UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

Self Learning Material

M.Sc. Mathematics

Second Semester

(2019 Admission Onwards)

MTH2CO7: REAL ANALYSIS II

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MTH2CO7: REAL ANALYSIS II

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Preface

The Self Learning Material (SLM) Real Analysis II is prepared based on the syllabus for M.Sc. Mathematics (CBCSS) PG Programme of University of Calicut effective from 2019 admission onwards. The material is mainly intended for helping the students who are studying M.Sc. Mathematics course under the School of Distance of Education, University of Calicut.

The material is prepared based on the text book Real Analysis (*Fourth Edition*) by H.L. Royden and P. M. Fitzpatrick (*Prentice Hall of lndia (2000)*).

In Real Analysis II we present the fundamental concepts of the Lebesgue theory of measure and integration. The Riemann integral, dealt with in calculus courses, is well suited for computations but less suited for dealing with limit processes. In this course we will introduce the so called Lebesgue integral, which keeps the advantages of the Riemann integral and eliminates its drawbacks.

In Chapter 1, we discuss definitions and examples of Sigma Algebras and Borel Sets. We see that the collection $\mathcal{B}$ of Borel sets of real numbers is the smallest σ-algebra of sets of real numbers that contains all of the open and closed sets of real numbers (Definition 1.1.12 and Example 1.1.13). In Chapter 2, we lay the foundation of Lebesgue theory by describing the concept of measurable set and the Lebesgue measure of such a set. In Chapter 3, we study *measurable functions.* We establish that all continuous functions on a measurable domain are measurable (Proposition 3.1.5), as are all monotone and step functions on a closed, bounded interval. In Chapter 4, we study Lebesgue integration.

In Chapter 5, we consider a generalization of the Vitali

Convergence Theorem to sequences of integrable functions on a set of infinite measure; for a pointwise convergent sequence of integrable functions, tightness must be added to uniform integrablity in order to justify passage of the limit under the integral sign.

The fundamental theorems of integral and differential calculus, with respect to the Riemann integral, are the workhorses of calculus. In Chapter 6, we formulate these two theorems for the Lebesgue integral. We answer the question: for a function f on the closed, bounded interval $[a, b]$, when is $\int_a^b f' = f(b) - f(a)$?

In Chapter 7, we discuss completeness for Lebesgue integrable functions.

Throughout this course we use the following notations:

$\mathbb{N} = \{0,\ 1,\ 2,\ \ldots\}$, the set of natural numbers
$\mathbb{Z} = \{\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots\}$, the set of integers
$\mathbb{Q}$, the set of rational numbers
$\mathbb{R}$, the set of real numbers

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Chapter 1

Sigma Algebra and Borel Sets

A Borel set is an element of a Borel σ-algebra. Roughly speaking, Borel sets are the sets that can be constructed from open or closed sets by repeatedly taking countable unions and intersections. This is an advantage over open sets for which countable intersections of open sets need not be an open set.

1.1 Sigma Algebra

Definition 1.1.1. Given a set X, a collection $\mathcal{A}$ of subsets of X is called a σ**-algebra (of subsets of** X **)** provided

1. the empty set, $\emptyset$, belongs to $\mathcal{A}$;
2. the complement in X of a set in $\mathcal{A}$ also belongs to $\mathcal{A}$;
3. the union of a countable collection of sets in $\mathcal{A}$ also belongs to $\mathcal{A}$.

Example 1.1.2. Given a set X, the collection $\mathcal{A} = \{\emptyset, X\}$ is a σ-algebra because of the following observations:

1. the empty set, $\emptyset$, belongs to $\mathcal{A}$;
2. $\emptyset^c = X - \emptyset = X$, and $X^c = X - X = \emptyset$, showing that the complement in X of the sets $\emptyset$ and X in $\mathcal{A}$ are X and $\emptyset$, respectively, and also belongs to $\mathcal{A}$;
3. $\emptyset \cup \emptyset = \emptyset$, $\emptyset \cup X = X$, and $X \cup X = X$, showing that the unions of collection of sets in $\mathcal{A}$ also belong to $\mathcal{A}$.

Remark 1.1.3. ***[$\{\emptyset, X\}$ is the smallest σ-algebra]*** The σ-algebra $\{\emptyset, X\}$ (discussed in Example 1.1.2), which has only two members, is contained in every σ-algebra of subsets of X.

Example 1.1.4. Given a set X, the collection $\mathcal{P}(X)$ of all subsets of X, called the power set of X (which is also denoted by 2^X), is a σ-algebra because of the following observations:

1. the empty set, $\emptyset$, belongs to $\mathcal{A}$;

2. the complement in X of any subset of X is a subset of X and so the complement in X of a set in $\mathcal{P}(X)$ also belongs to $\mathcal{P}(X)$;

3. the countable unions of collection of sets in $\mathcal{P}(X)$ also belong to $\mathcal{P}(X)$.

Remark 1.1.5. *[$\mathcal{P}(X)$ is the largest σ-algebra]* The σ-algebra $\mathcal{P}(X)$ (discussed in Example 1.1.4), which is the collection of all subsets of X, is a superset of every σ-algebra of subsets of X.

Remark 1.1.6. Combining remarks 1.1.3 and 1.1.5, we have

$$\underbrace{\{\emptyset,\ X\}}_{\substack{\text{smallest } \sigma \text{ algebra of} \\ \text{subsets of } X}} \subseteq \mathcal{A} \subseteq \underbrace{P(X)}_{\substack{\text{largest } \sigma \text{ algebra of} \\ \text{subsets of } X}}$$

for any σ-algebra $\mathcal{A}$ of subsets of X.

For any σ-algebra $\mathcal{A}$, we infer from De Morgan's Identities that $\mathcal{A}$ is closed under intersections of countable collections of sets that belong to $\mathcal{A}$; moreover, since the empty-set belongs to $\mathcal{A}$, $\mathcal{A}$ is closed with respect to the formation of finite unions and finite intersections of sets that belong to $\mathcal{A}$.

Definition 1.1.7. [Difference of Sets, Complement of A in B] Let A and B be two sets. The complement of A in B, denoted by $B \sim A$, is the set of all points in B that are not in A; that is,

$$B \sim A = \{x \mid x \in B,\ x \notin A\}.$$

Remark 1.1.8. If X is the universal set, or the reference set, then

$$X \sim A = \{x \mid x \in X,\ x \notin A\} = A^c,$$

the complement of A.

We observe that a σ-algebra is closed with respect to relative complements since if A_1 and A_2 belong to $\mathcal{A}$, so does $A_1 \sim A_2$. This is because of the following observation: We know that

$$A_1 \sim A_2 = A_1 \cap A_2^c. \tag{1.1}$$

Since A_2 belongs to $\mathcal{A}$, its complement $X \sim A_2$ also belongs to $\mathcal{A}$. Now A_1 and $X \sim A_2$ belong to $\mathcal{A}$, implies their intersection $A_1 \cap [X \sim A_2]$ belongs to $\mathcal{A}$. Hence from (1.1) it follows that $A_1 \sim A_2$ belongs to $\mathcal{A}$.

Proposition 1.1.9. *Let $\mathcal{F}$ be a collection of subsets of a set X. Then the intersection $\mathcal{A}$ of all σ-algebras of subsets of X that contain $\mathcal{F}$ is a σ-algebra that contains $\mathcal{F}$. Moreover, it is the smallest σ-algebra of subsets of X that contains $\mathcal{F}$ in the sense that any σ-algebra that contains $\mathcal{F}$ also contains $\mathcal{A}$.*

Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of sets that belong to a σ-algebra $\mathcal{A}$. Since $\mathcal{A}$ is closed with respect to the formation of countable intersections and unions, the following two sets belong to $\mathcal{A}$:

$$\limsup\{A_n\}_{n=1}^{\infty} = \bigcap_{k=1}^{\infty}\left[\bigcup_{n=k}^{\infty} A_n\right]$$

and

$$\liminf\{A_n\}_{n=1}^{\infty} = \bigcup_{k=1}^{\infty}\left[\bigcap_{n=k}^{\infty} A_n\right].$$

Remark 1.1.10.

1. The set $\limsup\{A_n\}_{n=1}^{\infty}$ is the set of points that belong to A_n for countably infinitely many indices n;

2. The set $\liminf\{A_n\}_{n=1}^{\infty}$ is the set of points that belong to A_n except for at most finitely many indices n.

Remark 1.1.11. Although the union of any collection of open sets is open and the intersection of any finite collection of open sets is open, as we have seen, the *intersection of a countable collection of open sets need not be open.* In our development of Lebesgue measure and integration on the real line, we will see that the smallest σ-algebra of sets of real numbers that contains the open sets is a natural object of study.

Definition 1.1.12. The collection $\mathcal{B}$ of Borel sets of real numbers is the smallest σ-algebra of sets of real numbers that contains all of the open sets of real numbers.

Example 1.1.13.

1. Every open set is a Borel set and since a σ-algebra is closed with respect to the formation of complements, every closed set is a Borel set.

2. As singleton sets are closed, by the above remark, every singleton set is a Borel set.

3. Since each singleton set is closed and since every countable set is the countable union of singleton sets, it follows that every countable set is a Borel set.

Definition 1.1.14. A countable intersection of open sets is called a **G_δ set.** A countable union of closed sets is called an **F_σ set.**

Example 1.1.15. ***[G_δ sets and F_σ sets are Borel sets]*** Since a σ-algebra is closed with respect to the formation of countable unions and countable intersections, each G_δ set and each F_σ set is a Borel set.

Example 1.1.16. Both the $\liminf$ and $\limsup$ of a countable collection of sets of real numbers, each of which is either open or closed, is a Borel set.

Chapter 2

Lebesgue Measure

The Riemann integral of a bounded function over a closed, bounded interval is defined using approximations of the function that are associated with partitions of its domain into finite collection of subintervals. The generalization of the Riemann integral to the Lebesgue integral will be achieved by using approximations of the function that are associated with decompositions of its domain into finite collections of sets which we call *Lebesgue measurable sets*. Each interval (Definition 2.1.1) is Lebesgue measurable (Proposition 2.3.20).

The richness of the collection of Lebesgue measurable sets provides better upper and lower approximations of a function, and therefore of its integral, than are possible by just employing intervals. This leads to a larger class of functions that are Lebesgue integrable over very general domains and an integral that has better properties. In this chapter we establish the basis for the forthcoming study of Lebesgue measurable functions and the Lebesgue integral: the basis is the concept of measurable set and the Lebesgue measure of such a set.

2.1 Intervals

The simplest sets of real numbers are the intervals.

Definition 2.1.1. Intervals

1. We define **open interval** $(a,\, b)$ to be the set

$$(a,\, b) = \{x \in \mathbb{R} : a < x < b\}\,.$$

2. The **half open interval** $(a,\, b]$ is the set

$$(a,\, b] = \{x \in \mathbb{R} : a < x \le b\}\,.$$

3. The **half open interval** $[a,\, b)$ is the set

$$[a,\, b) = \{x \in \mathbb{R} : a \le x < b\}\,.$$

Remark 2.1.2. We always take $a < b$, but we also consider the infinite intervals

$$(a,\, \infty) = \{x \in \mathbb{R} : a < x\}$$

and

$$(-\infty,\, a\,) = \{x \in \mathbb{R} : x < a\}\,.$$

We write $(-\infty,\, \infty\,)$ for $\mathbb{R}$, the set of all real numbers.

Closed interval $[a,\, b]$ is the set

$$[a,\, b] = \{x \in \mathbb{R} : a \le x \le b\}\,.$$

Definition 2.1.3. ***[Bounded and Unbounded Intervals]***

Intervals of the form

$$(a,\, b), \quad [a,\, b], \quad (a,\, b], \text{ and } [a,\, b)$$

are **bounded intervals.** Intervals of the form

$$(a,\, \infty), \quad (-\infty,\, a\,), \quad [a,\, \infty), \text{ and } (-\infty,\, a\,]$$

$(-\infty,\, \infty\,)$ can also be regarded as an interval. are **unbounded intervals.**

2.1.1 Length of an Interval

Let I be an interval of any type given above. Then we have the following definition:

Definition 2.1.4. The **length** $l(I)$ of the interval I is defined

1. to be the difference of the endpoints of the interval I if I is bounded, and

2. ∞ if I is unbounded.

Example 2.1.5.

1. The length of the bounded interval (a, b) is $b - a$. Lengths of the intervals $[a, b]$, $(a, b]$ and $[a, b)$ are also $b - a$. In particular, the length of $(0, 1)$ is 1 and that of $[0, 1]$ is also 1.

2. The lengths of the unbounded intervals (a, ∞), $(-\infty, a)$, $[a, \infty)$ and $(-\infty, a]$ are the same and is ∞.

Remark 2.1.6. (We recall that the system of real numbers $\mathbb{R}$ can be extended by including two elements $+\infty$ and $-\infty$. This enlarged set is called the **extended real number system.)** If we let S be the collection of all intervals (bounded and unbounded), then the length is the function

$$l : S \to \mathbb{R} \cup \{ \infty \}$$

from the collection of all intervals to the extended real number system. That is, *length is an example of a* **set function,** (Recall that a *set function* is a function that associates an extended real number to each set in some collection of sets.)

So far, the domain of the set function length is the collection of all intervals.

Aim of this chapter is to extend the notion of length to more complicated sets than intervals.

As a generalization, we could define the ***length* of an open set** to be the sum of the lengths the countable number of open intervals of which it is composed.

Example 2.1.7. The length of the open set $(0,\ 1)\cup(3,\ 8)$ is the sum of the lengths the of open intervals $(0,\ 1)$ and $(3,\ 8)$ of which it is composed and is given by

$$\begin{aligned} l(\,(0,\ 1)\cup(3,\ 8)\,) &= l((0,\ 1)) + l((3,\ 8)\,) \\ &= \underbrace{(1-0)}_{1} + \underbrace{(8-3)}_{5} = 6. \end{aligned}$$

However, the collection of sets consisting of intervals and open sets is still too limited for our purposes.

Aim: We construct a collection of sets called Lebesgue measurable sets, and a set function of this collection called Lebesgue measure which is denoted by m.

2.1.2 Collection of Lebesgue measurable sets is a σ-algebra

Definition 2.1.8. [Definition 1.1.1 (Page 2) revisited] A collection of subsets of $\mathbb{R}$ is called a **σ-algebra** provided it contains $\mathbb{R}$ and is closed with respect to the formation of complements and countable unions.

Remark 2.1.9. By De Morgan's Identities, a collection as in the definition above is closed with respect to the formation of countable intersections. That is, σ-algebra is closed with respect to the formation of countable intersections.

Definition 2.1.10. [**Collection of disjoint sets**] A collection of sets is said to be **disjoint** if the members of the collection are pairwise disjoint; that is, that each pair of sets in the collection has empty intersection.

The collection of Lebesgue measurable sets (which we will construct in coming sections) is a σ *-algebra which contains all open sets and all closed sets* The set function m, called **measure (or Lebesgue measure),** possesses the following three properties:

1. **(The measure of an interval is its length)** Each nonempty interval I is Lebesgue measurable and

$$m(I) = l(I).$$

2. **(Measure m is translation invariant)** If E is Lebesgue measurable and y is any number, then the translate of E by y, given by

$$E + y = \{x + y \mid x \in E\}$$

is also Lebesgue measurable and

$$m(E + y) = m(E).$$

3. **(Measure is countably additive over countable disjoint unions of sets)** If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of Lebesgue measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)\,.$$

2.1.3 Construction of a set function satisfying the above three properties

It is not possible to construct a set function having the above three properties and is defined for all sets of real numbers. In fact, there is not even a set function defined for all sets of real numbers that possesses the first two properties and is finitely additive. **We overcome this limitation by constructing a set function on a very rich class of sets that does possess the above three properties. The construction has two stages.**

Stage One: We first construct a set function called **outer measure,** which we denote by m^*. ***It is defined for any set of real numbers (any subset of*** $\mathbb{R}$), *and thus, in particular, for any interval.* We will see that

1. The outer measure of an interval is its length.

2. Outer measure is translation invariant.

3. However, *outer measure is not finitely additive.* But it is countably subadditive in the sense that if $\{E_k\}_{k=1}^{\infty}$ is

any countable collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k)\,.$$

Stage Two: The second stage in the construction is to determine what it means for a set to be **Lebesgue measurable** and show that the collection of Lebesgue measurable sets is a σ -*algebra* containing the open and closed sets. We then restrict the set function m^* to the collection of Lebesgue measurable sets, denote it by m, and prove m is countably additive. We call m, the **Lebesgue measure**.

2.2 Lebesgue Outer Measure

We have seen that the length $l(I)$ of the interval I is (Definition 2.2) defined

1. to be the difference of the endpoints of the interval I if I is bounded, and
2. ∞ if I is unbounded.

Definition 2.2.1. ***[Outer measure]*** For a set A of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover A, that is, collection for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths of nonempty intervals are positive numbers, this sum is uniquely defined independently of the order of the terms. We define the **outer measure** of A, denoted by $m^*(A)$, to be infimum of all such sums, that is,

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k) \middle| A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}. \tag{2.1}$$

Remark 2.2.2.

1. For any set A of real numbers, $m^*(A) \geq 0$, because by the Definition 2.2.1 (Equation (2.1)) $m^*(A)$ is the infimum of a set of positive real numbers.

2. For the empty set $\emptyset$, $m^*(\emptyset) = 0$.

3. Since any cover of a set B is also a cover of any subset

of B, **outer measure is monotone** in the sense that

$$\text{if } A \subset B, \text{ then } m^*(A) \leq m^*(B).$$

Example 2.2.3. Each set consisting of a single point has outer measure 0 (Ref. Exercise 2.8.1 in Page 106)

Example 2.2.4. A countable set has outer measure 0 (Ref. Exercise 2.8.2.)

Proposition 2.2.5. *The outer measure of an interval is its length.*

Proof. **CASE 1:** We begin with the case of a closed, bounded interval, say, $[a, b]$.
Claim:

$$m^*[a, b] = b - a. \tag{2.2}$$

i.e., we claim that

$$m^*[a,\ b] = l\,(\,[a,\ b]\,).$$

Subclaim 1: $m^*[a, b] \leq b - a$. To prove this subclaim, let $\varepsilon > 0$. Since the open interval $(a - \varepsilon,\ b + \varepsilon)$ contains $[a, b]$

(Fig.2.1),

$$m^*[a,\ b] \le l(a-\varepsilon,\ b+\varepsilon) = b - a + 2\varepsilon. \tag{2.3}$$

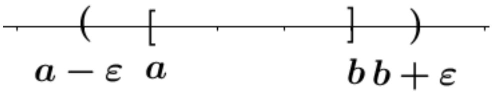

Figure 2.1: The open interval $(a-\varepsilon,\ b+\varepsilon)$ contains $[a,\ b]$ and $l(a-\varepsilon,\ b+\varepsilon) = (b+\varepsilon) - (a-\varepsilon) = b - a + 2\varepsilon$.

[[Details: By the definition of outer measure,

$$m^*([a,\ b]) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k) | A \subseteq \bigcup_{k=1}^{\infty} I_k \right\} \tag{2.4}$$

where infimum is taken over all countable coverings $\{I_k\}_{k=1}^{\infty}$ of $[a,\ b]$ (where I_k's are nonempty open, bounded intervals). In the above, we have seen that the open interval $(a-\varepsilon,\ b+\varepsilon)$ contains $[a,\ b]$, so that the collection $\{\,(a-\varepsilon,\ b+\varepsilon)\,\}$ (containing the single element, which is the open interval $(a -$

$\varepsilon,\ b+\varepsilon))$ covers C. Hence, from (2.4), we have

$$m^*([a,\,b]) = \inf\left\{\sum_{k=1}^{\infty} l(I_k) | A \subseteq \bigcup_{k=1}^{\infty} I_k\right\} \leq \underbrace{l(a-\varepsilon,\ b+\varepsilon)}_{b+\varepsilon-(a-\varepsilon)=b-a+2\varepsilon}\ ]]$$

Since $\varepsilon > 0$ is arbitrary, (2.3) shows that

$$m^*[a,\,b] \leq b-a. \tag{2.5}$$

Subclaim 2:

$$m^*[a,\,b] \geq b-a. \tag{2.6}$$

This is equivalent to showing that if $\{I_k\}_{k=1}^{\infty}$ is *any* countable collection of open, bounded intervals covering $[a,\,b]$, then

$$\sum_{k=1}^{\infty} l(I_k) \geq b-a. \tag{2.7}$$

[[Why (2.6) and (2.8) are equivalent? If we show that

$$\sum_{k=1}^{\infty} l(I_k) \geq b-a$$

for any countable collection $\{I_k\}_{k=1}^{\infty}$of open intervals covering

$[a,\ b]$, then

$$\inf\left\{\sum_{k=1}^{\infty} l(I_k) \mid A \subseteq \bigcup_{k=1}^{\infty} I_k\right\} \geq b-a$$

where infimum is taken over all countable coverings $\{I_k\}_{k=1}^{\infty}$ of $[a,\ b]$ (where I_k's are nonempty open, bounded intervals). Then

$$m^*([a,\ b]) = \inf\left\{\sum_{k=1}^{\infty} l(I_k) | A \subseteq \bigcup_{k=1}^{\infty} I_k\right\} \geq b-a$$

proving the subclaim 2.]]

Now to show that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of open, bounded intervals covering $[a,\ b]$, then

$$\sum_{k=1}^{\infty} l(I_k) \geq b-a, \tag{2.8}$$

we use the Heine-Borel Theorem, which gives that the closed bounded interval $[a,\ b]$ is compact and hence, any collection of open intervals covering $[a,\ b]$ contains a **finite subcollection** that also covers $[a,\ b]$. So for the given collection $\{I_k\}_{k=1}^{\infty}$ of open intervals covering $[a, b]$ has a finite subcollection, say, $\{ I_k \}_{k=1}^n$ that covers $[a, b]$. We will **show** that

$$\sum_{k=1}^{n} l(I_k) \geq b - a \tag{2.9}$$

and therefore

$$\sum_{k=1}^{\infty} l(I_k) \geq \sum_{k=1}^{n} l(I_k) \geq b - a. \tag{2.10}$$

Since the *finite* collection $\{ I_k \}_{k=1}^n$ covers $[a, b]$ (i.e., that $[a, b] \subseteq \bigcup_{k=1}^n I_k$) and since $a \in [a, b]$, we have $a \in \bigcup_{k=1}^n I_k$ and so there must be one of the I_k's that contains a. Select such an interval I_P from the collection $\{ I_k \}_{k=1}^n$ and denote it by $I_P = (a_1, \ b_1)$.

Case 1: If $b_1 \geq b$, then

$$l(I_P) = l(a_1, \ b_1) = b_1 - a_1 > b - a.$$

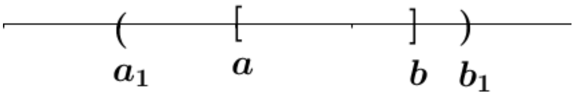

Figure 2.2: By the choice of the interval $(a_1,\ b_1)$, $a \in (a_1,\ b_1)$. If also $b_1 \geq b$, then we have the situation as in the figure.

Hence

$$\sum_{k=1}^{n} l(I_k) \geq l(\, I_P\,) > b - a.$$

Case 2: Consider the case $b_1 < b$. Then $b_1 \in [a,\ b)$ (i.e., b_1 is an element in the left closed right open interval $[a,\ b)$) and since $b_1 \notin (a_1, b_1)$, there must be an interval in the collection $\{I_k\}_{k=1}^{n}$, which we label $I_Q = (a_2,\ b_2)$, distinct from $(a_1,\ b_1)$, for which $b_1 \in (a_2, b_2)$; that is $a_2 < b_1 < b_2$.

Case 2a: If $b_2 \geq b$, then (Fig. 2.3)

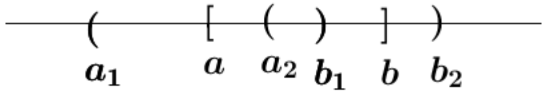

Figure 2.3: By the choice of the interval $(a_1,\ b_1)$, $a \in (a_1,\ b_1)$. If $b_1 < b$, then we choose $(a_2,\ b_2)$ such that $b_1 \in (a_2,\ b_2)$. Figure consider the Case 2a where $b_2 \geq b$.

$$\begin{aligned} l(\,I_P\,) + l(\,I_Q\,) &= l(a_1,\ b_1) + l(a_2,\ b_2) \\ &= (b_1 - a_1) + (b_2 - a_2) \\ &= b_2 - (a_2 - b_1) - a_1 \\ &= \underbrace{b_2 - a_1}_{} - \underbrace{(a_2 - b_1)}_{>0} \\ &> b_2 - a_1 \\ &> b - a. \end{aligned}$$

Hence

$$\sum_{k=1}^{n} l(I_k) \geq l(\,I_P\,) + l(\,I_Q\,) > b - a.$$

We continue this selection process until it terminates, as it must since there only n intervals in the collection $\{I_k\}_{k=1}^{n}$. Thus, we obtain a subcollection $\{(a_k,\ b_k)\}_{k=1}^{N}$ of $\{I_k\}_{k=1}^{n}$ for which the following three conditions are satisfied:

1. $a_1 < a$
2. $a_{k+1} < b_k$ for $1 \leq k \leq N-1$, and
3. $b_N > b$.

(Third condition holds since the selection process terminated. Since $\{I_k\}_{k=1}^n$ is a finite collection, our process must terminate with some interval $(a_N,\ b_N)$. But it terminates only if $b \in (a_N,\ b_N)$; that is if $a_N < b < b_N$). Thus,

$$\begin{aligned}
\sum_{k=1}^{n} l(I_k) &\geq \sum_{k=1}^{N} l(a_k - b_k) \\
&= (b_N - a_N) + (b_{N-1} - a_{N-1}) + \ \cdots \ + (b_1 - a_1) \\
&= b_N - \underbrace{(a_k - b_{k-1})}_{>0} - \underbrace{(a_{k-1} - b_{k-2})}_{>0} \\
&\quad - \ \cdots \ - \underbrace{(a_2 - b_1)}_{>0} - a_1 \\
&= b_N - a_1 - \underbrace{(a_k - b_{k-1})}_{>0} \\
&\quad - \underbrace{(a_{k-1} - b_{k-2})}_{>0} - \ \cdots \ - \underbrace{(a_2 - b_1)}_{>0}
\end{aligned}$$

$$\begin{aligned} &> b_N - a_1 \\ &> b - a. \end{aligned}$$

Thus the inequality (2.9) holds. Hence $m^*[a,\ b] \geq b - a$, proving the subclaim 2 given by the inequality (2.6). Now from (2.5) and (2.6) it follows that

$$m^*[a,\ b] = b - a.$$

This proves the claim (2.2).

CASE 2: If I is **any** bounded interval (need not be a closed interval), then given $\varepsilon > 0$, there are two closed, bounded intervals (Fig. (2.4)) J_1 and J_2 such that

$$J_1 \subseteq I \subseteq J_2$$

while

$$l(I) - \varepsilon < l(J_1) \ \text{ and } \ l(J_2) < l(I) + \varepsilon. \tag{2.11}$$

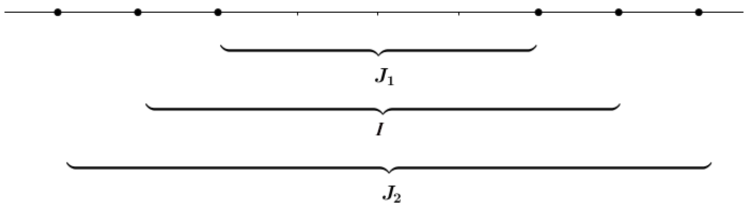

Figure 2.4: We choose J_1 and J_2 such that $J_1 \subseteq I \subseteq J_2$ and $l(I) - \varepsilon < l(J_1)$ and $l(J_2) < l(I) + \varepsilon$.

By CASE 1, for closed, bounded intervals J_1 and J_2,

$$m^*(J_1) = l(J_1) \text{ and } m^*(J_2) = l(J_2). \tag{2.12}$$

Also, by the montonicity of outer measure,

$$J_1 \subseteq I \subseteq J_2 \Rightarrow m^*(J_1) \leq m^*(I) \leq m^*(J_2). \tag{2.13}$$

Hence using (2.12),

$$\underbrace{l(J_1)}_{m^*(J_1)} \leq m^*(I) \leq \underbrace{l(J_2)}_{m^*(J_2)}$$

and using (2.11),

$$l(I)-\varepsilon < l\,(\,J_1\,) \leq m^*(I) \leq l\,(\,J_2\,) < l(I)+\varepsilon$$

As the choice of $\varepsilon > 0$ is arbitrary, it follows that for any $\varepsilon > 0$,

$$l(I)-\varepsilon < m^*(I) < l(I)+\varepsilon$$

and hence

$$l(I) = m^*(I).$$

Remark: Case 1 and Case 2 together shows that for any bounded interval I,

$$l(I) = m^*(I).$$

CASE 3: If I is an **unbounded** interval, then for each natural number n, there is an interval $J \subset I$ with $l\,(J) = n\,(Fig.\ (2.5))$.

By the monotonicity of outer measure, $J \subset I$ implies

$$m^*(I) \geq m^*(J).$$

Since J is a bounded interval, by the Remark just above,

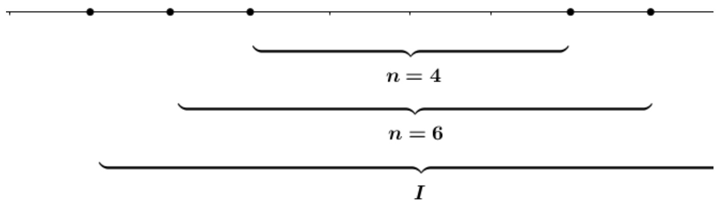

Figure 2.5: If I is an unbounded interval, then for each natural number n it is possible to find an interval of length n which is a subset of I. In the figure, two such intervals (of lengths 4 and 6) are shown.

$m^*(J) = l(J)$. Also, by the choice of J, $l(J) = n$. Thus,

$$m^*(I) \geq m^*(J) = l(J) = n\,.$$

That is,

$$m^*(I) \geq n\,.$$

The above holds for each natural number n. Therefore

$$m^*(I) = \infty\,. \tag{2.14}$$

Being the length of an unbounded interval (by Definition)

$$l(I) = \infty .$$

This together with (2.14) gives

$$m^*(I) = \infty \ = l(I) .$$

Considering all cases we have *outer measure of an interval is its length.* This completes the proof. □

Definition 2.2.6. For any set A and number y, $A + y$ is the set given by

$$A + y = \{a + y : a \in A\} \ .$$

Example 2.2.7. If $A = [1,\ 5]$, then $A{+}3 = \{a + 3 : a \in A\} = [4,\ 8]$ (Fig. (2.6)).

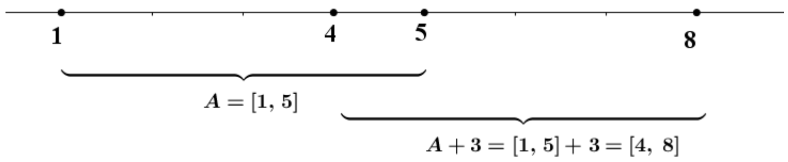

Figure 2.6: $A = [1,\ 5]$ and $A + 3 = [4,\ 8]$.

Example 2.2.8. If $A = [-3,\ 1]$, then

$$A + (-3) = \{a + (-3) : a \in A\} = [-6,\ -2] \ (Fig.(\ 2.7)).$$

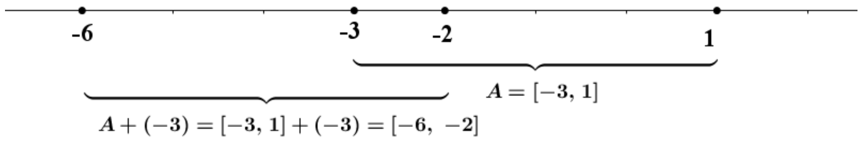

Figure 2.7: $A = [-3,\ 1]$ and $A + (-3) = [-6,\ -2]$.

Example 2.2.9. If $A = (-2, \infty)$, (Fig. (2.8)) then

$$A + (-6) = \{a + (-6) : a \in A\} = (-8, \infty).$$

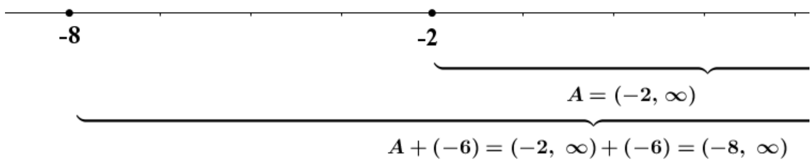

Figure 2.8: $A = [1, 5]$ and $A + 3 = [4, 8]$.

Proposition 2.2.10. *Outer measure is translation invariant, that is, for any set A and number y,*

$$m^*(A + y) = m^*(A).$$

Proof. Observe that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of sets, then

$$\{I_k\}_{k=1}^{\infty} \text{ covers } A \text{ if and only if } \{I_k + y\}_{k=1}^{\infty} \text{ covers } A + y.$$

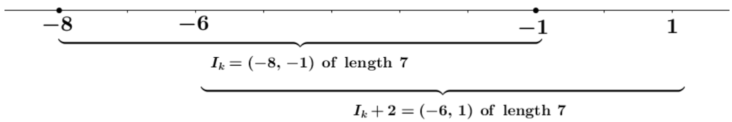

Figure 2.9: $A = (-8, -1)$ and $A + 2 = (-6, 1)$ are open intervals.

[[Details: We know that

$$A + y = \{a + y : a \in A\} \ .$$

$\{I_k\}_{k=1}^{\infty}$covers A if and only if $A \subseteq \bigcup_{k=1}^{\infty} I_k$ if and only if $A + y \subseteq \bigcup_{k=1}^{\infty}(I_k + y)$ where $I_k + y = \{z + y : z \in I_k\}$ for $k = 1, 2, \ldots$ if and only if $\{I_k+y\}_{k=1}^{\infty}$ covers $A+y$.]] Moreover, if each I_k is an open interval, then each $I_k + y$ is an open interval of the same length (An example is shown in Fig. 2.9) and so

$$\sum_{k=1}^{\infty} l(I_k) = \sum_{k=1}^{\infty} l(I_k + y).$$

The conclusion follows from these two observations. □

Proposition 2.2.11. *Outer measure is countably subadditive.*

That is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k). \tag{2.15}$$

Proof. **Case 1: If one of** the sets E_k' s has infinite outer measure, say $m^*(E_p) = \infty$, the inequality (2.15) holds trivially. [[Reason: $m^*(E_p) = \infty$ implies $\sum_{k=1}^{\infty} m^*(E_k) = \infty$, so that the right hand side of (2.15) is ∞; also we know that always $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \infty$. Thus (2.15) holds.]]

Case 2: Suppose $m^*(E_k)$ is finite **for all** k (i.e., we assume that each of the E_k's has finite outer measure). Let $\varepsilon > 0$. For each natural number k, there is a countable collection $\{I_{k,\,i}\}_{i=1}^{\infty}$ of open, bounded intervals for which

$$E_k \subseteq \bigcup_{i=1}^{\infty} I_{k,\,i} \text{ and } \sum_{k=1}^{\infty} l(I_{k,\,i}) \leq m^*(E_k) + \frac{\varepsilon}{2^k}. \tag{2.16}$$

[Reason: Fix a natural number k. By the definition of outer measure,

$$m^*(E_k) = \inf\left\{\sum_{k=1}^{\infty} l(J_k) | E_k \subseteq \bigcup_{k=1}^{\infty} J_k\right\}$$

where infimum is taken by considering all countable collections $\{J_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover E_k, that is, we consider all collections $\{J_k\}_{k=1}^{\infty}$ for which $E_k \subseteq \bigcup_{k=1}^{\infty} J_k$. Then $m^*(E_k) + \frac{\varepsilon}{2^k}$ is **not** the infimum of the set

$$\left\{\sum_{k=1}^{\infty} l(J_k) | E_k \subseteq \bigcup_{k=1}^{\infty} J_k\right\}$$

and hence there is a countable collection $\{I_{k,\,i}\}_{i=1}^{\infty}$ of nonempty open, bounded intervals for which $E_k \subseteq \bigcup_{i=1}^{\infty} I_{k,\,i}$, $\sum_{k=1}^{\infty} l(I_{k,\,i}) \in \{\sum_{k=1}^{\infty} l(J_k) | E_k \subseteq \bigcup_{k=1}^{\infty} J_k\}$ and $\sum_{k=1}^{\infty} l(I_{k,\,i}) \leq m^*(E_k) + \frac{\varepsilon}{2^k}$.
] Now we vary k over the set of natural numbers and obtain the collection

$$\{I_{k,\,i}\}_{1 \leq k,\,i < \infty} = \{I_{1,\,i}\}_{i=1}^{\infty} \cup \{I_{2,\,i}\}_{i=1}^{\infty} \cup \{I_{3,\,i}\}_{i=1}^{\infty} \cup \ \cdots .$$

Being the countable union of countable collections,

$$\{I_{k,\,i}\}_{1\,\leq\,k,\;i\;<\infty}$$

is a countable collection. For each natural number k, E_k is covered by $\{I_{k,\,i}\}_{i=1}^{\infty}$. Hence $\{I_{k,\,i}\}_{1\,\leq\,k,\;i\;<\infty}$ is a countable collection of open, bounded intervals that covers $\bigcup_{k=1}^{\infty} E_k$. That is,

$$\bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{1\,\leq\,k,\;i\,<\,\infty} I_{k,\,i}.$$

Thus, by the definition of outer measure,

$$
\begin{aligned}
m^*\left(\bigcup_{k=1}^{\infty} E_k\right) &\leq \sum_{1 \leq k,\ i < \infty} l(I_{k,\,i}) \\
&= \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} l(I_{k,\,i})\right] \\
&< \sum_{k=1}^{\infty}\left[m^*(E_k) + \frac{\varepsilon}{2^k}\right], using\ (2.16) \\
&= \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon \underbrace{\sum_{k=1}^{\infty}\frac{1}{2^k}}_{=\,1} \\
&= \left[\sum_{k=1}^{\infty} m^*(E_k)\right] + \varepsilon
\end{aligned}
$$

In the above

$$\sum_{k=1}^{\infty}\frac{1}{2^k} = 1$$

being the geometric series with initial term $a = \frac{1}{2}$ and common ratio $r = \frac{1}{2}$; we have $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$.

Since ε is an arbitrary positive number, we have

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) < \left[\sum_{k=1}^{\infty} m^*(E_k)\right] + \varepsilon$$

holds for every $\varepsilon > 0$. Hence

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Thus 2.15 is obtained and the proof is complete. □

The following result follows from Proposition 2.2.11 (Example 2.2.3 revisited) and its proof is given in Exercise 2.8.3.

Corollary 2.2.12. *A countable set has outer measure zero.*

Example 2.2.13. The set $[0,\ 1]$ is not countable.

Solution. If $[0,\ 1]$ is countable, then by Corollary (2.2.12), its measure must be 0, which is a contradiction to the fact that the outer measure of the interval $[0,\ 1]$ is its length given by $1 - 0 = 1$. Hence $[0,\ 1]$ is not countable.

Corollary 2.2.14. *Outer measure is finite subadditive. That is, if $\{E_k\}_{k=1}^{n}$ is any finite collection of sets, disjoint or not,* *then*

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} m^*(E_k). \tag{2.17}$$

Proof. The result is obtained by taking $E_k = \emptyset$ for $k > n$ in Proposition 2.2.11. □

2.3 The σ-Algebra of Lebesgue Measurable Sets

In the previous section we have seen that outer measure has four virtues:

1. it is defined for all sets of real numbers,
2. the outer measure of an interval is its length,
3. outer measure is countably subadditive, and
4. outer measure is translation invariant.

But outer measure *fails to be countably additive.* In fact, it is not even finitely additive because there are disjoint sets A

and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B)\,. \tag{2.18}$$

To solve this fundamental defect and make things better, *we identify a σ-algebra of sets, called the* **Lebesgue measurable sets,** *which contains all intervals and open sets and has the property that the restriction of the set function outer measure to the collection of Lebesgue measurable sets is countably additive.*

There are a number of ways to define what it means for a set to be measurable. We follow an approach due to the Greek mathematician Constantine Caratheodory.

Constantine Caratheodory
(1873-1950)
Greek Mathematician

Definition 2.3.1. A set E is said to be **measurable** if for each set A, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C). \tag{2.19}$$

Remark 2.3.2. 1. One advantage possessed by measurable sets is that the strict inequality (2.18)

$$m^*(A \cup B) < m^*(A) + m^*(B)$$

cannot occur if one of the sets is measurable. To exhibit this, *suppose* A *is measurable and* B *is any set disjoint from* A. Then, since A is measurable, by the definition of measurability given above, *for any set* Q,

$$m^*(Q) = m^*(Q \cap A) + m^*(Q \cap A^C)\,.$$

In particular, taking $Q = A \cup B$, we have

$$m^*(A \cup B) = m^*([A \cup B] \cap A) + m^*([A \cup B] \cap A^C) \tag{2.20}$$

By assumption B is disjoint from A, and hence (Fig.

(2.10))

$$[A \cup B] \cap A = \underbrace{[A \cap A]}_{A} \cup \underbrace{[B \cap A]}_{\emptyset} = A$$

and

$$[A \cup B] \cap A^C = \underbrace{[A \cap A^C]}_{\emptyset} \cup \underbrace{[B \cap A^C]}_{B} = B.$$

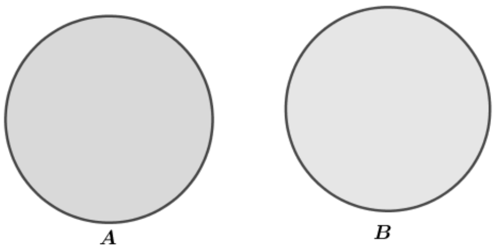

Figure 2.10: If $A \cap B = \emptyset$, then $[A \cup B] \cap A = A$ and $[A \cup B] \cap A^C = B$.

