Complex Analysis and Applications

Second Edition

Alan Jeffrey

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Preface

This volume develops complex analysis for students of applied mathematics and engineering, with special attention directed toward its applications. The second edition of this book has been restructured and completely revised. By rearranging the order of the chapters, the general development of complex analysis now forms the first three chapters of the book, while conformal mapping and its application to boundary value problems for the two-dimensional Laplace equation form the subject matter of the last two chapters. This separation of material makes the book more convenient for readers whose interests lie mainly in complex analysis and also for those who wish to gain an understanding of the important geometrical interpretation of complex analysis, and its applications to Dirichlet and Neumann boundary value problems.

This new edition, with very few exceptions, retains all of the topics in the first edition of the book and the revisions of the text have been made to emphasize important points in both the theory and application of complex analysis. As in the first edition, the introduction of each new idea is followed by examples illustrating its application. Sections are supported by large sets of exercises, the working of a selection of which is essential to develop an understanding of complex analysis. The purpose of the more routine exercises is to develop a familiarity with the manipulation that is essential when working with complex analysis. However, many other exercises are more demanding and they may involve small extensions of ideas found in the text. In the main, the more challenging exercises are collected at the end of an exercise set.

Although computers play no part in the analytical development of complex analysis, the use of a sophisticated computer algebra system is invaluable when making applications. The more complicated graphical plots in the text have all been produced using such a system and both the need for, and the benefits of the use of such systems is reflected in the small number of straightforward exercises that require a computer algebra system for their solution.

As in the first edition, the application of complex analysis to two-dimensional boundary value problems has been confined to the consideration of temperature distribution, fluid flow, and electrostatic problems. To limit the size of the book, other applications of complex analysis, for example elasticity, have been omitted. In each case, in order to show the relevance of complex analysis, each application is preceded by a concise introduction to the mathematical background of the subject, designed to show how a real valued potential function and its related complex potential can be derived from the mathematics that describe the physical situation.

No book can be free from external influences, and this one is no exception because it reflects the many books that have influenced the author, discussions with colleagues over the years, and the responses of students to the courses on which this book is based. The suggested reading and bibliography list at the end of the book is not intended to be comprehensive. The first group of books has been chosen because they are in some ways similar to this one, and so can serve to present different accounts of much of the material found here, while the second more advanced group lists some of the books the author has found useful.

Acknowledgments

It has been a pleasure to have collaborated with the many individuals who worked on this project. I wish to express my thanks to Lisa Van Horn, whose copyediting determined the final style of the book; and to the staff of Macmillan India, Ltd.: John Sollami, production manager, and the production team in Bangalore for converting rough files and diagrams into elegant pages. Thanks are also due to the staff of Taylor & Francis: Julie Spadaro, project editor, whose efficiency and guidance have expedited the various stages of production; Helena Redshaw, manager, Editorial Project Development, for her help during pre-production; and, finally, Sunil Nair, publisher, for agreeing to publish the second edition of my book.

Contents

To the memory of my dear wife Lisl
and to our children and grandchildren
who have given us so much happiness

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1

Analytic Functions

1.1 Review of Complex Numbers

This section reviews the elementary properties of complex numbers that are encountered in any first course on the subject. Its purpose is to collect, in a concise form, the basic concepts and algebraic operations on complex numbers that will be needed in what is to follow.

In the Cartesian representation of a complex number, also called the real and imaginary form of a complex number, the general complex number z is written

$$z = x + iy, \tag{1.1}$$

where x and y are real numbers, and i is the imaginary unit in the complex number system with the property that

$$i^2 = -1. \tag{1.2}$$

The quantity iy in Equation (1.1) is to be interpreted as the imaginary unit i scaled (multiplied) by the real number y; the symbol i is placed before the number y in Equation (1.1) to emphasize the role played by the imaginary unit. Hereafter when y has a specific value, such as 3, it will be more natural to write $3i$ instead of $i3$, although $i3$ and $3i$ have the same meaning.

In Equation (1.1), the real number x is called the real part of z, which is shown by writing

$$x = \text{Re}\{z\}. \tag{1.3}$$

Correspondingly, the real number y in Equation (1.1) is called the imaginary part of z, which is shown by writing

$$y = \text{Im}\{z\}. \tag{1.4}$$

As usual in analysis, the set of all real numbers will be denoted by $\mathbb{R}$. When a is a real number, it will be shown symbolically by writing $a \in \mathbb{R}$. Here we have used the symbol $\in$ to denote membership in a set, which is read either as "is a member of the set" or, more simply, as "belongs to the set." Using this notation in Equation (1.1), we can write $x, y \in \mathbb{R}$. The negated symbol $\notin$ is to be read "is not a member of the set," so in Equation (1.1) we can write $i \notin \mathbb{R}$ because the square of every real number is non-negative, so the imaginary unit i cannot be a real number because $i^2 = -1$.

A complex number z in which $y = 0$ is said to be purely real, whereas one in which $x = 0$ is said to be purely imaginary. For conciseness, the set of all complex numbers will be denoted by $\mathbb{C}$, so in Equation (1.1) we have $z \in \mathbb{C}$. As the set of all real numbers is obtained from $\mathbb{C}$ by excluding all numbers z that are purely imaginary, it follows that $\mathbb{R} \subset \mathbb{C}$, where the set theoretic symbol $\subset$ is to be read as "is a proper subset of." Here, by a *proper subset*, we mean that all numbers in $\mathbb{R}$ belong in $\mathbb{C}$ as special cases, but some numbers in $\mathbb{C}$ do not belong to $\mathbb{R}$.

The equality of two complex numbers is defined as the equality of their respective real and imaginary parts, so if $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, writing $z_1 = z_2$ implies the *two* real results $a_1 = a_2$ and $b_1 = b_2$.

Associated with every complex number $z = x + iy$ is another complex number called its complex conjugate, denoted by $\overline{Z}$, to be read "z bar," and defined as

$$\overline{z} = x - iy. \tag{1.5}$$

A comparison of Equations (1.1) and (1.5) shows that the operation of forming the complex conjugate of a complex number, called the *operation of conjugation*, simply involves reversing the sign of the imaginary part of z, leading to the obvious result that

$$\overline{(\overline{z})} = z. \tag{1.6}$$

An immediate consequence of the definition of a complex conjugate is that when z is purely real, $\overline{z} = z$; but when z is purely imaginary, $\widetilde{z} = -z$.

The algebra of complex numbers becomes clearer if they are represented geometrically. This is accomplished by representing a complex number $z = a + ib$ as a point (a, b) with respect to the ordinary rectangular Cartesian axes $O(x, y)$, with a the x-coordinate of z and b its y-coordinate. Thus the complex numbers represented by points on the x-axis are purely real numbers, while those represented by points on the y-axis are purely imaginary numbers. With this representation in mind, a point with coordinate b on the y-axis is understood to be the complex number ib. For obvious reasons the x-axis is called the *real axis* and the y-axis is called the *imaginary axis*. When using this geometrical approach, the (x, y)-plane is called the *complex plane*, which for convenience is often denoted by $\mathbb{C}$, although originally the graphical representation of a complex number was called an argand diagram. Another name for the complex plane, which will be used later, is the z-plane because complex numbers are usually represented by the symbol z. It will be seen later that the z-plane plays an important role throughout complex analysis, and that it is particularly important when *conformal transformations* are used to solve boundary value problems for a partial differential equation called the *Laplace equation*.

The zero or null complex number 0 is defined as the complex number with zero real and imaginary parts. So $z = 0$ if and only if, $\mathrm{Re}\{z\} = 0$ and $\mathrm{Im}\{z\} = 0$, so when written out in full the complex number $z = 0$ has the form $z = 0 + 0i$.

The sum of two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, written $z_1 + z_2$, is defined as the complex number whose real part is the sum of the real parts of z_1 and z_2, and whose imaginary part is the sum of the imaginary parts of z_1 and z_2, so that

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2). \tag{1.7}$$

To proceed further, it will be convenient to anticipate two special cases of the multiplication of complex numbers, the general definition of which will be given shortly. Let us agree that $1 \times z = z$; so if $z = a + ib$ then $1 \times z = a + ib$, while the negative of z, denoted by $-z$, is given by $-z = -a - ib$.

The difference of two complex numbers z_1 and z_2 now follows from Equation (1.7) by replacing the $+$ sign on the left of the equality sign by a $-$ sign, and the $+$ signs inside each of the two brackets on the right by $-$ signs, so that

$$z_1 - z_2 = (a_1 + ib_1) + (-a_2 - ib_2) = (a_1 - a_2) + i(b_1 - b_2). \tag{1.8}$$

It follows directly from Equation (1.8) that $z + (-z) = z - z = 0$.

The Cartesian representation of complex numbers in the z-plane allows them to be interpreted as two-dimensional vectors (directed quantities) which can be seen by examination of Figure 1.1(a), where the origin represents the initial point, or base of the vector $z = a + ib$, and the point with the Cartesian coordinates (a, b) represents the terminal point, or tip, of the vector. The vector z itself is the straight line segment directed from the base of the vector to its tip. Figure 1.1(a) also shows that geometrically, the vector $\overline{z} = a - ib$ is obtained from vector $z = a + ib$ by reflecting z in the real axis, and conversely that $\overline{(\overline{z})} = z$.

The parallelogram rule for the addition of vectors is shown in Figure 1.1(b) for vectors $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. There, vector z_2 is added to vector z_1 by translating vector z_2 parallel to itself, without change of length, until the base of z_2 coincides with the tip of z_1. The vector $z_1 + z_2$ then becomes the straight line segment with its base at the base of z_1(the origin) and its tip at the tip of the translated vector z_2, the coordinates of which are $(a_1 + a_2, b_1 + b_2)$. Figure 1.1(b) also shows that the addition of complex numbers is commutative because $z_1 + z_2 = z_2 + z_1$. Figure 1.1(c) shows how the difference of the

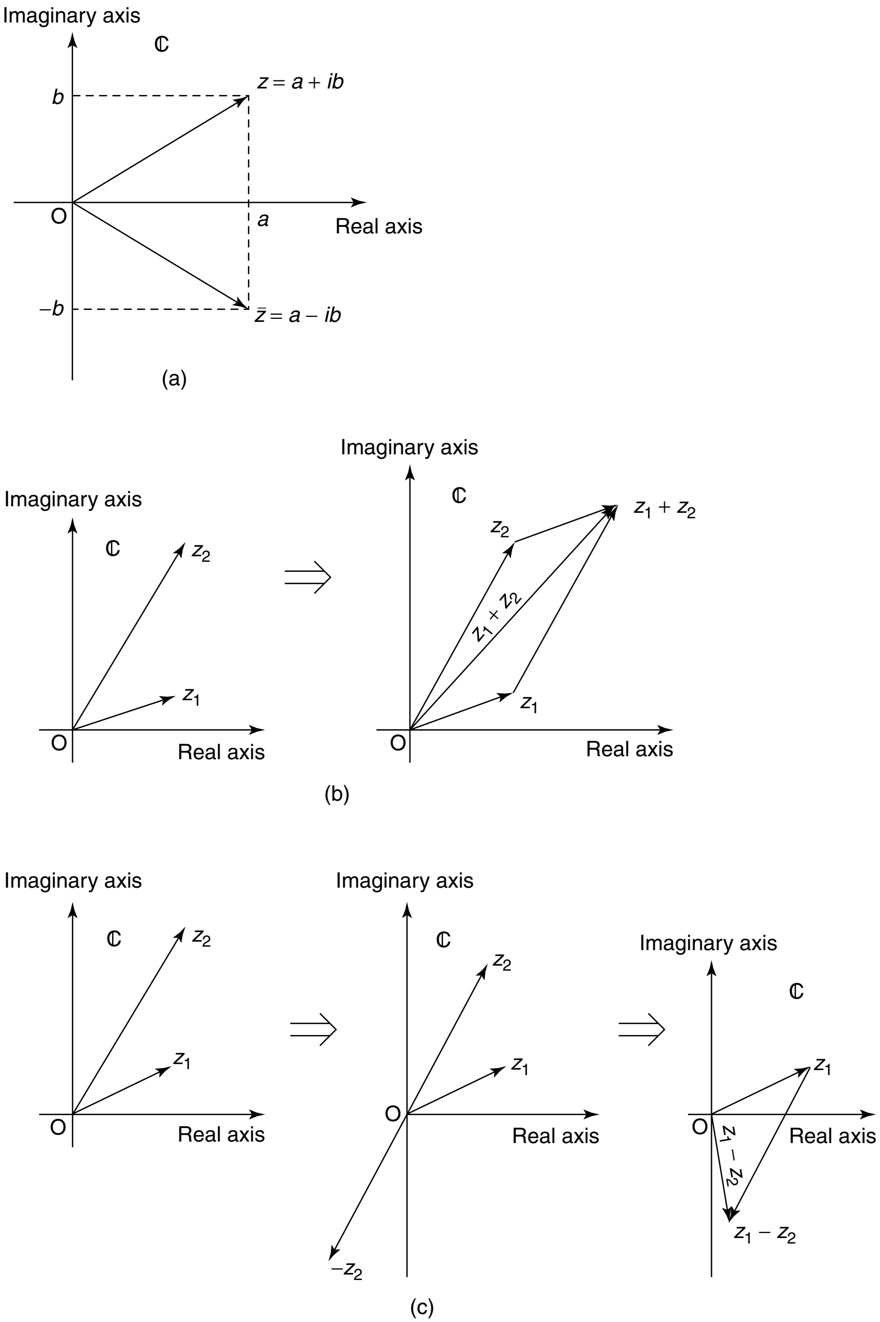

FIGURE 1.1
(a) z and $\bar{z}$ as vectors, (b) the sum $z_1 + z_2$, (c) the difference $z_1 - z_2$.

complex numbers $z_1 - z_2$ is represented geometrically as the sum $z_1 - z_2 = z_1 + (-z_2)$.

Unlike the real numbers in $\mathbb{R}$, complex numbers have no natural order, so if z_1 and z_2 are complex, it is meaningless to write $z_1 < z_2$. However, associated with any complex number $z = a + ib$ is a real number $|z|$ called its modulus, and defined as

$$|z| = \sqrt{a^2 + b^2}, \tag{1.9}$$

where the *positive* square root is always taken, so that $|z| \geqslant 0$. We remark here that $|z|$ is the analogue of the absolute value of a real number. Inspection of Figure 1.1(a) shows $|z| = \sqrt{a^2 + b^2}$ to be the length of the vector z drawn from its base (the origin) to its tip; and because $\overline{z} = a - ib$, it follows that $|z| = |\overline{z}|$. As $|z|$ is a real number, the moduli of complex numbers *can* be ordered, though this does not impose any natural order on the complex numbers themselves because all complex numbers with the same modulus, such as $|z| = r$, lie on a circle of radius r in the z-plane centered on the origin.

Given that $z = a + ib$, it is a routine matter to verify that

$$\begin{aligned}
\text{Re}\{z\} &= \tfrac{1}{2}(z + \overline{z}),\\
\text{Im}\{z\} &= \tfrac{1}{2i}(z - \overline{z}) = \tfrac{i}{2}(\overline{z} - z),\\
\overline{z_1 + z_2} &= \overline{z}_1 + \overline{z}_2,\\
\overline{z_1 z_2} &= \overline{z}_1 \overline{z}_2.
\end{aligned} \tag{1.10}$$

We will define the multiplication of the complex numbers $z_1 = a + ib$ and $z_2 = c + id$ in Cartesian form as the result of expanding the product $z_1 z_2 = (a + ib)(c + id)$ in the usual way, using the result $i^2 = -1$, and then collecting the real and imaginary parts of the product, so that

$$\begin{aligned}
z_1 z_2 &= (a + ib)(c + id)\\
&= ac + iad + ibc + i^2 bd\\
&= (ac - bd) + i(ad + bc).
\end{aligned} \tag{1.11}$$

In actual computations, necessitating the multiplication of complex numbers, rather than using the formal result $z_1 z_2 = (ac - bd) + i(ad + bc)$ in Equation (1.11), it is simpler to arrive at the product using the steps leading to result Equation (1.11).

Example 1.1.1

Given $z_1 = 2 + 3i$ and $z_2 = -1 + 4i$, find $z_1 z_2$, $z_1 \overline{z}_2$ and $|z_1 z_2|$.

SOLUTION

$$\begin{aligned}
z_1 z_2 &= (2 + 3i)(-1 + 4i)\\
&= (2)(-1) + (2)(4i) + (3i)(-1) + (3i)(4i)\\
&= -2 + 8i - 3i - 12 = -14 + 5i.
\end{aligned}$$

A similar calculation gives $z_1\overline{z}_2 = 10 - 11i$, while $|z_1z_2| = \sqrt{10^2 + (-11)^2} = \sqrt{221}$. ◇

If z_1 is purely real, so that $b = 0$ in Equation (1.11), it follows that $az_2 = ac + iad$, while if z_1 is purely imaginary, so $a = 0$ in Equation (1.11), then $ibz_2 = -bd + ibc$. Thus, for real λ and μ,

$$\lambda z = \lambda(a + ib) = \lambda a + i\lambda b \quad \text{and} \quad i\mu z = i\mu(a + ib) = -\mu b + i\mu a, \tag{1.12}$$

justifying our earlier use of the results $1 \times z = 1 \times (a + ib) = (a + ib)$, and $-z = -a - ib$.

An important consequence of multiplication is that if $z = a + ib$, then

$$z\overline{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2 = |\overline{z}|^2, \tag{1.13}$$

showing that the product $z\overline{z}$ is always real and such that $z\overline{z} \geqslant 0$. This simple result finds many applications, one of which occurs when complex numbers are divided.

When defining the quotient (division) z_1/z_2 of the complex numbers $z_1 = a + ib$ and $z_2 = c + id$ with $z_2 \neq 0$, we will again use the approach similar to the one used when deriving the product in Equation (1.11). The method involves first multiplying both the numerator and denominator of the quotient by $\overline{z}_2 = c - id$, when because of Equation (1.13) the denominator becomes the real number $c^2 + d^2$. The product in the numerator is then evaluated, after which the result of the quotient follows by dividing the real and imaginary parts in the numerator by $c^2 + d^2$. This is, of course, equivalent to *multiplying* the complex number in the numerator by $1/(a^2 + b^2)$. Carrying out these steps we gives

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} \\ &= \frac{(a + ib)(c - id)}{c^2 + d^2} \\ &= \frac{ac + bd}{|z_2|^2} + i\frac{bc - ad}{|z_2|^2}.\end{aligned}$$

So the formal definition of the quotient z_1/z_2 is

$$\frac{z_1}{z_2} = \frac{ac + bd}{|z_2|^2} + i\frac{bc - ad}{|z_2|^2}, \quad \text{when } z_2 \neq 0, \tag{1.14}$$

where the quotient z_1/z_2 is not defined when $z_2 = 0$.

This definition of a quotient is difficult to remember, so in practice when calculating z_1/z_2 it is usual to arrive at the result step by step, as in the derivation of Equation (1.14).

Example 1.1.2
Find the quotient $(2 + 3i)/(1 - 4i)$.

SOLUTION

$$\begin{aligned}\frac{2+3i}{1-4i} &= \frac{(2+3i)(1+4i)}{(1-4i)(1+4i)} = \frac{(2+3i)(1+4i)}{17} \\ &= \frac{-10+11i}{17} = -\tfrac{10}{17} + \tfrac{11}{17}\,i.\end{aligned}$$

◇

Notice that in the Examples 1.1.1 and 1.1.2 we have used of the convention introduced earlier by writing $2 + 3i$ and $1 + 4i$, in place of the notation $2 + i3$ and $1 + i4$ used in Equation (1.1), as this seems more natural when working with specific numbers.

The following important and very useful result involving the modulus of complex numbers is called the *triangle inequality*. If z_1 and z_2 are any two complex numbers, the *triangle inequality* asserts that

$$|z_1 + z_2| \leqslant |z_1| + |z_2|. \tag{1.15}$$

To prove this result notice that

$$\begin{aligned}|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) \\ &= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2 \\ &= |z_1|^2 + (z_1\overline{z}_2 + \overline{z_1\overline{z}_2}) + |z_2|^2.\end{aligned}$$

Now

$$z_1\overline{z}_2 + \overline{z_1\overline{z}_2} = 2\mathrm{Re}\{z_1\overline{z}_2\} \leqslant 2|z_1\overline{z}_2| = 2|z_1|\,|z_2|,$$

so substituting this inequality into the previous result we obtain

$$|z_1 + z_2|^2 \leqslant |z_1|^2 + 2|z_1|\,|z_2| + |z_2|^2 = \big(|z_1| + |z_2|\big)^2.$$

The expressions on each side of the inequality are non-negative, so taking the square root of both sides of this inequality we arrive at the triangle inequality

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

Replacing z_1 by $z_1 - z_2$ and z_2 by $z_2 - z_3$ in the triangle inequality gives another form of the triangle inequality that is often useful

$$|z_1 - z_3| \leqslant |z_1 - z_2| + |z_2 - z_3|. \tag{1.16}$$

The triangle inequality can be used to obtain another useful inequality. Applying the inequality in Equation (1.15) to the identity $z_1 = z_2 + (z_1 - z_2)$ gives

$$|z_1| \leqslant |z_2| + |z_1 - z_2|, \quad \text{so } |z_1| - |z_2| \leqslant |z_1 - z_2|.$$

Repeating the argument, but this time starting from the identity $z_2 = z_1 + (z_2 - z_1)$, we find that

$$|z_2| - |z_1| \leqslant |z_2 - z_1|.$$

Combining these two inequalities, and using the result $|z_1 - z_2| = |z_2 - z_1|$, shows that

$$-(|z_1| - |z_2|) \leqslant |z_1 - z_2| \leqslant |z_1| - |z_2|,$$

which is equivalent to the inequality

$$|z_1 - z_2| \geqslant \big||z_1| - |z_2|\big|, \tag{1.17}$$

that was to be established. Because Equation (1.17) has been derived from Equation (1.15) the two inequalities are equivalent, though the triangle inequality is used more frequently than Equation (1.17).

Example 1.1.3
Given $z_1 = 1 - \sqrt{3}i$ and $z_2 = 3 + 4i$, verify the inequalities in Equations (1.15) and (1.17).

SOLUTION

$$|z_1| = 2,\ |z_2| = 5,\ |z_1 + z_2| = \sqrt{9 - (3 - \sqrt{3})^2} = 3.2569,$$

and

$$|z_1 - z_2| = \sqrt{1 + (3 + \sqrt{3})^2} = 4.8366.$$

Equation (1.15) is satisfied because $3.2569 < 2 + 5 = 7$, while Equation (1.17) is satisfied because $4.8366 > |2 - 5| = 3$. ◇

The triangle inequality in Equation (1.15) illustrated in Figure 1.2(a) states that complex numbers can be interpreted as vectors, which is equivalent to the familiar result due to Euclid that the length of a side of a triangle is less than or equal to the sum of the lengths of the other two sides (hence the name *triangle inequality*). Clearly, equality can only occur in result Equation (1.15) when the origin, z_1 and z_2 are all collinear, as in Figure 1.2(b).

The inequality in Equation (1.15) can be generalized by mathematical induction to the case of n complex numbers $z_1, z_2, \ldots, z_n$, when it takes the form

$$|z_1 + z_2 + \cdots + z_n| \leqslant |z_1| + |z_2| + \cdots + |z_n|. \tag{1.18}$$

Complex numbers have a different representation called the *polar form*, arising from the use of the plane polar coordinates r and θ to describe z. This

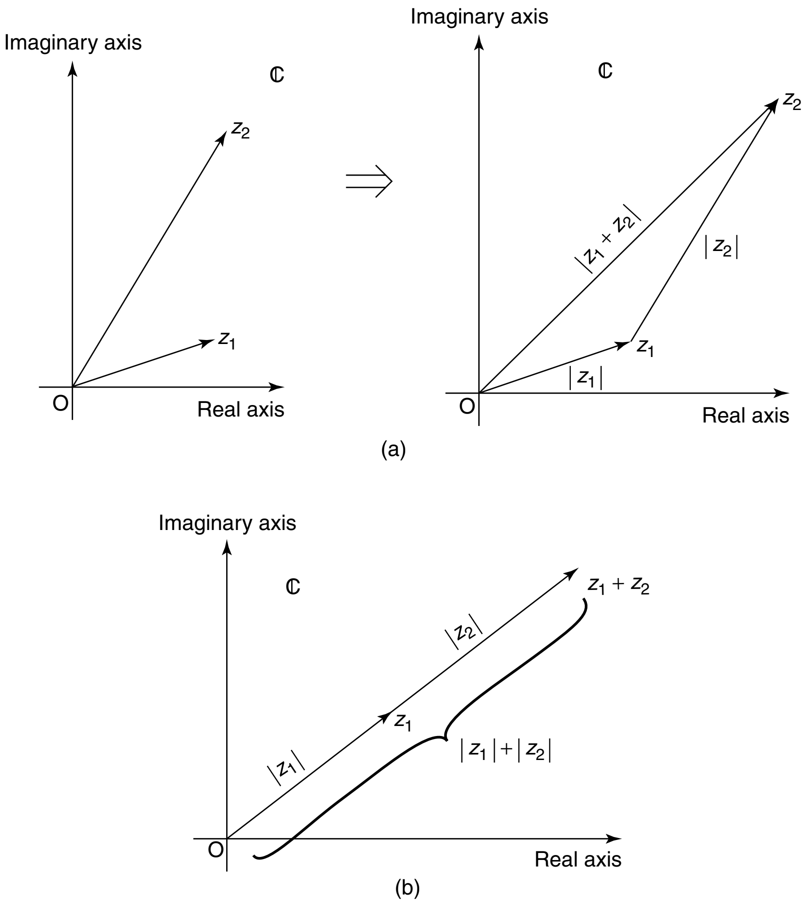

FIGURE 1.2
(a) $|z_1 + z_2| \leqslant |z_1| + |z_2|$, (b) $|z_1 + z_2| = |z_1| + |z_2|$.

form is illustrated in Figure 1.3, where point P in its real and imaginary form $z = a + ib$ is identified by its radial polar coordinate $r = |z| \geqslant 0$, measured from the origin, and the polar angle θ, that is always measured counterclockwise from the positive real axis.

