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Integer Programming

1 Introduction

In many programming problems optimal solution is sought in terms of integral values of the variables, nonintegral answers not being meaningful in the context of the situation which gives rise to the problem. For example, if the variables are the numbers of buses on different routes in a town or the numbers of bank branches in different regions of a country, fractional answers have no meaning. Mathematical programming subject to the constraint that the variables are integers is called integer programming. If some of the variables are restricted to be integers while others are real numbers, the problem is said to be mixed integer programming.

Strictly speaking, if in an LP problem we restrict the variables to integers, the problem becomes nonlinear. But it is convenient to call it an integer linear programming problem (ILP) because the constraints and the objective function remain linear if the integral restriction on the variables is ignored. If not all but some of the variables are restricted to be integers, we have a mixed integer linear programming problem (MILP). In general we may have an integer or a mixed integer nonlinear programming problem if it is obtained by imposing integer restriction on an otherwise nonlinear problem. In this chapter we shall consider only the integer and the mixed integer linear programming problems.

One obvious way of getting an answer to an ILP or MILP is to ignore the integer restrictions on variables and solve it as an ordinary LP problem, and then to round off the optimal solution to nearest integers. When the answers are in the neighbourhood of large integers, the method gives satisfactory results. For example, if the problem is concerned with human population in a town, a fractional answer giving the number of persons as 3548.68 can be rounded off to 3549 or even to 3550 without any significant error. But if the answer is in the neighbourhood of small integers such rounding off may give a totally wrong answer. We illustrate this by an example in the next section.

We have seen in chapters 4 and 5 that in transportation or network type of problems with integral data, the answers are always in terms of integers. If in an LP problem the optimal solution turns out to be integral, it is obviously the optimal solution to the related ILP problem also, and nothing more need be done. Special methods have to be derived if this is not so.

2 ILP in two-dimensional space

As in the case of an LP problem, it is easy to obtain a graphical solution of an

ILP problem if the number of variables is only two. We therefore take a two-dimensional ILP problem as an example to bring out the important features of a general ILP problem.

Consider the problem:

Maximise $\phi(X) = 3x_1 + 4x_2;$ (1)

subject to

$$2x_1 + 4x_2 \le 13,$$

$$-2x_1 + x_2 \le 2,$$

$$2x_1 + 2x_2 \ge 1, \quad (2)$$

$$6x_1 - 4x_2 \le 15,$$

$$x_1,\ x_2 \ge 0; \quad (3)$$

$$x_1,\ x_2 \text{ integers.} \quad (4)$$

This is an ILP problem. If we drop (4) we obtain the related LP problem.

Fig. 1 shows the graphical solution of the related LP problem. The polygon *ABCDEF* is the convex set of feasible solutions and the point $C(x_1 = 7/2, x_2 = 3/2)$ is the optimal solution with the maximum value of $\phi = 33/2$. If we round off (7/2, 3/2) to nearest integers, assuming that 1/2 may be rounded off to 0 or 1 with equal justification, we get the four points (3, 1), (4, 1), (4, 2), (3, 2). Of these the last three are not feasible. So the only feasible point obtained by rounding off is (3, 1), which makes $\phi = 13$.

Consider now the given ILP problem. We restrict x_1, x_2 to be integers, and so the set of feasible solutions are the points in the polygon *ABCDEF* whose coordinates are integers. Such points, marked × in Fig. 1, are (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (3, 1), (0, 2), (1, 2), (2, 2). Among these points the objective function ϕ is maximum at (2, 2) with $\phi = 14$. Thus rounding off gives a wrong answer.

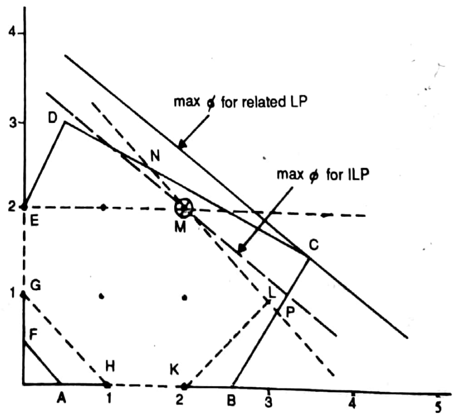


Fig. 1

The set of feasible solutions of the ILP problem is not convex, because it consists of the isolated nine points given above. If we obtain the convex hull of this nonconvex set of feasible solutions, we get the polygon *EGHKLM*. Every vertex of this convex polygon is a feasible solution of the ILP problem. Let us consider the LP problem:

Maximise $\phi(\mathbf{X}) = 3x_1 + 4x_2$ (1)

subject to $\mathbf{X} \in$ convex hull of feasible solutions of (2), (3) and (4).

The optimal solution of this problem can be seen to be the point $M(2, 2)$ which is the optimal solution of the given ILP. We may therefore conjecture that the optimal solution of an ILP problem is the same as the optimal solution to an LP problem whose objective function is the same as that of the ILP problem but whose constraints are such that the convex set of feasible sollutions turns out to be the convex hull of the set of feasible solutions of the ILP problem. We proceed to prove this conjecture.

3 General ILP and MILP problems

DEFINITION 1. *A vector* $\mathbf{X} \in E_n$ *shall be called an integer vector if its components* x_i *for all* i, $i = 1, 2, \ldots, n$, *are integers; it shall be called a mixed integer vector if* x_i *is integer for* $i \in J$ *where* $J \subset \{1, 2, 3, \ldots, n\}$.

With symbols as defined in section 3, chapter 3, we enunciate the general ILP or MILP problem as follows:

$$\text{Minimise} \quad f(\mathbf{X}) = \mathbf{CX}, \tag{5}$$

$$\text{subject to} \quad \mathbf{AX} = \mathbf{B}, \tag{6}$$

$$\mathbf{X} \geq 0, \tag{7}$$

$$\mathbf{X} \text{ an integer or a mixed integer vector.} \tag{8}$$

If we drop constraint (8) we are left with the related LP problem. A solution of (6), (7), (8) is obviously a solution of (6), (7). Therefore, if T_F denotes the set of feasible solutions of the ILP or the MILP problem, and S_F the set of feasible solutions of the related LP problem, then $T_F \subseteq S_F$. Since S_F, if nonempty, is a convex set (theorem 1, chapter 3), and every point of T_F is in S_F, the convex linear combinations of points in T_F are also in S_F. Hence $[T_F]$, the convex hull of T_F, is a subset of S_F (theorem 10, chapter 1). Thus

$$T_F \subseteq [T_F] \subseteq S_F. \tag{9}$$

The ILP or MILP problem (5)-(8) may now be stated as follows also:

$$\left.\begin{array}{ll}\text{Minimise} & f(\mathbf{X}) = \mathbf{CX},\\ \text{subject to} & \mathbf{X} \in T_F.\end{array}\right\} \tag{10}$$

The related LP problem is:

$$\left.\begin{array}{ll}\text{Minimise} & f(\mathbf{X}) = \mathbf{CX},\\ \text{subject to} & \mathbf{X} \in S_F.\end{array}\right\} \tag{11}$$

We state another LP problem associated with the above:

$$\left.\begin{array}{ll}\text{Minimise} & f(\mathbf{X}) = \mathbf{CX}, \\ \text{subject to} & \mathbf{X} \in [T_F].\end{array}\right\} \quad (12)$$

We prove three theorems concerning the solutions of these problems.

THEOREM 1. *If an optimal solution of* (11) *exists and* T_F *is nonempty, then optimal solutions of* (10) *and* (12) *exist. Also the optimal solution of* (11) *is a lower bound for the optimal solutions of* (10) *and* (12).

Proof. Let $\mathbf{X}_0$ be an optimal solution of (11). Then for all $\mathbf{X}$ in S_F

$$f(\mathbf{X}_0) \le f(\mathbf{X}).$$

Let $\mathbf{Y} \in T_F$. Then, from (9), $\mathbf{Y} \in S_F$, and so

$$f(\mathbf{X}_0) \le f(\mathbf{Y}).$$

This means that $f(\mathbf{Y})$, $\mathbf{Y} \in T_F$, has a lower bound, and so (10) has an optimal solution. Similarly we prove that (12) has an optimal solution. The second part of the theorem
also stands proved. Proved.

THEOREM 2. *If an optimal solution of* (11) *is an integer or a mixed integer vector as required by* (8), *then it is also an optimal solution of* (10).

Proof. Let $\mathbf{X}_0$ be an optimal solution of (11) satisfying (8). Then $\mathbf{X}_0 \in T_F$ and so T_F is nonempty. Let, if possible, $\mathbf{X}_0$ be not an optimal solution of (10). But, from theorem 1, an optimal solution exists. Let it be $\mathbf{Y}_0$. Then $\mathbf{Y}_0 \in T_F$ and

$$f(\mathbf{Y}_0) < f(\mathbf{X}_0).$$

Since $T_F \subseteq S_F$, $\mathbf{Y}_0 \in S_F$. Also $\mathbf{X}_0 \in S_F$. The above inequality then implies that $\mathbf{X}_0$ is not an optimal solution of (11) which contradicts our hypothesis. Hence $\mathbf{X}_0$ is an optimal solution of (10). Proved.

THEOREM 3. *An optimal solution of* (10) *is an optimal solution of* (12). *Conversely, a basic optimal solution of* (12) *is an optimal solution of* (10).

Proof. Let $\mathbf{X}_0 \in T_F$ be an optimal solution of (10). Then for all $\mathbf{X} \in T_F$,

$$f(\mathbf{X})_0 \le f(\mathbf{X}). \quad (13)$$

Let $\mathbf{Y}$ be any point in $[T_F]$. Then $\mathbf{Y}$ is a convex linear combination of some points $\mathbf{X}_i$, $i = 1, 2, \ldots, r$, of T_F, that is

$$\mathbf{Y} = \sum_{i=1}^{r} \lambda_i \mathbf{X}_i,\ \lambda_i \ge 0,\ \sum_{i=1}^{r} \lambda_i = 1.$$

Since $\mathbf{X}_0 \in T_F$, $\mathbf{X}_0$ is also in $[T_F]$. Let $\mathbf{Y}$ be different from $\mathbf{X}_0$, and if possible, let

$$f(\mathbf{Y}) < f(\mathbf{X}_0), \quad (14)$$

$$\Rightarrow f\left(\sum_{i=1}^{r} \lambda_i \mathbf{X}_i\right) < f(\mathbf{X}_0),$$

$$\Rightarrow \sum_{i=1}^{r} \lambda_i f(\mathbf{X}_i) < f(\mathbf{X}_0), \text{ since } f(\mathbf{X}) \text{ is linear,}$$

$$\Rightarrow f(\mathbf{X}_k) \sum_{i=1}^{r} \lambda_i < f(\mathbf{X}_0),$$

$$\Rightarrow f(\mathbf{X}_k) < f(\mathbf{X}_0), \tag{15}$$

where

$$f(\mathbf{X}_k) = \min_i f(\mathbf{X}_i).$$

But $\mathbf{X}_k$, being one of the $\mathbf{X}_i$'s, is in T_F, and so (15) contradicts (13). Therefore (14) is not true, and consequently

$$f(\mathbf{Y}) \geq f(\mathbf{X}_0)$$

which means $\mathbf{X}_0$ is an optimal solution of (12).

To prove the converse, let $\mathbf{X}_0$ be an optimal solution of (12). Then $\mathbf{X}_0$ is a vertex of $[T_F]$, and so $\mathbf{X}_0$ is in T_F (chapter 1, theorem 15). Let $\mathbf{X}$ be any other point in T_F. Then it is in $[T_F]$, and so

$$f(\mathbf{X}_0) \leq f(\mathbf{X}),$$

which means $\mathbf{X}_0$ is an optimal solution of (10). Proved.

The above theorem leads to the conclusion that to solve an ILP or MILP problem one has only to solve the associated LP problem whose set of feasible solutions is the convex hull of the set of feasible solutions of the original problem. It is, however, not easy to find the required convex hull, and therefore the theorem provides only theoretical insight and not any practical method of solution. The practical methods generally recommended fall in two categories, commonly called (i) the cutting plane method and (ii) the branch and bound method. We proceed to discuss these methods, in each case first explaining the underlying idea with the help of the numerical example of section 2.

4 Example of section 2 continued

We go back to the example of section 2 to illustrate the cutting plane method. Suppose we introduce an additional constraint in the problem which has the effect of cutting out the portion NPC from the polygon *ABCDEF* (Fig. 1). Since the equation of the straight line ML is $x_1 + x_2 = 4$, such a constraint is

$$x_1 + x_2 \leq 4.$$

This cuts out the optimal solution *C* of the LP problem without cutting out any of its integral feasible solutions. The solution to the LP problem with this additional constraint is the point *N* (3/2, 5/2) which again is nonintegral. Let us cut this out by introducing the constraint corresponding to the line EM which is

$$x_2 \leq 2.$$

This again does not cut out any of the integral feasible solutions.

With the two additional constraints, the original LP problem is now modified so that its set of feasible solutions is the polygon *ABPMEF* which still contains all the integral feasible solutions of the original problem. The optimal solution of this modified problem is the point *M* (2, 2). Since this is integral, it is the solution of the original ILP problem.

The additional constraints are called cuts. By introducing suitable cuts one by one and solving an LP problem every time, we could hope to arrive at the solution of the ILP problem. The important question now is – How to find suitable cuts.

5 Cutting planes

We confine our discussion to the general ILP problem (5)-(8), leaving the MILP problem for later comment. We therefore assume that **X** is an integer vector. The related LP problem is (5)-(7).

Let a basic optimal solution of the related LP problem be $[x_1, x_2, \ldots, x_m, 0 \ldots 0]$, and let the corresponding canonical form of equations (6) (see section 10, chapter 3), be

$$\begin{aligned} x_1 + a'_{1,m+1}\, x_{m+1} + \ldots + a'_{1n}\, x_n &= b'_1, \\ x_2 + a'_{2,m+1}\, x_{m+1} + \ldots + a'_{2n}\, x_n &= b'_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m + a'_{m,m+1}\, x_{m+1} + \ldots + a'_{mn}\, x_n &= b'_m \end{aligned} \tag{16}$$

Since the solution is necessarily feasible,

$$x_i = b_i' \geq 0,\ i = 1, 2, \ldots, m.$$

If all the b_i' are integers, we have the solution of the ILP problem, and nothing more need be done. In general this may not be so. Let a particular b_i' be noninteger. The corresponding equation in (16) is

$$x_i + a'_{i,m+1} x_{m+1} + \ldots + a'_{in}\, x_n = b_i'. \tag{17}$$

Let
$$b_i' = [b_i'] + \beta_i, \tag{18}$$

and
$$a'_{ij} = [a'_{ij}] + \alpha_{ij},\ j = m+1, \ldots, n, \tag{19}$$

where $[b_i']$ is the greatest integer less than b_i', and $[a'_{ij}]$ is the greatest integer less than or equal to a'_{ij}. Then

$$[b_i'] \geq 0,\ 0 < \beta_i < 1,\ 0 \leq \alpha_{ij} < 1.$$

Equation (17) can now be written as

$$x_i - [b_i'] + \sum_{j=m+1}^{n} [a'_{ij}]\, x_j = \beta_i - \sum_{j=m+1}^{n} \alpha_{ij}\, x_j.$$

This equation, being one of the constraints, must be satisfied by every feasible solution of the ILP as well as the related LP problem. For an integer feasible solution the left side should be an integer and so the right side too should be an integer. Also, since $0 \leq \alpha_{ij} < 1$ and x_j, being feasible, is non-negative,

$$\sum_{j=m+1}^{n} \alpha_{ij}\, x_j \geq 0.$$

Hence
$$\beta_i - \sum_{j=m+1}^{n} \alpha_{ij}\, x_j \text{ is an integer } \leq \beta_i.$$

But $0 < \beta_i < 1$. Therefore for an integer feasible solution

$$\beta_i - \sum_{j=m+1}^{n} \alpha_{ij}\, x_j \leq 0,$$

or

$$-\sum_{j=m+1}^{n} \alpha_{ij}x_j \leq -\beta_i. \tag{20}$$

But for the optimal solution of the related problem with which we started, $x_j = 0$, $j = m + 1, \ldots, n$, and so

$$\beta_i - \sum_{j=m+1}^{n} \alpha_{ij}x_j = \beta_i > 0.$$

Thus we have discovered a linear constraint (20) which is satisfied by integer solutions of the problem but cuts out the optimal solution of the LP problem provided it is nonintegral. This, therefore, provides a suitable new constraint. (20) with equality sign is the corresponding cutting plane. We note that β_i and α_{ij} in (20) are defined by (18) and (19) respectively.

We add the constraint (20) to the set of constraints (6) and solve the modified problem. If its optimal solution is integral, we stop, otherwise we again obtain a cutting plane to cut out this optimal solution but not any of the integer feasible solutions. We go on doing this till we get an integer optimal solution. It has been proved that the cutting plane method terminates in a finite number of iterations either with the integer optimal solution or with the conclusion that the given problem is not feasible.

The successively modified LP problems obtained after adding each time a constraint of the type (20) are best solved by the dual simplex method (section 20, chapter 3). Constraint (20) leads to the constraint equation

$$-\sum_{j=m+1}^{n} \alpha_{ij}x_j + y = -\beta_i,$$

where y is a slack variable. We add this constraint to the simplex tableau as it stands at the optimal stage of the preceding LP problem. A basic solution of the modified problem consists of the basic solution at the preceding stage along with $y = -\beta_i$. But this solution is not feasible. It is, however, dual feasible because $c_j \geq 0$, $j = m + 1, \ldots, n$, at this stage. Hence we apply the dual simplex algorithm to proceed further to obtain a solution which is both primal and dual feasible and therefore optimal.

6 Example

We illustrate the cutting plane method explained in the preceding section through the example of section 2. Introducing the slack variables x_3, x_4, x_5, x_6 and the artificial variable x_7 (which we introduce to obtain a basic feasible solution by first solving the Phase I problem which minimises w), the problem can be written as

$$\begin{array}{lll}
\text{Minimise } f = & -3x_1 - 4x_2, & \text{(Phase II)} \\
\text{minimise } w = & x_7, & \text{(Phase I)} \\
\text{subject to} & 2x_1 + 4x_2 + x_3 & = 13, \\
& -2x_1 + x_2 \quad + x_4 & = 2, \\
& 6x_1 - 4x_2 \quad + x_5 & = 15, \\
& 2x_1 + 2x_2 \quad - x_6 + x_7 & = 1; \\
& x_1, x_2, \ldots, x_7 \geq 0 \text{ and integers.} &
\end{array}$$

Table 1 is the simplex tableau for the complete solution. Phase I ends after the iteration $I = 2$ when we get a basic feasible solution of the related LP problem. The end of Phase II at iteration $I = 4$ gives the optimal solution of this problem. This is nonintegral, and so cutting planes in the form of additional constraints are included, one at a time, in subsequent iterations till the integer optimal solution is reached.

In the optimal solution of the related LP problem $x_1 = 7/2$. The corresponding equation of type (17) is

$$x_1 + \frac{2}{16}x_3 + \frac{2}{16}x_5 = \frac{7}{2}$$

or

$$x_1 + \left(0 + \frac{1}{8}\right)x_3 + \left(0 + \frac{1}{8}\right)x_5 = 3 + \frac{1}{2}.$$

The required constraint, by (20), is therefore

$$-\frac{1}{8}x_3 - \frac{1}{8}x_5 \leq -\frac{1}{2},$$

or, after introducing the slack variable x_8,

$$-\frac{2}{16}x_3 - \frac{2}{16}x_5 + x_8 = -\frac{1}{2}.$$

The constraint is added to the problem as it stands at stage $I = 4$ producing the problem at stage $I = 5$. The dual simplex algorithm is used to proceed further and the optimal solution of the current problem is obtained in $I = 6$. This is also non-integral. Hence from the equation

$$x_2 + \frac{1}{4}x_3 - \frac{1}{2}x_8 = \frac{7}{4},$$

or

$$x_2 + \left(0 + \frac{1}{4}\right)x_3 + \left(-1 + \frac{1}{2}\right)x_8 = 1 + \frac{3}{4},$$

a second constraint is obtained as

$$-\frac{1}{4}x_3 - \frac{1}{2}x_8 \leq -\frac{3}{4},$$

or

$$-\frac{1}{4}x_3 - \frac{1}{2}x_8 + x_9 = -\frac{3}{4}.$$

Again the optimal solution of the current problem is obtained by the dual simplex method ($I = 8$). The optimal solution is not yet integral. A third cutting plane is added in $I = 9$ which finally gives the integral optimal solution in $I = 10$.

TABLE 1

I	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7
	x_3	13	2	4	1				
	x_4	2	−2	1		1			
1	x_5	15	6	−4			1		
	x_7	1	2	2				−1	1
	$-f$	0	−3	−4					
	$-w$	−1	−2	−2				1	

I	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
	x_3	12		2	1			1	−1	
	x_4	3		3		1		−1	1	
2	x_5	12		−10			1	3	−3	Initial b.f.s. of related LP
	x_1	1/2	1	1				−1/2	1/2	
	$-f$	3/2		−1				3/2	3/2	
	$-w$	0		0				0	1	End of Phase I
	x_3	8		16/3	1		−1/3			
	x_4	7		−1/3		1	1/3			
3	x_6	4		−10/3			1/3	1		
	x_1	5/2	1	−2/3			1/6			
	$-f$	15/2		−6			1/2			
	x_2	3/2		1	3/16		−1/16			Optimal solution of related
4	x_4	15/2			1/16	1	5/16			LP
	x_6	9			10/16		2/16	1		
	x_1	7/2	1		2/16		2/16			Eq. giving cutting plane
	$-f$	33/2			9/8		1/8			End of Phase II

I	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_8	x_9	
	x_2	3/2		1	3/16		−1/16				
	x_4	15/2			1/16	1	5/16				
5	x_6	9			10/16		2/16	1			
	x_1	7/2	1		2/16		2/16				
	x_8	−1/2			−2/16		−2/16		1		First cutting plane
	$-f$	33/2			9/8		1/8				
	x_2	7/4		1	1/4				−1/2		Eq. giving cutting plane
	x_4	25/4			−1/4	1			5/2		
6	x_6	17/2			1/2			1	1		
	x_1	3	1		0				1		
	x_5	4			1		1		−8		
	$-f$	16			1				1		
	x_2	7/4		1	1/4				−1/2		
	x_4	25/4			−1/4	1			5/2		
	x_6	17/2			1/2			1	1		
7	x_1	3	1		0				1		
	x_5	4			1		1		−8		
	x_9	−3/4			−1/4				−1/2	1	Second cutting plane
	$-f$	16			1				1		

I	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_8	x_9	
	x_2	5/2		1	1/2					−1	
	x_4	5/2			−3/2	1				5	
	x_6	7			0			1		2	
8	x_1	3/2	1		−1/2					2	
	x_5	16			5		1			−16	
	x_8	3/2			1/2				1	−2	Gives cutting plane
	$-f$	29/2			1/2					2	

I	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_8	x_9	x_{10}	
	x_2	5/2		1	1/2					−1		
	x_4	5/2			−3/2	1				5		
	x_6	7			0			1		2		
9	x_1	3/2	1		−1/2					2		
	x_5	16			5		1			−16		
	x_8	3/2			1/2				1	−2		
	x_{10}	−1/2			−1/2					0	1	Third cutting plane
	$-f$	29/2			1/2					2		
	x_2	2		1						−1	1	
	x_4	4				1				5	−3	
	x_6	7						1		2	0	
10	x_1	2	1							2	−1	Integer optimal
	x_5	11					1			−16	10	solution
	x_8	1							1	−2	1	
	x_3	1			1					0	−2	
	$-f$	14								2	1	

7 Remarks on cutting plane methods

The above method of obtaining cutting planes is only one of the several methods of generating cutting planes which have been proposed by various authors and which can be found in the vast literature on integer programming. It has been proved that the cutting plane method solves the ILP problem in a finite number of steps, either giving an integer optimal solution or indicating that a feasible solution does not exist.