Thus, from (2.20), we obtain

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

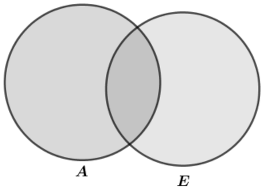

Figure 2.11: $A \cap E \neq \emptyset$.

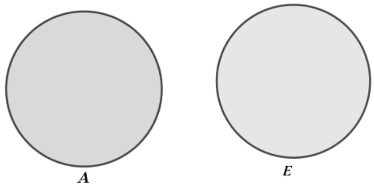

Figure 2.12: $A \cap E = \emptyset$.

2. Since $A = \underbrace{[A \cap E] \cup [A \cap E^C]}_{A \cap (E \cup E^C) \ = A}$ (two cases are shown in Fig.2.11 and Fig.2.12) and since outer measure m^* is finitely subadditive (Corollary 2.2.14), we always have

$$
\begin{aligned}
m^*(A) &= m^*\left([A \cap E] \cup [A \cap E^C]\right) \\
&\underset{\substack{\uparrow \\ \text{as } m^* \text{ is finitely} \\ \text{subadditive}}}{\leq} m^*(A \cap E) + m^*(A \cap E^C)
\end{aligned}
$$

Hence we see that *E is measurable if (and only if) for* *each A we have*

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^C). \tag{2.21}$$

This last inequality trivially holds if $m^*(A) = \infty$. Thus *it suffices to establish* (2.21) *for sets A that have finite outer measure.*

Remark 2.3.3. ***[Method of verifying measurability of a set]*** In view of the observation in Part 2 of the Remark 2.3.2 above, *to show the measurability of a set* E *we proceed as follows*:

1. if $m^*(A) = \infty$, then there is nothing to prove.
2. if A that has finite outer measure, then it suffices to establish

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^C). \tag{2.22}$$

Proposition 2.3.4. *A set is measurable if and only if its complement is measurable.*

Proof. By the Definition 2.3.1 of measurability, E is measurable if and only if for any set A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$$

By reordering the summands, we obtain

$$m^*(A) = m^*(A \cap E^C) + m^*(A \cap E). \tag{2.23}$$

Since $(E^C)^C = E$, (2.23) can be rewritten as, for any set A,

$$m^*(A) = m^*(A \cap E^C) + m^*(A \cap (E^C)^C). \tag{2.24}$$

(i.e., the definition of measurability is symmetric in E and E^C)

By Definition 2.3.1 of measurability, the last equality (which is true for any set A) shows that E^C is measurable. □

Proposition 2.3.5. *The empty set $\emptyset$ is measurable.*

Proof. The empty set $\emptyset$ is measurable, since for each set A

we have

$$m^*(\underbrace{A \cap \emptyset}_{\Phi}) + m^*(\underbrace{A \cap \emptyset^C}_{A}) = \underbrace{m^*(\emptyset)}_{0} + m^*(A) = m^*(A).$$

□

Proposition 2.3.6. *The set $\mathbb{R}$ of real numbers is measurable.*

Proof. The set $\mathbb{R}$ of real numbers is measurable, since for each set A we have

$$m^*(\underbrace{A \cap \mathbb{R}}_{A}) + m^*(\underbrace{A \cap \mathbb{R}^C}_{A \cap \emptyset = \emptyset}) = m^*(A) + \underbrace{m^*(\emptyset)}_{0} = m^*(A).$$

□

Theorem 2.3.7. *If $m^*(E) = 0$, then E is measurable. i.e., if E is a set of outer measure zero, then E is measurable.*

Proof. Let the set E have outer measure zero. i.e., suppose $m^*(E) = 0$. By Remark 2.3.3 (See inequality (2.22)), *to show that E is measurable, it is enough to show that for any set A,*

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^C). \tag{2.25}$$

For this, let A be any set. Then

$$A \cap E \subseteq E,$$

and so, by the monotonicity of outer measure,

$$m^* (A \cap E) \leq m^*(E) = 0.$$

Hence

$$m^* (A \cap E) = 0.$$

Also

$$A \cap E^C \subseteq A,$$

and so by the monotonicity of outer measure,

$$m^*(A \cap E^C) \leq m^*(A).$$

Thus,

$$\begin{aligned} m^*(A) &\geq m^*(A \cap E^C) \\ &= \underbrace{0}_{m^*(A\cap E)} + m^*(A \cap E^C) \\ &= m^*(A \cap E) + m^*(A \cap E^C). \end{aligned}$$

Hence (2.25) holds for any set A. This shows that E is measurable and proof is complete. □

Corollary 2.3.8. *Any countable set is measurable.*

Proof. By Example 2.2.4, outer measure of any countable set is zero. By Theorem 2.3.7, if E is a set of outer measure zero, then E is measurable. Hence any countable set is measurable. □

Proposition 2.3.9. *If E_1 and E_2 are measurable, so is $E_1 \cup E_2$.*

Proof. By the Remark 2.3.3 (See inequality (2.22)), to show that $E_1 \cup E_2$ is measurable, *it is enough to show that for any* *set* A,

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^C)\,. \quad (2.26)$$

To show this, let A be any set. Since E_1 is measurable, we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^C). \quad (2.27)$$

Since E_2 is measurable, considering the set $A \cap E_1^C$,

$$m^*(A \cap E_1^C) = m^*([A \cap E_1^C] \cap E_2) + m^*([A \cap E_1^C] \cap E_2^C). \quad (2.28)$$

Hence (2.27) becomes

$$m^*(A) = m^*(A \cap E_1) + \underbrace{m^*([A \cap E_1^C] \cap E_2) + m^*([A \cap E_1^C] \cap E_2^C)}_{m^*(A \cap E_1^C)}\,. \quad (2.29)$$

Also we use the following two set identities:

$$[A \cap E_1^C] \cap E_2^C = A \cap \underbrace{[E_1^C \cap E_2^C]}_{[E_1 \cup E_2]^C} = A \cap [E_1 \cup E_2]^C$$

and

$$\begin{aligned}[A\cap E_1]\cup[A\cap E_1^C\cap E_2] &= A\cap[\underbrace{E_1\cup(E_1^C\cap E_2)}_{E_1\cup E_2}] \\ &= A\cap(E_1\cup E_2)\end{aligned}$$

Hence

$$\begin{aligned}m^*(A\cap(E_1\cup E_2)) &= m^*\left(\underbrace{[A\cap E_1]}\cup\underbrace{[A\cap E_1^C\cap E_2]}\right) \\ &\leq m^*([A\cap E_1])+m^*\big([A\cap E_1^C\cap E_2]\big],\end{aligned}$$

where the last inequality follows by the finite subadditivity of outer measure (Corollary 2.2.14).

Hence (2.29) gives

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) \\ &\quad + \underbrace{m^*([A \cap E_1^C] \cap E_2^C)}_{m^*(A \cap [E_1 \cup E_2]^C)}, \text{ using } (2.28) \\ &= \underbrace{m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2)}_{\geq m^*(A \cap (E_1 \cup E_2))} \\ &\quad + m^*(A \cap [E_1 \cup E_2]^C) \\ &\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap [E_1 \cup E_2]^C). \end{aligned}$$

This inequality is true for any set A. Hence (2.26) is proved. Thus, $E_1 \cup E_2$ is measurable. □

Proposition 2.3.10. *The union of a finite collection of measurable sets is measurable.*

Proof. By Proposition 2.3.9, if E_1 and E_2 are measurable, so is $E_1 \cup E_2$.

Now let $\{E_k\}_{k=1}^n$ be any finite collection of measurable sets. We prove the measurability of the union $\bigcup_{k=1}^n E_k$, for general n, by induction.

Step 1. The result is trivially true for $n = 1$. (i.e., if consider a set $\{E\}$ consisting of a singleton measurable set E, then its union is E itself, which is measurable.)

Step 2. As induction argument, assume the result is true for $n - 1$. That is, we assume that for any collection of $n - 1$ measurable sets, its union is measurable. Now we show that the union $\bigcup_{k=1}^{n} E_k$ (the union of n measurable sets) is measurable. We note that

$$\bigcup_{k=1}^{n} E_k = \left[\bigcup_{k=1}^{n-1} E_k\right] \cup E_n. \tag{2.30}$$

By the induction argument, being the union of $n-1$ measurable sets, $\bigcup_{k=1}^{n-1} E_k$ is measurable. By the assumption, E_n also is a measurable set. Hence, by Proposition 2.3.9, the union of the two measurable sets $\bigcup_{k=1}^{n-1} E_k$ and E_n is also measurable. That is, $\left[\bigcup_{k=1}^{n-1} E_k\right] \cup E_n$ is measurable. Hence by (2.30), $\bigcup_{k=1}^{n} E_k$ is measurable. □

Proposition 2.3.11. *Let A be any set, and $\{E_1, \ldots, E_n\}$*

be a disjoint collection of measurable sets. Then

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

Proof. We prove the Proposition by induction on n.

Step 1. The result is clearly true for $n = 1$, because

$$m^*\left(A \cap \left[\bigcup_{k=1}^{1} E_k\right]\right) = m^*(A \cap E_k) = \sum_{k=1}^{1} m^*(A \cap E_1).$$

Step 2. As induction argument, we assume the result is true for $n - 1$. Since the E_k, $k = 1, 2, \ldots, n$ are disjoint sets, we have (a special case is shown in Figure 6.8)

$$\left[\bigcup_{k=1}^{n} E_k\right] \cap E_n = E_n$$

and

$$\left[\bigcup_{k=1}^{n} E_k\right] \cap E_n^C = \bigcup_{k=1}^{n-1} E_k$$

and hence

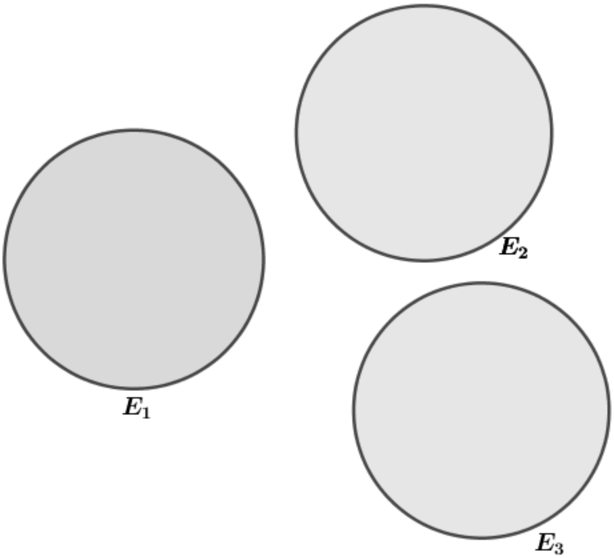

Figure 2.13: If E_1, E_2, and E_3 are disjoint sets, then $\left[\bigcup_{k=1}^{3} E_k\right] \cap E_3 = E_3$ and $\left[\bigcup_{k=1}^{3} E_k\right] \cap E_3^C = \bigcup_{k=1}^{2} E_k$.

$$A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n = A \cap E_n \tag{2.31}$$

and

$$A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n^C = A \cap \left[\bigcup_{k=1}^{n-1} E_k\right]. \tag{2.32}$$

Now by the measurability of E_n (using the Definition 2.3.1 of measurability of a set), for the set $A \cap [\bigcup_{i=1}^{n} E_i]$,

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_i\right]\right) = m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n\right) + m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n^C\right)$$

Using (2.31) and (2.32), the above equation becomes

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_i\right]\right) = m^*\left(A \cap E_n\right) + m^*\left(A \cap \left[\bigcup_{k=1}^{n-1} E_i\right]\right). \tag{2.33}$$

By the induction assumption, the result in the statement of the Proposition is true for $n-1$ sets, and hence

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n-1} E_i\right]\right) = \sum_{k=1}^{n-1} m^*\left(A \cap E_k\right).$$

Thus, (2.33) gives

$$\begin{aligned} m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_i\right]\right) &= m^*\left(A \cap E_n\right) + \sum_{k=1}^{n-1} m^*\left(A \cap E_k\right) \\ &= \sum_{k=1}^{n} m^*\left(A \cap E_k\right). \end{aligned}$$

This completes the proof. □

Corollary 2.3.12. *Let* $\{E_1, \ldots, E_n\}$ *be a disjoint collection of measurable sets. Then*

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(E_k).$$

Proof. By taking $A = \mathbb{R}$ in the statement of Proposition 2.3.11, we obtain

$$m^*\left(\mathbb{R} \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} m^*(\mathbb{R} \cap E_k).$$

Since

$$\mathbb{R} \cap \left[\bigcup_{k=1}^{n} E_k\right] = \bigcup_{k=1}^{n} E_k$$

and

$$\mathbb{R} \cap E_k = E_k, \text{ for } k = 1, \ldots, n,$$

we conclude that

$$m^* \left(\bigcup_{k=1}^{n} E_k \right) = \sum_{k=1}^{n} m^*(E_k).$$

□

Definition 2.3.13. A collection of subsets of $\mathbb{R}$ is called an **algebra** provided it contains $\mathbb{R}$ and is closed with respect to the formation of complements and finite unions; by De Morgan's Identities, such a collection is closed with respect to the formation of finite intersections.

Remark 2.3.14. [collection of measurable sets is an algebra] We infer from Proposition 2.3.10, together with the measurability of the complement of a measurable set (Proposition 2.3.4), that the collection of measurable sets is an algebra.

It is useful to observe that *the union of a countable collection of measurable sets is also the union of a countable disjoint collection of measurable sets.* We state this as follows:

Lemma 2.3.15. *Let* $\{A_k\}_{k=1}^n$ *be a countable collection of measurable sets. Define*

$$A'_1 = A_1$$

and for each $k \geq 2$, *define*

$$A'_k = A_k \sim \bigcup_{i=1}^{k-1} A_i.$$

Then $\{A'_k\}_{k=1}^\infty$ *is a disjoint collection of measurable sets whose union is the same as that of* $\{A_k\}_{k=1}^\infty$.

Proof. Since the collection of measurable sets is an algebra, $\{A'_k\}_{k=1}^\infty$ is a disjoint collection of measurable sets whose union is the same as that of $\{A_k\}_{k=1}^\infty$. □

Proposition 2.3.16. *The union of a countable collection of measurable sets is measurable.*

Proof. Let E be the union of a countable collection of measurable sets. As we observed in Lemma 2.3.15, there is a countable *disjoint* collection of measurable sets $\{E_k\}_{k=1}^\infty$ for which

$$E = \bigcup_{k=1}^{\infty} E_k.$$

Let A be *any* set. Let n be a natural number. Define

$$F_n = \bigcup_{k=1}^{n} E_k.$$

That is,

$$F_n = E_1 \cup E_2 \cdots \cup E_n.$$

Being the finite union of measurable sets, F_n is measurable (Proposition 2.3.10). Hence, by the definition of measurable set, for the measurable set F_n and the given set A,

$$m^*(A) = m^*(A \cap F_n) + m^*\left(A \cap F_n^C\right). \tag{2.34}$$

Since

$$F_n = \bigcup_{k=1}^{n} E_k \subseteq \bigcup_{k=1}^{\infty} E_k = E$$

we have

$$F_n^C \supseteq E^C,$$

and hence

$$A \cap F_n^C \supseteq A \cap E^C,$$

and so the monotonicity of outer measure gives

$$m^*\left(A \cap F_n^C\right) \geq m^*\left(A \cap E^C\right).$$

Thus, (2.34) gives

$$\begin{aligned} m^*(A) &= m^*(A \cap F_n) + m^*\left(A \cap F_n^C\right) \\ &\geq m^*(A \cap F_n) + m^*\left(A \cap E^C\right). \end{aligned} \tag{2.35}$$

Also,

$$\begin{aligned} m^*(A \cap F_n) &= m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) \\ &= \sum_{k=1}^{n} m^*(A \cap E_k), \text{ using Proposition 2.3.11.} \end{aligned}$$

Thus, (2.35) gives

$$m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap E^C).$$

Since *the left side of this inequality is independent of* n, it follows that *the above result is true for any natural number* n and hence we have (letting $n \to \infty$)

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap E^C). \tag{2.36}$$

By the countable subadditivitiy of the outer measure m^* (Proposition 2.2.11), we have

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \geq m^* \left(\bigcup_{k=1}^{\infty} (A \cap E_k) \right)$$

Since

$$\bigcup_{k=1}^{\infty} (A \cap E_k) = A \cap \bigcup_{k=1}^{\infty} E_k$$

the above inequality gives

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \geq m^*(A \cap \left[\bigcup_{k=1}^{\infty} E_k \right]) = m^*(A \cap E),$$

and so (2.36) gives

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^C).$$

Thus, (2.22) in the Remark 2.3.3 is satisfied for any set A. Hence E is measurable. This completes the proof. □

Definition 2.3.17. [σ-algebra] A collection of subsets of $\mathbb{R}$ is called an **σ-algebra** provided it contains $\mathbb{R}$ and is closed with respect to the formation of complements and countable unions.

Remark 2.3.18. By De Morgan's Identities, σ-algebra is also closed with respect to the formation of countable intersections.

Proposition 2.3.19. *The collection of measurable sets is a σ-algebra.*

Proof. Let $\mathcal{M}$ be the collection of measurable sets.

1. By Proposition 2.3.6, $\mathbb{R} \in \mathcal{M}$.

2. By Proposition 2.3.4, $A \in \mathcal{M}$ implies $A^c \in \mathcal{M}$; i.e., $\mathcal{M}$ is closed with respect to the formation of complements.

3. By Proposition 2.3.16, the union of a countable collection of measurable sets is measurable. Hence $\mathcal{M}$ is closed with respect to the formation of countable unions.

Hence, by Definition 2.3.17, $\mathcal{M}$ is a σ-algebra. That is, the collection of measurable sets is a σ-algebra. □

Proposition 2.3.20. *Every interval is measurable.*

Proof. By Proposition 2.3.19, the set of measurable sets are a σ-algebra. Therefore to show that every interval is measurable it suffices to show that every interval of the form (a, ∞) is measurable [Because then, by the Definition of σ-algebra or by Part 2 of the proof of Proposition 2.3.19, its complement $(-\infty, a]$ is measurable. Using De-Morgan's laws, Remark 2.3.18 and various set identities it can be shown that intervals of the form $[a, b]$, $[a, b)$, $(a, b]$, (a, b), $(-\infty, a]$, and (a, ∞) are also measurable].

Consider such an interval (a, ∞). Let A be any set. *We assume a does not belong to A.* Otherwise, replace A by $A \sim \{a\}$ (then $a \notin A \sim \{a\}$) leaving the outer measure unchanged, because of the following observation:

$$m^*(A) = m^*(\underbrace{[A \sim \{a\}] \cup \{a\}}_{A}) \leq m^*(A \sim \{a\}) + \underbrace{m^*(\{a\})}_{=0}$$

(where the last inequality is obtained by the countable sub-additivity of outer measure) and so

$$m^*(A) \leq m^*(A \backslash \{a\})$$

As outer measure is monotonic increasing, since $A \sim \{a\} \subset A$, we also have

$$m^*(A \backslash \{a\}) \leq m^*(A)$$

and thus

$$m^*(A \backslash \{a\}) = m^*(A).$$

It is enough to prove (2.22) in the Remark 2.3.3. That is, we must show that

$$m^*(A) \geq m^*(A \cap (a, \ \infty)) + m^*(A \cap (a, \ \infty)^C).$$

$(a,\ \infty)^C = (-\infty,\ a]$. So

$$A \cap (a,\ \infty)^C = A \cap (-\infty,\ a]$$

and since $a \notin A$, it follows that

$$A \cap (a,\ \infty)^C = A \cap (-\infty,\ a).$$

Hence it we let

$$A_1 = A \bigcap (-\infty,\ a), \quad \text{and} \quad A_2 = A \bigcap (a\,,\ \infty).$$

then it must be shown that

$$m^*(A_1) + m^*(A_2) \leq m^*(A\,). \tag{2.37}$$

By the definition of the outer measure $m^*(A\,)$ as an infimum (Definition 2.2.1), to verify (2.37) it is necessary and sufficient to show that for any countable collection $\{I_k\}_{k=1}^{\infty}$ of open, bounded intervals that covers A,

$$m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} l(I_k). \tag{2.38}$$

Indeed, for such a covering, for each index k, define

$$I'_k = I_k \bigcap (-\infty, \ a), \quad \text{and} \quad I''_k = I_k \bigcap (a\,, \ \infty).$$

Then I_k and I''_k are intervals and

$$l(I_k) = l(I'_k\,) + l(I''_k\,). \tag{2.39}$$

Since $\{I'_k\}_{k=1}^{\infty}$ and $\{I''_k\}_{k=1}^{\infty}$ are countable collections of open, bounded intervals that cover A_1 and A_2, respectively, by the definition of outer measure (Definition 2.2.1),

$$m^*(A_1) \le \sum_{k=1}^{\infty} l(I'_k), \quad \text{and} \quad m^*(A_2) \le \sum_{k=1}^{\infty} l(I''_k).$$

Therefore,

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\le \sum_{k=1}^{\infty} l(I'_k) + \sum_{k=1}^{\infty} l(I''_k) \\ &= \sum_{k=1}^{\infty} [l(I'_k) + l(I''_k)] \\ &= \sum_{k=1}^{\infty} l(I_k), \text{ using (2.39)}. \end{aligned}$$

Thus (2.38) holds and the proof is complete. □

Proposition 2.3.21. *Every open set is measurable.*

Proof. Every open set is the disjoint union of a countable collection of open intervals. By Proposition 2.3.20, *every open interval is measurable.* Hence, by Proposition 2.3.16, every open set is measurable. □

Proposition 2.3.22. *Every closed set is measurable.*

Proof. By Proposition 2.3.21, every open set is measurable. Since the collection of measurable sets is a σ-algebra (Proposition 2.3.19), it follows that complement of an open set is also measurable. Every closed set is the complement of an open set. Hence closed sets are measurable. This completes the proof. □

Recall that (Definition 1.1.14) a set of real numbers is said to be a G_δ **set** provided it is the intersection of a countable collection of open sets and said to be an F_σ **set** provided it is the union of a countable collection of closed sets.

Proposition 2.3.23. *Every G_δ set is measurable.*

Proof. Since open sets are measurable (Proposition 2.3.21) and since the collection of measurable sets is a σ-algebra (Proposition 2.3.19), using Remark 2.3.18, it follows that G_δ set, which is the intersection of a countable collection of open sets, is measurable. □

Proposition 2.3.24. *Every F_σ set is measurable.*

Proof. An F_σ set is the union of a countable collection of closed sets. By Proposition 2.3.22, every closed set is measurable and by Proposition 2.3.16 the union of a countable collection of measurable sets is measurable. Hence every F_σ set is measurable. □

Definition 2.3.25. The intersection of all the σ-algebras of subsets of $\mathbb{R}$ that contain the open sets is a σ-algebra called the **Borel σ-algebra;** members of this collection are called **Borel sets.**

Notation 2.3.26. Borel σ-algebra is denoted by $\mathcal{B}$.

Remark 2.3.27. The Borel σ-algebra is contained in every σ-algebra *that contains all open sets.* i.e., if $\mathcal{A}$ is a σ-algebra that contains the open sets, then

$$\mathcal{B} \subset \mathcal{A}.$$

Theorem 2.3.28. *The collection $\mathcal{M}$ of measurable sets is a σ-algebra that contains the σ-algebra $\mathcal{B}$ of Borel sets. Each interval, each open set, each closed set, each G_δ set and each F_σ set is measurable.*

Proof.

- By Proposition 2.3.19, the collection $\mathcal{M}$ of measurable sets is a σ-algebra.
- By Proposition 2.3.21, $\mathcal{M}$ contains all open sets.
- Now $\mathcal{M}$ is a σ-algebra that contains all open sets. Hence, by Remark 2.3.27,

  $$\mathcal{B} \subset \mathcal{M}.$$

  Hence $\mathcal{M}$ is a σ-algebra that contains the σ-algebra $\mathcal{B}$ of Borel sets.
- Each interval, each open set, each closed set, each G_δ set and each F_σ set is measurable follows from various propositions given above.

□

Proposition 2.3.29. *The translate of a measurable set is measurable.*

Proof. Let E be a measurable set. Let A be any set and y be a real number. We show that $E + y$ is measurable, using the measurability of E and the translation invariance of outer measure (Proposition 2.2.10).

$$\begin{aligned} m^*(A) &= m^*(A - y), \\ &\quad \text{by the translation invariance of outer measure} \\ &= m^*([A - y] \cap E) + m^*([A - y] \cap E^C), \\ &\quad \text{by the measurability of } E \text{ and} \\ &\quad \text{using the set } A - y \\ &= m^*(A \cap [E + y]) + m^*(A \cap [E + y]^C). \end{aligned}$$

As the choice of the set A is arbitrary,

$$m^*(A) = m^*(A \cap [E + y]) + m^*(A \cap [E + y]^C)$$

holds for any set A and this shows that $E + y$ is measurable.

As the choice of the real number y is also arbitrary, it follows that any translate of E is measurable. As the choice of the measurable set E is also arbitrary, we have translate of a measurable set is measurable. □

2.4 Outer and Inner Approximation of Lebesgue Measurable Sets

We now present two characterizations of measurability of a set, one based on inner approximation by closed sets and the other on outer approximation by open sets, which provide alternate angles of vision on measurability. These characterizations will be essential tools for our forthcoming study of approximation properties of measurable and integrable functions.

Measurable sets possess the following **excision property:**

Proposition 2.4.1. ***Excision Property:*** *If A is a measurable set of finite outer measure that is contained in B, then*

$$m^*(B \sim A) = m^*(B) - m^*(A). \tag{2.40}$$

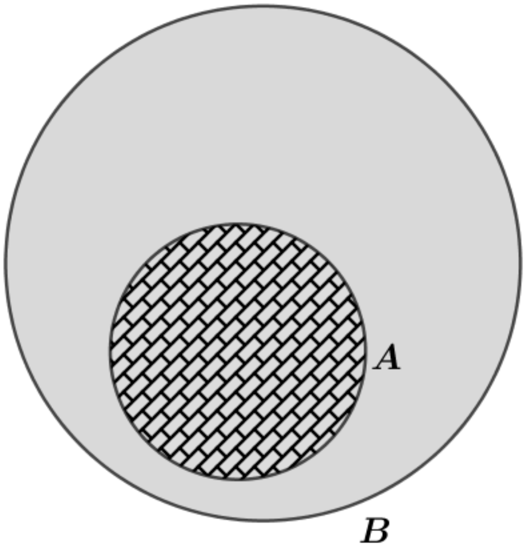

Figure 2.14: If $A \subseteq B$ then $B \cap A = A$ and $B \cap A^C = B \backslash A$.

Proof. By the measurability of A, for the set B,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^C).$$

Also, since $A \subseteq B$, we have $B \cap A = A$ and $B \cap A^C = B \sim A$ (Fig.(2.14)), and hence the above equation gives

$$m^*(B) = m^*(A) + m^*(B \sim A).$$

Since A is a measurable set of finite outer measure, $m^*(A) < \infty$, and subtracting $m^*(A)$ from both sides of the above equation, we get

$$m^*(B \sim A) = m^*(B) - m^*(A).$$

□

Theorem 2.4.2. *Let E be any set of real numbers. Then each one of the following four statements is equivalent to the measurability of E.*

(Outer Approximation by Open Sets and G_δ Sets)

(i) Given $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \sim E) < \varepsilon$.

(ii) There is a G_δ set G such that $E \subset G$, and $m^*(G \sim E) = 0$.

(Inner Approximation by Closed Sets and F_σ Sets)

(iii) *Given* $\varepsilon > 0$, *there is a closed set* $F \subset E$ *with* $m^*(E \sim F) < \varepsilon$.

(iv) *There is an* F_σ *set* F *with* $F \subset E$, *and* $m^*(E \sim F) = 0$.

Proof. We establish the equivalence of the measurability of E with each of the two outer approximation properties (i) and (ii). The remainder of the proof follows from De Morgan's Identities together with the observations that *a set is measurable if and only if its complement is measurable, is open if and only if its complement is closed, and is* F_σ *if and only if its complement is* G_δ.

We are now going to show that E **is measurable implies property (i) holds for** E **implies property (ii) holds for** E **implies** E **is measurable.** Assume E is measurable. Let $\varepsilon > 0$.

Case 1: First consider the case that $m^*(E) < \infty$. By the definition of outer measure (Definition 2.2.1), since

$$m^*(E) = \inf_{E \subset \bigcup J_k} \sum l(J_k),$$

where infimum is taken by considering nonempty open, bounded intervals $\{J_k\}_{k=1}^{\infty}$ that cover E. Hence, $m^*(E)+\varepsilon$ cannot be a lower bound of the set

$$\left\{\sum l\,(J_k) : E \;\subset\; \bigcup J_k, \text{ where } \{J_k\}_{k=1}^{\infty} \text{ covers } E\right\}$$

and hence there exists a countable collection of open intervals $\{\,I_k\}_{k=1}^{\infty}$ which covers E and for which

$$\sum_{k=1}^{\infty} l(I_k) < m^*(E)+\varepsilon\,. \tag{2.41}$$

Define

$$O=\bigcup_{k=1}^{\infty} I_k.$$

Then, being the union of open intervals, O is an open set . Since $\{\,I_k\}_{k=1}^{\infty}$ covers E, we have $E\subseteq\bigcup_{k=1}^{\infty} I_k$. Thus O is an open set containing E. By the definition of outer measure of O,

$$m^*(O)=\inf_{O\subset\bigcup J_k}\sum l\,(J_k)\,,$$

where infimum is taken by considering nonempty open, bounded intervals $\{J_k\}_{k=1}^{\infty}$ that covers O.

Thus, in particular,

$$m^*(O) \leq \sum_{k=1}^{\infty} l(I_k).$$

Using (2.41), this gives

$$m^*(O) \leq \sum_{k=1}^{\infty} l(I_k) < m^*(E) + \varepsilon\,.$$

Hence

$$m^*(O) - m^*(E) < \varepsilon\,.$$

However, E is measurable and has finite outer measure. Therefore, by the excision property of measurable sets (Proposition 2.4.1) ,

$$m^*(O \sim E) = m^*(O) - m^*(E) < \varepsilon\,.$$

Case 2: Now we assume that E is a measurable set of infinite measure; i.e., $m^*(E) = \infty$. For each $k \in \mathbb{N}$, let

$$E_k = E \cap (-k,\ k).$$

Then $E_k \subset (-k,\ k)$ and hence each E_k has finite measure and by what proved in Case 1 above (corresponding to $\frac{\varepsilon}{2^k}$) there is an open set, say O_k, $O_k \supset E_k$ such that

$$m^* \left(O_k \sim E_k\right) < \frac{\varepsilon}{2^k}. \tag{2.42}$$

Let $O = \bigcup_{k=1}^{\infty} O_k$. Then O is open, $O \supset E$ and since

$$O \sim E = \bigcup_{k=1}^{\infty} O_k \sim E \subset \bigcup_{k=1}^{\infty} [O_k \sim E_k]$$

we get (by the monotonicity of m^*)

$$m^* \left(O \sim E\right) \leq m^* \left(\bigcup_{k=1}^{\infty} \left[O_k \sim E_k\right]\right). \tag{2.43}$$

By the countable subadditivity of outer measure,

$$m^* \left(\bigcup_{k=1}^{\infty} \left[O_k \sim E_k\right]\right) \leq \sum_{k=1}^{\infty} m^* \left(O_k \sim E_k\right)$$

and hence (2.43) gives

$$\begin{aligned} m^*(O \sim E) &\leq \sum_{k=1}^{\infty} m^*(O_k \sim E_k) \\ &< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k}, \text{ using (2.42)} \\ &= \varepsilon. \end{aligned}$$

Thus property (i) holds for E.

Now assume property (i) holds for E. For each natural number k, choose an open set O_k that contains E and for which

$$m^*(O_k \sim E) < \frac{1}{k}.$$

Define

$$G = \bigcap_{k=1}^{\infty} O_k .$$

Then, being the countable intersection of open sets, G is a G_δ set that contains E. Moreover, since for each k, $G \sim E \subseteq O_k \sim E$, by the monotonicity of outer measure,

$$m^*(G \sim E) \leq m^*(O_k \sim E) < \frac{1}{k}.$$

That is, for each k,

$$0 \leq m^*(G \sim E) < \frac{1}{k}.$$

Hence

$$m^*(G \sim E) = 0$$

and so (ii) holds.

Now assume property (ii) holds for E. i.e.., assume that there is a G_δ set G such that $E \subset G$, and $m^*(G \sim E) = 0$. Since a set of measure zero is measurable (Theorem 2.3.7), it follows that $G \sim E$ is measurable. Also G_δ sets are measurable (Theorem 2.3.28). Hence G is also measurable. Also, since the collection of measurable sets is an algebra, intersection of the measurable sets $G \sim E$ and G is also measurable. That is, the set $G \cap [G \sim E]^C$ is measurable. Since

$$E = G \cap [G \sim E]^C$$

it follows that E is measurable.

Thus, we have shown that E **is measurable implies property (i) holds for** E **implies property (ii) holds for** E **implies** E

is measurable. This completes the proof. $\square$

Theorem 2.4.3. *[Measurable sets of finite outer measure are nearly equal to the disjoint union of a finite number of open interval] Let E be a measurable set of finite outer measure. Then for each $\varepsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which if $O = \bigcup_{k=1}^n I_k$, then*

$$m^*(E \sim O) + m^*(O \sim E) < \varepsilon.$$

Proof. According to Theorem 2.4.2(i), since E is measurable corresponding to $\varepsilon > 0$ there is an open set U such that

$$E \subseteq U \text{ , with } , m^*(U \sim E) < \frac{\varepsilon}{2}. \tag{2.44}$$

Since E is measurable and has finite outer measure, we infer from the excision property of outer measure that U also has finite outer measure [[Details: By excision property (Proposition 2.4.1), $m^*(U \sim E) = m^*(U) - m^*(E)$ and hence $m^*(U) = m^*(U \sim E) + m^*(E) < \infty$, since $m^*(E) < \infty$ and $m^*(U \sim E) < \frac{\varepsilon}{2}$.]] We note that *Every open set of real numbers is the disjoint union of a countable collection of open intervals* . Let

U be the union of the countable disjoint collection of open intervals $\{I_k\}_{k=1}^{\infty}$. Each interval is measurable and its outer measure is its length. Therefore, by Proposition 2.3.11 and the monotonicity of outer measure, for each natural number n,

$$\sum_{k=1}^{n} l(I_k) = m^* \left(\bigcup_{k=1}^{n} I_k \right) \leq m^*(U) < \infty.$$

The right-hand side of this inequality is independent of n. Therefore

$$\sum_{k=1}^{\infty} l(I_k) < \frac{\varepsilon}{2}.$$

Define $O = \bigcup_{k=1}^{n} I_k$. Since $O \sim E \subseteq U \sim E$, by the monotonicity of outer measure

$$m^*(O \sim E) \leq m^*(U \sim E)$$

and since $E \subseteq U$ with $m^*(U \sim E) < \frac{\varepsilon}{2}$ (using 2.44), the above gives

$$m^*(O \sim E) \leq m^*(U \sim E) < \frac{\varepsilon}{2}.$$

On the other hand, since $E \subseteq U$,

$$E \sim O \subseteq U \sim O = \bigcup_{k=n+1}^{\infty} I_k$$

so that by the definition of outer measure,

$$m^*(E \sim O) \leq \sum_{k=n+1}^{\infty} l(I_k) < \frac{\varepsilon}{2}.$$

Thus,

$$m^*(O \sim E) + m^*(E \sim O) < \varepsilon .$$

□

2.5 Countable Additivity, Continuity, and the Borel-Cantelli Lemma

2.5.1 Lebesgue Measure

Definition 2.5.1. The restriction of the set function outer measure to the class of measurable sets is called **Lebesgue**

measure. It is denoted by m, so that if E is a measurable set, its Lebesgue measure, $m(E)$, is defined by

$$m(E) = m^*(E).$$

Proposition 2.5.2. *Lebesgue measure is countably additive, that is, if* $\{E_k\}_{k=1}^{\infty}$ *is a countable disjoint collection of measurable sets, then its union* $\bigcup_{k=1}^{\infty} E_k$ *also is measurable and*

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)\,.$$

Proof. Proposition 2.3.16 tells us that $\bigcup_{k=1}^{\infty} E_k$ is measurable. According to Proposition 2.2.11, outer measure is countably subadditive. Thus,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k). \tag{2.45}$$

To get the equality, it remains to prove this inequality in the opposite direction. According to Proposition 2.3.11, for each natural number n,

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)\,.$$

Since $\bigcup_{k=1}^{\infty} E_k$ contains $\bigcup_{k=1}^{n} E_k$, by the monotonicity of outer measure and the preceding equality, for each n,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{n} m(E_k).$$

The left-hand side of this inequality if independent of n. Therefore

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k). \tag{2.46}$$

From the inequalities (2.45) and (2.46) it follows that

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

□

Theorem 2.5.3. *The set function* Lebesgue measure, *defined on the σ-algebra of Lebesgue measurable sets, assigns length to* *any interval, is translation invariant, and is countable additive.*

Proof. By Definition 2.5.1, the restriction of the set function outer measure to the class of measurable sets is called **Lebesgue measure** and denoted by m. Hence if A is a measurable set, then $m(A) = m^*(A)$.

- By Proposition 2.3.20, every interval is measurable. Hence its Lebesgue measure is the same as the outer measure. By Proposition 2.2.5, the outer measure of an interval is its length. Hence the Lebesgue measure of an interval is its length.

- By Proposition 2.2.10, outer measure is translation invariant. Hence on the σ-algebra of Lebesgue measurable sets, Lebesgue measure is translation invariant.

- By Proposition 2.5.2, Lebesgue measure is countably additive.

This completes the proof. □

Definition 2.5.4. A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is said to be

1. **ascending** provided for each k, $E_k \subseteq E_{k+1}$, and

2. **descending** provided for each k, $E_{k+1} \subseteq E_k$.

Remark 2.5.5. 1. $\{E_k\}_{k=1}^{\infty}$ is ascending if

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \subseteq E_k \subseteq E_{k+1} \subseteq \cdots$$

2. $\{E_k\}_{k=1}^{\infty}$ is descending if

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots \supseteq E_k \supseteq E_{k+1} \supseteq \cdots$$

Theorem 2.5.6. ***(The Continuity of Measure)*** *Lebesgue measure possesses the following continuity properties:*

1. *If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then*

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k). \tag{2.47}$$

2. *If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets*

and $m(B_1) < \infty$*, then*

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k). \tag{2.48}$$

Proof. We first prove (i).

Case 1. If there is an index k_0 for which $m(A_{k_0}) = \infty$, then, by the monotonicity of measure ($\bigcup_{k=1}^{\infty} A_k \supseteq A_{k_0}$ and $A_k \supseteq A_{k_0}$ for $k \geq k_0$ implies $m\left(\bigcup_{k=1}^{\infty} A_k\right) \geq m(A_{k_0})$ and $m(A_k) \supseteq m(A_{k_0})$ for$k \geq k_0$ so that) $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty$ and $m(A_k) = \infty$ for all $k \geq k_0$.

Therefore (2.47) holds since each side equals ∞.

Case 2. It remains to consider the case that $m(A_k) < \infty$ for all k. Define

$$A_0 = \emptyset$$

and then define

$$C_k = A_k \sim A_{k-1} \text{ for each } k \geq 1.$$

By construction, since the sequence $\{A_k\}_{k=1}^{\infty}$ is ascending,

$\{C_k\}_{k=1}^{\infty}$ is disjoint and $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} C_k$.

By the countable additivity of m (Proposition 2.5.2),

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} m(A_k \sim A_{k-1}). \tag{2.49}$$

Since $\{A_k\}_{k=1}^{\infty}$ is ascending, we infer from the excision property of measure that

$$\begin{aligned}\sum_{k=1}^{\infty} m(A_k \sim A_{k-1}) &= \sum_{k=1}^{\infty}[m(A_k) - m(A_{k-1})] \\ &= \lim_{n\to\infty} \sum_{k=1}^{n}[m(A_k) - m(A_{k-1})] \\ &= \lim_{n\to\infty} [m(A_n) - m(A_0)]\end{aligned} \tag{2.50}$$

Since $m(A_0) = m(\emptyset) = 0$, (2.47) follows from (2.49) and (2.50).

To prove (ii) we define

$$D_k = B_1 \sim B_k \text{ for each } k.$$

Since the sequence $\{B_k\}_{k=1}^{\infty}$ is descending, the sequence $\{D_k\}_{k=1}^{\infty}$ is ascending. By part (i),

$$m\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k\to\infty} m(D_k).$$

$$\begin{aligned}\bigcup_{k=1}^{\infty} D_k &= \bigcup_{k=1}^{\infty}[B_1 \sim B_k] \\ &= B_1 \sim \bigcap_{k=1}^{\infty} B_k, \text{ using De Morgan's identities.}\end{aligned}$$

On the other hand, by the excision property of measure, for each k, since $m\,(B_k) < \infty$,

$$m\,(D_k) = m\,(B_1) - m\,(B_k).$$

Therefore,

$$m\left(B_1 \sim \bigcap_{k=1}^{\infty} B_k\right) = \lim_{n\to\infty}[m(B_1) - m(B_n)].$$

Once more using excision we obtain the equality (2.48).

□

Definition 2.5.7. For a measurable set E, we say that a property holds **almost everywhere on** $\boldsymbol{E}$**,** or it holds for almost all $x \in E$, provided there is a subset E_0 of E for which $m(E_0) = 0$ and the property holds for all $x \in E \sim E_0$.

Remark 2.5.8. If f and g are extended real-valued functions on E, then

$$f = g \text{ a.e. on } E \text{ if } m\{x \in E : f(x) \neq g(x)\} = 0.$$

Lemma 2.5.9. ***[The Borel-Cantelli Lemma]***

Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.

