In the following theorems it is assumed that the primal and dual problems are in the form given in definition 4.

THEOREM 6. *The dual of the dual is the primal.*

Proof. The dual (46), (47), (48) written in the primal form is:

Minimize $$-\phi(Y) = -B'Y,$$

subject to $$-A'Y \geq -C', \; Y \geq 0.$$

Its dual, according to definition, is:

Maximize $$\psi(X) = -CX,$$

subject to $$-AX \leq -B, \; X \geq 0;$$

which may also be written as:

Minimize $$f(X) = -\psi(X) = CX,$$

subject to $$AX \geq B, \; X \geq 0.$$

This is the primal (43), (44), (45). Proved.

THEOREM 7. *The value of the objective function $f(X)$ for any feasible solution of the primal is not less than the value of the objective function $\phi(Y)$ for any feasible solution of the dual.*

Proof. Let us introduce the necessary slack variables in the primal (43), (44), (45) and the dual (46), (47), (48). We obtain

Primal: Minimize $f(X) = c_1x_1 + c_2x_2 + \ldots + c_nx_n,$

subject to

$$a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n - x_{n+1} = b_1,$$

$$a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n \quad - x_{n+2} = b_2,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 \ldots + a_{mn}x_n \quad - x_{n+m} = b_m,$$

$$x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m} \geq 0.$$

Dual: Maximize $\phi(Y) = b_1y_1 + b_2y_2 + \ldots + b_my_m,$

subject to

$$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m + y_{m+1} = c_1,$$

$$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \quad + y_{m+2} = c_2,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \quad + y_{m+n} = c_n,$$

$$y_1, y_2, \ldots, y_m, y_{m+1} \ldots, y_{m+n} \geq 0.$$

Let $x_1, x_2, \ldots, x_{n+m}$, and $y_1, y_2, \ldots, y_{m+n}$ be any feasible solutions of the primal and the dual respectively. Multiply the primal constraints by $y_1, y_2, \ldots, y_m$ respectively and add, also multiply the dual constraints by $x_1, x_2, \ldots, x_n$ respectively and add. Thus we obtain two equations. Subtracting one from the other we get

$$f - \phi = x_1y_{m+1} + x_2y_{m+2} + \ldots + x_ny_{m+n} + y_1x_{n+1} + y_2x_{n+2} + \ldots + y_mx_{n+m}.$$

Since all the variables on the right-hand side are non-negative (they are components of feasible solutions)

$$f - \phi \geq 0.$$ Proved.

COROLLARY. *It immediately follows from above that*

$$\min f(X) \geq \max \phi(Y).$$

THEOREM 8. *The optimum value of $f(X)$ of the primal, if it exists, is equal to the optimum value of $\phi(Y)$ of the dual.*

Proof. After introducing slack variables in (44) we get

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i, \; i = 1, 2, \ldots, m.$$

Let the primal have the optimal solution $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$. Since it has to be a basic feasible solution, at least n of these numbers are zero. Let $\pi_1, \pi_2, \ldots, \pi_m$ be the simplex multipliers for this solution. Then, as in (34), $f(X)$ for this b. f. s. is given by

$$f(X) + \sum_{i=1}^{m} \pi_i b_i = \sum_{j=1}^{n} \left(c_j + \sum_{i=1}^{m} a_{ij}\pi_i\right)x_j - \sum_{i=1}^{m} \pi_i x_{n+i}.$$

Since $f(X)$ is optimum, from (39),

$$\min f(X) = -\sum_{i=1}^{m} b_i\pi_i,$$

and all the relative cost coefficients are non-negative, that is,

$$c_j + \sum_{i=1}^{m} a_{ij}\pi_i \geq 0,\ j = 1, 2, \ldots, n;\ -\pi_i \geq 0,\ i = 1, 2, \ldots, m;$$

or

$$-\sum_{i=1}^{m} a_{ij}\pi_i \leq c_j, -\pi_i \geq 0.$$

The last two inequalities mean that $(-\pi_1, -\pi_2, \ldots, -\pi_m)$ is a solution of (47), (48), that is, a feasible solution of the dual. Corresponding to this solution, from (46),

$$\phi(Y) = -\sum_{i=1}^{m} b_i\pi_i = \min f(X).$$

Thus we have found a feasible solution of the dual such that

$$\min f(X) = \phi(Y)$$

which, by corollary of theorem 7, is possible only when

$$\min f(X) = \max \phi(Y).$$

Hence this solution of the dual is optimal. Proved.

THEOREM 9. *The negative of the simplex multipliers for the optimal solution of the primal are the values of the variables for the optimal solution of the dual; and the simplex multipliers for the optimal solution of the dual are the values of the variables for the optimal solution of the primal.*

The proof of the first part is implied in the proof of theorem 8, and the second part can be proved likewise.

THEOREM 10. *If the primal problem is feasible, then it has an unbounded optimum if and only if the dual has no feasible solution, and vice versa.*

Proof. Let the primal have an unbounded optimum. It means $f(X)$ has no lower bound, or in other words, there is no number which is less than all possible values of $f(X)$.

If possible, let the dual have a feasible solution. Then ϕ is a definite number corresponding to that solution, and by theorem 7 $\phi \leq f(X)$. This contradicts the conclusion in the last paragraph. So the dual has no feasible solution.

Conversely, let the primal be feasible and the dual infeasible. Let $f(X)$ have a minimum (not unbounded) for feasible X. By theorem 8, min $f(X)$ = max ϕ (Y) over feasible values of Y. Thus a feasible Y exists which contradicts the assumption that the dual is infeasible. Therefore $f(X)$ has an unbounded minimum.

Since the dual of the dual is the primal, the theorem is true if the words dual and primal are interchanged in its enunciation. Proved.

THEOREM 11. *If, in the optimal solutions of the primal and the dual, (i) a primal variable x_j is positive, then the corresponding dual slack variable y_{m+j} is zero; and (ii) if a primal slack variable x_{n+i} is positive, then the corresponding dual variable y_i is zero; and vice versa.*

Proof. It follows from theorems 7 and 8 that for the optimal solutions x_j, $j = 1, 2, \ldots n, n+1, \ldots n+m$, of the primal, and y_i, $i = 1, 2, \ldots m, m+1, \ldots m+n$, of the dual,

$$x_1 y_{m+1} + x_2 y_{m+2} + \ldots + x_n y_{m+n} + y_1 x_{n+1} + y_2 x_{n+2} + \ldots + y_m x_{n+m} = 0$$

Since an optimal solution is feasible, all $x_j \geq 0$, all $y_i \geq 0$. Hence all the terms in the expression on the left side above are non-negative, and since their sum is zero, each term separately should be zero. It follows that in a term like $x_j\, y_{m+j}$, if $x_j > 0$ then $y_{m+j} = 0$, and if $y_{m+j} > 0$ then $x_j = 0$. Also in a term like $y_i x_{n+i}$, if $x_{n+i} > 0$ then $y_i = 0$, and if $y_i > 0$ then $x_{n+i} = 0$. Proved.

These conditions are called the complementary slackness conditions. In words they can be stated as follows.

In the optimal solutions of the primal and the dual,

(i) if the jth primal variable $x_j > 0$, then the corresponding dual constraint is satisfied as an equation, or, in other words, the constraint is 'tight' (since its slack variable y_{m+j} is zero), and vice versa; and

(ii) if the ith primal constraint is satisfied as a strict inequality, or, in other words, the constraint is 'slack' (since its slack variable x_{n+i} is positive), then the corresponding dual variable y_i is zero, and vice versa.

This theorem is sometimes helpful in determining the optimal solution of the primal from the optimal solution of the dual, or vice versa.

As an example, consider the problem

Maximize $\quad f = 3x_1 + 2x_2 + x_3 + 4x_4,$

subject to $\quad 2x_1 + 2x_2 + x_3 + 3x_4 \leq 20,$

$3x_1 + x_2, +2x_3 + 2x_4 \leq 20,$

$x_1, x_2, x_3, x_4 \geq 0.$

Its dual is

Minimize $\quad \phi = 20y_1 + 20y_2,$

subject to $\quad 2y_1 + 3y_2 \geq 3,$

$2y_1 + y_2 \geq 2,$

$y_1 + 2y_2 \geq 1,$

$3y_1 + 2y_2 \geq 4,$

$y_1, y_2 \geq 0.$

This is a two-variable problem whose solution can be obtained geometrically as

$$y_1 = 1.2,\ y_2 = 0.2,\ \phi = 28.$$

After introducing the slack variables, the primal and dual constraints are

$$2x_1+2x_2+x_3+3x_4+x_5=20,$$

$$3x_1+x_2+2x_3+2x_4+x_6=20,$$

$$2y_1+3y_2-y_3 \qquad =3,$$

$$2y_1+y_2 \quad -y_4 \qquad =2,$$

$$y_1+2y_2 \qquad -y_5 \qquad =1,$$

$$3y_1+2y_2 \qquad -y_6 =4,$$

$$x_1, x_2, \ldots x_6, y_1, y_2, \ldots y_6 \geq 0.$$

Substituting the optimal values of y_1 (= 1.2) and y_2 (= 0.2) in the dual constraints, it follows that the slack variables

$$y_3=y_6=0, \; y_4>0, \; y_5>0.$$

Thus the second and the third constraints are satisfied as strict inequalities, and so the corresponding primal variables should be zero, that is, $x_2=0, x_3=0$. Also since the dual variables $y_1>0$, $y_2>0$, it follows that the corresponding primal constraints should be tight, that is, $x_5=x_6=0$. The primal constraints thus reduce to

$$2x_1+3x_4 = 20,$$

$$3x_1+2x_4 = 20,$$

which give $x_1=4, x_4=4$. The optimal solution of the primal is therefore

$$x_1=x_4=4, x_2=x_3=0, f=28.$$

19 Applications of duality

The existence of a dual to every LP problem and the primal-dual relationship are in conformity with the Kuhn-Tucker theory (see chapter 8, problem 12). Apart from this theoretical implication, the existence of the dual provides some practically useful suggestions which sometimes help in reducing the work in a straightforward application of the simplex method to the solution of the problem.