We see from Figure 1.3 that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \tag{1.19}$$

so $z = x + iy$ becomes the polar form

$$z = r(\cos\theta + i\sin\theta). \tag{1.20}$$

The radial distance $r = |z|$ from the origin to a complex number z represented by a point P in the complex plane is unique, but the polar angle θ is not because replacing θ by $\theta \pm 2k\pi$, with $k = 0, 1, 2, \ldots$, while keeping r unchanged, will always identify the same point P. The angle θ is called the *argument* of the complex number z, written $\arg(z)$, so $\theta = \arg(z)$, while $r = |z|$ is called the *modulus* of z. Representing z in terms of the polar coordinates (r, θ), as in Equation (1.20), is said to specify z in its *polar form* or, equivalently, in its *modulus-argument form*. Notice that the argument of the polar form of the zero complex number is undefined, though its modulus $r = 0$.

At first sight the ambiguity in the value of $\theta = \arg(z)$ might appear to cause a problem, but it will be seen shortly that it is in fact the many-valued nature of θ that provides the key to the resolution of situations where more than one solution is possible, as will be seen when finding roots of complex numbers. From among all possible arguments θ associated with a given complex number z, we identify one that is called the *principal argument*, or the *principal*

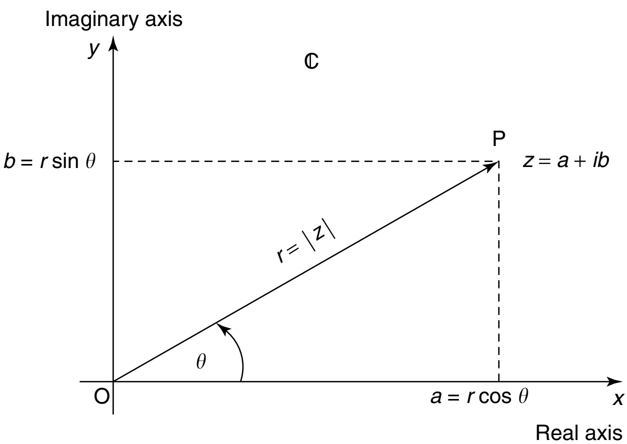

FIGURE 1.3
The polar representation of z.

value of z, and denoted by Arg(z). This is the value of $\theta = \text{Arg}(z)$ chosen such that $-\pi < \theta \leqslant \pi$, corresponding to

$$-\pi < \text{Arg}(z) \leqslant \pi. \tag{1.21}$$

Notice the convention used here, where arg(z) represents the set of all possible values of θ which differ one from the other by the addition of a multiple of 2π, whereas Arg(z) denotes the *unique* value of θ such that $-\pi < \text{Arg}(z) \leqslant \pi$. The connection between arg(z) and Arg(z) is given by

$$\arg(z) = \text{Arg}(z) \pm 2k\pi, \qquad k = 0, 1, 2, \ldots . \tag{1.22}$$

The *equality* of two complex numbers z_1 and z_2 in *polar form* is defined as the requirement that their moduli are equal, so that $|z_1| = |z_2|$, while their arguments are such that

$$\arg(z_1) = \arg(z_2) \pm 2k\pi,$$

for k zero or some positive integer. For example, if $|z_1| = |z_2| = 3$, $\arg(z_1) = 13\pi/6$ and $\arg(z_2) = -23\pi/6$, then $z_1 = z_2$, because the moduli of z_1 and z_2 are equal while $\arg(z_1) = \arg(z_2) + 6\pi$. This definition of the *equality* of the complex numbers z_1 and z_2 in polar form simply amounts to the obvious requirement that $|z_1| = |z_2|$, and $\text{Arg}(z_1) = \text{Arg}(z_2)$. Thus, in the preceding example, it is easily seen that $\text{Arg}(z_1) = \text{Arg}(z_2) = \pi/6$.

It follows directly from Equation (1.19) that

$$\theta = \tan^{-1}(y/x), \tag{1.23}$$

but θ is not unique because the inverse tangent function is many-valued, with each value differing from neighboring values by π. In addition, the interpretation of $\tan^{-1}(y/x)$ will depend on the interval chosen for the principal value of the inverse tangent function.

The standard convention used in analysis, and by calculators and computers when working with the inverse tangent function, is $-\pi/2 < \tan^{-1}(y/x) < \pi/2$. However this interval confines $\tan^{-1}(y/x)$ to the first and fourth quadrants, whereas z may also lie in either the second or the third quadrant, in which case $\tan^{-1}(y/x)$ will not satisfy the requirement that the principal value Arg(z) is such that $-\pi < \text{Arg}(z) \leqslant \pi$.

This difficulty in determining the correct value for Arg(z) has arisen because when forming the quotient y/x, no account was taken of the quadrant in which z was located. For example, if z lies in the *second* quadrant, then $x < 0$ and $y > 0$, so $y/x < 0$, in which case the convention $-\pi/2 < \tan^{-1}(y/x) < \pi/2$ will give a value for $\tan^{-1}(y/x)$ in the *fourth* quadrant. So to find the value of Arg(z), it will be necessary to *add* π to this value of $\tan^{-1}(y/x)$. Similarly, if z lies in the *third* quadrant $x < 0$ and $y < 0$, so $y/x > 0$, in which case the convention $-\pi/2 < \tan^{-1}(y/x) < \pi/2$ will give a value of $\tan^{-1}(y/x)$ in the *first* quadrant. So to find the value of Arg(z), it will be necessary to *subtract* π from the value of $\tan^{-1}(y/x)$. Given an arbitrary complex number $z = x + iy$, when using the standard convention $-\pi/2 < \tan^{-1}(y/x) < \pi/2$, the above arguments lead to the following rules for finding Arg(z):

***Rules for Computing* Arg(*z*)**

The convention $-\pi/2 < \tan^{-1}(y/x) < \pi/2$ will give the correct value $\text{Arg}(z) = \tan^{-1}(y/x)$ when $z = x + iy$ lies in the first quadrant, corresponding to $x > 0, y > 0$.

The convention $-\pi/2 < \tan^{-1}(y/x) < \pi/2$ will give the correct value $\text{Arg}(z) = \tan^{-1}(y/x)$ when $z = x + iy$ lies in the fourth quadrant, corresponding to $x > 0, y < 0$.

If z lies in the second quadrant, corresponding to $x < 0, y > 0$, then to find Arg(z) it is necessary to add π to the value of $\tan^{-1}(y/x)$ that is found when using the convention that $-\pi/2 < \tan^{-1}(y/x) < \pi/2$.

If z lies in the third quadrant, corresponding to $x < 0, y < 0$, then to find Arg(z) it is necessary to subtract π from the value of $\tan^{-1}(y/x)$ that is found when using the convention that $-\pi/2 < \tan^{-1}(y/x) < \pi/2$.

Example 1.1.4

Find $r = |z|$ and Arg(z) when (a) $z = 1 + \sqrt{3}i$, (b) $z = 1 - \sqrt{3}i$, (c) $z = -\sqrt{3} - i$, (d) $z = -1 + i$.

SOLUTION

(a) $r = 2$, and as z lies in the first quadrant the convention will give the correct value $\text{Arg}(z) = \tan^{-1}(\sqrt{3}/1) = \frac{1}{3}\pi$.

(b) $r = 2$, and as z lies in the fourth quadrant the convention will give the correct value $\text{Arg}(z) = \tan^{-1}(-\sqrt{3}/1) = -\frac{1}{3}\pi$.

(c) $r = 2$, but here z lies in the third quadrant, so to find the correct value of Arg(z) it will be necessary to subtract π from the value of θ found using the convention, so that $\text{Arg}(z) = \tan^{-1}\big((-1)/(-\sqrt{3})\big) - \pi = \frac{1}{6}\pi - \pi = -\frac{5}{6}\pi$.

(d) $r = \sqrt{2}$, but here z lies in the second quadrant, so to find the correct value of Arg(z) it will be necessary to add π to the value of θ found using the convention, so that $\text{Arg}(z) = \tan^{-1}(1/(-1)) + \pi = -\frac{1}{4}\pi + \pi = \frac{3}{4}\pi$. ◇

The polar form of complex numbers makes the operations of multiplication and division very simple. To see why this is, consider the two arbitrary complex numbers in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

where r_1 and r_2 are the moduli of z_1 and z_2, and $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$ are not necessarily the principal arguments of z_1 and z_2. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \end{aligned}$$

where the trigonometric identities $\cos(A + B) = \cos A \cos B - \sin A \sin B$ and $\sin(A + B) = \sin A \cos B + \cos A \sin B$ have been used.

A similar argument using the identities $\cos(A - B) = \cos A \cos B + \sin A \sin B$ and $\sin(A - B) = \sin A \cos B - \cos A \sin B$ shows that when $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 + \theta_2)].$$

1.1.1 Products and Quotients in Polar Form

We have the general results that when

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2), \tag{1.24}$$

the product

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \tag{1.25}$$

and when $z_2 \neq 0$ (that is $r_2 \neq 0$) the quotient

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 + \theta_2)]. \tag{1.26}$$

When expressed in words, results in Equations (1.25) and (1.26) show that when complex numbers in polar form are multiplied, their respective moduli are *multiplied* and their respective arguments are *added*; while when they are *divided* their respective moduli are *divided* and their respective arguments are *subtracted*. The relationships between $\arg(z_1)$ and $\arg(z_2)$ are thus

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \quad \text{and} \quad \arg(z_1/z_2) = \arg(z_1) - \arg(z_2), \tag{1.27}$$

but because of the definition of arg, these results will not necessarily be true if arg is replaced by Arg.

For future use we record the following results:

(a) The number 1 is represented by the point (1, 0) on the real axis, so its modulus is $|1| = 1$ and $\text{Arg}(1) = 0$.
(b) The number -1 is represented by the point $(-1, 0)$ on the real axis, so its modulus is $|-1| = 1$ and $\text{Arg}(-1) = \pi$.
(c) The imaginary unit i is represented by the point (0, 1) on the imaginary axis, so its modulus is $|i| = 1$ and $\text{Arg}(i) = \pi/2$. The *geometrical effect* of multiplying a vector z (a complex number) by i is to leave its length unchanged but to rotate the vector *counterclockwise* through an angle $\pi/2$, so $\text{Arg}(z)$ is increased by $\pi/2$.
(d) The negative imaginary unit $-i$ is represented by the point $(0, -1)$ on the imaginary axis, so its modulus is $|-i| = 1$ and $\text{Arg}(i) = -\pi/2$.

Setting $z_1 = z_2 = z$ in Equation (1.25), with $|z| = r$ and $\arg(z) = \theta$ shows that

$$z^2 = r^2(\cos 2\theta + i \sin 2\theta),$$

while setting $3\theta = \theta + 2\theta$ and repeating this reasoning shows that

$$z^3 = r^3(\cos 3\theta + i \sin 3\theta).$$

Routine mathematical induction then establishes the general result for integral n that

$$z^n = r^n(\cos n\theta + i \sin n\theta) \quad \text{for } n = 0, 1, 2, \ldots . \tag{1.28}$$

This result remains true when n is a negative integer because setting $z_1 = 1$ in Equation (1.26), using the fact that $\text{Arg}(1) = 0$ and writing z in place of z_2 and r in place of r_2, leads to the result

$$1/z = (1/r)(\cos \theta - i \sin \theta),$$

from which it follows that

$$1/z^n = (1/r^n)(\cos n\theta - i \sin n\theta), \quad \text{for } n = 1, 2, \ldots . \tag{1.29}$$

Thus result from Equation (1.29) is contained in Equation (1.28) if we allow $n = \pm 1, \pm 2, \ldots$. The special case of Equation (1.28) with $r = 1$ gives *de Moivre's theorem*

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \tag{1.30}$$

Example 1.1.5

Let $z = \cos\theta + i\sin\theta$. Find (a) z^{28} when $\theta = -\pi/6$ and (b) expressions for $\cos 3\theta$ and $\sin 3\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$, respectively.

SOLUTION

(a) From de Moivre's theorem with $n = 28$, $z^{28} = \cos(28\theta) + i\sin(28\theta)$, so setting $\theta = -\pi/6$ gives $z^{28} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Finding z^{28} in this way is straightforward, and far simpler than starting from the fact that $z = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ and raising z to the power 28 by repeated multiplication.

(b) From de Moivre's theorem with $n = 3$, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$, so expanding the expression the left by the binomial theorem and collecting its real and imaginary parts gives

$$\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos 3\theta + i\sin 3\theta.$$

Using the fact that the equality of complex numbers in their real and imaginary form requires the equality of their respective real and imaginary parts, this single equation involving complex numbers is seen to be equivalent to the two real trigonometric equations

$$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta \quad \text{and} \quad 3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta.$$

These results are not yet in the required form, because $\cos 3\theta$ needs to be expressed in terms of powers of $\cos\theta$ and $\sin 3\theta$ in terms of powers of $\sin\theta$. By using the identity $\cos^2\theta + \sin^2\theta = 1$ to eliminate $\sin^2\theta$ from the first result and $\cos^2\theta$ from the second result, we arrive at the required results

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \quad \text{and} \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta. \qquad \diamond$$

Result Equation (1.28) can be used to derive a useful result relating powers of z and $\overline{z}$. If $z = r(\cos\theta + i\sin\theta)$, then from Equation (1.28) $z^n = r^n(\cos n\theta + i\sin n\theta)$. Replacing θ by $-\theta$ in Equation (1.28), when $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, shows that $(\overline{z})^n = r^n(\cos n\theta - i\sin n\theta)$. A comparison of these results gives the useful result that

$$\overline{(z^n)} = (\overline{z})^n. \tag{1.31}$$

This simple result together with those in Equation (1.10) has many applications; one which concerns the roots of polynomial equations will be given at the end of this section.

The connection between z and $\overline{z}$ enables powers of $\cos\theta$ and $\sin\theta$ to be expressed in terms of cosines and sines of multiple angles. To show how this happens, notice that as any complex number z such that $|z| = 1$ can be written as $z = \cos\theta + i\sin\theta$, it follows that $\overline{z} = \cos\theta - i\sin\theta$, so combining these results gives

$$\cos\theta = \tfrac{1}{2}(z + \overline{z}) \quad \text{and} \quad \sin\theta = \tfrac{1}{2i}(z - \overline{z}),\ |z| = 1. \tag{1.32}$$

Thus

$$\cos^n\theta = \left(\frac{z + \overline{z}}{2}\right)^n \quad \text{and} \quad \sin^n\theta = \left(\frac{z - \overline{z}}{2i}\right)^n. \tag{1.33}$$

Example 1.1.6
Use result Equation (1.33) to find an expression for $\cos^3\theta$ in terms of cosines of multiples of θ.

SOLUTION

$$\cos^3\theta = \left(\frac{z + \overline{z}}{2}\right)^3 = \tfrac{1}{8}(z^3 + 3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3),$$

but $z\overline{z} = |z|^2 = 1$, so

$$\cos^3\theta = \tfrac{1}{4}\left(\frac{z^3 + \overline{z}^3}{2} + 3\,\frac{z + \overline{z}}{2}\right) = \tfrac{1}{4}(\cos 3\theta + 3\cos\theta),$$

or

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta. \qquad \diamond$$

A final task that remains is to solve an equation of the form

$$z^n = \zeta, \tag{1.34}$$

where ζ is an arbitrary complex number and n is a given integer. This is, of course, equivalent to finding the n roots of the equation $z = \zeta^{1/n}$. Setting $z = \rho(\cos\theta + i\sin\theta)$ and $\zeta = R(\cos\phi + i\sin\phi)$ in $z^n = \zeta$ gives

$$\rho^n(\cos\theta + i\sin\theta)^n = R(\cos\phi + i\sin\phi),$$

and after applying de Moivre's theorem this becomes

$$\rho^n(\cos n\theta + i\sin n\theta) = R(\cos\phi + i\sin\phi).$$

So to find the solutions z it will be necessary to find ρ and θ in terms of the known values of R, ϕ, and n.

Recalling that the equivalence of complex numbers expressed in polar form requires their moduli to be equal, and their arguments to be equal to within an additive multiple of 2π, we see that

$$\rho^n = R \quad \text{and} \quad n\theta = \phi + 2k\pi \quad \text{for } k = 0, \pm1, \pm2, \ldots . \tag{1.35}$$

Thus

$$\rho = R^{1/n} \quad \text{and} \quad \theta = \frac{\phi + 2k\pi}{n} \quad \text{for } k = 0, \pm1, \pm2, \ldots . \tag{1.36}$$

where ρ is the positive nth root of R.

The trigonometric functions $\cos\theta$ and $\sin\theta$ are periodic with period 2π, so as the integer k increases it follows that the n permissible values of θ will be determined from the second result in Equation (1.36) by allowing k to run through any set of n consecutive integers. For convenience it is usual to take this set of integers to be $k = 0, 1, 2, \ldots, n-1$. Allowing k to increase beyond the value $n-1$, or to decrease below zero, will simply generate the same set of n roots $z_0, z_1, \ldots, z_{n-1}$. So the roots $z_0, z_1, \ldots, z_{n-1}$ of $z = \zeta^{1/n}$ are given by

$$z_k = R^{1/p}\left(\cos\left(\frac{\phi + 2k\pi}{n}\right) + i\sin\left(\frac{\phi + 2k\pi}{n}\right)\right), \quad k = 0, 1, 2, \ldots, n-1, \tag{1.37}$$

where $R = |\zeta|$ and $\theta = \mathrm{Arg}(\zeta)$, or equivalently by $\theta = \arg(\zeta)$.

Example 1.1.7

For any fixed n, the n roots of $z^n = 1$, denoted sequentially by $\omega_0, \omega_1, \ldots, \omega_{n-1}$, are called the *nth roots of unity*. Find the form of these roots and use the result to find the cube roots of unity.

SOLUTION

In terms Equation (1.34), $\zeta = e^0 = 1$, so $R = |1| = 1$, while $\phi = \mathrm{Arg}(1) = 0$. So from Equation (1.37) the nth roots of unity are given by

$$\omega_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad \text{with } k = 0, 1, 2, \ldots, n-1.$$

When the nth roots of unity are plotted in the complex plane they are seen to lie on a circle of radius 1 with its center at the origin, called a *unit circle* (see Figure 1.4). The roots are uniformly distributed around the unit circle, and the vector from the origin to each root is inclined at an angle $2\pi/n$ radians to the vectors drawn to the adjacent roots, with the root ω_0 located at the point $z = 1$. Thus the polygon formed by chords joining adjacent roots on the unit circle is a regular n-sided polygon. Because all points on the unit circle are at a constant unit distance from the origin, the equation of this unit circle is $|z| = 1$.

To find the cube roots of unity we set $n = 3$ in the expression for ω_k, when we find that

$$\omega_0 = 1, \quad \omega_1 = -\tfrac{1}{2} + i\,\tfrac{\sqrt{3}}{2}, \quad \omega_2 = -\tfrac{1}{2} - i\,\tfrac{\sqrt{3}}{2}. \qquad \diamond$$

Notice that because the nth roots of unity are symmetrically spaced around the unit circle with $\omega_k = \cos(2k\pi/n) + i\sin(2k\pi/n)$, it follows from the polar representation of complex numbers that $\omega_k^2 = \omega_{k+1}$, $\omega_k^3 = \omega_{k+2}$, $\omega_k^4 = \omega_{k+3}, \ldots,$ $\omega_k^n = \omega_{k+n-1}$. Thus, knowing any one of the nth roots of unity other than $\omega_0 = 1$, ω_k, enables all of the other roots to be generated by raising it to suitable powers.

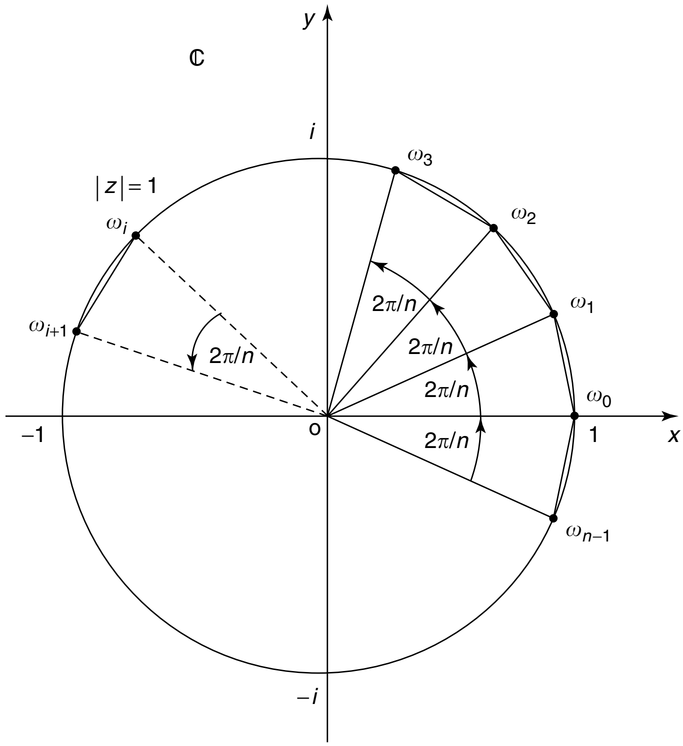

FIGURE 1.4
A plot of the nth roots of unity.

This property can be illustrated by considering $\omega_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, that is one of the cube roots of 1 found in Example 1.1.7 because a simple calculation shows that

$$\omega_1^2 = -\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2} = \omega_2, \ \omega_1^3 = 1, \quad \text{while } \omega_1^4 = -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2} = \omega_1.$$

Example 1.1.8

Find the values of z such that $z^3 = -4 - 4\sqrt{3}i$, and plot them in the complex plane.

SOLUTION

Using the notation of Equation (1.37), we have $n = 3$ and $\zeta = -4 - 4\sqrt{3}i$, so $R = |\zeta| = ((-4)^2 + (-4\sqrt{3})^2)^{1/2} = 8$, and $\phi = \text{Arg}(\zeta) = -\frac{2}{3}\pi$. Thus $\rho = R^{1/3} = 8^{1/3} = 2$, and $\theta = (-\frac{2}{3}\pi + 2k\pi)/3 = \frac{1}{3}(-2 + 6k)\pi$, with $k = 0$, 1 and 2. Thus the three cube roots of $\zeta = -4 - 4\sqrt{3}i$ are

$$z_0 = 2(\cos(2\pi/9) - i\sin(2\pi/9)) \approx 1.5321 - 1.2856i,$$
$$z_1 = 2(\cos(4\pi/9) + i\sin(4\pi/9)) \approx 0.3473 + 1.9696i,$$
$$z_2 = 2(\cos(10\pi/9) + i\sin(10\pi/9)) \approx -1.8794 - 0.6840i.$$

All three roots lie on the circle of radius 2 centered on the origin shown in Figure 1.5, with the equation $|z| = 2$. The roots are symmetrically spaced

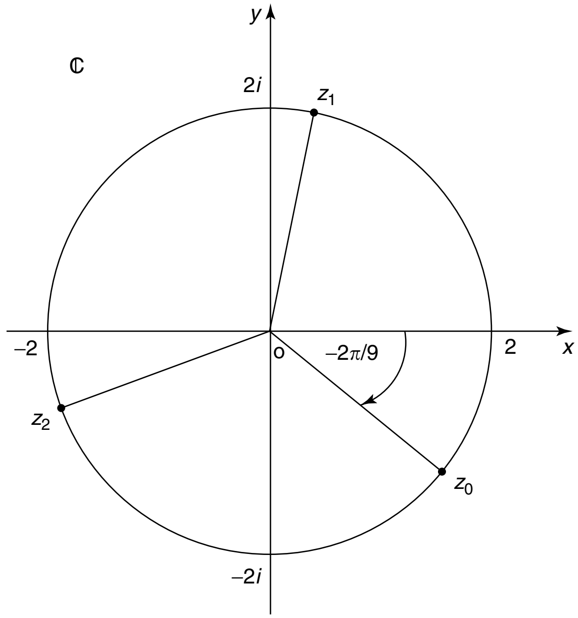

FIGURE 1.5
The cube roots of $\zeta = -4 - 4\sqrt{3}i$ spaced around the circle $|z| = 2$.

around the circle with their respective vectors inclined to one another at an angle of $2\pi/3$ radians. ◇

It is sometimes necessary to compute an expression such as $w^{q/p}$, where w is a complex number and p and q are integers. To do this we set $z = w^{q/p}$, and use the result that this is equivalent to $z^p = w^q$. As w and q are known, it is a simple matter to compute w^q, after which results in Equations (1.34) and (1.37) can be used with $\zeta = w^q$.

Example 1.1.9
Find the fourth roots of $(1+i)^{3/4}$.