Proof. For each n, by the countable subadditivity of m,

$$m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k) < \infty.$$

Hence, by the continuity of measure (Proposition 2.5.6,

$$m\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty}E_k\right]\right)=\lim_{n\to\infty}m(\bigcup_{k=n}^{\infty}E_k)\leq\lim_{n\to\infty}\sum_{k=n}^{\infty}m(E_k)=0.$$

Therefore almost all $x\in\mathbb{R}$ fail to belong to $\bigcap_{n=1}^{\infty}[\bigcup_{k=n}^{\infty}E_k]$ and therefore belong to at most finitely many E_k's. □

2.5.2 Properties of Lebesgue measure

The set function Lebesgue measure inherits the properties possessed by Lebesgue outer measure. We lsit some of these properties.

(Finite Additivity) For any *finite* disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets,

$$m\left(\bigcup_{k=1}^{n}E_k\right)=\sum_{k=1}^{n}m(E_k).$$

(Monotonicity) If A and B are measurable sets and $A\subseteq B$, then

$$m(A)\leq m(B).$$

(Excision) If, moreover, $A \subseteq B$ and $m(A) < \infty$, then

$$m(B \sim A) = m(B) - m(A)$$

so that if $m(A) = 0$, then

$$m(B \sim A) = m(B).$$

(Countable Monotonicity) For any countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets that covers a measurable set E,

$$m(E) \leq \sum_{k=1}^{\infty} m(E_k).$$

Countable monotonicity is a combination of the monotonicity and countable subadditivity properties of measure.

Remark 2.5.10. In our forthcoming study of Lebesgue integration it will be apparent that it is the countable additivity of Lebesgue measure that provides the Lebesgue integral with its decisive advantage over the Riemann integral.

2.6 Nonmeasurable Sets

In the previous sections, we defined Lebesgue measurable sets and studied properties of the class of measurable sets. It is only natural to ask if, in fact, there are any sets that fail to be measurable. The answer is not at all obvious.

We know that if a set E has outer measure zero, then it is measurable, and since any subset of E also has outer measure zero, every subset of E is measurable. This is the best that can be said regarding *the inheritance of measurability through the relation of set inclusion:* **we now show that if E is any set of real numbers with positive outer measure, then there are subsets of E that fail to be measurable.**

Lemma 2.6.1. *Let E be a bounded measurable set of real numbers. Suppose there is a bounded, countably infinite set of real numbers Λ for which the collection of translates of E, $\{\lambda + E\}_{\lambda \in \Lambda}$, is disjoint. Then $m(E) = 0$.*

Proof. The translate of a measurable set is measurable. Thus, by the countable additivity of measure over countable disjoint unions of measurable sets,

$$m\left[\bigcup_{\lambda\in\Lambda}(\lambda+E)\right]=\sum_{\lambda\in\Lambda}m(\lambda+E). \tag{2.51}$$

Since both E and Λ are bounded sets, the set

$$\bigcup_{\lambda\in\Lambda}(\lambda+E)$$

also is bounded and therefore has finite measure. Thus the left-hand side of (2.51) is finite. However, since measure is translation invariant,

$$m(\lambda+E)=m(E)>0$$

for each $\lambda\in\Lambda$. Thus, since the set Λ is countably infinite and the right-hand sum in (2.51) is finite, we must have $m(E)=0$. This completes the proof. □

Definition 2.6.2. For any nonempty set E of real numbers, we define two points in E to be **rationally equivalent** provided their difference belongs to $\mathbb{Q}$, the set of rational numbers. It is easy to see that this is an *equivalence relation,* that is, it is reflexive, symmetric, and transitive. We call it the **rational equivalence relation** on E. For this relation, there is the disjoint decomposition of E into the collection of equivalence classes. By a **choice set** for the rational equivalence relation on E we mean a set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class. We infer from the **Axiom of Choice** that there are such choice sets.

A choice set $\mathcal{C}_E$ is characterized by the following two properties:

1. the difference of two points in $\mathcal{C}_E$ is not rational;
2. for each point $x \in E$, there is a point $c \in \mathcal{C}_E$ for which $x = c + q$, with q rational.

This first characteristic property of $\mathcal{C}_E$ may be conveniently reformulated as follows:

$$\text{For any set } \Lambda \subseteq \mathbb{Q}, \ \{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda} \text{ is disjoint.} \tag{2.52}$$

Theorem 2.6.3. ***(Vitali)*** *Any set E of real numbers with positive outer measure contains a subset that fails to be mea-*

surable.

Proof. By the countable subadditivity of outer measure, we may suppose E is bounded. Let $\mathcal{C}_E$ be any choice set for the rational equivalence relation on E.

We claim that C_E is not measurable. To verify this claim, we assume it is measurable and derive a contradiction.

Let Λ_0 be any bounded, countably infinite set of rational numbers. Since $\mathcal{C}_E$ is measurable, and, by ((2.52)), the collection of translates of C_E by members of Λ_0 is disjoint, it follows from Lemma 2.6.1 that $m(C_E) = 0$. Hence, again using the translation invariance and the countable additivity of measure over countable disjoint unions of measurable sets,

$$m\left[\bigcup_{\lambda\in\Lambda_0}(\,\lambda + \mathcal{C}_E\,)\right] = \sum_{\lambda\in\Lambda_0} m(\,\lambda + \mathcal{C}_E\,) = 0.$$

To obtain a contradiction we make a special choice of Λ_0. Because E is bounded it is contained in some interval $[-b,\ b]$. We choose

$$\Lambda_0 = [-2b,\ \ 2b] \cap \mathbb{Q}.$$

Then Λ_0 is bounded, and is countably infinite since the rationals are countable and dense. **We claim that**

$$E \subseteq \bigcup_{\lambda \in [-2b,\ 2b] \cap \mathbb{Q}} (\lambda + \mathcal{C}_E). \tag{2.53}$$

Indeed, by the second characteristic property of $\mathcal{C}_E$, if $x \in E$, there is a number c in the choice set $\mathcal{C}_E$ for which $x = c + q$ with q rational. But x and c belong to $[-b,\ b]$, so that q belongs to $[-2b,\ 2b]$. Thus the inclusion (2.53) holds. This is a contradiction because E, a set of positive outer measure, is not a subset of a set of measure zero. The assumption that $\mathcal{C}_E$ is measurable has led to a contradiction and thus it must fail to be measurable. □

Theorem 2.6.4. *There are disjoint sets of real numbers A and B for which*

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof. We prove this by contradiction. Assume

$$m^*(A \cup B) = m^*(A) + m^*(B)$$

for every disjoint pair of sets A and B. Then, by the very definition of measurable set, every set must be measurable. This contradicts the preceding theorem. □

2.7 The Cantor Set

We have shown that *a countable set has measure zero* and *a Borel set is Lebesgue measurable.* These two assertions prompt the following two questions.

Question 1 If a set has measure zero, is it also countable?

Question 2 If a set is measurable, is it also Borel?

The answer to each of these questions is negative. In this section we give a detailed answer to the first question by constructing a set called the *Cantor set.*

2.7.1 The Cantor Set

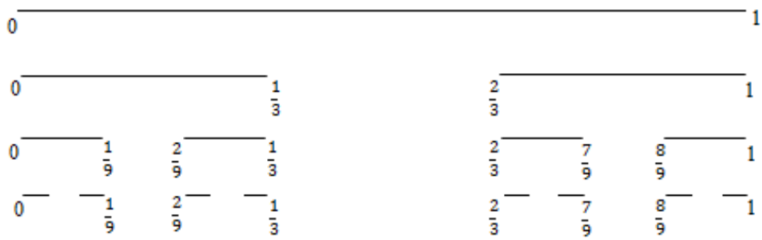

Figure 2.15: The Cantor set, produced by the iterated process of open middle one-third removal.

Consider the closed, bounded interval $I = [0, 1]$. The first step in the construction of the Cantor set is to subdivide I into three intervals of equal length $1/3$ and remove the interior of the middle interval, that is, we remove the open interval $(1/3, 2/3)$ from the interval $[0, 1]$ to obtain the closed set C_1, which is the union of two disjoint closed intervals, each of length $1/3$:

$$C_1 = [0, 1/3] \cup [2/3, 1].$$

We now repeat this *open middle one-third removal* on each of the two intervals in C_1 to obtain a closed set C_2, which is the union of $4 = 2^2$ closed intervals, each of length $1/3^2$:

$$C_2 = [0,\ 1/9] \cup [2/9,\ 1/3] \cup [2/3,\ 7/9] \cup [8/9, 1].$$

We now repeat this *open middle one-third removal* on each of the four intervals in C_2 to obtain a closed set C_3, which is the union of $8 = 2^3$ closed intervals, each of length $1/3^3$. We continue this removal operation countably many times to obtain the countable collection of sets $\{C_k\}_{k=1}^{\infty}$. We define the **Cantor set C** by

$$\mathbf{C} = \bigcap_{k=1}^{\infty} C_k.$$

The collection $\{C_k\}_{k=1}^{\infty}$ possesses the following two properties:

1. $\{C_k\}_{k=1}^{\infty}$ is a descending sequence of closed sets;
2. For each k, C_k is the disjoint union of 2^k closed intervals, each of length $1/3^k$.

We need the Nested Set Theorem in the proof of Proposition

2.7.2.

Theorem 2.7.1. [**The Nested Set Theorem**] *Let* $\{F_n\}_{n=1}^{\infty}$ *be a descending countable collection of nonempty closed sets of real numbers for which* F_1 *is bounded. Then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Proposition 2.7.2. *The Cantor set* $\mathbf{C}$ *is a closed, uncountable set of measure zero.*

Proof. 1. Each C_k considered in the construction of the Cantor set is closed. As the intersection of any collection of closed sets is closed, it follows that the Cantor set

$$\mathbf{C} = \bigcap_{k=1}^{\infty} C_k.$$

is closed.

2. Each closed set is measurable so that each C_k and $\mathbf{C}$ itself is measurable. Now each C_k is the disjoint union of 2^k intervals, each of length $1/3^k$, so that by the finite

additivity of Lebesgue measure,

$$m(C_k) = \left(\frac{2}{3}\right)^k.$$

By the monotonicity of measure, since

$$m(\mathbf{C}) \leq m(C_k) = \left(\frac{2}{3}\right)^k, \text{ for all } k,$$

we have

$$m(\mathbf{C}) = 0.$$

3. It remains to show that $\mathbf{C}$ is uncountable. To do so we argue by contradiction. Suppose $\mathbf{C}$ is countable. Let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbf{C}$. One of the two disjoint Cantor intervals whose union is C_1 fails to contain the point c_1; denote it by F_1. One of the two disjoint Cantor intervals in C_2 whose union is F_1 fails to contain the point c_2; denote it by F_2. Continuing in this way, we construct a countable collection of sets $\{F_k\}_{k=1}^{\infty}$, which, for each k, possesses the following three properties:

   (a) F_k is closed and $F_{k+1} \subseteq F_k$;

(b) $F_k \subseteq C_k$; and

(c) $c_k \notin F_k$.

From (i) and the Nested Set Theorem (Theorem 2.7.1) we conclude that the intersection $\bigcap_{k=1}^{\infty} F_k$ is nonempty. Let the point x belong to this intersection. By property (ii),

$$\bigcap_{k=1}^{\infty} F_k \subseteq \bigcap_{k=1}^{\infty} C_k = \mathbf{C},$$

and therefore the point x belongs to $\mathbf{C}$. However, $\{c_k\}_{k=1}^{\infty}$ is an enumeration of $\mathbf{C}$ so that $x = c_n$ for some index n. Thus

$$c_n = x \in \bigcap_{k=1}^{\infty} F_k \subseteq F_n.$$

This contradicts property (iii). Hence $\mathbf{C}$ must be uncountable.

□

2.8 Exercises

1. Prove that a set consisting of a single point has outer measure 0.

2. Prove that a countable set has outer measure 0.

3. Using Proposition 2.2.11 , prove that a countable set has outer measure zero.

4. Prove that a set of measure zero is measurable.

2.8.1 Answers to Exercises

1. Let $\{x\}$ be a singleton set. Then for $\varepsilon > 0$, $(x-\varepsilon,\ x+\varepsilon)$ is an open cover of $\{x\}$ so that

$$\begin{aligned} m^*(\{x\}) &= \inf\left\{\sum_{k=1}^{\infty} l(I_k) | A \subseteq \bigcup_{k=1}^{\infty} I_k\right\} \\ &\leq \underbrace{2\varepsilon}_{\substack{\uparrow \\ \text{length of the} \\ \text{interval } (x-\varepsilon,\ x+\varepsilon)}} . \end{aligned}$$

The above inequality is true for any $\varepsilon > 0$. That is, for any $\varepsilon > 0$, we have

$$0 \leq m^*(\{x\}) \leq 2\varepsilon.$$

Since ε is arbitrary, it follows that $m^*(\{x\}) = 0$.

2. Let C be a countable set. Then we can enumerate it (i.e., we can number the members of C by arranging it in an order) as

$$C = \{c_k\}_{k=1}^{\infty}.$$

Let $\varepsilon > 0$. For each natural number k, define

$$I_k = \left(c_k - \frac{\varepsilon}{2^{k+1}},\ c_k + \frac{\varepsilon}{2^{k+1}}\right).$$

For each k, $c_k \in I_k = \left(c_k - \frac{\varepsilon}{2^{k+1}},\ c_k + \frac{\varepsilon}{2^{k+1}}\right)$, so that the countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covers C. By the definition of outer measure,

$$m^*(C) = \inf\left\{\sum_{k=1}^{\infty} l(J_k) | A \subseteq \bigcup_{k=1}^{\infty} J_k\right\} \qquad (2.54)$$

where infimum is taken over all countable coverings $\{J_k\}_{k=1}^{\infty}$ of E (where J_k's are nonempty open, bounded intervals). In the above, we have constructed $\{I_k\}_{k=1}^{\infty}$ that covers C. Hence, from (2.54), we have

$$m^*(C) = \inf\left\{\sum_{k=1}^{\infty} l(J_k) | A \subseteq \bigcup_{k=1}^{\infty} J_k\right\} \leq \sum_{k=1}^{\infty} l(I_k) \tag{2.55}$$

Therefore,

$$
\begin{aligned}
0 \;\le\; & m^*(\,C\,) \le \sum_{k=1}^{\infty} l(I_k) \\
=\; & \sum_{k=1}^{\infty} \underbrace{\frac{\varepsilon}{2^k}}_{\substack{\uparrow \\ \text{length of the} \\ \text{interval } \left(c_k \,-\, \frac{\varepsilon}{2^{k+1}},\; c_k \,+\, \frac{\varepsilon}{2^{k+1}}\right)}} . \\
=\; & \varepsilon \sum_{k=1}^{\infty} \underbrace{\frac{1}{2^k}}_{\substack{\uparrow \\ \text{geometric series with intial term } \frac{1}{2} \\ \text{and common ratio } \frac{1}{2}}} \\
=\; & \varepsilon \frac{\frac{1}{2}}{1-\frac{1}{2}} = \varepsilon
\end{aligned}
$$

That is,

$$0 \le m^*(\,C\,) \le \varepsilon.$$

The above inequality is true for any $\varepsilon > 0$. That is, we have for any $\varepsilon > 0$,

$$0 \leq m^*(C) \leq \varepsilon.$$

Since ε is arbitrary, it follows that $m^*(C) = 0$.

3. This is an **alternate proof** of Exercise 2.8.2.If A is countable, then the set A is of the form

$$A = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}.$$

Then A can be expressed as the countable union of pair wise disjoint singleton sets $\{x_1\}, \{x_2\}, \{x_3\}, \ldots, \{x_n\}, \ldots$... i.e.,

$$A = \{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \ldots \cup \{x_n\} \cup \ldots.$$

Then

$$m^*(A) = m^*\left(\bigcup_n \{x_n\}\right)$$

$$\leq \sum_n m^*(\{x_n\}), \text{ using Proposition 2.2.11}$$

or inequality (2.15)

$$= \sum_n 0, \text{ since } m^*(\{x_n\}) = 0 \text{ for } n = 1, 2, 3, \ldots$$

using Example 2.2.3

$$= 0.$$

4. Ref. Theorem 2.3.7

Chapter 3

Lebesgue Measurable Functions

In this chapter we study *measurable functions* that will lay the foundation for the study of the *Lebesgue integral* (Chapter 4). We establish that all continuous functions on a measurable domain are measurable (Proposition 3.1.5), as are all monotone and step functions on a closed, bounded interval. Linear combinations of measurable functions are measurable. The pointwise limit of a sequence of measurable functions is measurable. We establish results regarding the approximation of measurable functions by simple functions (Definition 3.2.5) and by continuous functions.

3.1 Sums, Products, and Compositions

All the functions considered in this chapter take values in the extended real numbers, that is, the set $\mathbb{R} \cup \{\pm\infty\}$. Recall that a property is said to hold **almost everywhere** (abbreviated **a.e.)** on a measurable set E provided it holds on $E \sim E_0$, where E_0 is a subset of E for which $m(E_0) = 0$ (Definition 2.5.7).

Given two functions h and g defined on E, for notational brevity we often write

$$h \leq g \quad \text{to mean that} \quad h(x) \leq g(x) \quad \text{for all } x \in E\,.$$

Definition 3.1.1. A **sequence of functions** $\{ f_n \}$ **on** E **is increasing** provided $f_n \leq f_{n+1}$ on E for each index n. i.e., if

$$f_n(x) \le f_{n+1}(x) \text{ for all } x \in E \text{ and each index } n.$$

Proposition 3.1.2. *Let f be an extended real-valued function whose domain E is measurable. Then the following statements are equivalent:*

(i) *For each real number c, the set $\{x : f(x) > c\}$ is measurable.*

(ii) *For each real number c, the set $\{x : f(x) \ge c\}$ is measurable.*

(iii) *For each real number c, the set $\{x : f(x) < c\}$ is measurable.*

(iv) *For each real number c, the set $\{x : f(x) \le c\}$ is measurable.*

Each of these properties implies that for each extended real number c,

(v) *the set $\{x \in E : f(x) = c\}$ is measurable.*

Proof. Since the sets in (i) and (iv) are complementary in E, as are the sets in (ii) and (iii), and the complement in E of a measurable subset of E is measurable, (i) and (iv) are equivalent, as are (ii) and (iii). Details are given below:

(*i*) implies (*ii*): Note that

$$\{x \in E : f(x) \geq c\} = \bigcap_{k=1}^{\infty} \left\{x \in E : f(x) > c - \frac{1}{k}\right\}. \quad (3.1)$$

By the assumption (i),

$$\left\{x \in E : f(x) > c - \frac{1}{k}\right\}$$

is measurable for any natural number k, and, since intersection of a countable collection of measurable sets is measurable (Proposition 2.3.19 and), we have

$$\bigcap_{k=1}^{\infty} \left\{x \in E : f(x) > c - \frac{1}{k}\right\}$$

is measurable. This implies (using (3.1)) that $\{x \in E : f(x) \geq c\}$ is measurable. That is, (i) implies (ii) is proved.

(*ii*) implies (*i*): Note that

$$\{x \in E : f(x) > c\} = \bigcup_{k=1}^{\infty} \left\{x \in E : f(x) \geq c + \frac{1}{k}\right\}.$$

By the assumption (ii), $\left\{x \in E : f(x) \geq c + \frac{1}{k}\right\}$ is measurable for any natural number k , and, since union of a sequence of measurable sets is measurable (Proposition 2.3.16), we have $\bigcup_{k=1}^{\infty} \left\{x \in E : f(x) \geq c + \frac{1}{k}\right\}$ is measurable. This implies that $\{x \in E : f(x) > c\}$ is measurable. That is, (ii) implies (i) is proved.

(*i*) implies (*iv*): Note that

$$\{x \in E : f(x) \leq c\} = E \sim \{x \in E : f(x) > c\}.$$

E and $\{x \in E : f(x) > c\}$ are measurable implies the difference $E \sim \{x \in E : f(x) > c\}$ is also measurable. That is, $\{x \in E : f(x) \leq c\}$ is measurable. That is, (i) implies (iv).

(*iv*) implies (*i*): Note that

$$\{x \in E : f(x) > c\} = E \sim \{x \in E : f(x) \leq c\}.$$

E and $\{x \in E : f(x) \leq c\}$ are measurable sets implies the difference set $E \sim \{x : f(x) \leq c\}$ is also measurable. That is, $\{x \in E : f(x) > c\}$ is measurable. That is, (iv) implies (i).

As above, **(*ii*) implies (*iii*)** and **(*iii*) implies (*ii*)** can be proved.

That we have shown that the first four statements are equivalent.

Next we have to show that first four statements imply the fifth statement. For this assume one statement; and then by the discussion above, all the four statements hold. We have to show that this implies fifth statement.

Case 1) If c is a real number:

$$\{x \in E : f(x) = c\} = \{x \in E : f(x) \geq c\} \cap \{x \in E : f(x) \leq c\},$$

and so (ii) and (iv) implies sets on the right hand side are measurable and hence their intersection is measurable. i.e., $\{x \in E : f(x) = c\}$ is measurable.

Case 2) If $c = \infty$:

$$\{x \in E : f(x) = \infty\} = \bigcap_{k=1}^{\infty} \{x \in E : f(x) > k\}.$$

By (i), for any natural number k, $\{x \in E : f(x) > k\}$ is measurable, and hence the countable intersection

$$\bigcap_{k=1}^{\infty} \{x \in E : f(x) > k\}$$

is measurable. Thus, $\{x \in E : f(x) = \infty\}$ is measurable.

Case 3) If $c = -\infty$:

$$\{x \in E : f(x) = -\infty\} = \bigcap_{k=1}^{\infty} \{x \in E : f(x) < -k\},$$

and for any natural number k, $\{x \in E : f(x) < -k\}$ is measurable, and hence the countable intersection

$$\bigcap_{k=1}^{\infty} \{x \in E : f(x) < -k\}$$

is measurable. Hence $\{x \in E : f(x) = \infty\}$ is measurable.

Hence in any case (v) holds. This completes the proof. □

Definition 3.1.3. [Lebesgue measurable function] An extended real valued function f defined on a set E is said to be **Lebesgue measurable** or simply **measurable,** provided its domain E is measurable and it satisfies one of the first four statements of Proposition 3.1.2.

Proposition 3.1.4. *Let the function f be defined on a measurable set E. Then f is measurable if and only if for each open set O, the inverse image of O under f,*

$$f^{-1}(O) = \{x \in E | f(x) \in O\},$$

is measurable.

Proof Suppose the inverse image of each open set is measurable. *To prove that f is measurable*, by Definition 3.1.3, it is enough to show that one of the first four statements of Proposition 3.1.2 holds. We show that statement (i) of Proposition 3.1.2 holds. Fix a real number c. Then

$$\{x \in E | f(x) > c\} = f^{-1}((c, \ \infty)).$$

Since the interval $(c,\ \infty)$ is open, by the assumption the set $f^{-1}((c,\ \infty))$ is measurable. Hence, the set $\{x \in E | f(x) > c\}$ is also measurable. Since c is an arbitrary real number, this shows that, for each real number c, the set $\{x : f(x) > c\}$ is measurable. Hence statement (i) of Proposition 3.1.2 holds, and hence, by Definition 3.1.3, f is measurable.

Conversely, suppose f is measurable. Let O be open. Then we can express O as the union of a countable collection of open, bounded intervals $\{ I_k\}_{k=1}^{\infty}$ where each I_k may be expressed as

$$I_k = B_k \cap A_k$$

where $B_k = (-\infty,\ b_k)$ and $A_k = (a_k,\ \infty)$. That is,

$$O = \bigcup_{k=1}^{\infty} I_k = \left[\bigcup_{k=1}^{\infty} B_k \cap A_k\right].$$

Since f is a measurable function, by Definition 3.1.3, any one of the statements in Proposition 3.1.2 holds, and since the statements are equivalent, it follows that all statements in

Proposition 3.1.2 hold. Hence, noting that

$$f^{-1}(B_k) = \{x \in E | f(x) < b_k\}$$

and

$$f^{-1}(A_k) = \{x \in E | f(x) > a_k\}$$

we have (using statements in Proposition 3.1.2) each $f^{-1}(B_k)$ and $f^{-1}(A_k)$ are measurable sets. On the other hand, the measurable sets are a σ-algebra (Proposition 2.3.19) and therefore $f^{-1}(O)$ is measurable since

$$f^{-1}(O) = f^{-1}\left[\bigcup_{k=1}^{\infty} B_k \cap A_k\right] = \underbrace{\bigcup_{k=1}^{\infty} \underbrace{\underbrace{f^{-1}(B_k)}_{\text{measurable set}} \cap \underbrace{f^{-1}(A_k)}_{\text{measurable set}}}_{\text{measurable set}}}_{\text{measurable set}} .$$

The following proposition tells us that the most familiar functions from elementary analysis, the continuous functions, are measurable.

Proposition 3.1.5. [Continuous functions are measurable] *A real-valued function that is continuous on its measurable do-*

main is measurable.

Proof. Let the function f be continuous on the measurable set E. Let O be open. Since f is continuous, $f^{-1}(O)$ is an open subset of E. Hence this open subset can be expressed as the intersection of E with an open set U of $\mathbb{R}$ (Ref. The section Subspace Topology in any one of Topology text books). So

$$f^{-1}(O) = E \cap U \tag{3.2}$$

where U is open. Being open set, O is measurable. Thus, from (3.2), $f^{-1}(O)$, being the intersection of two measurable sets, is measurable. It follows from the preceding proposition that f is measurable. $\square$

Definition 3.1.6. A real-valued function that is either increasing or decreasing is said to be **monotone.**

We leave the proof of the next proposition as an exercise.

Proposition 3.1.7. *A monotone function that is defined on an interval is measurable.*

Proposition 3.1.8. *Let f be an extended real-valued function on E.*

1. *If f is measurable on E and $f = g$ a.e. on E, then g is measurable on E.*

2. *For a measurable subset D of E, f is measurable on E if and only if the restrictions of f to D and $E \sim D$ are measurable.*

Proof. First assume that f is measurable on E and $f = g$ a.e. on E. To prove that g is measurable on E (by Definition 3.1.3), it is enough to show that for each real number c, the set $\{x : g(x) > c\}$ is measurable. Define

$$A = \{x \in E : f(x) \neq g(x)\}.$$

Observe that

$$\begin{aligned}\{x \in E : g(x) > c\} \;&=\; \{x \in A : g(x) > c\} \\ &\quad \cup \left[\{x \in E : f(x) > c\} \cap [E \sim A]\right]. \end{aligned} \tag{3.3}$$

- Since $f = g$ a.e. E, we have $m(A) = 0$ (Remark 2.5.8). Thus, noting that

$$\{x \in E : g(x) > c\} \subseteq A$$

  it follows that $\{x \in E : g(x) > c\}$ is measurable, being a subset of the set A of measure zero.

- Since f is a measurable function on E, by Definition 3.1.3, the set $\{x \in E : f(x) > c\}$ is a measurable set.

- Since both E and A are measurable, and since the measurable sets are an algebra (Remark 2.3.14), the set $E \sim A$ is measurable.

Hence, the set on the right hand side of (3.3), being the union of measurable sets, is measurable. So, the set $\{x \in E : g(x) > c\}$ is measurable.

To verify (ii), just observe that for any c,

$$\begin{aligned}\{x \in E : f(x) > c\} &= \{x \in D : f(x) > c\} \\ &\quad \cup \{x \in E \sim D : f(x) > c\}\end{aligned}$$

and once more use the fact that the measurable sets are an algebra. □

Remark 3.1.9. The sum $f+g$ of two measurable extended real-valued functions f and g *is not properly defined at points at which* f *and* g *take infinite values of opposite sign.*
Assume f and g are finite a.e. on E. Define E_0 to be the set of points in E at which both f and g are finite. If the restriction of $f+g$ to E_0 is measurable, then,

- by the preceding proposition, any extension of $f+g$, as an extended real-valued function, to all of E also is measurable.

This is the sense in which we consider it unambiguous to state that the sum of two measurable functions that are finite a.e. is measurable. Similar remarks apply to products.

The following proposition tells us that *standard algebraic operations performed on measurable functions that are finite a.e. again lead to measurable functions.*

Theorem 3.1.10. *Let* f *and* g *be measurable functions on* E *that are finite a.e. on* E*.*

(a) (Linearity) *For any α and β,*

$$\alpha f + \beta g \text{ is measurable on } E.$$

(In particular,

Taking $\alpha = \beta = 1$, $f + g$ is measurable on E.

Taking $\alpha = 1$, $\beta = -1$, $f - g$ is measurable on E.

Taking $\beta = 0$, αf is measurable on E.)

(b) (Products) *fg is measurable on E.*

Proof. By the above remarks, we may assume f and g are finite on all of E. If $\alpha = 0$, then the function αf also is measurable. If $\alpha \neq 0$, observe that for a number c,

$$\{x \in E : \alpha f(x) > c\} = \left\{x \in E : f(x) > \tfrac{c}{\alpha}\right\} \text{ if } \alpha > 0$$

and $\{x \in E : \alpha f(x) > c\} = \left\{x \in E : f(x) < \tfrac{c}{\alpha}\right\}$ if $\alpha < 0$.

Thus the measurability of f implies the measurability of αf. Therefore to establish linearity it suffices to consider the case $\alpha = \beta = 1$.

For $x \in E$, if $f(x) + g(x) < c$, then $f(x) < c - g(x)$ and so, by the density of the set of rational numbers $\mathbb{Q}$ in $\mathbb{R}$, there is a rational number q for which

$$f(x) < q < c - g(x)\,.$$

Hence

$$\begin{aligned}\{x \in E : \quad & f(x) + g(x) < c\} \\ &= \bigcup_{q \in \mathbb{Q}} \{x \in E : g(x) < c - q\} \cap \{x \in E : f(x) < q\}\,.\end{aligned}$$

The rational numbers are countable. Thus

$$\{x \in E : f(x) + g(x) < c\,\}$$

is measurable, since it is the union of a countable collection of measurable sets. Hence $f + g$ is measurable.

(b) To prove that the product of measurable functions is measurable, first observe that

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right] \tag{3.4}$$

By (a), if f and g be measurable functions on E, then $f+g$ is measurable. **If we also prove that square of a measurable function is measurable,** then it follows that $(f+g)^2$, f^2 and g^2 are measurable. Then by the Linearity in (a), $(f+g)^2 - f^2$ is measurable. Then, $(f+g)^2 - f^2$ is measurable and g^2 is measurable implies, again by linearity, that $(f+g)^2 - f^2 - g^2$ is measurable. Again by linearity with $\alpha = \frac{1}{2}$ and $(f+g)^2 - f^2 - g^2$ in place of measurable function, linearity implies $\frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$ is measurable. That is, fg is measurable. So **we claim that square of a measurable function is measurable.** i.e., we show that if f is measurable, then f^2 is measurable. By Definition 3.1.3, it is enough to show that for any real number c, $\{x \in E : f^2(x) > c\}$ is measurable.

Case 1) For $c \geq 0$;

$$\begin{aligned}\left\{x \in E : f^2(x) > c\right\} &= \left\{x \in E : (f(x))^2 > c\right\} \\ &= \left\{x \in E : f(x) > \sqrt{c}\right\} \\ &\quad \cup \left\{x \in E : f(x) < -\sqrt{c}\right\}\end{aligned}$$

Since f is a measurable function, by Definition 3.1.3, the sets

$\{x \in E : f(x) > \sqrt{c}\}$ and $\{x \in E : f(x) < -\sqrt{c}\}$ are measurable sets. Hence their union

$$\left\{x \in E : f(x) > \sqrt{c}\right\} \cup \left\{x \in E : f(x) < -\sqrt{c}\right\}$$

is also measurable. i.e., $\{x \in E : f^2(x) > c\}$ is measurable.
Case 2) For $c < 0$,

$$\left\{x \in E : f^2(x) > c\right\} = \left\{x \in E : (f(x))^2 > c\right\} = E$$

is measurable, since E, the domain of f , is a measurable set. By Case 1 and Case 2, it follows that for any real number c, $\{x \in E : f^2(x) > c\}$ is measurable. This completes the proof.

□

Many of the properties of functions considered in elementary analysis, including continuity and differentiability, are preserved under the operation of composition of functions. However, *the composition of measurable functions may not be measurable.*

Proposition 3.1.11. [The Preservation of Measurability Under Composition] *Let g be a measurable real-valued func-*

tion defined on E and f a continuous real-valued function defined on all of $\mathbb{R}$. Then the composition $f \circ g$ is a measurable function on E.

Proof. According to Proposition 3.1.4, a function is measurable if and only if the inverse image of each open set is measurable. Let O be open. Then

$$(f \circ g)^{-1}(O) = g^{-1}(f^{-1}(O)) \,.$$

Since f is continuous and defined on an open set, the set $U = f^{-1}(O)$ is open. We infer from the measurability of the function g that $g^{-1}(U)$ is measurable. Thus the inverse image $(f \circ g)^{-1}(O)$ is measurable and so the composite function $f \circ g$ is measurable.

This completes the proof. □

Definition 3.1.12. If f is real valued function defined on a subset E of $\mathbb{R}$. Then $|f| : E \to \mathbb{R}$ is defined by

$$|f|\,(x) = |f(x)| \quad \text{for} \quad x \in E$$

and for $p > 0$, $|f|^p : E \to \mathbb{R}$ is defined by

$$|f|^p(x) = |f(x)|^p \quad \text{for} \quad x \in E.$$

Corollary 3.1.13. *From the definition above, we also have*

$$|f|(x) = |f(x)| = \max\{\, f(x),\ -f(x)\},$$

Corollary 3.1.14. *If f is a measurable function on E, then $|\, f \,|$ is a measurable function on E.*

Proof. If we take h be defined on E by $h(x) = |x|$ for $x \in E$, then h is continuous on E. Also, if we let f be a measurable function on E, then by the above Proposition, the composition $h \circ f$ is a measurable function on E. Since

$$|f| = h \circ f$$

it follows that $|\, f \,|$ is a measurable function on E.. □

Corollary 3.1.15. *If f is a measurable function on E, then $|\, f \,|^p$ is measurable with the same domain E for each $p > 0$.*

Definition 3.1.16. For a finite family $\{\, f_k \,\}_{k=1}^n$ of functions with common domain E, the function

$$\max\{ f_1, \ \ldots \ , \ f_n \}$$

is defined on E by

$$\max\{ f_1, \ldots \ f_n \} = \max\{ f_1(x), \ \ldots \ f_n(x) \} \, \text{for} \, x \in E.$$

The function

$$\min\{ f_1, \ \ldots \ , \ f_n \}$$

is defined on E by

$$\min\{ f_1, \ \ldots \ , \ f_n \} = \min\{ f_1(x), \ \ldots \ , \ f_n(x) \} \text{ for } x \in E.$$

Proposition 3.1.17. *For a finite family* $\{ f_k \}_{k=1}^{n}$ *of measurable functions with common domain E, the functions*

$$\max\{ f_1, \ \ldots \ , \ f_n \}$$

and

$$\min\{ f_1, \ \ldots \ , \ f_n \}$$

also are measurable.

Proof. For any c, we have

$$\{x \in E : \max\{ f_1, \dots , f_n \}(x) > c\} = \bigcup_{k=1}^{n} \{x \in E : f_k(x) > c\}$$

so this set is measurable since it is the finite union of measurable sets. Thus the function $\max\{ f_1, \dots , f_n \}$ is measurable. A similar argument shows that the function $\min\{ f_1, \dots , f_n \}$ also is measurable. □

Definition 3.1.18. For a function f defined on E, we have the associated functions $|f|$, f^+, and f^- defined on E by

$$|f|(x) = \max\{ f(x), -f(x)\},$$

$$f^+(x) = \max\{ f(x), 0\},$$

$$f^-(x) = \max\{ -f(x), 0\}.$$

Example 3.1.19. For the function

$$f(x) = \sin x, \quad 0 \le x \le 2\pi \text{ (Fig. (3.1))}$$

we have

$$-f(x) = -\sin x, \quad 0 \le x \le 2\pi \text{ (Fig. (3.2))}$$

$$|f|(x) = \max\{f(x), -f(x)\} = \begin{cases} \sin x, & 0 \le x \le \pi \\ -\sin x, & \pi < x \le 2\pi \end{cases}$$

(Fig. (3.3)

$$f^{+}(x) = \max\{f(x), 0\} = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \pi < x \le 2\pi \end{cases}$$ (Fig. (3.4))

$$f^{-}(x) = \max\{-f(x), 0\} = \begin{cases} 0, & 0 \le x \le \pi \\ -\sin x, & \pi < x \le 2\pi \end{cases}$$

(Fig. (3.5))

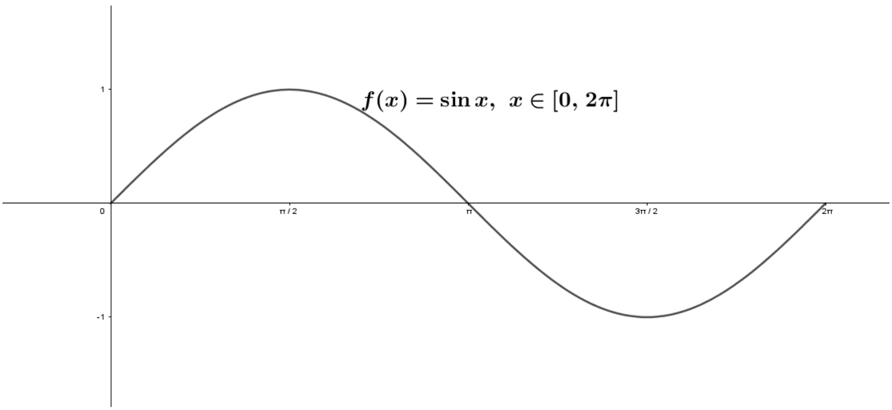

Figure 3.1: Graph of $f(x) = \sin x, \ \ 0 \le x \le 2\pi$

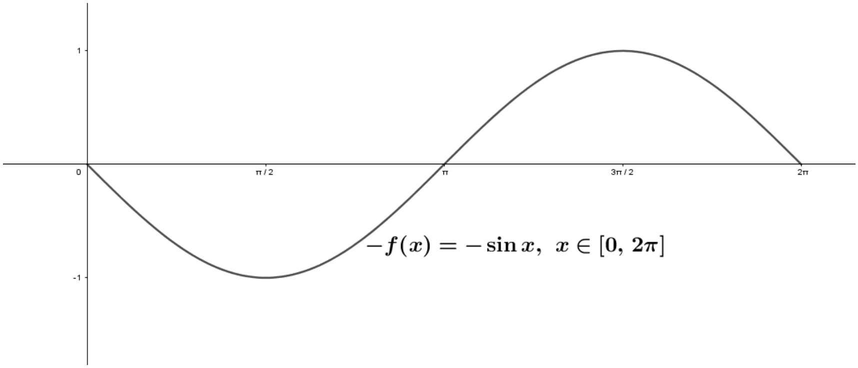

Figure 3.2: Graph of $-f(x) = -\sin x, \ 0 \le x \le 2\pi$

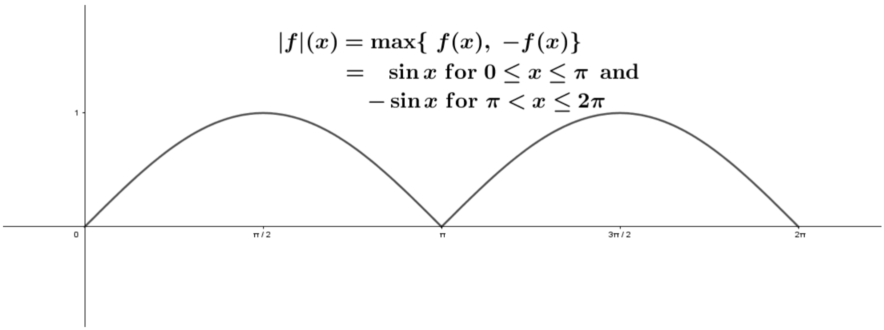

Figure 3.3: Graph of $|f|$

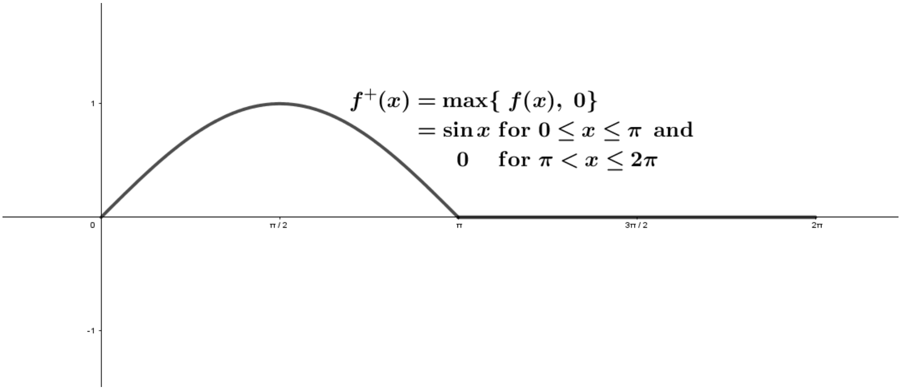

Figure 3.4: Graph of f^+

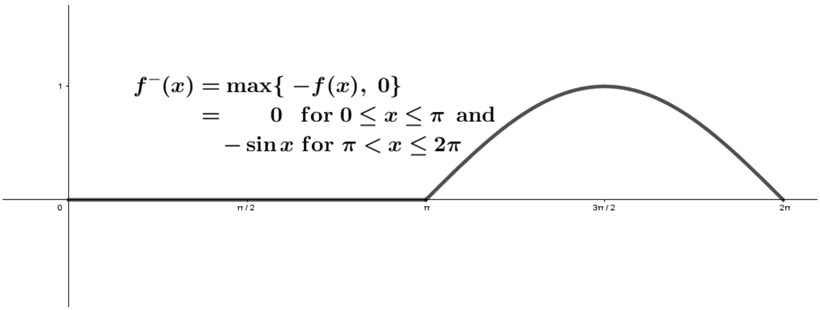

Figure 3.5: Graph of f^-

Corollary 3.1.20. *If f is measurable on E, then, so are the functions $|f|$, f^+, and f^-.*

Proof. If f is measurable on E, then, by the preceding proposition, so are the functions $|f|$, f^+, and f^-. □

This will be important when we study integration since the expression of f as the difference of two nonnegative functions, $f = f^+ - f^-$ on E, plays an important part in defining the Lebesgue integral.