It follows from the duality theorems that, given an LP problem, one may obtain its solution either by solving it or solving its dual. Sometimes the solution of the dual may involve less work. Usually in an LP problem numerical work increases more with the number of constraints than with the number of variables. Since the two get interchanged in the dual problem, if the constraints in the primal far outnumber the variables, then it is generally economical to solve the dual.

It is also possible under certain conditions to avoid the introduction of artificial variables to obtain an initial b.f.s. and thus avoid Phase I part in the simplex procedure. If the introduction of slack variables in the primal leads to a nonfeasible basic solution of the primal, but the introduction of slack variables in the dual provides a basic feasible solution of the dual, then also it may be economical to solve the dual. What is more interesting is that in such a case it is also possible to start on the simplex tableau of the primal with a nonfeasible basic solution, and proceed with the iterations with a modified algorithm which finally leads to the optimal solution, provided the cost coefficients satisfy a certain condition. The procedure which we explain in the next section is particularly useful when additional constraints are introduced in a problem after the optimal solution has been obtained under the original set of constraints, and the objective is to find the optimal solution to the modified problem without starting work from the very beginning. Such situations commonly arise in Sensitivity Analysis (chapter 7) and the cutting plane method of integer programming (chapter 6).

20 Dual simplex method

Consider the primal and the dual problems in the forms (43), (44), (45) and (46), (47), (48) respectively.

Suppose all $c_j \geq 0$ and all $b_i \leq 0$. Then the basis consisting of the basic variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ (which are the slack variables) is feasible and also optimal. Similarly the corresponding basis of the dual is feasible and also optimal. If, however, some or all $b_i > 0$ and all $c_j \geq 0$, then the basis $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ is not feasible for the primal, but the basis $y_{m+1}, y_{m+2}, \ldots, y_{m+n}$ is feasible for the dual. We call this situation *primal infeasible* and *dual feasible*. Suppose we start with the simplex algorithm on the dual. We shall be moving through a succession of basic feasible solutions of the dual (which means all $\bar{c}_j \geq 0$) till the final relative cost coefficients $\bar{b}_i$ of the dual are all non-positive. We would have then arrived at the optimal solution. To get the optimal basis of the primal from the optimal basis of the dual we shall have to use theorem 9.

It is possible to abridge this procedure by applying a slightly modified algorithm to the primal tableau wherein we start with a non-feasible basic solution but with non-negative cost coefficients. This modified procedure is called the *dual simplex method*.

Let us write the simplex tableau (table 5) for the primal problem (43)-(45) with a basis consisting of slack variables. We assume that some $b_i > 0$ (that is, the values of some basic variables are negative) and all $c_j \geq 0$. The dual simplex method consists in changing a negative basic variable in such a way that the value of the new basic variable in its place would be positive, and the relative cost coefficients for the changed basis still remain non-negative.

TABLE 5

Basis	Value	x_1	x_2	..	x_p	.	x_n	x_{n+1}	..	x_{n+m}
x_{n+1}	$-b_1$	$-a_{11}$	$-a_{12}$	..	$-a_{1p}$	..	$-a_{1n}$	1	..	0
x_{n+2}	$-b_2$	$-a_{21}$	$-a_{22}$	..	$-a_{2p}$	..	$-a_{2n}$	0	..	0
.	.	.	.		.		.	.		.
x_{n+r}	$-b_r$	$-a_{r1}$	$-a_{r2}$	..	$-a_{rp}$	..	$-a_{rn}$	0	..	0
.	.	.	.		.		.	.		.
x_{n+m}	$-b_m$	$-a_{m1}$	$-a_{m2}$	..	$-a_{mp}$	..	$-a_{mn}$	0	..	1
f	0	c_1	c_2	..	c_p	..	c_n	0	..	0

For example, let $b_r > 0$ so that the corresponding basic variable x_{n+r} is negative. Also let some coefficients $-a_{rj}$ be negative. Let, in particular, $-a_{rp} < 0$. We may replace x_{n+r} by x_p in the basis by dividing the rth equation by $-a_{rp}$ and eliminating x_p from all other equations and also from the last row giving the expression for f in terms of the nonbasic variables and relative cost coefficients. This change should be such that no relative cost coefficient becomes negative. This will be so when

$$c_j - \frac{a_{rj}}{a_{rp}} c_p \geq 0, \; j = 1, 2, \ldots, n+m,$$

or

$$\frac{c_j}{a_{rj}} \geq \frac{c_p}{a_{rp}} \text{ over all those } j \text{ for which } -a_{rj} < 0,$$

or

$$\min_j \frac{c_j}{a_{rj}} = \frac{c_p}{a_{rp}}, -a_{rj} < 0.$$

This leads to the determination of p. The value of the new basic variable x_p would be $(-b_r)/(-a_{rp})$ which is positive. If for $-b_r < 0$, there is no $-a_{rj} < 0$, the problem is infeasible.

We may change the basis in this way step by step, one basic variable in each iteration, till all the basic variables come to have non-negative values. Thus we shall arrive at a basic feasible solution which is optimal.

Notice that in this method we move through a set of points which are not primal feasible taking care all the time that the relative cost coefficients remain non-negative so that the moment we arrive at a feasible basis, we find ourselves at the optimal.

Example: Minimise $f = 3x_1 + 5x_2 + 2x_3$

subject to

$$-x_1 + 2x_2 + 2x_3 \geq 3,$$

$$x_1 + 2x_2 + x_3 \geq 2,$$

$$-2x_1 - x_2 + 2x_3 \geq -4$$

$$x_1, x_2, x_3 \geq 0.$$

Table 6 gives the iterations by the dual simplex algorithm leading to the optimal solution $x_1 = x_2 = 0$, $x_3 = 2$ and the optimum value of $f = 4$.

TABLE 6

Basis	Value	P_1	P_2	P_3	P_4	P_5	P_6
x_4	−3	1	−2	−2	1		
x_5	−2	−1	−2	−1		1	
x_6	4	2	1	−2			1
f	0	3	5	2			
x_3	3/2	−1/2	1	1	−1/2		
x_5	−1/2	−3/2	−1		−1/2	1	
x_6	7	1	3		−1		1
f	−3	4	3		1		
x_3	2	1	2	1		−1	
x_4	1	3	2		1	−2	
x_6	8	4	5			−2	1
f	−4	1	1			2	

The proof of the equivalence of two forms is left as an exercise (problem 32).

(viii) Go to (i)

22 Applications of LP

Linear programming finds extensive use in solving real life problems in economics, management, planning, industry and several other areas of human activity. Making mathematical models of real life situations is an important part of operations research, and the problem is often quite complex. There are no hard and fast rules about the way one goes about setting a mathematical model. It is more of an art than a science, and requires a thorough understanding of the system and the relative importance of various factors entering it. For, in real life there are far too many variables and parameters influencing decisions, and to set up a manageable mathematical problem representing the situation, approximations have necessarily to be made. Still a good linear programming model of a real problem may involve hundreds, even thousands, of variables and constraints. Whereas relatively small problems can be solved by using the computational algorithms presented in this chapter, more efficient algorithms are available for solving large scale problems. Even for special types of moderately sized problems more convenient algorithms are available. One such type we shall discuss in the next chapter.

In this section we give simple examples of formulating a LP problem in mathematical terms from a problem described in words. The important steps in setting up the mathematical model are the following.

(i) Identify the variables whose values are to be determined. Represent them by suitable symbols like x_1, x_2, etc.

(ii) Identify the objective or the criterion which is to be maximized or minimized, and express it as a linear function of the variables.

(iii) Identify all the constraints or restrictions and express them as linear equations or inequalities in terms of the variables.

The mathematical problem so formulated is solved and the results finally interpreted in the words of the original problem.

Example 1: A small manufacturing company produces one-band pocket and two-band table radios. Each two-band model requires twice as much time as one one-band model. If the company were to produce only two-band models, it could manufacture 150 units per week. The company is licensed to produce in all not more than 250 units per week. The market survey has shown that no more than 100 pieces of two-band model per week could be sold. The company is also committed to supply at least 50 pieces of one-band model per week. If the net profit on the sale of one-band model is Rs 10 per piece, and on the two-band model Rs 15 per piece, how should the company plan its production to maximize profit?