SOLUTION
Setting $z = (1+i)^{3/4}$, it follows that $z^4 = (1+i)^3$. Now $(1+i)^3 = -2+2i$, so we need to find the fourth roots of $\zeta = -2+2i$. We have $R = |\zeta| = 2\sqrt{2}$, and $\phi = \mathrm{Arg}(\zeta) = 3\pi/4$, so the roots z_k given by Equation (1.38) are

$$z_k = 2^{3/8}\left[\cos\frac{(3+8k)\pi}{16} + i\sin\frac{(3+8k)\pi}{16}\right], \quad k = 0,1,2,3.$$

Thus $z_0 \approx 1.0783 + 0.7205i$, $z_1 \approx -0.7205 + 1.0783i$, $z_2 \approx 1.0783 - 0.7205i$ and $z_3 = 0.7205 - 1.0783i$. ◇

We are now in a position to solve an arbitrary quadratic equation

$$az^2 + bz + c = 0, \tag{1.38}$$

where the coefficients a, b and c can be real or complex. Three cases are to be considered:

(i) If a, b and c are all real, and the discriminant $\Delta = b^2 - 4ac \geqslant 0$, the roots of the equation are given by the familiar elementary quadratic formula

$$z_{\pm} = \frac{-b \pm \sqrt{\Delta}}{2a}, \ \Delta \geqslant 0. \tag{1.39}$$

(ii) If the discriminant $\Delta = b^2 - 4ac < 0$, the two roots of the quadratic equation are given by

$$z_1 = \frac{b^2 - i\sqrt{-\Delta}}{2a} \quad \text{and} \quad z_2 = \frac{b^2 + i\sqrt{-\Delta}}{2a}, \ -\Delta = 4ac - b^2 > 0. \tag{1.40}$$

(iii) If, however, the discriminant $\Delta = b^2 - 4ac$ is complex with $\Delta = \alpha + i\beta$, the $\pm$ sign in the numerator of Equation (1.39) must be replaced by a $+$ sign, when the two roots of the quadratic equation are then given by

$$z = \frac{-b + \sqrt{\alpha + i\beta}}{2a}. \tag{1.41}$$

This is because finding the square root of a complex number automatically generates two values.

Example 1.1.10
Solve the quadratic equations:
(a) $z^3 - 3z + 4 = 0$ and (b) $4z^2 + 4z + 1 - i = 0$.

SOLUTION

(a) The coefficients are real, but the discriminant $\Delta = b^2 - 4ac = -7 < 0$, so by (ii) the two roots are $z_0 = \frac{3}{2} - \frac{1}{2}\sqrt{7}i$ and $z_1 = \frac{3}{2} + \frac{1}{2}\sqrt{7}i$.

(b) The coefficients a and b in the quadratic equation are real, but the coefficient $c = 1 - i$, and the discriminant $\Delta = i$, so by (iii) the roots are given by

$$z = \frac{-4 + \sqrt{16i}}{8}, \quad \text{which is equivalent to } z = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{i}.$$

Reasoning as in Example 1.1.8 is easily shown that the two square roots of i are given by

$$\sqrt{i} = \cos\left(\frac{(1+4k)\pi}{4}\right) + i \sin\left(\frac{(1+4k)\pi}{4}\right), \quad k = 0, 1$$

so $\sqrt{i} = -\frac{1}{\sqrt{2}}(1 + i)$ and $\sqrt{i} = \frac{1}{\sqrt{2}}(1 + i)$. Thus the two roots of the quadratic equation become

$$z_0 = -\tfrac{1}{2} - \tfrac{1}{2\sqrt{2}}(1 + i) \quad \text{and} \quad z_1 = -\tfrac{1}{2} + \tfrac{1}{2\sqrt{2}}(1 + i).$$ ◇

It is no coincidence that the roots of equation (a) in Example 1.1.10 are complex conjugates, while those of equation (b) are not. We now show why this is so by establishing a very useful property of polynomial equations in general.

The need to perform algebraic operations on complex numbers has been illustrated when solving the most general form of the quadratic equation in Equation (1.38), and the necessity will become even clearer when we come to work with *functions of a complex variable*. In anticipation of the forthcoming discussion of general functions of a complex variable, and to explain the relationships among the roots in Example 1.1.10, we will use an elementary argument to establish an important property of polynomial equations. We define a *polynomial of degree* $\boldsymbol{n}$ to be an expression of the form

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \tag{1.42}$$

where the term *degree* refers to the highest power of z that occurs in $P(z)$. The numbers $a_0, a_1, \ldots, a_n$ in Equation (1.42) are called the *coefficients* of the polynomial. A *root* of the polynomial equation $P(z) = 0$ refers to a specific number $z = \zeta$, say, that is a solution of the equation. The term *zero* when applied to the polynomial $P(z)$ refers to a value of z, say $z = \zeta$, with the property that $P(\zeta) = 0$. Thus a *root* is a solution of an *equation*, while a *zero* is a property of a *function*. To illustrate matters, the quadratic polynomial (function) $P(z) = z^2 + 4z + 3$ becomes zero when $z = -1$ and $z = -3$, so these are its zeros, whereas the quadratic equation $P(z) = 0$ has the roots $z = -1$ and $z = -3$. It is a fundamental algebraic result that a polynomial equation $P(z) = 0$ of degree n has n roots, where a root repeated p times is counted as p roots and called a *degenerate root*. This is a consequence of a result known as the *fundamental theorem of algebra*, and the result is true irrespective of whether the coefficients $a_0, a_1, \ldots, a_n$ of Equation (1.42) are real or complex. This result will be proved later; but in what follows we prove a useful theorem concerning polynomials because its proof only requires some of the elementary results concerning complex conjugates that have already been established.

THEOREM 1.1.1 A Property of Polynomials with Real Coefficients
If the coefficients $a_0, a_1, \ldots, a_n$ of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \tag{1.43}$$

are all real, then either all of the roots of $P(z) = 0$ are real or, if complex, they must occur in complex conjugate pairs, while if the degree of $P(z)$ is odd it must have at least one real root.

PROOF

We start from the equation determining the roots of $P(z) = 0$, namely

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0.$$

Taking the complex conjugate of this equation, and using the fact that all a_r are real $\overline{(a_r z^r)} = a_r \bar{z}^r$, it follows that

$$a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \cdots + a_n \bar{z}^n = 0,$$

showing that if z is a root, then so is its complex conjugate $\overline{z}$. The first part of the theorem is proved.

If $z = \zeta$ is a complex root of $P(z) = 0$, then $(z - \zeta)$ and $(z - \overline{\zeta})$ must be factors of $P(z)$, so the product

$$(z - \zeta)(z - \overline{\zeta}) = z^2 - (\zeta + \overline{\zeta})z + \zeta\overline{\zeta}$$

must also be a factor of $P(z)$. However, $\zeta + \overline{\zeta}$ and $\zeta\overline{\zeta} = |\zeta|^2$ are always real so the pair of complex conjugate roots corresponds to this quadratic factor with real coefficients. Thus to produce a polynomial with real coefficients, the roots must either all be real leading to real factors, or some may be real while the remaining pairs of complex conjugate roots will correspond to real quadratic factors.

If the degree of $P(z)$ is odd and equal to $2m + 1$, then $P(z)$ can have at most m real quadratic factors; so if the product of factors is to yield a polynomial with real coefficients, the remaining factor or factors must be real and $P(z)$ will have at least one real root. ◆

This theorem explains the behavior of the roots in Example 1.1.10 because in case (a), the quadratic polynomial had real coefficients so the roots of $P(z) = 0$ could either both be real or if complex, they must occur as a complex conjugate pair, as indeed they did. In case (b), the coefficients of the quadratic polynomial were *not* all real, so Theorem 1.1.1 did not apply, and the two complex roots found were *not* complex conjugates.

Example 1.1.11

Given the cubic polynomial $P(z) = z^3 - 3z^2 + 7z - 5$, find its roots and hence factor the polynomial.

SOLUTION

The polynomial has real coefficients so Theorem 1.1.1 applies. Because the degree of the polynomial is odd, it must have at least one real root, and inspection (trial and error) shows this to be $z = 1$, so $z - 1$ must be a factor. Dividing $P(z)$ by $z - 1$ gives

$$\frac{z^3 - 3z^2 + 7z - 5}{z - 1} = z^2 - 2z + 5,$$

so the remaining roots of $P(z) = 0$ must be the roots of $z^2 - 2z + 5 = 0$. The quadratic formula shows these to be $1 + 2i$ and $1 - 2i$, so when factored we find that

$$\begin{aligned} P(z) &= z^3 - 3z^2 + 7z - 5 \\ &= (z - 1)(z^2 - 2z + 5) \\ &= (z = 1)(z = 1 - 2i)(z = 1 + 2i). \end{aligned}$$ ◇

A final example of the way an elementary argument can provide information about the roots of a polynomial is shown by Theorem 1.1.2.

THEOREM 1.1.2 The Eneström–Kakeya Theorem
Let the polynomial

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

have real coefficients such that $a_0 > a_1 > \cdots > a_n > 0$*. Then the polynomial* $P(z)$ *has no zeros inside the unit circle* $|z| = 1$*.*

PROOF

We start from the identity

$$(1 - z)P(z) = a_0 - \sum_{k=0}^{n-1} (a_k - a_{k+1})z^{k+1} - a_n z^{n+1}.$$

Then as $a_k - a_{k+1} > 0$, we have $|a_k - a_{k+1}| = a_k - a_{k-1}$ for $k = 0, 1, \ldots, n - 1$, so an application of Equation (1.17) gives

$$|(1 - z)P(z)| \geqslant a_0 - \sum_{k=0}^{n-1} (a_k - a_{k+1})|z|^k - a_n |z|^{n+1}.$$

When $|z| < 1$ we have $|z|^k < 1$, so this inequality can be strengthened to

$$|(1 - z)P(z)| > a_0 - \sum_{k=0}^{n-1} (a_k - a_{k+1}) - a_n,$$

but after expanding the right-hand side, all terms are found to cancel so we have shown that $|(1 - z)P(z)| > 0$, when $|z| < 1$. Thus $P(z)$ has no zeros when $|z| < 1$ and consequently the equation $P(z) = 0$ has no roots strictly inside the unit circle $|z| = 1$. The theorem offers no information about the behavior of the function $P(z)$ or the roots of $P(z) = 0$ on the unit circle. ◆

Example 1.1.12

Given that $P(z) = z^3 + 3z^2 + 8z + 12$, find its zeros. Check that Theorem 1.1.2 (the Eneström–Kakeya Theorem) applies, and verify that the roots of $P(z) = 0$ all lie outside the unit circle $|z| = 1$.

SOLUTION

Inspection shows that $z = -2$ is a zero of $P(z)$, so $(z + 2)$ must be a factor. As $P(z)/(z + 2) = z^2 + z + 6$, the other two zeros of $P(z) = 0$ must be the zeros of $z^2 + z + 6$, that is, the roots of $z^2 + z + 6 = 0$, and from the quadratic formula these are found to be $-\frac{1}{2} + \frac{\sqrt{23}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{23}}{2}i$. Thus the three roots of $P(z) = 0$ are $z_1 = -2$, $z_2 = -\frac{1}{2} - \frac{\sqrt{23}}{2}i$ and $z_3 = -\frac{1}{2} + \frac{\sqrt{23}}{2}i$. The coefficients of $P(z)$ satisfy the conditions of Theorem 1.1.2, so $P(z)$ can have no zeros inside the unit circle, which is confirmed by the fact that $|z_1|$, $|z_2|$ and $|z_3|$ are all greater than 1. Equivalently, $P(z) = 0$ has no roots inside the unit circle. ◇

Exercises 1.1

1. Given $z_1 = 2 - 3i$ and $z_2 = 1 + 4i$, find: (a) $z_1 + 2z_2$ (b) $3z_1 - 4z_2$ (c) $2z_1 - 3\overline{z}_2$ (d) $4\overline{z}_1 - 2\overline{z}_2$.
2. Given $z_1 = 2 + i$ and $z_2 = 1 - 2i$, find: (a) z_1/z_2 (b) $\overline{z}_2/z_1$ (c) $z_2/\overline{z}_1$ (d) $\overline{z}_1/\overline{z}_2$.
3. Given $z_1 = 2 - 2\sqrt{3}i$ and $z_2 = 1 + i$, find: (a) $|z_1/z_2|$ (b) $z_2/|z_1|$ (c) $|z_1|/|z_2|$.
4. Find the Cartesian form of $(1 + 3i)/(1 + 2i) + (1 + 2i)/(1 - 3i)$.
5. Find the Cartesian form of $(1 - 2i)/(2 + i) + 3i/(1 - 2i)$.
6. Given $z_1 = 2 + 3i$, $z_2 = 1 - i$, $z_3 = -3 - 2i$, find (a) $z_1^2 z_3$ (b) $z_1\overline{z}_2 z_3$ (c) $(z_2 - z_3)/z_1$.
7. Find z, given that $(z - 1)/(2z + 3i) = 2 - 3i$.
8. Given $z_1 = 3 + 2i$ and $z_2 = 2 - 4i$, verify the triangle inequality in Equations (1.15) and (1.17).
9. Give an example of complex numbers z_1 and z_2 for which $|z_1 + z_2| = |z_1| + |z_2|$.
10. Using three complex numbers of your own choice, verify the generalization of the triangle inequality in Equation (1.18) when $n = 3$. When can the inequality sign be replaced by an equality sign?
11. Find the form of z if $z^2 = (\overline{z})^2$.
12. Is it true that $|z^2| = |\overline{z}|^2$?
13. When is it true that $|(a + ib)z| = |a||z| + |b||z|$?
14. Given that $z_1 = -\frac{1}{2}(1 + i)$ and $z_2 = 1 - 2i$, find the modulus and principal argument of (a) z_1 and z_2 (b) $z_1 z_2$ (c) z_1/z_2 (d) $z_2 z_3/z_1$.
15. Convert $w = (-2\sqrt{3} + 2i)/(1 - i)$ to polar form.
16. Convert $w = (2 + 4i)/(-1 + i)$ to polar form.
17. Find an expression for $\sin 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.
18. Find an expression for $\cos 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.
19. Find an expression for $\sin^4\theta$ in terms of cosines of multiples of θ.
20. Find an expression for $\cos^4\theta$ in terms of cosines of multiples of θ.
21. Is it true that the six roots of the equation $z^6 + z^3 + 1 = 0$ are of the form $z = \cos(2k\theta/\pi) + i\sin(2k\theta/\pi)$ with $k = 0, 1, 2, 3, 4, 5$? Justify your answer.
22. If for any fixed n the number ω is any nth root of unity, show that $1 + \omega + \omega^2 + \omega^3 + \cdots + \omega^{n-1} = 0$.

Use the polar representation of the nth roots of unity to explain this result.

23. Using cross multiplication, or otherwise, verify the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Multiply the numerator and denominator of the expression on the right by $z^{-1/2}$, set $z = \cos\theta + i\sin\theta$, use de Moivre's theorem and equate the real parts on each side of the result to obtain the *Lagrange identity*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\left(\frac{1}{2}\theta\right)}.$$

24. Using the method of Exercise 23, equate the imaginary parts on each side of the transformed identity to show that

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \tfrac{1}{2}\cot\left(\tfrac{1}{2}\theta\right) - \frac{\cos\left(n + \frac{1}{2}\right)\theta}{2\sin\left(\frac{1}{2}\theta\right)}.$$

25. Find the values of z such that $z^4 = -\frac{1}{2} + \frac{1}{2}i$.
26. Find the values of z such that $z^5 = 16 - 16\sqrt{3}i$.
27. Find the fourth roots of $-i^{3/4}$.
28. Find the two square roots of $1 - 2i$.
29. Show that if $z = x + iy$ and $w^2 = z$, that the square roots $w_\pm$ of z are given by

$$w_\pm = \pm\left(\sqrt{\frac{|z| + x}{2}} + i\,\mathrm{sgn}(y)\sqrt{\frac{|z| - x}{2}}\right), \quad \text{where } \mathrm{sgn}(y) = \begin{cases} 1 \text{ if } y \geqslant 0 \\ -1 \text{ if } y < 0 \end{cases}.$$

In this result the upper and lower + and − signs on the left are to be taken with the corresponding upper and lower + and − signs on the right.

30. Use the result of Exercise 29 to find the two square roots $w_\pm$ of $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and check the result using multiplication to show that $(w_\pm)^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.
31. Use the result of Exercise 29 to find the square roots of i.
32. Use the result of Exercise 29 to find the fourth roots of i.
33. Use Theorem 1.1.1 to find the roots of $z^3 + 5z^2 + 9z + 5 = 0$.
34. Use Theorem 1.1.1 to find the roots of $z^3 + 3z^2 + 3z + 2 = 0$.

35. Use Theorem 1.1.1 to find the roots of $z^3 + 5z^2 + 10z + 12 = 0$, and hence confirm the result of Theorem 1.1.2.
36. Use Theorem 1.1.1 to find the roots of $z^3 + 5z^2 + 14z + 24 = 0$, and hence confirm the result of Theorem 1.1.2.

1.2 Curves, Domains, and Regions

The need to consider curves and areas in the complex plane occurs throughout the study of complex analysis, so the nature of curves and the way they are defined must be described. An unbroken curve, or path, of finite length comprising a continuous set of points in the complex plane with distinct end points is called an *arc*. A closed curve formed by joining end to end a number of arcs, with the initial point of the first arc joined to the end point of the last arc, is called a *closed curve* or *contour* in the complex plane.

Of special importance are *simple arcs* that do not intersect themselves and contain no loops. A *simple closed curve* is a closed curve that forms a single loop such as a circle, an ellipse, or a triangular path; whereas a curve such as a figure eight, although a closed curve, is *not* a simple closed curve. We take it as axiomatic that a simple closed curve in the complex plane has points (complex numbers) that lie *inside* and points that lie *outside* **the curve**; by convention when a point in the complex plane is moved around a simple closed curve in the *positive sense*, it does so in a *counterclockwise* direction. When considering contour integrals in subsequent chapters, it will be important to distinguish between a point moving around a simple closed curve in the positive sense, and one moving around the curve in the *negative sense*, which is in a *clockwise* direction.

Figure 1.6 shows some typical simple closed curves which are formed by connecting end-to-end circular arcs and straight line segments, with the

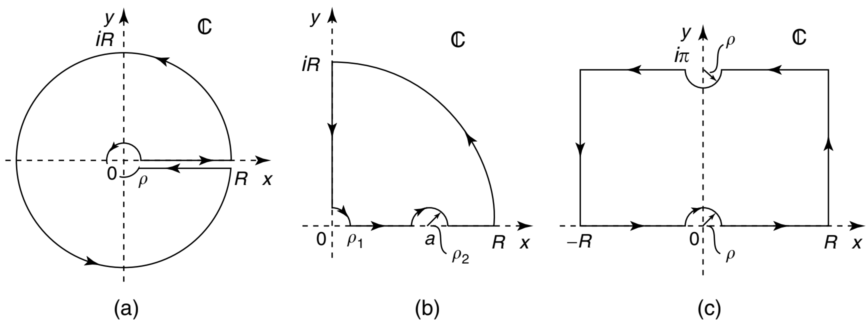

FIGURE 1.6
Examples of simple closed contours formed by joining up end-to-end circular arcs and straight line segments.

positive sense around each curve shown by arrows. Each of the small circular arcs in Figure 1.6(b, c) is called an *indentation* in the contour and when considering contour integration, indentations are necessary to avoid the contour passing through a singularity of a complex function.

When developing the calculus of complex functions, it is necessary to consider complex numbers as *complex variables*, and this happens in a different context when examining the geometrical properties of complex functions in connection with *conformal mappings*. In a conformal mapping, an arc (path) traced out as a point z moves in the z-plane is transformed by a complex function $w = f(z)$ into a related arc traced out by the variable w. So when displaying the geometrical consequences of such a transformation, called a *mapping* from z to w, it is necessary to introduce two different complex planes, one the z-plane, and the other the w-plane, though each will be contain the same set of points $\mathbb{C}$. Figure 1.7 shows how points on a circle with its center slightly displaced from the origin in the z-plane are mapped to points in the w-plane by the complex function $w = z + i/z$. The airfoil like closed simple curve in the w-plane is called a *Joukowski profile*, and is studied in Chapter 4 on conformal mapping.

A significant difference between real and complex numbers, already mentioned in Section 1.1, is that complex numbers have no natural order. So when working with the real number system $\mathbb{R}$, it makes sense when referring to limiting operations involving infinity to write $+\infty$ or $-\infty$, but no such distinction can be made with the complex number system $\mathbb{C}$, although some concept of infinity must still be retained. This is achieved by defining an idealized "*point at infinity*," written ∞ without a $\pm$ sign, to be the set of all numbers z that lie outside a circle of an arbitrarily large radius in the z-plane with its center at the origin. A more precise definition of the point at infinity is: the set of all complex numbers z that lie within an ε-neighborhood of infinity, defined as those z such that $|z| > 1/\varepsilon$, where ε is an arbitrarily small positive number. As each point in the complex plane represents a specific complex number, from now on the terms point and complex number will be used interchangeably because they are synonymous.

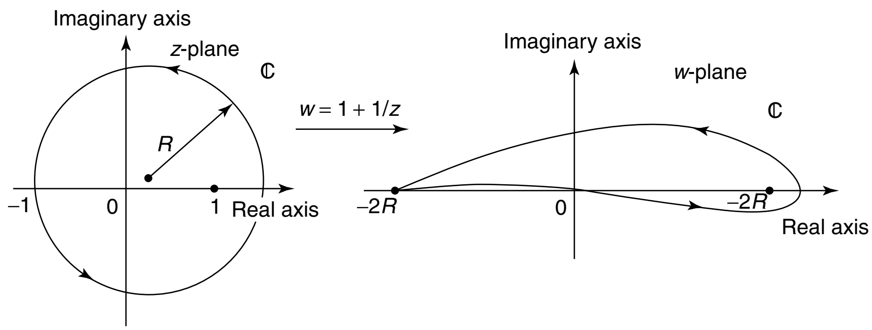

FIGURE 1.7
Two simple closed curves related by the Joukowski transformation.

The points in the complex plane $\mathbb{C}$ represent every complex number with the exception of the point at infinity. When the point at infinity is added to $\mathbb{C}$ the result is called *the* ***extended complex plane***, and it will always be this plane that will be used whenever limiting operations involving infinity arise.

A convenient finite geometrical representation of the extended complex plane involves using a stereographic projection in which the points in the extended complex plane are brought into one-to-one correspondence with points on a sphere of unit diameter. The idea is illustrated in Figure 1.8, where the sphere of unit diameter stands tangent to the extended complex plane at its origin O, with the point on the sphere vertically above O denoted by N for the *north pole* of the sphere, though the point O is not called the south pole.

A set of three-dimensional axes O(ξ, η, ζ) is located at O with the ξ- and η-axes coinciding, respectively, with the x- and y-axes in the complex plane, while the positive ζ-axis is drawn vertically above O through N. A point representing an arbitrary complex number $z = x + iy$ in the extended complex plane is then joined to the point N on the sphere by a straight line that intercepts the sphere at a point P(ξ, η, ζ). The coordinates (ξ, η, ζ) corresponding to $z = x + iy$ are the *stereographic coordinates* of z, and it can be seen that every finite point in the complex plane corresponds to a unique point on the sphere. The point at infinity in the extended complex plane, that is where the points outside an arbitrarily large circle in the complex plane centered on O correspond,

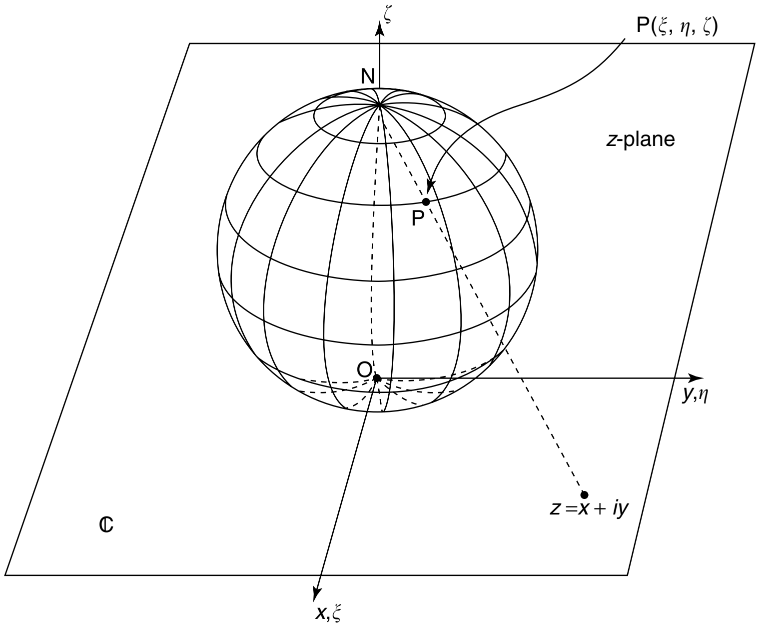

FIGURE 1.8
Stereographic projection of points on the Riemann sphere.

in the limit as the radius of the circle becomes arbitrarily large, to the point N on the sphere. This sphere is called the *Riemann sphere*, so with the sole exception of point N, every point on the Riemann sphere corresponds to a unique complex number z in the extended complex plane.

A detailed discussion of arcs and curves involves using the branch of mathematics called *topology*, but it will be sufficient to outline a few of its most important ideas. One property of any simple closed curve Γ which has already been mentioned is that it separates points in the complex plane into those that lie *inside* the curve and those that lie *outside* it. That is a simple closed curve has an *interior* and an *exterior*. This seemingly obvious result that we take as axiomatic is remarkably difficult to prove. The first attempt at a proof was given by the French mathematician Camille Jordan (1838–1922), though his proof turned out to be incomplete in some respects, and only later in 1905, was a rigorous proof given by the American topologist Oswald Veblen (1880–1960). To honor the work of Camille Jordan, the fact that a simple closed curve has both an inside and an outside is now called the *Jordan curve theorem*, and simple closed curves are also called *Jordan curves*.