3.2 Sequential Pointwise Limits and Simple Approximation

For a sequence $\{f_n\}$ of functions with common domain E and a function f on E, there are several distinct ways in which it is necessary to consider what it means to state that

"the sequence $\{f_n\}$ converges to f."

In this section we consider the concepts of pointwise convergence and uniform convergence, which are familiar from elementary analysis.

Definition 3.2.1. For a sequence $\{f_n\}$ of functions with common domain E, a function f on E and a subset A of E, we say that

1. The **sequence** $\{f_n\}$ **converges to** f **pointwise on** A provided

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in A.$$

2. The **sequence** $\{f_n\}$ **converges to** f **pointwise a.e. on** A provided it converges pointwise on $A \sim B$, where $m(B) = 0$.

3. The **sequence** $\{f_n\}$ **converges to** f **uniformly on** A provided for each $\varepsilon > 0$, there is an index N for which

$$|f - f_n| < \varepsilon \text{ on } A \text{ for all } n \geq N\,.$$

When considering sequences of functions $\{f_n\}$ and their convergence to a function f, we often implicitly assume that all of the functions have a common domain. We write "$\{f_n\} \to f$ pointwise on A " to indicate the sequence $\{f_n\}$ converges to f pointwise on A and use similar notation for uniform convergence.

The pointwise limit of continuous functions may not be continuous. The pointwise limit of Riemann integrable functions may not be Riemann integrable. The following proposition is the first indication that the measurable functions have much better stability properties.

Proposition 3.2.2. *Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a e. on E to the function f . Then f is measurable.*

Proof. Let E_0 be a subset of E for which $m(E_0) = 0$ and $\{f_n\}$ converges to f pointwise on $E \sim E_0$. Since $m(E_0) = 0$, it follows from Proposition 3.1.8 that f is measurable if and only if its restriction to $E \sim E_0$ is measurable. Therefore, by possibly replacing E by $E \sim E_0$, we may assume the sequence converges pointwise on all of E.

Fix a number c. We must show that $\{x \in E | f(x) < c\}$ is measurable. Observe that for a point $x \in E$, since

$\lim_{n\to\infty} f_n(x) = f(x)\,,$

$$f(x) < c$$

if and only if

there are natural numbers n and k for which $f_j(x) < c - \frac{1}{n}$ for all $j \geq k\,.$

But for any natural numbers n and j, since the function f_j is measurable, the set $\left\{x \in E | f_j(x) < c - \frac{1}{n}\right\}$ is measurable. Therefore, for any k, the intersection of the countably collection of measurable sets

$$\bigcap_{j=k}^{\infty} \left\{x \in E | f_j(x) < c - \frac{1}{n}\right\}$$

also is measurable. Consequently, since the union of a countable collection of measurable sets is measurable,

$$\{x \in E | f(x) < c\} = \bigcup_{1 \leq k,\ n < \infty} \left[\bigcap_{j=k}^{\infty} \left\{x \in E | f_j(x) < c - \frac{1}{n}\right\}\right]$$

is measurable. This completes the proof. □

Definition 3.2.3. If A is any set, we define the **characteristic function** χ_A of the set A to be the function given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Remark 3.2.4. The function χ_A is measurable if and only if A is measurable.

Thus the existence of a nonmeasurable set implies the existence of a nonmeasuarble function. Linear combinations of *characteristic functions of measurable sets play a role in Lebesgue integration similar to that played by step functions in Riemann Integration*, and so we name these functions.

Definition 3.2.5. A real-valued function φ defined on a measurable set E is called **simple** if it is measurable and assumes only a finite number of values.

If φ is simple, has domain E and assumes only the finite number of values $c_1, \ \ldots, \ c_n$, then

$$\varphi = \sum_{k=1}^{n} c_k \chi_{E_k}$$

on E, where $E_k = \{x \in E : \varphi(x) = c_k\}$.

This particular expression of φ as a linear combination of characteristic functions is called the **canonical representation of the simple function** φ.

It is easy to show that the sum, product, and difference of two simple functions are simple.

Lemma 3.2.6. ***The Simple Approximation Lemma*** *Let f be a measurable real-valued function on E. Assume f is bounded on E, that is, there is an $M \geq 0$ for which $|f| \leq M$ on E. Then for each $\varepsilon > 0$, there are simple functions φ_ε and ψ_ε defined on E which have the following approximation properties:*

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E.$$

Proof. Let (c, d) be an open, bounded interval that contains the image of E, $f(E)$, and

$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$$

be a partition of the closed, bounded interval $[c, d]$ such that

$y_k - y_{k-1} < \varepsilon$ for $1 \le k \le n$. Define

$$I_k = [y_{k-1},\ y_k) \text{ and } E_k = f^{-1}(I_k) \text{ for } 1 \le k \le n.$$

Since each I_k is an interval and the function f is measurable, each set E_k is measurable. Define the simple functions φ_ε and ψ_ε on E by

$$\varphi_\varepsilon = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{E_k} \text{ and } \psi_\varepsilon = \sum_{k=1}^{n} y_k \cdot \chi_{E_k}.$$

Let $x \in E$. Since $f(E) \subseteq (c,\ d)$, there is a unique k, $1 \le k \le n$, for which $y_{k-1} \le f(x) < y_k$ and therefore

$$\varphi_\varepsilon(x) = y_{k-1} \le f(x) < y_k = \psi_\varepsilon(x).$$

But $y_k - y_{k-1} < \varepsilon$, and therefore φ_ε and ψ_ε have required approximation properties. This completes the proof. □

To the several characterizations of measurable functions that we already established, we add the following one.

Theorem 3.2.7. ***The Simple Approximation Theorem*** *An extended real-valued function f on a measurable set E is* *measurable if and only if there is a sequence* $\{\varphi_n\}$ *of simple functions on* E *which converges pointwise on* E *to* f *and has the property that*

$$|\varphi_n| \leq |f| \text{ on } E \text{ for all } n.$$

If f *is nonnegative, we may choose* $\{\varphi_n\}$ *to be increasing.*

Proof. Since each simple function is measurable, Proposition 3.2.2 tells us that a function is measurable if it is the pointwise limit of a sequence of simple functions. It remains to prove the converse.

Assume f is measurable. We also assume $f \geq 0$ on E. The general case follows by expressing f as the difference of nonnegative measurable functions. Let n be a natural number. Define

$$E_n = \{x \in E | f(x) \leq n\}.$$

Then E_n is a measurable set and the restriction of f to E_n is a nonnegative bounded measurable function. By the Simple Approximation Lemma, applied to the restriction of f to

E_n and with the choice of $\varepsilon = \frac{1}{n}$, we may select simple functions φ_n and ψ_n defined on E_n which have the following approximation properties:

$$0 \le \varphi_n \le f \le \psi_n \text{ on } E_n \quad \text{and } 0 \le \psi_n - \varphi_n < \tfrac{1}{n} \text{ on } E_n \, .$$

Observe that

$$0 \le \varphi_n \le f \text{ and } 0 \le f - \varphi_n \le \psi_n - \varphi_n < \tfrac{1}{n} \text{ on } E_n \, . \quad (3.4)$$

Extend φ_n to all of E by setting

$$\varphi_n(x) = n \text{ if } f(x) > n \, .$$

The function φ_n is a simple function defined on E and $0 \le \varphi_n \le f$ on E. We claim that the sequence $\{\psi_n\}$ converges to f pointwise on E.

Let $x \in E$.

Case 1: Assume $f(x)$ is finite. Choose a natural number N for which $f(x) < N$. Then

$$0 \le f(x) - \varphi_n(x) < \tfrac{1}{n} \text{ for } n \ge N \, ,$$

and therefore $\lim_{n\to\infty} \psi_n(x) = f(x)$.

Case 2: Assume $f(x) = \infty$. Then $\varphi_n(x) = n$ for all n, so that $\lim\limits_{n\to\infty} \varphi_n(x) = f(x)$.

By replacing each φ_n with $\max\{\varphi_1, \ldots, \varphi_n\}$ we have $\{\varphi_n\}$ increasing. This completes the proof. □

3.3 Littlewood's Three Principles, Egoroff's Theorem, and Lusin's Theorem

Theorem 3.3.1. ***Egoroff's Theorem*** *Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\varepsilon > 0$, there is a closed set F contained in E for which*

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \sim F) < \varepsilon.$$

To prove Egoroff's Theorem it is convenient to first establish the following lemma.

Lemma 3.3.2. *Under the assumptions of Egoroff's Theorem, for each $\eta > 0$ and $\delta > 0$, there is a measurable subset A of E and an index N for which*

$$|f_n - f| < \eta \text{ on } A \text{ for all } n \geq N \text{ and } m(E \sim F) < \delta.$$

Proof. For each k, the function $|f - f_k|$ is properly defined, since f is real-valued, and it is measurable, so that the set $\{x \in E | \, |f(x) - f_k(x)| < \eta\}$ is measurable. The intersection of a countable collection of measurable sets is measurable. Therefore

$$E_n = \{x \in E | \, |f(x) - f_k(x)| < \eta \text{ for all } k \geq n\}$$

is a measurable set. Then $\{E_n\}_{n=1}^{\infty}$ is an ascending condition of measurable sets, and

$$E = \bigcup_{n=1}^{\infty} E_n,$$

since $\{f_n\}$ converges pointwise to f on E. We infer from the continuity of measure that

$$m(E) = \lim_{n \to \infty} m(E_n).$$

Since $m(E) < \infty$, we may choose an index N for which

$$m(E_N) > m(E) - \varepsilon.$$

Define $A = E_n$ and observe that, by the excision property of measure,

$$m(E \sim A) = m(E) - m(E_N) < \varepsilon.$$

This completes the proof of the Lemma. □

Proof of Egoroff's Theorem For each natural number n, let A_n be a measurable subset of E and $N(n)$ an index which satisfy the conclusion of the preceding lemma with $\delta = \frac{\varepsilon}{2^{n+1}}$ and $\eta = \frac{1}{n}$, that is,

$$m(E \sim A_n) < \frac{\varepsilon}{2^{n+1}} \tag{3.5}$$

and

$$|f_k - f| < \frac{1}{n} \text{ on } A_n \text{ for all } k \geq N(n). \tag{3.6}$$

Define

$$A = \bigcap_{n=1}^{\infty} A_n.$$

By De Morgan's Identities, the countably subadditivity of measure and (6.4),

$$\begin{aligned} m(E \sim A) &= m\left(\bigcup_{n=1}^{\infty}[E \sim A_n]\right) \\ &\leq \sum_{n=1}^{\infty} m(E \sim A_n) \\ &< \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} \\ &= \varepsilon \frac{\frac{1}{4}}{1-\frac{1}{2}} \\ &= \frac{\varepsilon}{2}. \end{aligned}$$

We claim that $\{f_n\}$ converges to f uniformly on A. Indeed, let $\varepsilon > 0$. Choose an index n_0 such that $\frac{1}{n_0} < \varepsilon$. Then, by (6.5),

$$|f_k - f| < \tfrac{1}{n_0} \text{ on } A_{n_0} \text{ for all } k \geq N(n_0).$$

However, $A \subseteq A_{n_0}$ and $1/n_0 < \varepsilon$ and therefore

$$|f_k - f| < \varepsilon \text{ on } A \text{ for } k \geq N(n_0).$$

Thus $\{f_n\}$ converges to f uniformly on A and $m(E \sim A) < \frac{\varepsilon}{2}$.

Finally, by Theorem 2.4.2, we may choose a closed set F contained in A for which $m(A \sim F) < \frac{\varepsilon}{2}$. Thus $m(E \sim F) < \varepsilon$ and $\{f_n\} \to f$ uniformly on F. This completes the proof of Egoroff's Theorem. □

It is clear that Egoroff's Theorem also holds if the convergence is pointwise a.e. and the limit function is finite a.e.

We now present a precise version of Littlewood's second principle in the case the measurable function is simple and then use this special case to prove the general case of the principle, Lusin's Theorem.

Proposition 3.3.3. *Let f be a simple function defined on E. Then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which*

$$f = g \text{ on } F \text{ and } m(A \sim F) < \varepsilon.$$

Proof. Let $a_1,\ a_2,\ \ldots\ ,a_n$ be the finite number of distinct values taken by f, and let them be taken on the sets

$$E_1,\ E_2,\ \ldots\ ,E_n\,,$$

respectively. The collection $\{E_k\}_{k=1}^{\infty}$ is disjoint since the a_k 's are distinct. According to Theorem 2.4.2, we may choose closed sets $F_1,\ F_2,\ \ldots\ ,F_n$ such that for each index k, k, $1 \leq k \leq n$,

$F_k \subseteq E_k$ and $m(E_k \sim F_k) < \frac{\varepsilon}{n}$.

Then

$$F = \bigcup_{k=1}^{\infty} F_k\,,$$

being the union of a finite collection of closes sets, is closed. Since $\{E_k\}_{k=1}^{\infty}$ is disjoint,

$$m(E \sim F) = m\left(\bigcup_{k=1}^{n}[E_k \sim F_k]\right) = \sum_{k=1}^{n} m(E_k \sim F_k) < \varepsilon.$$

Define g on F to take the value a_k on F_k for $1 \leq k \leq n$. Since the collection $\{F_k\}_{k=1}^{n}$ is disjoint, g is properly defined. Moreover, g is continuous on F since for a point $x \in F_i$, there is an open interval containing x which is disjoint from the closed set $\bigcup_{k\neq i} F_k$ and hence on the intersection of this interval with F the function g is constant. But g can be extended from a continuous function on the closed set F to a continuous function on all of $\mathbb{R}$. The continuous function g on $\mathbb{R}$ has the required approximation properties. □

Theorem 3.3.4. *[**Lusin's Theorem**] Let f be a real-valued measurable function on E. Then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which*

$$f = g \text{ on } F \text{ and } m(E \sim F) < \varepsilon.$$

Proof. We consider the case $m(E) < \infty$ and leave the extension to $m(E) = \infty$ as an exercise. According to the Simple Approximation Theorem, there is a sequence $\{f_n\}$ of simple functions defined on E that converges to f pointwise on E. Let n be a natural number. By the preceding proposition, with f replaced by f_n and ε replaced by $\frac{\varepsilon}{2^{n+1}}$, we may choose a continuous function g_n on $\mathbb{R}$ and a closed set F_n contained in E for which

$f_n = g_n$ on F_n and $m(E \sim F_n) < \frac{\varepsilon}{2^{n+1}}$.

According to Egoroff's Theorem, there is a closed set F_0 contained in E such that $\{f_n\}$ converges to f uniformly on F_0 and $m(E \sim F_0) < \frac{\varepsilon}{2}$. Define

$$F = \bigcap_{n=0}^{\infty} F_n.$$

Observe that, by De Morgan's Identities and the countable subadditivitiy of measure,

$$\begin{aligned} m(E \sim F) &= m\left([E \sim F_0] \cup \bigcup_{n=1}^{\infty} [E \sim F_n]\right) \\ &\leq \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} \\ &= \varepsilon. \end{aligned}$$

The set F is closed since it is the intersection of closed sets. Each f_n is continuous on F since $F \subseteq F_n$ and $f_n = g_n$ on F_n. Finally, $\{f_n\}$ converges to f uniformly on F since $F \subseteq F_0$. However, the uniform limit of continuous functions is continuous, so the restriction of f to F is continuous on

F. Finally, there is a continuous function g defined on all of $\mathbb{R}$ whose restriction to F equals f. This function g has the required approximation properties. $\square$

Chapter 4

Lebesgue Integration

In this chapter we study Lebesgue integration. We define this integral in four stages.

- We first define the integral for simple functions (Definition 3.2.5) over a set of finite measure.
- Then define the integral for bounded measurable functions f over a set of finite measure, in terms of integrals of upper and lower approximations of f by simple functions.

- We define the integral of a general nonnegative measurable function f over E to be the supremum of the integrals of lower approximations of f by bounded measurable functions that vanish outside a set of finite measure; the integral of such a function is nonnegative.

- Finally, a general measurable function is said to be integrable over E provided $\int_E |f| < \infty$.

We prove that linear combinations of integrable functions are integrable and that, on the class of integrable functions, the Lebesgue integral is a monotone, linear functional. A principal virtue of the Lebesgue integral, beyond the extent of the class of integrable functions, is the availability of quite general criteria which guarantee that if a sequence of integrable functions $\{f_n\}$ converge pointwise almost everywhere on E to f, then

$$\lim_{n\to\infty} \int_E f_n = \int_E [\lim_{n\to\infty} f_n] \equiv \int_E f.$$

We refer to that as *passage of the limit under the integral sign* or *interchanging of limit and integration.* Based on Egoroff's Theorem, a consequence of the countable additivity of

Lebesgue measure, we prove four theorems that provide criteria for justification of this passage: the Bounded Convergence Theorem, the Monotone Convergence Theorem, the Lebesgue Dominated Convergence Theorem, and the Vitali Convergence Theorem.

4.1 The Riemann Integral

Definition 4.1.1. Let f be a bounded real-valued function defined on the closed, bounded interval $[a, b]$. Let

$$P = \{x_0,\ x_1,\ \dots,\ x_n\}$$

be a **partition** of $[a, b]$, that is,

$$a = x_0 < x_1 < \cdots < x_n = b.$$

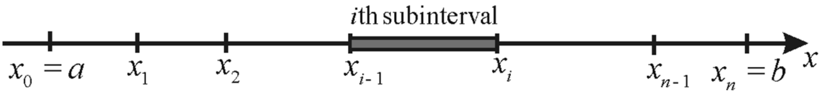

Figure 4.1: A partition $P = \{x_0,\ x_1,\ \ldots,\ x_n\}$ of $[a,\ b]$

The **lower Darboux sum for f with respect to the partition** P, denoted by $L(f,\ P)$, is defined by

$$L(f,\ P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$$

and the **upper Darboux sum for f with respect to the partition** P, denoted by $U(f,\ P)$, is defined by

$$U(f,\ P) = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1})$$

where, for $1 \leq i \leq n$,

$$m_i = \inf \{f(x) | x_{i-1} < x < x_i\}$$

and

$$M_i = \sup \{f(x) | x_{i-1} < x < x_i\}.$$

The **lower Riemann integral of f over** $[a, b]$, denoted by $(R)\underline{\int_a^b} f$, is defined by

$$(R)\underline{\int_a^b} f = \sup\{L(f,\ P) \mid P \text{ a partition of } [a,\ b]\}$$

and the **upper Riemann integral of f over** $[a, b]$, denoted by $(R)\overline{\int_a^b} f$, is defined by

$$(R)\overline{\int_a^b} f = \inf\{U(f,\ P) \mid P \text{ a partition of } [a,\ b]\}$$

Since f is assumed to be bounded and the interval $[a, b]$ has finite length, the lower and upper Riemann integrals are finite.

Proposition 4.1.2.

$$(R)\underline{\int_a^b} f \leq (R)\overline{\int_a^b} f.$$

That is, the upper integral is always at least as large as the lower integral.

Definition 4.1.3. A bounded real-valued function f defined on the closed, bounded interval $[a, b]$ is **Riemann integrable**

over $[a,\ b]$ if

$$(R)\underline{\int_a^b} f = (R)\overline{\int_a^b} f$$

and in that case the common value, called **Riemann integral of f over** $[a,\ b]$, is denoted by

$$(R)\int_a^b f$$

Notation 4.1.4. The notation of Riemann integral of f over $[a,\ b]$ as $(R)\int_a^b f$ is just to distinguish it from the Lebesgue integral, which we consider in the next section.

Remark 4.1.5. If we define

$$m_i = \inf\{f(x)|x_{i-1} \leq x \leq x_i\}$$

and

$$M_i = \sup\{f(x)|x_{i-1} \leq x \leq x_i\}$$

so the infima and suprema are taken over closed subintervals, we arrive at the same value of the upper and lower Riemann integral.

Definition 4.1.6. A real-valued function ψ defined on $[a,\ b]$

is called a **step function** provided there is a partition $P = \{x_0,\ x_1,\ \dots,\ x_n\}$ of $[a,\ b]$ and numbers $c_1,\ \dots,\ c_n$ such that for $1 \le i \le n$,

$$\psi(x) = c_i \ \text{ if } x_{i-1} < x < x_i\,.$$

Remark 4.1.7.

$$L(\psi,\ P) = \sum_{i=1}^{n} c_i\,(x_i - x_{i-1}) = U(\psi,\ P).$$

From the above remark and the definition of upper and lower Riemann integrals, we infer that a step function ψ is Riemann integrable and

$$(R)\int_a^b \psi = \sum_{i=1}^{n} c_i\,(x_i - x_{i-1}).$$

Therefore, we may reformulate the definition of the lower and upper Riemann integrals as follows:

$$
\begin{aligned}
&(R)\underline{\int_a^b} f \\
= \quad & \sup\left\{ (R)\int_a^b \varphi \mid \varphi \text{ a step function and } \varphi \le f \text{ on } [a,\ b] \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&(R)\overline{\int_a^b} f \\
= \quad & \inf\left\{ (R)\int_a^b \psi \mid \psi \text{ a step function and } \psi \ge f \text{ on } [a,\ b] \right\}.
\end{aligned}
$$

Example 4.1.8. [**Dirichlet's Function - An Example of a Function which is Not Riemann Integrable**] Define f on $[0,\ 1]$ by setting

$$
f(x) = \begin{cases} 1 \text{ if } x \text{ is rational,} \\ 0 \text{ if } x \text{ is irrational,} \end{cases}
$$

Let P be any partition of $[0,\ 1]$. By the density of the rationals and the irrationals,

$$L(f,\ P) = 0 \quad \text{and} \quad U(f,\ P) = 1.$$

Thus

$$(R)\underline{\int_0^1} f = 0 < 1 = (R)\overline{\int_0^1} f,$$

so f is **not** Riemann integrable. The set of rational numbers in $[0,\ 1]$ is countable. Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers in $[0,\ 1]$. For a natural number n, define f_n on $[0,\ 1]$ by setting

$$f_n(x) = \begin{cases} 1 \text{ if } x = q_k \text{ for some } \ q_k \text{ with } 1 \leq k \leq n, \\ \quad 0 \quad \text{otherwise.} \end{cases}$$

Then each f_n is a step function, so it is Riemann integrable. Thus, $\{f_n\}$ is an increasing sequence of Riemann integrable functions on $[0,\ 1]$,

$$|f_n| \leq 1 \ \text{ on}\,[0,\ \ 1] \ \text{ for all } \ n$$

and

$$\{f_n\} \rightarrow f \ \text{ pointwiseon}\,[0,\ \ 1].$$

However, the limit function f fails to be Riemann integrable on $[0, 1]$.

4.2 The Lebesgue Integral of a Bounded Function over a Set of Finite Measure

The Dirichlet function (Example 4.1.8) exhibits *one of the principal shortcomings of the Riemann integral: a uniformly bounded sequence of Riemann integrable functions on a closed, bounded interval can converge pointwise to a function that is not Riemann integrable.* We will see that *Lebesgue integral does not suffer from this shortcoming.*

Notation 4.2.1. Henceforth we only consider the Lebesgue integral, unless explicitly mentioned otherwise, and so we use the pure integral symbol to denote the Lebesgue integral.

Theorem 4.2.8 tells us that *any bounded function that is Riemann integrable over $[a, b]$ is also Lebesgue integrable over $[a, b]$ and two integrals are equal.*

We recall Definition 3.2.5 of simple function over a measurable set with the function ψ in place of φ :

Definition 4.2.2. A real-valued function ψ defined on a set E is called **simple** if it is measurable and assumes only a finite number of real values.

If ψ is simple, has domain E and assumes only the finite number of distinct values $a_1, \ldots, a_n$, then, by the measurability of ψ, its level sets (or inverse images) $\psi^{-1}(a_i)$ are measurable [1] and we have the canonical representation of ψ on E as

$$\psi = \sum_{k=1}^{n} c_k \chi_{E_k} \text{ on } E \qquad (4.1)$$
$$\text{where each } E_i = \psi^{-1}(a_i) = \{x \in E : \psi(x) = a_i\} .$$

This particular expression of ψ as a linear combination of

[1] Using Definition 3.1.3 and Proposition 2.3.10, $\psi^{-1}(a_i)$ is measurable, since

$$\psi^{-1}(a_i) = \underbrace{\underbrace{\{x \in E : \psi(x) \geq a_i\}}_{\text{measurable set since } \psi \text{ is measurable}} \cap \underbrace{\{x \in E : \psi(x) \leq a_i\}}_{\text{measurable set since } \psi \text{ is measurable}}}_{\text{measurable , beingtheintersectionofmeasurablesets}}$$

characteristic functions is called the **canonical representation of the simple function** ψ, and it is characterized by the fact that a_i distinct and nonzero (and hence the E_i are disjoint also).

Definition 4.2.3. (Integral of a simple function) For a simple function ψ defined on a set of finite measure $E,$ we define the **integral of** ψ over E by

$$\int_E \psi = \sum_{i=1}^{n} a_i \cdot m(E_i)$$

when ψ has the canonical representation (4.1)

$$\psi = \sum_{i=1}^{n} a_i \cdot \chi_{E_i}$$

on E, where each

$$E_i = \psi^{-1}(a_i) = \{x \in E : \psi(x) = a_i\} .$$

Lemma 4.2.4. *Let* $\{E_i\}_{i=1}^{n}$ *be a finite disjoint collection of measurable subsets of a set of finite measure* E*. For* $1 \leq i \leq n$*, let* a_i *be a real number.*

If $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ *on* E, *then* $\int_E \varphi = \sum_{i=1}^{n} a_i \cdot m(E_i)$.

Proof. The collection $\{E_i\}_{i=1}^{n}$ is disjoint but the above may not be the canonical representation since the a_i's may not be distinct. We must account for possible repetitions. Let $\{\lambda_1, \ldots, \lambda_m\}$ be distinct values taken by φ. For $1 \leq j \leq m$, set

$$A_j = \{x \in E : \varphi(x) = \lambda_j\} \text{ .(Fig.4.2)}$$

By definition of the integral in terms of canonical representations,

$$\int_E \varphi = \sum_{j=1}^{m} \lambda_j \cdot m(A_j) .$$

For $1 \leq j \leq m$, let I_j be the set of indices $i \in \{1, \ldots, n\}$ for which $a_i = \lambda_j$. Then

$$\{1, \ldots, n\} = \bigcup_{j=1}^{m} I_j,$$

and the union is disjoint. Moreover, by finite additivity of

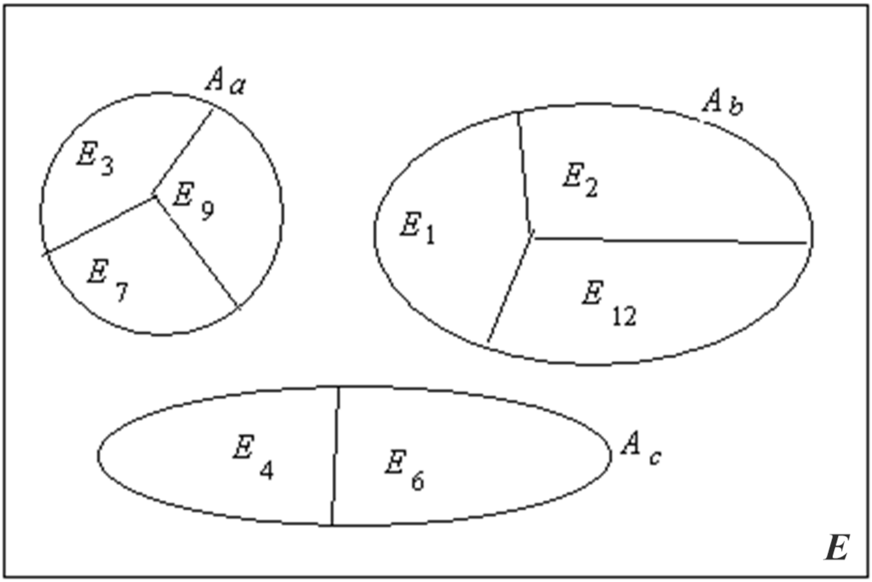

Figure 4.2: In the figure, assuming that $a_3 = a_7 = a_9 = \lambda_a$, we have for $x \in E_3$, $\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x) = a_3 = \lambda_a$. Similarly, $\varphi(x) = \lambda_a$ for $x \in E_7$ and $x \in E_9$. Also, $\varphi(x) = \lambda_b$ for $x \in E_1$, $x \in E_2$ and $x \in E_{12}$. Also, $\varphi(x) = \lambda_c$ for $x \in E_4$ and $x \in E_6$.

measure,

$$m(A_j) = \sum_{i \in I_j} m(E_i) \text{ for } 1 \leq j \leq m.$$

Therefore

$$\sum_{i=1}^{n} a_i \cdot m(E_i) = \sum_{j=1}^{m} \left[\sum_{i \in I_j} a_i \cdot m(E_i) \right] = \sum_{j=1}^{m} \lambda_j \left[\sum_{i \in I_j} m(E_i) \right]$$

$$= \sum_{j=1}^{m} \lambda_j \cdot m(A_j) = \int_E \varphi$$

This completes the proof. □

Proposition 4.2.5. *[**Linearity and Monotonicity of Integration**]Let φ and ψ be simple functions defined on a set of finite measure. Then for any α and β,*

(a) (Linearity of Integration)

$$\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi;$$

and,

(b) (Monotonicity of Integration)

$$\text{if } \varphi \leq \psi \text{ on } E, \text{ then } \int_E \varphi \leq \int_E \psi\,.$$

Proof. Since both φ and ψ take only a finite number of values on E, we may choose a finite disjoint collection $\{E_i\}_{i=1}^n$ of measurable subsets of E, the union of which is E, such that φ and ψ are constant on each E_i. For each i, $1 \leq i \leq n$, let a_i and b_i, respectively, be the values taken by φ and ψ on E_i. By the preceding lemma,

$$\textstyle\int_E \varphi = \sum_{i=1}^n a_i \cdot m(E_i) \text{ and } \int_E \psi = \sum_{i=1}^n b_i \cdot m(E_i)\,.$$

However, the simple function $\alpha\varphi+\beta\psi$ takes the constant value $\alpha a_i + \beta b_i$ on E_i. Thus, again by the preceding lemma,

$$\begin{aligned}\int_E (\alpha\varphi + \beta\psi) &= \sum_{i=1}^n (\alpha a_i + \beta b_i) \cdot m(E_i) \\ &= \alpha \sum_{i=1}^n a_i \cdot m(E_i) + \beta \sum_{i=1}^n b_i \cdot m(E_i) \\ &= \alpha \int_E \varphi + \beta \int_E \psi.\end{aligned}$$

(b) To prove monotonicity, assume $\varphi \leq \psi$ on E. Define $\eta = \psi - \varphi$ on E. By linearity,

$$\int_E \psi - \int_E \varphi = \int_E (\psi - \varphi) = \int_E \eta \geq 0\,,$$

since the nonnegative simple function η has a nonnegative integral. This completes the proof. □

The linearity of integration over sets of finite measure of simple functions shows that the restriction in the statement of Lemma 4.2.4 that the collection $\{E_i\}_{i=1}^n$ be disjoint is unnecessary.

A step function takes only a finite number of values and each interval is measurable. Thus a step function is simple. Since the measure of a singleton set is zero and the measure of an interval is its length, we infer from the linearity of Lebesgue integration for simple functions defined on sets of finite measure that the Riemann Integral over a closed, bounded interval of a step function agrees with the Lebesgue integral.

Definition 4.2.6. Let f be a bounded real-valued function defined on a set of finite measure E. By analogy with the Riemann Integral, we define the **lower Lebesgue integral**

of f over E to be

$$\sup\left\{\int_E \varphi \ : \varphi \text{ simple and } \varphi \leq f \text{ on } E\right\}$$

the **upper Lebesgue integral** of f over E to be

$$\inf\left\{\int_E \psi \ : \psi \text{ simple and } f \leq \psi \text{ on } E\right\}.$$

Since f is assumed to be bounded, by the monotonicity property of the integral for simple functions, the lower and upper integrals are finite and the upper integral is always at least as large as the lower integral.

Definition 4.2.7. A bounded function f on a domain E of finite measure is said to be **Lebesgue integrable** over E provided its upper and lower Lebesgue integrals over E are equal. The common value of the upper and lower integrals is called the **Lebesgue integral,** or simply the integral, of f over E and is denoted by $\int_E f$.

Theorem 4.2.8. *Let f be a bounded function defined on the closed, bounded interval $[a, b]$. If f is Riemann integrable over*

$[a, b]$, then it is Lebesgue integrable over $[a, b]$ and the two integrals are equal.

Proof. The assertion that f is Riemann integrable means that, setting $I = [a, b]$,

$$\sup\left\{(R)\int_I \varphi \ : \varphi \text{ a step function, } \varphi \leq f\right\}$$
$$= \inf\left\{(R)\int_I \psi \ : \psi \text{ a step function, } f \leq \psi\right\}.$$

To prove that f is Lebesgue integrable we must show that

$$\sup\left\{\int_I \varphi \ : \varphi \text{ simple and on } \varphi \leq f \text{ on } I\right\}$$

$$= \inf\left\{\int_I \psi \ : \psi \text{ simple and } f \leq \psi \text{ on } I\right\}$$

However, each step function is a simple function and, as we have already observed, ***for a step function, the Riemann integral and the Lebesgue integral are the same.*** Therefore the first equality implies the second and also the equality of the Riemann and Lebesgue integrals. □

Notation 4.2.9. We are now fully justified in using the symbol $\int_E f$, without any preliminary (R), to denote the integral of a bounded function that is Lebesgue integrable over a set of finite measure. In the case of an interval $E = [a, b]$, we sometimes use the familiar notation $\int_a^b f$ to denote $\int_{[a, b]} f$ and sometimes it is useful to use the classic Leibniz notation $\int_a^b f(x)\,dx$.

Example 4.2.10. The set E of rational numbers in $[0, 1]$ is a measurable set of measure zero. The Dirichlet function f is the restriction to $[0, 1]$ of the characteristic function of E, χ_E. Thus f is integrable over $[0, 1]$ and

$$\int_{[0,1]} f = \int_{[0,1]} 1 \cdot \chi_E = 1 \cdot m(E) = 0\,.$$

We have shown that f is not Riemann integrable over $[0, 1]$.

Theorem 4.2.11. *Let f be a bounded measurable function on a set of finite measure E. Then f is integrable over E.*

Proof. Let n be a natural number. By the Simple Approximation Lemma, with $\varepsilon = \frac{1}{n}$, there are two simple functions φ_n and ψ_n defined on E for which

$$\varphi_n \leq f \leq \psi_n \text{ on } E,$$

and

$$0 \leq \psi_n - \varphi_n \leq \frac{1}{n} \text{ on } E.$$

By the monotonicity and linearity of the integral for simple functions,

$$0 \leq \int_E \psi_n - \int_E \varphi_n = \int_E [\psi_n - \varphi_n] \leq \frac{1}{n} \cdot m(E).$$

However,

$$\begin{aligned} 0 \quad &\leq \quad \inf\left\{ \int_E \psi \,|\, \psi \text{ simple}\,,\ \psi \geq f \right\} \\ &\qquad - \sup\left\{ \int_E \varphi \,|\, \varphi \text{ simple}\,,\ \varphi \leq f \right\} \\ &\leq \quad \int_E \psi_n - \int_E \varphi_n \leq \frac{1}{n} \cdot m(E). \end{aligned}$$

This inequality holds for every natural number n and $m(E)$ is finite. Therefore the upper and lower Lebesgue integrals are equal and thus the function f is integrable over E. $\square$

It turns out that the converse of the preceding theorem is true; *a bounded function on a set of finite measure is Lebesgue integrable if and only if it is measurable* (We will see this in Theorem 5.3.2 in Page 249). This shows, in particular, that not every bounded function defined on a set of finite measure is Lebesgue integrable. In fact, for any measurable set E of finite positive measure, the restriction to E of the characteristic function of each nonmeasurable subset of E fails to be Lebesgue integrable over E.

Theorem 4.2.12. ***(Linearity and Monotonicity of Integration)*** *Let f and g be bounded measurable functions on a set of finite measure E. Then for any α and β,*

$$\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g. \tag{4.2}$$

Moreover,

$$\text{if } f \leq g \text{ on } E, \text{ then } \int_E f \leq \int_E g. \tag{4.3}$$

Proof. A linear combination of measurable bounded functions is measurable and bounded. Thus, by Theorem 4.2.11, $\alpha f + \beta g$ is integrable over E. We first prove linearity for $\beta = 0$. If ψ is a simple function so is $\alpha\psi$, and conversely (if $\alpha \neq 0$). We established linearity of integration for simple functions. Let $\alpha > 0$. Since the Lebesgue integral is equal to the upper Lebesgue integral,

$$\int_E \alpha f = \inf_{\psi \geq \alpha f} \int_E \psi = \alpha \inf_{\frac{\psi}{\alpha} \geq f} \int_E \frac{\psi}{\alpha} = \alpha \int_E f.$$

For $\alpha < 0$, since the Lebesgue integral is equal both to the upper Lebesgue integral and the lower Lebesgue integral,

$$\int_E \alpha f = \inf_{\varphi \geq \alpha f} \int_E \varphi = \alpha \sup_{\frac{\varphi}{\alpha} \leq f} \int_E \frac{\varphi}{\alpha} = \alpha \int_E f.$$

It remains to establish linearity in the case that $\alpha = \beta = 1$. Let ψ_1 and ψ_2 be simple functions for which $f \leq \psi_1$ and $g \leq$

ψ_2 on E. Then $\psi_1+\psi_2$ is a simple function and $f+g \leq \psi_1+\psi_2$ on E. Hence, since $\int_E (f+g)$ is equal to the upper Lebesgue integral of $f+g$ over E, by the linearity of integration for simple functions,

$$\int_E (f+g) \leq \int_E (\psi_1+\psi_2) = \int_E \psi_1 + \int_E \psi_2.$$

The greatest lower bound for the sums of integrals on the right-hand side, as ψ_1 and ψ_2 vary among simple functions for which $f \leq \psi_1$ and $g \leq \psi_2$, equals $\int_E f + \int_E g$. These inequalities tell us that $\int_E (f+g)$ is a lower bound for these same sums. Therefore,

$$\int_E (f+g) \leq \int_E f + \int_E g.$$

It remains to prove this inequality in the opposite direction. Let φ_1 and φ_2 be simple functions for which $\varphi_1 \leq f$ and $\varphi_2 \leq g$ on E. Then $\varphi_1+\varphi_2 \leq f+g$ on E and $\varphi_1+\varphi_2$ is simple. Hence, since $\int_E (f+g)$ is equal to the lower Lebesgue integral of $f+g$ over E, by the linearity of integration for simple functions,

$$\int_E (f+g) \geq \int_E (\varphi_1 + \varphi_2) = \int_E \varphi_1 + \int_E \varphi_2.$$

The least upper bound bound for the sums of integrals on the right-hand side, as φ_1 and φ_2 vary among simple functions for which $\varphi_1 \leq f$ and $\varphi_2 \leq g$, equals $\int_E f + \int_E g$. These inequalities tell us that $\int_E (f+g)$ is an upper bound for these same sums. Therefore,

$$\int_E (f+g) \geq \int_E f + \int_E g.$$

This completes the proof of linearity of integration.

To prove monotonicity, assume $f \leq g$ on E. Define $h = g - f$ on E. By linearity,

$$\int_E g - \int_E f = \int_E (g-f) = \int_E h\,.$$

The function h is nonnegative and therefore $\psi \leq h$ on E, where $\psi = 0$ on E. Since the integral of h equals its lower integral, $\int_E h \geq \int_E \psi = 0$. Therefore,

$$\int_E f \leq \int_E g.$$

□

Corollary 4.2.13. *Let f be a bounded measurable function on a set of finite measure E. Suppose A and B are disjoint measurable subsets of E. Then*

$$\int_{A\cup B} f = \int_A f + \int_B f. \tag{4.4}$$

Proof. Both $f\cdot\chi_A$ and $f\cdot\chi_B$ are bounded measurable functions on E. Since A and B are disjoint,

$$f \cdot \chi_{A\cup B} = f \cdot \chi_A + f \cdot \chi_B.$$

Furthermore, for any measurable subset E_l of E,

$$\int_{E_1} f = \int_E f \cdot \chi_{E_1}.$$

Therefore, by the linearity of integration,

$$\int\limits_{A\cup B} f = \int\limits_{E} f \cdot \chi_{A\cup B} = \int\limits_{E} f \cdot \chi_A + \int\limits_{E} f \cdot \chi_B = \int\limits_{A} f + \int\limits_{B} f.$$

□

Corollary 4.2.14. *Let f be a bounded measurable function on a set of finite measure E. Then*

$$\left| \int\limits_{E} f \right| \leq \int\limits_{E} |f|. \tag{4.5}$$

Proof. The function $|f|$ is measurable and bounded. Now

$$-|f| \leq f \leq |f| \text{ on } E.$$

By the linearity and monotonicity of integration,

$$-\int\limits_{E} |f| \leq \int\limits_{E} f \leq \int\limits_{E} |f|,$$

that is, (4.5) holds. □

Proposition 4.2.15. *Let $\{fn\}$ be a sequence of bounded measurable functions on a set of finite measure E.*

$$\text{If } \{ f_n\} \to f \text{ uniformly on } E, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Since the convergence is uniform and each f_n is bounded, the limit function f is bounded. The function f is measurable since it is the pointwise limit of a sequence of measurable functions. Let $\varepsilon > 0$. Choose an index N for which

$$|f - f_n| < \frac{\varepsilon}{m(E)} \text{ on } E \text{ forall } n \geq N. \tag{4.6}$$

By the linearity and monotonicity of integration and the preceding corollary (Corollary 4.2.14), for each $n \geq N$,

$$\left| \int_E f - \int_E f_n \right| = \left| \int_E [f - f_n] \right| \leq \int_E |f - f_n| \leq \frac{\varepsilon}{m(E)} \cdot m(E) = \varepsilon.$$

Therefore

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

□

This proposition is rather weak since frequently a sequence will be presented that converges pointwise but not uniformly. It is important to understand when it is possible to infer from

$$\{f_n\} \to f \text{ pointwise a.e. .on } E$$

that

$$\lim_{n\to\infty}\left[\int_E f_n\right] = \int_E \left[\lim_{n\to\infty} f_n\right] = \int_E f.$$

We refer to this equality as **passage of the limit under the integral sign** or **interchange of limit and integration.** Before proving first result regarding this passage, we consider an example.

