Lecture notes

String Theory I

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Chapter 1

Introduction

As it became clear that general relativity and Maxwell's theory are both intimately tied to the concept of local symmetries, a unified description of the forces of nature became conceivable. In general relativity (GR), a local choice of coordinates has to be made and the action is a local functional of the metric and of the matter fields that is independent of this choice. Electrodynamics (ED) can be described very efficiently by a vector potential $A^\mu = (\phi, \vec{A})$, with the field strengths $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being invariant under gauge transformations $\delta A_\mu = \partial_\mu \Lambda(x^\mu)$ [H. Weyl, 1918]. While classical physics can be formulated in terms of the field strengths, a local description of the coupling to quantum mechanical wave functions requires the gauge potentials, as is illustated by the Aharonov–Bohm effect [ah59].

A promising framework for unification was suggested by Kaluza and Klein [ka21]: They proposed that space-time is 5-dimensional, but with only 4 approximately flat directions and with one direction curled up on a small circle. Then the off-diagonal entry of the metric $A_\mu := g_{4\mu}$ transforms as a vector from the 4-dimensional point of view and serves as the gauge potential. Surprisingly, with this simple ansatz one recovers the complete set of Maxwell equations as a subset of the 5-dimensional Einstein equations. In addition, a scalar field $\phi = g_{44}$ shows up, but since the prediction of a new particle was beyond the scope of pre-quantum physics the consequences of this fact were discussed only much later. Moreover, it was not clear what mechanism could make the 5th dimension curl up and in the 1920s it was not even possible to pose this question within any proper framework, which certainly has to incorporate a quantum mechanical treatment of the gravitational interactions.

After a long development of quantum field theory, techniques for a perturbative analysis of quantum gravity became available in the 60s [fe63,De65]. At about the same time the standard model of strong and electroweak particle interactions, an $SU_3 \times SU_2 \times U_1$ gauge theory that is spontaneously broken to $SU_3 \times U_1$ below the Fermi scale $G_F^{-\frac{1}{2}} = 292.8\,GeV$ [data], was constructed. This led to the discovery of asymptotic freedom [co73] of QCD and – a decade later – to the detection of the W and Z bosons that mediate weak interactions. Attempts at a group theoretical 'grand unification' of the standard model by a gauge theory with SU_5 [ge74] or even larger gauge groups [sl81] produced surprisingly good predictions for the ratio of the W and Z masses and, at the same time, lead to a unification scale of $10^{14} - 10^{16} GeV$, far above the Fermi scale $300GeV$ of weak interactions. Since quantum gravity is bound to become important at the Planck mass[1] $M_{Pl} = \sqrt{\hbar c/G_N} = 1.22 \times 10^{19} GeV/c^2$, this may be regarded as an indication that gravity should no longer be ignored in particle physics.

It turned out that a perturbative quantization of gravity is spoiled by non-renormalizability, i.e. an infinite number of divergent quantum corrections that cannot be controlled by symmetries. An important example of such a correction is the one that modifies the cosmological constant Λ (the energy density of the vacuum, which becomes observable in gravitational interactions). The experimental bound for its physical value is best characterized by the tiny dimensionless ratio $|\Lambda|/M_{Pl}^2 < 10^{-121}$ [data].[2] It is clear that such a tiny quantity should be explained by a symmetry, the only known candidate for which is supersymmetry (SUSY) [WE83]: Note that the energy of a harmonic oscillator is $E = \frac{1}{2}\omega(a^\dagger a \pm aa^\dagger) = \omega(a^\dagger a \pm \frac{1}{2})$ for excitation modes that are quantized according to bosonic/fermionic statistics $a^\dagger a \mp aa^\dagger = 1$. The zero point energies are, therefore, of equal size and opposite sign. The energy operators for second quantized free fields consist of an infinite sum of such oscillator terms. In order to have a cancellation of zero point energies we should thus have an equal number of bosonic and fermionic degrees of freedom and a symmetry that controlls the cancellation when interactions are turned on. Due to the spin statistics theorem [ST64] a *physical* symmetry that transforms commuting into anti-commuting fields should be in a spin 1/2 representation of the Lorentz group and should be implemented by an anticommuting operator, the supercharge Q_α, in Hilbert space.

Because the anti-commutator of two SUSY transformations generates translations [WE83], the local (or gauged) version of supersymmetry automatically contains gravity and is hence called supergravity (SUGRA). In the late 70s and early 80s it was hoped that SUGRA might cure the divergences of quantum gravity. This also lead to a revival of the old ideas of Kaluza and Klein, but now with a higher dimensional compactification space in order to be able to incorporate the whole standard model of particle interactions into a (super)geometrical picture. It was shown that the standard model can only be obtained from at least 11 dimensions, which, at the same time, is the maximal dimension allowed for supergravity.[3] But it is hard to get chiral fermions by starting in an odd number of dimensions [ba87]. The alternative of adding

[1] In cgs units $M_{Pl} = 2.2 \times 10^{-5} g$; the corresponding length scale is $\sqrt{\hbar G_N/c^3} = 1.6 \times 10^{-33} cm$.

[2] The experimental bound for the photon mass is $m_\gamma < 3 \times 10^{-33} MeV$, so that electromagnetic gauge invariance appears to be an exact symmetry of nature.

[3] A Weyl fermion has $2^{d/2-1}$ components in d (even) dimensions, while a massless vector field and a spin 2 graviton have $d-2$ and $\mathcal{O}(d^2)$ degrees of freedom, respectively. Accordingly, the structure of SUSY and SUGRA strongly depends on the space-time dimension and SUGRA in $d > 11$ would imply the existence of fundamental particles with spin $j > 2$, for which no consistent interactions are known.

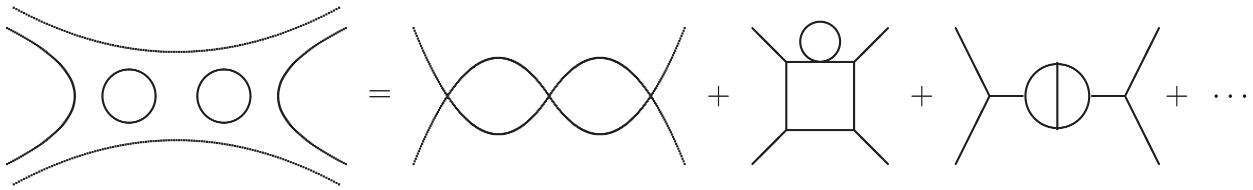

Fig. 1: $g = 2$ world sheet and some corresponding 2-loop Feynman graphs.

gauge symmetries to a 10-dimensional theory by hand goes against the original spirit of the ideas of Kaluza and Klein. Even worse, it turned out that SUGRA could not solve the problem with divergences.

1.1 String unification

String unification apparently works in a rather different way: Here the fundamental object is a thread or a loop in space-time which, during time evolution, sweeps out a surface that is called **world sheet** (in analogy to the world line of a point particle). The dynamics is described by an action that is proportional to the area of that surface, and hence in purely geometrical terms. Particles are oscillation modes of the string, and interactions occur by joining and splitting of string configurations, as is shown in Fig. 1. This has two important consequences:

- There is no interaction *point* (the apparent splitting point changes under Lorentz transformations), which avoids the UV divergences of second quantized point particle theory.
- There is a unique geometrical interaction, which unifies an a priory infinite number of independent couplings among different fields.

Alltogether, string theory leads to a unification of interactions *and* to a unification of particles.

From a more modern point of view we may think of the world sheet as an independent two-dimensional space with local coordinates σ^m. The string coordinate functions $X^\mu(\sigma)$ are quantum fields on that space and describe its embedding into a **target space**, which may itself be a topologically non-trivial manifold with local coordinates X^μ. The geometrical description of the action ensures that it can be constructed in a coordinate invariant way as a sum over terms that are defined via local coordinates. For historical reasons such a quantum field theory is called a **sigma model**. Unfortunately, there are two big problems with this approach:

- Scattering amplitudes are only defined as an infinite sum over different world sheet topologies, i.e. we do not have a non-perturbative definition of string theory. The sum over topologies may be badly divergent.
- To define a string theory we need to choose some fixed **background** target space geometry (or, in a more abstract description, a conformal field theory). Although different choices of this background should lead to equivalent physics, this is not manifest in the construction.

It may or may not be the case that some more elaborate version of string field theory [zw93], which is sometimes referred to by the name 'third quantization', will eventually solve this problem. At present it is not at all clear what string theory really is.

In any case, we do have a well-defined prescription for constructing a finite perturbative expansion of scattering amplitudes for the particles that effectively describe the physics of a string model at large distances (i.e. distances larger than $10^{-33}cm$). This is done in terms of 2-dimensional conformally invariant quantum field theories and a lot has been learned about how the properties of these world sheet models are related to the resulting space-time physics that we can proble in accelerator experiments or astrophysical observations. An incomplete dictionary is compiled at this point for later reference:

Space-time	$\longleftrightarrow$	**world sheet**
Einstein equations	$\longleftrightarrow$	conformal invariance
gauge invariance	$\longleftrightarrow$	current algebra (Kac–Moody)
anomaly cancellation	$\longleftrightarrow$	modular invariance
$N = 1$ supergravity	$\longleftrightarrow$	$N = 2$ supersymmetry
space-time geometry	$\searrow$	conformal field theory
particle spectrum	$\swarrow$	

In the last entry of this table it is indicated that a given background space-time geometry leads to a well-defined conformal field theory on the world sheet. Note, however, that this arrow cannot be reversed: In case of strong curvature quantum corrections can be large so that classically different background geometries can lead to quantum mechanically equivalent string theories (and therefore to the same space time physics).

It was already indicated above that we actually need a supersymmetric version of string theory. This has two reasons: The bosonic string has a tachyonic excitation in its spectrum, which indicates that it is unstable and which leads to IR divergences in perturbation theory. Furthermore, we want to describe spin 1/2 particles like electrons or nucleons, and these are missing in the excitation spectrum of the bosonic theory. This leads to the superstring whose conformal anomaly vanishes in 10 dimensions, which is also consistent with an effective low energy supergravity theory.

Actually, there seem to be 5 consistent supersymmetric string theories in 10 dimensions [GR87]: For a flat target space the coordinate fields satisfy the 2D wave equation $\Box X^\mu = (\partial_1 + \partial_0)(\partial_1 - \partial_0)X^\mu = 0$, whose general solution is a superposition of left-moving and right-moving excitations. In case of open superstrings, called type I, boundary conditions lead to a reflection of these modes at the string ends (the type I theory also contains closed string states in its spectrum since they can be formed by interactions, and its consistency requires to consider unoriented world sheets and Chan–Paton factors for the gauge group SO_{32} [GR87]). For oriented closed strings we have to make a choice in the relative chirality of the left and right moving supersymmetries. This leads to the type IIA and type IIB theories with $N = 2$ space-time SUSY, the latter of which is chiral. Moreover, in the closed string case we may even chose to combine a left-moving bosonic string with a right-moving fermionic string. The $D_{bos.} - D_{ferm.} = 26 - 10 = 16$ single left-moving bosons cannot be interpreted as space-time coordinates but rather show up as gauge degrees of freedom. This asymmetric construction is strongly constrained by potential quantum violation of symmetries (space-time anomalies coming from a violation of WS modular invariance), so that only two consistent choices exist: The heterotic strings with gauge groups $E_8 \times E_8$ and SO_{32}, respectively.

An important phenomenon in string theory (and many of its building blocks and effective theories) is duality, which means that different classical theories can lead to the identical quantum mechanical models. The oldest example of this type – except for bosonization – is the $R \longleftrightarrow 1/R$ duality of strings compactified on a circle with radius R. This duality exchanges winding modes and oscillation modes and is a stringy phenomenon that has no analogue in Kaluza–Klein compactification. Mirror symmetry is a generalization of this duality to certain 3-dimensional curved complex manifolds that can be used to construct more realistic models. In that case quantum mechanically equivalent backgrounds differ not only in size but also in shape and even topology, which leads to exciting implications for both, mathematics and physics [as94, mo95].

Quite recently, this duality business has even been extended to dualities among the above 5 different string theories, or rather their lower dimensional relatives which are continuously connected to 10-dimensional theories by letting some compactification radii go to infinity [as95, fe95, ka95, va95, wi95]. While most of these string-string dualities are still hypothetical, they already survived a number of non-trivial tests [ka95$_1$] and they may well teach us some important lessons towards understanding what string theory really is. There are attempts to understand these dualities in terms of hypothetical 11- or 12-dimensional theories, called M and F theory, respectively [wi96$_1$, va96, be96$_3$, ma96].

The present lecture notes on strings are largely based on the books by Green, Schwarz and Witten [GR87] and by Lüst and Theisen [LU89]. There are many other good sources, like the book by Kaku [KA88] and the lecture notes by Kiritsis [Ki97], which can be obtained via internet. In particular I recommend the excellent books by Polchinski [PO98]. Most books on string theory use the sign convention $\eta_{\mu\nu} = \mathrm{diag}(-, +, +, +)$ for the Minkowski space metric, so that mass and momentum are related by $m^2 = -p^2$ (we use natural units $\hbar = c = 1$). This convention facilitates to keep equations consistent while performing the Wick rotation to

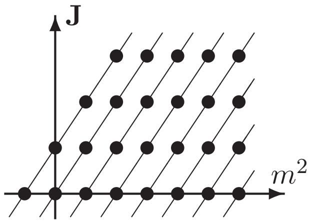

Fig. 2: Regge trajectories

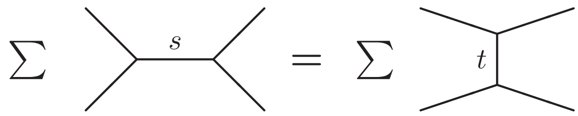

Fig. 3: Duality

Euclidean space. We will, however, use the convention $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$, which is mostly preferred in QFT textbooks, so that $t = x^0 = x_0$, which is somewhat nicer in the Hamilton formalism.

1.2 History of string theory

String theory was discovered in the late 60s as a model for hadron resonances, large numbers of which were found with a spin–mass relation described by Regge trajectories $J = \alpha_0 + m^2\alpha'$, as shown in Fig. 2. Renormalizable QFTs, however, were and are known only for spin $J \in \{0, \frac{1}{2}, 1\}$: Scalars interacting by exchange of a spin J particle, for example, have an amplitude $A_J(s,t) \sim \frac{s^J}{t-m^2}$ where $s = (p_1+p_2)^2$, $t = (p_1+p_3)^2$ and $u = (p_1+p_4)^2$ are the Mandelstam variables[4] for scattering of two particles with momenta p_1 and p_2 to particles with outgoing momenta $-p_3$ and $-p_4$, because there are J derivatives in the interaction term[5] [GR87]. This generated doubts that hadron resonances were really fundamental particles.

At that time analytical properties of the S-matrix, like the relation between s and t channel amplitudes, were studied extensively, and the idea of duality was born [do68]. It states that s and t channel contributions should be equal, instead of being added as in QFT (see Fig. 3). This hypothesis had only marginal experimental support, but Veneziano [ve68] guessed an amplitude with the desired property, namely $A(s,t) + A(t,u) + A(u,s)$ with

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} = \int_0^1 dz\, z^{-1-\alpha(s)}(1-z)^{-1-\alpha(t)}, \qquad \alpha(s) = \alpha_0 + \alpha' s. \tag{1.1}$$

It has exponentially soft UV behaviour, whereas for QFTs cross sections only decrease like inverse powers, and it has infinitely many poles, i.e. describes infinitely many particles.

It turned out that this dynamics can be described by a string picture, with the observed particles being the excitation modes of the string. The Nambu-Goto action for the string is proportional to the area of the world sheet, just as the action for a relativistic point particle

[4] $s/t/u$ is the total energy squared in the rest frame of the $s/t/u$ chanel, and $s+t+u = \sum m_i^2$.
[5] Loop amplitudes $\sim \int dp^n A^2(p)/p^4$ are UV finite for $J < 1$ and have a potentially renormalizable logarithmic divergence for $J = 1$ in $n = 4$ dimensions.

is proportional to its proper time $S[x] = -m\int ds = -m\int d\tau\,\sqrt{\dot{x}^2}$, where τ parametrizes the world line $x^\mu(\tau)$. From this geometrical picture of string interactions (see Fig. 1) duality is now apparent. Furthermore, the UV behaviour of string amplitudes is exponentially soft because there are no localizable interaction points on a smooth surface: The symmetries of the string organize the contributions of infinitely many massive particles of high spin in such a way that the sum of an infinite number of terms with polynomial growth is exponentially small, like in the Taylor expansion of $\exp(-x)$.

In the early 70s QCD turned out to do better in describing hadron interactions[6] (asymptotic freedom in 1973, etc.). But Scherk and Schwarz showed that strings provide a promising theory for quantum gravity [sc74]: There always is a massless spin 2 excitation – the graviton – and there are no UV divergences, because there are no point-like interactions. The bad news, however, was that, in light-cone quantization [go73], Lorentz invariance is broken in $D \neq 26$, and that the intercept α_0 turned out to be positive so that the squared mass of the ground state is negative (tachyonic) and the theory is, at best, formulated in an unstable 'vacuum'.

This inconsistency was eventually cured by fermions, which had already been introduced into dual models by Ramond [ra71] and by Neveu and Schwarz [ne71] in order to describe fermionic hadron resonances. A generalization of the Nambu action to the 'spinning string' [br76, de76] was possible, however, only after some development of supersymmetry. Due to the additional fermionic degrees of freedom on the world sheet the critical dimension of the spinning string reduces to $D = 10$. But this model still is plagued by inconsistencies related to a tachyon, which eventually was thrown out by the GSO projection [gl76]. The resulting spectrum of states then turned out to be space-time supersymmetric, i.e. contains an equal number of bosonic and fermionic degrees of freedom, which are related by an anticommuting symmetry.[7] There is an alternative formulation, called the Green–Schwarz superstring [GR87], which is manifestly space-time supersymmetric. We will, however, mainly consider the RNS model with the *manifest* supersymmetry living on the world sheet.

After almost 10 years of underground development of string theory and many fruitless efforts to find a viable model for SUGRA Kaluza–Klein unification it was time for the string revolution, which came in 1984 with the discovery of the Green–Schwarz mechanism [gr84]: In the 'zero slope' limit the superstring leads to a chiral 10-dimensional supergravity theory, and anomaly cancellation fixes the gauge group almost uniquely. The Kaluza–Klein scenario thus eventually obtained a solid basis, but this time including an (almost) unique additional

[6] 'Color strings' are, however, still in use for describing quark interaction at long distances; 'cosmic strings' could form from topological defects in spontaneous symmetry breaking. In both cases, the Nambu–Goto action is only an approximation. We will only be interested in 'fundamental' strings.

[7] It turned out that the GSO projection is not only possible but is mandatory in order to avoid global anomalies at higher genus (this requirement is called 'modular invariance'). Thus supersymmetry, and in particular the presence of fermions, presumably is an unavoidable consequence of string unification.

gauge group $E_8 \times E_8$ (or SO_{32}). As it turned out, however, the vacuum structure is not so unique after compactification or when string theories are constructed directly in 4 dimensions. There remain many open problems concerning the quantum mechanics that (hopefully) selects a ground state resembling the observable universe (which includes a small cosmological term *after* SUSY breaking). Moreover, it is still not at all clear what string theory really is.

