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Handbook of Normal Frames and Coordinates

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List of Conventions

References. The book is divided into chapters which have a sequential Roman enumeration. The chapters are divided into sections with a sequential Arabic enumeration, which is independent in each chapter. Some sections are divided into subsections.

In each chapter the subsections, equations, propositions, theorems, lemmas, and so on have a double independent enumeration of the form **m.n** or **(m.n)** for the equations, **m,n=1,2,...** where **m** is the number of the section in which the designated item appears and **n** is its sequential number in it. So, Proposition 4.7 and (3.12) (or equation (3.12)) mean respectively Proposition 7 in Section 4 and equation 12 in Section 3 of the current chapter. A suitable item from a chapter different from the current one is referred as **R.m**, or **R.m.n** or **(R.m.n)** for equations, where **R=I,II,...** is the Roman number of the chapter in which the item appears; e.g., Remark II.5.3 and IV.4 (or Section IV.4) mean respectively Remark 3 in Section 5 of Chapter II and Section 4 in Chapter IV.

The footnotes are indicated as superscripts in the main text and have independent Arabic enumeration in each section. When we refer to a footnote, it is on the current page if the page on which it appears is not explicitly indicated.

Citations. An Arabic number in square brackets, e.g., [27], directs the reader to the list of references, i.e., in this example [27] means the 27th item from the Bibliography list beginning on page 401.

The ends of the proofs are marked by empty square sign, viz. with $\square$.

Indices. The Latin indices refer to an arbitrary linear (vector) space, in particular to the tangent and cotangent spaces. If in a given problem are presented the tangent and cotangent spaces to a manifold and other vector space(s), then the indices referring to the first two spaces are denoted with small Greek letters; for the rest one(s) the Latin letters will be used.

Einstein's summation convention: in a product of quantities or in a single expression, a summation over indices repeated on different levels is assumed over the whole range in which they change. Any exception of this rule is explicitly stated.

Symmetrization and antisymmetrization. On indices included in (or surrounded by) round (resp. square) brackets a symmetrization (resp. antisymmetrization) with coefficient one over the factorial of their number is assumed. If some indices in such a group have to be excluded from this operation, they are included in (surrounded by) vertical bars.

Matrix of linear mapping with respect to a given basis, or bases, or field of, possibly local, bases: the same symbol but the kernel letter is in **boldface**. Exception: the matrix of a derivation (derivative operator) is denoted by boldface capital Greek letter gamma, i.e., by $\boldsymbol{\Gamma}$, possibly with some indices.

Matrix elements. When the elements of a (two-dimensional) matrix are labeled by superscript and subscript, the superscript is considered as a first index, numbering the matrix's rows, and the subscript as a second one, numbering the matrix's columns. In this way the matrix of composition of linear mappings is equal to the product of the matrices of the mappings in the same order in which they appear in the composition and this does not depend on the way the matrix's indices are situated.

Free arguments. If we want to show explicitly the argument(s) of some mapping or to single out it (them) as arbitrary while the other arguments, if any, are considered as fixed ones, we denote it (them) by (centered) dot, i.e., by $\cdot$. E.g., if $f\colon A \to C$ and $g\colon A \times B \to C$, then $f(\,\cdot\,) \equiv f$, $g(\,\cdot\,,\,\cdot\,) \equiv g$, and $g(\,\cdot\,,b)$, $b \in B$, means $g(\,\cdot\,,b)\colon A \to C$ with $g(\,\cdot\,,b)\colon a \mapsto g(a,b)$ for all $a \in A$.

Preface

The main subject of this book is an up-to-date and in-depth survey of the theory of normal frames and coordinates in differential geometry. The existing results, as well as new ones obtained lately by the author, on the theme are presented.

The text is so organized that it can serve equally well as a reference manual, introduction to and review of the current research on the topic. Correspondingly, the possible audience ranges from graduate and post-graduate students to scientists working in differential geometry and theoretical/mathematical physics. This is reflected in the bibliography which consists mainly of standard (text)books and journal articles.

The present monograph is the first attempt for collecting the known facts concerting normal frames and coordinates into a single publication. For that reason, the considerations and most of the proofs are given in details.

Conventionally local coordinates or frames, which can be holonomic or not, are called *normal* if in them the coefficients of a linear connection vanish on some subset, usually a submanifold, of a differentiable manifold. Until recently the existence of normal frames was known (proved) only for symmetric linear connections on submanifolds of a manifold. Now the problems concerning normal frames for derivations of the tensor algebra over a differentiable manifold are well investigate; in particular they completely cover the exploration of normal frames for arbitrary linear connections on a manifold. These rigorous results are important in connection with some physical applications. They may be applied for rigorous analysis of the equivalence principle. This results in two general conclusions: the (strong) equivalence principle (in its 'conventional' formulations) is a provable theorem and the normal frames are the mathematical realization of the physical concept of 'inertial' frames. The normal frames find other important physical application in the bundle formulation of quantum mechanics. It turns out that in a normal frame the bundle Heisenberg and Schrödinger pictures of motion coincide.

Applying some freedom of language, we can state the general physical idea: the normal frames are the most suitable ones for describing free objects and events, i.e., such that on them do not act any forces. Regardless of the different realizations of that idea in general relativity and its generalizations, quantum mechanics, gauge theories etc., there is an underlying mathematical background for the general description of such situations: the existence (or non-existence) of normal frames in vector bundles. This observation fixes to a great extend the mathematical tools required for the description of some fundamental physical theories.

In the book, formally, may be distinguished three parts: The first one includes Chapters I–III and deals with a variety of mathematical problems concerning normal frames and coordinates on differentiable manifolds. The second part consists of Chapters IV and V and investigates normal frames (and possibly coordinates) in vector bundles and differentiable bundles, respectively. The last part, involving the text after Chapter V, contains inquiry material.

The requisite mathematical language required for the description of normal frames is spread over the initial sections of the chapters. In particular, Sections I.2– I.4, III.2, IV.9, IV.2, IV.14.1 and V.2–V.5 can be collected into an introductory chapter under the title “Mathematical preliminaries”[1] but this is not done by pedagogical reasons.[2] The normal coordinates and frames, in the case of linear connections on a manifold, are initially introduced in Chapter I. It contains our basic preliminary material and a review of the Riemannian coordinates. Chapter II is devoted to the existence, uniqueness, construction and other related problems concerning normal frames and coordinates in manifolds endowed with linear connection. It presents, in historical order, a detailed review of the existing literature as well as generalization of a number of results, e.g., for connections with torsion. Further, in Chapter III, problems connected with the existence, uniqueness, holonomicity etc. of normal frames for arbitrary derivations of the tensor algebra over a manifold are investigated. Next (Chapter IV), the same range of problems is explored for normal frames for linear transports in vector bundles. This material covers completely the special case of normal frames for linear connections in vector bundles or on a differentiable manifold. The main aim of Chapter V is the exploration of normal frames (and coordinates, if any) for general connections on differentiable fibre bundles which, in particular, can be vector ones.

The general approach of the book is essentially coordinate-dependent or basis-dependent. This is due to its basic subject: frames, bases or coordinates with some special properties. However, if possible and suitable, the coordinate-free notation and methods are not neglected.

The basic mathematical prerequisites vary from chapter to chapter but generically they include the grounds of vector (linear) spaces, differentiable manifolds, vector bundles, connection theory, and a firm belief in the existence and uniqueness theorems of ordinary differential equations. Some of the corresponding concepts and results are reproduced in our text but the acquaintance with adequate literature is required. Appropriate references are given in the Introductions to the chapters and directly in the main text.

[1] As (practically) any ‘preliminary’ knowledge requires for its understanding some other ‘preliminary’ to it knowledge, in the corresponding sections are cited a number of works containing this second kind of mathematical ‘luggage’.

[2] The material is so organized, that the required concepts and results appear in the logical order in which they are necessary for some particular purpose(s).

The material is so organized that a successive chapter generalizes the preceding one(s) and refers to it (them).

Any suggestions and comments are welcome. The author's postal address is

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11 July, 2006
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Chapter I

Manifolds, Normal Frames and Riemannian Coordinates

The basic differential-geometric concepts, such as differentiable manifolds and mappings, tensors and tensor fields, and linear connections, on which the book rests, are introduced. Partially the notation and terminology employed are fixed. The normal frames and coordinates are defined as ones in which the coefficients of a linear connection in them vanish on some set. Certain their general properties are mentioned. The Riemannian coordinates, which are normal at their origin, are described.

1. Introduction

The goal of this chapter is twofold: it introduces most of the basic preliminary definitions and results on which our investigation rests (Sections 2, 3, and 4) and it begins the study of the normal frames and coordinates (Sections 5 and 6).

The main concepts of differential geometry required for the understanding of the book are: differentiable manifolds and mappings, submanifolds, Riemannian manifolds, tangent vectors and vector fields, tensors and tensor fields, linear connections. The readers acquainted with them may only look over the corresponding sections for our notation, omitting the major text to which they may wish to return later, following the references to it.

In more details, the contents of the chapter is as follows.

The purpose of Section 2 is to fix our terminology and notation concerning differentiable manifolds and some typical to them natural structures. This is not a summary of the differential geometry, only certain basic concepts and particular relations between them required for our future aims are presented. At first the concepts of topological and differentiable manifolds are introduced, then tangent vectors, cotangent vectors, and tensors and the corresponding fields of them on a manifold are defined. Also some expressions in local bases (or frames) and coordinates are given. If the reader is acquainted with all this, he/she can simply look over this Section for our notation skipping the main text. A reader interested in deeper understanding of these concepts, as well as in differential geometry as a whole, should consult with the specialized literature. Here is a (random) selection of such titles. An elementary introduction to differential geometry, with 'physical' orientation, can be found in [1–6]. The same purpose can serve the books [7–10] which are more 'mathematically' oriented. Our text follows the excellent (text)books [11, 12]. At last, the advanced works [13–16] can be recommended. A brief synopsis of the mathematics preceding the introduction of manifolds is given it [9, 14, 16] while [12, 17] contain an expanded presentation of the 'preliminary' to manifolds material. Of course, the reading of all of the above-mentioned serious books is not necessary for the understanding of what follows. For this end, the reading of Section 2 is sufficient and the references cited may be consulted for more detains and proofs of some assertions. The knowledge of the tensor analysis in coordinate-dependent language is desirable [18, 19]. It is almost sufficient for the most of this and subsequent chapters.

In Section 3, we introduce the concept of linear connection on a manifold. The approach chosen is, in a sense, middle between elementary books on general relativity, such as [20, 21], and pure mathematical ones on differential geometry, like [11, 22]. We have tried to follow closely [9, 11, 12] but the abstracting material is adapted to the goals of the present book. After a motivation for what the connections are needed for, we introduce the linear connections via a system of axioms for the covariant derivative of the algebra of tensor fields over a given manifold. We employ this method since the theory of vector bundles, which is not required for Chapters I–III, will be involve into action only at the beginning of Chapter IV. In this connection, let us mention that the linear connections can be defined only on the algebra of vector fields on a manifold (i.e., to the tangent to it bundle), and then they admit a unique extension on the whole algebra of tensor fields [11, Chapter 3, Proposition 7.5]. A more advanced and deep treatment of the theory of linear connections on manifolds and vector bundles can be found in [10, 13, 15, 22, 23]. We also present the notion of a parallel transport (induced by a linear connection) which will practically step on scene in Chapter IV but here is a natural place for it to appear. It will be used in Chapters I–III for proving and formulating some results. Section 3 ends with a brief consideration of the geodesics and exponential mapping.

The concept of Riemannian metric and Riemannian connection are given in Section 4. If the reader is interested in essence of Riemannian geometry, he/she is referred, for example, to [8–12, 19, 24–27].

In Section 5, we introduce the main objects of our investigation, the *normal frames and coordinates*. We define them as ones in which the coefficients of a linear connection vanish on a given set. Some considerations on the uniqueness and (an)holonomicity of the normal frames are presented too.

Section 6 contains a complete description of normal frames at a given point on (C^∞) Riemannian manifolds. This is done on the base of Riemannian coordinates which turn to be normal at their origin. The geodesic coordinates are pointed as other example of coordinates normal at a point. Some general results, proved further in Chapter II, concerning the existence of normal frames on submanifolds are quoted. An expanded presentation of the problem of existence of normal coordinates at a point of a C^∞ Riemannian manifold is given in [19, 24], where also a list of original early works on this topic can be found.

In Section 7 are presented a number of examples and exercises of concrete Riemannian connection and coordinates/frames normal for them on different sets. At first, the (locally) Euclidean and one-dimensional manifolds are considered. The (pseudo)spherical coordinates on (pseudo)spheres are (partially) investigated for sets on which they are normal for the Riemannian connection induced on them by the metric on them generated by the Euclidean one of the Euclidean space in which the (pseudo)spheres are embedded. Similar instance on the two-dimensional torus is presented. The cosmological models of Einstein, de Sitter and Schwarzschild are considered (in concrete coordinates) from the view-point of normal frames/coordinates on them. Some peculiarities of the light cone in Minkowski spacetime are pointed too.

Section 8 deals with certain terminological problems concerning bases and frames. Some links between these concepts are explicitly formulated and/or derived.

The chapter ends with some general remarks and conclusions in Section 9.

2. Differentiable manifolds

In this section we introduce the basic concepts of differential geometry on which the present and next chapters rest. First of all, attention is paid on the differentiable manifolds and the algebras of tensors and tensor fields on them.

2.1. Basic definitions

As we shall see a little further, the fields of real, $\mathbb{R}$, and complex, $\mathbb{C}$, numbers are the simplest examples of respectively real and complex (one-dimensional) differentiable manifolds. Correspondingly, their nth-Cartesian powers (direct products),

$$\mathbb{R}^n := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{-times}}, \quad \mathbb{C}^n := \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n\text{-times}}$$

where $n \in \mathbb{N}$, $\mathbb{N} := \{1, 2, \dots\}$ being the set of integers,[1] and $\times$ is the Cartesian product sign (of sets),[2] are the simplest examples of respectively real and complex n-dimensional manifolds.

The silent and, perhaps the most important idea in the concept of an n-dimensional manifold M over a field $\mathbb{K}$ is that it is a set which is 'locally Euclidean' or which 'locally looks like $\mathbb{K}^n$' where, to save writing, $\mathbb{K}$ denotes some of the fields $\mathbb{R}$ or $\mathbb{C}$. The intuitive meaning of the last phrase is that M can be divided into (or, rewording, can be presented as a union of) sets each of which is into a bijective (one-to-one onto) correspondence with some subset of $\mathbb{K}^n$, in particular, possibly, with the whole $\mathbb{K}^n$.

Now, following the books [11, 12], the precise definitions are in order.

Definition 2.1. An *n-dimensional topological $\mathbb{K}$-manifold* is a Hausdorff topological space such that every its point has a neighborhood homeomorphic to an open subset of $\mathbb{K}^n$.[3]

Notice, if we say that U is a neighborhood of a set $V \subseteq M$, M being a topological space, we mean that U is an open set in M containing V. Otherwise by a neighborhood (in M) we understand any open set in M (which set is a neighborhood of any its point in the just pointed sense).

For $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$) the manifold is called real (resp. complex). If $\mathbb{K}$ is clear from the context, the n-dimensional $\mathbb{K}$-manifold is referred as an n-manifold. In this work only the finite-dimensional case will be considered, i.e., $n \in \mathbb{N}$, with

[1] When writing $x \in X$, X being a set, we mean "for all x in X" if the point x is not specified (fixed, given) and is considered as an argument or a variable.

[2] For some purposes the sign $\oplus$ is used instead of $\times$; the result is called a direct sum or product in this case.

[3] For more general purposes, the Hausdorffness can be dropped and/or $\mathbb{K}$ can be (locally) replaced with an arbitrary dimensional vector space(s) [14, pp. 21–22], [10, pp. 2–4]. Some authors [13, p. 32] add the additional requirement for existence of a countable base. These cases will not be considered in the present book.

$n < \infty$. If both n and $\mathbb{K}$ are evident from the context, we speak simply of a topological manifold.

The number $n \in \mathbb{N}$ of an n-dimensional $\mathbb{K}$-manifold M is called its *dimension* and is denoted by $\dim M$. If $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$), the dimension of M is also denoted by $\dim_{\mathbb{R}}$ (resp. $\dim_{\mathbb{C}}$). (See p. 7 for the meaning of the symbol $\dim_{\mathbb{R}} M$ for a complex manifold M.)

A *chart* of (for) a topological $\mathbb{K}$-manifold M is an ordered pair (U, φ) of an open subset $U \subseteq M$, where $\subseteq$ is the contained in or equal sign, *domain* of the chart, and homeomorphism $\varphi\colon U \to V$ onto an open subset $V \subseteq \mathbb{K}^n$. Since any open set $V \subseteq \mathbb{K}^n$ is homeomorphic to the whole $\mathbb{K}^n$, we can always put $V = \mathbb{K}^n$. By Definition 2.1, for every point $p \in M$ there exists a chart in whose domain it is contained. This is the rigorous expression of the phrases that 'M locally looks like (a neighborhood of) $\mathbb{K}^n$' or 'M is locally Euclidean'.

Example 2.1. The set $\mathbb{K}^n$ is an n-manifold and $(\mathbb{K}^n, \mathsf{id}_{\mathbb{K}^n})$ is a (global) chart for it. Here and henceforth id_A denotes the identity mapping of a set A.

Let $\{r^i : i = 1, \dots, n\}$ be the standard Cartesian coordinate functions on $\mathbb{K}^n$, viz. if $c = (c^1, \dots, c^n) \in \mathbb{K}^n = \mathbb{K} \times \cdots \times \mathbb{K}$ (n-times), then $r^i(c) := c^i$ is the ith Cartesian coordinate of c. If (U, φ) is a chart of M, then $\{\varphi^1, \dots, \varphi^n\}$, $\varphi^i := r^i \circ \varphi\colon U \to \mathbb{K}$, $\circ$ being the sign of mapping's composition, is called *local coordinate system* on U (with respect to the chart (U, φ)) and we say that U is (local) coordinate neighborhood; the functions φ^i are called *coordinate functions*. As $\{r^i\}$ are fixed, sometimes φ is also called (local) coordinate system [10, p. 2]. The reason is that if we define the n-tuple $(\varphi^1, \dots, \varphi^n)$ as a mapping $U \to \mathbb{K}^n$ by $(\varphi^1, \dots, \varphi^n)(p) := (\varphi^1(p), \dots, \varphi^n(p))$, $p \in U$ then $\varphi \equiv (\varphi^1, \dots, \varphi^n)$.

The numbers $\varphi^1(p) = r^1(\varphi(p)), \dots, \varphi^n(p) = r^n(\varphi(p)) \in \mathbb{K}$ (or the ordered n-tuple $(\varphi^1(p), \dots, \varphi^n(p))$) are (is) called *local coordinates* of $p \in U$ with respect to (U, φ). Often $\varphi^i(p)$ is abbreviated to p^i.

If for two charts (U, φ) and (U', φ') is fulfilled $U \cap U' \neq \varnothing$, $\varnothing$ being the empty set and $\cap$ denotes intersections of sets, in $U \cap U'$ are defined two coordinate systems, $\{\varphi^i = r^i \circ \varphi\}$ and $\{\varphi'^i = r^i \circ \varphi'\}$. Obviously, the connection between them is $\varphi'^i = (r^i \circ \varphi' \circ \varphi^{-1}) \circ \varphi$ which implies $\varphi'^i(p) = (r^i \circ \varphi' \circ \varphi^{-1})\big(\varphi^1(p), \dots, \varphi^n(p)\big)$ for $p \in U \cap U'$. So, we can write $p'^i = p'^i(p^1, \dots, p^n)$, the explicit dependence being just given.

A family $\{(U_\alpha, \varphi_\alpha) | \alpha \in A\}$ of charts on M is called an *atlas* if $\{U_\alpha\}$ is an open cover of M, $M = \bigcup_{\alpha \in A} U_\alpha$. Here the sign $\cup$ mean union of sets. We speak of an atlas of class (of smoothness) C^k, $k \in \mathbb{N} \cup \{0\}$, or of a C^k atlas if the homeomorphisms φ_α are C^k compatible in a sense that if $U_\alpha \cap U_\beta \neq \varnothing$ for some $\alpha, \beta \in A$, then the mappings $\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are of class C^k as mappings, which actually are bijections, between open subsets of $\mathbb{K}^n$. A C^k mapping between open subsets of $\mathbb{K}^n$ is a one having continuous partial derivatives for all orders $r \leq k$, $k \in \{0\} \cup \mathbb{N} \cup \{\omega\}$; C^0 means continuous, C^ω stands for (real or complex) analytic, and if $k \in \mathbb{N}$ is arbitrary we speak of C^∞ mappings. The values $k = 3, 2, 1$ will be sufficient for the most problems to be considered in this book.

Obviously, any topological manifold M admits atlases. A chart (U, φ) of M is said to be (C^k) compatible with an atlas $\{(U_\alpha, \varphi_\alpha)|\alpha \in A\}$ if $\varphi_\alpha \circ \varphi^{-1} \colon \varphi(U_\alpha \cap U) \to \varphi_\alpha(U_\alpha \cap U)$ is a C^k mapping for every α such that $U_\alpha \cap U \neq \varnothing$. Two atlases are (C^k) compatible if each chart of one of them is (C^k) compatible with the other atlas. The compatibility relation between atlases is an equivalence relation. An equivalence class of atlases on M is called a C^k *differentiable structure* (on M). A differentiable structure on M can also be defined as the maximal (complete) atlas C^k compatible with (and containing) a given atlas $\{(U_\alpha, \varphi_\alpha)|\alpha \in A\}$. So, any atlas, when expanded to a complete atlas, gives rise to a C^k structure.

Definition 2.2. A *differentiable manifold of class* C^k is a topological manifold with fixed C^k differentiable structure on it.

Remark 2.1. To be quite precise, the object described in Definition 2.2 is called differentiable manifold *without boundary* of class C^k [14, p. 38]. There are a number of equivalent or similar definitions of a manifold *with boundary* (see, e.g., [14, Chapter II, § 4], [16, p. 139], [12, pp. 208–209]), one of them being the following. Let $\lambda \colon \mathbb{K}^n \to \mathbb{R}$ be continuous linear mapping into $\mathbb{R}$. A Euclidean ('upper') half space or plane H_λ^+ is the closed subset of $\mathbb{K}^n$ given by the non-negative values of λ, $H_\lambda^+ := \{h|h \in \mathbb{K}^n,\ \lambda(h) \geq 0\}$. For instance, in the real case, $\mathbb{K} = \mathbb{R}$, one usually takes $\lambda = r^n$, $\lambda(c^1, \dots, c^n) = c^n$ for $c^i \in \mathbb{R}$, which results in $H_\lambda^+ = \mathbb{R}_+^n := \{c|c \in \mathbb{R}^n,\ r^n(c) \geq 0\}$, or $\lambda \colon c \mapsto \sum_1^n a^i c^i$ for a fixed non-zero $a \in \mathbb{R}^n$. A chart (with boundary) in a $\mathbb{K}$-topological manifold M is a pair (U, φ) of an open subset $U \subseteq M$ and homeomorphism $\varphi \colon U \to V$ from U on an open subset V of H_λ^+. There are charts (U, φ) for which $\varphi(U)$ is open set in H_λ^+ homeomorphic to $\mathbb{K}^n$ and others for which $\varphi(U)$ is open set in H_λ^+ but not in $\mathbb{K}^n$. If we make this modification in all of the above text, i.e., take a chart with boundary for a chart, we obtain the notions of atlas and differentiable structure with boundary and correspondingly a *manifold with boundary.* An example of such a manifold, which is frequently met, is a real interval J closed from one of both ends with (J, id_J) as a differentiable structure. The manifolds without boundary are evident special case of the ones with boundary. Most of the results concerning manifolds without boundary can be transferred on manifolds with boundary; general directions on how this can be done are given in [14, pp. 38–40]. In a general context, a manifold should mean manifold with boundary. However, in this book we shall deal exclusively with manifolds without boundary which, for brevity, will be called simply manifolds.

Often a differentiable manifold of class C^k is denoted as C^k-manifold. So, a B-manifold with $B = n, \mathbb{K}, C^k$ means different things depending on the context. In this book only C^k-manifolds will be employed; they will, for brevity, be referred simply as manifolds, the class of smoothness, usually C^3, or C^2, or C^1, will be clear from the context.

Example 2.2. The sets $\mathbb{R}^n$ and $\mathbb{C}^n$ are C^∞ manifold with $(\mathbb{R}^n, \mathsf{id}_{\mathbb{R}^n})$ and $(\mathbb{C}^n, \mathsf{id}_{\mathbb{C}^n})$, respectively, as C^∞ differentiable structures.

It is well known [10,11,26], every n-dimensional *complex* manifold has also a structure of $2n$-dimensional *real* manifold and, consequently, can be considered as real manifold of dimension $2n$. Since the complex and real differentiable structures on a complex manifold are in bijective correspondence, both points of view, the real and complex ones, on it are equivalent. The dimension of a complex manifold M considered as a real manifold is denoted by $\dim_{\mathbb{R}} M$. Hence, we have $\dim_{\mathbb{R}} M = \dim M$ if M is real and $\dim_{\mathbb{R}} M = 2\dim M = 2\dim_{\mathbb{C}} M$ if M is complex.

A subset N of m-dimensional manifold M is called an n-dimensional *submanifold* of M, $n \le m$, if for every point $p \in N$ there exists a chart (U, φ) of M with p in its domain, $U \ni p$, and such that

$$\varphi\colon N \cap U \to \mathbb{K}^n \times \{a\}, \qquad \varphi\colon q \mapsto (\varphi^1(q), \dots, \varphi^n(q), a^1, \dots, a^{m-n})$$

for some fixed $a = (a^1, \dots, a^{m-n}) \in \mathbb{K}^{m-n}$ and all $q \in N \cap U$. The set of charts $\{(\bar{U}, \bar{\varphi})\}$, given by $\bar{U} := U \cap N$ and $\bar{\varphi}\colon \bar{U} \to \mathbb{K}^n$ with $\bar{\varphi}(q) := (\varphi^1(q), \dots, \varphi^n(q))$, is an atlas on N of the same class as the original atlas $\{(U, \varphi)\}$ on M. So, in more precise terms, a subset $N \subseteq M$ with just described differentiable stricture is an *n-dimensional submanifold* of the m-manifold M, $m \ge n$.[4] One should be aware of the fact that presently subtle versions of different definitions of a 'submanifold' are in current use.

Example 2.3. The open subsets of a manifold M (which will be referred as neighborhoods) and the sets consisting of finite number of its points are respectively $\dim M$- and zero-dimensional submanifolds of M.

Important examples of submanifolds are generated according to the theorem in [12, p. 228].

2.2. Differentiable mappings

Let M and N be C^k-manifolds of dimension m and n, respectively, and a mapping $f\colon M \to N$ be given. If (U, φ) and (V, ψ) are charts in the respective complete atlases on M and N such that $f(U) \subseteq V$, the mapping $\bar{f} := \psi \circ f \circ \varphi^{-1}$ from $\varphi(U) \subseteq \mathbb{K}^m$ into $\psi(V) \subseteq \mathbb{K}^n$ (locally) represents f in them.

A mapping f is *differentiable* of class C^r, $r \le k$ at $p \in M$ if $\psi \circ f \circ \varphi^{-1}$ is a C^r mapping at $\varphi(p)$. Such mappings are also called C^r differentiable or C^r mappings (at p).[5] Obviously, the C^r differentiability at a point is independent of the particular charts utilized above. If $r^1, \dots, r^m$ and $v^1, \dots, v^n$ are the standard

[4]If N already has a manifold structure, we suppose it to be equivalent to the described submanifold one. Otherwise one should clearly specify with respect to which differentiable structure N is considered as a manifold.

[5]For manifolds with boundary (see Remark 2.1) the following modifications must be done [16, p. 139]. If O is an open subset of H_λ^+, the mapping $g\colon O \to H_\mu^+$, with $\lambda, \mu\colon \mathbb{K}^n \to \mathbb{R}$ being C^0

coordinate functions on $\mathbb{K}^m$ and $\mathbb{K}^n$ respectively, in the local coordinate systems $\{\varphi^i := r^i \circ \varphi,\ i = 1, \dots, m\}$ and $\{\psi^j := v^j \circ \psi,\ j = 1, \dots, n\}$ the C^r differentiability means that the set of n functions

$$f^j := \psi^j \circ f \circ \varphi^{-1} = v^j \circ \psi \circ f \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{K}$$

are of class C^r at the point $\varphi(p) = (p^1, \dots, p^m) \in \mathbb{K}^m$, $p^i := \varphi^i(p)$. Sometimes this is expressed by the assertion that the set of functions $\bar{q}^j = f^j(q^1, \dots, q^m)$, $j = 1, \dots, n$, $q^i \in \mathbb{K}$, called *expression* of f in coordinates, are of class C^r at $q^i = p^i$.

The mapping $f \colon M \to N$ is of class C^r, $r \leq k$ (is a C^r mapping, is C^r differentiable) if it is C^r differentiable at every point $p \in M$. Analogously, f is a C^r mapping on $M' \subset M$ if it is a C^r at every point $p \in M'$.

The above definitions are *mutatis mutandis* transferred on mappings like $f \colon M' \to N'$ with $M' \subset M$ and $N' \subset N$: at the corresponding places U and V have to be replace by $U \cap M'$ and $V \cap N'$ respectively.

Under a differentiable mapping or simply mapping between C^k manifolds, we shall understand a C^k mapping between them, i.e., a C^r mapping with maximal r, $r = k$.

Example 2.4. A mapping $f \colon M \to \mathbb{K}^m$ is called function on M. If it is of class C^r (at $p \in M$), it is called a C^r function (at $p \in M$). It is real or complex depending on $\mathbb{K} = \mathbb{R}, \mathbb{C}$. If (U, φ) is a chart of M and we take $(\mathbb{K}^m, \mathsf{id}_{\mathbb{K}^m})$ as a natural atlas of $\mathbb{K}^m$, then f has the local representation $\bar{f} = f \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{K}^m$. The algebra of C^r, $r \leq k$ (resp. all) functions on (one and the same neighborhood of) a subset $U \subseteq M$ of a C^k manifold with $k \geq 0$ will be denoted by $\mathfrak{F}^r(U)$ (resp. by $\mathfrak{F}(U)$). In particular, $\mathfrak{F}^r(p)$, $p \in M$ and $\mathfrak{F}^r(M)$ are the algebras of C^r functions on a (fixed) neighborhood of p and on the whole M respectively. Let us note that the neighborhoods (if $U \neq M$) in these definitions must be fixed as otherwise the sets $\mathfrak{F}^r(U)$ and $\mathfrak{F}(U)$ are not quite algebras as they will not have unique zero elements: two zero functions $f + (-f)$ and $g + (-g)$ coincide iff the domains of f and g are identical.

A mapping $f \colon M \to N$ between C^k manifolds is called C^r, $r \leq k$, *diffeomorphism* if it is bijective and f and f^{-1} are of class C^r. A C^k diffeomorphism is referred as simply diffeomorphism. With respect to the manifolds, the diffeomorphisms play the same role as the isomorphisms (resp. homeomorphisms) in the theory of vector (resp. topological) spaces.

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linear functions, is of class C^r if it extends to a mapping $g' \colon O' \to H^+_\mu$, $g'|O = g$, of class C^r, where O' is open subset of H^+_λ containing O. A mapping between manifolds with boundary is of class C^r if it is locally of class C^r, i.e., if $\psi \circ f \circ \varphi^{-1}$ is such in the above notation.

2.3. Tangent vectors and vector fields

A number of different but equivalent definitions of a tangent vector can be found in the literature [7, 11–13]. Below we reproduce the most direct one.

Definition 2.3. A *tangent vector* on a differentiable manifold M of class C^k, with $k \geq 1$, at a point $p \in M$ is a linear mapping $X_p \colon \mathfrak{F}^1(p) \to \mathbb{K}$ satisfying the Leibnitz rule:

$$X_p(af + bg) = aX_p(f) + bX_p(g), \qquad a, b \in \mathbb{K} \quad f, g \in \mathfrak{F}^1(p) \tag{2.1}$$

$$X_p(fg) = g(p)X_p(f) + f(p)X_p(g), \qquad f, g \in \mathfrak{F}^1(p). \tag{2.2}$$

Thus a tangent vector at p is a derivation of the algebra $\mathfrak{F}^1(p)$.[6]

Example 2.5. A mapping $\gamma \colon J \to M$, J being $\mathbb{R}$-*interval* of arbitrary type (open or closed from one or the both ends), is called *path* (or *parameterized curve*).[7] If γ is a C^1 path, the mapping $\dot\gamma(s) \colon \mathfrak{F}^1(\gamma(s)) \to \mathbb{K}$ given by

$$\big(\dot\gamma(s)\big)(f) := \frac{\mathrm{d} f(\gamma(t))}{\mathrm{d}t}\Big|_{t=s}, \qquad s, t \in J \quad f \in \mathfrak{F}^1(\gamma(s)), \tag{2.3}$$

where $\cdots|_A$ means the restriction of $\cdots$ to the set A, is an important example of a tangent vector at $p = \gamma(s) \in \gamma(J)$.[8] So, $\dot\gamma(s)(f)$ is the derivative of f along γ at the parameter value s. The vector $\dot\gamma(s)$ is called *tangent to* γ *at* $\gamma(s)$ *or, more precisely, at the parameter value* s as it may happen that, for a non-injective path, there are $s, t \in J$ such that $s \neq t$, $\gamma(s) = \gamma(t)$ and $\dot\gamma(s) \neq \dot\gamma(t)$. Sometimes $\dot\gamma(s)$ is denoted by $\frac{\mathrm{d}}{\mathrm{d}s}$ which can be confusing. As a result of (2.3), the vector $\dot\gamma(s)$ is often denoted by $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=s}\gamma(t)$.

The set of all tangent vectors at $p \in M$ is a linear space (over $\mathbb{K} = \mathbb{R}, \mathbb{C}$) called *tangent* (vector) *space to* M *at* p. It is denoted by $T_p(M)$ or, for brevity, by M_p or T_p if there is no risk of ambiguity. The dimensions of $T_p(M)$ and M coincide for every p, $\dim T_p(M) \equiv \dim M$. Moreover, if (U, φ) is a chart with

[6] Generally, a mapping $\partial \colon R \to L$ from a ring R to a left R-module L is *derivation* if $\partial(x \cdot y) = x\partial(y) + y\partial(x)$, $x, y \in R$. The case when R is an algebra, which in turn is a linear space over $\mathbb{K}$, is frequently met in the differential geometry. In the last case on ∂ is added the restriction to be $\mathbb{K}$-linear.

[7] We make a clear distinction between the concepts 'path' and 'curve'. By a *curve* or *geometric curve* we understand an equivalence class of paths with respect to some set of changes of the path's parameter. More precisely, let T be some set of bijective mappings between the $\mathbb{R}$-intervals, two paths $\gamma_a \colon J_a \to A$, with $a = 1, 2$ and A being a non-empty set, are equivalent (with respect to T) if there is $\tau \in T$ such that $\gamma_2 = \gamma_1 \circ \tau$ and $J_2 = \tau(J_1)$. This is an equivalence relation on the set of paths in A. Any equivalence class $[\gamma]$ of paths equivalent to $\gamma \colon J \to A$ is called a curve. Any path of a curve $[\gamma]$ provides a 'parameterization' of the set $\gamma(J) \subseteq A$ which sometimes is also called a curve. Most often as T is chosen the class of (C^0, C^1, ...) diffeomorphisms between the real intervals.

[8] If J has end point(s) and s is such a point, the derivative in (2.3) is considered as one-sided, resp. $\dot\gamma(s)$ is one-sided tangent vector to γ at the endpoint(s).

$U \ni p$, $\{x^i = r^i \circ \varphi\}$ is the corresponding coordinate system, and $\{r^i\}$ are the standard Cartesian coordinates on $\varphi(U)$, then the mappings ($i = 1, \dots, \dim M$)

$$\frac{\partial}{\partial x^i}\Big|_p : \mathfrak{F}^1(p) \to \mathbb{K}, \qquad \frac{\partial}{\partial x^i}\Big|_p : f \mapsto \frac{\partial f}{\partial x^i}\Big|_p := \frac{\partial(f \circ \varphi^{-1})}{\partial r^i}\Big|_{\varphi(p)} \tag{2.4}$$

form a basis $\{\frac{\partial}{\partial x^i}|_p\}$ of $T_p(M)$, called associated (coordinate or holonomic [5, p. 110] basis) with the given chart or local coordinates.[9] Therefore every $X_p \in T_p(M)$ has a unique decomposition

$$X_p = X_p^i \frac{\partial}{\partial x^i}\Big|_p := \sum_{i=1}^{i=n} X_p^i \frac{\partial}{\partial x^i}\Big|_p \tag{2.5}$$

where

$$X_p^i := X_p(x^i) = X_p(r^i \circ \varphi) \in \mathbb{K} \tag{2.6}$$

are the (local) *components* of X_p (with respect to (U, φ) or $\{x^i\}$). Here, as well as throughout the whole book, the Einstein's summation convention is assumed (see the list of conventions, p. xi).

Example 2.6. Let $\gamma\colon J \to M$ be a C^1 path, $s \in J$ and (U, φ) be a chart such that $U \ni \gamma(s)$. Then

$$\dot\gamma(s) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s}\gamma(t) = \frac{\mathrm{d}\gamma^i(s)}{\mathrm{d}s}\frac{\partial}{\partial x^i}\Big|_{\gamma(s)}, \tag{2.7}$$

where $\gamma^i(s) = x^i(\gamma(s))$ are the local coordinates of $\gamma(s)$ in $\{x^i\}$.

For every (tangent) vector $X_p \in T_p(M)$ there exists a path $\gamma\colon J \to M$ such that for some $s \in J$ is fulfilled $\gamma(s) = p$ and $\dot\gamma(s) = X_p$.[10] In such representation $X_p^i = X_p(x^i) = \mathrm{d}x^i(\gamma(t))/\mathrm{d}t\big|_{t=s}$.

A change $(U, \varphi) \mapsto (U', \varphi')$ of the chart containing p, $p \in U \cap U' \neq \varnothing$, or equivalently, of the local coordinates $x^i \mapsto x'^i$ implies corresponding transformation of the basic vectors (2.4) and the vector's components in (2.5). For the components of $X_p \in T_p(M)$, due to (2.5) and (2.4), we have

$$X_p^{'i} = X_p(x'^i) = \sum_j X_p^j \frac{\partial(x'^i \circ \varphi^{-1})}{\partial r^j}\Big|_{\varphi(p)} = \frac{\partial x'^i}{\partial x^j}\Big|_p X_p^j \tag{2.8}$$

and, analogously, $X_p^i = \partial x^i/\partial x'^j\big|_p X_p^{'j}$. Very often the restrictions $|_{\varphi(p)}$ and $|_{\varphi'(p)}$ in these formulae are abbreviate to $|_p$ or even are not written at all. According to (2.6) the components of $\frac{\partial}{\partial x'^i}\big|_p$ with respect to $\frac{\partial}{\partial x^i}\big|_p$ are $\big(\frac{\partial}{\partial x'^i}\big|_p\big)^j =$

[9] Notice, in (2.4) enters the local representation $\bar f := f \circ \varphi^{-1}$ of f with respect to the charts (U, φ) of M and $(\mathbb{K}^n, \mathsf{id}_{\mathbb{K}^n})$ of $\mathbb{K}^n$.

[10] This path is not unique. Actually, there is a family of such paths having a common tangent vector at p.

$\frac{\partial}{\partial x'^i}\Big|_p(x^j) = \frac{\partial(x^j\circ\varphi'^{-1})}{\partial r^i}\Big|_{\varphi'(p)} = \frac{\partial x^j}{\partial x'^i}\Big|_p$, i.e.,

$$\frac{\partial}{\partial x'^i}\Big|_p = \frac{\partial x^j}{\partial x'^i}\Big|_p \frac{\partial}{\partial x^i}\Big|_p = \frac{\partial(x^j\circ\varphi')}{\partial r^i}\Big|_{\varphi'(p)} \frac{\partial}{\partial x^i}\Big|_p. \tag{2.9}$$

More generally, if $\{E_i|_p\}$ is an arbitrary basis of $T_p(M)$ and the expansion of $X_p \in T_p(M)$ over $\{E_i|_p\}$ is

$$X_p = X_p^i E_i|_p, \qquad X_p^i \in \mathbb{K}, \tag{2.10}$$

the change

$$\{E_i|_p\} \mapsto \{E'_i|_p = A_i^j(p)E_j|_p\}, \tag{2.11}$$

with a nondegenerate matrix $A(p) := \left[A_i^j(p)\right]$, $A_i^j(p) \in \mathbb{K}$, implies

$$X_p^i \mapsto X'^i_p = \left(A^{-1}(p)\right)^i_j X_p^j \tag{2.12}$$

as one must have $X_p = X'^i_p E'_i|_p$. Clearly, the equations (2.8) and (2.9) are special case of (2.12) and (2.11), respectively.