The problem is to determine the number of one-band and two-band model radios which the company should produce per week to earn maximum profit. Let these be x_1, x_2 respectively.

Since the profit from the sale per piece of one-band model is Rs 10, and of two-band is Rs 15, the total profit per week on x_1 and x_2 pieces of the two models respectively will be $10x_1 + 15x_2$, which it is sought to maximize. This therefore is the objective function to be maximized.

As for the constraints:

(i) The production capacity of the company is such that if only two-band radios were to be produced, it will be able to produce only 150 units per week. It takes twice as much time to produce a two-band model as to produce a one-band model. Therefore x_1 pieces of one-band model use the same manufacturing capacity as $\frac{1}{2}x_1$ pieces of two-band model. The total capacity used is therefore $\frac{1}{2}x_1 + x_2$ which cannot exceed 150. Hence

$$\tfrac{1}{2}x_1 + x_2 \le 150$$

(ii) Since the company is licensed to produce in all not more than 250 pieces per week,

$$x_1 + x_2 \le 250$$

(iii) Since the demand per week of two-band radios is not more than 100, the company should not produce more than 100 of this type. Hence

$$x_2 \le 100$$

(iv) Also since the manufacturer has a commitment to supply at least 50 one-band models per week,

$$x_1 \ge 50$$

(v) Finally since a solution in which either x_1 or x_2 has negative values has no practical significance (making negative number of articles is senseless), x_1, x_2 should be non-negative.

$$x_1 \ge 0,\ x_2 \ge 0$$

Summing up, the mathematical model of the problem which should be solved to provide answer to the company's problem is

Maximize $\quad f = 10x_1 + 15x_2$

subject to

$$\tfrac{1}{2}x_1 + x_2 \le 150$$

$$x_1 + x_2 \le 250$$

$$x_2 \le 100$$

$$x_1 \ge 50$$

$$x_1 \ge 0,\ x_2 \ge 0$$

This is a two-variable LP problem easily solvable graphically or by the simplex method. Its solution can be found to be $x_1 = 200$, $x_2 = 50$ with the maximum value of f as 2750. Interpreting in the words of the original problem, the company should manufacture 200 one-band models and 50 two-band models per week to earn the maximum possible profit of Rs 2750.

Example 2: A company manufactures three products, A, B and C. Each product has to undergo operations on three types of machines, M_1, M_2, M_3 before they are ready for sale. The time that each product requires on each machine, and the total time per day available on each machine are given in the following table. The table also shows the net profit per unit on the sale of the three products. Formulate the mathematical model for this problem to maximize the total net profit of the company per day, and obtain its solution.

Machine \ Product	Time per unit in minutes A	B	C	Total time available per day in minutes
M_1	1	2	1	480
M_2	2	1	0	540
M_3	1	0	3	510
Profit per unit in Rs	4	3	5	

The problem here is to determine the number of items of the products A, B, C which must be manufactured per day to maximize profits. Let these be x_1, x_2, x_3 respectively. Since there is a profit of Rs 4 per unit on A, Rs 3 on B and Rs 5 on C, the total profit on x_1 units of A, x_2 of B and x_3 of C is

$$f = 4x_1 + 3x_2 + 5x_3$$

The objective is to maximize this function.

As for the constraints, on machine M_1 time required for processing one unit of A and C each separately is 1 minute, and for one unit of B it is 2 minutes. The total time in minutes required on machine M_1 is therefore $x_1 + 2x_2 + x_3$ which should not exceed 480 minutes. Hence

$$x_1 + 2x_2 + x_3 \le 480$$

Similarly considering the limitations of time on machines M_2 and M_3 we should ensure that

$$2x_1 + x_2 \le 540$$

$$x_1 + 3x_3 \le 510$$

Also since negative values of the variables will be meaningless, we should have

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

Combining all these, the mathematical model of the given problem is

Maximize $\quad f = 4x_1 + 3x_2 + 5x_3$

subject to

$$x_1 + 2x_2 + x_3 \le 480$$

$$2x_1 + x_2 \le 540$$

$$x_1 + 3x_3 \le 510$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

(The solution of this problem by the simplex method is left to the student).

The solution is $x_1 = 231$, $x_2 = 78$, $x_3 = 93$, with $f = 1623$. Thus the company should manufacture 231 units of A, 78 units of B and 93 units of C per day. With this production schedule it will earn the maximum possible profit of Rs 1623 per day.

Example 3: The manager of an agricultural farm of 80 hectares learns that for effective protection against insects, he should spray at least 15 units of chemical A and 20 units of chemical B per hectare. Three brands of insecticides are available in the market which contain these chemicals. One brand contains 4 units of A and 8 units of B per kg and costs Rs 5 per kg, the second brand contains 12 and 8 units respectively and costs Rs 8 per kg, and the third contains 8 and 4 units respectively and costs Rs 6 per kg. It is also learnt that more than 2.5 kg per hectare of insecticides will be harmful to the crops. Determine the quantity of each insecticide he should buy to minimize the total cost for the whole farm.

Let the quantity of each of the three insecticides used be x_1, x_2, x_3 kg per hectare. Since the cost of these three is Rs 5, 8 and 6 per kg respectively, the total cost per hectare would be $5x_1 + 8x_2 + 6x_3$. This is the objective function to be minimized.

The first brand of insecticide contains 4 units of chemical A per kg, the second 12 units and the third 8 units. Hence the total content of chemical A is $4x_1 + 12x_2 + 8x_3$ units which should not be less than 15. Hence

$$4x_1 + 12x_2 + 8x_3 \geq 15$$

Similarly the constraint provided by the content of chemical B is

$$8x_1 + 8x_2 + 4x_3 \geq 20$$

Further, not more than a total of 2.5 kg per hectare of insecticides should be sprayed. Hence

$$x_1 + x_2 + x_3 \leq 2.5$$

Also, from the physical nature of the variables

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

Summing up, the mathematical model of this problem is

$$\begin{aligned} \text{Minimize} \quad & f = 5x_1 + 8x_2 + 6x_3 \\ \text{subject to} \quad & 4x_1 + 12x_2 + 8x_3 \geq 15 \\ & 8x_1 + 8x_2 + 4x_3 \geq 20 \\ & x_1 + x_2 + x_3 \leq 2.5 \\ & x_1 \geq 0, x_2 \geq 0,\ x_3 \geq 0 \end{aligned}$$

This LP problem can be solved using the big-M or two-phase simplex method. Also, since it is a minimization problem in which all the cost coefficients are positive and two of the constraints are $\geq$ type, on introducing the slack variables an optimal but infeasible basis can be found. Hence the problem can also be solved by the dual simplex method. The following table gives the solution by this method.

Basis	Values	x_1	x_2	x_3	x_4	x_5	x_6
x_4	− 15	− 4	− 12	− 8	1		
x_5	− 20	− 8	− 8	− 4		1	
x_6	5/2	1	1	1			1
f	0	5	8	6			
x_4	− 5		− 8	− 6	1	− 1/2	
x_1	5/2	1	1	1/2		− 1/2	
x_6	0		0	1/2		1/8	1
f	− 25/2		3	7/2		5/8	
x_2	5/8		1	3/4	− 1/8	1/16	
x_1	15/8	1		− 1/4	1/8	− 3/16	
x_6	0			1/2	0	1/8	1
f	− 115/8			5/4	3/8	7/16	

The optimal solution is $x_1 = 15/8, x_2 = 5/8, x_3 = 0$, with $f = 115/8$. Hence the manager must buy for each hectare 15/8 kg of the first brand of the insecticide and 5/8 kg of the second brand and none of the third. The cost will be Rs 115/8 per hectare, or Rs 1150 for the whole farm.

HISTORICAL NOTE

Linear programming began formally in 1947 when under the compulsions of World War II a United States Air Force project called SCOOP (Scientific Computation of Optimum Programs) was setup under the leadership of G.B. Dantzig. The simplex algorithm and much of the related theory was developed by Dantzig and his team in 1947 and further work on special problems and methods continued throughout the next decade by the Dantzig group in the U.S.A. and others in Europe. In the U.S.S.R. L.V. Kantorovitch published in 1939 a monograph in which the possibilities of applying linear mathematical models to increase the efficiency in organization and planning of production were suggested. Unfortunately the suggestions were not taken up otherwise much work might have been done in the U.S.S.R. in LP before Dantzig.

Contacts of the Dantzig team with John von Neumann (see note after chapter 8) led to fundamental insight into the mathematical theory of LP. Neumann emphasized the importance of duality and could immediately see the connection between LP and the Theory of Games on which he had done fundamental work in 1928.

In the last few decades LP has become a very important tool of analysis in the hands of economists. T.C. Koopmans in the U.S.A. and L.V. Kantorovitch in the U.S.S.R. have been pioneers in this field, for which they were jointly awarded Nobel prize in economics in 1975. Ragnar Frisch of Norway and Paul Samuelson of the U.S.A. are other outstanding economists, both Nobel prize winners, who have made significant contributions.

