\frac{1}{2}\|h\|^2} e^{-\frac{1}{2}q_\mu(f-i\tilde{h}, f-i\tilde{h})} \\ &= e^{if(h)} e^{-\frac{1}{2}q_\mu(f,f)}. \end{aligned}$$

Thus $(T_h)_* \mu$ is absolutely continuous with respect to μ and (133) holds. □

Remark 11.4.4. $h \in W \setminus \mathcal{H}(\mu)$ implies that μ and μ_h are mutually singular. Hence $h \in \mathcal{H}(\mu)$ iff $(T_h)_*\mu \ll \mu$.

Example 11.4.1 (Cameron-Martin space of the classical Wiener space). Recall the classical Wiener space: we had $W = C_0([0, 1])$ endowed with the Wiener measure μ together with the covariance

$$q_\mu(\text{ev}_t, \text{ev}_s) = \min\{s, t\},$$

where ev denotes the evaluation map. One can check that $E = \text{span}\{\text{ev}_t \mid t \in [0, 1]\}$ is dense in W^* . First, let us compute $J : E \rightarrow W$. By definition we have

$$(J(\text{ev}_t))(s) = \int_{C_0([0,1])} x(s) \text{ev}_t(x) d\mu(x) = \int_{C_0([0,1])} x(s)x(t) d\mu(x) = \min\{s, t\},$$

and thus $\frac{d}{ds}(J(\text{ev}_t)) = \chi_{[0,t]}$. Note that $q_\mu(\text{ev}_t, \text{ev}_s) = \min\{s, t\}$. On the other hand

$$\int_0^1 \dot{J}(\text{ev}_t)(u) \dot{J}(\text{ev}_s)(u) du = \int_0^1 \chi_{[0,t]}(u) \chi_{[0,s]}(u) du = \min\{s, t\}.$$

This shows that if $x, y \in \mathcal{H}(\mu)$, then

$$(135) \quad \langle x, y \rangle_{\mathcal{H}(\mu)} = \int_0^1 \dot{x}(u) \dot{y}(u) du.$$

Thus we can write down the space $\mathcal{H}(\mu) = \{x \in C_0([0, 1]) \mid \dot{x} \in L^2([0, 1])\}$ and the inner product on $\mathcal{H}(\mu)$ is given by (135). This means that the Wiener measure is the standard Gaussian measure on $H^1([0, 1])$, which is the space of 1-Sobolev paths. This observation is due to Cameron-Martin, which was later generalized by Gross.

12. WICK ORDERING

12.1. Motivating example and construction. Let $H_n(x)$ be the degree n Hermite polynomial on \mathbb{R} . It can be defined recursively as follows

$$\begin{aligned} H_0(x) &= 1 \\ \frac{d}{dx} H_n(x) &= n \frac{d}{dx} H_{n-1}(x) \\ \int_{\mathbb{R}} H_n(x) d\mu(x) &= 0, \quad \text{where } \mu \text{ is the standard Gaussian measure on } \mathbb{R}. \end{aligned}$$

Moreover, Hermite polynomials are given by the generating function

$$(136) \quad e^{tx - \frac{1}{2}t^2} = \sum_{n=0}^{\infty} t^n H_n(x),$$

i.e. $e^{tx - \frac{1}{2}t^2}$ is the generating function for $H_n(x)$. This is used as the best way to study properties of $H_n(x)$.

Exercise 12.1.1. Show that $H_n(x) = e^{-\frac{\Delta}{2}}(x^n)$, where $\Delta = -\frac{d^2}{dx^2}$ (this is a conceptual way to think about Wick ordering).

The following statements hold:

(1) $H_n(x)$ form an orthonormal basis of $L^2(\mathbb{R}, \mu)$.

(2)

$$\int_{\mathbb{R}} H_n(x)^2 d\mu(x) = n!$$

Note that here $L^2(\mathbb{R}, \mu) = \bigoplus_{n=0}^{\infty} \text{span}(H_n(x))$, i.e. $H_n(x) \perp \bigoplus_{k=0}^{n-1} \text{span}(H_k(x))$. More generally,

$$H_n(x^{t_1}, \dots, x^{t_r}) = H_{t_1}(x) \cdots H_{t_r}(x), \quad t_1 + \dots + t_r = n,$$

and one can check that Hermite polynomials form an orthogonal basis of $L^2(\mathbb{R}^n, \mu)$. On the other hand \mathbb{R} is a Hilbert space, in fact it is also a Cameron-Martin space (CMS) for standard Gaussian measure μ . We can then talk about the Bosonic Fock space of \mathbb{R} . We have

$$\begin{array}{ccc} \widetilde{\text{Sym}}^{\bullet}(\mathbb{R}) & \xlongequal{\quad} & L^2(\mathbb{R}, \mu) \\ \parallel & & \parallel \\ \bigoplus_{n \geq 0} \text{Sym}^n(\mathbb{R}) & \longrightarrow & \bigoplus_{n \geq 0} \text{span}(H_n(x)) \end{array}$$

where the arrow on the bottom represents a canonical isomorphism. More generally, we want to prove

$$L^2(W, \mu) \cong \widetilde{\text{Sym}}^{\bullet}(H(\mu)),$$

where the isomorphism is canonical. Let thus W be a separable Banach space and μ a centered Gaussian measure on W and $H(\mu)$ be its Cameron-Martin space with covariance q_{μ} . Let $f \in W^*$ (or $f \in H(\mu)$, since it doesn't matter). Define⁷ then $(: f^n :)$ recursively as follows:

$$\begin{aligned} (: f^0 :) &= 1 \\ \frac{\partial}{\partial f} (: f^n :) &= n (: f^{n-1} :) \\ \int_W (: f^n :) d\mu &= 0. \end{aligned}$$

This definition says that $(: f^n :)$ comes from f^n in the same way as $H_n(x)$ comes from x^n . Let us use generating functions to observe properties of $(: f^n :)$. Define

$$(: e^{\alpha f} :) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} (: f^k :)$$

⁷In the physics literature, this is usually written without the brackets, i.e. $: f^n :$. We write the brackets to avoid confusion with the "double dot" of a function or when two such objects are multiplied.

One can then check that

$$(137) \quad \int_W (: e^{\alpha f} :) d\mu = 1.$$

Moreover, by definition

$$\frac{\partial}{\partial f} (: e^{\alpha f} :) = \alpha (: e^{\alpha f} :),$$

which implies that $(: e^{\alpha f} :) = ce^{\alpha f}$, where c is a constant, which needs to be determined. Using (137), we see that $c = \frac{1}{\int_W e^{\alpha f} d\mu}$.

Exercise 12.1.2. Use Wick's theorem to show that

$$\begin{aligned} \int_W f^{2n+1} d\mu &= 0, \\ \int_W f^{2n} d\mu &= \frac{(2n)!}{2^n n!} q_\mu(f, f)^n. \end{aligned}$$

Hence, regarding $e^{\alpha f}$ as a formal power series in α , we see that

$$\int_W e^{\alpha f} d\mu = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \int_W f^k d\mu = \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{(2k)!} \frac{(2k)!}{k!} (q_\mu(f, f))^k = e^{\frac{1}{2}\alpha^2 q_\mu(f, f)},$$

because $(: e^{\alpha f} :) = e^{\alpha f} e^{-\frac{1}{2}\alpha^2 q_\mu(f, f)}$, which is exactly (136) when $W = \mathbb{R}$ and $f(x) = x$. One can use the generating function for $(: f^n :)$ to show that

$$\begin{aligned} (: f^n :) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!(n-2k)!} f^{n-2k} \left(-\frac{1}{2} q_\mu(f, f) \right)^k \\ f^n &= \sum_{k=0}^{\frac{n}{2}} \frac{n!}{k!(n-2k)!} (: f^{n-2k} :) \left(\frac{1}{2} q_\mu(f, f) \right)^k \end{aligned}$$

Proposition 12.1.1.

$$\int_W (: f^n :) (: g^m :) d\mu = \delta_{nm} n! \langle f, g \rangle^n.$$

Proof. We have

$$\begin{aligned} (: e^{\alpha f} :) &= e^{\alpha f - \frac{1}{2}\alpha^2 q_\mu(f, f)} \\ (: e^{\beta g} :) &= e^{\beta g - \frac{1}{2}\beta^2 q_\mu(g, g)}. \end{aligned}$$

Then

$$(: e^{\alpha f} :) (: e^{\beta g} :) = (: e^{\alpha f + \beta g} :) e^{\alpha\beta q_\mu(f, g)}.$$

Thus

$$\int_W (: e^{\alpha f} :) (: e^{\beta g} :) d\mu = e^{\alpha\beta q_\mu(f, g)} \underbrace{\int_W (: e^{\alpha f + \beta g} :) d\mu}_{=1} = \sum_{k=0}^{\infty} \frac{(\alpha\beta)^k}{k!} q_\mu(f, g)^k.$$

Comparing the coefficients of $(\alpha\beta)^k$ we get

$$\frac{1}{(k!)^2} \int_W (: f^k :)^k (: g^k :) d\mu = \frac{1}{(k!)!} q_\mu(f, g)^k.$$

□

The question is how to define $(: f^{k_1} g^{k_2} :)$? The idea is to use the recursive definition. Therefore we get for $n = n_1 + \dots + n_k$

$$\begin{aligned} (: f_1^0 \dots f_k^0 :) &= 1, \\ \int_W (: f_1^{n_1} \dots f_k^{n_k} :) d\mu &= 0, \\ \frac{\partial}{\partial f_i} (: f_1^{n_1} \dots f_k^{n_k} :) &= n_i (: f_1^{n_1} \dots f_i^{n_i-1} \dots f_k^{n_k} :). \end{aligned}$$

One can easily check that $(: (\alpha f + \beta g)^n :) = \sum_{k=1}^n \binom{n}{k} \alpha^k \beta^{n-k} (: f^k g^{n-k} :)$ by using generating functions.

Exercise 12.1.3. Show that

$$\int_W (: f_1 - f_n :) (: g_1 - g_m :) d\mu = \begin{cases} 0, & \text{if } m \neq n \\ \sum_{\sigma \in S_n} \prod_{i=1}^n \langle f_i, g_{\sigma(i)} \rangle, & \text{if } m = n \end{cases}$$

As in the finite dimensional case, given a polynomial function P on W , i.e.

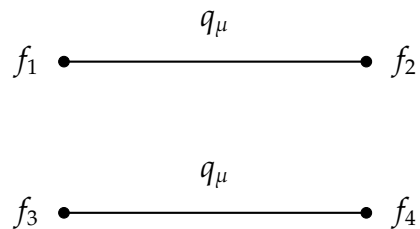
$$P(w) = \sum_{i_1, \dots, i_k} a_{i_1} \dots a_{i_k} f_{i_1}^{\alpha_1}(w) \dots f_{i_k}^{\alpha_k}(w),$$

where $f_{i_1}, \dots, f_{i_k} \in W^*$, formally

$$(: P :) = e^{-\frac{\Delta}{2}} P,$$

where we need to check what Δ is. One can use the Cameron-Martin space to make sense of Δ . Take an orthonormal basis $\{e_n\}$ of $\mathcal{H}(\mu)$. Then loosely speaking $\Delta = -\sum_{n=1}^{\infty} \frac{\partial^2}{\partial e_n^2}$.

12.2. Wick ordering as a value of Feynman diagrams. Given $f_1, f_2, f_3, f_4 \in W^*$, and $\gamma \in \{\{1, 2\}, \{3, 4\}\}$, we can construct a diagram as follows



The value of such a diagram will be $q_\mu(f_1, f_2)q_\mu(f_3, f_4)$.

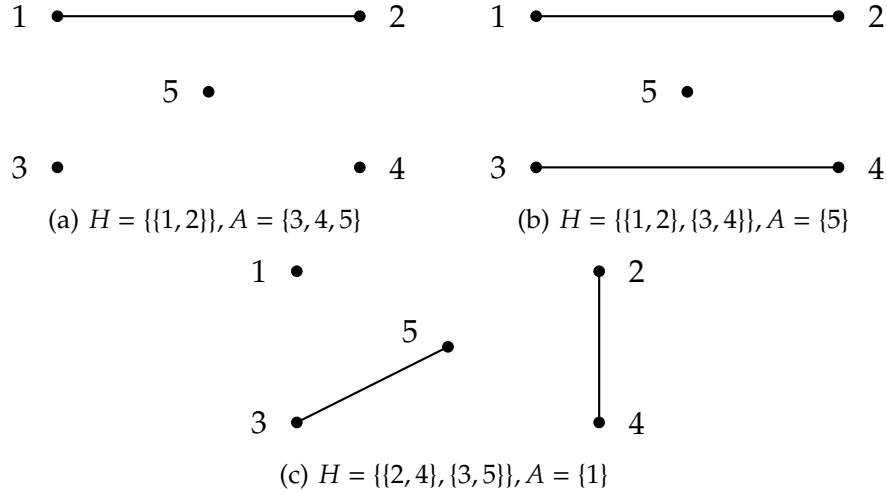


FIGURE 3. Examples of Feynman graphs with $V = \{1, 2, 3, 4, 5\}$ and different H and A .