The basis $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$ is usually referred as a *coordinate* one (associated with or generated by the local coordinates $\{x^i\}$) while $\{E_i|_p\}$ is called non-coordinate. For any basis $\{E_i|_p\}$ there exist local coordinates with respect to which it is coordinate (see Lemma II.5.2 on page 116).

The set of all tangent vectors on a manifold M is denoted by $T(M)$, i.e., $T(M) := \cup_{p\in M} T_p(M)$, and is called tangent bundle space.

Definition 2.4. A (*tangent*) *vector field* on $U \subseteq M$ is a mapping X assigning to each $p \in U$ a (tangent) vector X_p at p, i.e., $X : p \mapsto X_p \in T_p(M)$.

If U is clear from the context or if $U = M$, we speak simply of a vector field.

Example 2.7. If $\gamma\colon J \to M$ is a C^1 path without self-intersections, the mapping $\dot\gamma\colon \gamma(s) \mapsto \dot\gamma(s)$, $\dot\gamma(s)$ being the vector tangent to γ at $\gamma(s)$ (see (2.3)), is an important example of a vector field over $\gamma(J)$. It is called *tangent to* γ *vector field.*[11]

The set $\mathfrak{X}(U)$ of the vector fields on U is a linear space over $\mathbb{K}$; for the purpose we put $(\alpha X + \beta Y)\colon p \mapsto \alpha X_p + \beta Y_p$ with $\alpha, \beta \in \mathbb{K}$, $p \in U$, and $X, Y \in \mathfrak{X}(U)$. Moreover, this set can naturally be turned into a (left) module over the ring (algebra in this case) $\mathfrak{F}(U)$ of functions on U; this is done by defining $(fX + gY)\colon p \mapsto f(p)X_p + g(p)Y_p$ with $f, g\colon U \to \mathbb{K}$.

There is a naturally defined action of the vector fields on C^1 functions, viz. a vector field X on $U \subseteq M$ can be regarded as a mapping $\mathfrak{F}^1(U) \to \mathfrak{F}(U)$ by setting $X : f \mapsto Xf := X(f)$ with $X(f)\colon p \mapsto X_p(f)$ for all $p \in U$ and $f \in \mathfrak{F}^1(U)$.

[11] If γ is not injective, the mapping $\dot\gamma\colon \gamma(s) \to \dot\gamma(s)$ may turn to be multiple-valued at the points of self-intersection.

In terms of local coordinates $\{x^i\}$ defined via a chart (U,φ), we have

$$X = X^i\frac{\partial}{\partial x^i}, \qquad \frac{\partial}{\partial x^i}\colon p\mapsto \frac{\partial}{\partial x^i}\Big|_p := \frac{\partial(\,\cdot\,\circ\varphi^{-1})}{\partial r^i}\Big|_{\varphi(p)}, \tag{2.13}$$

where the functions

$$X^i := X(x^i) = X(r^i\circ\varphi)\colon U\to\mathbb{K} \tag{2.14}$$

are called *components* of X with respect to $\{x^i\}$ or (U,φ).

It is important to be noted that the set of vector fields $\mathfrak{X}(U)$ is a finitely generated (left) $\mathfrak{F}(U)$-module of rank $n=\dim M$ (see (2.13)) but it is also infinitely dimensional linear (vector) space over $\mathbb{K}$. A basis in the former case is formed by the vector fields $\{\frac{\partial}{\partial x^i}\}$. This basis is called associated to (generated by) the local coordinates $\{x^i\}$ or the chart (U,φ).

By virtue of (2.8), (2.9), (2.13), and (2.14), a change $(U,\varphi)\mapsto(U',\varphi')$ of the chart implies the transforms

$$\frac{\partial}{\partial x^i}\mapsto\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}, \qquad X^i\mapsto X'^i = \frac{\partial x'^i}{\partial x^j}X^j \tag{2.15}$$

where $\frac{\partial x^j}{\partial x'^i}\colon p\mapsto\frac{\partial x^j}{\partial x'^i}\big|_p = \frac{\partial(x^j\circ\varphi'^{-1})}{\partial r^i}\big|_{\varphi'(p)}$ and $\frac{\partial x'^i}{\partial x^j}\colon p\mapsto\frac{\partial x'^i}{\partial x^j}\big|_p = \frac{\partial(x'^i\circ\varphi^{-1})}{\partial r^j}\big|_{\varphi(p)}$, $p\in U\cap U'$.

We call a basis $\{E_i\}$ of the module of vector fields on (over) U a *frame* on (over) U.[12] So, a frame on U is a collection of $\dim M$ vector fields on U such that at every point $p\in U$ they form a basis of $T_p(M)$. At a single point p, $U=\{p\}$, the concepts 'basis in $T_p(M)$' and 'frame on (over, at) p' are synonyms.

The frame $\{\frac{\partial}{\partial x^i}\}$ on U is referred as a *coordinate* (or natural) one (associated with or generated by the local coordinates $\{x^i\}$) while a general frame $\{E_i\}$ is referred as a *noncoordinate* frame. A frame $\{E_i\}$ is called *holonomic* on U if there exist local coordinates $\{x^i\}$ on U such that $E_i = \partial/\partial x^i$, i.e., if $\{E_i\}$ is the associated with $\{x^i\}$ coordinate frame. Otherwise, if such $\{x^i\}$ do not exist, the frame is called *anholonomic*.[13] Equivalently, the holonomic (resp. anholonomic) frames can be defined as ones whose basic vector fields commute (resp. do not commute); for details *vide infra* Section 8.

Any change

$$\{E_i\}\mapsto\{E'_i = A_i^j E_j\} \tag{2.16}$$

of a frame $\{E_i\}$ on U with a non-degenerate matrix-valued function $A=[A_i^j]$, where $A_i^j\colon U\to\mathbb{K}$, implies the transformation

$$X^i\mapsto X'^i = \left(A^{-1}\right)_j^i X^j \tag{2.17}$$

of the local components of some vector field $X = X^iE_i = X'^iE'_i$.

[12] For details, see Section 8.

[13] A frame given only on U can be extended outside U, if $U\subset M$, in a holonomic as well as in anholonomic way. (Cf. Remark 5.1 on page 40 or see Lemma II.5.2 on page 116.)

A vector field X on $U \subseteq M$, M being a C^k differentiable manifold, is said to be differentiable of class C^r, or simply is called C^r (vector) field, with $r \leq k$ if for every C^{r+1} function $f\colon U \to \mathbb{K}$ the function $X(f)\colon U \to \mathbb{K}$ is of class C^r. A vector field is of class C^r if and only if its local components are C^r functions in any local chart (see (2.13)). The (left) module of all C^r vector fields on U will be denoted by $\mathfrak{X}^r(U)$.

Define the bracket, called also Lie bracket or commutator, of two C^1 vector fields $X, Y \in \mathfrak{X}^1(U)$ by[14]

$$[X, Y]_{\_} := X \circ Y - Y \circ X. \tag{2.18}$$

In local coordinates $\{x^i\}$, the local components of $[X,Y]_{\_}$ are $[X,Y]_{\_}^j = X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}$; in an arbitrary frame, we have $[X,Y]_{\_}^j = X(Y^i) - Y(X^i)$. The set $\mathfrak{X}^1(U)$ is an infinitely dimensional Lie algebra over the field $\mathbb{K}$ under the operation $[\,\cdot\,,\,\cdot\,]_{\_}\colon \mathfrak{X}^1(U) \times \mathfrak{X}^1(U) \to \mathfrak{X}(U)$.

A frame $\{E_i\}$ on a neighborhood (open set) $U \subseteq M$ is holonomic if and only if $[E_j, E_k]_{\_}|_U = 0$ (see Section 8 for some details).

Further, in Subsection IV.2.4, we shall look on the tangent vector fields as sections of the tangent bundle over a manifold.

Let $X \in \mathfrak{X}^0(U)$ and $\gamma\colon J \to U$ be a C^1 path in a neighborhood $U \subseteq M$. The path γ is called *integral path* (*curve*) for X through $p \in U$ if

$$\dot\gamma(s) = X_{\gamma(s)}, \quad \gamma(s_0) = p \tag{2.19}$$

for some fixed $s_0 \in J$ end every $s \in J$. Through every $p \in U$ passes a (locally) unique integral path for X; so if two integral paths have a common point, they (locally) coincide.[15] On the opposite, if U is filled with a family of non-intersecting C^1 paths passing through every its point, there exists a unique C^1 vector field on U whose integral paths form the given family of paths. It coincides with the field of vectors tangent to the paths of the family.

Let $f\colon M \to N$ be a C^1 mapping between C^1 manifolds M and N and $p \in M$. The *differential or induced tangent mapping* $f_*|_p$ of f at p, denoted also by $\mathrm{d}f_p \equiv \mathrm{d}f|_p$ and $f'_p \equiv f'|_p$, is a linear mapping

$$f_*|_p\colon T_p(M) \to T_{f(p)}(N) \tag{2.20}$$

such that, for all $X_p \in T_p(M)$ and all $g\colon N \to \mathbb{K}$ differentiable at $f(p)$,

$$(f_*|_p(X_p))(g) := X_p(g \circ f). \tag{2.21}$$

Evidently, $f_*|_p$ is a linear mapping. Its matrix, in the coordinate bases associated to charts (U, φ) of M and (V, ψ) of N with $U \ni p$ and $V \ni f(p)$ can be found

[14] Often the Lie bracket is denoted by $[\,\cdot\,,\,\cdot\,]$, omitting the minus sign as a subscript.

[15] For details on integral paths, see, e.g., [9, p. 131ff], [7, Sections 1.46–1.53] or [8, Sections 8.5–8.7].

as follows. Let $\{x^i | i = 1, \ldots, \dim M\}$ and $\{y^j | j = 1, \ldots, \dim N\}$ be the respective coordinate systems and $\{r^i\}$ and $\{v^j\}$ be the standard Cartesian coordinate systems in $\mathbb{K}^{\dim M}$ and $\mathbb{K}^{\dim N}$, respectively. Then

$$\Big(f_*|_p\Big(\frac{\partial}{\partial x^i}\Big|_p\Big)\Big)(g) = \frac{\partial}{\partial x^i}\Big|_p (g \circ f) = \frac{\partial (g \circ f \circ \varphi^{-1})}{\partial r^i}\Big|_{\varphi(p)}$$
$$= \sum_{j=1}^{\dim N} \frac{\partial (g \circ f \circ \varphi^{-1})}{\partial v^j}\Big|_{\varphi(p)} \frac{\partial (y^j \circ f \circ \varphi^{-1})}{\partial r^i}\Big|_{\varphi(p)} = \sum_{j=1}^{\dim N} \frac{\partial (g \circ \psi^{-1})}{\partial v^j}\Big|_{\psi(f(p))}$$
$$\times \frac{\partial (y^j \circ f \circ \varphi^{-1})}{\partial r^i}\Big|_{\varphi(p)} = \Big(\sum_{j=1}^{\dim N} \frac{\partial (y^j \circ f)}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{f(p)}\Big)(g),$$

so that

$$f_*|_p\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_{j=1}^{\dim N} \frac{\partial (y^j \circ f \circ \varphi^{-1})}{\partial r^i}\Big|_{\varphi(p)} \frac{\partial}{\partial y^j}\Big|_{f(p)}. = \sum_{j=1}^{\dim N} \frac{\partial (y^j \circ f)}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{f(p)}. \tag{2.22}$$

Therefore the matrix of $f_*|_p$ in the bases chosen has elements

$$\frac{\partial (y^j \circ f \circ \varphi^{-1})}{\partial r^i}\Big|_{\varphi(p)} = \frac{\partial}{\partial x^i}\Big|_p (y^j \circ f) = \frac{\partial (y^j \circ f)}{\partial x^i}\Big|_p, \tag{2.23}$$

where $i = 1, \ldots, \dim M$ and $j = 1, \ldots, \dim N$, and, consequently, coincides with the Jacobi matrix of the mapping f (with respect to the given coordinate systems).

Example 2.8. If $\gamma \colon \mathbb{R} \to M$ is a C^1 path in a C^1 manifold M, $s \in \mathbb{R}$, and we choose a chart (U, φ) of M with $U \ni \gamma(s)$ and the standard chart $(\mathbb{R}, \mathsf{id}_{\mathbb{R}})$ of $\mathbb{R}$, with coordinate function $\mathsf{id}_{\mathbb{R}}$ (as $r^1 = \mathsf{id}_{\mathbb{R}}$ in the one-dimensional case), then the differential $\gamma_*|_s$ at s maps the standard basic vector $\frac{\partial}{\partial r^1}\big|_s$ of $T_s(\mathbb{R})$ into the tangent vector $\dot{\gamma}(s)$ of γ at s and the matrix of $\gamma_*|_s$ consists of the components of $\dot{\gamma}(s)$:

$$\gamma_*|_s\Big(\frac{\partial}{\partial r^1}\Big|_s\Big) = \sum_i \frac{\partial (x^i \circ \gamma)}{\partial r^1}\Big|_s \frac{\partial}{\partial x^i}\Big|_{\gamma(s)} = \dot{\gamma}^i(s) \frac{\partial}{\partial x^i}\Big|_{\gamma(s)} = \dot{\gamma}(s).$$

Note that often r^1 is identified with s and one writes $\dot{\gamma}^i(s) = \frac{\partial (x^i \circ \gamma(s))}{\partial s}$.

Analogously, one can prove that the matrix of the differential at $p \in M$ of a function $f \colon M \to \mathbb{K}$ is formed from the gradient components of f, i.e., they are $\frac{\partial (f \circ \varphi^{-1})}{\partial r^i}\big|_{\varphi(p)}$ in appropriate bases.

A C^1 mapping $f \colon M \to N$ between C^1 manifolds is called *regular* if its differential $f_*|_p$ at $p \in M$ is one-to-one (injective) for every $p \in M$. The above example (with interval J for $\mathbb{R}$) shows that a C^1 path $\gamma \colon J \to M$ is regular iff its tangent vector $\dot{\gamma}$ is non-zero everywhere. Generally, a mapping $f \colon M \to N$ is regular iff $\dim_{\mathbb{R}} M \le \dim_{\mathbb{R}} N$ and the Jacobi matrix of f in some (and hence in any) bases has maximum rank, equal to $\dim_{\mathbb{R}} M$ (in real terms).

Details concerning the differential of C^1 mappings can be found in most of the standard (text)books on differential geometry, for instance, in [10, 11, 15].

2.4. Covectors and covector fields

The vector space $T_p^*(M)$ dual to the tangent space $T_p(M)$, $p \in M$, consists of the linear forms on $T_p(M)$. It is called the *cotangent* (*vector*) *space* to M at p and is of dimension $n = \dim M$. The elements of $T_p^*(M)$ are referred as *covectors*, *covariant vectors*, or *covariant tangent vectors* at p. For $X_p \in T_p(M)$, the covector $\omega_p \in T_p^*(M)$ is a linear mapping[16]

$$\omega_p \colon X_p \mapsto \omega_p(X_p) \in \mathbb{K}$$

If $\{E_i|_p\}$ is a basis in $T_p(M)$, its *dual basis* $\{E^i|_p\}$ in T_p^* is uniquely defined by

$$E^i|_p\big(E_j|_p\big) := \delta^i_j \tag{2.24}$$

where δ^j_i are the Kronecker (delta-)symbols, $\delta^j_i = 1$ for $i = j$ and $\delta^j_i = 0$ for $i \neq j$. If $\{x^i\}$ are local coordinates in a neighborhood of p, the (full) differentials $\mathrm{d}x^i|_p$ at p form the natural cobasis $\{\mathrm{d}x^i|_p\}$ with respect to $\{\partial/\partial x^i|_p\}$; $\mathrm{d}x^j\big(\partial/\partial x^i|_p\big) = \delta^j_i$. For the expansion of $\omega|_p \in T_p^*(M)$ with respect to $\{E^i|_p\}$, we write $\omega_p = (\omega_p)_i E^i|_p \equiv \omega_{p\,i} E^i|_p$. So, the components of covectors/vectors are labeled by subscripts/superscripts.

The change $\{E^i|_p\} \mapsto \{E'^i|_p = B^i_j(p) E^j|_p\}$ with a nondegenerate matrix $B(p) = [B^i_j(p)]$ leads to $\omega_{p\,i} \mapsto \omega'_{p\,i} = \big(B^{-1}(p)\big)^j_i \omega_{p\,j}$. Such a transformations are rarely used. The wide spread case is a transform $\{E^i|_p\} \mapsto \{E'^i|_p\}$ induced by a change $\{E_i|_p\} \mapsto \{E'_i|_p = A^j_i(p) E_j|_p\}$ of the dual basis in $T_p(M)$. In this situation, by (2.24), we have

$$E^i\big|_p \mapsto E'^i\big|_p = \big(A^{-1}(p)\big)^i_j(p) E^j|_p \tag{2.25}$$

while the covectors' components transform like (cf. (2.12))

$$\omega_{p\,i} \mapsto \omega'_{p\,i} = A^j_i(p) \omega_{p\,j}. \tag{2.26}$$

Hence, under (2.11) the covectors' components transform like a basis of vectors and the vectors' components change like a cobasis (basis of covectors).

[16] Sometimes the notation $\omega_p \colon X_p \mapsto \langle \omega_p, X_p \rangle$ is used. It expresses the action of the Kronecker (unit) tensor (see below) on the ordered pair (ω_p, X_p). We prefer to reserve the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ for the scalar products in quantum mechanics as accepted in this theory. More precisely, at present, in it the symbol $\langle\,\cdot\,|\,\cdot\,\rangle$ for the Hermitian scalar product is used which sometimes is replaced by $\langle\,\cdot\,,\,\cdot\,\rangle$ or $(\,\cdot\,,\,\cdot\,)$.

If local coordinate $\{x^i\}$ and $\{x'^{\,i}\}$ in charts (U,φ) and (U',φ'), respectively, are employed and $U\cap U'\neq\varnothing$, then it is fulfilled (see (2.9))[17]

$$A^j_i(p)=\frac{\partial x^j}{\partial x'^{\,i}}\Big|_p:=\frac{\partial(x^j\circ\varphi'^{\,-1})}{\partial r^i}\Big|_{\varphi'(p)},\qquad p\in U\cap U'\neq\varnothing. \tag{2.27}$$

Therefore (2.25) and (2.26) now read (cf. (2.9) and (2.8) resp.)

$$\mathrm{d}x^i|_p\mapsto \mathrm{d}x'^{\,i}|_p=\frac{\partial x'^{\,i}}{\partial x^j}\Big|_p\mathrm{d}x^j|_p:=\frac{\partial(x'^{\,i}\circ\varphi^{-1})}{\partial r^j}\Big|_{\varphi(p)}\mathrm{d}x^j|_p \tag{2.28}$$

$$\omega_{p\,i}\mapsto\omega'_{p\,i}=\frac{\partial x^j}{\partial x'^{\,i}}\Big|_p\omega_{p\,j}:=\frac{\partial(x^j\circ\varphi'^{\,-1})}{\partial r^i}\Big|_{\varphi'(p)}\omega_{p\,j}. \tag{2.29}$$

A *covariant vector* (*covector*) *field* ω on $U\subseteq M$ is a mapping assigning to each $p\in U$ a covector at p, $\omega\colon p\mapsto\omega_p\in T^*_p(M)$. It is often called a *one-form* (abbreviated to *1-form*), or *differential form* of degree 1.[18] Equivalently, a 1-form ω on U can be considered (defined) as $\mathfrak{F}(U)$-linear mapping from the (left) $\mathfrak{F}(U)$-module $\mathfrak{X}(U)$ into $\mathfrak{F}(U)$ given by

$$\omega\colon X\mapsto\omega(X),\quad \omega(X)\colon p\mapsto\omega_p(X_p),\qquad X\in\mathfrak{X}(U),\ p\in U. \tag{2.30}$$

In a chart (U,φ), the full differentials $\mathrm{d}x^i$ of the local coordinates $\{x^i\}$ form a natural basis $\{\mathrm{d}x^i\}$ in the set of 1-forms on U which is dual to the basis $\{\partial/\partial x^i\}$ in $\mathfrak{X}(U)$:

$$\omega=\omega_i\mathrm{d}x^i,\qquad \mathrm{d}x^i\big(\partial/\partial x^j\big):=\delta^i_j. \tag{2.31}$$

More generally, if $\{E_i\}$ is a frame on U, i.e., a basis in the module $\mathfrak{X}(U)$, then in the dual to it *coframe* $\{E^i\}$ we have

$$\omega=\omega_iE^i,\quad E^i:=(E_i)^*,\quad E^i(E_j):=\delta^i_j. \tag{2.32}$$

In such a pair $(\{E_i\},\{E^i\})$ of frame and coframe, the mapping $\omega\colon X\mapsto\omega(X)$ is given by $\omega(X)=\omega_iX^i\in\mathfrak{F}(U)$. The change (2.16) of the frame implies a change of the dual coframe and of the component of a covector field in it:

$$E^i\mapsto E'^{\,i}=\big(A^{-1}\big)^i_jE^j,\quad \omega_i\mapsto\omega'_i=A^j_i\omega_j. \tag{2.33}$$

In a case of a coordinate frame $\{\partial/\partial x^i\}$ and its dual coframe $\{\mathrm{d}x^i\}$ we, obviously, obtain

$$A^i_j=\frac{\partial x^i}{\partial x'^{\,j}}\colon p\mapsto\frac{\partial x^i}{\partial x'^{\,j}}\Big|_p=\frac{\partial(x^i\circ\varphi'^{\,-1})}{\partial r^j}\Big|_{\varphi'(p)}. \tag{2.34}$$

A covector field ω on U is said to be differentiable of class C^r if for $X\in\mathfrak{X}^r(U)$ the function $\omega(X)$ is of class C^r or, equivalently, the components of ω (in one and hence in any coframe) are in $\mathfrak{F}^r(U)$. For brevity such a covector field is called C^r covector field (or C^r 1-form).

[17]Recall that $x^i:=r^i\circ\varphi$ and $r^i\colon\mathbb{K}^n\to\mathbb{K}$ is defined by $r^i(c^1,\dots,c^n):=c^i$, $c^i\in\mathbb{K}$.

[18]With a few exceptions, in this work differential forms of degree greater than one (and the exterior algebra) will not be considered.

2.5. Tensors and tensor fields. Tensor algebras

Denote by $\otimes$ the tensor product sign (see, e.g., [11, Chapter 1, § 2]) and let $\otimes^r V := V \otimes \cdots \otimes V$ where the vector or vector space V over $\mathbb{K}$ is taken $r \in \mathbb{N}$ times; by definition $\otimes^0 V := \mathbb{K}$.

The *tensor space* of type (r, s), $r, s \in \mathbb{N} \cup \{0\}$, (and order (rank) $r + s$) at (over) $p \in M$ is

$$T_p{}^r_s(M) := \big[\otimes^r \big(T_p^*(M)\big)\big] \otimes \big[\otimes^s \big(T_p(M)\big)\big] \tag{2.35}$$

and consists (for $r + s \geq 1$) of all multilinear forms

$$\big[\times^r \big(T_p(M)\big)\big] \times \big[\times^s \big(T_p^*(M)\big)\big] \to \mathbb{K}, \tag{2.36}$$

with $\times^r V := V \times \cdots \times V$ (r-times). It is a $\mathbb{K}$-linear space of dimension $(\dim M)^{r+s}$. Since the tensor product is not commutative, the order of the multipliers in (2.35) is essential.

Remark 2.2. More generally, the tensor space of type $(r_1, \dots, r_m; s_1, \dots, s_m) \in \times^{2m}\big(\mathbb{N} \cup \{0\}\big)$ at p is defined as

$$T_p{}^{r_1}{}_{s_1}{}^{\dots}{}_{\dots}{}^{r_m}{}_{s_m} := \big[\otimes^{r_1} T_p^*(M)\big] \otimes \big[\otimes^{s_1} T_p(M)\big] \otimes \cdots \otimes \big[\otimes^{r_m} T_p^*(M)\big] \otimes \big[\otimes^{s_m} T_p(M)\big].$$

All such space with $\sum_{a=1}^m (r_a + s_a) = \text{const}$ are isomorphic but different unless their types coincide. In the complex case, $\mathbb{K} = \mathbb{C}$, some of the space $T_p(M)$ and $T_p^*(M)$ can be replaced with their complex conjugate spaces $\bar{T}_p(M)$ and $\bar{T}_p^*(M)$, respectively; for instance, a Hermitian metric on a complex manifold M at p is a Hermitian form on $T_p(M) \times T_p(M)$ but, equivalently it is a bilinear mapping $T_p(M) \times \overline{T}_p(M) \to \mathbb{C}$ whose transpose is equal to its complex conjugate mapping. In this book only tensor spaces of the type (2.35) will be involved. All results (and definitions) in it can *mutatis mutandis* be transferred to the above general cases which is a simple technical problem.

The elements of (2.35) are referred as tensors, or, if their type must be specified, tensors of type (r, s) (and rank $r + s$). They are also called r-contravariant (r times contravariant) and s-covariant (s times covariant) tensors, or tensors of contravariant degree r and covariant degree s.

In (2.35) can, of course, be introduced an arbitrary basis $E_{i_1 \dots i_r}^{j_1 \dots j_s}\big|_p$, all indices running from 1 to $\dim M$, but this is done very rarely. The interesting case, which is practically the only one considered in the literature and in our book, is when the basis is induced by some basis $\{E_i|_p\}$ in $T_p(M)$ and its dual basis $\{E^i|_p := (E_i|_p)^*\}$ in $T_p^*(M)$:

$$E_{i_1 \dots i_r}^{j_1 \dots j_s}\big|_p = E^{j_1}\big|_p \otimes \cdots \otimes E^{j_s}\big|_p \otimes E_{i_1}\big|_p \otimes \cdots \otimes E_{i_r}\big|_p. \tag{2.37}$$

The *components* $K^{i_1\dots i_r}_{j_1\dots j_s}(p)\in\mathbb{K}$ of $K_p\in T_{p}{}^{r}_{s}$ are defined by the expansion

$$K_p = K^{i_1\dots i_r}_{j_1\dots j_s}(p)E^{j_1}\big|_p\otimes\dots\otimes E^{j_s}\big|_p\otimes E_{i_1}\big|_p\otimes\dots\otimes E_{i_r}\big|_p. \tag{2.38}$$

where the Einstein's summation convention (see p. xi) is used. It is an elementary exercise to verify that a transform $\{E_i|_p\}\mapsto\{E'_i|_p = A^j_i(p)E_j|_p\}$ of the basis in $T_p(M)$ induces the changes:

$$E^{j_1\dots j_s}_{i_1\dots i_r}\big|_p\mapsto E'^{j_1\dots j_s}_{i_1\dots i_r}\big|_p = A^{k_1}_{i_1}(p)\cdots A^{k_r}_{i_r}(p)\big(A^{-1}(p)\big)^{j_1}_{l_1}\cdots\big(A^{-1}(p)\big)^{j_s}_{l_s}E^{l_1\dots l_s}_{k_1\dots k_r}\big|_p \tag{2.39}$$

$$\begin{aligned}K^{i_1\dots i_r}_{j_1\dots j_s}(p)\mapsto{}& K'^{i_1\dots i_r}_{j_1\dots j_s}(p)\\ &= \big(A^{-1}(p)\big)^{i_1}_{k_1}\cdots\big(A^{-1}(p)\big)^{i_r}_{k_r}A^{l_1}_{j_1}(p)\cdots A^{l_s}_{j_s}(p)K^{k_1\dots k_r}_{l_1\dots l_s}(p)\end{aligned} \tag{2.40}$$

where $A(p)=[A^j_i(p)]$ is a nondegenerate (constant) matrix.

The *tensor algebra* $\boldsymbol{T}_p(M)$ at $p\in M$ is the direct sum of all tensor spaces $T_{p}{}^{r}_{s}$ for $r,s\geq 0$ with the ordinary tensor multiplication[19] as the algebra's multiplication. This algebra is associative but non-commutative algebra.

A *tensor field* K of type (r,s) on $U\subseteq M$ is a mapping $K\colon p\mapsto K_p\in T_{p}{}^{r}_{s}(M)$, $p\in U$. The set $\mathfrak{T}^r_s(U)$ of tensor fields on U consists of all $\mathfrak{F}(U)$-multilinear mappings from the Cartesian product of r copies if the set of covectors on U and s copies of $\mathfrak{X}(U)$ into $\mathfrak{F}(U)$. It is a natural left $\mathfrak{F}(U)$-module: for $K\in\mathfrak{T}^r_s(U)$ and $f\in\mathfrak{F}(U)$, we define $fK\colon U\to\mathfrak{T}^r_s(U)$ with $fK\colon p\mapsto(fK)_p := f(p)K_p$. It is also a $\mathbb{K}$-linear space: for $\alpha,\beta\in\mathbb{K}$ and $K,L\in\mathfrak{T}^r_s(U)$, the field $\alpha K+\beta L\in\mathfrak{T}^r_s(U)$ is defined pointwise, viz. $(\alpha K+\beta L)\colon p\mapsto(\alpha K+\beta L)_p := \alpha K_p+\beta L_p$. As a linear space $\mathfrak{T}^r_s(U)$ is infinitely dimensional but as a module its rank is $(\dim M)^{r+s}$.

A frame $\{E_i\}$ on U induces the *tensor frame* (cf. (2.37))

$$E^{j_1\dots j_s}_{i_1\dots i_r} = E^{j_1}\otimes\dots\otimes E^{j_s}\otimes E_{i_1}\otimes\dots\otimes E_{i_r}. \tag{2.41}$$

where $\{E^i=(E_i)^*\}$ is the coframe on U dual to $\{E_i\}$. With respect to (2.41) a field $K\in\mathfrak{T}^r_s(U)$ has the representation

$$K = K^{i_1\dots i_r}_{j_1\dots j_s}E^{j_1}\otimes\dots\otimes E^{j_s}\otimes E_{i_1}\otimes\dots\otimes E_{i_r}. \tag{2.42}$$

where $K^{i_1\dots i_r}_{j_1\dots j_s}\in\mathfrak{F}(U)$ are called *components* of K with respect to (2.41) (or with respect to $\{E_i\}$, or to some local coordinate $\{x^i\}$ on U if $E_i=\frac{\partial}{\partial x^i}$). Analogously to (2.39) and (2.40), a change $\{E_i\}\mapsto\{E'_i = A^j_iE_j\}$ of the frame $\{E_i\}$ with a non-degenerate matrix-valued function $A=[A^j_i]$ implies

$$E^{j_1\dots j_s}_{i_1\dots i_r}\mapsto E'^{j_1\dots j_s}_{i_1\dots i_r} = A^{k_1}_{i_1}\cdots A^{k_r}_{i_r}\big(A^{-1}\big)^{j_1}_{l_1}\cdots\big(A^{-1}\big)^{j_s}_{l_s}E^{l_1\dots l_s}_{k_1\dots k_r} \tag{2.43}$$

[19]See, e.g, [11, Chapter 1, § 2] for definition of tensor multiplication. Note that $\alpha\otimes K_p = K_p\otimes\alpha = \alpha K_p$ for $\alpha\in\mathbb{K}$ and a tensor K_p at p

$$K^{i_1\dots i_r}_{j_1\dots j_s} \mapsto K'^{i_1\dots i_r}_{j_1\dots j_s} = \left(A^{-1}\right)^{i_1}_{k_1} \cdots \left(A^{-1}\right)^{i_r}_{k_r} A^{l_1}_{j_1} \cdots A^{l_s}_{j_s} K^{k_1\dots k_r}_{l_1\dots l_s}. \tag{2.44}$$

A tensor field is of class C^r, or is a C^r (tensor) field, if its components are of class C^r in some, and hence in any, tensor frame. The set of all C^k tensor fields of type (r, s) on U will be denote by $\mathfrak{T}^{r;k}_s(U)$.

Example 2.9. An example of C^∞ tensor field on M is the *unit tensor* (*field*), called also the *Kronecker tensor* (*field*). In any local frame $\{E_i\}$ it is given by

$$E_i \otimes E^i = \sum_{i=1}^{\dim M} E_i \otimes E^i. \tag{2.45}$$

and, consequently, its components are the Kronecker (delta-)symbols:

$$\delta^j_i = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}. \tag{2.46}$$

The *algebra* $\boldsymbol{T}(U)$ *of the tensor fields* on $U \subseteq M$ is the direct sum of all $\mathfrak{T}^r_s(U)$ for $r, s \geq 0$ with the tensor product of tensor fields as algebra's multiplication.[20] This algebra over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is associative but not commutative. If we restrict ourselves to C^r tensor fields, the corresponding algebra will be denoted by $\boldsymbol{T}^r(U)$.

At the end of this section, we define the *contraction operator(s)*. Let the integers m, n, r, and s be such that $1 \leq m \leq r$ and $1 \leq n \leq s$. The *contraction operator* C^m_n (of type (m, n), acting on the mth superscript and nth subscript) maps a tensor or tensor field K of type (r, s) into respectively a tensor or tensor field $C^m_n K$ of type $(r - 1, s - 1)$ such that in any local basis or frame respectively, we have

$$\left(C^m_n K\right)^{j_1\dots j_{r-1}}_{i_1\dots i_{r-1}} := \sum_{k=1}^{\dim M} K^{j_1\dots j_{n-1} k j_{n+1}\dots j_{r-1}}_{i_1\dots i_{m-1} k i_{m+1}\dots i_{r-1}}. \tag{2.47}$$

This definition is independent of the basis or frame in which the last equation is written; for example, $C^1_1(E^i \otimes E_i) = \delta^i_i = \dim M$ and $C^1_1(\omega \otimes X) = \omega(X)$ for vector (field) X and covector (field) ω. Generally two contraction operators do not commute. If the numbers m and n are arbitrary (and insignificant) for some problem, we shall speak simply of a contraction operator and denote it by C.

3. Linear connections on a differentiable manifold

Below we define the linear connections on a differentiable manifold by means of axiomatic description of the properties of covariant derivative. We give also an idea of a parallel transport and the connected to it concepts of geodesics and exponential mapping.

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[20]Note that $f \otimes K = K \otimes f = fK$ for $f \in \mathfrak{F}(U)$ and a tensor field K on U. (Cf. footnote 19 on the preceding page.)

3.1. Motivation

The comparison of objects defined or given at different points is the main idea and task of the theory of connections.[1] The mathematical structures by means of which this is done are known as *connections*. In more free and general language, by this term is understand any rule for implicit or explicit 'transportation' of objects from one point to another. A typical physical situation where these kind of problems arise is the following. Given an electromagnetic field in a region U of the space-time M and two (e.g., identical) charged point particles moving in U. The problems are to compare the velocities of the particles, the forces acting on them, and to describe (in an invariant way) the relative motion of one of the particles with respect to the other one. Leaving aside the last problem, the first two ones reduce to the comparison of two vectors, say X and Y, defined at points $p, q \in M$ and representing the particle's velocities or the forces acting on them. If for the space-time model M is taken $\mathbb{R}^3$, as in the classical physics, we can simply form the difference $X - Y$ of $X, Y \in \mathbb{R}^3$.[2] But if M is a more general manifold, for instance the one of general relativity, then the difference $X - Y$ is meaningless unless it is explicitly defined. Just for this purpose a connection is needed; in this case it must establish a link between the tangent spaces $T_p(M) \ni X$ and $T_q(M) \ni Y$. If this is done appropriately, we can *define* $X - Y$ as, e.g., $X - \bar{Y} \in T_p(M)$ where $\bar{Y} \in T_p(M)$ is the vector corresponding to $Y \in T_q(M)$ with respect to the connection. As one can expect, such a procedure is not unique (if it exists) and when physical problems, like the above one, are investigated, the correct correspondence between the theory's predictions and the Nature is the only criteria that can select the 'right' connection.

Without any doubts, the natural scene where the connections 'live' are the (fibre) bundles (see Section V.3) which is clearly reflected in the modern mathematical literature [10–12, 16] where the connection theory is primary described in terms of fibre bundles and only then is specified on manifolds. But we shall not follow this general approach for the following reasons. The historical order of events is just the opposite and for the main purpose of this book, the description of normal frames, is better if it is followed. This is justified and from pedagogical view-point: beginning with simple concepts, we step by step generalize them, the positive results at a lower level being a motivation for further developments of the theory. Besides, until recently (1992) the normal frames ware known only for symmetric linear connections and the presentation of general results (with a few applications at the moment) at the beginning may cause some phycological problems and push away the reader, especially if he/she is a physicists. At last, the

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[1] In the old literature (see, e.g., [19]) the word 'connexion' instead of 'connection' is used. Nowadays this is rarely done [10, 28].

[2] More precisely, the velocities X and Y are elements of the tangent spaces $T_p(\mathbb{R}^3)$ and $T_q(\mathbb{R}^3)$ which are naturally identified with $\mathbb{R}^3$, where, after the identification is done, the difference $X-Y$ is formed. In fact, this implicit convention is a connection on $\mathbb{R}^3$. The situation, when X and Y represent forces, is similar to the one when these vector fields represent velocities.

physical applications of the normal frames concern at present mainly the gravity physics.

In the connection theory there are two basic concepts, a 'connection' and 'parallel transport', the latter being called sometimes 'parallel translation' [22, Chapter VII, § 5]. In the majority of the literature, the connection is taken as a primitive one and the parallel transport is defined on its base. We shall follow this approach in the present and the next two chapters. It is also possible to take as an initial concept the parallel transport and on its ground to define what a connection is (see, e.g., [23]). This point of view is, by our opinion, more general (and difficult) and we utilize it in Chapter IV.

3.2. Basic definitions

The shortest way to come to the concept 'connection' on a manifold M is to try to define what a 'derivative of a C^1 vector field' is. The direct transferring of the definition of the derivative of a C^1 function $\mathbb{K} \to \mathbb{K}$ is impossible since the difference $X_p - X_q$, $X \in \mathfrak{X}^1(U)$, $p, q \in U \subseteq M$ is not defined for $q \neq p$. If p and q belong to one and the same coordinate neighborhood U, the quantities $X_p^i - X_q^i$ and $\partial X_p^i / \partial x^i$, with $\{x^i\}$ being local coordinate system in U, are well defined but are not components of a vector (field). It is well known that these quantities can be 'repaired' (redefined) in such a way that this results in the (equivalent in this context) concepts parallel transport [20] and covariant derivative($\equiv$ connection here) [21], respectively. This approach is typical for the elementary textbooks on general relativity, as the above-cited. Below we present an equivalent to these methods axiomatical point of view which is almost standard for the modern differential geometry.[3]

Definition 3.1. A *linear connection* (*or covariant derivative*) ∇ in (on) a neighborhood $U \subseteq M$, M being C^1 manifold, is a mapping assigning to every vector field X on U a mapping ∇_X, called *covariant derivative along* X, of the tensor algebra of C^1 tensor fields on U into the algebra of tensor fields on U, i.e., $\nabla\colon X \mapsto \nabla_X$, $X \in \mathfrak{X}(U)$ with $\nabla_X\colon \boldsymbol{T}^1(U) \to \boldsymbol{T}(U)$, such that:

- **(i)** $\nabla_{X+Y} = \nabla_X + \nabla_Y, \quad X, Y \in \mathfrak{X}(U);$
- **(ii)** $\nabla_{fX} = f\nabla_X, \quad f \in \mathfrak{F}(U);$
- **(iii)** $\nabla_X(K + L) = \nabla_X K + \nabla_X L, \quad K, L \in \mathfrak{T}_s^{r;1}(U);$
- **(iv)** $\nabla_X(K \otimes L) = (\nabla_X K) \otimes L + K \otimes (\nabla_X L), \quad K, L \in \boldsymbol{T}^1(U);$
- **(v a)** $\nabla_X\colon \mathfrak{F}^1(U) \to \mathfrak{F}^0(U);$
- **(v b)** $\nabla_X\colon \mathfrak{T}_0^{1;1}(U) \to \mathfrak{T}_0^{1;1}(U);$
- **(v c)** $\nabla_X\colon \mathfrak{T}_1^{0;1}(U) \to \mathfrak{T}_{1;1}^{0}(U);$
- **(vi)** $\nabla_X(g) = X(g), \quad g \in \mathfrak{F}^1(U);$
- **(vii)** $\nabla_X(\omega(Z)) = C_1^1\big(\nabla_X(\omega \otimes Z)\big), \quad \omega \in \mathfrak{T}_1^{0;1}(U),\ Z \in \mathfrak{T}_0^{1;1}(U).$

[3] Other equivalent definitions, as well as extended comments on them, can be found in the specialized literature on differential geometry; see, e.g., [9, 11, 12].