One disadvantage of the method is that the number of steps can be very large sometimes even in problems which apparently look simple. The number of constraints goes on increasing leading to increased volume of numerical work. Some relief can be obtained by dropping out a cutting plane from the simplex tableau once it becomes superfluous due to subsequent addition of other cutting planes. This happens when the slack variable in that cutting plane becomes a basic variable with a positive value in the simplex tableau. For example, in table 1, let us follow the part played by the first cutting plane introduced at the stage $l = 5$. We find that the slack variable x_8 in the constraint added at stage $l = 5$ is nonbasic for the optimal solution at stage $l = 6$. This means that at this stage $x_8 = 0$ and so the optimal solution lies on the cutting plane

$$-\frac{1}{8}x_3 - \frac{1}{8}x_5 = -\frac{1}{2}$$

Later on after a second cutting plane has been introduced at $l = 7$, x_8 appears as a basic variable with positive value at $l = 8$. This means that the optimal solution at this stage does not lie on the first cutting plane but within the region

$$-\frac{1}{8}x_3 - \frac{1}{8}x_5 < -\frac{1}{2}.$$

It remains so throughout subsequent work. The first cutting plane plays no active part in the remaining stages of the solution. It could therefore as well be erased from the simplex tableau after $l = 8$.

Cutting plane methods can be applied to MILP problems also. There are rules by which cutting planes for mixed integer problems can be obtained. We, however, omit these as, in general, the cutting plane method has been found to be less suitable than the other method, the branch and bound method, which we proceed to discuss.

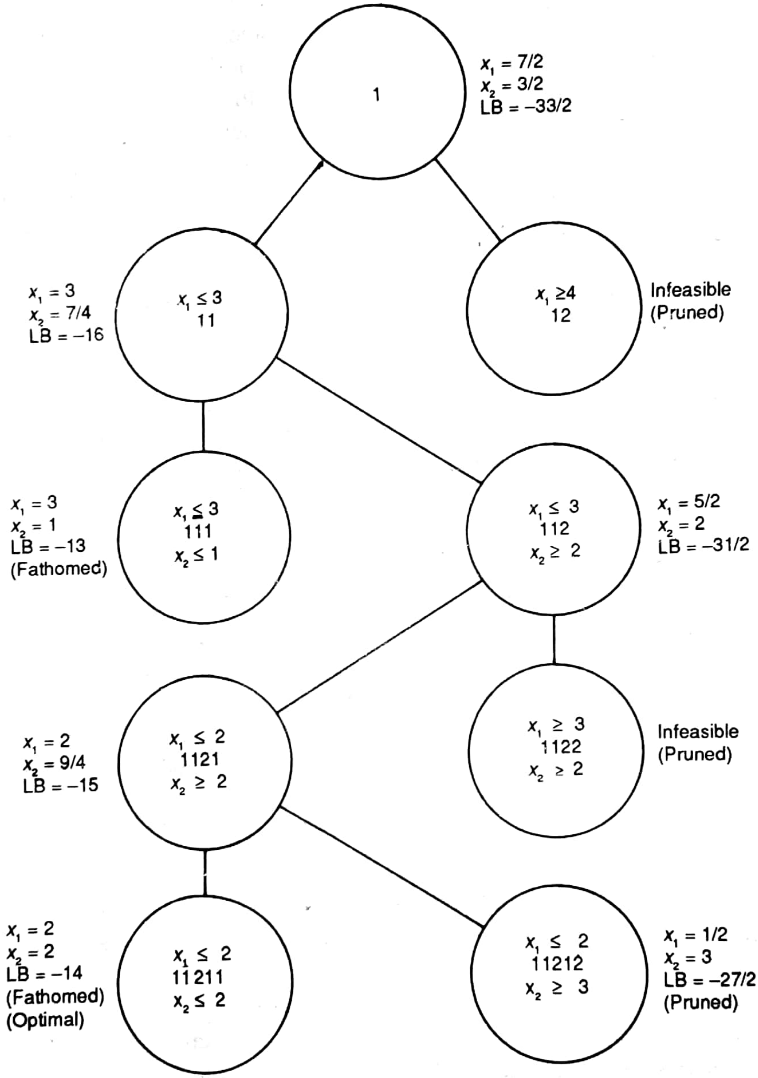
8 Branch and bound method—examples

It will be more helpful to the understanding of the branch and bound method if, before discussing the method in its generality, it is illustrated through simple examples.

Example 1: We again take the ILP problem (1)-(4) of section 2, but with the sign of the objective function changed, so that the problem is to

$$\text{minimise} \quad f = -3x_1 - 4x_2.$$

Dropping the constraint (4) we get the related LP problem. Its solution is $f = -33/2$ with $x_1 = 7/2$, $x_2 = 3/2$ (as can be easily obtained graphically or otherwise). Obviously $-33/2$ is a lower bound (LB) for the objective function f of this problem. Let us call the related LP as problem 1, and say that a LB of the objective function f of this problem is $-33/2$ with $x_1 = 7/2$, $x_2 = 3/2$. (We adopt this phraseology because, as we shall see later, what is essential to the branch and bound method is not the exact minimum value of the objective function but a lower bound to it). In Fig. 2 circle 1 at the top with information regarding the LB and the corresponding values of the variables indicates this situation.



1
$x_1 = 7/2$
$x_2 = 3/2$
LB = −33/2
$x_1 \le 3$
11
$x_1 = 3$
$x_2 = 7/4$
LB = −16
$x_1 \ge 4$
12
Infeasible
(Pruned)
$x_1 \le 3$
111
$x_2 \le 1$
$x_1 = 3$
$x_2 = 1$
LB = −13
(Fathomed)
$x_1 \le 3$
112
$x_2 \ge 2$
$x_1 = 5/2$
$x_2 = 2$
LB = −31/2
$x_1 \le 2$
1121
$x_2 \ge 2$
$x_1 = 2$
$x_2 = 9/4$
LB = −15
$x_1 \ge 3$
1122
$x_2 \ge 2$
Infeasible
(Pruned)
$x_1 \le 2$
11211
$x_2 \le 2$
$x_1 = 2$
$x_2 = 2$
LB = −14
(Fathomed)
(Optimal)
$x_1 \le 2$
11212
$x_2 \ge 3$
$x_1 = 1/2$
$x_2 = 3$
LB = −27/2
(Pruned)

Fig. 2

Let us divide problem 1 into two subproblems, problem 11 and problem 12, by imposing the constraints $x_1 \leq 3$ and $x_1 \geq 4$ respectively on problem 1. These constraints are suggested by the solution $x_1 = 7/2$. Since we are looking for integer solutions, the interval $3 < x_1 < 4$ in which $x_1 = 7/2$ lies can be left out, and further probes need be made only in regions $x_1 \geq 4$ and $x_1 \leq 3$. (We could, with equal justification, impose the constraints $x_2 \leq 1$ and $x_2 \geq 2$ first). This operation of replacing a problem by two subproblems is called *branching*. Problem 1 which may be called the parent problem has been branched into problems 11 and 12.

Solution to problem 11 can be obtained, again graphically or otherwise, as $x_1 = 3$, $x_2 = 7/4$, with $f = -16$. Following the phraseology explained above, -16 is the LB of the problem with values $x_1 = 3$, $x_2 = 7/4$. Problem 12 is easily found to be infeasible. It is therefore left out of further consideration, or, in standard branch and bound terminology, *pruned*. Problem 11 is further branched into two problems, 111 and 112, by imposing the additional constraints $x_2 \leq 1$ and $x_2 \geq 2$, which are suggested by the nonintegral value $x_2 = 7/4$. The LB of problem 111 is found to be -13 for $x_1 = 3$, $x_2 = 1$. This is an integer solution, and is therefore a possible candidate for the optimal solution of the original ILP problem. Moreover, no other feasible integer solution of problem 111 need be found out as the one already found gives the lowest value of f. We say that problem 111 has been *fathomed*. There is no need to branch it further, but its LB integer solution should be kept in view as a possible candidate for the optimal solution of the given ILP problem.

Problem 112 is found to have an LB $= -31/2$ with $x_1 = 5/2$, $x_2 = 2$. Since its LB is lower than the LB of problem 111, it may be concealing integer solutions which may give lower values of f than problem 111. Hence it should be further branched into problem 1121 and 1122. The latter is found to be infeasible and is therefore pruned. The former gives the LB -15 for $x_1 = 2$, $x_2 = 9/4$. Since this LB is lower than the LB of the fathomed problem 111, we branch problem 1121 into problems 11211 and 11212. The former has the LB -14 for $x_1 = 2$, $x_2 = 2$. Since this solution is integral, this problem stands fathomed, and its solution gives a possible candidate for the optimal solution of the original ILP problem. Problem 11212 is found to have the LB $-27/2$ for $x_1 = 1/2$, $x_2 = 3$. This problem is not fathomed, but since its LB is higher than the LB of the fathomed problem 11211, it cannot possibly conceal an integer solution which may be a candidate for optimality. Hence this problem is also left out of consideration or pruned.

Now all the subproblems have been fathomed or pruned or branched. The fathomed problem which gives the lowest LB for the objective function gives the optimal solution of the original ILP problem. Thus $x_1 = 2$, $x_2 = 2$, $f = -14$ is the required solution.

Example 2:

Minimise $$f = 3x_4 + 4x_5 + 5x_6,$$

subject to
$$\begin{aligned} 2x_1 \quad + 2x_4 - 4x_5 + 2x_6 &= 3, \\ 2x_2 + 4x_4 + 2x_5 - 2x_6 &= 5, \\ x_3 - x_4 + x_5 + x_6 &= 4, \end{aligned}$$

$$x_1, x_2, \ldots, x_6 \geq 0; x_1, x_2 \text{ integers.}$$

Since only two of the six variables are constrained to be integers, the problem is of mixed integer programming.

Deleting the integer constraints, we get the related LP problem, whose minimal solution is easily found as $x_1 = 3/2$, $x_2 = 5/2$, $x_3 = 4$ $x_4 = x_5 = x_6 = 0$, giving the LB of f as zero. This is problem 1 of Fig. 3. Since x_1 is required to be an integer, we branch problem 1 into problems 11 and 12 by introducing respectively the constraints $x_1 \leq 1$ and $x_1 \geq 2$ indicated by the value $x_1 = 3/2$ which lies between 1 and 2. The solutions to these two problems can be found by the dual simplex method, and are shown in the figure. Since these problems have optimal solutions in which the variable x_2 is non-integral, none of the problems has been fathomed. Nor any of them has been pruned (that is, not to be considered further). So both problems 11 and 12 are branched into problems 111, 112, 121, 122, with additional constraints respectively as $x_2 \leq 1$, $x_2 \geq 2$, $x_2 \leq 2$, $x_2 \geq 3$, indicated by the value $x_2 = 3/2$ in problem 11 and $x_2 = 9/4$ in problem 12. Again the four problems can be solved by the dual simplex method to give the solutions as written in the figure. Problems 112, 121 and 122 stand fathomed as the optimal solution in each case is integral in x_1, x_2. Problem 111 is not fathomed, but the LB in it is 9/4 which is greater than 3/2, the LB of problem 121. Hence it is pruned. Among the fathomed problems the least LB is provided by problem 121. This therefore gives the solution of the original problem.

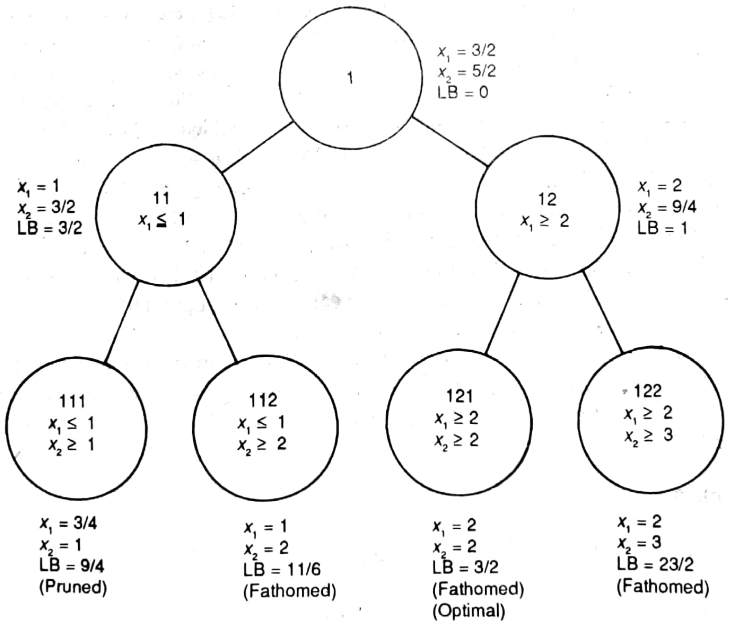


Fig. 3

9 Branch and bound method—general description

As the name implies and as is also clear from the above examples, the branch and bound method consists of two strategies, alternately followed, till the desired solution is obtained. One strategy consists in branching a problem into two subproblems, and the other in solving each of the two subproblems to obtain the minimum or a suitable lower bound of the objective function, if the original problem is to minimise the objective. (If the problem is to maximise an objective function, the lower bound is replaced by the upper bound).

Let the problem be of the MILP type in which the variables x_j, $j = 1, 2, ..., r$, are integers and $j = r + 1, ..., n$, are real numbers. The problem of ILP, by the branch and bound method, is only a special case of MILP, with $r = n$, and needs no separate discussion.

We start with the related LP problem, hereafter designated as problem 1, and solve it to obtain a lower bound of its objective function. Let us suppose, for the present, that it is the actual minimum that we are able to determine. We presume that this minimum and the corresponding optimal (minimal) solution can be found without much difficulty. If the optimal solution happens to satisfy the integer constraint also, it is the optimal solution of the given MILP, and nothing more need be done. If not, then the value of at least one of the variables x_j, $j = 1, 2, ..., r$, in that optimal solution is not integral. Let x_p be one such variable, and at the optimal let $x_p = b$, where b is not an integer. Let $[b]$ be the largest integer less than b. Since b, being feasible, is non-negative, $[b]$ is also non-negative.

Formulate two subproblems, designed as problems 11 and 12, by imposing on problem 1 the additional constraints $x_p \le [b]$ and $x_p \ge [b] + 1$ respectively. This operation is called *branching*. In effect, the set of feasible solutions of the MILP is partitioned into two subsets, and the optimal solution which we are seeking is in one subset or the other, provided it exists.

Each of the two subproblems 11 and 12 is now treated as an independent problem, and subjected to the same operation as problem 1, namely, obtaining the minimum of the objective function, and then, if necessary, branching. This 'branch and bound' process is continued through resulting subproblems which fan out from problem 1 as a *tree*. Branching terminates when any of the following three conditions arise.

(i) The subproblem yields an optimal solution which satisfies the integer constraint on all the variables x_j, $j = 1, 2, ..., r$; the subproblem is then said to have been *fathomed*.

(ii) The optimum (minimum) value of the objective function in the subproblem is not lower than the minimum value of the objective function in a subproblem which has been fathomed.

(iii) The subproblem turns out to be infeasible.

The reasons to terminate branching in the above three cases are as follows. In case (i) the optimal solution with required integer constraint out of the subset of feasible solutions of that subproblem has been obtained, and no further probe in that subproblem is necessary. In case (ii), since an integer optimal solution which is lower than the optimal solution of the subproblem has been discovered in the set of feasible solutions of another subproblem, the former subproblem needs no further probe, as it cannot be concealing a solution which would make the objective function lower than what has been discovered in the latter subproblem. In case (iii) the subproblem obviously cannot contain the required solution. Subproblems falling under cases (ii) and (iii) are said to be *pruned.*

When all the subproblems obtained through branching have been either fathomed or pruned, the branch and bound algorithm terminates. The fathomed subproblem with the lowest minimum gives the answer to the original problem.

The branch and bound method is partially enumerative. The set of feasible solutions is successively partitioned into subsets and those subsets which cannot contain the optimal solution are deleted from further consideration. The criterion for deletion is provided by the lower bound of the objective function for the feasible values in that subset.

It is sometimes difficult or strenuous to determine the minimum of the objective function in a problem. The reason why in the branch and bound method the stress is on a lower bound and not the minimum of the objective function is that any suitable lower bound and not necessarily the exact minimum is needed to decide whether a subset of feasible solutions should be further probed or deleted. Of course the closer a lower bound is to the minimum the better, but one has to balance the time spent in determining the minimum against the time spent in going ahead with further branching. If a suitable lower bound is more easily determined than the minimum, it is worthwhile saving time here. Branching being easier than finding the minimum, bulk of the total time spent in solving a problem by the branch and bound method is spent in the latter operation, and so whatever time can be saved on it should be saved. It may result in more branching, but the overall effort is less.

There are several strategies recommended for determining a lower bound. We shall briefly mention only two of them. The detailed discussion would be omitted. One method consists in ignoring the constraints which appear to be difficult, and minimising the objective function subject to the remaining constraints. This minimum is certainly not higher than the minimum under all the constraints, and so can serve as the required lower bound. Another method is to construct another objective function which is not greater than the given objective function for any feasible solution of the original problem, and determination of whose minimum is comparatively easier than that of the original function.

10 The 0–1 variable problems

Many problems in Operations Research can be formulated as mixed integer programmes with some (or all) variables constrained to have value 0 or 1. We shall call such variables the 0–1 variables. Mathematically such variables present no new problem, because

$$x_j = 0 \text{ or } 1$$

is equivalent to

$$0 \le x_j \le 1,\ x_j \text{ an integer.}$$

Introducing the 0–1 variables in a formulation is a very useful device through which a variety of conditions can be expressed. Most logical constraints can be

PROBLEMS VI

Solve problems 1 to 5 by the cutting plane method.

1. Minimise $4x_1 + 5x_2$ subject to $3x_1 + x_2 \geq 2$, $x_1 + 4x_2 \geq 5$, $3x_1 + 2x_2 \geq 7$; x_1, x_2 non-negative integers.

[13; (2, 1)]

2. Maximise $x_1 + x_2$ subject to $7x_1 - 6x_2 \leq 5$, $6x_1 + 3x_2 \geq 7$, $-3x_1 + 8x_2 \leq 6$; x_1, x_2 non-negative integers.

[2; (1, 1)]

3. Maximise $x_1 + x_2$ subject to $2x_1 \leq 3$, $2x_1 + 2x_2 \geq 5$, $-2x_1 + 2x_2 \leq 1$; x_1, x_2 non-negative integers.

[Infeasible]

4. Minimise $3x_1 - x_2$ subject to $-10x_1 + 6x_2 \leq 15$, $14x_1 + 18x_2 \geq 63$; x_1, x_2 non-negative integers.

[− 1; (1, 4)]

5. Minimise $-2x_1 - 3x_2$ subject to $2x_1 + 2x_2 \leq 7$, $0 \leq x_1 \leq 2$, $0 \leq x_2 \leq 2$; x_1, x_2 integers.

[− 8; (1, 2)]

Solve problems 6 to 10 by the branch and bound method.

6. Maximise $11x_1 + 21x_2$ subject to $4x_1 + 7x_2 + x_3 = 13$; x_1, x_2, x_3 non-negative integers.

[33; (3, 0, 1)]

7. Minimise $9x_1 + 10x_2$ subject to $0 \leq x_1 \leq 10$, $0 \leq x_2 \leq 8$, $3x_1 + 5x_2 \geq 45$; x_2 integer.

[95; (5/3, 8)]

8. Maximise $13x_1 + 3x_2 + 3x_3$ subject to $7x_1 + 6x_2 - 3x_3 \leq 8$, $7x_1 - 3x_2 + 6x_3 \leq 8$; x_1, x_2, x_3 non-negative integers.

[13; (1, 0, 0)]

9. Maximise $x_1 + 2x_2$ subject to $5x_1 + 7x_2 \leq 21$, $-x_1 + 3x_2 \leq 8$; x_1, x_2 non-negative integers.

[5; (1, 2)]

10. Same as 5.

11. Formulate the following knapsack problem as an ILP.
There are n objects, $j = 1, 2, \ldots, n$, whose weights are w_j and values v_j. They have to be chosen to be packed in a knapsack so that the total value of the objects chosen is maximum subject to their total weight not exceeding W.

[Maximise $\sum_j v_j x_j$ subject $\sum_j w_j x_j \leq W$, $x_j = 0$ or 1]

12. Solve the knapsack problem (as formulated above) with the following data.

(i)

Object j	Weight w_j	Value v_j
1	2	10
2	2	14
3	3	18
4	6	48
5	8	80

Knapsack capacity $W = 12$

[max value 98, with objects 3 and 5]

(ii)

Object j	Weight w_j	Value v_j
1	3	12
2	4	12
3	3	9
4	6	30
5	10	20
6	12	12

Knapsack capacity $W = 14$

[max value 54, with objects 1, 2, 4]

13. Maximise $2x_1 + 5x_2$ subject to $0 \le x_1 \le 8$, $0 \le x_2 \le 8$, and either $4 - x_1 \ge 0$ or $4 - x_2 \ge 0$.

[48; (4, 8)]

[Hint: Introduce two $0-1$ variables y_1, y_2 such that $4 - x_1 + 10y_1 \ge 0$, $4 - x_2 + 10y_2 \ge 0$, $y_1 + y_2 = 1$, 10 being a suitably large number].

14. In a network of streets and junctions, the junctions are denoted by $j = 1, 2, \ldots$ and the streets connecting the junctions by (i, j). Fire-hydrants have to be installed at some of the junctions such that every street connected to a junction has access to the fire-hydrant at that junction. The cost of installing the fire-hydrant at junction j is c_j. Formulate the problem as an integer program to minimise the cost of installing the fire-hydrants so that each street has access to at least one hydrant. Solve the problem for the following data.

$j = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

$c_j = 4 \quad 6 \quad 10 \quad 8 \quad 7 \quad 9$

$(i, j) = (1, 2), (1, 4), (1, 6), (2, 4), (3, 5), (3, 6), (4, 5).$

(Hint: Minimise $\Sigma c_j x_j$ subject to $x_j = 0$ or 1, $x_i + x_j \ge 1$, for every given (i, j)).

[Hydrants at $j = 1, 5$ at cost 11]

15. Minimise $3x_1 + 2x_2 + f_1 + f_2$,

subject to

$$5x_1 + 2x_2 \ge 10,$$

$$3x_1 + 5x_2 \ge 15,$$

$$f_1 = 5 \text{ if } x_1 > 0,\ f_1 = 0 \text{ if } x_1 = 0,$$

$$f_1 = 2 \text{ if } x_2 > 0,\ f_2 = 0 \text{ if } x_2 = 0.$$

[12; (0, 5)]

16. In a factory 4000 units of a certain product are to be manufactured. There are three machines on which it can be manufactured. The set up cost, the production cost per unit and the maximum production capacity for each machine are tabulated below. The objective is to minimize the total cost of producing the entire lot. Formulate the problem as an integer programme, and solve it.