Two important simple closed curves that often arise in practice are a rectangle and a circle. Figure 1.9(a) shows a set of points S in the complex plane lying strictly inside the shaded rectangular area defined by $a < \text{Re}\{z\} < b$ and $c < \text{Im}\{z\} < d$, where a, b, c and d are real numbers. Figure 1.9(b) shows a set of points S lying strictly inside a circle of radius ρ centered on the point z_0, that can be described analytically by writing $|z - z_0| < \rho$. When expressed in words this last inequality says that the points z in S are all such that the modulus of $z - z_0$ (that is the distance of z from z_0) is always strictly less than the radius ρ of the circle with its center at z_0. This circular area is called an *open disk* in the complex plane, while an area S such that $|z - z_0| \leqslant \rho$ where points on the bounding circle are included in S is called a *closed disk*. Thus the difference between an open and a closed disk is that in an open disk the points on the circular boundary are *excluded* from S, whereas in a closed disk the points on the circular boundary are *included* in S. Figure 1.9 uses the standard graphical convention that points to be excluded from S are shown as points

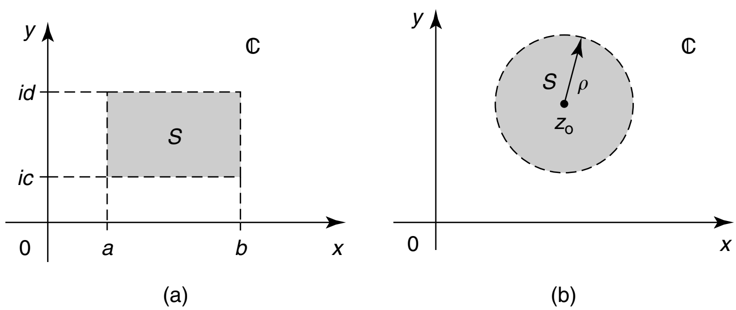

FIGURE 1.9
(a) An open rectangle. (b) A closed circular disk.

on a *dashed line*, whereas points to be *included* in S are shown as points on a *solid* line.

Often the radius ρ of an open disk is small and equal to a positive number δ, in which case the set of points $|z - z_0| < \delta$ inside the disk is called a *δ-neighborhood* of point z_0, sometimes denoted symbolically by writing $N(z_0, \delta)$. On occasion, when a singularity of a complex function arises at the point z_0 in $N(z_0, \delta)$, it is necessary to exclude the single point z_0 itself. When this occurs, we write $N'(z_0, \delta)$ and define this by the inequality $0 < |z - z_0| < \delta$. This set of points from which only the point z_0 is excluded, called a *deleted neighborhood* of z_0, and the deleted disk is then called a *punctured disk*.

A *boundary point* of a set S is a point where *every* neighborhood of the point contains points belonging to S and points not belonging to S. A set of points S with a boundary comprising a simple closed curve Γ is said to be *open* if every neighborhood of a point in S only contains points of S, irrespective of how close the point is to Γ. Correspondingly, a set of points S with a boundary comprising a simple closed curve Γ is said to be *closed* if it contains all of its boundary points. Figure 1.10(a) illustrates an example of an open set S, which shows the open sector

$$\alpha < \text{Arg}\{z\} < \beta \quad \text{and} \quad |z| < R, \quad \text{with } 0 < \alpha < \beta < \pi/2,\ R > 0,$$

while Figure 1.10(b) shows the closed sector

$$\alpha \leqslant \text{Arg}\{z\} \leqslant \beta \quad \text{and} \quad |z| \leqslant R, \quad \text{with } 0 < \alpha < \beta < \pi/2,\ R > 0.$$

Figure 1.10(b) also shows a typical boundary point P on the sector and an associated δ-neighborhood of P.

Often the shape of an arc or closed curve Γ in the complex plane is sufficiently complicated that it needs to be represented in *parametric form*. Such a

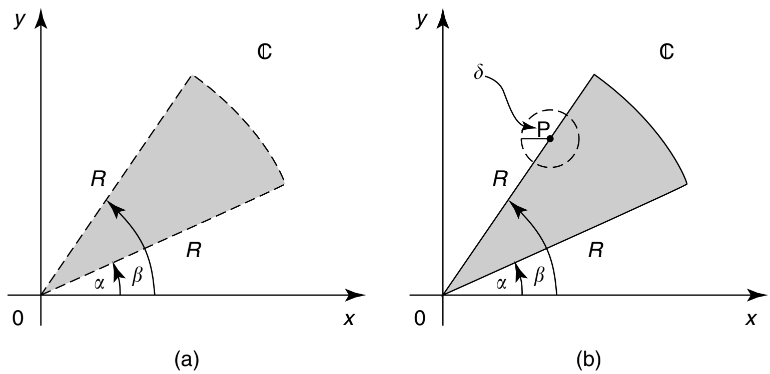

FIGURE 1.10
(a) An open sector. (b) A closed sector and a δ-neighborhood of P.

representation of Γ is accomplished in the z-plane by defining the x- and y-coordinates of points on Γ in the parametric form

$$x = x(s), \qquad y = y(s) \qquad \text{for } \alpha \leqslant s \leqslant \beta, \tag{1.44}$$

where $x(s)$ and $y(s)$ are monotonic functions and s is a parameter. In terms of the complex variable z the curve Γ has the representation

$$z(s) = x(s) + iy(s), \quad \text{for } \alpha \leqslant s \leqslant \beta, \tag{1.45}$$

so as s increases the point $z(s)$ moves in a particular direction along the curve Γ. A typical example of a parametric representation involves a circle of radius r centered on the point $z = a + ib$, which can be written as

$$x = a + r\cos\theta, \quad y = b + r\sin\theta, \quad \text{with } 0 \leqslant \theta \leqslant 2\pi, \tag{1.46}$$

where θ is the parameter. It should be recognized that parametric representations of curves are not unique and an equally good parametric representation of the circle described by Equation (1.46) is

$$x = a + \cos 2\theta + i\sin 2\theta, \quad \text{with } 0 \leqslant \theta \leqslant \pi, \tag{1.47}$$

Others are possible. In practice, the parametrization chosen is always the one that is easiest to use in subsequent calculations. The fact that different parametrizations are possible is unimportant when used to describe curves (*contours*) around which complex integration is to be performed because there the nature of the parametrization is taken into account automatically.

A curve represented by Equation (1.45) is said to be *smooth* if $x(s)$ and $y(s)$ are continuously differentiable and to be *piecewise smooth*, or *sectionally smooth*, if the curve is continuous (unbroken) but formed by joining end to end a finite number of piecewise smooth curves. It follows immediately from elementary calculus that when the length l of the smooth arc or curve Γ represented by Equation (1.45) is required, it is given by

$$l = \int_{\alpha}^{\beta} \sqrt{[dx/ds]^2 + [dy/ds]^2}\, ds, \tag{1.48}$$

where the positive square root is always taken.

The sets of points S (areas of the complex plane) that concern us here are those which are *connected.* This means for sets S in which any two points can be connected by an unbroken arc (not necessarily smooth), the points of which all lie in S. Two sets of points S_1 and S_2 are said to be *disconnected* if any arc joining an arbitrary point in each set contains points in S_1 and S_2, and also points *outside* both of these sets. The points in an open or closed disk and in a rectangular area are connected, whereas the points in two nonintersecting disks are not connected. A set of points within some neighborhood of the origin (which may be large or small) is said to be *bounded*, otherwise the set of points is said to be *unbounded*. The points in the sectors shown in Figures 1.10(a) and (b) are bounded, while the sets of points belonging to any parallel strip of finite width in the extended complex plane, or in the first quadrant of the plane, are unbounded.

Two terms that will often be used are *domain* and *region*. A *domain* is an open connected set of points, and a *region* is a domain together with all of its boundary points. Figure 1.10(a) shows a typical domain and Figure 1.10(b) a typical region.

A connected set of points S will be said to be *simply connected* if the interior of every simple closed curve drawn in S only contains points of S, otherwise the set will be said to be *nonsimply connected*. Thus a set S will be nonsimply connected if inside it simple closed curves are drawn entirely in S, with the property that the interior of some of the curves *do not* contain points of S. Figure 1.11(a) shows an example of a simply connected set of points S, and Figure 1.11(b) shows a nonsimply connected set of points S, where although the closed curve γ_1 can be contracted to a single arc joining the points P_1 and P_2, the curve γ_2 cannot be contracted to a single arc joining the points Q_1 and Q_2 because it encloses the domain D that does not contain points of S.

Exercises 1.2

In Exercises 1 through 11, shade the required domain or region and indicate a boundary belonging to it by a solid line and one that is excluded from it by a dashed line.

1. $|z - 3| \geqslant 1$.
2. $|\text{Arg}\{z\}| \leqslant \pi/6$, $\text{Re}\{z\} < 1$.
3. $\text{Im}\{z\} > 0$, $R_1 \leqslant |z| < R_2$ $(0 < R_1 < R_2)$.
4. $R_1 < |z| \leqslant R_2$, $\text{Re}\{z\} \geqslant \alpha$, $\text{Re}\{z\} \geqslant \alpha$, $\text{Re}\{z\} \leqslant \beta$ $(0 < \alpha < R_1 < \beta < R_2)$.
5. $\alpha < \text{Arg}(z) < \beta$, $a < \text{Im}\{z\} < b$ $(0 < \alpha < \beta < \pi/2$, $0 < a < b)$.

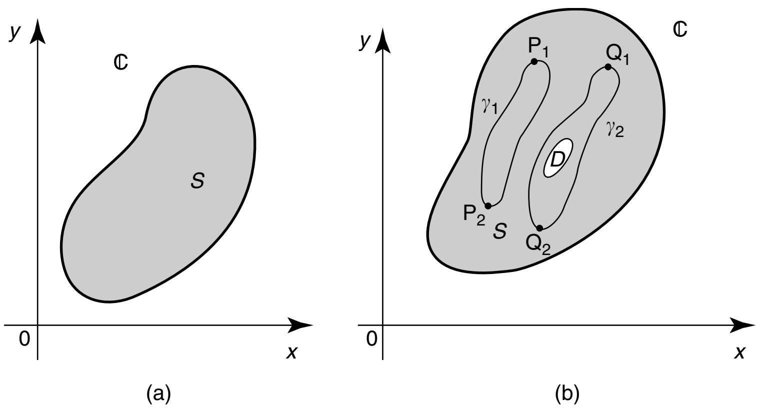

FIGURE 1.11
(a) A simply connected set of points S. (b) A nonsimply connected set of points S.

6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \geqslant 1,\ y > x, y > -x\ (0 < b < a)$.
7. $|z| \geqslant 1, -2 \leqslant \text{Re}\{z\} \leqslant 3, -1 \leqslant \text{Im}\{z\} \leqslant 2$.
8. $|z - 2| + |z + 2| \geqslant 6, |z - 2| + |z + 2| \leqslant 8$.
9. $\text{Re}\{z\} < \cos\xi, \text{Im}\{z\} < b\sin\xi, a > b > 0\ (-\pi < \xi \leqslant \pi)$.
10. $|z| \leqslant 1 + \cos\theta, \theta = \text{Arg}(z)\ (-\pi < \theta \leqslant \pi)$.
11. $\text{Re}\{z\} > a\cosh\xi, 0 < \text{Im}\{z\} \leqslant b\sinh\xi, a > b > 0\ (0 \leqslant \xi < \infty)$.

In Exercises 12 through 16, sketch the curves in $\mathbb{C}$ defined by the given equation.

12. $|z - 1| = |z + 2|$.
13. $\left|\dfrac{z+1+i}{z-1-i}\right| = 1$.
14. $4|z + 2| = |z - 2|$.
15. $\text{Re}\{z + 1/z\} = \alpha x\ (\alpha > 1)$.
16. $\text{Im}\{z + 1/z\} = \alpha y\ (0 < \alpha < 1)$.
17. For what range of values of a are there points of $\mathbb{C}$ common to both $|z - 1| \leqslant 1$ and $|z - a| \leqslant 1$.
18. Give inequalities specifying the set of points S in $\mathbb{C}$ that lies inside but not on the boundary of an annulus with inner radius 1 and outer radius 2 with its center at $z = i$, from which points z have been removed such that $0 < \text{Arg}(z) \leqslant \pi/3$. Sketch the boundaries of S, shade its interior points and state if it is a region or a domain.
19. Give inequalities specifying the set of points S in $\mathbb{C}$ that are exterior to an ellipse with semimajor axis 3 and semiminor axis 2 centered on the point $z = 2 - i$, with the semimajor axis parallel to the imaginary axis. Sketch the boundaries of S, shade its interior points, and state if it is a region or a domain.
20. Let the set S of points in $\mathbb{C}$ be those in an open unit circle centered on the origin, from which have been removed the points belonging to any three different diameters. Explain why S is not connected. Give two ways in which S may be modified to make it connected, and justify your answers.

1.3 Analytic Functions

We call z a complex variable if $z \in \mathbb{C}$ is allowed to assume values in some set D in the complex plane. Let us now denote by f a rule (usually a mathematical expression) that assigns to each $z \in D$ a unique complex number, w. Then f is called a *function of the complex variable* z defined on set D, that is usually a region, and we will show this by writing

$$w = f(z), \quad z \in D. \tag{1.49}$$

Expressed more precisely, $f(z)$ in Equation (1.49), is the *value* of the function f at z: that is the complex number w assigned by the function f to the number $z \in D$. By analogy with the real variable case, the set D is called the *domain of definition* of f, while the set R of all the complex numbers w is called the *range* of the function f. Although f is the *function* and $f(z)$ is its *value* for $z \in D$, it is an accepted convention to sometimes misuse this notation by referring to $f(z)$ as the function (rather than f) because this has the advantage of making explicit the independent variable z involved.

The set D on which function f is defined forms part of the definition of f and if D is not specified, it is taken to be the largest part of the complex plane in which f has meaning. Thus $f(z) = z^2$ for $|z| < R$ is a function defined for all points of the open disk of radius R centered on the origin, whereas writing $f(z) = z^2$ implies that z is any point in the complex plane.

Let a complex variable w depending on z be expressed in the real and imaginary form $w = u + iv$. Then if z is expressed in the Cartesian form $z = x + iy$, it follows that in general u and v will depend on x and y, so that

$$w = f(z) = u(x, y) + iv(x, y). \tag{1.50}$$

Here the functions $u(x, y)$ and $v(x, y)$ are called the *real* and *imaginary parts* of $f(z)$, respectively and that Equation (1.50) is the *Cartesian representation* of $f(z)$ which displays the explicit dependence of $f(z)$ on x and y.

Alternatively, if z is expressed in the polar form $z = r(\cos\theta + i\sin\theta)$, the analogous result is

$$w = f(z) = u(r, \theta) + iv(r, \theta), \tag{1.51}$$

which is called the *polar representation* of $f(z)$ where, of course, the functions u and v in Equation (1.51) are not the same as those in Equation (1.50).

Example 1.3.1 A Function Expressed in Cartesian and Polar Form
Let

$$w = f(z) = z^2 + 4z + 3, \quad z \in \mathbb{C},$$

then in this case set D is the set of all complex numbers. The Cartesian representation $f(z) = u(x, y) + iv(x, y)$ is found by setting $z = x + iy$ in $f(z)$ to obtain

$$\begin{aligned} w = f(z) = u(x, y) + iv(x, y) &= (x + iy)^2 + 4(x + iy) + 3 \\ &= x^2 + 2ixy - y^2 + 4x + 4iy + 3. \end{aligned}$$

Equating the respective real and imaginary parts on each side of this result gives

$$u(x, y) = \text{Re}\{f(z)\} = x^2 - y^2 + 4x + 3$$

and

$$v(x, y) = \text{Im}\{f(z)\} = 2xy + 4y.$$

Similarly, the polar representation $f(z) = u(r, \theta) + iv(r, \theta)$ follows by setting $z = r(\cos\theta + i\sin\theta)$ in $f(z)$ to obtain

$$\begin{aligned} f(z) &= r^2(\cos\theta + i\sin\theta)^2 + 4r(\cos\theta + i\sin\theta) + 3 \\ &= r^2(\cos 2\theta + i\sin 2\theta) + 4r(\cos\theta + i\sin\theta) + 3. \end{aligned}$$

Expanding the expression on the right, collecting its real and imaginary parts and equating the real and imaginary parts on each side of the equation gives

$$u(r, \theta) = \text{Re}\{f(z)\} = r^2 \cos 2\theta + 4r\cos\theta + 3 \tag{1.52}$$

and

$$v(r, \theta) = \text{Im}\{f(z)\} = r^2 \sin 2\theta + 4r\sin\theta. \tag{1.53}$$

If, for example, $z = 1 - i$ for which $r = |z| = \sqrt{2}$ and $\text{Arg}\{z\} = -i\pi/4$, the Cartesian representation becomes $f(1 - i) = 7 - 6i$ and, of course, the polar representation with $r = \sqrt{2}$ and $\theta = -\pi/4$ also gives the same result. ◇

Example 1.3.2 A Function Defined on a Disk

Let

$$f(z) = \begin{cases} z & \text{for } |z| < 1 \\ 1/\bar{z} & \text{for } 1 < |z| < 3. \end{cases}$$

In this example the set D on which $f(z)$ is defined is an open disk of radius 3 centered on the origin. Notice that $f(z)$ is defined differently within the open disk centered on the origin, from its definition in the annular domain $1 < |z| < 3$. Setting $z = x + iy$ in $f(z)$ gives

$$f(z) = \begin{cases} x + iy, & \text{for } |z| < 1 \\ \dfrac{x}{x^2 + y^2} + i\dfrac{y}{x^2 + y^2}, & \text{for } 1 < |z| < 3. \end{cases}$$

This shows that

$$u(x, y) = \text{Re}\{f(z)\} = \begin{cases} x, & \text{for } |z| < 1 \\ \dfrac{x}{x^2 + y^2}, & \text{for } 1 < |z| < 3, \end{cases}$$

and

$$v(x, y) = \text{Im}\{f(z)\} = \begin{cases} y, & \text{for } |z| < 1 \\ \dfrac{y}{x^2 + y^2}, & \text{for } 1 < |z| < 3. \end{cases}$$

Inspection of the different forms of $u(x, y)$ and $v(x, y)$ inside and outside the unit circle shows that they coincide on $|z| = 1$, so instead of omitting to define $f(z)$ on the unit circle, the function could have been defined as

$$f(z) = \begin{cases} z & \text{for } |z| \leqslant 1 \\ 1/\bar{z} & \text{for } 1 < |z| < 3. \end{cases}$$ ◇

1.3.1 Limits and Continuity

The real variable definition of a limit of a function of two variables can be extended in a natural manner to the case of a complex function. Starting with an intuitive definition, suppose that the complex function $f(z)$ defined in a domain D is such that by taking z sufficiently close to a point z_0 in D, it is possible to make $f(z)$ as close as we wish to some complex number L. Then, provided this approach to L is *independent* of the way in which z tends to z_0, written $z \to z_0$, the function will be said to have the *limit* L as z tends to z_0, and we will write

$$\lim_{z \to z_0} f(z) = L. \tag{1.54}$$

Notice that like the real variable case, this definition of a limit assumes nothing about the behavior of the function $f(z)$ at $z = z_0$, where it may or may not be defined, and when it is its value is not necessarily such that $f(z_0) = L$. Notice also the requirement that when taking the limit, it is necessary that the result is *independent* of the way in which z tends to z_0. When we define the derivative of a complex function, based as would be expected on a limit, this condition will be seen to play a fundamental role when arriving at a condition that ensures a complex function has a unique derivative.

The weakness of this intuitive definition is due to its failure to say in what sense two complex numbers are *close*, and this in turn is due to the fact that complex numbers have no natural order. The difficulty is overcome by defining

z to be close to z_0 if $z \neq z_0$ lies within a δ-neighborhood $N(z_0, \delta)$, where $\delta > 0$ is an arbitrarily small real number. The number δ then provides a direct measure of the closeness of z to z_0.

We now formulate the rigorous definition of a limit.

1.3.2 Definition of a Limit

The function $f(z)$ has the **limit** L as $z \to z_0$, denoted by

$$\lim_{z \to z_0} f(z) = L,$$

if for any real number $\varepsilon > 0$, however small, it is possible to find a real number $\delta > 0$, depending on $\varepsilon > 0$ such that for every $z \neq z_0$ in the punctured disk $0 < |z - z_0| < \delta$, $|f(z) - L| < \varepsilon$.

A punctured neighborhood has been used because the value of $f(z)$ at z_0 does not enter into the definition of a limit. It is this use of a punctured neighborhood of z_0 that imposes the condition that z must tend to z_0 *independently* of the way in which this occurs because it means that z may follow *any* path in $N(z_0, \delta)$ as it tends to z_0.

Let us be quite clear about what is meant when we say that z tends to z_0 independently of the way this occurs because this condition must hold in any neighborhood of z_0, and not only in a circular disk centered on the point. Consider Figure 1.12 in which an arbitrary neighborhood D of a point z_0 is shown, together with two arbitrary simple arcs Γ_1 and Γ_2 drawn from points P_1 and P_2 to z_0. Then to say z tends to z_0 independently of the way this happens means that z may move along any simple arc like Γ_1 or Γ_2, with an arrow showing the way z must move.

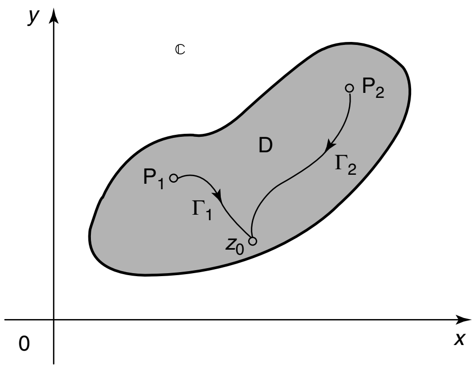

FIGURE 1.12
Two paths Γ_1 and Γ_2 along which $z \to z_0$.

Let us show that this definition of a limit implies that when a limit exists it is unique. Proving this is straight forward will provide an example of how the rigorous definition of a limit can be used. Suppose, if possible, that $f(z)$ has two different limits L_1 and L_2 as $z \to z_0$. Then by definition we can find an $\varepsilon > 0$ such that

$$|f(z) - L_1| < \varepsilon \quad \text{and} \quad |f(z) - L_2| < \varepsilon,$$

for any $z \neq z_0$ in $|z - z_0| < \delta$. Then

$$|L_1 - f(z)| + |L_2 - f(z)| < 2\varepsilon \quad \text{for } |z - z_0| < \delta \quad \text{and} \quad z \neq z_0.$$

To proceed further we will apply the version of the triangle inequality given in Equation (1.16) with $z_1 = L_1$, $z_2 = f(z)$ and $z_3 = L_2$ to the above result. This strengthens and simplifies the inequality to

$$|L_1 - L_2| < 2\varepsilon \quad \text{for all } z \neq z_0 \text{ in } |z - z_0| < \delta.$$

Because ε is arbitrary, so also is 2ε and we conclude that $L_1 = L_2$, thereby establishing the uniqueness of the limit.

The Theorem 1.3.1 summarizes the most important consequences of combining two functions $f(z)$ and $g(z)$ when each has a limit as $z \to z_0$. Only result (i) will be proved, as the other proofs follow in similar fashion and so are left as exercises.

THEOREM 1.3.1 Limit Theorems

If

$$\lim_{z \to z_0} f(z) = \alpha \quad \text{and} \quad \lim_{z \to z_0} g(z) = \beta, \text{ then}$$

$$\lim_{z \to z_0} [f(z) + g(z)] = \alpha + \beta,$$

$$\lim_{z \to z_0} [f(z)g(z)] = \alpha\beta,$$

$$\lim_{z \to z_0} [f(z)/g(z)] = \alpha/\beta, \text{ provided } \beta \neq 0,$$

Let

$$\lim_{w \to \beta} f(w) = k, \text{ then } \lim_{z \to z_0} f(g(z)) = k.$$

PROOF

To prove (i), by definition we can find an $\varepsilon > 0$ and $\delta_1, \delta_2 > 0$ such that

$$|f(z) - \alpha| < \tfrac{1}{2}\varepsilon \quad \text{for all } z \neq z_0 \text{ in } |z - z_0| < \delta_1,$$

and

$$|g(z) - \beta| < \tfrac{1}{2}\varepsilon \quad \text{for all } z \neq z_0 \text{ in } |z - z_0| < \delta_2.$$

Taking $\delta = \min(\delta_1, \delta_2)$, both inequalities are true for all $z \neq z_0$ in $|z - z_0| < \delta$. Adding these two inequalities gives

$$|f(z) - \alpha| + |f(z) - \beta| < \varepsilon.$$

Using triangle Inequality [Equation (1.15)] of Section 1.1 with $z_1 = f(z) - \alpha$ and $z_2 = g(z)$ gives

$$|[f(z) + g(z)] - (\alpha + \beta)| < \varepsilon \quad \text{for all } z \neq z_0 \text{ in } |z - z_0| < \delta.$$

As $\varepsilon > 0$ is arbitrary, this shows that

$$\lim_{z \to z_0} \{[f(z) + g(z)] - (\alpha + \beta)\} = 0,$$

from which result (i) follows immediately. ◆

When using the results of this theorem to evaluate specific limits, it is usually easiest to work directly with the complex functions $f(z)$ and $g(z)$, though sometimes it is convenient to use the related limits of real functions that are implied by the theorem. To see how this works, starting from $\lim_{z \to z_0} f(z) = L$, by writing $z_0 = x_0 + iy_0$, $L = a + ib$ and $f(z) = u + iv$ we have

$$\lim_{z \to z_0} f(z) = \lim_{\substack{x \to x_0, \\ y \to y_0}} u(x, y) + i \lim_{\substack{x \to x_0, \\ y \to y_0}} v(x, y), \tag{1.55}$$

so equating the respective real and imaginary parts on each side of the equation gives

$$\lim_{\substack{x \to x_0, \\ y \to y_0}} u(x, y) = a \quad \text{and} \quad \lim_{\substack{x \to x_0, \\ y \to y_0}} v(x, y) = b. \tag{1.56}$$

Example 1.3.3 Limits of Complex Functions
Find the following limits when they exist:

$$\text{(i)} \lim_{z\to 1+i}\left(\frac{z^2+4z-1}{z+2}\right), \quad \text{(ii)} \lim_{z\to i}\left(\frac{z^3+2z^2+z+2}{z-i}\right), \quad \text{(iii)} \lim_{z\to 0}\left[(1+z)\sin\left(\frac{1}{|z|}\right)\right].$$

SOLUTION

(i) The polynomials in the quotient have the nonzero values

$$\lim_{z\to 1+i}(z^2+4z-1) = 3+6i \quad \text{and} \quad \lim_{z\to 1+i}(z+2) = 3+i,$$

so

$$\lim_{z\to 1+i}\left(\frac{z^2+4z-1}{z+2}\right) = \frac{3+6i}{3+i} = \tfrac{3}{2}(1+i).$$

(ii) In this case both polynomials in the quotient vanish as $z \to i$, so we cannot use the result of Theorem 1.3.1 directly. However, the fact that each polynomial vanishes when $z = i$ means that $(z-i)$ is a factor of both polynomials. So dividing both numerator and denominator by $(z-i)$ gives

$$\lim_{z\to i}\left(\frac{z^3+2z^2+z+2}{z-i}\right) = \lim_{z\to i}\left[z^2+(2+i)z+2i\right] = -2+4i.$$

(iii) No limit exists in this case, because although the factor $(1+z)$ becomes 1 in the limit as $z \to 0$, the bounded factor $\sin 1/|z|$ has no limit as it merely oscillates between ± 1 with increasing rapidity as $z \to 0$.