Comments 3.1. There exist a number of equivalent definitions of a linear connection on a manifold. For this reason, we make below some remarks on the above one:

(1) Conditions (i) and (ii) specify that ∇ is $\mathfrak{F}(U)$-linear with respect to the vector fields along which (in the direction of which) it acts.

(2) Condition (iii) means that ∇_X is compatible with the additive structure of the tensor algebra on U.

(3) According to the Leibnitz rule/formula (iv), ∇_X is a derivation with respect to the tensor multiplication, i.e., it is a derivation of $\mathfrak{T}^1(U)$.

(4) The conditions (v) mean that the covariant derivative maps C^1 functions in functions and C^1 vector/covector fields in vector/covector fields. Taking this into account, one can derive from (iii) and (iv) that ∇_X preserves the types of the tensor fields. That is why often (va)–(vc) are replaced by the demand ∇_X to be type-preserving, but this is partially contained in the preceding axioms.

(5) Conditions (i)–(v) uniquely define ∇_X up to its action on C^1 functions, vector and covector fields. This arbitrariness in the definition of ∇_X is considerably reduced by (vi) and (vii): Condition (vi) defines ∇_X on C^1 functions while from (vii) follows that it is uniquely defined if its action on C^1 vector (or, equivalently, covector) fields is fixed.

(6) Often condition (vii) is formulated as: ∇_X commutes with the contraction operator C, i.e., $[\nabla_X, C^r_s]_- = 0$ for every $r, s \in \mathbb{N}$. Such a general demand is not needed as it follows from the presented axioms.

For the conditions under which a manifold admits linear connections, the reader is referred to the books on differential geometry cited in Section 1; in particular, to those problems are devoted [11, Chapter II, § 2] and [22, Chapter VII, § 3].

Below we suppose on a manifold M (of class at least C^1) to be given a linear connection ∇. Everything of what follows can be specified in an evident way on a neighborhood $U \subset M$.

Let $\{E_i\}$ be a C^1 frame on $U \subseteq M$. The (*local*) *coefficients* $\Gamma^i{}_{jk}$ in $\{E_i\}$ of a linear connection ∇ on U are defined by the expansion (see Definition 3.1, condition (v b))

$$\nabla_{E_k} E_j =: \Gamma^i{}_{jk} E_i \tag{3.1}$$

where all indices run from 1 to $\dim M$ and summation on repeated indices is assumed.[4]

[4]The functions $\Gamma^i{}_{jk}$ are sometimes called Christoffel symbols but we shall preserve this term for a special kind of $\Gamma^i{}_{jk}$ on a Riemannian manifold (see Section 4 and footnote 9 on page 36). Note the order of the indices j and k in the both sides of (3.1). In some works, like [11, 12], in the right-hand side of (3.1) instead of $\Gamma^i{}_{jk}$ stands Γ^i_{kj}. This results in the change of signs before certain terms in some equations.

If K is a C^1 tensor field of type (r,s) with expansion (2.42) in $\{E_i\}$, then it is a simple exercise to verify the formula

$$\nabla_X K = K^{i_1\dots i_r}_{j_1\dots j_s;k} X^k E^{j_1}\otimes\dots\otimes E^{j_s}\otimes E_{i_1}\otimes\dots\otimes E_{i_r}. \tag{3.2}$$

where

$$\begin{aligned} K^{i_1\dots i_r}_{j_1\dots j_s;k} := E_k\bigl(K^{i_1\dots i_r}_{j_1\dots j_s}\bigr) &+ \sum_{a=1}^{r} \Gamma^{i_a}{}_{lk} K^{i_1\dots i_{a-1} l i_{a+1}\dots i_r}_{j_1\dots j_s} \\ &- \sum_{b=1}^{s} \Gamma^{l}{}_{j_b k} K^{i_1\dots i_r}_{j_1\dots j_{b-1} l j_{b+1}\dots j_s} \end{aligned} \tag{3.3}$$

are the *components* of the so-called *covariant differential* $\nabla K \in \mathfrak{T}^r_{s+1}(U)$ of K defined by (see (2.36))

$$\begin{aligned} (\nabla K)\bigl(\omega^{(1)},\dots,\omega^{(r)},X_{(1)},\dots,X_{(s)};X\bigr) \\ := \bigl(\nabla_X K\bigr)\bigl(\omega^{(1)},\dots,\omega^{(r)},X_{(1)},\dots,X_{(s)}\bigr) \end{aligned} \tag{3.4}$$

with $\omega^{(1)},\dots,\omega^{(r)} \in \mathfrak{T}^0_1(U)$ and $X_{(1)},\dots,X_{(s)} \in \mathfrak{T}^1_0(U)$.

If we make a change $\{E_i\}\mapsto\{E'_i = A^j_i E_j\}$ of the frame with a non-degenerate C^1 matrix-valued function $A = [A^j_i]$, equation (3.1) implies the transformation[5]

$$\Gamma^i{}_{jk} \mapsto \Gamma'^{\,i}{}_{jk} = \bigl(A^{-1}\bigr)^i_l A^m_j A^n_k \Gamma^l{}_{mn} + \bigl(A^{-1}\bigr)^i_l E'_k\bigl(A^l_j\bigr) \tag{3.5}$$

of the coefficients of a linear connection ∇. Because of the importance of this result for the present book, we will rewrite it in the case of frames associated with two local C^2 coordinate systems $\{x^i\}$ and $\{x'^{\,i}\}$ in their common domain (see (2.34)):

$$\Gamma^i{}_{jk} \mapsto \Gamma'^{\,i}{}_{jk} = \frac{\partial x'^{\,i}}{\partial x^l}\frac{\partial x^m}{\partial x'^{\,j}}\frac{\partial x^n}{\partial x'^{\,k}}\Gamma^l{}_{mn} + \frac{\partial x'^{\,i}}{\partial x^l}\frac{\partial^2 x^l}{\partial x'^{\,j}\partial x'^{\,k}}. \tag{3.6}$$

Note, in this result enter the second partial derivatives of the local coordinates, so it is meaningful if the manifold M is of class not less than C^2 which will be supposed in this and the next chapters.

Consequently, a linear connection ∇ defines in any frame $\{E_i\}$ a family of functions $\{\Gamma^i{}_{jk}\}$ which transform according to (3.5). The opposite statement is also true (see, e.g., [11, Chapter III, Proposition 7.3]): if in every frame $\{E_i\}$ is given a family of functions $\{\Gamma^i{}_{jk}\}$ satisfying the transformation law (3.5), then there exists a unique linear connection ∇ whose local coefficients in $\{E_i\}$ coincide with $\{\Gamma^i{}_{jk}\}$.

[5]Notice, here and below we suppose the frames and the matrix-valued function A to be defined on a neighborhood (open set) in M of each point in U, i.e., on a neighborhood of U if U is not a neighborhood. Otherwise expressions like $E_k(A^i_j)$ may turn to be not (uniquely) defined.

The set of linear connections on M is not $\mathfrak{F}(M)$-linear or even $\mathbb{K}$-linear space. In fact, if ∇ and $\bar{\nabla}$ are linear connections with coefficients $\Gamma^i{}_{jk}$ and $\bar{\Gamma}^i{}_{jk}$ respectively and $f, g \in \mathfrak{F}(M)$, then we can define $f\nabla + g\bar{\nabla}$ as a 'linear connection' whose local coefficients are $f\Gamma^i{}_{jk} + g\bar{\Gamma}^i{}_{jk}$ but, by virtue of (3.5), one verifies easily (see the last term in (3.5)) that these functions form coefficients of a linear connection if and only if $f + g = 1$.[6] So, the only $\mathbb{K}$-linear (resp. $\mathfrak{F}(M)$-linear) combination under which the set of linear connection is closed is

$$\Gamma^i{}_{jk}, \bar{\Gamma}^i{}_{jk} \mapsto {}^f\Gamma^i{}_{jk} := f\Gamma^i{}_{jk} + (1-f)\bar{\Gamma}^i{}_{jk} \tag{3.7}$$

where $f \in \mathbb{K}$ (resp. $f \in \mathfrak{F}(M)$). This result is often mentioned when $f = 1/2$. If we restrict ourselves to coordinate frames, i.e., to ones associated with local coordinates, then, due to (3.6), the quantities $f\Gamma^i{}_{jk} + g\bar{\Gamma}^i{}_{kj}$ form coefficients of some linear connection iff $f + g = 1$. Hence, the mapping

$$\Gamma^i{}_{jk}, \bar{\Gamma}^i{}_{jk} \mapsto {}^f\bar{\Gamma}^i{}_{jk} = f\Gamma^i{}_{jk} + (1-f)\bar{\Gamma}^i{}_{kj} \tag{3.8}$$

defines a linear connection if only coordinate frames are considered. In particular, from every linear connection ∇ with local coefficients $\Gamma^i{}_{jk}$ we can form a linear connection ${}^{\mathrm{s}}\nabla$ with coefficients

$${}^{\mathrm{s}}\Gamma^i{}_{jk} := \frac{1}{2}\left(\Gamma^i{}_{jk} + \Gamma^i{}_{kj}\right) =: \Gamma^i{}_{(jk)} \tag{3.9}$$

in some coordinate frame. This is an example of symmetric linear connection. Generally a linear connection is called *symmetric* if in one (and hence in any – see (3.6)) coordinate frame $\{\partial/\partial x^i\}$ its local coefficients $\Gamma^i{}_{jk}$ are symmetric in their subscripts:

$$\Gamma^i{}_{jk} = \Gamma^i{}_{kj}. \tag{3.10}$$

Often these equations are written as

$$\Gamma^i{}_{kj} = \Gamma^i{}_{(kj)} \quad \text{or} \quad \Gamma^i{}_{[kj]} = 0 \tag{3.10$'$}$$

where symmetrization (resp. antisymmetrization) is understood over the indices included in round (resp. square) brackets (see the list of conventions, page xii). It is important to be emphasized, in the general case it is meaningful to be spoken on the symmetry properties of the coefficients of a linear connection only in *coordinate* frames. The same is valid with respect to the transform (3.8), or its special case (3.9): if (anholonomic) *non-coordinate* frames are involved, the quantities ${}^f\bar{\Gamma}^i{}_{jk}$ generally do not define coefficients of some linear connection.

Now we want to define what a C^r linear connection, called also linear connection of class C^r or C^r differentiable linear connection, on a C^k manifold is.[7]

[6] Notice, for $g = -f$ we get the quantities $f\left(\Gamma^i{}_{jk} - \bar{\Gamma}^i{}_{jk}\right)$ which are components of a tensor field of type $(1, 2)$.

[7] In the literature the problem of defining the differentiability of a linear connection does not arise because usually only the C^∞ case is considered and/or only the class of coordinate frames is employed (see below and, e.g., [12, p. 295]).

Our intention is to call ∇ a C^r differentiable if its local coefficients are of class C^r but this is not quite rigorous. Actually, if its coefficients $\Gamma^i{}_{jk}$ in a frame $\{E_i\}$ are C^r functions, then in another frame $\{E'_i = A^j_i E_j\}$ the differentiability of the new coefficients $\Gamma'^i{}_{jk}$ will depend, in conformity with (3.5), on of what class the transformation nondegenerate matrix $A = [A^j_i]$ is. If A^j_i are of class $C^{r'}$, then that of $\Gamma'^i{}_{jk}$ will be $C^{\min(r,r'-1)}$. Consequently the differentiability of the coefficients of a linear connection is *essentially frame-dependent* concept. For this reason, we adopt the following definition. Given a class of frames connected with linear transformations whose matrices are $C^{r'}$, $r' \geq 1$ matrix-valued functions, a linear connection is said to be C^r differentiable with $r \leq r' - 1$ with respect to it if its local coefficients are of class C^r with respect to one (and hence relative to any) frame in the given set of frames. Beginning with this point henceforth, when speaking of a C^r linear connection, we shall suppose, by default if the opposite is not explicitly stated, that the above class of $C^{r'}$, $r' \geq r+1$, frames is fixed and consists of all frames associated to (one or all systems of) local coordinates on the C^k, $k \geq r'+1 \geq r+2$, manifold or some its open subset and all frames that can be obtained from them by means of $C^{r'}$ linear transformations. For example, talking of a C^1 linear connection, we have in mind that it is a linear connection on C^3 manifold and such that its local coefficients in (one or) any frame, which is coordinate or obtainable from a coordinate one by means of C^2 changes, are C^1 functions.[8]

We shall now introduce two basic characteristics of the linear connections which will be mentioned frequently in this book.

The *curvature tensor field* (or simply *curvature*), R, and the *torsion tensor field* (or simply *torsion*), T, of a linear connection ∇ are tensor fields of types respectively $(1,3)$ and $(1,2)$, i.e.,

$$R\colon \mathfrak{T}^0_1(M) \times \big(\times^3 \mathfrak{X}(M)\big) \to \mathbb{K}$$
$$T\colon \mathfrak{T}^0_1(M) \times \big(\times^2 \mathfrak{X}(M)\big) \to \mathbb{K},$$

defined as follows. The curvature R for a C^1 linear connection ∇ on a C^3 manifold M is considered as an $\mathfrak{F}(M)$-trilinear mapping $\times^3\mathfrak{X}(M) \to \mathfrak{X}(M)$ such that $(X,Y,Z) \mapsto R(X,Y)Z$, $X,Y,Z \in \mathfrak{X}(M)$, where the *curvature operator* (called also simply *curvature*) $R(X,Y)$ (along the pair (X,Y)) is

$$R(X,Y) := \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{X\circ Y - Y\circ X} \equiv \big[\nabla_X, \nabla_Y\big]_{\_} - \nabla_{[X,Y]_{\_}}. \quad (3.11)$$

Analogously, the torsion T of an arbitrary linear connection ∇ on a manifold M is treated as an $\mathfrak{F}(M)$-bilinear mapping $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that $(X,Y) \mapsto T(X,Y)$, $X,Y, \in \mathfrak{X}(M)$, where the *torsion operator* (called also simply *torsion*) $T(X,Y)$ (along the pair (X,Y)) is

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]_{\_}. \quad (3.12)$$

[8]For a general and coordinate/frame independent definition of a differentiability class of a connection, see Section V.3.2.

Remark 3.1. Nevertheless that the torsion is defined for arbitrary linear connection on every manifold, this concept is intrinsically linked to the module of vector fields on the manifold, or, equivalently, to the tangent bundle to it, and can not be generalized on more general bundles. On the other hand, the concept of curvature survives a generalizations on vector bundles endowed with linear connections (or, more generally, linear transports along paths), as well as for connections on differentiable bundles.

In a local frame $\{E_i\}$, the components of the curvature R, $R^i{}_{jkl}$, and torsion T, $T^i{}_{jk}$, are defined by the expansions

$$R(E_k, E_l)E_j =: R^i{}_{jkl} E_i, \qquad T(E_j, E_k) =: T^i{}_{jk} E_i$$

and their explicit form is:

$$\begin{aligned} R^i{}_{jkl} &= -2\Gamma^i{}_{j[k,l]} - 2\Gamma^m{}_{j[k}\Gamma^i{}_{|m|l]} - \Gamma^i{}_{jm} C^m_{kl} \\ &= -\Gamma^i{}_{jk,l} + \Gamma^i{}_{jl,k} - \Gamma^m{}_{jk}\Gamma^i{}_{ml} + \Gamma^m{}_{jl}\Gamma^i{}_{mk} - \Gamma^i{}_{jm} C^m_{kl} \end{aligned} \tag{3.13}$$

$$T^i{}_{jk} = -2\Gamma^i{}_{[jk]} - C^i_{jk} = -\Gamma^i{}_{jk} + \Gamma^i{}_{kj} - C^i_{jk}. \tag{3.14}$$

Here the summation and (anti)symmetrization conventions are used (see the list of conventions), $f_{,i} := E_i(f)$ for $f \in \mathfrak{F}^1(M)$ and the (structure) functions $C^i_{jk} \in \mathfrak{F}(M)$ define the commutators of the basis vector fields,

$$[E_j, E_k]_{\_} =: C^i_{jk} E_i. \tag{3.15}$$

Notice, in a coordinate or holonomic frame $C^i_{jk} \equiv 0$ (see Section 8 for some details).

A linear connection with zero curvature (resp. torsion) on $\subseteq M$ is called *flat*, or *curvature free*, or *integrable* (resp. *torsionless* or *torsion free*) on U. If $U = M$ it is call simply flat (resp. torsionless).

Comparing (3.10) with (3.14), we see that a linear connection is symmetric on $U \subseteq M$ iff it is torsion free on U. So, the vanishment of the torsion is the *invariant* way to describe what a symmetric linear connection is.

The properties and geometric interpretation of curvature and torsion, as well as that of linear connections as a whole, are well known and investigated at length in the books on differential geometry pointed in Section 1.

3.3. Parallel transport

The concept of a 'parallel transport', first introduced on surfaces by Levi-Civita in 1917 [29], will be employed in the present and subsequent chapters mainly for many purposes, e.g., for proving and formulating some results. Besides this, we include here a small subsection devoted to it also for the following reasons:

(i) The parallel transport is the practical realization of the ideas underlying the connection theory (see subsection 3.1);

(ii) Up to this point we have introduce the full 'machinery' required for describing what a parallel transport associated to (defined by) a linear connection is;

(iii) This concept can be defined independently of the connection theory by a system of axioms and on its base the (linear) connection can (equivalently) be introduced [17, 23, 30–33] (see also Section IV.11, Definitions IV.11.2 and IV.11.3);

(iv) As we shall see in Chapter IV, the 'transport theory' is extremely suitable for exploring normal frames in vector bundles, in particular for arbitrary linear connections in them.

Let $\gamma\colon J \to M$, J being $\mathbb{R}$-interval, be a C^1 path in a manifold M endowed with a linear connection ∇. Let $V \in \mathfrak{X}(U)$ be arbitrary vector field on a neighborhood $U \supset \gamma(J)$ and such that $V|_{\gamma(J)} = \dot\gamma$, $\dot\gamma$ being the vector field tangent to γ (see (2.3)), i.e., $V_{\gamma(s)} = \dot\gamma(s)$, $s \in J$.[9] Consider the restriction of the action of ∇_V on $\gamma(J)$:

$$\nabla_{\dot\gamma}(K) := (\nabla_V K)|_{\gamma(J)}, \qquad (\nabla_{\dot\gamma} K)|_{\gamma(s)} := \frac{DK}{\mathrm{d}s} := (\nabla_V K)|_{\gamma(s)}. \tag{3.16}$$

The operator $\nabla_{\dot\gamma}$ is called *covariant derivative along the path* γ. In an arbitrary frame on U, due to (2.3), (3.2), and (3.3), we see that, for any tensor field $K \in \mathfrak{T}^{r;1}_{r'}(U)$, the tensor field $\nabla_{\dot\gamma} K$ given by $\nabla_{\dot\gamma} K\colon \gamma(s) \mapsto (\nabla_V K)_{\gamma(s)}$ is independent of U and of the values V_p and K_p for $p \in U\backslash\gamma(J)$:

$$\begin{aligned}(\nabla_{\dot\gamma} K)_{\gamma(s)} = &\Big\{ \frac{\mathrm{d}}{\mathrm{d}s}\Big(K^{i_1\ldots i_{r'}}_{j_1\ldots j_r}\big(\gamma(s)\big)\Big) \\ &+ \sum_{a=1}^{r} \Gamma^{i_a}{}_{lk}\big(\gamma(s)\big) K^{i_1\ldots i_{a-1} l i_{a+1}\ldots i_r}_{j_1\ldots j_s}\big(\gamma(s)\big)\dot\gamma^k(s) \\ &- \sum_{b=1}^{r'} \Gamma^{l}{}_{j_b k}\big(\gamma(s)\big) K^{i_1\ldots i_r}_{j_1\ldots j_{b-1} l j_{b+1}\ldots j_s}\big(\gamma(s)\big)\dot\gamma^k(s)\Big\} \\ &\times \big(E^{j_1}\otimes\cdots\otimes E^{j_{r'}}\otimes E_{i_1}\otimes\cdots\otimes E_{i_r}\big)\big|_{\gamma(s)}.\end{aligned} \tag{3.17}$$

Definition 3.2. The *parallel transport (along paths or curves)* assigned to a C^0 linear connection ∇ is a mapping $\mathsf{P}\colon \beta \mapsto \mathsf{P}^\beta$ which, for every C^1 path $\beta\colon [a,b] \to M$,[10] $a, b \in \mathbb{R}$, $a \le b$, puts into correspondence a mapping $\mathsf{P}^\beta\colon \boldsymbol{T}_{\beta(a)} \to \boldsymbol{T}_{\beta(b)}$ from the tensor algebra at $\beta(a)$ into the one at $\beta(b)$ such that if $K_0 \in \boldsymbol{T}_{\beta(a)}$, then

[9] Here we implicitly suppose γ to be injective as otherwise $\dot\gamma\colon \gamma(s) \mapsto \dot\gamma(s)$ may not be a single-valued mapping (see footnote 11 on page 11). However (see Remark 3.3 on the following page), taking (3.17) as a definition for $\nabla_{\dot\gamma} K$, the next considerations hold for arbitrary, injective or non-injective, paths.

[10] With $[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$ we denote a real closed interval with end points $a, b \in \mathbb{R}$ such that $a \le b$. Similarly, by $(a,b] = \{x \in \mathbb{R} : a < x \le b\}$ and $[a,b) = \{x \in \mathbb{R} : a \le x < b\}$ we denote real intervals closed from right and left, respectively, and opened from left and right, respectively, and with end points $a, b \in \mathbb{R}$ such that $a < b$. The notation $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ will stand for an open from both end points $a, b \in \mathbb{R}$, with $a < b$, real interval.

$\mathsf{P}^{\beta}(K_0) := K_{\beta(b)}$, where K is a tensor field on $\beta([a,b])$ of the same type as K_0 and is defined as the unique solution of the initial-value problem

$$\nabla_{\dot\beta} K = 0, \qquad K_{\beta(a)} = K_0. \tag{3.18}$$

Remark 3.2. According to (3.17), the problem (3.18) reduces in a local frame to a first-order system of ordinary differential equations with respect to the local components of K with initial condition at $s = a$. This initial-value problem has a unique solution along β by virtue of the existence and uniqueness theorems for such systems [34].

Remark 3.3. By (3.17), the solution K of (3.18) depends, generally, explicitly on β, i.e., the value $K_{\beta(s)}$ for $s \in [a,b]$ generally depends on s and β separately, not only on the combination $\beta(s)$; in particular, at the points of self-intersection of β, if any, K may be multiple-valued (as a tensor field). The right and natural interpretation of K is as a lifting of β from M to the bundle space of the tensor bundle of the same type as K_0. This fact will become important in Chapter IV where it will be treated appropriately. If β is injective, i.e., without self-intersections, K is ordinary (single-valued) tensor field over $\beta([a,b])$. So, to be more precise, one should write K_β and $K_\beta(s)$ for K and $K(\beta(s))$ respectively, and, according to (3.17), the symbol $(\nabla_{\dot\beta} K)(s)$, denoting the covariant derivative of (a lifting of paths) K along β at $s \in [a,b]$, should be defined by

$$\begin{aligned}
(\nabla_{\dot\beta} K)(s) = \Big\{ & \frac{\mathrm{d}}{\mathrm{d}s} \Big(K_{\beta}{}^{i_1\dots i_{r'}}_{j_1\dots j_r}(s) \Big) \\
& + \sum_{a=1}^{r} \Gamma^{i_a}{}_{lk}\big(\beta(s)\big) K_{\beta}{}^{i_1\dots i_{a-1} l i_{a+1}\dots i_r}_{j_1\dots j_s}(s) \dot\beta^k(s) \\
& - \sum_{b=1}^{r'} \Gamma^{l}{}_{j_b k}\big(\beta(s)\big) K_{\beta}{}^{i_1\dots i_r}_{j_1\dots j_{b-1} l j_{b+1}\dots j_s}(s) \dot\beta^k(s) \Big\} \\
& \times \big(E^{j_1} \otimes \cdots \otimes E^{j_{r'}} \otimes E_{i_1} \otimes \cdots \otimes E_{i_r} \big)\big|_{\beta(s)}.
\end{aligned} \tag{3.19}$$

The reader is referred to Chapter IV, in particular to Sections IV.11, IV.14, and IV.3 for further details concerning the definition of a parallel transport and its links with the concept of lifting (of paths).

Remark 3.4. Since in the present book, with an exception of Section IV.14 and Chapter V, only linear connection on manifolds and respectively parallel transports assigned to such connections will be employed, we shall implicitly understood, when talking about parallel transports, that they are with respect to (assigned to) some linear connection on a manifold, if this is not explicitly stated.

The properties of the parallel transport are well-known and can be found in practically every book on differential geometry covering the connection theory, e.g in [11, 22]. We will mention only some of them: (i) The transport along β depends on the set $\beta([a.b])$, not on its particular parameterization; (ii) It depends only on

$\beta(a)$ and $\beta(b)$ iff ∇ is flat; (iii) To product of paths corresponds the composition of the corresponding transports; (iv) The mapping P^β is a linear isomorphism.

Thus the parallel transport generated by a linear connection realizes the general ideas presented in Subsection 3.1. In particular, by its help we can give sense to operations with tensors defined at different points, say $p, q \in M$, $p \neq q$, by parallelly transporting, e.g., the tensors at p to q along some path connecting p and q.[11] For instance, we can compare two vectors $X \in T_p(M)$ and $Y \in T_q(M)$ by forming, e.g, the difference $Y - \mathsf{P}^\beta(X)$ for some path $\beta\colon [a,b] \to M$ such that $\beta(a) = p$ and $\beta(b) = q$ (see footnote 11).

Remark 3.5. Above we implicitly supposed the existence of a C^1 path β in M connecting the (arbitrary) points p and q. This is true if M is simply connected. If M is multiply connected, then always exist paths joining every $p, q \in M$ but such a path can be chosen in a smooth way, i.e., of class C^1 or higher, iff p and q belong to a simply connected region of M. Since, by definition, the parallel transport is defined only along C^1 paths, when talking of a parallel transport along a path joining two points in M, we always presuppose that they are situated in a simply connected subset of M. Moreover, when saying that the parallel transport between two points p and q is path-independent, we have in mind that this is with respect to reconcilable paths, i.e., ones forming (belonging to) a smooth, C^1, homotopy $\eta\colon [a,b] \times W \to M$ where W is inessential for us non-empty set,[12] $\eta(a, W) = p$, and $\eta(b, W) = q$. Said more freely, the paths connecting p and q are suppose such that any closed loop formed from every pair of them can smoothly be deformed to a point in M. This will be presupposed and further in this book. Otherwise the parallel transport may turn to be path-dependent even in the flat case: such situation may happen if M is multiply connected and there is a smooth closed path connection p and q which can not be smoothly contracted to a single point in M.

Remark 3.6. The concept of a parallel transport has a natural generalization in arbitrary vector bundles, called linear transport along paths. For details, see Sections IV.3 and IV.11, in particular Definition IV.11.1 and Proposition IV.11.1. For some links between (parallel) transports (along paths) and connections on differentiable bundles, see Section V.8.

Definition 3.3. A *tensor field* $K \in \boldsymbol{T}^1(U)$, $U \subseteq M$ is called *parallel along a C^1 path* $\gamma\colon J \to U$ with respect to a linear connection ∇ if $\nabla_{\dot\gamma} K = 0$. The field K is called parallel (with respect to ∇) on U if it is parallel along every path $\gamma\colon J \to U$.

A parallel (along γ) field is uniquely defined by fixing its value K_0 at a point $p = \gamma(c)$ for some $c \in J$; in fact we have $K_{\gamma(s)} = \mathsf{P}^{\gamma|[c,s]}(K_0)$, $s \in J$, with $\gamma|[c,s]$ being the restriction of γ to $[c,s] \subseteq J$, for $c \geq s$ and $K_{\gamma(s)} = \big(\mathsf{P}^{\gamma|[s,c]}\big)^{-1}(K_0)$ for $c \leq s$. The field K is parallel along γ iff $K_{\gamma(t)} = \mathsf{P}^{\gamma|[t,s]}(K_{\gamma(s)})$ for $t \geq s$ and

[11] This operation is path dependent unless the connection is flat.

[12] The set W is a manifold whose *real* dimension is $(\dim M - 1)$ if $\mathbb{K} = \mathbb{R}$ and $(2\dim_{\mathbb{C}} M - 1)$ for $\mathbb{K} = \mathbb{C}$.

$K_{\gamma(t)} = \big(\mathsf{P}^{\gamma|[s,t]}\big)^{-1}(K_{\gamma(s)})$ for $t \le s$ for all $s, t \in J$. A tensor field K is parallel on $U \subseteq M$ if and only if $(\nabla_X K)|_U = 0$ for all $X \in \mathfrak{X}(U)$.

Definition 3.4. A *frame* $\{E_i\}$ on $U \subseteq M$ (resp. along a path $\gamma\colon J \to M$) is called *parallel* (*on* U, resp. *along* γ) if its basic vector fields $E_1, \dots, E_{\dim M}$ are parallel on U(resp. along γ).

Frames parallel on $(\dim M)$-dimensional submanifolds U, i.e., on neighborhoods, exist iff ∇ is flat on U.

An almost standard way for construction of a (parallel) frame on some path-connected flat set U (with respect to a linear connection) is to fix a basis in $T_p(M)$ for some arbitrarily chosen point $p \in U$ and then to transport parallelly its vectors, i.e., the whole basis, to every $q \in U$ along some path connecting p and q. A frame parallel on U is analogous to a Cartesian coordinate system on $\mathbb{R}^n$ (see Chapter IV).

3.4. Geodesics and exponential mapping

The geodesic paths, called in different contexts also geodesic curves, geodesic lines, or simply geodesics, in a manifold with connection are in many aspects analogues of the straight lines in $\mathbb{R}^n$. Historically the introduction of normal frames/coordinates is primary related to them.

Definition 3.5. A *geodesic* or a *geodesic path* in a C^2 manifold endowed with C^0 linear connection ∇ is a C^1 path $\gamma\colon J \to M$ whose tangent vector $\dot\gamma$ is parallel along γ, i.e.,

$$\dot\gamma(s_2) = \begin{cases} \mathsf{P}^{\gamma|[s_1,s_2]}\dot\gamma(s_1) & \text{for } s_1, s_2 \in J, \quad s_1 \le s_2 \\ \big(\mathsf{P}^{\gamma|[s_2,s_1]}\big)^{-1}\dot\gamma(s_1) & \text{for } s_1, s_2 \in J, \quad s_2 \le s_1 \end{cases} \tag{3.20}$$

where P is the assigned to ∇ parallel transport, or, equivalently (however, see below Remark 3.8),

$$\nabla_{\dot\gamma}\dot\gamma = 0. \tag{3.21}$$

This equation, as well as its equivalent local forms (3.22) and (3.23) presented below, are known as the *geodesic equation* or *equation of the geodesics*.

Remark 3.7. In some works, like [18], the paths satisfying (3.21) are called autoparallel, the term ‘geodesic’ being reserved for paths satisfying certain Euler-Lagrange equations (derived from a variational principle) [12, 18]. This is more typical for the ‘physically oriented’ literature [35] which is connected with the development of theories based on manifolds endowed with (more or less independent) linear connection and Riemannian metric [36, 37]. The more widely accepted modern mathematical terminology is to call the former paths geodesics and the latter ones extremals [10, 11, 19]. That last convention is followed in the present book. We reserve the term “autoparallels” or “autoparallel paths” for a certain generalization of the geodesics to which is devoted Section IV.15.

Remark 3.8. If the C^1 path $\gamma\colon J \to M$ is not injective, the mapping $\dot\gamma\colon s \mapsto \dot\gamma(s) \in T_{\gamma(s)}(M)$ cannot be considered as a vector field over $\gamma(J)$ assigning to $\gamma(s)$ the value $\dot\gamma(s)$ as the mapping $\gamma(s) \mapsto \dot\gamma(s)$ is generally multiple-valued at the points of self-intersection of γ.[13] For this reason, the symbol $\nabla_{\dot\gamma}\dot\gamma$ is not well-defined if γ is not injective. In that case, as well as in the general one, it should be defined as a path in $T(M)$ whose local components are *defined* by the left-hand side of equation (3.22) below.

In the domain U of some local frame $\{E_i\}$, the *geodesic equation* (3.21) reads

$$\frac{\mathrm{d}\dot\gamma^i}{\mathrm{d}s} + \Gamma^i{}_{jk}\dot\gamma^j\dot\gamma^k = 0, \qquad s \in J \tag{3.22}$$

where (3.17) was used. If some local coordinate system $\{x^i\}$ is employed and $E_i = \partial/\partial x^i$, then $\dot\gamma^i = \mathrm{d}\gamma^i/\mathrm{d}s$ with $\gamma^i = x^i \circ \gamma$, so (3.22) reduces to

$$\frac{\mathrm{d}^2\gamma^i(s)}{\mathrm{d}s^2} + \Gamma^i{}_{jk}(\gamma(s))\frac{\mathrm{d}\gamma^j(s)}{\mathrm{d}s}\frac{\mathrm{d}\gamma^k(s)}{\mathrm{d}s} = 0 \qquad s \in J. \tag{3.23}$$

From here a number of immediate observations can be derived:

(i) If $\gamma\colon J \to M$ is a geodesic, then so is the path $\gamma \circ \tau$, with $\tau\colon J' \to J$ and $\tau\colon t \mapsto at + b$, $a, b \in \mathbb{R}$, $a \neq 0$, $t \in J'$, i.e., a geodesic is mapped into geodesic under nondegenerate affine change of its parameter.[14]

(ii) If $\gamma\colon J \to M$ is a geodesic, its restriction $\gamma|J'$ to any subinterval $J' \subset J$ is also geodesic.

(iii) If the connection is of class C^k, $k \geq 0$ (on C^{k+2} manifold), i.e., if $\Gamma^i{}_{jk}$ are of class C^k, any geodesic, which is a C^1 path by definition, is automatically a C^{k+2} path; in particular the geodesics of C^∞ linear connection on C^∞ manifold are C^∞ paths.

(iv) Since (3.23) is a second-order system of ordinary differential equations, from the theorems of existence and uniqueness of the solutions of such systems [34] follows that for every point $x \in M$ and every vector $X \in T_x(M)$ there exists a geodesic $\gamma\colon J \to M$ such that

$$\gamma(s_0) = x, \qquad \dot\gamma(s_0) = X \tag{3.24}$$

for a fixed $s_0 \in J$ and some interval J. Besides, there is a subinterval $J_0 \subseteq J$ such that $J_0 \ni s_0$ and on J_0 all such geodesics coincide. Hence locally, on $J_0 \ni s_0$, there is a unique geodesic with initial conditions (3.24). Generally $J_0 \subset \mathbb{R}$. If for every geodesic $J_0 = \mathbb{R}$, the connection/manifold is called (geodesically) complete.

[13] In fact, $\dot\gamma$ is a lifting of γ in the tangent bundle over M, not a vector field on $\gamma(J)$; it can be considered as a vector field iff γ is injective. For details, see Section IV.2.

[14] If $\tau(J') = J$, the sets $\gamma(J)$ and $(\gamma \circ \tau)(J')$ coincide, i.e., they represent one and the same (geometric, unparametrized geometric) curve $\sigma = \gamma(J)$; see [11, Chapter III, § 6] and footnote 7 on page 9.

(v) The geodesics depend only on the 'symmetric' part of the connection, i.e., on the symmetric part ${\Gamma^i}_{(jk)}$ of its coefficients ${\Gamma^i}_{jk}$ in any frame, coordinate or not: in particular, since $\dot\gamma^j\dot\gamma^k = \dot\gamma^k\dot\gamma^j$, we have the equivalent to (3.23) equation

$$\frac{\mathrm{d}^2\gamma^i(s)}{\mathrm{d}s^2} + {\Gamma^i}_{(jk)}(\gamma(s))\frac{\mathrm{d}\gamma^j(s)}{\mathrm{d}s}\frac{\mathrm{d}\gamma^k(s)}{\mathrm{d}s} = 0 \qquad s \in J. \tag{3.23'}$$

So, if two (or more) linear connections on M generate one and the same symmetric connection via (3.9), their geodesics coincide.

(vi) A geodesic is either regular or degenerate path: if $\dot\gamma(s_0) = 0$ for some $s_0 \in J$, then $\dot\gamma(s) \equiv 0$ for every $s \in J$ as the parallel transport is an isomorphism.

Now we are going to formulate a result which is often used for the proof of existence of normal frames at a single point.

A geodesic is called *maximal* if it is not a restriction of other geodesic, i.e., $\gamma\colon J \to M$ is maximal geodesic if there does not exist a geodesic $\bar\gamma\colon \bar J \to M$ such that $J \subset \bar J$ and $\gamma = \bar\gamma|J$. For every point $x \in M$, every vector $X \in T_x(M)$, and any number $s_0 \in \mathbb{R}$ there exists a unique maximal geodesic $\gamma^{s_0}_{x,X}\colon J_m \to M$ defined on some interval $J_m \ni s_0$ and such that[15]

$$\gamma^{s_0}_{x,X}(s_0) = x, \quad \dot\gamma^{s_0}_{x,X}(s_0) = X. \tag{3.25}$$

Using the uniqueness of the maximal geodesics, one easily verifies that

$$\gamma^{s_0}_{x,aX}(s) = \gamma^{s_0}_{x,X}(a(s - s_0) + s_0) \qquad a \in \mathbb{R}. \tag{3.26}$$

In what follows, we put $s_0 = 0$ (see Remark 3.9 on the next page) and write $\gamma_{x,X}$ for $\gamma^0_{x,X}$. For this choice, equation (3.26) reads

$$\gamma_{x,aX}(s) = \gamma_{x,X}(as) \qquad s \in J^m \ni 0 \in \mathbb{R}, \quad a \in \mathbb{R}. \tag{3.27}$$

Let $V_x \subseteq T_x(M)$ be such that $\gamma_{x,X}$, with X in V_x, is defined on interval J_m containing the point $s = 1$, $J_m \ni 1 \in \mathbb{R}$. The set V_x contains, evidently, the zero vector; besides, by (3.27) it contains also non-zero vectors. We define a mapping $\exp_x\colon V_x \to M$, called *exponential mapping at the point* x, by[16]

$$\exp_x X := \gamma_{x,X}(1), \qquad X \in V_x. \tag{3.28}$$

Due to (3.27), it is fulfilled

$$\exp_x(sX) = \gamma_{x,X}(s), \qquad s \in J_m \ni 1, \quad X \in V_x. \tag{3.29}$$

The following important result is valid.

[15] For the proof, see [38, p. 380]. Generally $\gamma^{s_1}_{x,X}$ and $\gamma^{s_2}_{x,X}$ are different unless $s_1 = s_2$.

[16] The *exponential mapping*, exp, itself is defined via $\exp\colon x \mapsto \exp_x$, $x \in M$, or by $\exp\colon X \mapsto \exp_x X$ for $X \in V_x \subseteq T_x(M)$.

Proposition 3.1. *For every point $x \in M$ there exists a neighborhood V_x^0 of the zero vector in $T_x(M)$ such that:*

(i) *It is star-shaped (star-like), i.e., if $X \in V_x^0$, then $aX \in V_x^0$ for $0 \leq a \leq 1$;*
(ii) *The exponential mapping $\exp_x$ at x is defined on $V_x^0 \subseteq V_x \subseteq T_x(M)$;*
(iii) *There exists a neighborhood $V(x)$ of x in M such that $\exp_x \colon V_x^0 \to V(x)$ is diffeomorphism.*

Proof. See [38, pp. 381–384], [11, Chapter III, § 8] or [39, Section 1.6] □

Remark 3.9. The choice $s_0 = 0$ made above is not necessary, it only saves some writing, simplifies the formulae, and eliminates from the theory an arbitrary number (which is insignificant at present). For $s_0 \neq 0$ the above construction goes like this: Let $V_{s_0,x} \subseteq T_x(M)$ be such that $\gamma_{x,X}^{s_0}$ with $X \in V_{s_0,x}$ is defined on $J_m \subseteq \mathbb{R}$ containing the point $s = s_0 + 1$, $J_m \ni (s_0 + 1) \in \mathbb{R}$. The exponential mapping at x (depending on s_0) is $\exp_{s_0,x} \colon V_{s_0,x} \to M$ with $\exp_{s_0,x} X := \gamma_{x,X}^{s_0}(s_0 + 1)$, $X \in V_{s_0,x}$. So, $\exp_{s_0,x}(sX) = \gamma_{x,X}^{s_0}(s_0 + s)$, $(s_0 + s) \in J_m$. Proposition 3.3 remains true if we replace in it V_x^0, V_x, $\exp_x$, and $V(x)$ respectively by $V_{s_0,x}^0$, $V_{s_0,x}$, $\exp_{s_0,x}$, and $V_{s_0}(x)$. At present this generalization does not find applications.