4

Transportation and Assignment Problems

1 Introduction

We shall consider in this chapter some linear programming problems which have special mathematical structure. The general method of solving an LP problem, namely, the simplex method, can be applied to them. But their special features have led to the discovery of simpler algorithms for their solution. Because of the occurrence of fairly large number of physical situations whose mathematical formulation conform or can be made to conform to these special structures, these problems have assumed considerable importance and have been given special names—the *transportation problem* and the *assignment problem*. While the names do indicate the physical situations in which the problems most obviously arise, they are used to refer to particular forms of mathematical models rather than any physical situations.

2 Transportation problem

The transportation model most apparently arises when we want to determine the minimum cost at which goods can be transported from given origins to specified destinations. Suppose there are m sources (or origins or supply centres) O_i, $i = 1, 2, \ldots, m$, of a certain commodity and n sinks (or destinations or demand centres) D_j, $j = 1, 2, \ldots, n$, where it is required. The quantity produced at source O_i is a_i (> 0), and the quantity required at sink D_j is b_j (> 0). Let us for the present assume that the total supply equals the total demand, that is

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{1}$$

If this condition is satisfied, the transportation problem is said to be balanced. The cost of unit flow (or transportation of the unit quantity) from each source to each sink is known. The problem is: How to meet the demand that the cost of transportation is minimum? Or in more general terms, what is the flow with minimum cost?

Let x_{ij} be the flow from O_i to D_j. Then the total outflow at O_i and the total inflow at D_j are respectively

$$\sum_{j=1}^{n} x_{ij} \text{ and } \sum_{i=1}^{m} x_{ij},$$

and therefore

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m, \tag{2}$$

and

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n. \tag{3}$$

Also since the flow, in order to be meaningful, should be either zero or positive, we further impose the condition

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j. \tag{4}$$

The cost of flow is

$$f = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij} \tag{5}$$

where c_{ij} is the cost of unit flow from O_i to D_j. The problem is to find x_{ij} subject to (1), (2), (3) and (4) which minimise the objective function (5). Equations (2) and (3) and the objective function are linear in x_{ij}. Therefore it is an *LP* problem. It can be solved by the simplex method. Because of the special structure of the problem, a simpler algorithm has been discovered by which a basic feasible solution can be obtained, its optimality or non-optimality can be tested, and, if non-optimal, a change can be made to another basic feasible solution which is nearer the optimal. We proceed to discuss the special structure of the transportation problem and the special algorithm to solve it.

3 Transportation array

Some special features of the constraint equations in the transportation problem are very well revealed when the equations are visualized in the array form (table 1). Also, as we shall see later, the simplex method of solution when applied to the transportation problem reduces to very simple rules of computation if the equations are written in the array form. Therefore for theoretical as well as practical reasons the transportation array is useful.

Visualised in the array form, equations (2) may be called the row equations and equations (3) the column equations. There are m rows and n columns in the array, providing mn number of cells, one for each of the variables. The cell in the ith row and the jth column, which we call the (i, j) cell, is the position of the variable x_{ij}. The constants a_i and b_j are placed respectively in an additional column on the right

TABLE 1

	D_1	D_2	...	D_j	...	D_n	
O_1	x_{11}	x_{12}	...	x_{1j}	...	x_{1n}	a_1
O_2	x_{21}	x_{22}	...	x_{2j}	...	x_{2n}	a_2
.	.	.	...	.	...	.	..
O_i	x_{i1}	x_{i2}	...	x_{ij}	...	x_{in}	a_i
.	.	.	...	.	...	.	..
O_m	x_{m1}	x_{m2}	...	x_{mj}	...	x_{mn}	a_m
	b_1	b_2	...	b_j	...	b_n	$\Sigma a_i = \Sigma b_j$

and an additional row below. In some discussions it may not be necessary to refer to a_i and b_j, and in such cases we shall consider the array to consist of only the mn number of (i, j) cells. It is also not necessary to explicitly write x_{ij} in the (i, j) cell. For example, the transportation array for $m = 3$, $n = 4$ would be as in table 2.

TABLE 2

				a_1
				a_2
				a_3
b_1	b_2	b_3	b_4	

4 Transportation matrix

Though the array is the most useful form of representation of the transportation equations, their matrix representation is also useful to bring out some of their important features. To put them in their matrix form, it is convenient to multiply one set of constraint equations, say equations (3), by -1, and put them as

$$\left.\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i, \; i = 1, 2, \ldots, m, \\ -\sum_{i=1}^{m} x_{ij} &= -b_j, \; j = 1, 2, \ldots, n. \end{aligned}\right\} \tag{6}$$

The advantage of this form will become evident as we proceed. The above equations, in the matrix form, are

$$\mathbf{TX} = \mathbf{B}, \tag{7}$$

where $\mathbf{X}$ is the column vector of elements x_{ij} which are mn in number. The column vector $\mathbf{B}$ has $m + n$ elements, m of the type a_i and n of the type $-b_j$. The matrix $\mathbf{T}$ is of order $(m + n) \times (mn)$.

To get a clear idea of the form of $\mathbf{T}$, let us write equations (6) more explicitly for the particular case $m = 3$, $n = 4$, as follows.

$$\begin{array}{cccccccccccccc}
x_{11}+x_{12}+x_{13}+x_{14} & & & & & & & & & & & & & = a_1 \\
& & & & x_{21}+x_{22}+x_{23}+x_{24} & & & & & & & & & = a_2 \\
& & & & & & & & x_{31}+x_{32}+x_{33}+x_{34} & & & & & = a_3 \\
-x_{11} & & & & -x_{21} & & & & -x_{31} & & & & & = -b_1 \\
\quad -x_{12} & & & & \quad -x_{22} & & & & \quad -x_{32} & & & & & = -b_2 \\
\qquad -x_{13} & & & & \qquad -x_{23} & & & & \qquad -x_{33} & & & & & = -b_3 \\
\qquad\quad -x_{14} & & & & \qquad\quad -x_{24} & & & & \qquad\quad -x_{34} & & & & & = -b_4
\end{array} \tag{8}$$

The matrix **T** for the above particular case is

$$\begin{array}{cccccccccccc} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} & \mathbf{P}_{14} & \mathbf{P}_{21} & \mathbf{P}_{22} & \mathbf{P}_{23} & \mathbf{P}_{24} & \mathbf{P}_{31} & \mathbf{P}_{32} & \mathbf{P}_{33} & \mathbf{P}_{34} \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (9)$$

$\mathbf{P}_{11}, \mathbf{P}_{12}, \ldots$ have been written above the matrix to denote the column vectors of **T** corresponding to $x_{11}, x_{12}, \ldots$ for future reference. It is obvious that each column of **T** has one entry +1, another −1, and all others zero. The general form of **T** can be written as

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_1 & \mathbf{I}_2 & . & . & \mathbf{I}_m \\ -\mathbf{U}_1 & -\mathbf{U}_2 & . & . & -\mathbf{U}_m \end{bmatrix} \qquad (10)$$

where $\mathbf{I}_i$, $i = 1, 2, \ldots, m$, is an $m \times n$ matrix in which all entries in the ith row are +1 and all other entries are zero, and $\mathbf{U}_i$ is an $n \times n$ unit matrix. It is important to note that each column of **T** contains two and only two nonzero entries, one +1 and the other −1.

The correspondence between a variable x_{ij} and a column in **T** should also be borne in mind. For every x_{ij} there is a column in **T** which gives the coefficients of that variable in the set of equations (6).

5 Triangular basis

The constraint equations (2, 3) or (6) are not linearly independent if (1) is satisfied. If (1) does not hold, the equations become inconsistent. This can be easily seen by adding all the equations (6). Unless otherwise mentioned we shall assume that the transportation problem is balanced, that is, (1) holds. Then the $m + n$ equations (6) are linearly dependent.

The same conclusion can be easily drawn from matrix (10). Since each column contains two and only two nonzero entries, +1 and −1, the sum of all rows of **T** is a zero row, which means that equations (6) are linearly dependent. It is also obvious that deletion of one, any one, row would leave the remaining set of rows linearly independent. Hence the rank of the matrix **T** is $m + n - 1$, or, in other words, any $m + n - 1$ of the equations (6) are linearly independent. The number of basic variables (chapter 1, section 9) in these equations is therefore $m + n - 1$. A basic solution will consist of at most $m + n - 1$ of the variables having nonzero values.

We now enunciate and prove a theorem about the transportation problem which provides a simple method of obtaining a basic solution.

THEOREM 1. *The transportation problem has a triangular basis.*

By *triangular basis* we mean that the system of equations, when put in terms of the basic variables only, the nonbasic variables having been put as zero, is triangular, that is, the matrix of coefficients of such equations, if necessary, after a permutation of its rows and columns, is triangular. In other words, there is an equation in which only one basic variable occurs; in another equation there is one more basic variable with the total number of basic variables being not more than two; in a third equation another basic variable occurs with the total now being not more than three, and so on.

The theorem can be proved equally easily by referring to the array or referring to the matrix. We prove it first by considering the array.