Example 1.3.4 A Limiting Process that Depends on the Direction of Approach
The function of two real variables

$$h(x,y) = \frac{xy}{2x^2+3y^2}$$

is a typical example of a function of two real variables that might form the real or imaginary part of a complex function. Let us show that this function has no limit at the origin. To demonstrate this we set $y = mx$, and let x and y tend to zero along this line by letting x tend to zero. The value of m will determine the direction of approach to the origin. So if the result of this operation depends on m, the function can have no limit because when a limit exists it must be independent of the way the general point $(x, y) \to (0, 0)$. Setting $y = mx$ in $h(x, y)$ gives

$$h(x, mx) = \frac{mx^2}{2x^2 + 3m^2x^2} = \frac{m}{2 + 3m^2},$$

which is independent of x, so in fact $h(x, y)$ has the constant value $m/(2 + 3m^2)$ along the line $y = mx$. Thus

$$\lim_{\substack{x \to 0,\\ y = mx}} \left(\frac{xy}{2x^2 + 3y^2} \right) = \lim_{x \to 0} h(x, mx) = \frac{m}{2 + 3m^2}.$$

As this result depends of m the function can have no limit at the origin. ◇

To develop the study of complex functions, it is now necessary to introduce the important concept of *continuity*. Simply stated, a complex function $f(z)$ is said to be *continuous* at $z = z_0$ if it possesses a limit L as $z \to z_0$ and, furthermore, $f(z_0)$ exists and is equal to L. Thus continuity is a property of a function $f(z)$ in a neighborhood of any point where it is continuous. The formal definition now follows.

1.3.4 Definition of Continuity

The complex function $f(z)$ is said to be *continuous* at z_0 if $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If the limit exists but does not equal $f(z_0)$, or if it is not defined, the function $f(z)$ is said to be *discontinuous* at z_0.

The following text addresses functions that are continuous throughout some domain of the complex plane. We see from this that the function

$$f(z) = \frac{z^2 - 4z - 1}{z + 2}$$

in (i) of Example 1.3.3 is continuous away from the point $z = -2$ because that is the only point where the quotient is undefined. Both the numerator and denominator of the function

$$f(z) = \frac{z^3 + 2z^2 + z + 2}{z - i}$$

in (ii) of Example 1.3.3 vanish when $z = i$, so the function is not properly defined there. However, there the difficulty was resolved by canceling the factor $(z - i)$ that was common to both the numerator and the denominator. The point $z = i$ is called a *singularity* of the function and when the limit at a singularity such as this can be removed by cancellation of a common factor in the numerator and denominator it is called a ***removable singularity***.

THEOREM 1.3.2 Properties of Continuous Functions
Let $f(z)$ and $g(z)$ be continuous at each point of a domain D. Then

$f(z) \pm g(z)$ is continuous throughout D;
$f(z)g(z)$ is continuous throughout D;
$f(z)/g(z)$ is continuous throughout D except at points where $g(z)$ vanishes but $f(z)$ remains finite and nonvanishing.

PROOF

The proofs of these properties are almost the same as the proofs of the results of Theorem 1.3.1, so as one of those proofs has been given in detail the proofs here will be left as exercises. ◆

As a typical illustration of a consequence of continuity we will prove that $f_n(z) = z^n$ is continuous for $n = 1, 2, \ldots$, and all $z \in \mathbb{C}$. Taking $\delta = \varepsilon$ in the definition of a limit it follows trivially that $f_1(z)$ is continuous for all $z \in \mathbb{C}$. Now suppose for some positive integer m that $f_m(z_0)$ is continuous for all $z \in \mathbb{C}$. Then by (ii) of Theorem 1.3.2 the function $f_{m+1}(z) = z^{m+1} = z^m z = f_m(z)f_1(z)$ must be continuous for all $z \in \mathbb{C}$.

That $f_n(z) = z^n$ is continuous now follows by mathematical induction, because $f_1(z)$ is continuous, and the continuity of $f_{m+1}(z)$ follows from the continuity of $f_m(z)$, so the result must be true for $n = 1, 2, \ldots$. Clearly, from this result and Theorem 1.3.2(i), polynomials $P_n(z)$ must also be continuous for all $z \in \mathbb{C}$.

1.3.5 Differentiability and Derivatives: Analytic Functions

A function $f(z)$ of a complex variable that is defined at $z = z_0$, and in some neighborhood of z_0, will be said to be *differentiable* at z_0 with *derivative* $f'(z_0)$ at that point, if the limit

$$f'(z_0) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) \tag{1.57}$$

exists.

It is clear from Equation (1.57) that the first derivative of $f(z)$ with respect to z, denoted as in the real variable calculus by $f'(z_0)$, is a complex number specific to the point $z = z_0$. When $f(z)$ has a derivative at every point $z = z_0$ of some domain D, the suffix zero is omitted from z_0 and $f(z)$ is said to be differentiable throughout D with the *derivative* $f'(z)$ for $z \in D$. A function $f(z)$ that possesses a derivative at every point of a domain D is said to be *analytic* in D, although other names used in place of analytic are *regular* and *holomorphic*. A function $f(z)$ that is analytic throughout the whole of the finite complex plane is said to be an *entire function*. Example 1.3.5 shows that a simple example of an entire function is the polynomial

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

for $n = 1, 2, \ldots,$ where the coefficients a_i are complex numbers.

When the nature of the domain D need not be specified it is usual to omit all reference to it, and simply to refer to $f(z)$ as an analytic function. It often happens that a function $f(z)$ is analytic everywhere in some domain D with the exception of a finite number of isolated points $z_1, z_2, \ldots, z_n$ where $f'(z)$ does not exist. Such points are called *isolated singular points*, or *isolated singularities* of $f(z)$. A typical example of a function with isolated singularities is

$$f(z) = \frac{z+4}{(z+1)(z-3)},$$

that Example 1.3.7(ii) shows to be analytic for all z with the exception of the two isolated singularities at $z = -1$ and $z = 3$ where differentiability fails because at each point the numerator is finite and nonzero, while the denominator vanishes.

An important consequence of the differentiability of $f(z)$ at a point z_0 is that it implies the continuity of $f(z)$ in a neighborhood of z_0, which can be seen by writing the difference $f(z) - f(z_0)$, with $z \neq z_0$, as

$$f(z) - f(z_0) = \left(\frac{f(z) - f(z_0)}{z - z_0}\right)(z - z_0),$$

because from Equation (1.57) and (ii) in Theorem 1.3.1(ii) we have

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \left(\frac{f(z) - (z_0)}{z - z_0}\right) \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0.$$

This shows that

$$\lim_{z \to z_0} f(z) = f(z_0),$$

that is just the definition of continuity given earlier.

An equivalent and often more useful form of the definition of the derivative given in Equation (1.57) is

$$f'(z_0) = \lim_{h \to z_0} \left(\frac{f(z_0 + h) - f(z_0)}{h} \right) \tag{1.58}$$

where, of course, h is a complex variable.

The next example shows that this is a working definition.

Example 1.3.5 Some Elementary Entire Functions

Show that:

(i) The function $f(z) = k$ (a complex constant) is everywhere differentiable and such that

$$f'(z) = \frac{dk}{dz} = 0 \quad \text{for all } z \in \mathbb{C}.$$

(ii) The function $f(z) = z^n$ with $n = 1, 2, \ldots,$ is everywhere differentiable and such that

$$f'(z) = \frac{d[z^n]}{dz} = nz^{n-1} \quad \text{for all } z \in \mathbb{C}.$$

(iii) The complex polynomial of degree n

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n,$$

in which $a_0, a_1, \ldots, a_n$ are arbitrary complex constants with $a_n \neq 0$, is everywhere differentiable and such that

$$P'(z) = \frac{d[P_n(z)]}{dz} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots + na_n z^{n-1} \quad \text{for all } z \in \mathbb{C}.$$

SOLUTION

Result (i) follows trivially from definition Equation (1.58) because for any given $z_0 \in \mathbb{C}$,

$$f'(z_0) = \left(\frac{d[k]}{dz} \right)_{z=z_0} = \lim_{h \to 0} \left(\frac{k(z_0 + h) - k(z_0)}{h} \right) = \lim_{h \to 0} \left(\frac{k - k}{h} \right) = 0.$$

The arbitrary nature of z_0 allows the suffix zero to be dropped so that z becomes a complex variable instead of a complex number. As a result the derivative of $f(z)$ as a function of z becomes

$$f'(z) = \frac{dk}{dz} = 0 \quad \text{for all } z \in \mathbb{C}.$$

Result (ii) follows in similar fashion after use of the binomial theorem. For any given $z = z_0$ and $n = 1, 2, \ldots$, we have

$$f'(z_0) = \left(\frac{d[z^n]}{dz}\right) = \lim_{h\to 0}\left(\frac{(z_0+h)^n - z_0^n}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{z_0^n + nz_0^{n-1}h + \frac{1}{2!}n(n-1)z_0^{n-2}h^2 + \cdots + h^n - z_0^n}{h}\right)$$
$$= \lim_{h\to 0}\left(nz_0^{n-1} + \tfrac{1}{2!}n(n-1)z_0^{n-2}h + \cdots + h^{n-1}\right) = nz_0^{n-1}.$$

Again dropping the suffix zero and allowing z to become an arbitrary complex variable shows that when $f(z)$ is regarded as a function of z it follows that

$$f'(z) = \frac{d[z^n]}{dz} = nz^{n-1} \qquad \text{for all } z \in \mathbb{C}.$$

This result is true for arbitrary n, and not simply for integral values of n, but to establish this it is necessary to modify this proof, although a description of this process is omitted.

To establish result (iii) it is necessary to show that $d[kz^n]/dz = knz^{n-1}$, where k is a constant, and that $d[Kz^m + kz^n]/dz = mKz^{m-1} + nkz^{n-1}$ for m, $n = 1, 2, \ldots$, for all $z \in \mathbb{C}$. The first result follows trivially from result (ii) by replacing z^n by kz^n and noticing that the constant factor k can then be removed from the subsequent calculations and replaced by a multiplication factor k. The second result also follows from result (ii) by replacing z^n by $Kz^m + kz^n$, separating out the two terms and using the result that $d[kz^n]/dz = knz^{n-1}$. Dropping the suffix zero to make $P_n(z)$ a function of z, and making repeated use of the derivative of a sum, we find that

$$P_n'(z) = \frac{d}{dz}[P_n(z)] = a_1 + 2a_2z + 3a_3z^2 + \cdots + na_nz^{n-1} \qquad \text{for all } z \in \mathbb{C}. \quad \diamond$$

Example 1.3.6 A Complex Function Only Differentiable at the Origin

This example provides an illustration of a function of a complex variable whose real and imaginary parts are both continuously differentiable for all x and y, yet the function is only differentiable at the origin. Consider the function $f(z) = z\bar{z}$. Taking an arbitrary point $z_0 \in \mathbb{C}$ and using Equation (1.58) gives

$$f'(z_0) = \lim_{h\to 0}\left(\frac{(z_0+h)(\bar{z}_0+\bar{h}) - z\bar{z}}{h}\right) = \lim_{h\to 0}\left[\bar{z}_0 + \bar{h} + z_0(\bar{h}/h)\right].$$

If $h \to 0$ through purely real values $\bar{h} = h$, s in the limit as $h \to 0$ we find $f'(z_0) = z_0 + \bar{z}_0$ which is a *real* number. However, if $h \to 0$ through purely imaginary values $\overline{h} = -h$, and now in the limit as $h \to 0$ we find $f'(z_0) = z_0 - \overline{z}_0$ which is a *purely imaginary* number. These two limits are equal only when $z_0 = 0$, so elsewhere the value of the limits depends on the way $h \to 0$ and this function of z is only differentiable at the origin.

A closer look at this function shows that the result should not be surprising because $z\overline{z}$ is always a real function, and if $f(z) = u + iv$, it follows that

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) \equiv 0.$$

Although the functions *u and v* are continuous and differentiable everywhere, the function $f(z) = z\overline{z}$ is not differentiable in the complex sense. This example illustrates the need for the test for complex differentiability given in the next theorem.

THEOREM 1.3.3 The Cauchy–Riemann Equations and Analyticity
Let the functions $u(x, y)$ and $v(x, y)$ in $f(z) = u(x, y) + iv(x, y)$ together with their first order partial derivatives be defined and continuous throughout some domain D, and let $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at every point of D. Then $f(z)$ is analytic in D, and its derivative in D when expressed in Cartesian form is given by either of the expressions

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or} \quad f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

To avoid interrupting the development of this section, the proof of this important theorem is deferred until Section 1.4 where after proving the theorem, another consequence of the Cauchy–Riemann equations is established for later use, and it is shown how, when either of the functions u or v is known, it is possible to find the other related function, and hence to construct an analytic function $f(z) = u(x, y) + iv(x, y)$.

Example 1.3.7 An Application of Theorem 1.3.3 to Some Complex Functions

Apply Theorem 1.3.3 to the following complex functions to determine if they are analytic, and in case (i) find $f'(z)$ as a function of x and y:

(i) $f(z) = 2z^3 + z^2 - z + 4;$

(ii) $f(z) = \dfrac{z + 4}{(z + 1)(z - 3)};$

(iii) $f(z) = \dfrac{x^4 - y^4}{x^3 + y^3} + i\,\dfrac{x^4 + y^4}{x^3 + y^3}$ with $f(0) = 0$,

(iv) $f(z) = z\bar{z}$.

SOLUTION

(i) Setting $z = x + iy$ in $f(z) = u + iv$, and separating the real and imaginary parts gives

$$u(x, y) = 2x^3 - 6xy^2 + x^2 - y^2 - x + 4$$

and

$$v(x, y) = 6x^2y - 2y^3 + 2xy - y.$$

Routine differentiation then shows that

$$\frac{\partial u}{\partial x} = 6x^2 - 6y^2 + 2x - 1, \quad \frac{\partial u}{\partial y} = -12xy - 2y,$$
$$\frac{\partial v}{\partial x} = 12xy + 2y, \quad \frac{\partial v}{\partial y} = 6x^2 - 6y^2 + 2x - 1.$$

Clearly $u(x, y)$ and $v(x, y)$ are continuous for all x and y and their derivatives satisfy the Cauchy–Riemann equations, so by Theorem 1.3.3 the function $f(z)$ is analytic for all z (it is, of course, an entire function).

Using either of the last results of Theorem 1.3.3 it follows that in Cartesian form the derivative of the function $f(z)$ is

$$f'(z) = 6x^2 - 6y^2 + 2x - 1 + i(12xy + 2y).$$

A simple way of arriving at this result in terms of z is provided by Result (i) of the next theorem, while in Section 1.4 it will be shown how the above result can be converted rapidly into the equivalent expression involving z.

(ii) The function $f(z)$ is the quotient of two polynomials, so it is continuous away from the points $z = -1$ and $z = 3$ where the denominator vanishes. Thus its real and imaginary parts u and v share these same properties where $z = -1$ corresponds to $x = -1$ and $z = 3$ corresponds to $x = 3$. Some routine though tedious differentiation, the details of which are left as an exercise, confirm that the Cauchy–Riemann equations are satisfied, so the function $f(z)$ is analytic everywhere away from the two isolated singularities at $z = -1$ and $z = 3$.

(iii) This example shows the necessity of the requirements in Theorem 1.3.3 that u, v and their first order partial derivatives are continuous in D.

From the definition of differentiation, using the standard suffix notation,

$$u_x = \partial u/\partial x,\ u_y = \partial u/\partial y,\ v_x = \partial v/\partial x \quad \text{and} \quad v_y = \partial v/\partial y,$$

we have

$$\begin{aligned} u_x(0,0) &= \lim_{x\to 0}\left(\frac{u(x,0)-u(0,0)}{x}\right) = \lim_{x\to 0}\left(\frac{u(x,0)}{x}\right) \\ &= \lim_{x\to 0}\left[\left(\frac{x^4-0}{x^3+0}\right)\Big/x\right] = 1. \end{aligned}$$

Similar arguments show that $u_y(0,0) = -1, v_x(0,0) = 1$ and $v_y(0,0) = 1$. Thus the function satisfies the Cauchy–Riemann equations at the origin, so assuming it is permissible to apply the last result of Theorem 1.3.3 it would seem that $f'(0) = 1 + i$.

However, if the limit is taken along the line $y = x$, that can be parametrized as $h = (1 + i)x$ with $f(h) = ix$ we have

$$\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}\right) = \lim_{x\to 0}\left(\frac{ix-0}{(1+i)x}\right) = \tfrac{1}{2}(1+i).$$

This is not equal to the value $1 + i$ determined previously, so the function is not differentiable at the origin, which is a singular point, though it is differentiable for all $z \neq 0$. The failure of differentiability at the origin is because this is the only point where u and v and their partial derivatives are not continuous, thereby violating the conditions of Theorem 1.3.3 at the origin and invalidating the application of the last result in the theorem.

(iv) The function $f(z) = \overline{z} = x - iy$ fails to satisfy the Cauchy–Riemann equation at any point in the complex plane. To see this set $f(z) = u + iv$, when $u = x$ and $v = -y$, so that $u_x = 1$, $u_y = 0$, $v_x = 0$ and $v_y = -1$. This shows that throughout the complex plane $u_x \neq v_y$. A function such as this which is said to be *nonanalytic*. Whereas Example (iii) failed to be analytic at one point, this one is nonanalytic for all z. ◇

Theorem 1.3.4 simplifies the task of finding $f'(z)$ as a function of z, thereby rendering it unnecessary to use either of the last results of Theorem 1.3.3. The definition of a complex derivative has the same form as the corresponding definition of the derivative of a real function of a real variable, aside from the fact that the result must be independent of the way $h \to 0$. Thus, formally, the rules for differentiating complex functions are the same as those for differentiating real functions, the difference being that in the complex case, before applying the rules, it is first necessary to ensure the functions involved are differentiable in the complex sense. We state these rules without further justification.

THEOREM 1.3.4 Differentiation Rules for Analytic Functions
If $f(z)$ and $g(z)$ are analytic functions in some domain D, then

(i) $\dfrac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$ is analytic in D,

(ii) $\dfrac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$ is analytic in D,

(iii) $\dfrac{d}{dz}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$ is analytic for all z in D such that $g(z) \neq 0$.

(iv) $\dfrac{d}{dz}(f[g(z)]) = f'(g(z))g'(z)$ is analytic for all z for which $f(g(z))$ is defined. ◆

Example 1.3.8 An Application of Theorem 1.3.4
Use Theorem 1.3.4 to find $f'(z)$ for the functions (i) and (ii) in Example 1.3.7.

SOLUTION

(i) From Theorem 1.3.4(i) we have $\dfrac{d}{dz}[2z^3 + z^2 - z + 4] = 6z^2 + 2z - 1$;

(ii) From Theorem 1.3.4(iii) we have $\dfrac{d}{dz}\left[\dfrac{z+4}{(z+1)(z-3)}\right] = \dfrac{5 - 8z - z^2}{(z+1)^2(z-3)^2}$. ◇

THEOREM 1.3.5 L'Hospital's Rule
Let $f(z)$ and $g(z)$ be analytic functions in some domain D such that $z_0 \in D, f(z_0) = 0$ and $g(z_0) = 0$, while $\lim_{z\to z_0} g'(z) \neq 0$ and $\lim_{z\to z_0} [f'(z)/g'(z)]$ exists. Then

$$\lim_{z\to z_0}\left(\frac{f(z)}{g(z)}\right) = \lim_{z\to z_0}\left(\frac{f'(z)}{g'(z)}\right).$$

PROOF

The result is almost immediate, because as $f(z_0) = g(z_0) = 0$, using the definition of a derivative, the limit can be rewritten as

$$\lim_{z\to z_0}\left(\frac{f(z)}{g(z)}\right) = \lim_{z\to z_0}\left(\frac{[f(z) - f(z_0)]/(z - z_0)}{[g(z) - g(z_0)]/(z - z_0)}\right) = \lim_{z\to z_0}\left(\frac{f'(z)}{g'(z)}\right).$$

The last limit is well defined and unique because $f(z)$ and $g(z)$ are analytic functions in some neighborhood of z_0, and by hypothesis $g'(z_0) \neq 0$. ◆

Example 1.3.9 An Application of L'Hospital's Rule

Evaluate $\lim_{z\to 1}\left(\dfrac{1-z^{n+1}}{z^2+3z-4}\right)$ when n is a positive integer.

SOLUTION

This is an indeterminate form to which Theorem 1.3.5 applies, with $f(z) = 1 - z^{n+1}$ and $g(z) = z^2 + 3z - 4$, with $z_0 = 1$. A direct application of Theorem 1.3.5 gives

$$\lim_{z\to 1}\left(\frac{1-z^{n+1}}{z^2+3z-4}\right) = \lim_{z\to 1}\left(\frac{-(n+1)z^n}{2z+3}\right) = -\tfrac{1}{5}(n+1). \qquad \diamond$$

Example 1.3.10 A Combination of L'Hospital's Rule with Other Reasoning

Evaluate $\lim_{z\to i}\left(\dfrac{z^2+2z+1-2i}{2z^2+z+2-i}\right)^2$.

SOLUTION

To determine the limit we apply L'Hospital's rule to the expression in parentheses which is an indeterminate form of the type to which Theorem 1.3.5 applies, and then raise the result to the power 2.

$$\lim_{z\to i}\left(\frac{z^2+2z+1-2i}{2z^2+z+2-i}\right) = \lim_{z\to i}\left(\frac{2z+2}{4z+1}\right) = \frac{2+2i}{1+4i} = \tfrac{1}{17}(10-6i),$$

so

$$\lim_{z\to i}\left(\frac{z^2+2z+1-2i}{2z^2+z+2-i}\right)^2 = \left[\tfrac{1}{17}(10-6i)\right]^2 = \tfrac{1}{289}(64-120i). \qquad \diamond$$

Exercises 1.3

1. Find $f(1+i)$ and $f(i\sqrt{3})$ if $f(z) = z^2 - 3$.
2. Find $f(-i)$ and $f(-\frac{1}{2}+i)$ if $f(z) = 1/(1+2z)$.
3. Find $f(2-i)$ and $f(3+i)$ if $f(z) = z|z|$.

Find the real and imaginary parts u and v of $f(z) = u + iv$ (a) in the Cartesian representation and (b) in the polar representation given that:

4. $f(z) = z^2 - z + 1$.
5. $f(z) = 1/z^2$.
6. $f(z) = z|z|^2$.

In each of the following exercises, by considering both the z-plane and the w-plane where $w = f(z)$, show graphically the domain in the w-plane that corresponds to the given domain in the z-plane.

7. $f(z) = 2z + 1, 0 \leqslant \text{Re}\{z\} \leqslant 2$.
8. $f(z) = z^2, |\text{Arg}\{z\}| \leqslant \pi/4$.
9. $f(z) = 1/z, \frac{1}{4} \leqslant |z| \leqslant \frac{1}{2}$ and $|z| \geqslant 1$.
10. Use the vector property of complex numbers together with geometrical arguments to sketch the path in the w-plane followed by $w = 2z + i$ as z moves around the unit circle $|z| = 1$ in the z-plane.

    When they exist, find the limits of the following functions $f(z)$, and state whether the functions are continuous at the points where the limits are to be determined.