Definition 3.6. A *neighborhood* V_x^0 of the zero vector in $T_x(M)$ having the properties (i)–(iii) described in Proposition 3.1 is called *normal* (*neighborhood* of $0 \in T_x(M)$). A neighborhood $V(x)$ of $x \in M$ is called *normal* if it is the image of a normal neighborhood of the zero vector in $T_x(M)$ under the exponential mapping (at x), $V(x) = \exp_x(V_x^0)$.

Therefore the essence of Proposition 3.3 is that every point $x \in M$ admits normal neighborhoods.

The normal neighborhoods possess a number of remarkable properties described, for instance, in [11, 38]. They have a straightforward relation to the existence of normal frames that will be revealed in Section 6 and in Chapter II.

4. Riemannian manifolds

Freely said, a Riemannian manifold is a real differentiable manifold on each tangent space, considered as a vector space, of which a scalar product is given. When metrics (scalar or inner products) are concerned, one should clearly distinguish the real, $\mathbb{K} = \mathbb{R}$, and complex, $\mathbb{K} = \mathbb{C}$, cases. In the former case the scaler product is understand as Riemannian metric [10, 11], while in the latter case it is assumed to be a Hermitian metric [10, 26, 40].[1] That is why, usually a real manifold with a

[1] Formally one can consider a 'Riemannian metric' on a complex manifold as a non-degenerate bilinear form on it. Such a form is complex-valued and does not define a metric or inner product in the accepted sense [16, p. 2], [10, Chapter I, Section 11]; in particular such a 'metric' does not define a distance function on the manifold and, correspondingly, there is no associated with it 'metric topology', etc. Regardless of this, if in what follows one considers such forms, all remains true *mutatis mutandis.*

Riemannian metric is called Riemannian manifold [11] while a complex manifold with a Hermitian metric is called Hermitian manifold [26].

In this book only Riemannian metrics will be considered as the Hermitian once require somewhat different treatment which is not primary related to its main subject.

Definition 4.1. A *Riemannian metric* g on a subset $U \subseteq M$ of a real manifold M is a field of symmetric non-degenerate bilinear forms on the tangent spaces at the points in U, i.e., g is a symmetric non-degenerate tensor field in $\mathfrak{F}^0_2(U)$:[2]

$$g\colon x \mapsto g_x, \quad x \in U, \qquad g_x\colon T_x(M) \times T_x(M) \to \mathbb{R}, \tag{4.1a}$$

$$g_x(X_x, Y_x + Z_x) = g_x(X_x, Y_x) + g_x(X_x, Z_x) \qquad X_x, Y_x, Z_x \in T_x(M), \tag{4.1b}$$

$$g_x(X_x, aY_x) = a g_x(X_x, Y_x) \qquad a \in \mathbb{R}, \quad X_x, Y_x \in T_x(M), \tag{4.1c}$$

$$g_x(X_x, Y_x) = g_x(Y_x, X_x) \qquad X_x, Y_x \in T_x(M), \tag{4.1d}$$

$$g_x(X_x, Y_x) = 0 \text{ for all } X_x \in T_x(M) \text{ and some } Y_x \in T_x(M) \iff Y_x = 0. \tag{4.1e}$$

A real manifold endowed with a Riemannian metric is called *Riemannian manifold*. Below we suppose the existence of a Riemannian metric on the real manifold M.[3]

Example 4.1. The n-dimensional Euclidean space $\mathbb{E}^n$, $n \in \mathbb{N}$, is an almost trivial example of a Riemannian manifold. It is defined as $\mathbb{R}^n$ endowed with *Euclidean* metric e such that $e(X, Y) := \sum_{i=1}^n X^i Y^i$ for $X = (X^1, \dots, X^n) \in \mathbb{R}^n$ and $Y = (Y^1, \dots, Y^n) \in \mathbb{R}^n$ (see footnote 2 on page 20 and Example 7.1 below).

Let g denotes a Riemannian metric on a real manifold M. For any $U \subseteq M$, it induces a Riemannian metric on the set $\mathfrak{X}(U)$ of vector fields on U: we define $g\colon \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{F}(U)$ by $g\colon (X, Y) \mapsto g(X, Y) \in \mathfrak{F}(U)$, $X, Y \in \mathfrak{X}(U)$ with $g(X, Y)\colon x \mapsto g_x(X_x, Y_x)$, $x \in U$.

If $X, Y \in T_x(M)$ (resp. $X, Y \in \mathfrak{X}(U)$), the number $g_x(X, Y) \in \mathbb{R}$ (resp. the function $g(X, Y) \in \mathfrak{F}(U)$) is called *scalar product* of the vectors (resp. vector fields) X and Y in $T_x(M)$ (resp. in $\mathfrak{X}(U)$). In [11, Chapter IV, § 1] is shown how the scalar product in $T_x(M)$ (resp. in $\mathfrak{X}(U)$) can be extended to a scalar product in $\boldsymbol{T}_{x}{}^{r}_{s}$ (resp. in $\mathfrak{T}^r_s$) for every $r, s \geq 0$ with $r + s \geq 1$ and that it induces an isomorphism $\boldsymbol{T}_{x}{}^{r}_{s} \to \boldsymbol{T}_{x}{}^{r\pm1}_{s\mp1}$ for every $r, s \geq 0$ such that $r \pm 1, s \mp 1 \geq 0$ but these, otherwise useful, results will not find applications in our investigation.

A Riemannian metric g on U is called C^r metric (C^r differentiable, (differentiable) of class C^r) if it is a C^r tensor field on U, i.e., $g \in \mathfrak{T}^{0;r}_2(U)$.

The Riemannian manifolds of class C^2 are interesting for us due to the existence on them of a 'natural' linear connection induced by the metric. This unique linear connection associate with a C^1 Riemannian metric is called *Riemannian*

[2] The condition (4.1e) is equivalent to the non-degeneracy only in the finite-dimensional case, as we presuppose here.

[3] See, e.g., [11,12,26] for the existence of such metrics, in particular on paracompact manifolds.

connection, known also as *Levi-Civita connection*, and it is defined as a metric-preserving torsionless linear connection on the respective manifolds. Explicitly this means that the Riemannian connection ∇ on U for a C^1 metric g on U is defined as a C^0 linear connection satisfying the system of equations[4]

$$\nabla_X g = 0 \tag{4.2}$$

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]_{\_} = 0 \tag{4.3}$$

for all vector fields $X, Y \in \mathfrak{X}(U)$. Its solution with respect to $\nabla_X Y$ is implicitly given by [11, Chapter IV, § 2][5]

$$\begin{aligned} 2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ + g([X,Y]_{\_}, Z) + g([Z,X]_{\_}, Y) + g(X, [Z,Y]_{\_}) \end{aligned} \tag{4.8}$$

where $X, Y, Z \in \mathfrak{X}(U)$ are arbitrary. The explicit solution of this equation for the Riemannian connection is easily found in some local frame $\{E_i\}$ on U.

Let

$$g_{ij} := g(E_i, E_j) \tag{4.9}$$

be the *local* (*covariant*) *components* of g in $\{E_i\}$.[6] From Definition 4.1 follows that $[g_{ij}]$ is a symmetric non-degenerate matrix-valued function on U:

$$g_{ij} = g_{ji} \qquad \det[g_{ij}] \neq 0, \infty. \tag{4.10}$$

---

[4]The metric preserving condition (4.2) can be expressed in a number of equivalent ways; e.g., by $\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, $Z \in \mathfrak{X}(U)$ or by saying that the scalar product of every two vectors does not change after they are parallelly transported along an arbitrary path in U. A linear connection satisfying only the condition (4.2) is called metric connection; generally it depends on g and the torsion tensor.

[5] Since $g(Y,Z) = C_1^1(C_1^1(g \otimes Y \otimes Z))$, we have

$$X(g(Y,Z)) = \nabla_X(g(Y,Z)) = (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{4.4}$$

Making here the cyclic permutations $(X,Y,Z) \mapsto (Y,Z,X) \mapsto (Z,X,Y)$, summing the first two equations obtained and subtracting the last one from the result, we get

$$\begin{aligned} 2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + (\nabla_Z g)(X,Y)) \\ - (\nabla_X)(Y,Z)) - (\nabla_Y g)(Z,X)) + g([X,Y]_{\_}, Z) + g([Z,Y]_{\_}, X) + g([Z,X]_{\_}, Y), \end{aligned} \tag{4.5}$$

where the torsionless condition (4.3) was used. Now the metricity condition (4.2) reduces this equation to (4.8). Equation (4.5) is useful, for example, in the Weyl spaces in which (4.2) is replaced with

$$\nabla_X g = \omega(X) g \tag{4.6}$$

for some one-form ω. A torsionless connection ∇ satisfying the equation (4.6) is called *Weyl connection* [41, 42]; the choice $\omega = 0$ reduce a Weyl connection to Riemannian one. Let us note that for a Weyl connection equation (4.13) below is replaced by

$$\Gamma^i{}_{jk} = \left\{ {i \atop jk} \right\} + \frac{1}{2}(g^{im} C^l_{mj} g_{lk} + g^{im} C^l_{mk} g_{lj} - C^i_{jk}) + \frac{1}{2}(g_{jk} g^{il} \omega_l - \delta^i_j \omega_k - \delta^i_k \omega_j) \tag{4.7}$$

where $\omega_i := \omega(E_i)$ is the ith component of ω.

[6]Note, in the real case $g(X,Y) = g_{ij} X^i Y^j$, while in the complex case, for a Hermitian metric h, we have $h(X,Y) = g_{ij} X^i \overline{Y^j}$ where $\overline{Y^j}$ is the complex conjugate to Y^j.

Define g^{ij}, the *contravariant components of the metric* g, as the elements of the matrix inverse to $[g_{ij}]$, $[g^{ij}] := [g_{ij}]^{-1}$, i.e.,

$$g^{ij} g_{jk} = \delta^i_k. \tag{4.11}$$

The matrix $[g^{ij}]$ is also symmetric and non-degenerate.

Now, if we set $X = E_i$, $Y = E_j$, and $Z = E_k$ in (4.8) and take into account (3.1) and (3.15), we get

$$2g_{lk}\Gamma^l{}_{ji} = g_{jk,i} + g_{ik,j} - g_{ij,k} + C^l_{ij} g_{lk} + C^l_{ki} g_{lj} + C^l_{kj} g_{il}. \tag{4.12}$$

From here and (4.11), we obtain

$$\Gamma^i{}_{jk} = \begin{Bmatrix} i \\ jk \end{Bmatrix} + \frac{1}{2}\big(g^{im} C^l_{mj} g_{lk} + g^{im} C^l_{mk} g_{lj} - C^i_{jk}\big) \tag{4.13}$$

where

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} := \begin{Bmatrix} i \\ jk \end{Bmatrix}_g := \frac{1}{2} g^{il}\big(g_{lj,k} + g_{lk,j} + g_{jk,l}\big) \tag{4.14}$$

are the so-called *Christoffel symbols* [19, p. 132] [7] of the metric g (induced by g, assigned to g).[8] If we restrict ourselves to the class of (C^0) holonomic frames on U (or on the whole C^2 manifold M), then $C^i_{jk} \equiv 0$ and the Christoffel symbols coincide with the local coefficients of the Riemannian connection in them.[9] That is why the Riemannian connection is often denoted by $\nabla^{\{\}}$ (and also by ${}^{\{\}}\nabla$ or $\overset{\{\}}{\nabla}$ if the position of the superscripts to ∇ is reserved for other purposes).

If the Christoffel symbols are known, from (4.14) we can find [18, p. 81] the derivatives $g_{ij,k} = E_k(g_{ij})$ of the metric tensor:

$$g_{ij,k} = g_{il}\begin{Bmatrix} l \\ jk \end{Bmatrix} + g_{jl}\begin{Bmatrix} l \\ ik \end{Bmatrix}. \tag{4.16}$$

This equality, which in the context of (4.14) is identity, can be interpreted in the following way. Let ∇ be a torsionless connection with coefficients $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$

[7] The right-hand side of (4.12) without the last three terms is known as Christoffel symbols of the first kind and is denoted by $[ij, k]$. That is why (4.14) are also known as Christoffel symbols of the second kind. For detains see, e.g., [18] and [24, p. 17]. Nowadays this old terminology, introduced by E. B. Christoffel is practically out of usage.

[8] If the metric g happens to be diagonal in $\{E_i\}$, i.e., $[g_{ij}] = \operatorname{diag}(g_1, \dots, g_{\dim M})$ where $\operatorname{diag}(a_1, \dots, a_n)$ means a diagonal matrix with diagonal elements $a_1, \dots, a_n$, then $[g^{ij}] = \operatorname{diag}(1/g_1, \dots, 1/g_{\dim M})$ and (do not sum over i!)

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{1}{2g_i}(\delta_{ij} g_{i,k} + \delta_{ik} g_{i,j} - \delta_{jk} g_{j,i}). \tag{4.15}$$

[9] For this reason, sometimes, the coefficients of an arbitrary linear connection are called Christoffel symbols. This is not quite fair as E.B. Christoffel introduced his symbols in 1869 [19, p. 132, footnote 1], [24, p. 17], while the general theory of linear (affine) connections was developed more then half a century later, mainly due to the works of E. Cartan.

in some coordinate frame. *Define* $\left\{ {i \atop jk} \right\} = {\Gamma^i}_{kj}$, then all metrics g whose Christoffel symbols coincide with ${\Gamma^i}_{kj}$ in the given frame are solutions of (4.16).

On a Riemannian manifold whose metric is of class C^2 exist two types of 'privilege' paths [11,24]: the geodesics assigned to the Riemannian connection and the extremals.[10] A remarkable result of the Riemannian geometry [11,12,24] is that in it the geodesics and extremals (defined as paths of minimal, maximal, or zero length between two points) *coincide*. Consequently, by (3.23) and (4.13), they satisfy the equations

$$\frac{\mathrm{d}^2\gamma^i}{\mathrm{d}s^2} + \left\{ {i \atop jk} \right\} \frac{\mathrm{d}\gamma^j}{\mathrm{d}s} \frac{\mathrm{d}\gamma^k}{\mathrm{d}s} = 0, \tag{4.17}$$

in any coordinate frame $\{\partial/\partial x^i\}$, with $\left\{ {i \atop jk} \right\}$ being defined by (4.14).

The geometry of Riemannian manifolds is one of the best developed branches of the differential geometry. To it is devoted a huge number of woks, some of which were mentioned in Section 1. The reader is referred to them for deeper concepts and results in this region.

5. Normal frames: general ideas uniqueness, and holonomicity

Any manifold, as we saw in Section 2, is locally Euclidean in a sense that every its point has a neighborhood homeomorphic to $\mathbb{K}^n$ (or its open subset). In subsection 3.2, we introduced linear connections on a manifold by means of covariant derivatives along vector fields on it. Are the linear connections 'locally Euclidean'? Are the covariant derivatives 'locally looking' like ordinary derivatives (along vector fields or along paths)? At this point these questions are only heuristic and their rigorous meaning will be revealed further in the present book. Since in any local frame the components of the covariant derivative of a tensor field are formed from the ordinary derivatives (along the basic vector fields) of its components (see (3.2) and the first term in (3.3)) and a 'correction' which is linear in them (see (3.2) and the sums in (3.3)), we may pose a problem for choosing a local frame so that this 'correction' to vanish. Since a vector field along it's integral paths can be replace by the integral paths' tangent vectors, we, by virtue of (3.17), could expect that such special frames may, possibly, exist along C^1 paths. We would like the above 'correction' to vanish for all C^1 tensor fields of arbitrary type. Due to (3.2) and (3.3), this is equivalent to the demand the (local) coefficients of the linear connection to vanish in the mentioned special kind of frames. Thus we have come to the following definition.

Definition 5.1. Given a linear connection ∇ on a differentiable manifold M and a subset $U \subseteq M$. A *frame* $\{E_i\}$, define on an open subset of M containing U

[10] The extremals are defined as paths for which some functional on the space of paths connecting two fixed points is stationary; for details see, e.g., [12, p. 309ff] or [24, p. 48ff].

or equal to it, is called *normal for* ∇ *on* U if in it the coefficients of ∇ vanish everywhere on U. Respectively, $\{E_i\}$ is called *normal for* ∇ *along* $g\colon Q \to M$, Q being non-empty set, if it is normal for ∇ on $g(Q)$.

We can rephrase the first part of this definition by writing

$$\{E_i\} \quad \text{is } \textit{normal} \text{ on } U \qquad \Longleftrightarrow \qquad \Gamma^i{}_{jk}|_U = 0 \quad \text{in } \{E_i\}. \tag{5.1}$$

Therefore if $K \in \mathfrak{T}^r_s(M)$, in a frame normal on U, if it exists, according to (3.2) and (3.3), we have on U

$$\big(\nabla_X K\big)^{i_1\dots i_r}_{j_1\dots j_s} = \big[E_k\big(K^{i_1\dots i_r}_{j_1\dots j_s}\big)\big]X^k = X\big(K^{i_1\dots i_r}_{j_1\dots j_s}\big) \tag{5.2}$$

which looks exactly like a directional derivative (along X) in $\mathbb{R}^n$. Just in this sense one can say that the linear connections and covariant derivatives are locally Euclidean but this is not quite rigorous because, at this point, we still do not know anything on the existence of normal frames. That problem will be investigated at length further in the present monograph.

By virtue of (3.1), the equalities $\Gamma^i{}_{jk}|_U = 0$, $i,j,k = 1,\dots,\dim M$, on a neighborhood U are equivalent to $(\nabla_X E_i)|_U = 0$ for every $X \in \mathfrak{X}(M)$, i.e., the basic vector fields E_i must be parallel on U (see Definition 3.3). Thus we have proved the following simple result.

Proposition 5.1. *A frame is normal on a neighborhood U in M iff it is parallel on U.*

Therefore the concepts ‘parallel frame’ and ‘normal frame’ are equivalent on neighborhoods.

Example 5.1. A frame normal for a linear connection on a neighborhood U on which the connection is flat is provided by a frame obtained from a basis at a fixed point in U by its parallel transportation to the other points in U along paths lying in U.

We want to emphasize on the fact that a frame $\{E_i\}$ normal for ∇ on a set $U \subseteq M$ is always supposed to be defined on a neighborhood $\overline{U}$ containing or equal to U, $\overline{U} \supseteq U$, and the equality $\overline{U} = U$ is admissible if U is itself a neighborhood. This is quite essential assumption as the property of $\{E_i\}$ to be normal at p for ∇ for every $p \in U$ depends on the properties of $\{E_i\}$ and ∇ in a neighborhood of p, not only at p. (See, e.g., equation (5.4) below in which the action of E_i on A appears which is not defined if A is not given on a neighborhood of each $p \in U$.)

Let $\{E_i\}$ be a frame on $U \subseteq M$. On U exists a normal frame $\{E'_i\}$ iff there is a C^1 matrix-valued function $A := [A^j_i]$ transforming $\{E_i\}$ into $\{E'_i\}$, $E'_i = A^j_i E_j$, and such that in $\{E'_i\}$ the coefficients on ∇ are $\Gamma'^i{}_{jk} = 0$ which is a system of partial differential equations with respect to A. To write it in a compact form, as well as for saving writing and for purposes that will be clear in the next chapters, we introduce the *matrices of the connection coefficients* $\Gamma_k := \big[\Gamma^i{}_{jk}\big]^{\dim M}_{i,j=1}$, $k = 1,\dots,\dim M$.

In their terms, the equation (3.5) reads[1]

$$\Gamma_k \mapsto \Gamma'_k = A^l_k A^{-1}\big(\Gamma_l A + E_l(A)\big) \tag{5.3}$$

where $E_l(A) := \big[E_l(A^j_i)\big]_{i,j=1}^{\dim M}$. Consequently $\{E'_i\}$ is normal on U if the first-order system of partial differential equations

$$\big(\Gamma_l A + E_l(A)\big)\big|_U = 0 \tag{5.4}$$

has solution(s) on U with respect to the matrix A.

The above considerations show that normal frames $\{E'_i\}$ exist on a set U if and only if the system (5.4) has solutions in some (and hence in any) frame $\{E_i\}$. Besides, if such solutions exist, their properties are completely responsible for the properties of the frames normal on U. Moreover, to any solution A of (5.4) there corresponds a unique frame normal on U and v. v., i.e., there is a bijective correspondence between the solutions of (5.4) and the frames normal on U. For these reasons, the equation (5.4) is called the *equation of the normal* (*on* U) *frames* or simply the *normal frame equation.*

To the problems of existence and uniqueness of the solutions of (5.4) for linear connections is devoted Chapter II. It contains a complete description of all frames and coordinates normal at a point, along path, or on submanifolds (if such exist at all in the last case).

Now we shall prove two simple propositions, the first concerning the uniqueness of normal frames while the second one reveals the role of the torsion for the existence of normal coordinates (see below Definition 5.2).

Proposition 5.2. *Let a linear connection admits a frame normal on $U \subseteq M$. The set of all frames normal for this connection on U consists of all frames that can be obtained from a fixed normal frame by means of linear transformations whose matrices vanish on U under the action of the basic vector fields of the normal frames.*

Proof. Let $\{E_i\}$ be a frame normal on U, i.e., $\Gamma_l = 0$ and $\{E'_i = A^j_i E_j\}$ be a frame on U. By (5.3), in $\{E'_i\}$ the matrices of the connection's coefficients are $\Gamma'_k = A^l_k A^{-1} E_l(A)$, $A = [A^j_i]$. Hence, $\Gamma'_k|_U = 0$ iff $E_l(A)|_U = 0$. □

Let us mention the following evident consequence of this proposition (or of its proof).

Corollary 5.1. *All frames normal on U for a linear connection, if any, are connected via linear transformations whose coefficients vanish on U under the action of the basic vector fields of the normal frames.*

Proposition 5.3. *Let a linear connection admits frames normal on some neighborhood $U \subseteq M$. All of these frames are either holonomic or anholonomic on U depending on is the torsion of the connection zero or non-zero on U respectively.*

[1]When writing the elements of a matrix A in the form A^j_i (or $A^j{}_i$, or $A_i{}^j$), we consider the superscript as a first index and the subscript as a second index (see the list of conventions, page xii). So, the product of A and B is $AB = [A^i_j B^j_k]$ if $A = [A^j_i]$ and $B = [B^j_i]$.

Proof. Suppose $\{E_i\}$ is a normal frame on U. In it, by (3.14), the torsion's components are $T^i{}_{jk}|_U = -C^i_{jk}|_U$ and, consequently (see (3.15)), $[E_j, E_k]_{\_}|_U = -\big(T^i{}_{jk}E_i\big)|_U$. Therefore $[E_j, E_k]_{\_}|_U = 0$ iff $T|_U = 0$. □

Remark 5.1. The holonomicity of the frames normal on $U \subseteq M$ does not imply any conclusions for their holonomicity outside U if $U \neq M$ and the frames are defined on a larger set $V \supset U$, $V \subseteq M$. Moreover, on $V\backslash U$ the frames (normal on U, but not on $V\backslash U$) can be either holonomic or anholonomic (or both on different subsets of $V\backslash U$). In fact, let $\{E_i\}$ be a frame defined on $V \subseteq M$ which is normal on $U \subset V$ and $A = [A^j_i]$ be non-degenerate matrix-valued function on V. According to (3.5), in the frame $\{E'_i := A^j_i E_j\}$ the connection's coefficients on U are $\Gamma'^i{}_{jk} = \big(A^{-1}\big)^i_l E'_k\big(A^l_j\big)$. Thereof $\{E'_i\}$ is normal on U iff $E'_k\big(A^l_j\big)|_U = 0$ or, equivalently, iff $E_k\big(A^l_j\big)|_U = 0$ which do not imply any restrictions on A (or on $\{E'_i\}$) on the set $V\backslash U$. Hence, choosing appropriately A on $V\backslash U$, we can force $\{E'_i\}$, which is normal on U, on the set $V\backslash U$, i.e., outside U, to be holonomic as well as anholonomic depending on the properties of A on $V\backslash U$.

Remark 5.2. Proposition 5.3 has the following generalization. Call a frame $\{E_i\}$, define on a neighborhood containing or equal to $U \subseteq M$, commuting on U if $[E_j, E_k]_{\_}|_U = 0$. (If U is a neighborhood, a frame commuting on U is holonomic on U and *vice versa*.) Then a frame normal for a connection ∇ is commuting on U iff ∇ is torsion free on U. (The proof is identical with the one of Proposition 5.3.) This result will not be used further as the commutativity of $\{E_i\}$ on U does not, generally, imply some assertions concerning the holonomicity of $\{E_i\}$ on U.

Remark 5.3. If U is not a neighborhood, from the relation $[E_j, E_k]_{\_}|_U = 0$, generally, does not follow the holonomicity of the frame $\{E_i\}$ on U as, by definition, this is a property defined on neighborhoods.

In particular, Proposition 5.3 means that only the symmetric, i.e., torsionless, linear connections may admit holonomic frames normal on a neighborhood and, consequently, normal coordinates. Moreover, these connections do not admit anholonomic normal frames.

Definition 5.2. Given a linear connection ∇ on a differentiable manifold M. A *chart* (V, x) of M and the associated to it *coordinate system* $\{x^i\}$ are called *normal on a subset* $U \subseteq V$ *for* ∇ if the coordinate frame $\big\{\frac{\partial}{\partial x^i}\big\}$ is normal on U for ∇.

Rewording, normal coordinates are those the associated to which local frames are normal. A trivial consequence of Definition 5.2 and Propositions 5.2 and 5.3 is the following result.

Corollary 5.2. *Coordinates normal on a neighborhood may exist only for torsion free linear connections and they generate all normal frames for them, if such exist at all.*

Remark 5.4. The theorem what spaces (with linear connection) admit normal coordinates (at a fixed point), which is a special case of Proposition 5.3 and Corollary 5.2, dates back to the first third of the twentieth century; see [43, 44].

We can also paraphrase the above in the assertion that the linear connections with non-vanishing torsion do not admit holonomic frames normal on a neighborhood and, consequently, coordinates normal for them do not exist; if for them normal frames exist, these frames are anholonomic with necessity.

For future purposes, we reformulate equation (5.4) in terms of local coordinates. Let $U \subseteq M$, and $\{x^i\}$ be local coordinates associated with a chart (V, x) such that $V \cap U \neq \varnothing$. Then, due to (3.6), the defined on V coordinates $x'^{\,i} = x'^{\,i}(x^1, \dots, x^{\dim M})$ are normal on $U \cap V$ iff

$$\left(\frac{\partial^2 x^i}{\partial x'^{\,j} \partial x'^{\,k}} + \frac{\partial x^m}{\partial x'^{\,j}} \frac{\partial x^n}{\partial x'^{\,k}} \Gamma^i{}_{mn} \right)\Bigg|_{U \cap V} = 0. \tag{5.4$'$}$$

Hence, normal coordinates exist iff the last equation has solution(s) with respect to $\{x'^{\,i}\}$ provided $\{x^i\}$ is given. Obviously, (5.4′) is a special case of (5.4) for $E_i = \frac{\partial}{\partial x^i}$, $E'_i = \frac{\partial}{\partial x'^{\,i}}$, and U replaced with $U \cap V$, the last reflecting the local character of the coordinates. We call (5.4′) the *normal coordinates equation* or the (*system of*) *equation*(*s*) *of normal coordinates*. Thus we see that the normal coordinates can be found by solving a second order system of partial differential equations while the analogous system for the normal frames is of first order. Correspondingly, if we are interested only in normal frames, the smoothness class of the connection admitting them is generally with one less then the one required for the existence of normal coordinates.

If the subset $U \subseteq M$ is not a neighborhood, results slightly weaker than Proposition 5.3 and Corollary 5.2 hold:

Proposition 5.4. *If a linear connection ∇ admits coordinates normal on a set U, it is torsionless on U. Said differently, if ∇ admits frame normal and holonomic on U, it is torsion free on U.*

Proof. If $q \in U$ and (V, x) with $V \ni q$ is a normal chart on U for ∇, equation (5.4) holds and, taking its antisymmetric part with respect to j and k, we get $\Gamma^i{}_{[mn]}|_{U \cap V} = 0$. Since $\{\frac{\partial}{\partial x^i}\}$ is a holonomic frame, $C^i_{jk} = 0$, the last equality is equivalent to $T^i_{jk}|_{U \cap V} = 0$ (see (3.14)). Hence $T^i_{jk}(q) = 0$ for every $q \in U$, i.e., ∇ is torsion free on U. □

Corollary 5.3. *Normal coordinates and holonomic normal frames may exist on $U \subseteq M$ only for linear connections which are torsionless on U.*

Proof. See Proposition 5.4. □

Further, we shall see that if frames normal on a submanifold exist, the vanishment of the torsion is also a sufficient conditions for the existence of normal coordinates.

Just in the above arguments we see the reasons, why in the majority of the literature only the case of normal coordinates for torsionless (symmetric) linear connections is investigated.

Let $\{E_i\}$ be a frame normal on U, defined on larger set $V \supset U$, $V \subseteq M$, and not normal on $V \backslash U \neq \varnothing$. In such a case it should be clearly understood that the connection coefficients in $\{E_i\}$ vanish *solely* on U and generally are non-zero on $V \backslash U$. This implies the non-vanishment of the derivatives $\Gamma^i{}_{jk,l} := E_l\big(\Gamma^i{}_{jk}\big)$, $\Gamma^i{}_{jk,lm} := E_m\big(\,E_l\big(\Gamma^i{}_{jk}\big)\big)$, etc., if they exist, i.e., some of the partial derivatives of the connection's coefficients may vanish on U but not all of them. If $\dim U = \dim M$, the equalities $\Gamma^i{}_{jk}|_U = 0$, due to (3.13), imply $R|_U = 0$, i.e., the flatness of the connection on U. Therefore on a neighborhood of M normal frames may exist if on it the connection is flat; in Section II.4 we shall see that this condition is also sufficient. By virtue of the above remarks, the normal frames, if any, can sometimes be used to simplify certain calculations. For instance, in a normal frame the components of curvature and torsion are (in the domain where the frame is normal):

$$R^i{}_{jkl} = -2\Gamma^i{}_{j[k,l]} \qquad T^i{}_{jk} = -C^i_{jk}.$$

Now, let us see what happens with the parallel transport in a set $U \subseteq M$ admitting a normal frame $\{E_i\}$. In it, due to (3.17), the parallel transport initial-value problem (3.18) reads

$$\frac{\mathrm{d}}{\mathrm{d}s} K^{i_1 \ldots i_{r'}}_{j_1 \ldots j_r}(\gamma(s)) = 0, \qquad K^{i_1 \ldots i_{r'}}_{j_1 \ldots j_r}(\gamma(s_0)) = K_0{}^{i_1 \ldots i_{r'}}_{j_1 \ldots j_r} \tag{5.5}$$

and hence $K^{i_1 \ldots i_{r'}}_{j_1 \ldots j_r}(\gamma(s)) = K_0{}^{i_1 \ldots i_{r'}}_{j_1 \ldots j_r} = \mathrm{const}$. This implies an interesting conclusion: *the parallel transport along paths lying entirely in the domain of a normal frame preserves the tensors' components in such a frames.* A trivial, but important, corollary from here is the following result.

Proposition 5.5. *If on a subset $U \subseteq M$ exist normal frames, then the parallel transport between every two points in U is independent of the path along with it is performed provided this path lies entirely in U.*

So, in a normal frame the parallel transport looks exactly as the 'parallel transport' in $\mathbb{R}^n$ which preserves the vectors' components while it changes only the points at which they are 'attached'. This rigorous result is the strict meaning of the assertion that 'locally the parallel transport is Euclidean'. Here the problem what exactly 'locally' means is open. Strictly speaking, by 'locally' one should understand a set on which normal frames or coordinates exist.

The assertion inverse to Proposition 5.5 is (locally) valid if U is a submanifold. This is important and highly untrivial result that will be proved in Subsection II.5.2 (see Corollary II.5.1 on page 123).

Proposition 5.6. *If frames normal on $U \subseteq M$ exist, then all of them are parallel on the set U.*

Proof. Let the frame $\{E_i\}$ be normal on a set U and $X = X^i E_i \in \mathfrak{X}(U)$. Applying condition (ii) of Definition 3.1, (3.1), and (5.1), we obtain $(\nabla_X E_j)|_U = (X^k \nabla_{E_k} E_j)|_U = X^k|_U \big(\Gamma^i{}_{jk}|_U\big) E_i|_U = 0$. So, according to Definition 3.4, the frame $\{E_i\}$ is parallel on U. □

The statement inverse to Proposition 5.6 is partially valid provided U is a submanifold of M: if on U exist parallel frames (with respect to paths in U), it admits also normal ones but, generally, not all parallel on U frames are normal. This essential assertion will be proved in Subsection II.5.2 (see Corollary II.5.2 on page 123). From more general positions, the links between parallel transports and normal frames will be studied in Chapter IV.

Running a few steps forward, we want briefly to stress on the importance of the normal frames for the physics. Above we saw that in a normal frame the mathematical structures related to a linear connection look like the ones in $\mathbb{R}^n$ (or in the Euclidean space $\mathbb{E}^n$). Similar situation is observed in the theoretical physics: the consideration of a physical phenomenon in a suitable *reference frame*, usually called *inertial*, makes it looking like a 'free' one, i.e., as in the absence of forces.[2] The analogy between normal frames and frames of reference is most obvious in the general theory of relativity: in it the gravitational field strength is (locally) described via the coefficients of some Riemannian linear connection, so the gravity force (locally) 'disappears' in a frame normal for this connection and, consequently, this frame is inertial. The same situation can be discovered in the classical physics but it is so natural and (almost?) trivial that it is practically nowhere mentioned in this context. Recently it was shown that the Heisenberg picture of (nonrelativistic) quantum mechanics, which is something like a 'quantum mechanics in an inertial frame', is identical to the representation of this theory in Schrödinger in an appropriate normal frame (in a Hilbert bundle – see Section IV.16 below). These and other examples push forward the idea of identifying the *mathematical* concept 'normal frame' with the *physical* concept 'inertial frame (of reference)'. Generally, in the identification

$$\text{NORMAL FRAME} \equiv \text{INERTIAL FRAME}$$

we see the reason why the manifolds with linear connection find a broad application in a lot of fundamental physical theories. The above explains why the investigation of the normal frames is essential for the theoretical physics, not only for pure mathematical purposes.

[2] A simple, but typical, example is the representation of the Klein-Gordon equation in normal coordinates which leads to the decomposition of the initial scalar field as a sum of independent harmonic oscillators [45, Chapter XXI, § 2]. A role similar to the normal coordinates play the normal modes or, more generally, normal waves in some linear dynamical systems [46, pp. 360–362].

6. Normal coordinates on Riemannian manifolds

In local coordinates $\{x^i\}$ on $U \subseteq M$, M being a Riemannian C^2 manifold with Riemannian metric g, the local coefficients of the Riemannian connection ∇ are (see (4.13) and (4.14))

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) = \begin{Bmatrix} i \\ jk \end{Bmatrix} \tag{6.1}$$

Proposition 6.1. *Local coordinates $\{x^i\}$ are normal on U if and only if in them*

$$\left. \frac{\partial g_{ij}}{\partial x^k} \right|_U = 0. \tag{6.2}$$

Proof. If $\{x^i\}$ are normal on U, then $\Gamma^i{}_{jk}|_U = 0$ by definition and, from equation (4.16), we get $g_{ij,k}(x) = 0$, $x \in U$. Conversely, if (6.2) holds, equation (6.1) implies $\Gamma^i{}_{jk}|_U = 0$. □

Therefore the normal coordinates on a Riemannian manifold can be defined as ones in which the partial derivatives of the components of the Riemannian metric vanish. Notice, equation (6.2) generally does not imply the vanishment of (all of) the second derivatives of the metric's components on U.

Regardless that nowadays we have at our disposal refine and powerful methods for proving the existence and construction of normal frames/coodinates on spaces far more general than the Riemannian ones, considered in the next chapters, below we present, following [24, pp. 51–57], possibly the first known such method which goes back to B. Riemann in 1854 [19, 24, resp. p. 155 and p. 53]. Besides from historical positions, this method is interesting due to the fact that, at least at a level of ideas, it is explicitly or implicitly presented as a part or underling background of some of the modern methods in that field.

Let M be a C^∞ manifold endowed with a C^∞ Riemannian metric g. Take an arbitrary point $p \in M$ and let $\{x^i\}$ be local coordinate system in a neighborhood $U' \ni p$. Consider on some subneighborhood $U \ni p$ of U' the unique geodesic $\gamma\colon J \to M$ passing through p, $\gamma(s_0) = p$ for some $s_0 \in J$, in the direction of arbitrarily chosen vector $X \in T_p(M)$, $\dot\gamma(s_0) = X$. This geodesic is necessarily of class C^∞ (see Section 4) and we can expand $\gamma^i(s)$, $s \in J$ in a power series with respect to $s - s_0$:

$$\gamma^i(s) = \gamma^i(s_0) + \left. \frac{\mathrm{d}\gamma^i}{\mathrm{d}s} \right|_{s_0} (s - s_0) + \sum_{n=2}^{\infty} \frac{1}{n!} \left. \frac{\mathrm{d}^n \gamma^i}{\mathrm{d}s^n} \right|_{s_0} (s - s_0)^n. \tag{6.3}$$

Taking into account $\gamma^i(s_0) = p^i = x^i(p)$, $\dot\gamma^i(s_0) = X^i$ and the geodesic equation (4.17), as well as the infinite number of equations obtained from it by differentiation with respect to s, we derive the expansion

$$\gamma^i(s) = p^i + X^i(s-s_0) - \frac{1}{2}\left\{ {i \atop jk} \right\}\Big|_p X^j X^k (s-s_0)^2 \\ - \sum_{n=3}^{\infty} \frac{1}{n!} \Gamma^i{}_{i_1\dots i_n}(p) X^{i_1} \cdots X^{i_n} (s-s_0)^n. \quad (6.4)$$

Here the C^1 functions $\Gamma^i{}_{i_1\dots i_n} : U \to \mathbb{R}$, which are symmetric in $i_1, \dots, i_n$, are given via the recurrent relations

$$\Gamma^i{}_{i_1\dots i_{n+1}} = \frac{\partial}{\partial x^{(i_1}} \Gamma^i{}_{i_2\dots i_{n+1})} - n \Gamma^k{}_{(i_1 i_2} \Gamma^i{}_{i_3\dots i_{n+1})k}, \qquad n \in \mathbb{N}\backslash\{1\}, \quad (6.5)$$

where

$$\Gamma^i{}_{jk} = \left\{ {i \atop jk} \right\} \quad (6.6)$$

and the round brackets denote symmetrization according to our convention on page xii.

If the series (6.4) is convergent and γ is without self-intersections, the quantities

$$y^i = X^i(s-s_0) \quad (6.7)$$

can be used as coordinates *along* γ (i.e., of the current point $\gamma(s)$) since the mapping $\varphi\colon \gamma(s) \mapsto \big(X^1(s-s_0), \dots, X^{\dim M}(s-s_0)\big)$ is homeomorphism (between 1-dimensional manifolds). These coordinates can be extended in a whole neighborhood of p. Let $V(p)$ be a normal neighborhood of p (see Definition 3.6 and Proposition 3.1). Since $U \cap V(p)$ is, evidently, normal neighborhood, every $q \in U \cap V(p)$ can be connected with p with a unique geodesic, say $\gamma\colon J \to U \cap V(p)$, lying entirely in $U \cap V(p)$ [38, p. 385], [11, Chapter IV, Proposition 3.4]. So, there exist unique $s_0, s \in J$ such that $\gamma(s_0) = p$ and $\gamma(s) = q$. In $U \cap V(p)$ we define a coordinate system $\{y^i\}$ such that

$$y^i(q) := \dot\gamma^i(s_0)(s-s_0). \quad (6.8)$$

In particular $y^i(p) = 0$. By virtue of (6.4), the link between $\{x^i\}$ and $\{y^i\}$ in $U \cap V(p)$ is

$$x^i(q) = x^i(p) + y^i(q) - \frac{1}{2}\left\{ {i \atop jk} \right\}\Big|_p y^j(q) y^k(q) - \sum_{n=3}^{\infty} \frac{1}{n!} \Gamma^i{}_{i_1\dots i_n}(p) y^{i_1}(q) \cdots y^{i_n}(q). \quad (6.9)$$

The transition $\{x^i\} \mapsto \{y^i\}$ is regular in a neighborhood of p as its Jacobian at p is $\det\big[\partial y^i/\partial x^j|_p\big] = \det\big[\partial x^i/\partial y^j|_p\big]^{-1} = \det[\delta^j_i] = 1 \neq 0$. Below we shall prove that $\{y^i\}$ are normal at p.