Proof. There cannot be an equation in which there is no basic variable because then the equation cannot be satisfied for $a_i \neq 0$ or $b_j \neq 0$. If possible, let every equation have at least two basic variables. Then there will be at least two basic variables in each row, and so the total number of basic variables will be at least $2m$. Also each column equation will have at least two basic variables, and so there will be in all at least $2n$ basic variables. Thus, if N is the total number of basic variables,

$$N \geq 2m, \; N \geq 2n.$$

If $\quad m > n$, then $N \geq 2m = m + m > m + n$;

if $\quad m < n$, then $N \geq 2n = n + n > n + m$;

if $\quad m = n$, then $N \geq 2m = m + n$.

So in every case $N \geq m + n$. But, as we have already seen, $N = m + n - 1$. This is a contradiction. Therefore the assumption that there are at least two basic variables in each row and column is wrong. There is therefore at least one equation, row or column, in which there is only one basic variable.

Let the rth row equation be such an equation and let x_{rc}, the variable in the rth row and the cth column, be the only basic variable in it. Then $x_{rc} = a_r$. Eliminate this equation from the system by deleting the rth row equation and putting $x_{rc} = a_r$ in the cth column equation. The rth row then stands cancelled, and b_c is replaced by $b_c' = b_c - a_r$.

The resulting system consists of $m - 1$ row equations and n column equations, of which $m + n - 2$ are linearly independent. Therefore, the number of basic variables in this system is $m + n - 2$. Repeating the earlier argument we conclude that there is an equation in this reduced system which has only one basic variable. If this equation happens to be the cth column equation, in the original system the cth column equation now contains two basic variables. So we conclude that the original system has an equation which has at most two basic variables. Continuing with this line of reasoning we next prove that there is an equation with at most three basic variables, and so on. We thus prove the theorem.

Alternative proof. Referring to the matrix **T**, it has $m + n$ rows but is of rank $m + n - 1$. Deleting a row from **T** we are left with a matrix $\overline{\mathbf{T}}$ with $m + n - 1$ rows, and it should be possible to find $m + n - 1$ columns in this matrix which are linearly independent. Let **A** be the $(m + n - 1) \times (m + n - 1)$ matrix with such linearly independent columns. Each of these columns can at most have two nonzero entries, one $+1$ and the other -1. If all the columns have two nonzero entries, then the sum of the rows will be a zero row, and so the matrix **A** will be singular which would mean that its columns are not linearly independent (chapter 1, section 9). This will be a contradiction. Hence all the columns cannot have two nonzero entries. The total number of nonzero entries in A should therefore be less than 2 $(m + n - 1)$. Since there are only $m + n - 1$ rows in A and each row must contain at least one nonzero entry (otherwise A will not be nonsingular), there should be at least one row with only one nonzero entry. This means there should be an equation with only one basic variable. Eliminating this equation from the system, we are left with a nonsingular matrix of order $m + n - 2$, and repeating the argument we must find an equation in this system containing only one basic variable, the original system then having an equation with at most two basic variables. Repeating the argument we prove that the basic variables constitute a triangular system of equations.

We have given two proofs because it is important for the reader to clearly understand the correspondence between the matrix and the array of the transportation problem.

The theorem provides a very simple method of testing whether a given set of $m + n - 1$ variables is a set of basic variables. For example, for $m = 3$, $n = 4$, consider the two different sets of six variables shown in tables 3 and 4. We shall test in each case whether they form a triangular set of equations. Considering table 3 first, there is an equation containing only one variable; it is the column equation $j = 4$. Let us

TABLE 3

x_{11}	x_{12}		
x_{21}		x_{23}	
		x_{33}	x_{34}

TABLE 4

x_{11}	x_{12}		
x_{21}	x_{22}		
		x_{33}	x_{34}

cross out this column, implying thereby that the variable x_{34} is eliminated from the equations. In the remaining array, the row equation $i = 3$ contains only one variable, namely x_{33}. Crossing out this row, we are left with an array in which the column equation $j = 3$ contains only one variable x_{23}. Crossing out the column $j = 3$, in the remaining array the row equation $i = 2$ has only one variable x_{21}. Crossing out this row, the column equations of the remaining array contain only one variable each. These variables therefore form a triangular set of equations, and so the variables are basic.

Turning to table 4, we cross out the column $j = 4$ which contains only one variable. In the remaining array the column $j = 3$ contains only one variable, so we cross it out. Now there is no row or column in the remaining array having only one variable. The variables therefore do not form a triangular system of equations and are not basic.

6 Finding a basic feasible solution

Theorem 1 also provides a practical method of finding a b.f.s. Let us arbitrarily choose x_{rc} as the basic variable which occurs alone in an equation. It can so occur either in the rth row or the cth column. If we choose the rth row equation, then $x_{rc} = a_r$; if we choose the cth column equation, then $x_{rc} = b_c$. Suppose $a_r > b_c$. Then if we choose $x_{rc} = a_r$, some other variable in the cth column will have to have a negative value in order that the column equation may be satisfied. This will mean going to a non-feasible solution. If on the other hand we choose $x_{rc} = b_c$, the rth row equation will need another variable with a positive value for satisfaction which will create no such difficulty. If $a_r < b_c$, the position will be reversed and it will be necessary to choose $x_{rc} = a_r$. The rule is to put $x_{rc} = min\,(a_r, b_c)$. If the two are equal, it is immaterial which choice is made. Just now let $a_r < b_c$. Then, following the above rule, put $x_{rc} = a_r$. This satisfies the rth row equation. We turn our attention to the cth column equation. Eliminating x_{rc} from it, we replace b_c by $b_c - a_r$, and then obtain an array with one row (or column) less. With this reduced array we proceed as we did with the first.

The procedure is continued till $m + n - 1$ rows and columns are crossed out and an equal number of variables evaluated. The procedure ensures that the solution so obtained is basic feasible. The last row or column left uncrossed will be automatically satisfied.

It is important to cross out one and not more than one row or column at each stage after choosing a basic variable. In the case when at any stage of the above procedure $a_r = b_c$, we may put $x_{rc} = a_r$ or b_c and may cross out the rth row or the cth column but not both. If we choose to cross out the rth row, then b_c is replaced by $b_c - a_r = 0$, and the cth column has still to be satisfied by choosing some other variable in the cth column to be included in the basis. The value of this variable would be zero. On the other hand if the cth column is crossed out first, the rth row should be kept open for choosing another basic variable in it whose value would be zero. The resulting basis in either case would be degenerate. We illustrate the method by two numerical examples, the second involving degeneracy.

Example 1: In the numerical problem of table 5

TABLE 5

	D_1	D_2	D_3	D_4	a_i
O_1	10	15			25
O_2		3	20	12	35
O_3				30	30
b_j	10	18	20	42	90

TABLE 6

	D_1	D_2	D_3	D_4	a_i
O_1			20	5	25
O_2				35	35
O_3	10	18		2	30
b_j	10	18	20	42	90

the three origins have capacities 25, 35 and 30, and the four destinations have demands 10, 18, 20 and 42. The total capacity equals the total demand. The number of linearly independent equations and so the number of basic variables is six. Starting with x_{11} as a basic variable, we put $x_{11} = 10$ because $a_1 = 25 > b_1 = 10$. This satisfies the first column equation, and $a_1' = 25 - 10 = 15$. Turning to the first row equation, we put $x_{12} = \min\,(15, 18) = 15$ and $b_2' = 18 - 15 = 3$. The first row equation is also satisfied. Next we put $x_{22} = 3$ which satisfies the second column equation. The second row equation should not be satisfied by putting $x_{23} = 32$, because $b_3 = 20 < a_2' = 32$. Instead we put $x_{23} = 20$ thus satisfying the third column equation. Proceeding further we put $x_{24} = 12$ and finally $x_{34} = 30$, thus satisfying all the equations and getting a b.f.s. as $x_{11} = 10, x_{12} = 15, x_{22} = 3, x_{23} = 20, x_{24} = 12, x_{34} = 30$, and all other variables zero.

It is not necessary that we should always start with x_{11}. Any variable may be selected to make a start. For example, starting with $x_{13} = 20$ the b.f.s. as given in table 6 is obtained.

Example 2: The example in table 7 illustrates a degenerate case. After setting $x_{11} = 10$, the first column is crossed and in the first row $a_1 = 25$ is replaced by $25 - 10 = 15$ which is equal to b_2. Putting $x_{12} = 15$ satisfies both the first row and the second column. But we cross the first row only, and put $x_{22} = 0$ and then cross the second column. Proceeding further we get a b.f.s. as shown in table 7.

TABLE 7

	D_1	D_2	D_3	D_4	a_i
O_1	10	15			25
O_2		0	10		10
O_3			10	40	50
b_j	10	15	20	40	85

7 Testing for optimality

To test whether a particular b.f.s. is optimal or not, we recall from chapter 3, section 11 that the objective function should be expressed in terms of the nonbasic variables only by eliminating the basic variables with the help of the constraint equations. The coefficients of the nonbasic variables in the new expression for the objective function are called the relative cost coefficients for the current b.f.s. If all the relative cost coefficients are non-negative, the solution is optimal and the corresponding value of the objective function is minimum. If a relative cost coefficient is negative, the value of the objective function can be further reduced by bringing the corresponding nonbasic variable in the basis in place of some basic variable which is dropped out of the basis.