Chapter 2

The bosonic string

The action of a relativistic point particle in a curved space with local coordinates x^μ and metric $g_{\mu\nu}(x)$ is proportional to its proper time $S = -m\int ds$, where the line element is $ds = \sqrt{\dot{x}^2}dt$ with $\dot{x}^2 = \dot{x}^\mu\dot{x}^\nu g_{\mu\nu}$ for some parametrization $x^\mu = x^\mu(t)$ of its world line. Variation with respect to x^μ yields the equation of motion (EoM)

$$\delta\int dt\sqrt{\dot{x}^2} = \int \delta x^\rho\left(\tfrac{1}{2\sqrt{\dot{x}^2}}(\dot{x}^\mu\dot{x}^\nu\partial_\rho g_{\mu\nu} - \partial_t(2\dot{x}^\mu g_{\mu\rho})) - \dot{x}^\mu g_{\mu\rho}\,\partial_t(\tfrac{1}{\sqrt{\dot{x}^2}})\right) = 0. \qquad (2.1)$$

We avoided the evaluation of the time derivative of the line element in the last term because we should first note that the action is invariant under reparametrization $t \to t'(t)$ of the world line. We thus cannot expect an EoM that uniquely fixed the time evolution. We can, however, take advantage of that fact and choose a parametrization for which ds is proportional to dt, i.e. set $\partial_t\sqrt{\dot{x}^2} = 0$. With this so-called affine parametrization the last term of the above equations goes away and we find the well-known geodesic equation

$$\ddot{x}^\alpha + \Gamma_{\mu\nu}{}^\alpha\dot{x}^\mu\dot{x}^\nu = 0 \qquad \text{with} \qquad \Gamma_{\mu\nu}{}^\alpha = \tfrac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})g^{\rho\alpha}. \qquad (2.2)$$

The *Christoffel symbol* $\Gamma_{\mu\nu}{}^\alpha$ thus determines the "acceleration" $\ddot{x}^\alpha$ away from a straight line *in coordinate space* for a particle with "velocity" $\dot{x}^\mu$. Since (2.2) originates from the variation of the argument of the square root the same classical EoM can be obtained from the simpler action $S = -\frac{1}{2}\,m\int\dot{x}$, which looks physically more appealing (because the "kinetic energy" becomes a quadratic form) at the expense of manifest reparametrization invariance.

The Nambu–Goto action for the bosonic string is, analogously, given by the area of the world sheet that is embedded in some D-dimensional space. If the target space is itself a general manifold with a metric $G_{\mu\nu}(X)$ depending on local coordinates X^μ then the resulting theory is called a (non-linear) σ-model. So we start with the action

$$S_N[X] = -T\int d^2\sigma\sqrt{-\det G^*} \qquad \text{with} \quad G^*_{mn} := (X^*G)_{mn} = \frac{\partial X^\mu}{\partial\sigma^m}\frac{\partial X^\nu}{\partial\sigma^n}G_{\mu\nu}(X), \qquad (2.3)$$

where σ^0 and σ^1 are local coordinates of the world sheet and the embedding is described by D coordinate functions $X^\mu(\sigma)$. The *induced metric* $G^*_{mn}(\sigma)$ on the world sheet is the *pull back* X^*G of the target space metric $G_{\mu\nu}(X)$ to the parameter space of the embedded surface. The *string tension* T is a constant with the dimension of an inverse length squared and the sign of the action is chosen such that the kinetic energy will be positive for the space-like coordinates $X^1, \ldots, X^{D-1}$ of the target space (see below).

2.1 The Polyakov action

From the σ-model point of view it is natural to consider the *Polyakov action* [po81]

$$S_P[X,g] = -\frac{T}{2}\int d^2\sigma\sqrt{-g}\; g^{mn}\frac{\partial X^\mu}{\partial\sigma^m}\frac{\partial X^\nu}{\partial\sigma^n}G_{\mu\nu}. \tag{2.4}$$

To establish classical equivalence with the Nambu-Goto action (2.3) we calculate the variation of S_P with respect to the (inverse) metric g^{mn}, which by definition[1] is proportional to the energy–momentum tensor T_{mn}. Using $\delta(\ln\det M) = \delta(\operatorname{tr}\ln M) = \operatorname{tr}(M^{-1}\delta M)$ we obtain

$$T_{mn} := \frac{2}{T}\frac{1}{\sqrt{-g}}\,\frac{\delta S_P}{\delta g^{mn}} = \frac{1}{2}g_{mn}g^{kl}G^*_{kl} - G^*_{mn} = 0. \tag{2.5}$$

The equation of motion $T_{mn} = 0$ implies that the world sheet metric must be proportional to the induced metric, i.e. $g_{mn} = \rho G^*_{mn}$, where the factor $\rho = 2/(g^{kl}G^*_{kl})$ drops out of all equations and remains arbitrary. Since the Polyakov action does not depend on derivatives of the metric, $\frac{\delta S_P}{\delta g^{mn}} = 0$ is algebraic in g_{mn} and we may insert it back into the action without changing the equations of motion for the 'matter fields' X^μ. Taking determinants, we thus observe classical equivalence.[2] For quantization, however, S_P is more convenient because the

[1] In string theory it is common practice to deviate from the usual normalization $T_{mn} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{mn}}$, which is consistent with the Noether formula $\hat{T}_l{}^m = \partial_l\phi^i\frac{\partial\mathcal{L}}{\partial\partial_m\phi^i} - \delta^m_l\mathcal{L}$ for the canonical energy–momentum tensor in Minkowski space. (Note that the Belinfante improvement terms $T_l{}^m - \hat{T}_l{}^m$ to the canonical energy-momentum tensor $\hat{T}_l{}^m$ vanish for the bosonic string because all matter fields X^μ are world-sheet scalars.)

[2] We can generalize these considerations to higher dimensions and consider membranes or arbitrary so-called p-branes, where p is the number of space-like dimensions, i.e. $p = 0$ corresponds to point particles and $p = 1$ to strings. In order to get classical equivalence to a dynamics that extremizes the "world-volume" of the embedding it turns out that one has to include a cosmological term $\sqrt{-g}$ into the Polyakov-type brane action,

$$\mathcal{L} = -\frac{t}{2}\int d^k\sigma\sqrt{-g}\,g^{mn}\partial_m X\cdot\partial_n X + \frac{\mu}{2}\int d^k\sigma\sqrt{-g}\,, \tag{2.6}$$

where $k = p+1$ is the space-time dimension of the brane. Variation with respect to the auxiliary world-volume metric yields the EoM

$$\delta\mathcal{L} \sim \delta g^{mn}(G^*_{mn} - \frac{1}{2}g_{mn}(g^{ij}G^*_{ij} - \mu/t)) + \delta X^\rho(\ldots) = 0. \tag{2.7}$$

Taking the trace of T_{mn} we find $g^{mn}G^*_{mn}(k-2) = k\mu/t$, which implies $\mu = 0$ and scale invariance in the string case $p = 1$. For $k > 2$ and $\mu \neq 0$ a simple calculation yields $g_{mn} = \frac{k-2}{\mu}tG^*_{mn}$ and

$$\mathcal{L} = -T_p\int\sqrt{-G^*} \qquad \text{with} \qquad T_p = t^{\frac{k}{2}}\left(\tfrac{k-2}{\mu}\right)^{\frac{k-2}{2}}. \tag{2.8}$$

world sheet scalars X^μ now have their usual kinetic terms rather than appearing in the square root of a determinant.

Now we compute the total variation of the action to obtain the equations of motion[3] for minimal area surfaces (since $G_{\mu\nu}$ depends on $X^\alpha(\sigma^m)$ we have $\partial_m G_{\mu\nu} = \partial_m X^\alpha \partial_\alpha G_{\mu\nu}$):

$$-\tfrac{2}{T}\,\delta S_P = \int d^2\sigma\,\sqrt{-g}\,\Big(\delta X^\alpha \partial_\alpha G_{\mu\nu} D_n X^\mu D^n X^\nu + 2D_n(\delta X^\alpha) D^n X^\rho\, G_{\alpha\rho} - \delta g^{mn} T_{mn}\Big) \tag{2.9}$$
$$= \int d^2\sigma\,\sqrt{-g}\,\Big(\delta X^\alpha\Big(\partial_\alpha G_{\mu\nu} D_n X^\mu D^n X^\nu - 2D^2 X^\rho G_{\alpha\rho} - 2D^n X^\rho D_n X^\nu \partial_\nu G_{\alpha\rho}\Big) - \delta g^{mn} T_{mn}\Big)$$

Here we ignored surface terms which have to be taken into account for open strings (see below). The last term $\partial_\nu G_{\alpha\rho}$ is symmetrized in ν and ρ, hence all derivatives of the target space metric combine to give the Christoffel symbol $\hat\Gamma_{\mu\nu\alpha} = \frac{1}{2}(\partial_\mu G_{\alpha\nu} + \partial_\nu G_{\alpha\mu} - \partial_\alpha G_{\mu\nu})$ of the target space metric. Contracting $\delta S_P/\delta X^\alpha$ with $G^{\lambda\alpha}$ we thus arrive at the equations of motion

$$\Delta X^\lambda + g^{mn}\partial_m X^\mu \partial_n X^\nu \hat\Gamma_{\mu\nu}{}^\lambda = 0, \tag{2.10}$$

$\Delta := D^2 = g^{mn} D_m D_n = g^{mn}(\partial_m\partial_n - \hat\Gamma_{mn}{}^l\partial_l)$ is the Laplace–Beltrami operator for scalars on the world sheet. (In a non-covariant evaluation of the variational derivative the Christoffel symbol $\hat\Gamma_{mn}{}^l$ of the world sheet metric comes from the term $\partial_m(\sqrt{g}\,g^{mn}) = -\sqrt{g}\,g^{kl}\hat\Gamma_{kl}{}^n$.) Note that we recover the geodesic equation if we let the string collapse to a point, so that all derivatives with respect to the space coordinate σ^1 are zero, and use an affine parametrization $g_{00} = 1$ of the resulting world line.

2.2 Local symmetries and gauge fixing

Before we try to solve the equations of motion we should have a look at the symmetries of the Polyakov action $\mathcal{L}_P$. By construction the Nambu–Goto action is coordinate invariant in the target space as well as on the world sheet. This carries over to $\mathcal{L}_P$, but for that action we have, in addition, the Weyl invariance $g_{mn}(\sigma) \to e^{2\Lambda(\sigma)}\, g_{mn}(\sigma)$ on the world sheet. Together with two coordinate functions $\tilde\sigma^m(\sigma)$ this gives a total of 3 functions of σ^m that we are free to choose.

The number of gauge degrees of freedom thus coincides with the degrees of freedom in the world sheet metric. This suggests that we should be able to use a flat metric $g_{mn} = \eta_{mn}$ on

(t,μ) have target space and world sheet dimensions $(2,0)$ and $(k-2,k)$, respectively. For the dimensions of the brane tension T_p we thus find $(D,0)$. Ignoring world-sheet dimensions people often set $t=-1$ and $\mu = m^2$ for the point particle. We note that world-line metric cannot be eliminated in the massless case $m=0$.

[3] The variational derivatives of an action $S = \int \mathcal{L}(\phi, \partial_m\phi)$ are defined by $\frac{\delta S}{\delta\phi^i} := \frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_m \frac{\partial\mathcal{L}}{\partial\partial_m\phi^i}$. In curved space it is, however, often more efficient to compute the variation directly with covariant partial integration in scalar densities using $\int \partial_{m_1}(\sqrt{g}\,A_{m_2\ldots m_I} B^{m_1\ldots m_I}) = \int \sqrt{g}\,D_{m_1}(A_{m_2\ldots m_I} B^{m_1\ldots m_I}) = \int \sqrt{g}\,(D_{m_1}A_{m_2\ldots m_I} B^{m_1\ldots m_I} + A_{m_2\ldots m_I} D_{m_1} B^{m_1\ldots m_I})$.

the world sheet, which indeed is true locally. To see this note that in two dimensions for any two linearly independent vector fields there exists a coordinate system whose coordinate lines coincide with the integral curves of the vector fields. Having a metric with Lorentzian signature, there are two natural vector fields defined by the two independent null vectors at each point. In a corresponding coordinate system with coordinates σ^+ and σ^- the metric has only off-diagonal entries. With $\partial_\pm := \partial/\partial\sigma^\pm$ we thus have

$$g_{+-} = g(\partial_+, \partial_-) = \tfrac{1}{2}e^\varphi, \qquad g_{++} = g_{--} = 0. \tag{2.11}$$

$\sigma^\pm$ are called light-cone coordinates. They are unique up to reparametrizations $\sigma^\pm \to f_\pm(\sigma^\pm)$ and we choose them in such a way that $\tau = \sigma^0 := (\sigma^+ + \sigma^-)/2$ is time-like and increasing with the target-space time X^0, whereas $\sigma = \sigma^1 := (\sigma^+ - \sigma^-)/2$ is space-like. $g_{+-} > 0$ is required by $g_{00} = g_{++} + 2g_{+-} + g_{--} > 0$ and $g_{11} = g_{++} - 2g_{+-} + g_{--} < 0$. These equations, as well as $g_{01} = g_{++} - g_{--}$, follow from $\partial_\tau = \partial_+ + \partial_-$ and $\partial_\sigma = \partial_+ - \partial_-$. We also find

$$\sigma^\pm = \tau \pm \sigma, \qquad g^{+-} = 2e^{-\varphi} = 1/g_{+-}, \qquad g_{+-} = \tfrac{1}{4}(g_{00} - g_{11}), \tag{2.12}$$

$$\partial_\pm = \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma), \qquad g_{mn} = e^\varphi \eta_{mn}, \qquad g_{\pm\pm} = \tfrac{1}{4}(g_{00} \pm 2g_{01} + g_{11}). \tag{2.13}$$

Now we can perform a Weyl rescaling to get a flat world sheet metric $g_{mn} = \eta_{mn}$.

In light-cone coordinates we obtain very simple expressions for the Christoffel symbol, whose only non-vanishing components are

$$\hat\Gamma_{++}{}^{+} = \partial_+ \ln\sqrt{-g} = \partial_+\varphi, \qquad \hat\Gamma_{--}{}^{-} = \partial_- \ln\sqrt{-g} = \partial_-\varphi, \tag{2.14}$$

since $\hat\Gamma_{m++} = \hat\Gamma_{m--} = 0$. For the energy–momentum tensor (2.5) we find

$$T_{++} = -\partial_+X\cdot\partial_+X, \qquad T_{+-} = 0, \qquad T_{--} = -\partial_-X\cdot\partial_-X. \tag{2.15}$$

There also is a simple geometrical interpretation of the minimal area equation: Observing that ∂_+X^μ and ∂_-X^μ are the light-like tangent vectors defined by the coordinate lines σ_+ and σ_- it is easy to see that (2.10) is nothing but $D_{\partial_-X}\partial_+X = 0$, i.e. the covariant derivative (with respect to the Levi–Civita connection in target space) of ∂_+X along ∂_-X has to vanish. For a flat target space $G_{\mu\nu} = \eta_{\mu\nu}$ this reduces to the wave equation $\Box X^\mu = \partial_+\partial_-X^\mu = 0$, whose general solution is a superposition of a left-moving mode $X_L^\mu(\sigma^+)$ and a right-moving mode $X_R^\mu(\sigma^-)$.

We now turn to global properties of our choice of parametrization. The basic assumption for our local construction was that the metric has Minkowski signature. This cannot be true globally for interacting strings, as can be seen for a 'pant' representing a smooth joining of two closed strings: For such a world sheet there is always some region where the induced metric is Euclidean. It is therefore convenient to restrict our attention to free strings with 'generic'

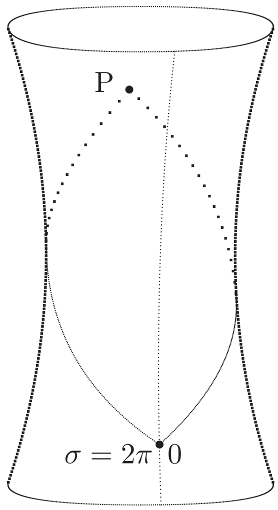

Fig. 4: Closed string with $0 < \sigma < 2\pi$

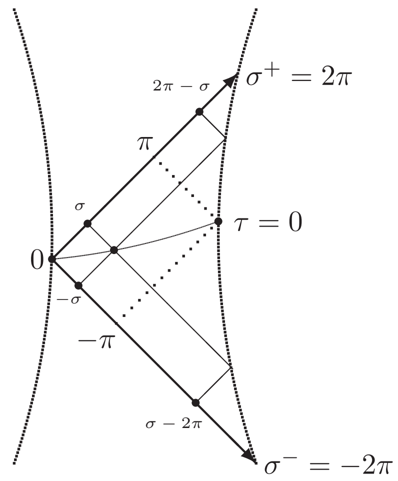

Fig. 5: Open string with $0 < \sigma < \pi$

world sheets and to postpone the study of the interacting case and the rigorous treatment of global questions till after a Wick rotation to Euclidean space. In particular, we exclude world sheets with closed time-like curves and degenerations of the light cone.

The choice of light cone coordinates still allows for reparametrizations of the coordinate lines $\sigma^\pm \to f^\pm(\sigma^\pm)$. This freedom can now be used to choose a parametrization that is 2π-periodic in σ for closed strings:

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \tag{2.16}$$

(the case of open strings will be discussed in section 2.3). In order to see that (2.16) is consistent with the conformal gauge $g_{mn} \propto \eta_{mn}$ we choose an arbitrary point on the closed string surface as the coordinate origin $\sigma = \tau = 0$ (see Fig. 4). Then we go along the two light-like curves in positive time direction till we arrive at the first intersection point P and choose some parametrization of these two pieces of coordinate lines in such a way that the coordinate labels are smooth and monotonic and reach 2π at P. We can now assign coordinates $\sigma^\pm$ modulo 2π to any point on the surface by going along the two light cones till we meet one of the sections of coordinate lines where the coordinates have been chosen. In order to fix the coordinates completely we cut the surface along a time-like curve through the origin and demand that the coordinate functions are continuous on the resulting strip. In order to see what happens at the cut we go from the origin to the intersection point along the σ^+ coordinate line in positive direction till we reach the point P with $\sigma^+ = 2\pi$ and $\sigma^- = 0$. Then we continue along the other coordinate line till we return to the origin, with σ^+ constant and σ^- decreasing by 2π till we arrive at the origin. We thus observe that the coordinate σ jumps by 2π and that τ is continuous if we cross the cut at the origin. Because of continuity of the coordinates away from the cut the same has to happen everywhere along the cut.

If we restrict ourselves to parametrizations satisfying (2.16) then we are only free to choose a parametrization of the coordinate lines $\sigma^\pm$ in the intervals $0 < \sigma^\pm < 2\pi$. Taking into account the additional freedom of choosing the origin and imposing smoothness of the coordinate transformation at the origin we end up with a residual reparametrization freedom $\sigma^\pm \to \sigma^\pm - \xi^\pm$ with smooth 2π-periodic functions $\xi^+(\sigma^+)$ and $\xi^-(\sigma^-)$, where we also require $\partial_\pm \xi^\pm < 1$ in order that the new coordinates be monotonic.