The just introduced coordinates $\{y^i\}$ are known as *Riemannian coordinates* with origin (pole) p.[1] Their domain coincides with that for which the series (6.9) may be inverted or, equivalently, it is a neighborhood of p in which no two geodesics through p meet again in it. The above results can be formulate as the following assertion.

Proposition 6.2. *For every point of a C^∞ Riemannian manifold with C^∞ metric there exists its normal neighborhood in which Riemannian coordinates can be introduces.*

Evidently, the equation (6.7) is nothing else but the equation of the unique geodesic with initial conditions (3.24). So, in Riemannian coordinates, the geodesic equation (4.17) looks exactly as the equation of the straight lines in $\mathbb{R}^n$. Denoting the Christoffel symbols with respect to $\{y^i\}$ with $\left\{ {i \atop jk} \right\}^y$, from (6.7) and (4.17), we derive

$$\left\{ {i \atop jk} \right\}^y y^j y^k \equiv 0$$

in the domain of $\{y^i\}$. On the opposite, if these equations hold, $\{y^i\}$ are Riemannian coordinates.

The following proposition expresses the most important for us result of this section.

Proposition 6.3. *The Riemannian coordinates are normal at their origin.*

Proof. Using (6.9), we calculate

$$\left.\frac{\partial x^i}{\partial y^j}\right|_p = \left.\frac{\partial y^i}{\partial x^j}\right|_p = \delta^i_j, \qquad \left.\frac{\partial^2 x^i}{\partial y^j \partial y^k}\right|_p = -\left.\left\{ {i \atop jk} \right\}\right|_p$$

The substitution of these equations in the transformation law (3.6) results in $\left.\left\{ {i \atop jk} \right\}^y\right|_p = 0$. □

Corollary 6.1. *The partial derivatives of the metric's components g^y_{ij} in the Riemannian coordinate system $\{y^i\}$ vanish at the their origin p:*

$$\left.\frac{\partial g^y_{ij}}{\partial y^k}\right|_p = 0. \tag{6.10}$$

Proof. The result is a consequence of Propositions 6.1 and 6.3 □

The Riemannian coordinates are 'more than normal' at their origin, viz. at it vanish not only the Christoffel symbols, but also all of the quantities (6.5) and

[1] The Riemannian (normal) coordinates find a vast field of application in physics, e.g., for approximate calculation of the metric tensor [47].

their symmetrized partial derivatives:

$$ {}^{y}\Gamma^{i}{}_{i_1\dots i_{n+1}}(p) = 0, \qquad n\in\mathbb{N}, \tag{6.11a} $$

$$ \frac{\partial}{\partial y^{(i_1}}\,{}^{y}\Gamma^{i}{}_{i_2\dots i_{n+2})}(p) = 0, \tag{6.11b} $$

$$ \frac{\partial}{\partial y^{(i_1}}\cdots\frac{\partial}{\partial y^{i_n}}\,{}^{y}\Gamma^{i}{}_{jk)}(p) = 0. \tag{6.11c} $$

Here the left superscript y indicates that the corresponding quantities are computed in $\{y^i\}$. The equalities (6.11a) follow from (6.9) if we put in them $x^i = y^i$ (the initial choice of $\{x^i\}$ is completely arbitrary) and take into account the symmetry of $\Gamma^{i}{}_{i_1\dots i_{n+1}}$ in the subscripts. The other equations, (6.11b) and (6.11c), are consequence of (6.11a) and (6.5). So, at the origin of the Riemannian coordinates vanish the Christoffel symbols together with their symmetrized derivatives of all orders.

If we (C^∞) change the initial local coordinates $\{x^i\}$, $\{x^i\}\mapsto\{x'^i\}$, then, due to (6.8) and (6.9), this results in the change of the Riemannian coordinates with a constant matrix:

$$ y^i(q)\mapsto y'^i(q) = a^i_j y^j(q) \tag{6.12} $$

with $a^i_j = \frac{\partial x'^i}{\partial x^j}\big|_p$. The converse is also true: a change (6.12) can be described via analytic change $\{x^i\}\mapsto\{x'^i\}$ which is, of course, not unique.

If $\{y^i\}$ are Riemannian coordinates with origin at some $p\in M$ in a neighborhood $U\ni p$, the holonomic frame $\{\partial/\partial y^i\}$ on U is normal at p. As we know from Proposition 5.2, all other frames $\{E_i\}$ normal at p are given by $E_i = A^j_i\frac{\partial}{\partial y^j}$ on U where the non-degenerate matrix-valued function $A = [A^j_i]$ is such that $\frac{\partial A}{\partial y^i}|_p = 0$. According to Proposition 5.3, these frames are holonomic at p, $[E_i, E_j]_\_|_p = 0$, but outside p they need not to be such, i.e., generally $[E_i, E_j]_\_|_q \neq 0$ for $q\in U\backslash\{p\}$ (see Remark 5.1 on page 40).

In this way we have obtained a *complete* description of the frames normal at a single point of a C^∞ Riemannian manifold.[2]

According to [19, pp. 155, 158] and [24, p. 53] normal coordinates (in a sense of Riemannian ones) ware first introduce by B. Riemann in 1854 [48].

Let $\{y^i\}$ be Riemannian coordinates with origin at $p\in M$. Since $[g^y_{ij}(p)]$ is a constant symmetric non-degenerate matrix, it can be diagonalized via transformation of the form $A[g^y_{ij}(p)]A^\top$ with constant orthogonal matrix $A = [a^j_i]$, $A^\top = A^{-1}$, $A^\top$ being the transposed to A matrix [49, Chapter 4, § 7]. So we can define new Riemannian coordinates $y'^i = a^i_j y^j$ in which $[g^{y'}_{ij}(p)] = A[g^y_{ij}(p)]A^\top = \mathrm{diag}(\varepsilon_1,\dots,\varepsilon_{\dim M})$ where $\varepsilon_i = \pm 1$ and $\mathrm{diag}(a_1,\dots,a_n)$ means a diagonal matrix with diagonal elements $a_1,\dots,a_n$. These particular Riemannian coordinates were

[2] At the moment this description is implicit. The explicit formula for the matrix A transforming $\{\partial/\partial y^i\}$ to arbitrary normal frame will be derived in Subsection II.2.2; see Theorem II.2.3.

first introduced by G.D. Birkhoff in 1923 [50] under the name 'normal coordinates' (see [19, p. 155] and [24, p. 55]). The associated to these coordinates frame $\{\partial/\partial y^i\}$ is such that at p it is orthonormal, $g(\partial/\partial y^i, \partial/\partial y^j)|_p = \pm\delta_{ij}$, and at the same time it is normal, $\nabla_{\partial/\partial y^i}(\partial/\partial y^j)|_p = 0$ (see [11, Chapter IV, § 3]). Nowadays these special Riemannian coordinates are constructed by means of the exponential mapping; see the end of Subsection II.2.3 and [8].

Besides the Riemannian coordinates, there are also other classes of coordinates normal at a single point. Their complete description on a manifold with arbitrary symmetric linear connection will be given in Subsection II.2.2. An example of such coordinates are the *geodesic coordinates*.[3] If $\{x^i\}$ are local coordinates in a neighborhood U of $p \in M$, the geodesic coordinates $\{z^i\}$ with origin at p of a point q belonging to some subneighborhood of U, are given via the series [24, p. 56]

$$x^i(q) = x^i(p) + z^i(q) - \frac{1}{2}\left\{ {i \atop jk} \right\}\Big|_p z^j(q) z^k(q) - \sum_{n=3}^{\infty} \frac{1}{n!} c^i{}_{i_1\dots i_n} z^{i_1}(q) \cdots z^{i_n}(q) \quad (6.13)$$

where $c^i{}_{i_1\dots i_n} = c^i{}_{(i_1\dots i_n)} \in \mathbb{R}$ and $z^i(p) := 0$. Analogously to (6.9), it is convergent and invertible with respect to $z^i(q)$ in some neighborhood of p.

Repeating the proof of Proposition 6.3, we see that the geodesic coordinates are normal at their origin p, $\left\{ {i \atop jk} \right\}\big|_p = 0$. In them the equation of the geodesics through the origin reads (see (6.4))

$$\gamma^i(s) = \gamma^i(s_0)(s - s_0) - \sum_{n=3}^{\infty} \frac{1}{n!} \Gamma^i{}_{i_1\dots i_n}(p) X^{i_1} \cdots X^{i_n} (s - s_0)^n$$

with $p = \gamma(s_0)$, $X = \dot\gamma(s_0)$, and the Γ's being defined by (6.5). Hence the Riemannian coordinates are geodesic ones in which the Γ's vanish at the origin (cf. (6.11)).

A number of examples of coordinates/frames normal for Riemannian connections will be presented below in Section 7. Instances of some applications of the Riemannian and geodesic coordinates can be found in [19, 24, 51].

At this point a natural question arises: are there subsets of a Riemannian manifold, different from single points, on which normal coordinates/frames exist?

The first result in this field was obtained by E. Fermi in 1922 [52]: a coordinate system can be chosen so that in it the partial derivatives of the components of the metric tensor vanish along a given smooth path without self-intersections (see also [53,54]). Now the Fermi's proof of this theorem is only of historical interest and we are not going to reproduce it here. Further, in Section II.3, we will establish an analogous result for arbitrary linear connections from which the above assertion is an evident special case (see, in particular, Proposition II.3.2 and Corollary II.3.1).

[3] In [51] the geodesic coordinates are defined as ones in which the partial derivatives of the metric vanish at a given point. By Proposition 6.1, the concepts 'geodesic coordinates' in Fock's sense and 'normal coordinates' in our sense are synonyms.

Till 1958 the problem for existence of normal coordinates/frames on sets other than points and curves (without self-intersections) in a general Riemannian manifold, as well as for other manifolds with linear connection, was open. In 1958 O'Raifeartaigh [55] proved a general theorem concerning the existence of normal frames on arbitrary submanifolds of manifolds with symmetric, i.e., torsionless, linear connection. This result will be reviewed in Section II.5. When applied to the above problem, it asserts that the points and curves are the only submanifolds of a Riemannian manifold on which normal coordinates/frames always exists. Only special types of Riemannian connections admit frames normal on other submanifolds; in particular, coordinates/frames normal on the whole manifold or its open subset (neighborhood) exist iff the connection is flat on it.

7. Examples of normal coordinates for Riemannian connections

The purpose of this section is to exemplify the general theory of Sections 5 and 6 on manifolds endowed with Riemannian connections generated by Riemannian metrics. The section contains also several exercises.

Most of the examples presented below will follow the following scheme. Let M be a submanifold of $\mathbb{R}^N$ for some $N \in \mathbb{N}$ [1] and $f\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a scalar product (see, e.g., (7.1) and (7.16) below) which induces a Riemannian metric $\bar{f}$ on $T(\mathbb{R}^N)$; one usually identifies f and $\bar{f}$ as $T(\mathbb{R}^N)$ and $\mathbb{R}^N$ are isomorphic vector spaces. Suppose g is the Riemannian metric on M induced (generated) by $\bar{f}$, viz. $g = \bar{f}|_M$ is the restriction of $\bar{f}$ to M.[2] Then the metric g induces on M a Riemannian connection ∇ as described in Section 4 (see, in particular, equations (4.13) and (4.14)). We shall calculate the coefficients of ∇ in concrete coordinate systems and will look for subsets of M on which these systems are normal for ∇. Besides, some constructions of geodesic coordinates will be pointed out.

Suppose the manifold under consideration has some symmetry and the coordinates/frames on it are chosen in a way that 'reflects' this symmetry. As we shall see below, in such a situation it is 'quite likely' that these coordinates/frames may happen to be normal on some ('natural' for the symmetry) subset for the connection induced on the manifold by a Riemannian metric which, in its turn, is generated by the Euclidean metric of some $\mathbb{R}^n$ for some $n \in \mathbb{N}$ in which the initial manifold is embedded.

---

[1] Recall the Whitney embedding theorem [56, p. 44, Theorem 1.9.12]: every C^2 n-dimensional manifold can be embedded in $\mathbb{R}^{2n}$, i.e., can be considered as a submanifold of $\mathbb{R}^N$ for some $N \in \mathbb{N}$; see also [56, p. 119, Remark 3.2.9].

[2] The easiest way to find g explicitly is to use differential forms. Let d denotes the exterior derivative operator and $\wedge$ is the wedge (exterior) product sign. If $\bar{f} = f_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$ in some coordinate system $\{x^i\}$ on $\mathbb{R}^N$, then $g = f_{ij}|_M \mathrm{d}x^i|_M \wedge \mathrm{d}x^j|_M$ and it is quite convenient to use as a coordinate system on M the restricted coordinate system $\{x^i|_M\}$, which agrees with the definition of a submanifold given on page 7.

Example 7.1 ((Locally) (pseudo-)Euclidean spaces). The Euclidean space $\mathbb{E}^n$, $n \in \mathbb{N}$, is a collection of $\mathbb{R}^n$ and (standard Euclidean) scalar product $e\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathrm{e}(p,q) = \sum_{i=1}^{n} p^i q^i \tag{7.1}$$

for all $p = (p^1, \dots, p^n) \in \mathbb{R}^n$ and $q = (q^1, \dots, q^n) \in \mathbb{R}^n$. If $\{u^i\}$ is the standard coordinate system in $\mathbb{R}^n$,[3] $u^i(p) := p^i$, the linear mapping $I_q\colon T_q(\mathbb{E}^n) \to \mathbb{E}^n$ with $I_q\big(p^i \frac{\partial}{\partial u^i}\big|_q\big) = p$ is an isomorphism and transfers the scalar product e from $\mathbb{E}^n$ to $T_x(\mathbb{E}^n)$ by $e \mapsto \bar{e}\colon p \mapsto \bar{e}_p := e \circ (I_p \times I_p)$; hence $\bar{e}$ is a metric on $\mathbb{E}^n$ (or $\mathbb{R}^n$) such that

$$\bar{e}\Big(p^i \frac{\partial}{\partial u^i}, p^i \frac{\partial}{\partial u^i}\Big) = e(p,q) \tag{7.2}$$

and the components of $\bar{e}$ in the global frame $\big\{\frac{\partial}{\partial u^i}\big\}$ are $\bar{e}_{ij} = \delta_{ij}$ (see (4.9)).[4] The metric $\bar{e}$ induces on $\mathbb{E}^n$ a (flat) Riemannian connection ∇^E whose coefficients in $\big\{\frac{\partial}{\partial u^i}\big\}$ are (see (4.13))

$$\Gamma^i{}_{jk} = \begin{Bmatrix} i \\ jk \end{Bmatrix} = 0. \tag{7.3}$$

Consequently the (global) coordinate system $\{u^i\}$ and the (global) frame $\big\{\frac{\partial}{\partial u^i}\big\}$ are normal for ∇^E on the whole space $\mathbb{E}^n$.

Similarly, if on a manifold M is given a Riemannian metric g such that its components g_{ij} are constant functions relative to a frame $\big(\frac{\partial}{\partial x^i}\big)$ induced by the coordinates x^i of a chart (V, x) of M, then the Riemannian connection ∇^g generated by g is such that (see (4.13))

$$\Gamma^i{}_{jk}|_V = \begin{Bmatrix} i \\ jk \end{Bmatrix}\Big|_V = 0. \tag{7.4}$$

Consequently the coordinate system $\{x^i\}$ and the corresponding to it natural frame $\big\{\frac{\partial}{\partial x^i}\big\}$ are normal for ∇^g on V; the afore considered case of $\mathbb{E}^n$ corresponds to the choice $(M, g) = (\mathbb{R}^n, \bar{e})$, $V = \mathbb{R}^n$ and $x^i = u^i$. All other frames $\{E_i\}$ normal on V for ∇^g are such that $E_i = A_i^j \frac{\partial}{\partial x^j}$ with $\det[A_i^j] \neq 0, \infty$ and $\frac{\partial A_i^j}{\partial x^k}\big|_V = 0$ (see Proposition 5.2), so that A_i^j are constant on V. Evidently $E_i = \frac{\partial}{\partial y^i}$ for some coordinates y^i normal on V and such that $x^i = A_j^i y^j + A^i$ for some numbers $A^i \in \mathbb{K}$.

Example 7.2 (One-dimensional manifolds). Let M be one-dimensional manifold, $\dim M = 1$, endowed with a C^1 Riemannian metric g. Let (V, x) be a chart of M

[3] In this section, we denote the standard coordinates on $\mathbb{R}^n$ by $u^1, \dots, u^n$ instead by $r^1, \dots, r^n$ to distinguish them from some radii (e.g., of (pseudo)spheres or torii), typically denoted by r with possible indices, and their powers.

[4] Usually one identifies $T_p(\mathbb{E}^n)$ with $\mathbb{E}^n$, $p^i \frac{\partial}{\partial u^i}$ with p, and e with $\bar{e}$.

and $\{x^1\}$ be the corresponding local coordinate system. The sole coefficient of the Riemannian connection ∇ induced by g is (see (4.13) or (4.15))

$$\Gamma^1{}_{11} = \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{1}{2}\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial x^1} = \frac{1}{2}\frac{\partial \ln g_{11}}{\partial x_1}, \tag{7.5}$$

where $g_{11} = 1/g^{11}$ is the sole component of g in $\{\frac{\partial}{\partial x^1}\}$. Therefore $\{x^1\}$ and $\{\frac{\partial}{\partial x^1}\}$ are normal on a subset $U \subseteq V$, i.e., $\Gamma^1{}_{11}|_U = 0$, iff

$$\frac{\partial g_{11}}{\partial x^1}\Big|_U = 0. \tag{7.6}$$

Suppose U is an open set, i.e., $U = x^{-1}(J)$ for some open real interval J. Define a chart (V, x') with local coordinate x'^1 such that, for a fixed $p_0 \in V$ and all $p \in V$,

$$x'^1(p) = x^1(p_0) + \int_{p_0}^{p} |g_{11}|^{-\frac{1}{2}}\,\mathrm{d}x^1 \tag{7.7}$$

where $|\lambda|$ is the absolute value of $\lambda \in \mathbb{R}$, viz. $|\lambda| = (\operatorname{sign}\lambda)\lambda$ with $\operatorname{sign}\lambda := \pm 1$ for $\lambda \lessgtr 0$ and $\operatorname{sign}\lambda = 0$ for $\lambda = 0$. The transformation $x^1 \mapsto x'^1$ is invertible in V as its Jacobian is $\frac{\partial x'^1}{\partial x^1} = |g_{11}|^{-\frac{1}{2}} \neq 0$. The component of g in $\{\frac{\partial}{\partial x^1}\}$ is (see (2.44) with $A^1_1 = \frac{\partial x^1}{\partial x'^1}$)

$$g'_{11} = g\Big(\frac{\partial}{\partial x'^1}, \frac{\partial}{\partial x'^1}\Big) = \operatorname{sign} g_{11}. \tag{7.8}$$

It is constant on V and consequently

$$\frac{\partial g'_{11}}{\partial x'^1}\Big|_V = 0 \qquad \Gamma'^i{}_{jk}|_V = 0 \tag{7.9}$$

so that $\{x^i\}$ and $\{\frac{\partial}{\partial x^1}\}$ are normal on V for ∇.

The main conclusion from the above is that for any point in one-dimensional Riemannian space there exists local coordinates in some its neighborhood which coordinates are normal on their domain for the Riemannian connection induced via the initial Riemannian metric. Since the curvature tensor identically vanishes for a C^1 connection on C^3 1-manifold (see (3.11) and (3.13)), this result is also a consequence of Theorem II.4.1, presented below on page 104, in a case of a C^2 metric on a C^3 1-dimensional Riemannian manifold.[5]

Example 7.3 (The two-sphere $\mathbb{S}^2$). Let $\mathbb{S}^2 := \{(v^1, v^2, v^3) \in \mathbb{R}^3 : (v^1)^2 + (v^2)^2 + (v^3)^2 = r^2\}$ be a two-dimensional sphere of radius $r \in \mathbb{R}$, $r > 0$, in $\mathbb{R}^3$. Let on $\mathbb{S}^2$ minus one point (e.g., the 'North pole') be given the standard spherical coordinates (θ, φ), with range $(0, \pi] \times [0, 2\pi)$, obtained from the spherical coordinates in $\mathbb{R}^3$ via restriction to $\mathbb{S}^2$ (see equations (7.12) below or/and [57, Section 3.1-6]).

[5]See also Example II.6.7.

The Euclidean metric in $\mathbb{R}^3$ induces on $\mathbb{S}^2$ a metric g whose components g_{ij}, $i, j = 1, 2$, in the coordinates $\{x^1 = \theta, x^2 = \varphi\}$ are such that [57, Table 6.5-1]

$$[g_{ij}] = \operatorname{diag}(r^2, r^2 \sin^2\theta) \quad [g^{ij}] = \operatorname{diag}\Big(\frac{1}{r^2}, \frac{1}{r^2 \sin^2\theta}\Big). \tag{7.10}$$

Let ∇ be the Riemannian connection generated by g (see Section 4). An elementary calculation by means of (4.15) shows that the non-vanishing coefficients of ∇ in $\{x^1, x^2\}$ are (see also [57, Table 6.5-1])

$$\Gamma^1{}_{22} = -\sin\theta\cos\theta \quad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot\theta. \tag{7.11}$$

Since $\cos\frac{\pi}{2} = \cot\frac{\pi}{2} = 0$, the spherical coordinates $\{x^1 = \theta, x^2 = \varphi\}$ and hence the frame $\left\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\right\}$ on $\mathbb{S}^2$ are normal for ∇ along the path $\gamma\colon [0, 2\pi) \to \mathbb{S}^2$ such that $\theta \circ \gamma = \frac{\pi}{2}$ and $\varphi \circ \gamma = \mathsf{id}_{[0,2\pi)}$, i.e., they are normal on the great circle obtained by intersecting $\mathbb{S}^2$ with the equatorial plane $\{u^3(p) = 0 : p \in \mathbb{R}^3\}$. The general frame $\{E_1, E_2\}$ normal for ∇ along γ is such that $E_i = A_i^j \frac{\partial}{\partial x^j}$ where the 2×2 matrix-valued function A satisfies the equation (see Proposition 5.2) $\frac{\partial A_i^j}{\partial x^k}\big|_{\gamma([0,2\pi))} = 0$. The explicit form of A will be presented in Example II.6.2 below; for details concerning frames/coordinates normal along paths for linear connections, see Section II.3 in the Chapter II.

Due to the symmetry of the sphere $\mathbb{S}^2$, it is clear that for any great circle on it there exist (local) coordinates on $\mathbb{S}^2$ which are normal on this circle for the Riemannian connection on $\mathbb{S}^2$ generated by the metric g (induced from $\mathbb{R}^3$).

Example 7.4 (The spheres $\mathbb{S}^n$ for any $n \in \mathbb{N}$). The circle $\mathbb{S}^1 := \{(v^1, v^2) \in \mathbb{R}^2 : (v^1)^2 + (v^2)^2 = r^2\}$ of radius $r \in \mathbb{R}$, $r > 0$, is a 1-dimensional manifold. The polar coordinate function φ, with range $[0, 2\pi)$ and induced from polar coordinates in $\mathbb{R}^2$, provides a coordinate function on a subset V equal to $\mathbb{S}^1$ without some arbitrarily fixed point. The standard Euclidean metric on $\mathbb{E}^2$ induces on $\mathbb{S}^1$ a metric g whose only component in $\{x^1 = \varphi\}$ is $g_{11} = r^2 = \text{const}$. Hence the metric g generates on $\mathbb{S}^2$ a Riemannian connection ∇ whose only coefficient in $\{x^1\}$ is $\Gamma^1{}_{11}|_V = 0$. Consequently, the polar coordinate system $\{x^1 = \varphi\}$ on $\mathbb{S}^2$ is normal for ∇ on V. This result agrees with the results of Examples 7.1 (g is a 'constant' metric in $\{x^1\}$) and 7.2 ($\mathbb{S}^1$ is a one-dimensional manifold).

Consider now the general case of an n-dimensional, $n \geq 2$, sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ of radius $r > 0$, viz.

$$\mathbb{S}^n := \{(v^1, \ldots, v^n) \in \mathbb{R}^{n+1} : (v^1)^2 + \cdots + (v^n)^2 = r^2\}.$$

Let $(R, \theta_{n-1}, \ldots, \theta_1, \varphi)$ be the (hyper-)spherical coordinates in $\mathbb{R}^{n+1}$; the range of R is $(0, \infty)$, the one of $\theta_{n-1}, \ldots, \theta_1$ is $(0, \pi]$ and φ takes values in $[0, 2\pi)$. They are connected with the standard coordinate system $\{u^1, \ldots, u^{n+1}\}$ in $\mathbb{R}^{n+1}$ by the following equations:

$$
\begin{aligned}
u^1 &= R\sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_2\sin\theta_1\cos\varphi\\
u^2 &= R\sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_2\sin\theta_1\sin\varphi\\
u^3 &= R\sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_2\cos\theta_1 \qquad (\text{if } n\geq 3)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
u^k &= R\sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_{k-1}\cos\theta_{k-2} \qquad \text{for } 3\leq k\leq n\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
u^n &= R\sin\theta_{n-1}\cos\theta_{n-2} \qquad (\text{if } n\geq 3)\\
u^{n+1} &= R\cos\theta_{n-1}.
\end{aligned}
\tag{7.12}
$$

The equation of $\mathbb{S}^n$ in these coordinates is $R^2 = r^2$ or $R = r$ (as $r > 0$ and R takes non-negative values). The functions

$$\{x^1 = \theta_{n-1}, x^2 = \theta_{n-2}, \dots, x^{n-1} = \theta_1, x^n = \varphi\} \tag{7.13}$$

provide an internal coordinates system on $\mathbb{S}^n$ (without one point) in which the metric g induced on $\mathbb{S}^n$ from $\mathbb{E}^n$ has components g_{ij}, $i, j = 1, \dots, n$, such that $(2 \leq k \leq n-1)$

$$[g_{ij}] = \operatorname{diag}(r^2, r^2\sin^2 x^1, \dots, r^2\sin^2 x^1 \dots \sin^2 x^{k-1}, \dots, r^2 \sin^2 x^1 \cdots \sin^2 x^{n-1}). \tag{7.14}$$

The metric g induces on $\mathbb{S}^n$ a Riemannian metric ∇. Using equations (4.15) and (4.13), we can calculate that ∇ has the following non-vanishing coefficients in the coordinate system (7.13) (do not sum over i and k!):

$$\Gamma^i{}_{ik}|_{k<i} = \Gamma^i{}_{ki}|_{k<i} = \cot x^k \tag{7.15a}$$

$$\Gamma^i{}_{kk}|_{k>i} = -\sin x^i \cos x^i \prod_{l=i+1}^{k-1} \sin^2 x^l \tag{7.15b}$$

for $1 \leq i, k \leq n$. (We set $\prod_{l=a}^{b}(\cdots) := 1$ for $b < a$.) For instance, these coefficients for $n = 2$ (resp. for $n = 3$) are given by (7.11) (resp. by (7.43) below).

Let $\gamma\colon [0, 2\pi) \to \mathbb{S}^n$ be a path in $\mathbb{S}^n$ such that $x^i \circ \gamma = \frac{\pi}{2}$ for $i = 1, \dots, n-1$ and $x^n \circ \gamma = \mathsf{id}_{[0,2\pi]}$. Since $\cos\frac{\pi}{2} = \cot\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$, we have

$$\Gamma^i{}_{jk}|_{\gamma([0,2\pi))} = 0$$

and, consequently, the coordinate system $\{x^i\}$ and the frame $\{\frac{\partial}{\partial x^i}\}$ are normal for ∇ along the path γ. From (7.12) is clear that the set $\gamma([0, 2\pi))$ is a circle obtained by intersecting $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with the (u^1, u^2)-plane $\{v \in \mathbb{R}^{n+1} : u^i(p) = 0 \text{ for } i \geq 3\}$ in $\mathbb{R}^{n+1}$; in fact, we have

$$\gamma([0, 2\pi)) = \{(v^1, v^1, \underbrace{0, \dots, 0}_{(n-1)-\text{times}}) \in \mathbb{R}^{n+1} : (v^1)^2 + (v^2)^2 = r^2\} = \mathbb{S}^1 \times (\underbrace{0, \dots, 0}_{(n-1)-\text{times}}).$$

It is clear, if C is a circle on $\mathbb{S}^n$ obtained by intersecting $\mathbb{S}^n$ by a 2-plane through its origin, then there are coordinates on $\mathbb{S}^n$ normal along C for the Riemannian connection considered above; this is a consequence of the afore-presented material in which u^1 and u^2 should be standard coordinates in the 2-plane to which C belongs.

The case of $\mathbb{S}^2$, investigated in Example 7.3, corresponds to the choice $n = 2$.

Example 7.5 (The pseudospheres $\mathbb{S}^n_q$). The Euclidean space $\mathbb{R}^n_q := \mathbb{R}^n_{p,q} := (\mathbb{R}^n, e^n_q)$ of index q,where $p, q \in \mathbb{N} \cup \{0\}$ and $p + q = n \in \mathbb{N}$, consists of the space $\mathbb{R}^n$ endowed with a scalar product (metric) (of index q) $e^n_q := e^n_{p,q}$ such that

$$e^n_q(v, w) := \sum_{a=1}^{p} v^a w^a - \sum_{b=p+1}^{n} v^b w^b \tag{7.16}$$

for $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$ and $w = (w^1, \ldots, w^n) \in \mathbb{R}^n$.[6]

Special cases of the spaces $\mathbb{R}^n_q$ are well know in physics. For instance, the (Lorentzian) space $\mathbb{R}^5_4 = \mathbb{R}^4_4 \times \mathbb{R}$ and the spherical space $\mathbb{S}^3 \subset \mathbb{R}^4$ of (special) radius R find application in the geometrical interpretation of the Einstein model of the Universe [58, § 138] (see also Example 7.9 below). However, the most famous example is the Minkowski spacetime $M_4 := \mathbb{R}^4_3$ (or the isomorphic to it space $\mathbb{R}^4_1$) on which the special theory of relativity and the whole relativistic physics rests; for instance, see [20, 21, 51, 59–64].

The pseudosphere (pseudo-Riemannian spherical manifold) $\mathbb{S}^n_q$ of index q, with $n \geq 1$ and $0 \leq q \leq n$, of radius $r \in \mathbb{R}$, $r > 0$, in $\mathbb{R}^{n+1}_q$ is defined by [39, Section 2.4], [27, Chapter 1, § 4]

$$\mathbb{S}^n_q := \{v \in \mathbb{R}^{n+1}_q : e^{n+1}_q(v, v) = r^2\}.$$

It is a C^ω 2-connected manifold. Since the pseudospheres reduce in the case $q = 0$ to ordinary spheres, $\mathbb{S}^n_0 = \mathbb{S}^n$, investigated in Example 7.4, further we shall suppose that $q \geq 1$. Since the pseudosphere $\mathbb{S}^1_1$ is a 2-connected 1-dimensional manifold, which case is covered by Example 7.2, we shall suppose below $n \geq 2$.[7]

By g below will be denoted the Riemannian metric on $\mathbb{S}^n_q$ induced by the metric e^{n+1}_q, $g = e^{n+1}_q|_{\mathbb{S}^n_q}$. Respectively, by ∇ will be denoted the Riemannian connection on $\mathbb{S}^n_q$ generated by g as described in Section 4.

[6] By replacing e and $\bar{e}$ in (7.2) by e^n_q and $\bar{e}_q$, respectively, we obtain a Riemannian (pseudo-Euclidean) metric $\bar{e}_q$ of index q on $\mathbb{R}^n$ induced by e^n_q; for evident reasons, one usually writes e^n_q for $\bar{e}^n_q$.

[7] Example 7.1 is also applicable in the concrete case. Briefly this can be shown as follows. On $\mathbb{R}^2_1$ can be introduced pseudospherical coordinates (ρ, χ), with respective ranges $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}$, connected with the standard coordinates (u^1, u^2) on $\mathbb{R}^2$ by

$$u^1 = \rho \cosh \chi \qquad u^2 = \rho \sinh \chi. \tag{7.17}$$

In them $\mathbb{S}^1_1 = \{v \in \mathbb{R}^2_1 : \rho(v) = \pm r\}$. The function χ provides a coordinate system $\{\chi\}$ with domain $\{v \in \mathbb{R}^2_1 : \rho(v) = +r\}$ or $\{v \in \mathbb{R}^2_1 : \rho(v) = -r\}$ on $\mathbb{S}^1_1$. In it the metric g induced on $\mathbb{S}^1_1$ by e^1_1 has a single component equal to r^2. Hence g induces a flat Riemannian connection on $\mathbb{S}^1_1$ for which $\{\chi\}$ is normal on its domain. Cf. the last part of Example 7.1.

Let $n \geq 2$, $q \geq 1$ and $p := n - q \geq 1$; hence $q \leq n-1$. Then in $\mathbb{R}^{n+1}_q$ can be introduced pseudospherical coordinates

$$(\rho, \chi_q, \dots, \chi_1, \theta_{p-1}, \dots, \theta_1, \varphi) \tag{7.18}$$

where for $p = 1$ the θ's and the terms containing them below should be missing/deleted. The range of ρ is $\mathbb{R} \setminus \{0\}$, the one of the χ's is $\mathbb{R}$, the range of the θ's is $(0, \pi]$, and φ takes values in $[0, 2\pi)$. The connection of the coordinates (7.18) with the standard ones $(u^1, \dots, u^{n+1})$ in $\mathbb{R}^{n+1}$ (or $\mathbb{R}^{n+1}_q$) is:

$$\begin{aligned}
u^1 &= \rho \cosh \chi_q \cdots \cosh \chi_1 \sin \theta_{p-1} \cdots \sin \theta_2 \sin \theta_1 \cos \varphi \\
u^2 &= \rho \cosh \chi_q \cdots \cosh \chi_1 \sin \theta_{p-1} \cdots \sin \theta_2 \sin \theta_1 \sin \varphi \\
u^3 &= \rho \cosh \chi_q \cdots \cosh \chi_1 \sin \theta_{p-1} \cdots \sin \theta_2 \cos \theta_1 \qquad (\text{if } p \geq 3) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u^{2+k} &= \rho \cosh \chi_q \cdots \cosh \chi_1 \sin \theta_{p-1} \cdots \sin \theta_{k+1} \cos \theta_k \qquad \text{for } 1 \leq k \leq p-2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u^p &= \rho \cosh \chi_q \cdots \cosh \chi_1 \sin \theta_{p-1} \cos \theta_{p-2} \qquad (\text{if } p \geq 3) \\
u^{p+1} &= \rho \cosh \chi_q \cdots \cosh \chi_1 \cos \theta_{p-1} \qquad (\text{if } p \geq 2) \\
u^{p+1+1} &= \rho \cosh \chi_q \cdots \cosh \chi_2 \sinh \chi_1 \qquad (\text{if } q \geq 3) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u^{p+1+k} &= \rho \cosh \chi_q \cdots \cosh \chi_{k+1} \sinh \chi_k \qquad \text{for } 2 \leq k \leq q-1 \ (\text{if } q \geq 3) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u^n &= \rho \cosh \chi_q \sinh \chi_{q-1} \qquad (n = p + 1 + q - 1) \\
u^{n+1} &= \rho \sinh \chi_q .
\end{aligned} \tag{7.19}$$

The comparison between (7.19) and (7.12) reveals that $(\theta_p, \dots, \theta_1, \varphi)$ are spherical coordinates on $\mathbb{S}^p \subset \mathbb{R}^{p+1} \subset \mathbb{R}^{p+1} \times \mathbb{R}^q_q \cong \mathbb{R}^{n+1}_q$, where $\cong$ is the 'isomorphic to' sign.

Suppose now that $n \geq 2$ and $q \geq 2$; hence $p := n - q \geq 0$ and this case covers the possibility $q = n$, i.e., the pseudosphere $\mathbb{S}^n_n$ in the Lorentz manifold $\mathbb{R}^{n+1}_n$. Then in $\mathbb{R}^{n+1}_q$ can be introduced phseudospherical coordinates

$$(\rho, \theta_{p-1}, \dots, \theta_1, \chi, \tau_{q-1}, \dots, \tau_1, \varphi) \tag{7.20}$$

where for $p = 1$ (resp. $p = 0$) the θ's (resp. the θ's and τ_{q-1}) and the terms containing them below should be missing/deleted. The range of ρ is $\mathbb{R} \setminus \{0\}$, the one of χ is $\mathbb{R}$, the range of the θ's and τ's is $(0, \pi]$, and φ takes values in $[0, 2\pi)$. The explicit connection of these coordinates with the standard ones $\{u^1, \dots, u^{n+1}\}$ on

$\mathbb{R}^{n+1}$ (or $\mathbb{R}^{n+1}_{q+1}$) is (cf. (7.12) and (7.19)):

$$
\begin{aligned}
u^{p+1} &= \rho \sin\theta_{p-1} \cdots \sin\theta_1 \sinh\chi \sin\tau_{q-1} \cdots \sin\tau_2 \sin\tau_1 \cos\varphi \\
u^{p+2} &= \rho \sin\theta_{p-1} \cdots \sin\theta_1 \sinh\chi \sin\tau_{q-1} \cdots \sin\tau_2 \sin\tau_1 \sin\varphi \\
u^{p+3} &= \rho \sin\theta_{p-1} \cdots \sin\theta_1 \sinh\chi \sin\tau_{q-1} \cdots \sin\tau_2 \cos\tau_1 \qquad (\text{if } q \geq 3) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u^{p+k} &= \rho \sin\theta_{p-1} \cdots \sin\theta_1 \sinh\chi \sin\tau_{q-1} \cdots \sin\tau_{k-1} \cos\tau_{k-2} \qquad \text{for } 3 \leq k \leq q \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u^{n} &= \rho \sin\theta_{p-1} \cdots \sin\theta_1 \sinh\chi \sin\tau_{q-1} \cos\tau_{q-2} \qquad (p+q=n;\ \text{if } q \geq 3) \\
u^{n+1} &= \rho \sin\theta_{p-1} \cdots \sin\theta_1 \sinh\chi \cos\tau_{q-1} \\
u^{1} &= \rho \sin\theta_{p-1} \cdots \sin\theta_1 \cosh\chi \\
u^{2} &= \rho \sin\theta_{p-1} \cdots \sin\theta_2 \cos\theta_1 \qquad (\text{if } p \geq 3) \\
&\cdots\cdots\cdots\cdots\cdots \\
u^{k} &= \rho \sin\theta_{p-1} \cdots \sin\theta_k \cos\theta_{k-1} \qquad \text{for } 2 \leq k \leq p-1 \\
&\cdots\cdots\cdots\cdots \\
u^{p-1} &= \rho \sin\theta_{p-1} \cos\theta_{p-2} \qquad (\text{if } p \geq 3) \\
u^{p} &= \rho \cos\theta_{p-1}.
\end{aligned}
\tag{7.21}
$$

The only peculiarity here is the case $q = n\,(\geq 2)$ and $p = 0$, when we have (see above; for $q = n = 2$, the terms containing the τ's should be missing/deleted):

$$
\begin{aligned}
u^{2} &= \rho \sinh\chi \sin\tau_{q-2} \cdots \sin\tau_1 \cos\varphi \\
u^{3} &= \rho \sinh\chi \sin\tau_{q-2} \cdots \sin\tau_1 \sin\varphi \\
u^{4} &= \rho \sinh\chi \sin\tau_{q-2} \cdots \sin\tau_2 \cos\tau_1 \qquad (\text{if } q \geq 3) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u^{k} &= \rho \sinh\chi \sin\tau_{q-2} \cdots \sin\tau_{k-1} \cos\tau_{k-2} \qquad \text{for } 4 \leq k \leq q-1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
u^{n} &= \rho \sinh\chi \sin\tau_{q-2} \cos\tau_{q-3} \qquad (q=n;\ \text{if } q \geq 4) \\
u^{n+1} &= \rho \sinh\chi \cos\tau_{q-2} \qquad (\text{if } q \geq 3) \\
u^{1} &= \rho \cosh\chi.
\end{aligned}
\tag{7.22}
$$

The coordinates (7.20) are also a modification of the spherical coordinates given via (7.12); indeed, the coordinates $(\tau_{q-1}, \ldots, \tau_1, \varphi)$ are spherical coordinates on $\mathbb{S}^q_q \subset \mathbb{R}^{q+1}_q \subset \mathbb{R}^{q+1}_q \times \mathbb{R}^p \cong \mathbb{R}^{n+1}_q$.