Machine	Set up cost	Production cost/unit	Capacity units
I	400	10	2400
II	600	4	1600
III	200	20	1200

17. Express the following conditions as simultaneous constraints using $0-1$ variables.
(i) Either $x_1 + 2x_2 \le 4$ or $2x_1 + 3x_2 \ge 12$.
(ii) If $x_3 \le 4$ then $x_4 \ge 5$, otherwise $x_4 \le 2$.
(iii) $x_5 = 1$ or 3 or 5 only.
(iv) At least two of the following constraints are satisfied.

$$x_6 + x_7 \le 3,\ x_6 \le 2,\ x_7 \le 4,\ x_6 + x_7 \ge 5.$$

18. Maximize $5x_1 + 2x_2 + x_3$, subject to $x_1 + x_2 + 2x_3 \le 10$, $|-3x_1 + 10x_2 - x_3| \ge 15$, $x_1, x_2, x_3 \ge 0$.
(Hint. Treat the second constraint as 'either or' constraint).

[[(10, 0, 0); 50]

7

Additional Topics in Linear Programming

1 Introduction

In the preceding four chapters we primarily discussed methods for solving linear programming problems. However, solving an *LP* in itself is not the end of the story. In most real life problems we want to find not only an optimal solution but also to know as to what happens to this optimal solution when changes are made in the initial system. It would be preferable to determine the effect of these changes on the optimal solution without having to solve a modified problem from the very beginning. In sensitivity analysis (also called post-optimality analysis) we develop methods to do this. A more general problem is to study the effects on the optimal solution of an *LP* as some parameter of it undergoes continuous change in its value. The procedures developed for doing this are known as parametric programming techniques.

In linear programming our aim so far has been to get as large (or small) a value of the objective function as is possible without violating any of the constraints. It may happen that in doing so other considerations which may also be important are ignored. In many practical problems, instead of maximizing or minimizing the objective function, it may be considered better to be satisfied with setting up a certain value of the objective function as a reasonable goal, and then try to achieve it as closely as possible. This approach is known as goal programming.

There can also be multiobjective linear programming problems in which it is desirable to optimize simultaneously more than one objective function satisfying the same set of constraints. The objectives may be conflicting, and it may not be possible to find a solution that accomplishes their simultaneous optimization. But one may still try to get the *best* solution, defining the best in some satisfactory manner. One can also visualize multiobjective goal programming problems in which different goals are set for different objective functions, it being desired that the objective functions achieve these goals as closely as possible.

In the present chapter we propose to study these topics in some detail.

2 Sensitivity analysis

In an *LP* the optimal solution is dependent on the values of the cost coefficients c_j, the constants b_i occurring on the right side of the constraint equations, and the coefficients a_{ij} in the constraints. In real life problems the values of these coefficients are seldom known with certainty because many of them are functions of some uncontrolled parameters. For instance, the future demands, the cost of raw materials, or the cost of energy resources cannot be accurately predicted. Hence the problem is not satisfactorily solved with the mere determination of the optimal solution. Each variation in the values of the data coefficients changes the problem which may affect the optimal solution found earlier. However, it is not always necessary to solve the whole problem afresh to determine the new optimal solution. In the following sections we discuss methods of starting out from the optimal solution already obtained to determine the new optimal solution under the following modifications.

(i) changes in the values of b_i;
(ii) changes in the values of c_j;
(iii) changes in the values of a_{ij};
(iv) introduction of new variables;
(v) introduction of new constraints;
(vi) deletion of certain variables;
(vii) deletion of some constraints.

3 Changes in b_i

For the *LP* problem:

Minimize $f(\mathbf{X}) = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{B}, \mathbf{X} \geq 0,$ (1)

let the optimal basis be

$$\mathbf{X}_0 = [x_1\, x_2 \ldots, x_m]', \mathbf{X}_0 \geq 0,$$

$x_{m+1}, x_{m+2}, \ldots, x_n$ being the nonbasic variables for this solution. The corresponding relative cost coefficients $\bar{c}_j$ given by (see section 10, chapter 3)

$$\bar{c}_j = c_j - \sum_{i=1}^{m} c_i \bar{a}_{ij}, \; j = 1, 2, \ldots, n, \tag{2}$$

are all nonnegative. Also since the nonbasic variables have zero values in the optimal solution, the constraints in (1) reduce to

$$\mathbf{A}_0\mathbf{X}_0 = \mathbf{B},$$

so that

$$\mathbf{X}_0 = \mathbf{A}_0^{-1}\mathbf{B},$$

where

$$\mathbf{A}_0 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

Let $\mathbf{B}$ change to $\mathbf{B} + \Delta\, \mathbf{B}$ where $\Delta\mathbf{B} = [\Delta b_1\, \Delta b_2 \ldots \Delta b_m]'$, everything else in (1) remaining the same. Then the new values of the variables of the earlier optimal basis are given by

$$\mathbf{X}_0 + \Delta\mathbf{X}_0 = \mathbf{A}_0^{-1}(\mathbf{B} + \Delta\mathbf{B}).$$

Now if $\qquad \mathbf{A}_0^{-1}(\mathbf{B} + \Delta\mathbf{B}) \geq 0$ (3)

the original optimal basis continues to be feasible for the new problem. It will also be an optimal basis if all the relative cost coefficients of the modified problem for this basis are nonnegative. The relative cost coefficients, by (2), are independent of **B**, and so remain unchanged. So they remain nonnegative. Hence, if (3) holds, the original optimal basis is still optimal. The new value of the objective function is given by $f(\mathbf{X}_0 + \Delta\mathbf{X}_0)$.

If, however, $\Delta\mathbf{B}$ is such that (3) does not hold, that is, the new values of the variables in the basis $\mathbf{X}_0$ are not all nonnegative, then the new solution $\mathbf{X}_0 + \Delta\mathbf{X}_0$ is not feasible. In such a case we may replace the values of the basic variables in the earlier optimal solution by their new values and proceed further by the big *M* or the two-phase or the dual simplex method to obtain the new optimal solution. If too many components of $\mathbf{X}_0 + \Delta\mathbf{X}_0$ are negative, it may be more economical to solve the new problem *ab initio*.

Example: Consider the problem of section 13, chapter 3. Its optimal solution, as obtained there, is given in table 1.

TABLE 1

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6
x_1	2/3	1				−4/3	1/3
x_2	1/3		1			1/3	−1/3
x_3	13/3			1		7/3	−1/3
x_4	11/3				1	2/3	1/3
f	−13/3					11/3	1/3

From this we find that

$$\mathbf{X}_0 = [x_1\ x_2\ x_3\ x_4]' = [2/3\ \ 1/3\ \ 13/3\ \ 11/3]'$$

Also, from the original problem,

$$\mathbf{C}_0 = [4\ \ 5\ \ 0\ \ 0],\ \mathbf{B} = [6\ \ 5\ \ 1\ \ 2]'$$

and, as explained in section 15, chapter 3,

$$\mathbf{A}_0 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{bmatrix},\quad \mathbf{A}_0^{-1} = \begin{bmatrix} 0 & 0 & 4/3 & -1/3 \\ 0 & 0 & -1/3 & 1/3 \\ 1 & 0 & -7/3 & 1/3 \\ 0 & 1 & -2/3 & -1/3 \end{bmatrix}$$

We proceed to consider three cases of changes in **B**.

(i) Let $\mathbf{B} + \Delta\mathbf{B} = [7\ 4\ 1\ 2]'$. (Notice that only b_1 and b_2 change).

Then $\mathbf{A}_0^{-1}(\mathbf{B} + \Delta\mathbf{B}) = [2/3\ \ 1/3\ \ 16/3\ \ 8/3]' \geq 0$. Thus the original optimal basis remains feasible and hence optimal. The new optimal values of the basic variables are $x_1 = 2/3$, $x_2 = 1/3$, $x_3 = 16/3$, $x_4 = 8/3$, and the new optimal value of the objective function is $f = 13/3$ which is the same as before. This is so because the values of only those basic variables have changed which do not occur in the original form of the objective function. In fact, from the form of $\mathbf{A}_0^{-1}$ it follows that any change in the value of b_1 or b_2 will not affect the values of x_1 and x_2, and therefore the value of the objective function, as long as the new basis is feasible.

(ii) Let $\mathbf{B} + \Delta\,\mathbf{B} = [6\ 5\ 1\ 1]'$.

In this case $\mathbf{A}_0^{-1}(\mathbf{B} + \Delta\mathbf{B}) = [1\ \ 0\ \ 4\ \ 4]'$.

Thus the original optimal basis is still feasible with the optimal values of the basic variables as $x_1 = 1$, $x_2 = 0$, $x_3 = 4$, $x_4 = 4$. However, the optimal value of the objective function is now 4 which is different from the earlier value.

(iii) Let $\mathbf{B} + \Delta\,\mathbf{B} = [6\ 5\ 2\ 1]'$.

Now $\mathbf{A}_0^{-1}(\mathbf{B} + \Delta\mathbf{B}) = [7/3\ \ -1/3\ 5/3\ 10/3]'$.

In this case the original optimal basis $\mathbf{X}_0$ becomes infeasible. Since the relative cost coefficients remain unchanged and so nonnegative, from this point onwards we may proceed by the dual simplex method to obtain the optimal solution of the modified problem. The new values of the basic variables are $x_1 = 7/3$, $x_2 = -1/3$, $x_3 = 5/3$, $x_4 = 10/3$, and the new value of f is 23/3. With these values in the simplex table, and doing one iteration of the dual simplex method, we get the new optimal solution as $x_1 = 2$, $x_2 = 0$, $x_3 = 2$, $x_4 = 3$, $x_5 = 0$, $x_6 = 1$; $f = 8$.

4 Changes in c_j

If c_j are changed to c_j^*, everything else in the problem remaining the same, then the changed relative cost coefficients of the nonbasic variables are given by (2) as

$$\bar{c}_j^* = c_j^* - \sum_{i=1}^{m} c_i^* \bar{a}_{ij},\ j = m+1, \ldots, n.$$

These may not all be nonnegative. If $\bar{c}_j^*$ is negative for some j, then this would mean that the basic feasible solution $\mathbf{X}_0$ which was earlier optimal is now not optimal. So from this point onwards further iterations may be done with $\bar{c}_j$ in the simplex table replaced by $\bar{c}_j^*$, $j = m+1, \ldots, n$, to obtain the new optimal solution.

If c_j^* are such that all $\bar{c}_j^*$ are nonnegative, then the original optimal basis remains optimal and the optimal values of the basic variables also remain unchanged. Optimum value of the objective function, however, will be different since the cost coefficients have changed. In the particular case when $c_j^* = c_j$ for the basic variables, even the value of the objective function will not change.

Example: Suppose in the example of section 3 $c_1 = 4$, $c_2 = 5$ are changed to $c_1^* = 5, c_2^* = 6$. Then, using (2),

$$\begin{aligned} \bar{c}_5^* &= -(-4/3)c_1^* - (1/3)c_2^* = (1/3)(4c_1^* - c_2^*) = 14/3 \\ \bar{c}_6^* &= -(1/3)c_1^* - (-1/3)c_2^* = (1/3)(c_2^* - c_1^*) = 1/3 \end{aligned} \tag{4}$$

As $\bar{c}_5^*$ and $\bar{c}_6^*$ are both nonnegative, the original optimal basis is still optimal, and there is no change in the optimal values of the basic variables. The new value of the objective function is $f = c_1^* x_1 + c_2^* x_2 = 16/3$. From (4) it is evident that $\bar{c}_5^*$ and $\bar{c}_6^*$ will be nonnegative as long as $c_1^* \le c_2^* \le 4c_1^*$. So the original optimal basis will remain optimal so long as the changed cost coefficients satisfy this condition.

Next suppose that c_1 and c_2 change to $c_1^* = 5$, $c_2^* = 1$. Then $\bar{c}_5^* = 19/3$ and $\bar{c}_6^* = -4/3$. Since $\bar{c}_6^*$ is negative, the original optimal basis ceases to be optimal. Replacing the previous value 11/3 of $\bar{c}_5$ by $\bar{c}_5^* = 19/3$, and the value 1/3 of $\bar{c}_6$ by $\bar{c}_6^* = -4/3$, and the original entry 23/3 for the value of f by its new value 11/3 in table 1, we can do one more iteration to obtain the new optimal solution $x_1 = 0$, $x_2 = 1$, $x_3 = 5$, $x_4 = 3$, $x_5 = 0$, $x_6 = 2$; $f = 1$.

5 Changes in a_{ij}

If the changes are in a_{ik}, where x_k is a nonbasic variable of the original optimal solution, then the modified value $\bar{c}_k^*$ of $\bar{c}_k$ may be found by using equation (37), chapter 3:

$$\bar{c}_k^* = c_k + \sum_{i=1}^{m} a_{ik}^* \pi_i, \tag{5}$$

where a_{ik}^* are new values of a_{ik}. If $\bar{c}_k^* \geq 0$, the original optimal solution is still optimal. If $\bar{c}_k^* < 0$, then further iterations of the simplex method may be done to find the new optimal solution. For this purpose the values of $\bar{a}_{ik}^*$, the modified values of a_{ik}^* in the original optimal table, may be calculated by the formula (see section 10, chapter 3)

$$[\bar{a}_{1k}^* \bar{a}_{2k}^* \ldots \bar{a}_{mk}^*]' = A_0^{-1}[a_{1k}^* a_{2k}^* \ldots a_{mk}^*]'. \tag{6}$$

If x_k is a basic variable in the original optimal solution, then the procedure may be as follows.

Introduce a new variable x_k^* in the system with coefficients a_{ik}^* and $c_k^* = c_k$. In this new problem treat the original variable x_k as an artificial variable and use phase I of the two-phase simplex method to eliminate it, and then proceed to phase II to get the new optimal solution.

Example: To illustrate the above procedures, we take the example of section 20, chapter 3. There the problem has been solved by the dual simplex method, and its optimal solution (table 6, chapter 3) is given in table 2.

TABLE 2

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6
x_3	2	1	2	1		−1	
x_4	1	3	2		1	−2	
x_6	8	4	5			−2	1
f	−4	1	1			2	

From the above, the optimal basis is

$$\mathbf{X}_0 = [x_3\ x_4\ x_6]' = [2\ \ 1\ \ 8]', \text{ with } \mathbf{C}_0 = [2\ \ 0\ \ 0],$$

and

$$A_0 = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ Hence } A_0^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

and, by (38), chapter 3, the simplex multipliers are

$$\Pi' = -C_0A_0^{-1} = [0 \ -2 \ 0] \tag{7}$$

We consider three cases of changes in a_{ij}.

(i) Let the values of a_{11}, a_{21}, a_{31} change from (-1, 1, 2) to (1, -1, -2). The changes are in the coefficients of x_1 which is a nonbasic variable. In this case

$$[a_{11}^* \ a_{21}^* \ a_{31}^*] = [1 \ -1 \ -2],$$

and so $$\bar{c}_1^* = c_1 + \sum_{i=1}^{3} a_{i1}^* \pi_i = 3 + 1\times 0 + (-1)(-2) + (-2)\times 0 = 5 > 0.$$

Therefore the original optimal solution is still optimal.

(ii) Suppose the changed values of a_{11}, a_{21}, a_{31} are 1, 2, − 2.

In this case $[a_{11}^* \ a_{21}^* \ a_{31}^*] = [1 \ 2 \ -2]$, and so $\bar{c}_1^* = -1$. The original optimal solution, therefore, ceases to be optimal. The entries in the column for $\mathbf{P}_1$ in table 2 are modified to

$$[\bar{a}_{11}^* \ \bar{a}_{21}^* \ \bar{a}_{31}^*]' = A_0^{-1}[a_{11}^* \ a_{21}^* \ a_{31}^*]' = A_0^{-1}[1 \ 2 \ -2]' = [2 \ 3 \ 2]'$$

The modified table appears as

Basis	B	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$
x_3	2	2	2	1		−1	
x_4	1	3	2		1	−2	
x_6	8	2	5			−2	1
f	−4	−1	1			2	

On performing one iteration of the simplex method we find the new optimal solution to be $x_1 = 1/3, x_2 = 0, x_3 = 4/3; f = 11/3$.

(iii) Let a_{13}, a_{23}, a_{33} change from (2, 1, −2) to (1, −1, 2). Since the change is in the coefficients of x_3 which is a basic variable in the original optimal solution, to study the effect of this change we introduce a new variable x_3^* with coefficients $a_{13}^* = 1$, $a_{23}^* = -1$, $a_{33}^* = 2$ and $c_3^* = c_3 = 2$. From these $\bar{c}_3^* = 4$, $\bar{a}_{13}^* = -1$, $\bar{a}_{23}^* = -3$, $a_{33}^* = 0$. Now the simplex table (given below) for finding the new optimal solution by the two-phase method, treating the original variable x_3 as an artificial variable, is obtained by introducing a column $\mathbf{P}_3^*$ in table 2. The objective function g for phase I is given by $g = x_3$, which in terms of nonbasic variables is $g = 2 - x_1 - 2x_2 + x_5 + x_3^*$.

Basis	B	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_3^*$
x_3	2	1	2	1		−1		−1
x_4	1	3	2		1	−2		−3
x_6	8	4	5			−2	1	0
f	−4	1	1			2		4
g	−2	−1	−2			1		1

Proceeding from here onwards, the new optimal solution is found to be $x_1 = 0$, $x_2 = 5/4$, x_3 (or x_3^*) = 1/2, $x_4 = 0$, $x_5 = 0$, $x_6 = 7/4$; $f = 29/4$.

6 Introduction of new variables

Let the new variables be x_{n+k}, $k = 1,2,3,\ldots$, and their coefficients be $a_{i,n+k}, i = 1,2,\ldots,m$, and c_{n+k}. Since the number of constraints remains the same, the number of basic variables also remains the same, and so the original optimal solution gives a basic feasible solution of the new problem. The relative cost coefficients corresponding to the newly introduced cost coefficients c_{n+k} would be given by (5) as

$$\bar{c}_{n+k} = c_{n+k} + \sum_{i=1}^{m} a_{i,n+k}\pi_i, k = 1,2,3,\ldots$$

If all these are nonnegative, the original optimal solution remains optimal for the new problem. If not, then from this point onwards iterations may be done to obtain the new optimal solution taking into account the new variables.

Example: Suppose in the example of section 3 we introduce a new variable x_7 such that (i) $c_7 = 2$, $a_{17} = 1$, $a_{27} = -1$, $a_{37} = -3$, $a_{47} = 3$, and (ii) $c_7 = 2$, $a_{17} = 1$, $a_{27} = -1$, $a_{37} = 3$, $a_{47} = 3$, and wish in each case to determine the new optimal solution.

For the optimal basis of the original problem (table 1)

$$\mathbf{X}_0 = [x_1\ x_2\ x_3\ x_4]' = [2/3\ \ 1/3\ \ 13/3\ \ 11/3]',$$

the simplex multipliers, by (7), are

$$\Pi' = -\mathbf{C}_0\mathbf{A}_0^{-1} = -[4\ \ 5\ \ 0\ \ 0]\mathbf{A}_0^{-1} = [0\ \ 0\ \ -11/3\ \ -1/3].$$

Hence, by (5), the value of $\bar{c}_7$ corresponding to c_7 for case (i) is

$$\bar{c}_7 = c_7 + \sum_{i=1}^{4} a_{i7}\pi_i = 12 > 0.$$

Therefore in case (i) the original optimal basis remains optimal, and the optimum value of the objective function remains unchanged.

In case (ii), proceeding similarly, $\bar{c}_7 = -10$. This being negative, the original optimal basis is now not optimal, and further iterations are necessary to get the new optimal. The starting table for finding the new optimal solution will be the same as of the original optimal solution, (table 1), with an additional column $\mathbf{P}_7$ in which

$$[\bar{a}_{17}\ \bar{a}_{27}\ \bar{a}_{37}\ \bar{a}_{47}]' = \mathbf{A}_0^{-1}\ [a_{17}\ a_{27}\ a_{37}\ a_{47}]' = [3\ \ 0\ \ -5\ \ -4]' \text{ and } \bar{c}_7 = -10.$$

Thus the starting table for further iterations is

Basis	B	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_7$
x_1	2/3	1				−4/3	1/3	3
x_2	1/3		1			1/3	−1/3	0
x_3	13/3			1		7/3	−1/3	−5
x_4	11/3				1	2/3	1/3	−4
f	−13/3					11/3	1/3	−10

The new optimal solution after two iterations turns out to be $x_1 = x_2 = 0$, $x_3 = 16/3$, $x_4 = 17/3$, $x_5 = 1$, $x_6 = 0$, $x_7 = 2/3$; $f = 4/3$.

7 Introduction of new constraints

If K is the set of feasible solutions of the original problem and K' the set of feasible solutions of the modified problem obtained by introducing new constraints, then $K' \subseteq K$. If the original optimal solution $\mathbf{X}_0$ satisfies the new constraints, then $\mathbf{X}_0$ is in K', and since $f(\mathbf{X}_0)$ is minimum in K, it is also minimum in K'. In this case, therefore, the original optimal solution continues to be optimal.

If some or all of the new constraints are violated by $\mathbf{X}_0$, then the problem has to be solved further by taking into consideration the new constraints. Each new constraint in the form of an inequality gives rise to a slack variable, and, if necessary, also an artificial variable. For a constraint in the form of an equation, an artificial variable may be introduced. A start is made with the feasible basis consisting of the variables in the original optimal solution and the slack or artificial variables (as the need be) of the new constraints. The problem may now be solved by the two-phase or the big M method. If all the additional constraints are inequalities, the problem may also be solved, without introducing artificial variables, by the dual simplex method.

Example: Let us introduce the additional constraint $3x_1 - 2x_2 \leq 2$ in the example of section 3. The original optimal solution, $x_1 = 2/3$, $x_2 = 1/3$, does not violate this constraint. Hence it continues to be the optimal solution of the modified problem.

However, if the new constraint is $3x_1 - 2x_2 \geq 2$, the situation becomes different. The original optimal solution, (table 1) $x_1 = 2/3$, $x_2 = 1/3$, violates this constraint. In order to obtain the new optimal solution, we introduce the slack variable x_7 in the new constraint, and write it as

$$3x_1 - 2x_2 - x_7 = 2.$$

Eliminating the basic variables x_1, x_2 from this equation with the help of the first two equations in the original optimal table, we put this equation as

$$-\frac{14}{3}x_5 + \frac{5}{3}x_6 + x_7 = -\frac{2}{3},$$

and introduce it in table 1 as follows.

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7
x_1	2/3	1				−4/3	1/3	
x_2	1/3		1			1/3	−1/3	
x_3	13/3			1		2/3	1/3	
x_4	11/3				1	2/3	1/3	
x_7	−2/3					−14/3	5/3	1
f	−13/3					11/3	1/3	

Doing one iteration of the dual simplex method, we obtain the new optimal solution as $x_1 = 6/7$, $x_2 = 2/7$; $f = 34/7$.

8 Deletion of variables

If the deleted variable is a nonbasic variable or a basic variable with a zero value in the optimal basis, then the original optimal solution remains unchanged, because the zero value of the variable in the optimal solution makes the variable nonexistent in effect.