In the transportation problem the relative cost coefficients can be worked out very easily. Let us for simplicity consider the problem with $m = 3, n = 4$, and write equations (2) and (3) and the objective function (5) in the following extended form.

$$
\begin{array}{llllllllllllll}
x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & & = a_1 \\
 & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & & = a_2 \\
 & & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & & = a_3 \\
x_{11} & & & & +x_{21} & & & & +x_{31} & & & & & = b_1 \\
 & x_{12} & & & & +x_{22} & & & & +x_{32} & & & & = b_2 \\
 & & x_{13} & & & & +x_{23} & & & & +x_{33} & & & = b_3 \\
 & & & x_{14} & & & & +x_{24} & & & & +x_{34} & & = b_4
\end{array}
\qquad (11)
$$

$$c_{11}x_{11}+c_{12}x_{12}+c_{13}x_{13}+c_{14}x_{14}+c_{21}x_{21}+c_{22}x_{22}+$$
$$+c_{23}x_{23}+c_{24}x_{24}+c_{31}x_{31}+c_{32}x_{32}+c_{33}x_{33}+c_{34}x_{34}=f \qquad (12)$$

Following table 5, let a feasible basis be x_{11}, x_{12}, x_{22}, x_{23}, x_{24}, x_{34}. To find the relative cost coefficients for this case we have to eliminate these variables from (12). Let π_1, π_2, π_3 be the simplex multipliers (chapter 3, section 15) for the three row equations and σ_1, σ_2, σ_3, σ_4 for the four column equations respectively. Then in order to eliminate the basic variables we must evaluate these multipliers from the following equations.

$$\pi_1+\sigma_1+c_{11}=0,$$

$$\pi_1+\sigma_2+c_{12}=0,$$

$$\pi_2+\sigma_2+c_{22}=0,$$

$$\pi_2+\sigma_3+c_{23}=0,$$

$$\pi_2+\sigma_4+c_{24}=0,$$

$$\pi_3+\sigma_4+c_{34}=0.$$

These are six equations but seven unknowns to be evaluated. In general these are respectively $m+n-1$ and $m+n$ in number. Obviously there are infinitely many solutions, but any one would serve our purpose. We may therefore choose any one of the simplex multipliers arbitrarily. The simplest way would be to choose one of these, say π_1, as zero. Putting $\pi_1=0$, the others are evaluated quite easily. In tabulated form the rule for evaluation turns out to be very simple. The simplex multipliers for a column and a row should be such that the sum of the two multipliers plus the cost coefficient in the intersecting cell should be zero provided the square is occupied by a basic variable. This rule enables us to evaluate all the simplex multipliers.

Now to evaluate the relative cost coefficients c_{ij}' (the coefficients of the nonbasic variables x_{ij} in f after the basic variables have been eliminated),

$$c_{ij}'=\pi_i+\sigma_j+c_{ij}$$

where the cell (i, j) corresponds to nonbasic variable. This is calculated for all nonbasic variables. If c_{ij}' is negative for any (i, j), the present basis is not optimal and the value of f can be improved by bringing the variable x_{ij} in the basis.

Example: Let the cost coefficients c_{ij} in example 1 (table 5) be as shown in the right-hand bottom corner of each square of table 8. We shall test whether the b.f.s. of table 5 is optimal. Let $\pi_1=0$ [written in the second row of the square containing a_1 (= 25)]. Since x_{11} (= 10) is a basic variable,

$$\pi_1+\sigma_1+c_{11}=0 \Rightarrow 0+\sigma_1+3=0 \Rightarrow \sigma_1=-3.$$

Also x_{12} (= 15) is a basic variable, and so

$$\pi_1+\sigma_2+c_{12}=0 \Rightarrow 0+\sigma_2+2=0 \Rightarrow \sigma_2=-2.$$

TABLE 8

	D_1	D_2	D_3	D_4	a_i
O_1	10 3	15−u 2	u −3 5	 −3 4	25 0
O_2	 2 4	3+u 1	20−u 7	12 6	35 1
O_3	 6 7	 8 8	 −3 3	30 5	30 2
b_j	10 −3	18 −2	20 −8	42 −7	90

Similarly, since x_{22} (=3) is a basic variable,

$$\pi_2+\sigma_2+c_{22}=0 \Rightarrow \pi_2-2+1=0 \Rightarrow \pi_2=1.$$

In this way all the simplex multipliers are easily evaluated. These are written in the second row in each square for a_i and b_j. Now, since the relative cost coefficient for the nonbasic variable x_{ij} is

$$c_{ij}'=\pi_i+\sigma_i+c_{ij},$$

$$c_{13}'=0-8+5=-3,\ c_{14}'=0-7+4=-3,$$

$$c_{21}'=2,\ c_{31}'=6,\ c_{32}'=8,\ c_{33}'=-3.$$

We write these in the left bottom corners of the nonbasic (vacant) cells. Since there are negative c_{ij}', the present basis is not optimal. The value of the objective function for the present solution is

$$f=10\times3+15\times2+3\times1+20\times7+12\times6+30\times5=425.$$

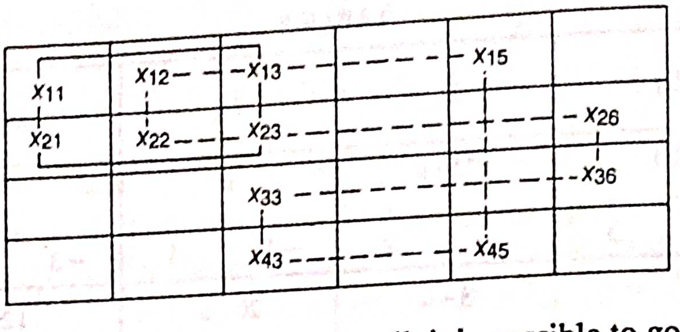
This can be reduced by a change of basis. The candidates to enter the basis are x_{13}, x_{14}, x_{33}.

8 Loop in transportation array

The procedure for changing the basis is based theoretically on a notion in the transportation array called the *loop* which we proceed to define and discuss.

DEFINITION. *A set of cells L in the transportation array is said to constitute a loop if in every row or column of the array the number of cells belonging to the set is either zero or two.*

Suppose (i_1, j_1) is a cell of the loop L. Then there must be cells (i_1, j_2), (i_2, j_2), (i_2, j_3) ...., (i_k, j_1) belonging to L. Examples of loops are shown in table 9. One loop consisting of x_{11}, x_{13}, x_{23}, x_{21} is shown by continuous lines, another consisting of x_{13}, x_{15}, x_{45}, x_{43}, x_{33}, x_{36}, x_{26}, x_{23} is shown by dotted lines. The idea is that in a set of



cells forming a loop, starting from any one cell, it is possible to go through all the other cells of the loop and back to the starting cell, visiting each intermediate cell once only and moving alternately along rows and columns of the array.

THEOREM 2. *The necessary and sufficient condition for a set of column vectors* P_{ij} *in the matrix* $\overline{T}$ *(section 5) to be linearly dependent is that the corresponding variables* x_{ij} *in the transportation array occupy cells a subset of which constitutes a loop.*

Proof. To prove that the condition is necessary, let a set of column vectors P_{ij} of the matrix $\overline{T}$ (obtained by deleting a row from T) be linearly dependent. It is more convenient in the present case to denote the column vectors as $P(i, j)$, $i = i_1, i_2, ..., i_p, ..., i_q$, $j = j_1, j_2, ..., j_r, ..., j_r$. Since they are linearly dependent, there exist numbers $\alpha(i, j)$, not all zero, such that

$$\sum_j \sum_i \alpha(i, j)\, P(i, j) = 0. \tag{13}$$

Pick up a nonzero multiplier $\alpha(i_p, j_r)$ of the column vector $P(i_p, j_r)$. The corresponding variable $x(i_p, j_r)$ occupies the cell (i_p, j_r) in the array. The row of the column vector $P(i_p, j_r)$ corresponding to the i_pth row of the array contains the entry +1. Therefore in order that (13) is satisfied for this row of the matrix of column vectors, there must be at least one more nonzero entry in that very row of the matrix, that is, there must be another column vector of type $P(i_p, j_s)$ in the given set of column vectors, and its multiplier $\alpha(i_p, j_s)$ should be nonzero. The corresponding variable $x(i_p, j_s)$ occupies the cell (i_p, j_s). Thus the i_pth row of the array contains at least two variables $x(i_p, j_r)$ and $x(i_p, j_s)$ corresponding to the column vectors of the given set.

Also in the column vector $P(i_p, j_s)$ the entry in the row corresponding to the j_sth column of the array is −1. Again, in order that (13) is satisfied for this row of the matrix of the given column vectors, there must be another column of the type $P(i_q, j_s)$ whose multiplier $\alpha(i_q, j_s)$ is nonzero. This means there is a variable $x(i_q, j_s)$ in the cell (i_q, j_s) of the array, making the number of variables in the j_sth column at least two.