The pseudosphere $\mathbb{S}^n_q$ has the following representation in any one of the coordinates (7.18) or (7.20) (when they are applicable for given n, q and $p = n - q$)

$$\mathbb{S}^n_q = \{v \in \mathbb{R}^{n+1}_q : \rho(v) = \pm r\}. \tag{7.23}$$

Consider now the case $n \geq 2$ and $q \geq 2$ when the coordinates (7.20) are applicable; to exclude the special case $(p, q) = (0, n)$, one can impose the additional restriction $p \geq 1$, but this does not change the essence of the calculations and the final results. They induce on the pseudosphere $\mathbb{S}^n_q$ the coordinate system

$$\{x^1 = \theta_{p-1}, \dots, x^{p-1} = \theta_1, x^p = \chi, x^{p+1} = \tau_{q-1}, \dots, x^{p+q-1} = \tau_1, x^n = \varphi\}. \tag{7.24}$$

The Riemannian metric g induced on $\mathbb{S}^n_q$ by e^{n+1}_q has in $\{x^i\}$ components g_{ij}, $i, j = 1, \dots, n$, forming the diagonal matrix

$$\begin{aligned}[g_{ij}] &= \operatorname{diag}(g_1, \dots, g_n) \\ &= r^2 \operatorname{diag}\bigl(1, \sin^2 x^1, \sin^2 x^1 \sin^2 x^2, \dots, \sin^2 x^1 \cdots \sin^2 x^{p-1}, \underbrace{0, \dots, 0}_{q\text{-times}}\bigr) \\ &\quad - r^2 \sin^2 x^1 \cdots \sin^2 x^{p-1} \sinh^2 x^p \\ &\quad \times \operatorname{diag}\bigl(\underbrace{0, \dots, 0}_{p\text{-times}}, 1, \sin^2 x^{p+1}, \sin^2 x^{p+1} \sin^2 x^{p+2}, \dots, \sin^2 x^{p+1} \cdots \sin^2 x^{p+q-1}\bigr).\end{aligned} \tag{7.25}$$

Let ∇ be the Riemannian connection on $\mathbb{S}^n_q$ generated by g. The coefficients ${\Gamma^i}_{jk}$ of ∇ in $\{x^i\}$ can be calculated via equations (4.13) and (4.15). The non-vanishing of them are (cf. equations (7.15), which remain true for $1 \leq i, j, k \leq n$ with $i, j, k \neq p$)

$$\left.{\Gamma^i}_{ik}\right|_{\substack{k<i \\ i,k\neq p}} = \left.{\Gamma^i}_{ki}\right|_{\substack{k<i \\ i,k\neq p}} = \cot x^k \qquad {\Gamma^p}_{pk}|_{k>p} = {\Gamma^p}_{kp}|_{k>p} = \cot x^k \tag{7.26a}$$

$$\left.{\Gamma^i}_{kk}\right|_{\substack{k>i \\ i,k\neq p}} = -\sin x^i \cos x^i \prod_{l=i+1}^{k-1} \sin^2 x^l \qquad \left.{\Gamma^i}_{pp}\right|_{i<p} = +\sin x^i \cos x^i \prod_{l=i+1}^{p-1} \sin^2 x^l \tag{7.26b}$$

$$\left.{\Gamma^p}_{kk}\right|_{k>p} = -\coth x^p \qquad \left.{\Gamma^i}_{ip}\right|_{i>p} = \left.{\Gamma^i}_{pi}\right|_{i>p} = +\coth x^p \tag{7.26c}$$

These formulae show that the pseudospherical coordinates (7.24) on $\mathbb{S}^n_q$ are nowhere normal for ∇ as $\coth s \neq 0$ for all $s \in \mathbb{R}$. (This result could be expected as the coordinates employed are 'quite near' to a direct sum of two sets of spherical coordinates, which are suitable of other kind of a symmetry.) However, on the subset $C := \{v \in \mathbb{S}^n_q : x^i(v) = \frac{\pi}{2} \text{ for } i \neq p\}$ (which is 2-connected 2-manifold), the coordinate system $\{x^i\}$ is 'very near' to a normal one as only the coefficients (7.26c) of ∇ are non-vanishing on C. For this reason, one can expect that coordinates normal on a subset of C can be found by a suitable change of the coordinate system $\{x^i\}$.

Let us look for geodesic coordinate system $\{z^i\}$ on $\mathbb{S}^n_q$ in a neighborhood of a point $v_0 \in C \subset \mathbb{S}^n_q$. We set (see (6.13) with $c^i_{\dots} = 0$)

$$x^i(v) = x^i(v_0) + z^i(v) - \frac{1}{2}{\Gamma^i}_{jk} z^j(v) z^k(v)$$

where $v \in C$ and $z^i(v_0) := 0$.

The equations (7.26) reduce this system of equations to:

$$x^i(v) = x^i(v_0) + z^i(v) \qquad \text{for } i < p$$
$$x^p(v) = x^p(v_0) + z^p(v) - \frac{1}{2}\coth(x^p(v_0)) \sum_{k=p+1}^{n} (z^k(v))^2$$
$$x^i(v) = x^i(v_0) + z^i(v) + \frac{1}{2}\coth(x^p(v_0)) z^i(v) z^p(v) \qquad \text{for } i > p.$$

We get from here:

$$z^i(v) = x^i(v) - x^i(v_0) \qquad \text{for } i < p \tag{7.27a}$$
$$\begin{aligned} z^p(v)[1 + \coth(x^p(v_0)) z^p(v)]^2 & \\ - [x^p(v) - x^p(v_0)][1 + \coth(x^p(v_0)) z^p(v)] & \\ - \frac{1}{2}\coth(x^p(v_0)) \sum_{k=p+1}^{n} [x^i(v) - x^i(v_0)]^2 &= 0 \end{aligned} \tag{7.27b}$$
$$z^i(v) = \frac{x^i(v) - x^i(v_0)}{1 + \coth(x^p(v_0)) z^p(v)} \qquad \text{for } i > p. \tag{7.27c}$$

Since (7.27b) is an algebraic cubic equation relative to $z^p(v)$, it has at least one real solution that can explicitly be found (in radicals) by the known methods [57, Section 1.8-3], [65]. Substituting this solution into (7.27a) and (7.27c), we can obtain a geodesic coordinate system $\{z^i\}$ on some subset of the pseudosphere $\mathbb{S}_q^n$ and which is normal at a point $v_0 \in C := \{v \in \mathbb{S}_q^n : x^i(v) = \frac{\pi}{2} \text{ for } i \neq p\}$.

Exercise 7.1. Let $\mathbb{H}_{q-1}^n := \{v \in \mathbb{R}_q^{n+1} : e_q^{n+1}(v, v) = -r^2\}$ be a hyperbolic space of radius $r \in \mathbb{R}$, $r > 0$, and index $q - 1$ and g be the metric on it induced by e_q^{n+1}. Suppose the coordinates (7.20) are defined via (7.21) in which the replacements $\sin\theta_i \mapsto \cosh\theta_i$ for $i = 1, \dots, p-1$ and $\cosh\chi \mapsto \sinh\chi$ are made. Show that $\mathbb{H}_{q-1}^n := \{v \in \mathbb{R}_q^{n+1} : \rho(v) = \pm r\}$. Calculate the components of g and the coefficients of ∇ in the coordinates (7.24). Construct geodesic coordinate system $\{z^i\}$ on $\mathbb{H}_{q-1}^n$ using the results obtained and following the above considerations for $\mathbb{S}_q^n$.

We shall now return to the coordinates (7.18), defined via (7.19), to study in them coordinates normal on the pseudosphere $\mathbb{S}_q^n$ when $n \geq 2$, $q \geq 1$ and $p = n - q \geq 1$. First of all, we notice that the set of functions

$$\{x^1 = \theta_{p-1}, \dots, x^{p-1} = \theta_1, x^p = \varphi, x^{p+1} = \chi_q, \dots, x^{p+q-1} = \chi_2, x^n = \chi_1\} \tag{7.28}$$

provides a coordinate system on $\mathbb{S}_q^n$ (precisely on any one of its two connected components). The metric g has in these coordinates components g_{ij} forming the diagonal matrix

$$\begin{aligned}[g_{ij}] &= \operatorname{diag}(g_1,\dots,g_n)\\ &= r^2\cosh^2 x^{p+1}\cdots\cosh^2 x^{p+q}\\ &\times \operatorname{diag}\big(1,\sin^2 x^1,\sin^2 x^1\sin^2 x^2,\dots,\sin^2 x^1\cdots\sin^2 x^{p-1},\underbrace{0,\dots,0}_{q\text{-times}}\big)\\ &- r^2\operatorname{diag}\big(\underbrace{0,\dots,0}_{p\text{-times}},1,\cosh^2 x^{p+1},\cosh^2 x^{p+1}\cosh^2 x^{p+2},\dots,\\ &\dots,\cosh^2 x^{p+1}\cdots\cosh^2 x^{p+q-1}\big).\end{aligned} \tag{7.29}$$

The Riemannian connection ∇ generated by g has the following non-vanishing components in $\{x^i\}$ (cf. (7.15) and (7.26)):

$$\begin{aligned}
&\Gamma^i{}_{ik}\big|_{\substack{k<i\\ i,k\le p}} = \Gamma^i{}_{ki}\big|_{\substack{k<i\\ i,k\le p}} = \cot x^k \prod_{l=p+1}^{p+q}\cosh^2 x^l\\
&\Gamma^i{}_{kk}\big|_{\substack{k>i\\ i,k\le p}} = -\sin x^i\cos x^i\left(\prod_{l=i+1}^{k-1}\sin^2 x^l\right)\prod_{l=p+1}^{p+q}\cosh^2 x^l\\
&\Gamma^i{}_{ik}\big|_{\substack{k<i\\ i,k\ge p+1}} = \Gamma^i{}_{ki}\big|_{\substack{k<i\\ i,k\ge p+1}} = \coth x^k\\
&\Gamma^i{}_{kk}\big|_{\substack{i<k\\ i,k\ge p+1}} = -\coth x^i\prod_{l=i+1}^{k-1}\cosh^2 x^l\\
&\Gamma^i{}_{ik}\big|_{\substack{i\le p\\ k\ge p+1}} = \Gamma^i{}_{ki}\big|_{\substack{i\le p\\ k\ge p+1}} = \coth x^k\\
&\Gamma^i{}_{kk}\big|_{\substack{i\ge p+1\\ k\le p}} = \cosh x^i\sinh x^i\left(\prod_{l=i+1}^{p+q}\cosh^2 x^l\right)\prod_{l=1}^{k-1}\sin^2 x^l.
\end{aligned} \tag{7.30}$$

Since $\cos\frac{\pi}{2} = \cot\frac{\pi}{2} = 0$ and $\sinh 0 = \tanh 0 = 0$, the coordinate system $\{x^i\}$ is normal for ∇ on the subset

$$\{v\in\mathbb{S}^n_q : x^1(v) = \dots = x^{p-1}(v) = \frac{\pi}{2} \text{ and } x^{p+1}(v) = \dots = x^{p+q}(v) = 0\} \subset \mathbb{S}^n_q.$$

We can also say that the coordinate system $\{x^i\}$ is normal for ∇ along the path $\gamma\colon [0,2\pi)\to\mathbb{S}^n_q$ such that $x^k\circ\gamma = \frac{\pi}{2}$ for $k = 1,\dots,p-1$, $x^k\circ\gamma = 0$ for $k = p+1,\dots,p+q$ and $x^p\circ\gamma = \mathsf{id}_{[0,2\pi)}$; we can equivalently express this by writing $\Gamma^i{}_{jk}\circ\gamma = 0$ in $\{x^i\}$. Since

$$\gamma([0,2\pi)) = \{(r\cos s, r\sin s, \underbrace{0,\dots,0}_{(n-1)\text{-times}}) : s\in[0,2\pi)\} = \mathbb{S}^1\times(\underbrace{0,\dots,0}_{(n-1)\text{-times}})\subset\mathbb{R}^{n+1},$$

the circle $\gamma([0,2\pi))$ can be obtained by intersection $\mathbb{S}^n_q$ with the (u^1,u^2)-plane in $\mathbb{R}^{n+1}$.

Example 7.6 (The torus $\mathbb{T}^2$). The torus $\mathbb{T}^2$ of radii $r_1, r_2 \in \mathbb{R}$, $r_1 > 0$ and $r_2 > 0$, in $\mathbb{R}^3$ is a product of two circles $\mathbb{S}^1$ and $\mathbb{S}^1$ of radii $r_1 > 0$ and $r_2 > 0$, respectively, $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$,[8] If φ_1 and φ_2 are the respective polar coordinates, with range $[0, 2\pi)$, on these circles, then

$$\begin{aligned} &\mathbb{T}^2 = \{(r\cos\varphi_1, r\sin\varphi_1, r_2\sin\varphi_2) \in \mathbb{R}^3 : \varphi_1, \varphi_2 \in [0, 2\pi)\} \\ &\quad r := r_1 + (1 + \cos\varphi_2)r_2. \end{aligned} \tag{7.31}$$

The standard Euclidean metric on $\mathbb{R}^3$ (see Example 7.1) induces on $\mathbb{T}^2$ a metric g whose components in the coordinate system $\{x^1 = \varphi_1, x^2 = \varphi_2\}$ of $\mathbb{T}^2$ has components g_{ij}, $i, j = 1, 2$, forming the diagonal matrix

$$[g_{ij}] = \operatorname{diag}(r^2, r_2^2). \tag{7.32}$$

The Riemannian connection ∇ induced by g on $\mathbb{T}^2$ has the following non-vanishing components in $\{x^1, x^2\}$ (see (4.13) and (4.15))

$$\Gamma^1{}_{12} = \Gamma^1{}_{21} = -\Gamma^2{}_{11} = -\frac{r_2}{r}\sin x^2. \tag{7.33}$$

From here immediately follows that the coordinates system $\{x^1 = \varphi_1, x^2 = \varphi_2\}$ is normal along the path $\gamma\colon [0, 2\pi) \to \mathbb{T}^2$ with $x^1 \circ \gamma = \mathsf{id}_{[0,2\pi)}$ and $x^2 \circ \gamma = 0$ (or $\gamma(s) = (r\sin s, r\cos s, 0)|_{r=r_1+2r_2} \in \mathbb{T}^2 \subset \mathbb{R}^3$ with $s \in [0, 2\pi)$). Obviously, $\gamma([0, 2\pi))$ is the the greater circle obtained by intersecting $\mathbb{T}^2$ with the so-called equatorial plane (i.e., $\{(v^1, v^2, 0) \in \mathbb{R}^2 : v^1, v^2 \in \mathbb{R}\}$ in the representation we are using).

Exercise 7.2. Generalize the above results for multidimensional tori, i.e., for

$$\mathbb{T}^n := \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n\text{-times}}$$

where $n \geq 2$ and the ith, $i = 1, \dots, n$, circle has radius $r_i \in \mathbb{R}$ with $r_i > 0$. Hint: (i) parameterize the ith circle with a polar angle φ_i with range $[0, 2\pi)$ and obtain a representation similar to (7.31),[9] (ii) then, in the coordinate system $\{x^1 = \varphi_1, \dots, x^n = \varphi_n\}$ on $\mathbb{T}^n$, find the components of the metric g induced on $\mathbb{T}^n$ by the standard one in $\mathbb{R}^{n+1}$ and (iii) at last, calculate in $\{x^i\}$ the components of the Riemannian connection generated by g and investigate the existence of a subset of $\mathbb{T}^n$ on which $\{x^i\}$ is normal for this connection.

Example 7.7 (Surfaces of revolution). A surface of revolution in $\mathbb{R}^3$ is obtained by rotating a plane path $\gamma\colon J \to \mathbb{R}^2 \subset \mathbb{R}^3$ around an axis in the plane of the path. If

[8] For more details regarding tori, see [42].

[9] For $n \geq 3$, a representation like $\mathbb{T}^n := \mathbb{T}^2 \times \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{(n-2)\text{-times}}$ may turn to be useful.

$s \in J$ is the path's parameter and $\rho(s) \geq 0$ is the distance of $\gamma(s)$ from the axis, say the third axis in $\mathbb{R}^3$, then the surface of revolution is [66, p. 767]

$$S = \{(\rho(s)\cos\alpha, \rho(s)\sin\alpha, z(s)) \in \mathbb{R}^3 : s \in J,\ \alpha \in [0, 2\pi)\} \tag{7.34}$$

where $z(s)$ defines γ by $\gamma(s) = (\rho(s), 0, z(s))$ and α is the angle of rotation; below we shall suppose that ρ and z are C^2 functions. The functions $x^1 \colon v \to x^1(v) = s$ and $x^2 \colon v \to x^2(v) = \alpha$ for $v = (\rho(s)\cos\alpha, \rho(s)\sin\alpha, z(s))$ provide a coordinate system $\{x^1, x^2\}$ on S.

The restriction of the Euclidean metric in $\mathbb{R}^3$ to S results into a tensor field g which has in $\{x^1, x^2\}$ components g_{ij}, $i, j = 1, 2$, such that

$$[g_{ij}] = \operatorname{diag}(\rho'^2 + z'^2, \rho^2), \tag{7.35}$$

where the prime means derivative relative to s, i.e., $\rho' := \frac{\mathrm{d}\rho}{\mathrm{d}s}$ and $z' := \frac{\mathrm{d}z}{\mathrm{d}s}$. Consequently, the tensor field g defines on the set $S \backslash \{\text{points at which } \rho'(s) = z'(s) = 0\}$ a Riemannian metric g. The metric g induces on S a Riemannian connection ∇. The non-vanishing coefficients of ∇ in $\{x^1, x^2\}$ are (see (4.13) and (4.15))

$$\Gamma^1{}_{11} = \frac{\rho'\rho'' + z'z''}{\rho'^2 + z'^2} \quad \Gamma^1{}_{22} = -\frac{\rho\rho'}{\rho'^2 + z'^2} \quad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{\rho'}{\rho}. \tag{7.36}$$

Thus the coordinate system $\{x^1, x^2\}$ is normal for ∇ on the set

$$C := \{v = (\rho(s)\cos\alpha, \rho(s)\sin\alpha, z(s)) \in S : \rho'(s) = 0,\ z'(s) \neq 0 \text{ and } z''(v) = 0\} \tag{7.37}$$

if it is non-empty.

For instance, $\{x^1, x^2\}$ is normal on the whole surface of revolution S if, and only if, it is a cylinder defined via $\rho(s) = c$ and $z(s) = as + b$ for some $a, b, c \in \mathbb{R}$ with $a \neq 0$ and $c \neq 0$. This result can be expected from the considerations in Example 7.1.

However, the set (7.37) may turn to be the empty set in which case the coordinates $\{x^1, x^2\}$ are nowhere normal on S for ∇. Examples of surfaces of revolution on which this happens are provided by the choices $\rho(s) = \mathrm{e}^s, s^3 + s$ and/or $z(s) = \mathrm{e}^s$.

Exercise 7.3. Construct explicitly geodesic coordinates system $\{z^1, z^2\}$ with $c^i_{\dots} = 0$ (see (6.13)) from $\{x^1, x^2\}$ such that $\{z^1, z^2\}$ is normal for ∇ at an arbitrarily chosen point $v_0 \in S$. (This construction is completely independent from is the set (7.37) empty or non-empty.)

Exercise 7.4. Obtain the results of Example 7.3 from the above ones. Hint: a sphere can be obtained by rotating a semi-circle around the axis passing through its ends.

Example 7.8 (Geodesic coordinates in Schwarzschild spacetime). The manifold of Schwarzschild is a 4-dimensional (pseudo-)Riemannian manifold endowed with a Riemannian connection ∇ induced by the Schwarzschild metric g which in suitably chosen (Schwarzschild) coordinates ($x^1 = r, x^2 = \theta, x^3 = \varphi, x^4 = ct$), c being the velocity of light in vacuum, has components $[g_{ij}]$, $i, j = 1, \dots, 4$, such that[10]

$$[g_{ij}] = \operatorname{diag}(-e^{-\lambda}, -r^2, -r^2 \sin^2\theta, \mathrm{e}^{\nu}) \tag{7.38}$$

where[11]

$$\mathrm{e}^{\nu} := 1 - \frac{r_g}{r} \quad \mathrm{e}^{\lambda} := \Big(1 - \frac{r_g}{r}\Big)^{-1} \tag{7.39}$$

with r_g being a constant (known as the (Schwarzschild) gravitational radius). Using (4.15), we find after some tedious calculations the following non-vanishing coefficients of the Riemannian connection induced by g [58, eq. (83.2)]

$$\begin{aligned}
&\Gamma^1{}_{11} = \frac{1}{2}\frac{\mathrm{d}\lambda}{\mathrm{d}r} \qquad \Gamma^1{}_{22} = -r\mathrm{e}^{-\lambda} \quad \Gamma^1{}_{33} = -r\mathrm{e}^{-\lambda}\sin^2\theta \quad \Gamma^1{}_{44} = \frac{1}{2}\mathrm{e}^{\nu-\lambda}\frac{\mathrm{d}\nu}{\mathrm{d}r}\\
&\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r} \quad \Gamma^2{}_{33} = -\sin\theta\cos\theta\\
&\Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r} \quad \Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta\\
&\Gamma^4{}_{14} = \Gamma^4{}_{41} = \frac{1}{2}\frac{\mathrm{d}\nu}{\mathrm{d}r},
\end{aligned} \tag{7.40}$$

where $\frac{\mathrm{d}\nu}{\mathrm{d}r} = +\big(1 - \frac{r_g}{r}\big)^{-1}\frac{r_g}{r^2}$ and $\frac{\mathrm{d}\lambda}{\mathrm{d}r} = +\big(1 - \frac{r_g}{r}\big)^{-1}\frac{r_g}{r^2}$, due to (7.39).

The coordinate system $\{x^i\}$ is not normal at all points of the Schwarzschild spacetime. However, from it can be constricted Riemannian or geodesic coordinates (with origin at any arbitrarily fixed point p) according to the procedures of Section 6. In particular, setting $c^i_{i_1\dots i_n} \equiv 0$ for all $n \geq 3$ in (6.13), we see that the equations

$$\begin{aligned}
x^1(q) = x^1(p) + z^1(q) - \frac{1}{4}\frac{\mathrm{d}\lambda}{\mathrm{d}r}(z^1(q))^2 + \frac{1}{2}r\mathrm{e}^{-\lambda}(z^2(q))^2\\
+ \frac{1}{2}r\mathrm{e}^{-\lambda}\sin^2(x^2(p))(z^3(q))^2 - \frac{1}{4}\mathrm{e}^{\nu-\lambda}\frac{\mathrm{d}\nu}{\mathrm{d}r}(z^4(q))^2
\end{aligned} \tag{7.41a}$$

$$x^2(q) = x^2(p) + z^2(q) - \frac{1}{r}z^1(q)z^2(q) + \frac{1}{2}\sin(x^2(p))\cos(x^2(p))(z^3(q))^2 \tag{7.41b}$$

$$x^3(q) = x^3(p) + z^3(q) - \frac{1}{r}z^1(q)z^3(q) - \cot(x^2(p))z^2(q)z^3(q) \tag{7.41c}$$

$$x^4(q) = x^4(p) + z^4(q) - \frac{1}{2}\frac{\mathrm{d}\nu}{\mathrm{d}r}z^1(q)z^4(q), \tag{7.41d}$$

[10] For details and the physical significants of the Schwarzschild metric, see (text)books on general relativity and gravitation, like [61, Chapter VII, § 4], [67, Chapter 3], [21, Chapter 8, § 2 and § 8], [68, § 100], or [58, § 83 and § 83].

[11] If we admit that λ and ν depend on r and do not fix this dependence, we obtain the class of the so-called spherically symmetric metrics [21, Chapter 13, § 5 B]. In particular, the choice $\mathrm{e}^{\nu} = \mathrm{e}^{-\lambda} = 1 - \frac{r_g}{r} + \frac{Q^2}{r^2}$, for some constant $Q \in \mathbb{R}$ (having a sense of electrical charge), selects the Reissner-Nordström metric [67, Chapter 5, in particular, § 39]. The particular setting (7.39) will not be used below; see, for instance (7.40) and (7.41).

where $z^i(p) := 0$, define $z^1, \dots, z^4$ as geodesic coordinates with origin at a point p in some neighborhood of p. Equations (7.41c) and (7.41d) yield

$$z^3(q) = (x^3(q) - x^3(p))\Big(1 - \frac{1}{r} z^1(q) - \cot(x^2(p)) z^2(q)\Big)^{-1}$$
$$z^4(q) = (x^4(q) - x^4(p))\Big(1 - \frac{1}{2}\frac{\mathrm{d}\nu}{\mathrm{d}r} z^1(q)\Big)^{-1}$$

which, when inserted into (7.41a) and (7.41b), give

$$\begin{aligned} x^1(q) - x^1(p) = {} & z^1(q) - \frac{1}{4}\frac{\mathrm{d}\lambda}{\mathrm{d}r}(z^1(q))^2 + \frac{1}{2} r \mathrm{e}^{-\lambda}(z^2(q))^2 \\ & + \frac{1}{2} r \mathrm{e}^{-\lambda} \sin^2(x^2(p)) \frac{(x^3(q)) - x^3(p))^2}{\big(1 - \frac{1}{r} z^1(q) - \cot(x^2(p)) z^2(q)\big)^2} \\ & - \frac{1}{4} \mathrm{e}^{\nu - \lambda} \frac{\mathrm{d}\nu}{\mathrm{d}r} \frac{(x^4(q)) - x^4(p))^2}{\big(1 - \frac{1}{2}\frac{\mathrm{d}\nu}{\mathrm{d}r} z^1(q)\big)^2} \\ x^2(q) - x^2(p) = {} & \Big(1 - \frac{1}{r} z^1(1)\Big) z^2(q) \\ & + \frac{1}{2} \sin(x^2(p)) \cos(x^2(p)) \frac{(x^3(q) - x^3(p))^2}{\big(1 - \frac{1}{r} z^1(q) - \cot(x^2(p)) z^2(q)\big)^2} \ . \end{aligned}$$

The latter equation is a cubic algebraic equation relative to $z^2(q)$ and hence it has at least one real solution that can be found by the known methods [57,65,69]. Substituting this solution into the former equation, one obtains an equation for only $z^1(q)$ which has at least one real solution z^1 (proof this!). Substituting this last function into the previous expressions for z^2, z^3 and z^4, one derives the required geodesic coordinate system $\{z^i\}$ as a function of the initial one $\{x^i\}$.

Example 7.9 (The Einstein Universe). The Einstein Universe [58, §§ 134–141] is a homogeneous and static model of the Universe[12] whose underlying geometrical structure is the (4-dimensional) Einstein manifold. This manifold is a Riemannian 4-dimensional manifold with metric g which, in suitable coordinates ($x^1 = \chi, x^2 = \theta, x^3 = \varphi, x^4 = ct$) ($c$ is the velocity of light in vacuum), with respective ranges $\mathbb{R}$, $(0, \pi]$, $[0, 2\pi)$ and $(-\infty, +\infty)$, has components g_{ij}, $i, j = 1, \dots, 4$, forming the diagonal matrix (see, e.g, [58, eq. (138.6)] or [62, eq. (12.135)])

$$[g_{ij}] = \operatorname{diag}(-R^2, -R^2 \sin^2 x^1, -R^2 \sin^2 x^1 \sin^2 x^2, 1) \tag{7.42}$$

where R is a constant (known as the radius of the spherical space representing the spacial part of the Einstein Universe). This metric induces a Riemannian

[12] There are possible only three versions of a homogeneous and static Universe in general relativity [58, § 134]: Einstein, de Sitter and Minkowski spacetimes.

connection ∇ whose nonvanishing coefficients in $\{x^i\}$ are (see (4.13) and (4.15); cf. (7.15))

$$\begin{aligned} &\Gamma^1{}_{22} = -\sin x^1 \cos x^1 && \Gamma^1{}_{33} = -\sin x^1 \cos x^1 \sin^2 x^2 \\ &\Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot x^1 && \Gamma^2{}_{33} = -\sin x^2 \cos x^2 \\ &\Gamma^3{}_{13} = \Gamma^3{}_{31} = \cot x^1 && \Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot x^2. \end{aligned} \tag{7.43}$$

(Notice, these coefficients are exactly the coefficients (7.15) of the sphere $\mathbb{S}^3$ in $\mathbb{R}^4$.) Consequently, the coordinate system $\{x^i\}$ is normal for ∇ on the 2-dimensional subspace $\{v : x^1(v) = x^2(v) = \frac{\pi}{2}\}$ of the Einstein Universe.

Example 7.10 (The de Sitter Universe). The de Sitter Universe [58, §§ 136, 142–145], [62, § 12.7] is also a homogeneous and static model of the Universe. It is based on a 4-dimensional Riemannian manifold, called the de Sitter manifold, whose metric g in suitable coordinates $(x^1 = \chi, x^2 = \theta, x^3 = \varphi, x^4 = ct)$ has components g_{ij} such that [58, eq. (142.3)]

$$[g_{ij}] = \operatorname{diag}(-R^2, -R^2 \sin^2 x^1, -R^2 \sin^2 x^1 \sin^2 x^2, \cos^2 x^1) \tag{7.44}$$

for some constant R. Applying (4.13) and (4.15) (see also (7.43)), one can verify that the Riemannian connection ∇ induced by g has in $\{x^i\}$ the following non-vanishing coefficients (cf. (7.43))

$$\begin{aligned} &\Gamma^1{}_{22} = -\sin x^1 \cos x^1 && \Gamma^1{}_{33} = -\sin x^1 \cos x^1 \sin^2 x^2 \\ &\Gamma^1{}_{44} = -\frac{1}{R^2} \sin x^1 \cos x^1 && \\ &\Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot x^1 && \Gamma^2{}_{33} = -\sin s^2 \cos x^2 \\ &\Gamma^3{}_{13} = \Gamma^3{}_{31} = \cot x^1 && \Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot x^2 \\ &\Gamma^4{}_{14} = \Gamma^4{}_{41} = -\cot x^1. && \end{aligned} \tag{7.45}$$

Evidently, the coordinate system $\{x^i\}$ is normal for ∇ on the 2-dimensional submanifold $\{v : x^1(v) = x^2(v) = \frac{\pi}{2}\}$ of the de Sitter Universe.

Via an appropriate change $\{x^i\} \mapsto \{y^i\}$ of the local coordinates, one can transform the metric's components g_{ij} to g^y_{ij} such that (see, e.g., [58, § 142, eq. (142.11)] or [62, eq. (12.161)])

$$[g_{ij}] = \operatorname{diag}(-e^{2\varkappa y^4}, -e^{2\varkappa y^4}, -e^{2\varkappa y^4}, y^4) \tag{7.46}$$

for some number $\varkappa \neq 0$. The non-vanishing coefficients ${}^y\Gamma^i{}_{jk}$ of ∇ in $\{y^i\}$ are

$$\begin{aligned} &{}^y\Gamma^i{}_{i4} = {}^y\Gamma^i{}_{4i} = -\varkappa && \text{for } i = 1,2,3 \\ &{}^y\Gamma^4{}_{kk} = \varkappa e^{2\varkappa y^4} && \text{for } k = 1,2,3. \end{aligned} \tag{7.47}$$

Thus the coordinates $\{y^i\}$ are nowhere normal for ∇ regardless that the metric looks 'simpler' in them.

The above conclusion is due to the main difference between the coordinates systems $\{x^i\}$ and $\{y^i\}$: the former reflects the symmetry of the de Sitter spacetime (its spacial part is spherically symmetric) while the latter one 'tends to represent' this space 'as near as possible' to the flat Minkowski space $\mathbb{R}^4_3$.

Exercise 7.5. Using (6.13) with $c^i_{\dots} = 0$, construct from $\{y^i\}$ a geodesic coordinate system $\{z^i\}$ normal for ∇ at some fixed point.

Example 7.11 (Non-static spacially homogeneous Universes). If we admit the constant R to change with the time, $R^2 = R_0^2 \mathrm{e}^{f(x^4)}$ for some $R_0 \in \mathbb{R}$ and C^1 function f, in the Einstein metric (7.42), we obtain a 4-dimensional Riemannian manifold with a metric g which in some fixed coordinates system $\{x^i\}$ has components g_{ij}, $i, j = 1, \dots, 4$, forming the diagonal matrix

$$\begin{aligned}[g_{ij}] &= \operatorname{diag}(-R_0^2 \mathrm{e}^{f(x^4)}, -R_0^2 \mathrm{e}^{f(x^4)} \sin^2 x^1, -R_0^2 \mathrm{e}^{f(x^4)} \sin^2 x^1 \sin^2 x^2, 1) \\ &=: \operatorname{diag}(g_1, g_2, g_3, g_4).\end{aligned} \tag{7.48}$$

This manifold represents a non-static (and spacially homogeneous) model of the Universe in the general theory of relativity [58, §§ 147–149; eq. (149.7)], [62, § 12.8]. The Riemannian connection generated by g has in $\{x^i\}$ the following non-vanishing coefficients (see (4.13) and (4.15); cf. (7.43))

$$\begin{array}{lll}
\Gamma^1{}_{22} = -\sin x^1 \cos x^1 & \Gamma^1{}_{33} = -\sin x^1 \cos x^1 \sin^2 x^2 & \Gamma^1{}_{14} = \Gamma^1{}_{41} = \dfrac{1}{2} f'(x^4) \\
\Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot x^1 & \Gamma^2{}_{33} = -\sin s^2 \cos x^2 & \Gamma^2{}_{24} = \Gamma^2{}_{42} = \dfrac{1}{2} f'(x^4) \\
\Gamma^3{}_{13} = \Gamma^3{}_{31} = \cot x^1 & \Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot x^2 & \Gamma^3{}_{34} = \Gamma^3{}_{43} = \dfrac{1}{2} f'(x^4) \\
\Gamma^4{}_{kk} = -\dfrac{1}{2} f'(x^4) g_k & \Gamma^i{}_{i4} = \Gamma^i{}_{4i} = \dfrac{1}{2} f'(x^4) & \text{for } i, k = 1, 2, 3,
\end{array} \tag{7.49}$$

where $f'(x^4) := \frac{\mathrm{d}f(x^4)}{\mathrm{d}x^4}$. Therefore the function g (which together with the cosmological constant determine the pressure and energy density [58, § 150, eqs. (150.7) and (150.8)]) is responsible for the existence of set(s) on which the coordinate system $\{x^i\}$ is normal. Indeed, if the equation $f'(x^4) = 0$ has a real solution x^4_0, $f'(x^4_0) = 0$, then $\{x^i\}$ is normal on the 1-dimensional submanifold $\{v : x^1(v) = x^2(v) = \frac{\pi}{2},\ x^4(v) = x^4_0\}$.

Exercise 7.6. If $f'(a) \neq 0$ for all $a \in \mathbb{R}$, construct from $\{x^i\}$ a coordinate system $\{z^i\}$ which is normal at a fixed spacetime point.

Example 7.12 (The light cone in Minkowski spacetime). The 4-dimensional manifold $M_4 = \mathbb{R}^4_3$ (or the isomorphic to it manifold $\mathbb{R}^4_1$) is know as the Minkowski spacetime and it is the geometrical base for the special theory of relativity and the whole relativistic physics [20, 21, 51, 59–64]. The causal structure of the physical theories based on it is determined by the so-called light cone C^3_3 which is defined as a 'pseudosphere' of index 3 and zero radius in it,

$$\begin{aligned} C^3_3 &= \{v \in \mathbb{R}^4_3 : e^4_3(v) = 0\} \\ &= \{(v^1, v^2, v^3, v^4) \in \mathbb{R}^4_3 : (v^1)^2 - (v^2)^1 - (v^3)^2 - (v^4)^2 = 0\}. \end{aligned} \tag{7.50}$$

The light cone is a 3-dimensional 1-connected C^ω manifold whose geometry is very well explored due to its physical significants. From the view-point of the considerations in Example 7.5, it is described by the choice $n = 3$ $(n + 1 = 4)$, $q = 3$, $p = 0$, and $r = 0$. The metric e^4_3, when restricted to C^3_3, induces a (1-time) degenerate tensor field on C^3_3 which can further be restricted on other subsets of C^3_3 to give a Riemannian metric g, which generates a connection ∇ on them.

The different kinds of pseudospherical coordinates in $\mathbb{R}^4_3$ are not suitable for studding the light cone due to its vanishing radius when it is consider as a pseudosphere. For instance, in the coordinates $(\rho, \chi, \tau_1, \varphi)$ in $\mathbb{R}^4_3$, provided by (7.22) and such that

$$\begin{aligned} u^2 &= \rho \sinh\chi \sin\tau_1 \cos\varphi \quad & u^4 &= \rho \sinh\chi \cos\tau_1 \\ u^3 &= \rho \sinh\chi \sin\tau_1 \sin\varphi \quad & u^1 &= \rho \cosh\chi, \end{aligned}$$

we have $C^3_3 = \{v \in \mathbb{R}^4_3 : \rho(v) = 0\}$ and the components of e^4_3 in it are given by the diagonal matrix $\operatorname{diag}(1, -\rho^2, -\rho^2 \sinh\chi, -\rho^2 \sinh\chi \sin\tau_1)$. Therefore $u^1|_{C^3_3} \equiv 0$ and the 'metric' induced by e^4_3 will have components forming the degenerate matrix $\operatorname{diag}(1, 0, 0, 0)$; the cause for this is that the change $(u^1, u^2, u^3, u^4) \mapsto (\rho, \chi, \tau_1, \varphi)$ is degenerate on the light cone.

Below we shall investigate two concrete coordinates systems in $\mathbb{R}^4_3$ which are suitable for description of the light cone C^3_3.

Define on $\mathbb{R}^4_3$ coordinates (H, R, θ, φ), with respective ranges $\mathbb{R}$, $(0, \infty)$, $(0, \pi]$ and $[0, 2\pi)$, such that

$$\begin{aligned} u^2 &= R \sin\theta \cos\varphi \\ u^3 &= R \sin\theta \sin\varphi \\ u^4 &= R \cos\theta \\ u^1 &= H. \end{aligned} \tag{7.51}$$

The Jacobian of the change to the new coordinates equals $-R^2 \sin\theta \neq 0$, so that they are well defined. These coordinates agree with the splitting $\mathbb{R}^4_3 = \mathbb{R}^1 \times \mathbb{R}^3_3$ and (R, θ, φ) are spherical coordinates in $\mathbb{R}^3$. In them $C^3_3 = \{v \in \mathbb{R}^4_3 : H = -R \text{ or } H = +R\}$ and the components of e^4_3 form the diagonal matrix $\operatorname{diag}(1, -1, -R^2, -R^2 \sin\theta)$. The set of functions

$$\{x^1 = R, x^2 = \theta, x^3 = \varphi\} \tag{7.52}$$

is a coordinate system on the future part $\{v \in \mathbb{R}^4_3 : H = +R\}$ or the past part $\{v \in \mathbb{R}^4_3 : H = -R\}$ of the light cone C^3_3. The components of the tensor g obtained by restricting e^4_3 to C^3_3 has in $\{x^1, x^2, x^3\}$ components g_{ij}, $i, j = 1, 2, 3$, such that[13]

$$[g_{ij}] = \operatorname{diag}(0, -(x^1)^2, -(x^1)^2 \sin^2 x^2). \tag{7.53}$$

Consequently g is degenerate and hence does not define on the light cone a Riemannian metric; one can prove that this result is independent of the particular coordinates used. However, on the set $C^3_3|_{R=R_0} = \{v \in \mathbb{R}^4_3 : H = \pm R_0,\ R = R_0\} \subset C^3_3$, where R_0 is a fixed positive real number, the restriction of e^4_3 or of g results into a Riemannian metric $\bar{g}$ which in the coordinates $\{x^1 = \theta, x^2 = \varphi\}$ on $C^3_3|_{R=R_0}$ has components $\bar{g}_{ij}$, $i, j = 1, 2$ such that

$$[\bar{g}_{ij}] = \operatorname{diag}(-R_0^2, -R_0^2 \sin^2 x^1) = -\operatorname{diag}(R_0^2, R_0^2 \sin^2 x^1). \tag{7.54}$$

Consequently $\bar{g} = -g_{\mathbb{S}^2}$, where $g_{\mathbb{S}^2}$ is the metric on the 2-sphere $\mathbb{S}^2$ of radius R_0 in $\mathbb{R}^3$ (see (7.10)). So, if $\bar{\nabla}$ is the Riemannian connection on $C^3_3|_{R=R_0}$ induced by $\bar{g}$, then its non-vanishing coefficients in $\{x^1, x^2\}$ are (7.11). Hence $\{x^1, x^2\}$ is normal along the path $\gamma\colon [0, 2\pi) \to C^3_3|_{R=R_0}$ such that $x^1 \circ \gamma = \frac{\pi}{2}$ and $x^2 \circ \gamma = \mathsf{id}_{[0,2\pi)}$. Geometrically $C^3_3|_{R=R_0}$ is the intersection of C^3_3 with the 3-plane $u^1 = \pm R_0$; physically this 3-plane represents the spacial part of the spacetime at the moment $t = \pm r_0/c$, c being the velocity of light in vacuum.