Definition 12.2.1 (Feynman diagram). A Feynman diagram with n vertices and rank r , where $r \leq \frac{n}{2}$, consists of a set V called the set of vertices (thus $|V| = n$), and a set H called the set of half edges, which consists of r disjoint pair of vertices. The remaining vertices are called unpaired vertices, which will be denoted by A .

Example 12.2.1.

If we are given $f_1, \dots, f_n \in W^*$, we can think of them as vertices of a Feynman diagram. Hence, given $f_1, \dots, f_n \in W^*$ and a Feynman diagram of rank r , i.e.

$$\gamma(H) = \{\{i_1, j_1\}, \dots, \{i_r, j_r\}\},$$

we define the value $F(f_1, \dots, f_n; \gamma)$ of the Feynman diagram as

$$F(f_1, \dots, f_n; \gamma) = \left(\prod_{k=1}^r q_\mu(f_{i_k}, f_{j_k}) \right) \prod_{i \in A} f_i.$$

Moreover, we say that γ is complete if $n = 2r$. With this notation, we can rephrase Wick's theorem as

$$\int f_1 \cdots f_n d\mu = \sum_{\substack{\gamma \text{ complete} \\ \text{Feynman diagram}}} F(\gamma).$$

12.3. Abstract point of view on Wick ordering.

Definition 12.3.1 (Gaussian Hilbert space). A Gaussian Hilbert space \mathcal{H} is a Hilbert space of random variables on some probability space $(\Omega, \sigma(\Omega), \mu)$ such that each $f \in \mathcal{H}$ is a Gaussian on \mathbb{R} , i.e. $f \in \mathcal{H}$, then $f_*\mu$ is Gaussian on \mathbb{R} .

Example 12.3.1. If K denotes the completion of W^* in $L^2(W, \mu)$, then K is a Gaussian Hilbert space. We assume that $\sigma(\Omega)$ is generated by elements of \mathcal{H} .

Given a Gaussian Hilbert space $\mathcal{H} \subseteq L^2(\Omega, \sigma(\Omega), \mu)$, define the set

$$P_n(\mathcal{H}) = \{P(\xi_1, \dots, \xi_n) \mid \xi_1, \dots, \xi_n \in \mathcal{H}, P \text{ is a polynomial of degree } \leq n\},$$

and define

$$\mathcal{H}^{:n:} = \overline{P_n(\mathcal{H})} \cap (\overline{P_{n-1}(\mathcal{H})})^\perp.$$

We can then observe⁸ that

(1)

$$\overline{P_n(\mathcal{H})} = \bigoplus_{k=0}^n \mathcal{H}^{:k:},$$

(2)

$$\bigoplus_{n=0}^{\infty} \mathcal{H}^{:n:} = \overline{\bigcup_{n=0}^{\infty} P_n(\mathcal{H})}.$$

Theorem 12.3.1.

$$L^2(\Omega, \sigma(\Omega), \mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{:n:}.$$

Proof. See □

Remark 12.3.1. This is just a way to say that “polynomial” random variables are dense in $L^2(\Omega, \sigma(\Omega), \mu)$.

Theorem 12.3.2. Let $\xi_1, \dots, \xi_n \in \mathcal{H}$. Then

$$(: \xi_1 \cdots \xi_n :) = \pi_n(\xi_1 \cdots \xi_n),$$

where $\pi_n : L^2(\Omega, \sigma(\Omega), \mu) \rightarrow \mathcal{H}^{:n:}$ is the projection.

Hence, we saw that $(: \xi_1, \dots, \xi_n :)$ is nothing but an orthogonal projection of $\xi_1 \cdots \xi_n$ onto $\mathcal{H}^{:n:}$. The idea here is that $\xi_1 \cdots \xi_n$ is not orthogonal to lower degree polynomials where as $(: \xi_1 \cdots \xi_n :)$ is orthogonal to lower degree polynomials. Hence, Wick ordering can be thought of as taking a polynomial and changing it into a new polynomial in such a way that the result is orthogonal to lower degree polynomials.

13. BOSONIC FOCK SPACES

Let $\mathcal{H}_1, \mathcal{H}_2$ be a separable Hilbert spaces. Then we can look at the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, where we have the inner product defined as

$$\langle h_1 \otimes h_2, h'_1 \otimes h'_2 \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} := \langle h_1, h'_1 \rangle_{\mathcal{H}_1} \langle h_2, h'_2 \rangle_{\mathcal{H}_2}.$$

Moreover, we denote by $\widehat{\mathcal{H}_1 \otimes \mathcal{H}_2}$ the completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$ with respect to $\langle \cdot, \cdot \rangle$. We call $\widehat{\otimes}$ the Hilbert-Schmidt tensor product. The space $\mathcal{H}_1^* \widehat{\otimes} \mathcal{H}_2$ is isomorphic to the space of

⁸We will define the completed direct sum at some later point.

Hilbert-Schmidt operators from \mathcal{H}_1 to \mathcal{H}_2 . Let $\{\mathcal{H}_i\}_{i=0}^\infty$ be a family of Hilbert spaces. Then $\widehat{\bigotimes}_i \mathcal{H}_i$ is the completion of $\bigotimes_i \mathcal{H}_i$ with respect to the norm $\sum_i \|x_i\|_{\mathcal{H}_i}^2$, i.e.

$$\widehat{\bigotimes}_i \mathcal{H}_i = \left\{ (x_i) \mid \sum_{i=0}^\infty \|x_i\|_{\mathcal{H}_i}^2 < \infty \right\}.$$

We will drop $\widehat{}$ from now on. We can also define the space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ and so on. Let now \mathcal{H} be a real and separable Hilbert space. Then $T^n \mathcal{H} = \mathcal{H}^{\otimes n}$. Define the map $P : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$ by $P(h_1 \otimes \cdots \otimes h_n) = \frac{1}{n!} \sum_{\sigma \in S_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}$. Then P is a projection. We define $Sym^n(\mathcal{H}) = P(\mathcal{H}^{\otimes n})$. Now we can see that S_n acts on $T^n \mathcal{H}$ and thus $Sym^n(\mathcal{H})$ is the invariant subspace of $T^n \mathcal{H}$ under this action. Given h_1, \dots, h_n , define

$$h_1 \otimes_s \cdots \otimes_s h_n = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)},$$

i.e. $h_1 \otimes_s \cdots \otimes_s h_n = \sqrt{n!} P(h_1 \otimes \cdots \otimes h_n)$.

Exercise 13.0.1. Show that

$$\langle h_1 \otimes_s \cdots \otimes_s h_n, h'_1 \otimes_s \cdots \otimes_s h'_n \rangle_{\mathcal{H}^{\otimes n}} = \sum_{\sigma \in S_n} \prod_{i=1}^n \langle h_i, h_{\sigma(i)} \rangle_{\mathcal{H}}.$$

In particular $\|h^{\otimes_s n}\|_{\mathcal{H}^{\otimes n}}^2 = n! \|h\|_{\mathcal{H}}^{2n}$ (i.e. $\|h^{\otimes_s n}\|_{\mathcal{H}^{\otimes n}} = \sqrt{n!} \|h\|_{\mathcal{H}}$).

Let us try to give an alternative definition for $Sym^n(\mathcal{H})$. For this, recall that closed subspaces are generated by elements of the form $h_1 \otimes_s \cdots \otimes_s h_n$.

Remark 13.0.1. From now on we will not indicate the inner products.

Definition 13.0.1 (Bosonic Fock space). The space $\widetilde{Sym}^\bullet(\mathcal{H}) = \bigoplus_{n=0}^\infty Sym^n(\mathcal{H})$ is called the Bosonic Fock space of \mathcal{H} .

Remark 13.0.2. We sometimes also write $\Gamma(\mathcal{H})$ or $\text{Exp}(\mathcal{H})$ for the Bosonic Fock space.

Remark 13.0.3. Similarly, one can define the Fermionic Fock space of \mathcal{H} by

$$u_1 \wedge \cdots \wedge u_n = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \text{sgn}(\sigma) u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}.$$

One can check that $\Gamma(\mathcal{H}_1 \otimes \mathcal{H}_2) = \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$. Now one can ask whether there is a functor $\mathcal{H} \mapsto \Gamma(\mathcal{H})$. Thus, given a bounded operator $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, we need to know whether $\Gamma(A)$ is bounded. As a matter of fact, this is not the case. On the other hand, if $\|A\| \leq 1$ then $\|\Gamma(A)\| \leq 1$. This leads to the functor

$$\Gamma : \mathbf{Hilb}_B^{\leq 1} \longrightarrow \mathbf{Hilb},$$

where $\mathbf{Hilb}_B^{\leq 1}$ is the category with Hilbert spaces as objects and bounded linear operators of norm ≤ 1 as morphisms and \mathbf{Hilb} the category of Hilbert spaces.

Remark 13.0.4. In the Fermionic case, no restriction on A is required.

Given $h \in \mathcal{H}$, we can define

$$\text{Exp}(h) = \sum_{n=0}^{\infty} \frac{h^{\otimes n}}{n!} \in \Gamma(\mathcal{H}).$$

Then, for $h_1, h_2 \in \mathcal{H}$, we observe

$$\langle \text{Exp}(h_1), \text{Exp}(h_2) \rangle = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \langle h_1^{\otimes n}, h_2^{\otimes n} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle h_1, h_2 \rangle^n = \text{Exp}(\langle h_1, h_2 \rangle).$$

Exercise 13.0.2. Show that $\text{Exp} : \mathcal{H} \rightarrow \Gamma(\mathcal{H})$ is continuous (*be aware that it is not linear*). Moreover, show that Exp is injective.

Lemma 13.0.1. The elements $\{\text{Exp}(h) \mid h \in \mathcal{H}\} \subseteq \Gamma(\mathcal{H})$ are linearly independent in $\Gamma(\mathcal{H})$.

Proof. Let $h_1, \dots, h_n \in \mathcal{H}$. We want to show

$$\sum_{i=1}^n \lambda_i \text{Exp}(h_i) = 0 \Rightarrow \lambda_i = 0, \quad \forall i \geq 1, 2, \dots, n.$$

For this, choose $h \in \mathcal{H}$ such that

$$\langle h, h_i \rangle \neq \langle h, h_j \rangle, \quad \forall i \neq j.$$

Then we get that $\sum_{i=1}^n \lambda_i \text{Exp}(h_i) = 0$ implies $\sum_{i=1}^n \lambda_i e^{\langle h_i, h \rangle} = 0$ for all $h \in \mathcal{H}$. Thus $\sum_{i=1}^n \lambda_i e^{z \langle h_i, h \rangle} = 0$ for all $z \in \mathbb{C}$ if we choose h as above. Hence, $\lambda_i = 0$ for all $i = 1, 2, \dots, n$. \square

Exercise 13.0.3. Show that $\{\text{Exp}(h) \mid h \in \mathcal{H}\}$ span $\Gamma(\mathcal{H})$.

Recall that

$$(1) \quad L^2(W, \mu) = \bigoplus_{n=0}^{\infty} K^n.$$

(2) there is a canonical isomorphism of Hilbert spaces $\mathcal{H}(\mu) \xrightarrow{T} K$.