11. $\lim_{z \to 1-i} f(z)$, with $f(z) = \frac{1}{2}(z^2 + 2z + 4)$.
12. $\lim_{z \to i} f(z)$, with $f(z) = \dfrac{z^2 - 2(1+i)z - 1 + 2i}{z - i}$, for $z \neq i$ and $f(i) = 1 + 2i$.
13. $\lim_{z \to 1} f(z)$, with $f(z) = (1 - z)/(1 - \overline{z})$.
14. $\lim_{z \to 0} f(z)$, with $f(z) = (\overline{z})^2/z$ for $z \neq 0$ and $f(0) = 1$.
15. Show from first principles that if $f(z) = (a + bz)^{-1}$, then

$$f'(z) = \frac{d}{dz}[(a+bz)^{-1}] = \frac{-b}{(a+bz)^2}, \quad z \neq -a/b.$$

16. Show from first principles that if $f(z)$ is analytic and k = const. that

$$\frac{d}{dz}[k + f(z)] = f'(z) \quad \text{and} \quad \frac{d}{dz}[kf(z)] = kf'(z).$$

17. Given that $f(z)$ and $g(z)$ are entire functions, state with reasons which of the following combinations is an entire function:
    (a) $f(z) + g(z)$
    (b) $f(z)g(z)$
    (c) $[f(z)]^2 + [g(z)]^2 + 1$

(d) $f(z)/g(z)$
(e) $f(z+i)g(z+2)$
(f) $f[g(z)]$
(g) $g[f(z) - 2i + 1)]/[g(z) + f(z)]$.

18. Differentiate the following functions once with respect to z:
$f(z) = 4z^6 - 3z^2 + iz - 2$;
$f(z) = (z^3 + 2z^2 - 1)(z^2 - 2 + i)$;
$f(z) = (z+2)/(3z-1)^2$;
$f(z) = (z^2 + 7z - i)^6$.

Use L'Hospital's rule, together with other arguments where necessary, to determine the following limits:

19. $\lim_{z\to -i}\left(\dfrac{z^{10} + z^2 + 2iz}{z+i}\right)$.

20. $\lim_{z\to 1-i}\left(\dfrac{z^2 - 2(1-i)z - 2i}{z^2 - 2z + 2}\right)$.

21. $\lim_{z\to -i}\left(\dfrac{z^6+1}{z^2+1}\right)^2$.

22. $\lim_{z\to i}\left(\dfrac{z^4-1}{z^2+1}\right)^2 \Big/ \lim_{z\to 3}\left(\dfrac{z^2-4z+3}{2z^2-13z+21}\right)$.

1.4 The Cauchy–Riemann Equations: Proof and Consequences

It is a straightforward matter to establish that for a complex function $f(z) = u + iv$ to be analytic is *necessary* that u and v satisfy the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$. However, to show that *sufficient* conditions for analyticity involve the additional requirements that u, v and all of their first order partial derivatives must be continuous is a little harder, so readers may wish to delay studying that part of the proof until later.

We recall first that when a function $f(z)$ is defined for z in some domain D, the derivative $f'(z)$ is

$$f'(z) = \lim_{s\to 0}\left(\frac{f(z+s) - f(z)}{s}\right), \quad \text{for all } z \in D, \tag{1.59}$$

provided the limit exists and is independent of the way $s \to 0$. Before proceeding to the details of the proof of Theorem 1.3.3 we will outline the basic steps used in the first part to show the necessity of the Cauchy–Riemann equations. These involve setting

$$f(z) = u(x, y) + iv(x, y), \tag{1.60}$$

using Equation (1.60) to express Equation (1.59) in terms of u and v, and then finding the limit in Equation (1.59) by letting $s \to 0$ in two different ways. This will lead to two different looking expressions for $f'(z)$ that must be equal if the derivative is to be independent of the way $s \to 0$. The Cauchy–Riemann equations follow by equating the respective real and imaginary parts of these two expressions, as do the expressions for $f'(z)$ given in Theorem 1.3.3.

1.4.1 Proof of Theorem 1.3.3

Necessity of the Cauchy–Riemann Equations

To derive the Cauchy–Riemann equations, we first proceed to the limit in Equation (1.59) using a purely real value of s by setting $s = h + 0i$, where h is real. Then, after using Equation (1.60), the expression for the derivative in Equation (1.59) becomes

$$\begin{aligned} f'(z) &= \lim_{h\to 0}\left(\frac{u(x+h, y) + iv(x+h, y) - [u(x, y) + iv(x, y)]}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{u(x+h, y) - u(x, y)}{h}\right) + i\lim_{h\to 0}\left(\frac{v(x+h, y) - v(x, y)}{h}\right). \end{aligned}$$

The existence of the derivative $f'(z)$ implies that each of the two limits on the right must exist, so recalling the definition of a partial derivative in real analysis, in the limit this last result is seen to reduce to

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{1.61}$$

We now repeat this process, although this time proceeding to the limit in Equation (1.59) using purely imaginary values of s by setting $s = 0 + ik$, where k is real. After grouping terms, the derivative in Equation (1.59) becomes

$$f'(z) = \lim_{k\to 0}\left(\frac{u(x, y+k) - u(x, y)}{ik}\right) + i\lim_{k\to 0}\left(\frac{v(x, y+k) - v(x, y)}{ik}\right).$$

Proceeding to the limit this becomes

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{1.62}$$

If the derivative in Equation (1.59) is independent of the way s tends to zero, results Equations (1.61) and (1.62) must be identical, which can only occur when their respective real and imaginary parts are equal. So equating the respective real and imaginary parts of Equations (1.61) and (1.62) shows that for $f(z)$ to be analytic, a *necessary* condition is that the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.63}$$

are satisfied. The two expressions for the derivative $f'(z)$ given in Theorem 1.3.3 are simply results Equations (1.61) and (1.62).

Sufficiency of the Cauchy–Riemann Equations

The Cauchy–Riemann equations in Theorem 1.3.3 are *necessary* for the analyticity of $f(z)$ and it remains for us to establish the *sufficiency* conditions. Before proceeding, recall the simplest form of Taylor's theorem for a real function $F(x, y)$ of the two real variables x and y. This is that if $F(x, y)$ together with its first-order partial derivatives exist and are continuous in a domain D containing the region $x_0 \leqslant x \leqslant x_0 + h, y_0 \leqslant y \leqslant y_0 + k$, with h, k arbitrary, then

$$\begin{aligned} F(x_0 + h, y_0 + k) = F(x_0, y_0) &+ hF_x(x_0 + \xi h, y_0 + \eta k) \\ &+ kF_y(x_0 + h\xi, y_0 + \eta k), \end{aligned}$$

where although unknown, the real numbers ξ, η are such that $0 < \xi < 1$, $0 < \eta < 1$. It will be recognized that the last two terms on the right represent the remainder term in this form of Taylor's theorem. For conciseness, in what follows we will use the abbreviations

$$\langle F_x \rangle_{\xi,\eta} = F_x(x_0 + \xi h, y_0 + \eta k) \quad \text{and} \quad \langle F_y \rangle_{\xi,\eta} = F_y(x_0 + \xi h, y_0 + \eta k).$$

If $f(z) = u + iv$ *is analytic in a domain* D containing the points (x_0, y_0) and $(x_0 + h, y_0 + k)$ we write

$$u(x_0 + h, y_0 + k) = h\langle u_x \rangle_{\xi_1,\eta_1} + k\langle u_y \rangle_{\xi_1,\eta_1},$$

and

$$v(x_0 + h, y_0 + k) = h\langle v_x \rangle_{\xi_2,\eta_2} + k\langle v_y \rangle_{\xi_2,\eta_2},$$

for some $0 < \xi_1 < 1$, $0 < \eta_1 < 1$, $0 < \xi_2 < 1$ and $0 < \eta_2 < 1$.

Setting $z_0 = x_0 + iy_0$, and $s = h + ik$, we have

$$f(z_0 + s) - f(z_0) = h\langle u_x \rangle_{\xi_1,\eta_1} + k\langle u_y \rangle_{\xi_1,\eta_1} + i\left[h\langle v_x \rangle_{\xi_2,\eta_2} + k\langle v_y \rangle_{\xi_2,\eta_2}\right].$$

After using the Cauchy–Riemann equations which are *necessary* for analyticity and adding and subtracting terms where appropriate, this result can be rearranged to give

$$\frac{f(z_0+s)-f(z_0)}{s} = \langle u_x \rangle_{\xi_1,\eta_1} + i\langle v_x \rangle_{\xi_2,\eta_2} + i\left(\frac{\varepsilon_1 h}{s} + \frac{\varepsilon_2 k}{s}\right),$$

where

$$\varepsilon_1 = \langle v_x \rangle_{\xi_2,\eta_2} - \langle v_x \rangle_{\xi_1,\eta_1} \quad \text{and} \quad \varepsilon_2 = \langle u_x \rangle_{\xi_2,\eta_2} - \langle u_x \rangle_{\xi_1,\eta_1}.$$

As $h, k \to 0$, so $x_0 + \xi_1 h \to x_0$, $x_0 + \xi_2 h \to x_0$, $y_0 + \xi_1 h \to y_0$ and $y_0 + \xi_2 h \to y_0$, thereby causing $\varepsilon_1, \varepsilon_2 \to 0$, because the *continuity* of the partial derivatives ensures that all of the functions involved tend to their respective limiting values at (x_0, y_0). And because $s = h + ik$ it follows that $|h/s| < 1$, $|k/s| < 1$, showing that h/s and k/s remain bounded as $s \to 0$. So proceeding to the limit we have

$$\begin{aligned} f'(z_0) &= \lim_{s\to 0}\left(\frac{f(z_0+s)-f(z_0)}{s}\right) \\ &= \lim_{s\to 0}\left(\langle u_x \rangle_{\xi_1,\eta_1} + i\langle v_x \rangle_{\xi_2,\eta_2}\right) + i\lim_{s\to 0}\left(\frac{\varepsilon_1 h}{s} + \frac{\varepsilon_2 k}{s}\right) \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0). \end{aligned}$$

The limit is independent of the way $s \to 0$, and z_0 was any point in D, so the result is true throughout D, and the proof of the sufficiency conditions is complete. ◆

Example 1.3.7(iii) involved a function that although satisfying the Cauchy–Riemann equations at the origin was *not* differentiable at that point because u, v and their first-order partial derivatives were not continuous at the origin and therefore did not satisfy the additional conditions of continuity necessary to guarantee analyticity. In fact, with the exception of the point $z = 0$, the function in Example 1.3.7(iii) satisfies the Cauchy–Riemann equations throughout the finite complex plane, and u, v and their first-order partial derivatives are continuous everywhere except for that one point; in fact the origin was the *only* point where the function is not differentiable.

1.4.2 The Cauchy–Riemann Equations in Polar Form

A routine change of variables from the Cartesian variables x and y to the polar coordinates r and θ through the transformation

$$x = r\cos\theta, \;\; y = r\sin\theta \tag{1.64}$$

shows that in terms of the polar representation, if $f(z) = u(r, \theta) + iv(r, \theta)$, the Cauchy–Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}, \tag{1.65}$$

with

$$f'(z) = \left(\frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta\right) + i\left(\frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta\right) \tag{1.66}$$

or, equivalently (after using the Cauchy–Riemann equations),

$$f'(z) = \left(\frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta\right) - i\left(\frac{\partial u}{\partial r}\sin\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta\right). \tag{1.67}$$

The derivation of these last results is left as an exercise.

Example 1.4.1 An Application of the Cauchy–Riemann Equations in Polar Form

Use the polar form of the Cauchy–Riemann equations to show that $f(z) = z^2 + 3z + 1$ is analytic.

SOLUTION

The analyticity of $f(z)$ for all z has already been established in Section 1.3, but to establish the result using the polar form of the Cauchy–Riemann equations we set $z = r(\cos\theta + i\sin\theta)$ in $f(z)$ and use De Moivre's theorem to obtain

$$f(z) = (u + iv) = r^2\cos 2\theta + 3r\cos\theta + 1 + i(r^2\sin 2\theta + 3r\sin\theta).$$

This shows that $u(r, \theta) = r^2\cos 2\theta + 3r\cos\theta + 1$ and $v(r, \theta) = r^2\cos 2\theta + 3r\sin\theta$, and routine differentiation shows these functions satisfy Equations (1.65) for all $z \neq 0$, while they and their partial derivatives are all continuous away from the origin. So from Theorem 1.3.3, using the polar form of the Cauchy–Riemann equations, the function $f(z)$ has been shown to be analytic everywhere except at $z = 0$.

As already mentioned in Section 1.3, $f(z)$ has been shown to be an entire function (analytic for all z), so a discrepancy appears to be between that result at $z = 0$ and the result obtained using the polar representation. This apparent discrepancy is easily resolved once it is recalled that in the polar form $\theta = \text{Arg}\{f(z)\}$ is *undefined* at the origin, so the polar form of the Cauchy–Riemann equations can tell us nothing about the analyticity of $f(z)$ at the origin. ◇

Differentiating the first of the Cartesian form of the Cauchy–Riemann equations partially with respect to x and the second partially with respect to y gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right).$$

However, when a function $v(x, y)$ of the two real variables x and y together with its first and second order partial derivatives are all continuous, its second order mixed partial derivatives are equal. Thus it follows directly by equating the results in these equations that $u(x, y)$ is a solution of the *partial differential equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.68}$$

called *the* ***Laplace equation***, and this result is often written in the abbreviated form $\Delta u = 0$. Here the symbol Δ represents the *partial differential operator* $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, and when Δ acts on u it shows the differentiation operations that are to be performed in order to arrive at the expression on the left of Equation (1.68).

The symbol Δ itself is called the *Laplacian operator* or, more simply, the *Laplacian*, and when written in terms of Cartesian coordinates x and y, it is called the *Cartesian form* of the Laplacian. Notice that by itself, the Laplacian is simply an instruction to perform certain differentiation operations on a suitably differentiable function, but it is *not* itself a function. Only when Δ acts on a suitably differentiable function $\phi(x, y)$ of two variables x and y does $\Delta\phi$ become a function.

If the differentiation of the Cauchy–Riemann equations had been performed in the reverse order, that is differentiation the first equation with respect to y and then the second with respect to x, it would have shown that in addition to Equation (1.68) being true, it also follows that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \tag{1.69}$$

so when $f(z) = u + iv$ is analytic, both $u(x, y)$ and $v(x, y)$ are solutions of Laplace's equation. A solution of Laplace's equation is called an *harmonic function*, and the functions u and v in an analytic function $f(z) = u + iv$ are called *conjugate harmonic functions*. In many parts of mathematics, and especially in applications to physical problems, the Laplace equation and its solutions are of considerable importance. For example, solutions of Laplace's equation describe the steady-state distribution of heat in a heat conducting solid, certain aspects of steady incompressible fluid flow, the electrostatic potential in a cavity, and many other physical situations.

Notice that u and v are only conjugate harmonic functions if they are respectively the real and imaginary parts of the *same* analytic function $f(z) = u + iv$. So the real part u_1 of an analytic function $f_1(z) = u_1 + iv_1$, and the imaginary part v_2 of a *different* analytic function $f_2(z) = u_2 + iv_2$ are *not* conjugate harmonic functions. The significance of Laplace's equation and the role played by conjugate harmonic functions become clear when boundary value problems have been defined and related to conformal mapping. It will suffice to remark here that for the specific analytic function $f(z) = z^2 = u + iv$, if we set $z = x + iy$, the conjugate harmonic functions are found to be

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

When the families of curves $u = const.$ and $v = const.$ are superimposed, they intersect each other at right angles, and families of curves possesing this property are called *mutually orthogonal trajectories*, or more simply, *orthogonal trajectories*. Figure 1.13 shows plots of these trajectories in the first quadrant of the (x, y)-plane. It will be shown later that the mutual orthogonality of plots of families of curves corresponding to conjugate harmonic functions is a general property of conjugate harmonic functions and not just a property of this particular choice of analytic function.

It is appropriate at this stage we formulate two simple rules by which to convert an analytic function $f(z) = u(x, y) + iv(x, y)$ into a function of z.

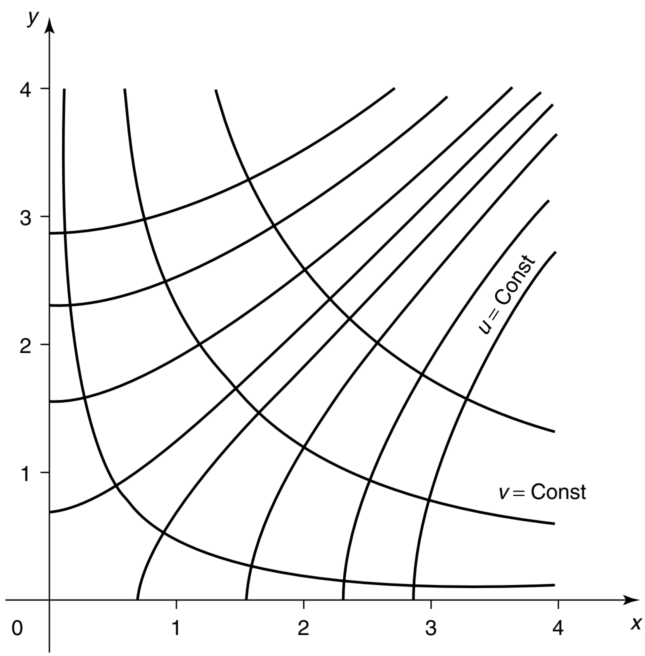

FIGURE 1.13
Typical orthogonal trajectories.

A little thought shows that if $f(z)$ is analytic, the functional form of $f(z)$ as a function of z must, in the Cartesian representation, be the form obtained from $f(z) = u(x, y) + iv(x, y)$ by setting $y = 0$ and replacing x by z in both u and v. Similarly, in the polar representation, the functional form of $f(z)$ as a function of z follows from $f(z) = u(r, \theta) + iv(r, \theta)$ by setting $\theta = 0$ and replacing r by z in both u and v.

Rules for Expressing f(z) = u + iv as a Function of z

If, and only if, $f(z) = u + iv$ is analytic, it can be expressed as a function of z either by

(i) Setting $y = 0$ and replacing x by z in both $u(x, y)$ and $v(x, y)$ in the Cartesian case; or by
(ii) Setting $\theta = 0$ and replacing r by z in both $u(r, \theta)$ and $v(r, \theta)$ in the polar case.

By way of example, applying Rule (i) to the analytic function $f(z) = x^2 - y^2 + 2ixy$ just used to illustrate orthogonal trajectories we find, as expected, that $f(z) = z^2$.

Later, given one harmonic function (either u or v), it will be necessary to find its harmonic conjugate (v or u) in order to construct the analytic function $f(z) = u + iv$, so we now show how this can be done. Suppose $u(x, y)$ is known, then we can find u_x and relate it to v_y through the Cauchy–Riemann equation $v_y = u_x$. Recalling the definition of partial differentiation with respect to y, during which x is regarded as a constant, after integrating the Cauchy–Riemann equations partially with respect to y (i.e., regarding x as a constant) we fine the general result

$$v(x, y) = \int \frac{\partial u}{\partial x}\, dy + h(x). \tag{1.70}$$

At this stage $h(x)$ is an arbitrary function of x, and it has been included because when Equation (1.70) is differentiated partially with respect to y its derivative will vanish, leaving only u_x on the right. A similar argument, starting from the second Cauchy–Riemann equation $v_x = -u_y$, using the fact that u_y can be found and integrating partially with respect to x (that is by regarding y as a constant) gives

$$v(x, y) = -\int \frac{\partial u}{\partial y}\, dx + k(y), \tag{1.71}$$

where this time an arbitrary function $k(y)$ must be added to the general result. Both Equations (1.70) and (1.71) are forms of the *same* function $v(x, y)$, so they must be identical for all x and y. The unknown function $h(x)$ is found by identifying it with terms in Equation (1.71) containing *only* functions of x, while the unknown function $k(y)$ is found by identifying it with terms in

Equation (1.70) containing *only* functions of y. A similar approach leads to the determination of $u(x, y)$ when $v(x, y)$ is known and, similarly, to $v(r, \theta)$ when $u(r, \theta)$ is known, or to $u(r, \theta)$ when $v(r, \theta)$ is known.

Example 1.4.2 Finding an Harmonic Conjugate

Given $u = 3x^2 + x - 3y^2 + 4$, find its harmonic conjugate, and hence find $f(z) = u + iv$ in terms of z.

SOLUTION

First it is necessary to check that u is harmonic because $u_{xx} = 6$ and $u_{yy} = -6$, showing that $u_{xx} + u_{yy} = 0$. Then as $u_x - 6x + 1$ and $u_y = -6y$, result Equation (1.70) becomes

$$v(x, y) = \int (6x + 1)dy = 6xy + y + h(x) + c,$$

while result Equation (1.71) becomes

$$v(x, y) = -\int (-6y)dx + k(y) = 6xy + k(y) + d,$$

where c and d are arbitrary real constants of integration. Equating these two forms of $v(x, y)$, that must be identical for all x and y, we find that as the second result contains no terms containing only functions of x, we must set $h(x) \equiv 0$. Then, as the only term in the first result containing only a function of y *is* y itself, we must set $k(y) = y$ and, finally, the arbitrary constants must be equal, so $d = c$. Thus the required harmonic conjugate function is

$$v(x, y) = 6xy + y + c.$$

Combining results to find the analytic function $f(z) = u + iv$ we find that

$$f(z) = 3x^2 + x - 3y^2 + 4 + c + i(6xy + y + c).$$

Rule (i) can be applied to express $f(z)$ in terms of z because $f(z)$ is analytic. Setting $y = 0$ and replacing x by z gives

$$f(z) = 3z^2 + z + 4 + c(1 + i).$$

It should come as no surprise that $f(z)$ contains an arbitrary additive constant because when either u + const. or v + const. is substituted into the Laplace equation, the constant will vanish. The role played by this arbitrary constant will be better understood when boundary value problems are considered in Chapter 5.

Exercises 1.4

In Exercises 1 through 10, use Theorem 1.3.3 to find if the following functions are analytic, and if so for what z this is true.

1. $f(z) = x^3 - 3xy^2 + x - 4 + i(3x^2y - y^3 + y)$.
2. $f(z) = 1/(z - 2)$.
3. $f(z) = z + 1/z$.
4. $f(z) = 1/(z^2 - 1)$.
5. $f(z) = z^2|z|^2$.
6. $f(z) = \mathrm{Re}\{z\} + i$.
7. $f(z) = x^2 - y^2 + x + i(2xy + |y|)$.
8. $f(z) = (\cos\theta - i\sin\theta)/r$.
9. $f(z) = r(\cos\theta - i\sin\theta)$.
10. $f(z) = 2r\cos\theta + (3/r^2)\cos 2\theta + i(2r\sin\theta - (3/r^2)\sin 2\theta)$.

In Exercises 11 through 13, use Theorem 1.3.3 to show the following functions are analytic and find $f'(z)$. Express $f(z)$ and $f'(z)$ in terms of z.

11. $f(z) = e^{2x}(\cos 2y + i\sin 2y)$.
12. $f(z) = \sin x\cosh y + i\cos x\sinh y$.
13. $f(z) = \cosh x\cos y + i\sinh x\sin y$.
14. Derive the polar form of the Cauchy–Riemann equations from the Cartesian form by using elementary calculus methods to make the change of variables $x = r\cos\theta$ and $y = r\sin\theta$.
15. Show that the derivatives $f'(z)$ in Equations (1.66) and (1.67) can be written in the form

$$f'(z) = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

and

$$f'(z) = \frac{1}{r}(\cos\theta - i\sin\theta)\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right).$$

16. Use the polar form of the Cauchy–Riemann equations to show that u and v must both satisfy the ***polar form of Laplace's equation***

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

17. Show that although $u = \sin x\cosh y$ and $v = \sinh x\sin y$ are both harmonic functions, they are not conjugate harmonic functions.

In Exercises 18 through 24, show that each of the functions is harmonic and find its harmonic conjugate. Use the result to write down the analytic function $f(z) = u + iv$, and hence determine $f(z)$ in terms of z.

18. $u = 3xy$.
19. $v = \cosh x \sin y$.
20. $v = e^x(y \cos y + x \sin y)$.
21. $u = x \sin x \cosh y - y \cos x \sinh y$.
22. $v = -\sin x \sinh y$ with $f(0) = 3$.
23. $v = -3\theta$ with $f(1) = 4$.
24. $v = k\left(r - \dfrac{a^2}{r}\right)\cos\theta$ (a, k real).

Exercises of Greater Difficulty

25. Prove that if $f(z)$ is analytic in D, and either $\operatorname{Re}\{f(z)\} =$ const. in D or $\operatorname{Im}\{f(z)\} =$ const. in D, then $f(z) \equiv$ const. in D.
26. Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic in D, then $\Delta\{|f(z)|\} = |f'(z)|^2/|f(z)|$ throughout D.
27. Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic in D, then $\Delta\{[u(x, y)]^n\} = n(n-1)[u(x, y)]^{n-2}\ |f'(z)|^2$, for $n = 1, 2, \ldots$.
28. Prove that if v is analytic in D then $\phi = uv$ is analytic in D, where $f(z) = u + iv$.
29. Prove that if $f(z) = u + iv$ is analytic in D and $|f(z)|$ is harmonic in D, then $f(z) =$ const. in D.
30. Let $f(z) = u + iv$ be analytic in a bounded domain D, with $|f(z)| = M =$ const. on the boundary of D. By considering the behavior of $\Delta|f(z)|$ prove that $f(z) \equiv M$ in D.

When establishing this result, make use of the standard results from elementary calculus: that if $\phi(x, y)$ is a twice differentiable function in D, then at a local extremum of ϕ inside D:

(i) ϕ has a maximum if $\partial\phi/\partial x = \partial\phi/\partial y = 0$, $\phi_{xx}\phi_{yy} - \phi_{xy}^2 > 0$ and $\phi_{xx} < 0$, and
(ii) ϕ has a minimum if $\partial\phi/\partial x = \partial\phi/\partial y = 0$, $\phi_{xx}\phi_{yy} - \phi_{xy}^2 > 0$ and $\phi_{xx} > 0$.

