11 Convex functions

DEFINITION 10. *Let* $X \in K \subseteq E_n$ *where* K *is a convex set. A function* $f(X)$ *is said to be convex if for any two points* X_1 and X_2 in K,

$$f(X) \le (1-\lambda)f(X_1) + \lambda f(X_2), 0 \le \lambda \le 1,$$

for every $$X = (1-\lambda)X_1 + \lambda X_2.$$

The function is said to be concave if the inequality sign is reversed or if $-f(X)$ *is convex.*

Interpreting in E_1, let x_1, x_2 be two points M and N respectively on the real line (Fig. 1) and let a convex linear combination of x_1 and x_2 be x which is any point R on the segment MN. With $f(x)$ as the curve shown, $MA = f(x_1)$, $NB = f(x_2)$, $RP = f(x)$. If $x = (1-\lambda)x_1 + \lambda x_2$, it is easy to see that $RC = (1-\lambda)f(x_1) + \lambda f(x_2)$. Now $RP \le RC$ for all R lying in MN means that the curve is bulging out or convex towards the real line. If this happens for all x_1, x_2 in a convex domain $[a, b]$ we say the curve $f(x)$ is convex. If $RP \ge RC$, $f(x)$ is concave.

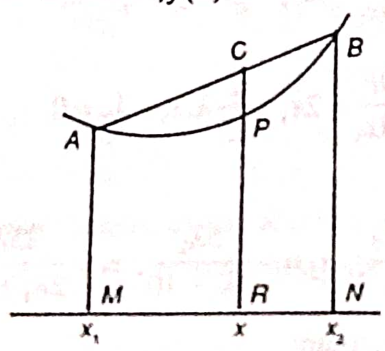

Fig. 1

It is important to note that the convexity or concavity of a function is defined only when its domain is a convex set. The use of convex sets in this context gives us regions in E_n which are in a sense 'unbroken' in each variable x_j, $j = 1, 2, \ldots, n$.

The following theorems are proved for convex functions. The corresponding theorems for concave functions can be easily enunciated.

THEOREM 3. *Let* $X \in E_n$ *and let* $f(X) = X'AX$ *be a quadratic form. If* $f(X)$ *is positive semidefinite, then* $f(X)$ *is a convex function.*

Proof. Let X_1, X_2 be any two points in E_n, and let $X = (1-\lambda)X_1 + \lambda X_2, 0 \le \lambda \le 1$. Also let $f(X) = X'AX$ be positive semidefinite, that is, $X'AX \ge 0$ for any $X \in E_n$. Then

$$\begin{aligned}
&(1-\lambda)f(X_1) + \lambda f(X_2) - f(X) \\
&= (1-\lambda)X'_1AX_1 + \lambda X'_2AX_2 - ((1-\lambda)X_1 + \lambda X_2)'A((1-\lambda)X_1 + \lambda X_2) \\
&= (1-\lambda)X'_1AX_1 + \lambda X'_2AX_2 - (1-\lambda)^2X'_1AX_1 - \lambda^2X'_2AX_2 - 2\lambda(1-\lambda)X'_1AX_2 \\
&= \lambda(1-\lambda)(X'_1AX_1 + X'_2AX_2 - 2X'_1AX_2) \\
&= \lambda(1-\lambda)(X_1 - X_2)'A(X_1 - X_2) \ge 0
\end{aligned}$$

because $0 \leq \lambda \leq 1$ and $X_1 - X_2$ is any vector in E_n. Hence

$$f(X) \leq (1-\lambda)f(X_1) + \lambda f(X_2)$$

which means $f(X)$ is a convex function. Proved.

THEOREM 4. *Let $K \subseteq E_n$ be a convex set, $X \in K$, and $f(X)$ a convex function. Then if $f(X)$ has a relative minimum, it is also a global minimum. Also if this minimum is attained at more than one point, the minimum is attained at the convex linear combination of all such points.*

Proof. Let $f(X)$ have a relative minimum at X_0. Let $X_1 \in K$. Then for any $\delta > 0$ it is possible to choose $\lambda, 0 < \lambda < 1$, such that there exists $X = \lambda X_0 + (1-\lambda)X_1$ lying in the δ-neighbourhood of X_0. By the definition of relative minimum, with X in this neighbourhood

$$f(X_0) \leq f(X)$$

$$\Rightarrow \quad f(X_0) \leq f(\lambda X_0 + (1-\lambda)X_1) \leq \lambda f(X_0) + (1-\lambda)f(X_1), \text{ since } f(X) \text{ is convex}$$

$$\Rightarrow \quad (1-\lambda) f(X_0) \leq (1-\lambda)f(X_1)$$

$$\Rightarrow \quad f(X_0) \leq f(X_1), \text{ since } 1-\lambda \text{ is positive,}$$

$$\Rightarrow \quad f(X_0) \text{ is a global minimum.}$$

Let Y_0 be another point where the minimum is attained. Then

$$f(X_0) = f(Y_0).$$

Since Y_0 is a point in K, what is true of X_1 is also true of Y_0, and so

$$f(X_0) \leq f(\lambda X_0 + (1-\lambda)Y_0)$$

$$\leq \lambda f(X_0) + (1-\lambda)f(Y_0) = f(Y_0)$$

$$\Rightarrow \quad f(X_0) = f(\lambda X_0 + (1-\lambda)Y_0)$$

which means minimum is also attained at the convex linear combination of X_0 and Y_0. Thus the set of points where $f(X)$ is minimum is a convex set and is therefore a convex linear combination of points (not necessarily only two) in it. Proved.

THEOREM 5. *Let $f(X)$ be defined in a convex domain $K \subseteq E_n$ and be differentiable. Then $f(X)$ is a convex function if and only if*

$$f(X_2) - f(X_1) \geq (X_2 - X_1)'\nabla f(X_1)$$

for all X_1, X_2 in K.

Proof. First, for any X_1, X_2 in K let

$$f(X_2) - f(X_1) \geq (X_2 - X_1)'\nabla f(X_1).$$

Let X_3 be any point in K such that

$$X_1 = \lambda X_2 + (1-\lambda)X_3, 0 \leq \lambda \leq 1.$$

Then, from hypothesis,

$$f(X_3) - f(X_1) \geq (X_3 - X_1)'\nabla f(X_1).$$

From the above two inequalities

$$\lambda f(X_2) - \lambda f(X_1) + (1-\lambda) f(X_3) - (1-\lambda) f(X_1)$$

$$\geq [\lambda (X_2 - X_1)' + (1-\lambda)(X_3 - X_1)'] \nabla f(X_1)$$

$$\Rightarrow \lambda f(X_2) + (1-\lambda) f(X_3) - f(X_1) \geq [\lambda X_2 + (1-\lambda) X_3 - X_1]' \nabla f(X_1) = 0$$

$$\Rightarrow \lambda f(X_2) + (1-\lambda) f(X_3) \geq f(X_1) = f(\lambda X_2 + (1-\lambda) X_3)$$

which means $f(X)$ is a convex function.

To prove the converse, let $f(X)$ be a convex function. Then for X_1, X_2 in K and $0 < \lambda < 1$,

$$(1-\lambda) f(X_1) + \lambda f(X_2) \geq f((1-\lambda) X_1 + \lambda X_2)$$

$$\Rightarrow \lambda f(X_2) - \lambda f(X_1) \geq f((1-\lambda) X_1 + \lambda X_2) - f(X_1)$$

$$\Rightarrow f(X_2) - f(X_1) \geq \frac{f(X_1 + \lambda (X_2 - X_1)) - f(X_1)}{\lambda}$$

Taking limit as $\lambda \to 0$, (see definition 5),

$$f(X_2) - f(X_1) \geq (X_2 - X_1)' \nabla f(X_1)$$

Proved.

THEOREM 6. *Let $f(X)$ be a convex differentiable function defined in a convex domain $K \subseteq E_n$. Then $f(X_0)$, $X_0 \in K$, is a global minimum if and only if*

$$(X - X_0)' \nabla f(X_0) \geq 0 \textit{ for all } X \textit{ in } K.$$

Proof. First, let $f(X_0)$ be a global minimum. Then for all X in K

$$f(X) \geq f(X_0).$$

Also, since for any X in K, $\lambda X + (1-\lambda) X_0$ is also in K,

$$f(\lambda X + (1-\lambda) X_0) \geq f(X_0), 0 < \lambda < 1,$$

$$\Rightarrow f(X_0 + \lambda (X - X_0)) \geq f(X_0)$$

$$\Rightarrow f(X_0 + \lambda (X - X_0)) - f(X_0) \geq 0.$$

Dividing by λ and taking limit as $\lambda \to 0$, (see definition 5),

$$(X - X_0)' \nabla f(X_0) \geq 0.$$

It should be noticed that if X_0 is an interior point in K, $f(X_0)$ is also a local minimum and so $\nabla f(X_0) = 0$ and then necessarily $(X - X_0)' \nabla f(X_0) = 0$. It is only when X_0 is a boundary point that $\nabla f(X_0)$ may not be zero, but even then necessarily $(X - X_0)' \nabla f(X_0) \geq 0$.

To prove the converse, let for every X in K

$$(X - X_0)' \nabla f(X_0) \geq 0.$$

Since $f(X)$ is convex, from theorem 5,

$$f(X) - f(X_0) \geq (X - X_0)' \nabla f(X_0) \geq 0$$

which means $f(X_0)$ is a global minimum.

Proved.

The geometrical interpretation of theorems 4, 5, 6 in E_1 is instructive. In E_1 $\nabla f(x)$ is df/dx which is the slope of the curve $f(x)$ with respect to the positive direction of x, and is positive, negative or zero according as the curve slopes up or down or remains steady as x increases. We assume $f(x)$ to be a convex function defined in the convex domain $a \le x \le b$. It is easy to see the truth of theorem 4 that any relative minimum is a global minimum, and if there are minima at two points, there is a minimum at every point on the line joining the two points (Fig. 2). To illustrate theorem 5, in figure 3, $MQ = f(x_2) - f(x_1)$ and $MN = (x_2 - x_1)(df/dx_1)$, and

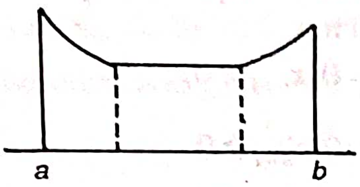

Fig. 2

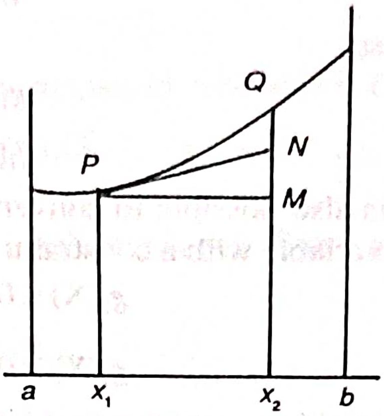

Fig. 3

apparently $f(x_2) - f(x_1) \ge (x_2 - x_1)(df/dx_1)$. In Fig. 4, $QN = f(x_1) - f(x_2)$ and $MN = (x_2 - x_1)(-df/dx_1)$, and so the apparent relation $MN \ge QN$ leads again to $f(x_2) - f(x_1) \ge (x_2 - x_1)(df/dx_1)$. To illustrate theorem 6, if min $f(x)$ lies at $x = x_0$, where $a < x_0 < b$, obviously $df/dx_0 = 0$ and so it is true that $(x - x_0)(df/dx_0) \ge 0$. If, however, $x_0 = a$, (Fig. 5), then $(x - x_0)(df/dx_0) \ge 0$ as $x \ge x_0$ and $(df/dx_0) \ge 0$. If, on the other hand, $x_0 = b$, (Fig. 6), then again $(x - x_0)(df/dx_0) \ge 0$ because $x \le x_0$ and $(df/dx_0) \le 0$.