Thus we can observe that there is a canonical isomorphism of Hilbert spaces

$$\begin{aligned} \text{Sym}^n(K) &\longrightarrow K^n \\ \xi_1 \otimes_s \cdots \otimes_s \xi_n &\longmapsto (: \xi_1 \cdots \xi_n :), \end{aligned}$$

which leads to a map

$$\begin{aligned} \Gamma(K) &\longrightarrow \bigoplus_{n=0}^{\infty} K^n \\ \text{Exp}(\xi) &\longmapsto \sum_{n=0}^{\infty} \frac{(: \xi^n :)}{n!} = (: \text{Exp}(\xi) :) = e^{\xi} e^{-\frac{1}{2}\|\xi\|^2}, \end{aligned}$$

that comes from the Segal-Ito isomorphism

$$\begin{aligned} \Gamma(\mathcal{H}) &\longrightarrow \Gamma(K) \cong L^2(W, \mu) \\ \text{Exp}(h) &\longmapsto e^{\langle h, \cdot \rangle} e^{-\frac{1}{2}\|h\|^2}. \end{aligned}$$

Part 4. Construction of Quantum Field Theories

14. FREE SCALAR FIELD THEORY

Recall that a classical scalar field theory on \mathbb{R}^n consists of the following data. A space of fields $\mathcal{F} = C^\infty(\mathbb{R}^n)$ and an action functional, which is a map $S : \mathcal{F} \rightarrow \mathbb{R}$ that is local, i.e. it only depends on fields and derivatives of fields. In free theory we are interested in the action functional S which is of the form

$$S(\phi) = \int_{\mathbb{R}^n} \phi(\Delta + m^2)\phi dx,$$

where dx denotes the Lebesgue measure on \mathbb{R}^n . In quantum theory, we are interested in defining a measure of the form

$$(138) \quad e^{-\frac{1}{2}S(\phi)} \mathcal{D}\phi$$

on \mathcal{F} . We will see that it is possible to define a measure of the form (138) but it lives on a much larger space than \mathcal{F} . Next, we will discuss Gaussian measures on locally convex spaces and as a consequence we will define a measure of the form (138).

14.1. Locally convex spaces.

Definition 14.1.1 (separating points). Let V be a vector space. A family $\{\rho_\alpha\}_{\alpha \in A}$ of seminorms on V is said to separate points if $\rho_\alpha(x) = 0$ for all $\alpha \in A$ implies $x = 0$.

Definition 14.1.2 (Natural topology). Given a family of seminorms $\{\rho_\alpha\}_{\alpha \in A}$ on V there exists a smallest topology for which each ρ_α is continuous and the addition operation is continuous. This topology, which is denoted by $\mathcal{O}(\{\rho_\alpha\})$, is called the natural topology on V .

Definition 14.1.3 (Locally convex space). A locally convex space is a vector space V together with a family $\{\rho_\alpha\}$ of seminorms that separate points.

Exercise 14.1.1. Show that the natural topology on a locally convex space is Hausdorff.

Let $\varepsilon > 0$ and $\alpha_1, \dots, \alpha_n \in A$. Define the set

$$N(\alpha_1, \dots, \alpha_n; \varepsilon) = \{v \in V \mid \rho_{\alpha_i}(v) < \varepsilon, \forall i = 1, 2, \dots, n\}.$$

One can check that

$$(1) \quad N(\alpha_1, \dots, \alpha_n; \varepsilon) = \bigcap_{i=1}^n N(\alpha_i; \varepsilon).$$

$$(2) \quad N(\alpha_1, \dots, \alpha_n; \varepsilon) \text{ is convex.}$$

Exercise 14.1.2. Check that the elements of

$$\{N(\alpha_1, \dots, \alpha_n; \varepsilon) \mid \alpha_1, \dots, \alpha_n \in A, n \in \mathbb{N}, \varepsilon > 0\}$$

form a neighbourhood basis at $0 \in V$.

From (2) it follows that a locally convex space V has a neighbourhood basis at $0 \in V$, where each open set in this basis is convex. This justifies the name **locally convex space**. One can define the notion of a Cauchy sequence and the notion of convergence in a locally convex space. Let V be a locally convex space. The following are equivalent:

- (1) V is metrizable.
- (2) $0 \in V$ has a countable neighbourhood basis.
- (3) The natural topology on V is generated by some countable family of seminorms.

Definition 14.1.4 (Fréchet space). A complete metrizable locally convex space is called a **Fréchet space**.

Example 14.1.1 (Schwartz space). Let $\phi \in C^\infty(\mathbb{R}^n)$ and let $\alpha = (\alpha_1, \dots, \alpha_k) \in (\mathbb{N} \cup \{0\})^k$ and $\beta = (\beta_1, \dots, \beta_\ell) \in (\mathbb{N} \cup \{0\})^\ell$. Let $D^\beta = \frac{\partial^{\beta_1 + \dots + \beta_\ell}}{\partial x_1^{\beta_1} \dots \partial x_\ell^{\beta_\ell}}$. Moreover, define

$$\|\phi\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)|,$$

and

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) \mid \|\phi\|_{\alpha, \beta} < \infty, \forall \alpha, \beta\}.$$

The space $\mathcal{S}(\mathbb{R}^n)$ is called the **Schwartz space** on \mathbb{R}^n . One can easily check that $\mathcal{S}(\mathbb{R}^n)$ is a locally convex space. In general $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space.

14.2. Dual of a locally convex space. Let V be a locally convex space and V^* be the set of continuous linear functionals on V , i.e. $\ell \in v^*$ iff $\ell : V \rightarrow \mathbb{R}$ is linear and continuous. Given $x \in V$, define $\rho_x : V^* \rightarrow \mathbb{R}$ by $\rho_x(\ell) = |\ell(x)|$. One can easily check that ρ_x is a seminorm. In fact $\{\rho_x \mid x \in V\}$ is a family of seminorms on V^* that separates points. Hence, $(V^*, \{\rho_x \mid x \in V\})$ is a locally convex space. The natural topology on V^* induces by $\{\rho_x \mid x \in V\}$ is called the **weak*-topology** on V^* . A sequence $\{\ell_n\}$ in V^* converges to $\ell \in V^*$ in the weak*-topology iff $\rho_x(\ell_n) \rightarrow \rho_x(\ell)$ for all $x \in V$, i.e. $\ell_n(x) \rightarrow \ell(x)$ for all $x \in V$. The weak*-topology on V^* is denoted by $\mathcal{O}(V^*, V)$.

Remark 14.2.1. The space of linear functionals on $(V^*, \mathcal{O}(V^*, V))$ is exactly V .

14.3. Gaussian measures on the dual of Fréchet spaces.

Theorem 14.3.1. *Let V be a Fréchet space. Then there is a bijection between the following sets*

$$\left\{ \text{Continuous positive definite symmetric bilinear forms on } V \right\} \\ \longleftrightarrow \left\{ \text{Centered Gaussian measures on } (V^*, \mathcal{O}(V^*, V)) \right\}$$

Proof. See [2, 6]. □

Let C be a continuous positive definite symmetric bilinear form on V . The construction of the associated Gaussian measure on V^* goes as follows. Let $F \subseteq V$ be a finite dimensional subspace of V . Let C_F be the restriction of C on F . Then C_F is a symmetric bilinear form

on F , which is positive definite. Hence C_F defines a Gaussian measure μ_{C_F} on F^* (F^* can be identified with F) of the form

$$d\mu_{C_F}(x) = \left(\det \left(\frac{C_F}{2\pi} \right) \right)^{-\frac{1}{2}} e^{-\frac{1}{2}C_F(x,x)},$$

where C_F is identified with a positive definite matrix. In fact, one can think of μ_{C_F} to be a measure on the F cylinder subsets of V^* . One can check that if $E \subseteq F$, then μ_{C_F} agrees with μ_{C_E} when restricted to the E cylinder subsets of V^* . Now we can proceed as in the construction of the Wiener measure and show that there is a Gaussian measure μ_C on the σ -algebra of V^* generated by cylinder sets. This gives the construction of μ_C .

Corollary 14.3.1. Let C be the bilinear form on $\mathcal{S}(\mathbb{R}^n)$ defined by

$$C(f, g) = \int_{\mathbb{R}^n} f(\Delta + m^2)^{-1} g dx.$$

Then there is a Gaussian measure μ on $\mathcal{S}(\mathbb{R}^n)$ whose covariance is C .

In this example the reproducing kernel space $K(\mu)$ of μ is $H^{-1}(\mathbb{R}^n)$, where $H^{-1}(\mathbb{R}^n)$ is the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to C . Hence, we have succeeded in defining the measure of the form $e^{-S(\phi)} \mathcal{D}\phi$, where

$$S(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} \phi(\Delta + m^2)\phi dx.$$

In other words, we have constructed the Gaussian measure associated to the free theory.

Remark 14.3.1. In this example, the Cameron-Martin space of μ is $H^1(\mathbb{R}^n)$, which is the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the map

$$(f, g) \mapsto \int_{\mathbb{R}^n} f(\Delta + m^2)g dx.$$

14.4. The operator $(\Delta + m^2)^{-1}$. We regard $(\Delta + m^2)^{-1}$ as an operator on $L^2(\mathbb{R}^n)$. It is known that $(\Delta + m^2)^{-1}$ is a positive operator, and that it is an integral operator. Let $C(x, y)$ be the integral kernel of $(\Delta + m^2)^{-1}$. Then

$$C(f, g) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x)C(x, y)g(y)dx dy.$$

In fact, one can show that $C(x, y)$ is the unique solution of

$$(139) \quad \Delta_y C(x, y) = \delta_x(y).$$

Using Fourier transform, we can give an explicit representation of $C(x, y)$. Formally we have the following chain of implications.

$$(\Delta + m^2)^{-1} f = g \Rightarrow (\Delta + m^2)g = f \Rightarrow (\xi^2 + m^2)g^{-1} = f^{-1} \Rightarrow g = \mathcal{F}^{-1} \left(\frac{1}{\xi^2 + m^2} \widehat{f} \right).$$

These hold since

$$g(x) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{e^{i\xi(x-y)}}{\xi^2 + m^2} f(y) dy d\xi,$$

and thus

$$C(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\xi(x-y)}}{\xi^2 + m^2} d\xi.$$

For $x \neq y$ one can show that

$$C(x, y) = (2\pi)^{-\frac{1}{2}} \left(\frac{m}{\|x - y\|} \right)^{\frac{n-2}{2}} K_{\frac{n-2}{2}}(m\|x - y\|),$$

where K_ν is a modified Bessel function. Next we will study $C(x, y)$ in more details. In particular the behaviour of $C(x, y)$ when $\|x - y\|$ is large and $\|x - y\|$ is small.

Remark 14.4.1. For $n = 1$ we have $C(x, y) = \frac{e^{-m|x-y|}}{m}$ and for $n = 3$ we have $C(x, y) = \frac{e^{-m\|x-y\|}}{4\pi\|x-y\|}$.

Proposition 14.4.1 (Properties of $C(x, y)$). The following hold:

- (1) For every $m|x - y|$ bounded away from zero, there exists some $M \geq 0$ such that we have

$$C(x, y) \leq Mm^{\frac{n-3}{2}} |x - y|^{\frac{n-1}{2}} e^{-m|x-y|}.$$

- (2) For $n \geq 3$ and $m|x - y|$ in a neighbourhood of zero we get

$$C(x, y) \sim |x - y|^{-n+2}$$

- (3) For $n = 2$ and $m|x - y|$ in a neighborhood of zero we get

$$C(x, y) \sim -\log(m|x - y|).$$

Proof. Recall first

$$(140) \quad C(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi(x-y)}}{\xi^2 + m^2} d\xi.$$

Exercise 14.4.1. Show that in general we have

$$C(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\xi\|x-y\|}}{\xi^2 + m^2} d\xi.$$

Hint: Choose an orthonormal basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n such that $e_1 = \frac{x-y}{\|x-y\|}$ and do a change of variables.

Now, using the residue theorem, we have

$$\begin{aligned} C(x, y) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{it\xi_1}}{\xi_1^2 + \left(\sqrt{m^2 + \xi_2^2 + \dots + \xi_n^2} \right)^2} d\xi_1 d\xi_2 \dots d\xi_n \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \frac{\pi e^{-t\sqrt{m^2 + \xi_2^2 + \dots + \xi_n^2}}}{\sqrt{m^2 + \xi_2^2 + \dots + \xi_n^2}} d\xi_2 \dots d\xi_n. \end{aligned}$$

Without loss of generality, assume that $m = 1$. Recall that for $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $f(x) = g(|x|)$ we have

$$(141) \quad \int_{\mathbb{R}^n} f(x) dx = v(S^{n-1}) \int_0^\infty r^{n-1} g(r) dr,$$

where $v(S^{n-1})$ is the volume of S^{n-1} . Using (142), we can write

$$(142) \quad C(x, y) = \frac{\pi A_{n-1}}{(2\pi)^{\frac{n}{2}}} \int_0^\infty \frac{r^{n-2} e^{-t\mu(r)}}{\mu(r)} dr,$$

where $\mu(r) = \sqrt{1+r^2}$.