The restriction that the parametrization of the surface should be 2π-periodic in σ with the metric g_{mn} being proportional to η_{mn} is called *conformal gauge*. A diffeomorphism $\tilde{\sigma}^m = \sigma^m - \xi^m(\sigma)$ that changes the metric only by a Weyl rescaling $\tilde{g}_{mn}(\tilde{\sigma}) = e^{2\Lambda(\sigma)} g_{mn}(\sigma)$ is called a *conformal transformation* (in other words, this is a reparametrization that preserves angles between coordinate lines; of course we may also think about such a transformation in an active way as giving us a new surface parametrized by the old coordinates). Such transformations respect the conformal gauge. Considering infinitesimal transformations (i.e. keeping only terms that are linear in the small quantities ξ^m and Λ) this leads to the *conformal Killing equation*

$$\delta g_{mn} = \mathcal{L}_\xi g_{mn} - 2\Lambda g_{mn} = D_m \xi_n + D_n \xi_m - 2\Lambda g_{mn} = 0. \tag{2.17}$$

This means that variation of the metric under infinitesimal coordinate transformations, which is given by the Lie derivative $\mathcal{L}_\xi g_{mn} = D_m \xi_n + D_n \xi_m$ with respect to the vector field ξ^m, can be compensated by a Weyl transformation. Taking the trace we find $D_n \xi^n = \Lambda g^{mn} g_{mn} = \Lambda\, d$, so that the Weyl factor becomes proportional to the covariant divergence of ξ. In $d = 2$ dimensions this yields $D_m \xi_n + D_n \xi_m - g_{mn} D_l \xi^l = 0$. Using light cone coordinates this equation is an identity for $(m,n) = (\pm, \mp)$ and we recover the conditions $D_\pm \xi_\pm = g_{+-} D_\pm \xi^\mp = 0 \;\Leftrightarrow\; \partial_\pm \xi^\mp = 0$.

2.3 Open strings

For open strings the Euler-Lagrange equations of motion still have to be supplemented by boundary conditions that come from the surface terms of a general variation of the action. But first we construct, in analogy to our discussion of closed strings, a conformally flat coordinate system whose space coordinate σ ranges from 0 to π for all τ. To this end we choose a point at the left boundary[4] as the origin and choose coordinate labels $0 < \sigma^+ < 2\pi$ along the future light cone, as shown in Fig. 5. Imposing the condition that the left and the right boundary of the string is parametrized by $\sigma = 0$ and $\sigma = \pi$, respectively, we can assign coordinates to all points on the string surface by following the light rays till we intersect the original piece of σ^+ coordinate line. The only difference to the case of closed strings is that this time the coordinate

[4] 'left' refers to decreasing space-like coordinate σ^1 for an oriented parametrization with X^0 increasing with a time-like coordinate σ^0.

lines are 'reflected' at the boundary. On the σ^- coordinate line through the origin, for example, we find coordinate labels between 0 and -2π ($\sigma^\pm$ is constant along the $\sigma^\mp$ coordinate lines).

From this construction it follows that the functions $f^+(\sigma^+)$ and $f^-(\sigma^-)$ that correspond to the residual gauge invariance in conformal gauge must be identical $f^+ \equiv f^-$ and 2π periodic to be consistent with a σ coordinate that runs from 0 to π. The freedom of parametrizing the σ^+ coordinate line (and thereby also the σ^- line) can also be interpreted in a different way: As a consequence of the choice of $\sigma^\pm$ labels the line $\tau = (\sigma^+ + \sigma^-)/2 = 0$ and the σ coordinate labels on that line are fixed. In turn, we can first choose the line of vanishing time $\tau = 0$ and assign σ coordinate labels between 0 and π on that line. Then the $\sigma^\pm$ coordinate labels can be constructed as shown in Fig. 5. Hence the choice of the line of equal time $\tau = \tau_0$ corresponds to the even part of 2π-periodic infinitesimal reparametrizations $\delta\sigma^\pm = f(\sigma^\pm)$, and the freedom of assigning the σ coordinate labels corresponds to f_{odd} in the unique decomposition $f = f_{even} + f_{odd}$.

Now we turn to the derivation of the boundary conditions at the ends of the string. We require that the action should be stationary if the variation vanishes at the initial and final times, but is arbitrary at the string ends. To avoid terms coming from a variation of the integration domain we assume a parametrization with $0 < \sigma < \pi$. We thus pick up a surface term

$$\int d^2\sigma \partial_m \left(\delta X^\mu \frac{\partial S}{\partial \partial_m X^\mu} \right) = -T \int_{t_0}^{t_1} d\tau \, \left(\delta X^\mu \sqrt{-g}\, G_{\mu\nu}\, g^{1n} \partial_n X^\nu \right) \Big|_{\sigma=0}^{\sigma=\pi} . \qquad (2.18)$$

If we require this expression to vanish for arbitrary variations δX^μ we conclude that for $\sigma = 0$ and for $\sigma = \pi$ we must have

$$\sqrt{-g}\,(g^{10}\dot{X} + g^{11}X') = (g_{01}\dot{X} - g_{00}X')/\sqrt{-g} \,= 0. \qquad (2.19)$$

This implies that the induced metric G^*_{mn} becomes singular at the boundary, which can be seen as follows: First assume that $\dot{X}$ and X' are linearly independent, implying that $g_{01}/\sqrt{-g} = g_{00}/\sqrt{-g} = 0$. The equations of motion imply that g_{mn} is proportional to the induced metric $G^*_{mn} = \rho g_{mn}$ everywhere on the world sheet, with ρ dropping out in the ratio $g_{mn}/\sqrt{-g}$. This cannot happen if G^*_{mn} has a non-singular limit at the boundary. If, on the other hand, $\dot{X}$ and X' become proportional at the boundary, then the matrix $G^*_{mn} = \partial_m X \cdot \partial_n X$ also is singular. The discussion of boundary conditions is therefore very delicate in general and we better first choose a convenient gauge.

In conformal gauge the induced metric G^*_{mn}, with entries $\dot{X}^2$, $\dot{X} \cdot X'$ and $(X')^2$, becomes proportional to the flat metric, i.e. $G^*_{mn} = \rho\,\eta_{mn}$, and ρ has to vanish at the boundary. Inserting this back into eq. (2.19) we find the boundary conditions $X' = 0$ and $\dot{X}^2 = 0$. The second condition has the geometrical interpretation that the string ends move with the speed of light, and therefore is independent of the gauge. The D conditions $\partial_\sigma X^\mu = 0$, on the other hand, are valid only in the conformal gauge, as can be seen by choosing a gauge for which the coordinate lines are not orthogonal.[5] Sticking to the conformal gauge, we have *Neumann* boundary conditions. We therefore can continue the coordinate functions $X^\mu(\tau,\sigma)$ beyond $0<\sigma<\pi$ to get even and 2π-periodic functions of σ. Hence all open string solutions can be obtained in the conformal gauge as special cases of closed string solutions.

Neumann boundary conditions (in conformal gauge) are the only Lorentz-invariant possibility to make surface terms vanish. If we relax that condition, however, it is also possible to make (2.19) vanish by choosing Dirichlet boundary conditions $X^\mu = const.$ for $D-p-1$ of the space-like string coordinates (in a flat target space). The string ends are then constrained to move on a p-dimensional submanifold, a so-called D-brane (a p-brane is an extended object of space-time dimension $p+1$, so that a 2-brane is a membrane and a 1-brane is a string; here, however, the 'D' stands for 'Dirichlet', indicating that open strings have to end on that brane without specifying its dimension). The consistency and importance of these boundary conditions was discovered in the context of T-duality [da89, ho89] (see below) and the presence and dynamics of the associated (solitonic) extended objects, i.e. p-branes acting as D-branes, plays an important role in recent results on non-perturbative string dualities [po95₁, po96₁].

2.4 Target space symmetries and conservation laws

By construction, the Polyakov action is invariant under arbitrary coordinate transformations of the world sheet and of the target space. The local world sheet invariances imply gauge symmetries of the action, as we discussed above. Target space coordinate invariance, on the other hand, in general does not imply any symmetry of the σ model, since $\mathcal{L}_P$ is only invariant if we also transform $G_{\mu\nu}$. Since the functions $G_{\mu\nu}(X)$ can be interpreted as (an infinite number of) coupling constants, target space coordinate transformations relate *different* σ models by a reparametrization of the dynamical fields X^μ. We do, however, have a symmetry of the σ model if the new functions $G'_{\mu\nu}(X')$ turn out to be identical to the old metric $G_{\mu\nu}(X)$. Then the target manifold has a (geometrical) symmetry, which corresponds to a global symmetry of the σ model, because the transformation $X\to X'$ of the dynamical fields is independent of σ.

Continuous target space symmetries are equivalent to the global existence of Killing vector fields $\Xi^\mu(X)$ with $\mathcal{L}_\Xi G_{\mu\nu} = D_\mu\Xi_\nu + D_\nu\Xi_\mu = 0$. According to the Noether theorem[6] they imply

[5] Consider, for example, the solution $X^\mu = (\tau, \cos\sigma\cos\tau, \cos\sigma\sin\tau, 0, \ldots)$ to the equations of motion in conformal-gauge, which satisfies all boundary conditions. Changing the parametrization by $\tau = \overline{\tau} + a\sigma$ we find $\dot{X}^2 = \sin^2\sigma$, $X'^2 = (a^2-1)\sin^2\sigma$, and $\dot{X}\,X' = a\sin^2\sigma$. Then $X' = (a, -a\cos\sigma\sin\overline{\tau} - \sin\sigma\cos\overline{\tau}, a\cos\sigma\cos\overline{\tau} - \sin\sigma\sin\overline{\tau}, \ldots)$, which shows that X' does not have to vanish at the boundary in a general gauge.

[6] The first Noether theorem states that continuous symmetries are in one-to-one correspondence with conserved charges: In a field theory with an action $S = \int d^4x \mathcal{L}(\phi^i, \partial\phi^i)$ that is invariant under the infinitesimal transformations $\delta_I\phi^i = R_I{}^i(\phi, \partial\phi)$, i.e. with $\mathcal{L}$ transforming into total derivatives $\delta_I\mathcal{L} = \partial_m K_I{}^m$, the

the existence of conserved quantities. In the case of a flat target space, for example, we have $G_{\mu\nu} = \eta_{\mu\nu}$ and the general solution to the Killing equation is

$$\Xi^\mu = A^\mu + X^\nu \Omega_\nu{}^\mu \tag{2.20}$$

with $\Omega_{\mu\nu}$ antisymmetric. Invariance under the D independent translations $\delta_\mu X^\rho = -\delta_\mu^\rho$ implies conservation of the target space energy–momentum currents P_μ^m with the corresponding conserved charges $P_\mu = \int d\sigma P_\mu^0$ (for convenience we use the conformal gauge):

$$P_\mu^m = -\delta_\mu^\rho \frac{\partial \mathcal{L}}{\partial \partial_m X^\rho} = T\eta^{mn}\partial_n X_\mu, \qquad P_\mu = T\int d\sigma\, \dot{X}_\mu \tag{2.21}$$

(the sign of the momentum has been chosen such that the energy P_0 is positive.) Note that the object P_μ^m is different from the canonical (flat) *world sheet* energy–momentum tensor $\hat{T}_l{}^m = \delta_l X^\mu \frac{\partial \mathcal{L}}{\partial \partial_m X^\mu} - K_l{}^m = -T(\partial_l X^\mu \partial^m X_\mu - \frac{1}{2}\delta_l^m \delta_k^j \partial_j X^\mu \partial^k X_\mu)$, where $\delta_l \mathcal{L} = \partial_m K_l{}^m = \partial_l \mathcal{L}$ is the infinitesimal change of the Lagrangian under translations $\delta_l \phi^i = \partial_l \phi^i$, so that $K_l{}^m = \delta_l{}^m \mathcal{L}$. In particular, $H = \int d\sigma\, \hat{T}_0{}^0 = \int d\sigma (\dot{X}^\mu \Pi_\mu - \mathcal{L})$, with the canonical momenta $\Pi_\mu = \partial\mathcal{L}/\partial\dot{X}^\mu = -T\dot{X}_\mu$, is the Hamiltonian of our 2-dimensional field theory, which generates time translations (up to a factor T, which is due to our convention in eq. (2.5), $\hat{T}_l{}^m$ is the flat limit of $T_l{}^m$).

Infinitesimal Lorentz transformations $\delta_{\mu\nu} X^\rho = \delta_\mu^\rho X_\nu - \delta_\nu^\rho X_\mu$ yield the angular momenta

$$J_{\mu\nu}^m = (X_\mu \delta_\nu^\rho - X_\nu \delta_\mu^\rho)\frac{\partial \mathcal{L}}{\partial \partial_m X^\rho} = T\left(X_\mu \partial^m X_\nu - X_\nu \partial^m X_\mu\right), \quad J_{\mu\nu} = T\int d\sigma \left(X_\mu \dot{X}_\nu - X_\nu \dot{X}_\mu\right). \tag{2.22}$$

The Poisson brackets of these charges represent the Poincaré algebra (we include a factor i since $i\{A,B\}_{PB}$ will be replaced by the commutator $[A,B]$ upon quantization):

$$\{P_\alpha, P_\beta\}_{PB} = 0, \qquad i\{J_{\mu\nu}, P_\alpha\}_{PB} = i\eta_{\mu\alpha}P_\nu - i\eta_{\nu\alpha}P_\mu, \tag{2.23}$$

$$i\{J_{\mu\nu}, J_{\alpha\beta}\}_{PB} = i\eta_{\mu\alpha}J_{\nu\beta} - i\eta_{\nu\alpha}J_{\mu\beta} - i\eta_{\mu\beta}J_{\nu\alpha} + i\eta_{\nu\beta}J_{\mu\alpha}. \tag{2.24}$$

The brackets among coordinates and momenta are $\{\Pi_\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = \delta(\sigma-\sigma')\delta_\mu{}^\nu$.

2.5 Classical solutions and light cone gauge

Now we want to solve the equations of motion, so we restrict our discussion to the case of a flat target space and use the conformal gauge with the appropriate boundary conditions. Then the

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explicit formula for the corresponding Noether currents is $J_I{}^m := \delta_I \phi \frac{\partial \mathcal{L}}{\partial \partial_m \phi} - K_I{}^m$. Since $\delta_I \mathcal{L} = \partial K_I = \delta_I \phi^i \left(\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_m \frac{\partial \mathcal{L}}{\partial(\partial_m \phi^i)}\right) + \partial_m \left(\delta_I \phi^i \frac{\partial \mathcal{L}}{\partial(\partial_m \phi^i)}\right)$ the equations of motion imply that the divergence $\partial_m J_I{}^m$ vanishes, i.e. the currents J_I are *conserved*, so that the charges $Q_I = \int d^3x\, J_I{}^0$ are time independent on shell up to surface terms $\dot{Q}_I = \int d^3x\, \vec{\partial}\vec{J}_I$.

coordinate functions fulfill $\partial_+\partial_-X^\mu = 0$ with $\sigma^\pm = \tau \pm \sigma$, hence $X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$ with

$$\partial_+X^\mu = \partial_+X^\mu_L = \sum_{n=-\infty}^{\infty} \frac{\alpha^\mu_n}{\sqrt{4\pi T}} e^{-in\sigma^+}, \qquad \partial_-X^\mu = \partial_-X^\mu_R = \sum_{n=-\infty}^{\infty} \frac{\tilde\alpha^\mu_n}{\sqrt{4\pi T}} e^{-in\sigma^-}. \tag{2.25}$$

Integrating these equations we obtain an integration constant, which we choose to be equal $x^\mu_L = x^\mu_R = x^\mu$ for X_L and X_R. The boundary conditions imply that the zero modes α^μ_0 and $\tilde\alpha^\mu_0$ must be equal, because there can be no linear σ dependence for a periodic function, and we set $\alpha^\mu_0 = \tilde\alpha^\mu_0 = p^\mu/\sqrt{4\pi T}$. Hence

$$X^\mu_L(\tau+\sigma) = \frac{1}{2}x^\mu_L + \frac{1}{4\pi T}p^\mu\,\sigma^+ + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^+}, \tag{2.26}$$

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu_R + \frac{1}{4\pi T}p^\mu\,\sigma^- + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\tilde\alpha^\mu_n e^{-in\sigma^-} \tag{2.27}$$

is the most general solution.[7] Reality of the coordinate functions X^μ implies $\alpha^*_n = \alpha_{-n}$. From the closed string solutions we obtain all solutions for open strings by restricting to even functions. This means that for open strings the left-moving and the right-moving modes must be equal $\tilde\alpha_n = \alpha_n$ and we have much less freedom.

We must be careful to remember that so far we only fulfilled the equations of motion $\delta S/\delta X^\mu = 0$. We still have to set the energy–momentum tensor to zero, i.e. we must impose $T_{++} = -(\partial_+X)^2 = 0$ and $T_{--} = -(\partial_-X)^2 = 0$: For the left-moving part this means that

$$T_{++} = \frac{1}{2\pi T}\sum_{n=-\infty}^{\infty} L_n e^{-in\sigma^+} = 0, \qquad T_{--} = \frac{1}{2\pi T}\sum_{n=-\infty}^{\infty} \tilde L_n e^{-in\sigma^-} = 0. \tag{2.28}$$

The Virasoro generators $L_n := T\int_0^{2\pi} d\sigma^+ T_{++}e^{in\sigma^+}$ and $\tilde L_n := T\int_0^{2\pi} d\sigma^- T_{--}e^{in\sigma^-}$ are given by

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty}\eta_{\mu\nu}\alpha^\mu_m\alpha^\nu_{n-m}, \qquad \tilde L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty}\eta_{\mu\nu}\tilde\alpha^\mu_m\tilde\alpha^\nu_{n-m}. \tag{2.29}$$

The constraints $L_0 = 0 = \tilde L_0$, which generate global translations on the world sheet (see below), play a special role. Inserting the definition $p^\mu = \sqrt{4\pi T}\alpha^\mu_0$ for the zero mode, they read

$$p^2 = -8\pi T\sum_{n>0}\alpha^*_n\cdot\alpha_n = -8\pi T\sum_{n>0}\tilde\alpha^*_n\cdot\tilde\alpha_n. \tag{2.30}$$

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[7] The situation is different for compactified dimensions: If X^μ lives on a circle of radius R then we must allow $X^\mu(\sigma+2\pi) = 2R\pi n + X^\mu(\sigma)$ since the string may wind n times around the loop. Then $\frac{1}{2T}(p^\mu_L - p^\mu_R) = n\,2\pi R$ for some $n\in\mathbb{Z}$. The total momentum $P^\mu = p^\mu_L + p^\mu_R$, on the other hand, will be quantized in units of $1/R$ in the quantum theory because $\exp(2\pi i\,RP^\mu)$ generates a translation by $2\pi R$ and thus has to be the identity operator. This is our first indication of the large/small radius duality $R\to 1/(4\pi T\,R)$. Upon quantization $p_L - p_R$ becomes the winding number operator. Choosing arbitrary integration constants x^μ_L and x^μ_R, the *collective coordinates* are $x^\mu = \frac{1}{2}(x^\mu_L + x^\mu_R)$ and we may use the combination $x^\mu_L - x^\mu_R$, which does not contribute to X^μ, as the conjugate variable for the winding number. In this way we decompose the operator algebra into a left-moving and a right-moving part. For uncompactified dimensions, on the other hand, left-movers and right-movers are always coupled through the momentum zero modes $p^\mu_L = p^\mu_R$.