If the variable to be deleted is a basic variable with positive value in the optimal solution, its removal will affect the optimal solution. To obtain the new optimal solution, we should delete from the original optimal table the column corresponding to the deleted variable. Also this variable should be dropped from the basis column. This leaves the equation against the deleted basic variable in a form which is not canonical, and the number of basic variables in the system one short. We may now introduce an artificial variable in this equation, and proceed to obtain the solution by the two-phase or the big M method. As an alternative, another approach is also possible which involves the dual simplex method. After dropping the deleted variable from the table, the sign of all the entries of the row corresponding to that variable are changed. This leaves the equation essentially unchanged. Then a new basic variable is introduced in this equation with +1 as its coefficient. This makes the new basis, which includes this variable, infeasible, but the relative cost coefficients remain unchanged as nonnegative. Therefore from here onwards the dual simplex method may be employed to obtain the new optimal solution, with the new variable becoming nonbasic, which can then be dropped without affecting the optimal solution.

Example: In the example of section 5, if we delete the variable x_1 which is nonbasic in the optimal solution, (table 2), the modified problem has the same optimal solution.

However, if x_3 is deleted, then in table 2 column $\mathbf{P}_3$ disappears and from the basis column x_3 goes. Following the second method suggested above, we change the signs of all entries in the row which corresponded to x_3. Then introduce a new variable x_7 with coefficient +1 in that row, treating it as a basic variable. Thus we get the following table, from which after two iterations of the dual simplex method we get the new optimal solution (ignoring the entries in column $\mathbf{P}_7$) as $x_1 = 0$, $x_2 = 3/2$; $f = 15/2$.

Basis	B	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_7$
x_7	-2	-1	-2		1		1
x_4	1	3	2	1	-2		
x_6	8	4	5		-2	1	
f	-4	1	1		2		
x_2	3/2	$-1/2$	1	$-1/2$			-1
x_5	1	-2		-1	1		-1
x_6	5/2	5/2		1/2		1	3
f	$-15/2$	11/2		5/2			3

9 Deletion of constraints

If the constraint to be deleted is such that its slack variable has a positive value in the optimal solution, then its deletion leaves the optimal solution unchanged. This is so because the constraint is not being satisfied as an equality by the optimal solution, and therefore it is ineffective in determining the optimal solution. There are other constraints, those which are satisfied as equations, which determine it. Therefore the ineffective constraint may be deleted from the problem without doing any damage to the optimal solution.

If the constraint to be deleted has zero value for its slack variable in the optimal solution, that is, if it is being satisfied as an equality, then the modified problem may have a different optimal solution. Let the constraint to be deleted be of the type

$$\sum_{j=1}^{n} a_{kj}x_j \leq b_k,$$

so that after introducing the slack variable it becomes

$$\sum_{j=1}^{n} a_{kj}x_j + u_k = b_k, u_k \geq 0.$$

There are two ways of looking at the process of deletion of this constraint. The obvious one is to delete it from the problem, but in that case we shall have to solve the problem *ab initio*. The other is to say that

$$\text{either } \sum_{j=1}^{n} a_{kj}x_j \leq b_k \text{ or } \sum_{j=1}^{n} a_{kj}x_j \geq b_k.$$

(Notice that it is one constraint *or* the other, not one *and* the other.) This also, in effect, removes the constraint. The two alternative constraints can be combined into a single equation by introducing a slack variable which is not restricted in sign:

$$\sum_{j=1}^{n} a_{kj}x_j + s_k = b_k, s_k \text{ unrestricted in sign,}$$

which is equivalent to

$$\sum_{j=1}^{n} a_{kj}x_j + u_k - v_k = b_k,\ u_k \geq 0,\ v_k \geq 0.$$

u_k can be identified with the slack variable already occurring in the original constraint, and so now we have only to introduce another variable v_k with coefficient -1 in the original constraint to get the modified problem. The problem so modified is equivalent to the problem obtained by deleting the constraint from the original problem. The problem with the additional variable can now be solved by the method of section 6.

Similarly, if the constraint to be deleted is of the type

$$\sum_{j=1}^{n} a_{kj}x_j \geq b_k,$$

or, on introducing the slack variable,

$$\sum_{j=1}^{n} a_{kj}x_j - v_k = b_k, v_k \geq 0,$$

we have to introduce a variable u_k with coefficient $+1$, so that the constraint becomes

$$\sum_{j=1}^{n} a_{kj}x_j + u_k - v_k = b_k, u_k \geq 0, v_k \geq 0,$$

which, in effect, deletes the constraint.

Finally, if the constraint to be deleted is of the type

$$\sum_{j=1}^{n} a_{kj}x_j = b_k,$$

we may introduce two new variable, u_k, v_k, with coefficients +1 and −1 respectively, in the constraint to get the desired effect of deleting the constraint.

Example: In the optimal solution of the example of section 5, (table 2), the variables x_4 and x_6, which are respectively the slack variables of the first and the third constraint of the original problem, are positive. Hence the deletion of the first or the third constraint from the problem leaves the optimal solution unchanged. But the slack variable x_5 of the second constraint is zero in the original optimal solution. Therefore if the second constraint is deleted, the optimal solution will change. Since this constraint is of ≥ type, to determine the optimal solution of the modified problem, we introduce a new variable x_7, with coefficient +1, in addition to the old slak variable x_5 (with coefficient −1), to get the constraint

$$x_1 + 2x_2 + x_3 - x_5 + x_7 = 2.$$

The other two constraints remain the same. The coefficients of x_7 in the three constraint equations and the objective function are $a_{17} = 0$, $a_{27} = 1$, $a_{37} = 0$, $c_7 = 0$. The corresponding values in the original optimal solution, (table 2), can be calculated to be $\bar{a}_{17} = 1$, $\bar{a}_{27} = 2$, $\bar{a}_{37} = 2$, $\bar{c}_7 = -2$, so that the starting simplex table for finding the new optimal solution is

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7
x_3	2	1	2	1		−1		1
x_4	1	3	2		1	−2		2
x_6	8	4	5			−2	1	2
f	−4	1	1			2		−2

One iteration on the above table gives the new opcimal solution as $x_1 = 0$, $x_2 = 0$, $x_3 = 3/2$; $f = 3$. It can be verified that this solution satisfies the first and the third constraint but not the second, as it now stands deleted.

10 Parametric linear programming

So far in this chapter we discussed the effect of changes in the values of the input data of an *LP* problem on its original optimal solution. The changes considered were discrete. We shall now assume that the coefficients in the problem vary continuously as a function of some parameter. The analysis of the effect of this functional dependence, hereafter called parametric variation, on the optimal solution of the problem, is called parametric linear programming. Parametric variation can be linear or nonlinear. The nonlinear case will not be considered here, as the computations in that case become too cumbersome.

In the subsequent sections we shall consider linear parametric variations in (i) the cost coefficients c_j, (ii) the right hand entries b_i of the constraints, (iii) the coefficients a_{ij}, and (iv) c_j, b_i and a_{ij} simultaneously.

Parametric linear programming is essentially based on the same concepts as sensitivity analysis. Assuming that the coefficients which are varying are linear functions of a parameter λ, the general strategy adopted is the following. We first compute the optimal solution for $\lambda = 0$. Then using optimality and feasibility conditions, we find the range of values of λ for which this optimal solution remains optimal and feasible. Suppose this range is $(0, \lambda_1)$. This means that any increase in the value of λ beyond λ_1 will make the present optimal solution nonoptimal or infeasible. At $\lambda = \lambda_1$ we determine a new optimal solution and find the range (λ_1, λ_2) of the values of λ for which this new optimal solution remains feasible and optimal. The process is repeated at λ_2 and continued till a value of λ is reached beyond which either the optimal solution does not change or does not exist. A similar strategy is adopted for investigating the effect of variations for the negative values of λ.

11 Parametric variations in c_j

Consider the problem:

Minimise $\quad f(\lambda) = (\mathbf{C} + \lambda \mathbf{C}^*)\mathbf{X},$

subject to
$$\mathbf{AX} = \mathbf{B}, \qquad (8)$$
$$\mathbf{X} \geq 0,$$

where **C** is the original cost vector, $\mathbf{C}^*$ the cost variation vector and λ a parameter which can have any real value. The problem is to determine the optimal solutions for all possible values of λ. It will be useful to put

$$f(\lambda) = f_0 + \lambda f^*, \text{ where } f_o = \mathbf{CX}, f^* = \mathbf{C}^*\mathbf{X}.$$

We first solve this problem for $\lambda = 0$. Let $\mathbf{X}_0 = [x_1\, x_2 \ldots x_m]'$ represent its optimal basis and $\mathbf{A}_0$ the corresponding coefficient matrix. Then $\bar{c}_j$, the relative cost coefficients of the nonbasic variables in the optimal solution, given by (2), namely,

$$\bar{c}_j = c_j - \sum_{i=1}^{m} c_i \bar{a}_{ij}, j = m+1, \ldots, n,$$

are all nonnegative. Now let λ be nonzero. Corresponding to the same basis the relative cost coefficients of the nonbasic variables are, by the same formula, given by

$$\bar{c}_j(\lambda) = (c_j + \lambda c_j^*) - \sum_{i=1}^{m} (c_i + \lambda c_i^*)\bar{a}_{ij}$$
$$= \left(c_j - \sum_{i=1}^{m} c_i \bar{a}_{ij}\right) + \lambda\left(c_j^* - \sum_{i=1}^{m} c_i^* \bar{a}_{ij}\right)$$
$$= \bar{c}_j + \lambda \bar{c}_j^*, j = m+1, \ldots, n.$$

Table 6 gives the solution of this problem by the simplex method. The optimal solution is $x_1 = 14$, $x_2 = 24$, with $f_1 = -38$, $f_2 = -440$. Thus in the optimal solution f_2 achieves its minimum value, but f_1 falls short by -7.

TABLE 6

Basis	Values	x_1	x_2	v_1	v_2	x_3	x_4
v_1	45	1	1	1			
v_2	440	16	9		1		
x_3	90	3	2			1	
x_4	80	4	1				1
f	-485	-17	-10				
v_1	25		3/4	1			$-1/4$
v_2	120		5		1		-4
x_3	30		5/4			1	$-3/4$
x_1	20	1	1/4				1/4
f	-145		$-23/4$				17/4
v_1	7			1		$-3/5$	1/5
v_2	0				1	-4	-1
x_2	24		1			4/5	$-3/5$
x_1	14	1				$-1/5$	2/5
f	-7	1				23/5	4/5

BIBLIOGRAPHICAL NOTE

(For references see bibliography)

Most books on linear programming include sensitivity analysis and parametric analysis. Gal (1979) discusses them in greater detail. Lee (1972) gives a nice introduction to multiobjective and goal programming. A brief introduction to these topics is also given in Gass (1985, fifth edition). Osyczka (1985) discusses the use of multicriterion optimization in engineering design problems, while Steuer (1986) gives some of the latest developments on these fast developing topics.

PROBLEMS VII

Sensitivity analysis

1. Solve graphically the LP problem: maximize $f = 4x_1 + 8x_2$, subject to $x_1 + 2x_2 \geq 20$, $2x_1 + 2x_2 \leq 100$, $x_1 - 3x_2 \leq 0$, $4x_1 - x_2 \geq 0$, $x_1 \geq 0$, $x_2 \geq 0$. Also analyse graphically how the optimal solution is modified when the following changes are introduced in the problem, (one at a time);
   (i) objective function is replaced by $8x_1 + 4x_2$;
   (ii) right hand side of the second constraint is changed to 50;
   (iii) the coefficients of x_2 in the constraints are changed from (2, 2, $-$ 3, $-$ 1) to (2, 1, $-$2, $-$1);
   (iv) fourth constraint is deleted;
   (v) a new constraint $2x_1 + x_2 \geq 10$ is introduced.

   [(10, 40), 360; (i) (37.5, 12.5), 350; (ii) (5, 20), 180; (iii) (50/3, 200/3), 600; (iv) (0, 50), 400; (v) no change.]

2. Solve the above problem using simplex method, and analyse the effects of the changes using sensitivity analysis methods.

3. Solve by simplex method the problem: maximize $f = -5x_1 + 13x_2 + 5x_3$, subject to $12x_1 + 10x_2 + 4x_3 \le 90$, $-x_1 + 3x_2 + x_3 \le 20$, $x_1, x_2, x_3 \ge 0$. Use the sensitivity analysis approach to investigate the effects on the optimal solution of the following changes introduced one at a time:

   (i) right side of the second constraint is changed to 30;
   (ii) coefficient of x_2 in the objective function changes to 8;
   (iii) coefficient of x_1 in the objective function changes to – 2, and in the constraints from 12, – 1 to 5, 10 respectively;
   (iv) a new variable is introduced with coefficient 10 in the objective function and 5 and 3 respectively in the first and second constraints;
   (v) variable x_3 is deleted from the problem;
   (vi) a new constraint $2x_1 + 5x_2 + 3x_3 \le 50$ is introduced.

   Verify your answers by solving the modified problems *ab initio* by the simplex method.
   [(0, 0, 20), 100; (i) (0, 9, 0), 117; (ii, iii, iv) no change; (v) (0, 20/3), 260/3; (vi) (0, 5/2, 25/2), 95].

4. For the problem: maximize $f = x_1 - x_2 + 2x_3$, subject to $x_1 - x_2 + x_3 \le 4$, $x_1 + x_2 - x_3 \le 3$, $2x_1 - 2x_2 + 3x_3 \le 15$, $x_1, x_2, x_3 \ge 0$, assuming x_4, x_5, x_6 respectively as the slack variables for the three constraints, the optimal table is the following.

Basis	Values	x_1	x_2	x_3	x_4	x_5	x_6
x_3	21	4		1		2	1
x_4	7	2			1	1	0
x_2	24	5	1			3	1
$-f$	18	2				1	1

   Carry out the sensitivity analysis for each of the following changes:

   (i) coefficient of x_1 in the objective function changes to 2;
   (ii) coefficients of x_1 in the problem become $c_1 = 4$, $a_{11} = 1$, $a_{21} = 2$, $a_{31} = 3$;
   (iii) coefficients of x_2 and x_3 change to $c_2 = -2$, $a_{12} = 2$, $a_{22} = 3$, $a_{32} = -1$, $c_3 = 1$, $a_{13} = 3$, $a_{23} = -2$, $a_{33} = 1$;
   (iv) right hand side vector changes from [4 3 15] to [2 4 20];
   (v) objective function changes to $3x_1 + x_2 + 5x_3$;
   (vi) first constraint is deleted;
   (vii) a new constraint $2x_1 + x_2 + 2x_3 \le 60$ is introduced;
   (viii) third constraint changes to $4x_1 - x_2 + 2x_3 \le 12$.

   [(i), (ii) no change; (iii) (17/5, 0, 1/5), 18/5; (iv) (0, 32, 28), 24; (v) (0, 24, 21), 129; (vi) no change; (vii) (0, 150/7, 135/7), 120/7; (viii) (0, 18, 15), 12; or (0, 4, 8), 12].

5. A company manufactures three products, A, B and C, using the same raw material and the same labour force. The mathematical model of the problem formulated to maximize the profits is:

Maximize	$f = 5x_1 + 3x_2 + x_3$,
subject to	$5x_1 + 6x_2 + 3x_3 \le 45$, (labour)
	$5x_1 + 3x_2 + 4x_3 \le 30$, (material)
	$x_1, x_2, x_3 \ge 0$;

   where x_1, x_2, x_3 are the amounts of the products A, B, C. The optimal solution to this problem is given in the following table, where x_4, x_5 are the slack variables in the first and second constraints respectively.

Basis	Values	x_1	x_2	x_3	x_4	x_5
x_2	5		1	$-1/3$	$1/3$	$-1/3$
x_1	3	1		1	$-1/5$	$2/5$
$-f$	30			3	0	1

Use sensitivity analysis to find the new optimal solution if the following changes, one at a time, are made in the data:

(i) coefficient of x_2 in the expression for f is changed to 2;

(ii) available material increases from 30 to 60 units;

(iii) per unit requirement of the material for the production of C is reduced from 4 to 2 units;

(iv) a constraint $3x_1 + 2x_2 + x_3 \leq 20$, expressing limitation of supervisory staff is added.

[(i) (6, 0, 0), 30; (ii) (9, 0, 0), 45; (iii, iv) no change].

6. In problem 2 determine the effect on the optimal solution if changes (i) and (ii) are introduced in succession, (that is, the solution obtained after introducing change (i) is subjected to change (ii)). Also solve the problem when both the changes are introduced simultaneously. Is the optimal solution in the two cases the same?

Repeat this analysis for the successive and simultaneous occurrence of changes (ii) and (iii).

What inference can be drawn regarding the technique to carry out sensitivity analysis when changes of more than one type are introduced simultaneously in the data?

[Changes may be introduced in succession.]

7. Use the approach suggested in problem 6 to determine the effect on the optimal solution of problem 5 of introducing simultaneously (a) changes (i) and (ii), (b) changes (ii) and (iii).

[In both cases (9, 0, 0), 45].

8. The following table gives the optimal solution to a LP problem of the type: Maximize $f = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{B}$, $\mathbf{X} \geq 0$.

Basis	Values	x_1	x_2	x_3	x_4	x_5
x_1	1	1		1	3	-1
x_2	2		1	1	-1	2
$-f$	8			4	3	4

x_4, x_5 are the slack variables respectively in the two constraints with right hand sides b_1 and b_2. The values of the cost coefficients are $c_1 = 2$, $c_2 = 3$, $c_3 = 1$.

(i) How much can the coefficient c_1 be increased before the current basis ceases to be optimal? Answer the same question with respect to c_3.

(ii) How much can the value of b_1 be varied before the present basis (x_1, x_2) ceases to be feasible? (It is not necessary to know the value of b_1 to answer this question).

(iii) Find the optimal solution by the dual simplex method when b_1 is increased by 3.

[(i) c_1, c_3 separately can be increased by 4; (ii) $7/15 \leq b_1 \leq 14/5$; (ii) (7, 0, 0), 14].

Parametric linear programming

9. Solve the following parametric programming problems for variations in the values of the coefficients of the objective function for values of parameter $\lambda \geq 0$.

(i) Maximize $(4 + 2\lambda)x_1 + 3(1 + \lambda)x_2$, subject to $3x_1 + 4x_2 \leq 12$, $3x_1 + 3x_2 \leq 10$, $4x_1 + 2x_2 \leq 8$, $x_1, x_2 \geq 0$.

(ii) Maximize $(2 + 3\lambda)\, x_1 + (3 + \lambda)\, x_2 + (1 + 2\lambda)\, x_3$, subject to $x_1 + x_2 + 2x_3 \leq 4$, $x_1 + 3x_2 - 2x_3 \leq 6$, $4x_1 + 2x_2 + x_3 \leq 10$, $x_1, x_2, x_3 \geq 0$.

(iii) Maximize $2(1 - \lambda)\, x_1 + (2 + \lambda)\, x_2 + (1 + 2\lambda)\, x_3$, subject to $-x_1 + x_2 + 2x_3 \leq 2$, $2x_1 - x_2 + 2x_3 \leq 8$. $x_1, x_2, x_3 \geq 0$.

(iv) Minimize $(1 + \lambda)\, x_1 + (1 + \lambda)\, x_2 + 2\,(2 + 3\lambda)\, x_3$, subject to $x_1 - x_2 + 2x_3 \geq 3$, $-2x_1 + 4x_2 - x_3 \geq 1$, $x_1 + x_2 + 2x_3 \leq 11$, $x_1, x_2, x_3 \geq 0$.

[(i) $0 \le \lambda \le 7$: (4/5, 12/5), $(52 + 44\lambda)/5$; $\lambda \ge 7$: (0, 3), $9 + 9\lambda$; (ii) $0 \le \lambda \le 13/6$: (17/11, 19/11, 4/11), $(95 + 78\lambda)/11$; $\lambda \ge 13/6$: (16/7, 0, 6/7), $(38 + 60\lambda)/7$; (iii) $0 \le \lambda \le 4$: (10, 12, 0), $44 - 8\lambda$; $\lambda \ge 4$: (0, 2, 0), $4 + 2\lambda$; (iv) $0 \le \lambda \le 1$: (0, 5/7, 13/7), $(57 + 83\lambda)/7$; $\lambda \ge 1$: (13/2, 7/2, 0), $10 + 10\lambda$.]

10. Solve the following problems for parametric variations in the values of the right side constants for $\lambda \ge 0$.
   (i) Maximize $4x_1 + 3x_2$, subject to $3x_1 + 4x_2 \le 12 + 2\lambda$, $3x_1 + 3x_2 \le 10 + 2\lambda$, $4x_1 + 2x_2 \le 8 + 3\lambda$, $x_1, x_2 \ge 0$.
   (ii) Minimize $2x_1 + 3x_2 + x_3$, subject to $x_1 + x_2 + 2x_3 \le 4 + \lambda$, $x_1 + 3x_2 - 2x_3 \le 6 + 3\lambda$, $4x_1 + 2x_2 + x_3 \le 10 + 2\lambda$, $x_1, x_2, x_3 \ge 0$.
   (iii) Maximize $2x_1 + 2x_2 + x_3$, subject to $-x_1 + x_2 + 2x_3 \le 2 + \lambda$, $2x_1 - x_2 + 2x_3 \le 8 - 5\lambda$, $x_1, x_2, x_3 \ge 0$.

[(i) $0 \le \lambda \le 4$: $(8(1+\lambda)/10, (24-\lambda)/10)$, $(104 + 29\lambda)/10$; $4 \le \lambda \le 16$: $((4+5\lambda)/6, (16-\lambda)/6)$, $(64 + 17\lambda)/6$; $\lambda \ge 16$: $((10 + 2\lambda)/3, 0)$, $(40 + 8\lambda)/3$. (ii) $\lambda \ge 0$: $(17/11, (19/11) + \lambda, 4/11)$, $(95/11) + 3\lambda$. (iii) $0 \le \lambda \le 5/2$: $(10 - 4\lambda, 12 - 3\lambda, 0)$, $44 - 14\lambda$; $\lambda \ge 5/2$: infeasible.]

11. Write the duals of 9(i) and 10(i), and carry out the parametric analysis of these duals for values of parameter $\lambda \ge 0$.

12. Show that the optimal solution of the following problem for $\lambda = 0$ remains optimal for $0 < \lambda \le 2/3$, and find that solution.
Maximize $3x_1 + 6x_2$, subject to $(1 + 2\lambda)x_1 \le 4$, $3(1 - \lambda)x_1 + 2x_2 \le 18$, $x_1, x_2 \ge 0$. [(0, 9), 54.]

13. Carry out the parametric analysis of the following problems for simultaneous variation in the cost coefficient and right hand side vectors.
   (i) Maximize $(4 - 10\lambda)x_1 + (8 - 4\lambda)x_2$, subject to $x_1 + x_2 \le 4$, $2x_1 + x_2 \le 3 - \lambda$, $x_1, x_2 \ge 0$, for $-\infty < \lambda < \infty$.
   (ii) Maximize $2(1 - \lambda)x_1 + (2 + \lambda)x_2 + (1 + 2\lambda)x_3$, subject to $-x_1 + x_2 + 2x_3 \le 2 + \lambda$, $2x_1 - x_2 + 2x_3 \le 8 - 5\lambda$, $x_1, x_2, x_3 \ge 0$, for $\lambda \ge 0$.