Example 4.2.16. For each natural number n, define f_n on $[0,\ 1]$ to have the value 0 if $x \geq 2/n$, have $f(1/n) = n$, $f(0) = 0$ and to be linear on the intervals $[0,\ 1/n]$ and $[1/n,\ 2/n]$. Observe that

$$\int_0^1 f_n = 1 \text{ for each } n.$$

Define $f = 0$ on $[0, 1]$. Then

$$\{f_n\} \to f \text{ pointwise on } [0,1], \text{ but } \lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f.$$

Thus, pointwise convergence alone is not sufficient to justify passage of the limit under the integral sign.

Theorem 4.2.17. ***[The Bounded Convergence Theorem]*** *Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E, that is, there is a number $M \geq 0$ for which*

$$|f_n| \leq M \text{ on } E \text{ for all } n\,.$$

$$\text{If } \{f_n\} \to f \text{ pointwise on } [0,1], \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. The proof of this theorem furnishes a nice illustration of Littlewood's Third Principle. If the convergence is uniform, we have the easy proof of the preceding proposition. However, Egoroff's Theorem tells us, roughly, that pointwise convergence is *nearly* uniform.

The pointwise limit of a sequence of measurable functions is measurable. Therefore f is measurable. Clearly $|f| \leq M$ on E. Let A be any measurable subset of E and n a natural number. By the linearity and additivity over domains of the integral,

$$\int_E f_n - \int_E f = \int_E [f_n - f] = \int_A [f_n - f] + \int_{E \sim A} f_n + \int_{E \sim A} (-f)$$

Therefore, by Corollary 4.2.14 and the monotonicity of integration,

$$\left| \int_E f_n - \int_E f \right| \leq \int_A [f_n - f] + 2M \cdot m(E \sim A). \tag{4.7}$$

To prove convergence of the integrals, let $\varepsilon > 0$. Since $m(E) < \infty$ and f is real-valued, Egoroff's Theorem tells us that there is a measurable subset A of E for which $\{f_n\} \to f$ uniformly on A and $m(E \sim A) < \frac{\varepsilon}{4M}$.

By uniform convergence, there is an index N for which

$$|f_n - f| < \frac{\varepsilon}{2 \cdot m(E)} \text{ on } A \text{ for all } n \geq N.$$

Therefore, for $n \geq N$, we infer from (4.7) and the monotonicity of integration that

$$\left| \int_E f_n - \int_E f \right| \leq \frac{\varepsilon}{2 \cdot m(E)} \cdot m(A) + 2M \cdot m(E \sim A) < \varepsilon.$$

Hence the sequence of integrals $\left\{ \int_E f_n \right\}$ converges to $\int_E f$. □

4.3 The Lebesgue Integral of a Measurable Nonnegative Function

A measurable function f on E is said to vanish outside a set of finite measure provided there is a subset E_0 of E for which $m(E_0) < \infty$ and $f \equiv 0$ on $E \sim E_0$. It is convenient to say that a function that vanishes outside a set of finite measure has finite support and define its support to be $\{x \in E | f(x) \neq 0\}$.

In the previous section, we defined the integral of a bounded measurable function f over a set of finite measure E. However, even if $m(E) < \infty$, if f is bounded and measurable on E but has finite support, we can define its integral over E by

$$\int_E f = \int_{E_0} f,$$

where E_0 has finite measure and $f \equiv 0$ on $E \sim E_0$. This integral is properly defined, that is, it is independent of the choice of set of finite measure E_0 outside of which f vanishes. This is a consequence of the additivity over domains property of integration for bounded measurable functions over a set of finite measure.

Definition 4.3.1. *(Lebesgue Integral of a Measurable Nonnegative Function)* If f is a nonnegative measurable function defined on a measurable set E, we define **the integral of f over E** by

$$\int_E f = \sup \left\{ \int_E h \,\middle|\, h \text{ bounded, measurable, of finite support and } 0 \le h \le f \text{ on } E \right\}.$$

Theorem 4.3.2. ***Chebychev's Inequality*** *Let f be a non-negative measurable function on E. Then for any $\lambda > 0$,*

$$m\{x \in E \mid f(x) \geq \lambda\} \leq \frac{1}{\lambda} \cdot \int_E f. \tag{4.8}$$

Proof. Define

$$E_\lambda = \{x \in E \mid f(x) \geq \lambda\}.$$

Case 1. Suppose $m(E_\lambda) = \infty$. Let n be a natural number. Define

$$E_{\lambda,\, n} = E_\lambda \cap [-n,\ n] \text{ and } \psi_n = \lambda \cdot \chi_{E_{\lambda,\, n}}.$$

Then ψ_n is a bounded measurable function of finite support,

$$\lambda \cdot m(E_{\lambda,\, n}) = \int_E \psi_n \ \text{ and } \ 0 \leq \psi_n \leq f \ \text{ on } \ E \ \text{ for all } \ n.$$

We infer from the continuity of measure that

$$\infty = \lambda \cdot m(E_\lambda) = \lambda \cdot \lim_{n\to\infty} m(E_{\lambda,\, n}) = \lim_{n\to\infty} \int_E \psi_n \leq \int_E f.$$

Thus inequality (4.8) holds since both sides equal ∞.

Case 2. Now consider the case $m(E_\lambda) < \infty$. Define

$$h = \lambda \cdot \chi_{E_\lambda}.$$

Then h is a bounded measurable function of finite support and $0 \leq h \leq f$ on E. By the definition of the integral of f over E,

$$\lambda \cdot m(E_\lambda) = \int_E h \leq \int_E f.$$

Divide both sides of this inequality by λ to obtain Chebychev's Inequality. This completes the proof. □

Proposition 4.3.3. *Let f be a nonnegative measurable function on E. Then*

$$\int_E f = 0 \text{ if and only if } f = 0 \text{ a.e. on } E. \tag{4.9}$$

Proof. First assume

$$\int_E f = 0.$$

Then, by Chebychev's Inequality, for each natural number n,

$$m\left\{x \in X \mid f(x) \geq \frac{1}{n}\right\} = 0\,.$$

By the countable additivity of Lebesgue measure,

$$m\,\{x \in X \mid f(x) > 0\} = 0\,.$$

Conversely, suppose $f = 0$ a.e. on E. Let φ be a simple function and h a bounded measurable function of finite support for which $0 \leq \varphi \leq h \leq f$ on E. Then $\varphi = 0$ a.e. on E and hence $\int_E \varphi = 0$. Since this holds for all such φ, we infer that $\int_E h = 0$. Since this holds for all such h, we infer that $\int_E f = 0$. □

Theorem 4.3.4. ***[Linearity and Monotonicity of Integration]*** *Let f and g be nonnegative measurable functions on E. Then for any $\alpha > 0$ and $\beta > 0$,*

(Linearity)

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g. \tag{4.10}$$

Moreover,

(Monotonicity)

$$\text{if } f \leq g \text{ on } E, \text{ then } \int_E f = \int_E g. \tag{4.11}$$

Proof. For $\alpha > 0$, $0 \leq h \leq f$ on E if and only if $0 \leq \alpha h \leq \alpha f$ on E. Therefore, by the linearity of the integral of bounded functions of finite support,

$$\int_E \alpha f = \alpha \int_E f.$$

Thus, to prove linearity we need only consider the case $\alpha = \beta = 1$. Let h and g be bounded measurable functions of finite support for which $0 \leq h \leq f$ and $0 \leq k \leq g$ on E. We have $0 \leq h{+}k \leq f{+}g$ on E, and $h{+}k$ also is a bounded measurable function of finite support. Thus, by the linearity of integration for bounded measurable functions of finite support,

$$\int_E h + \int_E k = \int_E (h+k) \leq \int_E (f+g).$$

The least upper bound for the sums of integrals on the left-hand side, as h and k vary among bounded measurable functions of finite support for which $h \leq f$ and $k \leq g$, equals $\int_E f + \int_E g$. These inequalities tell us that $\int_E (f+g)$ is an upper bound for these same sums. Therefore,

$$\int_E f + \int_E g \leq \int_E (f+g).$$

It remains to prove this inequality in the opposite direction, that is,

$$\int_E (f+g) \leq \int_E f + \int_E g.$$

By the definition of $\int_E (f+g)$ as the supremum of $\int_E l$ as l ranges over all bounded measurable functions of finite support for which $0 \leq l \leq f+g$ on E, to verify this inequality it is necessary and sufficient to show that for any such function l

,

$$\int_E l \leq \int_E f + \int_E g. \tag{4.12}$$

For such a function l, define the functions h and k on E by

$$h = \min\{ f,\ l \} \text{ and } k = l - h \text{ on } \ E.$$

Let x belongs to E. If $l(x) \leq f(x)$, then $k(x) = 0 \leq g(x)$;

if $l(x) > f(x)$, then $h(x) = l(x) - f(x) \leq g(x)$. Therefore, $h \leq g$ on E. Both h and k are bounded measurable functions of finite support. We have

$$0 \leq h \leq f,\ 0 \leq k \leq g \text{ and } l = h + k \text{ on } E.$$

Hence, again using the linearity of integration for bounded measurable functions of finite support and the definitions of $\int_E f$ and $\int_E g$, we have

$$\int_E l = \int_E h + \int_E k \leq \int_E f + \int_E g$$

Thus (4.12) holds and the proof of linearity is complete.

In view of the definition of $\int_E f$ as a supremum, to prove the monotonicity inequality (6.16) it is necessary and sufficient to show that if h is a bounded measurable function of finite support for which $0 \leq h \leq f$ on E, then

$$\int_E h \leq \int_E g \tag{4.13}$$

Let h be such a function. Then $h \leq g$ on E. Therefore, by the definition of $\int_E g$ as a supremum, $\int_E h \leq \int_E g$. This completes the proof of monotonicity. □

Theorem 4.3.5. ***[Additivity Over Domains of Integration]*** *Let f be a nonnegative measurable function on E. If A and B are disjoint measurable subsets of E, then*

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

In particular, if E_0 is a subset of E of measure zero, then

$$\int_E f = \int_{E \sim E_0} f. \tag{4.14}$$

Proof. Additivity over domains of integration follows from linearity as it did for bounded functions on sets of finite measure. The excision formula (4.14) follows from additivity over domains and the observation that, by Proposition 4.3.3, the integral of a nonnegative function over a set of measure zero is zero. □

The following lemma will enable us to establish several criteria to justify passage of the limit under the integral sign.

Lemma 4.3.6. ***[Fatou's Lemma]*** *Let $\{ f_n \}$ be a sequence* *of nonnegative measurable functions on E.*

$$\text{If } \{f_n\} \to f \text{ pointwise a.e. on } E, \text{ then } \int_E f \leq \liminf \int_E f_n. \tag{4.15}$$

Proof. In view of (4.14), by possibly excising from E a set of measure zero, we assume the pointwise convergence is on all of E. The function f is nonnegative and measurable since it is the pointwise limit of a sequence of such functions. To verify the inequality in (4.15) it is necessary and sufficient to show that if h is any bounded measurable function of finite support for which $0 \leq h \leq f$ on E, then

$$\int_E h \leq \liminf \int_E f_n. \tag{4.16}$$

Let h be such a function. Choose $M \geq 0$ for which $|h| \leq M$ on E. Define

$$E_0 = \{x \in E | h(x) \neq 0\}.$$

Then $m(E_0) < \infty$. Let n be a natural number. Define a function h_n on E by

$$h_n = \min\{h,\ f_n\} \text{ on E.}$$

Observe that the function h_n is measurable, that

$$0 \leq h_n \leq M \text{ on } E_0 \text{ and } h_n \equiv 0 \text{ on } E \sim E_0.$$

Furthermore, for each $x \in E$, since $h(x) \leq f(x)$ and $\{f_n(x)\} \to f(x)$, $\{h_n(x)\} \to h(x)$. We infer from the Bounded Convergence Theorem applied to the uniformly bounded sequence of restrictions of h_n to the set of finite measure E_0, and the vanishing of each h_n on $E \sim E_0$, that

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h = \int_E h.$$

However, for each n, $h_n \leq f_n$ on E and therefore, by the definition of the integral of f_n over E, $\int_E h_n \leq \int_E f_n$. Thus,

$$\int_E h = \lim_{n\to\infty} \int_E h_n \leq \liminf \int_E f_n.$$

□

The inequality in Fatou's Lemma may be strict.

Example 4.3.7. Let $E = (0, 1]$ and for a natural number n, define $f_n = n \cdot \chi_{\left(0,\, \frac{1}{n}\right)}$. Then $\{f_n\}$ converges pointwise on E to $f \equiv 0$ on E. However,

$$\int_E f = 0 < 1 = \lim_{n \to \infty} \int_E f_n.$$

As another example of strict inequality in Fatou's Lemma, let $E = \mathbb{R}$ and for a nautral number n, define $g_n = \chi_{(n,\, n+1)}$. Then $\{g_n\}$ converges pointwise on E to $g \equiv 0$ on E. However,

$$\int_E g = 0 < 1 = \lim_{n \to \infty} \int_E g_n.$$

However, the inequality in Fatou's Lemma is an equality if the sequence $\{f_n\}$ is increasing.

Theorem 4.3.8. ***The Monotone Convergence Theorem*** *Let $\{f_n\}$ be an increasing sequence of nonnegative mea-*

surable functions on E.

$$\text{If } \{f_n\} \to f \text{ pointwise a.e. on } E, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. According to Fatou's Lemma (Lemma 4.3.6),

$$\int_E f \leq \liminf \int_E f_n.$$

However, for each index n, $f_n \leq f$ a.e.on E, and so, by the monotonicity of integration for nonnegative measurable functions and (4.14),

$$\int_E f_n \leq \int_E f.$$

Therefore

$$\limsup \int_E f_n \leq \int_E f.$$

Hence

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

□

Corollary 4.3.9. *Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E.*

$$\text{If } f = \sum_{n=1}^{\infty} u_n \text{ pointwise a.e. on } E, \text{ then } \int_E f = \sum_{n=1}^{\infty} \int_E u_n.$$

Proof. Apply the Monotone Convergence Theorem with $f_n = \sum_{k=1}^{n} u_k$, for each index n, and then use the linearity of integration for nonnegative measurable functions. □

Definition 4.3.10. A nonnegative measurable function f on a measurable set E is said to be **integrable over** E provided

$$\int_E f < \infty.$$

Proposition 4.3.11. *Let the nonnegative function f be integrable over E. Then f is finite a.e. on E.*

Proof. Let n be a natural number. Chebychev's Inequality and the monotonicity of measure tell us that

$$m\{x \in E | f(x) = \infty\} \leq m\{x \in E | f(x) \geq n\} \leq \frac{1}{n} \int_E f.$$

But $\int_E f$ is finite and therefore $m\{x \in E | f(x) = \infty\} = 0$. □

Lemma 4.3.12. ***Beppo Levi's Lemma*** *Let* $\{f_n\}$ *be an increasing sequence of nonnegative measurable functions on* E*. If the sequence of integrals* $\left\{\int_E f_n\right\}$ *is bounded, then* $\{f_n\}$ *converges pointwise on* E *to a measurable function* f *that is finite a.e. on* E *and*

$$\lim_{n\to\infty} \int_E f_n = \int_E f < \infty.$$

Proof. Every monotone sequence of extended real numbers converges to an extended real number. Since $\{f_n\}$ is an increasing sequence of extended real-valued functions on E, we may define the extended real-valued nonnegative function f pointwise on E by

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ for all } x \in E.$$

According to the Monotone Convergence Theorem, $\left\{\int_E f_n\right\} \to \int_E f$. Therefore, since the sequence of real numbers $\left\{\int_E f_n\right\}$ is bounded, its limit is finite and so $\int_E f < \infty$. We infer from the preceding proposition that f is finite a.e. on E. □

4.4 The General Lebesgue Integral

We first recall Definition 3.1.18 (Also ref. Example 3.1.19):

Definition 4.4.1. For an extended real-valued function f on E, **positive part** f^+ of f is given by

$$f^+(x) = \max\{f(x),\ 0\} \quad \text{for all}\, x \in E;$$

and the **negative part** f^- of f is given by

$$f^-(x) = \max\{-f(x),\ 0\} \quad \text{for all}\, x \in E.$$

$|f|$ is defined by

$$|f|(x) = \max\{ f(x), -f(x)\} \text{ for all } x \in E.$$

Example 4.4.2. **[Example 3.1.19 revisited]** For the function

$$f(x) = \sin x, \quad 0 \leq x \leq 2\pi \text{ (Fig. (4.3))}$$

we have

$$-f(x) = -\sin x, \quad 0 \leq x \leq 2\pi \text{ (Fig. (4.4))}$$

$$f^+(x) = \max\{ f(x), 0\} = \begin{cases} \sin x, & 0 \leq x \leq \pi \\ 0, & \pi < x \leq 2\pi \end{cases}$$ (Fig. (4.5))

$$f^-(x) = \max\{ -f(x), 0\} = \begin{cases} 0, & 0 \leq x \leq \pi \\ -\sin x, & \pi < x \leq 2\pi \end{cases}$$

(Fig. (4.6))

Also,

$$|f|(x) = \max\{ f(x), -f(x)\} = \begin{cases} \sin x, & 0 \leq x \leq \pi \\ -\sin x, & \pi < x \leq 2\pi \end{cases}$$

(Fig. (4.7)

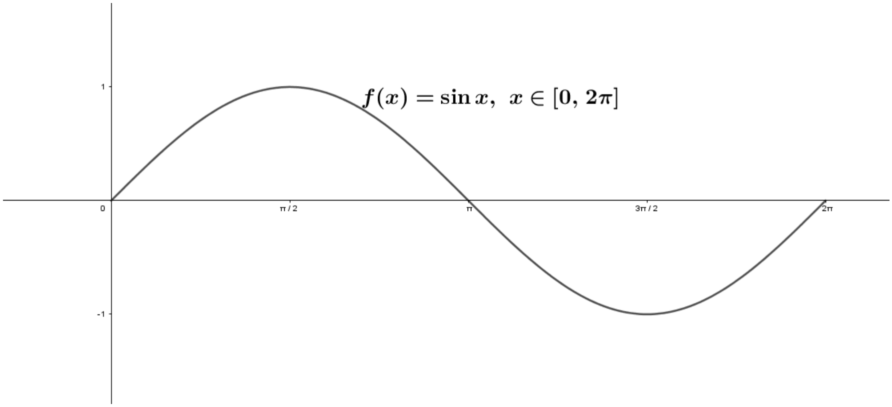

Figure 4.3: Graph of $f(x) = \sin x, \;\; 0 \le x \le 2\pi$

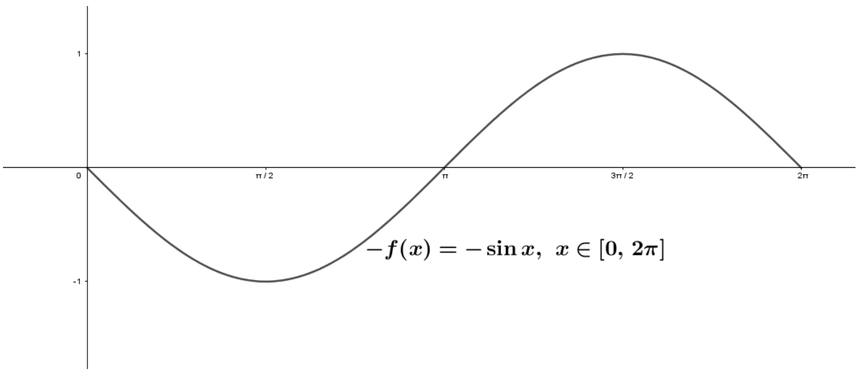

Figure 4.4: Graph of $-f(x) = -\sin x, \;\; 0 \le x \le 2\pi$

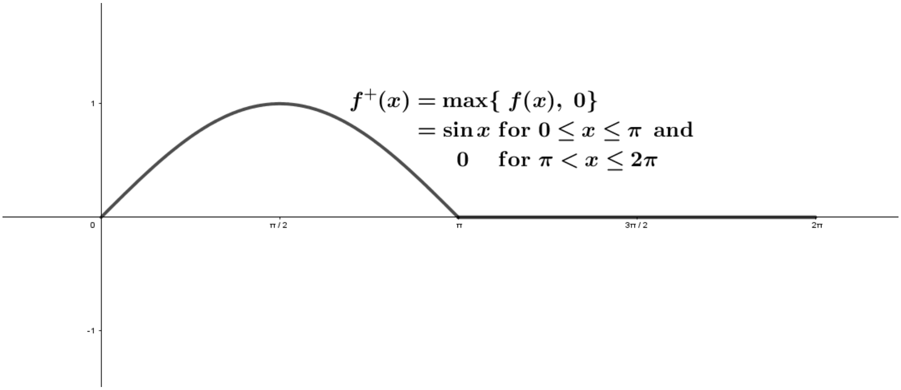

Figure 4.5: Graph of f^+

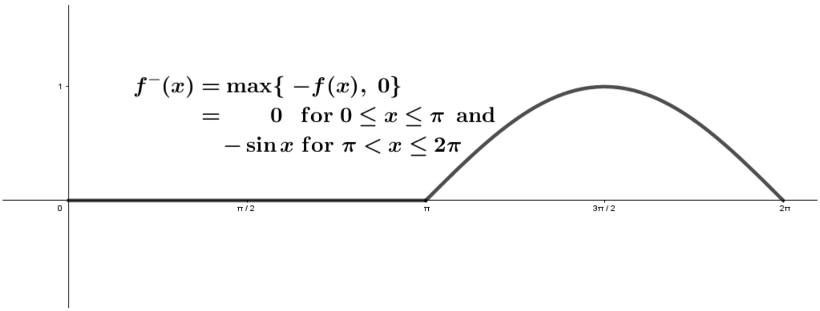

Figure 4.6: Graph of f^-

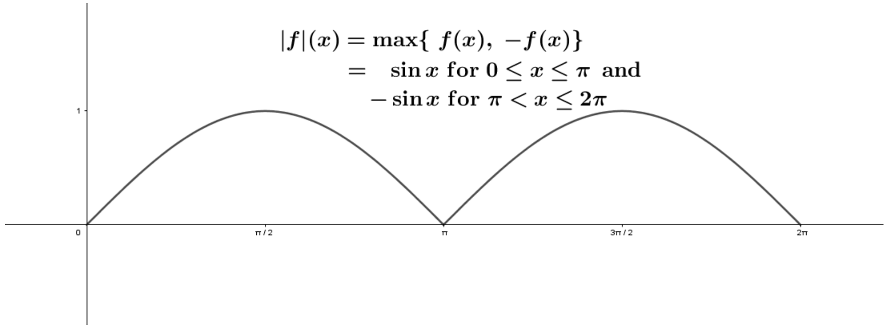

Figure 4.7: Graph of $|f|$

Remark 4.4.3. f^+ and f^- are nonegative functions on E, and

$$f = f^+ - f^- \text{ on } E$$

and

$$|f| = f^+ + f^- \text{ on } E.$$

An example of this can seen in Example 4.4.2. Observe that f is measurable if and only if both f^+ and f^- are measurable.

Proposition 4.4.4. *Let f be a measurable function on E. Then f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.*

Proof. Assume f^+ and f^- are integrable nonnegative functions. By the linearity of integration for nonnegative functions,

$$f^+ + f^-$$

is integrable over E. Hence

$$|f| = f^+ + f^-$$

is integrable over E.

Conversely, suppose $|f|$ is integrable over E. Since

$$0 \le f^+ \le |f|$$

and

$$0 \le f^- \le |f|$$

on E, we infer from the monotonicity of integration for nonnegative functions that both f^+ and f^- are integrable over E. □

Definition 4.4.5. A measurable function f is said to be **integrable** over E provided $|f|$ is integrable over E. In this case we define the **integral of f over E** by

$$\int_E f = \int_E f^+ - \int_E f^-.$$

Of course, for a nonnegative function f, since $f = f^+$ and $f^- \equiv 0$ on E, this definition of integral coincides with the one just considered. By the linearity of integration for bounded measurable functions of finite support, the above definition of integral also agrees with the definition of integral for this class of functions.

Proposition 4.4.6. *Let f be integrable over E. Then f is finite a.e. on E and*

$$\int\limits_E f = \int\limits_{E \sim E_0} f \quad \text{if} \quad E_0 \subseteq E \text{ and } m(E_0) = 0. \tag{4.17}$$

Proof. Proposition 4.3.11, tells us that $|f|$ is finite a.e. on E. Thus f is finite a.e. on E.

Moreover, (4.17) follows by applying (4.14) to the positive and negative parts of f. $\square$

The following criterion for integrability is the Lebesgue integral correspondent of the comparison test for the convergence of series of real numbers.

Proposition 4.4.7. ***[The Integral Comparison Test]*** *Let f be a measurable function on E. Suppose there is a nonnegative function g that is integrable over E and dominates f in the sense that*

$$|f| \leq g \text{ on } E.$$

Then f is integrable over E and

$$\left| \int_E f \right| \leq \int_E |f|.$$

Proof. By the monotonicity of integration for nonnegative functions, $|f|$, and hence f, is integrable. By the triangle inequality for real numbers and the linearity of integration for nonnegative functions,

$$\left| \int_E f \right| = \left| \int_E f^+ - \int_E f^- \right| \leq \int_E f^+ + \int_E f^- = \int_E |f|.$$

□

We have arrived our final stage of generality for the Lebesgue integral for functions of a single real variable. Before proving the linearity property for integration, we need to address, with respect to integration, a point already addressed with respect to measurability. The point is that for two functions f and g which are integrable over E, the sum $f+g$ is not properly defined at points in E where f and g take infinite values of opposite sign. However, by Proposition 4.4.6, if we define A to be the set of points in E at which both f and g are finite, then $m(E \sim A) = 0$. Once we show that $f+g$ is integrable over A, we define

$$\int_E (f+g) = \int_A (f+g)\,.$$

We infer from (4.17) that $\int_E (f+g)$ is equal to the integral over E of any extension of $(f+g)|_A$ to an extended real-valued function on all of E.

Theorem 4.4.8. ***[Linearity and Monotonicity of In-***

tegration] *Let the functions f and g be integrable over E. Then for any α and β, the function $\alpha f + \beta g$ is integrable over E and*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Moreover,

$$\text{if } f \leq g \text{ on } E \text{ then} \int_E f \leq \int_E g.$$

Proof. If $\alpha > 0$, then $[\alpha f]^+ = \alpha f^+$ and $[\alpha f]^- = \alpha f^-$, while if $\alpha < 0$, $[\alpha f]^+ = -\alpha f^-$ and $[\alpha f]^- = -\alpha f^+$. Therefore

$$\int_E \alpha f = \alpha \int_E f$$

since we established this for nonnegative functions f and $\alpha > 0$. So it suffices to establish linearity in the case $\alpha = \beta = 1$. By the linearity of integration for nonnegative functions, $|f| + |g|$ is integrable over E. Since

$$|f + g| \leq |f| + |g|$$

on E, by the integral comparison test, $f+g$ also is integrable over E. Proposition 4.4.6 tells us that f and g are finite a.e. on E. According to the same proposition, by possibly excising from E a set of measure zero, we may assume that f and g are finite on E. To verify linearity is to show that

$$\int_E [f+g]^+ - \int_E [f+g]^- = \left[\int_E f^+ - \int_E f^-\right] + \left[\int_E g^+ - \int_E g^-\right]. \tag{4.18}$$

But

$$(f+g)^+ - (f+g)^- = f+g = (f^+ - f^-) + (g^+ - g^-) \text{ on } E,$$

and therefore, since each of these six functions takes real values on E,

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+ \text{ on } E.$$

We infer from linearity of integration for nonnegative functions that

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+.$$

Since f, g and $f+g$ are integrable over E, each of these six integrals is finite. Rearrange these integrals to obtain (4.18). This completes the proof of linearity.

To establish monotonicity we again argue as above that we may assume g and f are finite on E. Define $h = g - f$ on E. Then h is a properly defined nonnegative measurable function on E. By linearity of integration for integrable functions and monotonicity of integration for nonnegative functions,

$$\int_E g - \int_E f = \int_E (g-f) = \int_E h \geq 0.$$

$\square$

Corollary 4.4.9. ***[Additivity Over Domains of Integration]*** *Let f be integrable over E. Assume A and B are* *disjoint measurable subsets of E. Then*

$$\int_{A\cup B} f = \int_A f + \int_B f. \tag{4.19}$$

Proof. Observe that

$$|f \cdot \chi_A| \leq |f|$$

and

$$|f \cdot \chi_B| \leq |f|$$

on E. By the integral comparison test, the measurable functions $f \cdot \chi_A$ and $f \cdot \chi_B$ are integrable over E. Since A and B are disjoint

$$f \cdot \chi_{A\cup B} = f \cdot \chi_A + f \cdot \chi_B \text{ on } E. \tag{4.20}$$

But for any measurable subset C of E,

$$\int_C f = \int_E f \cdot \chi_C.$$

Thus (4.19) follows from (4.20) and the linearity of integration. □

The following generalization of the Bounded Convergence Theorem provides another justification for passage of the limit under the integral sign.

Theorem 4.4.10. ***[The Lebesgue Dominated Convergence Theorem]*** *Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \leq g$ on E* for all *n. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and*

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Since $|f_n| \leq g$ and $|f| \leq g$ a.e. on E and g is integrable over E, by the integral comparison test, f and each f_n also are integrable over E. We infer from Proposition 4.4.6 that, by possibly excising from E a countable collection of sets of measure zero and using the countable additivity of Lebesgue measure, we may assume that f and each f_n is finite on E.

The function $g - f$ and for each n, the function $g - f_n$, are properly defined, nonnegative and measurable. Moreover, the sequence $\{g - fn\}$ converges pointwise a.e. on E to $g - f$. Fatou's Lemma (Lemma 4.3.6) tells us that

$$\int_E (g - f) \leq \liminf \int_E (g - f_n) \,.$$

Thus, by the linearity of integration for integrable functions,

$$\int_E g - \int_E f = \int_E (g - f) \leq \liminf \int_E (g - f_n) = \int_E g - \limsup \int_E f_n,$$

that is,

$$\limsup \int_E f_n \leq \int_E f.$$

Similarly, considering the sequence $\{g + f_n\}$, we obtain

$$\int_E f \leq \liminf \int_E f_n.$$

This completes the proof. $\square$

The following generalization of the Lebesgue Dominated Convergence Theorem, the proof of which we leave as an exercise, is often useful.

Theorem 4.4.11. ***(General Lebesgue Dominated Convergence Theorem)*** *Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ on E in the sense that*

$$|f_n| \leq |g_n| \text{ on } E \text{ for all } n.$$

If

$$\lim_{n\to\infty} \int_E g_n = \int_E g < \infty,$$

then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Remark 4.4.12. In Fatou's Lemma and the Lebesgue Dominated Convergence Theorem, the assumption of pointwise convergence a.e. on E rather than on all of E is not a decoration pinned on to honor generality. It is necessary for future applications of these results. We provide one illustration of this necessity. Suppose f is an increasing function on all of $\mathbb{R}$. A forthcoming theorem of Lebesgue (Lebesgue's Theorem (Theorem 6.4.6 in Page 284) tells us that

$$\lim_{n\to\infty} \frac{f(x+1/n)-f(x)}{1/n} = f'(x) \text{ for almost all } x. \tag{4.21}$$

From this and Fatou's Lemma we will show that for any closed, bounded interval $[a, b]$,

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

In general, given a nondegenerate closed, bounded interval $[a, b]$ and a subset A of $[a, b]$ that has measure zero, there is an increasing function f on $[a, b]$ for which the limit in (4.21) fails to exist at each point in A.

4.5 Countable Additivity and Continuity of Integration

The linearity and monotonicity properties of the Lebesgue integral, which we established in the preceding section, are extensions of familiar properties of the Riemann integral. **We now establish two properties of the Lebesgue integral which have no counterpart for the Riemann integral.** *The following countable additivity property for Lebesgue integration is a comparison of the countable additivity property of Lebesgue measure.*

Theorem 4.5.1. ***(The Countable Additivity of Integration)*** *Let f be integrable over E and $\{E_n\}_{n=1}^{\infty}$ a disjoint countable collection of measurable subsets of E whose union is E. Then*

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f. \tag{4.22}$$

Proof. Let n be a natural number. Define

$$f_n = f \cdot \chi_n$$

where χ_n is the characteristic function of the measurable set $\bigcup_{k=1}^{n} E_k$. Then f_n is a measurable function on E and

$$|f_n| \leq |f| \text{ on } E.$$

Observe that $\{f_n\} \to f$ pointwise on E. Thus, by the Lebesgue Dominated Convergence Theorem,

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

On the other hand, since $\{E_n\}_{n=1}^{\infty}$ is disjoint, it follows from *the additivity over domains property of the integral* that for each n,

$$\int_E f_n = \sum_{k=1}^{n} \int_{E_k} f.$$

Thus

$$\int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \int_{E_k} f \right] = \sum_{k=1}^{\infty} \int_{E_k} f.$$

□

We leave it to the reader to use the countable additivity of integration to prove the following result regarding the continuity of integration: use a pattern similar to the proof of continuity of measure based on countable additivity of measure.

Theorem 4.5.2. ***(The Continuity of Integration)*** *Let f be integrable over E.*

1. *If $\{E_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of E, then*

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f. \tag{4.23}$$

2. *If $\{E_n\}_{n=1}^{\infty}$ is an descending countable collection of measurable subsets of E, then*

$$\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f. \tag{4.24}$$

4.6 Uniform Integrability: Vitali Convergence Theorem

We conclude this chapter on Lebesgue integration by establishing, for functions that are integrable over a set of finite measure, a criterion for justifying passage of the limit under the integral sign which is suggested by the following lemma and proposition.

Lemma 4.6.1. *Let E be a set of finite measure and $\delta > 0$. Then E is the disjoint union of a finite collection of sets, each of which has measure less than δ.*

Proof. By the continuity of measure,

$$\lim_{n\to\infty} m(E \sim [-n,\ n]) = m(\emptyset) = 0.$$

Choose a natural number n_0 for which $m(E \sim [-n_0, n_0]) < \delta$. By choosing a fine enough partition of $[-n_0, n_0]$, express $E \cap [-n_0,\ n_0]$ as the disjoint union of a finite collection of sets, each of which has measure less than δ. □

Proposition 4.6.2. *Let f be a measurable function on E. If f is integrable over E, then for each $\varepsilon > 0$, there is a $\delta > 0$ for which*

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \int_A |f| < \varepsilon. \tag{4.25}$$

Conversely, in the case $m(E) < \infty$, if for each ε, there is a $\delta > 0$ for which (4.25) holds, then f is integrable over E.

Proof. The theorem follows by establishing it separately for the positive and negative parts of f. We therefore suppose $f \geq 0$ on E. First assume f is integrable over E. Let $\varepsilon > 0$. By the definition of the integral of a nonnegative integrable function, there is a measurable bounded function f_ε of finite support for which

$$0 \leq f_\varepsilon \leq f \text{ on } E \text{ and } 0 \leq \int_E f - \int_E f_\varepsilon < \frac{\varepsilon}{2}.$$

Since $f - f_\varepsilon \geq 0$ on E, if $A \subseteq E$ is measurable, then, by the linearity and additivity over domains of the integral,

$$\int_A f - \int_A f_\varepsilon = \int_A [f - f_\varepsilon] \leq \int_E [f - f_\varepsilon] = \int_E f - \int_A f_\varepsilon < \frac{\varepsilon}{2}.$$

But f_ε is bounded. Choose $M > 0$ for which $0 \leq f_\varepsilon < M$ on E_0. Therefore, if $A \subseteq E$ is measurable, then

$$\int_A f < \int_A f_\varepsilon + \frac{\varepsilon}{2} \leq M \cdot m(A) + \frac{\varepsilon}{2}.$$

Define

$$\delta = \frac{\varepsilon}{2M}.$$

Then (4.25) holds for this choice of δ.

Conversely, suppose $m(E) < \infty$ and for each $\varepsilon > 0$, there is a $\delta > 0$ for which (4.25) holds. Let $\delta_0 > 0$ respond to the $\varepsilon = 1$ challenge. Since $m(E) < \infty$, according to the preceding lemma, we may express E as the disjoint union of a finite collection of measurable subsets $\{E_k\}_{k=1}^N$, each of which has measure less than δ. Therefore

$$\sum_{k=1}^{N} \int_{E_k} f < N.$$

By the additivity over domains of integration it follows that if h is a nonnegative measurable function of finite support and $0 \leq h \leq f$ on E, then

$$\int_{E} h < N.$$

Therefore f is integrable. □

Definition 4.6.3. A family $\mathcal{F}$ of measurable functions on E is said to be **uniformly integrable over *E*** provided for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \int_{A} |f| < \varepsilon. \tag{4.26}$$

Example 4.6.4. Let g be a nonnegative integrable function over E. Define

$$\mathcal{F} = \{f \mid f \text{ is measurable on } E \text{ and } |f| \leq g \text{ on } E\}.$$

Then $\mathcal{F}$ is uniformly integrable. This follows from Proposition 4.6.2, with f replaced by g, and the observation that for any measurable subset A of E, by the monotonicity of integration, if $f \in \mathcal{F}$, then

$$\int_A |f| \leq \int_A g.$$

Proposition 4.6.5. *Let $\{f_k\}_{k=1}^n$ be a finite collection of functions, each of which is integrable over E. Then $\{f_k\}_{k=1}^n$ is uniformly integrable.*

Proof. Let $\varepsilon > 0$. For $1 \leq k \leq n$, by Proposition 4.6.2, there is a $\delta_k > 0$ for which

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta_k, \text{ then } \int_A |f_k| < \varepsilon. \tag{4.27}$$

Define

$$\delta = \min\{\delta_1, \ \ldots\ , \ \delta_n\}.$$

This δ responds to the ε challenge regarding the criterion for the collection $\{f_k\}_{k=1}^n$ to be uniformly integrable. □

Proposition 4.6.6. *Assume E has finite measure. Let the sequence of functions $\{f_n\}$ be uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E.*

Proof. Let $\delta_0 > 0$ respond to the $\varepsilon = 1$ challenge in the uniform integrability criteria for the sequence $\{f_n\}$. Since $m(E) < \infty$, by Lemma 4.21, we may express E as the disjoint union of a finite collection of measurable subsets $\{E_k\}_{k=1}^N$ such that $m(E_k) < \delta_0$ for $1 \leq k \leq N$. For any n, by the *monotonicity* and *additivity over domains property of the integral*,

$$\int_E |f_n| = \sum_{k=1}^N \int_{E_k} |f_n| < N.$$

We infer from Fatou's Lemma that

$$\int_E |f| \leq \liminf \int_E |f_n| \leq N.$$

Thus $|f|$ is integrable over E. □

Theorem 4.6.7. ***[The Vitali Convergence Theorem]*** *Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E.*

$$\text{If} \quad \{f_n\} \quad \to f \text{ pointwise a.e. on } E, \text{ then}$$

$$f \text{ is integrable over } E \text{ and } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Propositions 4.24 tells us that f is integrable over E and hence, by Proposition 4.14, is finite a.e. on E. Therefore, using Proposition 4.14 once more, by possibly excising from E a set of measure zero, we suppose the convergence is pointwise on all of E and f is real-valued. We infer from the integral comparison test and the linearity, monotonicity, and additivity over domains property of integration that, for any measurable subset A of E and any natural number n,

$$\begin{aligned}\left|\int_E f_n - \int_E f\right| &= \left|\int_E (f_n - f)\right| \\ &\leq \int_E |f_n - f| \\ &= \int_{E\sim A} |f_n - f| + \int_A |f_n - f| \\ &\leq \int_{E\sim A} |f_n - f| + \int_A |f_n| + \int_A |f|. \quad (4.28)\end{aligned}$$

Let $\varepsilon > 0$. By the uniform integrability of $\{f_n\}$, there is a $\delta > 0$ such that

$$\int_A |f_n| < \frac{\varepsilon}{3}$$

for any measurable subset of E for which $m(A) < \delta$. Therefore, by Fatou's Lemma (Lemma 4.3.6), we also have

$$\int_A |f| \leq \frac{\varepsilon}{3}$$

for any measurable subset of A for which $m(A) < \delta$. Since f is real-valued and E has finite measure, Egoroff's Theorem tells us that there is a measurable subset E_0 of E for which $m(E_0) < \delta$ and $\{f_n\} \to f$ uniformly on $E \sim E_0$. Choose a natural number N such that

$$|f_n - f| < \frac{\varepsilon}{3 \cdot m(E)}$$

on $E \sim E_0$ for all $n \geq N$. Take $A = E_0$ in the integral inequality (4.28). If $n \geq N$, then

$$\begin{aligned} \left| \int_E f_n - \int_E f \right| &\leq \int_{E \sim E_0} |f_n - f| + \int_{E_0} |f_n| + \int_{E_0} |f| \\ &< \frac{\varepsilon}{3 \cdot m(E)} \cdot m(E \sim E_0) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &\leq \varepsilon. \end{aligned}$$

This completes the proof. □

The following theorem shows that the concept of uniform integrability is an essential ingredient in the justification, for a sequence $\{h_n\}$ of nonnegative functions on a set of finite measure that converges pointwise to $h \equiv 0$, of passage of the limit under the integral sign.