Let us now look on the light cone C^3_3 from the view-point of coordinates (ρ, χ, R, φ) in $\mathbb{R}^4_3$ such that

$$\begin{aligned} u^1 &= \rho \cosh \chi \\ u^2 &= \rho \sinh \chi \\ u^3 &= R \cos \varphi \\ u^4 &= R \sin \varphi. \end{aligned} \tag{7.55}$$

The range of ρ is $\mathbb{R} \setminus \{0\}$, the one of χ is $\mathbb{R}$, R ranges in $(0, \infty)$, and φ takes values in $[0, 2\pi)$. The Jacobian of the change to the new coordinates equals ρR and hence they are well defined. The light cone has in these coordinates the representation $C^3_3 = \{v \in \mathbb{R}^4_3 : \rho^2 = R^2\}$ and the components of e^4_3 in them form the diagonal matrix $\operatorname{diag}(1, -\rho^2, -1, -R^2)$. The set $\{\rho, \chi, \varphi\}$ is a coordinates system on C^3_3 in which the tensor g, to which e^4_3 reduces on C^3_3, has components g_{ij}, $i, j = \rho, \chi, \varphi$, such that

$$[g_{ij}] = \operatorname{diag}(0, -\rho^2, -\rho^2). \tag{7.56}$$

(The same result can be obtained in the coordinate system $\{\pm R, \chi, \varphi\}$.) Thus g is degenerate and does not define a Riemannian metric on the light cone. However, when restricted to the set $C^3_3|_{\rho=\rho_0} = \{v \in \mathbb{R}^4_3 : \rho(v) = \rho_0 \text{ and } R = |\rho_0|\} \subset C^3_3$ for

[13] To derive (7.53), write e^4_3 as a differentials form, $e^4_3 = \mathrm{d}H^2 - \mathrm{d}R^2 - R^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)$ and restrict it to the light cone where $H^2 = R^2$.

a fixed number $\rho_0 \in \mathbb{R} \setminus \{0\}$, this tensor (or e^4_3) reduces to a Riemannian metric $\bar{g}$ whose components $\bar{g}_{ij}$, $i, j = 1, 2$, in $\{x^1 = \chi, x^2 = \varphi\}$ are given by

$$[\bar{g}_{ij}] = \operatorname{diag}(-\rho_0^2, -\rho_0^2) = -\rho_0^2 \operatorname{diag}(1, 1). \tag{7.57}$$

Hence $\bar{g} = -\rho_0^2 e_0^2$, where e_0^2 is the standard Euclidean metric in $\mathbb{R}^2_0 = \mathbb{R}^2$ (see Example 7.1). In this way, we see that, if $\bar{\nabla}$ is the Riemannian connection induced by $\bar{g}$ on $C^3_3|_{\rho=\rho_0}$, then the coordinate system $\{x^1 = \chi, x^2 = \varphi\}$ is everywhere normal on $C^3_3|_{\rho=\rho_0}$ for $\bar{\nabla}$. This conclusion is clear also from the geometrical interpretation of $C^3_3|_{\rho=\rho_0}$ as an intersection of the '3-cylinder' $\mathbb{R}^2_1 \times \mathbb{S}^1 \subset \mathbb{R}^4_3$, of 'radius' $\sqrt{-1}\rho_0$ with the pseudosphere $\mathbb{S}^2_1 \subset \mathbb{S}^2_1 \times \mathbb{R}^2_2 \subset \mathbb{R}^4_3$ of radius ρ_0.

Exercise 7.7. Generalize the above considerations for a general cone $C^n_q = \{v \in \mathbb{R}^{n+1}_q : e^{n+1}_q(v, v) = 0\}$ of index q, $1 \le q \le n$, in $\mathbb{R}^{n+1}_q$. (Hint: Relying on the $\mathbb{R}^{n+1}_q \cong \mathbb{R}^{p+1} \times \mathbb{R}^q_q$ for $p = n - q$, use standard coordinates on $\mathbb{R}^{p+1}$ and spherical coordinates on $\mathbb{R}^q_q$.) Show that the different pseudospherical coordinates in $\mathbb{R}^{n+1}_q$ are not applicable for investigating the cones. Find different analogues in this case of the coordinates given via (7.55).

8. Terminology 1: Bases and frames. Holonomicity

It is beyond any doubt, all authors define a *basis* of an n-dimensional, $n \in \mathbb{N}$, vector space as a set of n linearly independent vectors in it (see, for instance, [1,12, resp. Section 1.5 and p. 9] or any book on vector spaces[1]). Sometimes the term 'frame' is used as a synonym of basis [13, p. 8]. But when (tangent) vector fields over a subset $U \subseteq M$ of a differentiable manifold of dimension n are concerned, the situation slightly changes: a frame (n-frame, comoving (or moving) frame, or vielbein)[2] is defined as a set of n linearly independent vector fields over U [5, Sections 2.2 and 7.6] (see also [7]). It is easily seen that a frame over U is, in fact, a basis in the module $\mathfrak{X}(U)$ of vector fields on U over the ring (algebra) $\mathfrak{F}(U)$ of real or complex, if the complex case is considered, functions on U [15, p. 10]. Furthermore, when restricted to a single point $x \in U$ (or if $U = \{x\}$), any frame becomes a basis of $T_x(M)$. An evident example of a frame on a coordinate neighborhood U is the frame $\{\partial/\partial x^i\}$ associated with some coordinates $\{x^i\}$ on U (see [15, p. 10], [5, p. 10], or Section 2).

Taking into account the above-said, as well as the argumentation in the references cited, we accept the following definitions (cf. Subsection 2.3). If a single vector space of finite dimension n is concerned, the concepts 'basis' and 'frame' are synonyms and mean an arbitrary set on n linearly independent vectors in it. A *frame* on (over) $U \subseteq M$, M being manifold, is a set of $\dim M$ vector fields on U

[1] Note, the given in [12, p. 9] definition (of Hamel) basis covers the case of infinite, countable or not, dimension too.

[2] For $n = 3$ (resp. $n = 4$) a frame is often called triad (resp. tetrad) or dreibein (resp. vierbein) depending one prefers Greek or German.

such that at each $x \in U$ they form a basis in $T_x(M)$, i.e., it is a field of bases over U.[3] So, a frame at (on, over) x and basis in $T_x(M)$ are synonyms, i.e., equivalent concepts. Further, in Chapter IV, we shall see that the notion of 'frame' admits a natural generalization on vector bundles.[4]

Let $\{E_i,\ i = 1, \dots, \dim M\}$ be a frame on an open subset U in M. It is called *holonomic* (resp. *anholonomic*) [19, p. 99ff], [11] if the basic fields E_i commute (resp. do not commute), i.e., if $C^i_{jk} = 0$ (resp. if $C^i_{jk} \neq 0$ for at least one triple (i, j, k)) where C^i_{jk} define the commutators

$$[E_j, E_k]_{\_} := E_j \circ E_k - E_k \circ E_j =: C^i_{jk} E_i. \tag{8.1}$$

The functions C^i_{jk} are known as *structure functions* of the frame $\{E_i\}$ (of the Lie algebra of vector fields). The frame $\{\partial/\partial x^i\}$, associated with local coordinates $\{x^i\}$ in a neighborhood U, is holonomic,

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]_{\_} \equiv 0. \tag{8.2}$$

Therefore, if we expand a frame $\{E_i\}$ on U over $\{\partial/\partial x^i\}$, $E_i = B^j_i \partial/\partial x^j$ with a $B := [B^j_i]$ being nondegenerate matrix-valued C^1 function on U, the holonomicity of $\{E_i\}$ depends entirely on the functions

$$\begin{aligned} C^i_{jk} &= (B^{-1})^i_m \Big(B^l_j \frac{\partial}{\partial x^l} B^m_k - B^l_k \frac{\partial}{\partial x^l} B^m_j \Big) \\ &= (B^{-1})^i_m 2 B^l_{[j} \frac{\partial}{\partial x^{|l|}} B^m_{k]} = (B^{-1})^i_m 2 E_{[j} B^m_{k]} \\ &= 2 B^m_{[j} E_{k]} (B^{-1})^i_m = -2 B^m_j B^n_k \frac{\partial}{\partial x^{[m}} (B^{-1})^i_{n]} \end{aligned} \tag{8.3}$$

where over indices included in square brackets antisymmetrization is assumed, which up to a constant, equal to (-2), coincides with the components of the (field of the) object of anholonomicity (anholonomy) [19, p. 100], [70, Chapter IV, § 7].[5]

From (8.1) one can easily find the transformation law of the functions C^i_{jk} under a change $\{E_i\} \mapsto \{E'_i = A^j_i E_j\}$, $A = [A^j_i]$ being non-degenerate matrix-valued function, of the frame $\{E_i\}$:

$$C^i_{jk} \mapsto C'^i_{jk} = (A^{-1})^i_l [A^m_j A^n_k C^l_{mn} + 2 E'_{[j} (A^l_{k]})] \tag{8.4}$$

---

[3] Said differently, in terms of vector bundles, a frame on U is a section of the frame bundle (= bundle of frames) restricted to U. The frame bundle over M is (cf. [23, 27], see also [5, p. 139 and Section 9.11]) a vector bundle with base M and whose fibre over $x \in M$ consists of all bases in $T_x(M)$.

[4] The just presented definition of frame corresponds to the case of the tangent bundle $(T(M), \pi, M)$ over M (see Subsection IV.2.4). Here $T(M) := \bigcup_{x \in M} T_x(M)$ and $\pi\colon T(M) \to M$ is such that $\pi(X) := x$ if $X \in T_x(M)$.

[5] The (index) notation of J.A. Schouten is practically out of usage our days.

where $[E'_j, E'_k]_{\_} := E'_j \circ E'_k - E'_k \circ E'_j =: C'^{\,i}_{jk} E_i$. Notice, (8.3) corresponds to (8.4) with $E_i = \frac{\partial}{\partial x^i}$, $C^i_{jk} = 0$, $A = B$, and C^i_{jk} for $C'^{\,i}_{jk}$.

We have to emphasize that the holonomicity is a property of the frames, not of a particular basis. This is clear from (8.3) in which partial derivatives with respect to the local coordinates (or to frame's vector fields) enter. The holonomicity characterizes the frames with respect to the local existence of coordinates the associated to which (holonomic) frame coincides with the initial one (in some open subset of the manifold). Evidently (see (8.2)), only the holonomic frames can admit such coordinates. The opposite assertion is (locally) true too.

Proposition 8.1. *A frame $\{E_i\}$ on a neighborhood U is holonomic iff locally exist coordinates $\{y^i\}$ such that locally $E_i = \partial/\partial y^i$.*

Proof. Let $\{E^i\}$ be the frame dual to $\{E_i\}$,[6] i.e., $E^i = (E_i)^*$, $E^i(E_j) := \delta^i_j$. Hence, if locally $E_i = A^j_i \partial/\partial x^j$, then $E^j = (A^{-1})^j_k \mathrm{d}x^k$, $\mathrm{d}x^k := (\partial/\partial x^k)^*$. We are interested in the existence of local coordinate $\{y^i\}$ such that $E_i = \partial/\partial y^i$ which is equivalent to $E^i = \mathrm{d}y^i$, i.e., E^i must be locally exact 1-forms [13, p. 108], [2, p. 55]. A necessary and sufficient condition for this is $\mathrm{d}E^i = 0$ (cf. Poincaré's lemma [2, p. 55], [13, p. 121]), i.e.,

$$\frac{\partial}{\partial x^l}\left(A^{-1}\right)^i_k - \frac{\partial}{\partial x^k}\left(A^{-1}\right)^i_l = 0$$

which expresses the fact that we must have $\partial y^i/\partial x^k = \left(A^{-1}\right)^i_k$. A simple matrix computation verifies that the last conditions hold iff $C^i_{jk} = 0$. □

Sometimes a frame is called holonomic if it coincides with the coordinate frame generated by some local coordinates. Respectively, a frame is called *locally holonomic* if every point in its domain has a neighborhood in which the frame coincides with the coordinate frame generated by some local coordinates in this neighborhood. By Proposition 8.1, the concept 'holonomic frame' in our sense and 'locally holonomic frame' in the above sense are identical. and will be used further. The difference is that the latter emphasizes on the local link between the holonomic frames and the local coordinates.

Proposition 8.1 suggests a definition of the property 'holonomicity' on arbitrary set U, not only on neighborhoods.

Definition 8.1. A *frame* $\{E_i\}$ defined on a neighborhood $\overline{U}$ containing or equal to a set U, $\overline{U} \supseteq U$, is called *holonomic on* U, if there exists a chart (V, x) of M such that $V \supseteq U$ and $E_i|_U = \frac{\partial}{\partial x^i}\big|_U$, i.e., the frame $\{E_i\}$ coincides on U with the coordinate frame generated by some local coordinates.

If U is a neighborhood and $\overline{U} = U$, by virtue of Proposition 8.1, this definition agrees with the afore-presented one.

[6]It is a frame in the cotangent bundle $(T^*(M), \pi, M)|_U$ with $\pi^{-1}(p) := T^*_p(M)$, $p \in M$ (see Subsection IV.2.4 below).

Definition 8.2. A *frame* $\{E_i\}$ defined on an open subset of M containing or equal to a given set U is called *locally holonomic on* U if for every $q \in U$ exists a chart (V, x) of M such that $V \ni q$ and $E_i|_{U\cap V} = \frac{\partial}{\partial x^i}\big|_{U\cap V}$, i.e., on U the frame $\{E_i\}$ locally coincides with the frame assigned to some local coordinates.

If U is a neighborhood, the local holonomicity agrees with the holonomicity, as defined above. But if U is not a neighborhood, e.g., if it is a submanifold of M, the local holonomicity is weaker concept than the holonomicity.

The concept of a frame is especially useful in physics where the concept of a basis (of a single space) is rarely utilized. More precisely, in physics the term frame of reference or reference frame is applied. This is a set of (real or mathematical) objects with respect to which is described the behavior, e.g., the evolution in time, of some physical system(s). Such reference objects are practically always defined on some subset of the space(-time) which is frequently a neighborhood or a curve (path) and very seldom chosen as a single point. Taking into account that the frames are used for referring (describing) of vector fields,[7] which in turn may represent some physical fields, the analogy between frames and reference frames is evident. The general idea is that to the *physical* concept reference frame there corresponds the *mathematical* concept frame (in some vector bundle). (See, e.g., [71], where a good analysis of different kinds of reference frames can be found.)

9. Conclusion

This chapter, as we saw, has an introductory character. It does not contain new original material except the implicit description of the frames normal at a single point of a Riemannian manifold (Section 6) and partially the investigation of normal frames/coordinates in Section 7.

After the presentation of the minimum knowledge from the differential geometry, required for our work, we started with the initial ideas concerning normal frames and coordinates. The basic results here are: only torsionless linear connections (may) admit normal coordinates; if the torsion is non-zero, normal frames (may) exit, but normal coordinates do not. If normal frames exist, they are parallel and are connected with linear transformations whose matrices are constant under the action of their basic vector fields.

The Riemannian and geodesic coordinates, which are normal at their origins, were pointed out as first examples of normal coordinates.

[7] In the general case they are replaced with sections of vector bundles.

Chapter II

Existence, Uniqueness and Construction of Normal Frames and Coordinates for Linear Connections

An in-depth investigation of existence, uniqueness and construction of frames and coordinates normal for linear connections on manifolds is given. Detailed review of the literature dealing with normal coordinates is presented. Some proofs are improved/generalized which entails a number of new results. Similar problems in the case with non-zero torsion are studied. Main results: For arbitrary (resp. torsionless) connections frames (resp. coordinates) normal at a single point and along path exist; they exist on submanifolds of higher dimensions iff the parallel transport along paths lying in them is path-independent. Complete constructive description of all, if any, frames and coordinates normal for arbitrary linear connections.
♡

1. Introduction

This chapter presents a complete exploration of the problems linked to the existence, uniqueness, and construction of normal coordinates and frames for manifolds endowed with a linear connection, with or without torsion. The review of the literature dealing with normal coordinates is mixed with new results. Such are, first of all, the ones concerning normal frames, connections with non-vanishing torsion, and the complete constructive description of the normal coordinates, if any.

The methods for description of normal coordinates/frames on Riemannian manifolds can *mutatis mutandis* be transferred on arbitrary manifolds, real or complex ($\mathbb{K} = \mathbb{R}, \mathbb{C}$),[1] endowed with linear connection. The possibility for this is hidden in the fact that the existence and properties of the normal coordinates/frames on a Riemannian manifold is intrinsically connected with the properties of the Christoffel symbols, i.e., with the Riemannian connection, not with the particular metric generating them. After this situation was clearly understood, somewhere in 1922–1927 [50, 72–74] (see [19, p. 155] for other references), the attention of the mathematicians, working in the field, was completely switched to the exploration of normal coordinates on manifolds with linear connections. Practically only the symmetric (torsionless) case has been investigate (see the comments after Remark I.5.4 on page 41). Some random works, like [44, 75], dealing with the asymmetric case (non-zero torsion) do not add nothing new as they simply note that the symmetric parts (I.3.9) of the connection coefficients (in coordinate frame) are coefficients of a symmetric linear connection to which the known results for torsionless connections are applicable.

Below in this chapter, in more or less modern terms and notation, are reviewed all results concerning the existence of normal coordinates/frames on manifolds endowed with symmetric linear connection. It contains a number of original new results too.

At first (Section 2), we concentrate on coordinates or frames normal at a single point. We present the known classical methods in this field [18, 19, 70] and then, modifying the methods that will be given in Chapter III in full generality, we present a full description of these coordinates/frames.

In Section 3 the attention is turned on the coordinates or frames normal along paths without self-intersections. For symmetric linear connections, we give a detailed description of the Fermi coordinates as the first known coordinates of this kind with [19] being our basic reference. Then, modifying the methods developed for similar but more general problems (see Chapter III and [76]), we derive a complete description of all coordinates or frames normal along paths without self-intersections or along locally injective paths in manifolds with symmetric or, respectively, arbitrary linear connections.

Several pages deal with problems concerning normal frames and coordinates on submanifolds with maximum dimensionality (Section 4), in particular on neigh-

[1] In the literature is often supposed $\mathbb{K} = \mathbb{R}$ but this does not influence the results.

borhoods and on the whole manifold. We prove that such frames or coordinates exist iff the connection is (locally) respectively flat or flat and torsionless. A complete description of the normal frames and coordinates in these cases is presented. We also point to some links between normal frames and parallel transports for flat linear connections.

Section 5 explores the problems of existence, uniqueness, and construction of frames or coordinates normal on arbitrary submanifolds. The classical results of [55] are reproduced in details using modern notation. Meanwhile, the corresponding proofs are improved, some results are generalized for arbitrary connections, with or without torsion, and new ones are presented. Next, we provide a complete constructive description of all frames (resp. coordinates) normal on submanifolds of a manifold with arbitrary (resp. torsionless) linear connection. Amongst a number of general results, we prove that normal on a submanifold frames (resp. coordinates) exist iff the parallel transport is path-independent along paths lying entirely in it (resp. and the connection is torsionless).

Section 6 contains instances and exercises illustrating the general theory of this chapter. Explicit expressions for frames and coordinates normal at a single point in and along a great circle on a two-dimensional sphere are presented in a case of the Riemannian connection induced from the Euclidean space in which the sphere is embedded. Some problems connected with frames/coordinates normal for Weyl connections are investigated. All frames/coordinates normal in the one-dimensional case are explicitly described. A similar problem is solved along a geodesic path in a 2-dimensional manifold. All coordinates normal at a point in Einstein-de Sitter spacetime are found.

A brief recapitulation of the above items can be found in Section 7.

2. The case at a single point

The coordinates/frames that are normal at a single point are the most simple and widely known ones. The proof of existence of coordinates/frames normal at a single point of Riemannian manifold presented in Section I.6 can be transferred, practically without changes, to manifolds endowed with symmetric linear connections. The main steps of the so-obtained proof are outlined below in Subsection 2.1; for details see [19, p. 155–159], [70, Chapter V, Section 3], or [18, Section 4.3].[1]

2.1. Old classical method

Let M be a C^∞ $\mathbb{K}$-manifold, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, endowed with C^∞ linear connection ∇. Let $p \in M$, $V(p)$ be a normal neighborhood of p, (U, x) be a chart with $U \ni p$, and $\{x^i\}$ be the local coordinates on U associated with (U, x). For every

[1]In the last book, the symmetry condition is dropped but this does not effect the general conclusions.

$q \in U \cap V(p)$ there exists a unique geodesic $\gamma\colon J \to U \cap V(p)$ such that $\gamma(s_0) = p$ and $\gamma(s) = q$ for some (unique) $s_0, s \in J$. As the connection is of class C^∞, such are all geodesics, in particular γ. Therefore in $\{x^i\}$ the expansion (I.6.3) with $\gamma^i = x^i \circ \gamma$ is valid. Successively differentiating the equation of geodesics (I.3.23), we can express the derivatives $\frac{\mathrm{d}^n \gamma^i}{\mathrm{d}s^n}\big|_{s_0}$, $n \in \mathbb{N}\backslash\{1\}$ via $\dot\gamma(s_0)$ and the partial derivatives of the connection's coefficients $\Gamma^i{}_{jk}$ at $p = \gamma(s_0)$. The result is (cf. (I.6.4))

$$\gamma^i(s) = \gamma^i(s_0) + \dot\gamma^i(s_0)(s - s_0) - \frac{1}{2}\Gamma^i{}_{jk}(\gamma(s_0))\dot\gamma^j(s_0)\dot\gamma^k(s_0)(s - s_0)^2 - \sum_{n=3}^{\infty} \frac{1}{n!}\Gamma^i{}_{i_1\dots i_n}(\gamma(s_0))\dot\gamma^{i_1}(s_0)\dots\dot\gamma^{i_n}(s_0)(s - s_0)^n \quad (2.1)$$

where the Γ's are defined through the recurrent relations (I.6.5) in which $\Gamma^i{}_{jk}$ must be replaced with $\Gamma^i{}_{(jk)} = \frac{1}{2}\bigl(\Gamma^i{}_{jk} + \Gamma^i{}_{kj}\bigr)$, i.e., with the symmetrized connection coefficients.

Define a coordinate system $\{y^i\}$ on $U \cap V(p) \ni p = \gamma(s_0), q = \gamma(s)$ by putting

$$y^i(q) = \dot\gamma(s_0)(s - s_0). \quad (2.2)$$

The coordinates y^i are similar to the Riemannian coordinates of Section I.6 and are called *Riemannian normal coordinates* with origin at the point p; obviously $y^i(p) = 0$. The explicit relations between $\{y^i\}$ and $\{x^i\}$ is (see (2.1) and cf. (I.6.9))

$$x^i(q) = x^i(p) + y^i(q) - \frac{1}{2}\Gamma^i{}_{jk}(p)y^j(q)y^k(q) - \sum_{n=3}^{\infty} \frac{1}{n!}\Gamma^i{}_{i_1\dots i_n}(p)y^{i_1}(q)\dots y^{i_n}(q). \quad (2.3)$$

Since the Jacobian of the change $\{x^i\} \mapsto \{y^i\}$ at p is

$$\det\Bigl[\frac{\partial y^i}{\partial x^j}\Bigr]\Big|_p = \det\Bigl[\frac{\partial x^i}{\partial y^j}\Bigr]^{-1}\Big|_p = \det\bigl[\delta^i_j\bigr]^{-1} = 1$$

and $y^i(p) = 0$, the transition $\{x^i\} \mapsto \{y^i\}$ is regular in some neighborhood W of p in which the series (2.3) is convergent and invertible with respect to the y's.[2] Consequently the Riemannian normal coordinates are well-defined in the domain $W \cap U \cap V(p)$ which is a normal neighborhood of p. In it the only common point of every two geodesics though p is the point p itself. Thus we have proved the following result which is an evident generalization of Proposition I.6.2.

Proposition 2.1. *Every point of a C^∞ manifold with C^∞ linear connection has a normal neighborhood in which Riemannian normal coordinates with origin at that point can be introduced.*

[2]This is a consequence of the implicit function theorem; e.g., see [77, Chapter III, § 8], [7, Sections 1.37 and 1.38], [78, Chapter 10, Section 2], [79, Theorem 9.18].

In the domain of Riemannian normal coordinates, any geodesic through their origin has an equation like (2.2), i.e., the geodesics in these coordinates are locally described by linear equations like the straight lines in $\mathbb{R}^n$.

The adjective 'normal' in the Riemannian normal coordinates is justified by the following results.

Proposition 2.2. *The symmetric parts of the connection coefficients vanish at the origin of the Riemannian normal coordinates*

Proof. Using (2.1), we get:

$$\frac{\partial x^i}{\partial y^j}\Big|_p = \frac{\partial y^i}{\partial x^j}\Big|_p = \delta^i_j, \qquad \frac{\partial^2 x^i}{\partial y^j \partial y^k}\Big|_p = -\Gamma^i{}_{(jk)}(p) = -\frac{1}{2}\big(\Gamma^i{}_{jk}(p) + \Gamma^i{}_{kj}(p)\big). \quad (2.4)$$

Inserting these equations in (I.3.6), we find the coefficients ${}^y\Gamma^i{}_{jk}(p)$ of ∇ in $\{y^i\}$ at p:

$${}^y\Gamma^i{}_{jk}(p) = \Gamma^i{}_{jk}(p) - \Gamma^i{}_{(jk)}(p) = \Gamma^i{}_{[jk]}(p) = \frac{1}{2}\big(\Gamma^i{}_{jk}(p) - \Gamma^i{}_{kj}(p)\big).$$

Thereof ${}^y\Gamma^i{}_{(jk)}(p) = \Gamma^i{}_{([jk])}(p) = 0$. □

Corollary 2.1. *If a C^∞ manifold is endowed with symmetric C^∞ linear connection, the connection's coefficients vanish at the origin of Riemannian normal coordinates.*

Proof. See Proposition 2.2. □

Thus the existence of normal coordinates/frames at a single point in the symmetric C^∞ case is proved.

Example 2.1. Consider a C^∞ symmetric linear connection on C^∞ manifold M. Let $p \in M$ and in a chart (U, x) with $U \ni p$ the only non-vanishing coefficient of the connection at p to be $\Gamma^1{}_{11}(p) = 1$. By (I.6.5), the only non-zero value at p of the functions $\Gamma^i{}_{i_1 \dots i_n}$, with $n \geq 3$, is $\Gamma^1{}_{\underbrace{1\dots1}_{n-\text{times}}}(p) = (-1)^n 1 \times \cdots \times (n - 1) = (-1)^n (n-1)!$. Substituting these equations into (2.3), we get the following explicit connection between the coordinates $\{x^i\}$, associated with (U, x), and the Riemannian normal coordinates $\{y^i\}$ with origin at p:

$$\begin{aligned} x^1(q) &= x^1(p) + \ln(1 + y^1(q)) \\ x^i(q) &= x^i(p) + y^i(q) \qquad \text{for } i \geq 2, \end{aligned}$$

from which the Riemannian normal coordinates can be expressed as

$$\begin{aligned} y^1(q) &= \exp(x^1(q) - x^1(p)) - 1 \\ y^i(q) &= x^i(q) - x^i(p) \qquad \text{for } i \geq 2. \end{aligned}$$

Notice, the Riemannian normal coordinates are 'more than normal' at their origin: at that point, the equations (I.6.11) hold with ${\Gamma^i}_{jk}$ replaced by ${\Gamma^i}_{(jk)}$. The proof of this assertion is identical with the one of (I.6.11) in Section I.6.

Taking the above into account, we see that the equations of geodesics through p in Riemannian normal coordinates is

$$\gamma^i(s) = \gamma^i(s_0) + \dot{\gamma}^i(s_0)(s - s_0). \tag{2.5}$$

Exercise 2.1. Prove that a path given via the last equation in Riemannian normal coordinates is a geodesic path through their origin.

A conclusion follows from the above: a path through the origin of the Riemannian normal coordinates is (locally) a geodesic iff in them it is represented with linear equations with respect to its parameter. Said differently, such paths are geodesics iff they are inverse images (with respect to the local coordinates) of the straight lines through a fixed point in $\mathbb{K}^{\dim M}$.

According to [19, p. 158] and [53, p. 59], the Riemannian normal coordinates for manifolds with symmetric linear connections, considered on them as coordinates normal at a given point, were first introduced in 1922 by O. Veblen [72].

Let $\{y^i\}$ be Riemannian normal coordinates on C^∞ manifold endowed with symmetric C^∞ linear connection and p and U be their origin and domain respectively. By the definition of normal coordinates (see Section I.5), the defined on U frame $\{\frac{\partial}{\partial y^i}\}$ is normal at p. Consequently, according to Proposition I.5.2, the set of *all* frames on U normal at p is $\{E_i = A_i^j \frac{\partial}{\partial y^j}\}$ where the non-degenerate matrix-valued function $A := [A_i^j]$ is such that $\frac{\partial A}{\partial y^i}|_p = 0$. (If required, the frames $\{E_i\}$ can be extended outside U in completely arbitrary ways.) As the point p is arbitrary, in this way we have obtained a *complete* description of the frames normal at a single point of a C^∞ manifold endowed with symmetric C^∞ linear connection.[3]

It is almost self-evident, the Riemannian normal coordinates are not the only local coordinates normal at some point in the symmetric case. For example, we can define in a coordinate neighborhood U of $p \in M$ the *geodesic normal coordinates* $\{z^i\}$ through the series (cf. (I.6.13))

$$x^i(q) = x^i(p) + z^i(q) - \frac{1}{2}{\Gamma^i}_{jk}(p)z^j(q)z^k(q) - \sum_{n=3}^{\infty} \frac{1}{n!}{c^i}_{i_1\dots i_n} z^{i_1}(q)\cdots z^{i_n}(q) \tag{2.6}$$

where $\{x^i\}$ are some coordinates in U, $q \in U$, $z^i(p) := 0$, and ${c^i}_{i_1\dots i_n} = {c^i}_{(i_1\dots i_n)} \in \mathbb{K}$. As $\frac{\partial x^i}{\partial y^j}|_p = \delta^i_j$, there is neighborhood $V \subseteq U$ of p in which (2.6) is convergent and invertible, i.e., $\{z^i\}$ are really coordinates in V. Repeating the proof of Proposition 2.2, we get (cf. (2.4)) ${}^z{\Gamma^i}_{jk}(p) = {\Gamma^i}_{[jk]}(p)$. Therefore ${}^z{\Gamma^i}_{jk}(p) = 0$

[3]This description of the frames normal at a single point is implicit; for the explicit one, see Theorem 2.3 on page 82.

iff ${\Gamma^i}_{[jk]}(p) = 0$, etc. Obviously, the choice ${c^i}_{i_1 \dots i_n} = {\Gamma^i}_{i_1 \dots i_n}(p)$ returns us to the Riemannian normal coordinates.

Example 2.2. Let the only non-vanishing values of ${\Gamma^i}_{jk}(p)$ and ${c^i}_{i_1 \dots i_n}$ be ${\Gamma^1}_{11} = {c^1}_{1 \dots 1} = 1$. Then (2.6) reduces to the system

$$
\begin{aligned}
x^1(q) &= x^1(p) + 1 + 2z^1(q) - \mathrm{e}^{z^1(q)} \\
x^i(q) &= x^i(p) + z^i(q) \qquad \text{for } i \geq 2,
\end{aligned}
$$

which is invertible relative to $\{z^i\}$ everywhere in the domain U of $\{x^i\}$.

2.2. Complete description

The above-presented way for introduction of normal coordinates is the historically established one. Now we are going to modify it in order to obtain a full description of all coordinates normal at a given point. The first step in this direction is to notice that, due to the definition of normal frames/coordinates, the utilization of the geodesics in the construction of coordinates normal at a single point is not necessary. This is a useful tool but it does not always work! For instance, as we shall see, the geodesics may not exist (e.g., if the connection is not continuous) while coordinates normal at a point exist. Also, we find too strong the requirement for the underlying manifold to be of class C^∞.

Proposition 2.3. *Every point p of a C^3 manifold with linear connection has a neighborhood in which exist coordinates such that the symmetric parts of the connection coefficients in them vanish at p. All such coordinates are given via equation* (2.11′) *below in which the coordinates $\{x^i\}$ are arbitrary, $[b^i_j]$ is non-degenerate constant matrix, and the C^3 functions b^i_{jkl} together with their partial derivatives are bounded in the domain of $\{y^i\}$. The inverse transformation $\{y^i\} \to \{x^i\}$ is given by equation* (2.11) *below with $[a^i_j] = [b^i_j]^{-1}$.*

Proof. Let M be a C^3 manifold endowed with linear connection ∇ on which we do not impose any differentiability conditions. Choose an arbitrary point $p \in M$ and charts (U, x) and (V, y) such that $p \in U \cap V \neq \varnothing$. As we know from Subsection I.2.1, the local coordinates $x^i(q)$ and $y^i(q)$ of every point $q \in U \cap V$ are connected via C^3 functions $y^i(q) = f^i\big(x^1(q), \dots, x^{\dim M}(q)\big)$, or $f^i = r^i \circ y \circ x^{-1}$ with r^i being the standard coordinate functions on $\mathbb{K}^{\dim M}$. Since f^i and $\big(f^i\big)^{-1}$ are C^3 functions, there exist numbers $a^i_j, a^i_{jk} \in \mathbb{K}$ and C^3 functions $a^i_{jkl} \colon U \cap V \to \mathbb{K}$, which together with their partial derivatives are bounded on $U \cap V$, such that

$$
\begin{aligned}
x^i(q) = x^i(p) + a^i_j[y^j(q) - y^j(p)] + a^i_{jk}[y^j(q) - y^j(p)][y^k(q) - y^k(p)] \\
+ a^i_{jkl}(q)[y^j(q) - y^j(p)][y^k(q) - y^k(p)][y^l(q) - y^l(p)]. \quad (2.7)
\end{aligned}
$$

By virtue of the above-said, this formula is invertible on $U \cap V$ with respect to $y^i(q)$ which can be expressed in a similar way as functions of $\{x^i(q)\}$:

$$y^i(q) = y^i(p) + b^i_j[x^j(q) - x^j(p)] + b^i_{jk}[x^j(q) - x^j(p)][x^k(q) - x^k(p)] \\ + b^i_{jkl}(q)[x^j(q) - x^j(p)][x^k(q) - x^k(p)][x^l(q) - x^l(p)], \quad (2.8)$$

where $b^i_j, b^i_{jk} \in \mathbb{K}$ are constants and $b^i_{jkl}\colon U \cap V \to \mathbb{K}$ are of class C^3 and they and their first partial derivatives are bounded on $U \cap V$.

Now the problem, which is central for us, is: given a point p and coordinates $\{x^i\}$, can a chart (V, y) be chosen so that $\{y^i\}$ are normal at p? As one can expect, the answer is positive if the connection is symmetric.

From (2.7) and (2.8), we derive:

$$\frac{\partial x^i}{\partial y^j}\Big|_p = a^i_j, \quad \frac{\partial y^i}{\partial x^j}\Big|_p = b^i_j, \qquad \frac{\partial^2 x^i}{\partial y^j y^k}\Big|_p = 2a^i_{(jk)}, \quad \frac{\partial^2 y^i}{\partial x^j x^k}\Big|_p = 2b^i_{(jk)} \qquad (2.9)$$

where the matrices $[a^i_j]$ and $[b^i_j]$ are non-degenerate as a consequence of the invertability of (2.7) and (2.8) at p. Using the equality $\frac{\partial x^i}{\partial y^k}\frac{\partial y^k}{\partial x^j} = \delta^i_j$ and the one obtained from it by differentiation with respect to x^l, we find from (2.9)

$$\begin{aligned} &a^i_k b^k_j = \delta^j_i \iff [b^i_k] = [a^i_k]^{-1}, \\ &a^i_{(mn)} b^m_l b^m_j + a^i_k b^k_{(jl)} = 0 \iff b^i_{(jk)} = -b^i_l a^l_{(mn)} b^m_j b^n_k. \end{aligned} \qquad (2.10)$$

Let ${}^x\Gamma^i{}_{jk}$ and ${}^y\Gamma^i{}_{jk}$ be the coefficients of ∇ in $\{x^i\}$ and $\{y^i\}$ respectively. Applying the transformation laws (I.3.6) and (2.9), we get at p:

$${}^y\Gamma^i{}_{jk}(p) = \big([a^i_j]^{-1}\big)^i_l \big(a^m_j a^n_k\, {}^x\Gamma^l{}_{mn}(p) + 2a^l_{(jk)}\big).$$

So, at p we can obtain ${}^y\Gamma^i{}_{(jk)}(p) = 0$ if and only if $2a^l_{(jk)} = -a^m_j a^n_k\, {}^x\Gamma^l{}_{mn}(p)$ which, due to (2.10), is equivalent to $2b^i_{(jk)} = b^i_l\, {}^x\Gamma^l{}_{jk}(p)$.[4] Substituting the last equation into (2.7), we see that all coordinates $\{y^i\}$ for which ${}^y\Gamma^i{}_{(jk)}(p) = 0$, if such exist, are obtainable from fixed coordinates $\{x^i\}$ by inverting the equation

$$x^i(q) = x^i(p) + a^i_j[y^j(q) - y^j(p)] \\ - {}^x\Gamma^i{}_{(mn)}(p) a^m_j a^n_k [y^j(q) - y^j(p)][y^k(q) - y^k(p)] \\ + a^i_{jkl}(q)[y^j(q) - y^j(p)][y^k(q) - y^k(p)][y^l(q) - y^l(p)] \quad (2.11)$$

on some subneighborhood $W \ni p$ of U. By virtue of (2.8), this inversion results in

$$y^i(q) = y^i(p) + b^i_j[x^j(q) - x^j(p)] \\ + b^i_l\, {}^x\Gamma^l{}_{(jk)}(p)[x^j(q) - x^j(p)][x^k(q) - x^k(p)] \\ + b^i_{jkl}(q)[x^j(q) - x^j(p)][x^k(q) - x^k(p)][x^l(q) - x^l(p)]. \quad (2.11')$$

[4]We cannot set ${}^y\Gamma^i{}_{jk}(p) = 0$ or ${}^y\Gamma^i{}_{[jk]}(p) = 0$ unless ∇ is torsionless at p.

Here $a^i_j, b^i_j \in \mathbb{K}$, $\det[a^i_j] \neq 0, \infty$, $[b^i_j] = [a^i_j]^{-1}$, and the C^3 functions $a^i_{jkl}, b^i_{jkl} : W \to \mathbb{K}$ together with their partial derivatives are bounded on W. Since the matrix (the Jacobian at p) $\left[\frac{\partial x^i}{\partial y^j}|_p\right] = \left[a^i_j\right]$ is non-degenerate, by the implicit function theorem (see [77, Chapter III, § 8], [7, Sections 1.37 and 1.38], or [78, Chapter 10, Section 2]), there exists a neighborhood W of p in which (2.11) defines y^i as unique C^3 functions of x^i. Hence, in the neighborhood $V = U \cap W \ni p$, the mappings $\{x^i\} \mapsto \{y^i\}$ and $\{y^i\} \mapsto \{x^i\}$, defined via (2.11) and (2.11′) in which $x^i(p)$ and $y^i(p)$ are fixed numbers, are C^3 diffeomorphisms. This ends to proof of Proposition 2.3. □

If we specify the connection to be symmetric, from Proposition 2.3, we obtain a theorem describing all local coordinates normal at a given point.

Theorem 2.1. *Every point p of a C^3 manifold endowed with symmetric linear connection has a neighborhood on which coordinates normal at p exist. All normal coordinates $\{y^i\}$ in the mentioned neighborhood are given via equation* (2.11′) *in which the coordinates $\{x^i\}$ are arbitrary, $[b^i_j]$ is non-degenerate constant matrix, and the C^3 functions b^i_{jkl} together with their partial derivatives are bounded in the domain of $\{y^i\}$. The inverse transformation $\{y^i\} \to \{x^i\}$ is given by* (2.11) *with $[a^i_j] = [b^i_j]^{-1}$.*

This theorem gives a *complete* description of all normal coordinates at a single point of a C^3 manifold with symmetric linear connection. Analogous result concerning the normal frames is provided by the following theorem.