Exercise 14.4.2. Show that there is some $\varepsilon > 0$ such that

$$\mu(r) \geq \begin{cases} 1 + \varepsilon r^2, & \text{if } r \leq 1 \\ 1 + \varepsilon r, & \text{if } r \geq 1 \end{cases}$$

We will claim that

$$(143) \quad C(x, y) \leq k e^{-t} (t^{-\frac{n-1}{2}} + t^{-(n-1)}),$$

where $t = \|x - y\|$ and k some constant. Note then that

$$\int_0^1 \frac{r^{n-2} e^{-t\mu(r)}}{\mu(r)} \leq \int_0^1 r^{n-2} e^{-t(1+\varepsilon r^2)} dr \leq k e^{-t} t^{-\frac{n-1}{2}}$$

and

$$\int_1^\infty \frac{r^{n-2} e^{-t\mu(r)}}{\mu(r)} \leq \int_1^\infty r^{n-2} e^{-t(1+\varepsilon r)} dr \leq k e^{-t} t^{-(n-1)}.$$

If $t \geq 1$ then

$$C(x, y) \leq k e^{-t} t^{-\frac{n-1}{2}}.$$

For $0 < t \leq 1$ we have

$$\begin{aligned} \int_0^\infty \frac{r^{n-2} e^{-t\sqrt{1+r^2}}}{1+r^2} dr &= t^{-(n-2)} \int_0^\infty \frac{s^{n-2} e^{-\sqrt{s^2+t^2}}}{\sqrt{s^2+t^2}} ds \\ &\sim_{\{\text{as } t \rightarrow 0 \text{ in the integral}\}} t^{-(n-2)} \int_0^\infty \frac{s^{n-2} e^{-s}}{s} ds \\ &= t^{-(n-2)} \int_0^\infty s^{n-3} e^{-s} ds \end{aligned}$$

If $n = 2$, let $s = t\mu(r)$. Then $\mu(r) = \frac{s}{t}$, $1+r^2 = \frac{s^2}{t^2}$ and $r = \sqrt{\frac{s^2-t^2}{t^2}}$. Thus

$$C(x, y) = \int_t^\infty \frac{e^{-s}}{\sqrt{s^2+t^2}} ds \sim \int_t^\infty \frac{1}{\sqrt{s^2+t^2}} ds \sim -\log(t).$$

□

15. CONSTRUCTION OF SELF INTERACTING THEORY

To construct a theory with polynomial interaction, we want to define a measure of the form

$$e^{-S(\phi)} \mathcal{D}\phi$$

rigorously where

$$S(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} \phi(\Delta + m^2)\phi dx + \int_{\mathbb{R}^n} P(\phi)dx = S_f(\phi) + S_I(\phi),$$

with $P(y) = \sum_i a_i y^i$ some polynomial function on \mathbb{R} . We have succeeded in defining a measure μ of the form $e^{-S_f(\phi)} \mathcal{D}\phi$, but the price we had to pay was that it lives on $\mathcal{S}(\mathbb{R}^n)$. In fact $\mu(\mathcal{S}(\mathbb{R}^n)) = 0$ because for such measures the Cameron-Martin space is $H^1(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n) \subseteq H^1(\mathbb{R}^n)$. Hence, it is not obvious that we have to view $\phi \mapsto \int_{\mathbb{R}^n} \phi^n(x)dx$ as a measurable function on $\mathcal{S}(\mathbb{R}^n)$. Let us now define a measure of the form

$$e^{-S_f(\phi)} e^{-S_I(\phi)} \mathcal{D}\phi.$$

First, we will try to “define” measurable functions of the form

$$(144) \quad \phi \mapsto \int_{\mathbb{R}^n} \phi(x)^k dx.$$

In particular, we will try to bypass the difficulties in making sense of (144). Let us pretend that we can define (144). Formally we have

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} \phi(x)^k dx \right\|_{L^2(\mathcal{S}(\mathbb{R}^n), \mu)}^2 &= \int_{\phi \in \mathcal{S}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \phi(x)^k dx \right) \left(\int_{\mathbb{R}^n} \phi(y)^k dy \right) d\mu(\phi) \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left(\int_{\mathcal{S}(\mathbb{R}^n)} \phi(x)^k \phi(y)^k d\mu(\phi) \right) dx dy. \end{aligned}$$

Now, formally thinking of $\phi(x)$ as $\delta_x[\phi] = \langle \delta_x | \phi \rangle$, which is defined in terms of the Heaviside step function, and using Wick’s theorem, we see that

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} \phi(x)^k dx \right\|_{L^2(\mathcal{S}(\mathbb{R}^n), \mu)} &= \text{Linear combination of integrals of the form} \\ &\iint_{\mathbb{R}^n \times \mathbb{R}^n} C(x, x)^\alpha C(y, y)^\beta C(x, y)^\gamma dx dy. \end{aligned}$$

The existence of $\int_{\mathbb{R}^n} \phi(x)^k dx$ depends on the properties of $C(x, y)$ and hence it is dimension sensitive. In fact we can not define (144) because $C(x, x)$ is not integrable. We want to try two different attempts to make sense of $\int_{\mathbb{R}^n} \phi(x)^k dx$:

- (1) (Approximation of delta function) Let $h \in \mathcal{S}(\mathbb{R}^n)$ such that $h \geq 0, h(0) > 0$ and $\int_{\mathbb{R}^n} h = 1$. Consider e.g. $h_\epsilon(y) = \frac{1}{\epsilon^2} h\left(\frac{y}{\epsilon}\right)$ for some $\epsilon > 0$. Then $h_\epsilon \rightarrow \delta_0$ as $\epsilon \rightarrow 0$.

Similarly⁹ we can construct $\delta_{x,\varepsilon} \in \mathcal{S}'(\mathbb{R}^n)$ (space of Schwartz distributions on \mathbb{R}^n) such that $\delta_{x,\varepsilon} \rightarrow \delta_x$. Then $\phi \mapsto \delta_{x,\varepsilon}[\phi] = \langle \delta_{x,\varepsilon} | \phi \rangle$ is a polynomial function on $\mathcal{S}(\mathbb{R}^n)$. We denote the polynomial type by $\phi(\delta_{x,\varepsilon})$. We know how to compute

$$\int_{\phi \in \mathcal{S}(\mathbb{R}^n)} \phi(\delta_{x,\varepsilon})^k \phi(\delta_{y,\varepsilon})^m d\mu(\phi),$$

which is equal to the sum of terms of the form $A_{\alpha\beta\gamma} C(\delta_{x,\varepsilon}, \delta_{x,\varepsilon})^\alpha C(\delta_{y,\varepsilon}, \delta_{y,\varepsilon})^\beta C(\delta_{x,\varepsilon}, \delta_{y,\varepsilon})^\gamma$. Formally we have that

$$\left\langle \int_{\mathbb{R}^n} \phi(\delta_{x,\varepsilon})^k dx, \int_{\mathbb{R}^n} \phi(\delta_{y,\varepsilon})^m \right\rangle_{L^2(\mathcal{S}(\mathbb{R}^n), \mu)}$$

is equal to sum of expressions of the form

$$A_{\alpha\beta\gamma} \iint_{\mathbb{R}^n \times \mathbb{R}^n} C(\delta_{x,\varepsilon}, \delta_{x,\varepsilon})^\alpha C(\delta_{y,\varepsilon}, \delta_{y,\varepsilon})^\beta C(\delta_{x,\varepsilon}, \delta_{y,\varepsilon})^\gamma dx dy,$$

and we can try to take the limit $\varepsilon \rightarrow 0$. The conclusion here is that this attempt does not lead to anywhere, since we still get a diagonal contribution.

- (2) (Redefine observables) Let us try to get rid of diagonal contribution (i.e. terms like $C(x, x)$). This is where the Wick ordering comes into the play. We can think of Wick ordering as a renormalization process. Consider the map

$$\phi \mapsto \int_{\substack{\Lambda \subset \mathbb{R}^n \\ \Lambda \text{ compact}}} (:\phi(\delta_{x,\varepsilon})^k:) dx.$$

Thus we get

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \left(\int_{\mathcal{S}(\mathbb{R}^n)} (:\phi(\delta_{x,\varepsilon})^k:) (:\phi(\delta_{y,\varepsilon})^k:) d\mu(\phi) \right) dx dy = k! \iint_{\mathbb{R}^n \times \mathbb{R}^n} C(\delta_{x,\varepsilon}, \delta_{y,\varepsilon})^k dx dy.$$

Taking the limit $\varepsilon \rightarrow 0$ formally, it converges to

$$(145) \quad k! \iint_{\mathbb{R}^n \times \mathbb{R}^n} C(x, y)^k dx dy.$$

Let us list what we know so far:

(i) Wick ordering allows us to get rid of diagonal contribution of $C(x, y)$.

(ii) If $\iint_{\mathbb{R}^n \times \mathbb{R}^n} C(x, y)^k dx dy < \infty$, then there is a hope that we can define “Wick ordered polynomial” functions of the form

$$\phi \mapsto \int_{\mathbb{R}^n} (:\phi(x)^k:) dx.$$

⁹With $x = \frac{1}{\varepsilon}$ we get $h_x(x) = x^2 h(\varepsilon x) \rightarrow \delta_0$ as $x \rightarrow \infty$.

Recall that if $n \geq 3$, then $C(x, y) \sim \frac{1}{\|x-y\|^{n-2}}$ for $\|x-y\| \rightarrow 0$ and hence the integral of the form (145) will diverge in general. This means that if $n \geq 3$, Wick ordering renormalization may not kill all the infinities appearing in Feynmann amplitudes. However, if $n = 2$, we get $C(x, y) \sim -\log(\|x-y\|)$ as $\|x-y\| \rightarrow 0$ and in this case it is possible to define observables of the form $\phi \mapsto \int_{\substack{\Lambda \subset \mathbb{R}^2 \\ \Lambda \text{ compact}}} (: P(\phi) :)$. Define $\widetilde{S}_{I,\Lambda}(\phi) = \int (: P(\phi)(x) :) dx$ as a measurable function on $S'(\mathbb{R}^2)$, where P is any polynomial. Let $P(x) = x^4$ and $\Lambda \subset \mathbb{R}^2$. Then we can consider $\widetilde{S}_{I,\Lambda}^\varepsilon(\phi) = \int_\Lambda (: P(\phi, \delta_{\varepsilon,x}) :) dx$, where $\delta_{\varepsilon,x}$ is a smooth approximation of δ_x . More precisely, $\delta_{\varepsilon,x}$ can be constructed as follows. Let $h \in C_0^\infty(\mathbb{R}^2)$ with $h \geq 0$, $h(0) > 0$, and $\int_{\mathbb{R}^2} h = 1$. Then consider $\delta_{\varepsilon,x}(y) = \varepsilon^{-2} h\left(\frac{x-y}{\varepsilon}\right)$. In fact, $\delta_{\varepsilon,x} \rightarrow \delta_x$ in $S'(\mathbb{R}^2)$. If we take $\varepsilon = \frac{1}{k}$, we will get $\delta_{\varepsilon,x} = \delta_{k,x}$ and thus one can observe that the sequence $\{\widetilde{S}_{I,\Lambda}^k\}$ is Cauchy in $L^2(S'(\mathbb{R}^2, \mu))$. Define $\widetilde{S}_{I,\Lambda}(\phi) := \lim_{k \rightarrow \infty} \widetilde{S}_{I,\Lambda}^k(\phi)$.

Remark 15.0.1. Recall

$$\left\langle \int_\Lambda (: \phi(\delta_{k,x}) :)^n, \int_\Lambda (: \phi(\delta_{k,y}) :)^n \right\rangle_{L^2(S'(\mathbb{R}^2, \mu))} = n! \int_{\Lambda \times \Lambda} C(\delta_{k,x}, \delta_{k,y})^n dx dy.$$

To see that $\{\widetilde{S}_{I,\Lambda}^k\}$ is Cauchy we only have to understand how C behaves. In the Fourier picture it is easier understood. We have the Fourier transform of $\delta_{k,x}$ is $\left(\frac{1}{\varepsilon^2+m^2}\right) \widehat{h}^2\left(\frac{\xi}{k}\right)$, which can be understood very easily. Then one can use the properties of $C_k(x, y) := C(\delta_{k,x}, \delta_{k,y})$, which is a smooth approximation of Green function, to show that $\int_\Lambda (: \phi(\delta_{k,x}) :)^n dx$ is Cauchy.

15.1. More random variables. Let $f \in C_0^\infty(\underbrace{\mathbb{R}^2 \times \dots \times \mathbb{R}^2}_k)$. Then

$$\widetilde{S}_{I,\Lambda}(f, k) = \int_{\mathbb{R}^2 \times \dots \times \mathbb{R}^2} (: \phi(x_1) \cdots \phi(x_k) :) f(x_1, \dots, x_k) dx_1 \cdots dx_k$$

can be defined as before. More generally we can take $f \in L^2(\mathbb{R}^2 \times \dots \times \mathbb{R}^2)$. Moreover, we can also define

$$A(\phi) = \prod_{i=1}^n S_I(f_i, k_i).$$

We want to know how we can compute $\int_{S'(\mathbb{R}^2)} A(\phi) d\mu(\phi)$. For that, we recall: For (W, μ) a measure space and $f \in W^*$ we have

$$(: f :)^n = \sum_{k=0}^{\lfloor \frac{n!}{2} \rfloor} \frac{n!}{k!(n-2k)!} f^{n-2k} \left(\frac{-q_\mu(f, f)}{2} \right)^k.$$

We would like to know if an expression of the form

$$(: f_1 \cdots f_k :) (: g_{k+1} \cdots g_n :)$$

can be written as a linear combination of Wick ordered polynomials. The answer is yes, and the advantage is that it allows us to compute integration of product of Wick ordered polynomials easily.