Also compare the result of 13 (ii) with the results of 9 (iii) and 10 (iii) to see if the solutions for separate variations in the cost coefficient vector and right hand side vector superimpose when these variations are considered simultaneously.

[(i) $-\infty < \lambda \le -5$: (4, 0), $16 - 40\lambda$; $-5 \le \lambda \le -1$: $(-1 - \lambda, 5 + \lambda)$, $36 - 6\lambda + 6\lambda^2$; $-1 \le \lambda \le 2$: $(0, 3 - \lambda)$, $24 - 20\lambda + 4\lambda^2$; $2 \le \lambda \le 3$: (0, 0), 0; $\lambda > 3$: infeasible. (ii) $0 \le \lambda \le 5/2$: $(10 - 4\lambda, 12 - 3\lambda, 0)$, $44 - 22\lambda + 5\lambda^2$; $\lambda > 5/2$: infeasible.]

14. In parametric analysis the starting basic feasible solution is calculated at $\lambda = 0$. Difficulty can arise if no feasible solution exists for $\lambda = 0$. Show that it is also possible to do parametric analysis by starting with some other value of λ. As an example, analyse 9 (i) by starting with $\lambda = 10$.

15. Solve the problem: minimize $(2 + \lambda)x_1 + (1 + 4\lambda)x_2$, subject to $4x_1 + 3x_2 \ge 6$, $3x_1 + x_2 \ge 3$, $x_1 + 2x_2 \le 3$, $x_1, x_2 \ge 0$, for $\lambda = 0$ by the dual simplex method, and then do its parametric analysis for $\lambda \ge 0$. (This example illustrates that parametric analysis of a linear programme can be done even if its starting solution is obtained by the dual simplex method).
[$0 \le \lambda \le 2/13$: (3/5, 6/5), $(12 + 27\lambda)/5$; $\lambda \ge 2/13$: (3/2, 0), $(6 + 3\lambda)/2$.]

Multiobjective and goal programming

16. Find the (i) extreme point solutions, (ii) attainable solutions and (iii) extreme point efficient solutions to the following multiobjective problem: maximize $(f_1 = x_1 + 2x_2, f_2 = 3x_1 - x_2)$, subject to $3x_1 + x_2 \le 9$, $x_1 + 3x_2 \le 18$, $x_1 - x_2 \le 3$, $x_1 + x_2 \le 9$, $x_1, x_2 \ge 0$. (iv) Does this problem have an ideal solution?
[(i) (3, 0), (6, 3), (9/2, 9/2), (9/8, 45/8); (ii) $f_1 = 27/2$ at (9/2, 9/2), $f_2 = 15$ at (6, 3); (iii) (6,3), (9/2, 9/2); (iv) no.]

17. Find all the extreme point solutions to the following multiobjective problem and show that out of these seven are efficient solutions. Maximize (f_1, f_2, f_3), where $f_1 = -x_1 + 5x_2 + 2x_3 + x_4$, $f_2 = 3x_1 + x_2 + x_3 + 2x_4$, $f_3 = x_1 - x_2 + 4x_3 + 2x_4$; subject to $2x_1 + x_2 + 3x_3 + 4x_4 \le 60$, $3x_1 + 4x_2 + 2x_3 + x_4 \le 60$, $x_1, x_2, x_3, x_4 \ge 0$.

5

Flow and Potential in Networks

1 Introduction

Networks are familiar diagrams in electrical theory; they are easily visualized in transportation or communication systems like roads, railways or pipelines, nerves or blood vessels. A large variety of mathematical problems are presented by networks, ranging from puzzles for children to intricate problems challenging mathematicians. Many problems, particularly those which involve sequential operations or different but related states or stages, are conveniently described diagrammatically as networks. Sometimes a problem with no such apparent structure assumes a mathematical form which is best understood and solved by interpreting it as a network.

A network, in its more generalized and abstract sense, is called a graph. In recent years graph theory has been a subject of much study and research by mathematicians, and has found more and more applications in diverse areas. In the field of operations research graph theory plays a particularly important role as quite often the problem of finding an optimal solution can be looked upon as a problem of choosing the best sequence of operations out of a finite number of alternatives which can be represented as a graph.

In this chapter we shall discuss some linear programming problems of such special forms that the ideas of graph theory help in their solution. We shall not introduce graph theory in its abstraction but shall take up the special problems and show how they can be looked upon and solved as networks or graphs. We start with some definitions of terms relating to graphs.

2 Graphs: definitions and notation

A *graph* $G(V, U)$ or simply G (when there is no ambiguity) is defined as a set V of elements v_j, $j = 1, 2, \ldots, n$, which can be represented as points, and a set U of pairs (v_j, v_k), $v_j, v_k \in V$, which can be represented as arcs joining points of V. The elements of V are called *vertices* and the elements of U *arcs*. We shall denote the elements of U as either u_i, $i = 1, 2, \ldots, m$, or as (v_j, v_k).

If (v_j, v_k) are ordered pairs, we represent them by *directed arcs*, that is, arcs carrying arrow marks on them denoting the direction v_j to v_k. A graph with directed arcs is called a *directed graph*. Unless otherwise stated, we shall assume that a graph $G(V, U)$ is directed.

The graph is said to be *finite* when V and U are finite sets. We shall restrict our discussion to finite graphs only.

We shall denote a graph diagrammatically as shown in Fig. 1. The vertices are shown as small circles, with vertex v_j denoted as j. The examples given in this section to explain the defined terms refer to this figure.

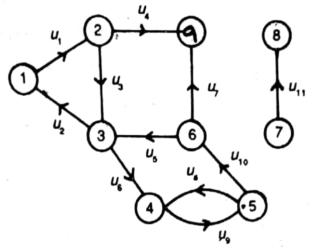


Fig. 1

An arc (directed or undirected) is said to be *incident with* a vertex which it joins to some other vertex. It *connects* the two vertices. The directed arc $u_i = (v_j, v_k)$ is said to be *incident from* or *going from* v_j and *incident to* or *going to* v_k; v_j is called the *initial* vertex and v_k the *terminal* vertex of the arc (v_j, v_k).

A *subgraph* of $G(V, U)$ (Fig. 1) is defined as a graph $G_1(V_1, U_1)$ with $V_1 \subseteq V$ and U_1 containing all those arcs of G which connect the vertices of G_1. For example, in the figure, if $V_1 = \{v_1, v_2, v_3\}$ and $U_1 = \{u_1, u_2, u_3\}$, then $G_1(V_1, U_1)$ is a subgraph of G. A *partial graph* of $G(V, U)$ is a graph $G_2(V, U_2)$ which contains all the vertices of G and some of its arcs ($U_2 \subseteq U$). For example, if we erase some arcs, say u_1, u_2, from Fig. 1 we shall be left with a partial graph of the original graph.

Let V_1 and V_2 be two subsets of V such that they have no common vertex, and let $u_i = (v_j, v_k)$ be an arc such that $v_j \in V_1$, $v_k \in V_2$. Then u_i is said to be *incident from* or *going from* V_1 and *incident to* or *going to* V_2. It is *incident with* both V_1 and V_2 and is said to *connect* them. In the figure, if $V_1 = \{v_2, v_3\}$ and $V_2 = \{v_6, v_9\}$, then u_4 connects V_1 and V_2. It goes from V_1 to V_2 and is incident with both. We shall denote by $\Omega(V_k)$ the set of arcs of $G(V, U)$ incident with a subset V_k of V, by $\Omega^+(V_k)$ the set of arcs incident to V_k, and by $\Omega^-(V_k)$ the set of arcs incident from V_k. In the figure, if $V_1 = \{v_2, v_3\}$, then

$$\Omega^+(V_1) = \{u_1, u_5\},\ \Omega^-(V_1) = \{u_2, u_4, u_6\} \text{ and } \Omega(V_1) = \{u_1, u_2, u_4, u_5, u_6\}.$$

A sequence of arcs $(u_1, u_2, ..., u_k, u_{k+1}, ..., u_q)$ of a graph such that every intermediate arc u_k has one vertex common with the arc u_{k-1} and another common with u_{k+1} is called a *chain*. For example, the sequence (u_2, u_3, u_4, u_7) in the figure is a chain. We may also denote a chain by the vertices which it connects, for example, the above chain may also be written as $(v_1, v_3, v_2, v_9, v_6)$.

A chain becomes a *cycle* if in the sequence of arcs no arc is used twice and the first arc has a vertex common with the last arc, and this vertex is not common with any intermediate arc. For example, the chain (u_3, u_5, u_7, u_4) in the figure is a cycle.

A *path* is a chain in which all the arcs are directed in the same sense such that the terminal vertex of the preceding arc is the initial vertex of the succeeding arc. In the figure the sequence of arcs (u_1, u_3, u_6, u_9) is a path. We may also denote the path in terms of the vertices as $(v_1, v_2, v_3, v_4, v_5)$. A path is a chain, but every chain is not a path.

A *circuit* is a cycle in which all the arcs are directed in the same sense. The cycles (u_1, u_3, u_2) and (u_8, u_9) are circuits.

A graph is said to be *connected* if for every pair of vertices there is a chain connecting the two. The graph in Fig. 1 is not connected because there is no chain connecting, for instance, v_6 to v_7 or v_2 to v_8. If we erase the vertices v_7, v_8 and the arc u_{11} we shall be left with a connected graph. If v_a is a vertex of a graph, then the set formed by v_a and all other vertices which are connected to v_a by chains, and the set of arcs connecting them, form a *component* of the graph. A connected graph has only one component. If a graph is not connected, it has at least two components. The graph of Fig. 1 has two components, one consisting of vertices v_7, v_8 and the arc u_{11}, and the other the remaining portion.

A graph is *strongly connected* if there is a path connecting every pair of vertices in it. Telephones in a town are the vertices of a strongly connected graph. Radio receivers and transmitters form a connected graph but not strongly connected, because there is a path from a transmitter to a receiver but not one from a receiver to a transmitter.

A *tree* is defined as a connected graph with at least two vertices and no cycles (Fig. 2). It can be proved that a tree with n vertices has $n - 1$ arcs, and that every pair of vertices is joined by one and only one chain. If we delete an arc from a tree, the resulting graph is not connected, and if we add an arc, a cycle is formed. As the name indicates, a natural tree is the best example of a graphical tree, the branches forming the arcs and the extremities of the branches forming the vertices.

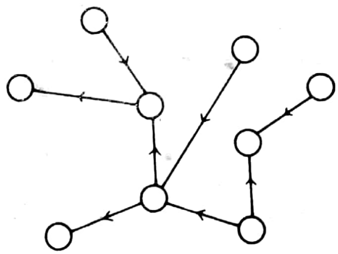


Fig. 2

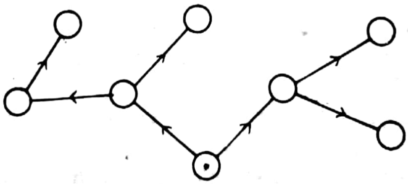


Fig. 3

A vertex which is connected to every other vertex of the graph by a path is called a *centre* of the graph. A graph may or may not have a centre, or may have many centres. Every vertex of a strongly connected graph is a centre. The tree of Fig. 2 has no centre. A tree can at the most have only one centre.

A tree with a centre is called an *arborescence* (Fig. 3). In the figure the centre is marked ⊙. In an arborescence all the arcs incident with the centre go from it and all the other arcs are directed in the same sense.

3 Minimum path problem

Let a number x_{jk} be associated with each arc (v_j, v_k) of a graph $G(V, U)$, and let v_a and v_b be two vertices of the graph. There may be a number of paths from v_a to v_b. For each path we define the *length* of the path as Σx_{jk} where the summation is over the sequence of arcs forming the path. The problem is to find the path of the smallest length.

The term *length* is used here in a generalized sense of any real number associated with the arc and should not be regarded as a geometrical distance. A road map connecting towns is a graph and the distance along a road between any two towns is the length of a path within the present definition of the term, but this is only a particular case. The time or the cost involved in going from one town to another is also a *length* under the present definition. There may be more abstract situations in which the length is not even non-negative. In general x_{jk} is a real number, unrestricted in sign.

Many methods and algorithms have been suggested for solving the problem of the minimum path. We shall describe two algorithms here, one applicable only to the case when $x_{jk} \geq 0$ for all arcs and the other for the general case when x_{jk} is unrestricted. A third method, using the principle of dynamic programming, will be given in chapter 10.

I All arc lengths non-negative. Let f_j denote the minimum path from v_a to v_j. We have to find f_b. Obviously $f_a = 0$.

Let V_p be a subset of V such that v_a is in V_p and v_b is not in V_p. Further suppose that f_j for every v_j in V_p has been determined. Now determine $f_j + x_{jk}$ for every v_j in V_p and v_k not in V_p such that (v_j, v_k) is an arc incident from V_p. Let

$$f_r + x_{rs} = \min\,(f_j + x_{jk})$$

where $v_r \in V_p$ and $v_s \notin V_p$. Then the minimum path from v_a to v_s is given by

$$f_s = f_r + x_{rs}.$$

This is so because to reach v_s we must leave V_p and $f_r + x_{rs}$ is the least of all paths going out of V_p along single arcs. Any alternative path to v_s can either be along some other single arc going out of V_p to v_s which would be larger, or along some other arc going out of V_p to some other point and then to v_s which would be larger still.

Now form an enlarged subset V_{p+1} of V defined by

$$V_{p+1} = V_p \cup \{v_s\},$$

and repeat the operation. Suppose we start with $p = 0$ with V_0 consisting of a single vertex v_a and $f_a = 0$. Following the procedure described above the sets $V_1, V_2, \ldots,$ $V_p, V_{p+1}, \ldots$ are formed. As soon as we arrive at a set in this sequence which includes v_b, we have found f_b. If no such set can be found, there is no path connecting v_a to v_b.

Example: Find the minimum path from v_0 to v_8 in the graph of Fig. 4 in which the number along a directed arc denotes its length.

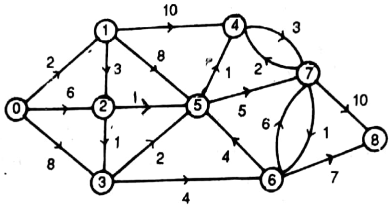


Fig. 4

Table 1 shows the iterations according to the algorithm explained above. In the V_p column are listed the vertices in the subset V_p. Under f are written the least distances to these vertices from v_0. $\Omega^-(V_p)$ are the arcs incident from V_p, (v_j, v_k) being written as (j, k). Under x are given the lengths of the arcs. f_s is the minimum of $f + x$, and v_s is the vertex to which this minimum distance leads and which in the next iteration is included in the enlarged subset V_{p+1}.

TABLE 1

p	V_p	f	$\Omega^-(V_p)$	x	$f+x$	f_s	v_s
0	0	0	(0, 1)	2	2	2	1
			(0, 2)	6	6		
			(0, 3)	8	8		
1	0	0	(0, 2)	6	6		
			(0, 3)	8	8		
	1	2	(1, 2)	3	5	5	2
			(1, 4)	10	12		
			(1, 5)	8	10		
2	0	0	(0, 3)	8	8		
	1	2	(1, 4)	10	12		
			(1, 5)	8	10		
	2	5	(2, 3)	1	6	6	3
			(2, 5)	1	6	6	5
3	0	0					
	1	2	(1, 4)	10	12		
	2	5					
	3	6	(3, 6)	4	10		
	5	6	(5, 4)	1	7	7	4
			(5, 7)	5	11		
4	0	0					
	1	2					
	2	5					
	3	6	(3, 6)	4	10	10	6
	4	7	(4, 7)	3	10	10	7
	5	6	(5, 7)	5	11		
5	0	0					
	1	2					
	2	5					
	3	6					
	4	7					
	5	6					
	6	10	(6, 8)	7	17	17	8
	7	10	(7, 8)	10	20		

The minimum path is found to be of length 17 and goes through the vertices (0, 1, 2, 3, 6, 8).

It should be appreciated that actually drawing the graph is not essential either to the description of the problem or to its solution. The problem is completely enunciated if all the vertices, arcs and arc lengths are specified. In fact in a large problem with many vertices and arcs drawing a figure may be neither practicable nor necessary.

II Arc lengths unrestricted in sign. Let v_a, v_b be two vertices in the graph $G(V, U)$ whose arc lengths are real numbers, positive, negative or zero. We have to find the minimum path from v_a to v_b. We assume that there are no circuits in the graph whose arc lengths add up to a negative number. For, if there is any such circuit, one can go around and round it and decrease the length of the path without limit, getting an unbounded solution.

Construct an arborescence $A_1(V_1, U_1)$, $V_1 \subseteq V$, $U_1 \subseteq U$, with centre v_a and V_1 containing all those vertices of V which can be reached from v_a along a path, and U_1 containing some arcs of U which are necessary to construct the arborescence. If V_1 contains v_b, a path connects v_a to v_b. In a particular arborescence this path is unique. There may be many arborescences and therefore many paths. A_1 is any one arborescence. If in any problem only one arborescence is possible, there is only one path from v_a to v_b, and that is the solution. If V_1 does not contain v_b there is no path from v_a to v_b and the problem has no solution.

The method of construction of the arborescence is straightforward. Mark out the arcs going from v_a. From the vertices so reached mark out the arcs (not necessarily all of them) going out to the other vertices. No vertex should be reached by more than one arc, that is, not more than one arc should be incident to any vertex. If there is a vertex to which no arc is incident, it cannot be reached from v_a and so is left out. No arc incident to v_a should be drawn.

Let f_j denote the length of the path from v_a to any vertex v_j in the arborescence. The arborescence determines f_j uniquely for each v_j in V_1, but f_j is not necessarily minimum. Let (v_k, v_j) be an arc in G but not in A_1. Consider the length $f_k + x_{kj}$ and compare it with f_j. If $f_j \leq f_k + x_{kj}$, make no change. If $f_j > f_k + x_{kj}$, delete the arc incident to v_j in A_1 and include instead the arc (v_k, v_j). This modifies the arborescence from A_1 to A_2 and reduces f_j to its new value $f_k + x_{kj}$, the reduction in the value of f_j being $f_j - f_k - x_{kj}$. The lengths of the paths to the vertices going through v_j are also reduced by the same amount. These adjustments are made and thus the new values of f_j for all v_j in A_2 are calculated.

Now repeat the operation in A_2, that is, select a vertex and see if any alternative arc gives a smaller path to it. If yes, modify A_2 to A_3 and adjust f_j accordingly. Ultimately an arborescence A_r is reached which cannot be further changed by the above procedure. A_r marks out the minimum path to each v_j from v_a, and f_b in this arborescence is the minimum path to v_b. The proof is as follows.

Proof. Let $(v_a, v_1, v_2, \ldots, v_b)$ be any path in G from v_a to v_b. Its length is $x_{a1} + x_{12} + \ldots + x_{pb}$. The vertices in this path are in A_r also because A_r contains all those vertices of G which can be reached from v_a. By the property of A_r given in the last paragraph, for every vertex v_j in A_r and for every arc (v_k, v_j) in G,

$$f_j \leq f_k + x_{kj},$$

or
$$f_j - f_k \leq x_{kj},$$

because otherwise A_r could have been further modified. Writing these inequalities for all vertices of the above path,

$$f_1 - f_a \leq x_{a1},$$

$$f_2 - f_1 \leq x_{12},$$

$$\ldots\ldots$$

$$f_b - f_p \leq x_{pb}$$

Adding, we get ,

$$f_b - f_a \leq x_{a1} + x_{12} + x_{23} + \ldots + x_{pb},$$

or, since $f_a = 0$,

$$f_b \leq x_{a1} + x_{12} + x_{23} + \ldots + x_{pb}.$$

Thus we prove that no path from v_a to v_b in G can be smaller than f_b. Since the path of length f_b is also in G, this path is the minimum. Proved.

The path of maximum length can be found either by changing the signs of the lengths of all arcs and then finding the minimum path, or by reversing the inequality $f_j > f_k + x_{kj}$ to $f_j < f_k + x_{kj}$ as the criterion for changing an arc in the arborescence, so that at every stage a greater path is selected against a smaller one.

Example: Find the minimum path from v_0 to v_7 in the graph G of Fig. 5. Notice that it has no circuit whose length is negative.

Draw an arborescence A_1 (Fig. 6) with centre v_0 consisting of all those vertices of the graph which can be reached from v_0, (v_8 is thus excluded), and the necessary number of arcs. Notice that there can be many such arborescences. A_1 is one of them.

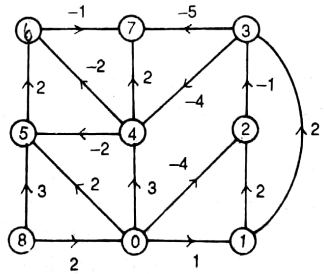


Fig. 5

The lengths f_j of the paths from v_0 to different vertices v_j of A_1 are as follows.

$$f_0 = 0, f_1 = 1, f_2 = -4, f_3 = 3, f_4 = 3, f_5 = 2, f_6 = 4, f_7 = 5.$$

Consider the vertex v_2. There is an arc (v_1, v_2) in G which is not in A_1, such that

$$f_2 = -4 < f_1 + x_{12} = 1 + 2 = 3.$$

So we leave A_1 unchanged.

Now consider the vertex v_3. There is an arc (v_2, v_3) in G which is not in A_1 such that

$$f_3 = 3 > f_2 + x_{23} = -4 - 1 = -5.$$

So we delete the arc (v_1, v_3) which is incident to v_3 in A_1 and instead include the arc (v_2, v_3). This gives us a new arborescence A_2 with $f_3 = -5$. Since no vertex is reached in A_1 through v_3, all other f_j remain unchanged.

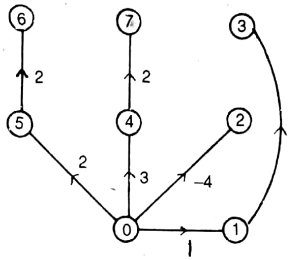


Fig. 6

Coming now to v_4 in A_2 (figure not drawn), arc (v_3, v_4) is in G but not in A_2 such that

$$f_4 = 3 > f_3 + x_{34} = -5 - 4 = -9.$$

So we delete the arc (v_0, v_4), include (v_3, v_4), get another arborescence A_3 with $f_4 = -9$ and consequently $f_7 = -7$.

Continuing like this we finally get the arborescence (Fig. 7) which cannot be further modified. No alternative arc decreases the length of the path from v_0 to any

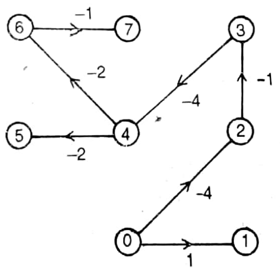


Fig. 7

vertex. This is seen by testing for every possible alternative arc. The minimum path from v_0 to v_7 is $(v_0, v_2, v_3, v_4, v_6, v_7)$ with length -12.

