Quantum Field Theory

Badis Ydri

Departement de Physique, Faculté des Sciences, Université d'Annaba,
Annaba, Algerie.

Contents

Introduction and References

References

- M.E.Peskin, D.V.Schroeder, An Introduction to Quantum Field Theory.
- J.Strathdee,Course on Quantum Electrodynamics, ICTP lecture notes.
- C.Itzykson, J-B.Zuber, Quantum Field Theory.
- J.Zinn-Justin,Quantum Field Theory and Critical Phenomena.
- D.Griffiths, Introduction to Elementary Particles.
- W.Greiner, Relativistic Quantum Mechanics.Wave Equations.
- W.Greiner, J.Reinhardt, Field Quantization.
- W.Greiner, J.Reinhardt, Quantum Electrodynamics.
- S.Randjbar-Daemi, Course on Quantum Field Theory, ICTP lecture notes.
- V.Radovanovic, Problem Book QFT.

Part I: Free Fields

1
Relativistic Quantum Mechanics

In this chapter we will follow the books of Griffiths (special relativity) and Peskin and Schroeder (relativistic wave equations).

1.1 Special Relativity

1.1.1 Postulates

Classical mechanics obeys the principle of relativity which states that the laws of nature take the same form in all inertial frames. An inertial frame is any frame in which Newton's first law holds. Therefore all other frames which move with a constant velocity with respect to a given inertial frame are also inertial frames.

Any two inertial frames O and $O^{'}$ can be related by a Galilean transformation which is of the general form

$$\begin{aligned} t^{'} &= t+\tau \\ \vec{x}^{'} &= R\vec{x}+\vec{v}t+\vec{d}. \end{aligned} \tag{1.1}$$

In above R is a constant orthogonal matrix, $\vec{d}$ and $\vec{v}$ are constant vectors and τ is a constant scalar. Thus the observer $O^{'}$ sees the coordinates axes of O rotated by R, moving with a velocity $\vec{v}$, translated by $\vec{d}$ and it sees the clock of O running behind by the amount τ. The set of all transformations of the form (1.1) form a 10-parameter group called the Galilean group.

The invariance/covariance of the equations of motion under these transformations which is called Galilean invariance/covariance is the precise statement of the principle of Galilean relativity.

In contrast to the laws of classical mechanics the laws of classical electrodynamics do not obey the Galilean principle of relativity. Before the advent of the theory of special relativity the laws of electrodynamics were thought to hold only in the inertial reference frame which is at rest with respect to an invisible medium filling all space known as the ether. For example electromagnetic waves were thought to propagate through the vacuum at a speed relative to the ether equal to the speed of light $c = 1/\sqrt{\mu_0 \epsilon_0} = 3 \times 10^8 m/s$.

The motion of the earth through the ether creates an ether wind. Thus only by measuring the speed of light in the direction of the ether wind we can get the value c whereas measuring it in any other direction will give a different result. In other words we can detect the ether by measuring the speed of light in different directions which is precisely what Michelson and Morley tried to do in their famous experiments. The outcome of these experiments was always negative in the sense that the speed of light was found exactly the same equal to c in all directions.

The theory of special relativity was the first to accommodate this empirical finding by postulating that the speed of light is the same in all inertial reference frames, i.e. there is no ether. Furthermore it postulates that classical electrodynamics (and physical laws in general) must hold in all inertial reference frames. This is the principle of relativity although now its precise statement can not be given in terms of the invariance/covariance under Galilean transformations but in terms of the invariance/covariance under Lorentz transformations which we will discuss further in the next section.

Einstein's original motivation behind the principle of relativity comes from the physics of the electromotive force. The interaction between a conductor and a magnet in the reference frame where the conductor is moving and the magnet is at rest is known to result in an emotional emf. The charges in the moving conductor will experience a magnetic force given by the Lorentz force law. As a consequence a current will flow in the conductor with an induced motional emf given by the flux rule $\mathcal{E} = -d\Phi/dt$. In the reference frame where the conductor is at rest and the magnet is moving there is no magnetic force acting on the charges. However the moving magnet generates a changing magnetic field which by Faraday's law induces an electric field. As a consequence in the rest frame of the conductor the charges experience an electric force which causes a current to flow with an induced transformer emf given precisely by the flux rule, viz $\mathcal{E} = -d\Phi/dt$.

So in summary although the two observers associated with the states of rest of the conductor and the magnet have different interpretations of the process their predictions are in perfect agreement. This indeed suggests as pointed out first by Einstein that the laws of classical electrodynamics are the same in all inertial reference frames.

The two fundamental postulates of special relativity are therefore:

- The principle of relativity: The laws of physics take the same form in all inertial reference frames.

- The constancy of the speed of light: The speed of light in vacuum is the same in all inertial reference frames.

1.1.2 Relativistic Effects

The gedanken experiments we will discuss here might be called "The train-and-platform thought experiments".

Relativity of Simultaneity We consider an observer $O^{'}$ in the middle of a freight car moving at a speed v with respect to the ground and a second observer O standing on a platform. A light bulb hanging in the center of the car is switched on just as the two observers pass each other.

It is clear that with respect to the observer $O^{'}$ light will reach the front end A and the back end B of the freight car at the same time. The two events "light reaches the front end" and "light reaches the back end" are simultaneous.

According to the second postulate light propagates with the same velocity with respect to the observer O. This observer sees the back end B moving toward the point at which the flash was given off and the front end A moving away from it. Thus light will reach B before it reaches A. In other words with the respect to O the event " light reaches the back end" happens before the event "light reaches the front end".

Time Dilation Let us now ask the question: How long does it take a light ray to travel from the bulb to the floor?

Let us call h the height of the freight car. It is clear that with respect to $O^{'}$ the time spent by the light ray between the bulb and the floor is

$$\Delta t^{'} = \frac{h}{c}. \tag{1.2}$$

The observer O will measure a time Δt during which the freight car moves a horizontal distance $v\Delta t$. The trajectory of the light ray is not given by the vertical distance h but by the hypotenuse of the right triangle with h and vdt as the other two sides. Thus with respect to O the light ray travels a longer distance given by $\sqrt{h^2+v^2\Delta t^2}$ and therefore the time spent is

$$\Delta t = \frac{\sqrt{h^2+v^2\Delta t^2}}{c}. \tag{1.3}$$

Solving for Δt we get

$$\Delta t = \gamma\frac{h}{c} = \gamma\Delta t^{'}. \tag{1.4}$$

The factor γ is known as Lorentz factor and it is given by

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{1.5}$$

Hence we obtain

$$\Delta t^{'} = \sqrt{1-\frac{v^2}{c^2}}\Delta t \leq \Delta t. \tag{1.6}$$

The time measured on the train is shorter than the time measured on the ground. In other words moving clocks run slow. This is called time dilation.

Lorentz Contraction We place now a lamp at the back end B of the freight car and a mirror at the front end A. Then we ask the question: How long does it take a light ray to travel from the lamp to the mirror and back?

Again with respect to the observer $O^{'}$ the answer is simple. If $\Delta x^{'}$ is the length of the freight car measured by $O^{'}$ then the time spent by the light ray in the round trip between the lamp and the mirror is

$$\Delta t^{'} = 2\frac{\Delta x^{'}}{c}. \tag{1.7}$$

Let Δx be the length of the freight car measured by O and Δt_1 be the time for the light ray to reach the front end A. Then clearly

$$c\Delta t_1 = \Delta x + v\Delta t_1. \tag{1.8}$$

The term $v\Delta t_1$ is the distance traveled by the train during the time Δt_1. Let Δt_2 be the time for the light ray to return to the back end B. Then

$$c\Delta t_2 = \Delta x - v\Delta t_2. \tag{1.9}$$

The time spent by the light ray in the round trip between the lamp and the mirror is therefore

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{\Delta x}{c-v} + \frac{\Delta x}{c+v} = 2\gamma^2 \frac{\Delta x}{c}. \quad (1.10)$$

The time intervals Δt and $\Delta t^{'}$ are related by time dilation, viz

$$\Delta t = \gamma \Delta t^{'}. \quad (1.11)$$

This is equivalent to

$$\Delta x^{'} = \gamma \Delta x \geq \Delta x. \quad (1.12)$$

The length measured on the train is longer than the length measured on the ground. In other words moving objects are shortened. This is called Lorentz contraction.

We point out here that only the length parallel to the direction of motion is contracted while lengths perpendicular to the direction of the motion remain not contracted.

1.1.3 Lorentz Transformations: Boosts

Any physical process consists of a collection of events. Any event takes place at a given point (x, y, z) of space at an instant of time t. Lorentz transformations relate the coordinates (x, y, z, t) of a given event in an inertial reference frame O to the coordinates $(x^{'}, y^{'}, z^{'}, t^{'})$ of the same event in another inertial reference frame $O^{'}$.

Let (x, y, z, t) be the coordinates in O of an event E. The projection of E onto the x axis is given by the point P which has the coordinates $(x, 0, 0, t)$. For simplicity we will assume that the observer $O^{'}$ moves with respect to the observer O at a constant speed v along the x axis. At time $t = 0$ the two observers O and $O^{'}$ coincides. After time t the observer $O^{'}$ moves a distance vt on the x axis. Let d be the distance between $O^{'}$ and P as measured by O. Then clearly

$$x = d + vt. \quad (1.13)$$

Before the theory of special relativity the coordinate $x^{'}$ of the event E in the reference frame $O^{'}$ is taken to be equal to the distance d. We get therefore the transformation laws

$$\begin{aligned} x^{'} &= x - vt \\ y^{'} &= y \\ z^{'} &= z \\ t^{'} &= t. \end{aligned} \quad (1.14)$$

This is a Galilean transformation. Indeed this is a special case of (1.1).

As we have already seen Einstein's postulates lead to Lorentz contraction. In other words the distance between $O^{'}$ and P measured by the observer $O^{'}$ which is precisely the coordinate $x^{'}$ is larger than d. More precisely

$$x^{'} = \gamma d. \tag{1.15}$$

Hence

$$x^{'} = \gamma(x - vt). \tag{1.16}$$

Einstein's postulates lead also to time dilation and relativity of simultaneity. Thus the time of the event E measured by $O^{'}$ is different from t. Since the observer O moves with respect to $O^{'}$ at a speed v in the negative x direction we must have

$$x = \gamma(x^{'} + vt^{'}). \tag{1.17}$$

Thus we get

$$t^{'} = \gamma(t - \frac{v}{c^2}x). \tag{1.18}$$

In summary we get the transformation laws

$$\begin{aligned} x^{'} &= \gamma(x - vt) \\ y^{'} &= y \\ z^{'} &= z \\ t^{'} &= \gamma(t - \frac{v}{c^2}x). \end{aligned} \tag{1.19}$$

This is a special Lorentz transformation which is a boost along the x axis.

Let us look at the clock found at the origin of the reference frame $O^{'}$. We set then $x^{'} = 0$ in the above equations. We get immediately the time dilation effect, viz

$$t^{'} = \frac{t}{\gamma}. \tag{1.20}$$

At time $t = 0$ the clocks in $O^{'}$ read different times depending on their location since

$$t^{'} = -\gamma\frac{v}{c^2}x. \tag{1.21}$$

Hence moving clocks can not be synchronized.

We consider now two events A and B with coordinates (x_A, t_A) and (x_B, t_B) in O and coordinates $(x^{'}_A, t^{'}_A)$ and $(x^{'}_B, t^{'}_B)$ in $O^{'}$. We can immediately compute

$$\Delta t^{'} = \gamma(\Delta t - \frac{v}{c^2}\Delta x). \quad (1.22)$$

Thus if the two events are simultaneous with respect to O, i.e. $\Delta t = 0$ they are not simultaneous with respect to $O^{'}$ since

$$\Delta t^{'} = -\gamma\frac{v}{c^2}\Delta x. \quad (1.23)$$

1.1.4 Spacetime

The above Lorentz boost transformation can be rewritten as

$$\begin{aligned} x^{0'} &= \gamma(x^0 - \beta x^1) \\ x^{1'} &= \gamma(x^1 - \beta x^0) \\ x^{2'} &= x^2 \\ x^{3'} &= x^3. \end{aligned} \quad (1.24)$$

In the above equation

$$x^0 = ct \ , \ x^1 = x \ , \ x^2 = y \ , \ x^3 = z. \quad (1.25)$$

$$\beta = \frac{v}{c} \ , \ \gamma = \sqrt{1-\beta^2}. \quad (1.26)$$

This can also be rewritten as

$$x^{\mu'} = \sum_{\nu=0}^{4} \Lambda^{\mu}_{\nu} x^{\nu}. \quad (1.27)$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.28)$$

The matrix Λ is the Lorentz boost transformation matrix. A general Lorentz boost transformation can be obtained if the relative motion of the two inertial reference frames O and $O^{'}$ is along an arbitrary direction in space. The transformation law of the coordinates x^μ will still be given by (1.27) with a more complicated matrix

Λ. A general Lorentz transformation can be written as a product of a rotation and a boost along a direction $\hat{n}$ given by

$$x^{'0} = x^0 \cosh\alpha - \hat{n}\vec{x}\sinh\alpha$$
$$\vec{x}^{'} = \vec{x} + \hat{n}\bigg((\cosh\alpha - 1)\hat{n}\vec{x} - x^0\sinh\alpha\bigg). \quad (1.29)$$

$$\frac{\vec{v}}{c} = \tanh\alpha\ \hat{n}. \quad (1.30)$$

Indeed the set of all Lorentz transformations contains rotations as a subset.

The set of coordinates (x^0, x^1, x^2, x^3) which transforms under Lorentz transformations as $x^{\mu'} = \Lambda^{\mu}_{\nu}x^{\nu}$ will be called a 4−vector in analogy with the set of coordinates (x^1, x^2, x^3) which is called a vector because it transforms under rotations as $x^{a'} = R^a_b x^b$. Thus in general a 4−vector a is any set of numbers (a^0, a^1, a^2, a^3) which transforms as (x^0, x^1, x^2, x^3) under Lorentz transformations, viz

$$a^{\mu'} = \sum_{\nu=0}^{4}\Lambda^{\mu}_{\nu}a^{\nu}. \quad (1.31)$$

For the particular Lorentz transformation (1.28) we have

$$a^{0'} = \gamma(a^0 - \beta a^1)$$
$$a^{1'} = \gamma(a^1 - \beta a^0)$$
$$a^{2'} = a^2$$
$$a^{3'} = a^3. \quad (1.32)$$

The numbers a^{μ} are called the contravariant components of the 4−vector a. We define the covariant components a_{μ} by

$$a_0 = a^0\ ,\ a_1 = -a^1\ ,\ a_2 = -a^2\ ,\ a_3 = -a^3. \quad (1.33)$$

By using the Lorentz transformation (1.32) we verify given any two 4−vectors a and b the identity

$$a^{0'}b^{0'} - a^{1'}b^{1'} - a^{2'}b^{2'} - a^{3'}b^{3'} = a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3. \quad (1.34)$$

In fact we can show that this identity holds for all Lorentz transformations. We recall that under rotations the scalar product $\vec{a}\vec{b}$ of any two vectors $\vec{a}$ and $\vec{b}$ is invariant, i.e.

$$a^{1'}b^{1'} + a^{2'}b^{2'} + a^{3'}b^{3'} = a^1b^1 + a^2b^2 + a^3b^3. \quad (1.35)$$

The 4-dimensional scalar product must therefore be defined by the Lorentz invariant combination $a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3$, namely

$$\begin{aligned} ab &= a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 \\ &= \sum_{\mu=0}^{3} a_\mu b^\mu \\ &= a_\mu b^\mu. \end{aligned} \tag{1.36}$$

In the last equation we have employed the so-called Einstein summation convention, i.e. a repeated index is summed over.

We define the separation 4−vector Δx between two events A and B occurring at the points $(x_A^0, x_A^1, x_A^2, x_A^3)$ and $(x_B^0, x_B^1, x_B^2, x_B^3)$ by the components

$$\Delta x^\mu = x_A^\mu - x_B^\mu. \tag{1.37}$$

The distance squared between the two events A and B which is called the interval between A and B is defined by

$$\Delta s^2 = \Delta x_\mu \Delta x^\mu = c^2 \Delta t^2 - \Delta \vec{x}^2. \tag{1.38}$$

This is a Lorentz invariant quantity. However it could be positive, negative or zero.

In the case $\Delta s^2 > 0$ the interval is called timelike. There exists an inertial reference frame in which the two events occur at the same place and are only separated temporally.

In the case $\Delta s^2 < 0$ the interval is called spacelike. There exists an inertial reference frame in which the two events occur at the same time and are only separated in space.

In the case $\Delta s^2 = 0$ the interval is called lightlike. The two events are connected by a signal traveling at the speed of light.

1.1.5 Metric

The interval ds^2 between two infinitesimally close events A and B in spacetime with position 4−vectors x_A^μ and $x_B^\mu = x_A^\mu + dx^\mu$ is given by

$$\begin{aligned} ds^2 &= \sum_{\mu=0}^{3} (x_A - x_B)_\mu (x_A - x_B)^\mu \\ &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\ &= c^2 (dt)^2 - (d\vec{x})^2. \end{aligned} \tag{1.39}$$

We can also write this interval as (using also Einstein's summation convention)

$$\begin{aligned} ds^2 &= \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu \\ &= \sum_{\mu,\nu=0}^{3} \eta^{\mu\nu} dx_\mu dx_\nu = \eta^{\mu\nu} dx_\mu dx_\nu. \end{aligned} \tag{1.40}$$

The 4×4 matrix η is called the metric tensor and it is given by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.41}$$

Clearly we can also write

$$ds^2 = \sum_{\mu,\nu=0}^{3} \eta_\mu^\nu dx^\mu dx_\nu = \eta_\mu^\nu dx^\mu dx_\nu. \tag{1.42}$$

In this case

$$\eta_\mu^\nu = \delta_\mu^\nu. \tag{1.43}$$

The metric η is used to lower and raise Lorentz indices, viz

$$x_\mu = \eta_{\mu\nu} x^\nu. \tag{1.44}$$

The interval ds^2 is invariant under Poincare transformations which combine translations a with Lorentz transformations Λ:

$$x^\mu \longrightarrow x^{'\mu} = \Lambda^\mu_\nu x^\nu + a^\mu. \tag{1.45}$$

We compute

$$ds^2 = \eta_{\mu\nu} dx^{'\mu} dx^{'\nu} = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{1.46}$$

This leads to the condition

$$\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma} \Leftrightarrow \Lambda^T \eta \Lambda = \eta. \tag{1.47}$$

1.2 Klein-Gordon Equation

The non-relativistic energy-momentum relation reads

$$E = \frac{\vec{p}^2}{2m} + V. \tag{2.48}$$

The correspondence principle is

$$E \longrightarrow i\hbar\frac{\partial}{\partial t}\ ,\ \vec{p} \longrightarrow \frac{\hbar}{i}\vec{\nabla}. \tag{2.49}$$

This yields immediately the Schrodinger equation

$$\bigg(-\frac{\hbar^2}{2m}\nabla^2 + V\bigg)\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{2.50}$$

We will only consider the free case,i.e. $V = 0$. We have then

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{2.51}$$

The energy-momentum $4-$vector is given by

$$p^\mu = (p^0, p^1, p^2, p^3) = (\frac{E}{c}, \vec{p}). \tag{2.52}$$

The relativistic momentum and energy are defined by

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\ ,\ E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{2.53}$$

The energy-momentum $4-$vector satisfies

$$p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^2 = m^2c^2. \tag{2.54}$$

The relativistic energy-momentum relation is therefore given by

$$\vec{p}^2c^2 + m^2c^4 = E^2. \tag{2.55}$$

Thus the free Schrodinger equation will be replaced by the relativistic wave equation

$$(-\hbar^2c^2\nabla^2 + m^2c^4)\phi = -\hbar^2\frac{\partial^2\phi}{\partial t^2}. \tag{2.56}$$

This can also be rewritten as

$$\bigg(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2-\frac{m^2c^2}{\hbar^2}\bigg)\phi=0. \tag{2.57}$$

This is Klein-Gordon equation. In contrast with the Schrodinger equation the Klein-Gordon equation is a second-order differential equation. In relativistic notation we have

$$E\longrightarrow i\hbar\frac{\partial}{\partial t}\Leftrightarrow p_0\longrightarrow i\hbar\partial_0\ ,\ \partial_0=\frac{\partial}{\partial x^0}=\frac{1}{c}\frac{\partial}{\partial t}. \tag{2.58}$$

$$\vec{p}\longrightarrow\frac{\hbar}{i}\vec{\nabla}\Leftrightarrow p_i\longrightarrow i\hbar\partial_i\ ,\ \partial_i=\frac{\partial}{\partial x^i}. \tag{2.59}$$

In other words

$$p_\mu\longrightarrow i\hbar\partial_\mu\ ,\ \partial_\mu=\frac{\partial}{\partial x^\mu}. \tag{2.60}$$

$$p_\mu p^\mu\longrightarrow -\hbar^2\partial_\mu\partial^\mu=\hbar^2\bigg(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\bigg). \tag{2.61}$$

The covariant form of the Klein-Gordon equation is

$$\bigg(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2}\bigg)\phi=0. \tag{2.62}$$

Free solutions are of the form

$$\phi(t,\vec{x})=e^{-\frac{i}{\hbar}px}\ ,\ px=p_\mu x^\mu=Et-\vec{p}\vec{x}. \tag{2.63}$$

Indeed we compute

$$\partial_\mu\partial^\mu\phi(t,\vec{x})=-\frac{1}{c^2\hbar^2}(E^2-\vec{p}^2c^2)\phi(t,\vec{x}). \tag{2.64}$$

Thus we must have

$$E^2-\vec{p}^2c^2=m^2c^4. \tag{2.65}$$

In other words

$$E^2=\pm\sqrt{\vec{p}^2c^2+m^2c^4}. \tag{2.66}$$

There exists therefore negative-energy solutions. The energy gap is $2mc^2$. As it stands the existence of negative-energy solutions means that the spectrum is not bounded from below and as a consequence an arbitrarily large amount of energy can be extracted. This is a severe problem for a single-particle wave equation. However these negative-energy solutions, as we will see shortly, will be related to antiparticles.

From the two equations

$$\phi^*\bigg(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2}\bigg)\phi=0, \tag{2.67}$$

$$\phi\bigg(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2}\bigg)\phi^*=0, \tag{2.68}$$

we get the continuity equation

$$\partial^\mu J_\mu=0, \tag{2.69}$$

where

$$J_\mu=\frac{i\hbar}{2m}[\phi^*\partial_\mu\phi-\phi\partial_\mu\phi^*]. \tag{2.70}$$

We have included the factor $i\hbar/2m$ in order that the zero component J_0 has the dimension of a probability density. The continuity equation can also be put in the form

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\vec{J}=0, \tag{2.71}$$

where

$$\rho=\frac{J_0}{c}=\frac{i\hbar}{2mc^2}[\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}]. \tag{2.72}$$

$$\vec{J}=-\frac{i\hbar}{2mc}[\phi^*\vec{\nabla}\phi-\phi\vec{\nabla}\phi^*]. \tag{2.73}$$

Clearly the zero component J_0 is not positive definite and hence it can be a probability density. This is due to the fact that the Klein-Gordon equation is second-order.

The Dirac equation is a relativistic wave equation which is a first-order differential equation. The corresponding probability density will therefore be positive definite. However negative-energy solutions will still be present.

1.3 Dirac Equation

Dirac equation is a first-order differential equation of the same form as the Schrodinger equation, viz

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi. \tag{3.74}$$

In order to derive the form of the Hamiltonian H we go back to the relativistic energy-momentum relation

$$p_\mu p^\mu-m^2c^2=0. \tag{3.75}$$

The only requirement on H is that it must be linear in spatial derivatives since we want space and time to be on equal footing. We thus factor out the above equation as follows

$$\begin{aligned} p_\mu p^\mu-m^2c^2 &= (\gamma^\mu p_\mu+mc)(\beta^\nu p_\nu-mc)\\ &= \gamma^\mu\beta^\nu p_\mu p_\nu-mc(\gamma^\mu-\beta^\mu)p_\mu-m^2c^2. \end{aligned} \tag{3.76}$$

We must therefore have $\beta^\mu=\gamma^\mu$, i.e.

$$p_\mu p^\mu = \gamma^\mu\gamma^\nu p_\mu p_\nu. \tag{3.77}$$

This is equivalent to

$$\begin{aligned} p_0^2-p_1^2-p_2^2-p_3^2 &= (\gamma^0)^2p_0^2+(\gamma^1)^2p_1^2+(\gamma^2)^2p_2^2+(\gamma^3)^2p_3^2\\ &+ (\gamma^1\gamma^2+\gamma^2\gamma^1)p_1p_2+(\gamma^1\gamma^3+\gamma^3\gamma^1)p_1p_3+(\gamma^2\gamma^3+\gamma^3\gamma^2)p_2p_3\\ &+ (\gamma^1\gamma^0+\gamma^0\gamma^1)p_1p_0+(\gamma^2\gamma^0+\gamma^0\gamma^2)p_2p_0+(\gamma^3\gamma^0+\gamma^0\gamma^3)p_3p_0. \end{aligned} \tag{3.78}$$

Clearly the objects γ^μ can not be complex numbers since we must have

$$\begin{aligned} &(\gamma^0)^2=1\ ,\ (\gamma^1)^2=(\gamma^2)^2=(\gamma^3)^2=-1\\ &\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=0. \end{aligned} \tag{3.79}$$

These conditions can be rewritten in a compact form as

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\eta^{\mu\nu}. \tag{3.80}$$

This algebra is an example of a Clifford algebra and the solutions are matrices γ^μ which are called Dirac matrices. In four-dimensional Minkowski space the smallest Dirac matrices must be 4×4 matrices. All 4×4 representations are unitarily equivalent. We choose the so-called Weyl or chiral representation given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \ , \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{3.81}$$

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.82}$$

Remark that

$$(\gamma^0)^+ = \gamma^0 \ , \ (\gamma^i)^+ = -\gamma^i \Leftrightarrow (\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0. \tag{3.83}$$

The relativistic energy-momentum relation becomes

$$p_\mu p^\mu - m^2c^2 \ = \ (\gamma^\mu p_\mu + mc)(\gamma^\nu p_\nu - mc) = 0. \tag{3.84}$$

Thus either $\gamma^\mu p_\mu + mc = 0$ or $\gamma^\mu p_\mu - mc = 0$. The convention is to take

$$\gamma^\mu p_\mu - mc = 0. \tag{3.85}$$

By applying the correspondence principle $p_\mu \longrightarrow i\hbar\partial_\mu$ we obtain the relativistic wave equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \tag{3.86}$$

This is the Dirac equation in a covariant form. Let us introduce the Feynamn "slash" defined by

$$\partial\!\!\!/ = \gamma^\mu\partial_\mu. \tag{3.87}$$

$$(i\hbar\partial\!\!\!/ - mc)\psi = 0. \tag{3.88}$$

Since the γ matrices are 4×4 the wave function ψ must be a four-component object which we call a Dirac spinor. Thus we have

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{3.89}$$

The Hermitian conjugate of the Dirac equation (4.100) is

$$\psi^+(i\hbar(\gamma^\mu)^+\overleftarrow{\partial_\mu} + mc) = 0. \tag{3.90}$$

In other words

$$\psi^+(i\hbar\gamma^0\gamma^\mu\gamma^0\overleftarrow{\partial_\mu}+mc)=0. \tag{3.91}$$

The Hermitian conjugate of a Dirac spinor is not ψ^+ but it is defined by

$$\bar{\psi}=\psi^+\gamma^0. \tag{3.92}$$

Thus the Hermitian conjugate of the Dirac equation is

$$\bar{\psi}(i\hbar\gamma^\mu\overleftarrow{\partial_\mu}+mc)=0. \tag{3.93}$$

Equivalently

$$\bar{\psi}(i\hbar\overleftarrow{\partial\!\!\!/}+mc)=0. \tag{3.94}$$

Putting (3.88) and (3.94) together we obtain

$$\bar{\psi}(i\hbar\overleftarrow{\partial\!\!\!/}+i\hbar\overrightarrow{\partial\!\!\!/})\psi=0. \tag{3.95}$$

We obtain the continuity equation

$$\partial_\mu J^\mu=0\ ,\ J^\mu=\bar{\psi}\gamma^\mu\psi. \tag{3.96}$$

Explicitly we have

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\vec{J}=0. \tag{3.97}$$

$$\rho=\frac{J^0}{c}=\frac{1}{c}\bar{\psi}\gamma^0\psi=\frac{1}{c}\psi^+\psi. \tag{3.98}$$

$$\vec{J}=\bar{\psi}\vec{\gamma}\psi=\psi^+\vec{\alpha}\psi. \tag{3.99}$$

The probability density ρ is positive definite as desired.

1.4 Free Solutions of The Dirac Equation

We seek solutions of the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu-mc)\psi=0. \tag{4.100}$$

The plane-wave solutions are of the form

$$\psi(x) = a\ e^{-\frac{i}{\hbar}px}u(p). \tag{4.101}$$

Explicitly

$$\psi(t,\vec{x}) = a\ e^{-\frac{i}{\hbar}(Et-\vec{p}\vec{x})}u(E,\vec{p}). \tag{4.102}$$

The spinor $u(p)$ must satisfy

$$(\gamma^\mu p_\mu - mc)u = 0. \tag{4.103}$$

We write

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}. \tag{4.104}$$

We compute

$$\gamma^\mu p_\mu - mc = \begin{pmatrix} -mc & \frac{E}{c} - \vec{\sigma}\vec{p} \\ \frac{E}{c} + \vec{\sigma}\vec{p} & -mc \end{pmatrix}. \tag{4.105}$$

We get immediately

$$u_A = \frac{\frac{E}{c} - \vec{\sigma}\vec{p}}{mc}u_B. \tag{4.106}$$

$$u_B = \frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc}u_A. \tag{4.107}$$

A consistency condition is

$$u_A = \frac{\frac{E}{c} - \vec{\sigma}\vec{p}}{mc}\frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc}u_A = \frac{\frac{E^2}{c^2} - (\vec{\sigma}\vec{p})^2}{m^2c^2}u_A. \tag{4.108}$$

Thus one must have

$$\frac{E^2}{c^2} - (\vec{\sigma}\vec{p})^2 = m^2c^2 \Leftrightarrow E^2 = \vec{p}^2c^2 + m^2c^4. \tag{4.109}$$

Thus we have a single condition

$$u_B = \frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc}u_A. \tag{4.110}$$

There are four possible solutions. These are

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow u^{(1)} = N^{(1)} \begin{pmatrix} 1 \\ 0 \\ \frac{\frac{E}{c}+p^3}{mc} \\ \frac{p^1+ip^2}{mc} \end{pmatrix}. \quad (4.111)$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow u^{(4)} = N^{(4)} \begin{pmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{mc} \\ \frac{\frac{E}{c}-p^3}{mc} \end{pmatrix}. \quad (4.112)$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow u^{(3)} = N^{(3)} \begin{pmatrix} \frac{\frac{E}{c}-p^3}{mc} \\ -\frac{p^1+ip^2}{mc} \\ 1 \\ 0 \end{pmatrix}. \quad (4.113)$$

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow u^{(2)} = N^{(2)} \begin{pmatrix} -\frac{p^1-ip^2}{mc} \\ \frac{\frac{E}{c}+p^3}{mc} \\ 0 \\ 1 \end{pmatrix}. \quad (4.114)$$

The first and the fourth solutions will be normalized such that

$$\bar{u}u = u^+\gamma^0 u = u_A^+ u_B + u_B^+ u_A = 2mc. \quad (4.115)$$

We obtain

$$N^{(1)} = N^{(2)} = \sqrt{\frac{m^2c^2}{\frac{E}{c}+p^3}}. \quad (4.116)$$

Clearly one must have $E \geq 0$ otherwise the square root will not be well defined. In other words $u^{(1)}$ and $u^{(2)}$ correspond to positive-energy solutions associated with particles. The spinors $u^{(i)}(p)$ can be rewritten as

$$u^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\xi^i \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^i \end{pmatrix}. \quad (4.117)$$

The 2−dimensional spinors ξ^i satisfy

$$(\xi^r)^+\xi^s = \delta^{rs}. \quad (4.118)$$

The remaining spinors $u^{(3)}$ and $u^{(4)}$ must correspond to negative-energy solutions which must be reinterpreted as positive-energy antiparticles. Thus we flip the signs of the energy and the momentum such that the wave function (4.102) becomes

$$\psi(t,\vec{x}) \quad = \quad a\ e^{\frac{i}{\hbar}(Et-\vec{p}\vec{x})}u(-E,-\vec{p}). \quad (4.119)$$

The solutions u^3 and u^4 become

$$v^{(1)}(E,\vec{p}) = u^{(3)}(-E,-\vec{p}) = N^{(3)}\begin{pmatrix} -\frac{\frac{E}{c}-p^3}{mc} \\ \frac{p^1+ip^2}{mc} \\ 1 \\ 0 \end{pmatrix}\ ,\ v^{(2)}(E,\vec{p}) = u^{(4)}(-E,-\vec{p}) = N^{(4)}\begin{pmatrix} 0 \\ 1 \\ -\frac{p^1-ip^2}{mc} \\ -\frac{\frac{E}{c}-p^3}{mc} \end{pmatrix}. \quad (4.120)$$

We impose the normalization condition

$$\bar{v}v = v^+\gamma^0 v = v_A^+v_B + v_B^+v_A = -2mc. \quad (4.121)$$

We obtain

$$N^{(3)} = N^{(4)} = \sqrt{\frac{m^2c^2}{\frac{E}{c}-p^3}}. \quad (4.122)$$

The spinors $v^{(i)}(p)$ can be rewritten as

$$v^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\eta^i \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta^i \end{pmatrix}. \quad (4.123)$$

Again the 2−dimensional spinors η^i satisfy

$$(\eta^r)^+\eta^s = \delta^{rs}. \quad (4.124)$$

1.5 Lorentz Covariance

In this section we will refer to the Klein-Gordon wave function ϕ as a scalar field and to the Dirac wave function ψ as a Dirac spinor field although we are still thinking of them as quantum wave functions and not classical fields.

Scalar Fields: Let us recall that the set of all Lorentz transformations form a group called the Lorentz group. An arbitrary Lorentz transformation acts as

$$x^\mu \longrightarrow x^{'\mu} = \Lambda^\mu{}_\nu x^\nu. \tag{5.125}$$

In the inertial reference frame O the Klein-Gordon wave function is $\phi = \phi(x)$. It is a scalar field. Thus in the transformed reference frame $O^{'}$ the wave function must be $\phi^{'} = \phi^{'}(x^{'})$ where

$$\phi^{'}(x^{'}) = \phi(x). \tag{5.126}$$

For a one-component field this is the only possible linear transformation law. The Klein-Gordon equation in the reference frame $O^{'}$ if it holds is of the form

$$\left(\partial_\mu^{'}\partial^{'\mu} + \frac{m^2c^2}{\hbar^2}\right)\phi^{'}(x^{'}) = 0. \tag{5.127}$$

It is not difficult to show that

$$\partial_\mu^{'}\partial^{'\mu} = \partial_\mu\partial^\mu \tag{5.128}$$

The Klein-Gordon (5.127) becomes

$$\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0. \tag{5.129}$$

Vector Fields: Let $V^\mu = V^\mu(x)$ be an arbitrary vector field (for example $\partial^\mu\phi$ and the electromagnetic vector potential A^μ). Under Lorentz transformations it must transform as a 4−vector, i.e. as in (5.125) and hence

$$V^{'\mu}(x^{'}) = \Lambda^\mu_\nu V^\nu(x). \tag{5.130}$$

This should be contrasted with the transformation law of an ordinary vector field $V^i(x)$ under rotations in three dimensional space given by

$$V^{'i}(x^{'}) = R^{ij}V^j(x). \tag{5.131}$$

The group of rotations in three dimensional space is a continuous group. The set of infinitesimal transformations (the transformations near the identity) form a vector space which we call the Lie algebra of the group. The basis vectors of this vector space are called the generators of the Lie algebra and they are given by the angular momentum operators J^i which satisfy the commutation relations

$$[J^i, J^j] = i\hbar\epsilon^{ijk}J^k. \tag{5.132}$$

A rotation with an angle $|\theta|$ about the axis $\hat{\theta}$ is obtained by exponentiation, viz

$$R = e^{-i\theta^i J^i}. \tag{5.133}$$

The matrices R form an $n-$dimensional representation with $n = 2j+1$ where j is the spin quantum number. The angular momentum operators J^i are given by

$$J^i = -i\hbar\epsilon^{ijk}x^j\partial^k. \tag{5.134}$$

This is equivalent to

$$\begin{aligned} J^{ij} &= \epsilon^{ijk}J^k \\ &= -i\hbar(x^i\partial^j - x^j\partial^i). \end{aligned} \tag{5.135}$$

Generalization of this result to 4$-$dimensional Minkowski space yields the six generators of the Lorentz group given by

$$J^{\mu\nu} = -i\hbar(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{5.136}$$

We compute the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\hbar\bigg(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}\bigg). \tag{5.137}$$

A solution of (5.137) is given by the 4×4 matrices

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\hbar(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha). \tag{5.138}$$

Equivalently we can write this solution as

$$(\mathcal{J}^{\mu\nu})^\alpha{}_\beta = i\hbar(\eta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta\eta^{\nu\alpha}). \tag{5.139}$$

This representation is the 4$-$dimensional vector representation of the Lorentz group which is denoted by $(1/2, 1/2)$. It is an irreducible representation of the Lorentz group. A scalar field transforms in the trivial representation of the Lorentz group denoted by $(0,0)$. It remains to determine the transformation properties of spinor fields.

Spinor Fields We go back to the Dirac equation in the form

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \tag{5.140}$$

This equation is assumed to be covariant under Lorentz transformations and hence one must have the transformed equation

$$(i\hbar\gamma^{'\mu}\partial^{'}_{\mu} - mc)\psi^{'} = 0. \tag{5.141}$$

The Dirac γ matrices are assumed to be invariant under Lorentz transformations and thus

$$\gamma^{'}_{\mu} = \gamma_{\mu}. \tag{5.142}$$

The spinor ψ will be assumed to transform under Lorentz transformations linearly, namely

$$\psi(x) \longrightarrow \psi^{'}(x^{'}) = S(\Lambda)\psi(x). \tag{5.143}$$

Furthermore we have

$$\partial^{'}_{\nu} = (\Lambda^{-1})^{\mu}\ {}_{\nu}\partial_{\mu}. \tag{5.144}$$

Thus equation (5.141) is of the form

$$(i\hbar(\Lambda^{-1})^{\nu}\ {}_{\mu}S^{-1}(\Lambda)\gamma^{'\mu}S(\Lambda)\partial_{\nu} - mc)\psi = 0. \tag{5.145}$$

We can get immediately

$$(\Lambda^{-1})^{\nu}\ {}_{\mu}S^{-1}(\Lambda)\gamma^{'\mu}S(\Lambda) = \gamma^{\nu}. \tag{5.146}$$

Equivalently

$$(\Lambda^{-1})^{\nu}\ {}_{\mu}S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda) = \gamma^{\nu}. \tag{5.147}$$

This is the transformation law of the γ matrices under Lorentz transformations. Thus the γ matrices are invariant under the simultaneous rotations of the vector and spinor indices under Lorentz transformations. This is analogous to the fact that Pauli matrices σ^i are invariant under the simultaneous rotations of the vector and spinor indices under spatial rotations.

The matrix $S(\Lambda)$ form a 4−dimensional representation of the Lorentz group which is called the spinor representation. This representation is reducible and it is denoted by $(1/2, 0) \oplus (0, 1/2)$. It remains to find the matrix $S(\Lambda)$. We consider an infinitesimal Lorentz transformation

$$\Lambda = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\ ,\ \Lambda^{-1} = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}. \tag{5.148}$$

We can write $S(\Lambda)$ as

$$S(\Lambda) = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}\ ,\ S^{-1}(\Lambda) = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}. \tag{5.149}$$

The infinitesimal form of (5.147) is

$$-(\mathcal{J}^{\alpha\beta})^{\mu}{}_{\nu}\gamma_{\mu} = [\gamma_{\nu}, \Gamma^{\alpha\beta}]. \tag{5.150}$$

The fact that the index μ is rotated with $\mathcal{J}^{\alpha\beta}$ means that it is a vector index. The spinor indices are the matrix components of the γ matrices which are rotated with the generators $\Gamma^{\alpha\beta}$. A solution is given by

$$\Gamma^{\mu\nu} = \frac{i\hbar}{4}[\gamma^{\mu}, \gamma^{\nu}]. \tag{5.151}$$

Explicitly

$$\begin{aligned}\Gamma^{0i} &= \frac{i\hbar}{4}[\gamma^0, \gamma^i] = -\frac{i\hbar}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\ \Gamma^{ij} &= \frac{i\hbar}{4}[\gamma^i, \gamma^j] = -\frac{i\hbar}{4}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix} = \frac{\hbar}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.\end{aligned} \tag{5.152}$$

Clearly Γ^{ij} are the generators of rotations. They are the direct sum of two copies of the generators of rotation in three dimensional space. Thus immediately we conclude that Γ^{0i} are the generators of boosts.