Example 15.1.1. We can write

$$(: f_1 \cdots f_n :) = f_1(: f_2 \cdots f_n :) - \sum_{j=2}^n q_n(f_1, f_j)(: f_2 \cdots \widehat{f_j} \cdots f_n :),$$

which is similar to integration by parts. Here the symbol $\widehat{}$ means that the element is omitted.

15.2. Generalized Feynman diagrams. Let $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n$, be the disjoint union of finite sets I_i for $i \in \{1, \dots, n\}$.

Definition 15.2.1 (Generalized Feynman diagram). A generalized Feynman diagram is a pair (I, E) , where

$$E \subseteq \{(a, b) \mid a \text{ and } b \text{ do not belong to some } I_i, i \in \{1, \dots, n\}\}.$$

We denote by A_E the remaining vertices. Let $F = (V, E)$ be a generalized Feynman diagram associated to $(: f_1 \cdots f_k :)(: g_{k+1} \cdots g_n :)$. Then

$$V(F) = \left(\prod_{e \in E} q_\mu(f_{\ell(e)}, g_{r(e)}) \right) \left(: \prod_{v \in A_E} \alpha_v : \right),$$

where α_v is either f_v or g_v and $\ell(e)$ is the left end point and $r(e)$ the right end point.

Corollary 15.2.1.

$$\int (: f_1 \cdots f_k :)(: g_{k+1} \cdots g_n :) d\mu = \text{sum of value of complete Feynman diagrams.}$$

Consider again the integral $\int_{S'(\mathbb{R}^2)} A(\phi) d\mu(\phi)$. This can be computed using generalized Feynman diagrams. Our goal is to show that $e^{-\widetilde{S}_{I,\Lambda}} \in L^1(S'(\mathbb{R}^n))$. Consider thus $\widetilde{S}_{I,\Lambda}^k(\phi) = \int_{\Lambda} (: \phi^{2k}(x) :) dx$. We want to know whether $e^{-\widetilde{S}_{I,\Lambda}^k} \in L^1(S'(\mathbb{R}^n))$. E.g. $(: x^4 :) = x^4 - 6x^2 + 3$, then $e^{-(x^4)}$ can behave bad.

Lemma 15.2.1.

$$\widetilde{S}_{I,\Lambda}^k \geq -b(\log k)^n,$$

as $k \rightarrow \infty$ for some $b > 0$.

Remark 15.2.1. This shows that $\widetilde{S}_{I,\Lambda}^k$ does not generalize to a polynomial, which is not bounded from below.

Proof. Let $Q(y) = \sum_{k=0}^{2n} a_k y^k$, for $a_{2n} > 0$. Then

$$\inf_{y \in \mathbb{R}} Q(y) \geq -b,$$

for some $0 \leq b < \infty$, and

$$\begin{aligned} (: \phi(\delta_{k,x})^{2n} :) &= \sum_{k=0}^{2n} = \frac{(2n)!}{k!} \phi(\delta_{k,x})^{2n-2k} \left((-1) \cdot \frac{C(\delta_{k,x}, \delta_{k,x})}{2} \right)^k \\ &= C_k(x, x)^n \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-2k)!} \frac{(-1)^k}{2^k} \left(\frac{\phi(\delta_{k,x})}{\sqrt{C_k(x, x)}} \right)^{2n-2k}. \end{aligned}$$

Thus $(: \phi(\delta_{k,x})^{2n} :) \geq -b \int_{\Lambda} C_k(x, x)^n$ for some $b > 0$ and hence $\widetilde{S}_{I,\Lambda}^k(\phi) \geq -b \int_{\Lambda} C_k(x, x)^n \geq -\widetilde{b}(\log k)^n$ as $k \rightarrow \infty$. \square

Corollary 15.2.2. $e^{-\widetilde{S}_{I,\Lambda}^k} \in L^p(\mathcal{S}'(\mathbb{R}^n))$ for all p .

Consider $S^I(P, f)(\phi) = \int_{\mathbb{R}^2} f(x) (: P(\phi(x)) :) dx$, where $P(x) = \sum_n a_n x^n$ is a function on $\mathcal{S}'(\mathbb{R}^2)$ and $f \in L^2(\mathbb{R}^2)$. We showed, $S^I(P, f) \in L^2(\mathcal{S}'(\mathbb{R}^2), \mu)$. Let $S^{I,k}(P, f) = \int f(x) P(\phi, \delta_{k,x}) dx$, where $\delta_{k,x}$ is a smooth approximation of δ_x . We showed, if $S^{I,k}$ is Cauchy, then $S^I(P, f) = \lim_{k \rightarrow \infty} S^{I,k}(P, f)$. In fact,

$$\|S^{I,k}(P, f) - S^I(P, f)\|_{L^2(\mathcal{S}'(\mathbb{R}^2), \mu)} \leq C f^{-\delta},$$

for some $\delta > 0$ as $k \rightarrow \infty$. Moreover, $e^{-S^{I,k}(P)} \in L^1(\mathcal{S}'(\mathbb{R}^n))$, where $S_{\Lambda}^{I,k}(P)(\phi) = \int_{\Lambda} (: P(\phi(x)) :) dx$ with $P(x) = x^{2k}$. The idea is that $S_{\Lambda}^{I,k}(\phi) \geq -C(\log k)^n$ for some $C > 0$. We can observe that $S_{\Lambda}^{I,k}(\phi) \geq 1 - \widetilde{C} \log(k)^n$ for some $\widetilde{C} > 0$ for large k (take $\widetilde{C} = \frac{2}{3}C$). The goal was to show that $e^{-S_{\Lambda}^I(P)} \in L^1(\mathcal{S}'(\mathbb{R}^2), \mu)$. The strategy is to study the sets, where $S_{\Lambda}^I(P)$ is bad, and then show that these sets have measure zero. Define a “bad set”

$$X(k) := \{\phi \in \mathcal{S}'(\mathbb{R}^2) \mid S_{\Lambda}^I(P)(\phi) \leq \widetilde{C}(\log k)^n\}.$$

Lemma 15.2.2.

$$X(k) \subseteq \{\phi \in \mathcal{S}'(\mathbb{R}^2) \mid |S_{\Lambda}^I(P)(\phi) - S_{\Lambda}^{I,k}(P)(\phi)| \geq 1\}.$$

Proof. Let $\phi \in X(k)$. Then

$$S_{\Lambda}^I(P)(\phi) - S_{\Lambda}^{I,k}(P)(\phi) \leq S_{\Lambda}^I(P)(\phi) - (1 - \widetilde{C}(\log k)^n) = \underbrace{S_{\Lambda}^I(P)(\phi) + \widetilde{C}(\log k)^n}_{\leq 1} - 1 \leq -1.$$

\square

Proposition 15.2.1. There is a $B > 0$ and $\delta > 0$ such that $\mu(X(k)) \leq Bk^{-\delta}$ as $k \rightarrow \infty$.

Proof. We have

$$\mu(X(k)) = \int_{X(k)} d\mu \leq \int_{X(k)} |S_{\Lambda}^I(P) - S_{\Lambda}^{I,k}(P)|^2 d\mu \leq \int_{\mathcal{S}'} |S_{\Lambda}^I(P) - S_{\Lambda}^{I,k}(P)|^2 d\mu \leq B_{\Lambda} k^{-\delta}$$

as $k \rightarrow \infty$. \square

Remark 15.2.2. One can show that $\mu(X(k)) \leq C \text{Exp}(-k^\alpha)$ for some $\alpha > 0$ as $k \rightarrow \infty$.

Let $(\Omega, \sigma(\Omega), \mu)$ be a probability space and $f: \Omega \rightarrow \mathbb{R}$ a measurable function on Ω . Denote by

$$\mu_f(x) = \mu(\{\omega \in \Omega \mid f(\omega) \geq x\}).$$

Let F be an increasing positive function on \mathbb{R} such that $\lim_{x \rightarrow \infty} F(x) = \infty$. Then

$$\int_{\Omega} F(f(\omega)) d\mu(\omega) = \int_{\mathbb{R}} F(x) \mu_f(x) dx$$

Theorem 15.2.1. *Let f be a measurable function on Ω such that $\mu(\{\omega \in \Omega \mid -f(\omega) \geq C(\log k)^n\}) \leq Ce^{-k_0}$ for $k \geq k_0$. Then*

$$\int_{\Omega} e^{-f(\omega)} d\mu(\omega) < \infty.$$

Proof. We have

$$\begin{aligned} \int_{\Omega} e^{-f(\omega)} d\mu(\omega) &= \int_{\{\omega \in \Omega \mid f(\omega) < C(\log k)^n\}} e^{-f(\omega)} d\mu(\omega) + \int_{\{\omega \in \Omega \mid -f(\omega) \geq C(\log k_0)^n\}} e^{-f(\omega)} d\mu(\omega) \\ &\leq B_1 \int e^x \mu_f(x) dx \leq B_1 + \int e^x \text{Exp}\left(-e^{\alpha(\frac{x}{C})^{1/n}}\right) dx < \infty. \end{aligned}$$

□

Corollary 15.2.3. $e^{-S_{\Lambda}^l(P)} \in L^1(\mathcal{S}'(\mathbb{R}^n))$.

Proof. Take $f = S_{\Lambda}^l(P)$ and $\Omega = \mathcal{S}'(\mathbb{R}^2)$. □

Remark 15.2.3. If P is a polynomial of the form $P(x) = \sum_{k=0}^{2n} a_k x^k$ with $a_{2n} > 0$, and $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ with $f \geq 0$, then we can show that $e^{-S^l(P,f)} \in L^1(\mathcal{S}'(\mathbb{R}^2), \mu)$.

Corollary 15.2.4. $\frac{e^{-S_{\Lambda}^l(P,f)}}{\int_{\mathcal{S}'(\mathbb{R}^2)} e^{-S_{\Lambda}^l(P,f)}}$ is a probability measure on $\mathcal{S}'(\mathbb{R}^2)$.

15.3. Theories with exponential interaction. Consider the potetial

$$V_g^\alpha(\phi) = \int g(x) (: \text{Exp}(\alpha\phi(x)) :) dx.$$

We want to define a theory for this type of interaction. Moreover, we want to show that $V_g^\alpha \in L^2(\mathcal{S}'(\mathbb{R}^2), \mu)$ with certain assumption on α and g . Define

$$V_k^\alpha(g) = \int_{\mathbb{R}^2} g(x) (: \text{Exp}(\alpha(\phi, \delta_{k,x})) :) dx.$$

Recall $(: \text{Exp}(\alpha f) :) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} (: f^k :)$ for $f \in \mathcal{S}(\mathbb{R}^2)$.

Lemma 15.3.1. We get $V_k^\alpha(g) \in L^2(\mathcal{S}'(\mathbb{R}^2), \mu)$, whenever $g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $0 \leq \alpha^2 \leq 4\pi$.

Proof. We have

$$\langle (: \text{Exp}(\alpha f) :), (: \text{Exp}(\alpha g) :) \rangle = \text{Exp}(\alpha^2 C(f, g)),$$

and thus

$$\|V_k^\alpha(g)\|^2 = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x)g(y)\text{Exp}(\alpha^2 C(\delta_{k,x}, \delta_{k,y}))dx dy = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x)g(y)\text{Exp}(\alpha^2 C_k(x, y))dx dy,$$

where $C_k(x, y) = C(\delta_{k,x}, \delta_{k,y})$. We know $C_k(x, y) \leq C(x, y)$ and $\iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x)g(y) \exp(\alpha^2 C(x, y))dx dy < \infty$, whenever $0 \leq \alpha^2 < 4\pi$, and $g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. The latter is true for $\|x - y\| \geq 1$ and $\|x - y\| < 1$ gives $\exp\left(\alpha^2 - \frac{\log\|x-y\|}{2\pi}\right) = \|x - y\| - \frac{\alpha^2}{4\pi}$. Hence $\|V_k^\alpha(g)\|^2 < \infty$. \square

Proposition 15.3.1. $\{V_k\}_k$ converges in $L^2(\mathcal{S}'(\mathbb{R}^2), \mu)$.