Vanishing of $H = (L_0 + \tilde{L}_0)/2$, the 2-dimensional Hamiltonian, tells us the mass $m^2 = P^2$ of a string in terms of the oscillators α_n with $n \neq 0$ (recall that $P^2 = p^2/4$ in case of open strings). This constraint is called the mass shell condition; the generator $L_0 - \tilde{L}_0$ of translations in the space direction equates the masses of left and right movers.

The Virasoro constraints $L_n = 0$ are infinitely many quadratic equations and hard to solve in general. It is therefore time to remember that we still have some gauge freedom left, which we may use to simplify these equations. Note that the periodic reparametrizations of the light-cone coordinates, which are still allowed, lead to the freedom $\tau \to \frac{1}{2}(f^+(\sigma^+) + f^-(\sigma^-))$, which just corresponds to a solution of the wave equation for the coordinate functions. We may therefore choose τ proportional to $c_\mu X^\mu$ for some fixed time-like or light-like vector c_μ (a space-like c_μ would lead to a space-like time direction on the world sheet). Because of the identity

$$V^\pm = V^0 \pm V^{D-1} \qquad \Rightarrow \qquad V_\mu W^\mu = \frac{1}{2}(V^+W^- + V^-W^+) - \sum_{i=1}^{D-2} V^i W^i \tag{2.31}$$

a light-like choice $c_\mu = (1, 0 \ldots, 0, 1)$ is particularly useful. In the resulting **light cone gauge** we impose $X^+ = p^+\tau\,/(2\pi T)$, which implies that all oscillator coefficients $\alpha_n^+ = \tilde{\alpha}_n^+ = 0$ vanish for $n \neq 0$. Now the Virasoro constraints obtain linear terms and can be solved explicitly for

$$\alpha_n^- = \frac{\sqrt{4\pi T}}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \alpha_m^i \alpha_{n-m}^i, \qquad \tilde{\alpha}_n^- = \frac{\sqrt{4\pi T}}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \tilde{\alpha}_m^i \tilde{\alpha}_{n-m}^i \tag{2.32}$$

(recall that $p^+ = \alpha_0^+ \sqrt{4\pi T}$). This gauge, however, abandons manifest Lorentz invariance in target space and it turns out that the quantum theory violates the Lorentz algebra (2.24) if $D \neq 26$ [go73]; historically this was the first derivation of the critical dimension of the bosonic string. Note that the light cone gauge assumes $p^+ \neq 0$, which can always be achieved by a Lorentz transformation unless $p^\mu \equiv 0$. This is o.k. for a single free string, but in case of interactions intermediate strings may have arbitrary momenta and we should expect subtle technical problems in perturbation theory [gr88].

In order to find a simple solution to the equations of motion (including the Virasoro constraints) we assume that only one frequency is excited (i.e. only one oscillator α_n and its complex conjugate α_{-n}, as well as the zero mode p^μ, are non-zero). Then the only relevant left-moving constraints are

$$-L_0 = \tfrac{1}{2}|\alpha_0|^2 + \alpha_{-n} \cdot \alpha_n = 0, \qquad -L_n = \alpha_0 \cdot \alpha_n = 0, \qquad -L_{2n} = \tfrac{1}{2}\alpha_n \cdot \alpha_n = 0. \tag{2.33}$$

The first constraint is the mass shell condition, $L_n = 0$ implies transversality with respect to the momentum p^μ, and the third condition implies a light-like polarization vector α_n; by complex conjugation $L_{-n} = 0$ and $L_{-2n} = 0$ are redundant. To simplify things further we keep the string oscillation in the $X^1 - X^2$ plane, and we go to the rest frame and set $x^\mu = \vec{p} = 0$. We now use the light cone gauge[8] and choose $\alpha_n^\mu = \rho\frac{n\sqrt{\pi T}}{2}(0,1,i,0\ldots)$ and $\tilde\alpha_n^\mu = \rho\frac{n\sqrt{\pi T}}{2}(0,1,\pm i,0\ldots)$. This is no further restriction since overall phases of α_n and $\tilde\alpha_n$ corresponds to shifts in $\sigma^\pm$ and the sign of α_n^2 corresponds to a choice of the X^2–direction. Then

$$X^\mu_{(\pm)} = \tfrac{\rho}{2}\Big(\tfrac{p^0\tau}{\pi T\rho}, \mathrm{Re}(ie^{-in\sigma^+}+ie^{-in\sigma^-}), \mathrm{Re}(-e^{-in\sigma^+}\mp e^{-in\sigma^-}), 0, \ldots\Big) \tag{2.34}$$

$$= \tfrac{\rho}{2}\Big(\tfrac{p^0\tau}{\pi T\rho},\ \sin n\sigma^+ + \sin n\sigma^-,\ -\cos n\sigma^+ \mp \cos n\sigma^-, 0, \ldots\Big). \tag{2.35}$$

Using the formulas $\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, $\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, and $\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$, we obtain

$$X^\mu_{(+)} = \rho\Big(\tfrac{p^0\tau}{2\pi T\rho},\ \sin n\tau\ \cos n\sigma,\ -\cos n\tau\ \cos n\sigma,\ 0, \ldots\Big), \tag{2.36}$$

$$X^\mu_{(-)} = \rho\Big(\tfrac{p^0\tau}{2\pi T\rho},\ \sin n\tau\ \cos n\sigma,\ \sin n\tau\ \sin n\sigma,\ 0, \ldots\Big). \tag{2.37}$$

$X^\mu_{(+)}$ is a solution of total length $4n\rho$ and $2n\rho$ for closed and open strings, respectively. It corresponds to a rotating (multiply covered) rod of length 2ρ. According to (2.30) we have $p^0 = 2\rho n\,\pi T$, so that the ends indeed move with the speed of light (the tangential vector $\dot X^\mu_{(+)}$ is light-like at the boundary). The solution $X^\mu_{(-)}$ only exists for closed strings and corresponds to a periodically collapsing (multiply covered) circle of maximal radius ρ, i.e. maximal length $2\rho n\pi$. At the maximal radius there is no kinetic energy and we can check that $(mass)/(length) = T$ is the string tension. For open strings we always have kinetic energy and the factor of 2 in the string length matches the relative factor of 2 in the ratio $\sqrt{-p^2}/m$ for the two types of strings.

Evaluation of (2.22) shows that the angular momentum tensor decomposes into an orbit contribution $x_\mu P_\nu - x_\nu P_\mu$ and the left- and right-moving spin contributions Σ and $\tilde\Sigma$,

$$J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \Sigma_{\mu\nu} + \tilde\Sigma_{\mu\nu}, \qquad \Sigma^{\mu\nu} = -i\sum_{n>0}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right), \tag{2.38}$$

where $\tilde\Sigma$ should be omitted in case of open strings (in that case the σ integral only extends from 0 to π and the spin contribution has to be divided by 2). Inserting the above solutions we find

$$p^2 = (2\rho n\pi T)^2, \qquad \Sigma^{12} = \pm\tilde\Sigma^{12} = 2\rho^2 n\pi T. \tag{2.39}$$

The classical spin is given by $J = \Sigma^{12}$ and $J = \Sigma^{12}+\tilde\Sigma^{12}$ for open and closed strings, respectively. The length scale ρ drops out in the ratio J/m^2, whose maximal value is obtained for the lowest frequency $n = 1$. This shows that the slope α' of the leading Regge trajectory is

$$\alpha'_{closed} = \frac{1}{4\pi T}, \qquad \alpha'_{open} = \frac{1}{2\pi T}, \qquad \frac{J}{m^2} \le \alpha'. \tag{2.40}$$

According to the literature it can be shown that all classical solutions obey this inequality. In the quantum theory it will be corrected by a constant shift α_0.

[8] There are spurious solutions to (2.33) that are missed by the light cone gauge condition: Consider, for example, $p^\mu = 0$ and $\alpha^\mu_{\pm n} = \tilde\alpha^\mu_{\pm n} = (1,1,0,\ldots,0)$.

2.6 Poisson brackets and Virasoro algebra

As a first step towards quantization we now compute the Poisson brackets among the oscillators α and $\tilde{\alpha}$, which follow from the canonical brackets $\{\Pi^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = \delta(\sigma-\sigma')\eta^{\mu\nu}$ with $\Pi^\mu = -T\dot{X}^\mu$, $\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = 0$ and $\{\Pi^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\}_{PB} = 0$ by a Fourier analysis. $\delta(\sigma-\sigma')$ is understood to be 2π–periodic. Inserting (2.26) and (2.27) we find

$$-\frac{\delta(\sigma-\sigma')\eta^{\mu\nu}}{T} = \{\frac{p^\mu}{2\pi T} + \sum_{m\neq 0} \frac{\alpha_m^\mu e^{-im(\tau+\sigma)} + \tilde{\alpha}_m^\mu e^{-im(\tau-\sigma)}}{\sqrt{4\pi T}}, \\ x^\nu + \frac{p^\nu \tau}{2\pi T} + \sum_{n\neq 0} \frac{i}{n} \frac{\alpha_n^\nu e^{-in(\tau+\sigma')} + \tilde{\alpha}_n^\nu e^{-in(\tau-\sigma')}}{\sqrt{4\pi T}}\}_{PB}. \tag{2.41}$$

Since the variables p^μ, x^μ, α_m^μ and $\tilde{\alpha}_m^\mu$ parametrize the general solution to the equations of motions, general results of the canonical formalism tell us that we have to fulfill these relations at a fixed time, say $\tau = 0$. This fixes all brackets among the coefficients in the Fourier representation of $X^\mu(\tau,\sigma)$ and guarantees the canonical brackets for all times.

We first consider the closed string and pick out the brackets among the individual coefficients by evaluating the double integrals $\iint d\sigma\, d\sigma' e^{i(k\sigma+k'\sigma')}$. For $k = k' = 0$ we obtain

$$\{p^\mu, x^\nu\}_{PB} = -\eta^{\mu\nu} \tag{2.42}$$

and $\{x^\mu, x^\nu\}_{PB} = \{p^\mu, p^\nu\}_{PB} = 0$. For $k = 0 \neq k'$ we obtain from the brackets $\{\dot{X}, \dot{X}\}_{PB}$ and $\{\dot{X}, X\}_{PB}$ at $\tau = 0$ that

$$\{p^\mu, \alpha_{k'}^\nu + \tilde{\alpha}_{-k'}^\nu\}_{PB} = 0 = \{p^\mu, \frac{1}{k'}\alpha_{k'}^\nu + \frac{1}{-k'}\tilde{\alpha}_{-k'}^\nu\}_{PB}, \tag{2.43}$$

hence $\{p^\mu, \alpha_n^\nu\}_{PB} = \{p^\mu, \tilde{\alpha}_n^\nu\}_{PB} = 0$. Similarly, for $k' = 0 \neq k$ the brackets $\{X, X\}_{PB}$ and $\{\dot{X}, X\}_{PB}$ imply that x^μ has vanishing brackets with all oscillators α and $\tilde{\alpha}$. Eventually, for k and k' non-zero we find that all brackets among α and $\tilde{\alpha}$ vanish and we conclude that

$$i\{\alpha_m^\mu, \alpha_n^\nu\}_{PB} = i\{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\}_{PB} = -n\,\delta_{m+n}\,\eta^{\mu\nu}, \qquad i\{x^\mu, p^\nu\}_{PB} = i\eta^{\mu\nu} \tag{2.44}$$

are the non-vanishing brackets, where δ_l is an abbreviation for $\delta_{l,0}$.

In case of open strings we must set $\tilde{\alpha}_n^\mu = \alpha_n^\mu$, which makes X^μ even and 2π periodic. It is convenient to integrate again over the interval $0 < \sigma, \sigma' < 2\pi$ in order that the exponentials provide a complete set of orthogonal vectors. But then we have to take into account a second contribution from the δ function, i.e. we must let $\delta(\sigma-\sigma') \to \delta(\sigma-\sigma') + \delta(\sigma+\sigma')$ in eq. (2.41). The double integral for the Fourier coefficients then gives $\frac{2\pi}{T}\eta^{\mu\nu}(\delta_{k-k'} + \delta_{k+k'})$ on the l.h.s. of that equation. We can now repeat the same calculation as above, with the only difference that the second δ function $\delta_{k+k'}$ now doubles the result for the bracket $\{x^\mu, p^\nu\}_{PB}$,

$$i\{x^\mu, p^\nu\}_{PB} = 2i\eta^{\mu\nu} \qquad \text{(open string).} \tag{2.45}$$

This can be understood easily because, with the ansatz (2.26), the total momentum P_μ is p_μ for closed strings and $p_\mu/2$ for open strings, so that $\{x_\mu, P_\nu\}_{PB} = \eta_{\mu\nu}$ in both cases, as we should expect. $\delta_{k+k'}$ also allows for a non-vanishing bracket $\{\alpha_n, \tilde\alpha_{-n}\}_{PB}$, which is necessary because of the identification of $\tilde\alpha$ and α. Otherwise the Poisson brackets are the same for open and closed strings.

Recall that the Virasoro generators $L_n := T\int_0^{2\pi} d\sigma^+ T_{++} e^{in\sigma^+}$ and $\tilde L_n := T\int_0^{2\pi} d\sigma^- T_{--} e^{in\sigma^-}$, which are the Fourier modes of the energy momentum tensor, are given by

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} \eta_{\mu\nu}\alpha_m^\mu \alpha_{n-m}^\nu, \qquad \tilde L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} \eta_{\mu\nu}\tilde\alpha_m^\mu \tilde\alpha_{n-m}^\nu. \tag{2.46}$$

They satisfy an infinite-dimensional Lie algebra, the **Virasoro algebra**

$$\{L_m, L_n\}_{PB} = i(m-n)L_{m+n}, \qquad \{\tilde L_m, \tilde L_n\}_{PB} = i(m-n)\tilde L_{m+n}, \tag{2.47}$$

as is easily verified using that $i\{L_n, \alpha_l^\rho\}_{PB} = l\alpha_{n+l}^\rho$.

A more direct way to understand this algebra is to observe that any *conformal* Killing vector field ξ defines a conserved current $J_\xi^m = \sqrt{-g}\,\xi^l T_l{}^m$ if the energy–momentum tensor is traceless and conserved (which can be shown to be consequences of Weyl invariance and the equations of motion of the matter fields, respectively). For left-moving conformal reparametrizations with $\xi = \xi^+(\sigma^+)\partial_+$ the corresponding conserved quantity is $L_\xi := \int d\sigma\,\xi^+ T_+{}^0 = \int d\sigma^+ \xi^+ (T_+{}^+ + T_+{}^-)/2 = \int d\sigma^+ \xi^+ T_{++}$ (recall that $\eta^{+-} = 2$; $d\sigma$ can be replaced by $d\sigma^+$ because the integrand only depends on $\sigma^+ = \sigma + \tau$). If we choose the basis $\xi_n = e^{in\sigma^+}\partial_+$ for periodic infinitesimal reparametrizations of σ^+, we find the Lie brackets $[\xi_m, \xi_n] = [e^{im\sigma^+}\partial_+, e^{in\sigma^+}\partial_+] = i(n-m)e^{i(m+n)\sigma^+}\partial_+$. Since L_ξ generates the Lie derivative $\{L_\xi, X\}_{PB} = \xi^m\partial_m X$, and because of the Jacobi identity, the Poisson algebra of the charges $L_n = L_{\xi_n}$ must have the same structure constants. Note that the Virasoro constraints L_n and $\tilde L_n$ are conserved quantities, i.e. it is sufficient to impose $T_{mn} = 0$ at some initial time. Obviously, the conformal algebra in two dimensions is the direct product of two identical, infinite dimensional Lie algebras.

Chapter 3

Quantization of bosonic strings

Before actually performing the quantization in section 3.1 let us discuss some general aspects of canonical quantization of gauge invariant systems. It is clear by now that the 'matter fields' X^μ are the dynamical fields of the bosonic string, and that, at least locally, the metric only consists of gauge degrees of freedom. In the Hamiltonian formalism this is indicated by the fact that the conjugate momenta $(\pi_g)^{mn} = \partial L/\partial \dot{g}_{mn}$ vanish identically. Therefore we cannot naively impose the Poisson brackets, at least not as a 'strong' identity: In the process of quantization it is certainly not consistent to impose a commutator $[g_{mn}, (\pi_g)^{kl}] = i(\delta_m^k \delta_n^l + \delta_n^k \delta_m^l)/2$ if $(\pi_g)^{kl} \equiv 0$.

Dirac and Bergmann [DI64] developed a method for obtaining a Hamiltonian description if the Legendre transformation is singular, i.e. if their are constraints like $(\pi_g)^{kl} \equiv 0$ that are inconsistent with the canonical brackets [fr75, HE92]. We will, however, first proceed on the Lagrangian side and fix the gauge. Faddeev and Popov observed that the gauge fixing modifies the path integral measure by a Jacobi factor, namely the determinant of the gauge variation of the gauge fixing term. Representing the determinant as a path integral over auxiliary fermionic *Faddeev–Popov ghosts* one can write that term as a contribution to the action. Later it was observed by Becchi, Rouet and Stora [be76] that the resulting action has a global fermionic nilpotent symmetry $s^2 = 0$ with $s\phi^i = c^I \delta_I \phi^i$, i.e. for a δ_I-invariant action the BRST transformation of a matter field ϕ^i is equal to its gauge transformation with the gauge parameters replaced by ghost fields.

The introduction of 'ghost fields' that have the same quantum numbers – including spin – as the gauge degrees of freedom, but the opposite statistics, was suggested already in 1963 by Feynman in the context of quantum gravity [fe63]. This was motivated by the observation that the gauge degrees of freedom propagate after gauge fixing and that their contribution to loop diagrams is not transversal, like in QED, and therefore spoils unitarity and gauge independence. The BRST symmetry is then necessary to control this compensation.

In order to have a well-defined ghost number that is consistent with the dynamics of the ghosts, we also need to introduce anti-ghost fields $\bar{c}^I$ with ghost number -1, whose BRST transform $b^I := s\bar{c}^I$ is called *lagrange multiplier* (or Nakanishi-Lautrup) field. For a general gauge theory with an irreducible closed gauge algebra $[\delta_I, \delta_J] = \mathcal{F}_{IJ}{}^K \delta_K$ it can be shown that $s^2\phi^i = 0$ implies that the BRST transformation of the ghosts is

$$sc^K = \tfrac{(-)^I}{2}\, c^I c^J \mathcal{F}_{JI}{}^K. \tag{3.1}$$

Nilpotency of s, i.e. the equation $s^2 c^I = 0$, is then equivalent to the Jacobi identity

$$\sum_{IJK} (-)^{IK} \left(\delta_I \mathcal{F}_{JK}{}^L + \mathcal{F}_{IJ}{}^M \mathcal{F}_{MK}{}^L\right) = 0, \tag{3.2}$$

which follows from $\sum_{IJK}(-)^{IK}[[\delta_I, \delta_J], \delta_K] = 0$. (We have an *open* gauge algebra if the graded commutator $[\delta_I, \delta_J]$ is proportional to δ_K only on shell, i.e. up to the equations of motion. In that case one needs the BV antibracket formalism [ba81]. Irreducibility of the gauge algebra means that the gauge transformations δ_I are linearly independent.) The BRST algebra thus encodes the structure of the symmetry algebra in a very efficient way.