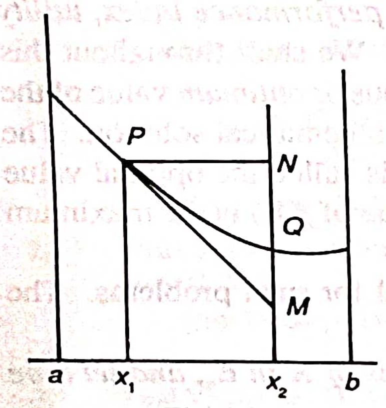

Fig. 4

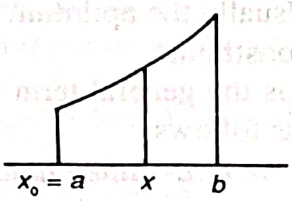

Fig. 5

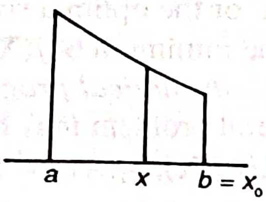

Fig. 6

12 General problem of mathematical programming

In operations research or systems analysis a system has to be studied and analyzed with a view to determine its most advantageous behaviour under certain limitations.

In the mathematical model of the system occur variables $X = (x_1, x_2, ..., x_n)$ which can be controlled and varied, and parameters over which there is no control. The latter are to be regarded as given constants. The limitations on X, when put in mathematical terms, take the form of *constraints* of the type

$$g_i(X) \leq 0,\ i = 1, 2, ..., p\ ;\ g_i(X) \geq 0,\ i = p+1, ...\ r\ ;\ g_i(X) = 0,\ i = r+1, ..., m\ ;$$

where $g_i(X)$ are real-valued functions of X. In general the constraints can always be put as

$$g_i(X) \leq 0, i = 1, 2, \ldots, m,$$

because

$$g_i(X) \geq 0 \Leftrightarrow -g_i(X) \leq 0,$$

and

$$g_i(X) = 0 \Leftrightarrow g_i(X) \geq 0, g_i(X) \leq 0.$$

It is also possible to convert an inequality into an equation by introducing an extra variable with a constraint imposed on it. Thus

$$g_i(X) \leq 0 \Leftrightarrow g_i(X) + x_{n+i} = 0, x_{n+i} \geq 0, \quad \text{(i)}$$

and

$$g_i(X) \geq 0 \Leftrightarrow g_i(X) - x_{n+i} = 0, x_{n+i} \geq 0 \quad \text{(ii)}$$

The variable x_{n+i} so introduced is called a *slack variable*. Many authors distinguish between cases (i) and (ii) by calling x_{n+i} in case (i) the *slack variable* and in case (ii) the *surplus variable*. We, however, propose to use the term *slack* to cover both cases.

The constraints $X \geq 0$, usually referred to as nonnegativity conditions, often occur in mathematical models of systems either because negative values of variables do not make any sense and are therefore excluded from consideration or because it is mathematically convenient to introduce some slack variables with this constraint.

The performance, return, utility or whatever other objective is sought to be achieved through the system is generally measured by a real-valued function $f(X)$. It is given various names in various problems, such as *performance index, utility measure, return function, objective function, gain*, etc. We shall throughout this book call it the objective function. The most advantageous or *optimum* value of the objective function is sought to be achieved through a mathematical solution. The value X_0 of the variable X which makes $f(X)$ optimum is called the optimal value of X or the optimal solution. Usually the optimum value of $f(X)$ is the maximum or the minimum of $f(X)$ under constraints.

Mathematical programming is the general term used for such problems. The general problem may be stated as follows.

Let $f(X)$, $g_i(X), i = 1, 2, \ldots, m$, be real-valued functions of X in E_n, and let S be the subset of E_n containing all points satisfying the constraints

$$g_i(X) \leq 0, i = 1, 2, \ldots, m; X \geq 0.$$

To find X_0 in S such that $f(X_0)$ is a global minimum in S.

As explained above, we may introduce slack variables x_{n+i} to put the constraints as

$$g_i(X) + x_{n+i} = 0,\ x_{n+i} \geq 0.$$

Then the total number of variables becomes $n + m$. We can state the general problem in the following alternative form also.

To find $X_0 \in S \subseteq E_n$ such that $f(X_0)$ is a global minimum in S and for all X in S

$$g_i(X) = 0,\ i = 1, 2, \ldots, m;\ X \geq 0.$$

It should however be remembered that n in the above statement is $n + m$ of the earlier statement and neither X nor g_i are identical in the two cases.

If S is a convex set and $f(X)$ and $g_i(X)$ are convex functions, the problem is said to be of *convex programming*. The following theorem is significant in this connection.

THEOREM 7. *Let* $X \in E_n$ and let $g_i(X)$, $i = 1, 2, \ldots, m$, *be convex functions in* E_n. Let $S \subseteq E_n$ *be the set of points satisfying the constraints* $g_i(X) \leq 0, i = 1, 2, \ldots, m$. *Then* S *is a convex set*

Proof. Let X_1, X_2 be in S, and let $X_3 = \lambda X_1 + (1-\lambda)X_2, 0 \leq \lambda \leq 1$. Since $g_i(X)$ is a convex function and $g_i(X_1) \leq 0, g_i(X_2) \leq 0$,

$$g_i(X_3) = g_i(\lambda X_1 + (1-\lambda)X_2)$$

$$\leq \lambda g_i(X_1) + (1-\lambda)g_i(X_2) \leq 0.$$

Hence X_3 is in S and so S is a convex set. Proved.

BIBLIOGRAPHICAL NOTE

(For references see bibliography)

Apostol [1969] gives a rigorous modern approach to the classical problem of extrema of functions of several variables. For functions of two variables Widder [1968] is sufficient.

Convex functions are discussed by Gass [1969], and in a more sophisticated manner by Berge and Ghouila-Houri [1965].

PROBLEMS II

Gradients and derivatives

1. Find $\nabla f(X)$ and $H(X)$ for
   (i) $f(X) = x_1^2 + 3x_1x_2 - 4x_2^2 + 4x_1 + 5x_2x_3 - x_3^2$;
   (ii) $f(X) = x_1^3 + 2x_2^3 + 3x_1x_2x_3 + x_3^2$.

2. Write Taylor series for $f(X)$ of problem 1 (i) about the origin and of problem 1 (ii) about the point (1, 1, 1).
   [(ii) $7 + 6(x_1-1) + 9(x_2-1) + 5(x_3-1) + 3(x_1-1)^2 + 6(x_2-1)^2 + (x_3-1)^2 + 3(x_1-1)(x_2-1)$ $+ 3(x_2-1)(x_3-1) - 3(x_3-1)(x_1-1) \ldots$]

3. Find $\nabla f(X_0)$ and $H(X_0)$ at $X_0 = (2, 0, -1)$ for

$$f(X) = 6x_2^2 - 18x_2x_3 - 6x_3x_1 + 2x_1x_2 - 7x_1 + 5x_2 - 6x_3 - 4,$$

   and write Taylor series for $f(X)$ about X_0.
   [$-(x_1-2) + 27x_2 - 18(x_3+1) + 2(x_1-2)x_2 - 6(x_1-2)(x_3+1) - 18x_2(x_3+1) + 6x_2^2$]

4. Find the directional derivative of $f(X)$ at X_0 of problem 3 in the direction of the vector $Y = [1\ 1\ 1]'$. [$8/\sqrt{3}$]

5. Find the unit vector in the direction of the steepest ascent of

$$f(X) = x_1^2 + 2x_1x_2 + x_1x_3 + x_2x_4 + x_4^2$$

at the point (1, 0, − 1, 1). Also find the directional derivative in the same direction.

[[1 3 1 2]$'/\sqrt{15}, \sqrt{15}$]

6. Find a point Y in E_4 such that $|Y - X_0| = 4$ and $Y - X_0$ is the vector of steepest descent for the function

$$f(X) = x_1^2 - 3x_1x_2 + 4x_1x_3 + x_3 + 4x_4^2$$

at the point $X_0 = (1, 0, -1, -1)$. [$1 + 8/\sqrt{102}, 12/\sqrt{102}, -1 - 20/\sqrt{102}, -1 + 32/\sqrt{102}$]

7. Find the directional derivative of $f(X) = 2x_1^3x_2 - 3x_2^2x_3$ at the point $X_0 = (1, 2, -1)$ in a direction towards the point Y = (3, − 1, 5). Find also the maximum directional derivative at X_0 and the vector of steepest descent. [− 90/7; 22, [−12 −14 12]]

Extrema of functions

8. Find the relative maxima and minima and saddle points, if any, of

$$f(X) = x_1^3 + x_2^3 - 3x_1 - 12x_2 + 25.$$

[Min 7 at (1, 2), max 43 at (− 1, − 2), saddle at (− 1, 2), (1, − 2)]

9. Show that the function $f(X) = x_1^2 + x_2^2 + x_3^2 - 6x_1x_2 + 8x_1x_3 - 10x_2x_3$ has a saddle point at the origin.

10. Show that $f(X) = 7x_1^2 + 10x_2^2 + 7x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$ has a local minimum at the origin which is also a global minimum.

11. Find the relative extrema of $(x_1^2 - 2x_1 + 4x_2^2 - 8x_2)^2$. [Max 25 at (1, 1)]

12. Find the least value of $|X|$, X in E_3, subject to constraints $x_1^2 + 8x_1x_2 + 7x_2^2 = 225$, $x_3 = 0$. [5]

13. Find the point on the surface $z = x^2 + y^2$ nearest to the point (3, − 6, 4). [(1, − 2, 6)]

14. Find the extreme values of x_3 such that

$$2x_1^2 + 3x_2^2 + x_3^2 - 12x_1x_2 + 4x_1x_3 - 35 = 0$$

[5, −5]

15. Give a geometrical interpretation of the solved example in section 10.

16. Find the volume of the largest rectangular solid inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

[$8abc/3\sqrt{3}$]

17. Find the extrema of $x_1x_2^2x_3^3$ under the constraints

$$x_1 + x_2 + x_3 = 6, x_1 > 0, x_2 > 0, x_3 > 0.$$

[108 at (1, 2, 3)]

18. Prove that the maximum and minimum distance from the origin to the curve of intersection defined by

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1, c_1x_1 + c_2x_2 + c_3x_3 = 0, \text{ in } E_3$$

can be obtained by solving for d the equation

$$\frac{c_1^2a_1^2}{a_1^2 - d^2} + \frac{c_2^2a_2^2}{a_2^2 - d^2} + \frac{c_3^2a_3^2}{a_3^2 - d^2} = 0.$$

19. Use the method of Lagrange multipliers to find the maxima and minima of $(x_1 - 4)^2 + (x_2 - 3)^2$ subject to $36(x_1 - 2)^2 + (x_2 - 3)^2 = 9$.

Verify the results through geometrical interpretation.
[Local minima at (5/2, 3) and (3/2, 3), the former being the global minimum; global maxima at (68/35, 5.98) and (68/35, 0.02).]

20. Use the method of Lagrange multipliers to find the maxima and minima of $x_2^2-(x_1+1)^2$ subject to $x_1^2+x_2^2 \leq 1$.