1.6 Exercises and Problems

Scalar Product Show explicitly that the scalar product of two 4−vectors in spacetime is invariant under boosts. Show that the scalar product is then invariant under all Lorentz transformations.

Relativistic Mechanics

- Show that the proper time of a point particle -the proper time is the time measured by an inertial observer flying with the particle- is invariant under Lorentz transformations. We assume that the particle is moving with a velocity $\vec{u}$ with respect to an inertial observer O.
- Define the 4−vector velocity of the particle in spacetime. What is the spatial component.
- Define the energy-momentum 4−vector in spacetime and deduce the relativistic energy.
- Express the energy in terms of the momentum.
- Define the 4−vector force.

Einstein's Velocity Addition Rule Derive the velocity addition rule in special relativity.

Weyl Representation

- Show that the Weyl representation of Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} , \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

  solves Dirac-Clifford algebra.

- Show that

$$(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0.$$

- Show that the Dirac equation can be put in the form of a schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi,$$

  with some Hamiltonian H.

Lorentz Invariance of the D'Alembertian Show that

$$\eta = \Lambda^T \eta \Lambda.$$

$$\Lambda_\rho{}^\mu = (\Lambda^{-1})^\mu{}_\rho.$$

$$\partial'_\nu = (\Lambda^{-1})^\mu{}_\nu \partial_\mu.$$

$$\partial'_\mu \partial'^\mu = \partial_\mu \partial^\mu.$$

Covariance of the Klein-Gordon equation Show that the Klein-Gordon equation is covariant under Lorentz transformations.

Vector Representations

- Write down the transformation property under ordinary rotations of a vector in three dimensions. What are the generators J^i. What is the dimensions of the irreducible representations and the corresponding quantum numbers.

- The generators of rotation can be alternatively given by

$$J^{ij} = \epsilon^{ijk} J^k.$$

  Calculate the commutators $[J^{ij}, J^{kl}]$.

- Write down the generators of the Lorentz group $J^{\mu\nu}$ by simply generalizing J^{ij} and show that

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\hbar\bigg(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}\bigg).$$

- Verify that

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\hbar(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha),$$

  is a solution. This is called the vector representation of the Lorentz group.

- Write down a finite Lorentz transformation matrix in the vector representation. Write down an infinitesimal rotation in the $xy-$plane and an infinitesimal boost along the $x-$axis.

Dirac Spinors

- Introduce $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. Show that

$$(\sigma_\mu p^\mu)(\bar{\sigma}_\mu p^\mu) = m^2c^2.$$

- Show that the normalization condition $\bar{u}u = 2mc$ for $u^{(1)}$ and $u^{(2)}$ yields

$$N^{(1)} = N^{(2)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} + p^3}}.$$

- Show that the normalization condition $\bar{v}v = -2mc$ for $v^{(1)}(p) = u^{(3)}(-p)$ and $v^{(2)}(p) = u^{(4)}(-p)$ yields

$$N^{(3)} = N^{(4)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} - p^3}}.$$

- Show that we can rewrite the spinors u and v as

$$u^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\xi^i \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^i \end{pmatrix}.$$

$$v^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\eta^i \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta^i \end{pmatrix}.$$

Determine ξ^i and η^i.

Spin Sums Let $u^{(r)}(p)$ and $v^{(r)}(p)$ be the positive-energy and negative-energy solutions of the free Dirac equation. Show that

-
$$\bar{u}^{(r)}u^{(s)} = 2mc\delta^{rs} \ , \ \bar{v}^{(r)}v^{(s)} = -2mc\delta^{rs} \ , \ \bar{u}^{(r)}v^{(s)} = 0 \ , \ \bar{v}^{(r)}u^{(s)} = 0.$$

-
$$u^{(r)+}u^{(s)} = \frac{2E}{c}\delta^{rs} \ , \ v^{(r)+}v^{(s)} = \frac{2E}{c}\delta^{rs}.$$

$$u^{(r)+}(E,\vec{p})v^{(s)}(E,-\vec{p}) = 0 \ , \ v^{(r)+}(E,-\vec{p})u^{(s)}(E,\vec{p}) = 0.$$

-
$$\sum_{s=1}^{2} u^{(s)}(E,\vec{p})\bar{u}^{(s)}(E,\vec{p}) = \gamma^\mu p_\mu + mc \ , \ \sum_{s=1}^{2} v^{(s)}(E,\vec{p})\bar{v}^{(s)}(E,\vec{p}) = \gamma^\mu p_\mu - mc.$$

Covariance of the Dirac Equation Determine the transformation property of the spinor ψ under Lorentz transformations in order that the Dirac equation is covariant.

Spinor Bilinears Determine the transformation rule under Lorentz transformations of $\bar{\psi}$, $\bar{\psi}\psi$, $\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\gamma^\mu\gamma^5\psi$ and $\bar{\psi}\Gamma^{\mu\nu}\psi$.

Clifford Algebra

- Write down the solution of the Clifford algebra in three Euclidean dimensions. Construct a basis for 2×2 matrices in terms of Pauli matrices.
- Construct a basis for 4×4 matrices in terms of Dirac matrices.
  Hint: Show that there are 16 antisymmetric combinations of the Dirac gamma matrices in $1+3$ dimensions.

Chirality Operator and Weyl Fermions

- We define the gamma five matrix (chirality operator) by
  $$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$
  Show that
  $$\gamma^5 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$
  $$(\gamma^5)^2 = 1.$$
  $$(\gamma^5)^+ = \gamma^5.$$
  $$\{\gamma^5,\gamma^\mu\} = 0.$$
  $$[\gamma^5,\Gamma^{\mu\nu}] = 0.$$
- We write the Dirac spinor as
  $$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$
  By working in the Weyl representation show that Dirac representation is reducible.
  Hint: Compute the eigenvalues of γ^5 and show that they do not mix under Lorentz transformations.
- Rewrite Dirac equation in terms of ψ_L and ψ_R. What is their physical interpretation.

1.7 Solutions

Scalar Product Straightforward.

Relativistic Mechanics

- The trajectory of a particle in spacetime is called a world line. We take two infinitesimally close points on the world line given by (x^0, x^1, x^2, x^3) and $(x^0 + dx^0, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$. Clearly $dx^1 = u^1 dt$, $dx^2 = u^2 dt$ and $dx^3 = u^3 dt$ where $\vec{u}$ is the velocity of the particle measured with respect to the observer O, viz

$$\vec{u} = \frac{d\vec{x}}{dt}.$$

The interval with respect to O is given by

$$ds^2 = -c^2 dt^2 + d\vec{x}^2 = (-c^2 + u^2)dt^2.$$

Let $O^{'}$ be the observer or inertial reference frame moving with respect to O with the velocity $\vec{u}$. We stress here that $\vec{u}$ is thought of as a constant velocity only during the infinitesimal time interval dt. The interval with respect to $O^{'}$ is given by

$$ds^2 = -c^2 d\tau^2. \tag{7.153}$$

Hence

$$d\tau = \sqrt{1 - \frac{u^2}{c^2}} dt.$$

The time interval $d\tau$ measured with respect to $O^{'}$ which is the observer moving with the particle is the proper time of the particle.

- The 4−vector velocity η is naturally defined by the components

$$\eta^\mu = \frac{dx^\mu}{d\tau}.$$

The spatial part of η is precisely the proper velocity $\vec{\eta}$ defined by

$$\vec{\eta} = \frac{d\vec{x}}{d\tau} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \vec{u}.$$

The temporal part is

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{c}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

- The law of conservation of momentum and the principle of relativity put together forces us to define the momentum in relativity as mass times the proper velocity and not mass time the ordinary velocity, viz

$$\vec{p} = m\vec{\eta} = m\frac{d\vec{x}}{d\tau} = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}\vec{u}.$$

This is the spatial part of the 4−vector momentum

$$p^\mu = m\eta^\mu = m\frac{dx^\mu}{d\tau}.$$

The temporal part is

$$p^0 = m\eta^0 = m\frac{dx^0}{d\tau} = \frac{mc}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{E}{c}.$$

The relativistic energy is defined by

$$E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}.$$

The 4−vector p^μ is called the energy-momentum 4−vector.

- We note the identity

$$p_\mu p^\mu = -\frac{E^2}{c^2} + \vec{p}^2 = -m^2c^2.$$

Thus

$$E = \sqrt{\vec{p}^2c^2 + m^2c^4}.$$

The rest mass is m and the rest energy is clearly defined by

$$E_0 = mc^2.$$

- The first law of Newton is automatically satisfied because of the principle of relativity. The second law takes in the theory of special relativity the usual form provided we use the relativistic momentum, viz

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

The third law of Newton does not in general hold in the theory of special relativity.

We can define a 4−vector proper force which is called the Minkowski force by the following equation

$$K^\mu = \frac{dp^\mu}{d\tau}.$$

The spatial part is

$$\vec{K} = \frac{d\vec{p}}{d\tau} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\vec{F}.$$

Einstein's Velocity Addition Rule We consider a particle in the reference frame O moving a distance dx in the x direction during a time interval dt. The velocity with respect to O is

$$u=\frac{dx}{dt}.$$

In the reference frame $O^{'}$ the particle moves a distance $dx^{'}$ in a time interval $dt^{'}$ given by

$$dx^{'}=\gamma(dx-vdt).$$

$$dt^{'}=\gamma(dt-\frac{v}{c^2}dx).$$

The velocity with respect to $O^{'}$ is therefore

$$u^{'}=\frac{dx^{'}}{dt^{'}}=\frac{u-v}{1-\frac{vu}{c^2}}.$$

In general if $\vec{V}$ and $\vec{V}'$ are the velocities of the particle with respect to O and $O^{'}$ respectively and $\vec{v}$ is the velocity of $O^{'}$ with respect to O. Then

$$\vec{V}'=\frac{\vec{V}-\vec{v}}{1-\frac{\vec{V}\vec{v}}{c^2}}.$$

Weyl Representation

- Straightforward.
- Straightforward.
- The Dirac equation can trivially be put in the form

  $$i\hbar\frac{\partial\psi}{\partial t}=(\frac{\hbar c}{i}\gamma^0\gamma^i\partial_i+mc^2\gamma^0)\psi. \tag{7.154}$$

  The Dirac Hamiltonian is

  $$H=\frac{\hbar c}{i}\vec{\alpha}\vec{\nabla}+mc^2\beta\ ,\ \alpha^i=\gamma^0\gamma^i\ ,\ \beta=\gamma^0. \tag{7.155}$$

  This is a Hermitian operator as it should be.

Lorentz Invariance of the D'Alembertian The invariance of the interval under Lorentz transformations reads

$$\eta_{\mu\nu}x^{\mu}x^{\nu} = \eta_{\mu\nu}x^{'\mu}x^{'\nu} = \eta_{\mu\nu}\Lambda^{\mu}\ _{\rho}x^{\rho}\Lambda^{\nu}\ _{\lambda}x^{\lambda}.$$

This leads immediately to

$$\eta = \Lambda^{T}\eta\Lambda.$$

Explicitly we write this as

$$\begin{aligned}\eta^{\mu}_{\nu} &= \Lambda_{\rho}\ ^{\mu}\eta^{\rho}_{\beta}\Lambda^{\beta}\ _{\nu}\\ &= \Lambda_{\rho}\ ^{\mu}\Lambda^{\rho}\ _{\nu}.\end{aligned}$$

But we also have

$$\delta^{\mu}_{\nu} = (\Lambda^{-1})^{\mu}\ _{\rho}\Lambda^{\rho}\ _{\nu}.$$

In other words

$$\Lambda_{\rho}\ ^{\mu} = (\Lambda^{-1})^{\mu}\ _{\rho}.$$

Since $x^{\mu} = (\Lambda^{-1})^{\mu}\ _{\nu}x^{'\nu}$ we have

$$\frac{\partial x^{\mu}}{\partial x^{'\nu}} = (\Lambda^{-1})^{\mu}\ _{\nu}.$$

Hence

$$\partial^{'}_{\nu} = (\Lambda^{-1})^{\mu}\ _{\nu}\partial_{\mu}.$$

Thus

$$\begin{aligned}\partial^{'}_{\mu}\partial^{'\mu} &= \eta^{\mu\nu}\partial^{'}_{\mu}\partial^{'}_{\nu}\\ &= \eta^{\mu\nu}(\Lambda^{-1})^{\rho}\ _{\mu}(\Lambda^{-1})^{\lambda}\ _{\nu}\partial_{\rho}\partial_{\lambda}\\ &= \eta^{\mu\nu}\Lambda_{\mu}\ ^{\rho}\Lambda_{\nu}\ ^{\lambda}\partial_{\rho}\partial_{\lambda}\\ &= (\Lambda^{T}\eta\Lambda)^{\rho\lambda}\partial_{\rho}\partial_{\lambda}\\ &= \partial_{\mu}\partial^{\mu}.\end{aligned}$$

Covariance of the Klein-Gordon equation Straightforward.

Vector Representations

- We have

$$V^{'i}(x^{'}) = R^{ij}V^{j}(x).$$

The generators are given by the angular momentum operators J^i which satisfy the commutation relations

$$[J^i, J^j] = i\hbar\epsilon^{ijk}J^k.$$

Thus a rotation with an angle $|\theta|$ about the axis $\hat{\theta}$ is obtained by exponentiation, viz

$$R = e^{-i\theta^i J^i}.$$

The matrices R form an $n-$dimensional representation with $n = 2j + 1$ where j is the spin quantum number. The quantum numbers are therefore given by j and m.

- The angular momentum operators J^i are given by

$$J^i = -i\hbar\epsilon^{ijk}x^j\partial^k.$$

  Thus

$$J^{ij} = \epsilon^{ijk}J^k = -i\hbar(x^i\partial^j - x^j\partial^i).$$

  We compute

$$[J^{ij}, J^{kl}] = i\hbar\bigg(\eta^{jk}J^{il} - \eta^{ik}J^{jl} - \eta^{jl}J^{ik} + \eta^{il}J^{jk}\bigg).$$

- Generalization to 4−dimensional Minkowski space yields

$$J^{\mu\nu} = -i\hbar(x^\mu\partial^\nu - x^\nu\partial^\mu).$$

  Now we compute the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\hbar\bigg(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}\bigg).$$

- A solution of is given by the 4×4 matrices

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\hbar(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha).$$

  Equivalently

$$(\mathcal{J}^{\mu\nu})^\alpha{}_\beta = i\hbar(\eta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta\eta^{\nu\alpha}).$$

We compute

$$(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}(\mathcal{J}^{\rho\sigma})^{\beta}{}_{\lambda}=(i\hbar)^2\bigg(\eta^{\mu\alpha}\eta^{\rho\nu}\delta^{\sigma}_{\lambda}-\eta^{\mu\alpha}\eta^{\sigma\nu}\delta^{\rho}_{\lambda}-\eta^{\nu\alpha}\eta^{\rho\mu}\delta^{\sigma}_{\lambda}+\eta^{\nu\alpha}\eta^{\sigma\mu}\delta^{\rho}_{\lambda}\bigg).$$

$$(\mathcal{J}^{\rho\sigma})^{\alpha}{}_{\beta}(\mathcal{J}^{\mu\nu})^{\beta}{}_{\lambda}=(i\hbar)^2\bigg(\eta^{\rho\alpha}\eta^{\mu\sigma}\delta^{\nu}_{\lambda}-\eta^{\rho\alpha}\eta^{\sigma\nu}\delta^{\mu}_{\lambda}-\eta^{\sigma\alpha}\eta^{\rho\mu}\delta^{\nu}_{\lambda}+\eta^{\sigma\alpha}\eta^{\nu\rho}\delta^{\mu}_{\lambda}\bigg).$$

Hence

$$\begin{aligned}[\mathcal{J}^{\mu\nu},\mathcal{J}^{\rho\sigma}]^{\alpha}{}_{\lambda} &= (i\hbar)^2\bigg(\eta^{\mu\sigma}\left[\eta^{\nu\alpha}\delta^{\rho}_{\lambda}-\eta^{\rho\alpha}\delta^{\nu}_{\lambda}\right]-\eta^{\nu\sigma}\left[\eta^{\mu\alpha}\delta^{\rho}_{\lambda}-\eta^{\rho\alpha}\delta^{\mu}_{\lambda}\right]-\eta^{\mu\rho}\left[\eta^{\nu\alpha}\delta^{\sigma}_{\lambda}-\eta^{\sigma\alpha}\delta^{\nu}_{\lambda}\right]\\ &+ \eta^{\nu\rho}\left[\eta^{\mu\alpha}\delta^{\sigma}_{\lambda}-\eta^{\sigma\alpha}\delta^{\mu}_{\lambda}\right]\bigg)\\ &= i\hbar\bigg[\eta^{\mu\sigma}(\mathcal{J}^{\nu\rho})^{\alpha}{}_{\lambda}-\eta^{\nu\sigma}(\mathcal{J}^{\mu\rho})^{\alpha}{}_{\lambda}-\eta^{\mu\rho}(\mathcal{J}^{\nu\sigma})^{\alpha}{}_{\lambda}+\eta^{\nu\rho}(\mathcal{J}^{\mu\sigma})^{\alpha}{}_{\lambda}\bigg].\end{aligned}$$

- A finite Lorentz transformation in the vector representation is

$$\Lambda=e^{-\frac{i}{2\hbar}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}.$$

$\omega_{\mu\nu}$ is an antisymmetric tensor. An infinitesimal transformation is given by

$$\Lambda=1-\frac{i}{2\hbar}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}.$$

A rotation in the $xy-$plane corresponds to $\omega_{12}=-\omega_{21}=-\theta$ while the rest of the components are zero, viz

$$\Lambda^{\alpha}{}_{\beta}=(1+\frac{i}{\hbar}\theta\mathcal{J}^{12})^{\alpha}{}_{\beta}=\begin{pmatrix}1&0&0&0\\0&1&\theta&0\\0&-\theta&1&0\\0&0&0&1\end{pmatrix}.$$

A boost in the $x-$direction corresponds to $\omega_{01}=-\omega_{10}=-\beta$ while the rest of the components are zero, viz

$$\Lambda^{\alpha}{}_{\beta}=(1+\frac{i}{\hbar}\beta\mathcal{J}^{01})^{\alpha}{}_{\beta}=\begin{pmatrix}1&-\beta&0&0\\-\beta&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Dirac Spinors

- We compute

$$\sigma_\mu p^\mu = \frac{E}{c} - \vec{\sigma}\vec{p} = \begin{pmatrix} \frac{E}{c} - p^3 & -(p^1 - ip^2) \\ -(p^1 + ip^2) & \frac{E}{c} + p^3 \end{pmatrix}.$$

$$\bar{\sigma}_\mu p^\mu = \frac{E}{c} + \vec{\sigma}\vec{p} = \begin{pmatrix} \frac{E}{c} + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & \frac{E}{c} - p^3 \end{pmatrix}.$$

Thus

$$(\sigma_\mu p^\mu)(\bar{\sigma}_\mu p^\mu) = m^2c^2.$$

- Recall the four possible solutions:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow u^{(1)} = N^{(1)} \begin{pmatrix} 1 \\ 0 \\ \frac{\frac{E}{c}+p^3}{mc} \\ \frac{p^1+ip^2}{mc} \end{pmatrix}.$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow u^{(4)} = N^{(4)} \begin{pmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{mc} \\ \frac{\frac{E}{c}-p^3}{mc} \end{pmatrix}.$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow u^{(3)} = N^{(3)} \begin{pmatrix} \frac{\frac{E}{c}-p^3}{mc} \\ -\frac{p^1+ip^2}{mc} \\ 1 \\ 0 \end{pmatrix}.$$

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow u^{(2)} = N^{(2)} \begin{pmatrix} -\frac{p^1-ip^2}{mc} \\ \frac{\frac{E}{c}+p^3}{mc} \\ 0 \\ 1 \end{pmatrix}.$$

The normalization condition is

$$\bar{u}u = u^+\gamma^0 u = u_A^+ u_B + u_B^+ u_A = 2mc.$$

We obtain immediately

$$N^{(1)} = N^{(2)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} + p^3}}.$$

- Recall that

$$v^{(1)}(E,\vec{p}) = u^{(3)}(-E,-\vec{p}) = N^{(3)}\left(\begin{array}{c} -\frac{\frac{E}{c}-p^3}{mc} \\ \frac{p^1+ip^2}{mc} \\ 1 \\ 0 \end{array}\right),$$

$$v^{(2)}(E,\vec{p}) = u^{(4)}(-E,-\vec{p}) = N^{(4)}\left(\begin{array}{c} 0 \\ 1 \\ -\frac{p^1-ip^2}{mc} \\ -\frac{\frac{E}{c}-p^3}{mc} \end{array}\right).$$

The normalization condition in this case is

$$\bar{v}v = v^+\gamma^0 v = v_A^+ v_B + v_B^+ v_A = -2mc.$$

We obtain now

$$N^{(3)} = N^{(4)} = \sqrt{\frac{m^2c^2}{\frac{E}{c}-p^3}}.$$

- Let us define

$$\xi_0^1 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \ , \ \xi_0^2 = \left(\begin{array}{c} 0 \\ 1 \end{array}\right).$$

We have

$$u^{(1)} = N^{(1)}\left(\begin{array}{c} \xi_0^1 \\ \frac{\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}\xi_0^1 \end{array}\right) = N^{(1)}\frac{1}{\sqrt{\sigma_\mu p^\mu}}\left(\begin{array}{c} \sqrt{\sigma_\mu p^\mu}\xi_0^1 \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi_0^1 \end{array}\right) = \left(\begin{array}{c} \sqrt{\sigma_\mu p^\mu}\xi^1 \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^1 \end{array}\right).$$

$$u^{(2)} = N^{(2)}\left(\begin{array}{c} \frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\xi_0^2 \\ \xi_0^2 \end{array}\right) = N^{(2)}\frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}}\left(\begin{array}{c} \sqrt{\sigma_\mu p^\mu}\xi_0^2 \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi_0^2 \end{array}\right) = \left(\begin{array}{c} \sqrt{\sigma_\mu p^\mu}\xi^2 \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^2 \end{array}\right).$$

The spinors ξ^1 and ξ^2 are defined by

$$\xi^1 = N^{(1)}\frac{1}{\sqrt{\sigma_\mu p^\mu}}\xi_0^1 = \sqrt{\frac{\bar{\sigma}_\mu p^\mu}{\frac{E}{c}+p^3}}\xi_0^1.$$

$$\xi^2 = N^{(2)}\frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}}\xi_0^2 = \sqrt{\frac{\sigma_\mu p^\mu}{\frac{E}{c}+p^3}}\xi_0^2.$$

They satisfy

$$(\xi^r)^+\xi^s = \delta^{rs}.$$

Similarly let us define

$$\eta_0^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ \eta_0^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we have

$$v^{(1)} = N^{(3)} \begin{pmatrix} -\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\eta_0^1 \\ \eta_0^1 \end{pmatrix} = -N^{(3)}\frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\eta_0^1 \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta_0^1 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\eta^1 \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta^1 \end{pmatrix}.$$

$$v^{(2)} = N^{(4)} \begin{pmatrix} \eta_0^2 \\ -\frac{\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}\eta_0^2 \end{pmatrix} = N^{(4)}\frac{1}{\sqrt{\sigma_\mu p^\mu}} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\eta_0^2 \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta_0^2 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\eta^2 \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta^2 \end{pmatrix}.$$

$$\eta^1 = -N^{(3)}\frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}}\eta_0^1 = -\sqrt{\frac{\sigma_\mu p^\mu}{\frac{E}{c}-p^3}}\eta_0^1.$$

$$\eta^2 = N^{(4)}\frac{1}{\sqrt{\sigma_\mu p^\mu}}\eta_0^2 = \sqrt{\frac{\bar{\sigma}_\mu p^\mu}{\frac{E}{c}-p^3}}\eta_0^2.$$

Again they satisfy

$$(\eta^r)^+\eta^s = \delta^{rs}.$$

Spin Sums

- We have

$$u^{(r)}(E,\vec{p}) = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\xi^r \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^r \end{pmatrix} \ , \ v^{(r)}(E,\vec{p}) = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\eta^r \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta^r \end{pmatrix}.$$

We compute

$$\bar{u}^{(r)}u^{(s)} = u^{(r)+}\gamma^0 u^{(s)} = 2\xi^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\xi^s = 2mc\xi^{r+}\xi^s = 2mc\delta^{rs}.$$

$$\bar{v}^{(r)}v^{(s)} = v^{(r)+}\gamma^0 v^{(s)} = -2\eta^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\eta^s = -2mc\eta^{r+}\eta^s = -2mc\delta^{rs}.$$

We have used

$$(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu) = m^2c^2.$$

$$\xi^{r+}\xi^s = \delta^{rs} \ , \ \eta^{r+}\eta^s = \delta^{rs}.$$

We also compute

$$\bar{u}^{(r)}v^{(s)} = u^{(r)+}\gamma^0 v^{(s)} = -\xi^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\eta^s + \xi^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\eta^s = 0.$$

A similar calculation yields

$$\bar{v}^{(r)}u^{(s)} = u^{(r)+}\gamma^0 v^{(s)} = 0.$$

- Next we compute

$$u^{(r)+}u^{(s)} = \xi^{r+}(\sigma_\mu p^\mu + \bar{\sigma}_\mu p^\mu)\xi^s = \frac{2E}{c}\xi^{r+}\xi^s = \frac{2E}{c}\delta^{rs}.$$

$$v^{(r)+}v^{(s)} = \eta^{r+}(\sigma_\mu p^\mu + \bar{\sigma}_\mu p^\mu)\eta^s = \frac{2E}{c}\eta^{r+}\eta^s = \frac{2E}{c}\delta^{rs}.$$

We have used

$$\sigma^\mu = (1, \sigma^i) \ , \ \sigma^\mu = (1, -\sigma^i).$$

We also compute

$$u^{(r)+}(E,\vec{p})v^{(s)}(E,-\vec{p}) = \xi^{r+}(\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)} - \sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)})\xi^s = 0.$$

Similarly we compute that

$$v^{(r)+}(E,-\vec{p})u^{(s)}(E,\vec{p}) = 0.$$

In the above two equation we have used the fact that

$$v^{(r)}(E,-\vec{p}) = \begin{pmatrix} \sqrt{\bar{\sigma}_\mu p^\mu}\eta^r \\ -\sqrt{\sigma_\mu p^\mu}\eta^r \end{pmatrix}.$$

- Next we compute

$$\begin{aligned}
\sum_s u^{(s)}(E,\vec{p})\bar{u}^{(s)}(E,\vec{p}) &= \sum_s u^{(s)}(E,\vec{p})u^{(s)+}(E,\vec{p})\gamma^0 \\
&= \sum_s \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\xi^s\xi^{s+}\sqrt{\sigma_\mu p^\mu} & \sqrt{\sigma_\mu p^\mu}\xi^s\xi^{s+}\sqrt{\bar{\sigma}_\mu p^\mu} \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^s\xi^{s+}\sqrt{\sigma_\mu p^\mu} & \sqrt{\bar{\sigma}_\mu p^\mu}\xi^s\xi^{s+}\sqrt{\bar{\sigma}_\mu p^\mu} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

We use

$$\sum_s \xi^s\xi^{s+} = 1.$$

We obtain

$$\sum_s u^{(s)}(E,\vec{p})\bar{u}^{(s)}(E,\vec{p}) = \begin{pmatrix} mc & \sigma_\mu p^\mu \\ \bar{\sigma}_\mu p^\mu & mc \end{pmatrix} = \gamma^\mu p_\mu + mc.$$

Similarly we use

$$\sum_s \eta^s \eta^{s+} = 1,$$

to calculate

$$\sum_s v^{(s)}(E,\vec{p})\bar{v}^{(s)}(E,\vec{p}) = \begin{pmatrix} -mc & \sigma_\mu p^\mu \\ \bar{\sigma}_\mu p^\mu & -mc \end{pmatrix} = \gamma^\mu p_\mu - mc.$$

Covariance of the Dirac Equation Under Lorentz transformations we have the following transformation laws

$$\psi(x) \longrightarrow \psi^{'}(x^{'}) = S(\Lambda)\psi(x).$$

$$\gamma_\mu \longrightarrow \gamma_\mu^{'} = \gamma_\mu.$$

$$\partial_\mu \longrightarrow \partial_\nu^{'} = (\Lambda^{-1})^\mu{}_\nu \partial_\mu.$$

Thus the Dirac equation $(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$ becomes

$$(i\hbar\gamma^{'\mu}\partial_\mu^{'} - mc)\psi^{'} = 0,$$

or equivalently

$$(i\hbar(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^{'\mu}S(\Lambda)\partial_\nu - mc)\psi = 0.$$

We must have therefore

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu,$$

or equivalently

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu.$$

We consider an infinitesimal Lorentz transformation

$$\Lambda = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta} \ , \ \Lambda^{-1} = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}.$$

The corresponding $S(\Lambda)$ must also be infinitesimal of the form

$$S(\Lambda) = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta} \ , \ S^{-1}(\Lambda) = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}.$$

By substitution we get

$$-(\mathcal{J}^{\alpha\beta})^{\mu}{}_{\nu}\gamma_{\mu}=[\gamma_{\nu},\Gamma^{\alpha\beta}].$$

Explicitly this reads

$$-i\hbar(\delta^{\beta}_{\nu}\gamma^{\alpha}-\delta^{\alpha}_{\nu}\gamma^{\beta})=[\gamma_{\nu},\Gamma^{\alpha\beta}],$$

or equivalently

$$\begin{aligned}[]
[\gamma_0,\Gamma^{0i}]&=i\hbar\gamma^{i}\\
[\gamma_j,\Gamma^{0i}]&=-i\hbar\delta^{i}_{j}\gamma^{0}\\
[\gamma_0,\Gamma^{ij}]&=0\\
[\gamma_k,\Gamma^{ij}]&=-i\hbar(\delta^{j}_{k}\gamma^{i}-\delta^{i}_{k}\gamma^{j}).
\end{aligned}$$

A solution is given by

$$\Gamma^{\mu\nu}=\frac{i\hbar}{4}[\gamma^{\mu},\gamma^{\nu}].$$

Spinor Bilinears The Dirac spinor ψ changes under Lorentz transformations as

$$\psi(x)\longrightarrow\psi^{'}(x^{'})=S(\Lambda)\psi(x).$$

$$S(\Lambda)=e^{-\frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}}.$$

Since $(\gamma^{\mu})^{+}=\gamma^0\gamma^{\mu}\gamma^0$ we get $(\Gamma^{\mu\nu})^{+}=\gamma^0\Gamma^{\mu\nu}\gamma^0$. Therefore

$$S(\Lambda)^{+}=\gamma^0S(\Lambda)^{-1}\gamma^0.$$

In other words

$$\bar{\psi}(x)\longrightarrow\bar{\psi}^{'}(x^{'})=\bar{\psi}(x)S(\Lambda)^{-1}.$$

As a consequence

$$\bar{\psi}\psi\longrightarrow\bar{\psi}^{'}\psi^{'}=\bar{\psi}\psi.$$

$$\bar{\psi}\gamma^5\psi\longrightarrow\bar{\psi}^{'}\gamma^5\psi^{'}=\bar{\psi}\psi.$$

$$\bar{\psi}\gamma^{\mu}\psi\longrightarrow\bar{\psi}^{'}\gamma^{\mu}\psi^{'}=\Lambda^{\mu}{}_{\nu}\bar{\psi}\gamma^{\nu}\psi.$$

$$\bar{\psi}\gamma^{\mu}\gamma^5\psi\longrightarrow\bar{\psi}^{'}\gamma^{\mu}\gamma^5\psi^{'}=\Lambda^{\mu}{}_{\nu}\bar{\psi}\gamma^{\nu}\gamma^5\psi.$$

We have used $[\gamma^5,\Gamma^{\mu\nu}]=0$ and $S^{-1}\gamma^{\mu}S=\Lambda^{\mu}{}_{\nu}\gamma^{\nu}$. Finally we compute

$$\begin{aligned}
\bar{\psi}\Gamma^{\mu\nu}\psi\longrightarrow\bar{\psi}^{'}\Gamma^{\mu\nu}\psi^{'}&=\bar{\psi}S^{-1}\Gamma^{\mu\nu}S\psi\\
&=\bar{\psi}\frac{i\hbar}{4}[S^{-1}\gamma^{\mu}S,S^{-1}\gamma^{\nu}S]\psi\\
&=\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\bar{\psi}\Gamma^{\alpha\beta}\psi.
\end{aligned}$$

Clifford Algebra

- The Clifford algebra in three Euclidean dimensions is solved by Pauli matrices, viz

$$\{\gamma^i,\gamma^j\}=2\delta^{ij}\ ,\ \gamma^i\equiv\sigma^i.$$

  Any 2×2 matrix can be expanded in terms of the Pauli matrices and the identity. In other words

$$M_{2\times 2}=M_0\mathbf{1}+M_i\sigma_i.$$

- Any 4×4 matrix can be expanded in terms of a 16 antisymmetric combinations of the Dirac gamma matrices.

  The 4−dimensional identity and the Dirac matrices provide the first five independent 4×4 matrices. The product of two Dirac gamma matrices yield six different matrices which because of $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$ can be encoded in the six matrices $\Gamma^{\mu\nu}$ defined by

$$\Gamma^{\mu\nu}=\frac{i\hbar}{4}[\gamma^\mu,\gamma^\nu].$$

  There are four independent 4×4 matrices formed by the product of three Dirac gamma matrices. They are

$$\gamma^0\gamma^1\gamma^2\ ,\ \gamma^0\gamma^1\gamma^3\ ,\ \gamma^0\gamma^2\gamma^3\ ,\ \gamma^1\gamma^2\gamma^3.$$

  These can be rewritten as

$$i\epsilon^{\mu\nu\alpha\beta}\gamma_\beta\gamma^5.$$

  The product of four Dirac gamma matrices leads to an extra independent 4×4 matrix which is precisely the gamma five matrix. In total there are $1+4+6+4+1=16$ antisymmetric combinations of Dirac gamma matrices. Hence any 4×4 matrix can be expanded as

$$M_{4\times 4}=M_0\mathbf{1}+M_\mu\gamma^\mu+M_{\mu\nu}\Gamma^{\mu\nu}+M_{\mu\nu\alpha}i\epsilon^{\mu\nu\alpha\beta}\gamma_\beta\gamma^5+M_5\gamma^5.$$

Chirality Operator and Weyl Fermions

- We have

$$\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3.$$

Thus

$$\begin{aligned}
-\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} &= -\frac{i}{4!}(4)\epsilon_{0abc}\gamma^0\gamma^a\gamma^b\gamma^c\\
&= -\frac{i}{4!}(4.3)\epsilon_{0ij3}\gamma^0\gamma^i\gamma^j\gamma^3\\
&= -\frac{i}{4!}(4.3.2)\epsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3\\
&= i\gamma^0\gamma^1\gamma^2\gamma^3\\
&= \gamma^5.
\end{aligned}$$

We have used

$$\epsilon_{0123} = -\epsilon^{0123} = -1.$$

We also verify

$$\begin{aligned}
(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3.\gamma^0\gamma^1\gamma^2\gamma^3\\
&= \gamma^1\gamma^2\gamma^3.\gamma^1\gamma^2\gamma^3\\
&= -\gamma^2\gamma^3.\gamma^2\gamma^3\\
&= 1.
\end{aligned}$$

$$\begin{aligned}
(\gamma^5)^+ &= -i(\gamma^3)^+(\gamma^2)^+(\gamma^1)^+(\gamma^0)^+\\
&= i\gamma^3\gamma^2\gamma^1\gamma^0\\
&= -i\gamma^0\gamma^3\gamma^2\gamma^1\\
&= -i\gamma^0\gamma^1\gamma^3\gamma^2\\
&= i\gamma^0\gamma^1\gamma^2\gamma^3\\
&= \gamma^5.
\end{aligned}$$

$$\{\gamma^5,\gamma^0\} = \{\gamma^5,\gamma^1\} = \{\gamma^5,\gamma^2\} = \{\gamma^5,\gamma^3\} = 0.$$

From this last property we conclude directly that

$$[\gamma^5,\Gamma^{\mu\nu}] = 0.$$

- Hence the Dirac representation is reducible. To see this more clearly we work in the Weyl or chiral representation given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}\ ,\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

In this representation we compute

$$\gamma^5 = i\begin{pmatrix} \sigma^1\sigma^2\sigma^3 & 0 \\ 0 & \sigma^1\sigma^2\sigma^3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence by writing the Dirac spinor as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

we get

$$\Psi_R = \frac{1+\gamma^5}{2}\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix},$$

and

$$\Psi_L = \frac{1-\gamma^5}{2}\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}.$$

In other words

$$\gamma^5\Psi_L = -\Psi_L \ , \ \gamma^5\Psi_R = \Psi_R.$$

The spinors Ψ_L and Ψ_R do not mix under Lorentz transformations since they are eigenspinors of γ^5 which commutes with Γ^{ab}. In other words

$$\Psi_L(x) \longrightarrow \Psi^{'}_L(x^{'}) = S(\Lambda)\Psi_L(x).$$

$$\Psi_R(x) \longrightarrow \Psi^{'}_R(x^{'}) = S(\Lambda)\Psi_R(x).$$

- The dirac equation is

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0.$$

In terms of ψ_L and ψ_R this becomes

$$i\hbar(\partial_0 + \sigma^i\partial_i)\psi_R = mc\psi_L \ , \ i\hbar(\partial_0 - \sigma^i\partial_i)\psi_L = mc\psi_R.$$

For a massless theory we get two fully decoupled equations

$$i\hbar(\partial_0 + \sigma^i\partial_i)\psi_R = 0 \ , \ i\hbar(\partial_0 - \sigma^i\partial_i)\psi_L = 0.$$

These are known as Weyl equations. They are relevant in describing neutrinos. It is clear that ψ_L describes a left-moving particles and ψ_R describes a right-moving particles.

2

Canonical Quantization of Free Fields

2.1 Classical Mechanics

2.1.1 D'Alembert Principle

We consider a system of many particles and let $\vec{r}_i$ and m_i be the radius vector and the mass respectively of the ith particle. Newton's second law of motion for the ith particle reads

$$\vec{F}_i = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji} = \frac{d\vec{p}_i}{dt}. \tag{1.1}$$

The external force acting on the ith particle is $\vec{F}_i^{(e)}$ whereas $\vec{F}_{ji}$ is the internal force on the ith particle due to the jth particle ($\vec{F}_{ii} = 0$ and $\vec{F}_{ij} = -\vec{F}_{ji}$). The momentum vector of the ith particle is $\vec{p}_i = m_i\vec{v}_i = m_i\frac{d\vec{r}_i}{dt}$. Thus we have

$$\vec{F}_i = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji} = m_i\frac{d^2\vec{r}_i}{dt^2}. \tag{1.2}$$

By summing over all particles we get

$$0\sum_i \vec{F}_i = \sum_i \vec{F}_i^{(e)} = \sum_i m_i\frac{d^2\vec{r}_i}{dt^2} = M\frac{d^2\vec{R}}{dt^2}. \tag{1.3}$$

The total mass M is $M = \sum_i m_i$ and the average radius vector $\vec{R}$ is $\vec{R} = \sum_i m_i \vec{r}_i / M$. This is the radius vector of the center of mass of the system. Thus the internal forces if they obey Newton's third law of motion will have no effect on the motion of the center of mass.