The role of the BRST symmetry in canonical quantization was eventually clarified by Kugo and Ojima [ku79]: Initially it was assumed that $\bar{c}$ is the complex conjugate of c, but then gauge fixing does not give a real Hamiltonian and thus formally spoils unitarity. Rather, c and $i\bar{c}$ are independent real fields.[1] In the quantum theory the conserved charge Q_{BRST} that corresponds to the BRST symmetry commutes with the Hamiltonian (up to anomalies), so it can be used to define a 'physical' subspace of the Fock space with the physical states defined by the condition $Q|phys\rangle = 0$, which is consistent with time evolution.

If there is no anomaly in the commutation relation $\{Q, Q\} = 2Q^2 = 0$ then all states fall into doublet and singlet representations $(|\psi\rangle, |Q\psi\rangle)$ and $|\psi_{phys}\rangle$ of Q. For the doublets the dual states must also form a doublet since the BRST charge Q, which generates a real symmetry transformation, should be hermitian. Furthermore, BRST-trivial states $Q|\psi\rangle$ have vanishing scalar product with all physical states, which therefore correspond to cohomology classes of Q-invariant states modulo Q-exact states. This is the 'quartet mechanism' by which doublet states cannot contribute to negative norm states in the physical Hilbert space. What remains to be checked for a given theory is that the 'physical Hilbert space' that we end up in this way does not contain any negative norm states.

Expectation values of physical operators, i.e. observables, should not depend on the representative we choose for a physical state. This is guaranteed if $\mathcal{O}$ (anti)commutes with Q, i.e. $[Q, \mathcal{O}_{phys}] = 0$. In turn, physical expectation values of Q-exact operators $\mathcal{O} = [Q, \mathcal{O}']$ vanish,

[1] With this assignment gauge transforms $s\phi^i$ of real fields, lagrange multiplier fields $s\bar{c}$, and gauge fixing terms $s\psi$ with imaginary anticommuting ψ are real. Note that $(XY)^* = (-)^{XY} X^* Y^*$ and $\mathcal{O}^*\phi = (-)^{\mathcal{O}\phi}(\mathcal{O}\phi^*)^*$.

so that physical observables also correspond to cohomology classes. In particular, the sum of gauge dependent and ghost dependent terms of an s-invariant classical action with vanishing ghost number can be shown to be s-exact: $\mathcal{L}(\phi,c,\bar c,b)=\mathcal{L}_{inv.}(\phi)+s\Psi(\phi,c,\bar c,b)$ (ψ is often called *gauge fermion*). This suggests that physical quantities should be independent of the choice of the gauge fixing term $s\psi$, which is known as the Fradkin–Vilkovisky theorem [HE92].

To illustrate the cohomological elimininination of negative norm states we consider the example of a gauge field, which will be reproduced by the BRST cohomology of open strings at the massless level: For covariant gauges, i.e. $\Psi=\frac{\alpha}{2}\bar cb-\bar c\partial A$, the transformations $sA=c$ and $s\bar c=b\sim\partial A$ lead to the transformations $s\bar c^\dagger\sim k^\mu a_\mu^\dagger$ and $sa_\mu^\dagger\sim k_\mu c^\dagger$ for free (asymptotic) photon and ghost creation operators with momentum $k^\mu=(k,0,0,k)$. Lorentz covariant commutation relations $[a_\mu,a_\nu^\dagger]=-\eta_{\mu\nu}$ then lead to negative norm states for timelike polarization. BRST invariance, however, restricts the photon polarization to transversal $a_{1,2}^\dagger$ or longitudinal $a_l^\dagger=a_0^\dagger+a_3^\dagger\sim s\bar c$. The states created by the latter decouple in the physical subspace $\mathcal{H}_{phys}$, as is easily seen in a basis with lightlike and transversal polarizations.

3.1 BRST quantization

Now we are ready to apply the above machinery to the case of the bosonic string. The Polyakov action is invariant under the nilpotent transformation

$$sX^\mu=c^l\partial_lX^\mu,\qquad sg_{mn}=D_mc_n+D_nc_m-2g_{mn}\lambda,\qquad sc^m=c^l\partial_lc^m,\qquad s\lambda=c^l\partial_l\lambda,\qquad(3.3)$$

where c^m are the diffeomorphism ghosts and λ is the Weyl ghost. We want to fix the metric to a background value $\hat g_{mn}$, which we initially keep arbitrary. We will see that the equations of motion for the antighost field b_{mn} imply that b_{++} is a function of σ^+, so that this field naturally has lower indices. It is thus convenient to fix the inverse metric $g^{mn}(\sigma)=\hat g^{mn}(\sigma)$ and we add the gauge fixing and ghost term $\int d^2\sigma\mathcal{L}^{(c)}$ with

$$\tfrac{2}{T}\mathcal{L}^{(c)}=s(\sqrt{-g}\,b_{mn}(g^{mn}-\hat g^{mn}))=\tilde L_{mn}(g^{mn}-\hat g^{mn})+2\sqrt{-g}\,b_{mn}(g^{ml}D_lc^n-g^{mn}\lambda),\qquad(3.4)$$

and $\tilde L_{mn}=\sqrt{-g}\,L_{mn}=s(\sqrt{-g}\,b_{mn})$ to the Polyakov action. Note that the quantum numbers of the anti-ghost field come from the gauge fixing term, whereas those of the ghosts are inherited from the gauge transformation (only the numbers of degrees of freedom must coincide for ghosts and anti-ghosts, but quantum numbers like the spin can be different). The factor $\sqrt{-g}$ is inserted to make b_{mn} a symmetric tensor rather than a tensor density.

Variation with respect to g^{mn}, L_{mn}, λ and b_{mn} implies the equations of motion

$$L_{mn}+T^{(X)}_{mn}+T^{(c)}_{mn}=0,\qquad g^{mn}=\hat g^{mn},\qquad b_{mn}g^{mn}=0,\qquad 2\lambda=D_nc^n,\qquad(3.5)$$

which are algebraic for the fields L_{mn}, g^{mn}, λ and for the trace of the anti-ghost.

$$T^{(c)}_{mn} = (b_{mj}D_nc^j + b_{nj}D_mc^j + D_j(b_{mn}c^j) - g_{mn}g^{ij}b_{jk}D_ic^k) + (g_{mn}g^{ij}b_{ij} - 2b_{mn})\lambda \qquad (3.6)$$

$$= b_{mj}D_nc^j + b_{nj}D_mc^j + D_jb_{mn}c^j - g_{mn}g^{ij}b_{jk}D_ic^k \qquad (3.7)$$

is the ghost contribution to the energy–momentum tensor[2] and $T^{(X)}_{mn}$ is the 'matter' contribution coming from the Polyakov action. $g^{mn}b_{mn} = 0$ implies that $T^{(c)}_{mn}$ is traceless. Furthermore, the total energy–momentum $T_{mn} = T^{(X)}_{mn} + T^{(c)}_{mn}$, corresponding to the action

$$\mathcal{L} = \mathcal{L}_P + T\sqrt{-g}\, b_{mn}(g^{ml}D_lc^n - \tfrac{1}{2}g^{mn}D_lc^l), \qquad (3.8)$$

is proportional to the BRST variation of the traceless anti-ghost b_{mn}. Note that we can eliminate a set of fields whose *own* equations of motion are algebraic by inserting their values back into the action.

In light-cone coordinates we find $T^{(c)}_{+-} = 0$ and $T^{(c)}_{++} = 2b_{++}D_+c^+ + D_+b_{++}c^+$, where we used the equation of motion $\delta S/\delta c^n = g^{ml}D_lb_{mn} = 0$, implying $D_-b_{++}c^- = 0$. The Christoffel symbols drop out of this expression so that

$$T^{(c)}_{++} = 2b_{++}\partial_+c^+ + \partial_+b_{++}c^+, \qquad T^{(c)}_{--} = 2b_{--}\partial_-c^- + \partial_-b_{--}c^-, \qquad (3.9)$$

Since $\hat\Gamma_{++}{}^{+}$ and $\hat\Gamma_{--}{}^{-}$ are the only non-vanishing components of the Christoffel symbol and $\sqrt{-g}\, g^{+-} = 1$, the complete Lagrangian in light-cone coordinates is

$$\mathcal{L} = T\partial_+X^\mu\partial_-X^\nu G_{\mu\nu} + T(b_{++}\partial_-c^+ + b_{--}\partial_+c^-) \qquad (3.10)$$

and the equations of motion imply that b_{++} and c^+ only depend on σ^+.

Like the action for Dirac fermions, our ghost action is only linear in time derivatives so that the Legendre transformation is singular. Now the constraints are, however, not a consequence of gauge symmetries but can be understood by a counting of Cauchy data for Euler-Lagrange and Hamiltonian equations of motion. In this situation the trick that avoids the Dirac procedure is to interpret the action as the result of an inverse Legendre transformation $L(q^i, \dot q^i, p_i) = \dot q^ip_i - H$ where the momenta have not been eliminated. This is often called first order formalism. The variational equations imply

$$\delta\mathcal{L} = \delta q^i\left(-\dot p_i - \frac{\partial H}{\partial q^i}\right) + \delta p_i\left((-)^i\dot q^i - \frac{\partial H}{\partial p_i}\right) = 0, \qquad p_i = \frac{\partial\mathcal{L}}{\partial\dot q^i}, \qquad (3.11)$$

[2] The most tedious part of the computation of $T^{(c)}_{mn} = \frac{1}{T}\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}^{(c)}}{\delta g^{mn}}$ is the variation of the Christoffel symbol contained in the covariant derivative: $\delta(D_lc^n) = \delta\hat\Gamma_{lm}{}^{n}c^m = \frac{1}{2}c^m\delta(g^{nk}(\partial_lg_{mk} + \partial_mg_{lk} - \partial_kg_{ml}))$. Since both sides of this equation are covariant, all terms linear in Christoffel symbols or partial derivatives of the metric must cancel and we immediately obtain $\delta(D_lc^n) = \frac{1}{2}g^{nk}c^m(D_l\delta g_{mk} + D_m\delta g_{lk} - D_k\delta g_{ml})$. Covariant partial integration of the variation of the action then leads to (3.6).

where we have taken into account signs $(-)^i$ due to an odd grading $i = |q^i|$ for fermionic phase space variables q^i, which have to be quantized by anticommutators. Consistency with the Hamiltonian equations of motion $\dot{f} = \{H, f\}_{PB}$ now fixes the signs for the graded Poisson bracket

$$\{p_i, q^j\}_{PB} = \delta_i{}^j, \qquad \{f, g\}_{PB} = (-)^{if} \left((-)^i \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) = -(-)^{fg} \{g, f\}_{PB} \tag{3.12}$$

for arbitrary phase space functions $f(q,p)$ and $g(q,p)$, which also satisfies the graded Jacobi identity. (It is quite common to define complex and hermitian conjugation of fermionic objects by $(fg)^* = g^* f^* = (-)^{fg} f^* g^*$ and $(AB)^\dagger = B^\dagger A^\dagger$, which implies that the product of real fermions is purely imaginary. In our convention fermionic momenta conjugate to real coordinates are therefore imaginary.)

Returning to the quantization of the ghost fields we thus do not introduce conjugate momenta but rather regard the antighost b as (proportional to) the momentum conjugate to c. From (3.8) it thus follows that the imaginary field $-Tb_{++}$ is the conjugate momentum to c^+,

$$i\{b_{++}(\tau+\sigma), c^+(\tau+\sigma')\}_{PB} = -\tfrac{i}{T}\delta(\sigma-\sigma') = i\{b_{--}(\tau-\sigma), c^-(\tau-\sigma')\}_{PB}. \tag{3.13}$$

For the Fourier modes $b_n = (b_{-n})^\dagger = -iT\int d\sigma\, b_{++} e^{in\sigma^+}$ and $c_n = (c_{-n})^\dagger = \frac{1}{2\pi}\int d\sigma\, c^+ e^{in\sigma^+}$, and their right-moving relatives with

$$b_{--} = \frac{i}{2\pi T} \sum_{n=-\infty}^{\infty} \tilde{b}_n e^{-in\sigma^-}, \qquad c^- = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{-in\sigma^-}, \tag{3.14}$$

this implies the Poisson brackets

$$\{b_n, c_m\}_{PB} = i\delta_{m+n} = \{\tilde{b}_n, \tilde{c}_m\}_{PB} \tag{3.15}$$

which have to be replaced by anticommutators for the quantized oscillator modes.

In light-cone coordinates the BRST current $J_S^m = -sX^\mu \frac{\partial \mathcal{L}}{\partial \partial_m X^\mu} - sc^l \frac{\partial \mathcal{L}}{\partial D_m c^l} + c^m \mathcal{L}_P$ reads

$$\begin{aligned} J_S^- &= -2Tc^l \partial_l X^\mu\, \partial_+ X^\nu G_{\mu\nu} - 2Tc^n \partial_n c^+\, b_{++} + 2Tc^-\, \partial_+ X^\mu \partial_- X^\nu G_{\mu\nu} && (3.16)\\ &= -2T(c^+ \partial_+ X^\mu \partial_+ X^\nu G_{\mu\nu} + b_{++} c^+ \partial_+ c^+). && (3.17) \end{aligned}$$

This suggests to define left- and right-moving BRST charges $Q_\pm$ with $Q = -\int d\sigma\, J_S^0 = -\frac{1}{2}(\int d\sigma^+ J_S^- + \int d\sigma^- J_S^+) = Q_+ + Q_-$,

$$Q_+ = T \int d\sigma^+\; c^+(\partial_+ X^\mu \partial_+ X^\nu G_{\mu\nu} + b_{++}\partial_+ c^+) = T \int d\sigma^+\; c^+(T_{++}^{(X)} + \frac{1}{2} T_{++}^{(c)}) \tag{3.18}$$

and its right-moving partner Q_-; note that $(c^+)^2 = 0$ and $c^+ D_+ c^+ = c^+ \partial_+ c^+$.

For a flat target space we can insert the solutions to the equations of motion. For the Fourier modes L_n of $T_{++} = \frac{1}{2\pi T} \sum L_n e^{-in\sigma^+}$ we obtain

$$L_n^{(X)} = -\tfrac{1}{2} \sum_{m=-\infty}^{\infty} :\alpha_{n-m} \cdot \alpha_m:, \qquad L_n^{(c)} = \sum_{m=-\infty}^{\infty} (n+m) : b_{n-m} c_m : \tag{3.19}$$

and for the BRST charge

$$Q_+ = \sum_{n=-\infty}^{\infty} :(L_n^{(X)} + \frac{1}{2} L_n^{(c)}) c_{-n} : -a c_0 \tag{3.20}$$

$$= \sum_{n=-\infty}^{\infty} L_n^{(X)}\, c_{-n} - \frac{1}{2} \sum_{n,m=-\infty}^{\infty} (m-n) : c_{-m} c_{-n} b_{m+n} : -a c_0, \tag{3.21}$$

where we introduced normal ordering symbol :: that puts creation operators (negative index) to the left and a coefficient a parametrizing the ordering ambiguity in Q_+. We find

$$[L_n, b_l] = (n-l) b_{n+l}, \qquad [L_n, c_l] = -(2n+l) c_{n+l} \tag{3.22}$$

with $\{Q_+, b_n\} = L_n := L_n^{(X)} + L_n^{(c)} - a\delta_n$, and $\{Q_+, c_l\} = \sum_n\ (n + l/2) c_{l+n} c_{-n}$.

Next we turn to the construction of a Fock space representation of our operator algebra. Recall the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = -m \delta_{m+n} \eta^{\mu\nu}, \qquad [x^\nu, P^\mu] = -i\eta^{\mu\nu}, \qquad \{b_m, c_n\} = \delta_{m+n}. \tag{3.23}$$

with $p^\mu = \sqrt{4\pi T} \alpha_0^\mu$. We define a vacuum state that is annihilated by all oscillators with positive mode number, i.e. $\alpha_n^\mu|0\rangle = b_n|0\rangle = c_n|0\rangle = 0$ for $n > 0$. The difficult part is the treatment of the zero modes. All states can be constructed as sums of tensor products of a coordinate factor and a ghost factor. For the bosonic part we can, for example, diagonalize the momentum and define eigenstates $P_\mu|k\rangle = k_\mu|k\rangle$, so that $|k\rangle = \ :e^{ikX}: |0\rangle$. In the ghost sector the zero mode algebra is $b_0^2 = c_0^2 = 0$ and $\{b_0, c_0\} = 1$. We cannot diagonalize a nilpotent operator, so we need to introduce a 2-dimensional representation space with

$$\begin{array}{lll} b_0|\uparrow\rangle = |\downarrow\rangle & c_0|\uparrow\rangle = 0 & \langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1 \\ b_0|\downarrow\rangle = 0 & c_0|\downarrow\rangle = |\uparrow\rangle & \langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0 \end{array} \tag{3.24}$$

and $\langle\downarrow|c_0 = \langle\uparrow|$, $\langle\uparrow|b_0 = \langle\downarrow|$, $\langle\uparrow|b_0|\uparrow\rangle = 1 = \langle\downarrow|c_0|\downarrow\rangle$.

3.2 Conformal anomaly and critical dimension

For the consistency of the BRST quantization program we have to check that $Q^2 = 0$. This will fix the constant a in eq. (3.21) and also gives us the critical dimension. First we observe that $Q_+^2=0$ implies that the Virasoro algebra has no anomalous contribution (the anticommutator of Q_+ and Q_- vanishes trivially, so we only need to consider left movers). Indeed, $0=[\{Q_+,Q_+\},b_n]=[Q_+,\{Q_+,b_n\}]-[\{Q_+,b_n\},Q_+]=2[Q_+,L_n]$, hence

$$[L_m,L_n]=[L_m,\{Q_+,b_n\}]=\{Q_+,[L_m,b_n]\}=(m-n)\{Q_+,b_{m+n}\}=(m-n)L_{m+n}. \qquad (3.25)$$

The converse is also true since one can show that

$$Q_+^2=\frac{1}{2}\sum c_{-m}c_{-n}([L_m,L_n]-(m-n)L_{m+n}). \qquad (3.26)$$

This calculation, however, is very tedious, so we postpone it till we have more efficient tools for computing commutators of normal ordered expressions when we come to operator products and contour integrals in the complex plane.