4 Spanning tree of minimum length

Let $G(V, U)$ be a connected graph with undirected arcs, and let $T(V, U')$ be a tree such that $U' \subseteq U$. The set of vertices of T is the same as that of G, while all the arcs of T are arcs of G also. Then $T(V, U')$ is said to be a *spanning* tree of $G(V, U)$ or T is said to *span* G. In Fig. 8 a spanning tree is shown in thick lines and the graph

TABLE 2

A	Path	f	Alternative path	f
1	(1, 2, 4)	4	(1, 4)	−1
	(1, 3)	2	(1, 2, 3)	1
2	(1, 4)	−1	(1, 2, 3, 4)	5
	(1, 2, 3)	1		
3	(1, 4)	−1		
	(1, 2, 3)	1		

A more general problem of maximum potential difference in a network is presented if the constraints are of the type

$$b_{jk} \le f_k - f_j \le c_{jk}$$

for all arcs (v_j, v_k). The method of solution remains the same because each inequality of the above type can be written as two inequalities

$$f_k - f_j \le c_{jk},$$

$$f_j - f_k \le -b_{jk}.$$

Example: Find the maximum potential difference between v_1 and v_4 in the graph $G(V, U)$ where

V	1	2	3	4		
U	(1, 2)	(1, 3)	(2, 3)	(3, 4)	(4, 2)	(1, 4)

subject to the constraints

$$-2 \le f_2 - f_1 \le 3,\ 6 \le f_3 - f_2 \le 10,\ f_4 - f_3 \le -2,\ -2 \le f_2 - f_4,$$

$$1 \le f_4 - f_1 \le 6,\ f_3 - f_1 \le 7.$$

The constraints can be written as

$$f_2 - f_1 \le 3,\ f_1 - f_2 \le 2,\ f_3 - f_2 \le 10,\ f_2 - f_3 \le -6,\ f_4 - f_3 \le -2,$$

$$f_4 - f_2 \le 2, f_4 - f_1 \le 6,\ f_1 - f_4 \le -1,\ f_3 - f_1 \le 7$$

The graph of the problem and the arborescence of minimum path are shown in Figs. 12 and 13. The maximum potential $f_4 - f_1$ is 3.

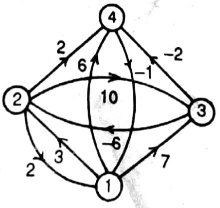


Fig. 12

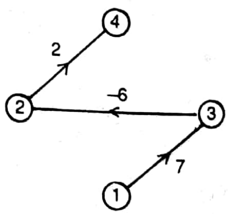


Fig. 13

6 Scheduling of sequential activities

The problem of minimum path finds an important application in scheduling and coordinating various activities in a project so as to complete it in minimum time at a given cost. Also it is possible to estimate the least rise in cost or the maximum saving possible if certain activities are speeded up or slowed down to finish the project within a prescribed period.

A project involves a number of activities, operations or jobs which we identify as vertices $v_a, \ldots, v_j, \ldots, v_k, \ldots, v_b$ of a graph, v_a represents the beginning and v_b the end of the project. Each job v_j requires some time for its completion. It may not be possible to start on a job unless some specified time has been spent on some other job or jobs. The problem is to find the minimum time in which the project can be finished and the time schedule for each job.

Let c_{jk} be the time required on job v_j before job v_k can start. It is the time interval between the start of the two jobs, v_j preceding v_k. We indicate this information by drawing the arc (v_j, v_k) and associating the length c_{jk} with it. The time required to complete v_j is represented by the arc (v_j, v_b) of length c_{jb}, as it would mean that the time c_{jb} should be spent on v_j before the end v_b can be reached. Also if v_j can start only after some time has passed from the beginning of the project, we may indicate it by c_{aj}. All arcs (v_j, v_k) with lengths c_{jk} will in this way denote a sequential relationship in terms of time among various jobs.

Each sequence of jobs which must be done before work on v_j can begin is represented by a path connecting v_a to v_j. The longest of these paths determines the earliest time v_j can start. In this way the longest path joining v_a to v_b gives the minimum time of completion of the project. The problem thus reduces to finding the maximum path with arc lengths c_{jk} (or the minimum path with arc lengths $-c_{jk}$). This path is called the *critical path.*

Example: A building activity has been analyzed as follows, v_j stands for a job.

(i) v_1 and v_2 can start simultaneously, each one taking 10 days to finish.
(ii) v_3 can start after 5 days and v_4 after 4 days of starting v_1.
(iii) v_4 can start after 3 days of work on v_3 and 6 days of work on v_2.
(iv) v_5 can start after v_1 is finished and v_2 is half done.
(v) v_3, v_4 and v_5 take respectively 6, 8 and 12 days to finish. Find the critical path and the minimum time for completion.

Fig. 14 is the graph of the activity, vertices v_a and v_b representing the start and the finish, and the other vertices the jobs to be done in between. The arc lengths denote the time between the start of two jobs.

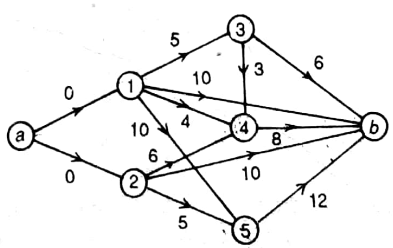


Fig. 14

The arborescence giving the maximum path is shown in Fig. 15. The critical path is (v_a, v_1, v_5, v_b) of length 22 days which is the minimum time of completion of the work.

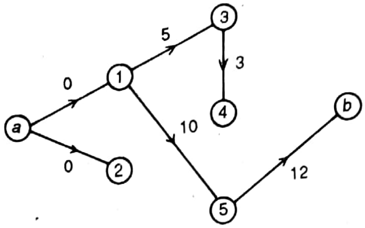


Fig. 15

Having determined the critical path, another type of question can be raised. Suppose it is possible to reduce time on some jobs but at an increased cost. To keep matters simple let us assume that cost of a job increases linearly as time of its completion decreases within certain limits. What is the least increase in cost if the time of completion of the project is decreased by a certain period? If the maximum path is to be reduced, some arc lengths must decrease. We have to determine the decrease which costs the least.

Let α_j be the increase in cost for a unit decrease in time for the completion of the job v_j. Normally α_j would be positive. But there can be situations when slowing down a job results in increased cost and speeding it up leads to some saving. Therefore α_j, in general, are real numbers unrestricted in sign.

We pick up the paths which need reduction in time and examine each vertex v_j in these paths in the order of α_j increasing, beginning with the smallest α_j, and reduce the time there as much as is necessary and possible. When the reduction in time in the concerned paths has been made, we stop further reduction and add up the cost. We illustrate by the following example.

Example: In the previous example the work is required to be finished in 16 days. The following table gives the normal values of c_{jb} (same as in the previous example), the minimum possible values of c_{jb}, and the increase α_j in cost at v_j for a unit decrease in time. Find the minimum additional cost at which the work can be completed.

	v_j	1	2	3	4	5
Normal	c_{jb}	10	10	6	8	12
Minimum	c_{jb}	7	8	4	6	8
	α_j	3	1	2	2	2

The paths from v_a to v_b exceeding the length of 16 days are (v_a, v_1, v_5, v_b) and (v_a, v_2, v_5, v_b). The former is of length 22 days and so needs a reduction of 6 days, while the latter is of length 17 and so needs to be reduced by 1 day. The jobs (or vertices) involved are v_1, v_2, v_5. Of these reduction at v_2 is the cheapest. So we start with $\alpha_2 = 1$. The path through v_2 needs reduction of 1 day only which we get by putting $c_{25} = 4$ days at a cost of 1. This brings the path (v_a, v_2, v_5, v_b) to 16. In the other path the reduction required is still 6, because the arc (v_2, v_5) is not in it. The vertices to be examined are now v_1 and v_5 of which reduction at v_5 is cheaper. So we should reduce c_{5b} by 6. That would make $c_{5b} = 6$, but the minimum possible value of c_{5b} is 8. So we reduce it to 8 days at a cost $4 \times 2 = 8$. The remaining reduction of 2 days can come only at v_1 at a cost $2 \times 3 = 6$. But reducing c_{5b} to 8 days makes reduction in c_{25} unnecessary. So we restore c_{25} to its original value of 5 days. The minimum additional cost of doing the work in 16 days is thus 14.

The method of coordinating and scheduling of activities described in this section is commonly referred to as CPM (critical path method), and is helpful in maintaining progress in construction projects, manufacturing and assembly works, etc. We have given an example of how increase or decrease in cost can be estimated for a specified completion time. There can be variations of this problem. For example, it should be possible to estimate the minimum completion time for a given total cost.

Another similar procedure, called PERT (project evaluation and review technique), goes further and takes into account chance variations in completion times of various jobs to estimate the total expected time of completion. Since we are keeping stochastic and probabilistic considerations completely out of the scope of the present work, we omit further discussion.

7 Maximum flow problem

Like potential, flow in a network is a familiar concept in electrical theory. Flow of liquid through a network of pipelines, or of traffic through a network of roads, or of production through assembly lines are other examples of network flows. In physical terms the basic condition of flow in a network is that at every vertex the total flow in should be equal to the total flow out, that is, there should be no accumulation of whatever stuff is flowing. To extend the idea to more abstract situations it is necessary to give a precise definition of flow in a graph.

Let x_i be a real number associated with every arc u_i, $i = 1, 2, \ldots, m$, of a graph $G(V, U)$ such that for every vertex v_j,

$$\Sigma_1 x_i = \Sigma_2 x_i,\ j = 1, 2, \ldots, n,$$

where the left-hand side summation Σ_1 is on all arcs going to v_j and the right-hand side summation Σ_2 is on all arcs going from v_j. Then x_i is said to be a *flow in the arc* u_i, and the set $\{x_i\}$, $i = 1, 2, \ldots, m$, is said to be a *flow in the graph G*.

To state the problem of maximum flow in a network we define a graph as follows.

Let $G(V, U)$ be a graph (Fig. 16) with V as the set of $n + 2$ vertices $v_a, v_1, v_2, v_3, \ldots, v_n, v_b$, and U as the set of $m + 1$ arcs $u_0, u_1, u_2, \ldots, u_m$. The vertices v_a and v_b and the arc u_0 play a special role in this graph. v_a is called the *source* and v_b the *sink*, and the arc u_0 connects v_b to v_a. It is the only arc going from v_b and also the only arc going to v_a. Other arcs incident with v_a are such that they all go from v_a. Similarly all other arcs incident with v_b go to v_b.

With every arc u_i, $i = 1, 2, \ldots, m$, (except u_0), is associated a real number $c_i \geq 0$ called the *capacity* of the arc.

Let $\{x_i\}$ be a flow in the graph G such that $0 \le x_i \le c_i$, $i = 1, 2, ..., m$. Notice that x_0 as the flow in the arc u_0 is defined but the capacity of the arc u_0 is not defined and so there is no constraint on x_0. By the definition of flow and because x_0 is the only flow out at v_b and in at v_a,

$$\begin{aligned}\text{total flow in at } v_b &= \text{total flow out at } v_b\\ &= x_0\\ &= \text{total flow in at } v_a\\ &= \text{total flow out at } v_a.\end{aligned}$$

All that flows out at v_a flows in at v_b. This explains why v_a and v_b have been called the source and the sink respectively. The arc u_0 serves as a mathematical device to bring the flow in the network within the definition of a flow in a graph. We shall call u_0 the *return arc*.

The problem is to determine the maximum flow out at the source (= maximum flow in at the sink). More precisely, the problem is to find the flow $\{x_i\}$ such that

$$x_0 \text{ is maximum}$$

subject to $$0 \le x_i \le c_i,\ i = 1, 2, .., m.$$

We first describe an algorithm which solves the problem. The proof will be given later.

(i) Start by assuming a feasible flow. In the absence of any better guess, it is always possible to start with $x_i = 0$ for all i.

(ii) Divide the set V of vertices into two subsets, W_1 and W_2, such that each vertex is either in W_1 or in W_2 but not in both. To begin with let $W_1 = \{v_a\}$, all other vertices being in W_2.

(iii) Adopt the following procedure of transferring a vertex from W_2 to W_1. Let $v_j \in W_1$, $v_k \in W_2$.

(a) if (v_j, v_k) is an arc u_i and $x_i < c_i$, transfer v_k to W_1;
(b) if (v_k, v_j) is an arc u_i and $x_i > 0$, transfer v_k to W_1;
(c) otherwise do not transfer v_k to W_1.

Go on transferring vertices from W_2 to W_1 like this. If v_b is transferred to W_1 by this procedure, the flow is not optimal.

(iv) If the flow is not optimal, increase x_i in arc of category (a) in which $x_i < c_i$ and decrease x_i in arc of category (b) in which $x_i > 0$ so that the flow remains feasible and at least one arc gets capacity flow. Go back to (ii).

Repeating operations (ii), (iii) and (iv) we shall come to a stage when v_b cannot be transferred to W_1 by operation (iii). The flow at that stage is optimal.

Example: In the graph of Fig. 16, numbers along arcs are values of c_i. Find the maximum flow in the graph.

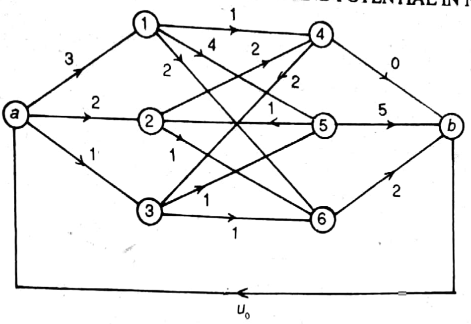


Fig. 16

Assuming the initial flow as zero in all arcs, let $W_1 = \{v_a\}$. There is an arc (v_a, v_1) from v_a in W_1 to v_1 in W_2 in which the flow (zero) is less than its capacity 3. Therefore by criterion (iiia) transfer v_1 to W_1. Now there is an arc (v_1, v_4) with $v_1 \in W_1$ and $v_4 \in W_2$ such that the flow in it is less than its capacity 1. So we transfer v_4 to W_1. In the arc (v_4, v_b), $v_4 \in W_1$, $v_b \in W_2$, the flow is zero which is equal to its capacity and so we cannot transfer v_b to W_1 for this reason. But there is another arc (v_4, v_3), $v_4 \in W_1$, $v_3 \in W_2$, which is such that v_3 is transferrable to W_1. Further, because the flow in arc (v_3, v_5), $v_3 \in W_1$, $v_5 \in W_2$ is below capacity, v_5 is transferred to W_1, and finally because the arc (v_5, v_b) satisfies the same criterion, v_b is transferred to W_1. Thus it is possible to transfer v_b to W_1 and so the flow is not optimal.

We have gone along the chain $(v_a, v_1, v_4, v_3, v_5, v_b)$. The least capacity in this chain is 1. So in each arc of this chain and also in the return arc (v_b, v_a) increase the flow to 1, keeping the flow as it was in all other arcs. The modified flow is feasible because in each arc it is less than or equal to its capacity, and also at every vertex the flow in equals the flow out.

The above reasoning is repeated with every modified feasible flow until it is not possible to transfer v_b to W_1. The iterations are shown in table 3. In each feasible

TABLE 3

Arcs	Capacity c_i	Feasible flows I	II	III	IV	V	VI
(a, 1)	3	(0)	(1)	3*	3*	3*	3*
(a, 2)	2	0	0	(0)	(1)	2*	2*
(a, 3)	1	0	0	0	0	(0)	1*
(1, 4)	1	(0)	1*	1*	1*	(1*)	0
(1, 5)	4	0	(0)	2	2	(2)	3
(1, 6)	2	0	0	0	0	0	0
(2, 4)	2	0	0	0	(0)	1	1
(2, 6)	1	0	0	(0)	1*	1*	1*
(3, 5)	1	(0)	1*	1*	1*	1*	1*
(3, 6)	1	0	0	0	(0)	1*	1*
(4, 3)	2	(0)	1	1	(1)	(2*)	1
(4, b)	0	0*	0*	0*	0*	0*	0*
(5, 2)	1	0	0	0	0	0	0
(5, b)	5	(0)	(1)	3	3	(3)	4
(6, b)	2	0	0	(0)	(1)	2*	2*
(b, a)		0	1	3	4	5	6

flow the numbers in ( ) indicate the chain along which it is possible to proceed to bring v_b into W_1. The asterisk indicates that the flow in the corresponding arc is equal to its capacity and cannot be further increased.

The change from flow V to flow VI deserves to be followed carefully. The chain in V the flow through which has been modified is $(v_a, v_3, v_4, v_1, v_5, v_b)$. We argue as follows. Starting with $W_1 = \{v_a\}$, v_3 can be transferred to W_1 by criterion (iiia). There is no unsaturated arc going out from v_3, both (v_3, v_6) and (v_3, v_5) carrying capacity flows. But (v_4, v_3) is an arc such that $v_3 \in W_1$, $v_4 \in W_2$, and the flow in it is 2 which is greater than zero. Hence, by criterion (iiib), v_4 is transferred to W_1. Again there is an arc (v_1, v_4) with $v_4 \in W_1$ and $v_1 \in W_2$ and with the flow in it greater than zero. So v_1 is also transferred to W_1 by criterion (iiib). This time there is an arc (v_1, v_5) with $v_1 \in W_1$, $v_5 \in W_2$ with flow 2 in it which is less than its capacity 4. Consequently, by criterion (iiia), v_5 is transferred to W_1, and finally, by the same criterion, v_b is transferred to W_1. So the flow is not optimal. In this chain arcs (v_4, v_3) and (v_1, v_4) occur in reverse directions. We reduce flows in them by 1 and increase flows in other arcs of the chain by 1 thereby saturating the arc (v_a, v_3).

The iterations stop at this stage because no matter how we try we cannot bring v_b into W_1. In fact we cannot even proceed one step from the initial position of W_1 containing only one point v_a. This is so because the arcs going out from v_a are all saturated and so neither v_1 nor v_2 nor v_3 can be brought in W_1. The maximum flow in the graph is 6.

We now proceed to prove the algorithm. We begin with a definition.

DEFINITION 1. *If in the graph $G(V, U)$ of the maximum flow problem, W_2 is a subset of V such that $v_b \in W_2$ $v_a \notin W_2$, then the set of arcs $\Omega^+(W_2)$ (arcs incident to W_2) is said to be a cut. The capacity of the cut is the sum of the capacities of the arcs contained in the cut.*

THEOREM 1. *For any feasible flow $\{x_i\}$, $i = 1, 2, \ldots, m$, in the graph, the flow x_0 in the return arc is not greater than the capacity of any cut in the graph.*

Proof. Let $\Omega^+(W_2)$ be any cut. Consider the flow in the arcs going to and going from W_2. The flow in should be equal to the flow out.
Therefore

$$\Sigma_1 x_i = x_0 + \Sigma_2 x_i,$$

where Σ_1 and Σ_2 respectively denote summations over the arcs going to and going from W_2 (except u_0). Since $x_i \geq 0$ for all i,

$$\Sigma_1 x_i \geq x_0.$$

Also $x_i \leq c_i$ for all i. Therefore

$$\Sigma_1 c_i \geq x_0.$$

where $\Sigma_1 c_i$ is the capacity of the cut $\Omega^+(W_2)$. .. Proved.

THEOREM 2. *The algorithm described earlier in this section solves the problem of the maximum flow.*

Proof. Suppose by the application of the algorithm a stage is reached when no vertex of W_2 can be transferred to W_1 by the prescribed procedure and $v_b \in W_2$. The set of arcs $\Omega^+(W_2)$ is a cut. Let $u_i \in \Omega^+(W_2)$. It means u_i is an arc (v_p, v_q) where $v_p \in W_1$, $v_q \in W_2$. The flow in this arc should be saturated, that is $x_i = c_i$, because if it were not so it would have been possible to transfer v_q from W_2 to W_1 by criterion (iiia), which is contrary to hypothesis.

Again, let $u_j \in \Omega^-(W_2), j \neq 0$. It means u_j is an arc (v_r, v_s) where $v_r \in W_2, v_s \in W_1$. The flow in this arc should be zero, because if it were not so, it would have been possible to transfer v_r from W_2 to W_1 by criterion (iiib), which again is contrary to hypothesis.

We conclude that the flow into W_2 is Σc_i, summation being over all $u_i \in \Omega^+(W_2)$, and the flow out of W_2 is only in the return arc u_0, because it is the only arc going from W_2 carrying a nonzero flow. Let the flow in u_0 be y_0. Then, since the flow in and out of W_2 should balance.

$$\Sigma c_i = y_0,$$

where Σc_i is the capacity of the cut obtained by the application of the algorithm. But from theorem 1, for any flow x_0 in u_0,

$$x_0 \leq \Sigma c_i,$$

where Σc_i is the capacity of any cut. It follows that

$$y_0 = \max x_0,$$

which means the algorithm leads to finding out the maximum flow. Proved.

THEOREM 3. *The maximum flow in a graph is equal to the minimum of the capacities of all possible cuts in it.*

Proof. By theorem 1, $x_0 \leq \Sigma c_i$,

Therefore $\max x_0 \leq \min \Sigma c_i$.

But we have seen in the course of the proof of theorem 2 that there is a cut corresponding to which the flow in u_0 is equal to cut capacity. Necessarily this flow should be maximum and the corresponding cut capacity should be the least of all cut capacities. Proved.

This theorem is generally known as the max-flow min-cut theorem.

8 Duality in the maximum flow problem

Theorems 1 and 3 above are basically the duality theorems 7 and 8 (chapter 3, section 18) of linear programming. It is therefore interesting to examine the dual of the maximum flow problem, particularly because the motivation for the algorithm for solving it has come from the dual.

To make the discussion easier to understand, we take a particular network (Fig. 17) with 5 vertices (including a source and a sink) and 8 arcs (including the return arc) rather than a general network with $n + 2$ vertices and $m + 1$ arcs.

Going back about another three quarters of a century, the Maxwell-Kirchoff theory of electric current distribution in networks had clarified the basic concepts governing flow and potential. But the electrical problem involved minimization of a quadratic function (heat generation) whereas the objective function in transportation problems is generally linear.

BIBLIOGRAPHICAL NOTE

(For references see bibliography)

Berge and Ghouila-Houri (1965) gives a good account of transportation network problems of different types with algorithms for solution and large number of utilitarian examples. The relevant basic ideas of graph theory are also introduced.

Ford and Fulkerson (1962) is an authoritative book on network flows and contains general mathematical theory as well as practically oriented concepts and problems.

Christofides (1975) deals with algorithmic and computational aspects of graph theory and presents the main techniques for the solution of important graph theory problems.

Murty (1976) and Bazaraa and Jarvis (1977) include good discussion of network problems.

PROBLEMS V

Minimum path

1. Explain why the algorithm for finding the minimum path when all arc lengths are non-negative, given in section 3, is not applicable to the general case when arc lengths can be negative also.

2. (i) Find the minimum path from v_1 to v_8 in the graph with arcs and arc lengths (i) given below. Solve the problem by both the algorithms given in section 3 and compare the numerical work involved.

   (ii) Find the minimum path from v_1 to v_8 in the same graph with arc lengths (ii).

   In the following table (i, j) denotes the arc (v_i, v_j).

Arc	(1,2)	(1,3)	(1,4)	(2,3)	(2,6)	(2,5)	(3,5)	(3,4)	(4,7)
Length (i)	1	4	11	2	8	7	3	7	3
Length (ii)	−1	4	−11	2	−8	7	−3	7	3
Arc	(5,6)	(5,8)	(6,3)	(6,4)	(6,7)	(6,8)	(7,3)	(7,8)	
Length (i)	1	12	4	2	6	10	2	2	
Length (ii)	1	12	4	2	6	−10	−2	2	

[(i) 15; (ii) − 22]

3. In each case of problem 2, is there a maximum path from v_1 to v_8? Explain with reasons. Identify circuits with positive lengths.