The goal of mechanics is to solve the set of second order differential equations (1.2) for $\vec{r}_i$ given the forces $\vec{F}_i^{(e)}$ and $\vec{F}_{ji}$. This task is in general very difficult and it is made even more complicated by the possible presence of constraints which limit the motion of the system. As an example we take the class of systems known as rigid bodies in which the motion of the particles is constrained in such a way that the distances between the particles are kept fixed and do not change in time. It is clear that constraints correspond to forces which can not be specified directly but are only known via their effect on the motion of the system. We will only consider holonomic constraints which can be expressed by equations of the form

$$f(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., t) = 0. \tag{1.4}$$

The constraints which can not be expressed in this way are called nonholonomic. In the example of rigid bodies the constraints are holonomic since they can be expressed as

$$(\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0. \tag{1.5}$$

The presence of constraints means that not all the vectors $\vec{r}_i$ are independent, i.e not all the differential equations (1.2) are independent. We assume that the system contains N particles and that we have k holonomic constraints. Then there must exist $3N - k$ independent degrees of freedom q_i which are called generalized coordinates. We can therefore express the vectors $\vec{r}_i$ as functions of the independent generalized coordinates q_i as

$$\begin{aligned}
&\vec{r}_1 = \vec{r}_1(q_1, q_2, ..., q_{3N-k}, t) \\
&. \\
&. \\
&. \\
&\vec{r}_N = \vec{r}_N(q_1, q_2, ..., q_{3N-k}, t).
\end{aligned} \tag{1.6}$$

Let us compute the work done by the forces $\vec{F}_i^{(e)}$ and $\vec{F}_{ji}$ in moving the system from an initial configuration 1 to a final configuration 2. We have

$$W_{12} = \sum_i \int_1^2 \vec{F}_i d\vec{s}_i = \sum_i \int_1^2 \vec{F}_i^{(e)} d\vec{s}_i + \sum_{i,j} \int_1^2 \vec{F}_{ji} d\vec{s}_i. \tag{1.7}$$

We have from one hand

$$W_{12}=\sum_i\int_1^2\vec{F}_i d\vec{s}_i = \sum_i\int_1^2 m_i\frac{d\vec{v}_i}{dt}\vec{v}_i dt$$
$$= \sum_i\int_1^2 d(\frac{1}{2}m_i v_i^2)$$
$$= T_2-T_1. \quad (1.8)$$

The total kinetic energy is defined by

$$T=\sum_i\frac{1}{2}m_i v_i^2. \quad (1.9)$$

We assume that the external forces $\vec{F}_i^{(e)}$ are conservative, i.e they are derived from potentials V_i such that

$$\vec{F}_i^{(e)}=-\vec{\nabla}_i V_i. \quad (1.10)$$

Then we compute

$$\sum_i\int_1^2\vec{F}_i^{(e)}d\vec{s}_i=-\sum_i\int_1^2\vec{\nabla}_i V_i d\vec{s}_i=-\sum_i V_i|_1^2. \quad (1.11)$$

We also assume that the internal forces $\vec{F}_{ji}$ are derived from potentials V_{ij} such that

$$\vec{F}_{ji}=-\vec{\nabla}_i V_{ij}. \quad (1.12)$$

Since we must have $\vec{F}_{ij}=-\vec{F}_{ji}$ we must take V_{ij} as a function of the distance $|\vec{r}_i-\vec{r}_j|$ only, i.e $V_{ij}=V_{ji}$. We can also check that the force $\vec{F}_{ij}$ lies along the line joining the particles i and j.

We define the difference vector by $\vec{r}_{ij}=\vec{r}_i-\vec{r}_j$. We have then $\vec{\nabla}_i V_{ij}=-\vec{\nabla}_j V_{ij}=\vec{\nabla}_{ij}V_{ij}$. We then compute

$$\sum_{i,j}\int_1^2\vec{F}_{ji}d\vec{s}_i = -\frac{1}{2}\sum_{i,j}\int_1^2(\vec{\nabla}_i V_{ij}d\vec{s}_i+\vec{\nabla}_j V_{ij}d\vec{s}_j)$$
$$= -\frac{1}{2}\sum_{i,j}\int_1^2\vec{\nabla}_{ij}V_{ij}(d\vec{s}_i-d\vec{s}_j)$$
$$= -\frac{1}{2}\sum_{i,j}\int_1^2\vec{\nabla}_{ij}V_{ij}d\vec{r}_{ij}$$
$$= -\frac{1}{2}\sum_{i\neq j}V_{ij}|_1^2. \quad (1.13)$$

Thus the work done is found to be given by

$$W_{12} \quad = \quad -V_2 + V_1. \tag{1.14}$$

The total potential is given by

$$V = \sum_i V_i + \frac{1}{2}\sum_{i\neq j} V_{ij}. \tag{1.15}$$

From the results $W_{12} = T_2 - T_1$ and $W_{12} = -V_2 + V_1$ we conclude that the total energy $T+V$ is conserved. The term $\frac{1}{2}\sum_{i\neq j} V_{ij}$ in V is called the internal potential energy of the system.

For rigid bodies the internal energy is constant since the distances $|\vec{r}_i - \vec{r}_j|$ are fixed. Indeed in rigid bodies the vectors $d\vec{r}_{ij}$ can only be perpendicular to $\vec{r}_{ij}$ and therefore perpendicular to $\vec{F}_{ij}$ and as a consequence the internal forces do no work and the internal energy remains constant. In this case the forces $\vec{F}_{ij}$ are precisely the forces of constraints, i.e. the forces of constraint do no work.

We consider virtual infinitesimal displacements $\delta\vec{r}_i$ which are consistent with the forces and constraints imposed on the system at time t. A virtual displacement $\delta\vec{r}_i$ is to be compared with a real displacement $d\vec{r}_i$ which occurs during a time interval dt. Thus during a real displacement the forces and constraints imposed on the system may change. To be more precise an actual displacement is given in general by the equation

$$d\vec{r}_i = \frac{\partial\vec{r}_i}{\partial t}dt + \sum_{j=1}^{3N-k}\frac{\partial\vec{r}_i}{\partial q_j}dq_j. \tag{1.16}$$

A virtual displacement is given on the other hand by an equation of the form

$$\delta\vec{r}_i = \sum_{j=1}^{3N-k}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j. \tag{1.17}$$

The effective force on each particle is zero, i.e $\vec{F}_{i\ \text{eff}} = \vec{F}_i - \frac{d\vec{p}_i}{dt} = 0$. The virtual work of this effective force in the displacement $\delta\vec{r}_i$ is therefore trivially zero. Summed over all particles we get

$$\sum_i(\vec{F}_i - \frac{d\vec{p}_i}{dt})\delta\vec{r}_i = 0. \tag{1.18}$$

We decompose the force $\vec{F}_i$ into the applied force $\vec{F}_i^{(a)}$ and the force of constraint $\vec{f}_i$, viz $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$. Thus we have

$$\sum_i(\vec{F}_i^{(a)} - \frac{d\vec{p}_i}{dt})\delta\vec{r}_i + \sum_i \vec{f}_i\delta\vec{r}_i = 0. \tag{1.19}$$

We restrict ourselves to those systems for which the net virtual work of the forces of constraints is zero. In fact virtual displacements which are consistent with the constraints imposed on the system are precisely those displacements which are prependicular to the forces of constraints in such a way that the net virtual work of the forces of constraints is zero. We get then

$$\sum_i (\vec{F}_i^{(a)} - \frac{d\vec{p}_i}{dt})\delta\vec{r}_i = 0. \tag{1.20}$$

This is the principle of virtual work of D'Alembert. The forces of constraints which as we have said are generally unknown but only their effect on the motion is known do not appear explicitly in D'Alembert principle which is our goal. Their only effect in the equation is to make the virtual displacements $\delta\vec{r}_i$ not all independent.

2.1.2 Lagrange's Equations

We compute

$$\begin{aligned}\sum_i \vec{F}_i^{(a)}\delta\vec{r}_i &= \sum_{i,j} \vec{F}_i^{(a)}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j \\ &= \sum_j Q_j\delta q_j.\end{aligned} \tag{1.21}$$

The Q_j are the components of the generalized force. They are defined by

$$Q_j = \sum_i \vec{F}_i^{(a)}\frac{\partial\vec{r}_i}{\partial q_j}. \tag{1.22}$$

Let us note that since the generalized coordinates q_i need not have the dimensions of lenght the components Q_i of the generalized force need not have the dimensions of force.

We also compute

$$\begin{aligned}\sum_i \frac{d\vec{p}_i}{dt}\delta\vec{r}_i &= \sum_{i,j} m_i\frac{d^2\vec{r}_i}{dt^2}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j \\ &= \sum_{i,j} m_i\left[\frac{d}{dt}\left(\frac{d\vec{r}_i}{dt}\frac{\partial\vec{r}_i}{\partial q_j}\right) - \frac{d\vec{r}_i}{dt}\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right)\right]\delta q_j \\ &= \sum_{i,j} m_i\left[\frac{d}{dt}\left(\vec{v}_i\frac{\partial\vec{r}_i}{\partial q_j}\right) - \vec{v}_i\frac{\partial\vec{v}_i}{\partial q_j}\right]\delta q_j.\end{aligned} \tag{1.23}$$

By using the result $\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$ we obtain

$$\sum_i \frac{d\vec{p}_i}{dt}\delta\vec{r}_i = \sum_{i,j} m_i \bigg[\frac{d}{dt}\bigg(\vec{v}_i \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\bigg) - \vec{v}_i \frac{\partial \vec{v}_i}{\partial q_j}\bigg]\delta q_j$$
$$= \sum_j \bigg[\frac{d}{dt}\bigg(\frac{\partial T}{\partial \dot{q}_j}\bigg) - \frac{\partial T}{\partial q_j}\bigg]\delta q_j. \quad (1.24)$$

The total kinetic term is $T = \sum_i \frac{1}{2} m_i v_i^2$. Hence D'Alembert's principle becomes

$$\sum_i (\vec{F}_i^{(a)} - \frac{d\vec{p}_i}{dt})\delta\vec{r}_i = -\sum_j \bigg[Q_j - \frac{d}{dt}\bigg(\frac{\partial T}{\partial \dot{q}_j}\bigg) + \frac{\partial T}{\partial q_j}\bigg]\delta q_j = 0. \quad (1.25)$$

Since the generalized coordinates q_i for holonomic constraints can be chosen such that they are all independent we get the equations of motion

$$-Q_j + \frac{d}{dt}\bigg(\frac{\partial T}{\partial \dot{q}_j}\bigg) - \frac{\partial T}{\partial q_j} = 0. \quad (1.26)$$

In above $j = 1, ..., n$ where $n = 3N - k$ is the number of independent generalized coordinates. For conservative forces we have $\vec{F}_i^{(a)} = -\vec{\nabla}_i V$, i.e

$$Q_j = -\frac{\partial V}{\partial q_j}. \quad (1.27)$$

Hence we get the equations of motion

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{q}_j}\bigg) - \frac{\partial L}{\partial q_j} = 0. \quad (1.28)$$

These are Lagrange's equations of motion where the Lagrangian L is defined by

$$L = T - V. \quad (1.29)$$

2.1.3 Hamilton's Principle: The Principle of Least Action

In the previous section we have derived Lagrange's equations from considerations involving virtual displacements around the instantaneous state of the system using the differential principle of D'Alembert. In this section we will rederive Lagrange's equations from considerations involving virtual variations of the entire motion between times t_1 and t_2 around the actual entire motion between t_1 and t_2 using the integral principle of Hamilton.

The instantaneous state or configuration of the system at time t is described by the n generalized coordinates q_1, q_2,...,q_n. This is a point in the n-dimensional configuration space with axes given by the generalized coordinates q_i. As time evolves the system changes and the point $(q_1, q_2, ..., q_n)$ moves in configuration space tracing out a curve called the path of motion of the system.

Hamilton's principle is less general than D'Alembert's principle in that it describes only systems in which all forces (except the forces of constraints) are derived from generalized scalar potentials U. The generalized potentials are velocity-dependent potentials which may also depend on time, i.e $U = U(q_i, \dot{q}_i, t)$. The generalized forces are obtained form U as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\bigg(\frac{\partial U}{\partial \dot{q}_j}\bigg). \tag{1.30}$$

Such systems are called monogenic where Lagrange's equations of motion will still hold with Lagrangians given by $L = T - U$. The systems become conservative if the potentials depend only on coordinates. We define the action between times t_1 and t_2 by the line integral

$$I[q] = \int_{t_1}^{t_2} Ldt \ , \ L = T - V. \tag{1.31}$$

The Lagrangian is a function of the generalized coordinates and velocities q_i and $\dot{q}_i$ and of time t, i.e $L = L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)$. The action I is a functional.

Hamilton's principle can be states as follows. The line integral I has a stationary value, i.e it is an extremum for the actual path of the motion. Therefore any first order variation of the actual path results in a second order change in I so that all neighboring paths which differ from the actual path by infintesimal displacements have the same action. This is a variational problem for the action functional which is based on one single function which is the Lagrangian. Clearly I is invariant to the system of generalized coordinates used to express L and as a consequence the equations of motion which will be derived from I will be covariant. We write Hamilton's principle as follows

$$\frac{\delta}{\delta q_i} I[q] = \frac{\delta}{\delta q_i} \int_{t_1}^{t_2} L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)dt. \tag{1.32}$$

For systems with holonomic constraints it can be shown that Hamilton's principle is a necessary and sufficient condition for Lagrange's equations. Thus we can take Hamilton's principle as the basic postulate of mechanics rather than Newton's laws when all forces (except the forces of constraints) are derived from potentials which can depend on the coordinates, velocities and time.

Let us denote the soultions of the extremum problem by $q_i(t, 0)$. We write any other path around the correct path $q_i(t, 0)$ as $q_i(t, \alpha) = q_i(t, 0) + \alpha\eta_i(t)$ where the

η_i are arbitrary functions of t which must vanish at the end points t_1 and t_2 and are continuous through the second derivative and α is an infinitesimal parameter which labels the set of neighboring paths which have the same action as the correct path. For this parametric family of curves the action becomes an ordinary function of α given by

$$I(\alpha) = \int_{t_1}^{t_2} L(q_i(t,\alpha), \dot{q}_i(t,\alpha), t)dt. \tag{1.33}$$

We define the virtual displacements δq_i by

$$\delta q_i = \left(\frac{\partial q_i}{\partial \alpha}\right)|_{\alpha=0} d\alpha = \eta_i d\alpha. \tag{1.34}$$

Similarly the infinitesimal variation of I is defined by

$$\delta I = \left(\frac{dI}{d\alpha}\right)|_{\alpha=0} d\alpha. \tag{1.35}$$

We compute

$$\begin{aligned}
\frac{dI}{d\alpha} &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha}\right)dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial}{\partial t}\frac{\partial q_i}{\partial \alpha}\right)dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\frac{\partial q_i}{\partial \alpha}\right)dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \alpha}\right)dt + \left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \alpha}\right)_{t_1}^{t_2}.
\end{aligned} \tag{1.36}$$

The last term vanishes since all varied paths pass through the points $(t_1, y_i(t_1, 0)$ and $(t_2, y_i(t_2, 0))$. Thus we get

$$\delta I = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i dt. \tag{1.37}$$

Hamilton's principle reads

$$\frac{\delta I}{d\alpha} = \left(\frac{dI}{d\alpha}\right)|_{\alpha=0} = 0. \tag{1.38}$$

This leads to the equations of motion

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\eta_i dt = 0. \tag{1.39}$$

This should hold for any set of functions η_i. Thus by the fundamental lemma of the calculus of variations we must have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{1.40}$$

Formaly we write Hamilton's principle as

$$\frac{\delta I}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{1.41}$$

These are Lagrange's equations.

2.1.4 The Hamilton Equations of Motion

Again we will assume that the constraints are holonomic and the forces are monogenic, i.e they are derived from generalized scalar potentials as in (1.30). For a system with n degrees of freedom we have n Lagrange's equations of motion. Since Lagrange's equations are second order differential equations the motion of the system can be completely determined only after we also supply $2n$ initial conditions. As an example of initial conditions we can provide the n q_is and the n $\dot{q}_i$'s at an initial time t_0.

In the Hamiltonian formulation we want to describe the motion of the system in terms of first order differential equations. Since the number of initial conditions must remain $2n$ the number of first order differential equation which are needed to describe the system must be equal $2n$, i.e we must have $2n$ independent variables. It is only natural to choose the first half of the $2n$ independent variables to be the n generalized coordinates q_i. The second half will be chosen to be the n generalized momenta p_i defined by

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}. \tag{1.42}$$

The pairs (q_i, p_i) are known as canonical variables. The generalized momenta p_i are also known as canonical or conjugate momenta.

In the Hamiltonian formulation the state or configuration of the system is described by the point $(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$ in the $2n$-dimensional space known as the phase space of the system with axes given by the generalized coordinates and momenta q_i and p_i. The $2n$ first order differential equations will describe how the point $(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$ moves inside the phase space as the configuration of the system evolves in time.

The transition from the Lagrangian formulation to the Hamiltonian formulation corresponds to the change of variables $(q_i, \dot{q}_i, t) \longrightarrow (q_i, p_i, t)$ which is an example of a Legendre transformation. Instead of the Lagrangian which is a function of q_i,$\dot{q}_i$ and t, viz $L = L(q_i, \dot{q}_i, t)$ we will work in the Hamiltonian formulation with the Hamiltonian H which is a function of q_i, p_i and t defined by

$$H(q_i, p_i, t) = \sum_i \dot{q}_i p_i - L(q_i, \dot{q}_i, t). \quad (1.43)$$

We compute from one hand

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt. \quad (1.44)$$

From the other hand we compute

$$\begin{aligned} dH &= \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \\ &= \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \\ &= \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt. \end{aligned} \quad (1.45)$$

By comparison we get the canonical equations of motion of Hamilton

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \; , \; -\dot{p}_i = \frac{\partial H}{\partial q_i}. \quad (1.46)$$

We also get

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \quad (1.47)$$

For a large class of systems and sets of generalized coordinates the Lagrangian can be decomposed as $L(q_i, \dot{q}_i, t) = L_0(q_i, t) + L_1(q_i, \dot{q}_i, t) + L_2(q_i, \dot{q}_i, t)$ where L_2 is a homogeneous function of degree 2 in $\dot{q}_i$ whereas L_1 is a homogeneous function of degree 1 in $\dot{q}_i$. In this case we compute

$$\dot{q}_i p_i = \dot{q}_i \frac{\partial L_1}{\partial \dot{q}_i} + \dot{q}_i \frac{\partial L_2}{\partial \dot{q}_i} = L_1 + 2L_2. \quad (1.48)$$

Hence

$$H = L_2 - L_0. \quad (1.49)$$

If the transformation equations which define the generalized coordinates do not depend on time explicitly, i.e $\vec{r}_i = \vec{r}_i(q_1, q_2, ..., q_n)$ then $\vec{v}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j$ and as a consequence $T = T_2$ where T_2 is a function of q_i and $\dot{q}_i$ which is quadratic in the

$\dot{q}_i$'s. In general the kinetic term will be of the form $T = T_2(q_i, \dot{q}_i, t) + T_1(q_i, \dot{q}_i, t) + T_0(q_i, t)$. Further if the potential does not depend on the generalized velocities $\dot{q}_i$ then $L_2 = T$, $L_1 = 0$ and $L_0 = -V$. Hence we get

$$H = T + V. \tag{1.50}$$

This is the total energy of the system. It is not difficult to show using Hamilton's equations that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$. Thus if V does not depend on time explicitly then L will not depend on time explicitly and as a consequence H will be conserved.

2.2 Classical Free Field Theories

2.2.1 The Klein-Gordon Lagrangian Density

The Klein-Gordon wave equation is given by

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2}\right)\phi(x) = 0. \tag{2.51}$$

We will consider a complex field ϕ so that we have also the independent equation

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2}\right)\phi^*(x) = 0. \tag{2.52}$$

From now on we will reinterpret the wave functions ϕ and ϕ^* as fields and the corresponding Klein-Gordon wave equations as field equations.

A field is a dynamical system with an infinite number of degrees of freedom. Here the degrees of freedom $q_{\vec{x}}(t)$ and $\bar{q}_{\vec{x}}(t)$ are the values of the fields ϕ and ϕ^* at the points $\vec{x}$, viz

$$\begin{aligned} q_{\vec{x}}(t) &= \phi(x^0, \vec{x}) \\ \bar{q}_{\vec{x}}(t) &= \phi^*(x^0, \vec{x}). \end{aligned} \tag{2.53}$$

Remark that

$$\begin{aligned} \dot{q}_{\vec{x}} &= \frac{dq_{\vec{x}}}{dt} = c\partial_0 \phi + \frac{dx^i}{dt}\partial_i \phi \\ \dot{\bar{q}}_{\vec{x}} &= \frac{d\bar{q}_{\vec{x}}}{dt} = c\partial_0 \phi^* + \frac{dx^i}{dt}\partial_i \phi^*. \end{aligned} \tag{2.54}$$

Thus the role of $\dot{q}_{\vec{x}}$ and $\dot{\bar{q}}_{\vec{x}}$ will be played by the values of the derivatives of the fields $\partial_\mu \phi$ and $\partial_\mu \phi^*$ at the points $\vec{x}$.

The field equations (2.51) and (2.52) should be thought of as the equations of motion of the degrees of freedom $q_{\vec{x}}$ and $\bar{q}_{\vec{x}}$ respectively. These equations of motion should be derived from a Lagrangian density $\mathcal{L}$ which must depend only on the fields and their first derivatives at the point $\vec{x}$. In other words $\mathcal{L}$ must be local. This is also the reason why $\mathcal{L}$ is a Lagrangian density and not a Lagrangian. We have then

$$\mathcal{L}=\mathcal{L}(\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*)=\mathcal{L}(x^0,\vec{x}). \tag{2.55}$$

The Lagrangian is the integral over $\vec{x}$ of the Lagrangian density, viz

$$L=\int d\vec{x}\mathcal{L}(x^0,\vec{x}). \tag{2.56}$$

The action is the integral over time of L, namely

$$S=\int dtL=\int d^4x\mathcal{L}. \tag{2.57}$$

The Lagrangian density $\mathcal{L}$ is thus a Lorentz scalar. In other words it is a scalar under Lorentz transformations since the volume form d^4x is a scalar under Lorentz transformations. We compute

$$\begin{aligned}\delta S &= \int d^4x\delta\mathcal{L}\\ &= \int d^4x\bigg[\delta\phi\frac{\delta\mathcal{L}}{\delta\phi}+\delta\partial_\mu\phi\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}+\text{h.c}\bigg]\\ &= \int d^4x\bigg[\delta\phi\frac{\delta\mathcal{L}}{\delta\phi}+\partial_\mu\delta\phi\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}+\text{h.c}\bigg]\\ &= \int d^4x\bigg[\delta\phi\frac{\delta\mathcal{L}}{\delta\phi}-\delta\phi\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}+\partial_\mu\bigg(\delta\phi\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\bigg)+\text{h.c}\bigg].\end{aligned} \tag{2.58}$$

The surface term is zero because the field ϕ at infinity is assumed to be zero and hence

$$\delta\phi=0\ ,\ x^\mu\longrightarrow\pm\infty. \tag{2.59}$$

We get

$$\delta S = \int d^4x\bigg[\delta\phi\bigg(\frac{\delta\mathcal{L}}{\delta\phi}-\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\bigg)+\text{h.c}\bigg]. \tag{2.60}$$

The principle of least action states that

$$\delta S=0. \tag{2.61}$$

We obtain the Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} = 0. \tag{2.62}$$

$$\frac{\delta \mathcal{L}}{\delta \phi^*} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^*} = 0. \tag{2.63}$$

These must be the equations of motion (2.52) and (2.51) respectively. A solution is given by

$$\mathcal{L}_{\rm KG} = \frac{\hbar^2}{2}\bigg(\partial_\mu \phi^* \partial^\mu \phi - \frac{m^2c^2}{\hbar^2}\phi^*\phi\bigg). \tag{2.64}$$

The factor $\hbar^2$ is included so that the quantity $\int d^3x \mathcal{L}_{\rm KG}$ has dimension of energy. The coefficient $1/2$ is the canonical convention.

The conjugate momenta $\pi(x)$ and $\pi^*(x)$ associated with the fields $\phi(x)$ and $\phi^*(x)$ are defined by

$$\pi(x) = \frac{\delta \mathcal{L}_{KG}}{\delta \partial_t \phi} \ , \ \pi^*(x) = \frac{\delta \mathcal{L}_{KG}}{\delta \partial_t \phi^*}. \tag{2.65}$$

We compute

$$\pi(x) = \frac{\hbar^2}{2c^2}\partial_t \phi^* \ , \ \pi^*(x) = \frac{\hbar^2}{2c^2}\partial_t \phi. \tag{2.66}$$

The Hamiltonian density $\mathcal{H}_{KG}$ is the Legendre transform of $\mathcal{L}_{KG}$ defined by

$$\begin{aligned}\mathcal{H}_{\rm KG} &= \pi(x)\partial_t\phi(x) + \pi^*(x)\partial_t\phi^*(x) - \mathcal{L}_{\rm KG} \\ &= \frac{\hbar^2}{2}\bigg(\partial_0\phi^*\partial_0\phi + \vec{\nabla}\phi^*\vec{\nabla}\phi + \frac{m^2c^2}{\hbar^2}\phi^*\phi\bigg).\end{aligned} \tag{2.67}$$

The Hamiltonian is given by

$$H_{\rm KG} = \int d^3x \mathcal{H}_{\rm KG}. \tag{2.68}$$

2.2.2 The Dirac Lagrangian Density

The Dirac equation and its Hermitian conjugate are given by

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \tag{2.69}$$

$$\bar{\psi}(i\hbar\gamma^{\mu}\overleftarrow{\partial_{\mu}}+mc)=0. \tag{2.70}$$

The spinors ψ and $\bar{\psi}$ will now be interpreted as fields. In other words at each point $\vec{x}$ the dynamical variables are $\psi(x^0,\vec{x})$ and $\bar{\psi}(x^0,\vec{x})$. The two field equations (2.69) and (2.70) will be viewed as the equations of motion of the dynamical variables $\psi(x^0,\vec{x})$ and $\bar{\psi}(x^0,\vec{x})$. The local Lagrangian density will be of the form

$$\mathcal{L}=\mathcal{L}(\psi,\bar{\psi},\partial_{\mu}\psi,\partial_{\mu}\bar{\psi})=\mathcal{L}(x^0,\vec{x}). \tag{2.71}$$

The Euler-Lagrange equations are

$$\frac{\delta\mathcal{L}}{\delta\psi}-\partial_{\mu}\frac{\delta\mathcal{L}}{\delta\partial_{\mu}\psi}=0. \tag{2.72}$$

$$\frac{\delta\mathcal{L}}{\delta\bar{\psi}}-\partial_{\mu}\frac{\delta\mathcal{L}}{\delta\partial_{\mu}\bar{\psi}}=0. \tag{2.73}$$

A solution is given by

$$\mathcal{L}_{\text{Dirac}}=\bar{\psi}(i\hbar c\gamma^{\mu}\partial_{\mu}-mc^2)\psi. \tag{2.74}$$

The conjugate momenta $\bar{\Pi}(x)$ and $\Pi(x)$ associated with the fields $\psi(x)$ and $\bar{\psi}(x)$ are defined by

$$\Pi(x)=\frac{\delta\mathcal{L}_{\text{Dirac}}}{\delta\partial_t\psi}\ ,\ \bar{\Pi}(x)=\frac{\delta\mathcal{L}_{\text{Dirac}}}{\delta\partial_t\bar{\psi}}. \tag{2.75}$$

We compute

$$\Pi(x)=\bar{\psi}i\hbar\gamma^0\ ,\ \bar{\Pi}(x)=0. \tag{2.76}$$

The Hamiltonian density $\mathcal{H}_{\text{Dirac}}$ is the Legendre transform of $\mathcal{L}_{\text{Dirac}}$ defined by

$$\begin{aligned}\mathcal{H}_{\text{Dirac}} &= \Pi(x)\partial_t\psi(x)+\partial_t\bar{\psi}(x)\bar{\Pi}(x)-\mathcal{L}_{\text{Dirac}}\\ &= \bar{\psi}(-i\hbar c\gamma^i\partial_i+mc^2)\psi\\ &= \psi^+(-i\hbar c\vec{\alpha}\vec{\nabla}+mc^2\beta)\psi. \end{aligned}\tag{2.77}$$

2.3 Canonical Quantization of a Real Scalar Field

We will assume here that the scalar field ϕ is real. Thus $\phi^*=\phi$. This is a classical field theory governed by the Lagrangian density and the Lagrangian

$$\mathcal{L}_{\text{KG}}=\frac{\hbar^2}{2}\bigg(\partial_{\mu}\phi\partial^{\mu}\phi-\frac{m^2c^2}{\hbar^2}\phi^2\bigg). \tag{3.78}$$

$$L_{\rm KG}=\int d^3x\mathcal{L}_{\rm KG}. \tag{3.79}$$

The conjugate momentum is

$$\pi=\frac{\delta\mathcal{L}_{\rm KG}}{\delta\partial_t\phi}=\frac{\hbar^2}{c^2}\partial_t\phi. \tag{3.80}$$

We expand the classical field ϕ as

$$\phi(x^0,\vec{x})=\frac{c}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}Q(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}. \tag{3.81}$$

In other words $Q(x^0,\vec{p})$ is the Fourier transform of $\phi(x^0,\vec{x})$ which is given by

$$\frac{c}{\hbar}Q(x^0,\vec{p})=\int d^3x\phi(x^0,\vec{x})e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \tag{3.82}$$

Since $\phi^*=\phi$ we have $Q(x^0,-\vec{p})=Q^*(x^0,\vec{p})$. We compute

$$\begin{aligned}L_{\rm KG}&=\frac{1}{2}\int\frac{d^3p}{(2\pi\hbar)^3}\bigg[\partial_tQ^*(x^0,\vec{p})\partial_tQ(x^0,\vec{p})-\omega(\vec{p})^2Q^*(x^0,\vec{p})Q(x^0,\vec{p})\bigg]\\&=\int_+\frac{d^3p}{(2\pi\hbar)^3}\bigg[\partial_tQ^*(x^0,\vec{p})\partial_tQ(x^0,\vec{p})-\omega(\vec{p})^2Q^*(x^0,\vec{p})Q(x^0,\vec{p})\bigg].\end{aligned} \tag{3.83}$$

$$\omega^2(\vec{p})=\frac{1}{\hbar^2}(\vec{p}^2c^2+m^2c^4). \tag{3.84}$$

The sign $\int_+$ stands for the integration over positive values of p^1, p^2 and p^3. The equation of motion obeyed by Q derived from the Lagrangian $L_{\rm KG}$ is

$$(\partial_t^2+\omega(\vec{p}))Q(x^0,\vec{p})=0. \tag{3.85}$$

The general solution is of the form

$$Q(x^0,\vec{p})=\frac{1}{\sqrt{2\omega(\vec{p})}}\bigg[a(\vec{p})\ e^{-i\omega(\vec{p})t}+a(-\vec{p})^*\ e^{i\omega(\vec{p})t}\bigg]. \tag{3.86}$$

This satisfies $Q(x^0,-\vec{p})=Q^*(x^0,\vec{p})$. The conjugate momentum is

$$\pi(x^0,\vec{x})=\frac{\hbar}{c}\int\frac{d^3p}{(2\pi\hbar)^3}P(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}\ ,\ P(x^0,\vec{p})=\partial_tQ(x^0,\vec{p}). \tag{3.87}$$

$$\frac{\hbar}{c}P(x^0,\vec{p}) = \int d^3x\pi(x^0,\vec{x})e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \quad (3.88)$$

Since $\pi^* = \pi$ we have $P(x^0,-\vec{p}) = P^*(x^0,\vec{p})$. We observe that

$$P(x^0,\vec{p}) = \frac{\delta L_{\rm KG}}{\delta\partial_t Q^*(x^0,\vec{p})}. \quad (3.89)$$

The Hamiltonian is

$$H_{\rm KG} = \int_+ \frac{d^3p}{(2\pi\hbar)^3}\bigg[P^*(x^0,\vec{p})P(x^0,\vec{p}) + \omega^2(\vec{p})Q^*(x^0,\vec{p})Q(x^0,\vec{p})\bigg]. \quad (3.90)$$

The real scalar field is therefore equivalent to an infinite collection of independent harmonic oscillators with frequencies $\omega(\vec{p})$ which depend on the momenta $\vec{p}$ of the Fourier modes.

Quantization of this dynamical system means replacing the scalar field ϕ and the conjugate momentum field π by operators $\hat{\phi}$ and $\hat{\pi}$ respectively which are acting in some Hilbert space. This means that the coefficients a and a^* become operators $\hat{a}$ and $\hat{a}^+$ and hence Q and P become operators $\hat{Q}$ and $\hat{P}$. The operators $\hat{\phi}$ and $\hat{\pi}$ will obey the equal-time canonical commutation relations due to Dirac, viz

$$[\hat{\phi}(x^0,\vec{x}),\hat{\pi}(x^0,\vec{y})] = i\hbar\delta^3(\vec{x}-\vec{y}). \quad (3.91)$$

$$[\hat{\phi}(x^0,\vec{x}),\hat{\phi}(x^0,\vec{y})] = [\hat{\pi}(x^0,\vec{x}),\hat{\pi}(x^0,\vec{y})] = 0. \quad (3.92)$$

These commutation relations should be compared with

$$[q_i,p_j] = i\hbar\delta_{ij}. \quad (3.93)$$

$$[q_i,q_j] = [p_i,p_j] = 0. \quad (3.94)$$

The field operator $\hat{\phi}$ and the conjugate momentum operator $\hat{\pi}$ are given by

$$\frac{\hbar}{c}\hat{\phi}(x^0,\vec{x}) = \int \frac{d^3p}{(2\pi\hbar)^3}\hat{Q}(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}} = \int_+ \frac{d^3p}{(2\pi\hbar)^3}\hat{Q}(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}} + \int_+ \frac{d^3p}{(2\pi\hbar)^3}\hat{Q}^+(x^0,\vec{p})e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \quad (3.95)$$

$$\frac{c}{\hbar}\hat{\pi}(x^0,\vec{x}) = \int \frac{d^3p}{(2\pi\hbar)^3}\hat{P}(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}} = \int_+ \frac{d^3p}{(2\pi\hbar)^3}\hat{P}(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}} + \int_+ \frac{d^3p}{(2\pi\hbar)^3}\hat{P}^+(x^0,\vec{p})e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \quad (3.96)$$

It is then not difficult to see that the commutation relations (3.91) and (3.92) are equivalent to the equal-time commutation rules

$$[\hat{Q}(x^0,\vec{p}),\hat{P}^+(x^0,\vec{q})] = i\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \quad (3.97)$$

$$[\hat{Q}(x^0,\vec{p}),\hat{P}(x^0,\vec{q})]=0. \tag{3.98}$$

$$[\hat{Q}(x^0,\vec{p}),\hat{Q}(x^0,\vec{q})]=[\hat{P}(x^0,\vec{p}),\hat{P}(x^0,\vec{q})]=0. \tag{3.99}$$

We have

$$\hat{Q}(x^0,\vec{p})=\frac{1}{\sqrt{2\omega(\vec{p})}}\bigg[\hat{a}(\vec{p})\ e^{-i\omega(\vec{p})t}+\hat{a}(-\vec{p})^+\ e^{i\omega(\vec{p})t}\bigg]. \tag{3.100}$$

$$\hat{P}(x^0,\vec{p})=-i\sqrt{\frac{\omega(\vec{p})}{2}}\bigg[\hat{a}(\vec{p})\ e^{-i\omega(\vec{p})t}-\hat{a}(-\vec{p})^+\ e^{i\omega(\vec{p})t}\bigg]. \tag{3.101}$$

Since $\hat{Q}(x^0,\vec{p})$ and $\hat{P}(x^0,\vec{p})$ satisfy (3.97), (3.98) and (3.99) the annihilation and creation operators $a(\vec{p})$ and $a(\vec{p})^+$ must satisfy

$$[\hat{a}(\vec{p}),\hat{a}(\vec{q})^+]=\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \tag{3.102}$$

The Hamiltonian operator is

$$\begin{aligned}\hat{H}_{\rm KG} &= \int_+\frac{d^3p}{(2\pi\hbar)^3}\bigg[\hat{P}^+(x^0,\vec{p})\hat{P}(x^0,\vec{p})+\omega^2(\vec{p})\hat{Q}^+(x^0,\vec{p})\hat{Q}(x^0,\vec{p})\bigg]\\ &= \int_+\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\bigg[\hat{a}(\vec{p})^+\hat{a}(\vec{p})+\hat{a}(\vec{p})\hat{a}(\vec{p})^+\bigg]\\ &= 2\int_+\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\bigg[\hat{a}(\vec{p})^+\hat{a}(\vec{p})+\frac{\hbar}{2}(2\pi\hbar)^3\delta^3(0)\bigg]\\ &= \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\bigg[\hat{a}(\vec{p})^+\hat{a}(\vec{p})+\frac{\hbar}{2}(2\pi\hbar)^3\delta^3(0)\bigg].\end{aligned} \tag{3.103}$$

Let us define the vacuum (ground) state $|0>$ by

$$\hat{a}(\vec{p})|0>=0. \tag{3.104}$$

The energy of the vacuum is therefore infinite since

$$\hat{H}_{\rm KG}|0> \ = \ \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\bigg[\frac{\hbar}{2}(2\pi\hbar)^3\delta^3(0)\bigg]|0>. \tag{3.105}$$

This is a bit disturbing. But since all we can measure experimentally are energy differences from the ground state this infinite energy is unobservable. We can ignore this infinite energy by the so-called normal (Wick's) ordering procedure defined by

$$:\hat{a}(\vec{p})\hat{a}(\vec{p})^+:=\hat{a}(\vec{p})^+\hat{a}(\vec{p})\ ,\ \ :\hat{a}(\vec{p})^+\hat{a}(\vec{p}):=\hat{a}(\vec{p})^+\hat{a}(\vec{p}). \tag{3.106}$$

We then get

$$: \hat{H}_{\rm KG} : \ = \ \int \frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\hat{a}(\vec{p})^+\hat{a}(\vec{p}). \tag{3.107}$$

Clearly

$$: \hat{H}_{\rm KG} : |0> \ = \ 0. \tag{3.108}$$

It is easy to calculate

$$[\hat{H}_{\rm KG}, \hat{a}(\vec{p})^+] = \hbar\omega(\vec{p})\hat{a}(\vec{p})^+ \ , \ [\hat{H}, \hat{a}(\vec{p})] = -\hbar\omega(\vec{p})\hat{a}(\vec{p}). \tag{3.109}$$

This establishes that $\hat{a}(\vec{p})^+$ and $\hat{a}(\vec{p})$ are raising and lowering operators. The one-particle states are states of the form

$$|\vec{p}>= \frac{1}{c}\sqrt{2\omega(\vec{p})}\hat{a}(\vec{p})^+|0> . \tag{3.110}$$

Indeed we compute

$$\hat{H}_{\rm KG}|\vec{p}>= \hbar\omega(\vec{p})|\vec{p}>= E(\vec{p})|\vec{p}> \ , \ E(\vec{p}) = \sqrt{\vec{p}^2c^2 + m^2c^4}. \tag{3.111}$$

The energy $E(\vec{p})$ is precisely the energy of a relativistic particle of mass m and momentum $\vec{p}$. This is the underlying reason for the interpretation of $|\vec{p}>$ as a state of a free quantum particle carrying momentum $\vec{p}$ and energy $E(\vec{p})$. The normalization of the one-particle state $|\vec{p}>$ is chosen such that

$$< \vec{p}|\vec{q}>= \frac{2}{c^2}(2\pi\hbar)^3E(\vec{p})\delta^3(\vec{p}-\vec{q}). \tag{3.112}$$

We have assumed that $< 0|0 >= 1$. The factor $\sqrt{2\omega(\vec{p})}$ in (3.110) is chosen so that the normalization (3.112) is Lorentz invariant.

The two-particle states are states of the form (not bothering about normalization)

$$|\vec{p},\vec{q}>= \hat{a}(\vec{p})^+\hat{a}(\vec{q})^+|0> . \tag{3.113}$$

We compute in this case

$$\hat{H}_{\rm KG}|\vec{p},\vec{q}>= \hbar(\omega(\vec{p})+\omega(\vec{q}))|\vec{p}> . \tag{3.114}$$

Since the creation operators for different momenta commute the state $|\vec{p},\vec{q}>$ is the same as the state $|\vec{q},\vec{p}>$ and as a consequence our particles obey the

Bose-Einstein statistics. In general multiple-particle states will be of the form $\hat{a}(\vec{p})^+\hat{a}(\vec{q})^+...\hat{a}(\vec{k})^+|0>$ with energy equal to $\hbar(\omega(\vec{p})+\omega(\vec{q})+...+\omega(\vec{k}))$.