Theorem 4.6.8. *Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a. e. on E to $h \equiv 0$. Then*

$$\lim_{n\to\infty} \int_E h_n = 0 \quad \text{if} \quad \text{and only if}$$

$$\{h_n\} \text{ is uniformly integrable over } E.$$

Proof. If $\{h_n\}$ is uniformly integrable, then, by the Vitali Convergence Theorem,

$$\lim_{n\to\infty} \int_E h_n = 0\,.$$

Conversely, suppose $\lim_{n\to\infty} \int_E h_n = 0$. Let $\varepsilon > 0$. We may choose a natural number N for which

$$\int_E h_n < \varepsilon \text{ if } n \geq N.$$

Therefore, since each $h_n \geq 0$ on E,

$$\text{if } A \subseteq E \text{ is measurable and } n \geq N, \text{ then} \int_E h_n < \varepsilon\,. \quad (4.29)$$

According to Propositions 4.6.2 and 4.6.5, the finite collection $\{h_n\}_{n=1}^{N-1}$ is uniformly integrable over E. Let δ respond to the ε challenge regarding the criterion for the uniform integrability of $\{h_n\}_{n=1}^{N-1}$. We infer from (4.29) that δ also responds to the ε challenge regarding the criterion for the uniform integrability of $\{h_n\}_{n=1}^{\infty}$. $\square$

Chapter 5

Lebesgue Integration: Further Topics

In this brief chapter, we first consider a generalization of the Vitali Convergence Theorem to sequences of integrable functions on a set of infinite measure; for a pointwise convergent sequence of integrable functions, tightness must be added to uniform integrablity in order to justify passage of the limit under the integral sign. We then consider a mode of sequential convergence for sequences of measurable functions called convergence in measure and examine its relationship to pointwise convergence and convergence of integrals. Finally, we prove that a bounded function is Lebesgue integrable over a set of finite measure if and only if it is measurable (Theorem 5.3.2), and that a bounded function is Riemann integrable over a closed, bounded interval if and only if it is continuous at almost all points in its domain.

5.1 Uniform Integrability and Tightness: A General Vitali Convergence Theorem

The Vitali Convergence Theorem (Theorem 4.6.7) of the preceding chapter tells us that if $m(E) < \infty$, $\{f_n\}$ is uniformly integrable over E and converges pointwise almost everywhere on E to f, then f is integrable over E and passage of the limit under the integral sign is justified, that is,

$$\lim_{n\to\infty}\left[\int_E f_n\right] = \int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty}\int_E f. \tag{5.1}$$

This theorem requires that E have finite measure. Indeed, for each natural number n, define $f_n = \chi_{[n,\, n+1]}$ and $f \equiv 0$ on $\mathbb{R}$. Then $\{f_n\}$ is uniformly integrable over $\mathbb{R}$ and converges pointwise on $\mathbb{R}$ to f. However,

$$\lim_{n\to\infty}\left[\int_E f_n\right] = 1 \neq 0 = \int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty}\int_E f.$$

The following property of functions that are integrable over sets of infinite measure suggests an additional property which should accompany uniform integrability in order to justify passage of the limit under the integral sign for sequences of functions on a domain of infinite measure.

Proposition 5.1.1. *Let f be integrable over E. Then for each*

$\varepsilon > 0$, *there is a set of finite measure* E_0 *for which*

$$\int_{E \sim E_0} |f| < \varepsilon\,.$$

Proof. Let $\varepsilon > 0$. The nonnegative function $|f|$ is integrable over E. By the definition of the integral of a nonnegative function, there is a bounded measurable function g on E, which vanishes outside a subset E_0 of E of finite measure, for which $0 \leq g \leq |f|$ and $\int_E |f| - \int_E g < \varepsilon$. Therefore, by the linearity and additivity over domains properties of integration,

$$\int_{E \sim E_0} |f| = \int_{E \sim E_0} [|f| - g] \leq \int_E [|f| - g] < \varepsilon.$$

□

Definition 5.1.2. A family $\mathcal{F}$ of measurable functions on E is said to be **tight** over E provided for each $\varepsilon > 0$, there is a subset E_0 of E of finite measure for which

$$\int_{E \sim E_0} |f| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

We infer from Proposition 4.6.2 of the preceding chapter that if $\mathcal{F}$ is a family of functions on E that is uniformly integrable and tight over E, then each function in $\mathcal{F}$ is integrable over E.

Theorem 5.1.3. ***[The Vitali Convergence Theorem]*** *Let $\{f_n\}$ be a sequence of functions on E that is uniformly integrable and tight over E. Then f is integrable over E and*

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Let $\varepsilon > 0$. By the tightness over E of the sequence $\{f_n\}$, there is a measurable subset E_0 of E which has finite measure and

$$\int_{E\sim E_0} |f_n| < \frac{\varepsilon}{4} \text{ for all } n.$$

We infer from Fatou's Lemma that

$$\int_{E\sim E_0} |f| < \frac{\varepsilon}{4}.$$

Therefore f is integrable over $E \sim E_0$. Moreover, by the linearity and monotonicity of integration,

$$\left|\int_{E\sim E_0}[f_n-f]\right| \leq \int_{E\sim E_0}|f_n| + \int_{E\sim E_0}|f| < \frac{\varepsilon}{2} \text{ for all } n. \quad (5.2)$$

But E_0 has finite measure and $\{f_n\}$ is uniformly integrable over E_0. Therefore, by the Vitali Convergence Theorem for functions on domains of finite measure, f is integrable over E_0 and we may choose an index N for which

$$\left|\int_{E_0}[f_n-f]\right| < \frac{\varepsilon}{2} \text{ for all } n \geq N. \quad (5.3)$$

Therefore f is integrable over E and, by (5.2) and (5.3),

$$\left|\int_{E}[f_n-f]\right| < \varepsilon \text{ for all } n \geq N.$$

The proof is complete. □

We leave the proof of the following corollary as an exercise.

Corollary 5.1.4. *Let* $\{h_n\}$ *be a sequence of nonnegative in-*

tegrable functions on E. Suppose $\{h_n(x)\} \to 0$ for almost all x in E. Then

$$\lim_{n\to\infty} \int_E h_n = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable and tight over } E.$$

5.2 Convergence in Measure

We have considered sequences of functions that converge uniformly, that converge pointwise, and that converge pointwise almost everywhere. To this list we add one more mode of convergence that has useful relationships both to pointwise convergence almost everywhere and to forthcoming criteria for justifying the passage of the limit under the integral sign.

Definition 5.2.1. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite a.e. on E. The sequence $\{f_n\}$ is said to **con-**

verge in measure on E to f provided for each $\eta > 0$,

$$\lim_{n\to\infty} m\{x \in E \mid |f_n(x) - f(x)| > \eta\} = 0.$$

When we write $\{f_n\} \to f$ in measure on E we are implicitly assuming that f and each f_n is measurable, and finite a.e. on E. Observe that if $\{f_n\} \to f$ uniformly on E, and f is a real-valued measurable function on E, then $\{f_n\} \to f$ in measure on E since for $\eta > 0$, the set

$$\{x \in E \mid |f_n(x) - f(x)| > \eta\}$$

is empty for n sufficiently large. However, we also have the following much stronger result.

Proposition 5.2.2. *Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f and f is finite a.e. on E. Then $\{f_n\} \to f$ in measure on E.*

Proof. First observe that f is measurable since it is the pointwise limit almost everywhere of a sequence of measurable functions. Let $\eta > 0$. To prove convergence in measure we let $\varepsilon > 0$ and seek an index N such that

$$m\{x \in E \mid |f_n(x) - f(x)| > \eta\} < \varepsilon \quad \text{for all } n \geq N. \tag{5.4}$$

Egoroff's Theorem tells us that there is a measurable subset F of E with $m(E \sim F) < \varepsilon$ such that $\{f_n\} \to f$ uniformly on F. Thus there is an index N such that

$$|f_n - f| > \eta \text{ on } F \text{ for all } n \geq N.$$

Thus, for $n \geq N$,

$$\{x \in E \mid |f_n(x) - f(x)| > \eta\} \subseteq E \sim F$$

and so (5.4) holds for this choice of N. □

The above proposition is false if E has infinite measure. The following example shows that the converse of this proposition also is false.

Example 5.2.3. Consider the sequence of subintervals of

$[0, 1]$, $\{I_n\}_{n=1}^{\infty}$, which has initial terms listed as

$$[0,\ 1],\ [0,\ 1/2],\ [1/2,\ 1],\ [0,\ 1/3],\ [1/3,\ 2/3],\ [2/3,\ 1],$$

$$[0,\ 1/4],\ [1/4,\ 1/2],\ [1/2,\ 3/4],\ [3/4,\ 1]\ \ldots$$

For each index n, define f_n to be the restriction to $[0, 1]$ of the characteristic function of I_n. Let f be the function that is identically zero on $[0, 1]$. We claim that $\{f_n\} \to f$ in measure. Indeed, observe that

$$\lim_{n \to \infty} l(I_n) = 0$$

since for each natural number m,

$$\text{if } n > 1 + \cdots + m = \frac{m(m+1)}{2}, \text{ then } l(I_n) < \frac{1}{m}.$$

Thus, for $0 < \eta < 1$, since

$$\{x \in E \mid |f_n(x) - f(x)| > \eta\} \subseteq I_n,$$

we have

$$0 \leq \lim_{n\to\infty} m\{x \in E|\,|f_n(x) - f(x)| > \eta\} \leq \lim_{n\to\infty} l(I_n) = 0.$$

However, it is clear that there is no point x in $[0, 1]$ at which $\{f_n(x))\}$ converges to $f(x)$ since for each point x in $[0, 1]$, $f_n(x) = 1$ for infinitely many indices n, while $f(x) = 0$.

Theorem 5.2.4. (Riesz) *If $\{f_n\} \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise a. e. on E to f.*

Proof. By the definition of convergence in measure, there is a strictly increasing sequence of natural numbers $\{n_k\}$ for which

$$m\left\{x \in E \mid |f_j(x) - f(x)| > \frac{1}{k}\right\} < \frac{1}{2^k} \text{ for all } j \geq n_k.$$

For each index k, define

$$E_k = \left\{x \in E \mid |f_{n_k}(x) - f(x)| > \frac{1}{k}\right\}.$$

Then $m(E_k) < 1/2^k$ and therefore

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

The Borel-Cantelli Lemma tells us that for almost all $x \in E$, there is an index $K(x)$ such that $x \notin E_k$ if $k \geq K(x)$, that is,

$$|f_{n_k}(x) - f(x)| \leq \frac{1}{k} \text{ forall } k \geq K(x).$$

Therefore

$$\lim_{k\to\infty} f_{n_k}(x) = f(x).$$

□

Corollary 5.2.5. *Let $\{f_n\}$ be a sequence of nonnegative integrable functions on E. Then*

$$\lim_{n\to\infty} \int_E f_n = 0 \tag{5.5}$$

if and only if

$$\{f_n\} \to 0 \text{ in measure on } E \text{ and } \{f_n\} \text{ is uniformly integrable and tight over } E. \quad (5.6)$$

Proof. First assume (5.5). Corollary 5.1.4 tells us that $\{f_n\}$ is uniformly integrable and tight over E. To show that $\{f_n\} \to 0$ in measure on E, let $\eta > 0$. By Chebychev's Inequality, for each index n,

$$m\{x \in E \mid f_n > \eta\} \leq \frac{1}{\eta} \cdot \int_E f_n.$$

Thus,

$$0 \leq \lim_{n\to\infty} m\{x \in E \mid f_n > \eta\} \leq \frac{1}{\eta} \cdot \lim_{n\to\infty} \int_E f_n = 0.$$

Hence $\{f_n\} \to 0$ in measure on E.

To prove the converse, we argue by contradiction. Assume (5.6) holds but (5.5) fails to hold. Then there is some $\varepsilon_0 > 0$

and a subsequence $\{f_{n_k}\}$ for which

$$\int_E f_{n_k} \geq \varepsilon_0 \text{ for all } k.$$

However, by Theorem 5.2.4, a subsequence of $\{f_{n_k}\}$ converges to $f \equiv 0$ pointwise almost everywhere on E and this subsequence is uniformly integrable and tight so that, by the Vitali Convergence Theorem, we arrive at a contradiction to the existence of the above $\varepsilon_0 > 0$. This completes the proof. □

5.3 Characterizations of Riemann and Lebesgue Integrability

Lemma 5.3.1. *Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of functions, each of which is integrable over E, such that $\{\varphi_n\}$ is increasing while $\{\psi_n\}$ is decreasing on E. Let the function f on E have the property that*

$$\varphi_n \leq f \leq \psi_n \text{ on } E \text{ for all } n.$$

If

$$\lim_{n\to\infty} \int_E [\psi_n - \varphi_n] = 0,$$

then

$$\{\varphi_n\} \to f \text{ pointwise a.e. on } E,$$
$$\{\psi_n\} \to f \text{ pointwise a.e. on } E,$$
$$f \text{ is integrable over } E,$$

and

$$\lim_{n\to\infty} \int_E \varphi_n = \int_E f \text{ and } \lim_{n\to\infty} \int_E \psi_n = \int_E f .$$

Proof. For x in E, define

$$\varphi^*(x) = \lim_{n\to\infty} \varphi_n(x) \text{ and } \psi^*(x) = \lim_{n\to\infty} \psi_n(x).$$

The functions are φ^* and ψ^* properly defined since monotone sequences of extended real valued numbers converge to an extended real number and they are measurable since each is the pointwise limit of a sequence of measurable functions. We have the inequalities

$$\varphi_n \leq \varphi^* \leq f \leq \psi^* \leq \psi_n \text{ on } E \text{ for all } n. \tag{5.7}$$

By the monotonicity and linearity of the integral of nonnegative measurable functions,

$$0 \leq \int_E (\psi^* - \varphi^*) \leq \int_E (\psi_n - \varphi_n) \text{ for all } n,$$

so that

$$0 \leq \int_E (\psi^* - \varphi^*) \leq \lim_{n\to\infty} \int_E (\psi_n - \varphi_n) = 0.$$

Since $\psi^* - \varphi^*$ is a nonnegative measurable function and

$$\int_E (\psi^* - \varphi^*) = 0,$$

Proposition 4.3.3 in Page 189 tells us that $\psi^* = \varphi^*$ a.e. on E. But

$$\varphi^* \leq f \leq \psi^* \text{ on } E.$$

Therefore

$$\{\varphi_n\} \to f \text{ and } \{\psi_n\} \to f \text{ pointwise a.e. on } E.$$

Therefore f is measurable. Observe that since

$$0 \le f - \varphi_1 \le \psi_1 - \varphi_1 \text{ on } E$$

and ψ_1 and ϕ_1 are integrable over E, we infer from the integral comparison test that f is integrable over E. We infer from inequality (5.7) that for all n,

$$0 \le \int_E \psi_n - \int_E f_n = \int_E (\psi_n - f) \le \int_E (\psi_n - \varphi_n)$$

and

$$0 \le \int_E f - \int_E \varphi_n = \int_E (f - \varphi_n) \le \int_E (\psi_n - \varphi_n)$$

therefore

$$\lim_{n\to\infty} \int_E \varphi_n = \int_E f = \lim_{n\to\infty} \int_E \psi_n.$$

□

Theorem 5.3.2. *Let f be a bounded function on a set of finite measure E. Then f is Lebesgue integrable over E if and only if it is measurable.*

Proof. We have already shown that *a bounded measurable function on a set of finite measure is Lebesgue integrable* (Theorem 4.2.11 in Page 174). It remains to prove the converse. Suppose f is integrable. From the equality of the upper and lower Lebesgue integrals we conclude that there are sequences of simple functions $\{\varphi_n\}$ and $\{\psi_n\}$ for which

$$\varphi_n \leq f \leq \psi_n \text{ on } E \text{ for all } n,$$

and

$$\lim_{n\to\infty} \int_E [\psi_n - \varphi_n] = 0.$$

Since the maximum and minimum of a pair of simple functions are again simple, using the monotonicity of integration and by possibly replacing φ_n by $\max_{1\leq i\leq n}\varphi_i$ and ψ_n by $\min_{1\leq i\leq n}\psi_i$, we may suppose $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing. By the preceding lemma, $\{\varphi_n\} \to f$ pointwise almost everywhere on E. Therefore f is measurable since it is the pointwise limit almost everywhere of a sequence of measurable functions. $\square$

At the very beginning of our consideration of integration, we showed that if a bounded function on the closed, bounded interval $[a, b]$ is Riemann integrable over $[a, b]$, then it is Lebesgue integrable over $[a, b]$ and the integrals are equal. We may therefore infer from the preceding theorem that if a bounded function on $[a, b]$ is Riemann integrable, then it is measurable. The following theorem is much more precise.

Theorem 5.3.3. (Lebesgue) *Let f be a bounded function on the closed, bounded interval $[a, b]$. Then f is Riemann integrable over $[a, b]$ if and only if the set of points in $[a, b]$ at which f fails to be continuous has measure zero.*

Proof. We first suppose f is Riemann integrable. We infer from the equality of the upper and lower Riemann integrals over $[a, b]$ that there are sequences of partitions $\{P_n\}$ and

$\{P_n'\}$ of $[a, b]$ for which

$$\lim_{n\to\infty} [U(f,\ P_n) - L(f,\ P_n')] = 0,$$

where $U(f,\ P_n)$ and $L(f,\ P_n')$ are upper and lower Darboux sums, respectively. Since, under refinement, lower Darboux sums increase and upper Darboux sums decrease, by possibly replacing each P_n by a common refinement of

$$P_1,\ \ldots,\ P_n,\ P_1',\ \ldots,\ P_n',$$

we may assume each P_{n+1} is a refinement of P_n and $P_n = P_n'$. For each index n, define φ_n to be the lower step function associated with f with respect to P_n, that is, which agrees with f at the partition points of P_n and which on each open interval determined by P_n has constant value equal to the infimum of f on that interval. We define the upper step function ψ_n in a similar manner. By definition of the Darboux sums,

$$L(P,\ f_n) = \int_a^b \varphi_n \ \text{ and } \ U(P,\ f_n) = \int_a^b \psi_n \ \text{ for all } n.$$

Then $\{\varphi_n\}$ and $\{\psi_n\}$ are sequences of integrable functions such that for each index n,

$$\varphi_n \leq f \leq \psi_n \text{ on } E.$$

Moreover, the sequence $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing, because each P_{n+1} is a refinement of P_n. Finally,

$$\lim_{n\to\infty} \int_a^b [\psi_n - \varphi_n] = \lim_{n\to\infty} [U(P,\ f_n) - L(P,\ f_n)] = 0.$$

We infer from the preceding lemma that

$$\{\varphi_n\} \to f \ \text{ and } \ \{\psi_n\} \to f \ \text{ pointwise a.e on } [a,\ b].$$

The set E of points x at which either $\{\psi_n(x)\}$ or $\{\varphi_n(x)\}$ fail to converge to $f(x)$ has measure 0. Let E_0 be the union of E

and the set of all the partition points in the P_n's. As the union of a set of measure zero and a countable set, $m(E_0) = 0$. We claim that f is continuous at each point in $E \sim E_0$. Indeed, let x_0 belong to $E \sim E_0$. To show that f is continuous at x_0, let $\varepsilon > 0$. Since $\{\psi_n(x_0)\}$ and $\{\varphi_n(x_0)\}$ converge to $f(x_0)$, we may choose a natural number n_0 for which

$$f(x_0) - \varepsilon < \varphi_{n_0}(x_0) \leq f(x_0) \leq \psi_{n_0}(x_0) < f(x_0) + \varepsilon. \quad (5.8)$$

Since x_0 is not a partition point of P_{n_0}, we may choose $\delta > 0$ such that the open interval $(x_0 - \delta, x_0 + \delta)$ is contained in the open interval I_{n_0} determined by P_{n_0} which contains x_0. This containment implies that

$$\text{if } |x - x_0| < \delta,$$
$$\text{then } \varphi_{n_0}(x_0) \leq \varphi_{n_0}(x) \leq f(x) \leq \psi_{n_0}(x) \leq \psi_{n_0}(x).$$

From this inequality and inequality (5.8) we infer that

$$\text{if } |x - x_0| < \delta, \text{ then } |f(x) - f(x_0)| < \varepsilon.$$

Thus f is continuous at x_0.

It remains to prove the converse. Assume f is continuous at almost all points in $[a, b]$. Let $\{P_n\}$ be any sequence of partitions of $[a, b]$ for which

$$\lim_{n\to\infty} \operatorname{gap} P_n = 0,$$

where the gap of a partition P is defined to be the maximum distance between consecutive points of the partition.

We claim that

$$\lim_{n\to\infty} [U(P,\ f_n) - L(P,\ f_n)] = 0. \tag{5.9}$$

If this is verified, then from the following estimate for the lower and upper Riemann integrals,

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le [U(P,\ f_n) - L(P,\ f_n)] \ \text{ for all } n,$$

we conclude that f is integrable over $[a, b]$. For each n, let φ_n and ψ_n be the lower and upper step functions associated with

f over the partition P_n. To prove (5.9) is to prove that

$$\lim_{n\to\infty} \int_a^b [\psi_n - \varphi_n] = 0. \tag{5.10}$$

The Riemann integral of a step function equals its Lebesgue integral. Moreover, since the function f is bounded on the bounded set $[a, b]$, the sequences $\{\varphi_n\}$ and $\{\psi_n\}$ are uniformly bounded on $[a, b]$. Hence, by the Bounded Convergence Theorem, to verify (5.10) it suffices to show that $\{\varphi_n\} \to f$ and $\{\psi_n\} \to f$ pointwise on the set of points in (a, b) at which f is continuous and which are not partition points of any partition P_n. Let x_0 be such a point. We show that

$$\lim_{n\to\infty} \varphi_n(x_0) = f(x_0) \text{ and } \lim_{n\to\infty} \psi_n(x_0) = f(x_0). \tag{5.11}$$

Let $\varepsilon > 0$. Let $\delta > 0$ be such that

$$f(x_0) - \frac{\varepsilon}{2} < f(x) < f(x_0) + \frac{\varepsilon}{2} \text{ if } |x - x_0| < \delta. \tag{5.12}$$

Choose an index N for which gap $P_n < \delta$ if $n \geq N$. If $n \geq N$ and I_n is the open partition interval determined by P_n, which contains x_0, then $I_n \subseteq (x_0 - \delta,\ x_0 + \delta)$. We infer from (5.12) that

$$f(x_0) - \frac{\varepsilon}{2} \leq \varphi_n(x_0) < f(x_0) < \psi_n(x_0) \leq f(x_0) + \frac{\varepsilon}{2}$$

and therefore

$$0 \leq \psi_n(x_0) - f(x_0) < \varepsilon \text{ for all } n \geq N$$

and

$$0 \leq f(x_0) - \varphi_n(x_0) < \varepsilon \text{ for all } n \geq N.$$

Thus (5.11) holds. This completes the proof. □

Chapter 6

Differentiation and Integration

6.1 Introduction

The fundamental theorems of integral and differential calculus, with respect to the Riemann integral, are the workhorses of calculus. In this chapter we formulate these two theorems for the Lebesgue integral. For a function f on the closed, bounded interval $[a, b]$, when is

$$\int_a^b f' = f(b) - f(a)\,? \tag{6.1}$$

Assume f is continuous. Extend f to take the value $f(b)$ on $(b,\ b+1]$, and for $0 < h \leq 1$, define (Definition 6.4.7) the **divided difference function** $\mathrm{Diff}_h f$ and **average value function** $\mathrm{Av}_h f$ on $[a,\ b]$ by

$$\mathrm{Diff}_h f(x) = \frac{f(x+h) - f(x)}{h}$$

and

$$\mathrm{Av}_h f = \frac{1}{h}\int_x^{x+h} f(t)dt \text{ for all } x \in [a,\ b].$$

A change of variables and cancellation provides the discrete formulation of (6.1) for the Riemann integral:

$$\int_a^b \mathrm{Diff}_h f = \mathrm{Av}_h f(b) - \mathrm{Av}_h f(a).$$

Figure 6.1: **Henri Lebesgue** (1875-1941) French Mathematician

The limit of the right-hand side as $h \to 0^+$ equals $f(b) - f(a)$. We prove a striking theorem of Henri Lebesgue which tells us that a monotone function on (a, b) has a finite derivative almost everywhere. We then define what it means for a function to be absolutely continuous and prove that if f is absolutely continuous, then f is the difference of monotone functions and the collection of divided differences, $\{\mathrm{Diff}_h f\}_{0<h\leq 1}$, is uniformly integrable. Therefore, by the Vitali Convergence Theorem, (6.1) follows for f absolutely continuous by taking

the limit as $h \to 0^+$ in its discrete formulation. If f is monotone and (6.1) holds, we prove that f must be absolutely continuous. From the *integral form of the fundamental theorem,* (6.1), we obtain the **differential form** of the fundamental theorem, namely, if f is Lebesgue integrable over $[a,\ b]$, then

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x) \text{ for almost all } x \text{ in } [a,\ b]. \tag{6.2}$$

6.2 Monotonic Functions

Definition 6.2.1. Let f be a real-valued function defined on a subset S of $\mathbb{R}$. Then f is said to be **increasing** (or **nondecreasing**) on S if for every pair of points x and y in S,

$$x < y \quad \text{implies} \quad f(x) \leq f(y).$$

If

$$x < y \quad \text{implies} \quad f(x) < f(y),$$

then f is said to be **strictly increasing** on S.

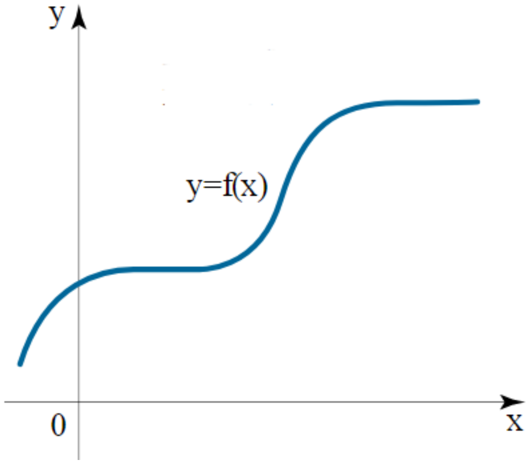

Figure 6.2: Graph of an increasing function.

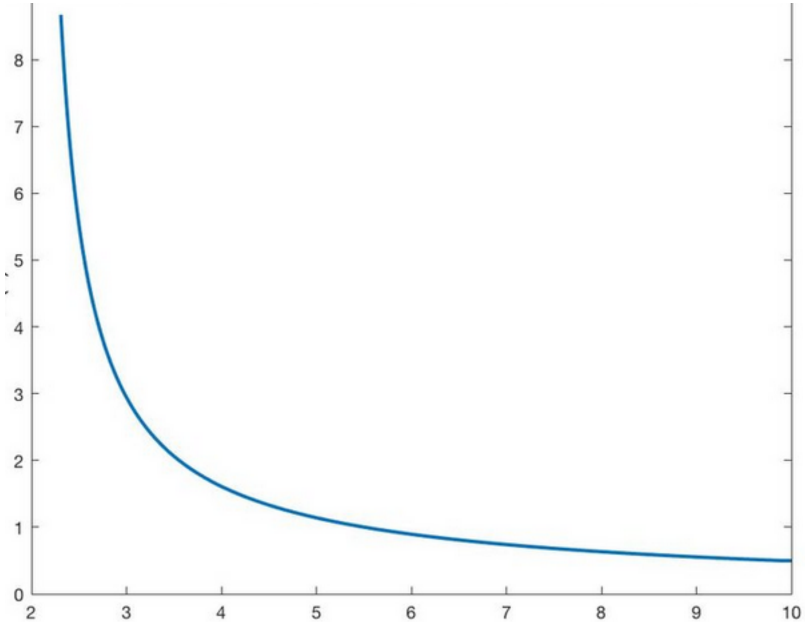

Figure 6.3: Graph of a decreasing function.

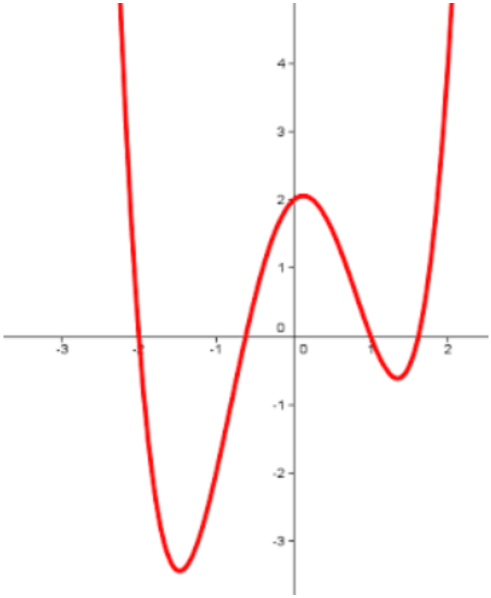

Figure 6.4: Graph of a function which is neither increasing or decreasing on the interval where it is defined. If we restrict the function to smaller domain that restriction functions can be made either increasing or decreasing.

Definition 6.2.2. Let f be a real-valued function defined on a subset S of $\mathbb{R}$. Then f is said to be **decreasing** (or

nonincreasing) on S if for every pair of points x and y in S,

$$x < y \quad \text{implies} \quad f(x) \geq f(y).$$

If

$$x < y \quad \text{implies} \quad f(x) > f(y),$$

then f is said to be **strictly decreasing** on S.

Definition 6.2.3. A function is called **monotonic** on S if it is increasing on S or decreasing on S.

Remark 6.2.4. If f is an increasing function, then $-f$ is a decreasing function. Because of this simple fact, in many situations involving monotonic functions it suffices to consider only the case of increasing functions.

Definition 6.2.5. [Right Hand Limit of a Function at a Point] Let f be defined on an interval (a, b). Assume $c \in [a, b)$. If $f(x) \to A$ as $x \to c$ through values greater than c, we say that A is the **righthand limit** of f at c and we indicate this by writing

$$\lim_{x \to c+} f(x) = A.$$

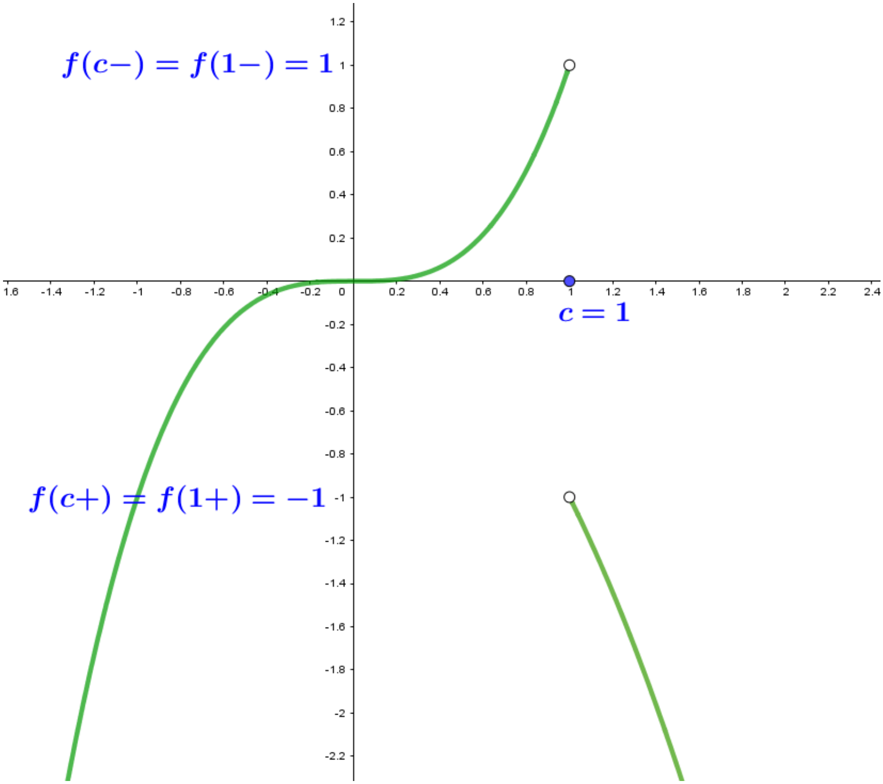

Figure 6.5: The function f is not defined at $c = 1$. Lefthand limit at 1 is 1 and righthand limit at 1 is -1, because $f(x) \to 1$ as $x \to 1$ through values less than 1, and $f(x) \to -1$ as $x \to 1$ through values greater than 1.

The righthand limit A is also denoted by $f(c+)$. In the ε, δ terminology this means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(c+)| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta < b.$$

Note that f need not be defined at the point c itself. If f is defined at c and if $f(c+) = f(c)$, we say that f **is continuous from the right at** c.

Definition 6.2.6. [LEFT HAND LIMIT OF A FUNCTION AT A POINT] Let f be defined on an interval (a, b). Assume $c \in (a, b]$. If $f(x) \to B$ as $x \to c$ through values less than c, we say that B is the **lefthand limit** of f at c and we indicate this by writing

$$\lim_{x \to c-} f(x) = B.$$

The righthand limit A is also denoted by $f(c-)$. In the ε, δ terminology this means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(c-)| < \varepsilon \quad \text{whenever} \quad a < c - \delta < x < c.$$

Note that f need not be defined at the point c itself. If f is defined at c and if $f(c-) = f(c)$, we say that f **is continuous from the left at** c.

If $a < c < b$, then f is *continuous* at c if, and only if,

$$f(c) = f(c+) = f(c-).$$

We say c is a *discontinuity* of f if f is not continuous at c. In this case one of the following conditions is satisfied :

(a) Either $f(c+)$ or $f(c-)$ does not exist.

(b) Both $f(c+)$ and $f(c-)$ exist but have different values.

(c) Both $f(c+)$ and $f(c-)$ exist and $f(c+) = f(c-) \neq f(c)$.

In case (c), the point c is called a **removable discontinuity,** since the discontinuity could be removed by redefining f at c to have the value $f(c+) = f(c-)$. In cases (a) and (b), we call c an **irremovable discontinuity** because the discontinuity cannot be removed by redefining f at c.

Definition 6.2.7. (Fig.6.5 and Fig.6.6) Let f be defined on a closed interval $[a, b]$. If $f(c+)$ and $f(c-)$ both exist at some interior point c, then

(a) $f(c) - f(c-)$ is called the **left hand jump** of f at c,

(b) $f(c+) - f(c)$ is called the **right hand jump** of f at c,

(c) $f(c+) - f(c-)$ is called the **jump** of f at c,

If any one of these three numbers is different from 0, then c is called a **jump discontinuity of** f.

For the end points a and b, only one-sided jumps are considered, the right hand jump at a, $f(a+) - f(a)$, and the left hand jump at b, $f(b) - f(b-)$.

Next theorem states that functions which are monotonic on compact (closed and bounded) intervals always have finite right hand and left hand limits. Hence their discontinuities (if any) must be jump discontinuities.

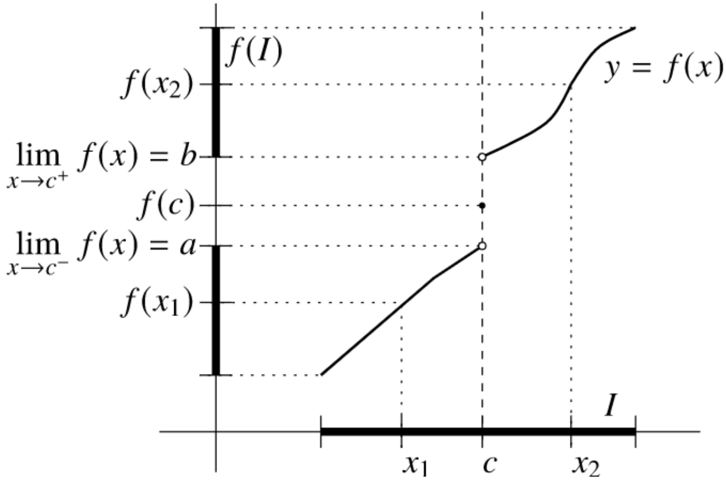

Figure 6.6: Functions which are monotonic on compact (closed and bounded) intervals always have finite right hand and left hand limits.

Theorem 6.2.8. *If f is increasing on $[a, b]$, then (the right hand limit) $f(c+)$ (the left hand limit) $f(c-)$ both exist* **for each** *c in (a, b) and we have*

$$f(c-) \leq f(c) \leq f(c+).$$

At the end points we have

$$f(a) \leq f(a+) \quad \text{and} \quad f(b-) \leq f(b).$$

6.3 Continuity of Monotone Functions

Definition 6.3.1. A function is defined to be **monotone** if it is either increasing or decreasing.

Monotone functions play a decisive role in resolving the question posed in the introductory section. There are two reasons for this. First, a theorem of Lebesgue (Theorem 6.4.6 in Page 284) asserts that a monotone function on an open interval is differentiable almost everywhere. Second, a theorem of Jordan (Theorem 6.5.6 in Page 299) tells us that a very general family of functions on a closed, bounded interval, those of bounded variation, which includes Lipschitz functions, may be expressed as the difference of monotone functions and therefore they also are differentiable almost everywhere on the interior of their domain. In this brief preliminary section we consider continuity properties of monotone functions.

Theorem 6.3.2. *Let f be a monotone function on the open interval (a, b). Then f is continuous except possibly at a countable number of points in (a, b).*

Proof. Assume f is increasing. Furthermore, assume (a, b) is bounded and f is increasing on the closed interval $[a, b]$. Otherwise, express (a, b) as the union of an ascending sequence of open, bounded intervals, the closures of which are contained in (a, b), and take the union of the discontinuities in each of this countable collection of intervals. For each $x_0 \in (a, b)$, f has a limit from the left and from the right at x_0. Define

$$f(x_0^-) = \lim_{x \to x_0^-} f(x) = \sup \{f(x) | a < x < x_0\},$$
$$f(x_0^+) = \lim_{x \to x_0^+} f(x) = \inf \{f(x) | x_0 < x < b\}.$$

Since f is increasing,

$$f(x_0^-) \leq f(x_0^+).$$

The function f fails to be continuous at x_0 if and only if $f(x_0^-) < f(x_0^+)$, in which case we define the open *jump* interval $J(x_0)$ by

$$J(x_0) = \{y | f(x_0^-) < y < f(x_0^+)\}.$$

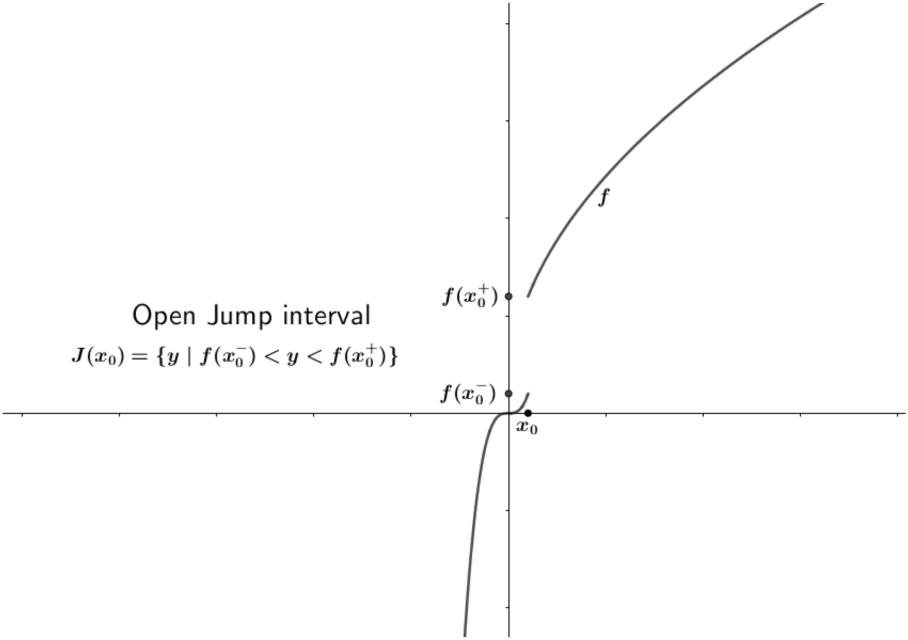

Figure 6.7: Figure shows open jump interval (on the y-axis) of the function f at the point x_0. A function may have (i) no jump interval (if the function is continuous on $[a, b]$) (ii) only one jump interval (if the function have only one point of discontinuity on $[a, b]$) or (iii) more than one jump interval (if the function have more than one point of discontinuity on $[a, b]$.) Each jump interval is contained in the bounded interval $[f(a), f(b)]$ and the collection of jump intervals is disjoint.

Each jump interval is contained in the bounded interval $[f(a), f(b)]$ and the collection of jump intervals is disjoint. Therefore, for each natural number n, there are only a finite number of jump intervals of length greater than $1/n$. Thus the set of points of discontinuity of f is the union of a countable collection of finite sets and therefore is countable. □

Proposition 6.3.3. *Let C be a countable subset of the open interval (a, b). Then there is an increasing function on (a, b) that is continuous only at points in $(a, b) \sim C$.*

Proof. **Case 1.** If C is finite the proof is clear.
Case 2. Assume C is countably infinite. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of C. Define the function f on (a, b) by setting

$$f(x) = \sum_{\{n \mid q_n \leq x\}} \frac{1}{2^n} \text{ for all } a < x < b,$$

where we use the convention that a sum over the empty-set is zero.