We can proceed with this type of argument till we find that for all nonzero multipliers $\alpha(i, j)$ of $P(i, j)$ there must be corresponding variables $x(i, j)$ in the array such that in a row or a column of the array if one of such variables occur, then at least one more of them also occurs. This proves the existence of a loop in the set of variables corresponding to the given set of linearly dependent column vectors.

To prove the converse, let the set of variables $x(i, j)$ corresponding to the column vectors $P(i, j)$, $i = i_1, i_2, ..., i_p, ..., i_q$, $j = j_1, j_2, ..., j_r, ..., j_s$, contain a subset forming a loop. Let the subset be $x(i_1, j_1)$, $x(i_1, j_2)$, $x(i_2, j_2)$, ..., $x(i_p, j_r)$, $x(i_p, j_1)$. Let the corresponding column vectors be multiplied by +1 and −1 alternatively and let the remaining column vectors of the given set be multiplied by zero. Then consider the following linear combination of the given set of column vectors.

$$\mathbf{P}(i_1, j_1)-\mathbf{P}(i_1, j_2)+\mathbf{P}(i_2, j_2)-.....-\mathbf{P}(i_p, j_1)$$

The row of this vector corresponding to the i_1th row of the array becomes zero because only two vectors $\mathbf{P}(i_1, j_1)$ and $\mathbf{P}(i_1, j_2)$ have nonzero entries in this row and they are each +1, and so their difference becomes zero. Similarly for all other rows of the vector corresponding to other rows and columns of the array. Hence the above linear combination is a zero vector, which means that the given column vectors are linearly dependent. Proved.

9 Changing the basis

Theorem 2 provides a method of changing the basis in a transportation array so as to bring into the basis any desired variable in place of another which is deleted from the basis without making the solution non-feasible.

In section 7 we have seen how to select a variable for entry into the basis. The existing $m + n - 1$ basic variables along with this new variable become $m + n$ in number. The corresponding $m + n$ column vectors in the matrix $\overline{\mathbf{T}}$ are linearly dependent because the matrix is of rank $m + n - 1$. Hence, by theorem 2, the $m + n$ variables in the transportation array have a loop within themselves. It can be proved that this loop is unique for a particular set of basic variables with a particular additional variable and includes the latter. (We omit the proof). This loop can be easily traced as illustrated in the example of table 8.

Let us, in table 8, decide to bring x_{13} into the basis. This variable x_{13} together with x_{12}, x_{22}, x_{23} which are variables of the existing basis, forms a loop. The values of these basic variables at this stage are

$$x_{12} = 15, x_{22} = 3, x_{23} = 20$$

If we put $x_{13} = u$ (a constant), and alternately subtract and add u from and to the other variables of the loop so that the equations are still satisfied, we get

$$x_{12} = 15 - u, x_{22} = 3 + u, x_{23} = 20 - u.$$

The value of u which would reduce the value of one of these variables to zero without making any of the others negative is $u = 15$, Then

$$x_{12} = 0, x_{22} = 18, x_{23} = 5.$$

Thus x_{12} goes out of the basis and x_{13} comes in. The new basis is shown in table 10. The value of f for this solution is 380 which is an improvement on the previous value (section 7).

TABLE 10

	D_1	D_2	D_3	D_4	a_i
O_1	10		15		25
O_2		18	5	12	35
O_3				30	30
b_j	10	18	20	42	90

TABLE 11

	D_1	D_2	D_3	D_4	a_i
O_1				25	25
O_2	10	18		7	35
O_3			20	10	30
b_j	10	18	20	42	90

The procedure outlined in sections 6, 7 & 9 constitutes the algorithm for the solution of the transportation problem. It is repeated till an optimal solution is obtained. It is left to the reader to proceed with the successive iterations. The optimal solution is shown in table 11 with the minimum value of f as 310.

It should be remembered that while the minimum value of the objective function has to be unique, the optimal basis may not be unique. There may be other solutions giving the same value of the objective function. If any $c_{ij}' = 0$ at the optimal stage, then an alternative solution exists with the corresponding variable in the basis (see problem 15, chapter 3).

A summary of the transportation algorithm is given in section 16.

10 Degeneracy

Degeneracy can occur in transportation problem also. The example in table 7 illustrates a degenerate case. $x_{22} = 0$ is also a basic variable. The general remarks on degeneracy in chapter 3, section 14 apply here also. In practical problems it has seldom proved to be a hurdle. One can proceed to the next b.f.s. according to the prescribed rule and hopefully one would get out of the loop to eventually arrive at the optimal solution. There can, however, be examples in which one is caught in the loop and is unable to get out by the ordinary rule. We shall omit the discussion of the method to overcome this difficulty.

11 Unbalanced problem

If we remove condition (1) and assume that

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j,$$

the problem (2)-(5) becomes infeasible. In physical sense, if $\sum a_i > \sum b_j$, there would be surplus left at the sources after all the demands are met, and if $\sum a_i < \sum b_j$, there would be deficit at the sinks after all the sources have exhausted their capacities. Problems involving surpluses and deficits are common and significant in practical life. They have only to be posed properly to have feasible solutions.

Let us look at the problem of surplus at the sources in the following way. If the total available supply is more than the total demand, the demands at the sinks can be fully met without exhausting the supplies. We may want to know the minimum cost of meeting the demands at all the sinks. The problem then may be

$$\left.\begin{aligned} &\text{Minimise} && \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}x_{ij},\\ &\text{subject to} && \sum_{j=1}^{n} x_{ij} \le a_i,\ i = 1, 2, \ldots, m,\\ &&& \sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n,\\ &&& \sum_{i=1}^{m} a_i \ge \sum_{j=1}^{n} b_j,\\ &&& x_{ij} \ge 0. \end{aligned}\right\} \qquad (14)$$

If equality holds in the third constraint, the total demand can be fully met only with all the supply, and so a feasible solution exists. The problem then reduces to a balanced problem already discussed. If, however, the third constraint is an inequality, the following artifice converts it into a balanced problem.

We create a fictitious sink $j = n + 1$ with demand

$$b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j. \tag{15}$$

The new problem with m sources and $n + 1$ sinks is balanced. Any amount going from a source to the fictitious sink is actually the surplus remaining at that source. Before this balanced problem can be solved, the cost coefficient $c_{i,n+1}$ from the ith source to the fictitious sink for all i, $i = 1, 2, ..., m$, should be known. It is obviously the cost of surplus lying at the source i. Depending upon the physical nature of the problem, it may be zero or some other number which should be estimated.

The problem of surplus at the sources may be posed in another way also. The demand may be flexible with prescribed minimum at each sink. The supply at each source is fixed and all of it must be transported. The problem is then as follows.

$$\left.\begin{aligned} &\text{Minimise} && \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \\ &\text{subject to} && \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m, \\ & && \sum_{i=1}^{m} x_{ij} \geq b_j,\ j = 1, 2, \ldots, n, \\ & && \sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j, \\ & && x_{ij} \geq 0. \end{aligned}\right\} \tag{16}$$

To solve it we again introduce a fictitious sink $j = n + 1$ with demand b_{n+1} given by (15). But now the cost coefficients for the $(n + 1)$th column in the array should be taken as

$$c_{i,n+1} = c_{i,r_i} = \min(c_{i1}, c_{i2}, c_{i3}, \ldots, c_{in}), i = 1, 2, \ldots, m.$$

The idea is that with the minimum demands at all the sinks been met, the surplus at a source is transported to that sink for which the cost is minimum. The optimal solution of the balanced problem with m sources and $n + 1$ sinks provides the optimal solution of the original problem after the value in the $(i, n + 1)$ cell is added to the value in the (i, r_i) cell for all i, $i = 1, 2, ..., m$. (See Problem 7).

The problem in which the total demand exceeds the total supply may be posed as follows.

$$\left.\begin{aligned} &\text{Minimise} && \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \\ &\text{subject to} && \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m, \\ & && \sum_{i=1}^{m} x_{ij} \leq b_j,\ j = 1, 2, \ldots, n, \\ & && \sum_{i=1}^{m} a_i \leq \sum_{j=1}^{n} b_j, \\ & && x_{ij} \geq 0. \end{aligned}\right\} \tag{17}$$

The demand at each sink is not necessarily fully met. The actual supply may fall short of the demand. To solve this problem a fictitious source $i = m + 1$ is introduced with capacity

$$a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i.$$

The cost coefficients in the $(m + 1)$th row should be the cost of deficit at each sink. It again depends upon the physical nature of the problem to estimate how much loss is suffered on account of deficit supply at a sink. The balanced problem of $m + 1$ sources and n sinks so obtained is solved for optimal solution. Supplies from the fictitious $(m + 1)$th source given by the optimal solution are to be interpreted as deficits in the original problem.

The following is an example of an unbalanced problem, also involving degeneracy during its solution.

Example: Table 12 gives the quantity of goods available at four origins O_i, $i = 1, 2, 3, 4$, and the minimum requirement at three destinations, D_j, $j = 1, 2, 3$, and the cost of transportation of unit quantity of goods from origins to destinations. The available goods exceed the minimum total requirement, and the excess can be transported to the destinations, but at minimum cost. Find the distribution of goods such that the total cost of transportation is minimum.