   [Maximum unbounded because of circuits with positive lengths]

4. Find the minimum spanning tree in the following undirected graph. Arc (v_i, v_j) is denoted as (i, j).

Arc	(1,2)	(1,3)	(1,4)	(2,3)	(2,8)	(2,10)	(3,4)	(3,8)	(4,5)	(4,6)
Length	7	4	8	3	9	14	4	10	15	12
Arc	(4,8)	(5,6)	(5,7)	(6,7)	(6,8)	(6,9)	(7,9)	(8,9)	(8,10)	(9,10)
Length	10	4	1	2	20	16	18	3	4	6

[42]

5. Five villages in a hilly region are to be connected by roads. The direct distance (in km) between each pair of villages along a possible road and its cost of construction per km (in 10^4 rupees) are given in the following table (distances are given in the upper triangle and costs in the lower triangle). Find the minimum cost at which all the villages can be connected, and the roads which should be constructed.

		Distances 1	2	3	4	5
	1		18	12	15	10
	2	3		15	8	22
Costs	3	4	3		6	20
	4	5	5	6		7
	5	2	2	5	7	

(Hint. Construct the spanning tree of minimum cost).

[Rs 140×10^4; (1, 5, 2, 4, 3)]

Project scheduling

6. A project consists of activities $A, B, C, ..., M$. In the following data $X - Y = c$ means Y can start after c days of work on X. A, B, C can start simultaneously. K and M are the last activities and take 14 and 13 days respectively. $A - D = 4, B - F = 6, B - E = 3, C - E = 4$. $D - H = 5$, $D - F = 3, E - F = 10, F - G = 4, G - I = 12, H - I = 3, H - J = 3, J - K = 8, I - K = 7, I - L = 7, L - M = 9$. Find the least time of completion of the project. If activities K and L both need a crane, and only one crane is available, how should the crane be used so that the project is completed with the least delay? [59; use crane first on L, resulting delay 1 day]

7. Tasks $A, B, C, ..., H, I$ constitute a project. The notation $X < Y$ means that the task X must be finished before Y can begin. With this notation

$$A < D,\ A < E,\ B < F,\ D < F,\ C < G,\ C < H,\ F < I,\ G < I.$$

The time (in days) of completion of each task in as follows.

Task	A	B	C	D	E	F	G	H	I
Time	8	10	7	9	16	7	8	14	9

Draw a graph to represent the sequence of tasks and find the minimum time of completion of the project. [33 days]

8. The project of problem 7 is required to be completed as early as possible. How soon can it be completed and at what additional minimum cost with the following data?

Task	A	B	C	D	E	F	G	H	I
Increase in cost for each day less	1	2	—	3	4	1	—	6	4
Minimum time	6	7	7	8	13	6	8	11	8

[28 days; cost 16]

Maximum flow

9. Find the maximum flow in the graph with the following arcs and arc capacities, flow in each arc being non-negative. Arc (v_j, v_k) is denoted as (j, k). v_a is the source and v_b the sink.

Arc	$(a,1)$	$(a,2)$	$(a,3)$	(1,4)	(1,5)	(1,6)	(2,4)	(2,5)	(2,6)
Capacity	2	2	2	1	1	1	1	1	1
Arc	(3,4)	(3,5)	(3,6)	$(4,b)$	$(5,b)$	$(6,b)$			
Capacity	1	1	1	2	2	2			

10. Find the maximum non-negative flow in the network described below, arc (v_j, v_k) being denoted as (j, k). v_a is the source and v_b the sink.

Arc	$(a,1)$	$(a,2)$	(1,2)	(1,3)	(1,4)	(2,4)	(3,2)	(3,4)	(4,3)	$(3,b)$	$(4,b)$
Capacity	8	10	3	4	2	8	3	4	2	10	9

[14]

11. Find the maximum flow in the network with the following data, flow in arcs not necessarily being non-negative. The arc (v_j, v_k) is denoted as (j, k) and the flow limit (b, c) means that the constraint on the flow x_i is $b_i \le x_i \le c_i$. v_a is the source and v_b the sink.

Arc	$(a,1)$	$(a,2)$	$(1,2)$	$(1,3)$	$(2,4)$	$(3,4)$	$(3,b)$	$(4,b)$
(b_i, c_i)	(0,10)	(0,5)	(−2,3)	(7,10)	(−3,5)	(−1,1)	(0,8)	(0,4)

[12]

12. Families $a_1, a_2, \ldots, a_m$ decide to go on a picnic in cars $b_1, b_2, \ldots, b_n$. The number of persons in family a_i is c_i, and the seating capacity of car b_j is k_j. Assuming that the total seating capacity is not less than the total number of persons, it is required to allot persons to cars such that in car b_j the number of persons from the same family should not exceed h_j. Formulate the problem as that of maximum flow, and solve it for the following data.

i	1	2	3	4	5
c_i	2	3	4	4	2

j	1	2	3
k_j	5	5	5
h_j	2	2	2

(Hint. Let a_0 be the source, b_0 the sink, (a_0, a_i) an arc with capacity c_i, (b_j, b_0) an arc with capacity k_j, (a_i, b_j) an arc with capacity h_j).

13. Convoys of army vehicles have to go from stations a_i, $i = 1, 2, 3, 4$, to b_j, $j = 1, 2, 3$, at night. The maximum number of vehicles leaving a_i or arriving at b_j is different for each station due to limited parking space, and is given in the following table. Each a_i is connected to each b_j by road. For secrecy reasons no convoy should consist of more than 15 vehicles.

Station	a_1	a_2	a_3	a_4	b_1	b_2	b_3
Parking capacity (no. of vehicles)	40	30	25	55	50	30	45

Find how the vehicles should be sent so that the total number of vehicles moved is maximum. Is the optimal solution unique? If not, find two alternatives.
(Hint. Let a_0 be a source connected to each a_i, and b_0 be a sink connected to each b_j, with capacity of arc (a_0, a_i) or (b_j, b_0) equal to parking capacity of a_i or b_j. Let the capacity of each arc (a_i, b_j) be 15. Find the maximum flow).

[Maximum vehicles 125, distribution not unique. See answer to problem 14]

14. Solve problem 13 if the convoy on each road should consist of not more than 15 and not less than 7 vehicles. Is there a solution to this problem if the least strength of each convoy is 8 vehicles?
[125 vehicles. (i) is the solution to this problem. Both (i) and (ii) are solutions of problem 13.

	(i)	(ii)		(i)	(ii)		(i)	(ii)
(a_1, b_1)	15	15	(a_1, b_2)	7	5	(a_1, b_3)	15	5
(a_2, b_1)	15	15	(a_2, b_2)	7	0	(a_2, b_3)	8	15
(a_3, b_1)	9	5	(a_3, b_2)	9	10	(a_3, b_3)	7	10
(a_4, b_1)	11	15	(a_4, b_2)	7	15	(a_4, b_3)	15	15]

15. Show that if $\{x_i\}$ and $\{y_i\}$ are two flows in a graph, then $\{ax_i + by_i\}$, where a and b are real constants, is also a flow.

16. Let ψ be a set of arcs forming a circuit in a graph $G(V, U)$, and let $\{x_i(\psi)\}$ be a set of numbers such that $x_i(\psi) = 1$ for $u_i \in \psi$ and $x_i(\psi) = 0$ for $u_i \notin \psi$, where $u_i \in U$. Show that $\{x_i(\psi)\}$ is a flow in G. (We call $\{x_i(\psi)\}$ the unit flow in the circuit ψ).

17. Let $\psi_1, \psi_2, \ldots, \psi_r$ be circuits in a graph G and let $\{x_i(\psi_j)\}$ be the unit flow in the circuit ψ_j, (see Problem 16). Show that a necessary and sufficient condition for $\{y_i\}$ to be a flow in G is that y_i is of the form

$$y_i = a_1 x_i(\psi_1) + a_2 x_i(\psi_2) + \ldots a_r x_i(\psi_r),$$

where $a_1, a_2, \ldots, a_r$ are non-negative real numbers.

12

Theory of Games

1 Introduction

In all the various types of optimization problems considered so far the assumption has been that there is a single decision maker whose interest lies in choosing the variables in such a way as to optimize the objective function, there being no conflict in deciding what the objective is. There are, however, situations in which there are two or more decision makers, each one making decisions (that is, choosing variables) to optimize *his* objective function which may be in conflict with the objectives of others. Trade and commerce, battles and wars, various types of games, and many other activities present situations in which different parties compete to achieve their own objective and prevent others from achieving theirs. Mathematical models of such situations and their solutions form the subject matter of the theory of games.

Game is defined as an activity between two or more persons involving *moves* by each person according to a set of rules, at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit).

The set of rules defines the game. Going through the set of rules once by the participants defines a *play*. There can be various types of games. They can be classified on the basis of the following characteristics.

(*i*) *Chance or strategy:* If in a game the moves are determined by chance, we call it a *game of chance*, if they are determined by skill, it is a *game of strategy*. In general a game may involve partly strategy and partly chance. We shall discuss the simplest models of games of strategy only.

(*ii*) *Number of persons:* A game is called an *n*-person game if the number of persons playing it is *n*. (Here 'person' means an individual or group aiming at one objective.)

(*iii*) *Number of moves:* These may be finite or infinite.

(*iv*) *Number of alternatives (or choices) available to each person per move:* These also may be finite or infinite.

A *finite game* has a finite number of moves, each involving a finite number of alternatives. Otherwise the game is infinite.

(*v*) *Information available to players of the past moves of other players:* The two extreme cases are, (a) no information at all, (b) complete information available. There can be cases in between in which information is partly available.

(*vi*) *Pay off:* It is a quantitative measure of the satisfaction a person gets at the end of the play. It is a real-valued function of the variables in the game.

Let p_i be the pay off to the person P_i, $i = 1, 2, ..., n$, in an n-person game. Then if $\sum_{i=1}^{n} p_i = 0$, the game is said to be a *zero-sum game*.

2 Matrix (or rectangular) games

A matrix game is a zero-sum two-person game with the following mathematical model. (The name 'rectangular' or 'matrix' has no other significance except that the game can be described in a rectangular matrix form).

The player P_1 has m choices i, $i = 1, 2, ..., m$, the rows of a matrix, while P_2 has n choices j, $j = 1, 2, ..., n$, the columns of the same matrix. The $m \times n$ matrix $\mathbf{A} = \{a_{ij}\}$ gives the *pay off* to P_1 for all possible combinations of the choices. Since it is a zero-sum game, the pay off to P_2 is the matrix $-\mathbf{A}$. Conventionally the pay off matrix to P_1, the player who chooses row-wise, is taken as the matrix of the game (table 1).

TABLE 1

	j / i	P_2: 1	2	...	n
P_1	1	a_{11}	a_{12}	...	a_{1n}
	2	a_{21}	a_{22}	...	a_{2n}
	—	—	—	—	—
	m	a_{m1}	a_{m2}	...	a_{mn}

The game is played as follows. P_1 chooses a value of i and P_2 choose a value of j without each knowing what the other has chosen. Then the choices are disclosed, and P_1 receives a_{ij} (or P_2 pays a_{ij}).

Here we shall discuss matrix games only.

3 Problem of game theory

To solve a mathematical model of a game is to investigate whether there is an optimal way to play it, that is, whether there exists any rational argument in favour of playing it one way or the other. Briefly, the problem is to discover, if any, the optimal strategy.

This is explained further in the following examples.

Example 1: Consider the following matrix game.

	j / i	P_2: 1	2	3	4
P_1	1	4	−2	−4	−1
	2	3	1	−1	2
	3	2	3	−2	−2
	4	−1	−3	−3	1
	5	−3	2	−2	−3

P_1 wishes to obtain the largest possible a_{ij} by choosing some i, $i = 1, 2, \ldots, 5$, while P_2 is determined to make P_1's gain the minimum possible by his choice of j, $j = 1, 2, 3, 4$. We shall call P_1 the maximizing player and P_2 the minimizing player. It would be rational for P_1 to argue as follows.

"If I choose $i = 1$, then it could happen that P_2 chooses $j = 3$ in which case I gain only -4. Similarly for my other choices $i = 2, 3, 4, 5$, P_2 can force me to get only $-1, -2, -3, -3$ respectively by his choice of j. Thus the best choice for me is to opt for $i = 2$, for this assures me at least the gain -1. In general, I should try to maximize my least gain, or find out

$$\max_i \min_j a_{ij}."$$

P_2 can argue similarly to keep P_1's gain the least. By his choosing $j = 1, 2, 3, 4$, P_1's gain can be respectively as high as 4, 3, -1, 2. So P_2 should settle for $j = 3$ because that would minimize P_1's gain. In general, he should find out

$$\min_j \max_i a_{ij}.$$

It turns out in the present problem that

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij}$$

and so the arguments of P_1 and P_2 lead to the same pay off. It may not always happen, as in the following example.

Example 2: Consider the following game.

	j \ i	P_2: 1	2	3
	1	2	-3	7
P_1	2	-7	4	-5
	3	5	-6	6

Arguing as in example 1, in this problem

$$\max_i \min_j a_{ij} = -3,$$

$$\min_j \max_i a_{ij} = 4.$$

The two are not equal. Notice that

$$\max_i \min_j a_{ij} < \min_j \max_i a_{ij}$$

If a matrix $\{a_{ij}\}$ is such that

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = a_{rs}$$

the matrix is said to have a *saddle point* at (r, s). In a game whose pay off matrix is of this type, the *optimal strategies* of players P_1 and P_2 are said to be $i = r$ and $j = s$ respectively, and a_{rs} is said to be the *value* of the game. Example 1 is of this type. But, in general, a matrix need not be of this type, as example 2 shows, and a saddle point as defined above may not exist. The above definitions of optimal strategy and value of the game are therefore not adequate to cover all cases and need to be generalized. The definition of a saddle point of a function of several variables and some theorems connected with it form the basis of such a generalization. We therefore first present these theorems.

4 Minimax theorem, saddle point

Let $f(\mathbf{X}, \mathbf{Y})$ be a real-valued function of two vectors $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X} \in E_n$, $\mathbf{Y} \in E_m$. Suppose this function is such that if $\mathbf{X}$ is kept fixed at some value and $\mathbf{Y}$ is varied, then $f(\mathbf{X}, \mathbf{Y})$ has a minimum for some value of $\mathbf{Y}$. We denote this value by

$$\phi = \min_{\mathbf{Y}} f(\mathbf{X}, \mathbf{Y}).$$

If we give to $\mathbf{X}$ some other fixed value, we may find another value of ϕ. Thus for different values of $\mathbf{X}$ we can obtain values of ϕ, assuming that ϕ exists in every case. This means that ϕ is a function of $\mathbf{X}$ and we may write

$$\phi(\mathbf{X}) = \min_{\mathbf{Y}} f(\mathbf{X}, \mathbf{Y}).$$

Let us now suppose that $\phi(\mathbf{X})$ has a maximum for some value of $\mathbf{X}$. We may write it as

$$\max_{\mathbf{X}} \phi(\mathbf{X}) = \max_{\mathbf{X}} \min_{\mathbf{Y}} f(\mathbf{X}, \mathbf{Y}).$$

Similarly the expression

$$\min_{\mathbf{Y}} \max_{\mathbf{X}} f(\mathbf{X}, \mathbf{Y})$$

is interpreted. Here we first find a maximum of $f(\mathbf{X}, \mathbf{Y})$ with respect to $\mathbf{X}$ keeping $\mathbf{Y}$ fixed, and then find the minimum of the function so obtained with respect to $\mathbf{Y}$.

THEOREM 1. *Let* $f(\mathbf{X}, \mathbf{Y})$ *be such that both* $\max_{\mathbf{X}} \min_{\mathbf{Y}} f(\mathbf{X}, \mathbf{Y})$ *and* $\min_{\mathbf{Y}} \max_{\mathbf{X}} f(\mathbf{X}, \mathbf{Y})$ *exist. Then*

$$\max_{\mathbf{X}} \min_{\mathbf{Y}} f(\mathbf{X}, \mathbf{Y}) \le \min_{\mathbf{Y}} \max_{\mathbf{X}} f(\mathbf{X}, \mathbf{Y}). \tag{1}$$

Proof. Let $\mathbf{X}_0$ and $\mathbf{Y}_0$ be some arbitrarily chosen points in E_n and E_m respectively. Then

$$\min_{\mathbf{Y}} f(\mathbf{X}_0, \mathbf{Y}) \le f(\mathbf{X}_0, \mathbf{Y}_0)$$

and

$$\max_{\mathbf{X}} f(\mathbf{X}, \mathbf{Y}_0) \ge f(\mathbf{X}_0, \mathbf{Y}_0).$$

Hence

$$\min_{\mathbf{Y}} f(\mathbf{X}_0, \mathbf{Y}) \le \max_{\mathbf{X}} f(\mathbf{X}, \mathbf{Y}_0).$$

But $\mathbf{Y}_0$ is arbitrarily chosen and could have been any point in E_m, and for every one of them the inequality should hold. Even if we had chosen $\mathbf{Y}_0$ to be that point for which $\max_{\mathbf{X}} f(\mathbf{X}, \mathbf{Y})$ has the least value, the inequality shall be true. So

$$\min_{\mathbf{Y}} f(\mathbf{X}_0, \mathbf{Y}) \le \min_{\mathbf{Y}} \max_{\mathbf{X}} f(\mathbf{X}, \mathbf{Y}).$$

Also since $\mathbf{X}_0$ is any point in E_n, the inequality will hold even if we choose that $\mathbf{X}_0$ which makes $\min_{\mathbf{Y}} f(\mathbf{X}, \mathbf{Y})$ maximum. Therefore

$$\max_X \min_Y f(X, Y) \le \min_Y \max_X f(X, Y).$$ Proved.

COROLLARY 1. *Let* $\{a_{ij}\}$ *be an* $m \times n$ *matrix. Then*

$$\max_i \min_j a_{ij} \le \min_j \max_i a_{ij}. \tag{2}$$

We have only to regard $\{a_{ij}\}$ as a real-valued function $f(i, j) = a_{ij}$ of two variables i and j where $i = 1, 2, ..., m$, and $j = 1, 2, ..., n$, and (2) follows immediately from (1).

We defined the saddle point of a function in chapter 2, section 7. Because of its importance in the present context we state the definition again.

DEFINITION 1. *A point* (X_0, Y_0), $X_0 \in E_n$, $Y_0 \in E_m$, *is said to be a saddle point of* $f(X, Y)$ if

$$f(X, Y_0) \le f(X_0, Y_0) \le f(X_0, Y). \tag{3}$$

We now prove the following theorem on the existence of a saddle point.

THEOREM 2. *Let* $f(X, Y)$ *be such that both* $\max_X \min_Y f(X, Y)$ *and* $\min_Y \max_X f(X, Y)$ *exist. Then the necessary and sufficient condition for the existence of a saddle point* (X_0, Y_0) of $f(X, Y)$ *is that*

$$f(X_0, Y_0) = \max_X \min_Y f(X, Y) = \min_Y \max_X f(X, Y). \tag{4}$$

Proof. (i) To prove that the condition is necessary, let (X_0, Y_0) be a saddle point such that (3) holds. Since

$$f(X, Y_0) \le f(X_0, Y_0) \text{ for all } X \in E_n,$$

$$\max_X f(X, Y_0) \le f(X_0, Y_0).$$

But $$\min_Y [\max_X f(X, Y)] \le \max_X f(X, Y_0).$$

Therefore $$\min_Y \max_X f(X, Y) \le f(X_0, Y_0).$$

Again, from (3), since

$$f(X_0, Y_0) \le f(X_0, Y) \text{ for all } Y \in E_m,$$

$$f(X_0, Y_0) \le \min_Y f(X_0, Y).$$

But $$\min_Y f(X_0, Y) \le \max_X [\min_Y f(X, Y)].$$

Hence $$f(X_0, Y_0) \le \max_X \min_Y f(X, Y).$$

Thus we find that

$$\min_Y \max_X f(X, Y) \le f(X_0, Y_0) \le \max_X \min_Y f(X, Y).$$

But from theorem 1,

$$\min_Y \max_X f(X, Y) \ge \max_X \min_Y f(X, Y).$$

The only conclusion from the above two statements is that

$$\max_{\mathbf{X}} \min_{\mathbf{Y}} f(\mathbf{X},\mathbf{Y}) = \min_{\mathbf{Y}} \max_{\mathbf{X}} f(\mathbf{X},\mathbf{Y}) = f(\mathbf{X}_0,\mathbf{Y}_0).$$

(ii) To prove that the condition is sufficient, let (4) be true. Let the maximum of $\min_{\mathbf{Y}} f(\mathbf{X},\mathbf{Y})$ occur at $\mathbf{X}_0$, and the minimum of $\max_{\mathbf{X}} f(\mathbf{X},\mathbf{Y})$ occur at $\mathbf{Y}_0$. Then, from (4)

$$\min_{\mathbf{Y}} f(\mathbf{X}_0,\mathbf{Y}) = \max_{\mathbf{X}} f(\mathbf{X},\mathbf{Y}_0). \tag{5}$$

But by definition of minimum,

$$\min_{\mathbf{Y}} f(\mathbf{X}_0,\mathbf{Y}) \le f(\mathbf{X}_0,\mathbf{Y}_0),$$

and so from (5),

$$\max_{\mathbf{X}} f(\mathbf{X},\mathbf{Y}_0) \le f(\mathbf{X}_0,\mathbf{Y}_0),$$

which means that

$$f(\mathbf{X},\mathbf{Y}_0) \le f(\mathbf{X}_0,\mathbf{Y}_0) \text{ for all } \mathbf{X}.$$

Also, by definition of maximum,

$$\max_{\mathbf{X}} f(\mathbf{X},\mathbf{Y}_0) \ge f(\mathbf{X}_0,\mathbf{Y}_0),$$

and so, again from (5),

$$\min_{\mathbf{Y}} f(\mathbf{X}_0,\mathbf{Y}) \ge f(\mathbf{X}_0,\mathbf{Y}_0),$$

which means

$$f(\mathbf{X}_0,\mathbf{Y}) \ge f(\mathbf{X}_0,\mathbf{Y}_0) \text{ for all } \mathbf{Y}.$$

Thus we find that

$$f(\mathbf{X},\mathbf{Y}_0) \le f(\mathbf{X}_0,\mathbf{Y}_0) \le f(\mathbf{X}_0,\mathbf{Y})$$

which, by definition, means that $(\mathbf{X}_0, \mathbf{Y}_0)$ is a saddle point of $f(\mathbf{X}, \mathbf{Y})$.

Proved.

COROLLARY 2. *Let $\{a_{ij}\}$ be an $m \times n$ matrix. Then the necessary and sufficient condition that $\{a_{ij}\}$ has a saddle point at $i = r, j = s$ is that*

$$a_{rs} = \max_{i} \min_{j} a_{ij} = \min_{j} \max_{i} a_{ij}. \tag{6}$$

As in corollary 1, regarding $\{a_{ij}\}$ as a real-valued function of two variables i and j, (6) follows immediately from (4).