Let us compute (with $px = cp^0t - \vec{p}\vec{x}$)

$$\begin{aligned} \frac{\hbar}{c}\hat{\phi}(x) &= \int \frac{d^3p}{(2\pi\hbar)^3}\hat{Q}(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}} \\ &= \int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\bigg(\hat{a}(\vec{p})e^{-\frac{i}{\hbar}px}+\hat{a}(\vec{p})^+e^{\frac{i}{\hbar}px}\bigg)_{p^0=E(\vec{p})/c}. \end{aligned} \tag{3.115}$$

Finally we remark that the unit of $\hbar$ is $[\hbar] = ML^2/T$, the unit of ϕ is $[\phi] = 1/(L^{3/2}M^{1/2})$, the unit of π is $[\pi] = (M^{3/2}L^{1/2})/T$, the unit of Q is $[Q] = M^{1/2}L^{5/2}$, the unit of P is $[P] = (M^{1/2}L^{5/2})/T$, the unit of a is $[a] = (M^{1/2}L^{5/2})/T^{1/2}$, the unit of H is $[H] = (ML^2)/T^2$ and the unit of momentum p is $[p] = (ML)/T$.

2.4 Canonical Quantization of Free Spinor Field

We expand the spinor field as

$$\psi(x^0,\vec{x}) = \frac{1}{\hbar}\int \frac{d^3p}{(2\pi\hbar)^3}\chi(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}. \tag{4.116}$$

The Lagrangian in terms of χ and χ^+ is given by

$$\begin{aligned} L_{\text{Dirac}} &= \int d^3x\mathcal{L}_{\text{Dirac}} \\ &= \int d^3x\bar{\psi}(i\hbar c\gamma^\mu\partial_\mu - mc^2)\psi \\ &= \frac{c}{\hbar^2}\int \frac{d^3p}{(2\pi\hbar)^3}\bar{\chi}(x^0,\vec{p})(i\hbar\gamma^0\partial_0 - \gamma^ip^i - mc)\chi(x^0,\vec{p}). \end{aligned} \tag{4.117}$$

The classical equation of motion obeyed by the field $\chi(x^0,\vec{p})$ is

$$(i\hbar\gamma^0\partial_0 - \gamma^ip^i - mc)\chi(x^0,\vec{p}) = 0. \tag{4.118}$$

This can be solved by plane-waves of the form

$$\chi(x^0,\vec{p}) = e^{-\frac{i}{\hbar}Et}\chi(\vec{p}), \tag{4.119}$$

with

$$(\gamma^\mu p_\mu - mc)\chi(\vec{p}) = 0. \tag{4.120}$$

We know how to solve this equation. The positive-energy solutions are given by

$$\chi_+(\vec{p}) = u^{(i)}(E,\vec{p}). \tag{4.121}$$

The corresponding plane-waves are

$$\chi_+(x^0,\vec{p}) = e^{-i\omega(\vec{p})t}u^{(i)}(E(\vec{p}),\vec{p}) = e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p}). \tag{4.122}$$

$$\omega(\vec{p}) = \frac{E}{\hbar} = \frac{\sqrt{\vec{p}^2c^2 + m^2c^4}}{\hbar}. \tag{4.123}$$

The negative-energy solutions are given by

$$\chi_-(\vec{p}) = v^{(i)}(-E,-\vec{p}). \tag{4.124}$$

The corresponding plane-waves are

$$\chi_+(x^0,\vec{p}) = e^{i\omega(\vec{p})t}v^{(i)}(E(\vec{p}),-\vec{p}) = e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p}). \tag{4.125}$$

In the above equations

$$E(\vec{p}) = E = \hbar\omega(\vec{p}). \tag{4.126}$$

Thus the general solution is a linear combination of the form

$$\chi(x^0,\vec{p}) = \sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})b(\vec{p},i) + e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})d(-\vec{p},i)^*\bigg). \tag{4.127}$$

The spinor field becomes

$$\psi(x^0,\vec{x}) = \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}e^{\frac{i}{\hbar}\vec{p}\vec{x}}\sum_i\bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})b(\vec{p},i) + e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})d(-\vec{p},i)^*\bigg). \tag{4.128}$$

The conjugate momentum field is

$$\begin{aligned}\Pi(x^0,\vec{x}) &= i\hbar\psi^+ \\ &= i\int\frac{d^3p}{(2\pi\hbar)^3}\chi^+(x^0,\vec{p})e^{-\frac{i}{\hbar}\vec{p}\vec{x}}.\end{aligned} \tag{4.129}$$

After quantization the coefficients $b(\vec{p},i)$ and $d(-\vec{p},i)^*$ and a s a consequence the spinors $\chi(x^0,\vec{p})$ and $\chi^+(x^0,\vec{p})$ become operators $\hat{b}(\vec{p},i)$, $\hat{d}(-\vec{p},i)^+$, $\hat{\chi}(x^0,\vec{p})$ and $\hat{\chi}^+(x^0,\vec{p})$ respectively. As we will see shortly the quantized Poisson brackets for a spinor field are given by anticommutation relations and not commutation relations.

In other words we must impose anticommutation relations between the spinor field operator $\hat{\psi}$ and the conjugate momentum field operator $\hat{\Pi}$. In the following we will consider both possibilities for the sake of completeness. We set then

$$[\hat{\psi}_\alpha(x^0,\vec{x}),\hat{\Pi}_\beta(x^0,\vec{y})]_\pm = i\hbar\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}). \tag{4.130}$$

The plus sign corresponds to anticommutator whereas the minus sign corresponds to commutator. We can immediately compute

$$[\hat{\chi}_\alpha(x^0,\vec{p}),\hat{\chi}^+_\beta(x^0,\vec{q})]_\pm = \hbar^2\delta_{\alpha\beta}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \tag{4.131}$$

This is equivalent to

$$[\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^+]_\pm = \hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}), \tag{4.132}$$

$$[\hat{d}(\vec{p},i)^+,\hat{d}(\vec{q},j)]_\pm = \hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}), \tag{4.133}$$

and

$$[\hat{b}(\vec{p},i),\hat{d}(\vec{q},j)]_\pm = [\hat{d}(\vec{q},j)^+,\hat{b}(\vec{p},i)]_\pm = 0. \tag{4.134}$$

We go back to the classical theory for a moment. The Hamiltonian in terms of χ and χ^+ is given by

$$\begin{aligned}
H_{\rm Dirac} &= \int d^3x\mathcal{H}_{\rm Dirac} \\
&= \int d^3x\bar{\psi}(-i\hbar c\gamma^i\partial_i + mc^2)\psi \\
&= \frac{c}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\bar{\chi}(x^0,\vec{p})(\gamma^ip^i+mc)\chi(x^0,\vec{p}) \\
&= \frac{c}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\chi^+(x^0,\vec{p})\gamma^0(\gamma^ip^i+mc)\chi(x^0,\vec{p}).
\end{aligned} \tag{4.135}$$

The eigenvalue equation (4.120) can be put in the form

$$\gamma^0(\gamma^ip^i+mc)\chi(x^0,\vec{p}) = \frac{E}{c}\chi(x^0,\vec{p}). \tag{4.136}$$

On the positive-energy solution we have

$$\gamma^0(\gamma^ip^i+mc)\chi_+(x^0,\vec{p}) = \frac{\hbar\omega(\vec{p})}{c}\chi_+(x^0,\vec{p}). \tag{4.137}$$

On the negative-energy solution we have

$$\gamma^0(\gamma^i p^i + mc)\chi_-(x^0,\vec{p}) = -\frac{\hbar\omega(\vec{p})}{c}\chi_-(x^0,\vec{p}). \tag{4.138}$$

Hence we have explicitly

$$c\gamma^0(\gamma^i p^i + mc)\chi(x^0,\vec{p}) = \frac{\hbar\omega(\vec{p})}{\sqrt{2\omega(\vec{p})}}\sum_i\bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})b(\vec{p},i) - e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})d(-\vec{p},i)^*\bigg). \tag{4.139}$$

The Hamiltonian becomes

$$\begin{aligned} H_{\rm Dirac} &= \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}E(\vec{p})\sum_i\bigg(b(\vec{p},i)^*b(\vec{p},i) - d(-\vec{p},i)d(-\vec{p},i)^*\bigg)\\ &= \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\sum_i\bigg(b(\vec{p},i)^*b(\vec{p},i) - d(\vec{p},i)d(\vec{p},i)^*\bigg). \end{aligned} \tag{4.140}$$

After quantization the Hamiltonian becomes an operator given by

$$\hat{H}_{\rm Dirac} = \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\sum_i\bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i) - \hat{d}(\vec{p},i)\hat{d}(\vec{p},i)^+\bigg). \tag{4.141}$$

At this stage we will decide once and for all whether the creation and annihilation operators of the theory obey commutation relations or anticommutation relations. In the case of commutation relations we see from the commutation relations (4.133) that $\hat{d}$ is the creation operator and $\hat{d}^+$ is the annihilation operator. Thus the second term in the above Hamiltonian operator is already normal ordered. However we observe that the contribution of the $d-$particles to the energy is negative and thus by creating more and more d particles the energy can be lowered without limit. The theory does not admit a stable ground state.

In the case of anticommutation relations the above Hamiltonian operator becomes

$$\hat{H}_{\rm Dirac} = \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\sum_i\bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i) + \hat{d}(\vec{p},i)^+\hat{d}(\vec{p},i)\bigg). \tag{4.142}$$

This expression is correct modulo an infinite constant which can be removed by normal ordering as in the scalar field theory. The vacuum state is defined by

$$\hat{b}(\vec{p},i)|0> = \hat{d}(\vec{p},i)|0> = 0. \tag{4.143}$$

Clearly

$$\hat{H}_{\rm Dirac}|0> = 0. \tag{4.144}$$

We calculate

$$[\hat{H}_{\rm Dirac}, \hat{b}(\vec{p}, i)^+] = \hbar\omega(\vec{p})\hat{b}(\vec{p}, i)^+ \ , \ [\hat{H}_{\rm Dirac}, \hat{b}(\vec{p}, i)] = -\hbar\omega(\vec{p})\hat{b}(\vec{p}, i). \quad (4.145)$$

$$[\hat{H}_{\rm Dirac}, \hat{d}(\vec{p}, i)^+] = \hbar\omega(\vec{p})\hat{d}(\vec{p}, i)^+ \ , \ [\hat{H}_{\rm Dirac}, \hat{d}(\vec{p}, i)] = -\hbar\omega(\vec{p})\hat{d}(\vec{p}, i). \quad (4.146)$$

Excited particle states are obtained by acting with $\hat{b}(\vec{p}, i)^+$ on $|0>$ and excited antiparticle states are obtained by acting with $\hat{d}(\vec{p}, i)^+$ on $|0>$. The normalization of one-particle excited states can be fixed in the same way as in the scalar field theory, viz

$$|\vec{p}, ib>= \sqrt{2\omega(\vec{p})}\hat{b}(\vec{p}, i)^+|0> \ , \ |\vec{p}, id>= \sqrt{2\omega(\vec{p})}\hat{d}(\vec{p}, i)^+|0> . \quad (4.147)$$

Indeed we compute

$$\hat{H}_{\rm Dirac}|\vec{p}, ib>= E(\vec{p})|\vec{p}, ib> \ , \ \hat{H}_{\rm Dirac}|\vec{p}, id>= E(\vec{p})|\vec{p}, id> . \quad (4.148)$$

$$<\vec{p}, ib|\vec{q}, jb>=<\vec{p}, id|\vec{q}, jd>= 2E(\vec{p})\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \quad (4.149)$$

Furthermore we compute

$$<0|\hat{\psi}(x)|\vec{p}, ib>= u^{(i)}(\vec{p})e^{-\frac{i}{\hbar}px}. \quad (4.150)$$

$$<0|\bar{\hat{\psi}}(x)|\vec{p}, id>= \bar{v}^{(i)}(\vec{p})e^{-\frac{i}{\hbar}px}. \quad (4.151)$$

The field operator $\bar{\hat{\psi}}(x)$ acting on the vacuum $|0>$ creates a particle at $\vec{x}$ at time $t = x^0/c$ whereas $\hat{\psi}(x)$ acting on $|0>$ creates an antiparticle at $\vec{x}$ at time $t = x^0/c$.

General multiparticle states are obtained by acting with $\hat{b}(\vec{p}, i)^+$ and $\hat{d}(\vec{p}, i)^+$ on $|0>$. Since the creation operators anticommute our particles will obey the Fermi-Dirac statistics. For example particles can not occupy the same state, i.e. $\hat{b}(\vec{p}, i)^+\hat{b}(\vec{p}, i)^+|0>= 0$.

The spinor field operator can be put in the form

$$\hat{\psi}(x) = \frac{1}{\hbar}\int \frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i \bigg(e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})\hat{b}(\vec{p}, i) + e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})\hat{d}(\vec{p}, i)^+\bigg). \quad (4.152)$$

2.5 Propagators

2.5.1 Scalar Propagator

The probability amplitude for a scalar particle to propagate from the spacetime point y to the spacetime x is

$$D(x-y) =< 0|\hat{\phi}(x)\hat{\phi}(y)|0 > . \tag{5.153}$$

We compute

$$\begin{aligned} D(x-y) &= \frac{c^2}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\int\frac{d^3q}{(2\pi\hbar)^3}\frac{e^{-\frac{i}{\hbar}px}}{\sqrt{2\omega(\vec{p})}}\frac{e^{\frac{i}{\hbar}qy}}{\sqrt{2\omega(\vec{q})}} < 0|\hat{a}(\vec{p})\hat{a}(\vec{q})^+|0 > \\ &= c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}. \end{aligned} \tag{5.154}$$

This is Lorentz invariant since $d^3p/E(\vec{p})$ is Lorentz invariant. Now we will relate this probability amplitude with the commutator $[\hat{\phi}(x), \hat{\phi}(y)]$. We compute

$$\begin{aligned} [\hat{\phi}(x), \hat{\phi}(y)] &= \frac{c^2}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\int\frac{d^3q}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\frac{1}{\sqrt{2\omega(\vec{q})}} \\ &\times \left(e^{-\frac{i}{\hbar}px}e^{\frac{i}{\hbar}qy}[\hat{a}(\vec{p}), \hat{a}(\vec{q})^+] - e^{\frac{i}{\hbar}px}e^{-\frac{i}{\hbar}qy}[\hat{a}(\vec{q}), \hat{a}(\vec{p})^+]\right) \\ &= D(x-y) - D(y-x). \end{aligned} \tag{5.155}$$

In the case of a spacelike interval, i.e. $(x-y)^2 = (x^0-y^0)^2 - (\vec{x}-\vec{y})^2 < 0$ the amplitudes $D(x-y)$ and $D(y-x)$ are equal and thus the commutator vanishes. To see this more clearly we place the event x at the origin of spacetime. The event y if it is spacelike it will lie outside the light-cone. In this case there is an inertial reference frame in which the two events occur at the same time, viz $y^0 = x^0$. In this reference frame the amplitude takes the form

$$D(x-y) = c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{\frac{i}{\hbar}\vec{p}(\vec{x}-\vec{y})}. \tag{5.156}$$

It is clear that $D(x-y) = D(y-x)$ and hence

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0 \ , \ \text{iff} \ (x-y)^2 < 0. \tag{5.157}$$

In conclusion any two measurements in the Klein-Gordon theory with one measurement lying outside the light-cone of the other measurement will not affect each other. In other words measurements attached to events separated by spacelike intervals will commute.

In the case of a timelike interval, i.e. $(x-y)^2 > 0$ the event y will lie inside the light-cone of the event x. Furthermore there is an inertial reference frame in which the two events occur at the same point, viz $\vec{y} = \vec{x}$. In this reference frame the amplitude is

$$D(x-y) = c^2 \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{2E(\vec{p})} e^{-\frac{i}{\hbar}p^0(x^0-y^0)}. \tag{5.158}$$

Thus in this case the amplitudes $D(x-y)$ and $D(y-x)$ are not equal. As a consequence the commutator $[\hat{\phi}(x), \hat{\phi}(y)]$ does not vanish and hence measurements attached to events separated by timelike intervals can affect each.

Let us rewrite the commutator as

$$\begin{aligned} < 0|[\hat{\phi}(x), \hat{\phi}(y)]|0 > &= [\hat{\phi}(x), \hat{\phi}(y)] \\ &= c^2 \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{2E(\vec{p})} \left(e^{-\frac{i}{\hbar}p(x-y)} - e^{\frac{i}{\hbar}p(x-y)} \right) \\ &= c^2 \int \frac{d^3p}{(2\pi\hbar)^3} \left(\frac{1}{2E(\vec{p})} e^{-\frac{i}{\hbar}\left(\frac{E(\vec{p})}{c}(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)} \right. \\ &+ \left. \frac{1}{-2E(\vec{p})} e^{-\frac{i}{\hbar}\left(-\frac{E(\vec{p})}{c}(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)} \right). \end{aligned} \tag{5.159}$$

Let us calculate from the other hand

$$\begin{aligned} \frac{1}{c}\int \frac{dp^0}{2\pi} \frac{i}{p^2 - m^2c^2} e^{-\frac{i}{\hbar}p(x-y)} &= \frac{1}{c}\int \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - \frac{E(\vec{p})^2}{c^2}} e^{-\frac{i}{\hbar}p(x-y)} \\ &= \frac{1}{c}\int \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - \frac{E(\vec{p})^2}{c^2}} e^{-\frac{i}{\hbar}\left(p^0(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)}. \end{aligned} \tag{5.160}$$

There are two poles on the real axis at $p^0 = \pm E(\vec{p})/c$. In order to use the residue theorem we must close the contour of integration. In this case we close the contour such that both poles are included and assuming that $x^0 - y^0 > 0$ the contour must be closed below. Clearly for $x^0 - y^0 < 0$ we must close the contour above which then yields zero. We get then

$$\begin{aligned} \frac{1}{c}\int \frac{dp^0}{2\pi} \frac{i}{p^2 - m^2c^2} e^{-\frac{i}{\hbar}p(x-y)} &= \frac{i}{2\pi c}(-2\pi i) \left[\left(\frac{p^0 - \frac{E(\vec{p})}{c}}{(p^0)^2 - \frac{E(\vec{p})^2}{c^2}} e^{-\frac{i}{\hbar}\left(p^0(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)} \right)_{p^0 = E(\vec{p})/c} \right. \\ &+ \left. \left(\frac{p^0 + \frac{E(\vec{p})}{c}}{(p^0)^2 - \frac{E(\vec{p})^2}{c^2}} e^{-\frac{i}{\hbar}\left(p^0(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)} \right)_{p^0 = -E(\vec{p})/c} \right] \\ &= \frac{1}{2E(\vec{p})} e^{-\frac{i}{\hbar}\left(\frac{E(\vec{p})}{c}(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)} + \frac{1}{-2E(\vec{p})} e^{-\frac{i}{\hbar}\left(-\frac{E(\vec{p})}{c}(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)}. \end{aligned} \tag{5.161}$$

Thus we get

$$D_R(x-y) = \theta(x^0-y^0) < 0|[\hat{\phi}(x),\hat{\phi}(y)]|0 >$$
$$= c\hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2-m^2c^2}e^{-\frac{i}{\hbar}p(x-y)}. \quad (5.162)$$

Clearly this function satisfies

$$(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2})D_R(x-y) = -i\frac{c}{\hbar}\delta^4(x-y). \quad (5.163)$$

This is a retarded (since it vanishes for $x^0<y^0$) Green's function of the Klein-Gordon equation.

In the above analysis the contour used is only one possibility among four possible contours. It yielded the retarded Green's function which is non-zero only for $x^0>y^0$. The second contour is the contour which gives the advanced Green's function which is non-zero only for $x^0<y^0$. The third contour corresponds to the so-called Feynman prescription given by

$$D_F(x-y) = c\hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2-m^2c^2+i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}. \quad (5.164)$$

The convention is to take $\epsilon>0$. The fourth contour corresponds to $\epsilon<0$.

In the case of the Feynman prescription we close for $x^0>y^0$ the contour below so only the pole $p^0=E(\vec{p})/c-i\epsilon^{'}$ will be included. The integral reduces to $D(x-y)$. For $x^0<y^0$ we close the contour above so only the pole $p^0=-E(\vec{p})/c+i\epsilon^{'}$ will be included. The integral reduces to $D(y-x)$. In summary we have

$$D_F(x-y) = \theta(x^0-y^0)D(x-y)+\theta(y^0-x^0)D(y-x)$$
$$= < 0|T\hat{\phi}(x)\hat{\phi}(y)|0 > . \quad (5.165)$$

The time-ordering operator is defined by

$$T\hat{\phi}(x)\hat{\phi}(y)=\hat{\phi}(x)\hat{\phi}(y)\ ,\ x^0>y^0$$
$$T\hat{\phi}(x)\hat{\phi}(y)=\hat{\phi}(y)\hat{\phi}(x)\ ,x^0<y^0. \quad (5.166)$$

By construction $D_F(x-y)$ must satisfy the Green's function equation (5.163). The Green's function $D_F(x-y)$ is called the Feynman propagator for a real scalar field.

2.5.2 Dirac Propagator

The probability amplitudes for a Dirac particle to propagate from the spacetime point y to the spacetime x is

$$S_{ab}(x-y)=< 0|\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0 > . \quad (5.167)$$

The probability amplitudes for a Dirac antiparticle to propagate from the spacetime point x to the spacetime y is

$$\bar{S}_{ba}(y-x) = <0|\bar{\hat{\psi}}_b(y)\hat{\psi}_a(x)|0> . \tag{5.168}$$

We compute

$$S_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D(x-y). \tag{5.169}$$

$$\bar{S}_{ba}(y-x) = -\frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D(y-x). \tag{5.170}$$

The retarded Green's function of the Dirac equation can be defined by

$$(S_R)_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D_R(x-y). \tag{5.171}$$

It is not difficult to convince ourselves that

$$(S_R)_{ab}(x-y) = \theta(x^0-y^0) < 0|\{\hat{\psi}_a(x), \bar{\hat{\psi}}_b(y)\}_+|0> . \tag{5.172}$$

This satisfies the equation

$$(i\hbar\gamma^\mu\partial_\mu^x - mc)_{ca}(S_R)_{ab}(x-y) = i\hbar\delta^4(x-y)\delta_{cb}. \tag{5.173}$$

Another solution of this equation is the so-called Feynman propagator for a Dirac spinor field given by

$$(S_F)_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D_F(x-y). \tag{5.174}$$

We compute

$$(S_F)_{ab}(x-y) = <0|T\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> . \tag{5.175}$$

The time-ordering operator is defined by

$$\begin{aligned} T\hat{\psi}(x)\hat{\psi}(y) &= \hat{\psi}(x)\hat{\psi}(y) \ , \ x^0 > y^0 \\ T\hat{\psi}(x)\hat{\psi}(y) &= -\hat{\psi}(y)\hat{\psi}(x) \ , x^0 < y^0. \end{aligned} \tag{5.176}$$

By construction $S_F(x-y)$ must satisfy the Green's function equation (5.173). This can also be checked directly from the Fourier expansion of $S_F(x-y)$ given by

$$(S_F)_{ab}(x-y) = \hbar\int \frac{d^4p}{(2\pi\hbar)^4}\frac{i(\gamma^\mu p_\mu + mc)_{ab}}{p^2 - m^2c^2 + i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}. \tag{5.177}$$

2.6 Discrete Symmetries

In the quantum theory corresponding to each continuous Lorentz transformation Λ there is a unitary transformation $U(\Lambda)$ acting in the Hilbert space of state vectors. Indeed all state vectors $|\alpha>$ will transform under Lorentz transformations as $|\alpha>\longrightarrow U(\Lambda)|\alpha>$. In order that the general matrix elements $<\beta|\mathcal{O}(\hat{\psi},\bar{\hat{\psi}})|\alpha>$ be Lorentz invariant the field operator $\hat{\psi}(x)$ must transform as

$$\hat{\psi}(x)\longrightarrow \hat{\psi}^{'}(x)=U(\Lambda)^{+}\hat{\psi}(x)U(\Lambda). \tag{6.178}$$

Hence we must have

$$S(\Lambda)\hat{\psi}(\Lambda^{-1}x)=U(\Lambda)^{+}\hat{\psi}(x)U(\Lambda). \tag{6.179}$$

In the case of a scalar field $\hat{\phi}(x)$ we must have instead

$$\hat{\phi}(\Lambda^{-1}x)=U(\Lambda)^{+}\hat{\phi}(x)U(\Lambda). \tag{6.180}$$

There are two discrete spacetime symmetries of great importance to particle physics. The first discrete transformation is parity defined by

$$(t,\vec{x})\longrightarrow P(t,\vec{x})=(t,-\vec{x}). \tag{6.181}$$

The second discrete transformation is time reversal defined by

$$(t,\vec{x})\longrightarrow T(t,\vec{x})=(-t,\vec{x}). \tag{6.182}$$

The Lorentz group consists of four disconnected subroups. The subgroup of continuous Lorentz transformations consists of all Lorentz transfomrations which can be obtained from the identity transformation. This is called the proper orthochronous Lorentz group. The improper orthochronous Lorentz group is obtained by the action of parity on the proper orthochronous Lorentz group. The proper nonorthochronous Lorentz group is obtained by the action of time reversal on the proper orthochronous Lorentz group. The improper nonorthochronous Lorentz group is obtained by the action of parity and then time reversal or by the action of time reversal and then parity on the proper orthochronous Lorentz group.

A third discrete symmetry of fundamental importance to particle physics is charge conjugation operation C. This is not a spacetime symmetry. This is a symmetry under which particles become their antiparticles. It is well known that parity P, time reversal T and charge conjugation C are symmetries of gravitational, electromagnetic and strong interactions. The weak interactions violate P and C and to a lesser extent T and CP but it is observed that all fundamental forces conserve CPT.

2.6.1 Parity

The action of parity on the spinor field operator is

$$\begin{aligned}U(P)^+\hat{\psi}(x)U(P) &= \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})U(P)^+\hat{b}(\vec{p},i)U(P)\\&+ e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})U(P)^+\hat{d}(\vec{p},i)^+U(P)\bigg)\\&= S(P)\hat{\psi}(P^{-1}x).\end{aligned}\tag{6.183}$$

We need to rewrite this operator in terms of $\tilde{x}=P^{-1}x=(x^0,-\vec{x})$. Thus $px=\tilde{p}\tilde{x}$ where $\tilde{p}=P^{-1}x=(p^0,-\vec{p})$. We have also $\sigma p=\bar{\sigma}\tilde{p}$ and $\bar{\sigma}p=\sigma\tilde{p}$. As a consequence we have

$$u^{(i)}(\vec{p})=\gamma^0u^{(i)}(\vec{\tilde{p}})\ ,\ v^{(i)}(\vec{p})=-\gamma^0v^{(i)}(\vec{\tilde{p}}).\tag{6.184}$$

Hence

$$\begin{aligned}U(P)^+\hat{\psi}(x)U(P) &= \gamma^0\frac{1}{\hbar}\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(i)}(\vec{\tilde{p}})U(P)^+\hat{b}(\vec{p},i)U(P)\\&- e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(i)}(\vec{\tilde{p}})U(P)^+\hat{d}(\vec{p},i)^+U(P)\bigg).\end{aligned}\tag{6.185}$$

The parity operation flips the direction of the momentum but not the direction of the spin. Thus we expect that

$$U(P)^+\hat{b}(\vec{p},i)U(P)=\eta_b\hat{b}(-\vec{p},i)\ ,\ U(P)^+\hat{d}(\vec{p},i)U(P)=\eta_d\hat{d}(-\vec{p},i).\tag{6.186}$$

The phases η_b and η_a must clearly satisfy

$$\eta_b^2=1\ ,\ \eta_d^2=1.\tag{6.187}$$

Hence we obtain

$$U(P)^+\hat{\psi}(x)U(P) = \gamma^0\frac{1}{\hbar}\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(\eta_be^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(i)}(\vec{\tilde{p}})\hat{b}(\vec{\tilde{p}},i)-\eta_d^*e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(i)}(\vec{\tilde{p}})\hat{d}(\vec{\tilde{p}},i)^+\bigg).\tag{6.188}$$

This should equal $S(P)\hat{\psi}(\tilde{x})$. Immediately we conclude that we must have

$$\eta_d^*=-\eta_b.\tag{6.189}$$

Hence

$$U(P)^+\hat{\psi}(x)U(P) = \eta_b\gamma^0\hat{\psi}(\tilde{x}).\tag{6.190}$$

2.6.2 Time Reversal

The action of time reversal on the spinor field operator is

$$\begin{aligned} U(T)^+\hat{\psi}(x)U(T) &= \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(U(T)^+e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})\hat{b}(\vec{p},i)U(T)\\ &+ U(T)^+e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})\hat{d}(\vec{p},i)^+U(T)\bigg)\\ &= S(T)\hat{\psi}(T^{-1}x). \end{aligned} \tag{6.191}$$

This needs to be rewritten in terms of $\tilde{x}=T^{-1}x=(-x^0,\vec{x})$. Time reversal reverses the direction of the momentum in the sense that $px=-\tilde{p}\tilde{x}$ where $\tilde{p}=(p^0,-\vec{p})$. Clearly if $U(T)$ is an ordinary unitary operator the phases $e^{\mp\frac{i}{\hbar}px}$ will go to their complex conjugates $e^{\pm\frac{i}{\hbar}px}$ under time reversal. In other words if $U(T)$ is an ordinary unitary operator the field operator $U(T)^+\hat{\psi}(x)U(T)$ can not be written as a constant matrix times $\hat{\psi}(\tilde{x})$. The solution is to choose $U(T)$ to be an antilinear operator defined by

$$U(T)^+c=c^*U(T)^+. \tag{6.192}$$

Hence we get

$$\begin{aligned} U(T)^+\hat{\psi}(x)U(T) &= \frac{1}{\hbar}\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(i)*}(\vec{p})U(T)^+\hat{b}(\vec{p},i)U(T)\\ &+ e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(i)*}(\vec{p})U(T)^+\hat{d}(\vec{p},i)^+U(T)\bigg). \end{aligned} \tag{6.193}$$

We recall that

$$u^{(1)}(\vec{p})=N^{(1)}\begin{pmatrix}\xi_0^1\\ \frac{\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}\xi_0^1\end{pmatrix}\ ,\ v^{(1)}=N^{(3)}\begin{pmatrix}-\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\eta_0^1\\ \eta_0^1\end{pmatrix}. \tag{6.194}$$

Hence (by using $\sigma^{i*}=-\sigma^2\sigma^i\sigma^2$) we obtain

$$u^{(1)*}(\vec{p}) = N^{(1)}\begin{pmatrix}\xi_0^{1*}\\ \sigma^2\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\sigma^2\xi_0^{1*}\end{pmatrix}=N^{(1)}\gamma^1\gamma^3\begin{pmatrix}-i\sigma^2\xi_0^{1*}\\ \frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}(-i\sigma^2\xi_0^{1*})\end{pmatrix}. \tag{6.195}$$

$$v^{(1)*}(\vec{p}) = N^{(3)}\begin{pmatrix}\sigma^2\frac{-\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}\sigma^2\eta_0^{1*}\\ \eta_0^{1*}\end{pmatrix}=N^{(3)}\gamma^1\gamma^3\begin{pmatrix}\frac{-\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}(-i\sigma^2\eta_0^{1*})\\ -i\sigma^2\eta_0^{1*}\end{pmatrix}. \tag{6.196}$$

We define

$$\xi_0^{-s} = -i\sigma^2 \xi_0^{s*} \ , \ \eta_0^{-s} = i\sigma^2 \eta_0^{s*}. \tag{6.197}$$

Note that we can take ξ_0^{-s} proportional to η_0^s. We obtain then

$$u^{(1)*}(\vec{p}) = N^{(1)}\gamma^1\gamma^3 \begin{pmatrix} \xi_0^{-1} \\ \frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\xi_0^{-1} \end{pmatrix} = \gamma^1\gamma^3 \begin{pmatrix} \sqrt{\sigma_\mu \tilde{p}^\mu}\xi^{-1} \\ \sqrt{\bar{\sigma}_\mu \tilde{p}^\mu}\xi^{-1} \end{pmatrix} = \gamma^1\gamma^3 u^{(-1)}(\vec{\tilde{p}}). \tag{6.198}$$

$$v^{(1)*}(\vec{p}) = -N^{(3)}\gamma^1\gamma^3 \begin{pmatrix} \frac{-\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}\eta_0^{-1} \\ \eta_0^{-1} \end{pmatrix} = -\gamma^1\gamma^3 \begin{pmatrix} \sqrt{\sigma_\mu \tilde{p}^\mu}\eta^{-1} \\ -\sqrt{\bar{\sigma}_\mu \tilde{p}^\mu}\eta^{-1} \end{pmatrix} = -\gamma^1\gamma^3 v^{(-1)}(\vec{\tilde{p}}). \tag{6.199}$$

Similarly we can show that

$$u^{(2)*}(\vec{p}) = \gamma^1\gamma^3 u^{(-2)}(\vec{\tilde{p}}) \ , \ v^{(2)*}(\vec{p}) = -\gamma^1\gamma^3 v^{(-2)}(\vec{\tilde{p}}). \tag{6.200}$$

In the above equations

$$\xi^{-s} = N^{(1)}(-\tilde{p}^3)\frac{1}{\sqrt{\sigma_\mu \tilde{p}^\mu}}\xi_0^{-s} \ , \ \eta^{-s} = -N^{(3)}(-\tilde{p}^3)\frac{1}{\sqrt{\bar{\sigma}_\mu \tilde{p}^\mu}}\eta_0^s. \tag{6.201}$$

Let us remark that if ξ_0^i is an eigenvector of $\vec{\sigma}\hat{n}$ with spin s then ξ_0^{-i} is an eigenvector of $\vec{\sigma}\hat{n}$ with spin $-s$, viz

$$\vec{\sigma}\hat{n}\xi_0^i = s\xi_0^i \ \Leftrightarrow \vec{\sigma}\hat{n}\xi_0^{-i} = -s\xi_0^{-i}. \tag{6.202}$$

Now going back to equation (6.193) we get

$$U(T)^+\hat{\psi}(x)U(T) = \frac{1}{\hbar}\gamma^1\gamma^3 \int \frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}} \sum_i \bigg(e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}} u^{(-i)}(\vec{\tilde{p}})U(T)^+\hat{b}(\vec{p},i)U(T) - e^{\frac{i}{\hbar}\tilde{p}\tilde{x}} v^{(-i)}(\vec{\tilde{p}})U(T)^+\hat{d}(\vec{p},i)^+U(T)\bigg). \tag{6.203}$$

Time reversal reverses the direction of the momentum and of the spin. Thus we write

$$U(T)^+\hat{b}(\vec{p},i)U(T) = \eta_b\hat{b}(-\vec{p},-i) \ , \ U(T)^+\hat{d}(\vec{p},i)U(T) = \eta_d\hat{d}(-\vec{p},-i). \tag{6.204}$$

We get then

$$U(T)^+\hat{\psi}(x)U(T) = \frac{1}{\hbar}\gamma^1\gamma^3\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(\eta_b e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(-i)}(\vec{\tilde{p}})\hat{b}(\vec{\tilde{p}},-i) - \eta_d^* e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(-i)}(\vec{\tilde{p}})\hat{d}(\vec{\tilde{p}},-i)^+\bigg). \quad (6.205)$$

By analogy with $\xi_0^{-s}=-i\sigma^2\xi_0^{s*}$ we define

$$\hat{b}(\vec{\tilde{p}},-i) = -(-i\sigma^2)_{ij}\hat{b}(\vec{\tilde{p}},j)\ ,\ \hat{d}(\vec{\tilde{p}},-i) = -(-i\sigma^2)_{ij}\hat{d}(\vec{\tilde{p}},j). \quad (6.206)$$

Also we choose

$$\eta_d^* = -\eta_b. \quad (6.207)$$

Hence

$$U(T)^+\hat{\psi}(x)U(T) = \frac{\eta_b}{\hbar}\gamma^1\gamma^3\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(-i)}(\vec{\tilde{p}})\hat{b}(\vec{\tilde{p}},-i) + e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(-i)}(\vec{\tilde{p}})\hat{d}(\vec{\tilde{p}},-i)^+\bigg)$$
$$= \eta_b\gamma^1\gamma^3\hat{\psi}(-x^0,\vec{x}). \quad (6.208)$$

2.6.3 Charge Conjugation

This is defined simply by (with $C^+=C^{-1}=C$)

$$C\hat{b}(\vec{p},i)C = \hat{d}(\vec{p},i)\ ,\ C\hat{d}(\vec{p},i)C = \hat{b}(\vec{p},i) \quad (6.209)$$

Hence

$$C\hat{\psi}(x)C = \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})\hat{d}(\vec{p},i) + e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})\hat{b}(\vec{p},i)^+\bigg). \quad (6.210)$$

Let us remark that (by choosing $N^{(1)}\xi_0^{-i}=-N^{(3)}\eta_0^i$ or equivalently $\xi^{-i}=\eta^i$Řş)

$$u^{(1)*}(\vec{p}) = iN^{(1)}\gamma^2\begin{pmatrix}-\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\xi_0^{-1}\\ \xi_0^{-1}\end{pmatrix} = -iN^{(3)}\gamma^2\begin{pmatrix}-\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\eta_0^{1}\\ \eta_0^{1}\end{pmatrix} = -i\gamma^2v^{(1)}(\vec{p}). \quad (6.211)$$

In other words

$$u^{(1)}(\vec{p}) = -i\gamma^2v^{(1)*}(\vec{p})\ ,\ v^{(1)}(\vec{p}) = -i\gamma^2u^{(1)*}(\vec{p}). \quad (6.212)$$

Similarly we find

$$u^{(2)}(\vec{p}) = -i\gamma^2 v^{(2)*}(\vec{p})\ ,\ v^{(2)}(\vec{p}) = -i\gamma^2 u^{(2)*}(\vec{p}). \tag{6.213}$$

Thus we have

$$\begin{aligned} C\hat{\psi}(x)C &= \frac{1}{\hbar}(-i\gamma^2)\int \frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i \left(e^{-\frac{i}{\hbar}px}v^{(i)*}(\vec{p})\hat{d}(\vec{p},i) + e^{\frac{i}{\hbar}px}u^{(i)*}(\vec{p})\hat{b}(\vec{p},i)^+\right) \\ &= \frac{1}{\hbar}(-i\gamma^2)\int \frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i \left(e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})\hat{d}(\vec{p},i)^+ + e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})\hat{b}(\vec{p},i)\right)^* \\ &= -i\gamma^2\psi^*(x). \end{aligned} \tag{6.214}$$

2.7 Exercises and Problems

Scalars Commutation Relations Show that

- $$\hat{Q}(x^0, -\vec{p}) = \hat{Q}^+(x^0, \vec{p}).$$

- $$[\hat{Q}(x^0, \vec{p}), \hat{P}^+(x^0, \vec{q})] = i\hbar(2\pi\hbar)^3\delta^3(\vec{p} - \vec{q}).$$

- $$[\hat{a}(\vec{p}), \hat{a}(\vec{q})^+] = \hbar(2\pi\hbar)^3\delta^3(\vec{p} - \vec{q}).$$

The One-Particle States For a real scalar field theory the one-particle states are defined by

$$|\vec{p}> = \frac{1}{c}\sqrt{2\omega(\vec{p})}\hat{a}(\vec{p})^+|0> .$$

- Compute the energy of this state.
- Compute the scalar product $<\vec{p}|\vec{q}>$ and show that it is Lorentz invariant.
- Show that $\hat{\phi}(x)|0>$ can be interpreted as the eigenstate $|\vec{x}>$ of the position operator at time x^0.

Momentum Operator

1) Compute the total momentum operator of a quantum real scalar field in terms of the creation and annihilation operators $\hat{a}(\vec{p})^+$ and $\hat{a}(\vec{p})$.
2) What is the total momentum operator for a Dirac field.