[Maxima at $(-1/2, \pm\sqrt{3}/2)$, minimum at (1, 0). Is there an extremum at (− 1, 0) ?]

Convex functions

21. Prove that $f(x)=x^2, x \in R$, is a convex function.
22. Prove that $f(X)=||X||, X \in E_n$, is a convex function.
23. Prove that the linear function $f(X)=CX, X \in E_n$, is both convex and concave.
24. Prove that $f(X)=2x_1^2+2x_2^2+4x_3^2+2x_1x_2+2x_1x_3+4x_2x_3$ is a convex function.
25. Show that the sum of two convex functions is a convex function.
26. Prove that every positive linear combination of convex functions in K is a convex function in K. (This is a generalization of the above problem.)
27. Prove that for $f(X)$ to be convex in K, it is necessary and sufficient that for any positive integer m and for $X_i \in K, i=1,2,\ldots,m$.

$$f(\lambda_1X_1+\lambda_2X_2+\ldots+\lambda_mX_m) \leq \lambda_1f(X_1)+\lambda_2f(X_2)+\ldots+\lambda_mf(X_m)$$

where
$$\sum_{i=1}^{m}\lambda_i=1, \lambda_i \geq 0.$$

(Compare with theorem 8, chapter 1).

3

Linear Programming

1 Introduction

The general problem of mathematical programming, described in chapter 2, section 12, reduces to linear programming (LP) when the functions $f(X)$, $g_i(X)$, $i = 1, 2, \ldots, m$, all are linear. So the problem of LP is to find a minimum (or maximum) of a linear function subject to linear constraints. Because of its relative simplicity LP was the first area of constrained optimization which attracted the mathematicians during World War II when problems arising in military operations were posed before them. Subsequently it found extensive use in problems of economics, management, planning and other complex operations in diverse areas of human study and activity.

2 LP in two-dimensional space

Since the basic features of LP can be illustrated in two-dimensional space, we first consider an LP problem in two variables.

Let $X \in E_2$, and

$$f(X) = 4x_1 + 5x_2 \tag{1}$$

Also let

$$\begin{aligned} x_1 - 2x_2 &\le 2, && \text{(i)} \\ 2x_1 + x_2 &\le 6, && \text{(ii)} \\ x_1 + 2x_2 &\le 5, && \text{(iii)} \\ -x_1 + x_2 &\le 2, && \text{(iv)} \\ x_1 + x_2 &\ge 1, && \text{(v)} \end{aligned} \tag{2}$$

$$x_1, x_2 \ge 0. \tag{3}$$

The problem is to find $X_0=(x_{10}, x_{20})$ which maximizes $f(X)$ and satisfies the constraints (2) and (3).

Let us try a graphical approach. The non-negativity conditions (3) restrict the point X_0 to the first quadrant of the E_2 space (x_1 x_2 plane). Constraint (i) gives the half-space bounded by the straight line $x_1-2x_2=2$. If (i) is to be satisfied, the point X_0 is either on this line or on the same side of it as the origin. Similarly (ii) is the half-space of points lying on the line $2x_1 + x_2 = 6$ or on the same side of it as the

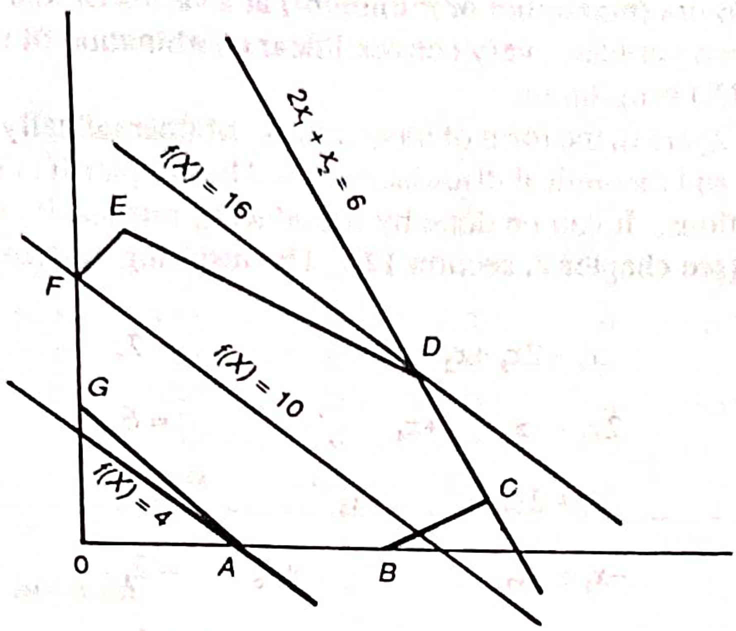

Fig. 1

origin. So with (iii) and (iv). Constraint (v) however is the half-space of points either on the line $x_1 + x_2 = 1$ or on that side of it which is opposite to the origin. The intersection of all constraints (2) and (3) is the convex polygon *ABCDEFG* (see Fig. 1). Any point **X** within the polygon or on its boundary satisfies constraints (2) and (3). An infinity of such points exist. Our problem is to find that point (or points) $\mathbf{X}_0$ within the polygon or on its boundary which makes $f(\mathbf{X}) = 4x_1 + 5x_2$ the maximum.

Consider the parallel straight lines that correspond to different values of $f(\mathbf{X})$. The lines $f(\mathbf{X}) = 8, 10, 12$ cut through the polygon and $f(\mathbf{X}) = 16$ just goes through the point *D*. The line for values of $f(\mathbf{X})$ greater than 16 will not intersect the polygon at all. Therefore the maximum value, subject to the constraints, that $f(\mathbf{X})$ can have is 16, and it is attained at the point $x_1 = 7/3, x_2 = 4/3$, which is one of the vertices of the polygon.

Suppose the problem were to find the minimum value of $f(\mathbf{X})$ subject to the same constraints. It can be seen that the line $f(\mathbf{X}) = 4$ just passes through the point $A(1, 0)$ and lines for smaller values of $f(\mathbf{X})$ do not intersect the polygon. The solution therefore would be $\min f(\mathbf{X}) = 4$ for $\mathbf{X} = (1, 0)$ which again is a vertex of the polygon.

In both cases the point where $f(\mathbf{X})$ attains its extreme value is unique and is a vertex of the polygon.

Let us consider a slightly different case. Let $f(\mathbf{X})$ be $2x_1 + x_2$, and suppose the maximum of $f(\mathbf{X})$ subject to the same constraints has to be determined. The answer in this case is $f(\mathbf{X}) = 6$ with any point on the side *DC* of the polygon giving this value. $f(\mathbf{X})$ thus attains its maximum value at the vertex $C(14/5, 2/5)$ and also at $D(7/3, 4/3)$ and also at every point on *CD*, that is, any convex linear combination of *C* and *D*.

The following features of the problem which are of fundamental significance deserve notice.

(i) The set of solutions of (2) and (3) is a convex set with vertices.

(ii) $f(X)$ is optimum (maximum or minimum) at a vertex of this convex set, and if there are two such vertices, every convex linear combination of the vertices is also a point where $f(X)$ is optimum.

Constraints (2) are in the form of inequalities. Mathematically it is easier to deal with equations, and theoretical discussion would be simpler if constraints could be written as equations. It can be done by introducing one slack variable in each of the constraints (see chapter 2, section 12).' The resulting system of equations and constraints is

$$\begin{aligned} x_1 - 2x_2 + x_3 \qquad\qquad\qquad\qquad &= 2, \\ 2x_1 + x_2 \quad + x_4 \qquad\qquad\qquad &= 6, \\ x_1 + 2x_2 \qquad\quad + x_5 \qquad\quad &= 5, \\ -x_1 + x_2 \qquad\qquad\quad + x_6 \quad &= 2, \\ x_1 + x_2 \qquad\qquad\qquad\quad - x_7 &= 1; \end{aligned} \tag{4}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0. \tag{5}$$

Note that we introduce the slack variables in such a way that all of them, along with the original variables, satisfy the non-negativity conditions (5).

We have replaced (2) and (3) with the equivalent conditions (4) and (5) but only by increasing the number of variables. But any complication caused by this increase is more than compensated by the fact that we have to deal with equations now.

Let us examine equations (4) closely. These are five equations in seven unknowns and therefore can have an infinity of solutions. Putting any two variables zero, we get unique values for the rest of the variables, thus getting a basic solution. There are ${}^7C_2$ (=21) ways of choosing two variables as zero. Table 1 gives all the 21 basic solutions of (4). The significant thing about these solutions is that there are exactly seven solutions which are non-negative, and these, so far as the values of x_1 and x_2 are concerned, correspond to the seven vertices of the convex polygon we have obtained graphically. One can suspect some theoretical relationship between such solutions of equations of constraints and the solution to the LP problem. We shall establish such a relationship in the following sections.

TABLE 1

No.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Vertices of Polygon
1	0	0	2	6	5	2	−1	
2	0	−1	0	7	7	3	−2	
3	0	6	14	0	−7	−4	5	
4	0	5/2	7	7/2	0	−1/2	3/2	
5	0	2	6	4	1	0	1	
6	0	1	4	5	3	1	0	F
7	2	0	0	2	3	4	1	G
8	3	0	−1	0	2	5	2	B

9	5	0	−3	−4	0	7	4	
10	−2	0	4	10	7	0	−3	
11	1	0	1	4	4	3	0	A
12	14/5	2/5	0	0	7/5	22/5	11/5	C
13	7/2	3/4	0	−7/4	0	19/4	−13/4	
14	−6	−4	0	22	19	0	11	
15	4/3	−1/3	0	11/3	13/3	11/6	0	
16	7/3	4/3	7/3	0	0	3	8/3	D
17	4/3	10/3	22/3	0	−3	0	−11/3	
18	5	−4	−11	0	8	11	0	
19	1/3	7/3	19/3	3	0	0	5/3	E
20	−3	4	13	8	0	−5	0	
21	−1/2	3/2	11/2	11/2	5/2	0	0	

3 General LP problem

We can now enunciate the general LP problem as follows.
Let $X \in E_n$ and $f(X)$ and $g_i(X)$ be linear functions defined as

$$f(X) = \sum_{j=1}^{n} c_j x_j,$$

$$g_i(X) = \sum_{j=1}^{n} a_{ij} x_j - b_i;\ c_j, a_{ij}, b_i \in R;\ i = 1, 2, \ldots, m, j = 1, 2, \ldots, n.$$

To find X_0 such that

$$f(X_0) \le f(X)$$

for all X satisfying the constraints

$$g_i(X) = 0, i = 1, 2, \ldots, m,$$

and $$X \ge 0$$

It is more customary to state this LP problem, in matrix notation, as

Minimize $$f(X) = CX, \tag{6}$$

subject to $$AX = B, \tag{7}$$

$$X \ge 0 \tag{8}$$

where C is a row vector and X and B are column vectors

$$C = [c_1\, c_2 \ldots c_n],\ X = [x_1\, x_2 \ldots x_n]',\ B = [b_1\, b_2 \ldots b_m]',$$

and A is an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \cdot & \ldots & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

The equivalent form of the problem in ordinary notation is

Minimize $$f = \sum_{j=1}^{n} c_j x_j, \tag{6}$$

subject to $$\sum_{j=1}^{n} a_{ij} x_j = b_i, \; i = 1, 2, \ldots, m, \tag{7}$$

$$x_j \geq 0, \; j = 1, 2, \ldots, n \tag{8}$$

Expression (6) is the objective function to be minimized, (7) are the constraints, and (8) the non-negativity conditions. Equations (8) are also constraints but, because of their simplicity, are treated separately from (7). The coefficients c_j are usually called the *cost coefficients*.