5 Strategies and pay off

As was mentioned at the end of section 3, in a matrix game, if the pay off matrix $\{a_{ij}\}$ has a saddle point (r, s), then $i = r, j = s$ are the optimal strategies of the game and the pay off a_{rs} is called the value of the game. If the matrix has no saddle point, the game has no optimal strategies in the above sense. By introducing probability with choice and mathematical expectation with pay off, the concept of optimal strategy can be extended to apply to all matrices. This we proceed to do.

Let P_1 choose a particular i, $i = 1, 2, \ldots, m$, with probability x_i. We may also interpret it as the relative frequency with which P_1 chooses i in a large number of plays of the game. The probabilities x_i, $i = 1, 2, \ldots, m$, constitute the strategy of P_1. Similarly if P_2 chooses a particular j with probability y_j, the probabilities y_j, $j = 1, 2, \ldots, n$, are the strategy of P_2.

DEFINITION 2. *The vector* $\mathbf{X} = \{x_i\}$ *of nonnegative numbers* x_i, *such that* $\sum_{i=1}^{m} x_i = 1$, *is defined as the mixed strategy of* P_1. *Similarly the vector* $\mathbf{Y} = \{y_j\}$ *of nonnegative numbers* y_j, *such that* $\sum_{j=1}^{n} y_j = 1$, *defines the mixed strategy of* P_2.

For the sake of brevity we define S_m as the set of ordered m-tuples of nonnegative numbers whose sum is unity, and say that $\mathbf{X} \in S_m$. Similarly $\mathbf{Y} \in S_n$. Unless otherwise mentioned it will be assumed throughout this chapter that $\mathbf{X} \in S_m$ and $\mathbf{Y} \in S_n$, where $\mathbf{X}$ and $\mathbf{Y}$ are mixed strategies of P_1 and P_2 respectively.

DEFINITION 3. *The mixed strategy* $\mathbf{X} = \xi_i$ *whose ith component is unity and all other components are equal to zero is called a pure strategy of* P_1. *Similarly* $\mathbf{Y} = \eta_j$, *where all the components of* $\mathbf{Y}$ *except the jth are zero, is called a pure strategy of* P_2.

DEFINITION 4. *The mathematical expectation or the payoff function* $E(\mathbf{X}, \mathbf{Y})$ *in the game whose payoff matrix is* $\mathbf{A} = \{a_{ij}\}$ *is defined as*

$$E(\mathbf{X},\mathbf{Y}) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j = \mathbf{X}'\mathbf{A}\mathbf{Y}$$

where $\mathbf{X}$ *and* $\mathbf{Y}$ *are the mixed strategies of* P_1 *and* P_2.

Following the argument given in section 3, it is reasonable to postulate that P_1 should choose $\mathbf{X}$ so as to maximize his least expectation and P_2 should choose $\mathbf{Y}$ so as to minimize P_1's greatest expectation. Thus P_1 aims at $\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y})$ and P_2 aims at $\min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y})$.

DEFINITION 5. *If* $\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) = \min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y}) = E(\mathbf{X}_0,\mathbf{Y}_0)$, *then* $(\mathbf{X}_0, \mathbf{Y}_0)$ *is defined as the strategic saddle point of the game,* $\mathbf{X}_0$ *and* $\mathbf{Y}_0$ *are defined as the optimal strategies, and* $v = E(\mathbf{X}_0, \mathbf{Y}_0)$ *is the value of the game.*

According to a theorem, known as the fundamental theorem of the theory of rectangular games, a strategic saddle point always exists. Before turning to the theoretical aspects let us consider the following example.

Example: Consider the following matrix game

		P_2	
	$i \backslash j$	1	2
P_1	1	5	1
	2	3	4

The above matrix is without a saddle point, as

$$\max_i \min_j a_{ij} = 3 \neq \min_j \max_i a_{ij} = 4.$$

Let the mixed strategies of P_1 and P_2 be $\mathbf{X} = [x_1 \ \ x_2]$ and $\mathbf{Y} = [y_1 \ \ y_2]$. Then

$$E(\mathbf{X},\mathbf{Y}) = 5x_1y_1 + 3x_2y_1 + x_1y_2 + 4x_2y_2,$$

where

$$x_1 + x_2 = 1, y_1 + y_2 = 1.$$

Eliminating x_2, y_2, we get

$$E(\mathbf{X},\mathbf{Y}) = 5x_1y_1 - 3x_1 - y_1 + 4$$

$$= 5(x_1 - \frac{1}{5})(y_1 - \frac{3}{5}) + \frac{17}{5}.$$

If P_1 chooses $x_1 = 1/5$, he ensures that his expectation is at least 17/5. He cannot be sure of more than 17/5, because by choosing $y_1 = 3/5$, P_2 can keep $E(\mathbf{X}, \mathbf{Y})$ down to 17/5. So P_1 might as well settle for 17/5 and play $\mathbf{X}_0 = [1/5, 4/5]$, and P_2 reconcile to $-$ 17/5 and play $\mathbf{Y}_0 = [3/5, 2/5]$. These are the optimal strategies for P_1 and P_2. The value of the game is 17/5, and $(\mathbf{X}_0, \mathbf{Y}_0)$ is a saddle point of $E(\mathbf{X}, \mathbf{Y})$.

6 Theorems of matrix games

We begin with a theorem which is required in the proof of the fundamental theorem of games.

THEOREM 3. *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix, and let* $\mathbf{P}_j$ *and* $\mathbf{Q}_i$, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, *be its column and row vectors respectively. Then either (i) there exists a* $\mathbf{Y}$ *in* S_n *such that* $\mathbf{Q}_i\mathbf{Y} \leq 0$ *for all i, or (ii) there exists an* $\mathbf{X}$ *in* S_m *such that* $\mathbf{X}'\,\mathbf{P}_j > 0$ *for all j.*

Proof. Let $\xi_i \in S_m$ be a vector such that its *i*th component is unity and all other components are zero. Consider the $m + n$ points

$$\xi_1, \xi_2, \ldots, \xi_m, \mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$$

belonging to E_m. Let C be the convex hull of the $m + n$ points. Then the origin $\mathbf{0}$ of E_m is either in C or not in C. We consider the two case separately.

(i) Let $\mathbf{0}$ be in C. Then $\mathbf{0}$ can be expressed as a convex linear combination of the $m + n$ points which span C (chapter 1, section 15). Hence there exist

$$[\lambda_1, \lambda_2, \ldots, \lambda_m, \mu_1, \mu_2, \ldots, \mu_n] \in S_{m+n} \tag{7}$$

such that

$$\sum_{i=1}^{m} \lambda_i \xi_i + \sum_{j=1}^{n} \mu_j \mathbf{P}_j = \mathbf{0},$$

or

$$\lambda_i + \sum_{j=1}^{n} \mu_j a_{ij} = 0, i = 1, 2, \ldots, m. \tag{8}$$

Since $\lambda_i \geq 0$,

$$\sum_{j=1}^{n} \mu_j a_{ij} \leq 0, i = 1, 2, \ldots, m. \tag{9}$$

Also

$$\sum_{j=1}^{n} \mu_j > 0,$$

for, if it is equal to zero, each μ_j should be zero, which from (8) would mean that each λ_i should also be zero. This would contradict hypothesis (7), and therefore is not possible. Dividing (9) by $\sum_{j=1}^{n} \mu_j$ we get

$$\left(\sum_{j=1}^{n} \mu_j a_{ij}\right) / \sum_{j=1}^{n} \mu_j \leq 0.$$

Putting

$$y_j = \mu_j / \sum_{j=1}^{n} \mu_j$$

we get

$$\sum_{j=1}^{n} y_j a_{ij} \leq 0,$$

or

$$\mathbf{Q}_i \mathbf{Y} \leq 0, \; i = 1, 2, \ldots, m \tag{10}$$

This proves alternative (i) of the theorem.

(ii) Let $\mathbf{0} \notin C$. Then by the theorem on separating hyperplanes (theorem 18, chapter 1) there exists a hyperplane containing $\mathbf{0}$, say $\mathbf{BZ} = 0$, such that C is contained in the halfspace $\mathbf{BZ} > 0$. In particular, since $\xi_i \in C$,

$$\mathbf{B}\xi_i > 0,$$

or

$$b_i > 0, i = 1, 2, \ldots, m,$$

and therefore

$$\sum_{i=1}^{m} b_i > 0,$$

where b_i is the ith component of $\mathbf{B}$. Also $\mathbf{P}_j \in C$ and so

$$\mathbf{BP}_j > 0, j = 1, 2, \ldots, n,$$

or

$$\sum_{i=1}^{m} b_i a_{ij} > 0.$$

Dividing by $\sum_{i=1}^{m} b_i$ and putting $x_i = b_i / \sum_{i=1}^{m} b_i$, we get

$$\sum_{i=1}^{m} x_i a_{ij} > 0,$$

or

$$\mathbf{X'P}_j > 0, j = 1, 2, \ldots, n, \tag{11}$$

which proves alternative (ii) of the theorem.

We now state and prove the fundamental theorem of rectangular games.

THEOREM 4. *For an* $m \times n$ *matrix game both* $\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X}, \mathbf{Y})$ *and* $\min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X}, \mathbf{Y})$ *exist and are equal.*

Proof. $E(\mathbf{X}, \mathbf{Y})$ is a continuous linear function of $\mathbf{X}$ defined over the closed and bounded subset S_m of E_m for each $\mathbf{Y}$ in S_n. Therefore $\max_{\mathbf{X}} E(\mathbf{X}, \mathbf{Y})$ exists and is a continuous function of $\mathbf{Y}$. Since S_n is also closed and bounded, $\min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X}, \mathbf{Y})$ exists. Similarly we prove that $\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X}, \mathbf{Y})$ also exists.

From theorem 3 either (10) or (11) holds. Let (11) hold. Then multiplying (11) by the component y_j of $\mathbf{Y}$ and summing for all j, we get

$$E(\mathbf{X},\mathbf{Y}) = \sum_{j=1}^{n} \sum_{i=1}^{m} x_i a_{ij} y_j > 0$$

for all $\mathbf{Y}$. Hence
$$\min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) > 0,$$

and consequently
$$\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) > 0.$$

If, on the other hand, (10) holds, then by a similar argument we conclude that

$$\min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y}) \le 0$$

At least one of the above two inequalities must hold, and so

$$\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) < 0 < \min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y}) \text{ is not true.} \qquad (12)$$

Let $\mathbf{A}_k$ be the matrix $\{a_{ij}-k\}$ formed by subtracting k from eac[illegible]ments of $\mathbf{A}$, and let its expectation function be $E_k(\mathbf{X}, \mathbf{Y})$. Then

$$E_k(\mathbf{X},\mathbf{Y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i(a_{ij}-k)y_j,$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j - k \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j,$$

$$= E(\mathbf{X},\mathbf{Y}) - k. \qquad (13)$$

Since $\mathbf{A}$ is any matrix, what is true for $\mathbf{A}$ is true for $\mathbf{A}_k$. Therefore, from (12),

$$\max_{\mathbf{X}} \min_{\mathbf{Y}} E_k(\mathbf{X},\mathbf{Y}) < 0 < \min_{\mathbf{Y}} \max_{\mathbf{X}} E_k(\mathbf{X},\mathbf{Y}) \text{ is not true,}$$

or using (13),
$$\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) < k < \min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y}) \text{ is also not true} \qquad (14)$$

for any value of k. The only conclusion from (12) and (14) is that

$$\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) < \min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y}) \text{ is false.}$$

Therefore

$$\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) \ge \min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y}).$$

But from theorem 1

$$\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) \le \min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y}).$$

Hence

$$\max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) = \min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y}). \tag{15}$$

Proved.

By theorem 2, (15) is a necessary and sufficient condition for a point $(\mathbf{X}_0, \mathbf{Y}_0)$, $\mathbf{X}_0 \in S_m$, $\mathbf{Y}_0 \in S_n$, to exist such that

$$E(\mathbf{X}_0,\mathbf{Y}_0) = \max_{\mathbf{X}} \min_{\mathbf{Y}} E(\mathbf{X},\mathbf{Y}) = \min_{\mathbf{Y}} \max_{\mathbf{X}} E(\mathbf{X},\mathbf{Y}),$$

and

$$E(\mathbf{X},\mathbf{Y}_0) \le E(\mathbf{X}_0,\mathbf{Y}_0) \le E(\mathbf{X}_0,\mathbf{Y}), \tag{16}$$

for all $\mathbf{X} \in S_m$, $\mathbf{Y} \in S_n$.

By definition 5, $(\mathbf{X}_0, \mathbf{Y}_0)$ is a strategic saddle point, $E(\mathbf{X}_0, \mathbf{Y}_0)$ is the value of the game and $\mathbf{X}_0$, $\mathbf{Y}_0$ are the optimal strategies.

We thus conclude that every matrix game has a value and an optimal strategy for each player.

THEOREM 5. *Condition* (16) *is equivalent to*

$$E(\xi_i,\mathbf{Y}_0) \le E(\mathbf{X}_0,\mathbf{Y}_0) \le E(\mathbf{X}_0,\eta_j) \tag{17}$$

where ξ_i, $i = 1, 2, \ldots, m$ and η_j, $j = 1, 2, \ldots, n$, *are the pure strategies.*

Proof. To prove the equivalence of (16) and (17) we have to prove that (17) is a necessary and sufficient condition for the existence of (16).

That the condition is necessary is obvious. For, (16) holds for all $\mathbf{X} \in S_m$ and $\mathbf{Y} \in S_n$, and ξ_i and η_j are in S_m and S_n respectively.

To prove that (17) is sufficient for (16), we notice that

$$E(\xi_i,\mathbf{Y}) = \sum_{j=1}^{n} a_{ij} y_j,$$

because the ith component of ξ_i is unity and all the other components are zero. Similarly

$$E(\mathbf{X},\eta_j) = \sum_{i=1}^{m} x_i a_{ij}.$$

Hence

$$\sum_{i=1}^{m} E(\xi_i,\mathbf{Y}) x_i = E(\mathbf{X},\mathbf{Y}),$$

and

$$\sum_{j=1}^{n} E(\mathbf{X},\eta_j) y_j = E(\mathbf{X},\mathbf{Y}).$$

Now let (17) be true, that is, let

$$E(\xi_i,\mathbf{Y}_0) \le E(\mathbf{X}_0,\mathbf{Y}_0) \le E(\mathbf{X}_0,\eta_j)$$

Then

$$E(\xi_i,\mathbf{Y}_0) x_i \le E(\mathbf{X}_0,\mathbf{Y}_0) x_i, E(\mathbf{X}_0,\mathbf{Y}_0) y_j \le E(\mathbf{X}_0,\eta_j) y_j$$

$$\Rightarrow \sum_{i=1}^{m} E(\xi_i,\mathbf{Y}_0) x_i \le \sum_{i=1}^{m} E(\mathbf{X}_0,\mathbf{Y}_0) x_i, \; \sum_{j=1}^{n} E(\mathbf{X}_0,\mathbf{Y}_0) y_j \le \sum_{j=1}^{n} E(\mathbf{X}_0,\eta_j) y_j$$

$$\Rightarrow E(\mathbf{X},\mathbf{Y}_0) \le E(\mathbf{X}_0,\mathbf{Y}_0) \le E(\mathbf{X}_0,\mathbf{Y}),$$

since

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1.$$

Proved.

The optimal strategy of P_2 is thus $\mathbf{Y}_0 = [0 \quad 11/16 \quad 0 \quad 5/16]$.

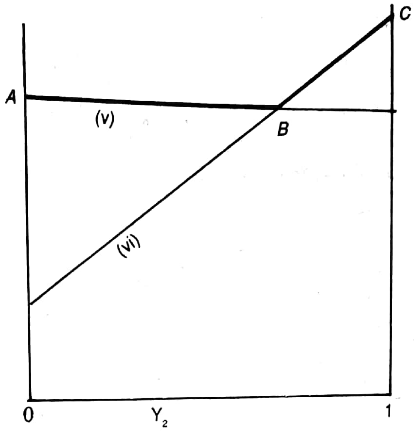


Fig. 2

It can be verified that $E(\mathbf{X}_0, \mathbf{Y}_0) = 245/16$. This completes the solution of the problem.

To solve an $m \times 2$ game graphically, piecewise linear curve representing the maximum expectation for any y_1 should be drawn and the point where it attains its minimum would give the optimal strategy of P_2 and the value of the game. The optimal strategy of P_1 can be found as above.

8 Notion of dominance

Sometimes a row or a column in the payoff matrix of a game is obviously ineffective in influencing the optimal strategies and the value of the game. For example, consider the game

		P_2			
	$i \backslash j$	1	2	3	4
	1	4	−8	7	−2
P_1	2	3	−9	2	−3
	3	−2	6	8	2

Notice the rows 1 and 2. For every j, $a_{1j} > a_{2j}$. Whatever the choice of P_2, P_1 will do better by choosing $i = 1$ rather than $i = 2$. The second row therefore should not play any part in the strategy of P_1, or, in other words, the probability associated with it should be zero. The solution of the above game would be the same as that of the game with the payoff matrix

$$\begin{bmatrix} 4 & -8 & 7 & -2 \\ -2 & 6 & 8 & 2 \end{bmatrix}$$

The problem is thus simplified. As another example, in the following matrix, the first column does not play any part in deciding the strategy of P_2 and so may be left out of consideration.

$$\begin{bmatrix} 3 & 1 & -1 \\ -1 & -2 & 3 \\ 4 & 3 & -3 \end{bmatrix}$$

9 Rectangular game as an LP problem

It can be shown that the problem of solving a rectangular game is equivalent to solving a problem of linear programming. This provides one of the methods of solving a matrix game problem. Since the simplex method of solving an LP problem has been described in chapter 3, we shall only explain here how to convert the problem of a matrix game into an equivalent LP problem.

Let the $m \times n$ matrix $\mathbf{A} = \{a_{ij}\}$ be the payoff matrix of the game and v its value. v is a real number. By increasing every a_{ij} by a suitable positive number k, we may form a matrix $\mathbf{A}_k = \{a_{ij} + k\} = \{a_{ij}'\}$ where every $a_{ij}' > 0$. The expectation function $E_k(\mathbf{X}, \mathbf{Y})$ of the game with payoff matrix $\mathbf{A}_k$ is given by equation (13) as

$$E_k(\mathbf{X},\mathbf{Y}) = E(\mathbf{X},\mathbf{Y}) + k.$$

By such a transformation the optimal strategies of the game do not change, but the value of the game is increased by k, and it is ensured that this new value is positive.

Let us assume that, if necessary, after this transformation, the matrix of the game is $\mathbf{A} = \{a_{ij}\}$, where $a_{ij} > 0$ for all i and j, and the value of the game is $v > 0$. Let $\mathbf{X}_0 = [x_1\ x_2\ \ldots\ x_m]$ and $\mathbf{Y}_0 = [y_1\ y_2\ \ldots\ y_n]$ be optimal strategies of P_1 and P_2 respectively. Then, from (17), for all j,

$$\left.\begin{aligned} & E(\mathbf{X}_0, \eta_j) \geq E(\mathbf{X}_0, \mathbf{Y}_0) = v, \\ \text{or} \quad & \sum_{i=1}^{m} a_{ij} x_i \geq v,\ j = 1,2,\ldots,n, \\ \text{subject to} \quad & \sum_{i=1}^{m} x_i = 1, \\ \text{and} \quad & x_i \geq 0, i = 1,2,\ldots,m. \end{aligned}\right\} \quad (19)$$

Since $v > 0$, dividing (19) throughout by v, we get

$$\sum_{i=1}^{m} a_{ij} x_i' \geq 1, \quad j = 1,2,\ldots,n,$$

$$\text{subject to} \quad \sum_{i=1}^{m} x_i' = 1/v, x_i' \geq 0.$$

The strategy of P_1 is to maximize v. Therefore he has to choose x_i', such that

$$\left.\begin{aligned} & f = \sum_{i=1}^{m} x_i' \text{ is minimum} \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} x_i' \geq 1, \quad j = 1,2,\ldots,n, \\ & x_i' \geq 0, \qquad i = 1,2,\ldots,m. \end{aligned}\right\} \tag{20}$$

This is an LP problem put in the standard primal form. The value of the game is $v = 1/f_{\min}$, and the optimal strategy of P_1 is $\{x_i\} = \{x_i'v\}$ where $\{x_i'\}$ is the optimal solution of the LP problem.

If we start from the inequality

$$E(\xi_i, Y_0) \leq E(X_0, Y_0) = v$$

of (17), we shall get the LP problem as

$$\left.\begin{aligned} & \phi = \sum_{j=1}^{n} y_j' \text{ is maximum} \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} y_j' \leq 1, \quad i = 1,2,\ldots,m, \\ & y_j' \geq 0, \qquad j = 1,2,\ldots,n, \end{aligned}\right\} \tag{21}$$

which is the dual of (20). One may either solve the primal or the dual to get the solution of the game.

HISTORICAL NOTE

Games of chance have been studied for a long time; in fact the theory of probabilitiy had its origins in this study. The first attempt to formulate a mathematical theory of games of strategy was made by Emile Borel in 1921. John von Neumann gave a sound foundation to the theory in 1928 when he proved the minimax theorem, the fundamental theorem in game theory. In 1944 Neumann and Morgenstern published the "Theory of Games and Economic Behaviour". Since then the subject has received much attention and has been applied to competitive situations in diverse fields including economics, politics and military.

BIBLIOGRAPHICAL NOTE

(For references see bibliography)

Mckinsey (1952) and Dresher (1961) are comprehensive introductions to games of strategy; the former confines more or less to theory while the latter includes applications also. For economists Karlin (1959) may be of greater interest and benefit. Chapters on game theory are often included in books on operations research.

PROBLEMS XII

1. Examine the following payoff matrices for saddle points. In case the saddle point exists, find the optimal strategies and value of the game. In every case verify that

$$\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}$$

(i) $\begin{bmatrix} 1 & 3 \\ -2 & 10 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 & -2 \\ 1 & 0 & 1 \\ -2 & -1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}$ (vii) $\begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix}$

2. Solve the games with the following payoff matrices

(i) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

3. Solve graphically the games whose payoff matrices are the following.

(i) $\begin{bmatrix} 2 & 7 \\ 3 & 5 \\ 11 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$

4. Use the notion of dominance to simplify the following payoff matrices and then solve the game.

(i) $\begin{bmatrix} 0 & 5 & -4 \\ 3 & 9 & -6 \\ 3 & -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 2 & 1 \\ 4 & 3 & 1 & 3 & 2 \\ 4 & 3 & 4 & -1 & 2 \end{bmatrix}$

5. Write both the primal and the dual LP problems corresponding to the rectangular games with the following payoff matrices. Solve the game by solving the LP problem by simplex method

(i) $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix}$

6. Show that an alternative formulation of an LP problem equivalent to the problem of strategy of P_1 in a rectangular game with payoff matrix

$$A = \{a_{ij}\}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \text{ is:}$$

Maximize $$v = \sum_{i=1}^{m} a_{in}x_i - x_{m+n}$$

subject to $$\sum_{i=1}^{m} (a_{ij} - a_{in})x_i - x_{m+j} + x_{m+n} = 0, \; j = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = 1, \quad x_i \geq 0.$$

Write the corresponding LP problem for the strategy of P_2 and show that it is the dual of the above.

7. Following the formulation suggested in the above problem, formulate the primal and dual LP problems equivalent to the matrix games of problems 3 and 5 above, and solve them by the simplex method.