TABLE 12

	D_1	D_2	D_3	
O_1	2	1	3	10
O_2	4	5	7	25
O_3	6	0	9	25
O_4	1	3	5	30
	20	20	15	

The total availability is 90, while the minimum requirement is 55. Hence we introduce a fictitious destination with demand 35. Whatever goes to the fictitious destination D_4 from O_1 should really go to one of the real destinations D_1, D_2 or D_3 such that the cost is minimum. Hence the cost coefficient $c_{14} = \min(c_{11}, c_{12}, c_{13}) = \min(2, 1, 3) = 1$. Similarly $c_{24} = 4$, $c_{34} = 0$, $c_{44} = 1$. The balanced transportation array obtained after introducing D_4 and the corresponding cost coefficients is given in table 13(i). It also indicates a basic feasible solution which serves as a starting point for search for the optimal solution. The simplex multipliers and the resulting relative cost coefficients of the nonbasic variables are also written in the table, as was done in table 8.

Noting that the relative cost coefficient c'_{23} is negative and the least (-7), we bring x_{23} into the basis. Putting $x_{23} = u$, we adjust $x_{33} = 5 - u$, $x_{32} = 15 + u$, $x_{22} = 5 - u$, and hence set $u = 5$. This makes $x_{33} = x_{22} = 0$. Treating only one of these, x_{33}, as the nonbasic variable, and retaining x_{22} as a basic variable, we get the new basic feasible solution as shown in table 13(ii). It is a degenerate solution.

TABLE 13(i)

	D_1	D_2	D_3	D_4	
O_1	 9 2	 7 1	10 3	 7 1	10 6
O_2	20 4	5−u 5	u −7 7	 −1 4	25 −5
O_3	 7 6	15+u 0	5−u 9	5 0	25 0
O_4	 1 1	 2 3	 −5 5	30 1	30 −1
	20 1	20 0	15 −9	35 0	

TABLE 13(ii)

	D_1	D_2	D_3	D_4	
O_1	 2 2	 0 1	10 3	 0 1	10 4
O_2	20 4	0−u 5	5 7	u −1 4	25 0
O_3	 7 6	20+u 0	 7 9	5−u 0	25 5
O_4	 1 1	 2 3	 2 5	30 1	30 4
	20 −4	20 −5	15 −7	35 −5	

TABLE 13(iii)

	D_1	D_2	D_3	D_4	
O_1	 2 2	 1 1	10 3	 1 1	10 4
O_2	20 4	 1 5	5 7	0 4	25 0
O_3	 6 6	20 0	 6 9	5 0	25 4
O_4	 0 1	 2 3	 1 5	30 1	30 3
	20 −4	20 −4	15 −7	35 −4	

For this basic feasible solution c'_{24} is the only negative relative cost coefficient. Therefore x_{24} is brought into the basis. This necessitates dropping x_{22} out of the basis. The new value of x_{24} is also zero. Table 13 (iii) gives the next basic feasible solution. As all the relative cost coefficients are non-negative in this table, we have obtained the optimal solution. It is $x_{13} = 10, x_{21} = 20, x_{23} = 5, x_{32} = 20, x_{34} = 5, x_{44} =$ 30. The value of x_{34} is to be interpreted as the excess quantity sent from O_3 to D_2, and similarly the value of x_{44} is the excess quantity from O_4 to D_1. Thus the optimal distribution in terms of the original problem is: $O_1 \rightarrow D_3$, 10; $O_2 \rightarrow D_1$, 20; $O_2 \rightarrow D_3$, 5; $O_3 \rightarrow D_2$, 25; $O_4 \rightarrow D_1$, 30. The supply to D_1 is 50, to D_2 it is 25, and to D_3

it is 15. The first two destinations receive more than the minimum required, while the third receives just the required minimum. The total cost of transportation is $10 \times 3 + 20 \times 4 + 5 \times 7 + 20 \times 0 + 5 \times 0 + 30 \times 1 = 175$.

13 Caterer problem

What is popularly known as the caterer problem in operations research first arose in connection with number of spare engines required to maintain a fleet of aeroplanes airworthy during a certain period. We shall describe the problem in general terms.

Suppose there is an article which is used once and then sent for repair or servicing before it can be used again. On a job $a_1, a_2, ..., a_n$ (positive integers) of these articles are required at times $T, 2T, ..., nT$ respectively. The job lasts till nT. The job begins at T with a_1 articles purchased new from the market at a certain price. But at successive times the requirement can be met partly by repaired articles and partly, if necessary, through purchase of new ones. The minimum time of repair is rT and maximum $(r + s)T$, r and s being positive integers with $r + s < n$. The quicker the service, the higher the cost of repair, which in any case is less than the price of a new article. The problem is: How to organize purchase and repair of articles so that the job is completed with minimum cost of the articles.

We can look at the problem as follows.

Let x_{ij} be the number of articles received back after repair which were sent for repair at time iT to be returned at time jT, and let c_{ij} be the cost of this repair per article. Then $\sum_{i=1}^{n} x_{ij}$ is the total number of repaired articles available at jT.

Of course x_{ij} is meaningless for $i \geq j$ and can have nonzero value only if $r \leq j - i \leq r + s$. The difficulty is easily overcome by putting $c_{ij} = \infty$ for inadmissible values of i, so that the minimum cost expression can never include a nonzero value of x_{ij} for any inadmissible i. Any shortage at time jT will have to be met by purchase of new articles. Let $x_{n+1,j}$ be this number. Then

$$\sum_{i=1}^{n} x_{ij} + x_{n+1,j} = a_j, \; j = 1, 2, \ldots, n.$$

The use of the symbol $x_{n+1,j}$ for the number of new articles purchased is convenient. It makes up the deficit in the inequality

$$\sum_{i=1}^{n} x_{ij} \leq a_j$$

and is therefore a slack variable. Moreover its introduction helps put the problem in the transportation form.

Also $\sum_{j=1}^{n} x_{ij}$ is the total number of articles sent out for repair at time iT. Again since x_{ij} can be nonzero only for $r \leq j - i \leq r + s$, we put $c_{ij} = \infty$ for inadmissible values of j. All the articles used at time iT need not be sent for repair, as the job is to last only up to nT and if they cannot be repaired before that time they may as well be left unrepaired. The cost of leaving an article unrepaired may be taken as zero. Let $x_{i,n+1}$ be the number of articles used but not sent for repair at time iT. Then

$$\sum_{j=1}^{n} x_{ij} + x_{i,n+1} = a_i, \; i = 1, 2, \ldots, n.$$

Also we have the non-negativity conditions

$$x_{ij} \geq 0, x_{i,n+1} \geq 0, x_{n+1,j} \geq 0$$

The objective function to be minimized is

$$f = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij} + c \sum_{j=1}^{n} x_{n+1,j}$$

where c is the price of the new article.

We may finally put the equations derived above in the standard transportation form

$$\sum_{i=1}^{n+1} x_{ij} = a_j, \; j = 1, 2, \ldots, n+1;$$

$$\sum_{j=1}^{n+1} x_{ij} = a_i, \; i = 1, 2, \ldots, n+1;$$

$$x_{ij} \geq 0;$$

$$f = \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} c_{ij} x_{ij};$$

provided we can give meanings to $x_{n+1,n+1}$, a_{n+1} and $c_{n+1,n+1}$. Let a_{n+1} be a sufficiently large number chosen arbitrarily. A convenient value will be Σa_i which is the total number of articles required on all the days. The variable $x_{n+1,\,n+1}$ can be interpreted as the number of new articles left without being used and so not purchased at all.

The cost of this fictitious transaction may be taken as $c_{n+1,\, n+1} = 0$. This finally puts the problem in the transportation form which can be solved by the standard procedure.

Example: A caterer needs clean table covers every day for six days to meet a contract according to the following schedule.

Days	1	2	3	4	5	6
Number of covers	50	60	80	70	90	100

The cost of a new cover is Rs 20 while washing charges are Re 1 for return on the fourth day or later, Rs 2 for return on the third day and Rs 3 for the next day. Find the minimum cost schedule for the purchase and washing of table covers, assuming that after the end of the contract the covers are rejected.

The problem, when put in the transportation form, is as shown in table 15. The table shows an initial b.f.s. and also the optimal solution (bold numbers). The minimum cost is Rs. 2950. The new purchases are 100 required on the first two days. Subsequently the used ones return after washing.

TABLE 15

		Sent for washing →						Soiled and rejected	
		1	2	3	4	5	6	7	
Received after washing	1	∞	**50** 10 3	20 2	20 1	1	1	0	50
	2	∞	∞	**60** 60 3	2	1	1	0	60
	3	∞	∞	∞	**70** 50 3	**10** 20 2	10 1	0	80
	4	∞	∞	∞	∞	**70** 70 3	2	0	70
	5	∞	∞	∞	∞	∞	**90** 90 3	0	90
	6	∞	∞	∞	∞	∞	∞	**100** 100 0	100
Purchased	7	**50** 50 20	**10** 50 20	**20** 20	20	**10** 20	**10** 20	**350** 350 0	450
		50	60	80	70	90	100	450	