In any case, absence of anomalies in the Virasoro algebra is necessary for $Q^2=0$. Since L_0 is the only mode for which there is an ordering ambiguity it is easy to see that

$$[L_m,L_n]=(m-n)L_{m+n}+A_m\delta_{m+n}. \qquad (3.27)$$

Obviously, $A_{-m}=-A_m$ and $A_0=0$. From the Jacobi identity $\sum_{lmn}[L_l,[L_m,L_n]]=0$ it follows for $l+m+n=0$ that

$$(m-n)A_l+(n-l)A_m+(l-m)A_n=0. \qquad (3.28)$$

For $l=1$ we get $(n-1)A_{n+1}=(n+2)A_n-(2n+1)A_1$ which determines all A_m in terms of A_1 and A_2. Since $A_m=m$ and $A_m=m^3$ solve this equation, we find

$$A_m=\tfrac{A_2-2A_1}{6}\,m^3-\tfrac{A_2-8A_1}{6}\,m. \qquad (3.29)$$

The final step in the calculation of the anomaly is to fix the two remaining constants by evaluating expectation values $A_m=\langle\uparrow|\,L_mL_{-m}-2mL_0\,|\downarrow\rangle$ of (3.27) where $m>0$.

$$\begin{aligned} A_1 &= \langle\uparrow|\,(\alpha_0\cdot\alpha_1)(\alpha_0\cdot\alpha_{-1})-2(-\tfrac{\alpha_0^2}{2}-a)+(b_1c_0+2b_0c_1)(-b_{-1}c_0-2b_0c_{-1})\,|\downarrow\rangle \\ &= \langle\uparrow|\,2a-(2b_0c_1)(b_{-1}c_0)\,|\downarrow\rangle=2a-2, \qquad (3.30)\\ A_2 &= \langle\uparrow|\,(\tfrac{\alpha_1^2}{2}+\alpha_0\cdot\alpha_2)(\tfrac{\alpha_{-1}^2}{2}+\alpha_0\cdot\alpha_{-2})+4(\tfrac{\alpha_0^2}{2}+a)\,|\downarrow\rangle \\ &\qquad -\langle\uparrow|\,(2b_2c_0+3b_1c_1+4b_0c_2)(2b_{-2}c_0+3b_{-1}c_{-1}+4b_0c_{-2})\,|\downarrow\rangle \\ &= \langle\uparrow|\,\tfrac{1}{4}\alpha_1^2\alpha_{-1}^2+4a-(3\cdot3+4\cdot2)\,|\downarrow\rangle=D/2+4a-17. \qquad (3.31)\end{aligned}$$

Putting the pieces together we find

$$A_m=\frac{m^3}{12}(D-26)-\frac{m}{12}(D-2-24a), \qquad (3.32)$$

so that $Q^2 = 0$ implies $a = 1$ and $D = 26$. Note that the term linear in m depends on a. This can be used to eliminate A_1 even if $Q^2 \neq 0$, i.e. to bring the anomaly into the form

$$A_m = \frac{m^3 - m}{12} c, \tag{3.33}$$

where c is called central charge (in our case $a = 1$ is the appropriate value for any D). Then the $SL(2)$ subalgebra of the Virasoro algebra that is generated by L_0 and $L_{\pm 1}$ is free of anomalies[3] (the non-vanishing commutators are $[L_{\pm 1}, L_0] = \pm L_{\pm 1}$ and $[L_1, L_{-1}] = 2L_0$).

In addition to the BRST charge there is another (classically) conserved quantity: We assign ghost number ± 1 to ghosts c^m, λ and antighosts b_{mn}, respectively, and observe that our BRST-invariant classical action has ghost number 0. It is thus invariant under the infinitesimal transformation $\delta c^n = c^n$, $\delta\lambda = \lambda$ and $\delta b_{mn} = -b_{mn}$. This leads to the conserved Noether current $J^m_{gh} = -\delta c^n \frac{\partial \mathcal{L}}{\partial \partial_m c^n} = T\sqrt{-g}\, g^{ml} b_{ln} c^n$, which again simplifies nicely in the light cone gauge: $J^+_{gh} = 2Tb_{--}c^-$. There is thus a left-moving and a right-moving contribution to the ghost number, $\mathcal{N} = -i \int d\sigma J^0_{gh} = -\frac{i}{2} \int d\sigma (J^+_{gh} + J^-_{gh}) = \mathcal{N}_+ + \mathcal{N}_-$,

$$\mathcal{N}_+ = \int \tfrac{d\sigma^+}{2i} J^-_{gh} = \sum_{n=-\infty}^{\infty} : c_n b_{-n} : \; + \text{const.} = \tfrac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n>0} (c_{-n} b_n - b_{-n} c_n) + \tfrac{3}{2}, \tag{3.34}$$

where we include a factor of i in the definition of the charge to make the eigenvalues real. The reason for our asymmetric choice of the constant coming from the operator ordering ambiguity will become clear below. It leads to $\mathcal{N}_+ |\!\uparrow\rangle = 2|\!\uparrow\rangle$ and $\mathcal{N}_+ |\!\downarrow\rangle = |\!\downarrow\rangle$. We will see later that J_{gh} is not conserved in the quantum theory. An anomaly of a global symmetry, however, does not spoil the consistency of a theory; the anomalous violation of ghost number conservation is related to the topology of the world sheet and will play an important role in interactions.

3.3 Physical states

The physical subspace of our Fock space is defined by the cohomology of Q. We first consider states of the form $P(\alpha)|k\rangle \otimes |\!\uparrow\rangle$ or $P(\alpha)|k\rangle \otimes |\!\downarrow\rangle$, where $P(\alpha)$ is a polynomial in the physical creation operators α^μ_{-m}. For such states

$$Q_+ \left(P(\alpha)|k\rangle \otimes |\!\downarrow\rangle \right) = \sum_{n\ge 0} \left(L^{(X)}_n - \delta_{n,0} \right) P(\alpha)|k\rangle \otimes c_{-n}|\!\downarrow\rangle, \tag{3.35}$$

$$Q_+ \left(P(\alpha)|k\rangle \otimes |\!\uparrow\rangle \right) = \sum_{n>0} L^{(X)}_n P(\alpha)|k\rangle \otimes c_{-n}|\!\uparrow\rangle. \tag{3.36}$$

[3] This is a special case of the following result: An anomalous term like the one in eq. (3.27) is called a central extension if it is consistent with the Jacobi identity. It is easy to show that semi-simple (finite-dimensional) Lie algebras only admit trivial central extensions, i.e. the 'central' terms in the algebra can be eliminated by adding constants to the generators.

This looks similar to Gupta–Bleuler in QED, where the annihilation part of the gauge condition is imposed as a constraint on physical states: In the present context we need to make sure that physical expectation values of T_{++} vanish. Since the states built on $|\downarrow\rangle$ are dual to the states built on $|\uparrow\rangle$ the above formulas imply that all expectation values of $:L_n^{(X)}: -\delta_{n,0}$ between physical states that do not contain ghost or antighost creation operators vanish. (For the complete $L_n = \{Q, b_n\}$ obviously all physical expectation values vanish). Hence our formalism reduces to the 'old covariant approach' [GR87] in the ghost-free sector.

Since the mass shell operator $L_0 = \{Q, b_0\}$, the momentum operator P^μ and Q all commute with one another, we can compute the cohomology for Q for eigenspaces with fixed eigenvalues. If $(L_0 - \lambda)|\Phi\rangle = 0$ with $\lambda \neq 0$ for some Q-invariant state $|\Phi\rangle$ then $|\Phi\rangle = Q(\frac{1}{\lambda}b_0|\Phi\rangle)$ is Q-exact, so that non-trivial physical states must be on-shell states. In this way we recover the mass shell condition $L_0 = 0$ also for the states of the form (3.36) that are built on $|\uparrow\rangle$. We will see that representatives of all propagating physical states can be chosen to be of the form (3.35) or of the form (3.36) [PO98]. There is a one-to-one correspondence of these states, which can be obtained from one another by application of b_0 or c_0. Since the physical states that are built on $|\downarrow\rangle$ are automatically on-shell, whereas those on $|\uparrow\rangle$ are off-shell null states or on-shell limits of null states, the states $P(\alpha)|\downarrow\rangle$ seem to be more natural. We may therefore choose the 'Siegel gauge' $b_0|\Phi\rangle = 0$ in addition to the physical state condition $Q|\Phi\rangle = 0$. For these states we can define a modified scalar product $\langle\varphi||\varphi'\rangle := \langle\varphi|c_0|\varphi'\rangle$. We could also modify the physical state condition by considering the so-called (semi)relative cohomology. These concepts play an important role in string field theory [zw93, be94, ne89, di91b, ga97].

We first consider the tachyon and the massless states in more detail. Since $[L_m^{(X)}, L_1^{(X)}] = (m-1)L_{m+1}^{(X)}$ for $m > 0$ it is sufficient to impose

$$L_0^{(X)} = -(\frac{1}{2}\alpha_0^2 + \alpha_{-1}\cdot\alpha_1 + \alpha_{-2}\cdot\alpha_2 + \ldots) = 1, \tag{3.37}$$

$$L_1^{(X)} = -(\alpha_0\cdot\alpha_1 + \alpha_{-1}\cdot\alpha_2 + \alpha_{-2}\cdot\alpha_3 + \ldots) = 0, \tag{3.38}$$

$$L_2^{(X)} = -(\frac{1}{2}\alpha_1^2 + \alpha_0\cdot\alpha_2 + \alpha_{-1}\cdot\alpha_3 + \alpha_{-2}\cdot\alpha_4 + \ldots) = 0 \tag{3.39}$$

on $P(\alpha)|k\rangle$. For $P(\alpha) = 1$ we obtain $\hat{k}^2 = k^2/(4\pi T) = -2$ with $\alpha_0^\mu|k\rangle = \hat{k}^\mu|k\rangle$, i.e. we find a scalar, tachyonic state in the string spectrum. On the next level $P(\alpha) = t_\mu\alpha_{-1}^\mu$ the mass shell condition is $k^2 = 0$ and $L_1^{(X)} = 0$ implies transversality $t_\mu k^\mu = 0$ of the polarization vector t_μ. The norm of this state is proportional to t^2, i.e. it vanishes for a longitudinal polarization $t_\mu \sim k_\mu$. We expect that such a state is Q-exact, and indeed,

$$Q(b_{-1}|\downarrow\rangle) = L_{-1}^{(X)}|\downarrow\rangle + L_{-1}^{(c)}|\downarrow\rangle - b_{-1}Q|\downarrow\rangle \tag{3.40}$$

$$= -\hat{k}\cdot\alpha_{-1}|\downarrow\rangle - b_{-1}c_0|\downarrow\rangle - b_{-1}c_0(:L_0: -1)|\downarrow\rangle \tag{3.41}$$

$$= -\hat{k}\cdot\alpha_{-1}|\downarrow\rangle + \tfrac{1}{2}\hat{k}^2 b_{-1}c_0|\downarrow\rangle. \tag{3.42}$$

In the case of open strings this is the whole story: We have a massless vector excitation in the target space, whose polarization must be transversal since $Q\alpha^\mu_{-1}|\downarrow\rangle \sim \hat{k}^\mu c_{-1}|\downarrow\rangle$ should vanish, and there is a null (i.e. zero norm) polarization, which is Q-exact. For closed strings we have to include the right-movers and therefore have a polarization tensor $t_{\mu\nu}$ which is transversal in both indices. Now the gauge invariance corresponds to $\delta t_{\mu\nu} = k_\mu v^{(R)}_\nu + k_\nu v^{(L)}_\mu$. The physical interpretation requires the decomposition of the polarization tensor into irreducible representations of the Lorentz group: $t_{\mu\nu}$ has a traceless symmetric part, the graviton, an antisymmetric tensor field $B_{\mu\nu} = t_{\mu\nu} - t_{\nu\mu}$, and a scalar degree of freedom due to the trace, which is called dilaton. The antisymmetric part of the gauge invariance implies that only the field strength $H_{\mu\nu\rho} = \sum \partial_\mu B_{\nu\rho}$ enters physical quantities.

Summarizing our observations, we recover the essential ingredients of QED and of linearized gravity. In addition, we have an infinite tower of gauge symmetries at higher levels which controll the interplay of the infinite set of massive string modes at and above the Planck mass, which, in our picture of string unification, is proportional to $\sqrt{T}$. In order to obtain interaction terms for gravitons and the other target-space fields we need to compute string interactions.

So far it seems that the 'old covariant approach' to string quantization is sufficient. Ghosts will become important in case of interactions. Before coming to this subject, however, we first continue to the Euclidean domain and set up the machinery of CFT. Since the $SL(2)$ subalgebra of the Virasoro algebra has no anomaly we could require $L_n = 0$ on the vacuum state for $n \geq -1$. Accordingly, it will turn out to be useful to work with the $SL(2)$ invariant ghost vacuum $|0\rangle_{gh}$ which is defined by

$$b_n|0\rangle_{gh} = 0 \quad n \geq -1, \qquad\qquad c_n|0\rangle_{gh} = 0 \quad n \geq 2. \tag{3.43}$$

It is related to our previous vacuum by $|\downarrow\rangle = c_1|0\rangle_{gh}$ and $|0\rangle_{gh} = b_{-1}|\downarrow\rangle$. The importance of this vacuum will become clear in the context of CFT on the complex plain.

That $D \leq 26$ is a necessary condition for consistent string quantization can be seen easily by computing the norm of a physical scalar state at the second mass level: We make the ansatz

$$|\phi\rangle = (\alpha_{-1}\cdot\alpha_{-1} + A(\alpha_0\cdot\alpha_{-1})^2 + B\alpha_0\cdot\alpha_{-2})|p\rangle \tag{3.44}$$

Since $L_{n+1} = \frac{1}{n-1}[L_n, L_1]$ it is sufficient to impose $L_0 = L_1 = L_2 = 0$ with $L_0 = L_0^{(X)} - 1$. Straightforward evaluation of the commutators gives $L_0|\phi\rangle = (-\frac{1}{2}\alpha_0^2 + 2 - 1)\alpha_{-1}^2|\phi\rangle$, $L_1|\phi\rangle = 2(1 + A\alpha_0^2 + B)\alpha_0\cdot\alpha_{-1})|p\rangle$ and $L_2|\phi\rangle = (-D - (A - 2B)\alpha_0^2)|p\rangle$, so that we find

$$\frac{\langle\phi|\phi\rangle}{\langle p|p\rangle} = 2D + 4\alpha_0^2 A + 2\alpha_0^4 A^2 - 2\alpha_0^2 B^2 = \tfrac{2}{25}(D-1)(26-D), \qquad B = \tfrac{D-1}{5},\ A = -\tfrac{D+4}{10} \tag{3.45}$$

More generally it can be shown that there are no negative norm states if $a = 1$ and $D = 26$ or if $a \leq 1$ and $D \leq 25$ (see [GR87]). A covariant quantization of string theory below the critical dimension has first been persued successfully by Polyakov [po81]. He found that the conformal anomaly makes the conformal mode ϕ of the metric $g_{mn} = e^{\phi}\eta_{mn}$ dynamical, with an effective Lagrangian (Wess-Zumino term)

$$\tfrac{26-D}{48\pi}\ \left(\tfrac{1}{2}(\partial\phi)^2 + \mu^2 e^{\phi}\right) \tag{3.46}$$

which is positive if $D < 26$. This action has been known for a long time under the name Liouville action and ϕ is thus called Liouville field.

3.4 Strings in background fields

To recover the full content of gravity it seems that we have to study graviton scattering order by order in perturbation theory, which would require the calculation of correlation functions with an arbitrary number of graviton vertex operator insertions. There is, however, an alternative approach that directly gives us the Einstein equations in curved space [fr85, ca85₁]. Recall that we can consider the string with an arbitrary target space metric $G_{\mu\nu}(X)$. We computed the equations of motion of the coordinate fields in such a background. It is not so obvious, however, how the dynamics of the background metric arises. As it turns out, this dynamics is fixed by the absence of conformal anomalies.

Here we should be more general: In principle, all massless fields in our theory can form condensates. Hence there should be a consistent movement of strings in curved target spaces with additional backgrounds that correspond to the dilaton and antisymmetric tensor fields. Indeed, if we write down the most general renormalizable action for the coordinate fields we find $S = S_P + S_B + S_\phi + S_\tau$ with

$$\mathcal{L} = \mathcal{L}_P - \tfrac{1}{4\pi\alpha'}\varepsilon^{mn}\partial_m X^\mu \partial_n X^\nu B_{\mu\nu}(X) + \tfrac{1}{4\pi}\sqrt{-g}\,\phi(X)R^{(2)} + \sqrt{-g}\,\tau(X), \tag{3.47}$$

where $R^{(2)}$ is the curvature scalar on the world sheet and the string tension is expressed in terms of the (closed string) Regge slope $\alpha' = 1/(2\pi T)$. The low energy limit of string theory, with the mass scale $\sqrt{T}$ going to ∞, then corresponds to $\alpha' \to 0$. The background field $\phi(X)$ describes the dilaton. Since the Einstein term is a total derivative in 2 dimensions a constant ϕ respects conformal invariance. A non-constant ϕ can, however, contribute to the cancellation of conformal anomalies at the quantum level. Similarly $\tau(X)$, which can be related to the tachyon, is needed at the quantum level as a counterterm for quadratic divergences.

The classical equations of motion for the coordinate fields are

$$\frac{G^{\alpha\rho}}{\sqrt{-g}\,T}\frac{\delta S}{\delta X^\rho} = \Delta X^\alpha + \partial_m X^\mu \partial_n X^\nu \left(g^{mn}\hat{\Gamma}_{\mu\nu}{}^{\alpha} - \tfrac{1}{2}\tfrac{\varepsilon^{mn}}{\sqrt{-g}} H_{\mu\nu}{}^{\alpha}\right) + O(\alpha') \tag{3.48}$$

where $H_{\mu\nu\lambda} = \sum_{\mu\nu\lambda}\partial_\mu B_{\nu\lambda}$, i.e. $H = dB$ with $B = \frac{1}{2}dx^\mu dx^\nu B_{\mu\nu}$ and $H = \frac{1}{3!}dx^\mu dx^\nu dx^\lambda H_{\mu\nu\lambda}$. In the case of a non-vanishing B field (3.47) is called a σ-model with torsion because $\pm\frac{1}{2}H_{\mu\nu}{}^{\rho}$ acts as a totally antisymmetric torsion that contributes to the connection for left and right moving modes with opposite signs (this can be seen using light-cone coordinates). The contributions of the last two terms in the action to (3.48) is of non-leading order $O(\alpha')$.

It can be shown that the resulting quantum theory is conformally invariant to leading order in α' iff the following 'β-functionals' for the coupling functions G, B and ϕ vanish [fr85, ca85₁],

$$0 = R_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda}{}^{\rho}H_{\nu\rho}{}^{\lambda} - 2D_\mu D_\nu\phi \tag{3.49}$$

$$0 = D_\lambda H_{\mu\nu}{}^{\lambda} - 2H_{\mu\nu}{}^{\lambda}D_\lambda\phi \tag{3.50}$$

$$0 = 4D_\mu\phi D^\mu\phi - 4D_\mu D^\mu\phi + R + \tfrac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} \tag{3.51}$$

which can be interpreted as equations of motion for the metric, the (field strength of) the antisymmetric tensor field, and the dilaton, respectively. Actually these equations are the Euler Lagrange equations for the (effective) action

$$S_{26} = \tfrac{1}{\kappa_0^2}\int d^{26}X\sqrt{-G}e^{-2\phi}(R - 4D_\mu\phi D^\mu\phi + \tfrac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}). \tag{3.52}$$

In 2 dimensions $\sqrt{-g}\,R$ is a total derivative whose integral is a topological invariant: The Gauß–Bonnet theorem states that

$$\tfrac{1}{4\pi}\int\sqrt{g}R^{(2)} = \chi \tag{3.53}$$

is equal to the Euler number of the manifold. Therefore the vacuum expectation value of a constant dilaton field ϕ determines the strength of the string coupling (i.e. the relative weights of loop diagrams with different genera). The factor $e^{-2\phi}$ in the effective action corresponds to the topology of the sphere, which is tree level diagram for closed strings. Note that the target space metric $G_{\mu\nu}$ (string frame) is related to the physical metric (Einstein frame) by a Weyl transformation with an appropriate power of e^ϕ.