The form (6), (7), (8) of an LP problem is general. If given in any other form it can always be converted to this form (see chapter 2, section 12). If $f(X)$ is to be maximized, then we may put $-f(X) = \psi(X)$ which is to be minimized. If a constraint is an inequality, then it can be converted to an equation by introducing a slack variable. If a variable x_j is unconstrained, that is, if it may vary from $-\infty$ to $+\infty$, then we may replace x_j by two other variables x_{j1} and x_{j2} such that $x_j = x_{j1} - x_{j2}$ where $x_{j1} \geq 0, x_{j2} \geq 0$.

Example: Write the following LP in the above standard form.

Maximize $$f = 2x_1 + x_2 - x_3,$$

subject to
$$2x_1 - 5x_2 + 3x_3 \leq 4,$$
$$3x_1 + 6x_2 - x_3 \geq 2,$$
$$x_1 + x_2 + x_3 = 4,$$
$$x_1 \geq 0, x_3 \geq 0, x_2 \text{ unrestricted}.$$

It has equality as well as inequality constraints, and one variable x_2 is unrestricted.

Replacing x_2 by two variables x_{21}, x_{22}, such that $x_2 = x_{21} - x_{22}$, $x_{21} \geq 0$, $x_{22} \geq 0$, putting all the constraints as equations by introducing slack variables x_4 and x_5, and changing the sign of the objective function, the problem takes the following standard form.

Minimize $$\psi = (-f) = -2x_1 - x_{21} + x_{22} + x_3,$$

subject to
$$2x_1 - 5x_{21} + 5x_{22} + 3x_3 + x_4 = 4,$$
$$3x_1 + 6x_{21} - 6x_{22} - x_3 - x_5 = 2,$$
$$x_1 + x_{21} - x_{22} + x_3 = 4,$$
$$x_1, x_{21}, x_{22}, x_3, x_4, x_5 \geq 0$$

4 Feasible solutions

DEFINITION 1. *A solution of* (7) *and* (8) *is called a feasible solution.*

We shall denote by S_F the set of feasible solutions. It is possible that there may be no feasible solution. In that case S_F is empty.

THEOREM 1. *The set S_F of feasible solutions, if not empty, is a closed convex set (polytope) bounded from below and so has at least one vertex.*

Proof. S_F is the intersection of the hyperplanes $g_i(X) = 0, i = 1, 2, \ldots, m$, and the set $H = \{X \mid X \geq 0\}$. All these are closed convex sets and H is bounded from below. Hence S_F is a closed convex set (polytope) bounded from below, and so it has a vertex (see chapter 1, section 18, theorem 21).

Alternatively, we can give a more direct proof of the convexity of S_F as follows. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two feasible solutions. Then

$$\mathbf{X}_1 \geq 0, \ \mathbf{X}_2 \geq 0; \tag{9}$$

and

$$\mathbf{AX}_1 = \mathbf{B}, \ \mathbf{AX}_2 = \mathbf{B}. \tag{10}$$

Let $\mathbf{X}$ be any convex linear combination of $\mathbf{X}_1$ and $\mathbf{X}_2$. Then

$$\mathbf{X} = (1-\lambda)\mathbf{X}_1 + \lambda\mathbf{X}_2, \ 0 \leq \lambda \leq 1,$$

$$\geq 0, \quad \text{from (9).} \tag{11}$$

Further

$$\mathbf{AX} = \mathbf{A}[(1-\lambda)\mathbf{X}_1 + \lambda\mathbf{X}_2]$$

$$= (1-\lambda)\mathbf{AX}_1 + \lambda\mathbf{AX}_2 = \mathbf{B}, \ \text{from (10).} \tag{12}$$

(11) and (12) mean that $\mathbf{X}$ is a feasible solution. Thus the convex linear combination of every two feasible solutions is a feasible solution. Therefore the set of feasible
solutions is a convex set. Proved.

5 Basic solutions

Equations (7), namely

$$\mathbf{AX} = \mathbf{B} \tag{7}$$

are m equations in n unknowns. We shall assume that $m < n$ and the equations are linearly independent (see chapter 1, section 9). Generally constraints appear as inequalities in mathematical models and the introduction of slack variables makes $m < n$. Therefore the assumption is justified.

If any of the $n-m$ variables x_j are given the values zero, the remaining system of m equations in m unknowns may have a unique solution (section 9, chapter 1). This solution along with the assumed zeros is a solution of (7). It is called a *basic* solution. The m variables remaining in the system after $n-m$ variables have been put equal to zero are called the *basic* variables or simply the *basis*. The rest of the variables may be called nonbasic. Since the unique solution of m equations in m variables may also contain zeros, a basic solution must contain at least $n-m$ zeros. (The case when the number of zeros is more than $n-m$ is called degenerate, and will be discussed later.)

Equations (7) may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ b_m \end{bmatrix} \tag{13}$$

or as

$$x_1\mathbf{P}_1 + x_2\mathbf{P}_2 + x_3\mathbf{P}_3 + \ldots + x_n\mathbf{P}_n = \mathbf{B} \tag{14}$$

where $\mathbf{P}_j, j = 1, 2, \ldots, n$, is the m-vector in the jth column of $\mathbf{A}$.

Since $\mathbf{P}_j$ is a vector in E_m, not more than m of the vectors $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ can be linearly independent. Since the equations are assumed linearly independent, exactly m of the vectors are linearly independent (see chapter 1, section 9). Let these m vectors (suffixes rearranged, if necessary) be

$$\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots, \mathbf{P}_m.$$

That they are linearly independent means that there do not exist $\alpha_j \in R, j = 1, 2, \ldots, m$, not all zero, such that

$$\alpha_1\mathbf{P}_1 + \alpha_2\mathbf{P}_2 + \ldots + \alpha_m\mathbf{P}_m = 0.$$

On the other hand it is possible to find α_j, not all zero, such that

$$\alpha_1\mathbf{P}_1 + \alpha_2\mathbf{P}_2 + \ldots + \alpha_m\mathbf{P}_m + \alpha_{m+r}\mathbf{P}_{m+r} = 0$$

where $\mathbf{P}_{m+r}$ is any of the remaining vectors of the set. Thus the vectors $\mathbf{P}_{m+1}, \mathbf{P}_{m+2}, \ldots, \mathbf{P}_n$ can be separately expressed as linear combinations of $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m$, and so (14) can be rewritten as

$$y_1\mathbf{P}_1 + y_2\mathbf{P}_2 + \ldots + y_m\mathbf{P}_m = \mathbf{B}$$

Suppose that the m-vector $[\xi_1\ \xi_2\ \xi_3 \ldots \xi_m]'$ is the solution of the above equations. Then the n-vector $[\xi_1\ \xi_2 \ldots \xi_m 0\ 0 \ldots 0]'$ is a solution of (14) or (7).

This is a basic solution of $\mathbf{AX} = \mathbf{B}$. The corresponding linearly independent vectors $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m$ are a basis and the variables $x_1, x_2, \ldots, x_m$ are the basic variables.

6 Basic feasible solutions

DEFINITION 2. *A basic solution of* (7) *satisfying* (8) *is called a basic feasible solution (b.f.s.)*

THEOREM 2. *A basic feasible solution of the LP problem is a vertex of the convex set of feasible solutions. Or, equivalently, if a set of vectors* $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m$ *can be found that are linearly independent such that*

$$\xi_1\mathbf{P}_1 + \xi_2\mathbf{P}_2 + \ldots + \xi_m\mathbf{P}_m = \mathbf{B}, \tag{15}$$

and $$\xi_j \geq 0,\ j = 1, 2, \ldots, m,$$

then $$\mathbf{X}_\xi = [\xi_1\ \xi_2 \ldots \xi_m\ 0\ 0 \ldots 0]'$$

which is a b.f.s. is an extreme point (or vertex) of S_F.

Proof. That the point $\mathbf{X}_\xi$ belongs to S_F is obvious. Suppose it is not an extreme point. Then two points $\mathbf{X}_1$ and $\mathbf{X}_2$ different from $\mathbf{X}_\xi$ exist in S_F such that

$$\mathbf{X}_\xi = \lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2,\quad 0 < \lambda < 1,$$

that is, $$\xi_j = \lambda x_{j2} + (1-\lambda)x_{j2},\ j = 1, 2, \ldots, m,$$

and $$0 = \lambda x_{j1} + (1-\lambda)\, x_{j2},\ j = m+1, \ldots, n.$$

Since $$\mathbf{X}_1, \mathbf{X}_2 \in S_F,\ x_{j1}, x_{j2} \geq 0.\ \text{Also } 0 < \lambda < 1.$$

Hence $$x_{j1} = x_{j2} = 0,\ j = m+1, \ldots, n.$$

Therefore
$$X_1 = [x_{11}\ x_{21}\ \ldots\ x_{m1}\ 0\ 0\ \ldots\ 0]',$$
$$X_2 = [x_{12}\ x_{22}\ \ldots\ x_{m2}\ 0\ 0\ \ldots\ 0]'.$$
Since X_1, X_2 are solutions of $AX = B$,
$$x_{11}P_1 + x_{21}P_2 + \ldots + x_{m1}P_m = B, \qquad (16)$$
$$x_{12}P_1 + x_{22}P_2 + \ldots + x_{m2}P_m = B. \qquad (17)$$
From (15) and (16),
$$(\xi_1 - x_{11})\,P_1 + (\xi_2 - x_{21})\,P_2 + \ldots + (\xi_m - x_{m1})\,P_m = 0.$$
But $P_1, P_2, \ldots, P_m$ are, by hypothesis, linearly independent. Therefore
$$\xi_1 = x_{11}, \xi_2 = x_{21}, \ldots, \xi_m = x_{m1},$$
or
$$X_\xi = X_1,$$
which contradicts the assumption. Hence X_ξ is an extreme point. Proved.

THEOREM 3. *A vertex of S_F is a basic feasible solution.*
(This is the converse of theorem 2.)

Proof. Let $X_\xi = [\xi_1, \xi_2, \ldots, \xi_n]'$ be a vertex of S_F. Then since $X_\xi \in S_F$, $X_\xi \geq 0$.

Let r of the ξ_j's, $j = 1, 2, \ldots, n$, be nonzero, where $r \leq n$. Since $m < n$, either $r \leq m$ or $r > m$. If $r \leq m$, X_ξ is obviously a b.f.s. and so the theorem holds.