Proof. Recall that $V_k = \sum_{k=0}^\infty \frac{\alpha^k}{k!} \int g(x)(: f_k(x)^k :)dx$. The Weierstrass M-test tells us that for any metric space (X, d) , a Banach space W , $f_k: X \rightarrow W$ with $|f_k(x)| \leq M_k$ with numbers $M_k > 0$ such that $\sum_{k=0}^\infty M_k < \infty$, then $\sum_{k=0}^\infty f_k(x)$ converges uniformly for x . \square

Exercise 15.3.1. Show that there is a $C_k > 0$ such that $\left\| \frac{\alpha^2}{k!} \int g(x)(: \phi_k(x)^k :)dx \right\|_{L^2(\mathcal{S}'(\mathbb{R}^2), \mu)}^2 \leq C_k$.

This implies that V_k converges uniformly on X (by the M-test). Recall that

$$\frac{\alpha^k}{k!} \int g(x)(: \phi_k(x)^k :)dx \rightarrow \frac{\alpha^k}{k!} \int g(x)(: \phi(x)^k :)dx,$$

where $\phi_k = (\phi, \delta_{k,x})$. If $V = \lim_{k \rightarrow \infty} V_k$, then

$$V = \sum_{k=0}^\infty \frac{\alpha^k}{k!} \int g(x)(: \phi(x)^k :)dx = \int g(x)(: \text{Exp}(\alpha\phi(x)) :)dx.$$

Thus we have shown that $V \in L^2(\mathcal{S}'(\mathbb{R}^2), \mu)$. We can observe the following:

- (1) $V_k \geq 0$ for all k , whenever $g \geq 0$. this implies that for such g , $V \geq 0$ and hence $e^{-V} \in L^1$.
- (2) For $0 \geq \alpha^2 < 4\pi$ and $g \geq 0$ with $g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, we showed $e^{-V_g^\alpha} \in L^1(\mathcal{S}'(\mathbb{R}^2), \mu)$. Let ν be a measure on $[-\alpha, \alpha]$. Then

$$\int_{[-\alpha, \alpha]} e^{-V_g^{\alpha'}} d\nu(\alpha') \in L^1(\mathcal{S}'(\mathbb{R}^2), \mu).$$

15.4. The Osterwalder-Schrader Axioms. Let μ be a Borel measure on $\mathcal{S}'(\mathbb{R}^n)$.

15.4.1. *Analyticity (OS0)*. Let $f_1, \dots, f_k \in \mathcal{S}'(\mathbb{R}^n)$. Define a function $\widehat{\mu}(f_1, \dots, f_k): \mathbb{C}^k \rightarrow \mathbb{C}$ by $\widehat{\mu}(f_1, \dots, f_k)(z_1, \dots, z_k) = \widehat{\mu}(\sum_j z_j f_j)$, where $\widehat{\mu}$ is the characteristic function of μ , i.e.

$$\widehat{\mu}(f) = \int_{\mathcal{S}'(\mathbb{R}^n)} e^{i\phi(f)} d\mu(\phi).$$

Definition 15.4.1 (Analyticity). We say that μ is analytic or μ has analyticity if $\widehat{\mu}(f_1, \dots, f_k)$ is entire on \mathbb{C}^k for all $f_1, \dots, f_k \in \mathcal{S}'(\mathbb{R}^n)$ and $k \in \mathbb{N}$. This means that μ decays faster than any exponential map.

Remark 15.4.1. An immediate consequence is that $\int_{\mathcal{S}'(\mathbb{R}^n)} \phi(f) d\mu(\phi) < \infty$ for all k and for all $f \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$\widehat{\mu}(if) = \int_{\mathcal{S}'(\mathbb{R}^n)} e^{-\phi(f)} d\mu(\phi) < \infty$$

and

$$\widehat{\mu}(-if) = \int_{\mathcal{S}'(\mathbb{R}^n)} e^{\phi(f)} d\mu(\phi) < \infty,$$

if μ is analytic.

Example 15.4.1. Let μ be the Gaussian measure on $\mathcal{S}'(\mathbb{R}^n)$, whose covariance is given by $(\Delta + m^2)^{-1}$, and let $C(f, g) = \int f(x)C(x, y)g(y)dx dy = \langle f, (\Delta + m^2)^{-1}f \rangle_{\mathcal{C}^2(\mathbb{R}^n)}$. We claim that μ has analyticity. We prove this via an example. Consider $\widehat{\mu}(z_1 f_1 + z_2 f_2) = \int_{\mathcal{S}'(\mathbb{R}^n)} e^{i\phi(z_1 f_1 + z_2 f_2)} d\mu(\phi) = e^{-\frac{1}{2}(2z_1 z_2 C(f_1, f_2) + z_1^2 C(f_1, f_1) + z_2^2 C(f_2, f_2))}$, which is obviously an entire function, and $\widehat{\mu}(z_1 f_1 + z_2 f_2) = \widehat{\mu}(f_1, f_2)(z_1, z_2)$ which is analytic.

Proposition 15.4.1. Let ν be any Gaussian measure on $\mathcal{S}'(\mathbb{R}^n)$. Then ν has analyticity.

15.4.2. *Euclidean invariance (OS1)*. Let $E(n)$ be the Euclidean group of \mathbb{R}^n , i.e. the group generated by rotations, reflections and translations. Let $R \in O(n)$ and $a \in \mathbb{R}^n$. Let $T(a, R) \in E(n)$ be defined by $(T(a, R))(x) = Rx + a$. Notice that $E(n)$ acts on $\mathcal{S}'(\mathbb{R}^n)$ by $(T(a, R)f)(x) = f(T(a, R)^{-1}x)$. $E(n)$ acts also on $\mathcal{S}'(\mathbb{R}^n)$ by $(T(a, R)\phi)(f) = \phi(T(a, R)f)$.

Definition 15.4.2 (Euclidean invariance I). We say that μ is Euclidean invariant if $(T(a, R))_* \mu = \mu$ for all $T(a, R) \in E(n)$.

Lemma 15.4.1. μ is Euclidean invariant if and only if $\widehat{\mu}(f) = \widehat{\mu}(T(a, R)f)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 15.4.3 (Euclidean invariance II). Let ν be a Gaussian measure on $\mathcal{S}'(\mathbb{R}^n)$ whose covariance is $C_\nu: \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathbb{R}$. We say C_ν is Euclidean invariant if $Cov(T(a, R)f, T(a, R)g) = C_\nu(f, g)$ for all $T(a, R) \in E(n)$ and $f, g \in \mathcal{S}'(\mathbb{R}^n)$.

Lemma 15.4.2. Let ν be Gaussian. Then ν is Euclidean invariant if and only if C_ν is Euclidean invariant.

Example 15.4.2. Let μ be the Gaussian measure on $\mathcal{S}'(\mathbb{R}^n)$ with covariance $(\Delta + m^2)^{-1}$. Then μ is Euclidean invariant. We have

$$C(x, y) = \frac{1}{(2\pi)^n} \int \frac{e^{i\xi_1 \|x-y\|}}{m^2 + \xi^2} d\xi.$$

Next, we want to construct a Hilbert space $\mathcal{E} = L^2(\mathcal{S}'(\mathbb{R}^n), \mu)$.

15.4.3. *Reflection positivity (OS3)*. Let $f_1, \dots, f_k \in \mathcal{S}(\mathbb{R}^n)$, such that $\text{supp}(f_i) \subseteq \mathbb{R}_+^n$. Write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ and $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times (0, \infty)$.

Definition 15.4.4 (Reflection positivity I). We say that μ has reflection positivity if for all $z_1, \dots, z_k \in \mathbb{C}$ we have $\sum_{i,j} \bar{z}_i \widehat{\mu}(f_i \cdot \text{im}(\theta) \cdot f_j) z_j \geq 0$, where $\theta(x, t) = (x, -t)$. for all $k \in \mathbb{N}$, $f_1, \dots, f_k \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp}(f_j) \subseteq \mathbb{R}_+^n$.

Assume that ν is Gaussian and let C_ν be its covariance.

Definition 15.4.5 (Reflection Positivity II). We say that C_ν has reflection positivity if

$$C_\nu(f, \theta f) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x, t) f(y, -s) C((x, t), (y, s)) dx dy dt ds \geq 0$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp}(f) \subseteq \mathbb{R}_+^n$.

Exercise 15.4.1. Let ν be a Gaussian measure on $\mathcal{S}'(\mathbb{R}^n)$. Then ν has reflection positivity if and only if C_ν has reflection positivity.

Example 15.4.3. Let μ be the Gaussian measure with covariance $(\Delta + m^2)^{-1}$. Then μ has reflection positivity.

Proof. We will show that $C(f, g) = \langle f, (\Delta + m^2)^{-1} g \rangle_{L^1(\mathbb{R}^n)}$ is reflection positive. Let $f \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp}(f) \subseteq \mathbb{R}_+^n$, and $x = (\bar{x}, t)$. Then

$$C(f, g) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x) C(x, y) g(y) dy = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{e^{i\xi(x-y)}}{\xi^2 + m^2} d\xi \right) g(y) dx dy = \int \frac{\widehat{f}(\xi) \widehat{g}(\xi)}{\xi^2 + m^2} d\xi.$$

Moreover, we have $C(f, \theta f) = \int_{\mathbb{R}^n} \frac{\overline{\widehat{\theta f}(\xi)} \widehat{f}(\xi)}{\xi^2 + m^2} d\xi$, which we want to be positive. We have $\widehat{f}(\xi, i\xi_n) = \frac{1}{(2\pi)^{n/2}} \int_0^\infty \left(\int_{\mathbb{R}^{n-1}} f(x, t) e^{i\xi \cdot x - \xi_n t} \right) dt$. Similarly $\overline{\widehat{\theta f}(\xi, i\xi_n)} = \frac{1}{(2\pi)^{n/2}} \int_0^\infty \left(\int_{\mathbb{R}^{n-1}} \overline{f(x, t)} e^{-i\xi \cdot x - \xi_n t} \right) dt$. Thus we get

$$\widehat{f}(\xi, i\xi_n) = \overline{\widehat{\theta f}(\xi, i\xi_n)}.$$

Using these relations, we can show

$$C(f, \theta f) = \int_{\mathbb{R}^{n-1}} \frac{|\widehat{f}(\xi, i\mu(\xi))|^2}{\sqrt{m^2 + \xi^2}} d\xi \geq 0,$$

where $\mu(\xi) = \sqrt{m^2 + \xi^2}$. □

Considering $\mathcal{E} = L^2(\mathcal{S}'(\mathbb{R}^n), \nu)$, we assume that ν has analyticity and Euclidean invariance. Consider the set

$$\mathcal{A} = \left\{ A(\phi) = \sum_{j=1}^k c_j e^{i\phi(f_j)} \mid c_j \in \mathbb{C}, k \in \mathbb{N} \right\}$$

In fact, $\mathcal{A} \subseteq \mathcal{E}$, because of analyticity, and \mathcal{A} is an algebra. Moreover, let $\mathcal{E}_+ = \{A(\phi) \in \mathcal{A} \mid \text{supp}(f_i) \subseteq \mathbb{R}_+^n\} \subseteq \mathcal{E}$. We define a bilinear form b on \mathcal{E}_+ by $b(A, B) = \int \overline{\theta A B} d\nu(\phi)$.

Exercise 15.4.2. The measure ν is reflection positive if and only if for all $A \in \mathcal{E}_+$, $b(A, A) \geq 0$. Let $N = \{A \in \mathcal{E}_+ \mid b(A, A) = 0\}$, and let \mathcal{H} be the completion of \mathcal{E}_+/N .

Definition 15.4.6. \mathcal{H} is called the physical Hilbert space.

We can observe that if we have $T: \mathcal{E} \rightarrow \mathcal{E}$ such that $T(\mathcal{E}_+) \subseteq \mathcal{E}_+$ and $T(N) \subseteq N$, then T induces a map $T(t): \mathcal{H} \rightarrow \mathcal{H}$, where $T(t)(\vec{x}, s) = (\vec{x}, s + t)$ for $t \geq 0$. We know that $T(t)$ acts on \mathcal{E} unitarily.

Lemma 15.4.3. We have $T(t)\mathcal{E}_+ \subseteq \mathcal{E}_+$ and $T(t)N \subseteq N$.