Since a geometric series with a ratio less than 1 converges,

f is properly defined. Moreover,

$$\text{if } a < u < v < b, \text{ then } f(v) - f(u) = \sum_{\{n | u < q_n \leq v\}} \frac{1}{2^n}. \tag{6.3}$$

Thus f is increasing. Let $x_0 = q_k$ belong to C. Then, by (6.3),

$$f(x_0) - f(x) \geq \frac{1}{2^k} \text{ for all } x < x_0.$$

Therefore f fails to be continuous at x_0. Now let x_0 belong to $(a, b) \sim C$. Let n be a natural number. There is an open interval I containing x_0 for which q_n does not belong to I for $1 \leq k \leq n$. We infer from (6.3) that

$$|f(x) - f(x_0)| < \frac{1}{2^n} \text{ for all } x \in I.$$

Therefore f is continuous at x_0. □

6.4 Differentiability of Monotone Functions: Lebesgue's Theorem

Definition 6.4.1. A closed, bounded interval $[c, d]$ is said to be nondegenerate provided $c < d$.

Definition 6.4.2. Let $\mathcal{F}$ be a collection of closed, bounded, nondegenerate intervals. We say that $\mathcal{F}$ **covers** a set E in **the sense of Vitali** if for each point x in E and $\varepsilon > 0$, there is an interval $I \in \mathcal{F}$ such that $x \in I$ and $l(I) < \varepsilon$.

Lemma 6.4.3. *[**The Vitali Covering Lemma**] Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that covers E in the sense of VItali. Then for each $\varepsilon > 0$, there is a finite disjoint subcollection $\{I_k\}_{k=1}^{n} = \{I_1, \ldots, I_n\}$ of $\mathcal{F}$ for which*

$$m^*\left[E \sim \bigcup_{n=1}^{N} I_n\right] < \varepsilon. \tag{6.4}$$

Proof. Since E is a set of finite outer measure, we have $m^*(E) < \infty$, and hence there is an open set O containing E for which

$m(O) < \infty$. Because $\mathcal{F}$ is a Vitali covering of E, we may assume that each interval in $\mathcal{F}$ is contained in O. By the countable additivity and monotonicity of measure,

$$\text{if } \{I_k\}_{k=1}^{n} \subseteq \mathcal{F} \text{ is disjoint, then} \sum_{k=1}^{\infty} l\left(I_k\right) \leq m(O) < \infty. \quad (6.5)$$

Moreover, since each I_k is closed and $\mathcal{F}$ is a Vitali covering of E,

$$\text{if } \{I_k\}_{k=1}^{n} \subseteq \mathcal{F}, \qquad \text{then } E \sim \bigcup_{k=1}^{\infty} I_k \subseteq \bigcup_{I \in \mathcal{F}_n} I \text{ where}$$
$$\mathcal{F}_n = \left\{ I \in \mathcal{F} \mid I \cap \bigcup_{k=1}^{n} I_k = \emptyset \right\}. (6.6)$$

Case 1: If there is a finite disjoint subcollection of $\mathcal{F}$ that covers E, the proof is complete.

Case 2: Otherwise, we inductively choose a disjoint countable subcollection $\{I_k\}_{k=1}^{\infty}$ of $\mathcal{F}$ which has the following property:

$$E \sim \bigcup_{k=1}^{n} I_k \subseteq \bigcup_{k=n+1}^{\infty} 5 * I_k \text{ for all } n, \qquad (6.7)$$

where, for a closed, bounded interval I, $5 * I$ denotes the closed interval that has the same midpoint as I and 5 times its length.

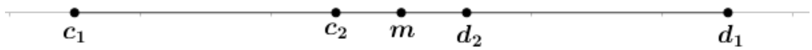

Figure 6.8: We choose points c_1, c_2, d_1, d_2, and m such that $I = [c_2, d_2]$; $5 * I = [c_1, d_1]$; m is the midpoint of both the intervals and that the closed interval $5 * I$ has 5 times length of I.

To begin this selection, let I_1 be any interval in $\mathcal{F}$. Suppose n is a natural number and the finite disjoint subcollection $\{I_k\}_{k=1}^{n}$ of $\mathcal{F}$ has been chosen. Since $E \sim \bigcup_{k=1}^{n} I_k \neq \emptyset$, the collection $\mathcal{F}_n$ defined in (6.6) is nonempty. Moreover, the supremum, s_n, of the lengths of the intervals in $\mathcal{F}_n$ is finite since $m(O)$ is an upper bound for these lengths. Choose I_{n+1} to be an interval in $\mathcal{F}_n$ for which $l(I_{n+1}) > s_n/2$. This inductively defines $\{I_k\}_{k=1}^{\infty}$, a countable disjoint subcollection of $\mathcal{F}$

such that for each n,

$$l(I_{n+1}) > \frac{l(I)}{2} \quad \text{if} \quad I \in \mathcal{F} \quad \text{and} \quad I \cap \bigcup_{k=1}^{n} I_k = \emptyset. \tag{6.8}$$

We infer from (6.5) that $l(I_k) \to 0$. Fix a natural number n. To verify the inclusion (6.7), let $x \in E \sim \bigcup_{k=1}^{n} I_k$. We infer from (6.6) that there is an $I \in \mathcal{F}$ which contains x and is disjoint from $\bigcup_{k=1}^{n} I_k$. Now I must have nonempty intersection with some I_k, for otherwise, by (6.8), $l(I_k) > \frac{l(I)}{2}$ for all k, which contradicts the convergence of $\{l(I_k)\}$ to 0. Let N be the first natural number for which $I \cap I_N \neq \emptyset$. Then $N > n$. Since $I \cap \bigcup_{k=1}^{N-1} I_k = \emptyset$, we infer from (6.8) that $l(I_N) > \frac{l(I)}{2}$ Since $x \in I$ and $I \cap I_N \neq \emptyset$, the distance from x to the midpoint of I_N is at most

$$l(I) + \frac{l(I_N)}{2}$$

and hence, since $l(I) < 2 \cdot l(I_N)$, the distance from x to the midpoint of I_N is less than $\frac{5}{2} \cdot l(I_N)$. This means that x belongs to $5 * I_N$. Thus,

$$x \in 5 * I_N \subseteq \bigcup_{k=n+1}^{\infty} 5 * I_k.$$

We have established the inclusion (6.7).

Let $\varepsilon > 0$. We infer from (6.5) that here is a natural number n for which $\sum_{k=n+1}^{\infty} l\left(I_k\right) < \frac{\varepsilon}{5}$. This choice of n, together with the inclusion (6.7) and the monotonicity and countable additivity of measure, establishes (6.4). This completes the proof. □

Definition 6.4.4. For a real-valued function f and an interior point x of its domain, the **upper derivative** of f at x, denoted by $\bar{D}f(x)$, is defined by

$$\bar{D}f(x) = \lim_{h \to 0} \left[\sup_{0<|t|\leq h} \frac{f(x+t) - f(x)}{t} \right];$$

and the **lower derivative** of f at x, $\underline{D}f(x)$ is defined by

$$\underline{D}f(x) = \lim_{h \to 0} \left[\inf_{0<|t|\leq h} \frac{f(x+t) - f(x)}{t} \right].$$

Clearly, we have

$$\bar{D}f(x) \geq \underline{D}f(x).$$

If

$$\bar{D}f(x) = \underline{D}f(x)$$

we say that ***f* is differentiable at *x*** and define $f'(x)$ to be the common value of the upper and lower derivatives of f at x.

The Mean Value Theorem of Calculus tells us that if a function f is continuous on the closed, bounded interval $[c,\ d]$ and differentiable on its interior $(c,\ d)$ with $f' \geq \alpha$ on $(c,\ d)$, then

$$\alpha \cdot (d - c) \leq [f(d) - f(c)].$$

The proof of the following generalization of this inequality, inequality (6.9), is a nice illustration of the fruitful interplay between the Vitali Covering Lemma and monotonicity properties of functions.

Lemma 6.4.5. *Let f be an increasing function on the closed,* *bounded interval* $[a,\ b]$*. Then for each* $\alpha > 0$,

$$m^*\left\{x \in (a,\ b) | \bar{D}f(x) \geq \alpha\right\} \leq \frac{1}{\alpha} \cdot [f(b) - f(a)] \qquad (6.9)$$

and

$$m^*\left\{x \in (a,\ b) | \bar{D}f(x) = \infty\right\} = 0. \qquad (6.10)$$

Proof. Let $\alpha > 0$. Define

$$E_\alpha = \left\{x \in (a,\ b) | \bar{D}f(x) \geq \alpha\right\}.$$

Choose $\alpha' \in (0,\ \alpha)$. Let $\mathcal{F}$ be the collection of closed, bounded intervals $[c,\ d]$ contained in $(a,\ b)$ for which $f(d) - f(c) \geq \alpha'(d - c)$. Since $\bar{D}f \geq \alpha$ on E_α, $\mathcal{F}$ is a Vitali covering of E_α. The Vitali Covering Lemma tells us that there is a finite disjoint subcollection $\{[c_k,\ d_k]\}_{k=1}^n$ of $\mathcal{F}$ for which

$$m^*\left[E_\alpha \sim \bigcup_{k=1}^n [c_k,\ d_k]\right] < \varepsilon.$$

Since

$$E_\alpha \subseteq \bigcup_{k=1}^n [c_k,\ d_k] \bigcup \left\{E_\alpha \sim \bigcup_{k=1}^n [c_k,\ d_k]\right\},$$

by the finite subadditivity of outer measure, the preceding inequality and the choice of the intervals $[c_k,\ d_k]$,

$$m^*(E_\alpha) < \sum_{k=1}^{n}(d_k - c_k) + \varepsilon \leq \frac{1}{\alpha'} \cdot \sum_{k=1}^{n}[f(d_k) - f(c_k)] + \varepsilon. \quad (6.11)$$

However, the function f is increasing on $[a,\ b]$ and $\{\,[c_k,\ d_k]\,\}_{k=1}^{n}$ is a disjoint collection of subintervals of $[a,\ b]$. Therefore

$$\sum_{k=1}^{n}[f(d_k) - f(c_k)] \leq f(b) - f(a).$$

Thus for each $\varepsilon > 0$, and each $\alpha' \in (0,\ \alpha)$,

$$m^*(E_\alpha) \leq \frac{1}{\alpha'} \cdot [f(b) - f(a)] + \varepsilon. \quad (6.12)$$

This proves (6.9). For each natural number n,

$$\{x \in (a,\ b) \mid \bar{D}f(x) = \infty\} \subseteq E_n$$

and therefore

$$m^*\{x \in (a,\ b) \mid \bar{D}f(x) = \infty\} \leq m^*(E_n) \leq \frac{1}{n} \cdot (f(b) - f(a)).$$

Letting $n \to \infty$ the right hand approaches 0 and hence this proves (6.10). □

Theorem 6.4.6. ***[Lebesgue's Theorem]*** *If the function f is monotone on the open interval $(a,\ b)$, then it is differentiable almost everywhere on $(a,\ b)$.*

Proof. Assume f is increasing. Furthermore, assume $(a,\ b)$ is bounded. Otherwise, express $(a,\ b)$ as the union of an ascending sequence of open, bounded intervals and use the continuity of Lebesgue measure. The set of points x in $(a,\ b)$ at which $\bar{D}f(x) > \underline{D}f(x)$ is the union of the sets

$$E_{\alpha,\ \beta} = \left\{x \in (a,\ b) | \bar{D}f(x) > \alpha > \beta > \underline{D}f(x)\right\}$$

where α and β are rational numbers. Hence, since this is a countable collection, by the countable subadditivity of outer measure, it suffices to prove that each $E_{\alpha,\ \beta}$ has outer measure zero. Fix rationals α and β with $\alpha > \beta$ and set $E = E_{\alpha,\ \beta}$. Let $\varepsilon > 0$. Choose an open set O for which

$$E \subseteq O \subseteq (a,\ b) \ \text{ and } \ m(O) < m^*(E) + \varepsilon. \tag{6.13}$$

Let $\mathcal{F}$ be the collection of closed, bounded intervals $[c, d]$ contained in O for which $f(d) - f(c) < \beta(d-c)$. Since $\underline{D}f < \beta$ on E, $\mathcal{F}$ is a Vitali covering of E. The Vitali Covering Lemma tells us that there is a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^{n}$ of $\mathcal{F}$ for which

$$m^*\left[E \sim \bigcup_{k=1}^{n} [c_k, d_k]\right] < \varepsilon. \tag{6.14}$$

By the choice of the intervals $[c_k, d_k]$, the inclusion of the union of the disjoint collection intervals $\{[c_k, d_k]\}_{k=1}^{n}$ in O and (6.13),

$$\begin{aligned} \sum_{k=1}^{n} [f(d_k) - f(c_k)] &< \beta\left[\sum_{k=1}^{n}(d_k - c_k)\right] \\ &\leq \beta \cdot m(O) \qquad (6.15) \\ &\leq \beta \cdot [m^*(E) + \varepsilon]. \qquad (6.16) \end{aligned}$$

For $1 \leq k \leq n$, we infer from the preceding lemma, applied to the restriction of f to $[c_k, d_k]$, that

$$m^*\left(E \cap (c_k, d_k)\right) \leq \frac{1}{\alpha}[f(d_k) - f(c_k)].$$

Therefore, by (6.14),

$$\begin{aligned} m^*(E) &\leq \sum_{k=1}^{n} m^*\left(E \cap (c_k,\ d_k)\right) + \varepsilon \\ &\leq \frac{1}{\alpha}\left[\sum_{k=1}^{n}[f(d_k) - f(c_k)]\right] + \varepsilon. \end{aligned} \tag{6.17}$$

We infer from (6.16) and (6.17) that

$$m^*(E) \leq \frac{\beta}{\alpha} \cdot m^*(E) + \frac{1}{\alpha} \cdot \varepsilon + \varepsilon \text{ for all } \varepsilon > 0.$$

Therefore, since $0 \leq m^*(E) < \infty$ and $\frac{\beta}{\alpha} < 1,\ m^*(E) = 0.$ □

Lebesgue's Theorem is the best possible in the sense that if E is a set of measure zero contained in the open interval $(a,\ b)$, there is an increasing function on $(a,\ b)$ that fails to be differentiable at each point in E.

Definition 6.4.7. Let f be integrable over the closed, bounded interval $[a,\ b]$. Extend f to take the value $f(b)$ on $(b,\ b{+}1)$. For $0 < h \leq 1$, define the **divided difference function** $\text{Diff}_h f$ of $[a,\ b]$ by

$$\text{Diff}_h f = \frac{f(x+h) - f(x)}{h} \text{ for all } x \in [a,\ b]$$

and the **average value function** $\text{Av}_h f$ of $[a,\ b]$ by

$$\text{Av}_h f = \frac{1}{h} \int_x^{x+h} f \text{ for all } x \in [a,\ b]$$

By a change of variables in the integral and cancellation, for all $a \leq u < v \leq b$,

$$\int_u^v \text{Diff}_h f = \text{Av}_h f(v) - \text{Av}_h f(u). \tag{6.18}$$

Corollary 6.4.8. *Let f be an increasing function on the closed, bounded interval $[a,\ b]$. Then f' is integrable over $[a,\ b]$ and*

$$\int_a^b f' \leq f(b) - f(a). \tag{6.19}$$

Proof. Since f is increasing on $[a,\ b+1]$, it is measurable and therefore the divided difference functions are also measurable. Lebesgue's Theorem tells us that f is differentiable almost everywhere on $(a,\ b)$. Therefore $\left\{ \text{Diff}_{1/n} f \right\}$ is a sequence of nonnegative measurable functions that converges pointwise almost everywhere on $[a, b]$ to f'. According to Fatou's Lemma 4.3.6,

$$\int_a^b f' \leq \liminf_{n\to\infty} \left[\int_a^b \mathrm{Diff}_{1/n} f\right] \tag{6.20}$$

By the change of variable formula (6.18), for each natural number n, since f is increasing,

$$\begin{aligned}\int_a^b \mathrm{Diff}_{1/n} f &= \frac{1}{1/n} \cdot \int_b^{b+1/n} f - \frac{1}{1/n} \cdot \int_a^{a+1/n} f \\ &= f(b) - \frac{1}{1/n} \cdot \int_a^{a+1/n} f \\ &\leq f(b) - f(a).\end{aligned}$$

Thus

$$\limsup_{n\to\infty} \left[\int_a^b \mathrm{Diff}_{1/n} f\right] \leq f(b) - f(a). \tag{6.21}$$

The inequality (6.19) follows from the inequalities (6.20) and (6.21). □

Remark 6.4.9. The integral in (6.19) is independent of the values taken by f at the endpoints. On the other hand, the right-hand side of this equality holds for the extension of any increasing extension of f on the open, bounded interval (a, b) to its closure $[a, b]$. Therefore a tighter form of equality (6.19) is

$$\int_a^b f' \leq \sup_{x\in(a,\, b)} f(x) - \inf_{x\in(a,\, b)} f(x). \tag{6.22}$$

The right-hand side of this inequality equals $f(b)-f(a)$ if and only if f is continuous at the endpoints. However, even if f is increasing and continuous on $[a, b]$, inequality (6.19) may be strict. It is strict for the Cantor-Lebesgue function φ on $[0, 1]$ since $\varphi(1) - \varphi(0) = 1$ while φ' vanishes almost everywhere on $(0, 1)$. We show that for an increasing function f on $[a, b]$, (6.19) is an equality if and only if the function is absolutely continuous on $[a, b]$ (see the forthcoming Corollary 6.7.3).

Remark 6.4.10. For a continuous function f on a closed, bounded interval $[a, b]$ that is differentiable on the open interval (a, b), in the absence of a monotonicity assumption on f we cannot infer that its derivative f' is integrable over $[a, b]$. We leave it as an exercise to show that for f defined on $[0, 1]$

by

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & \text{for } 0 < x \le 1, \\ 0 & \text{for } x = 0, \end{cases} \tag{6.23}$$

f' is not integrable over $[0,\ 1]$ where

$$f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right), \qquad 0 < x < 1.$$

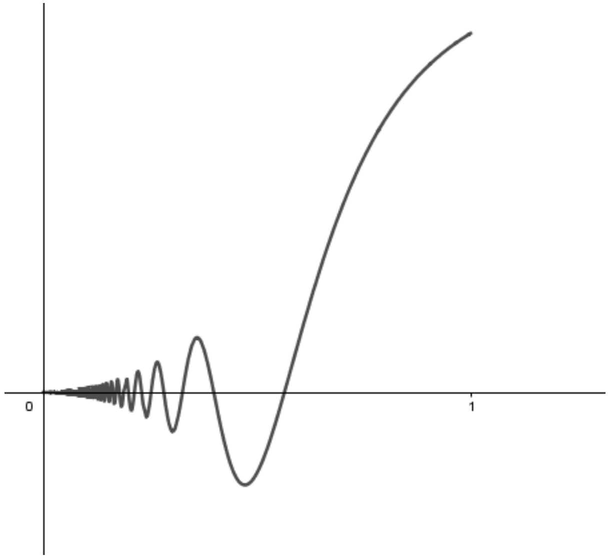

Figure 6.9: Graph of f given by (6.23)

6.5 Functions of Bounded Variation: Jordan's Theorem

Lebesgue's Theorem tells us that a monotone function on an open interval is differentiable almost everywhere. Therefore the **difference** of two increasing functions on an open interval also is differentiable almost everywhere. We now provide a characterization of the class of functions on a closed, bounded interval that may be expressed as the difference of increasing functions, which shows that this class is surprisingly large: it includes, for instance, all Lipschitz functions.

Let f be a real-valued function defined on the closed, bounded interval $[a, b]$ and $P = \{x_0, \ldots, x_k\}$ be a partition of $[a, b]$. Define the variation of f with respect to P by

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

and the **total variation** of $f on$ $[a, b]$ by

$$TV(f) = \sup \{V(f, P) | P \text{ is a partition of } [a, b]\}.$$

For a subinterval $[c, d]$ of $[a, b]$, $TV(f_{[c, d]})$ denotes the total variation of the restriction of f to $[c, d]$.

Definition 6.5.1. A real-valued function f on the closed, bounded interval $[a, b]$ is said to be of **bounded variation**

on $[a, b]$ provided

$$TV(f) < \infty.$$

Example 6.5.2. Let f be an increasing function on $[a, b]$. Then show that f is of bounded variation on $[a, b]$ and

$$TV(f) = f(b) - f(a).$$

Solution For any partition $P = \{x_0, \ \ldots, \ x_k\}$ of $[a, b]$,

$$\begin{aligned} V(f, \ P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1}^{k} (f(x_i) - f(x_{i-1})), \ \text{since}\, f \text{ is an increasing} \\ &\quad \text{function on } [a, b], \ f(x_i) \geq f(x_{i-1}), \ \text{and so} \\ &\quad |f(x_i) - f(x_{i-1})| = f(x_i) - f(x_{i-1}) \\ &= f(b) - f(a). \end{aligned}$$

Example 6.5.3. Let f be a Lipschitz function on $[a, b]$. Then show that f is of bounded variation on $[a, b]$ and

$$TV(f) \leq c \cdot (b - a),$$

where

$$|f(u) - f(v)| \leq c\,|u - v| \text{ for all } u,\ v \text{ in } [a,\ b]. \tag{6.24}$$

Solution For any partition $P = \{x_0,\ \ldots,\ x_k\}$ of $[a,\, b]$,

$$\begin{aligned}
V(f,\ P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\
&\leq c \cdot \sum_{i=1}^{k} |x_i - x_{i-1}|\,, \text{ using (6.24)} \\
&= c \cdot \sum_{i=1}^{k} [x_i - x_{i-1}], \text{ since } x_i \geq x_{i-1} \\
& \quad |x_i - x_{i-1}| = x_i - x_{i-1} \\
&= c \cdot (b - a).
\end{aligned}$$

Thus, $c \cdot [b - a]$ is an upper bound of the set of variations of f with respect to a partition of $[a,\, ,b]$ and hence

$$TV(f) \leq c \cdot (b - a).$$

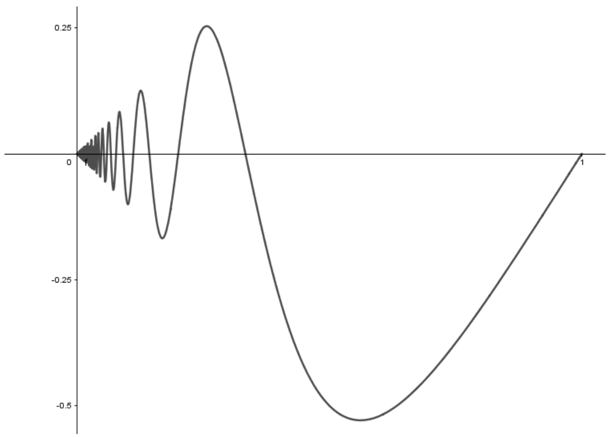

Figure 6.10: Graph of f mentioned in Example 6.5.4.

Example 6.5.4. Define the function f on $[0, 1]$ by (Fig. 6.10)

$$f(x) = \begin{cases} x\cos(\pi/2x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Then prove that f is continuous on $[0, 1]$, but not of bounded variation on $[0, 1]$.

Solution The verification that f is continuous on $[0, 1]$ is left as an exercise. To prove that f is not of bounded variation on $[0, 1]$, we proceed as follows:

For a natural number n, consider the partition

$$P_n = \left\{0,\ \frac{1}{2n},\ \frac{1}{2n-1},\ \ldots,\ \frac{1}{3},\ \frac{1}{2},\ 1\right\}$$

of $[0,\ 1]$. Then

$$V(f,\ P_n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Hence

$$\begin{aligned} TV(f) &= \sup\{V(f,\ P) | P \text{ is a partition of } [a,\ b]\} \\ &> 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \end{aligned}$$

That is, the inequality

$$TV(f) > 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

holds for any natural number n. Since the series

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots$$

is the divergent harmonic series the above shows that

$$TV(f) < \infty \text{ is } \textit{not} \text{ possible.}$$

Hence f is not of bounded variation on $[0,\ 1]$. □

Observe that if $c \in (a,\ b)$, P is a partition of $[a,\ b]$, and P' is the refinement of P obtained by adjoining c to P, then, by the triangle inequality,

$$V(f,\ P) \le V(f,\ P').$$

Thus, in the definition of the total variation of a function on $[a,\ b]$, the supremum can be taken over partitions of $[a,\ b]$ that contain the point c. Now a partition P of $[a,\ b]$ that contains the point c induces, and is induced by, partitions P_1 and P_2 of $[a,\ c]$ and $[c,\ b]$, respectively, and for such partitions

$$V(f_{|[a,\ b]},\ P) = V(f_{|[a,\ c]},\ P_1) + V(f_{|[c,\ b]},\ P_2). \tag{6.25}$$

Take the supremum among such parttions to conclude that

$$TV(f_{|[a,\ b]}) = TV(f_{|[a,\ c]}) + TV(f_{|[c,\ b]}). \tag{6.26}$$

We infer from this that if f is of bounded variation on $[a,\, b]$, then

$$TV(f_{|[a,\, v]}) - TV(f_{|[a,\, u]}) \;=\; TV(f_{|[u,\, v]\, \geq 0}) \qquad (6.27)$$
$$\text{for all } \; a \leq u < v \leq b.$$

Therefore the function $x \mapsto TV(f_{|[a,\, x]})$, which we call the **total variation function** for f, is a real valued increasing function on $[a,\, b]$. Moreover, for $a \leq u < v \leq b$, if we take the crudest partition

$$P = [u,\, v]$$

of $[u,\, v]$, we have

$$\begin{aligned} f(u) - f(v) &\leq |f(u) - f(v)| \\ &= V(f_{|[u,\, v]},\; P) \\ &\leq TV(f_{|[u,\, v]}) \\ &= TV(f_{|[a,\, v]}) - TV(f_{|[a,\, u]}). \end{aligned}$$

Thus,

$$f(v)+TV(f_{|[a,\ v]}) \geq f(u)+TV(f_{|[a,\ u]}) \text{ for all } a \leq u < v \leq b. \tag{6.28}$$

We have established the following lemma.

Lemma 6.5.5. *Let the function f be of bounded variation on the closed, bounded interval $[a,\ b]$. Then f has the following explicit expression as the difference of two increasing functions on $[a,\ b]$:*

$$f(x) = [f(x)+TV(f_{|[a,\ x]})] - TV(f_{|[a,\ x]}) \text{ for all } x \in [a,\ b]. \tag{6.29}$$

Theorem 6.5.6. ***[Jordan's Theorem]*** *A function f is of bounded variation on the closed, bounded interval $[a,\ b]$ if and only if it is the difference of two increasing functions on $[a,\ b]$.*

Proof. Let f be of bounded variation on $[a,\ b]$. The preceding lemma provides an explicit representation of f as the difference of increasing functions.

To prove the converse, let

$$f = g - h \quad \text{on} \quad [a,\, b],$$

where g and h are increasing functions on $[a,\, b]$. For any partition $P = \{x_0,\ \ldots,\ x_k\}$ of $[a,\ ,b]$,

$$\begin{aligned}
V(f,\ P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^{k} |(g(x_i) - h(x_i)) - (g(x_{i-1}) - h(x_{i-1}))| \\
&= \sum_{i=1}^{k} |(g(x_i) - g(x_{i-1})) + (h(x_{i-1}) - h(x_i))| \\
&\leq \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{k} |h(x_{i-1}) - h(x_i)| \\
&= \sum_{i=1}^{k} (g(x_i) - g(x_{i-1})) + \textstyle\sum_{i=1}^{k} (h(x_i) - h(x_{i-1})), \\
&\quad \text{since } |g(x_i) - g(x_{i-1})| = g(x_i) - g(x_{i-1}) \\
&\quad \text{and } |h(x_i) - h(x_{i-1})| = h(x_i) - h(x_{i-1}) \\
&= (g(b) - g(a)) + (h(b) - h(a)).
\end{aligned}$$

Thus, the set of variations f with respect to partitions of $[a, b]$ is bounded above by $(g(b)-g(a))+(h(b)-h(a))$ and therefore f is of bounded variation of $[a, b]$. □

Definition 6.5.7. We call the expression of a function of bounded variation f as the difference of increasing functions a **Jordan decomposition** of f.

Corollary 6.5.8. *If the function f is of bounded variation on the closed, bounded interval $[a, b]$, then it is differentiable almost everywhere on the open interval (a, b) and f' is integrable over $[a, b]$.*

Proof. According to Jordan's Theorem, f is the difference of two increasing functions on $[a, b]$. Thus Lebesgue's Theorem tells us that f is the difference of two functions which are differentiable almost everywhere on (a, b). Therefore f is differentiable almost everywhere on (a, b). The integrability of f' follows from Corollary 6.4.8. □

6.6 Absolutely Continuous Functions

Definition 6.6.1. A real-valued function f on a closed, bounded interval $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ provided for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b),

$$\text{if } \sum_{k=1}^{n} (b_k - a_k) < \delta, \quad \text{then} \quad \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon. \tag{6.30}$$

The criterion for absolute continuity in the case the finite collection of intervals consists of a single interval is the criterion for the uniform continuity of f on $[a, b]$. Thus absolutely continuous functions are continuous. The converse is false, even for increasing functions.

Example 6.6.2. The Cantor-Lebesgue function φ is increasing and continuous on $[0, 1]$, but it is not absolutely continuous. Indeed, to see that φ is not absolutely continuous, let n be a natural number. At the n-th stage of the construction of the Cantor set, a disjoint collection $\{[c_k, d_k]\}_{1 \le k \le 2^n}$

of 2^n subintervals of $[0, 1]$ have been constructed that cover the Cantor set, each of which bas length $(1/3)^n$. The Cantor-Lebesgue function is constant on each of the intervals that comprise the complement in $[0, 1]$ of this collection of intervals. Therefore, since φ is increasing and $\varphi(1) - \varphi(1) = 1$,

$$\sum_{1 \le k \le 2^n} [d_k - c_k] = \left(\frac{2}{3}\right)^n$$

while

$$\sum_{1 \le k \le 2^n} [\varphi(d_k) - \varphi(c_k)] = 1.$$

There is no response to the $\varepsilon = 1$ challenge regarding the criterion for φ to be absolutely continuous. □

Clearly linear combinations of absolutely continuous functions are absolutely continuous. However, the composition of absolutely continuous functions may fail to be absolutely continuous.

Proposition 6.6.3. *If the function f is Lipschitz on a closed, bounded interval $[a, b]$, then it is absolutely continuous on $[a, b]$.*

Proof. Let $c > 0$ be a Lipschitz constant for f on $[a, b]$, that is,

$$|f(u) - f(v)| \leq c\,|u - v| \quad \text{for all} \quad u, \ v \ \in [a, \ b].$$

Then, regarding the criterion for the absolute continuity of f, it is clear that $\delta = \varepsilon/c$ responds to any $\varepsilon > 0$ challenge. □

There are absolutely continuous functions that fail to be Lipschitz: the function f on $[0, 1]$, defined by

$$f(x) = \sqrt{x} \ \text{ for } \ 0 \leq x \leq 1,$$

is absolutely continuous but not Lipschitz.

Theorem 6.6.4. *Let the Junction f be absolutely continuous on the closed, bounded interval $[a, b]$. Then f is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation.*

Proof. We first prove that f is of bounded variation. Indeed, let $\delta > 0$ respond to the $\varepsilon = 1$ challenge regarding the criterion for the absolute continuity of f. Let P be a partition of $[a, b]$

into N closed intervals $\{[c_k,\ d_k]\}_{k=1}^N$, each of length less than δ. Then, by the definition of δ in relation to the absolute continuity of f, it is clear that

$$TV(f_{[c_k,\ d_k]}) \leq 1 \ \text{ for } \ 1 \leq k \leq n.$$

The additivity formula (6.25) extends to finite sums. Hence

$$TV(f) = \sum_{k=1}^{N} TV(f_{[c_k,\ d_k]}) \leq N.$$

Therefore f is of bounded variation. In view of (6.29) and the absolute continuity of sums of absolutely continuous functions, to show that f is the difference of increasing absolutely continuous functions it suffices to show that the total variation function for f is absolutely continuous. Let $\varepsilon > 0$. Choose δ as a response to the $\varepsilon/2$ challenge regarding the criterion for the absolute continuity of f on $[a,\ b]$. Let $\{(c_k,\ d_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of $(a,\ b)$ for which $\sum_{k=1}^{n} [d_k - c_k] < \delta$. For $1 \leq k \leq n$, let P_k be a partition of $[c_k,\ d_k]$. By the choice of δ in relation to the absolute continuity of f on $[a, b]$,

$$\sum_{k=1}^{n} TV(f_{[c_k,\ d_k]},\ P_k) < \frac{\varepsilon}{2}.$$

Take the supremum as, for $1 \leq k \leq n$, P_k vary among partitions of $[c_k, d_k]$, to obtain

$$\sum_{k=1}^{n} TV(f_{[c_k,\ d_k]}) \leq \frac{\varepsilon}{2} < \varepsilon.$$

We infer from (6.27) that, for $1 \leq k \leq n$,

$$TV(f_{[c_k,\ d_k]}) = TV(f_{[a,\ d_k]}) - TV(f_{[a,\ c_k]}).$$

Hence

$$\text{if} \quad \sum_{k=1}^{n} [d_k - c_k] < \delta, \quad \text{then} \quad \sum_{k=1}^{n} \left| TV(f_{[a,\ d_k]}) - TV(f_{[a,\ c_k]}) \right| < \varepsilon. \tag{6.31}$$

Therefore the total variation function for f is absolutely continuous on $[a, b]$. □

Theorem 6.6.5. *Let the function f be continuous on the* *closed, bounded interval* $[a, b]$. *Then* f *is absolutely continuous on* $[a, b]$ *if and only if the family of divided difference functions* $\{\text{Diff}_h f\}_{0<h\leq 1}$ *is uniformly integrable over* $[a, b]$.

Proof. First assume $\{\text{Diff}_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$. Let $\varepsilon > 0$. Choose $\delta > 0$ for which

$$\int_E |\text{Diff}_h| < \frac{\varepsilon}{2} \quad \text{if} \quad m(E) < \delta \;\text{ and }\; 0 < h \leq 1.$$

We claim that δ responds to the ε challenge regarding the criterion for f to be absolutely continuous. Indeed, let $\{(c_k,\ d_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of (a, b) for which $\sum_{k=1}^n [d_k - c_k] < \delta$. For $0 < h \leq 1$ and $1 \leq k \leq n$, by (6.18),

$$\text{Av}_h f(d_k) - \text{Av}_h f(c_k) = \int_{c_k}^{d_k} \text{Diff}_h f.$$

Therefore

$$\sum_{k=1}^n |\text{Av}_h f(d_k) - \text{Av}_h f(c_k)| \leq \sum_{k=1}^n \int_{c_k}^{d_k} |\text{Diff}_h f| = \int_E |\text{Diff}_h f|,$$

where $E = \bigcup_{k=1}^{n} (c_k,\ d_k)$ has measure less than δ. Thus, by the choice of δ,

$$\sum_{k=1}^{n} |\mathrm{Av}_h f(d_k) - \mathrm{Av}_h f(c_k)| < \frac{\varepsilon}{2} \quad \text{for all } 0 < h \leq 1.$$

Since f is continuous, take the limit as $h \to 0^+$ to obtain

$$\sum_{k=1}^{n} |f(d_k) - f(c_k)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence f is absolutely continuous.

To prove the converse, suppose f is absolutely continuous. The preceding theorem tells us that f is the difference of increasing absolutely continuous functions. We may therefore assume that f is increasing, so that the divided difference functions are nonnegative. To verify the uniformly integrability of $\{\mathrm{Diff}_h f\}_{0<h\leq 1}$, let $\varepsilon > 0$. We must show that there is a $\delta > 0$ such that for each measurable subset E of $(a,\ b)$,

$$\int_E \mathrm{Diff}_h f < \varepsilon \ \text{ if } \ m(E) < \delta \ \text{ and } \ 0 < h \leq 1. \tag{6.32}$$

According to Theorem 2.4.2 in Page 75, a measurable set E is contained in a G_δ set G for which $m(G \sim E) = 0$. But every G_δ set is the intersection of a descending sequence of open sets. Moreover, every open set is the disjoint union of a countable collection of open intervals, and therefore every open set is the union of an ascending sequence of open sets, each of which is the union of a finite disjoint collection of open intervals. Therefore, by the continuity of integration, to verify (6.32) it suffices to find a $\delta > 0$ such that for $\{(c_k,\ d_k)\}_{k=1}^{n}$ a disjoint collection of open subintervals of $(a,\, b)$,

$$\int_E \mathrm{Diff}_h \ < \ \frac{\varepsilon}{2} \ \text{ if } \ m(E) < \delta, \qquad (6.33)$$

$$\text{where } E = \bigcup_{k=1}^{n} (c_k,\ \ d_k), \ \text{ and } \ 0 < h \le 1.$$

Choose $\delta > 0$ as the response to the $\varepsilon/2$ challenge regarding the criterion for the absolute continuity of f on $[a,\, b+1]$. By a change of variables for the Riemann integral and cancellation,

$$\int_u^v \mathrm{Diff}_h f \quad = \quad \frac{1}{h}\int_0^h g(t)dt,$$

$$\text{where } g(t) = f(v+t) - f(u+t)$$

$$\text{for } \ 0 \le t \le 1 \text{ and } \ a \le u < v \le b.$$

Therefore, if $\{(c_k,\ d_k)\}_{k=1}^n$ is a disjoint collection of open subintervals of $(a,\ b)$,

$$\int_E \mathrm{Diff}_h = \frac{1}{h}\int_0^h g(t)dt,$$

where

$$E = \bigcup\nolimits_{k=1}^n (c_k,\ d_k)$$

and

$$g(t) = \sum_{k=1}^n [f(d_k+t) - f(c_k+t)] \text{ for all } \ 0 \le t \le 1.$$

If $\sum_{k=1}^{n} [d_k - c_k] < \delta$, then, for $0 \leq t \leq 1$,

$$\sum_{k=1}^{n} [(d_k + t) - (c_k + t)] < \delta,$$

and therefore $g(t) < \frac{\varepsilon}{2}$. Thus

$$\int_E \mathrm{Diff}_h = \frac{1}{h} \int_0^h g(t)dt < \frac{\varepsilon}{2}.$$

Hence (6.33) is verified for this choice of δ. □

Remark 6.6.6. For a nondegenerate closed, bounded interval $[a, b]$, let $\mathcal{F}_{Lip}$, $\mathcal{F}_{AC}$, and $\mathcal{F}_{BV}$ denote the families of functions on $[a, b]$ that are Lipschitz, absolutely continuous, and of bounded variation, respectively. We have the following strict inclusions:

$$\mathcal{F}_{Lip} \subseteq \mathcal{F}_{AC} \subseteq \mathcal{F}_{BV}$$

Proposition 6.6.3 tells us of the first inclusion, and the second inclusion was established in Theorem. Each of these collections is closed with respect to the formation of linear combinations. Moreover a function in one of these collections has its total variation function in the same collection. Therefore, by (6.29), a function in one of these collections may be expressed as the difference of two increasing functions in the same collection.

6.7 Integrating Derivatives: Differentiating Indefinite Integrals

Let f be a continuous function on the closed, bounded interval $[a, b]$. In (6.18), take $a = u$ and $b = v$ to arrive at the following discrete formulation of the fundamental theorem of integral calculus:

$$\int_a^b \mathrm{Diff}_h f = \mathrm{Av}_h f(b) - \mathrm{Av}_h f(a).$$

Since f is continuous, the limit of the right-hand side as $h \to 0^+$ equals $f(b) - f(a)$. We now show that if f is absolutely continuous, then the limit of the left-hand side as $h \to 0^+$ equals $\int_a^b f'$ and thereby establish the fundamental theorem of integral calculus for the Lebesgue integral.

Theorem 6.7.1. *Let the function f be absolutely continuous on the closed, bounded interval $[a, b]$. Then f is differentiable almost everywhere on (a, b), its derivative f' is integrable over $[a, b]$, and*

$$\int_a^b f' = f(b) - f(a). \tag{6.34}$$

Proof. We infer from the discrete formulation of the fundamental theorem of integral calculus that

$$\lim_{n\to\infty} \left[\int_a^b \mathrm{Diff}_{1/n} f \right] = f(b) - f(a). \tag{6.35}$$

Theorem 6.6.4 tells us that f is the difference of increasing functions on $[a, b]$ and therefore, by Lebesgue's Theorem, is differentiable almost everywhere on (a, b). Therefore $\{\mathrm{Diff}_{1/n} f\}$ converges pointwise almost everywhere on (a, b) to f'. On the other hand, according to Theorem 6.6.5, $\{\mathrm{Diff}_{1/n} f\}$ is uniformly integrable over $[a, b]$. The Vitali Convergence Theorem 4.6.7 in Page 227 permits passage of the limit under the integral sign in order to conclude that

$$\lim_{n\to\infty}\left[\int_a^b \mathrm{Diff}_{1/n} f\right] = \int_a^b \lim_{n\to\infty} \mathrm{Diff}_{1/n} f = \int_a^b f'. \tag{6.36}$$

Formula (6.34) follows from (6.35) and (6.36). □

In the study of calculus, indefinite integrals are defined with respect to the Riemann integral. We here call a function f on a closed, bounded interval $[a, b]$ the **indefinite integral** of g over $[a, b]$ provided g is Lebesgue integrable over $[a, b]$ and

$$f(x) = f(a) + \int_a^x g \quad \text{for all } x \in [a, b]. \tag{6.37}$$

Theorem 6.7.2. *A function f on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.*

Proof. First suppose f is absolutely continuous on $[a, b]$. For each $x \in (a, b]$, f is absolutely continuous over $[a, x]$ and hence, by the preceding theorem, in the case $[a, b]$ is replaced by $[a,\ x]$,

$$f(x) = f(a) + \int_a^x f'.$$

Thus f is the indefinite integral of f' over $[a,\ b]$.