1.5 Elementary Functions

Theorem 1.3.3 has established necessary and sufficient conditions for a function $f(z)$ to be analytic, so we now examine the basic properties of the most frequently occurring elementary analytic functions. Each is defined as an extension of its real variable counterpart in such a way that the two become identical when z is real. When examining these complex analytic functions we will be concerned with both their similarities and their dissimilarities, while also identifying the domain in which the complex function is analytic. Nonanalytic functions like $f(z) = \overline{z}$ that fail to satisfy the Cauchy–Riemann equations at any point of the complex plane are not used in this section.

1.5.1 Polynomials

The general complex polynomial $P_n(z)$ of degree n has the form

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \tag{1.72}$$

where the coefficients $a_0, a_1, \ldots, a_n$ are arbitrary complex numbers with $a_n \neq 0$. It has already been established that $P_n(z)$ is a continuous analytic function for all $z \in \mathbb{C}$, and so it is an entire function. Furthermore, its derivative

$$P_n'(z) = \frac{d}{dz}[P_n(z)] = a_1 + 2a_2 z + 3a_3 z^2 + \cdots + na_n z^{n-1} \quad \text{for all } z \in \mathbb{C}. \tag{1.73}$$

1.5.2 Rational Functions

By analogy with real functions, the quotient of two complex polynomials $P_m(z)$ and $Q_m(z)$ of respective degrees m and n, is called a *rational function*, and it will be assumed that any factors common to $P_m(z)$ and $Q_n(z)$ have been removed. The resulting function

$$f(z) = P_m(z)/Q_n(z) \tag{1.74}$$

is, by Theorems 1.3.2(iii) and 1.3.3, analytic for every z that is not a zero of $Q_n(z)$. If $Q_n(z)$ has n distinct zeros $z_1, z_2, \ldots, z_n$ the rational function in Equation (1.74) will only cease to be analytic at these n points. These n points are called *poles* of the function, where a function $f(z)$ will be said to have a *pole* at $z = z_0$ if

$$\lim_{z \to z_0} |f(z)| = \infty. \tag{1.75}$$

Functions other than rational functions can have poles but in applications of complex analysis, the poles of rational functions occur frequently and are important.

If one of the zeros of $Q_n(z)$ is repeated r times, the zero will be said to have *multiplicity* r, in which case the number of points where Equation (1.74) has a pole will be reduced to $n - r + 1$. A rational function like Equation (1.74) is an example of a *meromorphic function*, which is defined as a function that is analytic (holomorphic) throughout a domain D, except for a finite number of points at each of which it has a pole.

It is often helpful to simplify a rational function such as Equation (1.74) by expressing it as a sum of simpler rational functions known as its *partial fraction* expansion. In general, if $P_m(z)$ is of degree $m \leq n - 1$, and none of the zeros $z_1, z_2, \ldots, z_n$ of $P_n(z)$ is a multiple zero, the *partial fraction* expansion of Equation (1.74) takes the form

$$\frac{P_m(z)}{Q_n(z)} = \frac{A_1}{z - z_1} + \frac{A_2}{z - z_2} + \cdots + \frac{A_n}{z - z_n}, \tag{1.76}$$

where the numbers $A_1, A_2, \ldots, A_n$ are called *undetermined coefficients*. The undetermined coefficients are found precisely in the way they are found when using a partial fraction expansion in elementary calculus, with the exception that now, in general, the coefficients can be complex numbers. This partial fraction expansion has to be modified if a zero z_s of $Q_n(z)$ has multiplicity p, because then Equation (1.76) will have p terms, each with the same denominator $z = z_s$, in which case these p terms must be replaced by the set of p terms

$$\frac{B_1}{z - z_s} + \frac{B_2}{(z - z_s)^2} + \cdots + \frac{B_p}{(z - z_s)^p}. \tag{1.77}$$

Given a rational function $f(z)$ there are various ways of finding its partial fraction expansion, but the classical way, as used in elementary calculus, is illustrated in the next example.

Example 1.5.1 A Partial Fraction Expansion Involving a Repeated Zero

Use partial fractions to simplify

$$f(z) = \frac{3z^4 + 13z^3 + 19z^2 + 13z + 2 + i}{(z + 2)(z + 1)^2}.$$

SOLUTION

The difficult part of any partial fraction expansion involves factoring the denominator $Q_n(z)$, but this does not occur here because the denominator is already factored as $(z + 2)(z + 1)^2$, showing that zero $z = -1$ has multiplicity 1, while zero $z = -2$ has multiplicity 2. However, another difficulty with this example exists because the degree of the numerator is 4 but the degree of the denominator is 3, unlike the case discussed previously in which the degree of the numerator had to be *strictly less* than the degree of the denominator. The way around this difficulty adopted here is to divide the denominator into the numerator longhand, and then to apply partial fractions to the remaining rational function. The result of the division is easily seen to be

$$f(z) = 3z + 1 + \frac{2z + i}{(z + 2)(z + 1)^2},$$

so now it is only necessary to find the partial fraction expansion of the last term on the right. The appropriate form of the partial fraction expansion is

$$\frac{2z+i}{(z+2)(z+1)^2} = \frac{A_1}{z+2} + \frac{B_1}{z+1} + \frac{B_2}{(z+1)^2}.$$

Multiplying this by $(z + 2)(z + 1)^2$ it becomes

$$2z + i = A_1(z + 1)^2 + B_1(z + 1)(z + 2) + B_2(z + 2).$$

This must be an identity, and so it must be true for all z. Two obvious choices for z that lead quickly to undetermined coefficients are $z = -1$ and $x = -2$. Setting $z = -1$ gives $B_2 = -2 + i$, while $z = -2$ gives $A_1 = -4 + i$. To find B_1 we must make a different choice for z, so setting $z = 0$ gives $i = A_1 + 2B_1 + B_2$, from which it follows that $B_1 = 4 - i$. So the required partial fraction expansion is

$$\frac{3z^4 + 13z^3 + 19z^2 + 13z + 2 + i}{(z+2)(z+1)^2} = 3z + 1 + \frac{-4+i}{z+2} + \frac{4-i}{z+1} + \frac{-2+i}{(z+1)^2},$$

with a pole at $z = -1$ and another at $z = -2$. ◇

1.5.3 The Exponential Function

We define the *complex exponential function*, denoted either by e^z or by $\exp z$, in terms of $z = x + iy$, as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y). \tag{1.78}$$

Setting $e^z = u + iv$, we see that $u = e^x \cos y$ and $v = e^x \sin y$ and these u and v satisfy the Cauchy–Riemann equations (check this) and together with their first-order partial derivatives are continuous, showing that e^z is analytic for all z, and so is an entire function. From Theorem 1.3.3 we also have that

$$\frac{d}{dz}[e^z] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x(\cos y + i \sin y) = e^z.$$

This same form of argument establishes the more general result that

$$\frac{d}{dz}[e^{kz}] = ke^{kz}, \tag{1.79}$$

for any complex constant k.

It is left as an exercise to show that an immediate consequence of definition in Equation (1.78) is that

$$e^{z_1+z_2} = e^{z_1}e^{z_2}, \tag{1.80}$$

a special case of which is

$$e^{x+iy} = e^x(\cos y + i\sin y). \tag{1.81}$$

Thus the multiplication of complex exponential functions obeys the same rule as in the real variable case. When the result in Equation (1.80) is written in the equivalent form

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$$

it expresses more clearly the way in which $\exp(z_1)$ and $\exp(z_2)$ combine when multiplied.

It follows from Equation (1.78) that

$$|e^z| = e^x, \tag{1.81}$$

and that

$$\arg(e^z) = y + 2n\pi, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots . \tag{1.82}$$

Thus the modulus of e^z never vanishes, while the result in Equation (1.82) shows that e^z is periodic in y with period 2π. As a result, if the behavior of e^z is known in any infinite strip in the complex plane of width 2π drawn parallel to the real axis, the periodicity with respect to y will determine the behavior of e^z for all z. The semi-infinite strip in the z-plane

$$-\pi < y \leqslant \pi \tag{1.83}$$

is called the *fundamental strip* for e^z. By restricting y in this manner, one-to-one relationship exists between points z in the fundamental strip and the points $w = e^z$ in the w-plane. Here, a *one-to-one* relationship means that to one point in the fundamental strip in the z-plane there corresponds precisely one point in the w-plane, and conversely. It is important to remember that e^∞ has no meaning in the complex plane.

1.5.4 The Logarithmic Function

The function ln z is called *the **natural logarithm*** of the *complex variable* $z = x + iy$. So to avoid confusion, the natural logarithm of a *real* number r is denoted by $\ln_e r$. The function ln is defined as the function inverse to the exponential function so that

$$w = \ln z \qquad \text{when } e^w = z. \tag{1.84}$$

Setting $w = u + iv$ and $z = re^{i\theta}$ with $r = |z|$, the relationship $e^w = z$ becomes $e^{u+iv} = e^u e^{iv} = re^{i\theta}$, so equating the modulus on each side of this equation gives

$$e^u = r = |z|, \quad \text{showing that } u = \ln_e|z|, \tag{1.85}$$

where $\ln_e|z|$ is the natural logarithm of the real number $|z|$, defined for all $z \neq 0$.

Similarly, equating the arguments on each side of the equation gives

$$v = \theta = \arg(z) = \operatorname{Arg}(z) + 2n\pi, \quad \text{for } n = 0, \pm 1, \pm 2, \dots, \tag{1.86}$$

where

$$-\pi < \operatorname{Arg}(z) \leqslant \pi. \tag{1.87}$$

Combining Equations (1.85) and (1.86) gives

$$\ln z = \ln_e|z| + i(\operatorname{Arg}(z) + 2n\pi), \quad \text{for } z \neq 0,\ n = 0, \pm 1, \pm 2, \dots. \tag{1.88}$$

This last result shows that $\ln z$ is *infinitely many-valued* for any given z, with all real parts of $\ln z$ the same, but with the imaginary parts differing from one another by integral multiples of 2π.

The *principal part* of $\ln z$, denoted here by $\operatorname{Ln} z$, is chosen to correspond to the situation where $\arg(z)$ and $\operatorname{Arg}(z)$ coincide, and so $\operatorname{Ln} z$ is defined as

$$\operatorname{Ln} z = \ln_e|z| + i\operatorname{Arg}(z) \quad \text{for } z \neq 0. \tag{1.89}$$

Consequently, if $w = \operatorname{Ln} z = u + iv$, then

$$u = \ln_e|z| \quad \text{and} \quad -\pi < v \leqslant \pi \quad \text{for } z \neq 0. \tag{1.90}$$

We now prove that $\operatorname{Ln} z$ is a continuous function of z except when z lies on the negative real axis, across which it is discontinuous. Write $z = re^{i\theta}$, so that

$$\operatorname{Ln} z = \ln_e r + i\theta, \tag{1.91}$$

and let $\theta \downarrow \alpha$ signify that θ decreases to α, and let $\theta \uparrow \alpha$ signify that θ increases to α. Now consider $\lim_{z \to z_0} \operatorname{Ln} z$ at an arbitrary point $z_0 = r_0 e^{i\alpha}$ with

$-\pi < \alpha < \pi$ and $r_0 \neq 0$, so that z_0 is arbitrary, but not on the negative real axis. Then

$$\lim_{\substack{r \to r_0 \\ \theta \uparrow \alpha}} \mathrm{Ln}(re^{i\theta}) = \lim_{r \to r_0} \ln_e r + \lim_{\theta \uparrow \alpha}(i\theta) = \ln r_0 + \alpha i,$$

and, similarly,

$$\lim_{\substack{r \to r_0 \\ \theta \downarrow \alpha}} \mathrm{Ln}(re^{i\theta}) = \lim_{r \to r_0} \ln_e r + \lim_{\theta \downarrow \alpha}(i\theta) = \ln r_0 + \alpha i.$$

As z_0 was arbitrary, this has established the continuity of $\mathrm{Ln}\, z$ except when z lies on the negative real axis.

Finally, let us consider the limiting behavior of $\mathrm{Ln}\, z$ across any point $x = -r_0$ on the negative real axis with $r_0 > 0$. The limits then show that

$$\lim_{\substack{r \to r_0 \\ \theta \uparrow \pi}} \mathrm{Ln}(re^{i\theta}) = \lim_{r \to r_0} \ln_e r + \lim_{\theta \uparrow \pi}(i\theta) = \ln_e r_0 + \pi i,$$

and

$$\lim_{\substack{r \to r_0 \\ \theta \uparrow -\pi}} \mathrm{Ln}(re^{i\theta}) = \lim_{r \to r_0} \ln_e r + \lim_{\theta \uparrow -\pi}(i\theta) = \ln_e r_0 - \pi i.$$

Thus $\mathrm{Ln}\, z$ has a jump of $2\pi i$ across each point on the negative real axis. Thus, we have proved the assertion that $\mathrm{Ln}\, z$ is continuous everywhere except at the origin and across points on the negative real axis where it is discontinuous.

To examine the relationship between $w = \mathrm{Ln}\, z$ and its inverse function $z = e^w$, it is necessary to consider the behavior of these functions in the complex plane. If the point $z = 0$ and all of the points on the negative real axis are removed from the z-plane, a one-to-one correspondence exists between the remaining points in the z-plane and points in the strip $-\pi < v \leqslant \pi$ in the w-plane. Thus for any z in this modified z-plane, the function $w = \mathrm{Ln}\, z$ is determined uniquely. Conversely, for any point w in the strip $-\pi < v \leqslant \pi$ in the w-plane, the function $z = e^w$ is also determined uniquely for all w in this modified w-plane.

When a complex plane is modified in this manner by having removed from it all points on a line to make the correspondence between two functions one-to-one, the plane is said to be **cut**. In this case the z-plane was cut along the negative real axis. The fundamental strip in the w-plane and the cut z-plane are shown in Figure 1.14. The cut is to be regarded as a barrier that may not be crossed by z. In Figure 1.14, a full line shows points that belong to a region and a dashed line points that are excluded from the region.

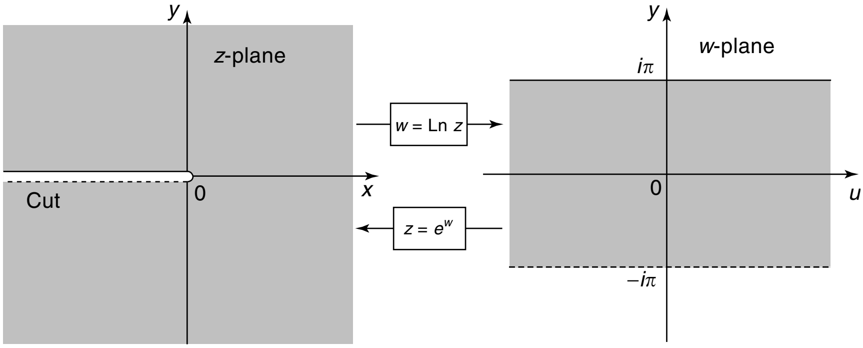

FIGURE 1.14
The cut z-plane for $w = \text{Ln}\, z$ and the fundamental strip in the w-plane.

The individual functions

$$w = \ln_e |z| + i(\text{Arg}(z) + 2n\pi),$$

corresponding to different choices of $n = 0, \pm 1, \pm 2, \ldots$, are called *branches* of $\ln z$, and the logarithmic function is seen to have infinitely many branches. In this context, the cut in the z-plane along the negative real axis is called a *branch cut*.

It is a straightforward matter to show that $\text{Ln}\, z = \ln_e r + i\theta$ satisfies the conditions of Theorem 1.1.3 provided θ is restricted so that $-\pi < \theta \leqslant \pi$, in which case $\text{Ln}\, z$ is an analytic function in the cut z-plane. The same form of argument establishes the fact that each branch of $\ln z$ is also analytic in the cut z-plane.

Writing

$$\begin{aligned}\ln z &= \ln_e[(x^2 + y^2)^{1/2}] + i \arg(z)\\ &= \tfrac{1}{2} \ln_e(x^2 + y^2) + i \tan^{-1}(y/x),\end{aligned}$$

it follows from Theorem 1.3.3 that for any z in the cut z-plane

$$\begin{aligned}\frac{d}{dz}[\ln z] &= \tfrac{1}{2}\frac{\partial}{\partial x} \ln_e(x^2 + y^2) + i\frac{\partial}{\partial x} \tan^{-1}(y/x)\\ &= \frac{x}{(x^2 + y^2)^{1/2}} + i\frac{(-y/x^2)}{1 + (y/x)^2} = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}.\end{aligned}$$

So we have proved that in the cut z-plane the derivative of the logarithmic function $\ln z$ is

$$\frac{d}{dz}[\ln z] = \frac{1}{z}. \tag{1.92}$$

The following familiar properties of ln z can also be proved in similar fashion, though this is left as an exercise:

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2, \tag{1.93}$$

$$\ln(z_1/z_2) = \ln z_1 - \ln z_2, \tag{1.94}$$

$$\ln(z^{p/q}) = (p/q)\ln z, \quad p, q \text{ integers.} \tag{1.95}$$

These last three results require interpretation because of the many-valued nature of ln z.

They are to be taken to mean that the value of the left-side of each result is included among the set of values of the right side. Results in Equations (1.93) and (1.94) are not necessarily true if $\ln z$ is replaced by the principal value $\operatorname{Ln} z$. This is because of the constraint laced on the arguments involved that must satisfy the principal value condition.

As a generalization of z^n we have z^c where $z = x + iy$ and c is an arbitrary complex constant. This is defined as

$$z^c = \exp(c \ln z), \quad \text{for } z \neq 0. \tag{1.96}$$

The infinitely many-valued nature of ln z makes the complex power z^c many-valued, and its principal value is defined as

$$z^c = \exp(c \operatorname{Ln} z), \quad \text{for } z \neq 0. \tag{1.97}$$

An immediate consequence of Equation (1.96) is that if α and β are complex number, then when $z \neq 0$,

$$\begin{aligned} z^\alpha z^\beta &= z^{\alpha+\beta}, \quad z^\alpha / z^\beta = z^{\alpha-\beta}, \\ 1/z^\alpha &= z^{-\alpha} \quad \text{and} \quad (z^\alpha)^n = z^{n\alpha} \text{ for integral } n. \end{aligned} \tag{1.98}$$

If the z-plane is cut along the negative real axis up to and including the origin, an application of Theorem 1.3.4(iv) to the principal value of $f(z) = z^c$ shows that in the cut z-plane

$$\frac{d}{dz}[\exp(c \operatorname{Ln} z)] = \frac{c}{z}\exp(c \operatorname{Ln} z). \tag{1.99}$$

This may be re-expressed in the more familiar, though less precise form

$$\frac{d}{dz}[z^c] = cz^{c-1}, \tag{1.100}$$

as long as it is understood that the principal value of z^c is to be used on the right.

For any complex number $a \neq 0$, we define

$$a^z = \exp(z \ln a). \tag{1.101}$$

Once the value to be assigned to the many-valued function ln a has been chosen, a branch of Equation (1.101) is identified that is analytic in the cut z-plane. It then follows as before that

$$\frac{d}{dz}[a^z] = a^z \ln a, \tag{1.102}$$

where here again it is to be understood that the principal value of a^z is to be used in the expression on the right. The next example illustrates these properties of the complex exponential and logarithmic functions.

Example 1.5.2 Special Values of ln z and z^c

(i) Find $\ln(-1)$ and $\mathrm{Ln}(-1)$. As $e^{i\pi} = -1$, it follows that $|e^{i\pi}| = 1$ and $\mathrm{Arg}(e^{i\pi}) = \pi$. So from Equation (1.88)

$$\begin{aligned}\ln(-1) &= \ln_e|e^{\pi i}| + i(\mathrm{Arg}(e^{i\pi}) + 2n\pi), \quad \text{for } n = 0, \pm1, \pm2, \ldots, \\ &= \ln_e 1 + (\pi + 2n\pi)i, \\ &= (1 + 2n)\pi i, \quad \text{for } n = 0, \pm1, \pm2, \ldots \quad (\text{because} \ln 1 = 0),\end{aligned}$$

and so

$$\ln(-1) = \pm\pi i, \pm3\pi i, \pm5\pi i, \ldots .$$

In particular, setting $n = 0$ the principal value is found to be

$$\mathrm{Ln}(-1) = \pi i.$$

(ii) Find $\ln z$ and $\mathrm{Ln}\, z$, when $z = \frac{5}{2}(1 + \mathrm{i}\sqrt{3})$. Because $z = 5e^{i\pi/3}$, we see that $|z| = 5$ and $\mathrm{Arg}(z) = \pi/3$, so from Equation (1.88)

$$\ln\left(\tfrac{5}{2}(1 + i\sqrt{3})\right) = \ln_e 5 + \tfrac{1}{3}\pi i + 2n\pi i, \quad \text{for } n = 0, \pm1, \pm2, \ldots .$$

Thus

$$\ln\left(\tfrac{5}{2}(1 + i\sqrt{3})\right) = \ln_e 5 + \tfrac{1}{3}\pi i,\ \ln 5 - \tfrac{5}{3}\pi i, \ln_e 5 + \tfrac{7}{3}\pi i, \ldots .$$

In particular, setting $n = 0$, the principal value is found to be

$$\ln\left(\tfrac{5}{2}(1 + i\sqrt{3})\right) = \ln_e 5 + \tfrac{1}{3}\pi i.$$

(iii) Find z^i, when $z = \frac{3}{\sqrt{2}}(1+i)$. Because $z = 3e^{i\pi/4}$, we see that $|z| = 3$ and $\mathrm{Arg}(z) = \pi/4$. From Equation (1.96) we have

$$
\begin{aligned}
z^i &= \exp(i \ln z) = \exp\left[i\left(\ln_e 3 + \tfrac{1}{4}\pi i + 2n\pi i\right)\right], \qquad \text{for } n = 0, \pm 1, \pm 2, \dots, \\
&= \exp\left[-\tfrac{1}{4}(1+8n)\pi + i \ln 3\right], \\
&= \exp\left[-\tfrac{1}{4}(1+8n)\pi\right][\cos(\ln_e 3) + i \sin(\ln_e 3)], \qquad \text{for } n = 0, \pm 1, \pm 2, \dots .
\end{aligned}
$$

This is infinitely many-valued and its principal value, corresponding to $n = 0$, is

$$z^i = e^{-\pi/4}[\cos(\ln_e 3) + i \sin(\ln_e 3)]. \qquad \diamond$$

1.5.5 Trigonometric Functions

For arbitrary $z = x + iy$, we define *the complex sine* and *cosine* functions as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{1.103}$$

These results provide a direct extension of the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$ which is only defined for real values of θ.

Following the pattern of real-valued trigonometric functions, the other complex trigonometric functions are defined in terms of $\sin z$ and $\cos z$ as:

$$\tan z = \frac{\sin z}{\cos z}, \qquad \csc z = \frac{1}{\sin z} \qquad (\text{also written } \operatorname{cosec} z)$$

$$\sec z = \frac{1}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}. \tag{1.104}$$

Using the results $e^{2\pi ni} = e^{-2\pi ni} = 1$, for $n = 0, \pm 1, \pm 2, \dots$, allows us to write

$$
\begin{aligned}
\sin z &= \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz}e^{2\pi ni} - e^{-iz}e^{-2\pi ni}}{2i} \\
&= \frac{e^{i(z+2n\pi)} - e^{-i(z+2n\pi)}}{2i} = \sin(z + 2n\pi).
\end{aligned}
$$

A similar argument applies to the cosine function, so we have shown that, as in the real variable case, these complex functions are periodic with period 2π, because

$$\sin(z + 2n\pi) = \sin z \quad \text{and} \quad \cos(z + 2n\pi) = \cos z, \tag{1.105}$$

for $n = 0, \pm 1, \pm 2, \dots$.

The fact that $\tan z$ and $\cot z$ are periodic with period π can be established in the same way, so that

$$\tan(z+n\pi) = \tan z \quad \text{and} \quad \cot(z+n\pi) = \cot z, \tag{1.106}$$

for $n = 0, \pm 1, \pm 2, \ldots$.

The following identities are direct consequences of the definitions in Equations (1.103) and (1.104):

$$\sin^2 z + \cos^2 z = 1, \tag{1.107}$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2, \tag{1.108}$$

$$\cos(z_1 \pm z_2) = \sin z_1 \cos z_2 \mp \cos z_1 \sin z_2, \tag{1.109}$$

$$\sinh(ix) = i \sinh x, \quad \cosh(ix) = \cosh x \ (x \text{ real}). \tag{1.110}$$

Euler's formula also holds for complex z when it becomes

$$e^{iz} = \cos z + i \sin z, \tag{1.111}$$

and it also follows that

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z, \quad \tan(-z) = -\tan z. \tag{1.112}$$

The fact that $\sin z$ and $\cos z$ are defined as linear combinations of e^{iz} and e^{-iz} taken together with the fact that the exponential functions are entire functions, means that $\sin z$ and $\cos z$ are also entire functions. Furthermore, the definitions of other complex trigonometric functions in terms of $\sin z$ and $\cos z$ mean that they are analytic except at the zeros of the denominators of their defining relations.