Proof. The first part is obvious. For the second part, observe that $\theta \circ T(t) = T(-t) \circ \theta$. Let $A \in N$. Then

$$\begin{aligned} \langle T(t)A, \theta T(t)A \rangle_{\mathcal{E}} &= \langle T(t)A, T(-t)\theta A \rangle_{\mathcal{E}} = \langle T(2t)A, \theta, \theta A \rangle_{\mathcal{E}} \\ &= b(T(2t)A, A) \leq \underbrace{b(A, A)^{1/2} b(T(2t)A, T(2t)A)^{1/2}}_{=0} = 0, \end{aligned}$$

which implies that $T(t)A \in N$ because of reflection positivity. \square

One can also check that the map $T(t)^\wedge: \mathcal{H} \rightarrow \mathcal{H}$ is a semigroup for $t \geq 0$.

Lemma 15.4.4. We have $\|T(t)\|_{\mathcal{H}} \leq 1$, for $t \geq 0$. Moreover, $t \mapsto T(t)$ is strongly continuous.

Corollary 15.4.1. $T(t) = e^{-tH}$, where H is a positive self adjoint operator on \mathcal{H} . Moreover, $H(1) = 0$.

Example 15.4.4 (Free massive scalar field theory). Consider a measure μ and the Green's functions $C(x, y)$. We want to know whether we can find an "explicit" representation of \mathcal{H} in terms of time zero hypersurfaces in \mathbb{R}^{n-1} . We can indeed write $\mathcal{H} \cong L^2(\mathcal{S}'(\mathbb{R}^{n-1}), \nu) \subseteq \Gamma(H^{-1}(\mathbb{R}^n)) = L^2(\mathcal{S}'(\mathbb{R}^n), \mu)$, where ν is a Gaussian measure.

Let $f \in \mathcal{S}(\mathbb{R}^{n-1})$. Define then $j_0 f = f \otimes \delta_0$, where $(f \otimes \delta_0)(\vec{x}, t) = f(\vec{x})\delta_0(t)$. We claim that $f \otimes \delta_0 \in H^{-1}(\mathbb{R}_+^n)$. Indeed, we have $\widehat{f \otimes \delta_0}(\vec{\xi}, \xi_n) = \widehat{f}(\vec{\xi})$, and we know

$$\begin{aligned} \langle f \otimes \delta_0, C(f \otimes \delta_0) \rangle &= \frac{1}{2\pi} \int_{\mathbb{R}^n} \frac{|\widehat{f}(\vec{\xi})|^2}{\xi^2 + m^2} d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^{n-1}} |\widehat{f}(\vec{\xi})|^2 \left(\int_{\mathbb{R}} \frac{1}{\xi^2 + m^2} d\xi_n \right) d\vec{\xi} = \frac{1}{2} \int_{\mathbb{R}^{n-1}} \frac{|\widehat{f}(\vec{\xi})|^2}{\sqrt{\xi^2 + m^2}} d\vec{\xi} \end{aligned}$$

Thus $f \otimes \delta_0 \in H^{-1}(\mathbb{R}^n)$. Moreover, $\langle f \otimes \delta_0, C(f \otimes \delta_0) \rangle_{L^2(\mathbb{R}^n)} = \frac{1}{2} \left\langle f, \left(\sqrt{\Delta_{\mathbb{R}^{n-1}} + m^2} \right)^{-1} f \right\rangle_{L^2(\mathbb{R}^{n-1})}$.

If we define $B(f, g) = \frac{1}{2} \int_{\mathbb{R}^{n-1}} f(\Delta_{\mathbb{R}^{n-1}} + m^2)^{-1/2} g dx$, we can see that j_0 defines an isometry $K_{B(\nu)} \rightarrow H^{-1}(\mathbb{R}^n)$, where $K_{B(\nu)}$ is the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to B .

Lemma 15.4.5. For $t \in \mathbb{R}$, define $(j_t f) = f \otimes \delta_t$, with $f \in \mathcal{S}(\mathbb{R}^{n-1})$. Then for $t \geq s$, $\langle j_t f, j_s g \rangle_{L^2(\mathbb{R}^n)} = \frac{1}{2} \left\langle f, (\Delta_{\mathbb{R}^n} + m^2)^{-1/2} e^{-(t-s)\sqrt{\Delta_{\mathbb{R}^n} + m^2}} g \right\rangle_{L^2(\mathbb{R}^n)}$.

Let ν be the Gaussian measure on $\mathcal{S}'(\mathbb{R}^{n-1})$ whose covariance is B . Denote by $H^{-1/2}(\mathbb{R}^{n-1}) := K_{B(\nu)}$. Then we know $L^2(\mathcal{S}'(\mathbb{R}^{n-1}), \nu) \cong \Gamma(H^{-1/2}(\mathbb{R}^{n-1}))$. Given an operator A on \mathcal{H} , one can define an operator $d\Gamma(A)$ on $\Gamma(\mathcal{H})$ as follows: on $Sym^n(\mathcal{H})$ we get $d\Gamma(A) = A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \dots$, and on $Sym^0(\mathcal{H}) = \mathbb{C}$ we get $d\Gamma(A) = 0$. If we identify \mathcal{H} with $L^2(\mathcal{S}'(\mathbb{R}^{n-1}), \nu)$ or $\Gamma(H^{-1/2}(\mathbb{R}^{n-1}))$, then

$$d\Gamma(\sqrt{\Delta_{\mathbb{R}^n} + m^2}).$$

16. QFT AS OPERATOR VALUED DISTRIBUTION

The motivation of this section is to get a better understanding of relativistic quantum mechanics. Recall the data for a quantum mechanical system:

- Hilbert space of states \mathcal{H} (e.g. $L^2(\mathbb{R}^n)$)
- Observables, which are represented by self adjoint operators on \mathcal{H} ,
- “symmetries”, which are unitary representations on \mathcal{H} , and 1-parameter group of symmetries, leading to specific observables (e.g. time translation \rightsquigarrow Hamiltonian of the system).
- Dynamics is controlled by the Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = \widehat{H}\psi$.

16.1. Relativistic quantum mechanics. In relativistic quantum mechanics we want to have unitary representation of the Poincaré group \mathcal{P} , which is the group of all “space-time” symmetries. Recall that Minkowski space-time is given by $\mathbb{M}^n = \mathbb{R}^{1,n-1}$, where we can have coordinates in position space (such as (t, \vec{x})) or in momentum space (such as $(\xi_0, \vec{\xi})$). Denote by \mathcal{L} the Lorentz group, which is the set of all linear isometries of \mathbb{M}^n , i.e. $\{(\Lambda_{ij}) \mid \Lambda^T g \Lambda = g\}$, thus for $\Lambda \in \mathcal{L}$ we have $\det \Lambda \in \{\pm 1\}$. Moreover, we can write \mathcal{L} as a union of subspaces:

$$\mathcal{L} = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow \cup \mathcal{L}_-^\uparrow \cup \mathcal{L}_-^\downarrow,$$

where the label \uparrow (\downarrow) means the determinant is $+1$ (-1), and the label $+$ ($-$) means $\Lambda_{00} > 0$ (< 0). Note that $I \in \mathcal{L}_+^\uparrow$, which we call the restricted Lorentz group. We define the Poincaré group by

$$\mathcal{P} = \{T(\Lambda, a) \mid \Lambda \in \mathcal{L}, a \in \mathbb{R}^n\},$$

where $T(\Lambda, a)(x) = \Lambda x + a$. Thus $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^n$. We can write \mathcal{P} as the union of subspaces in the same way as for \mathcal{L} . We call \mathcal{P}_+^\uparrow the restricted Poincaré group. We want to have a projective unitary representation of \mathcal{P}_+^\uparrow .

16.1.1. Bergmann’s construction ($n = 4$). In this construction, the Projective unitary representation of \mathcal{P}_+^\uparrow come from unitary representation of $\widetilde{\mathcal{P}}_+^\uparrow$, which is the universal cover of \mathcal{P}_+^\uparrow . In fact $SL(2, \mathbb{C})$ is the universal cover of \mathcal{L}_+^\uparrow . Hence, in this case $G = \widetilde{\mathcal{P}}_+^\uparrow = SL(2, \mathbb{C}) \ltimes \mathbb{R}^4$.

16.1.2. *Wigner's construction.* Take $p \in \mathbb{R}^4$, and let H_p be the stabilizer of p by the action of $SL(2, \mathbb{C})$. Moreover, take a unitary irreducible representation $\mathcal{H}_{\sigma,p}$ of H_p . One can then use the Mackey machine: choose a G invariant measure on G/H_p and define the Hilbert space \mathcal{H} to be the $\mathcal{H}_{\sigma,p}$ valued functions on G/H_p and use the invariant measure to define an inner product.

Proposition 16.1.1 (Wigner). \mathcal{H} is an irreducible unitary representation of G . Moreover, all irreducible unitary representations of G arise this way.

Remark 16.1.1. H_p can have $(2s + 1)$ -dimensional irreducible representations. Here $s \in \{0, \frac{1}{2}, 1\}$ represents the "spin" of the particle.

Assume $s = 0$. We start with the trivial representation, which is a 1-dimensional representation of H_p . Consider the sets

$$\begin{aligned} X_m^+ &= \{\xi^1 - m^2 = 0 \mid \xi_0 > 0\} \\ X_m^- &= \{\xi^1 - m^2 = 0 \mid \xi_0 < 0\} \end{aligned}$$

and write $X_m = X_m^+ \cup X_m^-$ and $X = \bigcup_{m \geq 0} X_m$. Take $p = (m, 0, 0, 0)$. Then we have $G/H_p = X_m^+$. We want to construct an invariant measure on X_m^+ . Let f be a positive function on $(0, \infty)$. Then $f(\xi^2)d\xi$ is an invariant measure on X . We would like to have an invariant measure

of the form $\delta(\xi^1 - m^2)d\xi$. We define $\phi: (0, \infty) \times \mathbb{R}^3 \rightarrow X_m^+$, $(y, \vec{\xi}) \mapsto (\sqrt{y^2 + |\vec{\xi}|^2}, \vec{\xi})$. Then $\phi^*(f(\xi^2)d\xi) = \frac{f(y)dyd\vec{\xi}}{\sqrt{y^2 + |\vec{\xi}|^2}}$. We want to have the pushforward of $\delta_m = \frac{dyd\vec{\xi}}{\sqrt{y^2 + |\vec{\xi}|^2}}$ to be our measure

on X_m^+ . More precisely, define $\alpha: \mathbb{R}^3 \rightarrow X_m^+$ by $\alpha(\vec{\xi}) = (\sqrt{m^2 + |\vec{\xi}|^2}, \vec{\xi})$. As an invariant measure, we want the pushforward of $\frac{1}{2\sqrt{m^2 + |\vec{\xi}|^2}}d\vec{\xi}$ on \mathbb{R}^3 to X_m^+ . Wigner's theorem gives us $\mathcal{H} = L^2(X_m^+, \mu_m) \cong L^2(\mathbb{R}^3, \nu)$, where $\frac{d\nu}{d\vec{\xi}} = \frac{1}{2\sqrt{m^2 + |\vec{\xi}|^2}}$. The position operator is then given by $\frac{1}{2}(\Delta + m^2)^{-1/2}$. One can summarize the result by saying that the Hilbert space for spin zero particles is given by $L^2(\mathbb{R}^3, \nu) \cong H^{-1/2}(\mathbb{R}^3)$.

16.2. **Garding-Wightman formulation of QFT.** We want to give the axioms of the so-called Garding-Wightman formulation of QFT. We have the following axioms:

(GW1) We have a Hilbert space \mathcal{H} , a vacuum state $\Omega \in \mathcal{H}$, and a unitary representation \mathcal{P}_+^\uparrow on \mathcal{H} .

(GW2) We have a field operator $\Phi: \mathcal{S}(\mathbb{R}^+) \rightarrow \text{Operators on } \mathcal{H}$ together with a dense subspace D of \mathcal{H} such that

- (a) $\Omega \in D$
- (b) $D \subseteq D(\Phi(f))$ for all f ,
- (c) $f \mapsto \Phi(f)|_D$ is linear,
- (d) for all $\Omega_1, \Omega_2 \in D$, the assignment $f \mapsto \langle \Phi(f)\Omega_1, \Omega_2 \rangle$ is a Schwarz distribution (regularity),

$$(e) \Phi(f)^* = \Phi(\bar{f}).$$

(GW3) (Covariance) We have

(a) $U(a, \Lambda)D \subseteq D$, where $U(a, \Lambda)$ is the unitary representation of $T(a, \Lambda) \in \mathcal{P}_+^\uparrow$ on \mathcal{H} ,

$$(b) U(a, \Lambda) \cdot \Phi(f) \cdot U(a, \Lambda)^{-1} = \Phi(T(a, \Lambda)f).$$

(GW4) (Spectrum) Since \mathbb{R}^4 acts unitarily on \mathcal{H} via $U(a, \Lambda)$, we can take P_1, \dots, P_4 to be the infinitesimal generators of this action. One can show that P_1, \dots, P_4 are essentially self adjoint. The axio is then given by: The joint spectrum of (P_1, \dots, P_4) lies in $X^+ = \{\xi^2 \geq 0 \mid \xi_0 > 0\}$, where physically $\xi^2 = E^2 - \vec{p}^2$.