If $r > m$, then we may put X_ξ as
$$X_\xi = [\xi_1\ \xi_2\ \ldots\ \xi_r,\ 0\ 0\ \ldots\ 0]'.$$
where $\xi_j > 0$ for $j = 1, 2, \ldots, r$. Since X_ξ is a solution of $AX = B$, we have
$$\xi_1P_1 + \xi_2P_2 + \ldots + \xi_rP_r = B. \qquad (18)$$
As $r > m$, the vectors $P_1, P_2, \ldots, P_r$ are not linearly independent.
Hence there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_r$ not all zero such that
$$\alpha_1P_1 + \alpha_2P_2 + \ldots + \alpha_rP_r = 0.$$
Multiplying by $c > 0$, we get
$$c\alpha_1P_1 + c\alpha_2P_2 + \ldots + c\alpha_rP_r = 0. \qquad (19)$$
From (18) and (19),
$$(\xi_1 + c\alpha_1)\,P_1 + (\xi_2 + c\alpha_2)\,P_2 + \ldots + (\xi_r + c\alpha_r)\,P_r = B, \qquad (20)$$
and
$$(\xi_1 - c\alpha_1)\,P_1 + (\xi_2 - c\alpha_2)\,P_2 + \ldots + (\xi_r - c\alpha_r)\,P_r = B. \qquad (21)$$
Choose $c > 0$ sufficiently small to make
$$\xi_j \pm c\alpha_j > 0 \text{ for } j = 1, 2, \ldots, r.$$
Then we conclude from (20) and (21) that
$$X_1 = [\xi_1 + c\alpha_1\ \ \xi_2 + c\alpha_2\ \ldots\ \xi_r + c\alpha_r\ \ 0\ 0\ \ldots\ 0]',$$
and
$$X_2 = [\xi_1 - c\alpha_1\ \ \xi_2 - c\alpha_2\ \ldots\ \xi_r - c\alpha_r\ \ 0\ 0\ \ldots\ 0]'$$
are feasible solutions. We have now three feasible solutions, X_ξ, X_1 and X_2 which are related through

$$X_\xi = \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2$$

Hence X_ξ is a convex linear combination of X_1 and X_2 which are both different from X_ξ. This means that X_ξ is not a vertex which contradicts our initial assumption.

Hence $r > m$, which means that X_ξ is a basic feasible solution. Proved.

COROLLARY. *Associated with every extreme point of S_F is a set of m linearly independent vectors $P_1, P_2, \dots, P_m$ of A.*

7 Optimal solutions

THEOREM 4. *If S_F is nonempty, the objective function $f(X)$ has either an unbounded minimum or it is minimum at a vertex of S_F.*

By unbounded minimum we mean that there is always an X in S_F such that $f(X) < -N$ where N is as large a positive number as we please. In other words $f(X)$ can be made as small as we please without violating the constraints.

For a clear understanding of the proof given below the reader is advised to read the sections on convex sets in chapter 1, particularly sections 14-19.

Proof. Two cases arise. Either S_F is bounded or unbounded.

Case (i). S_F is bounded. Then S_F has vertices and every point in S_F is a convex linear combination of its vertices. Let $X_1, X_2, \dots, X_p$ be the vertices of S_F.

Since S_F is closed and bounded, $f(X)$ is finite for all X in S_F, and so there is a point X_0 in S_F where $f(X_0)$ is minimum. X_0 can be expressed as a convex linear combination of $X_r, r = 1, 2, \dots, p$, and so

$$X_0 = \sum_{r=1}^{p} \alpha_r X_r, \quad \sum_{r=1}^{p} \alpha_r = 1, \alpha_r \geq 0.$$

Since $f(X)$ is linear,

$$f(X_0) = f\left(\sum_{r=1}^{p} \alpha_r X_r\right) = \sum_{r=1}^{p} \alpha_r f(X_r)$$

$$\geq \sum_{r=1}^{p} \alpha_r f(X_k) = f(X_k),$$

where $f(X_k)$ is the least of the values $f(X_r)$, $r = 1, 2, \dots p$. But by hypothesis

$$f(X_0) \leq f(X_k).$$

Therefore $$f(X_0) = f(X_k)$$

which means that $f(X)$ is minimum at X_k which is a vertex of S_F.

Case (ii). S_F is unbounded. Since it is bounded from below (see theorem 1), S_F has a vertex. Let X_1 be its vertex.

Consider the cone S_C with vertex X_1 and produced by the hyperplanes intersecting at X_1 (see Fig. 2). S_F is a subset of S_C. The edges of S_C are also wholly or partly edges of S_F in the following sense. If Y is a fixed point other than X_1 on an edge of S_C, then any point on the edge is $X=(1-\lambda) X_1 + \lambda Y, \lambda \geq 0$. If X is in S_F for all $\lambda \geq 0$, then the edge of S_C is also wholly an edge of S_F (as X_1A in figure). If

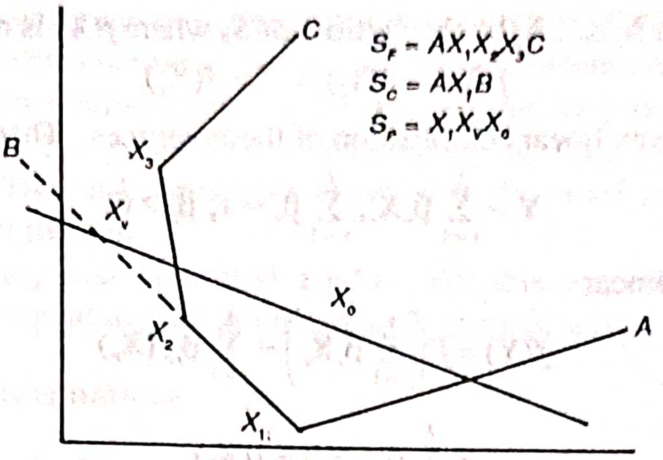

Fig. 2

X is in S_F for $\lambda \leq \lambda_0$ and not in S_F for $\lambda > \lambda_0$, then, the edge of S_C is partly an edge of S_F (as X_1B in figure), and the point $X_2 = (1-\lambda_0) X_1 + \lambda_0 Y$ is the other extremity of the edge of S_F, the first extremity being the point X_1. X_2 is also a vertex of S_F. Thus moving along any edge of the cone S_C from the vertex X_1, either we shall arrive at another vertex X_2 of S_F or not. In the latter case S_F has an unbounded edge.

Now consider $f(X_1)$. One of the two following situations can arise.

(a) For every X on any of the edges of S_C, $f(X_1) \leq f(X)$. In other words $f(X)$ does not decrease as λ increases where $X=(1-\lambda) X_1 + \lambda Y$, $\lambda > 0$, and Y is any fixed point on any of the edges.

If possible, let X_0 in S_C be a point different from X_1 such that $f(X_0)$ is minimum for all X in S_C. We can find a hyperplane containing X_0 intersecting all the edges of S_C and thereby producing a bounded polytope $S_P \subseteq S_C$. The vertices of S_P are contained in the edges of S_C. From case (i) $f(X)$ has a minimum in the bounded polytope S_P at one of the vertices of S_P. Let X_v be such a vertex. Then since $X_0 \in S_P \subseteq S_C$, $f(X_0) \leq f(X_v)$. Also $f(X_v) \leq f(X_0)$, for $f(X_v)$ is minimum in S_P. Hence $f(X_0) = f(X_v)$. But $f(X_1) \leq f(X_v)$ because X_v is on an edge of S_C. Also $f(X_v) \leq f(X_1)$ because $f(X_v)$ is minimum in S_P. Hence $f(X_v) = f(X_1) = f(X_0)$ which means $f(X)$ is minimum at X_1 in S_C. Since $S_F \subseteq S_C$, in S_F also $f(X)$ is minimum at X_1, a vertex of S_F.

(b) Along some edge of S_C $f(X)$ decreases as λ increases. If this edge is wholly the edge of S_F, then $f(X)$ decreases without limit along this edge, and so $f(X)$ has an unbounded minimum in S_F.

If the edge is partly in S_F, then for $\lambda = \lambda_0$ we arrive at another vertex X_2 of S_F with $f(X_2) < f(X_1)$. We can now apply the same reasoning to X_2 which we have been applying to X_1, namely that either $f(X_2)$ is minimum or $f(X)$ has an unbounded minimum or there is another vertex X_3 of S_F such that $f(X_3) < f(X_2)$. Since the number of vertices of S_F is finite, proceeding like this, if $f(X)$ has not an unbounded minimum, we shall arrive at some vertex of S_F for which $f(X)$ is minimum.

Proved.

THEOREM 5. *If $f(X)$ is minimum at more than one of the vertices of S_F, then it is minimum at all those points which are the convex linear combinations of these vertices.*

Proof. Let $X_1, X_2, \ldots, X_k$ be the vertices of S_F where $f(X)$ is minimum. Then

$$f(X_1) = f(X_2) = \ldots = f(X_k).$$

Let Y be any convex linear combination of these vertices. Then

$$Y = \sum_{r=1}^{k} \beta_r X_r, \ \sum_{r=1}^{k} \beta_r = 1, \ \beta_r \geq 0,$$

and since $f(X)$ is linear

$$f(Y) = f\left(\sum_{r=1}^{k} \beta_r X_r\right) = \sum_{r=1}^{k} \beta_r f(X_r)$$

$$= \sum_{r=1}^{k} \beta_r f(X_1) = f(X_1)$$

which means $f(X)$ is minimum at Y also.

Proved.

DEFINITION 3. *A solution of* (7) *and* (8) *which optimizes the objective function* (6) *is called an optimal solution of the LP problem.*

8 Summary

We may summarize some of the conclusions that can be drawn from the theoretical discussion in sections 4-7.

If the set S_F of feasible solutions is empty, the problem has no solution.

If S_F is nonempty, it is a convex set (polytope) with vertices corresponding to the basic feasible solutions. These are finite in number as they are a subset of basic solutions which are at most ${}^nC_m$ in number.

The convex set S_F may be bounded or unbounded. If bounded, it is a convex polyhedron, and the problem has a solution with $f(X)$ attaining its minimum value at a vertex.

If S_F is unbounded, $f(X)$ may have a finite minimum at a vertex. Or else $f(X)$ may tend to $-\infty$ in which case the solution is unbounded.

9 Simplex method

It has not been possible to find an analytic solution to the LP problem. The difficulty arises because the tools of analysis are not well suited to handle inequalities. Numerical methods which enable us to compute the solution for numerical values of a_{ij}, b_i and c_j for finite number of variables and constraints have been discovered. The most general and widely used of these methods is called the *simplex method.*

The simplex method provides an algorithm[1] which consists in moving from one vertex of S_F (one b.f.s.) to another in a prescribed manner such that the value of the objective function $f(X)$ at the succeeding vertex is less than at the preceding vertex. The procedure of jumping from vertex to vertex is repeated. If we can reduce $f(X)$

1. A rule of procedure usually involving repetitive application of an operation. The word is derived from the Arabic Al Khwarizmi (after the Arab mathematician of the same name, about 825 A.D.) which in Old French became algorismus and in Middle English algorism.

at each jump, then no basis can ever repeat and we can never go back to a vertex already covered. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number of steps. We shall explain the procedure through a numerical example. The general proof goes along the lines indicated in the numerical example. The basic principles involved in the proof are already discussed in the course of theorem 4.

The first step in any case is to find a b.f.s. For this purpose we define a canonical form of the equations which immediately gives us a b.f.s.