Differentiation of the complex trigonometric functions using the above definitions and the properties of e^z shows that

$$\frac{d}{dz}[\sin z] = \cos z, \quad \frac{d}{dz}[\cos z] = -\sin z \quad \text{for all } z\text{, Equation text is deleted.}$$

$$\frac{d}{dz}[\tan z] = \sec^2 z, \quad \frac{d}{dz}[\sec z] = \sec z \tan z, \quad \text{for } z \neq \tfrac{1}{2}\pi + n\pi, \ n = 0, \pm 1, \pm 2, \ldots, \tag{1.113}$$

$$\frac{d}{dz}[\csc z] = -\csc z \cot z, \quad \frac{d}{dz}[\cot z] = -\csc^2 z, \quad \text{for } z \neq n\pi, \ n = 0, \pm 1, \pm 2, \ldots .$$

Combining Equations (1.108) to (1.110) and using the real variable definitions of the hyperbolic sine and cosine functions shows that $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$, and

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y. \tag{1.114}$$

The nonvanishing of $\sinh y$ for $y \neq 0$ coupled with the results from Equations (1.114) implies that $\sin z$ and $\cos z$ can only vanish on the real axis. Thus the zeros of $\sin z$ occur at the points

$$z = n\pi, \quad \text{for } n = 0, \pm 1, \pm 2, \dots, \tag{1.115}$$

while the zeros of $\cos z$ can only occur at the points

$$z = \tfrac{1}{2}(2n + 1)\pi, \quad \text{for } n = 0, \pm 1, \pm 2, \dots. \tag{1.116}$$

Thus the zeros of the complex functions $\sin z$ and $\cos z$ occur at the same points as those of the corresponding real functions.

Example 1.5.3 Roots Involving a Complex Sine Function
Find all of the Roots of $\sin z = \cosh 3$.

SOLUTION
Combining the first result in Equation (1.114) with the equation $\sin z = \cosh 3$, and equating its real and imaginary parts gives

$$\sin x \cosh y = \cosh 3 \quad \text{and} \quad \cos x \sinh y = 0.$$

As $\cosh y \geqslant 1$ for all y, for the first equation to be true it is necessary that $\sin x > 0$ and $y \neq 0$. Using $y > 0$ in the second equation then shows that it can only be satisfied when x is a zero of $\cos x$. The only zeros of $\cos x = 0$ for which $\sin x > 0$ are seen to occur when $x = \frac{1}{2}(4n + 1)\pi$, for $n = 0, \pm 1, \pm 2, \dots$. Thus, for these values of x, the first equation becomes $\cosh y = \cosh 3$, which has two solutions, $y = \pm 3$. Consequently the roots of $\sin z = \cosh 3$ are infinite in number and are given by

$$z = \tfrac{1}{2}(4n + 1)\pi \pm 3i, \quad \text{for } n = 0, \pm 1, \pm 2, \dots. \qquad \diamond$$

1.5.6 Hyperbolic Functions

For arbitrary $z = x + iy$ the *complex hyperbolic sine* and *cosine* functions are defined as

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{1.117}$$

These are direct generalizations of the corresponding real variable functions and they are obviously entire functions. The following identities follow directly from these definitions:

$$\cosh^2 z + \sinh^2 z = 1, \tag{1.118}$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2, \tag{1.119}$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2, \tag{1.120}$$

$$\sinh(ix) = i \sin x, \quad \cosh(ix) = \cos x \ (x \text{ real}) \tag{1.121}$$

$$i \sinh z = \sin(iz), \quad \cosh z = \cos(iz) \ (z \text{ complex}), \tag{1.122}$$

and also

$$\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z, \quad \tanh(-z) = -\tanh z, \tag{1.123}$$

where $\tanh z$ and the other complex hyperbolic functions are defined as:

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z} \ (\text{also written cosech } z),$$

$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}. \tag{1.124}$$

The derivatives of these functions found in the usual manner are formally the same as those of the corresponding real variable functions, namely:

$$\frac{d}{dz}[\sinh z] = \cosh z, \quad \frac{d}{dz}[\cosh z] = \sinh z,$$

$$\frac{d}{dz}[\tanh z] = \operatorname{sech}^2 z, \quad \frac{d}{dz}[\operatorname{csch} z] = -\operatorname{csch} z \coth z, \tag{1.125}$$

$$\frac{d}{dz}[\operatorname{sech} z] = -\operatorname{sech} z \tanh z, \quad \frac{d}{dz}[\coth z] = -\operatorname{csch}^2 z.$$

Combining Equations (1.119) to (1.121) leads to the following useful results

$$\sinh(x \pm iy) = \sinh x \cos y \pm i \cosh x \sin y$$

and

$$\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y. \tag{1.126}$$

Inspection of these results shows that the zeros of $\sinh z$ occur at the points

$$z = n\pi i, \quad n = 0, \pm 1, \pm 2, \dots, \tag{1.127}$$

while the zeros of $\cosh z$ occur at the points

$$z = \tfrac{1}{2}(2n+1)\pi i, \quad n = 0, \pm 1, \pm 2, \dots. \tag{1.128}$$

Notice that unlike the corresponding real variable case, the hyperbolic functions $\sinh z$ and $\cosh z$ are periodic functions with period $2\pi i$. This can be seen by using the result $e^{2n\pi i} = e^{-2n\pi i} = 1$ for $n = 0, \pm 1, \pm 2, \ldots$, in the definitions of $\cosh z$ and $\sinh z$ because, for example,

$$\begin{aligned}\cosh z &= \tfrac{1}{2}(e^z + e^{-z}) = \tfrac{1}{2}(e^z e^{2\pi ni} + e^{-z}e^{-2\pi ni}) \\ &= \tfrac{1}{2}(e^{(z+2n\pi i)} + e^{-(z+2n\pi i)}) = \cosh(z + 2n\pi i),\end{aligned} \tag{1.129}$$

with the corresponding result

$$\sinh z = \sinh(z + 2n\pi i). \tag{1.130}$$

Example 1.5.4 Roots of the Hyperbolic Cosine Function
Find all of the roots of $\cosh z = -1$.

SOLUTION
As $\cosh z = -1$, from the second result in Equation (1.126) we have

$$\cosh x \cos y + i \sinh x \sin y = -1,$$

so equating the real and imaginary parts gives

$$\cosh x \cos y = -1 \quad \text{and} \quad \sinh x \sin y = 0.$$

The first result is only possible if $x = 0$ and $\cos y = -1$, so that $y = \frac{1}{2}(2n + 1)\pi$, in which case the second result is satisfied automatically, so the roots of the equation are

$$z = \tfrac{1}{2}(2n + 1)\pi i, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots. \qquad \diamond$$

1.5.7 Inverse Trigonometric and Hyperbolic Functions

The following text, for the sake of uniformity, denotes functions inverse to the trigonometric and hyperbolic functions just discussed by adding the prefix *arc* to the corresponding function. So, for example, we write

$$w = \arcsin z \text{ when } z = \sin w \quad \text{and} \quad w = \operatorname{arccosh} z \text{ when } z = \cosh w.$$

An equivalent notation also in use involves adding a superscript -1 to a function to denote the inverse function, so in this notation

$$w = \sin^{-1} z \text{ when } z = \sin w \quad \text{and} \quad w = \cosh^{-1} z \text{ when } z = \cosh w.$$

These inverse functions were first encountered in Examples 1.5.3 and 1.5.4, where they were many-valued, though at the time they were not identified as inverse functions. It was possible to evaluate these particular inverse functions by considering their real and imaginary parts separately because only an inverse function with a purely real argument was involved. When inverse trigonometric and hyperbolic functions involve arbitrary complex numbers a different approach becomes necessary. To show the approach that is required we now find all of the complex numbers $w = \arcsin z$ for an arbitrary fixed complex number z.

By definition,

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i},$$

so after multiplication by $2ie^{iw}$ this becomes

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0,$$

which is a quadratic equation for e^{iw}. Solving this gives

$$e^{iw} = iz + \sqrt{1 - z^2},$$

where it will be recalled that the $\pm$ usually associated with the square root sign is omitted since in complex analysis the square root operation is understood to be two-valued. Taking the natural logarithm shows that the inverse sine function $w = \arcsin z$ can be written in the form

$$w = -i \ln\left(\sqrt{1 - z^2} + iz\right).$$

Thus in this result the logarithmic function and the square root function both contribute to the infinitely many-valued behavior of this inverse function.

The same form of reasoning applied to the inverse cosine and tangent functions lead to the results

$$\arcsin z = -i \ln\left(iz + \sqrt{1 - z^2}\right), \quad z \in \mathbb{C}, \tag{1.131}$$

$$\arccos z = -i \ln\left(z + i\sqrt{1 - z^2}\right), \quad z \in \mathbb{C}, \tag{1.132}$$

$$\arctan z = \frac{i}{2} \ln\left(\frac{i + z}{i - z}\right), \quad z \neq \pm i. \tag{1.133}$$

If these results are differentiated using implicit differentiation, for any fixed branch the become analytic functions with the derivatives

$$\frac{d}{dz}[\arcsin z] = \frac{1}{\sqrt{1-z^2}}, \qquad z \in \mathbb{C}_2, \tag{1.134}$$

$$\frac{d}{dz}[\arccos z] = \frac{-1}{\sqrt{1-z^2}}, \qquad z \in \mathbb{C}_2, \tag{1.135}$$

$$\frac{d}{dz}[\arctan z] = \frac{1}{1+z^2}, \qquad z \in \mathbb{C}_1, \tag{1.136}$$

where $\mathbb{C}_1$ is the complex plane cut along the imaginary axis from $y = 1$ to $+\infty$ and from $y = -1$ to $-\infty$, and $\mathbb{C}_2$ is the complex plane cut along the real axis from $x = 1$ to $+\infty$ and from $x = -1$ to $-\infty$.

The functions in Equations (1.131) to (1.136) are infinitely many-valued, but they can be made analytic by restricting z in such a way that a particular branch of the logarithmic function is used together with a specific branch of the square root function. The branch of the square root function used in Equations (1.131) or (1.132) must, of course, be the one used in Equations (1.134) or (1.135) for the function and its derivative to be compatible.

Similar arguments applied to the inverse hyperbolic functions show that

$$\operatorname{arcsinh} z = \ln\left(z + \sqrt{z^2+1}\right), \qquad \text{for all } z, \tag{1.137}$$

$$\operatorname{arccosh} z = \ln\left(z + \sqrt{z^2-1}\right), \qquad \text{for all } z, \tag{1.138}$$

$$\operatorname{arctanh} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \qquad \text{for } z \neq \pm 1. \tag{1.139}$$

Implicit differentiation of these results for any fixed branch yield analytic functions with the derivatives

$$\frac{d}{dz}[\operatorname{arcsinh} z] = \frac{1}{\sqrt{z^2+1}}, \qquad z \in \mathbb{C}_1, \tag{1.140}$$

$$\frac{d}{dz}[\operatorname{arccosh} z] = \frac{1}{\sqrt{z^2-1}}, \qquad z \in \mathbb{C}_2, \tag{1.141}$$

$$\frac{d}{dz}[\operatorname{arctanh} z] = \frac{1}{1-z^2}, \qquad z \in \mathbb{C}_2, \tag{1.142}$$

Here also, the infinitely many-valued functions in Equations (1.137) to (1.139) can be made analytic by restricting z to a particular branch of the logarithmic function and using a specific branch of the square root function where it occurs in Equations (1.140) and (1.141).

Example 1.5.5 The Inverse Hyperbolic Sine and Its Derivative
Find all of the values of arcsinh z and its derivative when $z = i\sqrt{5}$.

SOLUTION
From Equation (1.137) we have

$$\operatorname{arcsinh}(i\sqrt{5}) = \ln(i\sqrt{5} + \sqrt{-4}) = \ln\left[i(\sqrt{5} \pm 2)\right].$$

However,

$$\left|i(\sqrt{5} \pm 2)\right| = \sqrt{5} \pm 2, \quad \text{and} \quad \operatorname{Arg}\left[i(\sqrt{5} \pm 2)\right] = \tfrac{1}{2}\pi,$$

so

$$\arg[i(\sqrt{5} \pm 2)] = \tfrac{1}{2}\pi + 2n\pi = \tfrac{1}{2}(1 + 4n\pi), \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

Thus

$$\operatorname{arcsinh}(\sqrt{5}) = \ln(\sqrt{5} \pm 2) + i\,\tfrac{1}{2}(1 + 4n)\pi, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

Taking the positive sign arising from the square root function gives the infinitely many-valued result

$$\operatorname{arcsinh}(\sqrt{5}) = \ln(\sqrt{5} + 2) + i\,\tfrac{1}{2}(1 + 4n)\pi, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots,$$

while taking the negative sign gives

$$\operatorname{arcsinh}(\sqrt{5}) = \ln(\sqrt{5} - 2) + i\,\tfrac{1}{2}(1 + 4n)\pi, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

If we now take the result corresponding to the positive sign together with the principal branch of the logarithmic function (corresponding to $n = 0$), we find that

$$\operatorname{arcsinh}(\sqrt{5}) = \ln(\sqrt{5} + 2) + \frac{\pi i}{2}, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

The compatible derivative corresponding to the positive sign follows from Equation (1.140), from which it is seen to be

$$\left[\frac{d}{dz}(\operatorname{arcsinh} z)\right]_{z=i\sqrt{5}} = \frac{1}{2i} = -\tfrac{1}{2}i.$$

If the negative branch of the square root had been taken with, say, the first branch of the logarithmic function (corresponding to $n = 1$) we would have obtained

$$\operatorname{arcsinh}(\sqrt{5}) = \ln(\sqrt{5} - 2) + \frac{5\pi i}{2},$$

and for the compatible derivative the result

$$\left[\frac{d}{dz}(\operatorname{arcsinh} z)\right]_{z=i\sqrt{5}} = \frac{1}{-2i} = \tfrac{1}{2}i. \qquad \diamond$$

The following text is a summary of the properties of analytic functions.

1.5.8 An Analytic Function and Its Derivatives

The function $f(z) = u + iv$ is analytic in a domain D with the derivative

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{(Cartesian representation)}$$

or, equivalently,

$$\begin{aligned} f'(z) &= \left(\frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta\right) + i\left(\frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta\right) \\ &= \left(\frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta\right) - i\left(\frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta\right) \end{aligned} \quad \text{(Polar representation)}$$

if

(i) the first order partial derivatives of u and v are continuous in D, and
(ii) the Cauchy–Riemann equations are satisfied in D so that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{(Cartesian representation)}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \quad \text{(Polar representation)}$$

Harmonic Functions

The function ϕ is said to be a harmonic function if

$$\Delta\phi = 0,$$

where in two dimensions $\Delta\phi$ is the Laplacian

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \qquad \text{(Cartesian representation)}$$

or, equivalently,

$$\Delta\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}. \qquad \text{(Polar representation)}$$

If $f(z) = u + iv$ is an analytic function in a domain D, then both u and v are harmonic in D, so that

$$\Delta u = 0 \quad \text{and} \quad \Delta v = 0.$$

The functions u and v belonging to the analytic function $f(z) = u + iv$ are called conjugate harmonic functions.

Rules for Differentiation

If D is a domain where $f(z)$ and $g(z)$ are analytic, then

1. $\dfrac{d}{dz}[kf(z)] = kf'(z) \qquad (k \text{ is a complex constant});$

2. $\dfrac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z);$

3. $\dfrac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z);$

4. $\dfrac{d}{dz}\{f[g(z)]\} = \dfrac{d}{dz}\{f[g(z)]\}\dfrac{d}{dz}\{g(z)\};$

5. $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}, \qquad (g(z) \neq 0).$

Derivatives of Elementary Functions

6. $\dfrac{d}{dz}[k] = 0, \qquad (k \text{ a complex constant});$

7. $\dfrac{d}{dz}[z^n] = nz^{n-1}$, $\quad$ (n an integer);

8. $\dfrac{d}{dz}[z^c] = cz^{c-1}$, $\quad$ (c an arbitrary constant, and the principal value of z^c used on the right);

9. $\dfrac{d}{dz}[a_0 + a_1 z + \cdots + a_n z^n] = a_1 + a_2 z + \cdots + a_n z^{n-1}$;

10. $\dfrac{d}{dz}[e^{kz}] = ke^{kz}$, $\quad$ (k a complex constant);

11. $\dfrac{d}{dz}[a^z] = a^z \ln a$, $\quad$ (a an arbitrary complex constant and the principal value of a^z used on the right);

12. $\dfrac{d}{dz}[\ln z] = \dfrac{1}{z}$, $\quad$ ($z \neq 0$).

Definitions of Trigonometric Functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \tan z = \frac{\sin z}{\cos z},$$

$$\csc z = \frac{1}{\sin z}, \qquad \sec z = \frac{1}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}.$$

$\sin z$, $\cos z$, $\sec z$ and $\csc z$ have a fundamental period of 2π, while $\tan z$ and $\cot z$ have a fundamental period of π.

The following results are true for $n = 0, \pm 1, \pm 2, \ldots$

13. $\dfrac{d}{dz}[\sin z] = \cos z$, $\quad$ all z.

14. $\dfrac{d}{dz}[\cos z] = -\sin z$, $\quad$ all z.

15. $\dfrac{d}{dz}[\tan z] = \sec^2 z$, $\quad z \neq \left(n + \frac{1}{2}\right)\pi$.

16. $\dfrac{d}{dz}[\csc z] = -\csc z \cot z$, $\quad z \neq n\pi$.

17. $\dfrac{d}{dz}[\sec z] = \sec z \tan z$, $\quad z \neq (n + \frac{1}{2})\pi$.

18. $\dfrac{d}{dz}[\cot z] = -\csc^2 z$, $\quad z \neq n\pi$.

Definitions of Hyperbolic Functions

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \quad \tanh z = \frac{\sinh z}{\cosh z},$$
$$\operatorname{csch} z = \frac{1}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \coth z = \frac{1}{\tanh z}.$$

$\sinh z$, $\cosh z$, $\operatorname{sech} z$ and $\operatorname{csch} z$ have a fundamental period of $2\pi i$, while $\tanh z$ and $\coth z$ have a fundamental period of πi.

The following results are true when $n = 0, \pm 1, \pm 2 \ldots$.

19. $\dfrac{d}{dz}[\sinh z] = \cosh z.$

20. $\dfrac{d}{dz}[\cosh z] = \sinh z.$

21. $\dfrac{d}{dz}[\tanh z] = \operatorname{sech}^2 z, \quad z \neq (n + \frac{1}{2})\pi.$

22. $\dfrac{d}{dz}[\operatorname{csch} z] = -\operatorname{csch} z \coth z, \quad z \neq n\pi i.$

23. $\dfrac{d}{dz}[\operatorname{csch} z] = -\operatorname{csch} z \coth z, \quad z \neq n\pi i.$

24. $\dfrac{d}{dz}[\operatorname{sech} z] = -\operatorname{sech} z \coth z, \quad z \neq (n + \frac{1}{2})\pi i.$

25. $\dfrac{d}{dz}[\coth z] = -\operatorname{csch}^2 z, \quad z \neq n\pi i.$

Definitions of Inverse Trigonometric Functions

$$\arcsin z = -i \ln\left(iz + \sqrt{1 - z^2}\right), \quad \arccos z = -i \ln\left(z + i\sqrt{1 - z^2}\right),$$

$$\arctan z = \frac{i}{2} \ln\left(\frac{i + z}{i - z}\right).$$

26. $\dfrac{d}{dz}[\arcsin z] = \dfrac{1}{\sqrt{1 - z^2}}, \quad$ (for a fixed branch).

27. $\dfrac{d}{dz}[\arccos z] = \dfrac{-1}{\sqrt{1 - z^2}}, \quad$ (for a fixed branch).

28. $\dfrac{d}{dz}[\arctan z] = \dfrac{1}{1 + z^2}, \quad$ (for a fixed branch).

Definitions of Inverse Hyperbolic Functions

$$\operatorname{arcsinh} z = \ln\left(z + \sqrt{z^2 + 1}\right), \quad \operatorname{arccosh} z = \ln\left(z + \sqrt{z^2 - 1}\right),$$

$$\operatorname{arctanh} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right).$$

29. $\dfrac{d}{dz}[\operatorname{arcsinh} z] = \dfrac{1}{\sqrt{z^2+1}}$, (for a fixed branch).

30. $\dfrac{d}{dz}[\operatorname{arccosh} z] = \dfrac{1}{\sqrt{z^2-1}}$, (for a fixed branch).

31. $\dfrac{d}{dz}[\operatorname{arctanh} z] = \dfrac{1}{1-z^2}$, (for a fixed branch).

Useful Identities and Properties of Complex Functions

32. $e^{iz} = \cos z + i \sin z$.
33. $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$.
34. If $z = x + iy$, then

$$|e^z| = e^x, \quad \arg(e^z) = y + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots .$$

35. If $z = x + iy$ and $w = \ln z = u + iv$, then

$$u = \ln|z|, \quad v = \operatorname{Arg}(z) + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots .$$

36. $\ln(z_1 z_2) = \ln z_1 + \ln z_2$.
37. $\ln(z_1/z_2) = \ln z_1 - \ln z_2$.
38. $\ln(z^{p/q}) = (p/q)\ln z$, ($p$, q integers).
39. $\sin^2 z + \cos^2 z = 1$.
40. $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.
41. $\sin 2z = 2\sin z \cos z$.
42. $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$.
43. $\cos 2z = \cos^2 z - \sin^2 z = 1 - 2\sin^2 z = 2\cos^2 z - 1$.
44. $\sin(iz) = i \sinh z$, $\cos(iz) = \cosh z$.
45. $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, $\tan(-z) = -\tan z$.
46. The zeros of $\sin z$ occur at $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.
47. The zeros of $\cos z$ occur at $z = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$.
48. The zeros of $\tan z$ occur at $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$
49. If $z = x + iy$, then $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$.
50. $\cosh^2 z - \sinh^2 z = 1$.
51. $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$.
52. $\sinh 2z = 2\sinh z \cosh z$.
53. $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$.
54. $\cosh 2z = \cosh^2 z + \sinh^2 z = 1 + 2\sinh^2 z = 2\cosh^2 z - 1$.
55. $\sinh(iz) = i \sin z$, $\cosh(iz) = \cos z$.

56. The zeros of $\sinh z$ occur at $z = n\pi, n = 0, \pm 1, \pm 2, \ldots$.
57. The zeros of $\cosh z$ occur at $z = \frac{1}{2}(2n+1)\pi, n = 0, \pm 1, \pm 2, \ldots$.
58. The zeros of $\tanh z$ occur at $z = n\pi i, n = 0, \pm 1, \pm 2, \ldots$.
59. If $z = x + iy$, then $|\sinh z|^2 = \sin^2 y + \sinh^2 x$, $|\cosh z|^2 = \cos^2 y + \sinh^2 x$

Exercises 1.5

Simplify the following rational functions by means of partial fractions.

1. $\dfrac{3z}{(z-1)(z+1)^2}$.
2. $\dfrac{1}{z^3 - 2z^2 + z}$.
3. $\dfrac{z^4 - 3}{z^2 + 2z + 1}$.
4. $\dfrac{z^2 + 2}{(z+1)^3(z-2)}$.

In each of the following exercises use the given value of z to find e^z, e^{-z}, and $\exp(z^2)$.

5. $z = \frac{3}{2}\pi i$.
6. $z = 2 + \frac{1}{2}\pi i$.
7. $z = 1 + 2i$.
8. $z = -\frac{1}{4}\pi i$.

In each of the following exercises use the given value of z to find $\ln z$ and $\operatorname{Ln} z$.

9. $z = i$.
10. $z = -i$.
11. $z = e^{-3}$.
12. $z = e^{5i}$.
13. $z = 4$.
14. $z = -5i$.

Find the values and the principal value of the following expressions.

15. $(-4)^i$.
16. $(i)^{-3i}$.
17. $i^{2/\pi}$.
18. $(-i)^{i}$.
19. $(-i)^{-2i}$.

20. 1^{2i}.
21. $3^{(1-i)}$.
22. $(i)^i$.
23. Find all of the roots of $\sin z = 5$.
24. Find all of the roots of $\cos z = 2$.
25. Find all of the roots of $\cos z = 3i$.
26. Find all of the roots of $\sinh z = i$.
27. Find all of the values of $\arctan \sqrt{7}$.
28. Find all of the values of $\arccos 4i$.
29. Find all of the values of $\operatorname{arccosh}(\sqrt{3}/2)$.
30. Find all of the values of $\operatorname{arctanh} i$.
31. By considering $|\sinh z|^2$ with $z = x + iy$, prove that

$$\sinh|y| \leqslant |\sin z| \leqslant \cosh y.$$

32. By considering $|\sinh z|^2$ with $z = x + iy$, prove that

$$\sinh|x| \leqslant |\sinh z| \leqslant \cosh x.$$

Exercises of Higher Difficulty

33. Derive the result that if $P(z)$ is a polynomial of degree m and $Q(z)$ is a polynomial of degree n, with $m > n$, and $Q(z)$ has the n simple zeros $z_1, z_2, \ldots, z_n$ (each with multiplicity 1), that the unknown coefficients A_r, $r = 1, 2, \ldots, n$ in the partial fraction expansion

$$\frac{P(z)}{Q(z)} = \frac{A_1}{z - z_1} + \frac{A_2}{z - z_2} + \cdots + \frac{A_n}{z - z_n},$$

are given by

$$A_r = \frac{P(z_r)}{Q'(z_r)}, \qquad r = 1, 2, \ldots, n.$$

Apply the result to the rational function

$$\frac{z^2 + z - 3i}{(z-2)(z-3)(z+i)}.$$

34. Given that $f(z) = z^2 = u + iv$, with $z = x + iy$, prove that the families of curves $u =$ const. and $v =$ const. form orthogonal trajectories (See Figure 1.13). Does the mutual orthogonality of these two families of curves hold everywhere in the (x, y)-plane, and if not where does it fail and why?

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