(GW5) (Locality) If f and g have space-like disjoint support, then $[\Phi(f), \Phi(g)] = 0$.

Remark 16.2.1. By the axioms, one can show that the vaccum is unique: If $U(a, \Lambda)\Omega' = \Omega'$ for all $T(a, \Lambda) \in \mathcal{P}_+^\uparrow$, then $\Omega' = c\Omega$, where $c \in \mathbb{C}$.

Given $f_1, \dots, f_k \in \mathcal{S}(\mathbb{R}^4)$, we can define $(f_1, \dots, f_k) \mapsto \langle \Phi(f_1) \cdots \Phi(f_k)\Omega, \Omega \rangle$. By (GW2) this assignment is a distribution in $\mathcal{S}'(\mathbb{R}^4)$, i.e.

$$\langle \Phi(f_1) \cdots \Phi(f_k)\Omega, \Omega \rangle = W_k(f_1 \otimes \cdots \otimes f_k) = \int_{\mathbb{R}^4} W_k(x_1, \dots, x_k) f_1(x_1) \cdots f_k(x_k) dx_1 \cdots dx_k.$$

Definition 16.2.1 (Wightman distribution). $W_k(x_1, \dots, x_k)$ are called Wightman distribution.

We can now formulate the Wightman axioms:

(W1) W_k are \mathcal{P}_+^\uparrow -invariant,

(W2) If $f_1 \in \mathcal{S}(\mathbb{R}^4), \dots, f_k \in \mathcal{S}(\mathbb{R}^{4k})$, then $\sum_{i,j=0}^k W_{i+j}(\bar{f}_i \otimes f_j) \geq 0$.

(W3) (Locality) $W_k(x_1, \dots, x_j, x_{j+1}, \dots, x_k) = W_k(x_1, \dots, x_{j+1}, x_j, \dots, x_k)$, whenever x_j and x_{j+1} are space-like seperated.

We recall the Euclidean setting: We have a measure $e^{-S(\phi)} \mathcal{D}(\phi)$ on $\mathcal{S}'(\mathbb{R}^4)$, a two point function $C(f, g)$ and for $f_1, \dots, f_k \in \mathcal{S}(\mathbb{R}^4)$ we have a map $(f_1, \dots, f_k) \mapsto \int \phi(f_1) \cdots \phi(f_k) e^{-S(\phi)} \mathcal{D}(\phi)$. We would like to know how we can relate the Minkowski to the Euclidean setting.

We can observe that $W_k(x_1, \dots, x_k) = \omega_k(x_1 - x_2, \dots, x_{k-1} - x_k)$ because of translation invariance.

(W4) (Spectral condition) The Fourier transform of $\widehat{\omega}_k$ of ω_k has support in $\underbrace{X^+ \times \cdots \times X^+}_k$.

Remark 16.2.2. There is one more Wightman axiom, called ‘‘Cluster property’’ (W6), which is related to uniqueness of vaccum.

Theorem 16.2.1 (Wightman reconstruction theorem). *If we have distributions $(W_k)_k$ satisfying (W1)-(W6), then there is a ‘‘unique’’ GW field theory, whose Wightman distributions are W_k .*

Sketch of the proof. Let $f \in V = \bigoplus_{k \geq 0} \mathcal{S}(\mathbb{R}^4)$ (for $k = 0$ we get $V = \mathbb{C}$), such that $f = (f_0, f_1, \dots, f_j, \dots)$, where everything is zero except for finitely many j s. Moreover, let $(f, g) = \sum_{i,j} W_{i+j}(f_i \otimes g_j)$ and $N = \{f \in V \mid (f, f) = 0\}$. Let \mathcal{H} be the completion of V/N and $\Omega = (1, 0, 0, \dots)$, and for $h \in \mathcal{S}(\mathbb{R}^4)$, we have $\Phi: V \rightarrow V, \Phi(h)(f_0, f_1, \dots) = (0, f_0 \otimes h, f_1 \otimes h, \dots)$. \square

16.2.1. Wick rotation. Consider the distribution $W_k(x_1, \dots, x_k)$ for $x_1, \dots, x_k \in \mathbb{M}^4$, where $x_1 = (t_1, \vec{x}_1), \dots, x_k = (t_k, \vec{x}_k)$ such that $x_i^2 = t_i^2 - \vec{x}_i^2$. Formally, we want to define $W_k(it_1, \vec{x}_1, it_2, \vec{x}_2, \dots, it_k, \vec{x}_k)$, with $x_i^2 = -t_i^2 - \vec{x}_i^2$. We want to do this by considering complex variables $z_i = x_i + iy_i$ and pass from $W_k(x_1, \dots, x_k)$ to $W_k(z_1, \dots, z_k)$. First, we analytically continue $W_k(x_1, \dots, x_k)$ to a holomorphic function. Next, we think of $W_k(x_1, \dots, x_k)$ as a boundary value of an analytic function.

Definition 16.2.2 (Boundary value). Let ϕ be a distribution in $\mathcal{S}'(\mathbb{R}^n)$. Let F be a holomorphic function. We say ϕ is a boundary value of F if for fixed $y_0 \in \mathbb{R}^n$, we have

$$\phi(f) = \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} F(x + ity_0) f(x) dx,$$

or equivalently we say F is an analytic continuation of ϕ .

Remark 16.2.3. It is not clear whether all $\phi \in \mathcal{S}'(\mathbb{R}^n)$ have analytic continuations. In fact, let $T \in \mathcal{S}'(\mathbb{R}^n)$ such that $\text{supp}(T) \subseteq$ some cone C , where C is the intersection of two hyperplanes. Then T is a boundary value of an analytic function on $\mathbb{R}^n - iC^*$, where C^* is the dual cone.

Corollary 16.2.1. Recall $\widehat{\omega}_k$ has support in $X^+ \times \dots \times X^+$. Then ω_k can be analytically continued to a holomorphic function $\omega_k(z_1, \dots, z_k)$ on $\underbrace{(\mathbb{R}^4 - iX^+) \times \dots \times (\mathbb{R}^4 - iX^+)}_{k-1}$.

Now we can observe that the W_k have analytic continuation to $W_k(z_1, \dots, z_k)$ on $\mathcal{T}_k = \{(z_1, \dots, z_k) \mid \text{Im}(z_{i+1} - z_i) \in X^+\}$.

16.2.2. Schwinger functions. We want to construct $W_k(it_1, \vec{x}_1, \dots, it_k, \vec{x}_k)$ having $W_k(x_1, \dots, x_k)$ with $x_j = (t_j, \vec{x}_j)$. The problem is that all the points $(it_1, \vec{x}_1), \dots, (it_k, \vec{x}_k) \notin \mathcal{T}_k$.

Exercise 16.2.1. Take $k = 1$ and show $(it, i) \in \mathcal{T}_1$ if and only if $t_1 > 0$.

Hence, we want to enlarge \mathcal{T}_k and extend W_k to this bigger set. Take

$$\mathcal{T}_k^e = \{(\Lambda z_1, \dots, \Lambda z_k) \mid \Lambda \in \mathcal{L}, \det \Lambda = 1\}.$$

E.g. we had $\Lambda = -I$ before. Then we can extend $W_k(w_1, \dots, w_k) = W_k(z_1, \dots, z_k)$ if $(w_1, \dots, w_k) = (\Lambda z_1, \dots, \Lambda z_k)$ for some $\Lambda \in \mathcal{L}$.

Lemma 16.2.1. It is possible to extend W_k as before (using a lot of assumptions).

Denote by

$$\Sigma_k(\mathcal{T}_k^e) = \{(z_{\sigma(1)}, \dots, z_{\sigma(n)}) \mid \sigma \in \Sigma_k, (z_1, \dots, z_k) \in \mathcal{T}_k^e\}$$

the permutation group of order k on \mathcal{T}_k^e .

Definition 16.2.3. $\mathcal{T}_k^{p,e} := \Sigma_n(\mathcal{T}_k^e)$.

Exercise 16.2.2. Show that the Euclidean points $\mathcal{E}_n \subseteq \mathcal{T}_n^{p,e}$, where $\mathcal{E}_n \subseteq \mathbb{C}$ and $(z_1, \dots, z_n) \in \mathcal{E}_n$ if and only if $z_j = (it_j, \vec{x}_j)$.

Remark 16.2.4. In fact, $W_n(z_1, \dots, z_n)$ can be extended to an analytic function on $\mathcal{T}_n^{p,e}$ (technical result).

Definition 16.2.4.

$$\widetilde{\mathcal{E}}_n = \{(\underbrace{(t_1, \vec{x}_1)}_{y_1}, \dots, \underbrace{(t_n, \vec{x}_n)}_{y_n}) \mid (it_1, \vec{x}_1, \dots, it_n, \vec{x}_n) \in \mathcal{E}_n\}.$$

We call (y_1, \dots, y_n) non-coincident if $y_k \neq y_\ell$ for all $k \neq \ell$.

Definition 16.2.5 (Schwinger function). For a non-coincident Euclidean point (y_1, \dots, y_n) , we define $S_n(y_1, \dots, y_n) = W_n(it_1, \vec{x}_1, \dots, it_n, \vec{x}_n)$. We call S_n Schwinger functions.

16.2.3. *Properties of Schwinger functions.* For a free massive scalar field theory we can compute $W_2(x, y)$ explicitly. It is given by

$$W_2(x, y) = C_{W_2} \int_{\mathbb{R}^3} \frac{e^{-i(\omega(\vec{\xi})(x_0 - y_0) + \vec{\xi}(\vec{x} - \vec{y}))}}{\omega(\vec{\xi})} d\vec{\xi},$$

where $\omega(\vec{\xi}) := \sqrt{m^2 + \vec{\xi}^2}$ and C_{W_2} some constant. Let

$$W_2(t, \vec{x}) = C_{W_2} \int_{\mathbb{R}^3} \frac{e^{it(\omega(\vec{\xi}) + \vec{\xi} \cdot \vec{x})}}{\omega(\vec{\xi})} d\vec{\xi}.$$

Thus we get

$$\begin{aligned} S_2(y) = W_2(it, \vec{x}) &= C_{W_2} \int_{\mathbb{R}^3} \frac{e^{t\omega(\vec{\xi})} e^{-i\vec{x} \cdot \vec{\xi}}}{\omega(\vec{\xi})} d\vec{\xi} \\ &= C_{W_2} \int_{\mathbb{R}^3} e^{-i\vec{\xi} \cdot \vec{x}} d\vec{\xi} \int_0^\infty \frac{e^{-it\xi_0}}{\xi_0^2 + m^2 + \vec{\xi}^2} d\xi_0 = C_{W_2} \int_{\mathbb{R}^4} \frac{e^{-iy\xi}}{m^2 + \xi^2} d\xi = G(y), \end{aligned}$$

where G is the Green's function. The Osterwalder-Schrader axioms can be reformulated with the Schwinger functions as follows:

(OS1) $S(y_1, \dots, y_n)$ defines a distribution on $\mathcal{S}_\neq(\mathbb{R}^{4n})$, where

$$\mathcal{S}_\neq(\mathbb{R}^{4n}) = \{f \in \mathcal{S}(\mathbb{R}^{4n}) \mid f(y_i - y_j) = 0, \forall i \neq j\}.$$

Moreover, $\overline{S_n(f)} = S_n(\overline{\theta f})$, where $\theta(t, \vec{x}) = (-t, \vec{x})$, and if $h \in \mathcal{S}((\mathbb{R}_+^4)^{n-1})$ we get

$$|S_n(h)(y_2 - y_1, \dots, y_n - y_{n-1})| \leq \|h\|,$$

where $\|\cdot\|$ is some norm on $\mathcal{S}((\mathbb{R}_+^4)^{n-1})$.

(OS2) (Euclidean invariance)

(OS3) Let $f_n \in \mathcal{S}((\mathbb{R}_+^4)^n)$. Then $\sum_{m,n} S_{n+m}(\overline{\theta f_n} \otimes f_m) \geq 0$.

(OS4) $S_n(y_{\sigma(1)}, \dots, y_{\sigma(n)}) = S_n(y_1, \dots, y_n)$ for all $\sigma \in \Sigma_n$.

(OS5) (Cluster property)

Theorem 16.2.2 (Reconstruction theorem). *If we have $S_n(y_1, y_n)$ satisfying (OS1)-(OS5), then there is an unique Garding-Wightman theory.*

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