Conversely, suppose that f is the indefinite integral over $[a,\ b]$ of g. For a disjoint collection $\{(a_k,\ b_k)\}_{k=1}^n$ of open intervals in $(a,\ b)$, if we define

$$E = \bigcup_{k=1}^n (a_k,\ b_k),$$

then, by the monotonicity and additivity over domains properties of the integral,

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^n \left| \int_{a_k}^{b_k} g \right| \le \sum_{k=1}^n \int_{a_k}^{b_k} |g| = \int_E |g|. \quad (6.38)$$

Let $\varepsilon > 0$. Since $|g|$ is integrable over $[a,\ b]$, according to

Proposition 4.6.2 in Page 221, there is a $\delta > 0$ such that

$$\int_E |g| < \varepsilon \text{ if } E \subseteq [a,\ b] \text{ is measurable and } m(E) < \delta.$$

It follows from (6.38) that this same δ responds to the ε challenge regarding the criterion for f to be absolutely continuous $[a, b]$. □

Corollary 6.7.3. *Let the function f be monotone on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if*

$$\int_a^b f' = f(b) - f(a). \tag{6.39}$$

Proof. Theorem 6.7.1 is the assertion that (6.39) holds if f is absolutely continuous, irrespective of any monotonicity assumption. Conversely, assume f is increasing and (6.39) holds. Let x belong to $[a, b]$. By the additivity over domains of integration,

$$\begin{aligned} 0 &= \int_a^b f' - [f(b) - f(a)] \\ &= \left\{ \int_a^x f' - [f(x) - f(a)] \right\} + \left\{ \int_x^b f' - [f(b) - f(x)] \right\}. \end{aligned}$$

According to Corollary 6.4.8 in Page 6.4.8,

$$\int_a^x f' - [f(x) - f(a)] \leq 0$$

and

$$\int_x^b f' - [f(b) - f(x)] \leq 0.$$

If the sum of two nonnegative numbers is zero, then they both are zero. Therefore

$$f(x) = f(a) + \int_a^x f'.$$

Thus f is the indefinite integral of f'. The preceding theorem tells us that f is absolutely continuous. □

Lemma 6.7.4. *Let f be integrable over the closed, bounded interval $[a, b]$. Then*

$$f(x) = 0 \text{ for almost all } x \in [a,\ b] \tag{6.40}$$

if and only if

$$\int_{x_1}^{x_2} f = 0 \text{ for all } (x_1,\ x_2) \subseteq [a,\ b]. \tag{6.41}$$

Proof. Clearly (6.40) implies (6.41). Conversely, suppose (6.41) holds. We claim that

$$\int_E f = 0 \text{ for all measurable sets } E \subseteq [a,\ b]. \tag{6.42}$$

Indeed, (6.42) holds for all open sets contained in (a, b) since integration is countably additive and every open set is the union of countable disjoint collection of open intervals. The continuity of integration then tells us that (6.42) also holds for all G_δ sets contained in $(a,\ b)$ since every such set is the intersection of a countable descending collection of open sets. But every measurable subset of $[a,\ b]$ is of the form $G \sim E_0$, where G is a G_δ subset of $(a,\ b)$ and $m(E_0) = 0$ (Theorem 2.4.2 in Page 75). We conclude from the additivity over domains of integration that (6.42) is verified. Define

$$E^+ = \{x \in [a,\ b] \mid f(x) \geq 0\}$$

and

$$E^- = \{x \in [a,\ b] \mid f(x) \leq 0\}.$$

These are two measurable subsets of $[a,\ b]$ and therefore, by (6.42),

$$\int_a^b f^+ = \int_{E^+} f = 0$$

and

$$\int_a^b (-f^-) = -\int_{E^-} f = 0.$$

According to Proposition 4.3.3 in Page 189, a nonnegative integrable function with zero integral must vanish almost everywhere on its domain. Thus f^+ and f^- vanish almost everywhere on $[a,\ b]$ and hence so does f. $\square$

Theorem 6.7.5. *Let f be integrable over the closed, bounded interval $[a,\ b]$. Then*

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x) \text{ for almost all } x \in (a,\ b). \tag{6.43}$$

Proof. Define the function F on $[a,\ b]$ by

$$F(x) = \int_a^x f \text{ for all } x \in [a,\ b].$$

Theorem 6.8.7 in Page 331 tells us that since F is an indefinite integral, it is absolutely continuous. Therefore, by Theorem 6.7.1, F is differentiable almost everywhere on $(a,\ b)$ and its derivative F is integrable. According to the preceding lemma, to show that the integrable function $F' - f$ vanishes almost everywhere on $[a,\ b]$ it suffices to show that its integral over every closed subinterval of $[a,\ b]$ is zero. Let the closed interval $[x_1,\ x_2]$ be contained in $[a,\ b]$. According to Theorem 6.7.1, in the case $[a, b]$ is replaced by $[x_1, x_2]$, and the linearity and additivity over domains properties of integration,

$$\begin{aligned}\int_{x_1}^{x_2} [F' - f] &= \int_{x_1}^{x_2} F' - \int_{x_1}^{x_2} f \\ &= F(x_2) - F(x_1) - \int_{x_1}^{x_2} f \\ &= \int_0^{x_2} f - \int_0^{x_1} f - \int_{x_1}^{x_2} f \\ &= 0.\end{aligned}$$

□

Definition 6.7.6. A function of bounded variation is said to be **singular** provided its derivative vanishes almost everywhere. The Cantor-Lebesgue function is a non-constant singular function.

We infer from Theorem 6.7.1 that an absolutely continuous function is singular if and only if it is constant. Let f be of bounded variation on $[a, b]$. According to Corollary 6.5.8, f'

is integrable over $[a, b]$. Define

$$g(x) = \int_a^x f' \text{ for all } x \in [a,\ b],$$

and

$$h(x) = f(x) - \int_a^x f' \text{ for all } x \in [a,\ b],$$

so that

$$f = g + h \text{ on } [a,\ b].$$

According to Theorem 6.7.2, the function g is absolutely continuous. We infer from Theorem 6.7.5 that the function h is singular. The above decomposition of a function of bounded variation f as the sum $g+h$ of two functions of bounded variation, where g is absolutely continuous and h is singular, is called a **Lebesgue decomposition** of f.

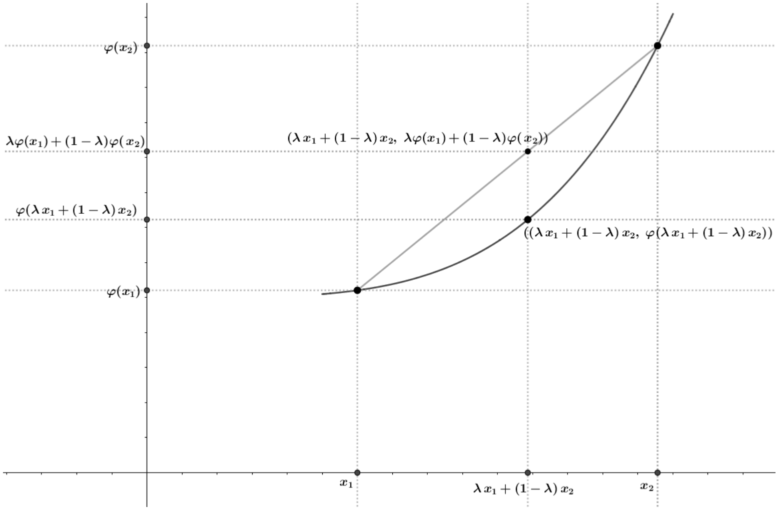

Figure 6.11: Graph of a convex function. For each pair of points x_1, x_2 in (a, b) and each λ with $0 \leq \lambda \leq 1$, $\varphi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$. The point $((\lambda\, x_1 + (1-\lambda)\, x_2,\ \varphi(\lambda\, x_1 + (1-\lambda)\, x_2))$ lies on the graph while $(\lambda\, x_1 + (1-\lambda)\, x_2,\ \lambda\varphi(x_1) + (1-\lambda)\varphi(\, x_2))$ lies on the line segment with end points $(x_1, \varphi(x_1))$ and $(x_2, \varphi(x_2))$.

6.8 Convex Functions

Throughout this section (a, b) is an open interval that may be bounded or unbounded.

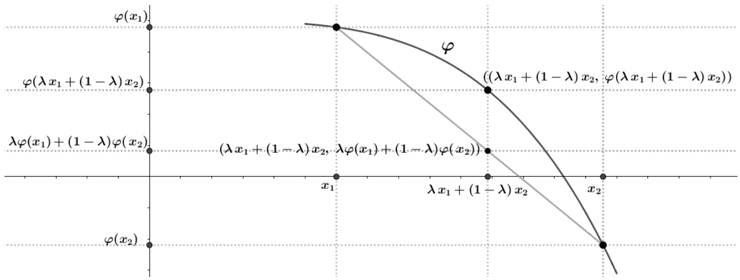

Figure 6.12: Curve in the figure is *not* the graph of a convex function. There are points x_1, x_2 in (a, b) and λ with $0 \le \lambda \le 1$, $\varphi(\lambda x_1 + (1-\lambda)x_2) > \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$.

Definition 6.8.1. A real-valued function φ on (a, b) is said to be **convex** provided for each pair of points x_1, x_2 in (a, b) and each λ with $0 \le \lambda \le 1$,

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \le \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2). \tag{6.44}$$

If we look at the graph of φ (Fig. 6.11), the convexity inequality can be formulated geometrically by saying that each point on the chord between $(x_1, \varphi(x_1))$ and $(x_2, \varphi(x_2))$ is above the graph of φ.

Observe that for two points $x_1 < x_2$ in (a, b), each point

x in $(x_1,\ x_2)$ may be expressed as

$$x = \lambda x_1 + (1-\lambda)x_2 \quad \text{where} \quad \lambda = \frac{x_2 - x}{x_2 - x_1}.$$

Thus the convexity inequality may be written as

$$\varphi(x) \le \left[\frac{x_2 - x}{x_2 - x_1}\right]\varphi(x_1) + \left[\frac{x - x_1}{x_2 - x_1}\right]\varphi(x_2)$$
$$\text{for} \quad x_1 < x < x_2 \text{ in } (,\ b).$$

Regathering terms, this inequality may also be rewritten as

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \le \frac{\varphi(x_2) - \varphi(x)}{x_2 - x} \quad \text{for } x_1 < x < x_2 \text{in } (a,\ b). \tag{6.45}$$

Therefore convexity may also be formulated geometrically by saying that for $x_1 < x < x_2$, the slope of the chord from $(x_1,\ \varphi(x_1))$ to $(x,\ \varphi(x))$ is no greater than the slope of the chord from $(x,\ \varphi(x))$ to $(x_2,\ \varphi(x_2))$.

Proposition 6.8.2. *If φ is differentiable on (a, b) and its derivative φ' is increasing, then φ is convex. In particular, φ is convex if it has a nonnegative second derivative φ'' on*

$(a,\, b)$.

Proof. Let $x_1,\, x_2$ be in $(a,\, b)$ with $x_1 < x_2$, and let x belong to $(x_1,\, x_2)$. We must show that

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

However, apply the Mean Value Theorem to the restriction of φ to each of the intervals $[x_1,\, x]$ and $[x,\, x_2]$ to choose points $c_1 \in (x_1,\, x)$ and $c_2 \in (x,\, x_2)$ for which

$$\varphi'(c_1) = \frac{\varphi(x) - \varphi(x_1)}{x - x_1} \quad \text{and } \varphi'(c_2) = \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

Thus, since φ' is increasing, ($c_1 < c_2$ implies $\varphi'(c_1) \leq \varphi'(c_2)$ and hence)

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} = \varphi'(c_1) \leq \varphi'(c_2) = \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

□

Example 6.8.3. Each of the following three functions is convex since each has a nonnegative second derivative:

(i) $\varphi_1(x) = x^p$ on $(0,\ \infty)$ for $p \geq 1$;

(ii) $\varphi_2(x) = e^{ax}$ on $(-\infty,\ \infty)$;

(iii) $\varphi_3(x) = \ln(1/x)$ on $(0,\ \infty)$.

The following final geometric reformulation of convexity will be useful in the establishment of differentiability properties of convex functions.

Lemma 6.8.4. The Chordal Slope Lemma *let φ be convex on $(a,\, b)$. If $x_1 < x < x_2$ belong to $(a,\, b)$, then for $p_1 = (x_1,\, \varphi(x_1))$, $p = (x,\, \varphi(x))$, $p_2 = (x_2,\, \varphi(x_1))$ (Fig. 6.13),*

$$\text{slope of } \overline{p_1 p} \ \leq \ \text{slope of } \overline{p_1 p_2} \ \leq \ \text{slope of } \overline{p p_2}$$

Proof. Regather terms in the inequality (6.45) to rewrite it in the following two equivalent forms:

$$\frac{\varphi(x_1) - \varphi(x)}{x_1 - x} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \quad \text{for } x_1 < x < x_2 \text{ in } (a,\ b);$$

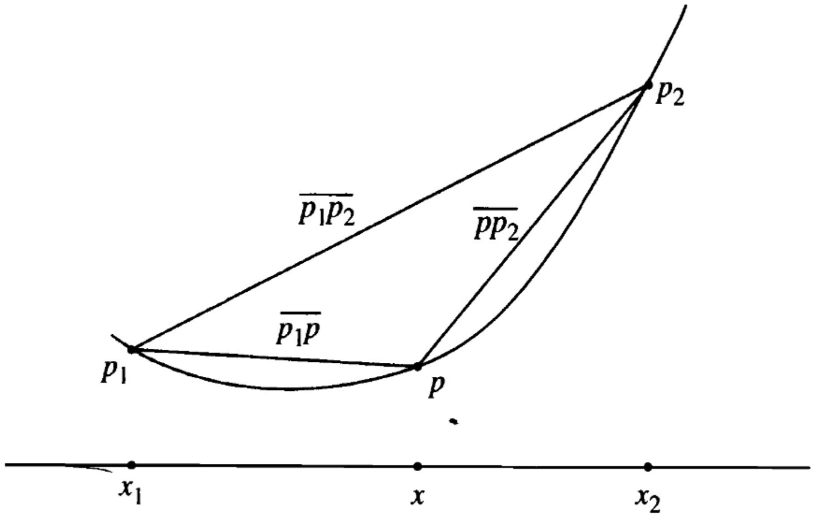

Figure 6.13: slope of $\overline{p_1p} \leq$ slope of $\overline{p_1p_2} \leq$ slope of $\overline{pp_2}$

and

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} \le \frac{\varphi(x_2)-\varphi(x)}{x_2-x} \quad \text{for } x_1 < x < x_2 \text{ in } (a,\ b).$$

□

For a function g on an open interval (a, b), and point $x_0 \in (a, b)$, if

$$\lim_{h\to 0,\ h<0} \frac{g(x_0+h)-g(x_0)}{h} \text{ exists and is finite,}$$

we denote this limit by $g'(x_0^-)$ and call it the **left-hand derivative of g at x_0**. Similarly, we define **right-hand derivative of g at x_0**.

$$g'(x_0^+) = \lim_{h\to 0,\ h>0} \frac{g(x_0+h)-g(x_0)}{h}$$

provided the limit on the right hand side exists. Of course, g is differentiable at x_0 if and only if it has left-hand and right-hand derivatives at x_0 that are equal. The continuity and differentiability properties of convex functions follow from the following lemma, whose proof follows directly from the Chordal Slope

Lemma.

Lemma 6.8.5. *Let φ be a convex function on (a, b). Then φ has left-hand and right-hand derivatives at each point $x \in (a, b)$. Moreover, for points u, v in (a, b) with $u < v$, these one-sided derivatives satisfy the following inequality:*

$$\varphi'(u^-) \le \varphi'(u^+) \le \frac{\varphi(v) - \varphi(u)}{v - u} \le \varphi'(v^-) \le \varphi'(v^+). \quad (6.46)$$

Corollary 6.8.6. *Let φ be a convex function on (a, b). Then φ is Lipschitz, and therefore absolutely continuous, on each closed, bounded subinterval $[c, d]$ of (a, b).*

Proof. According to the preceding lemma, for $c \le u < v \le d$,

$$\varphi'(c^+) \le \varphi'(u^+) \le \frac{\varphi(v) - \varphi(u)}{v - u} \le \varphi'(v^-) \le \varphi'(d^-) \quad (6.47)$$

and therefore

$$|\varphi(u) - \varphi(v)| \le M\,|u - v| \ \text{ for all } u,\ v \in [c,\ d],$$

where

$$M = \max\{\left|\varphi'(c^+)\right|, \ \left|\varphi'(d^-)\right|\}.$$

Thus the restriction of φ to $[u, v]$ is Lipschitz. A Lipschitz function on a closed, bounded interval is absolutely continuous. □

We infer from the above corollary and Corollary 6.5.8 in Page 301 that any convex function defined on an open interval is differentiable almost everywhere on its domain. In fact, much more can be said.

Theorem 6.8.7. *Let φ be a convex function on (a, b). Then φ is differentiable except at a countable number of points and its derivative φ' is an increasing function.*

Proof. □

We infer from the inequalities (6.46) that the functions

$$x \mapsto f'(x^-) \text{ and } x \mapsto f'(x^+)$$

are increasing real-valued functions on (a, b). But, according to Theorem 6.3.2 in 271, an increasing real-valued function is continuous except at a countable number of points. Thus, except on a countable subset $\mathcal{C}$ of (a, b), both the left-hand and right-hand derivatives of φ are continuous. Let x_0 belong to $(a, b) \sim \mathcal{C}$. Choose a sequence $\{x_n\}$ of points greater than x_0 that converges to x_0. Apply Lemma 6.8.5, with $x_0 = u$ and $x_n = v$, and take limits lo conclude that

$$\varphi'(x_0^-) \leq \varphi'(x_0^+) \leq \varphi'(x_0^-).$$

Then

$$\varphi'(x_0^-) = \varphi'(x_0^+)$$

so that φ is differentiable at x_0. To show that φ' is an increasing function on $(a, b) \sim \mathcal{C}$, let u, v belong to $(a, b) \sim \mathcal{C}$ with $u < v$. Then by Lemma 6.8.5,

$$\varphi'(u) \leq \frac{\varphi(u) - \varphi(v)}{u - v} \leq \varphi'(v).$$

Let φ be a convex function on (a, b) and x_0 belong to (a, b). For a real number m, the line $y = m(x - x_0) + \varphi(x_0)$, which passes through the point $(x_0, \varphi(x_0))$, is called a **supporting line** at x_0 for the graph of φ provided this line always lies below the graph of φ, that is, if

$$\varphi(x) \geq m(x - x_0) + \varphi(x_0) \text{ for all } x \in (a,\ b).$$

It follows from Lemma 6.8.5 that such a line is supporting if and only if its slope m lies between the left- and right-hand derivatives of φ at x_0. Thus, in particular, there is always at least one supporting line at each point. This notion enables us to give a short proof of the following inequality:

Theorem 6.8.8. ***[Jensen's Inequality]*** *Let φ be a convex function on $(-\infty, \infty)$, f an integrable function over $[0,\ 1]$, and $\varphi \circ f$ also integrable over $[0,\ 1]$. Then*

$$\varphi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 (\varphi \circ f)(x)dx. \tag{6.48}$$

Proof. Define $\alpha = \int_0^1 f(x)dx$. Choose m to lie between the left-hand and right-hand derivative of φ at the point α. Then

$$y = m(t - \alpha) + \varphi(\alpha)$$

is the equation of a supporting line at $(\alpha, \varphi(\alpha))$ for the graph of φ. Hence

$$\varphi(t) \geq m(t - \alpha) + \varphi(\alpha) \text{ for all } t \in \mathbb{R}\,.$$

Since f is integrable over $[0, 1]$, it is finite a.e.on $[0, 1]$ and therefore, substituting $f(x)$ for t in this inequality, we have

$$\varphi(f(x)) \geq m(f(x) - \alpha) + \varphi(\alpha) \text{ for almost all } x \in [0,\ 1]\,.$$

Integrate across this inequality, using the monotonicity of the Lebesgue integral and the assumption that both f and $\varphi \circ f$ are integrable over $[a, b]$, to obtain

$$\begin{aligned}
\int_0^1 \varphi(f(x))dx &\geq \int_0^1 [m(f(x) - \alpha) + \varphi(\alpha)]dx \\
&= m\left[\int_0^1 f(x)dx - \alpha\right] + \varphi(\alpha) \\
&= \varphi(\alpha).
\end{aligned}$$

□

A few words regarding the assumption, for Jensen's Inequality, of the integrability of $\varphi \circ f$ over $[0, 1]$ are in order. We have shown that a convex function is continuous and therefore Proposition 3.1.11 in Page 129 tells us that the composition $\varphi \circ f$ is measurable if φ is convex and f is integrable. If $\varphi \circ f$ is nonnegative, then it is unnecessary to assume the $\varphi \circ f$ is integrable since equality (6.48) trivially holds if the right-hand integral equals $+\infty$. In the case $\varphi \circ f$ fails to be nonnegative, if there are constants c_1 and c_2 for which

$$|\varphi(x)| \leq c_1 + c_2 |x| \quad \text{for all } x \in \mathbb{R}\,, \tag{6.49}$$

then we infer from the integral comparison test that $\varphi \circ f$ is integrable over $[0, 1]$ if f is. In the absence of the growth assumption (6.49), the function $\varphi \circ f$ may not be integrable over $[0, 1]$.

Chapter 7

L^P spaces

7.1 Introduction

We have already seen the notion of completeness in the case of real numbers $\mathbb{R}$, given a sequence of real numbers a_n, such that $\lim_{n,m\to\infty} |a_n - a_m| = 0$, then there exits a real number a such that $\lim_{n\to\infty} |a_n - a| = 0$. In this chapter we will see a corresponding completeness for Lebesgue integrable function.

7.2 Normed Linear space

Let E be a measurable set of real numbers. Define $\mathcal{F}$ to be the set of measurable extended real valued function of E, that is finite $a.e$ on E. Define an equivalence relation ' $\approx$ ' on $\mathcal{F}$ by $f \approx g$ if

$$f(x) = g(x), \quad for\ almost\ all\ x \in E$$

Check that ' $\approx$ ' indeed forms an equivalence relation on $\mathcal{F}$. Let $\mathcal{F}/\approx$ denote the collection of all equivalence classes of $\mathcal{F}$. $\mathcal{F}/\approx$ has a natural linear structure:given two functions f and g in $\mathcal{F}$, their equivalence classes $[f]$ and $[g]$ and real numbers α and β, we define the linear combination $\alpha[f] + \beta[g]$ to be the equivalence class of the functions in $\mathcal{F}$ that take the value $\alpha f(x)+\beta g(x)$ at points x in E at which both f and g are finite.

Definition 7.2.1. For $0 < p \leq \infty$, define $L^p(E)$ **to be the collection of equivalence classes** $[f]$ **of functions such that**

$$\int_E |f|^p < \infty.$$

The above property is well defined because if $f \approx g$, then $\int_E |f|^p = \int_E |g|^p$. Notice that, for any real number a, b, $|a+b|^p \le 2^p(|a^p + |b|^p)$. This fact along with the linearity and monotonicity of integration gives us that if $[f]$ and $[g]$ beongs to $L^p(E)$, then so does their linear combination $\alpha[f] + \beta[\beta]$. Therefore, $L^p(E)$ is a linear space.

We call a function $f \in \mathcal{F}$ **essentially bounded** if there is some $M \ge 0$, called an essential upper bound for f, for which

$$|f(x)| \le M \ for \ almost \ all \ x \in E$$

We define $L^\infty(E)$ to be the collection of equivalence classes $[f]$ for which f is essentially bounded. Check that $L^\infty(E)$ also forms a linear space.

Definition 7.2.2. Norm: Let X be a linear space. A real-valued functional ‘ $\| \|$ ’ on X is called a **norm** if for each f and g in X and each real number a, it satisfies the following properties:

1. $\|f + g\| \le \|f\| + \|g\|$ (The Triangle Inequality)
2. $\|af\| = |a|\|f\|$ (Positive Homogeneity)

3. $\|f\| \geq 0$ and $\|f\| = 0$ if anf only if $f = 0$ (Nonnegativity).

Definition 7.2.3. A **normed Linear space** is a linear space together with a norm. If X is a linear space normed by $\|\ \|$, we say that a function in X is a unit function provided $\|f\| = 1$. For any $f \in X$ and $f \neq 0$, the function $f/\|f\|$ is a unit function and it is a scalar multiple of f which we call the **normalization of f.**

Example 7.2.4. Normed Linear space $L^1(E)$
For $f \in L^1(E)$, define

$$\|f\|_1 = \int_E |f|$$

For $f, g \in L^1(E)$ and real number a,

1.

$$\begin{aligned} \|f+g\|_1 &= \int_E |f+g| \\ &\leq \int_E |f| + |g| \\ &= \int_E |f| + \int_E |g| \\ &= \|f\|_1 + \|g\|_1. \end{aligned}$$

2. $\|af\|_1 = |a|\|f\|_1$

3. $\|f\|_1 = 0$ if and only if $f = 0$ *a.e.*

Therefore,

$$\| \ \|_1 \text{ is a norm on } L^1(E).$$

Example 7.2.5. Normed Linear space $L^\infty(E)$ For $f \in L^\infty(E)$, define $\|f\|_\infty$ as the infimum of the essential upper bounds of f(we will call this the **essential supremum**)and we claim that $'\|\|'_\infty$ is a norm on $L^\infty(E)$. We will first show that $\|f\|_\infty$ is an essential upper bound of f on E i.e

$$|f| \leq \|f\|_\infty, \quad a.e \ on \ E$$

For each natural number n, there is subset E_n of E such that

$$|f| \leq \|f\|_\infty + \frac{1}{n} \ on \ E \sim E_n \ and \ m(E_m) = 0$$

Now define $E_\infty = \bigcup_{n=1}^\infty E_n$. Then, we have

$$|f| \leq \|f\|_\infty \ on \ E \sim E_\infty \ and \ m(E_\infty) = 0$$

Thus, proving that $\|f\|_\infty$ is a essential upper bound of f. $\| \ \|$ clearly satisfies positive homogeneity and non negativity. Now to show if satisfies triangle inequality. For $f, g \in L^\infty(E)$,

$$\begin{aligned} |f(x) + g(x)| &\leq |f(x)| + |g(x)| \\ &\leq \|f\|_\infty + \|g\|_\infty \ for \ almost \ all \ x \in E. \end{aligned}$$

Thus, $\|f\|_\infty + \|g\|_\infty$ is a essential upper bound of $f+g$. Therefore,

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

(since $\|f + g\|_\infty$ is the infimum of the essential upper bound of $f + g$.)

Example 7.2.6. The Normed Linear Spaces l_p and l_∞
We define l_p to be the collection of real sequences $a = (a_1, a_2, ...)$ such that

$$\sum_{k=1}^{\infty} |a_k|^p < \infty$$

Since for real number a, b, $|a + b|^p \leq 2^p(|a^p + |b|^p)$, the sum of two sequences in l_p also belongs to l_p. Also real multiple of a sequence in l_p belongs to l_p. Thus l_p is a linear space. For

$\{a_k\} \in l_p$ we define

$$\|\{a_k\}\|_p = \sum_{k=1}^{\infty} |a_k|^p$$

It is trivial to check that $\| \ \|$ is a norm on l_p.
We define l_∞ to be the linear space of real bounded sequences and for $\{a_k\} \in l_\infty$, define

$$\|\{a_k\}\|_\infty = \sup\{a_k\}$$

Clearly, $\| \ \|_\infty$ is a norm on l_∞.

Example 7.2.7. The Normed Linear Space $C[a, b]$
Let $[a, b]$ be a closed, bounded interval.Then the linear space of continuous real-valued functions on $[a, b]$ is denoted by $C[a, b]$. We define

$$\|f\|_{\max} = \max_{x \in [a,b]} |f(x)|$$

Clearly, $\| \ \|_{max}$ is satisfies positive homogeneity and non negativity. For $f, g \in C[a, b]$

$$\begin{aligned} \|f+g\|_{\max} &= \max_{x\in[a,b]} |f(x)+g(x)| \\ &\le \max_{x\in[a,d]} (|f(x)|+|g(x)|) \\ &= \max_{x\in[a,d]} |f(x)| + \max_{x\in[a,d]} |g(x)| \\ &= \|f\|_{\max} + \|g\|_{\max} \end{aligned}$$

Thus, $\| \, \|_{\max}$ satisfies traingle inequality. Therefore, $\| \, \|_{\max}$ is a norm on $C[a, b]$.

7.3 The inequalities of Young, Holder, and Minkowski

We have already seen the linear space $L^p(E)$ for $1 \le p \le \infty$ for a mesurable set E of real numbers. We have defined the norm in the case of $p = 1$ and $p = \infty$. In this section, we will define the norm for $1 < p < \infty$.

Definition 7.3.1. For E a measurable set, $1 < p < \infty$, and a function f in $L^p(E)$, define

$$\|f\|_p = [\int_E |f|^p]^{\frac{1}{p}}$$

We will show that $\|f\|_p$ is norm on $L^p(E)$. Clearly $\| \|_p$ satisfies positive homogeneity. If $\|f\|_p = 0$, then f vanishes $a.e$ in E. Thus, $[f]$ is a zero element in $L^p(E)$, which implies that $f = 0$. Therefore, non negativity is also satisfied. Now, it only remains to establish triangle inequality, which need some work.

Definition 7.3.2. The conjugate of a number $p \in (1, \infty)$ is the number $q = \frac{p}{p-1}$, which is the unique number $q \in (1, \infty)$ for which

$$\frac{1}{p} + \frac{1}{q} = 1$$

The conjugate of 1 is defined to be ∞ and the conjugate of ∞ defined to be 1.

Proposition 7.3.3. ***Young's Inequality*** *For $1 < p < \infty$, q the conjugate of p, and any two positive numbers a and b,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. Defined by $g(x) = (1/p)x^p + 1/q - x$ for $x > 0$. Observe that the derivative of $g(x)$ is positive for $x \in (1, \infty)$, negative for $x \in (0, 1)$ and $g(x)$ vanishes at $x = 1$. Therefore, $g(x)$ is non negative for $x \in (0, \infty)$.

$$(1/p)x^p + 1/q - x \geq 0 \quad for\ x \in (0, \infty)$$

$$\Longrightarrow (1/p)x^p + 1/q \geq x \quad for\ x \in (0, \infty)$$

Thus, we have

$$x \leq (1/p)x^p + 1/q > 0 \quad if\ x > 0$$

Put $x = \frac{a}{b^{q-1}}$. Then

$$\begin{aligned}
\frac{a}{b^{q-1}} &\leq \frac{1}{p}(\frac{a}{b^{q-1}})^p + \frac{1}{q} \\
a &\leq \frac{a^p}{p(b^{q-1})^{p-1}} + \frac{b^{q-1}}{q} \\
a &\leq \frac{a^p}{pb^{(q-1)(p-1)}} + \frac{b^{q-1}}{q} \\
a &\leq \frac{a^p}{pb} + \frac{b^{q-1}}{q} \quad \text{(using } \; p(q-1) = q) \\
ab &\leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{(multiplying by } \; b)
\end{aligned}$$

□

Theorem 7.3.4. *Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. If f belongs to $L^p(E)$ and g belongs to $L^q(E)$, then their product fg is integrable over E and*

Holder's Inequality $$\int_E |f \cdot g| \leq \|f\|_p \|g\|_q. \tag{7.1}$$

Moreover, if $f \neq 0$, the function f^* defined by

$$f^* = \|f\|_p^{1-p} \text{sgn}(f)|f|^{p-1} \tag{7.2}$$

is an element in $L^q(E)$.

$$\int_E ff^* = \|f\|_p \quad \text{and} \quad \|f^*\|_q = 1 \tag{7.3}$$

Proof. We will first consider the case when $p = 1$. Then (7.1) follows from the monotonicity of integration and the fact that $\|g\|_\infty$ is the essential upper bound of g in E. Since $f^* = \text{sgn}(f)$, (7.3) also follows trivially. Now consider the case when $p > 1$. Assume $f \neq 0$ and $g \neq 0$ as otherwise its trivial. Without loss of generality take $\|f\|_p = 1$ and $\|g\|_q = 1$(we can always replace f and g by their normalization). Thus, we have

$$\int_E |f|^p = 1 \quad \textit{and} \quad \int_E |g|^q = 1$$

Then the Holder's inequality becomes

$$\int_E |fg| \leq 1$$

Since f^p and g^q are integrable over E, f and g are finite *a.e* on E. Thus, by Young's inequality

$$|fg| = |f||g| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q} \quad \textit{a.e on } E$$

Thus, we get

$$\int_E |fg| \leq \frac{1}{p}\int_E |f|^p + \frac{1}{q}\int_E |g|^q = \frac{1}{p} + \frac{1}{q} = 1$$

Now, observe that

$$ff^* = \|f\|_p^{1-p}|f|^p$$

$$\implies \int_E ff^* = \|f\|_p^{1-p}\int_E |f|^p = \|f\|_p^{1-p}\|f\|_p = \|f\|_p$$

Also,

$$\begin{aligned}\int_E (|f^*|)^q &= \|f\|_p^{(1-p)q}\int_E |f|^p \ (using\ (p-1)q = p) \\ &= 1 \ (using\ \|f\|_p = 1)\end{aligned}$$

which proves (7.3). □

Remark 7.3.5. For $f \in L^p(E)$, f^* as defined in (7.2) is called the **conjugate function** of f.

Theorem 7.3.6. ***Minkowski's Theorem*** *Let E be a measurable set and $1 \leq p \leq \infty$. If the functions f and g belong* *to $L_p(E)$, then so does their sum $f + g$ and, moreover,*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. We have already seen the case when $p = 1$ and ∞. Therefore we will consider the case when $p \in (1, \infty)$. We have already inferred why $f + g$ also belongs to $L_p(E)$. Let $f + g \neq 0$ and $(f + g)^*$ be the conjugate function. Then by Holder's inequality,

$$\begin{aligned} \|f + g\|_p &= \int_E (f + g)(f + g)^* \\ &= \int_E f(f + g)^* + g(f + g)^* \\ &\leq \|f\|_p \|(f + g)^*\|_q + \|g\|_p \|(f + g)^*\|_q \\ &= \|f\|_p + \|g\|_p \end{aligned}$$

□

Definition 7.3.7. The Cauchy-Schwarz Inequality :. It is a special case of Holder inequality when $p = q = 2$. Let E be a measurable set and f and g measurable functions on E for which f^2 and g^2 are integrable over E. Then their product

fg also is integrable over E and

$$\int_E |fg| \leq \sqrt{\int_E f^2}\sqrt{\int_E g^2}$$

Corollary 7.3.8. *Let E be a measurable set and $1 < p < \infty$. Suppose F is a family of functions in $L^p(E)$ that is bounded in $L^p(E)$ in the sense that there is a constant M for which*

$$\|f\|_p \leq M \text{ for all } f \in F$$

Then the family F is uniformly integrable over E.

Corollary 7.3.9. *Let E be a measurable set of finite measure and $1 \leq p_1 < p_2 \leq \infty$. Then $L^{p_2}(E) \subseteq L^{p_1}(E)$. Furthermore,*

$$\|f\|_{p_1} \leq c\|f\|_{p_2} \text{ for all } f \in L^{p_2}(E)$$

.

7.4 L^p Space Completeness Theorem: The Riesz-Fischer Theorem

Definition 7.4.1. A sequence $\{F_n\}$ in a linear space X that is normed by $\| \ \|$ is said to converge to f in X provided that

$$\lim_{n\to\infty} \|f - f_n\| = 0$$

and we denote it by

$$\{f_n\} \to f \text{ in } X \quad \text{or} \quad \lim_{n\to\infty} f_n = f \text{ in } X$$

Remark 7.4.2.

1. For a sequence $\{f_n\}$ and function f in $C[a,b]$, $\{f_n\} \to f$ in $C[a,b]$, normed by the maximum norm, if and only if $\{f_n\}$ converges to f uniformly on $[a,b]$.

2. Similarly, for a sequence $\{f_n\}$ and function f in $L^\infty(E)$, $\{f_n\} \to f$ in $L^\infty(E)$ if and only if $\{f_n\} \to f$ uniformly $a.e$ on E.

3. For a sequence $\{f_n\}$ and function f in $L^p(E)$, where

$1 < p < \infty$,

$\{f_n\} \to f$ in $L^p(E)$ if and only if $\lim_{n\to\infty} \int_E |f - f_n|^p = 0.$

Definition 7.4.3. A sequence $\{f_n\}$ in a linear space X that is normed by $'\|\|'$ is said to be Cauchy in X provided for each $\epsilon > 0$, there is a natural number N such that

$$\|f_m - f_n\| < \epsilon \quad for\ all\ m, n \geq N$$

A normed linear space X is said to be complete provided every Cauchy sequence in X converges to a function in X. A complete normed linear space is called a Banach space.

Proposition 7.4.4. *Let X be a normed linear space. Then every convergent sequence in X is Cauchy. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.*

Proof. Let $\{f_n\} \to f$. Then, by the triangle inequality, for all

m, n,

$$\begin{aligned} \|f_m - f_n\| &= \|f - m - f = f - f_n\| \\ &\leq \|f_m - f\| \\ &= \|f_n - f\|. \end{aligned}$$

Therefore $\{f_n\}$ is cauchy.
Now, let $\{f_n\}$ be cauchy and $\{f_{n_k}\}$ be the convergent subsequence of $\{f_n\}$, which converges to f in X. Choose a N such that $|f_m - f_n| < \epsilon/2$ for all $m, n \geq N$. Now we choose k such that $n_k > N$ and $|f_{n_k} - f| < \epsilon/2$. By triangle inequality, for $n \geq N$,

$$\begin{aligned} \|f_n - f\| &= \|f_n - f_{n_k} + f_{n_k} - f\| \\ &\leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| \\ &< \epsilon. \end{aligned}$$

Thus $\{f_n\} \to f$ in X. □

Definition 7.4.5. Let X be a linear space normed by $\| \ \|$. A sequence $\{f_n\}$ in X is said to be **rapidly Cauchy** provided there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \epsilon_k$ for which

$$\|f_{k+1} - f_k\| \leq \epsilon_k^2 \;\; for\ all\ k$$

Observe that

$$\|f_{n+k} - f_n\| \leq \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\| \leq \sum_{j=n}^{\infty} \epsilon_j^2 \;\; for\ all\ n\ and\ k$$

Theorem 7.4.6. *Let X be a normed linear space. Then every rapidly Cauchy sequence in X is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.*

Proof. Let $\{f_n\}$ be a rapidly cauchy sequnce and $\sum_{i=1}^{\infty} \epsilon_i$ the convergent series of non negative numbers such that

$$\|f_{k+1} - f_k\| \leq \epsilon_k^2 \;\; for\ all\ k$$

Then,

$$\|f_{n+k} - f_n\| \leq \sum_{j=n}^{\infty} \epsilon_j^2 \;\; for\ all\ n\ and\ k$$

Since $\sum_{i=1}^{\infty} \epsilon_i$ converges, $\sum_{i=1}^{\infty} \epsilon_i^2$ also converges. Thus $\{f_n\}$ is cauchy.

Now assume that $\{f_n\}$ is a cauchy sequnce in X. We choose a strictly increasing sequence of natural numbers n_k such that

$$\|f_{n_{k+1}} - f_{n_k}\| \leq (1/2)^k \quad for\ all\ k$$

The subsequence $\{f_{n_k}\}$ is rapidly Cauchy.

□

Theorem 7.4.7. *Let E be a measurable set and $1 \leq p \leq \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.*

Theorem 7.4.8. ***The Riesz-Fischer Theorem:*** *Let E be a measurable set and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $f_n \to f$ in $L^p(E)$, a subsequence of f_n converges pointwise a.e. on E to I.*

Proof. Let f_n be a Cauchy sequence $L^p(E)$. Then there is a subsequence f_{n_k} of f_n that is rapidly Cauchy. Then by the previous theorem f_{n_k} converges to a function f in $L^p(E)$ both with respect to the $L^p(E)$ norm and pointwise *a.e.* on E. Therefore, the whole Cauchy sequence converges to f with respect to the $L^p(E)$ norm. □

Syllabus

SEMESTER 2

MTH2C07: REAL ANALYSIS II
No. of Credits: 4
No. of Hours of Lectures/week: 5

TEXT:
H. L.Royden, P. M. Fitzpatrick REAL ANALYSIS (4th Edn.), Prentice Hall of lndia, 2000.

Module 1

The Real Numbers: Sets, Sequences and Functions.
Chapter l : Sigma Algebra , Borel sets Section l.4: Proposition l3
Lebesgue Measure Chapter 2: Sections 2.l, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 upto Proposition l9.
Lebesgue Measurable Functions Chapter 3: Sections 3.l, 3.2 , 3.3

Module 2

Lebesgue Integration Chapter 4: Sections 4.1, 4.2, 4.3, 4.4, 4.5, 4.6
Lebesgue Integration: Further Topics Chapter 5: Sections: 5.1, 5.2,5.3

Module 3

Differentiation and Integration Chapter 6: Sections 6.l, 6.2, 6.3, 6.4, 6.5, 6.6. The L^p spaces : Completeness and Approximation Chapter 7: Sections 7.1,7.2,7.3

References

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2. R G Bartle:, *The Elements of Integration and Lebesgue Measure*, Wiley (1995).

3. S K Berberian: *Measure Theory and Integration*, The Mc Millan Company, New York (1965).

4. L M Graves: *The Theory of Functions of Real Variable*, Tata McGraw-Hill Book Co (1978).

5. P R Halmos: *Measure Theory*, GTM, Springer Verlag

6. W Rudin: *Real and Complex Analysis*, Tata McGraw Hill, New Delhi (2006)

7. I K Rana: *An Introduction to Measure and Integration*, Narosa Publishing Company, New York.

8. Terence Tao: *An Introduction to Measure Theory, Graduate Studies in Mathematics*, Vol 126 AMS AMS

YouTube Channels

- Krishna's Classroom
- Dr Bijumon R