10 Canonical form of equations

Let $x_1, x_2, \ldots, x_m$ be the basic variables corresponding to a certain basis of the equations

$$\mathbf{AX} = \mathbf{B} \tag{7}$$

These can then be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 - a_{1,m+1}x_{m+1} - \cdots - a_{1n}x_n \\ b_2 - a_{2,m+1}x_{m+1} - \cdots - a_{2n}x_n \\ \cdots \\ b_m - a_{m,m+1}x_{m+1} - \cdots - a_{mn}x_n \end{bmatrix}$$

The $m \times m$ matrix on the left side is nonsingular because the basic vectors which are the columns of this matrix are linearly independent. Premultiplying both sides by its inverse, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_m \end{bmatrix} = \begin{bmatrix} \bar{b}_1 - \bar{a}_{1,m+1}x_{m+1} - \cdots - \bar{a}_{1n}x_n \\ \bar{b}_2 - \bar{a}_{2,m+1}x_{m+1} - \cdots - \bar{a}_{2n}x_n \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ \bar{b}_m - \bar{a}_{m,m+1}x_{m+1} - \cdots - \bar{a}_{mn}x_n \end{bmatrix}$$

or

$$\begin{aligned} x_1 + \bar{a}_{1,m+1}x_{m+1} + \cdots + \bar{a}_{1n}x_n &= \bar{b}_1 \\ x_2 + \bar{a}_{2,m+1}x_{m+1} + \cdots + \bar{a}_{2n}x_n &= \bar{b}_2 \\ \cdots\cdots\cdots\cdots\cdots \\ x_m + \bar{a}_{m,m+1}x_{m+1} + \cdots + \bar{a}_{mn}x_n &= \bar{b}_m \end{aligned} \tag{22}$$

Equations (22) which are equivalent to (7) are called the canonical form of the equations provided $\bar{b}_i \geq 0$, $i = 1, 2, \ldots, m$. Corresponding to each feasible basis we can get a canonical form, and vice versa. The advantage of putting the equations in a canonical form is that the basis and the corresponding b.f.s. can be immediately known. Since the b.f.s. should have zero values of nonbasic variables, putting $x_{m+1} = x_{m+2} = \cdots = x_n = 0$ in (22), we get the b.f.s. as $(\bar{b}_1\, \bar{b}_2\, \bar{b}_3 \ldots \bar{b}_m\; 0\; 0 \ldots 0)$. Thus the right side of (22) gives the values of the basic variables.

Using (22) we can eliminate the basic variables from the objective function (6) and get

$$f(\mathbf{X}) = \sum_{i=1}^{m} \bar{b}_i c_i + \sum_{j=m+1}^{n} \bar{c}_j x_j \tag{23}$$

where $$\bar{c}_j = c_j - \sum_{i=1}^{m} c_i \bar{a}_{ij},\ j = m+1, \ldots, n.$$

It may be noted that the above formula for $\bar{c}_j$ holds even for $j = 1, 2, \ldots m$. It can be seen to give zero values for $\bar{c}_j, j = 1, 2, \ldots m$, which is right because $x_j, j = 1, 2, \ldots m$, have been eliminated from (6), and therefore their coefficients in (23) are zero. The advantage of this form is that the value of $f(X)$ for the present b.f.s. is immediately obtained as $\sum_{i=1}^{m} \bar{b}_i c_i$. The coefficients $\bar{c}_j, j = m+1, \ldots, n$, are called the relative cost coefficients.

11 Simplex method (numerical example)

We explain the simplex method through the example of section 2 with the modification that we delete constraint (v).

Introducing slack variables and converting the problem of maximizing $f(X)$ to minimizing $-f(X) = \psi(X)$, we put the problem in the following standard form.

Minimize $$\psi(X) = -4x_1 - 5x_2; \quad (24)$$

subject to
$$\begin{aligned} x_1 - 2x_2 + x_3 \quad &= 2, \\ 2x_1 + x_2 \quad + x_4 \quad &= 6, \\ x_1 + 2x_2 \quad + x_5 \quad &= 5, \\ -x_1 + x_2 \quad + x_6 &= 2; \end{aligned} \quad (25)$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \quad (26)$$

Equations (25) are four equations in six variables. Two of the variables, arbitrarily chosen, can be given zero values to obtain a basic solution. To obtain a b.f.s. (or a vertex of the convex set of feasible solutions) zero variables have to be so chosen that the other variables are non-negative.

I First canonical form. Equations (25) are in canonical form which gives a b.f.s as

$$x_1 = 0, x_2 = 0, x_3 = 2, x_4 = 6, x_5 = 5, x_6 = 2;$$

and the corresponding value of ψ as

$$\psi = -4x_1 - 5x_2 = 0. \quad (27)$$

Both x_1 and x_2, the nonbasic variables in terms of which ψ is expressed, are zero. If either of them is made positive, ψ will decrease because the coefficients of x_1 and x_2 in (27) are negative. So ψ at this stage is not minimum. It can be decreased by changing the basis so as to include x_1 or x_2 in place of some other variable which is in the present basis. Let us decide to bring x_2 into the basis. It would be equally reasonable to decide in favour of x_1. But which variable to drop (or be given zero value) ? Examining (25) the following are the alternatives (keeping $x_1 = 0$).

(i) Put $x_3 = 0$; then $x_2 = -1, x_4 = 7, x_5 = 7, x_6 = 3$. This is not a feasible solution.
(ii) Put $x_4 = 0$; then $x_2 = 6, x_3 = 14, x_5 = -7, x_6 = -4$. This is also not feasible.
(iii) Put $x_5 = 0$; then $x_2 = 5/2, x_3 = 7, x_4 = 7/2, x_6 = -1/2$. This is also not feasible.
(iv) Put $x_6 = 0$; then $x_2 = 2, x_3 = 6, x_4 = 4, x_5 = 1$. This is a feasible solution.

So the greatest value that can be given to x_2 without making the solution non-feasible is 2. Putting $x_2 = 2$ would mean putting $x_6 = 0$ which means x_6 goes out of the basis.

It is easy to discover a simple rule for deciding which variable to drop. Consider the ratios 2/(–2), 6/1, 5/2, 2/1 of the right-hand side constant of each of the equations (25) to the coefficient of x_2 in that equation. Of the positive ones the least is 2/1 corresponding to the last equation which determines the maximum value which can be given to x_2 bringing it into the basis without forcing any other variable to become negative. This also indicates that x_6 should be dropped from the basis. The negative ratio need not be considered because in the corresponding equation x_2 can be made as large as we please without forcing any other variable in that equation to become negative.

II Second canonical form. The new basic variables should therefore be x_3, x_4, x_5, x_2. The canonical form for this basis can be obtained by eliminating x_2 from the first three of the equations (25) with the help of the last, and writing the equations such that the coefficient of each basic variable in its respective equation is 1. The required form is

$$\begin{aligned} -x_1 + 2x_6 + x_3 \qquad\qquad &= 6, \\ 3x_1 - x_6 \quad + x_4 \qquad &= 4, \\ 3x_1 - 2x_6 \qquad + x_5 &= 1, \\ -x_1 + x_6 \qquad\qquad + x_2 &= 2. \end{aligned} \tag{28}$$

From this canonical form we get the second b.f.s. as

$$x_3 = 6, x_4 = 4, x_5 = 1, x_2 = 2, x_1 = 0, x_6 = 0.$$

Also we eliminate x_2 from (24) and express ψ in terms of the nonbasic variables x_1 and x_6 as

$$\psi + 10 = -9x_1 + 5x_6. \tag{29}$$

This gives the value of ψ for the present b.f.s. as –10. Notice that since ψ is expressed in terms of the nonbasic variables which are zero, the constant term occurring in (29) directly gives the value of ψ. It is an improvement on the first value. Can we reduce it further? Yes, by bringing x_1 into the basis because the coefficient of x_1 in (29) is negative. But which basic variable to drop? The relevant ratios to be examined in (28) are 6/(–1), 4/3, 1/3, 2/(–1). Keeping the negative ratios out of consideration for reasons already explained, the least ratio is 1/3 which corresponds to the third of equations (28). So to bring x_1 into the basis x_5 should be dropped.

III Third canonical form. The new basic variables are x_1, x_2, x_3, x_4 and the corresponding canonical form is

$$\begin{aligned}
\frac{1}{3}x_5+\frac{4}{3}x_6+x_3 &= \frac{19}{3},\\
-x_5+x_6+x_4 &= 3,\\
\frac{1}{3}x_5-\frac{2}{3}x_6+x_1 &= \frac{1}{3},\\
\frac{1}{3}x_5+\frac{1}{3}x_6+x_2 &= \frac{7}{3};
\end{aligned} \tag{30}$$

and ψ expressed in terms of the nonbasic variables is

$$\psi+13=3x_5-x_6. \tag{31}$$

The b.f.s. is $x_1=1/3, x_2=7/3, x_3=19/3, x_4=3, x_5=0, x_6=0$.

The coefficient of x_6 in (31) is negative and so ψ can be further decreased by bringing x_6 back into the basis. The ratios to be observed now are 19/4, 3/1, 7/1; the fourth one being negative is out of consideration. Out of these 3/1 is the least. So x_6 should replace x_4.

IV Fourth canonical form. The new basic variables are x_1, x_2, x_3, x_6 and the corresponding canonical form is

$$\begin{aligned}
-\frac{4}{3}x_4+\frac{5}{3}x_5+x_3 &= \frac{7}{3},\\
x_4-x_5+x_6 &= 3,\\
\frac{2}{3}x_4-\frac{1}{3}x_5+x_1 &= \frac{7}{3},\\
-\frac{1}{3}x_4+\frac{2}{3}x_5+x_2 &= \frac{4}{3};
\end{aligned} \tag{32}$$

and ψ expressed in terms of the nonbasic variables x_4, x_5 is

$$\psi+16=x_4+2x_5. \tag{33}$$

The value of ψ at this stage is -16, and it cannot be further reduced by any change of basis because the coefficients of x_4 and x_5 are positive.

We have come to the end of our search. The minimum value of ψ is -16 and so the maximum value of $f(X)$ is 16. The optimal solution is

$$x_1=7/3,\ x_2=4/3,\ x_3=7/3,\ x_4=0,\ x_5=0,\ x_6=3.$$

It is instructive to compare results with the graphical solution of the problem. The set of feasible solutions of (2) and (3) [excluding (v)] is the convex polygon *OBCDEFO* (see Fig. 1). $f(X)$ becomes maximum at the point D. It can be verified that the first, second, third and fourth basic feasible solutions correspond to the vertices O, F, E and D respectively.

12 Simplex tableau

The numerical work explained in the last section can be economically organized in a form known as the simplex tableau. The following is the simplex tableau for the preceding example.

TABLE 2

I	Basis	B	P_1	P_2	P_3	P_4	P_5	P_6
	x_3	2	1	−2	1			
	x_4	6	2	1		1		
1	x_5	5	1	2			1	
	x_6	2	−1	1				1
	ψ	0	−4	−5				
	x_3	6	−1		1			2
	x_4	4	3			1		−1
2	x_5	1	3				1	−2
	x_2	2	−1	1				1
	ψ	10	−9					5
	x_3	19/3			1		1/3	4/3
	x_4	3				1	−1	1
3	x_1	1/3	1				1/3	−2/3
	x_2	7/3		1			1/3	1/3
	ψ	13					3	−1
	x_3	7/3			1	−4/3	5/3	
	x_6	3				1	−1	1
4	x_1	7/3	1			2/3	1/3	
	x_2	4/3		1		−1/3	2/3	
	ψ	16				1	2	

The first column shows the iteration number I. The second and the third columns give the variables in the basis and their values (vector **B**). The succeeding columns give the vectors $\mathbf{P}_1$, $\mathbf{P}_2$, ... of the canonical form, which means that row-wise the entries in these columns are the coefficients of x_1, x_2, ... in the equations. (For this reason sometimes it is more convenient to write x_1, x_2, ... in place of $\mathbf{P}_1$, $\mathbf{P}_2$, ... at the top of these columns). For example, the first numerical row in the above table records the first equation in the first canonical form ($I = 1$) which is the equation with 1 as the coefficient of x_1, −2 as the coefficient of x_2, 1 as the coefficient of x_3, and 2 as the right side constant. The equation for the objective function ψ is also written as a row, putting ψ in the column for basic variables, its value with sign changed in the next column, and the relative cost coefficients of the nonbasic variables in the respective columns. The equation for ψ is read, say for $I = 2$, as

$$\psi + 10 = -9x_1 + 5x_6, \quad \text{[compare (29)]}.$$

The following sequence of steps constitutes one iteration leading from one b.f.s. to another. ($I = 1$ is taken as an example).