Fermions Anticommutation Relations Show that

- $$[\hat{\chi}_\alpha(x^0, \vec{p}), \hat{\chi}_\beta^+(x^0, \vec{q})]_+ = \hbar^2\delta_{\alpha\beta}(2\pi\hbar)^3\delta^3(\vec{p} - \vec{q}).$$

- $$[\hat{b}(\vec{p}, i), \hat{b}(\vec{q}, j)^+]_+ = [\hat{d}(\vec{p}, i)^+, \hat{d}(\vec{q}, j)]_+ = \hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p} - \vec{q}).$$

$$[\hat{b}(\vec{p}, i), \hat{d}(\vec{q}, j)]_+ = [\hat{d}(\vec{q}, j)^+, \hat{b}(\vec{p}, i)]_\pm = 0.$$

Retarded Propagator The retarded propagator is

$$D_R(x-y) = c\hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2-m^2c^2}e^{-\frac{i}{\hbar}p(x-y)}.$$

Show that the Klein-Gordon equation with contact term, viz

$$(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2})D_R(x-y) = -i\frac{c}{\hbar}\delta^4(x-y).$$

Feynman Propagator We give the scalar Feynman propagator by the equation

$$D_F(x-y) = c\hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2-m^2c^2+i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}.$$

- Perform the integral over p_0 and show that

$$D_F(x-y) = \theta(x^0-y^0)D(x-y)+\theta(y^0-x^0)D(y-x).$$

- Show that

$$D_F(x-y) = <0|T\hat{\phi}(x)\hat{\phi}(y)|0>,$$

  where T is the time-ordering operator.

The Dirac Propagator The probability amplitudes for a Dirac particle (antiparticle) to propagate from the spacetime point y (x) to the spacetime x (y) are

$$S_{ab}(x-y) =<0|\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)|0> .$$

$$\bar{S}_{ba}(y-x) =<0|\hat{\bar{\psi}}_b(y)\hat{\psi}_a(x)|0> .$$

1) Compute S and $\bar{S}$ in terms of the Klein-Gordon propagator $D(x-y)$ given by

$$D(x-y) = \int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}.$$

2) Show that the retarded Green's function of the Dirac equation is given by

$$(S_R)_{ab}(x-y) = <0|\{\hat{\psi}_a(x),\hat{\bar{\psi}}_b(y)\}|0> .$$

3) Verify that S_R satisfies the Dirac equation

$$(i\hbar\gamma^\mu\partial^x_\mu-mc)_{ca}(S_R)_{ab}(x-y) = i\frac{\hbar}{c}\delta^4(x-y)\delta_{cb}.$$

4) Derive an expression of the Feynman propagator in terms of the Dirac fields $\hat{\psi}$ and $\hat{\bar{\psi}}$ and then write down its Fourier Expansion.

Dirac Hamiltonian Show that the Dirac Hamiltonian

$$\hat{H}_{\rm Dirac} = \int \frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\sum_i \bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i)+\hat{d}(\vec{p},i)^+\hat{d}(\vec{p},i)\bigg),$$

satisfies

$$[\hat{H}_{\rm Dirac},\hat{b}(\vec{p},i)^+]=\hbar\omega(\vec{p})\hat{b}(\vec{p},i)^+ \ , \ [\hat{H}_{\rm Dirac},\hat{d}(\vec{p},i)^+]=\hbar\omega(\vec{p})\hat{d}(\vec{p},i)^+.$$

Energy-Momentum Tensor Noether's theorem states that each continuous symmetry transformation which leaves the action invariant corresponds to a conservation law and as a consequence leads to a constant of the motion.

We consider a single real scalar field ϕ with a Lagrangian density $\mathcal{L}(\phi,\partial_\mu\phi)$. Prove Noether's theorem for spacetime translations given by

$$x^\mu \longrightarrow x^{'\mu}=x^\mu+a^\mu.$$

In particular determine the four conserved currents and the four conserved charges (constants of the motion) in terms of the field ϕ.

Electric Charge

1) The continuity equation for a Dirac wave function is

$$\partial_\mu J^\mu=0 \ , \ J^\mu=\bar{\psi}\gamma^\mu\psi.$$

The current J^μ is conserved. According to Noether's theorem this conserved current (when we go to the field theory) must correspond to the invariance of the action under a symmetry principle. Determine the symmetry transformations in this case.

2) The associated conserved charge is

$$Q=\int d^3x J^0.$$

Compute Q for a quantized Dirac field. What is the physical interpretation of Q.

Chiral Invariance

1) Rewrite the Dirac Lagrangian in terms of ψ_L and ψ_R.

2) The Dirac Lagrangian is invariant under the vector transformations

$$\psi \longrightarrow e^{i\alpha}\psi.$$

Derive the conserved current j^μ.

3) The Dirac Lagrangian is almost invariant under the axial vector transformations

$$\psi \longrightarrow e^{i\gamma^5\alpha}\psi.$$

Derive the would-be current $j^{\mu 5}$ in this case. Determine the condition under which this becomes a conserved current.

4) Show that in the massless limit

$$j^\mu = j^\mu_L + j^\mu_R \ , \ j^{\mu 5} = -j^\mu_L + j^\mu_R.$$

$$j^\mu_L = \bar{\Psi}_L\gamma^\mu\Psi_L \ , \ j^\mu_R = \bar{\Psi}_R\gamma^\mu\Psi_R.$$

Parity and Time Reversal Determine the transformation rule under parity and time reversal transformations of $\bar{\psi}$, $\bar{\psi}\psi$, $i\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma^\mu\gamma^5\psi$.

Angular Momentum of Dirac Field

- Write down the infinitesimal Lorentz transformation corresponding to an infintesimal rotation around the z axis with an angle θ.
- From the effect of a Lorentz transformation on a Dirac spinor calculate the variation in the field at a fixed point, viz

$$\delta\psi(x) = \psi^{'}(x) - \psi(x).$$

- Using Noether's theorem compute the conserved current j^μ associated with the invariance of the Lagrangian under the above rotation. The charge J^3 is defined by

$$J^3 \ = \ \int d^3x j^0.$$

Show that J^3 is conserved and derive an expression for it in terms of the Dirac field. What is the physical interpretation of J^3. What is the charge in the case of a general rotation.

- In the quantum theory J^3 becomes an operator. What is the angular momentum of the vacuum.

- What is the angular momentum of a one-particle zero-momentum state defined by

$$|\vec{0}, sb >= \sqrt{\frac{2mc^2}{\hbar}}\hat{b}(\vec{0}, s)^+|0 > .$$

  Hint: In order to answer this question we need to compute the commutator $[\hat{J}^3, \hat{b}(\vec{0}, s)^+]$.

- By analogy what is the angular momentum of a one-antiparticle zero-momentum state defined by

$$|\vec{0}, sd >= \sqrt{\frac{2mc^2}{\hbar}}\hat{d}(\vec{0}, s)^+|0 > .$$

2.8 Solutions

Scalars Commutation Relations Straightforward.

The One-Particle States

- The Hamiltonian operator of a real scalar field is given by (ignoring an infinite constant due to vacuum energy)

$$\hat{H}_{\rm KG} = \int \frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\hat{a}(\vec{p})^+\hat{a}(\vec{p}).$$

It satisfies

$$\hat{H}_{\rm KG}|0> = 0.$$

$$[\hat{H}_{\rm KG},\hat{a}(\vec{p})^+] = \hbar\omega(\vec{p})\hat{a}(\vec{p})^+ \ , \ [\hat{H},\hat{a}(\vec{p})] = -\hbar\omega(\vec{p})\hat{a}(\vec{p}).$$

Thus we compute

$$\begin{aligned}
\hat{H}_{\rm KG}|\vec{p}> &= \frac{1}{c}\sqrt{2\omega(\vec{p})}\hat{H}_{\rm KG}\hat{a}(\vec{p})^+|0> \\
&= \frac{1}{c}\sqrt{2\omega(\vec{p})}[\hat{H}_{\rm KG},\hat{a}(\vec{p})^+]|0> \\
&= \frac{1}{c}\sqrt{2\omega(\vec{p})}\hbar\omega(\vec{p})\hat{a}(\vec{p})^+|0> \\
&= \hbar\omega(\vec{p})|\vec{p}> .
\end{aligned}$$

- Next we compute

$$< \vec{p}|\vec{q}>= \frac{2}{c^2}(2\pi\hbar)^3E(\vec{p})\delta^3(\vec{p}-\vec{q}).$$

We have assumed that $< 0|0 >= 1$. This is Lorentz invariant since $E(\vec{p})\delta^3(\vec{p}-\vec{q})$ is Lorentz invariant. Let us consider a Lorentz boost along the $x-$direction, viz

$$x^{0'} = \gamma(x^0-\beta x^1) \ , \ x^{1'} = \gamma(x^1-\beta x^0) \ , \ x^{2'} = x^2 \ , \ x^{3'} = x^3.$$

The energy-momentum $4-$vector $p^\mu = (p^0,p^i) = (E/c,p^i)$ will transform as

$$p^{0'} = \gamma(p^0-\beta p^1) \ , \ p^{1'} = \gamma(p^1-\beta p^0) \ , \ p^{2'} = p^2 \ , \ p^{3'} = p^3.$$

We compute

$$\begin{aligned}\delta(p^1-q^1) &= \delta(p^{1'}-q^{1'})\frac{dp^{1'}}{dp^1}\\ &= \delta(p^{1'}-q^{1'})\gamma(1-\beta\frac{dp^0}{dp^1})\\ &= \delta(p^{1'}-q^{1'})\gamma(1-\beta\frac{p^1}{p^0})\\ &= \delta(p^{1'}-q^{1'})\frac{p^{0'}}{p^0}.\end{aligned}$$

Hence we have

$$p^0\delta(\vec{p}-\vec{q}) = p^{0'}\delta(\vec{p}^{\,'}-\vec{q}^{\,'}).$$

- The completeness relation on the Hilbert subspace of one-particle states is

$$\mathbf{1}_{\rm one-particle}=c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}|\vec{p}><\vec{p}|. \quad (8.215)$$

It is straightforward to compute

$$\hat{\phi}(x^0,\vec{x})|0>=c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}|\vec{p}>\;\; e^{\frac{i}{\hbar}(E(\vec{p})t-\vec{p}\vec{x})}. \quad (8.216)$$

This is a linear combination of one-particle states. For small $\vec{p}$ we can make the approximation $E(\vec{p})\simeq mc^2$ and as a consequence

$$\hat{\phi}(x^0,\vec{x})|0>=\frac{e^{\frac{i}{\hbar}mc^2t}}{2m}\int\frac{d^3p}{(2\pi\hbar)^3}|\vec{p}>\;\; e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \quad (8.217)$$

In this case the Dirac orthonormalization and the completeness relations read

$$<\vec{p}|\vec{q}>=2m(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \quad (8.218)$$

$$\mathbf{1}_{\rm one-particle}=\frac{1}{2m}\int\frac{d^3p}{(2\pi\hbar)^3}|\vec{p}><\vec{p}|. \quad (8.219)$$

The eigenstates $|\vec{x}>$ of the position operator can be defined by

$$<\vec{p}|\vec{x}>=\sqrt{2m}e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \quad (8.220)$$

Hence

$$\hat{\phi}(x^0,\vec{x})|0>=\frac{e^{\frac{i}{\hbar}mc^2t}}{\sqrt{2m}}|\vec{x}>. \quad (8.221)$$

In other words in the relativistic theory the operator $\hat{\phi}(x^0,\vec{x})|0>$ should be interpreted as the eigenstate $|\vec{x}>$ of the position operator. Indeed we can compute in the relativistic theory

$$<0|\hat{\phi}(x^0,\vec{x})|\vec{p}>=e^{-\frac{i}{\hbar}px}\ ,\ px=E(\vec{p})t-\vec{p}\vec{x}. \quad (8.222)$$

We say that the field operator $\hat{\phi}(x^0,\vec{x})$ creates a particle at the point $\vec{x}$ at time $t=x^0/c$.

Momentum Operator

- For a real scalar field

$$\hat{P}_i = c\int d^3x\hat{\pi}\partial_i\hat{\phi} = \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\vec{p}\hat{a}(\vec{p})^+\hat{a}(\vec{p}).$$

- For a Dirac field

$$\hat{P}_i = \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\vec{p}\sum_i\bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i)+\hat{d}(\vec{p},i)^+\hat{d}(\vec{p},i)\bigg).$$

Fermions Anticommutation Relations

- We have

$$\hat{\chi}(x^0,\vec{p})=\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})\hat{b}(\vec{p},i)+e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})\hat{d}(-\vec{p},i)^+\bigg).$$

We compute

$$\begin{aligned}[\hat{\chi}_\alpha(x^0,\vec{p}),\hat{\chi}^+_\beta(x^0,\vec{q})]_\pm &= \frac{c}{2\sqrt{\omega(\vec{p})\omega(\vec{q})}}\sum_{i,j}e^{i(\omega(\vec{p})-\omega(\vec{q}))t}u^{(i)}_\alpha(\vec{p})u^{(j)*}_\beta(\vec{q})[\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^+]_\pm\\
&+ \frac{c}{2\sqrt{\omega(\vec{p})\omega(\vec{q})}}\sum_{i,j}e^{-i(\omega(\vec{p})+\omega(\vec{q}))t}u^{(i)}_\alpha(\vec{p})v^{(j)*}_\beta(-\vec{q})[\hat{b}(\vec{p},i),\hat{d}(-\vec{q},j)]_\pm\\
&+ \frac{c}{2\sqrt{\omega(\vec{p})\omega(\vec{q})}}\sum_{i,j}e^{i(\omega(\vec{p})+\omega(\vec{q}))t}v^{(i)}_\alpha(-\vec{p})u^{(j)*}_\beta(\vec{q})[\hat{d}(-\vec{p},i)^+,\hat{b}(\vec{q},j)^+]_\pm\\
&+ \frac{c}{2\sqrt{\omega(\vec{p})\omega(\vec{q})}}\sum_{i,j}e^{i(\omega(\vec{p})-\omega(\vec{q}))t}v^{(i)}_\alpha(-\vec{p})v^{(j)*}_\beta(-\vec{q})[\hat{d}(-\vec{p},i)^+,\hat{d}(-\vec{q},j)]_\pm.\end{aligned}$$

We impose

$$[\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^+]_\pm=\hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}),$$

$$[\hat{d}(\vec{p},i)^+,\hat{d}(\vec{q},j)]_\pm=\hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}),$$

and

$$[\hat{b}(\vec{p},i),\hat{d}(\vec{q},j)]_\pm=[\hat{d}(\vec{q},j)^+,\hat{b}(\vec{p},i)]_\pm=0.$$

Thus we get

$$\begin{aligned}[\hat{\chi}_\alpha(x^0,\vec{p}),\hat{\chi}^+_\beta(x^0,\vec{q})]_\pm &= \frac{c\hbar}{2\omega(\vec{p})}\sum_i u^{(i)}_\alpha(\vec{p})u^{(i)*}_\beta(\vec{p})(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})\\ &+ \frac{c\hbar}{2\omega(\vec{p})}\sum_i v^{(i)}_\alpha(-\vec{p})v^{(i)*}_\beta(-\vec{p})(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}).\end{aligned}$$

By using the completeness relations $\sum_s u^{(s)}(E,\vec{p})\bar{u}^{(s)}(E,\vec{p})=\gamma^\mu p_\mu+mc$ and $\sum_s v^{(s)}(E,\vec{p})\bar{v}^{(s)}(E,\vec{p})=\gamma^\mu p_\mu-mc$ we derive

$$\sum_i u^{(i)}_\alpha(E,\vec{p})u^{(i)*}_\beta(E,\vec{p})+\sum_i v^{(i)}_\alpha(E,-\vec{p})v^{(i)*}_\beta(E,-\vec{p})=\frac{2E(\vec{p})}{c}\delta_{\alpha\beta}.$$

We get then the desired result

$$[\hat{\chi}_\alpha(x^0,\vec{p}),\hat{\chi}^+_\beta(x^0,\vec{q})]_\pm=\hbar^2\delta_{\alpha\beta}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}).$$

- Straightforward.

Retarded Propagator Straightforward.

Feynman Propagator Straightforward.

The Dirac Propagator

- We compute

$$\begin{aligned}
S_{ab}(x-y) &= c\int \frac{d^3p}{(2\pi\hbar)^3}\int \frac{d^3q}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}\frac{1}{2E(\vec{q})}\sum_{i,j} e^{\frac{i}{\hbar}py}e^{-\frac{i}{\hbar}qx}u_a^{(i)}(\vec{q})\bar{u}_b^{(j)}(\vec{p}) < \vec{q},ib|\vec{p},jb > \\
&= c\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}\sum_i u_a^{(i)}(\vec{p})\bar{u}_b^{(i)}(\vec{p}) \\
&= c\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}(\gamma^\mu p_\mu + mc)_{ab} \\
&= c(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)} \\
&= \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D(x-y).
\end{aligned}$$

Similarly

$$\begin{aligned}
\bar{S}_{ba}(y-x) &= c\int \frac{d^3p}{(2\pi\hbar)^3}\int \frac{d^3q}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}\frac{1}{2E(\vec{q})}\sum_{i,j} e^{-\frac{i}{\hbar}py}e^{\frac{i}{\hbar}qx}v_a^{(i)}(\vec{q})\bar{v}_b^{(j)}(\vec{p}) < \vec{p},jd|\vec{q},id > \\
&= c\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{\frac{i}{\hbar}p(x-y)}\sum_i v_a^{(i)}(\vec{p})\bar{v}_b^{(i)}(\vec{p}) \\
&= c\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{\frac{i}{\hbar}p(x-y)}(\gamma^\mu p_\mu - mc)_{ab} \\
&= -c(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{\frac{i}{\hbar}p(x-y)} \\
&= -\frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D(y-x).
\end{aligned}$$

- The retarded Green's function of the Dirac equation can be defined by

$$(S_R)_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D_R(x-y).$$

We compute

$$\begin{aligned}
(S_R)_{ab}(x-y) &= \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}\bigg(\theta(x^0-y^0) < 0|[\hat{\phi}(x),\hat{\phi}(y)]|0 > \bigg) \\
&= \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab} < 0|[\hat{\phi}(x),\hat{\phi}(y)]|0 > \\
&+ \frac{i\hbar}{c}\gamma^0_{ab}\partial_0^x\theta(x^0-y^0). < 0|[\hat{\phi}(x),\hat{\phi}(y)]|0 > \\
&= \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab} < 0|[\hat{\phi}(x),\hat{\phi}(y)]|0 > \\
&+ \frac{i\hbar}{c}\gamma^0_{ab}\delta(x^0-y^0). < 0|[\hat{\phi}(x),\hat{\phi}(y)]|0 > .
\end{aligned}$$

By inspection we will find that the second term will vanish. Thus we get

$$\begin{aligned}(S_R)_{ab}(x-y) &= \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x+mc)_{ab} < 0|[\hat{\phi}(x),\hat{\phi}(y)]|0> \\ &= \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x+mc)_{ab}D(x-y) \\ &- \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x+mc)_{ab}D(y-x) \\ &= \theta(x^0-y^0) < 0|\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> + \theta(x^0-y^0) < 0|\bar{\hat{\psi}}_b(y)\hat{\psi}_a(x)|0> \\ &= \theta(x^0-y^0) < 0|\{\hat{\psi}_a(x),\bar{\hat{\psi}}_b(y)\}|0> .\end{aligned}$$

- From the Fourier expansion of the retarded Green's function $D_R(x-y)$ we obtain

$$(S_R)_{ab}(x-y) = \hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i(\gamma^\mu p_\mu+mc)_{ab}}{p^2-m^2c^2}e^{-\frac{i}{\hbar}p(x-y)}.$$

We can immediately compute

$$\begin{aligned}(i\hbar\gamma^\mu\partial_\mu^x-mc)_{ca}(S_R)_{ab}(x-y) &= \hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i(\gamma^\mu p_\mu-mc)_{ca}(\gamma^\mu p_\mu+mc)_{ab}}{p^2-m^2c^2}e^{-\frac{i}{\hbar}p(x-y)} \\ &= i\hbar\delta^4(x-y)\delta_{cb}.\end{aligned}$$

- The Feynman propagator is defined by

$$(S_F)_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x+mc)_{ab}D_F(x-y).$$

We compute

$$\begin{aligned}(S_F)_{ab}(x-y) &= \theta(x^0-y^0) < 0|\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> - \theta(y^0-x^0) < 0|\bar{\hat{\psi}}_b(y)\hat{\psi}_a(x)|0> \\ &+ \frac{i\hbar}{c}(\gamma^0)_{ab}\delta(x^0-y^0)(D(x-y)-D(y-x)).\end{aligned}$$

Again the last term is zero and we end up with

$$(S_F)_{ab}(x-y) = < 0|T\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> .$$

T is the time-ordering operator. The Fourier expansion of $S_F(x-y)$ is

$$(S_F)_{ab}(x-y) = \hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i(\gamma^\mu p_\mu+mc)_{ab}}{p^2-m^2c^2+i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}.$$

Dirac Hamiltonian Straightforward.

Energy-Momentum Tensor We consider spacetime translations

$$x^\mu \longrightarrow x^{'\mu} = x^\mu + a^\mu.$$

The field ϕ transforms as

$$\phi \longrightarrow \phi^{'}(x^{'}) = \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi.$$

The Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ is a scalar and therefore it will transform as $\phi(x)$, viz

$$\mathcal{L} \longrightarrow \mathcal{L}^{'} = \mathcal{L} + \delta\mathcal{L} \ , \ \delta\mathcal{L} = \delta x^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} = a^\mu \partial_\mu \mathcal{L}.$$

This equation means that the action changes by a surface term and hence it is invariant under spacetime translations and as a consequence Euler-Lagrange equations of motion are not affected.

From the other hand the Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ transforms as

$$\begin{aligned} \delta\mathcal{L} &= \frac{\delta \mathcal{L}}{\delta \phi}\delta\phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\delta\partial_\mu\phi \\ &= \left(\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta\phi\right). \end{aligned}$$

By using Euler-Lagrange equations of motion we get

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta\phi\right).$$

Hence by comparing we get

$$a^\nu \partial^\mu \left(- \eta_{\mu\nu}\mathcal{L} + \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)}\delta\phi \right) = 0.$$

Equivalently

$$\partial^\mu T_{\mu\nu} = 0.$$

The four conserved currents $j_\mu^{(0)} = T_{\mu 0}$ (which is associated with time translations) and $j_\mu^{(i)} = T_{\mu i}$ (which are associated with space translations) are given by

$$T_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)}\partial_\nu \phi.$$

The conserved charges are (with $\pi = \delta\mathcal{L}/\delta(\partial_t\phi)$)

$$Q^{(0)} = \int d^3x j_0^{(0)} = \int d^3x T_{00} = \int d^3x(\pi\partial_t\phi - \mathcal{L}).$$

$$Q^{(i)} = \int d^3x j_0^{(i)} = \int d^3x T_{0i} = c\int d^3x \pi\partial_i\phi.$$

Clearly T_{00} is a Hamiltonian density and hence $Q^{(0)}$ is the Hamiltonian of the scalar field. By analogy T_{0i} is the momentum density and hence $Q^{(i)}$ is the momentum of the scalar field. We have then

$$Q^{(0)} = H \ , \ Q^{(i)} = P_i.$$

We compute

$$\frac{dH}{dt} = \int d^3x \frac{\partial T_{00}}{\partial t} = -c\int d^3x \partial^i T_{i0} = 0.$$

Similarly

$$\frac{dP_i}{dt} = 0.$$

In other words H and P_i are constants of the motion.

Electric Charge

- The Dirac Lagrangian density and as a consequence the action are invariant under the global gauge transformations

$$\psi \longrightarrow e^{i\alpha}\psi.$$

  Under a local gauge transformation the Dirac Lagrangian density changes by

$$\delta\mathcal{L}_{\rm Dirac} = -\hbar c\partial_\mu(\bar{\psi}\gamma^\mu\psi\alpha) + \hbar c\partial_\mu(\bar{\psi}\gamma^\mu\psi)\alpha.$$

  The total derivative leads to a surface term in the action and thus it is irrelevant. We get then

$$\delta\mathcal{L}_{\rm Dirac} = \hbar c\partial_\mu(\bar{\psi}\gamma^\mu\psi)\alpha.$$

  Imposing $\delta\mathcal{L}_{\rm Dirac} = 0$ leads immediately to $\partial_\mu J^\mu = 0$.

- We compute

$$\hat{Q} = \frac{1}{\hbar}\int \frac{d^3p}{(2\pi\hbar)^3}\sum_i \bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i) - \hat{d}(\vec{p},i)^+\hat{d}(\vec{p},i)\bigg).$$

  $\hat{Q}$ is the electric charge.

Chiral Invariance

- The Dirac Lagrangian in terms of ψ_L and ψ_R reads

$$\begin{aligned}\mathcal{L}_{\rm Dirac} &= \bar{\psi}(i\hbar c\gamma^\mu\partial_\mu - mc^2)\psi \\ &= i\hbar c\bigg(\psi_R^+(\partial_0+\sigma^i\partial_i)\psi_R + \psi_L^+(\partial_0-\sigma^i\partial_i)\psi_L\bigg) - mc^2\bigg(\psi_R^+\psi_L + \psi_L^+\psi_R\bigg).\end{aligned}$$

- This Lagrangian is invariant under the vector transformations

$$\psi \longrightarrow e^{i\alpha}\psi \Leftrightarrow \psi_L \longrightarrow e^{i\alpha}\psi_L \text{ and } \psi_R \longrightarrow e^{i\alpha}\psi_R.$$

The variation of the Dirac Lagrangian under these transformations is

$$\delta\mathcal{L}_{\rm Dirac} = \hbar c(\partial_\mu j^\mu)\alpha + \text{surface term} \ , \ j^\mu = \bar{\psi}\gamma^\mu\psi.$$

According to Noether's theorem each invariance of the action under a symmetry transformation corresponds to a conserved current. In this case the conserved current is the electric current density

$$j^\mu = \bar{\psi}\gamma^\mu\psi.$$

- The Dirac Lagrangian is also almost invariant under the axial vector (or chiral) transformations

$$\psi \longrightarrow e^{i\gamma^5\alpha}\psi \Leftrightarrow \psi_L \longrightarrow e^{i\gamma^5\alpha}\psi_L \text{ and } \psi_R \longrightarrow e^{i\gamma^5\alpha}\psi_R.$$

The variation of the Dirac Lagrangian under these transformations is

$$\delta\mathcal{L}_{\rm Dirac} = \bigg(\hbar c(\partial_\mu j^{\mu 5}) - 2imc^2\bar{\psi}\gamma^5\psi\bigg)\alpha + \text{surface term} \ , \ j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

Imposing $\delta\mathcal{L}_{\rm Dirac}=0$ yields

$$\partial_\mu j^{\mu 5} = 2i\frac{mc}{\hbar}\bar{\psi}\gamma^5\psi.$$

Hence the current $j^{\mu 5}$ is conserved only in the massless limit.

- In the massless limit we have two conserved currents j^μ and $j^{\mu 5}$. They can be rewritten as

$$j^\mu = j^\mu_L + j^\mu_R \ , \ j^{\mu 5} = -j^\mu_L + j^\mu_R.$$

$$j^\mu_L = \bar{\Psi}_L\gamma^\mu\Psi_L \ , \ j^\mu_R = \bar{\Psi}_R\gamma^\mu\Psi_R.$$

These are electric current densities associated with left-handed and right-handed particles.

Parity and Time Reversal Under parity we have

$$U(P)^+\hat{\psi}(x)U(P) = \eta_b\gamma^0\hat{\psi}(\tilde{x}).$$

Immediately we get

$$U(P)^+\bar{\hat{\psi}}(x)U(P) = \eta_b^*\bar{\hat{\psi}}(\tilde{x})\gamma^0.$$

Hence

$$U(P)^+\bar{\hat{\psi}}\hat{\psi}(x)U(P) = |\eta_b|^2\bar{\hat{\psi}}\hat{\psi}(\tilde{x}) = \bar{\hat{\psi}}\hat{\psi}(\tilde{x}).$$

$$U(P)^+i\bar{\hat{\psi}}\gamma^5\hat{\psi}(x)U(P) = -|\eta_b|^2 i\bar{\hat{\psi}}\gamma^5\hat{\psi}(\tilde{x}) = -i\bar{\hat{\psi}}\gamma^5\hat{\psi}(\tilde{x}).$$

$$\begin{aligned} U(P)^+\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(x)U(P) &= +|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(\tilde{x}) = +\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(\tilde{x})\ ,\ \mu=0 \\ &= -|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(\tilde{x}) = -\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(\tilde{x})\ ,\ \mu\neq 0. \end{aligned}$$

$$\begin{aligned} U(P)^+\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(x)U(P) &= -|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(\tilde{x}) = -\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(\tilde{x})\ ,\ \mu=0 \\ &= +|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(\tilde{x}) = +\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(\tilde{x})\ ,\ \mu\neq 0. \end{aligned}$$

Under time reversal we have

$$U(T)^+\hat{\psi}(x)U(T) = \eta_b\gamma^1\gamma^3\hat{\psi}(-x^0,\vec{x}).$$

We get

$$U(T)^+\bar{\hat{\psi}}(x)U(T) = \eta_b^*\bar{\hat{\psi}}(-x^0,\vec{x})\gamma^3\gamma^1.$$

We compute

$$U(T)^+\bar{\hat{\psi}}\hat{\psi}(x)U(T) = |\eta_b|^2\bar{\hat{\psi}}\hat{\psi}(-x^0,\vec{x}) = \bar{\hat{\psi}}\hat{\psi}(-x^0,\vec{x}).$$

$$\begin{aligned} U(T)^+i\bar{\hat{\psi}}\gamma^5\hat{\psi}(x)U(T) &= -iU(T)^+\bar{\hat{\psi}}\gamma^5\hat{\psi}(x)U(T) \\ &= -|\eta_b|^2 i\bar{\hat{\psi}}\gamma^5\hat{\psi}(-x^0,\vec{x}) = -i\bar{\hat{\psi}}\gamma^5\hat{\psi}(-x^0,\vec{x}). \end{aligned}$$

$$\begin{aligned} U(T)^+\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(x)U(T) &= U(T)^+\bar{\hat{\psi}}(x)U(T).(\gamma^\mu)^*.U(T)^+\hat{\psi}(x)U(T) \\ &= +|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(-x^0,\vec{x}) = +\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(-x^0,\vec{x})\ ,\ \mu=0 \\ &= -|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(-x^0,\vec{x}) = -\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(-x^0,\vec{x})\ ,\ \mu\neq 0. \end{aligned}$$

$$\begin{aligned} U(T)^+\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(x)U(T) &= U(T)^+\bar{\hat{\psi}}(x)U(T).(\gamma^\mu)^*\gamma^5.U(T)^+\hat{\psi}(x)U(T) \\ &= +|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(-x^0,\vec{x}) = +\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(-x^0,\vec{x})\ ,\ \mu=0 \\ &= -|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(-x^0,\vec{x}) = -\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(-x^0,\vec{x})\ ,\ \mu\neq 0. \end{aligned}$$

Angular Momentum of Dirac Field

- An infintesimal rotation aroun the z axis with an angle θ is given by the Lorentz transformation

$$\Lambda = 1 + \frac{i}{\hbar}\theta\mathcal{J}^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly

$$t^{'} = t \ , \ x^{'} = x + \theta y \ , \ y^{'} = -\theta x + y \ , \ z^{'} = z.$$

- Under this rotation the spinor transforms as

$$\psi^{'}(x^{'}) = S(\Lambda)\psi(x).$$

From one hand

$$\begin{aligned} \psi^{'}(x^{'}) &= \psi^{'}(t, x + \theta y, y - \theta x, z) \\ &= \psi^{'}(x) - \theta(x\partial_y - y\partial_x)\psi^{'}(x) \\ &= \psi^{'}(x) - \frac{i\theta}{\hbar}(\vec{x} \times \vec{p})^3\psi^{'}(x). \end{aligned}$$

From the other hand

$$\begin{aligned} \psi^{'}(x^{'}) &= S(\Lambda)\psi^{'}(x) \\ &= \psi(x) - \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}\psi(x) \\ &= \psi(x) - \frac{i}{\hbar}\omega_{12}\Gamma^{12}\psi(x) \\ &= \psi(x) + \frac{i}{\hbar}\theta\Gamma^{12}\psi(x) \\ &= \psi(x) + i\theta\frac{\Sigma^3}{2}\psi(x), \end{aligned}$$

where

$$\Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}.$$

Hence

$$\delta\psi(x) = \psi^{'}(x) - \psi(x) = \frac{i\theta}{\hbar}[\vec{x} \times \vec{p} + \frac{\hbar}{2}\vec{\Sigma}]^3\psi.$$

The quantity $\vec{x} \times \vec{p} + \frac{\hbar}{2}\vec{\Sigma}$ is the total angular momentum.

- Under the change $\psi(x) \longrightarrow \psi^{'}(x) = \psi(x) + \delta\psi(x)$ the Dirac Lagrangian $\mathcal{L}_{\rm Dirac} = \bar{\psi}(i\hbar c\gamma^\mu\partial_\mu - mc^2)\psi$ changes by

$$\begin{aligned}\delta\mathcal{L}_{\rm Dirac} &= \partial_\mu\left(\frac{\delta\mathcal{L}_{\rm Dirac}}{\delta(\partial_\mu\psi)}\delta\psi\right) + {\rm h.c}\\ &= -c\theta\partial_\mu j^\mu + {\rm h.c.}\end{aligned}$$

The current j^μ is given by

$$j^\mu = \bar{\psi}\gamma^\mu[\vec{x}\times\vec{p} + \frac{\hbar}{2}\vec{\Sigma}]^3\psi.$$

Assuming that the Lagrangian is invariant under the above rotation we have $\delta\mathcal{L}_{\rm Dirac} = 0$ and as a consequence the current j^μ is conserved. This is an instance of Noether's theorem. The integral over space of the zero-component of the current j^0 is the conserved charge which is identified with the angular momentum along the z axis since we are considering the invariance under rotations about the z axis. Hence the angular momentum of the Dirac field along the z direction is defined by

$$\begin{aligned}J^3 &= \int d^3x j^0\\ &= \int d^3x\psi^+(x)[\vec{x}\times\vec{p} + \frac{\hbar}{2}\vec{\Sigma}]^3\psi.\end{aligned}$$

This is conserved since

$$\begin{aligned}\frac{dJ^3}{dt} &= \int d^3x\partial_t j^0\\ &= -\int d^3x\partial_i j^i\\ &= -\oint_S \vec{j}d\vec{S}.\end{aligned}$$

The surface S is at infinity where the Dirac field vanishes and hence the surface integral vanishes. For a general rotation the conserved charge will be the angular momentum of the Dirac field given by

$$\vec{J} = \int d^3x\psi^+(x)[\vec{x}\times\vec{p} + \frac{\hbar}{2}\vec{\Sigma}]\psi.$$

- In the quantum theory the angular momentum operator of the Dirac field along the z direction is

$$\hat{J}^3 = \int d^3x\hat{\psi}^+(x)[\hat{\vec{x}}\times\hat{\vec{p}} + \frac{\hbar}{2}\Sigma^3]\hat{\psi}(x).$$

It is clear that the angular momentum of the vacuum is zero, viz

$$\hat{J}^3|0>=0. \tag{8.223}$$

- Next we consider a one-particle zero-momentum state. This is given by

$$|\vec{0},sb>=\sqrt{\frac{2mc^2}{\hbar}}\hat{b}(\vec{0},s)^+|0>.$$

Hence

$$\begin{aligned}\hat{J}^3|\vec{0},sb> &= \sqrt{\frac{2mc^2}{\hbar}}\hat{J}^3\hat{b}(\vec{0},s)^+|0>\\ &= \sqrt{\frac{2mc^2}{\hbar}}[\hat{J}^3,\hat{b}(\vec{0},s)^+]|0>.\end{aligned}$$

Clearly for a Dirac particle at rest the orbital piece of the angular momentum operator vanishes and thus

$$\hat{J}^3 = \int d^3x\hat{\psi}^+(x)[\frac{\hbar}{2}\Sigma^3]\hat{\psi}(x).$$

We have

$$\hat{\psi}(x^0,\vec{x})=\frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\hat{\chi}(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}.$$

We compute

$$\hat{J}^3 = \frac{1}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\hat{\chi}^+(x^0,\vec{p})[\frac{\hbar}{2}\Sigma^3]\hat{\chi}(x^0,\vec{p}).$$

Next we have

$$\hat{\chi}(x^0,\vec{p})=\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})\hat{b}(\vec{p},i)+e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})\hat{d}(-\vec{p},i)^+\bigg).$$

We get

$$\begin{aligned}\hat{J}^3 &= \int\frac{d^3p}{(2\pi\hbar)^3}\frac{c}{4E(\vec{p})}\sum_i\sum_j\bigg[u^{(i)+}(\vec{p})\Sigma^3u^{(j)}(\vec{p})\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},j)+v^{(i)+}(\vec{p})\Sigma^3v^{(j)}(\vec{p})\hat{d}(\vec{p},i)\hat{d}(\vec{p},j)^+\\ &+ e^{2i\omega(\vec{p})t}u^{(i)+}(\vec{p})\Sigma^3v^{(j)}(-\vec{p})\hat{b}(\vec{p},i)^+\hat{d}(-\vec{p},j)^++e^{-2i\omega(\vec{p})t}v^{(i)+}(-\vec{p})\Sigma^3u^{(j)}(\vec{p})\hat{d}(-\vec{p},i)\hat{b}(\vec{p},j)\bigg].\end{aligned}$$

We can immediately compute

$$[\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},j),\hat{b}(\vec{0},s)^+] = \hbar\delta_{sj}(2\pi\hbar)^3\delta^3(\vec{p})\hat{b}(\vec{p},i)^+$$
$$[\hat{d}(\vec{p},i)\hat{d}(\vec{p},j)^+,\hat{b}(\vec{0},s)^+] = 0$$
$$[\hat{b}(\vec{p},i)^+\hat{d}(-\vec{p},j)^+,\hat{b}(\vec{0},s)^+] = 0$$
$$[\hat{d}(-\vec{p},i)\hat{b}(\vec{p},j),\hat{b}(\vec{0},s)^+] = \hbar\delta_{sj}(2\pi\hbar)^3\delta^3(\vec{p})\hat{d}(-\vec{p},i).$$

Thus (by using $u^{(i)+}(\vec{0})\Sigma^3u^{(s)}(\vec{0}) = (2E(\vec{0})\xi^{i+}\sigma^3\xi^s)/c$)

$$[\hat{J}^3,\hat{b}(\vec{0},s)^+]|0>=\sum_i\xi^{i+}\frac{\hbar\sigma^3}{2}\xi^s\hat{b}(\vec{0},i)^+|0> .$$

Hence

$$\hat{J}^3|\vec{0},sb>=\sum_i\xi^{i+}\frac{\hbar\sigma^3}{2}\xi^s|\vec{0},ib> .$$

Let us choose the basis

$$\xi_0^1=\begin{pmatrix}1\\0\end{pmatrix}\ ,\ \xi_0^2=\begin{pmatrix}0\\1\end{pmatrix}.$$

Thus one-particle zero-momentum states have spins given by

$$\hat{J}^3|\vec{0},1b>=\frac{\hbar}{2}|\vec{0},1b>\ \ ,\ \ \hat{J}^3|\vec{0},2b>=-\frac{\hbar}{2}|\vec{0},2b> .$$

- A similar calculation will lead to the result that one-antiparticle zero-momentum states have spins given by

$$\hat{J}^3|\vec{0},1d>=-\frac{\hbar}{2}|\vec{0},1d>\ \ ,\ \ \hat{J}^3|\vec{0},2d>=\frac{\hbar}{2}|\vec{0},2d> .$$

Part II: Canonical Quantization of Interacting Fields

3

The $S-$Matrix and Feynman Diagrams For $\phi-$Four Theory

In this chapter we will follow the ICTP lecture notes by Strathdee and the book by Peskin and Schroeder.