For open strings the relevant background field is a gauge field 1-form in target space and the σ-model interaction term is the integral of its pull-back over the boundary of the world sheet

$$\int_{\partial\Sigma} A = \int_{\partial\Sigma} ds\ \partial_s X^\mu A_\mu(X(s)). \tag{3.54}$$

Since an interacting open string theory always produces closed strings as bound states we should add this term to the σ-model action. Note that the torsion term in (3.47) can be written as an integral $\frac{1}{2\pi\alpha'}\int_\Sigma B$ of the pull-back of the 2-form B-field to the world sheet. In the presence of boundaries the gauge transformation $B \to B + d\Lambda$ of the B field induces a surface term which can be compensated by an accompanying gauge transformation $A \to A - \frac{1}{2\pi\alpha'}\Lambda$, so that the physical field strengths that are invariant under 0-form λ and 1-form Λ gauge transformations are $H = dB$ and $\mathcal{F} = B + 2\pi\alpha' F$, with $F = dA$. Indeed, quantum conformal invariance imposes (for constant dilaton) the generalized Maxwell equation

$$\partial_\mu(\sqrt{\det(G+\mathcal{F})}\ \theta^{\mu\nu}) = 0 = \overline{G}^{\rho\mu}D_\rho\mathcal{F}_{\mu\nu} - \tfrac{1}{2}\theta^{\rho\sigma}H_{\rho\sigma}{}^{\mu}\mathcal{F}_{\mu\nu} \qquad \text{with}\ \ \overline{G} + \theta = (G+\mathcal{F})^{-1}, \tag{3.55}$$

which is the variational equation of the Born–Infeld action

$$S_p = -T_p \int d^{p+1}X \,\mathrm{tr}\, e^{-\phi} \sqrt{G_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}} \tag{3.56}$$

with respect to the gauge field A. The dilaton dependence $e^{-\phi}$ is due to Euler number 1 of the disk. The symmetric part $\overline{G} = G^{-1}(1 - G^{-1}\mathcal{F})^{-1}$ of $(G + \mathcal{F})^{-1}$ is called open string metric and its antisymmetric part $\theta = -G^{-1}\mathcal{F}\overline{G}$ becomes a Poisson structure in a topolgical limit $\mathcal{F} \gg G$ that is related to noncommutative geometry [se99]. The relation between the so-called brane tension T_p to the string tension can be fixed by T-duality and loop calculations as explained in [PO98]. Whatever its value, T_p plays the role of a cosmological constant, which is another sign for the inconsistency of bosonic strings and the need for a compansating contribution that will come from supersymmetry. It will turn out that the only consistent solution in 10 dimensions is unoriented open strings with gauge group $SO(32)$ [PO98].

3.5 BRST cohomology and no ghost theorem

We now compute the complete cohomology and show that the norm is positive definite on the space of physical states. The name “no ghost theorem” traditionally refers to the absence of negative norm states (in the old covariant approach) and is not related to Faddeev–Popov ghosts. In the context of BRST quantization, one result of the analysis is that all representatives of propagating physical states can be chosen to contain no ghost or anti-ghost creation operators. Since $\{Q, b_n^\dagger\} = L_n^\dagger$, we expect that all $b_n^\dagger$ and $L_n^\dagger$ creation operators drop out of the cohomology if the $L_n^\dagger$ can be used as part of a basis for the algebra of creation operators. But then ghost creation operators $c_n^\dagger$ also must drop out since they can only contribute to zero-norm states.

We will see that this “linearization” of $L_n^\dagger$ is only possible for non-vanishing momentum $P^\mu \neq 0$. We thus first consider this case and use (target space) light cone coordinates in a reference frame where the eigenvalue $\hat{k}^+$ of α_0^+ is non-zero. In principle cohomology calculations are just linear algebra: We can choose a vector-space basis where Q has a triangluar form $Qe_n = f_n$ and $Qg_m = 0$. Here $\{f_n\}$ form a basis of $\mathrm{Im}\, Q$ and the $\{g_m\}$ have to be added to the set $\{f_n\}$ to get a basis of the kernel of Q. The subspace generated by $\{g_m\}$ is ambiguous because linear combinations of $\{f_n\}$ can be added, and similarly an arbitrary element of the kernel of Q can be added to e_n. We can then define a homotopy operator by $Kf_n = e_n$ and $Kg_n = 0$ with $K^2 = 0$. Obviously $N = \{K, Q\}$ is one on $\{e_n, f_n\}$ and vanishes on $\{g_m\}$ and the kernel of N is a possible choice for representatives of the cohomology classes of Q. In our case the Fock space is infinite dimensional, but we can use the algebra of the creation operators to organize the calculation. The action of Q on the operators is non-linear, but it is a standard tool of homological algebra [BO82, GE99] to use a suitable grading of the BRST complex that linearizes the calculation: We start with a (filtration) degree $N = N_{gh} + N_{lc}$ with

$$N_{lc} = \sum_{m\neq 0} \frac{1}{2m} \alpha^+_m \alpha^-_{-m}, \qquad \alpha^\pm_m = \alpha^0_m \pm \alpha^{D-1}_m, \quad [\alpha^\pm_m, \alpha^\mp_n] = 2n\delta_{m+n}, \tag{3.57}$$

which counts the number of positive (right) minus the number of negative (left) light-cone oscillators

$$[N_{lc}, (\alpha^\pm_n)^\dagger] = \pm(\alpha^\pm_n)^\dagger, \qquad n \neq 0. \tag{3.58}$$

Q then splits into a sum of terms Q_j with nonnegative degrees j, where $[N, Q_j] = jQ_j$,

$$Q = Q_0 + Q_1 + Q_2, \qquad Q_0 = -\hat k^+ \sum_{m\neq 0} \alpha^-_{-m} c_m, \quad Q_0^2 = \{Q_0, Q_1\} = \ldots = Q_2^2 = 0. \tag{3.59}$$

The next step is to compute the cohomology of Q_0, which acts as

$$\{Q_0, b_n\} = -\hat k^+ \alpha^-_n, \qquad [Q_0, \alpha^+_n] = -2n\hat k^+ c_n. \tag{3.60}$$

The appropriate homotopy operator is $K = \frac{1}{\hat k^+} \sum_{m\neq 0} \alpha^+_{-m} b_m$ with $[K, \alpha^-_n] = \frac{1}{\hat k^+} 2nb_n$ and $\{K, c_n\} = \frac{1}{\hat k^+}\alpha^+_n$. Then

$$E_0 = \{Q_0, K\} = \sum_{m>0} 2m(b^\dagger_m c_m + c^\dagger_m b_m) - \alpha^+_{-m}\alpha^-_m - \alpha^-_{-m}\alpha^+_m \tag{3.61}$$

is the Euler operator counting (with multiplicities $2n$) the numbers of creation operators $b^\dagger_n, c^\dagger_n$ and $\alpha^\pm_{-n}$. The normal ordering constant in E_0 vanishes because Q_0 and K vanish on the ground states. Moreover, $Q_0^2 = K^2 = 0$ implies

$$[E_0, Q_0] = [\{Q_0, K\}, Q_0] = [Q_0, \{K, Q_0\}] = [Q_0, E_0] = 0, \qquad [E_0, K] = 0. \tag{3.62}$$

E_0 has eigenvalue $2n$ on the creation operators $b^\dagger_n, c^\dagger_n, (\alpha^\pm_n)^\dagger$ and 0 for transversal polarizations. We may thus diagonalize E_0 and the cohomology is given by its kernel, i.e. the (linearized) physical states $|\psi_0\rangle$ can be chosen to satisfy $Q_0|\psi_0\rangle = K|\psi_0\rangle = E_0|\psi_0\rangle = 0$.

The cohomology of Q is now easily seen to be isomorphic to the cohomology of Q_0 because each $|\psi_0\rangle$ is in one-to-one correspondece to a state $|\psi\rangle$ in the kernel of Q and of K that satisfies

$$\{Q, K\}|\psi\rangle = (E_0 + E')|\psi\rangle = 0, \qquad |\psi\rangle = (1 - \frac{1}{E_0}E' + \frac{1}{E_0}E'\frac{1}{E_0}E' - \ldots)|\psi_0\rangle \tag{3.63}$$

and the degree 0 part of $|\psi\rangle$ is $|\psi_0\rangle$. The expression for $|\psi\rangle$ exists (at least as a formal series) because E_0 is positive and invertible on the image of E'. Moreover, $\langle\psi||\psi\rangle = \langle\psi_0||\psi_0\rangle$ because all the correction terms produce an excess of c_n and α^+_n operators and thus do not contribute to the scalar product. This establishes positivity and the no-ghost theorem.

For $P^\mu = 0$ there are only $2D+4$ on-shell states, which are the singlets $b_{-1}|\downarrow\rangle$, $\alpha^\mu_{-1}|\downarrow\rangle$ and their dual states $c_{-1}|\uparrow\rangle$, $\alpha^\mu_{-1}|\uparrow\rangle$, and the self-dual doublet $b_{-1}|\uparrow\rangle$ and $Q(b_{-1}|\uparrow\rangle) = -2c_{-1}|\downarrow\rangle$. The '$SL(2,\mathbb{C})$ invariant vacuum' $|0\rangle = b_{-1}|\downarrow\rangle$ and its dual are the only Lorentz-invariant physical states. It will play an important role in conformal field theory.

Chapter 4

Superstrings

So far we formulated a theory with very attractive properties: It describes the interaction of gravitons and, for open strings or after compactification to four dimensions, gauge bosons. The trouble is that it is inconsistent: The presence of the tachyon in the spectrum makes the ground state unstable and, even worse, it leads to a divergence of the integral over the modular parameter at genus 1. A second problem of the bosonic string is that it only describes space-time bosons. So it is natural to introduce additional fermionic fields. We expect that their presence makes the theory less divergent and hopefully consistent. But should it be world-sheet or space-time fermions? It turns out that both approaches are viable and lead to equivalent results, but in a very non-trivial way.

Anticipating that the cancellation of bosonic and fermionic contribution to harmful divergences should be controlled by a symmetry we may introduce spacetime fermions $\theta^A_{\underline{\alpha}}$ with $A = 1, 2$, which are 16-component Weyl spinors of equal/opposite chirality for type IIB/IIA superstrings, respectively. The two space-time supersymmetries with constant spinor parameters ε^A transform bosons into fermions [GR87]

$$\delta_\varepsilon X^\mu = i\bar\varepsilon^A\gamma^\mu\theta^A, \qquad \delta_\varepsilon\theta^A = \varepsilon^A. \tag{4.1}$$

The action can then be written as

$$S = -\int \frac{d^2\sigma}{4\pi\alpha'}\sqrt{-g}\, g^{mn}\Pi_m\Pi_n + S_{WZ}, \qquad \Pi^\mu_m = \partial_m X^\mu - i\bar\theta^A\gamma^\mu\partial_m\theta^A \tag{4.2}$$

where Π^μ_m is a fermionic extension of $\partial_m X^\mu$ that is supersymmetric, i.e. $\delta_\varepsilon\Pi^\mu_m = 0$. This is known as the Green–Schwarz (GS) superstring. A counting shows that there are too many fermionic degrees of freedom, which have to be eliminated by a fermionic gauge invariance called κ symmetry. The topological (i.e. metric independent) Wess-Zumino term S_{WZ} is necessary to create such a symmetry. Unfortunately, κ symmetry is a reducible symmetry of infinite stage, whose covariant gauge fixing leads to an infinite tower of ghosts for ghosts. This prohibited a covariant quantization of the GS superstring for many years and this problem of was only solved quite recently. We will therefore focus on the RNS formulation. Here the additional fields $\psi^\mu(\sigma,\tau)$ are spinors on the world sheet and transform as a vector in target space. Two such models, the Ramond model [ra71] and the Neveu–Schwarz model [ne71], were constructed already in 1971 in an attempt to include baryons into the dual model of meson interactions.

4.1 The RNS model

For simplicity we first write down the action in (super)conformal gauge and for flat target space,

$$S=-\int\frac{d^2\sigma}{4\pi\alpha'}(\eta^{mn}\partial_mX^\mu\partial_nX^\nu+i\overline{\psi}^\mu\partial\!\!\!/\psi^\nu)\eta_{\mu\nu} \tag{4.3}$$

Here γ^a are 2×2 matrices satisfying $\gamma^a\gamma^b=\eta^{ab}+\varepsilon^{ab}\gamma_*$ with $\varepsilon^{01}=1=-\varepsilon^{10}$, which we chose as

$$\gamma^0=\begin{pmatrix}0&i\\-i&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&i\\i&0\end{pmatrix},\qquad \gamma_*=\gamma^0\gamma^1=\begin{pmatrix}-1&0\\0&1\end{pmatrix} \tag{4.4}$$

in a Majorana-Weyl representation, and $\psi=\left(\begin{smallmatrix}\psi_+\\\psi_-\end{smallmatrix}\right)$ are real two-component spinors (note that the Minkowski signature is essential for the existence of a Majorana-Weyl representation in two dimensions). In light-cone coordinates $\gamma^0\gamma^\pm=\gamma^0(\gamma^0\pm\gamma^1)=1\pm\gamma_*$ and the kinetic term for the fermions becomes

$$\overline{\psi}\partial\!\!\!/\psi=\psi^T\gamma^0(\gamma^+\partial_++\gamma^-\partial_-)\psi=2\psi_+\partial_-\psi_++2\psi_-\partial_+\psi_- \tag{4.5}$$

so that left/right handed fermions become left/right movers by their equations of motion and

$$S=-\int\frac{d^2\sigma}{2\pi\alpha'}\left(\partial_+X\partial_-X+i\psi_+\partial_-\psi_++i\psi_-\partial_+\psi_-\right). \tag{4.6}$$

The action is invariant under supersymmetry transformations

$$\delta X^\mu=\varepsilon^\alpha\psi^\mu_\alpha,\qquad \delta\psi^\mu_\alpha=\frac{i}{2}\varepsilon^\beta(\gamma^mC)_{\beta\alpha}\partial_mX^\mu. \tag{4.7}$$

where the charge conjugation matrix $C_{\alpha\beta}$, which shifts indices in spinor space, can be chosen to be numerically identical to γ^0 in a Majorana representation. In light cone coordinates this becomes

$$\delta X^\mu=\varepsilon^+\psi^\mu_++\varepsilon^-\psi^\mu_-,\qquad \delta\psi^\mu_\pm=i\varepsilon^\pm\partial_\pm X^\mu. \tag{4.8}$$

In fact, the action is invariant not only under global supersymmetry with constant $\varepsilon^\pm$ but also under arbitrary superconformal transformations $\varepsilon^\pm(\sigma^\pm)$ since

$$\begin{aligned}\delta\left(\partial_+X\partial_-X+i\psi_+\partial_-\psi_++i\psi_-\partial_+\psi_-\right) &= \partial_+(\varepsilon^+\psi_+\partial_-X)+\partial_-(\varepsilon^-\psi_-\partial_+X)\\ &\quad+2\partial_-\varepsilon^+\psi_+\partial_+X+2\partial_+\varepsilon^-\psi_-\partial_-X.\end{aligned} \tag{4.9}$$

The Noether currents corresponding to conformal and superconformal transformations are

$$T_{\pm\pm} = -\partial_\pm X \partial_\pm X + i\psi_\pm \partial_\pm \psi_\pm, \qquad T_{F\pm} = -2\psi_\pm \partial_\pm X \tag{4.10}$$

in string conventions, which differ by a factor $\frac{1}{T} = 2\pi\alpha'$ from the usual normalization of currents; the supercurrent T_{F+} is the coefficient of $\partial_-\varepsilon^+$ in eq. (4.9). The canonical anticommutation relations are

$$\{\psi_+^\mu(\tau,\sigma), \psi_+^\nu(\tau,\sigma')\} = \pi\alpha'\,\delta(\sigma-\sigma')\eta^{\mu\nu} = \{\psi_-^\mu(\tau,\sigma), \psi_-^\nu(\tau,\sigma')\} \tag{4.11}$$

as is easily inferred by comparison with the ghost action.[1]

An essential new feature of this model is that we have some freedom in choosing the boundary conditions of fermions: When we go once around a closed string, only observable quantities have to be periodic. Such quantities, however, always contain an even number of spinors so that the sign of ψ^μ drops out. Therefore spinors may obey periodic or anti-periodic boundary conditions

$$\psi(\sigma + 2\pi) = \pm\psi(\sigma) =: e^{2\pi i\phi}\psi(\sigma) \tag{4.12}$$

with $\phi = 0$ for the Ramond (R) sector and $\phi = 1/2$ for the NS sector. For closed strings left and right movers do not couple and we may introduce four sectors with different boundary conditions, namely (R,R), (R,NS), (NS,R) and (NS,NS).

The mode expansion of the general solution of the equations of motion thus becomes

$$\begin{aligned} \psi_+^\mu(\sigma,\tau) &= \sqrt{\tfrac{\alpha'}{2}} \sum_{r\in\mathbb{Z}+\phi} \psi_r^\mu e^{-ir\sigma^+}, \\ \psi_-^\mu(\sigma,\tau) &= \sqrt{\tfrac{\alpha'}{2}} \sum_{r\in\mathbb{Z}+\phi} \tilde\psi_r^\mu e^{-ir\sigma^-}, \end{aligned} \qquad \begin{cases} \phi = 0 & \text{(R)} \\ \phi = \frac{1}{2} & \text{(NS)} \end{cases} \tag{4.13}$$

and for the anti-commutation relations of the quantized oscillators we find

$$\{\psi_r^\mu, \psi_s^\nu\} \;=\; \eta^{\mu\nu}\delta_{r+s,0} \;=\; \{\tilde\psi_r^\mu, \tilde\psi_s^\nu\}. \tag{4.14}$$

All oscillators transform as target-space vectors, so it seems that we cannot get the space-time fermions that were our original motivation. In the R sector, however, there are zero modes which form a representation of the Clifford algebra:

$$\{\psi_0^\mu, \psi_0^\nu\} \;=\; \eta^{\mu\nu} \;=\; \{\tilde\psi_0^\mu, \tilde\psi_0^\nu\} \tag{4.15}$$

Therefore the Ramond vacuum is degenerate and all states created from it by the action of $(\alpha_n^\mu)^\dagger$ and $(\psi_r^\mu)^\dagger$ transform in target-space spinor representations. A tensor product of two

[1] A factor $\frac{1}{2}$ comes from the fact that the conjugate momentum of ψ_+ is equal to ψ_+ itself, and not to a different field, which can only happen for fermions. This factor can be derived by comparison with the case of a complex fermion $\psi = \psi_1 + i\psi_2$, for which $\int \psi^*\dot\psi = \int \psi_1\dot\psi_1 + \int \psi_2\dot\psi_2$.

spinor representations only contains tensor representations of target space rotations. For closed strings space-time fermions therefore must arise from the sectors (R,NS) and (NS,R) with mixed boundary conditions. In any case, if we want to have space-time bosons and space-time fermions then we need to consider a combination of the Ramond model and of the Neveu–Schwarz model.