(i) Examine the relative cost coefficients. If all are non-negative, the current solution is optimal.

(ii) If not, pick out the numerically largest negative coefficient (– 5). The vector corresponding to it (P_2) is to be brought into the basis. The corresponding basic variable is x_2.

(iii) Divide each element of vector **B** by the corresponding elements of the chosen column vector (P_2). Out of the positive ratios choose the least (2/1). The corresponding basic variable (x_6) has to go out of the basis.

(iv) If all the ratios are negative, it means that the value of the incoming variable (whatever it is), can be made as large as we please without violating the feasibility condition. It follows that the problem has an unbounded solution. Iteration stops.

(v) Replace x_6 by x_2 in the basic variables column in the table for the next iteration $I = 2$ and rewrite the equation against it so that the coefficient of x_2 is 1. Eliminate x_2 from the rest of the equations in such a way that the coefficients of the basic variables x_3, x_4, x_5 remain 1.

(vi) Eliminate x_2 from the equation for ψ also so that it is expressed in terms of the new nonbasic variables x_1, x_6 only. The entry in the third column of the ψ equation gives the value of $-\psi$ at this stage.

(vii) Thus the table for $I = 2$ is complete. Go to (i).

13 Finding the first b.f.s.; artificial variables

In the example of section 11 the introduction of slack variables gave a canonical form which immediately led to a b.f.s. providing a starting point for the iterative procedure. This happened because all the constraints were of the type 'less than' and all the constants on the right-hand sides of the inequalities were non-negative. But if there is a 'greater than' constraint with non-negative right-hand side or 'less than' constraint with negative right-hand side, then a b.f.s. cannot be obtained right away.

To overcome this difficulty we first put the constraints so that the right-hand side constants are all non-negative. Then we introduce the necessary slack variables. To get a b.f.s. of this system we formulate an auxiliary LP problem whose one b.f.s. can be obtained straightaway as in the last example. This auxiliary problem has the property that its optimal solution may immediately give a b.f.s. of the original problem. The auxiliary problem is also solved by the simplex method.

We explain the procedure through the following example.

Minimize $\quad f(X) = 4x_1 + 5x_2;$

subject to

$$2x_1 + x_2 \le 6,$$

$$x_1 + 2x_2 \le 5,$$

$$x_1 + x_2 \ge 1,$$

$$x_1 + 4x_2 \ge 2,$$

$$x_1, x_2 \ge 0.$$

Introducing the slack variables,

$$\begin{aligned} 2x_1 + x_2 + x_3 \qquad\qquad &= 6, \\ x_1 + 2x_2 \quad + x_4 \qquad &= 5, \\ x_1 + x_2 \qquad\quad - x_5 \quad &= 1, \\ x_1 + 4x_2 \qquad\qquad - x_6 &= 2; \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

The solution $x_1 = 0, x_2 = 0, x_3 = 6, x_4 = 5, x_5 = -1, x_6 = -2$ is not a basic feasible solution. To get the first b.f.s. to serve as a starting point of iteration for this problem, let us formulate the following auxiliary problem.

Minimize $\quad g(X) = x_7 + x_8;$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 \qquad\qquad\qquad &= 6, \\ x_1 + 2x_2 \quad + x_4 \qquad\qquad &= 5, \\ x_1 + x_2 \qquad\quad - x_5 + x_7 \quad &= 1, \\ x_1 + 4x_2 \qquad\qquad - x_6 + x_8 &= 2; \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0.$$

We have introduced two more variables, x_7 and x_8, called *artificial variables*, both non-negative and with positive algebraic signs, one in each of the equations which arose from 'greater than' constraints. Also the objective function in this new problem is taken as the sum of the artificial variables.

The solution of this problem may be $g(X) = 0$ with $x_7 = x_8 = 0$ and the values of the other variables non-negative with at least two of them zero, because the optimal solution should be a b.f.s. with at most four variables having nonzero values. Then the values of the variables other than the artificial ones should constitute a b.f.s. of the original problem which can become the starting point of iterations for that problem.

Table 3 is the simplex tableau for this example. In phase *I* we solve the auxiliary problem. Its optimal solution gives the starting b.f.s. for the original problem. At the beginning of phase *II* we drop the columns for the artificial variables and the row for the function $g(X)$, and carry on the iterations for minimizing $f(X)$. It is convenient to carry the equation for $f(X)$ through phase *I* also, so that when we start on phase *II* the expression for $f(X)$ in terms of the nonbasic variables at that stage is readily available.

Phase *I* starts with the basic variables x_3, x_4, x_7, x_8, and so $g(X)$ should be expressed in terms of the nonbasic variables as

$$g(X) - 3 = -2x_1 - 5x_2 + x_5 + x_6$$

TABLE 3

	Basis	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
Phase I	x_3	6	2	1	1		0	0		
	x_4	5	1	2		1	0	0		
	x_7	1	1	1			−1	0	1	
	x_8	2	1	4			0	−1		1
	g	−3	−2	−5			1	1		
	f	0	4	5			0	0		
	x_3	11/2	7/4		1		0	1/4		−1/4
	x_4	4	1/2			1	0	1/2		−1/2
	x_7	1/2	3/4				−1	1/4	1	−1/4
	x_2	1/2	1/4	1			0	−1/4		1/4
	g	−1/2	−3/4				1	−1/4		5/4
	f	−5/2	11/4				0	5/4		−5/4
	x_3	5	1		1		1		−1	0
	x_4	3	−1			1	2		−2	0
	x_6	2	3				−4	1	4	−1
	x_2	1	1	1			−1		1	0
	g	0	0				0		1	1
End of Phase I	f	−5	−1				5		−5	0
Phase II	x_1	2/3	1				−4/3	1/3		
	x_2	1/3		1			1/3	−1/3		
	x_3	13/3			1		7/3	−1/3		
	x_4	11/3				1	2/3	1/3		
	f	−13/3					11/3	1/3		

In general we define the auxiliary (or phase *I*) problem for the LP problem (6) − (8) as

Minimize $$g(\mathbf{X}) = \sum_{i=1}^{m} x_{n+i}$$

subject to $$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i,\ i = 1, 2, \ldots, m,$$

$$\mathbf{X} \geq 0,$$

where $\mathbf{X} = [x_1\ x_2, \ldots x_n\ x_{n+1} \ldots x_{n+m}]'$, x_{n+i} being called the artificial variables.

In this problem min g(X) = 0 if and only if $x_{n+i} = 0$ for all i. Hence if we solve this problem by the simplex method we get its solution as g(X) = 0 only if in its optimal b.f.s. the artificial variables are zero. The optimal values of the rest of the variables, being non-negative, will then satisfy the constraints of the original problem. Moreover, not more than m of these being nonzero, they will constitute a basic feasible solution of the original problem providing a starting point for its solution by the simplex method.

If min g(X) > 0, the conclusion is that there is no feasible solution of auxiliary problem with the values of the artificial variables as zero, and consequently no feasible solution of the original LP problem.

As an alternative to solving the problem in two phases, it is also possible to solve it in one phase after the artificial variables have been introduced. We describe one such method, popularly called the *big M method.* In this method the original objective function f is replaced by

$$F = f + M \sum_{i=1}^{m} x_{n+i}$$

where x_{n+i} are the artificial variables and M is an arbitrary large number as compared to the coefficients in f. This modified objective function F is minimised subject to the constraints of the original problem. It can be shown that if in the optimal solution of the modified problem all the artificial variables are zero, then that is also the optimal solution of the original problem. If, however, in the optimal solution of the modified problem all the artificial variables are not zero, the conclusion is that the original problem is not feasible. If the modified problem is found to have an unbounded minimum, then the original problem too, if feasible, is unbounded.

To solve the numerical problem of this section by the big M method, we may take the objective function as

$$F = 4x_1 + 5x_2 + 100\,(x_7 + x_8).$$

The iterations would be the same as in table 3 except that the successive rows for F (instead of for f or g) will be as follows:

	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
F	−300	−196	−495			100	100		
F	−105/2	−289/4				100	−95/4		495/4
F	−5	−1				5		95	100
F	−13/3					11/3	1/3	289/3	299/3

The starting form of F in terms of the nonbasic variables is

$$F = 300 - 196x_1 - 495x_2 + 100x_5 + 100x_6.$$

The complete simplex tableau can be written out by inserting the rows for F above in place of the rows for g and f in table 3.

14 Degeneracy

The least of a set of non-negative ratios decides which variable is to be dropped from the basis at a particular stage. It may happen that two or more ratios are equal and the least. In that case a tie occurs as to which variable to drop. One can arbitrarily decide in favour of one, but then it turns out that the variables which tied with it and continue to remain in the basis also become zero. In other words, one or more of the basic variables too have zero value. Such a case is called *degenerate.*

The difficulty appears in the next iteration when we find that the variable to be brought into the basis and the variable to be dropped both are already zero. The basis, theoretically, is changed, but the value of the objective function remains the same. Geometrically it may be interpreted as the case of two coincident vertices. We change from one to the other but substantially remain where we were. In most cases we go ahead with our iterations and find that following the procedure we eventually change to a substantially different basis which gives an improved value of the objective function, and we finally get the optimal solution.

It may, however, happen that the successive iterations only make us go through a number of (one or more) degenerate bases to arrive back at the degenerate basis from which we started.

We get into a cycle with no apparent way to get out of it. This situation presents some difficulty, but procedures have been discovered to overcome it. Such a situation is very rare. It is claimed that in thousands of linear programming models solved by the simplex method, some of them very large, there is not a case when degeneracy has proved a hurdle. The procedure recommended to deal with such a hurdle, if it occurs, is therefore of theoretical interest, and so we shall omit its discussion.

15 Simplex multipliers

In the simplex method it is necessary at every iteration to express the objective function of $f(X)$ in terms of the nonbasic variables as given in a general form by (23). We get this equation for each successive canonical form from the preceding canonical form. It is possible to get it for any basis directly from the original equations (6) and (7). This we proceed to explain.

Suppose $(x_1, x_2, \ldots x_m, 0, 0, \ldots 0)$ is a b.f.s. To express $f(X)$ in terms of the nonbasic variables $x_{m+1}, x_{m+2}, \ldots, x_n$, we may eliminate the basic variables $x_1, x_2, \ldots x_m$ from (6) with the help of (7). With this object in view let us multiply each of the equations (7) by constants $\pi_1, \pi_2, \ldots, \pi_m$ respectively and add them to (6). We get

$$\left(c_1 + \sum_{i=1}^{m} a_{i1}\pi_i\right)x_1 + \left(c_2 + \sum_{i=1}^{m} a_{i2}\pi_i\right)x_2 + \ldots + \left(c_n + \sum_{i=1}^{m} a_{in}\pi_i\right)x_n = f + \sum_{i=1}^{m} b_i\pi_i. \quad (34)$$

Choose $\pi_1, \pi_2, \ldots, \pi_m$ such that the coefficients of $x_1, x_2, \ldots x_m$ vanish, that is, let

$$\sum_{i=1}^{m} a_{ij}\pi_i = -c_j,\ j = 1, 2, \ldots, m. \quad (35)$$

Then (34) reduces to

$$f = \sum_{j=m+1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} b_i\pi_i \quad (36)$$

where
$$\bar{c}_j = c_j + \sum_{i=1}^{m} a_{ij}\pi_i,\ j = m+1, \ldots, n. \quad (37)$$

(35) are m equations in m unknowns π_i, and on solution give the required values of π_i. In matrix notation we may put (35) as

$$A'_0\Pi = -C'_0$$

where

$$A_0 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}, \Pi = [\pi_1\ \pi_2 \dots \pi_m]', C_0 = [c_1\ c_2 \dots c_m].$$

Hence

$$\Pi = -[A'_0]^{-1}C'_0 = -[A_0^{-1}]'C'_0,$$

or
$$\Pi' = -C_0A_0^{-1}. \tag{38}$$

Here we have defined Π as a column vector and C_0 as a row vector. Π', C_0', A_0' are the transpose of the respective vectors and matrix. The vector Π is called the *multiplier vector* and its components the *simplex multipliers*. To calculate Π at any stage the inverse of A_0, the matrix of basic vectors at that stage, is needed.