3.1 Forced Scalar Field

3.1.1 Asymptotic Solutions

We have learned that a free neutral particle of spin 0 can be described by a real scalar field with a Lagrangian density given by (with $\hbar = c = 1$)

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2. \quad (1.1)$$

The free field operator can be expanded as (with $p^0 = E(\vec{p}) = E_{\vec{p}}$)

$$\begin{aligned}\hat{\phi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{p})}}\bigg(\hat{a}(\vec{p})e^{-ipx} + \hat{a}(\vec{p})^+ e^{ipx}\bigg) \\ &= \int \frac{d^3p}{(2\pi)^3}\hat{Q}(t,\vec{p})e^{i\vec{p}\vec{x}}.\end{aligned} \quad (1.2)$$

$$\hat{Q}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}(\vec{p})e^{-iE_{\vec{p}}t} + \hat{a}(-\vec{p})^+ e^{iE_{\vec{p}}t}\bigg). \quad (1.3)$$

The simplest interaction we can envisage is the action of an arbitrary external force $J(x)$ on the real scalar field $\phi(x)$. This can be described by adding a term of the form $J\phi$ to the Lagrangian density $\mathcal{L}_0$. We get then the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + J\phi. \tag{1.4}$$

The equations of motion become

$$(\partial_\mu\partial^\mu + m^2)\phi = J. \tag{1.5}$$

We expand the source in Fourier modes as

$$J(x) \quad = \quad \int \frac{d^3p}{(2\pi)^3} j(t,\vec{p}) e^{i\vec{p}\vec{x}}. \tag{1.6}$$

We get then the equations of motion in momentum space

$$(\partial_t^2 + E_{\vec{p}}^2) Q(t,\vec{p}) = j(t,\vec{p}). \tag{1.7}$$

By assuming that $j(t,\vec{p})$ vanishes outside a finite time interval we conclude that for early and late times where $j(t,\vec{p})$ is zero the field is effectively free. Thus for early times we have

$$\hat{Q}(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm in}(\vec{p})e^{-iE_{\vec{p}}t} + \hat{a}_{\rm in}(-\vec{p})^+ e^{iE_{\vec{p}}t}\bigg)\ ,\ t \longrightarrow -\infty. \tag{1.8}$$

For late times we have

$$\hat{Q}(t,\vec{p}) = \hat{Q}_{\rm out}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm out}(\vec{p})e^{-iE_{\vec{p}}t} + \hat{a}_{\rm out}(-\vec{p})^+ e^{iE_{\vec{p}}t}\bigg)\ ,\ t \longrightarrow +\infty. \tag{1.9}$$

The general solution is of the form

$$\hat{Q}(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p}) + \frac{1}{E_{\vec{p}}}\int_{-\infty}^{t} dt^{'} \sin E_{\vec{p}}(t-t^{'}) j(t^{'},\vec{p}). \tag{1.10}$$

Clearly for early times $t \longrightarrow -\infty$ we get $\hat{Q} \longrightarrow \hat{Q}_{\rm in}$. On the other hand since for late times $t \longrightarrow +\infty$ we have $\hat{Q} \longrightarrow \hat{Q}_{\rm out}$ we must have

$$\hat{Q}_{\rm out}(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p}) + \frac{1}{E_{\vec{p}}}\int_{-\infty}^{+\infty} dt^{'} \sin E_{\vec{p}}(t-t^{'}) j(t^{'},\vec{p}). \tag{1.11}$$

We define the positive-energy and the negative-energy parts of $\hat{Q}$ by

$$\hat{Q}^+(t,\vec{p}) \quad = \quad \frac{1}{\sqrt{2E_{\vec{p}}}}\hat{a}(\vec{p})e^{-iE_{\vec{p}}t}\ ,\ \hat{Q}^-(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\hat{a}(-\vec{p})^+ e^{iE_{\vec{p}}t}. \tag{1.12}$$

Equation (1.10) is equivalent to the two equations

$$\hat{Q}^{\pm}(t,\vec{p}) = \hat{Q}^{\pm}_{\rm in}(t,\vec{p}) \pm \frac{i}{2E_{\vec{p}}}\int_{-\infty}^{t} dt^{'} e^{\mp iE_{\vec{p}}(t-t^{'})} j(t^{'},\vec{p}). \quad (1.13)$$

The Feynman propagator in one-dimension is given by

$$G_{\vec{p}}(t-t^{'}) = \frac{e^{-iE_{\vec{p}}|t-t^{'}|}}{2E_{\vec{p}}} = \int \frac{dE}{2\pi}\frac{i}{E^2 - E^2_{\vec{p}} + i\epsilon} e^{-iE(t-t^{'})}. \quad (1.14)$$

Note that in our case $t - t^{'} > 0$. Hence

$$\hat{Q}^{+}(t,\vec{p}) = \hat{Q}^{+}_{\rm in}(t,\vec{p}) + i\int_{-\infty}^{t} dt^{'} G_{\vec{p}}(t-t^{'}) j(t^{'},\vec{p}). \quad (1.15)$$

$$\hat{Q}^{-}(t,\vec{p}) = \hat{Q}^{-}_{\rm in}(t,\vec{p}) - i\int_{-\infty}^{t} dt^{'} G_{\vec{p}}(t^{'}-t) j(t^{'},\vec{p}). \quad (1.16)$$

For late times we get

$$\hat{Q}^{+}_{\rm out}(t,\vec{p}) = \hat{Q}^{+}_{\rm in}(t,\vec{p}) + i\int_{-\infty}^{+\infty} dt^{'} G_{\vec{p}}(t-t^{'}) j(t^{'},\vec{p}). \quad (1.17)$$

$$\hat{Q}^{-}_{\rm out}(t,\vec{p}) = \hat{Q}^{-}_{\rm in}(t,\vec{p}) - i\int_{-\infty}^{+\infty} dt^{'} G_{\vec{p}}(t^{'}-t) j(t^{'},\vec{p}). \quad (1.18)$$

These two equations are clearly equivalent to equation (1.11).

The above two equations can be rewritten as

$$\hat{Q}^{\pm}_{\rm out}(t,\vec{p}) = \hat{Q}^{\pm}_{\rm in}(t,\vec{p}) \pm \frac{i}{2E_{\vec{p}}}\int_{-\infty}^{+\infty} dt^{'} e^{\mp iE_{\vec{p}}(t-t^{'})} j(t^{'},\vec{p}). \quad (1.19)$$

In terms of the creation and annihilation operators this becomes

$$\hat{a}_{\rm out}(\vec{p}) = \hat{a}_{\rm in}(\vec{p}) + \frac{i}{\sqrt{2E_{\vec{p}}}} j(p) \ , \ \hat{a}_{\rm out}(\vec{p})^{+} = \hat{a}_{\rm in}(\vec{p})^{+} - \frac{i}{\sqrt{2E_{\vec{p}}}} j(-p). \quad (1.20)$$

$$j(p) \equiv j(E_{\vec{p}},\vec{p}) = \int dt e^{iE_{\vec{p}}t} j(t,\vec{p}). \quad (1.21)$$

We observe that the "in" operators and the "out" operators are different. Hence there exists two different Hilbert spaces and as a consequence two different vacua $|0 \text{ in} >$ and $|0 \text{ out} >$ defined by

$$\hat{a}_{\rm out}(\vec{p})|0 \text{ out} >= 0 \ , \ \hat{a}_{\rm in}(\vec{p})|0 \text{ in} >= 0 \ \forall \vec{p}. \quad (1.22)$$

3.1.2 The Schrodinger, Heisenberg and Dirac Pictures

The Lagrangian from which the equation of motion (1.7) is derived is

$$\int_{+} \frac{d^3p}{(2\pi)^3}\bigg(\partial_t Q(t,\vec{p})^* \partial_t Q(t,\vec{p}) - E_{\vec{p}}^2 Q(t,\vec{p})^* Q(t,\vec{p}) + j(t,\vec{p})^* Q(t,\vec{p}) + j(t,\vec{p}) Q(t,\vec{p})^*\bigg). \tag{1.23}$$

The corresponding Hamiltonian is (with $P(t,\vec{p}) = \partial_t Q(t,\vec{p})$)

$$\int_{+} \frac{d^3p}{(2\pi)^3}\bigg(P(t,\vec{p})^* P(t,\vec{p}) + E_{\vec{p}}^2 Q(t,\vec{p})^* Q(t,\vec{p}) - j(t,\vec{p})^* Q(t,\vec{p}) - j(t,\vec{p}) Q(t,\vec{p})^*\bigg). \tag{1.24}$$

The operators $\hat{P}(t,\vec{p})$ and $\hat{Q}(t,\vec{p})$ are the time-dependent Heisenberg operators. The time-independent Schrodinger operators will be denoted by $\hat{P}(\vec{p})$ and $\hat{Q}(\vec{p})$. In the Schrodinger picture the Hamiltonian is given by

$$\int_{+} \frac{d^3p}{(2\pi)^3}\bigg(P(\vec{p})^* P(\vec{p}) + E_{\vec{p}}^2 Q(\vec{p})^* Q(\vec{p}) - j(t,\vec{p})^* Q(\vec{p}) - j(t,\vec{p}) Q(\vec{p})^*\bigg). \tag{1.25}$$

This Hamiltonian depends on time only through the time-dependence of the source. Using box normalization the momenta become discrete and the measure $\int d^3p/(2\pi)^3$ becomes the sum $\sum_{\vec{p}}/V$. Thus the Hamiltonian becomes

$$\sum_{p^1>0}\sum_{p^2>0}\sum_{p^3>0} \mathcal{H}_{\vec{p}}(t). \tag{1.26}$$

We recall the equal time commutation relations $[\hat{Q}(t,\vec{p}), \hat{P}(t,\vec{p})^+] = i(2\pi)^3\delta^3(\vec{p}-\vec{q})$ and $[\hat{Q}(t,\vec{p}), \hat{P}(t,\vec{p})] = [\hat{Q}(t,\vec{p}), \hat{Q}(t,\vec{p})] = [\hat{P}(t,\vec{p}), \hat{P}(t,\vec{p})] = 0$. Using box normalization the equal time commutation relations take the form

$$\begin{aligned} &[\hat{Q}(t,\vec{p}), \hat{P}(t,\vec{p})^+] = iV\delta_{\vec{p},\vec{q}} \\ &[\hat{Q}(t,\vec{p}), \hat{P}(t,\vec{p})] = [\hat{Q}(t,\vec{p}), \hat{Q}(t,\vec{p})] = [\hat{P}(t,\vec{p}), \hat{P}(t,\vec{p})] = 0. \end{aligned} \tag{1.27}$$

The Hamiltonian of a single forced oscillator which has a momentum $\vec{p}$ is

$$\mathcal{H}_{\vec{p}}(t) = \frac{1}{V}\bigg(P(\vec{p})^* P(\vec{p}) + E_{\vec{p}}^2 Q(\vec{p})^* Q(\vec{p})\bigg) + V(t,\vec{p}). \tag{1.28}$$

The potential is defined by

$$V(t,\vec{p}) \;=\; -\frac{1}{V}\bigg(j(t,\vec{p})^* Q(\vec{p}) + j(t,\vec{p}) Q(\vec{p})^*\bigg). \tag{1.29}$$

The unitary time evolution operator must solve the Schrodinger equation

$$i\partial_t U(t) = \hat{\mathcal{H}}_{\vec{p}}(t)U(t). \quad (1.30)$$

The Heisenberg and Schrodinger operators are related by

$$\hat{Q}(t,\vec{p}) \ = \ U(t)^{-1}\hat{Q}(\vec{p})U(t). \quad (1.31)$$

We introduce the interaction picture through the unitary operator Ω defined by

$$U(t) = e^{-it\hat{\mathcal{H}}_{\vec{p}}}\Omega(t). \quad (1.32)$$

In the above equation $\mathcal{H}_{\vec{p}}$ is the free Hamiltonian density, viz

$$\mathcal{H}_{\vec{p}} \ = \ \frac{1}{V}\bigg(P(\vec{p})^*P(\vec{p}) + E_{\vec{p}}^2 Q(\vec{p})^*Q(\vec{p})\bigg). \quad (1.33)$$

The operator Ω satisfies the Schrodinger equation

$$i\partial_t\Omega(t) = \hat{V}_I(t,\vec{p})\Omega(t). \quad (1.34)$$

$$\begin{aligned}\hat{V}_I(t,\vec{p}) &= e^{it\hat{\mathcal{H}}_{\vec{p}}}\hat{V}(t,\vec{p})e^{-it\hat{\mathcal{H}}_{\vec{p}}}\\ &= -\frac{1}{V}(j(t,\vec{p})^*\hat{Q}_I(t,\vec{p}) + j(t,\vec{p})\hat{Q}_I(t,\vec{p})^+). \end{aligned} \quad (1.35)$$

The interaction, Schrodinger and Heisenberg operators are related by

$$\begin{aligned}\hat{Q}_I(t,\vec{p}) &= e^{it\hat{\mathcal{H}}_{\vec{p}}}\hat{Q}(\vec{p})e^{-it\hat{\mathcal{H}}_{\vec{p}}}\\ &= \Omega(t)U(t)^{-1}\hat{Q}(\vec{p})U(t)\Omega(t)^{-1}\\ &= \Omega(t)\hat{Q}(t,\vec{p})\Omega(t)^{-1}. \end{aligned} \quad (1.36)$$

We write this as

$$\hat{Q}(t,\vec{p}) \ = \ \Omega(t)^{-1}\hat{Q}_I(t,\vec{p})\Omega(t). \quad (1.37)$$

It is not difficult to show that the operators $\hat{Q}_I(t,\vec{p})$ and $\hat{P}_I(t,\vec{p})$ describe free oscillators, viz

$$(\partial_t^2 + E_{\vec{p}}^2)\hat{Q}_I(t,\vec{p}) = 0 \ , \ (\partial_t^2 + E_{\vec{p}}^2)\hat{P}_I(t,\vec{p}) = 0. \quad (1.38)$$

3.1.3 The $S-$Matrix

Single Oscillator: The probability amplitude that the oscillator remains in the ground state is $<0\ \mathrm{out}|0\ \mathrm{in}>$. In general the matrix of transition amplitudes is

$$S_{mn} = < m\ \mathrm{out}|n\ \mathrm{in} > . \tag{1.39}$$

We define the $S-$matrix S by

$$S_{mn} = < m\ \mathrm{in}|S|n\ \mathrm{in} > . \tag{1.40}$$

In other words

$$< m\ \mathrm{out}| = < m\ \mathrm{in}|S. \tag{1.41}$$

It is not difficult to see that S is a unitary matrix since the states $|m\ \mathrm{in}>$ and $|m\ \mathrm{in}>$ are normalized and complete. Equation (1.41) is equivalent to

$$\begin{aligned} < 0\ \mathrm{out}|(\hat{a}_{\mathrm{out}}(\vec{p}))^m &= < 0\ \mathrm{in}|(\hat{a}_{\mathrm{in}}(\vec{p}))^m S \\ &= < 0\ \mathrm{out}|S^{-1}(\hat{a}_{\mathrm{in}}(\vec{p}))^m S \\ &= < 0\ \mathrm{out}|(S^{-1}\hat{a}_{\mathrm{in}}(\vec{p})S)^m. \end{aligned} \tag{1.42}$$

Thus

$$\hat{a}_{\mathrm{out}}(\vec{p}) = S^{-1}\hat{a}_{\mathrm{in}}(\vec{p})S. \tag{1.43}$$

This can also be written as

$$\hat{Q}_{\mathrm{out}}(t,\vec{p}) = S^{-1}\hat{Q}_{\mathrm{in}}(t,\vec{p})S. \tag{1.44}$$

From the other hand, the solution of the differential equation (1.34) can be obtained by iteration as follows. We write

$$\Omega(t) = 1 + \Omega_1(t) + \Omega_2(t) + \Omega_3(t) + ... \tag{1.45}$$

The operator $\Omega_n(t)$ is proportional to the nth power of the interaction $\hat{V}_I(t)$. By substitution we get the differential equations

$$i\partial_t\Omega_1(t) = \hat{V}_I(t,\vec{p}) \Leftrightarrow \Omega_1(t) = -i\int_{-\infty}^{t} dt_1 \hat{V}_I(t_1,\vec{p}). \tag{1.46}$$

$$i\partial_t\Omega_n(t) = \hat{V}_I(t,\vec{p})\Omega_{n-1}(t) \Leftrightarrow \Omega_n(t) = -i\int_{-\infty}^{t} dt_1 \hat{V}_I(t,\vec{p})\Omega_{n-1}(t_1)\ ,\ n\geq 2. \tag{1.47}$$

Thus we get the solution

$$\begin{aligned}\Omega(t) &= 1-i\int_{-\infty}^{t}dt_1\hat{V}_I(t_1,\vec{p})+(-i)^2\int_{-\infty}^{t}dt_1\hat{V}_I(t_1,\vec{p})\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_2,\vec{p})\\ &+ (-i)^3\int_{-\infty}^{t}dt_1\hat{V}_I(t_1,\vec{p})\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_2,\vec{p})\int_{-\infty}^{t_2}dt_3\hat{V}_I(t_3,\vec{p})+...\\ &= \sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2...\int_{-\infty}^{t_{n-1}}dt_n\hat{V}_I(t_1,\vec{p})...\hat{V}_I(t_n,\vec{p}).\end{aligned} \tag{1.48}$$

This expression can be simplified by using the time-ordering operator T. Let us first recall that

$$\begin{aligned}T(\hat{V}_I(t_1)\hat{V}_I(t_2))=\hat{V}_I(t_1)\hat{V}_I(t_2)\ ,\ \text{if}\ t_1>t_2\\ T(\hat{V}_I(t_1)\hat{V}_I(t_2))=\hat{V}_I(t_2)\hat{V}_I(t_1)\ ,\ \text{if}\ t_2>t_1.\end{aligned} \tag{1.49}$$

For ease of notation we have suppressed momentarily the momentum-dependence of $\hat{V}_I$. Clearly $T(\hat{V}_I(t_1)\hat{V}_I(t_2))$ is a function of t_1 and t_2 which is symmetric about the axis $t_1=t_2$. Hence

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2T(\hat{V}_I(t_1)\hat{V}_I(t_2)) &= \frac{1}{2}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2)+\frac{1}{2}\int_{-\infty}^{t}dt_2\int_{-\infty}^{t_2}dt_1\hat{V}_I(t_2)\hat{V}_I(t_1)\\ &= \int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2).\end{aligned} \tag{1.50}$$

The generalized result we will use is therefore given by

$$\frac{1}{n!}\int_{-\infty}^{t}dt_1...\int_{-\infty}^{t}dt_nT(\hat{V}_I(t_1)...\hat{V}_I(t_n)) = \int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2...\int_{-\infty}^{t_{n-1}}dt_n\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n). \tag{1.51}$$

By substituting this identity in (1.48) we obtain

$$\begin{aligned}\Omega(t) &= \sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2...\int_{-\infty}^{t}dt_nT(\hat{V}_I(t_1,\vec{p})\hat{V}_I(t_2,\vec{p})...\hat{V}_I(t_n,\vec{p}))\\ &= T\bigg(e^{-i\int_{-\infty}^{t}ds\hat{V}_I(s,\vec{p})}\bigg).\end{aligned} \tag{1.52}$$

It is clear that

$$\Omega(-\infty) = 1. \tag{1.53}$$

This can only be consistent with the assumption that $j(t,\vec{p})\longrightarrow 0$ as $t\longrightarrow -\infty$. As we will see shortly we need actually to assume the stronger requirement that the source $j(t,\vec{p})$ vanishes outside a finite time interval. Hence for early times $t \longrightarrow -\infty$ we have $\Omega(t) \longrightarrow 1$ and as a consequence we get $\hat{Q}(t,\vec{p}) \longrightarrow \hat{Q}_I(t,\vec{p})$ from (1.37). However we know that $\hat{Q}(t,\vec{p}) \longrightarrow \hat{Q}_{\rm in}(t,\vec{p})$ as $t \longrightarrow -\infty$. Since $\hat{Q}_I(t,\vec{p})$ and $\hat{Q}_{\rm in}(t,\vec{p})$ are both free fields, i.e. they solve the same differential equation we conclude that they must be the same field for all times, viz

$$\hat{Q}_I(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p}) \ , \ \forall t. \tag{1.54}$$

Equation (1.37) becomes

$$\hat{Q}(t,\vec{p}) \ = \ \Omega(t)^{-1}\hat{Q}_{\rm in}(t,\vec{p})\Omega(t). \tag{1.55}$$

For late times $t \longrightarrow \infty$ we know that $\hat{Q}(t,\vec{p}) \longrightarrow \hat{Q}_{\rm out}(t,\vec{p})$. Thus from the above equation we obtain

$$\hat{Q}_{\rm out}(t,\vec{p}) \ = \ \Omega(+\infty)^{-1}\hat{Q}_{\rm in}(t,\vec{p})\Omega(+\infty). \tag{1.56}$$

Comparing this equation with (1.44) we conclude that the $S-$matrix is given by

$$S = \Omega(+\infty) = T\bigg(e^{-i\int_{-\infty}^{+\infty} ds \hat{V}_I(s,\vec{p})}\bigg). \tag{1.57}$$

Scalar Field: Generalization of (1.57) is straightforward. The full $S-$matrix of the forced scalar field is the tensor product of the individual $S-$matrices of the forced harmonic oscillators one for each momentum $\vec{p}$. Since $\hat{Q}(t,-\vec{p}) = \hat{Q}(t,\vec{p})^+$ we only consider momenta $\vec{p}$ with positive components. In the tensor product all factors commute because they involve momenta which are different. We obtain then the evolution operator and the $S-$matrix

$$\begin{aligned}\Omega(t) \ &= \ T\bigg(e^{-i\int_{-\infty}^{t} ds \sum_{p^1>0}\sum_{p^2>0}\sum_{p^3>0}\hat{V}_I(s,\vec{p})}\bigg)\\ &= \ T\bigg(e^{\frac{i}{2}\int_{-\infty}^{t} ds \int \frac{d^3p}{(2\pi)^3}\Big(j(s,\vec{p})^*\hat{Q}_I(s,\vec{p})+j(s,\vec{p})\hat{Q}_I(s,\vec{p})^+\Big)}\bigg)\\ &= \ T\bigg(e^{i\int_{-\infty}^{t} ds \int d^3x J(x)\hat{\phi}_I(x)}\bigg)\\ &= \ T\bigg(e^{i\int_{-\infty}^{t} ds \int d^3x \mathcal{L}_{\rm int}(x)}\bigg).\end{aligned} \tag{1.58}$$

$$S = \Omega(+\infty) \ = \ T\bigg(e^{i\int d^4x \mathcal{L}_{\rm int}(x)}\bigg). \tag{1.59}$$

The interaction Lagrangian density depends on the interaction field operator $\hat{\phi}_I = \hat{\phi}_{\rm in}$, viz

$$\mathcal{L}_{\rm int}(x) = \mathcal{L}_{\rm int}(\hat{\phi}_{\rm in}) = J(x)\hat{\phi}_{\rm in}(x). \quad (1.60)$$

3.1.4 Wick's Theorem For Forced Scalar Field

Let us recall the Fourier expansion of the field $\hat{\phi}_{\rm in}$ given by

$$\hat{\phi}_{\rm in}(x) = \int \frac{d^3p}{(2\pi)^3}\hat{Q}_{\rm in}(t,\vec{p})e^{i\vec{p}\vec{x}}. \quad (1.61)$$

We compute immediately

$$\int d^3x\mathcal{L}_{\rm int}(x) = \frac{1}{V}\sum_{\vec{p}} j(t,\vec{p})^*\hat{Q}_{\rm in}(t,\vec{p}) = \frac{1}{V}\sum_{\vec{p}}\frac{j(t,\vec{p})^*}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm in}(\vec{p})e^{-iE_{\vec{p}}t} + \hat{a}_{\rm in}(-\vec{p})^+e^{iE_{\vec{p}}t}\bigg). \quad (1.62)$$

Also we compute

$$\Omega(t) = T\bigg(e^{\sum_{\vec{p}}\big(\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+ - \alpha_{\vec{p}}(t)^*\hat{a}_{\rm in}(\vec{p})\big)}\bigg) = T\prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+ - \alpha_{\vec{p}}(t)^*\hat{a}_{\rm in}(\vec{p})}\bigg). \quad (1.63)$$

$$\alpha_{\vec{p}}(t) = \frac{i}{V}\frac{1}{\sqrt{2E_{\vec{p}}}}\int_{-\infty}^t ds j(s,\vec{p})e^{iE_{\vec{p}}s}. \quad (1.64)$$

It is clear that the solution $\Omega(t)$ is of the form (including also an arbitrary phase $\beta_{\vec{p}}(t)$)

$$\Omega(t) = \prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+ - \alpha_{\vec{p}}(t)^*\hat{a}_{\rm in}(\vec{p}) + i\beta_{\vec{p}}(t)}\bigg). \quad (1.65)$$

We use the Campbell-Baker-Hausdorff formula

$$e^{A+B} = e^Ae^Be^{-\frac{1}{2}[A,B]}\ ,\ \text{if}\ [A,[A,B]] = [B,[A,B]] = 0. \quad (1.66)$$

We also use the commutation relations

$$[\hat{a}_{\rm in}(\vec{p}),\hat{a}_{\rm in}(\vec{q})^+]=V\delta_{\vec{p},\vec{q}}. \quad (1.67)$$

$$\begin{aligned}\Omega(t) &= \prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+}e^{-\alpha_{\vec{p}}(t)^*\hat{a}_{\rm in}(\vec{p})}e^{-\frac{1}{2}V|\alpha_{\vec{p}}(t)|^2+i\beta_{\vec{p}}(t)}\bigg)\\ &= \prod_{\vec{p}}\Omega_{\vec{p}}(t). \end{aligned} \quad (1.68)$$

In the limit $t\longrightarrow\infty$ we compute

$$-\frac{1}{2}V\sum_{\vec{p}}|\alpha_{\vec{p}}(+\infty)|^2 = -\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})\frac{1}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}. \quad (1.69)$$

We also need to compute the limit of $i\beta_{\vec{p}}(t)$ when $t\longrightarrow+\infty$. After some calculation, we obtain

$$i\sum_{\vec{p}}\beta_{\vec{p}}(+\infty) = \frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})\bigg(\frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}-\frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}\bigg). \quad (1.70)$$

Putting (1.69) and (1.70) together we get finally

$$\begin{aligned}-\frac{1}{2}V\sum_{\vec{p}}|\alpha_{\vec{p}}(+\infty)|^2+i\sum_{\vec{p}}\beta_{\vec{p}}(+\infty) &= -\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})\bigg(\frac{\theta(t^{'}-t)}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}\\ &+ \frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}\bigg)\\ &= -\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})D_F(x-x^{'}). \end{aligned} \quad (1.71)$$

From this last equation and from equation (1.68) we obtain the $S-$matrix in its pre-final form given by

$$S=\Omega(+\infty) = \prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+}e^{-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})}\bigg)e^{-\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})D_F(x-x^{'})}. \quad (1.72)$$

This expression is already normal-ordered since

$$:\bigg(e^{\sum_{\vec{p}}\big(\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})\big)}\bigg): \;=\; \prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+}e^{-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})}\bigg). \tag{1.73}$$

In summary we have

$$\begin{aligned} S=\Omega(+\infty) &= T\bigg(e^{\sum_{\vec{p}}\big(\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})\big)}\bigg)\\ &= :\bigg(e^{\sum_{\vec{p}}\big(\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})\big)}\bigg): e^{-\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})D_F(x-x^{'})}. \end{aligned} \tag{1.74}$$

More explicitly we write

$$S=T\bigg(e^{i\int d^4xJ(x)\hat{\phi}_{\rm in}(x)}\bigg)=:e^{i\int d^4xJ(x)\hat{\phi}_{\rm in}(x)}:e^{-\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})D_F(x-x^{'})}. \tag{1.75}$$

This is Wick's theorem.

3.2 The Φ−Four Theory

3.2.1 The Lagrangian Density

In this section we consider more general interacting scalar field theories. In principle we can add any interaction Lagrangian density $\mathcal{L}_{\rm int}$ to the free Lagrangian density $\mathcal{L}_0$ given by equation (1.1) in order to obtain an interacting scalar field theory. This interaction Lagrangian density can be for example any polynomial in the field ϕ. However there exists only one single interacting scalar field theory of physical interest which is also renormalizable known as the $\phi-$four theory. This is obtained by adding to (1.1) a quartic interaction Lagrangian density of the form

$$\mathcal{L}_{\rm int}=-\frac{\lambda}{4!}\phi^4. \tag{2.76}$$

The equation of motion becomes

$$\begin{aligned} (\partial_\mu\partial^\mu+m^2)\phi &= \frac{\delta\mathcal{L}_{\rm int}}{\delta\phi}\\ &= -\frac{\lambda}{6}\phi^3. \end{aligned} \tag{2.77}$$

Equivalently

$$(\partial_t^2+E_{\vec{p}}^2)Q(t,\vec{p}) \;=\; \int d^3x\frac{\delta\mathcal{L}_{\rm int}}{\delta\phi}e^{-i\vec{p}\vec{x}}. \tag{2.78}$$

We will suppose that the right-hand side of the above equation goes to zero as $t\longrightarrow\pm\infty$. In other words we must require that $\delta\mathcal{L}_{\rm int}/\delta\phi\longrightarrow 0$ as $t\longrightarrow\pm\infty$. If this is not true (which is generically the case) then we will assume implicitly an adiabatic switching off process for the interaction in the limits $t\longrightarrow\pm\infty$ given by the replacement

$$\mathcal{L}_{\rm int}\longrightarrow e^{-\epsilon|t|}\mathcal{L}_{\rm int}. \tag{2.79}$$

With this assumption the solutions of the equation of motion in the limits $t\longrightarrow -\infty$ and $t\longrightarrow +\infty$ are given respectively by

$$\hat{Q}_{\rm in}(t,\vec{p})=\frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm in}(\vec{p})e^{-iE_{\vec{p}}t}+\hat{a}_{\rm in}(-\vec{p})^+e^{iE_{\vec{p}}t}\bigg)\;,\;t\longrightarrow -\infty. \tag{2.80}$$

$$\hat{Q}_{\rm out}(t,\vec{p})=\frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm out}(\vec{p})e^{-iE_{\vec{p}}t}+\hat{a}_{\rm out}(-\vec{p})^+e^{iE_{\vec{p}}t}\bigg)\;,\;t\longrightarrow +\infty. \tag{2.81}$$

3.2.2 The $S-$Matrix

The Hamiltonian operator in the Schrodinger picture is time-independent of the form

$$\hat{H}=\hat{H}_0(\hat{Q},\hat{Q}^+,\hat{P},\hat{P}^+)+\hat{V}(\hat{Q},\hat{Q}^+). \tag{2.82}$$

$$\begin{aligned}\hat{H}_0(\hat{Q},\hat{Q}^+,\hat{P},\hat{P}^+) &= \int_+\frac{d^3p}{(2\pi)^3}\bigg[\hat{P}^+(\vec{p})\hat{P}(\vec{p})+E_{\vec{p}}^2\hat{Q}^+(\vec{p})\hat{Q}(\vec{p})\bigg]\\ &= \frac{1}{2}\sum_{\vec{p}}\hat{\mathcal{H}}_{\vec{p}}.\end{aligned} \tag{2.83}$$

$$\begin{aligned}\hat{V}(\hat{Q},\hat{Q}^+) &= (+\frac{\lambda}{4!})\frac{1}{V^3}\sum_{\vec{p}_1,\vec{p}_2,\vec{p}_3}\hat{Q}(\vec{p}_1)\hat{Q}(\vec{p}_2)\hat{Q}(\vec{p}_3)\hat{Q}(\vec{p}_1+\vec{p}_2+\vec{p}_3)^+\\ &= -\int d^3x\mathcal{L}_{\rm int}.\end{aligned} \tag{2.84}$$

The scalar field operator and the conjugate momentum field operator in the Schrodinger picture are given by

$$\hat{\phi}(\vec{x}) = \frac{1}{V}\sum_{\vec{p}} \hat{Q}(\vec{p})e^{i\vec{p}\vec{x}}. \tag{2.85}$$

$$\hat{\pi}(\vec{x}) = \frac{1}{V}\sum_{\vec{p}} \hat{P}(\vec{p})e^{i\vec{p}\vec{x}}. \tag{2.86}$$

The unitary time evolution operator of the scalar field must solve the Schrodinger equation

$$i\partial_t U(t) = \hat{H}U(t). \tag{2.87}$$

The Heisenberg and Schrodinger operators are related by

$$\hat{\phi}(t,\vec{x}) = U(t)^{-1}\hat{\phi}(\vec{x})U(t). \tag{2.88}$$

We introduce the interaction picture through the unitary operator Ω defined by

$$U(t) = e^{-it\hat{H}_0}\Omega(t). \tag{2.89}$$

The operator Ω satisfies the Schrodinger equation

$$i\partial_t\Omega(t) = \hat{V}_I(t)\Omega(t). \tag{2.90}$$

$$\hat{V}_I(t) \equiv \hat{V}_I(\hat{Q},\hat{Q}^+,t) = e^{it\hat{H}_0}\hat{V}(\hat{Q},\hat{Q}^+)e^{-it\hat{H}_0}. \tag{2.91}$$

The interaction, Schrodinger and Heisenberg operators are related by

$$\begin{aligned}\hat{\phi}_I(t,\vec{x}) &= e^{it\hat{H}_0}\hat{\phi}(\vec{x})e^{-it\hat{H}_0}\\ &= \Omega(t)U(t)^{-1}\hat{\phi}(\vec{x})U(t)\Omega(t)^{-1}\\ &= \Omega(t)\hat{\phi}(t,\vec{x})\Omega(t)^{-1}.\end{aligned} \tag{2.92}$$

We write this as

$$\hat{\phi}(x) = \Omega(t)^{-1}\hat{\phi}_I(x)\Omega(t). \tag{2.93}$$

Similarly we should have for the conjugate momentum field $\hat{\pi}(x) = \partial_t\hat{\phi}(x)$ the result

$$\hat{\pi}_I(x) = e^{it\hat{H}_0}\hat{\pi}(\vec{x})e^{-it\hat{H}_0}. \tag{2.94}$$

$$\hat{\pi}(x) = \Omega(t)^{-1}\hat{\pi}_I(x)\Omega(t). \tag{2.95}$$

It is not difficult to show that the interaction fields $\hat{\phi}_I$ and $\hat{\pi}_I$ are free fields. Indeed we can show for example that $\hat{\phi}_I$ obeys the equation of motion

$$(\partial_t^2 - \vec{\nabla}^2 + m^2)\hat{\phi}_I(t, \vec{x}) = 0. \tag{2.96}$$

Thus all information about interaction is encoded in the evolution operator $\Omega(t)$ which in turn is obtained from the solution of the Schrodinger equation (2.90). From our previous experience this task is trivial. In direct analogy with the solution given by the formula (1.52) of the differential equation (1.34) the solution of (2.90) must be of the form

$$\begin{aligned}\Omega(t) &= \sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2...\int_{-\infty}^{t}dt_n T(\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n))\\ &= T\bigg(e^{-i\int_{-\infty}^{t}ds\hat{V}_I(s)}\bigg)\\ &= T\bigg(e^{i\int_{-\infty}^{t}ds\int d^3x\mathcal{L}_{\rm int}(\hat{\phi}_I(s,\vec{x}))}\bigg).\end{aligned} \tag{2.97}$$

Clearly this satisfies the boundary condition

$$\Omega(-\infty) = 1. \tag{2.98}$$

As before this boundary condition can only be consistent with the assumption that $V_I(t) \longrightarrow 0$ as $t \longrightarrow -\infty$. This requirement is contained in the condition (2.79).

The $S-$matrix is defined by

$$\begin{aligned}S = \Omega(+\infty) &= T\bigg(e^{-i\int_{-\infty}^{+\infty}ds\hat{V}_I(s)}\bigg)\\ &= T\bigg(e^{i\int d^4x\mathcal{L}_{\rm int}(\hat{\phi}_I(x))}\bigg).\end{aligned} \tag{2.99}$$

Taking the limit $t \longrightarrow -\infty$ in equation (2.93) we see that we have $\hat{\phi}(x) \longrightarrow \phi_I(x)$. But we already know that $\hat{\phi}(x) \longrightarrow \hat{\phi}_{\rm in}(x)$ when $t \longrightarrow -\infty$. Since the fields $\hat{\phi}_I(x)$ and $\hat{\phi}_{\rm in}(x)$ are free fields and satisfy the same differential equation we conclude that the two fields are identical at all times, viz

$$\hat{\phi}_I(x) = \hat{\phi}_{\rm in}(x)\ ,\ \forall t. \tag{2.100}$$

The $S-$matrix relates the "in" vacuum $|0\ {\rm in}>$ to the "out" vacuum $|0\ {\rm out}>$ as follows

$$<0\ {\rm out}| =<0\ {\rm in}|S. \tag{2.101}$$

For the $\phi-$four theory (as opposed to the forced scalar field) the vacuum is stable. In other words the "in" vacuum is identical to the "out" vacuum, viz

$$|0\text{ out}>=|0\text{ in}>=|0>. \tag{2.102}$$

Hence

$$<0|=<0|S. \tag{2.103}$$

The consistency of the supposition that the "in" vacuum is identical to the "out" vacuum will be verified order by order in perturbation theory. In fact we will also verify that the same holds also true for the one-particle states, viz

$$|\vec{p}\text{ out}>=|\vec{p}\text{ in}>. \tag{2.104}$$

3.2.3 The Gell-Mann Low Formula

We go back to equation

$$\hat{\phi}(x) = \Omega(t)^+\hat{\phi}_I(x)\Omega(t). \tag{2.105}$$

We compute

$$\begin{aligned}
\hat{\phi}(x) &= \Omega(t)^+\hat{\phi}_I(x)\Omega(t)\\
&= S^{-1}T\bigg(e^{-i\int_t^{+\infty}ds\hat{V}_{\rm in}(s)}\bigg)\hat{\phi}_{\rm in}(x)T\bigg(e^{-i\int_{-\infty}^{t}ds\hat{V}_{\rm in}(s)}\bigg)\\
&= S^{-1}\bigg(1-i\int_t^{+\infty}dt_1\hat{V}_{\rm in}(t_1)+(-i)^2\int_t^{+\infty}dt_1\int_{t_1}^{+\infty}dt_2\hat{V}_{\rm in}(t_2)\hat{V}_{\rm in}(t_1)+...\bigg)\hat{\phi}_{\rm in}(x)\\
&\times \bigg(1-i\int_{-\infty}^{t}dt_1\hat{V}_{\rm in}(t_1)+(-i)^2\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)+...\bigg)\\
&= S^{-1}\bigg(\hat{\phi}_{\rm in}(x)-i\int_t^{+\infty}dt_1\hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x)+(-i)^2\int_t^{+\infty}dt_1\int_{t_1}^{+\infty}dt_2\hat{V}_{\rm in}(t_2)\hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x)\\
&- i\hat{\phi}_{\rm in}(x)\int_{-\infty}^{t}dt_1\hat{V}_{\rm in}(t_1)+(-i)^2\int_t^{+\infty}dt_1\int_{-\infty}^{t}dt_2\hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x)\hat{V}_{\rm in}(t_2)\\
&+ (-i)^2\hat{\phi}_{\rm in}(x)\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)+...\bigg).
\end{aligned}\tag{2.106}$$

We use the identities

$$\int_{-\infty}^{+\infty}dt_1T(\hat{\phi}_{\rm in}(x)\hat{V}_{\rm in}(t_1))=\hat{\phi}_{\rm in}(x)\int_{-\infty}^{t}dt_1\hat{V}_{\rm in}(t_1)+\int_t^{+\infty}dt_1\hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x). \tag{2.107}$$

$$\int_t^{+\infty} dt_1 \int_{t_1}^{+\infty} dt_2 T(\hat{V}_{\rm in}(t_2)\hat{V}_{\rm in}(t_1)) = \int_t^{+\infty} dt_1 \int_t^{t_1} dt_2 T(\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)). \quad (2.108)$$

$$\begin{aligned}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 T(\hat{\phi}_{\rm in}(x)\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)) &= \int_t^{+\infty} dt_1 \int_t^{t_1} dt_2 \hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)\hat{\phi}_{\rm in}(x)\\ &+ \int_t^{+\infty} dt_1 \int_{-\infty}^{t} dt_2 \hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x)\hat{V}_{\rm in}(t_2)\\ &+ \hat{\phi}_{\rm in}(x)\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2).\end{aligned} \quad (2.109)$$

We get

$$\begin{aligned}\hat{\phi}(x) &= S^{-1}T\bigg(\hat{\phi}_{\rm in}(x)\bigg(1 - i\int_{-\infty}^{+\infty} dt_1 \hat{V}_{\rm in}(t_1) + (-i)^2\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2) + ...\bigg)\bigg)\\ &= S^{-1}T\bigg(\hat{\phi}_{\rm in}(x)S\bigg).\end{aligned} \quad (2.110)$$

This result holds to all orders in perturbation theory. A straightforward generalization is

$$T(\hat{\phi}(x)\hat{\phi}(y)...) = S^{-1}T\bigg(\hat{\phi}_{\rm in}(x)\hat{\phi}_{\rm in}(y)...S\bigg). \quad (2.111)$$

This is known as the Gell-Mann Low formula.