The energy momentum tensor $T_{\pm\pm}$ and its superpartner $T_{F\pm}$ are conserved quantities whose Fourier modes (with $m \in \mathbb{Z}$ and $r \in \mathbb{Z}+\phi$)

$$T_{\pm\pm} = \alpha' \sum L_n e^{-in\sigma^+}, \qquad L_n^{X,\psi} = -\tfrac{1}{2}\sum :\alpha_m\alpha_{n-m}: -\tfrac{1}{2}\sum(\tfrac{n}{2}-r) :\psi_r\psi_{n-r}: -a, \tag{4.16}$$

$$T_{F\pm} = \alpha' \sum G_r e^{-ir\sigma^+}, \qquad G_r^{X,\psi} = \sum \alpha_m \psi_{r-m}, \tag{4.17}$$

obey the super-Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{4.18}$$

$$[L_m, G_r] = (\tfrac{m}{2}-r)G_{m+r}, \qquad \{G_r, G_s\} = 2L_{r+s} + \tfrac{c}{3}(r^2-\tfrac{1}{4})\delta_{r+s,0}. \tag{4.19}$$

Its form is strongly constrained by the Jacobi identities: If we start with an ansatz $\{G_r, G_s\} = 2L_{r+s}+c_r$ the commutator $[L_m, G_r]$ is fixed up to a normalization of the generators and $c_r = c_{-r}$ can be evaluated in terms of the central extension of the Virasoro algebra.[2]

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[2] For a dynamical system with first class constraints $\mathcal{G}_a$, ghosts η^a and conjugate ghost momenta $\mathcal{P}_a$ the BRSTcharge Ω is

$$\Omega = \eta^a\mathcal{G}_a - \tfrac{(-)^b}{2}\eta^b\eta^a f_{ab}{}^c\mathcal{P}_c, \qquad \{\mathcal{G}_a, \mathcal{G}_b\} = f_{ab}{}^c\mathcal{G}_c, \quad \{\mathcal{P}_a, \eta^b\} = -\delta_a{}^b. \tag{4.20}$$

For a closed algebra, i.e. for field independent structure constants $f_{ab}{}^c$ satisfying the graded Jacobi identity, it is a straightforward exercise to show that $\{\Omega,\Omega\} = 0$. For open algebras, i.e. field-dependent structure funcions $f_{ab}{}^c$, $\Omega^2 = 0$ requires additional terms of higer order in $\mathcal{P}$ and η, whose coefficients are called higher structure functions [HE92].

With $\Omega \to Q$ and the specifications $\mathcal{G}_a \to \{L_n, G_r\}$, $\eta^a \to \{c_{-n}, \gamma_{-r}\}$, and $\mathcal{P}_a \to \{-b_n, \beta_r\}$ we obtain

$$\begin{aligned} Q &= \sum(c_{-n}L_n^{X,\psi} + \gamma_{-r}G_r^{X,\psi}) + \sum\sum\big(\tfrac{m-n}{2}c_{-n}c_{-m}b_{m+n} + (\tfrac{n}{2}-r)c_{-n}\gamma_{-r}\beta_{n+r} - \gamma_{-s}\gamma_{-r}b_{r+s}\big) \\ &= \sum :c_{-n}(L_n^{X,\psi} + \tfrac{1}{2}L_n^{c,\gamma}): + \sum :\gamma_{-r}(G_r^{X,\psi} + \tfrac{1}{2}G_r^{c,\gamma}): -ac_0 \end{aligned} \tag{4.21}$$

with the familiar parametrization of the ordering ambiguity, $\{c_m, b_n\} = \delta_{m+n}$ and $[\gamma_r, \beta_s] = \delta_{r+s}$. The ghost contributions to the complete constraints $L_n := \{Q, b_n\}$ and $G_r := [Q, \beta_r]$ are

$$L_n^{c,\gamma} = \sum(n+m)b_{n-m}c_m + \sum(\tfrac{n}{2}+r)\beta_{n-r}\gamma_r, \qquad G_r^{c,\gamma} = -\sum(\tfrac{n}{2}+r)\beta_{r-n}c_n - 2\sum b_{r-s}\gamma_s. \tag{4.22}$$

$$[L_n, b_l] = (n-l)b_{n+l}, \quad \{G_r, b_l\} = (\tfrac{l}{2}-r)\beta_{r+l}, \quad [L_n, \beta_r] = (\tfrac{n}{2}-r)\beta_{n+r}, \quad [G_r, \beta_s] = -2b_{r+s}. \tag{4.23}$$

Jacobi identities now imply

$$\begin{aligned} &\{Q,b_n\}=L_n \Rightarrow [Q,L_n]=[Q^2,b_n] \Rightarrow [L_m,L_n]=(m-n)\{Q,b_{m+n}\}-\{[Q^2,b_m],b_n\},\ [L_m,G_r]=[L_m,[Q,\beta_r]]=(\tfrac{n}{2}-r)[Q,\beta_{m+r}]-[[Q^2,b_m],\beta_r] \\ &[Q,\beta_r]=G_r \Rightarrow \{Q,G_r\}=[Q^2,\beta_r] \Rightarrow \{G_r,G_s\}=2\{Q,b_{r+s}\}-\{[Q^2,\beta_r],\beta_s\} \quad \Rightarrow \quad Q^2=\tfrac{c}{12}(\textstyle\sum_n (n^3-n)c_nc_{-n}+\sum_r(4r^2-1)\gamma_r\gamma_{-r}) \end{aligned}$$

i.e. $Q^2 = 0$ implies the SCA with $c = 0$. For the ghosts we find

$$[L_n, c_l] = -(2n+l)c_{n+l},\ \{G_r, c_l\} = -2\gamma_{r+l},\ [L_n, \gamma_r] = -(\tfrac{3n}{2}+r)\gamma_{n+r},\ \{G_r, \gamma_s\} = (\tfrac{3r}{2}+s)c_{r+s}. \tag{4.24}$$

The BRST variations of the ghosts are $\{Q, c_n\} = \sum(\frac{n}{2}-m)c_{n-m}c_m$ and $\{Q, \gamma_r\} = \sum(r-\frac{3n}{2})c_n\gamma_{r-n}$.

In the bosonic case the Virasoro generators had to be used as constraints for physical states in order to eliminate negative norms from the Hilbert space. We now have additional negative norm states from the time-like components of the fermionic creation operators, and we need, in the terminology of the old covariant approach, the annihilation parts of T_F as additional constraints on physical states. The BRST charge whose cohomology imposes these constraints can be obtained from the gauge fixing procedure of 2-dimensional supergravity, which leads to the above action in the superconformal gauge. It could, however, also be written down directly in terms of the constraint algebra by using the Batalin-Fradkin-Vilkovisky formalism [HE92].

In order to discuss the physics of our model we still need to determine the normal ordering constant a, which turns out to be $\frac{1}{2}$ in the NS sector and 0 in the R sector. A quick way to evaluate this constant is its interpretation as a Casimir energy due to the finite length of the string [PO98]. In the case of the bosonic string the total contribution of the zero point energies yields $\sum_1^\infty \frac{n}{2}$, which is divergent. Regularization with $e^{-\alpha n}$ for $\alpha \to 0$ and subtraction of the continuum limit yields

$$\textstyle\sum_1^\infty n e^{-\alpha n} - \int_0^\infty dn\, n e^{-\alpha n} = -\frac{\partial}{\partial\alpha}\left(\sum e^{-\alpha n} - \int_0^\infty dn\, e^{-\alpha n}\right) = \frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha} - \frac{1}{1-e^{-\alpha}}\right) \to -\frac{1}{2}B_2 \quad (4.25)$$

where $B_2 = \frac{1}{6}$ is the Bernoulli number.[3] The same result can be obtained by zeta-function regularization, i.e. by analytic continuation of the Riemann zeta function $\zeta(s) = \sum_1^\infty \frac{1}{n^s}$, which yields $\sum_1^\infty n\big|_{\mathrm{reg}} = \zeta(-1) = -\frac{1}{12}$. For the sum of all transversal polarizations we thus obtain $-a = -\frac{1}{24}(D-2)$, which is the correct value for the bosonic string in 26 dimensions. We can, actually, turn the argument around and predict the critical dimension by requiring Lorentz invariance of the spectrum in light-cone quantization: At the first excited level $\alpha^\mu_{-1}|k\rangle$ we only have $(D-2)$ states, which therefore must be massless because they cannot for a representation of the little group $SO(D-1)$ of the center of mass frame of a massive particle.[4]

Returning to the NSR string we observe that boson and fermion vacuum energies cancel in the R sector, implying $a = 0$. In the NS sector we find $\sum n - \sum(n-\frac{1}{2}) = 2\sum n - \sum\frac{n}{2} = \frac{3}{2}\sum n$ which, yields $a = \frac{1}{2}$ for 8 (instead of 24) transversal dimensions. Again we can turn the argument around and conclude that the critical dimension must be 10 because now the first excited level $\psi^\mu_{-\frac{1}{2}}|k\rangle$ has to be massless, which requires $a = \frac{1}{2}$. At the first massive level in the NS sector $\alpha^\mu_{-1}|0\rangle$ and $\psi^\mu_{-1/2}\psi^\nu_{-1/2}|0\rangle$ combine into an antisymmetric tensor of SO(9). The true vacuum of the conformal field theory should thus be in the NS sector, while the degenerate Ramond vacua should be obtained from it by the action of a (so far formal) spin field.

The bad news is that, inspite of world-sheet supersymmetry, the NS sector still has a tachyonic ground state. At the massless level we have twice as many physical states in the

[3]The Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, ... are defined by $\frac{te^t}{e^t-1} = \sum B_n \frac{t^n}{n!}$.

[4]At the 2nd level the left-moving states $\alpha^\mu_{-2}|k\rangle$ and $\alpha^\mu_{-1}\alpha^\nu_{-1}|k\rangle$ form a traceless symmetric tensor of $SO(D-1)$. For closed strings this has to be tensored with the contribution from the right-movers.

Ramond sector. The level spacing is, on the other hand, quantized in half integer units in the NS sector. In 1976 Gliozzi, Scherk and Olive [gl76] observed that a space-time supersymmetric spectrum can be obtained by using the fermion number symmetry to project out all states with odd fermion number. In order that the massless level survives we need to asign odd fermion number to the NS vacuum, which eliminates the tachyon. In the Ramond sector the 0-modes ψ_0^μ are represented as γ-matrics so that half of the states have even fermion number and the other half is odd. The projector $\frac{1}{2}(1-(-)^F)$ thus removes half of the states, which amounts to a projection of Dirac spinors to Weyl spinors. (Note that the Lorentz transformations are represented by $\Sigma^{\mu\nu}=\frac{1}{4}[\gamma^\mu,\gamma^\nu]$, which has even fermion number, so that Lorentz invariance is preserved by the projection.)

Later it was discovered that the GSO projection is not only allowed but, in fact, required by consisteny. What we have done is to add two different Hilbert spaces by hand and it is not clear that this yields a consistent quantum field theory. The orbifold construction of conformal field theory states that we can use a symmetry of the model to project to invariant states. But in order to preserve modular invariance (absence of global anomalies on higher genus world sheets) we then need to introduce new sectors into the model where the boundary conditions of the fields are twisted by a group action. The RNS string and the GSO projection are a special case of this construction, where the discrete symmetry is the parity of the fermion number.

What we have done so far is to simply add by hand the two Hilbert spaces that provide representations for the oscillator algebras in the two sectors of the RNS model. It is not clear if such a procedure is consistent. Indeed, if we want to formulate the model on the torus, which we have to do if we want a theory of interacting strings, then we must do this in a modular invariant way. But it is easy to see that boundary conditions along the two homology cycles of the torus mix under modular transformations (only the completely periodic spin structure is invariant).[5] So if we want to have a Ramond sector then we are forced to also sum over the different boundary conditions in the 'time' directions. This amounts to a projection of the total Hilbert space to states that are even under a certain operator, known as the (world sheet) 'fermion number'. This projection is known as the GSO projection [gl76]. It eventually eliminates the tachyon from the spectrum and makes the theory space-time supersymmetric.

Since we need sectors with mixed boundary conditions (R,NS) and (NS,R) the considerations of the last paragraph apply to the left movers and to the right movers seperately, i.e. we have to sum independently over all spin structures of left mover and right movers. The same conlusion applies to higher genera, where the sum extends over all 2^{2g} different spin structures.

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[5] If we parametrize $z=\xi_1+\tau\xi_2$ and assign boundary conditions, or 'spin structures', $(\sigma_1,\sigma_2)=(\pm,\pm)$ with $\psi(\xi_1+1,\xi_2)=\sigma_1\psi(\xi_1,\xi_2)$ and $\psi(\xi_1,\xi_2+1)=\sigma_2\psi(\xi_1,\xi_2)$ then $S:\tau\to-1/\tau$ leaves $(+,+)$ and $(-,-)$ invariant and exchanges $(+,-)$ and $(-,+)$. The other $SL(2,\mathbb{Z})$ generator $T:\tau\to\tau+1$ leaves $(+,\pm)$ invariant and exchanges $(-,-)$ and $(-,+)$.

4.2 Consistent superstrings in 10 dimensions

Euler number $\chi = 2-2g = \int\sqrt{-g}R/4\pi$, $\quad g_{string} = e^{\langle\phi\rangle}$, $\quad$ Weyl: $G_{\mu\nu} = e^{\frac{4\phi}{D-2}} G_{\mu\nu}^{Einstein}$,
$\kappa = \sqrt{8\pi G_N} = \sqrt{8\pi}/M_{Pl} = 2.43\times 10^{18}GeV = \kappa_0 g_s$, $\quad\Rightarrow\quad G_N \sim g_s^2\alpha'^4$ $\qquad (\alpha' = \frac{1}{2\pi T}\,{}^{Regge}_{slope})$

Type II (closed, oriented): antisymmetric tensor fields: $C^{\alpha\gamma}(\Gamma^{\mu_1\ldots\mu_p})_\alpha{}^\beta|0\rangle^R_\beta\otimes|0\rangle^R_\gamma$
In a Weyl-basis the charge conjugation matrix is block diagonal in $D\in 4\mathbb{Z}\Rightarrow F_{even}$ in Type IIA.

RR-vertices $\sim$ field strengths, since $G_0 = 0$ implies the Dirac equation ($L_0 = 0\ \Rightarrow$ Klein-Gordon);
perturbative string states have no RR-charges: $dF = d*F = 0 \Rightarrow \begin{cases} F^{(p+2)} = dC^{(p+1)} \\ *F^{(p+2)} = dC^{(D-p-3)}\end{cases}$
D-branes: coupling $\int_{\Sigma_{p+1}} C^{(p+1)}$; magnetic charge for (NS) B-field: NS 5-brane.

The 5 consistent theories in 10 dimensions:

$$\int\!dV := \int d^{10}X\sqrt{-G}, \quad |F_p|^2 := \tfrac{1}{p!}F_{\mu_1\ldots\mu_p}F^{\mu_1\ldots\mu_p}, \quad S_{NS} = \int\!dV\ e^{-2\phi}(R+4(\partial\phi)^2-\tfrac12 H^2)$$

- **IIA:** $\quad S^{IIA}_{Bose} = S_{NS} - \frac{1}{4\kappa_{10}^2}\int\!dV\ (|F_2|^2+|\tilde F_4|^2) - \frac{1}{4\kappa_{10}^2}\int B_2\wedge F_4\wedge F_4$
  - with $\ \tilde F_4 = dC_3 - C_1\wedge H_3\ \Rightarrow\ $ BI: $d\tilde F_4 = -F_2\wedge H_3$ $\quad$ (normalization: $C_I = e^{-\phi}C_I^{RR}$)
  - by reduction of **11D SUGRA:** $\ 2\kappa_{11}^2 S^{11}_{Bose} = \int\!dV\ (R-\frac12|F_4|^2) - \frac16\int A_3\wedge F_4\wedge F_4$
    on a circle $[ds^2 = G_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}(dx^{10}+A_\nu dx^\nu)^2] \rightarrow\ \kappa_{10}^2 = \kappa_{11}^2/2\pi r$, $\quad G^{10} = e^{\frac23\phi}G^{11}$
- **IIB:** $\quad S^{IIB}_{Bose} = S_{NS} - \frac{1}{4\kappa_{10}^2}\int\!dV\ (|F_1|^2+|\tilde F_3|^2+\frac12|\tilde F_5|^2) - \frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3$
  - $\tilde F_3 = F_3 - C_0\wedge H_3$, $\quad \tilde F_5 = F_5 - \frac12 C_2\wedge H_3 + \frac12 B_2\wedge F_3$
  - C_4 is self dual: $\tilde F_5 = *\tilde F_5$ (don't insert into the action!!)
  - $SL(2,\mathbb{R})$-invariance: $\tau = C_0 + ie^{-\phi} \rightarrow \dfrac{a\tau+b}{c\tau+d}$, $\quad \begin{pmatrix}F_3\\H_3\end{pmatrix} \rightarrow \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}F_3\\H_3\end{pmatrix}$
    $SL(2,\mathbb{Z})$ subgroup after quantization ($\sim$ discrete Peccei-Quinn symmetry $\int aF\wedge F$ in 4D)
- **Type I**: $\quad S^{I}_{Bose} = \frac{1}{2\kappa_{10}^2}\int\!dV\ \left[e^{-2\phi}(R+4(\partial\phi)^2) - \frac12|\tilde F_3|^2\right]\ -\ \frac{1}{2g_{10}^2}\int\!dV\ e^{-\phi}\,\mathrm{Tr}(|F_2|^2)$
  open, unoriented, $SO(32)$ $\quad \tilde F_3 = dC_2 - (\frac{k_{10}}{2g_{10}})^2\omega_3$, $\quad \omega_3 = \mathrm{Tr}(A\wedge dA - \frac{2i}{3}A\wedge A\wedge A)$
- **Heterotic**: $\quad S^{Het.}_{Bose} = \frac{1}{2\kappa_{10}^2}\int\!dV\ e^{-2\phi}\left[R+4(\partial\phi)^2 - \frac12|\tilde H_3|^2\ -\ (\frac{k_{10}}{2g_{10}})^2\,\mathrm{Tr}(|F_2|^2)\right]$
  SO_{32} or $E_8\times E_8$ (phenomenologically preferred!)
- S-duality: $\phi^I = -\phi^H$, $\ G^I_{\mu\nu} = e^{-\phi^H}G^H_{\mu\nu}\ \ldots$ $\quad$ strong $\leftrightarrow$ weak coupling [for $SO(32)$]

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