Having calculated π_i for any basis, the value of the objective function for that basis is given by

$$f = -\sum_{i=1}^{m} b_i\pi_i, \tag{39}$$

because the terms in (36) involving the nonbasic variables $x_{m+1}, x_{m+2}, \dots, x_n$ are zero for the simple reason that the variables themselves are zero.

The above discussion is of theoretical interest in a general case when A_0 is any $m\times m$ matrix. For, to get A_0^{-1} or to solve equations (35) for $\pi_i, i = 1, 2, \dots m$, means essentially the same thing, and may not be easy. However, where A_0 arises from constraints converted into equations through slack variables, it becomes easy to read A_0^{-1} and Π from the tables of the original equations and equations in the canonical form with respect to the basis under consideration.

As an example, consider the problem of section 13 and its solution obtained in table 3. The table below shows the initial form of the problem after introducing slack variables, and the final canonical form which gives the optimal solution. We shall show how A_0^{-1} and the simplex multipliers for the optimal solution can be read off from the table.

Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_3	6	2	1	1			
x_4	5	1	2		1		
x_5	1	1	1			−1	
x_6	2	1	4				−1
f	0	4	5				
x_1	2/3	1				−4/3	1/3
x_2	1/3		1			1/3	−1/3
x_3	13/3			1		7/3	−1/3
x_4	11/3				1	2/3	1/3
f	−13/3					11/3	1/3

x_1, x_2, x_3, x_4 being the basic variables in the optimal solution, the problem is to find A_0^{-1} where A_0 is the matrix of the coefficients of these variables in the initial form, namely,

$$A_0 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{bmatrix}$$

A_0^{-1} operating on the initial matrix of coefficients produces the final matrix of coefficients, that is

$$A_0^{-1}\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 4 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -4/3 & 1/3 \\ 0 & 1 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 7/3 & -1/3 \\ 0 & 0 & 0 & 1 & 2/3 & 1/3 \end{bmatrix}$$

or, taking only the submatrix of the last four columns on either side,

$$A_0^{-1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4/3 & 1/3 \\ 0 & 0 & 1/3 & -1/3 \\ 1 & 0 & 7/3 & -1/3 \\ 0 & 1 & 2/3 & 1/3 \end{bmatrix}$$

Since the inverse of a diagonal matrix with diagonal entries 1 or -1 is the matrix itself, we get

$$A_0^{-1} = \begin{bmatrix} 0 & 0 & -4/3 & 1/3 \\ 0 & 0 & 1/3 & -1/3 \\ 1 & 0 & 7/3 & -1/3 \\ 0 & 1 & 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

or

$$A_0^{-1} = \begin{bmatrix} 0 & 0 & 4/3 & -1/3 \\ 0 & 0 & -1/3 & 1/3 \\ 1 & 0 & -7/3 & 1/3 \\ 0 & 1 & -2/3 & -1/3 \end{bmatrix}$$

The rule for determining A_0^{-1} boils down to this. Observe the columns in the original form which constitute a diagonal matrix with 1 or -1 on the diagonal. The matrix of columns corresponding to them in the final form multiplied by this diagonal matrix gives the required inverse.

To get the simplex multipliers, by (38)

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4] = -[4\ 5\ 0\ 0]\begin{bmatrix} 0 & 0 & 4/3 & -1/3 \\ 0 & 0 & -1/3 & 1/3 \\ 1 & 0 & -7/3 & 1/3 \\ 0 & 1 & -2/3 & -1/3 \end{bmatrix} = [0\ \ 0\ \ -11/3\ \ -1/3]$$

Even otherwise, we can directly observe that the row for f in the final form can only be obtained from the initial form by multiplying the initial rows of the coefficients by 0, 0, $-11/3$, $-1/3$ respectively and adding to the initial row for f.

17 Duality in LP problems

To every LP problem there corresponds another LP problem called its *dual.* The original problem is called the *primal.* There exists an important theoretical relationship between the primal and its dual which is of practical use also. Before defining the dual, we shall restate the LP problem in a standard form different from the form defined in equations (6), (7) and (8). This alternative statement is also quite general and is better suited to proving the duality theorems.

We state the general LP problem as

$$\text{Minimize} \qquad f(\mathbf{X}) = \mathbf{CX}, \tag{43}$$

$$\text{subject to} \qquad \mathbf{AX} \geq \mathbf{B}, \tag{44}$$

$$\mathbf{X} \geq 0, \tag{45}$$

where A is an $m \times n$ matrix, X is a column and C a row n-vector and B is a column m-vector. The above problem may also be written as

$$\text{Minimize} \qquad f(\mathbf{X}) = \sum_{j=1}^{n} c_j x_j, \tag{43}$$

$$\text{subject to} \qquad \sum_{j=1}^{n} a_{ij} x_j \geq b_i,\ i = 1, 2, \ldots, m, \tag{44}$$

$$x_j \geq 0,\ j = 1, 2, \ldots, n. \tag{45}$$

That the above form is quite general follows from the fact that constraints in other forms can always be put in form (44). For, an inequality of the type $\sum_j a_{ij} x_j \leq b_i$ can be put as $\sum_j (-a_{ij} x_i) \geq -b_i$, and an equation of the type $\sum_j a_{ij}\, x_j = b_i$ can be put as two inequalities $\sum_j a_{ij} x_j \geq b_i$, and $-\sum_j a_{ij} x_j \geq -b_i$.

For example, the example of section 3 can be written in the present standard form as

$$\text{Minimize} \quad f = -2x_1 - x_{21} + x_{22} + x_3,$$

$$\text{subject to} \quad -2x_1 + 5x_{21} - 5x_{22} - 3x_3 \geq -4,$$

$$3x_1 + 6x_{21} - 6x_{22} - x_3 \geq 2,$$

$$x_1 + x_{21} - x_{22} + x_3 \geq 4,$$

$$-x_1 - x_{21} + x_{22} - x_3 \geq -4,$$

$$x_1, x_{21}, x_{22}, x_3 \geq 0.$$

DEFINITION 4. *If the primal LP problem is in the form* (43), (44), (45), *then its dual is defined as the following LP problem.*

$$\text{Maximize} \quad \phi(\mathbf{Y}) = \mathbf{B}'\mathbf{Y} \tag{46}$$

$$\text{subject to} \quad \mathbf{A}'\mathbf{Y} \leq \mathbf{C}', \tag{47}$$

$$\mathbf{Y} \geq 0, \tag{48}$$

where $\mathbf{Y}$ is a column m-vector.
It may also be written as:

$$\text{Maximize} \quad \phi(\mathbf{Y}) = \sum_{i=1}^{m} b_i y_i, \tag{46}$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \; j = 1, 2, \ldots, n, \tag{47}$$

$$y_i \geq 0, \; i = 1, 2, \ldots, m. \tag{48}$$

The primal-dual pair of problems can be defined in other forms also (for example, see problem 28). The equivalence of various definitions can be easily established.

The above definition implies the following correspondence between the primal in the standard form (43), (44), (45), and its dual.

	Primal	Dual
n	n variables	n constraints
m	m constraints	m variables
$c_j, j = 1, 2, .. n$	cost coefficients	constraint constants
$b_i, i = 1, 2, .. m$	constraint constants	cost coefficients
variables	$x_j \geq 0, j = 1, 2, \ldots n$	$y_i \geq 0, i = 1, 2, .. m$
constraints	$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$	$\sum_{i=1}^{m} a_{ij} y_i \leq c_j$
objective function	minimize $\sum_{j=1}^{n} c_j x_j$	maximize $\sum_{i=1}^{m} b_i y_i$

As an example, to write the dual of the example of section 3 we first put it in the standard form (43), (44), (45), as has been done above. We then write the dual as

Maximise $\phi = -4y_1 + 2y_2 + 4y_3 - 4y_4,$

subject to

$$-2y_1 + 3y_2 + y_3 - y_4 \le -2,$$

$$5y_1 + 6y_2 + y_3 - y_4 \le -1,$$

$$-5y_1 - 6y_2 - y_3 + y_4 \le 1,$$

$$-3y_1 - y_2 + y_3 - y_4 \le 1,$$

$$y_1, y_2, y_3, y_4 \ge 0.$$

We can simplify the above statement of the problem by noticing that y_3 and y_4 occur throughout as $y_3 - y_4$, and so $y_3 - y_4$ can be regarded as a single variable. Let it be denoted by a single symbol, say y_3. (It does not mean that we are making the statement $y_3 - y_4 = y_3$. It is convenient to call the new variable y_3 because the other two variables are y_1 and y_2. We could use any other symbol). Since the original y_3 and y_4 are both non-negative, $y_3 - y_4$ is unrestricted, and so the new variable y_3 is unrestricted. The problem can now be written as

Maximize $\phi = -4y_1 + 2y_2 + 4y_3,$

subject to

$$-2y_1 + 3y_2 + y_3 \le -2,$$

$$5y_1 + 6y_2 + y_3 \le -1,$$

$$-5y_1 - 6y_2 - y_3 \le 1,$$

$$-3y_1 - y_2 + y_3 \le 1,$$

$$y_1, y_2 \ge 0, \ y_3 \text{ unrestricted.}$$

We further notice that the second and third constraints are equivalent to one equation. Therefore we may write the above problem as

Maximize $\phi = -4y_1 + 2y_2 + 4y_3,$

subject to

$$-2y_1 + 3y_2 + y_3 \le -2,$$

$$5y_1 + 6y_2 + y_3 = -1,$$

$$-3y_1 - y_2 + y_3 \le 1,$$

$$y_1, y_2 \ge 0, \ y_3 \text{ unrestricted.}$$

This is the dual of the primal problem of section 3 which, for better comparison, we write below.

Minimize $f = -2x_1 - x_2 + x_3,$

subject to

$$-2x_1 + 5x_2 - 3x_3 \ge -4,$$

$$3x_1 + 6x_2 - x_3 \ge 2,$$

$$x_1 + x_2 + x_3 = 4,$$

$$x_1, x_3 \geq 0, x_2 \text{ unrestricted.}$$

The interesting point to note in the above primal and dual problems is that the third constraint in the primal is an equation while the third variable in the dual is unrestricted, and the second variable in the primal is unrestricted while the second constraint in the dual is an equation. (For general statement of this property see problems 27, 28).

If we generalize the statement of the standard LP problem to admit that some of the constraints (44) may be equations and the rest inequalities of ≥ type, then we get the following rule regarding constraints and variables in the primal and the dual.

Primal	Dual
*i*th constraint ≥ type	$y_i \geq 0$
*i*th constraint = type	y_i unrestricted
$x_j \geq 0$	*j*th constraint ≤ type
x_j unrestricted	*j*th constraint = type