3.2.4 LSZ Reduction Formulae and Green's Functions

We start by writing equations (2.80) and (2.81) in the form

$$e^{iE_{\vec{p}}t}(i\partial_t + E_{\vec{p}})\hat{Q}_{\rm in}(t,\vec{p}) = \sqrt{2E_{\vec{p}}}\ \hat{a}_{\rm in}(\vec{p}). \quad (2.112)$$

$$e^{iE_{\vec{p}}t}(i\partial_t + E_{\vec{p}})\hat{Q}_{\rm out}(t,\vec{p}) = \sqrt{2E_{\vec{p}}}\ \hat{a}_{\rm out}(\vec{p}). \quad (2.113)$$

Now we compute trivially the integral

$$\int_{-\infty}^{+\infty} dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t + E_{\vec{p}})\hat{Q}(t,\vec{p})\bigg) = \sqrt{2E_{\vec{p}}}\ (\hat{a}_{\rm out}(\vec{p}) - \hat{a}_{\rm in}(\vec{p})). \quad (2.114)$$

From the other hand we compute

$$\int_{-\infty}^{+\infty}dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})\hat{Q}(t,\vec{p})\bigg) = i\int_{-\infty}^{+\infty}dte^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)\hat{Q}(t,\vec{p}) = i\int d^4x\frac{\delta\mathcal{L}_{\rm int}}{\delta\phi}e^{ipx}. \quad (2.115)$$

We obtain then the identity

$$i\int_{-\infty}^{+\infty}dte^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)\hat{Q}(t,\vec{p})=\sqrt{2E_{\vec{p}}}\;(\hat{a}_{\rm out}(\vec{p})-\hat{a}_{\rm in}(\vec{p})). \quad (2.116)$$

This is the first instance of LSZ reduction formulae. Generalizations of this result read

$$i\int_{-\infty}^{+\infty}dte^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) = \sqrt{2E_{\vec{p}}}\;\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)-T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\hat{a}_{\rm in}(\vec{p})\bigg). \quad (2.117)$$

Next we put to use these LSZ reduction formulae. We are interested in calculating the matrix elements of the $S-$matrix. We consider an arbitrary "in" state $|\vec{p}_1\vec{p}_2...\text{ in}>$ and an arbitrary "out" state $|\vec{q}_1\vec{q}_2...\text{ out}>$. The matrix elements of interest are

$$<\vec{q}_1\vec{q}_2...\text{ out}|\vec{p}_1\vec{p}_2...\text{ in}>=<\vec{q}_1\vec{q}_2...\text{ in}|S|\vec{p}_1\vec{p}_2...\text{ in}>. \quad (2.118)$$

We recall that

$$|\vec{p}_1\vec{p}_2...\text{ in}> = a_{\rm in}(\vec{p}_1)^+a_{\rm in}(\vec{p}_2)^+...|0>. \quad (2.119)$$

$$|\vec{q}_1\vec{q}_2...\text{ out}> = a_{\rm out}(\vec{q}_1)^+a_{\rm out}(\vec{q}_2)^+...|0>. \quad (2.120)$$

We also recall the commutation relations (using box normalization)

$$[\hat{a}(\vec{p}),\hat{a}(\vec{q})^+]=V\delta_{\vec{p},\vec{q}}\;,\;[\hat{a}(\vec{p}),\hat{a}(\vec{q})]=[\hat{a}(\vec{p})^+,\hat{a}(\vec{q})^+]=0. \quad (2.121)$$

We compute by using the LSZ reduction formula (2.116) and assuming that the $\vec{p}_i$ are different from the $\vec{q}_i$ the result

$$\begin{aligned}<\vec{q}_1\vec{q}_2...\text{ out}|\vec{p}_1\vec{p}_2...\text{ in}> &= <\vec{q}_2...\text{ out}|\hat{a}_{\rm out}(\vec{q}_1)|\vec{p}_1\vec{p}_2...\text{ in}>\\ &= <\vec{q}_2..\text{ out}|\bigg(\hat{a}_{\rm in}(\vec{q}_1)+\frac{i}{\sqrt{2E_{\vec{q}_1}}}\int_{-\infty}^{+\infty}dt_1e^{iE_{\vec{q}_1}t_1}(\partial_{t_1}^2+E_{\vec{q}_1}^2)\hat{Q}(t_1,\vec{q}_1)\bigg)\\ &\times\;|\vec{p}_1\vec{p}_2..\text{ in}>\\ &= \frac{1}{\sqrt{2E_{\vec{q}_1}}}\int_{-\infty}^{+\infty}dt_1e^{iE_{\vec{q}_1}t_1}i(\partial_{t_1}^2+E_{\vec{q}_1}^2)<\vec{q}_2...\text{ out}|\hat{Q}(t_1,\vec{q}_1)|\vec{p}_1\vec{p}_2...\text{ in}>.\end{aligned} \quad (2.122)$$

From the LSZ reduction formula (2.117) we have

$$i\int_{-\infty}^{+\infty} dt_2 e^{iE_{\vec{q}_2}t_2}(\partial_{t_2}^2+E_{\vec{q}_2}^2)T(\hat{Q}(t_2,\vec{q}_2)\hat{Q}(t_1,\vec{q}_1))=\sqrt{2E_{\vec{q}_2}}\bigg(\hat{a}_{\rm out}(\vec{q}_2)\hat{Q}(t_1,\vec{q}_1)-\hat{Q}(t_1,\vec{q}_1)\hat{a}_{\rm in}(\vec{q}_2)\bigg). \tag{2.123}$$

Thus immediately

$$i\int_{-\infty}^{+\infty} dt_2 e^{iE_{\vec{q}_2}t_2}(\partial_{t_2}^2+E_{\vec{q}_2}^2) < \vec{q}_3..\ {\rm out}|T(\hat{Q}(t_2,\vec{q}_2)\hat{Q}(t_1,\vec{q}_1))|\vec{p}_1\vec{p}_2..\ {\rm in}> \ = \sqrt{2E_{\vec{q}_2}}\ < \vec{q}_2..\ {\rm out}|\hat{Q}(t_1,\vec{q}_1)|\vec{p}_1\vec{p}_2..\ {\rm in}>. \tag{2.124}$$

Hence

$$< \vec{q}_1\vec{q}_2...\ {\rm out}|\vec{p}_1\vec{p}_2...\ {\rm in}> \ = \frac{1}{\sqrt{2E_{\vec{q}_1}}}\frac{1}{\sqrt{2E_{\vec{q}_2}}}\int_{-\infty}^{+\infty} dt_1 e^{iE_{\vec{q}_1}t_1}i(\partial_{t_1}^2+E_{\vec{q}_1}^2)\int_{-\infty}^{+\infty} dt_2 e^{iE_{\vec{q}_2}t_2}i(\partial_{t_2}^2+E_{\vec{q}_2}^2) \times < \vec{q}_3...\ {\rm out}|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2))|\vec{p}_1\vec{p}_2...\ {\rm in}>. \tag{2.125}$$

By continuing this reduction of all "out" operators we end up with the expression

$$< \vec{q}_1\vec{q}_2...\ {\rm out}|\vec{p}_1\vec{p}_2...\ {\rm in}> \ = \frac{1}{\sqrt{2E_{\vec{q}_1}}}\frac{1}{\sqrt{2E_{\vec{q}_2}}}...\int_{-\infty}^{+\infty} dt_1 e^{iE_{\vec{q}_1}t_1}i(\partial_{t_1}^2+E_{\vec{q}_1}^2)\int_{-\infty}^{+\infty} dt_2 e^{iE_{\vec{q}_2}t_2}i(\partial_{t_2}^2+E_{\vec{q}_2}^2)... \times < 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...)|\vec{p}_1\vec{p}_2...\ {\rm in}>. \tag{2.126}$$

In order to reduce the "in" operators we need other LSZ reduction formulae which involve the creation operators instead of the annihilation operators. The result we need is essentially the Hermitian conjugate of (2.117) given by

$$-i\int_{-\infty}^{+\infty} dt e^{-iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})^+\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...) \ = \sqrt{2E_{\vec{p}}}\bigg(\hat{a}_{\rm out}(\vec{p})^+T(\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...)-T(\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...)\hat{a}_{\rm in}(\vec{p})^+\bigg). \tag{2.127}$$

By using these LSZ reduction formulae we compute

$$< 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...)|\vec{p}_1\vec{p}_2...\ {\rm in}> \ = \frac{1}{\sqrt{2E_{\vec{p}_1}}}\int_{-\infty}^{+\infty} dt_1^{'} e^{-iE_{\vec{p}_1}t_1^{'}}i(\partial_{t_1^{'}}^2+E_{\vec{p}_1}^2) < 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...\hat{Q}(t_1^{'},\vec{p}_1)^+)|\vec{p}_2...\ {\rm in}>. \tag{2.128}$$

Full reduction of the "in" operators leads to the expression

$$
\begin{aligned}
&< 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...)|\vec{p}_1\vec{p}_2... \text{ in} > \;=\\
&\frac{1}{\sqrt{2E_{\vec{p}_1}}}\frac{1}{\sqrt{2E_{\vec{p}_2}}}...\int_{-\infty}^{+\infty}dt_1^{'}e^{-iE_{\vec{p}_1}t_1^{'}}i(\partial_{t_1^{'}}^2+E_{\vec{p}_1}^2)\int_{-\infty}^{+\infty}dt_2^{'}e^{-iE_{\vec{p}_2}t_2^{'}}i(\partial_{t_2^{'}}^2+E_{\vec{p}_2}^2)... \;\times\\
&< 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...\hat{Q}(t_1^{'},\vec{p}_1)^+\hat{Q}(t_2^{'},\vec{p}_2)^+...)|0> .
\end{aligned}
\tag{2.129}
$$

Hence by putting the two partial results (2.126) and (2.129) together we obtain

$$
\begin{aligned}
< \vec{q}_1... \text{ out}|\vec{p}_1... \text{ in} > \;&=\; \frac{1}{\sqrt{2E_{\vec{q}_1}}}...\frac{1}{\sqrt{2E_{\vec{p}_1}}}...\int_{-\infty}^{+\infty}dt_1e^{iE_{\vec{q}_1}t_1}i(\partial_{t_1}^2+E_{\vec{q}_1}^2)...\int_{-\infty}^{+\infty}dt_1^{'}e^{-iE_{\vec{p}_1}t_1^{'}}i(\partial_{t_1^{'}}^2+E_{\vec{p}_1}^2)...\\
&\times\; < 0|T(\hat{Q}(t_1,\vec{q}_1)...\hat{Q}(t_1^{'},\vec{p}_1)^+...)|0> .
\end{aligned}
\tag{2.130}
$$

The final (fundamental) result is that $S-$matrix elements $< \vec{q}_1... \text{ out}|\vec{p}_1... \text{ in} >$ can be reconstructed from the so-called Green's functions $< 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_1^{'})...)|0 >$. Indeed we can rewrite equation (2.130) as

$$
\begin{aligned}
< \vec{q}_1... \text{ out}|\vec{p}_1... \text{ in} > \;&=\; \frac{1}{\sqrt{2E_{\vec{q}_1}}}...\frac{1}{\sqrt{2E_{\vec{p}_1}}}...\int d^4x_1e^{iq_1x_1}i(\partial_1^2+m^2)...\int d^4x_1^{'}e^{-ip_1x_1^{'}}i(\partial_1^{'2}+m^2)...\\
&\times\; < 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_1^{'})...)|0> .
\end{aligned}
\tag{2.131}
$$

The factor $1/\sqrt{2E_{\vec{q}_1}}...1/\sqrt{2E_{\vec{p}_1}}$ is only due to our normalization of the one-particle states given in equations (2.119) and (2.120).

3.3 Feynman Diagrams For $\phi-$Four Theory

3.3.1 Perturbation Theory

We go back to our most fundamental result (2.111) and write it in the form (with $\mathcal{L}_{\rm int}(\hat{\phi}_{\rm in}(x)) = \mathcal{L}_{\rm int}(x)$)

$$
\begin{aligned}
< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0> \;&=\; < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)...S\bigg)|0>\\
&=\; < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)...e^{i\int d^4y\mathcal{L}_{\rm int}(y)}\bigg)|0>\\
&=\; \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4y_1..\int d^4y_n < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)..\mathcal{L}_{\rm int}(y_1)..\mathcal{L}_{\rm int}(y_n)\bigg)|0> .
\end{aligned}
\tag{3.132}
$$

These are the Green's functions we need in order to compute the $S-$matrix elements. They are written solely in terms of free fields and the interaction Lagrangian density. This expansion is the key perturbative series in quantum field theory.

Another quantity of central importance to perturbation theory is the vacuum-to-vacuum amplitude given by

$$< 0|0> = < 0|S|0> \;=\; \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4y_1..\int d^4y_n < 0|T\bigg(\mathcal{L}_{\rm int}(y_1)..\mathcal{L}_{\rm int}(y_n)\bigg)|0>. \quad (3.133)$$

Naively we would have thought that this norm is equal to 1. However it turns out that this is not the case and taking this fact into account will simplify considerably our perturbative calculations.

3.3.2 Wick's Theorem For Green's Functions

From the above discussion it is clear that the remaining task is to evaluate terms of the generic form

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0> . \quad (3.134)$$

To this end we rewrite the Wick's theorem (1.75) in the form

$$< 0|T\bigg(e^{i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}\bigg)|0> = e^{-\frac{1}{2}\int d^4x\int d^4x^{'} J(x)J(x^{'})D_F(x-x^{'})}. \quad (3.135)$$

Because the scalar field is real we also have

$$< 0|T\bigg(e^{-i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}\bigg)|0> = e^{-\frac{1}{2}\int d^4x\int d^4x^{'} J(x)J(x^{'})D_F(x-x^{'})}. \quad (3.136)$$

This means that only even powers of J appear. We expand both sides in powers of J we get

$$\sum_{n=0}\frac{i^{2n}}{2n!}\int d^4x_1...d^4x_{2n}J(x_1)...J(x_{2n}) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0> \;=\; \sum_{n=0}\frac{1}{n!}(-\frac{1}{2})^n\int d^4x_1\int d^4x_2...\int d^4x_{2n-1}\int d^4x_{2n} \;\times\; J(x_1)J(x_2)...J(x_{2n-1})J(x_{2n})D_F(x_1-x_2)...D_F(x_{2n-1}-x_{2n}). \quad (3.137)$$

Let us look at few examples. The first non-trivial term is

$$\frac{i^2}{2!}\int d^4x_1 d^4x_2 J(x_1)J(x_2) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\bigg)|0> \;=\; \frac{1}{1!}(-\frac{1}{2})^1\int d^4x_1\int d^4x_2 J(x_1)J(x_2)D_F(x_1-x_2). \quad (3.138)$$

Immediately we get the known result

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\bigg)|0 >= D_F(x_1-x_2). \quad (3.139)$$

The second non-trivial term is

$$\frac{i^4}{4!}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 J(x_1)J(x_2)J(x_3)J(x_4) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(x_3)\hat{\phi}_{\rm in}(x_4)\bigg)|0> \;=\; \frac{1}{2!}(-\frac{1}{2})^2\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4 \;\times\; J(x_1)J(x_2)J(x_3)J(x_4)D_F(x_1-x_2)D_F(x_3-x_4). \quad (3.140)$$

Equivalently

$$\frac{i^4}{4!}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 J(x_1)J(x_2)J(x_3)J(x_4) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(x_3)\hat{\phi}_{\rm in}(x_4)\bigg)|0> \;=\; \frac{1}{2!}(-\frac{1}{2})^2\frac{1}{3}\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4 \;\times\; J(x_1)J(x_2)J(x_3)J(x_4)\bigg(D_F(x_1-x_2)D_F(x_3-x_4) \;+\; D_F(x_1-x_3)D_F(x_2-x_4)+D_F(x_1-x_4)D_F(x_2-x_3)\bigg). \quad (3.141)$$

In the last equation we have symmetrized the right-hand side under the permutations of the spacetime points x_1, x_2, x_3 and x_4 and then divided by $1/3$ where 3 is the number of independent permutations in this case. This is needed because the left-hand side is already symmetric under the permutations of the x_i's. By comparing the two sides we then obtain

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(x_3)\hat{\phi}_{\rm in}(x_4)\bigg)|0> \;=\; D_F(x_1-x_2)D_F(x_3-x_4)+D_F(x_1-x_3)D_F(x_2-x_4) \;+\; D_F(x_1-x_4)D_F(x_2-x_3). \quad (3.142)$$

The independent permutations are called contractions and we write

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(x_3)\hat{\phi}_{\rm in}(x_4)\bigg)|0> \;=\; \sum_{\rm contraction}\prod D_F(x_i-x_j). \quad (3.143)$$

This generalizes to any Green's function. In equation (3.137) we need to symmetrize the right-hand side under the permutations of the spacetime points x_i's before comparing with the left-hand side. Thus we need to count the number of independent permutations or contractions. Since we have $2n$ points we have $(2n)!$ permutations not all of them independent. Indeed we need to divide by 2^n since $D_F(x_i - x_j) = D_F(x_j - x_i)$ and we have n such propagators. Then we need to divide by $n!$ since the order of the n propagators $D_F(x_1-x_2)$,...,$D_F(x_{2n-1}-x_{2n})$ is irrelevant. We get then $(2n)!/(2^n n!)$ independent permutations. Equation (3.137) becomes

$$\sum_{n=0}\frac{i^{2n}}{2n!}\int d^4x_1...d^4x_{2n}J(x_1)...J(x_{2n}) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0> \; = \; \sum_{n=0}\frac{1}{n!}(-\frac{1}{2})^n\frac{2^n n!}{(2n)!}\int d^4x_1\int d^4x_2...\int d^4x_{2n-1}\int d^4x_{2n} \times J(x_1)J(x_2)...J(x_{2n-1})J(x_{2n})\sum_{\rm contraction}\prod D_F(x_i-x_j). \quad (3.144)$$

By comparison we obtain

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0 >= \sum_{\rm contraction}\prod D_F(x_i-x_j). \quad (3.145)$$

This is Wick's theorem for Green's functions.

An alternative more systematic way of obtaining all contractions goes as follows. First let us define

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0 >=< 0|T\bigg(F(\hat{\phi}_{\rm in})\bigg)|0 > . \quad (3.146)$$

We introduce the functional Fourier transform

$$F(\hat{\phi}_{\rm in}) = \int \mathcal{D}J\tilde{F}(J)\; e^{i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}. \quad (3.147)$$

Thus

$$\begin{aligned} < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0 > &= < 0|T\bigg(\int \mathcal{D}J\tilde{F}(J)\; e^{i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}\bigg)|0 > \\ &= \int \mathcal{D}J\tilde{F}(J) < 0|T\bigg(e^{i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}\bigg)|0 > \\ &= \int \mathcal{D}J\tilde{F}(J)e^{-\frac{1}{2}\int d^4x\int d^4x' J(x)D_F(x-x')J(x')}. \end{aligned} \quad (3.148)$$

We use the identity (starting from here we only deal with classical fields instead of field operators)

$$f\left(\frac{\delta}{\delta\phi}\right)e^{i\int d^4xJ(x)\phi(x)}=f(iJ)e^{i\int d^4xJ(x)\phi(x)} \quad (3.149)$$

In particular we have

$$e^{\frac{1}{2}\int d^4x\int d^4x'\frac{\delta}{\delta\phi(x)}D_F(x-x')\frac{\delta}{\delta\phi(x')}}e^{i\int d^4xJ(x)\phi(x)}=e^{-\frac{1}{2}\int d^4x\int d^4x'J(x)D_F(x-x')J(x')}e^{i\int d^4xJ(x)\phi(x)} \quad (3.150)$$

Thus

$$\begin{aligned}<0|T\left(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\right)|0> &= \int\mathcal{D}J\tilde{F}(J)\left[e^{\frac{1}{2}\int d^4x\int d^4x'\frac{\delta}{\delta\phi(x)}D_F(x-x')\frac{\delta}{\delta\phi(x')}}e^{i\int d^4xJ(x)\phi(x)}\right]_{\phi=0}\\ &= \left[e^{\frac{1}{2}\int d^4x\int d^4x'\frac{\delta}{\delta\phi(x)}D_F(x-x')\frac{\delta}{\delta\phi(x')}}F(\phi)\right]_{\phi=0}. \quad (3.151)\end{aligned}$$

We think of F as a function in several variables which are the classical fields $\phi(x_i)$. Thus we have

$$\frac{\delta F}{\delta\phi(x)}=\delta^4(x-x_1)\frac{\partial F}{\partial\phi(x_1)}+\delta^4(x-x_2)\frac{\partial F}{\partial\phi(x_2)}+... \quad (3.152)$$

Hence

$$\begin{aligned}<0|T\left(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\right)|0> &= \left[e^{\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-x_j)\frac{\partial}{\partial\phi(x_j)}}F(\phi)\right]_{\phi=0}\\ &= \left[e^{\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-x_j)\frac{\partial}{\partial\phi(x_j)}}\left(\phi(x_1)...\phi(x_{2n})\right)\right]_{\phi=0} \quad (3.153)\end{aligned}$$

This is our last version of the Wick's theorem.

3.3.3 The 2−Point Function

We have

$$\begin{aligned}<0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0> &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4y_1..\int d^4y_n<0|T\left(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)..\mathcal{L}_{\rm int}(y_n)\right)|0>\\ &= <0|T\left(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\right)|0>+i\int d^4y_1<0|T\left(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)\right)|0>\\ &+ \frac{i^2}{2!}\int d^4y_1\int d^4y_2<0|T\left(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)\mathcal{L}_{\rm int}(y_2)\right)|0>+... \quad (3.154)\end{aligned}$$

By using the result (3.153) we have (since we are considering only polynomial interactions)

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)..\mathcal{L}_{\rm int}(y_n)\bigg)|0> \;=\; \bigg[e^{\partial D_F\partial}\bigg(\phi(x_1)\phi(x_2)\mathcal{L}_{\rm int}(y_1)..\mathcal{L}_{\rm int}(y_n)\bigg)\bigg]_{\phi=0}. \tag{3.155}$$

$$\partial D_F\partial = \frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-x_j)\frac{\partial}{\partial\phi(x_j)}+\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(y_i)}D_F(y_i-y_j)\frac{\partial}{\partial\phi(y_j)} + \sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-y_j)\frac{\partial}{\partial\phi(y_j)}. \tag{3.156}$$

The 0th order term is the free propagator, viz

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\bigg)|0>= D_F(x_1-x_2). \tag{3.157}$$

We represent this amplitude by a line joining the external points x_1 and x_2 (figure 1). This is our first Feynman diagram. Physically this represents a scalar particle created at x_2 then propagates in spacetime before it gets annihilated at x_1.

The first order is given by

$$i\int d^4y_1 < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)\bigg)|0> \;=\; i(-\frac{\lambda}{4!})\int d^4y_1 < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(y_1)^4\bigg)|0>. \tag{3.158}$$

We apply the Wick's theorem. There are clearly many possible contractions. For six operators we can have in total 15 contractions which can be counted as follows. The first operator can be contracted in 5 different ways. The next operator can be contracted in 3 different ways and finally the remaining two operators can only be contracted in one way. Thus we get 5.3.1 = 15. However there are only two distinct contractions among these 15 contractions. They are as follows

$a)-$ We can contract the two external points x_1 and x_2 together. The internal point $z=y_1$ which we will call a vertex since it corresponds to an interaction corresponds to 4 internal points (operators) which can be contracted in 3.1 = 3 different ways. We have therefore three identical contributions coming from these three contractions. We get

$$3\times i(-\frac{\lambda}{4!})D_F(x_1-x_2)\int d^4z D_F(0)^2=\frac{1}{8}(-i\lambda)\int d^4z D_F(x_1-x_2)D_F(0)^2. \tag{3.159}$$

$b)-$ We can contract one of the external points with one of the internal points. There are four different ways for doing this. The remaining external point must then be contracted with one of the remaining three internal points. There are three different ways for doing this. In total we have 4.3 = 12 contractions which lead to the same contribution. We have

$$12 \times i(-\frac{\lambda}{4!}) \int d^4 z D_F(x_1 - z) D_F(x_2 - z) D_F(0) = \frac{1}{2}(-i\lambda) \int d^4 z D_F(x_1 - z) D_F(x_2 - z) D_F(0). \quad (3.160)$$

The two amplitudes (3.159) and (3.160) stand for the 15 possible contractions which we found at first order. These contractions split into two topologically distinct sets represented by the two Feynman diagrams $a)$ and $b)$ on figure 2 with attached values given precisely by (3.159) and (3.160). We observe in constructing these diagrams the following

- Each line (internal or external) joining two spacetime points x and y is associated with a propagator $D_F(x-y)$.
- Interaction is represented by a vertex. Each vertex is associated with a factor $-i\lambda$.
- We multiply the propagators and vertices together then we integrate over the internal point.
- We divide by a so-called symmetry factor S. The symmetry factor is equal to the number of independent permutations which leave the diagram invariant.

  A diagram containing a line which starts and ends on the same vertex will be symmetric under the permutation of the two ends of such a line. This is clear from the identity

$$\int d^4 z D_F(0) = \int d^4 z \int d^4 u D_F(z-u) \delta^4(z-u). \quad (3.161)$$

  Diagram $b)$ contains such a factor and thus the symmetry factor in this case is $S = 2$. Diagram $a)$ contains two such factors and thus one must divide by 2.2. Since this diagram is also invariant under the permutation of the two $D_F(0)$ we must divide by an extra factor of 2. The symmetry factor for diagram $a)$ is therefore $S = 2.2.2 = 8$.

The second order in perturbation theory is given by

$$\frac{i^2}{2!} \int d^4 y_1 \int d^4 y_2 < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)\mathcal{L}_{\rm int}(y_2)\bigg)|0> \; =$$

$$-\frac{1}{2}(\frac{\lambda}{4!})^2 \int d^4 y_1 \int d^4 y_2 < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(y_1)^4\hat{\phi}_{\rm in}(y_2)^4\bigg)|0> . \quad (3.162)$$

Again we apply Wick's theorem. There are in total 9.7.5.3 = 9.105 contractions which can be divided into three different classes (figure 3) as follows

1) The first class corresponds to the contraction of the two external points x_1 and x_2 to the same vertex y_1 or y_2. These contractions correspond to the two topologically different contractions $a)_1$ and $b)_1$ on figure 3.

   In $a)_1$ we contract x_1 with one of the internal points in 8 different ways, then x_2 can be contracted in 3 different ways to the same internal point (say y_1). If the two remaining y_1 points are contracted together the remaining internal points y_2 can then be contracted together in 3 different ways. There are in total 8.3.3 contractions. The analytic expression is

$$-\frac{8.3.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(0)^3 = \frac{(-i\lambda)^2}{16}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(0)^3. \quad (3.163)$$

   In $b)_1$ we consider the case where one of the remaining y_1 points is contracted with one of the internal points y_2 in 4 different ways. The last y_1 must then also be contracted with one of the y_2 in 3 different ways. This possibility corresponds to 8.3.4.3 contractions. The analytic expression is

$$-\frac{8.3.4.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(y_1-y_2)^2D_F(0) = \frac{(-i\lambda)^2}{4}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(y_1-y_2)^2D_F(0) \quad (3.164)$$

2) The second class corresponds to the contraction of the external point x_1 to one of the vertices whereas the external point x_2 is contracted to the other vertex. These contractions correspond to the two topologically different contractions $a)_2$ and $b)_2$ on figure 3.

   In $a)_2$ we contract x_1 with one of the internal points (say y_1) in 8 different ways, then x_2 can be contracted in 4 different ways to the other internal point (i.e. y_2). There remains three internal points y_1 and three internal points y_2. Two of the y_1 can be contracted in 3 different ways. The remaining y_1 must be contracted with one of the y_2 in 3 different ways. Thus we have in total 8.4.3.3 contractions. The expression is

$$-\frac{8.4.3.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)D_F(0)^2 = \frac{(-i\lambda)^2}{4}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)D_F(0)^2 \quad (3.165)$$

In $b)_2$ we consider the case where the three remaining y_1 are paired with the three remaining y_2. The first y_1 can be contracted with one of the y_2 in 3 different ways, the second y_1 can be contracted with one of the remaining y_2 in 2 different ways. Thus we have in total 8.4.3.2 contractions. The expression is

$$-\frac{8.4.3.2}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)^3 = \frac{(-i\lambda)^2}{6}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)^3. \quad (3.166)$$

3) The third class corresponds to the contraction of the two external points x_1 and x_2 together. These contractions correspond to the three topologically different contractions $a)_3$, $b)_3$ and $c)_2$ on figure 3.

In $a)_3$ we can contract the y_1 among themselves in 3 different ways and contract the y_2 among themselves in 3 different ways. Thus we have 3.3 contractions. The expression is

$$-\frac{3.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(0)^4 = \frac{(-i\lambda)^2}{128}\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(0)^4. \quad (3.167)$$

In $b)_3$ we can contract two of the y_1 together in 6 different ways, then contract one of the remaining y_1 with one of the y_2 in 4 different ways, and then contract the last y_1 with one of the y_2 in 3 different ways. Thus we have 6.4.3 contractions. The expression is

$$-\frac{6.4.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(y_1-y_2)^2D_F(0)^2 = \frac{(-i\lambda)^2}{16}\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(y_1-y_2)^2D_F(0)^2. \quad (3.168)$$

In $c)_3$ we can contract the first y_1 with one of the y_2 in 4 different ways, then contract the second y_1 with one of the y_2 in 3 different ways, then contract the third y_1 with one of the y_2 in 2 different ways. We get 4.3.2 contractions. The expression is

$$-\frac{4.3.2}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(y_1-y_2)^4 = \frac{(-i\lambda)^2}{48}\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(y_1-y_2)^4. \quad (3.169)$$

The above seven amplitudes (3.163), (3.164), (3.165), (3.166), (3.167), (3.168) and (3.169) can be represented by the seven Feynman diagrams $a)_1$, $b)_1$, $a)_2$, $b)_2$, $a)_3$, $b)_3$ and $c)_3$ respectively. We use in constructing these diagrams the same rules as before. We will only comment here on the symmetry factor S for each diagram. We have

- The symmetry factor for the first diagram is $S = (2.2.2).2 = 16$ where the first three factors of 2 are associated with the three $D_F(0)$ and the last factor of 2 is associated with the interchange of the two $D_F(0)$ in the figure of eight.
- The symmetry factor for the second diagram is $S = 2.2 = 4$ where the first factor of 2 is associated with $D_F(0)$ and the second factor is associated with the interchange of the two internal lines $D_F(y_1 - y_2)$.
- The symmetry factor for the third diagram is $S = 2.2$ where the two factors of 2 are associated with the two $D_F(0)$.
- The symmetry factor of the 4th diagram is $S = 3! = 6$ which is associated with the permutations of the three internal lines $D_F(y_1 - y_2)$.
- The symmetry factor of the 5th diagram is $S = 2^7 = 128$. Four factors of 2 are associated with the four $D_F(0)$. Two factors of 2 are associated with the permutations of the two $D_F(0)$ in the two figures of eight. Another factor of 2 is associated with the interchange of the two figures of eight.
- The symmetry factor of the 6th diagram is $S = 2^4 = 16$. Two factors of 2 comes from the two $D_F(0)$. A factor of 2 comes from the interchange of the two internal lines $D_F(y_1 - y_2)$. Another factor comes from the interchange of the two internal points y_1 and y_2.
- The symmetry factor of the last diagram is $S = 4!.2 = 48$. The factor 4! comes from the permutations of the four internal lines $D_F(y_1 - y_2)$ and the factor of two comes from the interchange of the two internal points y_1 and y_2.

3.3.4 Connectedness and Vacuum Energy

From the above discussion we observe that there are two types of Feynman diagrams. These are

- Connected Diagrams: These are diagrams in which every piece is connected to the external points. Examples of connected diagrams are diagram $b)$ on figure 2) and diagrams $b)_1$, $a)_2$ and $b)_2$ on figure 4.

- Disconnected Diagrams: These are diagrams in which there is at least one piece which is not connected to the external points. Examples of disconnected diagrams are diagram $a)$ on figure 2) and diagrams $a)_1$, $a)_3$, $b)_3$ and $c)_3$ on figure 4.

We write the 2−point function up to the second order in perturbation theory as

$$<0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0> \; = \; D_0(x_1-x_2)[V_1+\frac{1}{2}V_1^2+V_2+V_3]+D_1(x_1-x_2)[1+V_1] + D_2^1(x_1-x_2)+D_2^2(x_1-x_2)+D_2^3(x_1-x_2). \quad (3.170)$$

The "connected" 2−point function at the 0th and 1st orders is given respectively by

$$D_0(x_1-x_2) = \text{diagram 1)} = D_F(x_1-x_2). \quad (3.171)$$

$$D_1(x_1-x_2) = \text{diagram } 2b) = \frac{1}{2}(-i\lambda)\int d^4y_1 D_F(x_1-y_1)D_F(x_2-y_1)D_F(0). \quad (3.172)$$

The "connected" 2−point function at the 2nd order is given by the sum of the three propagators D_2^1, D_2^2 and D_2^3. Explicitly they are given by

$$D_2^1(x_1-x_2) = \text{diagram } 4b)_1 = \frac{(-i\lambda)^2}{4}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(y_1-y_2)^2 D_F(0). \quad (3.173)$$

$$D_2^2(x_1-x_2) = \text{diagram } 4a)_2 = \frac{(-i\lambda)^2}{4}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)D_F(0)^2. \quad (3.174)$$

$$D_2^3(x_1-x_2) = \text{diagram } 4b)_2 = \frac{(-i\lambda)^2}{6}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)^3. \quad (3.175)$$

The connected 2−point function up to the second order in perturbation theory is therefore

$$<0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0>_{\rm conn} \; = \; D_0(x_1-x_2)+D_1(x_1-x_2)+D_2^1(x_1-x_2)+D_2^2(x_1-x_2)+D_2^3(x_1-x_2). \quad (3.176)$$

The corresponding Feynman diagrams are shown on figure 5. The disconnected diagrams are obtained from the product of these connected diagrams with the so-called vacuum graphs which are at this order in perturbation theory given by V_1, V_2 and V_3 (see (3.170)). The vacuum graphs are given explicitly by

$$V_1 = \frac{-i\lambda}{8}\int d^4y_1 D_F(0)^2. \tag{3.177}$$

$$V_2 = \frac{(-i\lambda)^2}{16}\int d^4y_1 \int d^4y_2 D_F(y_1-y_2)^2 D_F(0)^2. \tag{3.178}$$

$$V_3 = \frac{(-i\lambda)^2}{48}\int d^4y_1 \int d^4y_2 D_F(y_1-y_2)^4. \tag{3.179}$$

The corresponding Feynman diagrams are shown on figure 6. Clearly the "full" and the "connected" 2−point functions can be related at this order in perturbation theory as

$$<0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0> = <0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0>_{\rm conn}\exp(\text{vacuum graphs}). \tag{3.180}$$

We now give a more general argument for this identity. We will label the various vacuum graphs by V_i, $i = 1, 2, 3, ...$ A generic Feynman diagram will contain a connected piece attached to the external points x_1 and x_2 call it W_j, n_1 disconnected pieces given by V_1, n_2 disconnected pieces given by V_2, and so on. The value of this Feynman diagram is clearly

$$W_j \prod_i \frac{1}{n_i!} V_i^{n_i}. \tag{3.181}$$

The factor $1/n_i!$ is a symmetry factor coming from the permutations of the n_i pieces V_i among themselves. Next by summing over all Feynman diagrams (i.e, all possible connected diagrams and all possible values of n_i) we obtain

$$\begin{aligned}\sum_j \sum_{n_1,...,n_i,...} W_j \prod_i \frac{1}{n_i!} V_i^{n_i} &= \sum_j W_j \sum_{n_1,...,n_i,...} \prod_i \frac{1}{n_i!} V_i^{n_i}\\ &= \sum_j W_j \prod_i \sum_{n_i} \frac{1}{n_i!} V_i^{n_i}\\ &= \sum_j W_j \prod_i \exp(V_i)\\ &= \sum_j W_j \exp(\sum_i V_i).\end{aligned} \tag{3.182}$$

This is the desired result. This result holds also for any other Green's function, viz

$$< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >=< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >_{\rm conn} \exp({\rm vacuum~graphs}). \quad (3.183)$$

Let us note here that the set of all vacuum graphs is the same for all Green's functions. In particular the $0-$point function (the vacuum-to-vacuum amplitude) will be given by

$$< 0|0 >= \exp({\rm vacuum~graphs}). \quad (3.184)$$

We can then observe that

$$\begin{aligned} < 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >_{\rm conn} &= \frac{< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >}{< 0|0 >} \\ &= {\rm sum~of~connected~diagrams~with~} n {\rm ~external~points}. \end{aligned} \quad (3.185)$$

We write this as

$$< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >_{\rm conn} = < \Omega|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|\Omega > . \quad (3.186)$$

$$|\Omega >= \frac{|0 >}{\sqrt{< 0|0 >}} = e^{-\frac{1}{2}({\rm vacuum~graphs})}|0 > . \quad (3.187)$$

The vacuum state $|\Omega >$ will be interpreted as the ground state of the full Hamiltonian $\hat{H}$ in contrast to the vacuum state $|0 >$ which is the ground state of the free Hamiltonian $\hat{H}_0$. The vector state $|\Omega >$ has non-zero energy $\hat{E}_0$. Thus $\hat{H}|\Omega >= \hat{E}_0|\Omega >$ as opposed to $\hat{H}_0|0 >= 0$. Let $|n >$ be the other vector states of the Hamiltonian $\hat{H}$, viz $\hat{H}|n >= \hat{E}_n|n >$.

The evolution operator $\Omega(t)$ is a solution of the differential equation $i\partial_t\Omega(t) = \hat{V}_I(t)\Omega(t)$ which satisfies the boundary condition $\Omega(-\infty) = 1$. A generalization of $\Omega(t)$ is given by the evolution operator

$$\Omega(t, t^{'}) = T\bigg(e^{-i\int_{t^{'}}^t ds\hat{V}_I(s)}\bigg). \quad (3.188)$$

This solves essentially the same differential equation as $\Omega(t)$, viz

$$i\partial_t\Omega(t, t^{'}) = \hat{V}_I(t, t_0)\Omega(t). \quad (3.189)$$

$$\hat{V}_I(t, t_0) = e^{i\hat{H}_0(t-t_0)}\hat{V}e^{-i\hat{H}_0(t-t_0)}. \quad (3.190)$$

This evolution operator $\Omega(t,t^{'})$ satisfies obviously the boundary condition $\Omega(t,t)=1$. Furthermore it is not difficult to verify that an equivalent expression for $\Omega(t,t^{'})$ is given by

$$\Omega(t,t^{'})=e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t^{'})}e^{-i\hat{H}_0(t^{'}-t_0)}. \tag{3.191}$$

We compute

$$\begin{aligned} e^{-i\hat{H}T}|0> &= e^{-i\hat{H}T}|\Omega><\Omega|0>+\sum_{n\neq 0}e^{-i\hat{H}T}|n><n|0> \\ &= e^{-i\hat{E}_0T}|\Omega><\Omega|0>+\sum_{n\neq 0}e^{-i\hat{E}_nT}|n><n|0>\,. \end{aligned} \tag{3.192}$$

In the limit $T\longrightarrow\infty(1-i\epsilon)$ the second term drops since $\hat{E}_n>\hat{E}_0$ and we obtain

$$e^{-i\hat{H}T}|0> = e^{-i\hat{E}_0T}|\Omega><\Omega|0>\,. \tag{3.193}$$

Equivalently

$$e^{-i\hat{H}(t_0-(-T))}|0> = e^{-i\hat{E}_0(t_0+T)}|\Omega><\Omega|0>\,. \tag{3.194}$$

Thus

$$|\Omega>=\frac{e^{i\hat{E}_0(t_0+T)}}{<\Omega|0>}\Omega(t_0,-T)|0>\,. \tag{3.195}$$

By choosing $t_0=T$ and using the fact that $\Omega(T,-T)=S$ we obtain

$$|\Omega>=\frac{e^{i\hat{E}_0(2T)}}{<\Omega|0>}|0>\,. \tag{3.196}$$

Finally by using the definition of $|\Omega>$ in terms of $|0>$ and assuming that the sum of vacuum graphs is pure imaginary we get

$$\frac{\hat{E}_0}{\text{vol}}=i\frac{\text{vacuum graphs}}{2T.\text{vol}}. \tag{3.197}$$

Every vacuum graph will contain a factor $(2\pi)^4\delta^4(0)$ which in the box normalization is equal exactly to $2T$.vol where vol is the volume of the three dimensional space. Hence the normalized sum of vacuum graphs is precisely equal to the vacuum energy density.

3.3.5 Feynman Rules For $\Phi-$Four Theory

We use Feynman rules for perturbative $\phi-$four theory to calculate the nth order contributions to the Green's function $< 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_N))|0 >$. They are given as follows

1) We draw all Feynman diagrams with N external points x_i and n internal points (vertices) y_i.

2) The contribution of each Feynman diagram to the Green's function $< 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_N))|0 >$ is equal to the product of the following three factors

   - Each line (internal or external) joining two spacetime points x and y is associated with a propagator $D_F(x-y)$. This propagator is the amplitude for propagation between the two points x and y.
   - Each vertex is associated with a factor $-i\lambda$. Interaction is represented by a vertex and thus there are always 4 lines meeting at a given vertex. The factor $-i\lambda$ is the amplitude for the emission and/or absorption of scalar particles at the vertex.
   - We divide by the symmetry factor S of the diagram which is the number of permutations which leave the diagram invariant.

3) We integrate over the internal points y_i, i.e. we sum over all places where the underlying process can happen. This is the superposition principle of quantum mechanics.

These are Feynman rules in position space. We will also need Feynman rules in momentum space. Before we state them it is better we work out explicitly few concrete examples. Let us go back to the Feynman diagram $b)$ on figure 2. It is given by

$$\frac{1}{2}(-i\lambda)\int d^4z D_F(x_1-z)D_F(x_2-z)D_F(0). \tag{3.198}$$

We will use the following expression of the Feynman scalar propagator

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}. \tag{3.199}$$

We compute immediately

$$\frac{1}{2}(-i\lambda)\int d^4z D_F(x_1-z)D_F(x_2-z)D_F(0) = \int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\bigg(\frac{1}{2}(-i\lambda)(2\pi)^4\delta^4(p_1+p_2) \times e^{-ip_1x_1}e^{-ip_2x_2}\Delta(p_1)\Delta(p_2)\Delta(q)\bigg). \tag{3.200}$$

$$\Delta(p) = \frac{i}{p^2 - m^2 + i\epsilon}. \tag{3.201}$$

In the above equation p_1 and p_2 are the external momenta and q is the internal momentum. We integrate over all these momenta. Clearly we still have to multiply with the vertex $-i\lambda$ and divide by the symmetry factor which is here 2. In momentum space we attach to any line which carries a momentum p a propagator $\Delta(p)$. The new features are two things 1) we attach a plane wave e^{-ipx} to each external point x into which a momentum p is flowing and 2) we impose momentum conservation at each vertex which in this case is $(2\pi)^4\delta^4(p_1+p_2+q-q) = (2\pi)^4\delta^4(p_1+p_2)$. See figure 7.

We consider another example given by the Feynman diagram $b)_2$ on figure 4). We find

$$\frac{(-i\lambda)^2}{6}\int d^4y_1 \int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)^3 =$$
$$\int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4q_1}{(2\pi)^4}\int \frac{d^4q_2}{(2\pi)^4}\int \frac{d^4q_3}{(2\pi)^4}\bigg(\frac{1}{6}(-i\lambda)^2(2\pi)^4\delta^4(p_1+p_2)(2\pi)^4\delta^4(p_1-q_1-q_2-q_3) \times$$
$$e^{-ip_1x_1}e^{-ip_2x_2}\Delta(p_1)\Delta(p_2)\Delta(q_1)\Delta(q_2)\Delta(q_3)\bigg) \tag{3.202}$$

This expression can be reconstructed from the same rules we have discussed in the previous case. See figure 8.

In summary Feynman rules in momentum space read

1) We draw all Feynman diagrams with N external points x_i and n internal points (vertices) y_i.

2) The contribution of each Feynman diagram to the Green's function $< 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_N))|0 >$ is equal to the product of the following five factors

   - Each line (internal or external) joining two spacetime points x and y is associated with a propagator $\Delta(p)$ where p is the momentum carried by the line.
   - Each vertex is associated with a factor $-i\lambda$.
   - We attach a plane wave $\exp(-ipx)$ to each external point x where p is the momentum flowing into x.
   - We impose momentum conservation at each vertex.
   - We divide by the symmetry factor S of the diagram.

3) We integrate over all internal and external momenta.

3.4 Exercises and Problems

Asymptotic Solutions

- Show that

$$\hat{Q}(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p}) + \frac{1}{E_{\vec{p}}}\int_{-\infty}^{t} dt^{'} \sin E_{\vec{p}}(t-t^{'}) j(t^{'},\vec{p}),$$

  is a solution of the equation of motion

$$(\partial_t^2 + E_{\vec{p}}^2)Q(t,\vec{p}) = j(t,\vec{p}).$$

- Show that

$$\hat{Q}(t,\vec{p}) = \hat{Q}^{+}_{\rm in}(t,\vec{p}) + \hat{Q}^{-}_{\rm out}(t,\vec{p}) + i\int_{-\infty}^{+\infty} dt^{'} G_{\vec{p}}(t-t^{'}) j(t^{'},\vec{p}),$$

  is also a solution of the above differential equation.

- Express the Feynman scalar propagator $D_F(x-x^{'})$ in terms of $G_{\vec{p}}(t-t^{'})$.
- Show that this solution leads to

$$\hat{\phi}(x) \;=\; \hat{\phi}^{+}_{\rm in}(x) + \hat{\phi}^{-}_{\rm out}(x) + i\int d^4x^{'} D_F(x-x^{'}) J(x^{'}).$$

Hint: Use

$$\frac{d}{dt}\int_{-\infty}^{t} dt^{'} f(t^{'},t) = \int_{-\infty}^{t} dt^{'} \frac{\partial f(t^{'},t)}{\partial t} + f(t,t).$$

$$(\partial_t^2 + E_{\vec{p}}^2)G_{\vec{p}}(t-t^{'}) = -i\delta(t-t^{'}).$$

Feynman Scalar Propagator Verify that the Feynman propagator in one-dimension is given by

$$G_{\vec{p}}(t-t^{'}) = \int \frac{dE}{2\pi}\frac{i}{E^2 - E_{\vec{p}}^2 + i\epsilon} e^{-iE(t-t^{'})} = \frac{e^{-iE_{\vec{p}}|t-t^{'}|}}{2E_{\vec{p}}}.$$

Fourier Transform Show that the Fourier transform of the Klein-Gordon equation of motion

$$(\partial_\mu\partial^\mu + m^2)\phi = J$$

is given by

$$(\partial_t^2 + E_{\vec{p}}^2)Q(t,\vec{p}) = j(t,\vec{p}).$$

Forced Harmonic Oscillator We consider a single forced harmonic oscillator given by the equation of motion

$$(\partial_t^2+E^2)Q(t)=J(t).$$

- Show that the $S-$matrix defined by the matrix elements $S_{mn}=< m\,\text{out}|n\,\text{in}>$ is unitary.
- Determine S from solving the equation

$$S^{-1}\hat{a}_{\rm in}S=\hat{a}_{\rm out}=\hat{a}_{\rm in}+\frac{i}{\sqrt{2E}}j(E).$$

- Compute the probability $|< n\ out|0\ in>|^2$.
- Determine the evolution operator in the interaction picture $\Omega(t)$ from solving the Schrodinger equation

$$i\partial_t\Omega(t)=\hat{V}_I(t)\Omega(t)\ ,\ \hat{V}_I(t)=-J(t)\hat{Q}_I(t).$$

- Deduce from the fourth question the $S-$matrix and compare with the result of the second question.

Interaction Picture Show that the fields $\hat{Q}_I(t,\vec{p})$ and $\hat{P}_I(t,\vec{p})$ are free fields.

Time Ordering Operator Show that

$$\frac{1}{3!}\int_{-\infty}^t dt_1\int_{-\infty}^t dt_2\int_{-\infty}^t dt_3 T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) = \int_{-\infty}^t dt_1\int_{-\infty}^{t_1} dt_2\int_{-\infty}^{t_2} dt_3 \hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3).$$

Wick's Theorem For Forced Scalar Field Show that

$$i\sum_{\vec{p}}\beta_{\vec{p}}(+\infty) = \frac{1}{2}\int d^4x\int d^4x^{'}\,J(x)J(x^{'})\bigg(\frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}-\frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}\bigg).$$

Unitarity of The $S-$Matrix

- Show that

$$S^{-1} = \bar{T}\bigg(e^{i\int_{-\infty}^{+\infty}ds\hat{V}_I(s)}\bigg).$$

- Use the above result to verify that S is unitary.

Evolution Operator $\Omega(t)$ and Gell-Mann Low Formula Verify up to the third order in perturbation theory the following equations

$$\Omega(t) = \bar{T}\bigg(e^{i\int_t^{+\infty} ds \hat{V}_I(s)}\bigg)S.$$

$$\hat{\phi}(x) = S^{-1}\bigg(T\hat{\phi}_{\rm in}(x)S\bigg).$$

Interaction Fields are Free Fields Show that the interaction fields $\hat{\phi}_I(t,\vec{x})$ and $\hat{\pi}_I(t,\vec{x})$ are free fields.

LSZ Reduction Formulae

- Show the LSZ reduction formulae

$$i\int_{-\infty}^{+\infty} dt e^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) =$$
$$\sqrt{2E_{\vec{p}}}\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) - T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

- Show that

$$i\int d^4x e^{ipx}(\partial_\mu\partial^\mu+m^2)T(\hat{\phi}(x)\hat{\phi}(x_1)\hat{\phi}(x_2)...) =$$
$$\sqrt{2E_{\vec{p}}}\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{\phi}(x_1)\hat{\phi}(x_2)...) - T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

- Derive the LSZ reduction formulae

$$-i\int_{-\infty}^{+\infty} dt e^{-iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})^+\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...) =$$
$$\sqrt{2E_{\vec{p}}}\bigg(\hat{a}_{\rm out}(\vec{p})^+T(\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...) - T(\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...)\hat{a}_{\rm in}(\vec{p})^+\bigg).$$

  Hint: Start from

$$e^{-iE_{\vec{p}}t}(-i\partial_t+E_{\vec{p}})\hat{Q}_{\rm in}(t,\vec{p})^+ = \sqrt{2E_{\vec{p}}}\ \hat{a}_{\rm in}(\vec{p})^+.$$

$$e^{-iE_{\vec{p}}t}(-i\partial_t+E_{\vec{p}})\hat{Q}_{\rm out}(t,\vec{p})^+ = \sqrt{2E_{\vec{p}}}\ \hat{a}_{\rm out}(\vec{p})^+.$$

Wick's Theorem Show that

$$\left[e^{\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-x_j)\frac{\partial}{\partial\phi(x_j)}}\bigg(\phi(x_1)...\phi(x_{2n})\bigg)\right]_{\phi=0} = \sum_{\rm contraction}\prod D_F(x_i-x_j).$$

The 4−Point Function in Φ−Four Theory Calculate the 4−point function in ϕ−four theory up to the second order in preturbation theory.

Evolution Operator $\Omega(t,t^{'})$ Show that the evolution operators

$$\Omega(t,t^{'}) = T\bigg(e^{-i\int_{t^{'}}^{t}ds\hat{V}_I(s)}\bigg),$$

and

$$\Omega(t,t^{'}) = e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t^{'})}e^{-i\hat{H}_0(t^{'}-t_0)}.$$

solve the differential equation

$$i\partial_t\Omega(t,t^{'}) = \hat{V}_I(t,t_0)\Omega(t).$$

Determine $\hat{V}_I(t,t_0)$.

Φ−Cube Theory The ϕ−cube theory is defined by the interaction Lagrangian density

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{3!}\phi^3.$$

Derive Feynman rules for this theory by considering the 2−point and 4−point functions up to the second order in perturbation theory.

3.5 Solutions

Asymptotic Solutions

- Straightforward.
- Straightforward. This is a different solution in which we do not have the constraint $t-t^{'}>0$ in the Feynman Green's function $G_{\vec{p}}(t-t^{'})$.
- $$\int \frac{d^3p}{(2\pi)^3}G_{\vec{p}}(t-t^{'})e^{i\vec{p}(\vec{x}-\vec{x}^{'})} = \int \frac{d^3p}{(2\pi)^3}\int \frac{dp^0}{(2\pi)^3}\frac{i}{(p^0)^2-E^2_{\vec{p}}+i\epsilon}e^{-ip^0(t-t^{'})+i\vec{p}(\vec{x}-\vec{x}^{'})}$$
  $$= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-x^{'})}$$
  $$= D_F(x-x^{'}).$$
- Thus the second solution corresponds to the causal Feynman propagator. Indeed by integrating both sides of the equation over $\vec{p}$ we obtain

$$\hat{\phi}(x) = \hat{\phi}^{+}_{\rm in}(x)+\hat{\phi}^{-}_{\rm out}(x)+i\int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}\vec{x}}\int_{-\infty}^{+\infty} dt^{'}G_{\vec{p}}(t-t^{'})j(t^{'},\vec{p})$$
$$= \hat{\phi}^{+}_{\rm in}(x)+\hat{\phi}^{-}_{\rm out}(x)+i\int \frac{d^3p}{(2\pi)^3}\int d^4x^{'}G_{\vec{p}}(t-t^{'})J(x^{'})e^{i\vec{p}(\vec{x}-\vec{x}^{'})}.$$

In other words

$$\hat{\phi}(x) = \hat{\phi}^{+}_{\rm in}(x)+\hat{\phi}^{-}_{\rm out}(x)+i\int d^4x^{'}D_F(x-x^{'})J(x^{'}).$$

Feynman Scalar Propagator Perform the integral using the residue theorem.

Fourier Transform Straightforward.

Forced Harmonic Oscillator

- Verify that

$$\sum_l S^{*}_{lm}S_{ln}=\delta_{mn}.$$

- We get

$$S = \exp(\alpha\hat{a}_{\rm in}^+ - \alpha^*\hat{a}_{\rm in} + i\beta) = e^{\alpha\hat{a}_{\rm in}^+}e^{-\alpha^*\hat{a}_{\rm in}}e^{+i\beta-\frac{1}{2}|\alpha|^2}.$$

$$\alpha = \frac{i}{\sqrt{2E}}j(E).$$

In this result β is still arbitrary. We use $[\hat{a}_{\rm in}, \hat{a}_{\rm in}^+] = 1$ and the BHC formula

$$e^A e^B = e^{A+B}e^{\frac{1}{2}[A,B]}.$$

In particular

$$\hat{a}_{\rm in}e^{\alpha\hat{a}_{\rm in}^+} = e^{\alpha\hat{a}_{\rm in}^+}(\hat{a}_{\rm in} + \alpha).$$

- We find

$$| < n\ out|0\ in > |^2 = \frac{x^n}{n!}e^{-x}\ ,\ x = |\alpha|^2.$$

We use $|n \text{ in} >= ((\hat{a}_{\rm in}^+)^n/\sqrt{n!})|0 \text{ in} >$ and $< n \text{ in}|m \text{ in} >= \delta_{nm}$.

- We use

$$\hat{Q}_I(t) = \hat{Q}_{\rm in}(t) = \frac{1}{\sqrt{2E}}(\hat{a}_{\rm in}e^{-iEt} + \hat{a}_{\rm in}^+e^{iEt}).$$

We find

$$\Omega(t) = \exp(\alpha(t)\hat{a}_{\rm in}^+ - \alpha^*(t)\hat{a}_{\rm in} + i\beta(t)) = e^{\alpha(t)\hat{a}_{\rm in}^+}e^{-\alpha^*(t)\hat{a}_{\rm in}}e^{+i\beta(t)-\frac{1}{2}|\alpha(t)|^2}.$$

$$\alpha(t) = \frac{i}{\sqrt{2E}}\int_{-\infty}^t dsJ(s)e^{iEs}.$$

The Schrodinger equation $i\partial_t\Omega(t) = \hat{V}_I(t)\Omega(t)$ becomes

$$i\partial_t\Omega = i\bigg(\partial_t\alpha\hat{a}_{\rm in}^+ - \partial_t\alpha^*\hat{a}_{\rm in} + i\partial_t\beta - \frac{1}{2}\partial_t\alpha.\alpha^* + \frac{1}{2}\partial_t\alpha^*.\alpha\bigg)\Omega.$$

This reduces to

$$\partial_t\beta(t) = \frac{i}{2}(\alpha\partial_t\alpha^* - \alpha^*\partial_t\alpha).$$

Thus

$$\beta(t) = \frac{i}{2}\int_{-\infty}^t ds(\alpha\partial_s\alpha^* - \alpha^*\partial_s\alpha).$$

- In the limit $t \longrightarrow \infty$ we obtain

$$\alpha(+\infty) = \frac{i}{\sqrt{2E}}\int_{-\infty}^{+\infty} dsJ(s)e^{iEs} = \frac{i}{\sqrt{2E}}j(E) = \alpha.$$

$$-\frac{1}{2}|\alpha(+\infty)|^2 = -\frac{1}{4E}\int_{-\infty}^{+\infty} ds\int_{-\infty}^{+\infty} ds^{'} J(s)J(s^{'})e^{iE(s-s^{'})}.$$

Also

$$\begin{aligned} i\beta(+\infty) &= -\frac{1}{4E}\int_{-\infty}^{+\infty} ds\int_{-\infty}^{+\infty} ds^{'} J(s)J(s^{'})e^{-iE(s-s^{'})}\theta(s-s^{'}) \\ &+ \frac{1}{4E}\int_{-\infty}^{+\infty} ds\int_{-\infty}^{+\infty} ds^{'} J(s)J(s^{'})e^{iE(s-s^{'})}\theta(s-s^{'}). \end{aligned}$$

Hence (by using $1-\theta(s-s^{'}) = \theta(s^{'}-s)$)

$$\begin{aligned} i\beta(+\infty) - \frac{1}{2}|\alpha(+\infty)|^2 &= -\frac{1}{4E}\int_{-\infty}^{+\infty} ds\int_{-\infty}^{+\infty} ds^{'} J(s)J(s^{'})e^{-iE(s-s^{'})}\theta(s-s^{'}) \\ &- \frac{1}{4E}\int_{-\infty}^{+\infty} ds\int_{-\infty}^{+\infty} ds^{'} J(s)J(s^{'})e^{iE(s-s^{'})}\theta(s^{'}-s) \\ &= -\frac{1}{2}\int_{-\infty}^{+\infty} ds\int_{-\infty}^{+\infty} ds^{'} J(s)J(s^{'})G(s-s^{'}). \end{aligned}$$

The Feynman propagator in one-dimension is

$$G(s-s^{'}) = \frac{1}{2E}\bigg(e^{-iE(s-s^{'})}\theta(s-s^{'}) + e^{iE(s-s^{'})}\theta(s^{'}-s)\bigg).$$

The $S-$matrix is

$$S = e^{\alpha\hat{a}_{\rm in}^{+}}e^{-\alpha^*\hat{a}_{\rm in}}e^{-\frac{1}{2}\int_{-\infty}^{+\infty} ds\int_{-\infty}^{+\infty} ds^{'} J(s)J(s^{'})G(s-s^{'})}.$$

This is the same formula obtained in the second question except that β is completely fixed in this case.

Interaction Picture From one hand we compute that

$$i\partial_t\hat{Q}_I(t,\vec{p}) = -[\hat{Q}_I(t,\vec{p}),\hat{V}_I(t,\vec{p})] + \Omega(t)i\partial_t\hat{Q}_I(t,\vec{p})\Omega^{-1}(t).$$

From the other hand we compute

$$\begin{aligned} i\partial_t\hat{Q}(t,\vec{p}) &= U^{-1}(t)[\hat{Q}(\vec{p}),\hat{\mathcal{H}}_{\vec{p}}]U(t) + U^{-1}(t)[\hat{Q}(\vec{p}),\hat{V}(t,\vec{p})]U(t) \\ &= \Omega^{-1}(t)[\hat{Q}_I(t,\vec{p}),\hat{\mathcal{H}}_{\vec{p}}]\Omega(t) + \Omega^{-1}(t)[\hat{Q}_I(t,\vec{p}),\hat{V}_I(t,\vec{p})]\Omega(t). \end{aligned}$$

We can then compute immediately that

$$i\partial_t \hat{Q}_I(t,\vec{p}) = [\hat{Q}_I(t,\vec{p}),\hat{\mathcal{H}}_{\vec{p}}].$$

Next we compute

$$\begin{aligned} i\partial_t \hat{Q}_I(t,\vec{p}) = [\hat{Q}_I(t,\vec{p}),\hat{\mathcal{H}}_{\vec{p}}] &= e^{it\hat{\mathcal{H}}_{\vec{p}}}[\hat{Q}(\vec{p}),\hat{\mathcal{H}}_{\vec{p}}]e^{-it\hat{\mathcal{H}}_{\vec{p}}} \\ &= ie^{it\hat{\mathcal{H}}_{\vec{p}}}\hat{P}(\vec{p})e^{-it\hat{\mathcal{H}}_{\vec{p}}} \\ &= i\hat{P}_I(t,\vec{p}). \end{aligned}$$

Similarly we compute

$$\begin{aligned} i\partial_t \hat{P}_I(t,\vec{p}) = [\hat{P}_I(t,\vec{p}),\hat{\mathcal{H}}_{\vec{p}}] &= e^{it\hat{\mathcal{H}}_{\vec{p}}}[\hat{P}(\vec{p}),\hat{\mathcal{H}}_{\vec{p}}]e^{-it\hat{\mathcal{H}}_{\vec{p}}} \\ &= -iE_{\vec{p}}^2 e^{it\hat{\mathcal{H}}_{\vec{p}}}\hat{Q}(\vec{p})e^{-it\hat{\mathcal{H}}_{\vec{p}}} \\ &= -iE_{\vec{p}}^2\hat{Q}_I(t,\vec{p}). \end{aligned}$$

Thus the operators $\hat{Q}_I(t,\vec{p})$ and $\hat{P}_I(t,\vec{p})$ describe free oscillators.

Time Ordering Operator We have

$$\begin{aligned} T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) &= \hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)\ ,\ \text{if } t_1>t_2>t_3 \\ T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) &= \hat{V}_I(t_2)\hat{V}_I(t_1)\hat{V}_I(t_3)\ ,\ \text{if } t_2>t_1>t_3 \\ T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) &= \hat{V}_I(t_1)\hat{V}_I(t_3)\hat{V}_I(t_2)\ ,\ \text{if } t_1>t_3>t_2 \\ T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) &= \hat{V}_I(t_3)\hat{V}_I(t_1)\hat{V}_I(t_2)\ ,\ \text{if } t_3>t_1>t_2 \\ T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) &= \hat{V}_I(t_2)\hat{V}_I(t_3)\hat{V}_I(t_1)\ ,\ \text{if } t_2>t_3>t_1 \\ T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) &= \hat{V}_I(t_3)\hat{V}_I(t_2)\hat{V}_I(t_1)\ ,\ \text{if } t_3>t_2>t_1. \end{aligned}$$

Thus $T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))$ is a function of t_1, t_2 and t_3 which is symmetric about the axis $t_1=t_2=t_3$. Therefore the integral of $T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))$ in the different six regions $t_1>t_2>t_3$, $t_2>t_1>t_3$, etc gives the same result. Hence

$$\frac{1}{6}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2\int_{-\infty}^{t}dt_3 T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) = \int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt_3\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3).$$

Wick's Theorem For Forced Scalar Field In order to compute $i\beta_{\vec{p}}(t)$ when $t\longrightarrow+\infty$ we start from

$$\partial_t\Omega_{\vec{p}}(t)\Omega_{\vec{p}}(t)^{-1} = \dot{\alpha}_{\vec{p}}\ \hat{a}_{\rm in}(\vec{p})^+ - \dot{\alpha}_{\vec{p}}^*\ \hat{a}_{\rm in}(\vec{p}) + \frac{V}{2}\dot{\alpha}_{\vec{p}}^*\ \alpha_{\vec{p}} - \frac{V}{2}\dot{\alpha}_{\vec{p}}\ \alpha_{\vec{p}}^* + i\dot{\beta}_{\vec{p}}.$$

In deriving this last result we used

$$e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+}\hat{a}_{\rm in}(\vec{p})=(\hat{a}_{\rm in}(\vec{p})-V\alpha_{\vec{p}}(t))e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+}.$$

Clearly we must have

$$\partial_t\Omega_{\vec{p}}(t)\Omega_{\vec{p}}(t)^{-1}=-iV_I(t,\vec{p}).$$

From the second line of (1.58) we have

$$\Omega(t) = T\bigg(e^{\frac{i}{V}\int_{-\infty}^t ds\sum_{\vec{p}}\frac{1}{\sqrt{2E_{\vec{p}}}}\big(j(s,\vec{p})^*\hat{a}_{\rm in}(\vec{p})e^{-iE_{\vec{p}}s}+j(s,\vec{p})\hat{a}_{\rm in}(\vec{p})^+e^{iE_{\vec{p}}s}\big)}\bigg).$$

The potential $\hat{V}_I(t,\vec{p})$ can then be defined by

$$\hat{V}_I(t,\vec{p}) = -\frac{1}{V}\frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(j(t,\vec{p})^*\hat{a}_{\rm in}(\vec{p})e^{-iE_{\vec{p}}t}+j(t,\vec{p})\hat{a}_{\rm in}(\vec{p})^+e^{iE_{\vec{p}}t}\bigg).$$

The differential equation $\partial_t\Omega_{\vec{p}}(t)\Omega_{\vec{p}}(t)^{-1}=-iV_I(t,\vec{p})$ yields then the results

$$\dot{\alpha}_{\vec{p}}=\frac{i}{V}\frac{j(t,\vec{p})}{\sqrt{2E_{\vec{p}}}}e^{iE_{\vec{p}}t}.$$

$$\dot{\beta}_{\vec{p}}=\frac{iV}{2}\big(\dot{\alpha}^*_{\vec{p}}\,\alpha_{\vec{p}}-\dot{\alpha}_{\vec{p}}\,\alpha^*_{\vec{p}}\big).$$

The first equation yields precisely the formula (1.64). The second equation indicates that the phase $\beta(t)$ is actually not zero. The integration of the second equation gives

$$\beta_{\vec{p}} = \frac{1}{4iVE_{\vec{p}}}\int_{-\infty}^t ds\int_{-\infty}^s ds^{'}j(s,\vec{p})j(s^{'},\vec{p})^*e^{iE_{\vec{p}}(s-s^{'})} - \frac{1}{4iVE_{\vec{p}}}\int_{-\infty}^t ds\int_{-\infty}^s ds^{'}j(s,\vec{p})^*j(s^{'},\vec{p})e^{-iE_{\vec{p}}(s-s^{'})}.$$

By summing over $\vec{p}$ and taking the limit $t\longrightarrow\infty$ we obtain

$$i\sum_{\vec{p}}\beta_{\vec{p}}(+\infty) = \frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})\bigg(\frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}-\frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}\bigg).$$

Unitarity of The $S-$Matrix

- The solution $\Omega(t)$ can be written explicitly as

$$\Omega(t) = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 ... \int_{-\infty}^{t_{n-1}} dt_n \hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n).$$

The first few terms of this expansion are

$$\Omega(t) = 1 - i\int_{-\infty}^{t} dt_1 \hat{V}_I(t_1) + (-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2) + ...$$

Let us rewrite the different terms as follows

$$\int_{-\infty}^{t} dt_1 \hat{V}_I(t_1) = \int_{-\infty}^{+\infty} dt_1 \hat{V}_I(t_1) - \int_{t}^{+\infty} dt_1 \hat{V}_I(t_1).$$

$$\begin{aligned}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2) &= \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2) + \int_{t}^{+\infty} dt_1 \int_{t_1}^{+\infty} dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2)\\ &- \int_{t}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2).\end{aligned}$$

Hence to this order we have

$$\begin{aligned}\Omega(t) &= \left(1 + i\int_{t}^{+\infty} dt_1 \hat{V}_I(t_1) + i^2 \int_{t}^{+\infty} dt_1 \int_{t_1}^{+\infty} dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2) + ...\right)\\ &\times \left(1 - i\int_{-\infty}^{+\infty} dt_1 \hat{V}_I(t_1) + (-i)^2 \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2) + ...\right)\\ &= \bar{T}\left(e^{i\int_t^{+\infty} ds \hat{V}_I(s)}\right)S.\end{aligned}$$

The operator $\bar{T}$ is the anti time-ordering operator, i.e. it orders earlier times to the left and later times to the right. This result is actually valid to all orders in perturbation theory. Taking the limit $t \longrightarrow -\infty$ in this equation we obtain

$$S^{-1} = \bar{T}\left(e^{i\int_{-\infty}^{+\infty} ds \hat{V}_I(s)}\right).$$

- Recall that

$$\Omega(t) = T\bigg(e^{-i\int_{-\infty}^{t}ds\hat{V}_I(s)}\bigg).$$

By taking the Hermitian conjugate we obtain

$$S^+ = \bar{T}\bigg(e^{i\int_{-\infty}^{+\infty}ds\hat{V}_I(s)}\bigg).$$

In other words S is unitary as it should be. This is expected since by construction the operators $U(t)$ and $\Omega(t)$ are unitary.

Evolution Operator $\Omega(t)$ and Gell-Mann Low Formula Straightforward.

Interaction Fields are Free Fields We compute

$$\begin{aligned}
i\partial_t\hat{\phi}_I(t,\vec{x}) &= [\hat{\phi}_I(t,\vec{x}),\hat{H}_0] \\
&= e^{it\hat{H}_0}[\hat{\phi}(\vec{x}),\hat{H}_0]e^{-it\hat{H}_0} \\
&= e^{it\hat{H}_0}\int\frac{d^3\vec{p}}{(2\pi)^3}e^{i\vec{p}\vec{x}}\int_+\frac{d^3\vec{q}}{(2\pi)^3}[\hat{Q}(\vec{p}),\hat{P}^+(\vec{q})]\hat{P}(\vec{q})e^{-it\hat{H}_0} \\
&= ie^{it\hat{H}_0}\int\frac{d^3\vec{p}}{(2\pi)^3}e^{i\vec{p}\vec{x}}\hat{P}(\vec{p})e^{-it\hat{H}_0} \\
&= ie^{it\hat{H}_0}\hat{\pi}(\vec{x})e^{-it\hat{H}_0} \\
&= i\hat{\pi}_I(t,\vec{x}).
\end{aligned}$$

Similarly

$$\begin{aligned}
i\partial_t\hat{\pi}_I(t,\vec{x}) &= [\hat{\pi}_I(t,\vec{x}),\hat{H}_0] \\
&= e^{it\hat{H}_0}[\hat{\pi}(\vec{x}),\hat{H}_0]e^{-it\hat{H}_0} \\
&= e^{it\hat{H}_0}\int\frac{d^3\vec{p}}{(2\pi)^3}e^{i\vec{p}\vec{x}}\int_+\frac{d^3\vec{q}}{(2\pi)^3}E_{\vec{q}}^2[\hat{P}(\vec{p}),\hat{Q}^+(\vec{q})]\hat{Q}(\vec{q})e^{-it\hat{H}_0} \\
&= -ie^{it\hat{H}_0}\int\frac{d^3\vec{p}}{(2\pi)^3}E_{\vec{p}}^2e^{i\vec{p}\vec{x}}\hat{Q}(\vec{p})e^{-it\hat{H}_0} \\
&= i(\vec{\nabla}^2-m^2)e^{it\hat{H}_0}\hat{\phi}(\vec{x})e^{-it\hat{H}_0} \\
&= i(\vec{\nabla}^2-m^2)\hat{\phi}_I(t,\vec{x}).
\end{aligned}$$

These last two results indicates that the interaction field $\hat{\phi}_I$ is a free field since it obeys the equation of motion

$$(\partial_t^2-\vec{\nabla}^2+m^2)\hat{\phi}_I(t,\vec{x})=0.$$

LSZ Reduction Formulae

- Let us consider the integral

$$\int_{-\infty}^{+\infty} dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\bigg).$$

We compute

$$\int_{-\infty}^{+\infty} dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\bigg) = \sqrt{2E_{\vec{p}}}\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) - T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

On the other hand we compute

$$\int_{-\infty}^{+\infty} dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\bigg) = i\int_{-\infty}^{+\infty} dte^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...).$$

Hence we obtain the LSZ reduction formulae

$$i\int_{-\infty}^{+\infty} dte^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) = \sqrt{2E_{\vec{p}}}\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) - T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

- We use the identity (with the notation $\partial^2=\partial_\mu\partial^\mu$)

$$\int d^3xe^{-i\vec{p}\vec{x}}(\partial^2+m^2)\hat{\phi}(x) = (\partial_t^2+E_{\vec{p}}^2)\hat{Q}(t,\vec{p}).$$

The above LSZ reduction formulae can then be put in the form

$$i\int d^4xe^{ipx}(\partial_\mu\partial^\mu+m^2)T(\hat{\phi}(x)\hat{\phi}(x_1)\hat{\phi}(x_2)...) = \sqrt{2E_{\vec{p}}}\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{\phi}(x_1)\hat{\phi}(x_2)...) - T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

- Straightforward.

Wick's Theorem Straightforward.

The 4−Point Function in Φ−Four Theory The first order in perturbation theory is given by

$$i\int d^4y_1 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_4)\mathcal{L}_{\rm int}(y_1)\bigg)|0> \;=\; i(-\frac{\lambda}{4!})\int d^4y_1 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_4)\hat{\phi}_{\rm in}(y_1)^4\bigg)|0>$$

In total we $7.5.3=105$ contractions which we can divide into three classes

- We contract only two external points together and the other two external points are contracted with the internal points. Here we have six diagrams corresponding to contracting (x_1,x_2), (x_1,x_3), (x_1,x_4), (x_2,x_3), (x_2,x_4) and (x_3,x_4). Each diagram corresponds to 12 contractions coming from the 4 possibilities opened to the first external point to be contracted with the internal points times the 3 possibilities opened to the second external point when contracted with the remaining internal points. See figure 9a). The value of these diagrams is

$$\begin{aligned}12i(-\frac{\lambda}{4!})\int d^4y_1 D_F(0) \;\times\; & \bigg[D_F(x_1-x_2)D_F(x_3-y_1)D_F(x_4-y_1)\\ &+D_F(x_1-x_3)D_F(x_2-y_1)D_F(x_4-y_1)\\ &+D_F(x_1-x_4)D_F(x_3-y_1)D_F(x_2-y_1)\\ &+D_F(x_2-x_3)D_F(x_1-y_1)D_F(x_4-y_1)\\ &+D_F(x_2-x_4)D_F(x_3-y_1)D_F(x_1-y_1)\\ &+D_F(x_3-x_4)D_F(x_1-y_1)D_F(x_2-y_1)\bigg].\end{aligned}$$

  The corresponding Feynman diagram is shown on figure 10a).

- We can contract all the internal points among each other. In this case we have three distinct diagrams corresponding to contracting x_1 with x_2 and x_3 with x_4 or x_1 with x_3 and x_2 with x_4 or x_1 with x_4 and x_2 with x_3. Each diagram corresponds to 3 contractions coming from the three possibilities of contracting the internal points among each other. See figure 9b). The value of these diagrams is

$$3i(-\frac{\lambda}{4!})\int d^4y_1 D_F(0)^2\bigg[D_F(x_1-x_2)D_F(x_3-x_4) \;+\; D_F(x_1-x_3)D_F(x_2-x_4)+D_F(x_1-x_4)D_F(x_2-x_3)\bigg].$$

  The corresponding Feynman diagram is shown on figure 10b).

- The last possibility is to contract all the internal points with the external points. The first internal point can be contracted in 4 different ways with the external points, the second internal point will have 3 possibilities, the third internal point will have two possibilities and the fourth internal point will have one possibility. Thus there are 4.3.2 = 24 contractions corresponding to a single diagram. See figure 9c). The value of this diagram is

$$24i(-\frac{\lambda}{4!})\int d^4y_1\bigg[D_F(x_1-y_1)D_F(x_2-y_1)D_F(x_3-y_1)D_F(x_4-y_1)\bigg].$$

  The corresponding Feynman diagram is shown on figure 10c).

The second order in perturbation theory is given by

$$\frac{i^2}{2!}\int d^4y_1\int d^4y_2<0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_4)\mathcal{L}_{\rm int}(y_1)\mathcal{L}_{\rm int}(y_2)\bigg)|0> \ =$$
$$-\frac{1}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2<0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_4)\hat{\phi}_{\rm in}(y_1)^4\hat{\phi}_{\rm in}(y_2)^4\bigg)|0>.$$

There are in total 11.9.7.5.3 contractions.

- We contract two of the internal points together whereas we contract the other two with the external points. We have 6 possibilities corresponding to the 6 contractions (x_1,x_2), (x_1,x_3), (x_1,x_4), (x_2,x_3), (x_2,x_4) and (x_3,x_4). Thus we have (6).8.7.5.3 contractions in all involved. We focus on the contraction (x_3,x_4) since the other ones are similar. In this case we obtain 4 contractions which are precisely $a)_1$, $b)_1$,$a)_2$ and $b)_2$ shown on figure 3). The value of these diagrams is

$$-\frac{1}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2D_F(x_3-x_4)\times\bigg[8.3.3D_F(x_1-y_1)D_F(x_2-y_1)D_F(0)^3$$
$$+8.3.4.3D_F(x_1-y_1)D_F(x_2-y_1)D_F(y_1-y_2)^2D_F(0)$$
$$+8.4.3.3D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)D_F(0)^2$$
$$+8.4.3.2D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)^3\bigg].$$

  Clearly these diagrams are given by

$$D_F(x_3-x_4)\times\bigg(a)_1+b)_1+a)_2+b)_2\text{ of figure 4}\bigg).$$

  To get the other 5 possibilities we should permute the points x_1,x_2,x_3 and x_4 appropriately.

- Next we can contract the 4 internal points together giving

  $$D_F(x_1-x_2)D_F(x_3-x_4)+D_F(x_1-x_3)D_F(x_2-x_4)+D_F(x_1-x_4)D_F(x_2-x_3).$$

  This should be multiplied by the sum of 7.5.3 contractions of the external points given on figure 11. Compare with the contractions on figure $3a)_3$, $3b)_3$ and $3c)_3$. The value of these diagrams is

  $$-\frac{1}{2}(\frac{\lambda}{4!})^2\bigg(D_F(x_1-x_2)D_F(x_3-x_4)+D_F(x_1-x_3)D_F(x_2-x_4)\quad+\\ D_F(x_1-x_4)D_F(x_2-x_3)\bigg)\int d^4y_1\int d^4y_2\bigg(3.3D_F(0)^4\quad+\\ 6.4.3D_F(0)^2D_F(y_1-y_2)^2+4.3.2D_F(y_1-y_2)^4\bigg).$$

  The corresponding Feynman diagrams are shown on figure 12.

- There remains 48.7.5.3 contractions which must be accounted for. These correspond to the contraction of all of the internal points with the external points. The set of all these contractions is shown on figure 13. The corresponding Feynman diagrams are shown on figure 14. The value of these diagrams is

  $$-\frac{1}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1)\quad\times\quad\bigg[\\
  8.3.2.3.4D_F(x_2-y_1)D_F(x_3-y_1)D_F(x_4-y_2)D_F(y_1-y_2)D_F(0)\quad+\\
  8.3.2.3D_F(x_2-y_1)D_F(x_3-y_1)D_F(x_4-y_1)D_F(0)^2\quad+\\
  8.3.4.2.3D_F(x_2-y_1)D_F(x_3-y_2)D_F(x_4-y_1)D_F(y_1-y_2)D_F(0)\quad+\\
  8.3.4.3D_F(x_2-y_1)D_F(x_3-y_2)D_F(x_4-y_2)D_F(0)^2\quad+\\
  8.3.4.3.2D_F(x_2-y_1)D_F(x_3-y_2)D_F(x_4-y_2)D_F(y_1-y_2)^2\quad+\\
  8.4.3.3D_F(x_2-y_2)D_F(x_3-y_1)D_F(x_4-y_2)D_F(0)^2\quad+\\
  8.4.3.3.2D_F(x_2-y_2)D_F(x_3-y_1)D_F(x_4-y_2)D_F(y_1-y_2)^2\quad+\\
  8.4.3.2.3D_F(x_2-y_2)D_F(x_3-y_1)D_F(x_4-y_1)D_F(y_1-y_2)D_F(0)\quad+\\
  8.4.3.3D_F(x_2-y_2)D_F(x_3-y_2)D_F(x_4-y_1)D_F(0)^2\quad+\\
  8.4.3.2.3D_F(x_2-y_2)D_F(x_3-y_2)D_F(x_4-y_2)D_F(y_1-y_2)D_F(0)\quad+\\
  8.4.3.3.2D_F(x_2-y_2)D_F(x_3-y_2)D_F(x_4-y_1)D_F(y_1-y_2)^2\bigg].$$

Evolution Operator $\Omega(t,t^{'})$ Straightforward.

$\Phi-$**Cube Theory** Straightforward.