Theorem 2.2. *Let $\{y^i\}$ be coordinates normal at a point p in a C^3 manifold with symmetric linear connection. In the domain U of $\{y^i\}$ all frames normal at p have the form $\left\{E_i = A^j_i \frac{\partial}{\partial y^j}\right\}$, where the non-degenerate matrix-valued function $A = [A^i_j]$ is such that $\frac{\partial A}{\partial y^j}\Big|_p = 0$.*[5] *Outside U the frames can be extended arbitrarily. All of these frames normal at p are holonomic at p but in $U\backslash\{p\}$ they need not to be such.*

Proof. See Propositions I.5.2 and I.5.3. □

Remark 2.1. In the notation of Theorem 2.3 on the next page below, the explicit form of A is given via (2.14) with y^i for x^i and $\Gamma^i{}_{jk}(p) = 0$ (as $\{y^i\}$ is normal at p), i.e.,

$$A(q) = A_0 + A_{jk}(q)[y^j(q) - y^j(p)][y^k(q) - y^k(p)]. \tag{2.12}$$

For the proof, see the proof of Theorem 2.3 below. The frames $\{E_i\}$ holonomic on U and normal at p are such that $E_i = \frac{\partial}{\partial z^i}$ for some coordinates z^i on U which are

[5] In the notation of Theorem 2.3 on the following page, the explicit form of A is given via (2.14) with y^i for x^i; for the proof, see the proof of Theorem 2.3 below.

normal at p. In this case $A^i_j = \frac{\partial y^i}{\partial z^j}$ which, due to (2.12), implies

$$z^i(q) = y^i(p) + (A_0^{-1})^i_j(q)[y^j(q) - y^j(p)] \\ + b^i_{jkl}(q)[y^j(q) - y^j(p)][y^k(q) - y^k(p)][y^l(q) - y^l(p)], \quad (2.13)$$

where $b^i_{jkl}\colon U \to \mathbb{K}$ are of class C^1 and they and their partial derivatives are bounded on U. Of course, this result is a special case of (2.11′) as $\{y^i\}$ are normal at p and the connection considered is symmetric.

If the manifold and its connection are of class C^∞, the choice $a^i_j = \delta^i_j$ and the expansion of a^i_{jkl} into a power series, with suitable coefficients, brings us back to the results of Subsection 2.1.

Theorem 2.2 can easily be generalized to arbitrary linear connections, with or without torsion:

Theorem 2.3. *Let M be a C^2 manifold endowed with linear connection. For every point $p \in M$ there exist frames normal at p. Moreover, if (U, x) is a chart with $U \ni p$, then in U all frames normal at p are $\{E_i = A^j_i \frac{\partial}{\partial x^j}\}$ where $A := [A^i_j]\colon U \to$ $\mathrm{GL}(\dim M, \mathbb{K})$, with $\mathrm{GL}(n, \mathbb{K})$ being the group of all $n \times n$ invertible matrices with entries in $\mathbb{K}$, is non-degenerate, of class C^1, and its general form is*

$$A(q) = \{\mathbb{1} - \Gamma_j(p)[x^j(q) - x^j(p)]\}A_0 \\ + A_{jk}(q)[x^j(q) - x^j(p)][x^k(q) - x^k(p)] \qquad q \in U. \quad (2.14)$$

Here Γ_i are the matrices of the connection coefficients, A_0 is constant and non-degenerate matrix, and A_{jk} are C^1 matrix-valued functions on U such that they and their partial derivatives are bounded at p.

Proof. A frame $\{E'_i\}$ is normal at p iff A satisfies (I.5.4) with $\{p\}$ for U and $\frac{\partial}{\partial x^i}$ for E_i, i.e., $\frac{\partial A}{\partial x^i}\big|_p + \Gamma_i(p)A(p) = 0$. Supposing A to be of class C^2, we can write the expansion

$$A(q) = A_0 + A_k[x^k(q) - x^k(p)] + A_{jk}(q)[x^j(q) - x^j(p)][x^k(q) - x^k(p)],$$

where $A_0, A_i \in \mathrm{GL}(\dim M, \mathbb{K})$ are constant, A_0 is non-degenerate, and the C^1 matrix-valued functions A^i_{jk} on U together with their first partial derivatives are bounded at p. From the last equation, we get $A(p) = A_0$ and $\frac{\partial A}{\partial x^i}\big|_p = A_i$ and hence (see above) $A_i = -\Gamma_i(p)A_0$, the substitution of which into the last displayed equation gives (2.14). □

Theorem 2.3 gives a *complete* constructive description of all frames normal at a single point.

The above description of the normal coordinates/frames is an adaptation of the developed in [80] methods (see also Chapter III) for the case of manifolds with linear connections.

2.3. Modern classical method

In the modern books on differential geometry, the normal coordinates as described in Subsection 2.1 are most often not mentioned at all; see, e.g,: [11, Chapter III, § 8], [12, pp. 313–314], [10, p. 110], [28, p. 133], [1, § 6.7], and [8, § 3.8, (ii)]. Nowadays the coordinates normal at a point are introduced by means of the exponential mapping (see Subsection I.3.4), like in the references just cited. This method is practically identical with the introduction of Riemannian normal coordinates, presented in Subsection 2.1, only other terminology and concepts are involved. It is briefly outlined below.

Let M be a C^2 manifold, $p \in M$, and (U, φ) be a chart with $U \ni p$. The chart (U, φ) or the associated with it local coordinates $\{\varphi^i\}$ are called *normal at* p (with respect to p) if the inverse images of the straight lines through $\varphi(p) \in \mathbb{K}^{\dim M}$ are geodesics in M through p. This definition, as we shall see further, agrees with our previous terminology (see Definition I.5.2).

Let a basis $\{e_i\}$ in $T_p(M)$ be given. Since the mapping $\nu\colon T_p(M) \to \mathbb{K}^{\dim M}$ defined by $X \mapsto (X^1, \dots, X^{\dim M})$ for $X = X^i e_i \in T_p(M)$ is a linear isomorphism, the exponential mapping can be used to define normal coordinates in a neighborhood of p. The chart $(T_p(M), \nu)$ of $T_p(M)$ provides a natural coordinate system on $T_p(M)$. Let $V_p^0 \subseteq T_p(M)$ and $V(p) \subseteq M$ be normal neighborhoods such that the exponential mapping $\exp_p : V_p^0 \to V(p)$ is a diffeomorphism (see Definition I.3.6). Consider a local chart $(V(p), x)$ with

$$x := \nu \circ \exp_p^{-1} \colon V(p) \to \mathbb{K}^{\dim M}. \tag{2.15}$$

If $\gamma_{p,X} \colon J \to V(p)$, $J \ni 0 \in \mathbb{R}$ is the unique geodesic with $\gamma_{p,X}(0) = p$ and $\dot\gamma_{p,X}(0) = X$, its current coordinates in the coordinate system $\{x^i\}$ are

$$\begin{aligned}\gamma^i_{p,X}(s) := x^i(\gamma_{p,X}(s)) &= (r^i \circ x)(\gamma_{p,X}(s)) = (r^i \circ \nu \circ \exp_p^{-1}) \circ \exp_p(sX)\\ &= r^i(\nu(sX)) = r^i(sX^1, \dots, sX^{\dim M}) = sX^i, \qquad s \in J\end{aligned}$$

where (I.3.29) is used and $\{r^i\}$ are the standard coordinate functions on $\mathbb{K}^{\dim M}$. Reversing the last equalities, we see that any path given by $\gamma^i(s) = sX^i$ in $\{x^i\}$ is a geodesic through p. Consequently $\{x^i\}$ are normal coordinates at p according to the last definition. In case of symmetric connection, the coordinates $\{x^i\}$ are also normal according to Definition I.5.2. Indeed, if $\gamma\colon J \to M$ is a geodesic through p, in $\{x^i\}$ we have $\gamma^i(s) = sX^i$ for some $X^i \in \mathbb{K}$, the substitution of which in the geodesic equation (I.3.23′) on page 32 results in ${\Gamma^i}_{(jk)}\big(x^{-1}(sX^1, \dots, sX^{\dim M})\big)X^jX^k = {\Gamma^i}_{(jk)}(\gamma(s))X^jX^k \equiv 0$ with ${\Gamma^i}_{jk}$ being the connection coefficients in $\{x^i\}$. In particular we have

$$0 = {\Gamma^i}_{(jk)}(p)X^jX^k = {\Gamma^i}_{(jk)}(x^{-1}(\mathbf{0}))X^jX^k$$

as $p = x^{-1}(\mathbf{0})$, $\mathbf{0} \in \mathbb{K}^{\dim M}$. Since X is completely arbitrary, such are X^i, and consequently

$${\Gamma^i}_{(jk)}(p) := \frac{1}{2}\big({\Gamma^i}_{jk}(p) + {\Gamma^i}_{kj}(p)\big) = 0.$$

So, if the connection is symmetric, then ${\Gamma^i}_{jk}(p) = 0$, i.e., $\{x^i\}$ are normal coordinates at p in the sense of Definition I.5.2.

Comparing the above results with the ones in Subsection 2.1, we see that in the case of a C^∞ manifold with C^∞ linear connection the coordinates $\{x^i\}$ are just the Riemannian normal coordinates introduced in Subsection 2.1. For this reason, we find it appropriate and convenient to call the coordinates defined by (2.15) *Riemannian normal coordinates*, not simply normal as in the literature cited, with an exception of [8, § 3.8, (ii)], and at the beginning of this subsection; thus we reserve the adjective 'normal' for frames and coordinates described via Definitions I.5.1 and I.5.2.

At the end, let us note that in [8, § 3.8, (ii)] the local coordinates, described by (2.15) (with $\imath := \nu^{-1}$ being a linear isometry), are applied to a proper Riemannian manifold (the metric being positively definite) which results to a new construction of the particular Riemannian coordinates obtained by G. D. Birkhoff [50] (see p. 48). These coordinates appear in [8, § 3.8, (ii)] under the name Riemannian normal coordinates.

3. The case along paths without self-intersections

The first proof of existence of coordinates normal along curve without self-intersections on (C^∞) Riemannian manifold was given by E. Fermi in 1922 [52]. The explicit formula for transition to the coordinates discovered by E. Fermi was given by Levi-Civita in 1926 [81]. Analogous result on (C^∞) manifold with arbitrary symmetric (C^∞) connection was originally published by L. P. Eisenhart in 1927 [53, p. 64].[1] These special types of coordinates are widely known as *Fermi coordinates* [19, Chapter III, § 8], [8, § 3.8, (ii)]. Sometimes they are referred as Fermi geodesic coordinates [12, p. 327][2] and very rarely as geodesic (along a curve) coordinates [25, § 91], [44].

Since all of the proofs, known to the author, of the existence of such coordinates are more or less identical at a level of ideas, i.e., a construction of particular class of coordinates with the property required, we suggest to call *Fermi coordinates* the special kind of coordinates normal along a path described below in Subsection 3.1, thus reserving the term 'coordinates normal along a path' for a particular realization of Definition I.5.2.

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[1]For other original papers on this topic, see [19, p. 166].

[2]More precisely, in [12, p. 327] is done the following. Let $\gamma\colon J \to M$ be without self-intersections and for some $s_0 \in J$ a basis $\{E_i^0\}$ in $T_{\gamma(s_0)}(M)$ be fixed. Define along γ a frame $\{E_i\}$ such that $E_i|_{\gamma(s_0)} = E_i^0$ and $(\nabla_{\dot\gamma} E_i)|_{\gamma(J)} = 0$, i.e., $\{E_i\}$ is obtained from $\{E_i^0\}$ by means of parallel propagation along γ. Notice, $\{E_i\}$ is defined only on $\gamma(J)$; outside $\gamma(J)$ it can be extended arbitrarily. The Fermi geodesic coordinates with domain U, $U \cap \gamma(J) \neq \varnothing$, are local coordinates $\{x^i\}$ such that $\frac{\partial}{\partial x^i}|_p = E_i|_p$ for $p \in U \cap \gamma(J)$. Generally these coordinates are not normal along γ in U.

One must be aware of the fact that as ‘Fermi coordinates’ can be found coordinate systems entirely different from the ones we are dealing with. For instance, in the literature on general relativity under the name ‘Fermi coordinates’ are known completely different coordinates which generally are not normal at all but along a geodesic in a Riemannian manifold they are normal at their origin [62, Chapter II, § 10]. More precisely, in the last case the coordinates mentioned, as defined in [62, Chapter II, § 10], coincide with the normal coordinates introduced by Birkhoff (see p. 48). The cause for this is that these coordinates are defined by means of the Fermi-Walker transport [62, Chapter I, § 4] which along a geodesic coincides with the parallel transport along it. Another example of ‘Fermi coordinates’, which generally are not normal, may be found in [28, pp. 133–134]. In [55] the term ‘Fermi coordinates’ is a synonym of our notion of ‘coordinates normal on a submanifold’.

There are two basic methods for proving the existence of coordinates normal along paths without self-intersections. The first one is to construct a specific coordinate system in a neighborhood of the path and then, by explicit calculation, to show that along the path given the coordinates constructed are normal. The second one consist in finding a particular (class of) solution of equation (I.5.4′) on page 41 along a given path. A typical example of the former method is given in Subsection 3.1, while a modification of the latter method is presented in Subsection 3.2

Below in this section, the manifold M will be considered as real one, i.e., if it is complex, it will be regarded as real one of dimension $2\dim M = 2\dim_{\mathbb{C}} M = \dim_{\mathbb{R}} M$ (see page 7).[3] Formally we shall reflect this by writing $\dim_{\mathbb{R}} M (= \dim M$ if M is real and $= 2\dim_{\mathbb{C}} M$ if M is complex) instead of $\dim M$ for the dimension of M; respectively, all Latin indices, whose range is not specified, run from 1 to $\dim_{\mathbb{R}} M$ and the values of the coordinate homeomorphisms will be in $\mathbb{R}^{\dim_{\mathbb{R}} M}$.

3.1. Fermi coordinates

Below, following [19, pp. 166–169], we shall prove the existence of coordinates normal along a (part of a given) path without self-intersections by explicit construction of concrete such coordinates. Regardless of the technical difficulties, the idea of the proof is quite simple and consists in the following. In a neighborhood of the path $\gamma\colon J \to M$, the manifold M is (locally) represented as a direct sum of suitable $(\dim_{\mathbb{R}} M - 1)$-dimensional submanifolds V_s, one for each point $\gamma(s)$, $s \in J$. So, every point p of this neighborhood belongs to a single $V_{s(p)}$ for some unique $s(p) \in J$. Then the Fermi coordinates of p are $(s(p), \xi^2, \dots, \xi^{\dim_{\mathbb{R}} M})$ where $(\xi^2, \dots, \xi^{\dim_{\mathbb{R}} M})$ are the Riemannian normal coordinates of p with respect

[3] In this way we avoid some completely technical problems connected with the fact that the domain of every path is a *real* interval by definition. Roughly speaking, if $\gamma\colon J \to M$ is C^1 regular injective path in a complex manifold M, the set $\gamma(J)$ is a submanifold of M with real dimension $\dim_{\mathbb{R}} \gamma(J) = 1$ and complex one $\dim_{\mathbb{C}} \gamma(J) = 1/2$, the last case, of non-integer dimension, being out of the range of our work.

to $\gamma(s(p))$ in $V_{s(p)}$ considered as a $(\dim_{\mathbb{R}} M - 1)$-dimensional submanifold of M. The details of this construction are presented below and their aim is the formulation and proof of Proposition 3.1 below and of other results presented after it. If the reader is not interested in the technical details that follow below, he/she can jump to the definition of Fermi coordinates, given just after equation (3.9′) below, and next to proceed with Proposition 3.1 and the text after it.

Let $\gamma\colon J \to M$ be a C^1 regular[4] path without self-intersections[5] in a C^∞ manifold M endowed with C^∞ connection ∇. Let $s_0 \in J$ be arbitrarily fixed inner, i.e., not boundary, if any,[6] point and (U, x) be a chart in whose domain is $\gamma(s_0)$, $U \ni \gamma(s_0)$. Let $X \in \mathfrak{X}(\gamma(J))$ be a vector field parallel along γ with zero component along $\dot\gamma(s_0)$ at $\gamma(s_0)$, i.e.,

$$\nabla_{\dot\gamma} X = 0, \qquad X_{\gamma(s_0)} \in T_{\gamma(s_0)}(M) \backslash \{a\dot\gamma(s_0) | a \in \mathbb{R}\backslash\{0\}\}$$

or, equivalently, we can write $X_{\gamma(s_0)} = \sum_{i=2}^{\dim_{\mathbb{R}} M} X^i_{\gamma(s_0)} E^0_i$ for every basis $\{E^0_i\}$ in $T_{\gamma(s_0)}(M)$ with $E^0_1 = \dot\gamma(s_0)$.

At every point $\gamma(s) \in \gamma(J) \cap U$, $s \in J_U := \{u | u \in J,\ \gamma(u) \in U\}$, we consider the geodesics $\beta_s\colon J_X \to U$ with initial conditions $\beta_s(t_0) = \gamma(s)$ and $\dot\beta_s(t_0) = X_{\gamma(s)}$ for $s \in J_U$, some fixed $t_0 \in J_X$, and arbitrary X as defined above (i.e., for arbitrary its value $X_{\gamma(s_0)} \in T_{\gamma(s_0)}(M) \backslash \{a\dot\gamma(s_0) | a \in \mathbb{R}\backslash\{0\}\}$). According to (2.1), the expansion

$$\beta^i_s(t) = \gamma^i(s) + X^i_{\gamma(s)}(t - t_0) - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^i{}_{i_1 \dots i_n}(\gamma(s)) X^{i_1}_{\gamma(s)} \dots X^{i_n}_{\gamma(s)} (t - t_0)^n \quad (3.1)$$

is valid in the associated to (U, x) local coordinates $\{x^i\}$. Here $\beta^i_s := x^i \circ \beta_s$, $\gamma^i := x^i \circ \gamma$, $X = X^i \frac{\partial}{\partial x^i}\big|_{\gamma(J)}$, $\Gamma^i{}_{jk}$ are the coefficients of ∇ in $\{x^i\}$, and $\Gamma^i{}_{i_1 \dots i_n}$ for $n \geq 3$ are defined via (I.6.5).

Let $\{E^0_i\}$ be a basis in $T_{\gamma(s_0)}(M)$ with $E^0_1 = \dot\gamma(s_0)$ and $\{E'_i\}$ be a frame along γ obtained from $\{E^0_i\}$ by parallel transportation along γ:

$$\nabla_{\dot\gamma} E'_i = 0 \quad E'_i\big|_{\gamma(s_0)} = E^0_i, \quad E^0_1 = \dot\gamma(s_0).$$

[4]A C^1 path $\gamma\colon J \to M$ is regular if $\dot\gamma(s) \neq 0, \infty$ for all $s \in J$ (see p. 14).

[5]A point $p \in \gamma(J)$ is called self-intersection point of $\gamma\colon J \to M$ if there exist $s_1, s_2 \in J$ such that $s_1 \neq s_2$ and $\gamma(s_1) = \gamma(s_2) = p$. Given $s_0 \in J$, the number $k(p)$ of the different values $s \in J$ with $s \neq s_0$ and $\gamma(s) = \gamma(s_0) = p$ is called self-intersection number of γ at $p = \gamma(s_0)$; we also say that γ self-intersects (itself) $k(p)$ times at p. If $k(p) = 0$ for all $p \in \gamma(J)$, the path γ is said to be without self-intersections, i.e., γ is without self-intersections if for every $s_1 \in J$ there does not exist $s_2 \in J$, $s_1 \neq s_2$ with $\gamma(s_1) = \gamma(s_2)$. In other words, γ is without self-intersections if γ is an injective mapping.

[6]If J has left or/and right end (boundary) point(s) and s_0 happens to be its left or right end, some complications arise; e.g., with the definition of $\dot\gamma(s_0)$ as a corresponding one-sided (left or right) limit, the implicit function theorem [77, Chapter III, § 8] can not be applied in a neighborhood of $(s_0, \gamma(s_0))$, etc. Generally this is not a typical situation and we exclude it for a moment from our investigation. Later we will drop this assumption. These complications are due to the fact that if J is closed from one or both ends, the set $\gamma(J)$ is a manifold *with* boundary (see Remark I.2.1 on page 6) which we treat with methods for manifolds *without* boundary. They will not appear if from the beginning one starts with manifolds with boundary.

By construction, we have (see Subsection I.3.3) $X = X'^i E'_i = \sum_{i=2}^{\dim_{\mathbb{R}} M} X'^i E'_i$ where X'^i are constants, $X'^i \in \mathbb{R}$, and $X'^1 = 0$.

In $U \cap \gamma(J)$ the expansion $E'_i = A^j_i \frac{\partial}{\partial x^i}$ is valid for some non-degenerate matrix-valued function $A = [A^j_i]$, $A^j_i \colon U \cap \gamma(J) \to \mathbb{R}$. Applying (I.2.12), we get $X^i = A^i_j X'^j$ with $X'^1 = 0$, so (3.1) can be rewritten as

$$\begin{aligned}\beta^i_s(t) = \gamma^i(s) + \sum_{j=2}^{\dim_{\mathbb{R}} M} A^i_j(\gamma(s)) X'^j (t - t_0) \\ - \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=2}^{\dim_{\mathbb{R}} M} A^{j_1}_{i_1}(\gamma(s)) \cdots A^{j_n}_{i_n}(\gamma(s)) \Gamma'^i{}_{j_1 \dots j_n}(\gamma(s)) \\ \times X'^{i_1}_{\gamma(s)} \dots X'^{i_n}_{\gamma(s)} (t - t_0)^n. \quad (3.2)\end{aligned}$$

Now the idea is in (a subneighborhood of) U to be constructed local coordinates $\{x'^i\}$ such that $\frac{\partial}{\partial x'^i}\big|_p = E'_i|_p$ and $\Gamma'^i{}_{jk}(p) = 0$ for p in (a subset of) $U \cap \gamma(J)$. As $E'_i|_p = A^j_i(p) \frac{\partial}{\partial x^j}\big|_p$, $p \in U \cap \gamma(J)$, the only restriction on the connection $x'^i = x'^i(x^1, \dots, x^{\dim_{\mathbb{R}} M})$ is $\frac{\partial x^j}{\partial x'^i}\big|_p = A^j_i(p)$, $p \in U \cap \gamma(J)$. Below we will construct a particular realization of this link.

Let $f^i \colon J_U \to \mathbb{R}$, with $s \in J_U := \{u | u \in J,\ \gamma(u) \in U\}$, be C^1 functions and f^1 be C^1 diffeomorphism from J_U on the image $f^1(J_U)$. We shall find these functions from the requirement that[7]

$$x'^1\big(\beta_s(t)\big) := f^1(s), \qquad x'^i\big(\beta_s(t)\big) := f^i(s) + (t - t_0) X'^i, \quad i \geq 2 \qquad (3.3)$$

which is a special realization of the above general connection between $\{x^i\}$ and $\{x'^i\}$.

Remark 3.1. Here and below we implicitly suppose that the family of geodesic paths $\{\beta_s\}$ forms (for different X) an $(\dim_{\mathbb{R}} M - 1)$-dimensional foliation along the path γ, i.e., that the sets $V_s := \{\beta_s(J_X)$ where X, as defined above, is arbitrary$\}$ are such that: $\bigcup_{\gamma(s) \in U} V_s = U$, $V_s \cap V_{s'} = \varnothing$ for $s \neq s'$, $s, s' \in J$ and $V_s \cap \gamma(J) = \gamma(s)$. Then for every $p \in U$ there exists only one $s \in J$ such that $p \in V_s$. Equations (3.3) mean that at first we define the coordinates of p with respect to $\gamma(s)$ in V_s and then of $\gamma(s)$ along γ. The existence of such a foliation $\{V_s\}$ along γ, which is overlooked in [19], is natural but not trivial and requires corresponding proof. Rigorously we have to do the following. Let $\{y^i\}$ be any coordinate system in U. In Subsection 3.2 (see the text preceding equation (3.12)) will be proved that there exist an interval $J_1 \subseteq J$ and neighborhood $U_1 \subseteq U$ such that $\dot\gamma^1_y|_{J_1} \neq 0$ and $\gamma^1_y|_{J_1} \colon J_1 \to \gamma^1_y(J_1)$ is diffeomorphism $(\gamma^k_y := y^k \circ \gamma)$. Now, in a neighborhood U'_1 of $\gamma(J_1)$ in U_1, choose $\{y^i\}$ such that $\frac{\partial}{\partial y^i}\big|_{\gamma(s)} = E'_i\big|_{\gamma(s)}$ for $s \in J_1$ and define $V_s := \big\{p \in U'_1 | y^1(p) = \gamma^1(s)\big\}$.[8] Then we have $U'_1 = \bigcup_{s \in J_1} V_s$, $V_s \cap V_{s'} = \varnothing$

[7] Generally the last condition on f^1 can not be satisfied on the whole interval J_U if equations (3.3) hold (see below the comment after (3.5)). If this happens to be the case, the interval J_U must be replaced by some open subinterval $J'_U \subset J_U$ containing the initial point s_0, $J'_U \ni s_0$.

[8] Such coordinates $\{y^i\}$ always exist – see [76, Lemma 4.1] or Lemma 5.2 on page 116 for $\dim_{\mathbb{R}} N = 1$.

for $s \neq s'$, $s, s' \in J_1$ and $V_s \cap \gamma(J_1) = \gamma(s)$ for $s \in J_1$. The sets V_s are $(\dim_{\mathbb{R}} M - 1)$-dimensional submanifolds of M. At last, in each V_s we take a normal neighborhood V_s^0 of $\gamma(s) \in V_s$ and put $U_1^0 := \bigcup_{s \in j_1} V_s^0$. It is evident that U_1^0 is a neighborhood of $\gamma(J_1)$ and for every $p \in U_1^0$ there exists a unique $s(p) \in J_1$ such that there is unique geodesic $\beta_{s(p)}$ joining p and $\gamma(s(p))$ in $V_{s(p)}^0$. Thus, generally, we are forced to consider instead of U and J smaller sets $U_1^0 \subseteq U$ and $J_1 \subseteq J$, respectively, on which the foliation mentioned exists. Hence, to be quite precise, below we have to replace U with U_1^0 and J with J_1 where U_1^0 and J_1 are defined above. Since these 'details' do not change the main ideas and formulae, we leave them and follow [19] directly.

As $\beta_s(t_0) = \gamma(s)$, now the equation of γ is $x'^i(\gamma(s)) = f^i(g(\xi))$, $i \geq 2$, where $g := \left(f^1\right)^{-1}$ and $\xi := x'^1(\beta_s(t)) = x'^1(\gamma(s))$. Hence along γ the A's and Γ's may be considered as functions of $\xi = f^1(s)$ as $s = (f^1)^{-1}(\xi) =: g(\xi)$. Consequently, using (3.3), we transform (3.2) into

$$\begin{aligned} \beta_s^i(t) = \gamma^i(g(\xi)) + \sum_{j=2}^{\dim_{\mathbb{R}} M} A_j^i\big(\gamma(g(\xi))\big)\big[x'^j(\beta_s(t)) - f^j(g(\xi))\big] \\ - \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=2}^{\dim_{\mathbb{R}} M} A_{i_1}^{j_1}\big(\gamma(g(\xi))\big) \cdots A_{i_n}^{j_n}\big(\gamma(g(\xi))\big) \\ \times \Gamma^i{}_{j_1 \dots j_n}(\gamma(g(\xi)))\big[x'^{i_1}(\beta_s(t)) - f^{i_1}(g(\xi))\big] \cdots \big[x'^{i_n}(\beta_s(t)) - f^{i_n}(g(\xi))\big]. \end{aligned} \tag{3.4}$$

We will find the functions f^i from the condition $E_i'|_p = \frac{\partial}{\partial x'^i}\big|_p$ for $p \in U \cap \gamma(J)$. In this case $\frac{\partial}{\partial x'^i}\big|_p = A_i^j(p) \frac{\partial}{\partial x^j}\big|_p$, so that $A_i^j(p) = \frac{\partial x^j}{\partial x'^i}\big|_p$. Taking this into account, by differentiating (3.4) with respect to $\xi = x'^1$ and x'^i, $i \geq 2$, and restricting the results on $\gamma(J_U) = U \cap \gamma(J)$, we obtain:

$$A_1^i = \frac{\mathrm{d}f^i}{\mathrm{d}s}\left(\frac{\mathrm{d}f^1}{\mathrm{d}s}\right)^{-1} - \sum_{j \geq 2} A_j^i \frac{\mathrm{d}f^j}{\mathrm{d}s}\left(\frac{\mathrm{d}f^1}{\mathrm{d}s}\right)^{-1}, \qquad i = 1, \dots, \dim_{\mathbb{R}} M$$

and the identities $A_i^j = A_i^j$ for $i \geq 2$, $j \geq 1$. Hence the functions f^i must be chosen such that $\dot\gamma^i(s) = A_j^i(\gamma(s)) \frac{\mathrm{d}f^j(s)}{\mathrm{d}s}$, $i, j \geq 1$. Since in $\left\{E_i' = A_i^j \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x'^i}\right\}$ and $\left\{\frac{\partial}{\partial x^i}\right\}$ the tangent vector field $\dot\gamma$ has the expansions $\dot\gamma(s) = \dot\gamma'^i(s) E_i'|_{\gamma(s)} = \dot\gamma^i(s) \frac{\partial}{\partial x^i}|_{\gamma(s)} = A_j^i(\gamma(s)) \frac{\mathrm{d}f^j}{\mathrm{d}s}\big|_s \frac{\partial}{\partial x^i}|_{\gamma(s)}$, we find $\frac{\mathrm{d}f^i(s)}{\mathrm{d}s} = \dot\gamma'^i(s)$, i.e.,

$$f^i(s) = \int_{s_0}^{s} \dot\gamma'^i(\sigma)\,\mathrm{d}\sigma + c^i = \gamma'^i(s) - \gamma'^i(s_0) + c^i \tag{3.5}$$

with $c^i \in \mathbb{R}$ being constants. In particular, if γ is geodesic, then $\dot\gamma'^i(s) = \dot\gamma'^i(s_0) = \delta_0^1$ as now $E_1'|_{\gamma(s)} = \dot\gamma(s)$, so that $f^i(s) = \delta_1^i(s - s_0) + c^i$.

In (3.5) $f^1\colon J_U \to f^1(J_U) \subseteq \mathbb{R}$ is C^1 (and also C^∞) diffeomorphism but this is not the typical situation. Generally the invertability and differentiability of $f^1(s) = \gamma'^1(s) - \gamma'^1(s_0) + c^1$, $s \in J_U$ may be a problem. If we can find some $i_0 \in \{1, \dots, \dim_{\mathbb{R}} M\}$ for which $f^{i_0}\colon J_U \to f^{i_0}(J_U)$, given by (3.5), is C^1 diffeomorphism, we can simply renumber the coordinates and take f^{i_0} for f^1. But this is also an exception. In the most general case, we, on the base of the implicit function theorem,[9] can only assert that if $\gamma'^1 \neq 0$, $s \in J_U$, then there exists an open interval $J'_U \subseteq J_U$ such that the restriction $f^1|_{J'_U}\colon J'_U \to f^1(J'_U)$ is a C^1 diffeomorphism.[10] Thereof the special coordinates $\{x'^i\}$, given via (3.3), are defined only for $s \in J'_U$ with f^i given by (3.5).

Let us summarize the above discussion. We started with some $s_0 \in J$ and a chart (U, x) with $U \ni \gamma(s_0)$. Then a parallel frame $\{E'_i\}$ along γ was constructed, $\nabla_{\dot\gamma} E'_i = 0$, with $E'_1|_{\gamma(s_0)} = \dot\gamma(s_0)$, and coordinates $\{x'^i\}$ in U such that $E'_i|_p = \frac{\partial}{\partial x'^i}\big|_p$ for $p \in U \cap \gamma(J) = \gamma(J_U)$ and for which (3.3) holds with f^i given by (3.5).

Below we are going to prove that there exists a neighborhood of $\gamma(s_0)$ in which the coordinates $\{x'^i\}$ are normal along γ.

As the coordinate system $\{x^i\}$ is completely arbitrary, we can take $\{x'^i\}$ for it, $x^i = x'^i$. This results in $A(p) = [A^j_i(p)] = \mathbb{1}$, with $\mathbb{1} = [\delta^j_i]$ being the $\dim_{\mathbb{R}} M \times \dim_{\mathbb{R}} M$ identity (called also unit) matrix, and $\frac{\partial}{\partial x'^i} = \frac{\partial}{\partial x^i} = E'_i$. Inserting these equalities into (3.4) and omitting the primes, we get

$$\begin{aligned}\beta^i_s(t) = \gamma^i(g(\xi)) + \sum_{j=2}^{\dim_{\mathbb{R}} M} \delta^i_j \big[x^j(\beta_s(t)) - f^j(g(\xi))\big] \\ - \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=2}^{\dim_{\mathbb{R}} M} {\Gamma^i}_{i_1\dots i_n}(\gamma(g(\xi))) \\ \times \big[x^{i_1}(\beta_s(t)) - f^{i_1}(g(\xi))\big] \cdots \big[x^{i_n}(\beta_s(t)) - f^{i_n}(g(\xi))\big]. \quad (3.6)\end{aligned}$$

From the last equation, we get, by differentiation with respect to $\beta^i_s(t)$, $i \geq 2$, and restricting the result to γ (i.e., putting $t = t_0$), the equation

$$\sum_{k=2}^{\dim_{\mathbb{R}} M} {\Gamma^i}_{(jk)}(\gamma(s))[x^k(\gamma(s)) - f^k(g(\xi))] = 0$$

for $i \geq 1$, $j \geq 2$ and every $\gamma(s)$. Hence

$$ {\Gamma^i}_{(jk)}(\gamma(s)) = 0 \qquad \text{for } i \geq 1,\ j,k \geq 2, \text{ and } s \in J'_U. \quad (3.7)$$

Moreover, by construction

$$0 = \nabla_{\dot\gamma} E'_i|_{\gamma(s)} = \dot\gamma^j(s) \nabla_{E'_j} E'_i = \dot\gamma^j(s) {\Gamma^k}_{ij}(\gamma(s)) E'_k|_{\gamma(s)}$$

[9] For instance, see: [77, Chapter III, § 8], [7, Sections 1.37 and 1.38], [78, Chapter 10, Section 2], [79, Theorem 9.18]. Notice, at that precise place, the assumption that s_0 is not an end point of J, if any, is explicitly used. Also the assumption of regularity of γ at (every) s_0 is essential here.

[10] Here we suppose $\gamma'^1(s) \neq 0$ for all $s \in J'_U$. Below we point out that this may not be the case, but there always exist intervals $J_\delta \ni s_0$, $J_\delta \subseteq J_U$ such that $\gamma'^1|_{J_\delta} \neq 0$. One can define J'_U as the union of all such intervals, i.e., to take the maximal such interval for it.

or $\dot\gamma^j(s)\Gamma^k{}_{ij}(\gamma(s)) \equiv 0$, $s \in J$. Consequently, for *symmetric* connection ∇, we find

$$\dot\gamma^1(s)\Gamma^k{}_{11}(\gamma(s)) + \sum_{j=2}^{\dim_{\mathbb{R}} M} \dot\gamma^j(s)\Gamma^k{}_{1j}(\gamma(s)) = 0, \quad \dot\gamma^1(s)\Gamma^k{}_{1j}(\gamma(s)) = 0, \ j \geq 2 \quad (3.8)$$

where equation (3.7) was used, i.e., $\Gamma^k{}_{ij} = \Gamma^k{}_{(ij)} = 0$ for $i, j \geq 2$ and $k \geq 1$. From the last equations, we conclude that $\Gamma^k{}_{1j}(s) = 0$ for $j \geq 1$ if $\dot\gamma^1(s) \neq 0$. (If γ is a geodesic, then $E'_1|_{\gamma(s)} = \dot\gamma(s)$ and hence $\dot\gamma^1(s) = \dot\gamma'^1(s) = 1$ for all $s \in J$, but this is not the general case.) Now it is time to use the choice $E'_1|_{\gamma(s_0)} = \dot\gamma(s_0)$ again: from this it follows $\dot\gamma^1(s_0) = 1$ and $\dot\gamma^j(s_0) = 0$ for $j \geq 2$, as now $\frac{\partial}{\partial x^i}\big|_p = \frac{\partial}{\partial x'^i}\big|_p = E'_i|_p$,. Since $\gamma\colon J \to M$ is supposed to be of class C^1, the function $\dot\gamma^1\colon J \to \mathbb{R}$ is of class C^0, i.e., continuous. Thereof for every $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ there exists $\delta \in \mathbb{R}$, $\delta > 0$ (δ may depend on ε) such that $|\dot\gamma^1(s) - \dot\gamma^1(s_0)| < \varepsilon$ for $s \in J$ and $|s - s_0| < \delta$. So, in the interval $J_\delta := \{s \in J, \ |s - s_0| < \delta\} \subseteq J$ is fulfilled $1 - \varepsilon < \dot\gamma^1(s) < 1 + \varepsilon$ as $\dot\gamma^1(s_0) = 1$.

At the end, we choose some $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$ and fix corresponding $\delta \in \mathbb{R}$, $\delta > 0$. Then for $s \in J_0 := J'_U \cap J_\delta$, we have $0 < 1 - \varepsilon < \dot\gamma^1(s)$ and hence $\dot\gamma^1(s) \neq 0$. So equations (3.8) imply $\Gamma^k{}_{1j}(s) = 0$ for all $j \geq 1$. Combining this with (3.7) and taking into account that now the connection considered is symmetric, we finally obtain

$$\Gamma^k{}_{ij}(\gamma(s)) \equiv 0 \qquad \text{for } s \in J_0 := J'_U \cap J_\delta \quad (3.9)$$

or, equivalently,

$$\Gamma^k{}_{ij}\big|_{\gamma(J_0)} = 0, \qquad J_0 := J'_U \cap J_\delta \ni s_0. \quad (3.9')$$

The so-constructed coordinates $\{x'^i\}$ are called *Fermi coordinates* along γ (with a reference path γ). By definition, their domain is a subneighborhood $\overline{U} \subseteq U$ such that $\overline{U} \cap \gamma(J) \subseteq \gamma(J_0)$ as we supposed that s_0 is not a boundary point of J, if any.

Now a few words are in order for the case when the interval J is closed from one or both ends and s_0 is its end point. In such a case, the above investigation can be modified as follows. Let s_0 be the left (or, resp., right) end of J. Define $\dot\gamma(s_0) = \dot\gamma_+(s_0)$ (resp. $\dot\gamma(s_0) = \dot\gamma_-(s_0)$) via (cf. (I.2.3)) $(\dot\gamma_\pm(s_0))(f) := \lim_{\varepsilon\to\pm 0} \frac{1}{\varepsilon}[f(\gamma(s_0 + \varepsilon)) - f(\gamma(s_0))]$, $f \in \mathfrak{F}(\gamma(s_0))$ where $\varepsilon \to +0$ (resp. $\varepsilon \to -0$) means that ε tends to $0 \in \mathbb{R}$ with values greater (resp. less) than zero. Now take some interval $J' \supset J$ for which s_0 is inner, not boundary, point and certain regular C^1 path $\gamma'\colon J' \to M$ without self-intersections such that $\gamma'|_J = \gamma$ and $\dot\gamma'(s_0) = \dot\gamma(s_0)$.[11] All of the above results and conclusions are completely valid for γ' at $s = s_0 \in J'$. Consequently, applying them to γ' and restricting the results to $\gamma = \gamma'|_J$, we obtain their true versions for any point along γ, including the one(s) corresponding to the end(s) of

[11] The choice of J' and γ' can be made in infinitely many ways. The procedure just described is actually the definition of γ^i (or of γ) as C^1 mappings on a manifold with boundary, the interval J in the particular case, according to the definition in Footnote I.5 on page 7.

J. Of course, the construction of the Fermi coordinates at the end point(s) of J depends on the particular choice of $\gamma'\colon J' \to M$ but this is not essential as we are interested only in their existence, not in their concrete properties.

Generally the Fermi coordinates are local and are not defined along the whole path γ. Since the point s_0 is arbitrarily fixed in the above considerations, the main moral from them is:

Proposition 3.1. *Let $\gamma\colon J \to M$ be regular C^1 path without self-intersections in C^∞ manifold endowed with C^∞ linear connection. For every point $p \in \gamma(J)$ there is its neighborhood on which Fermi coordinates exist.*

Proposition 3.2. *In Fermi coordinates, the coefficients of a symmetric linear connection vanish along the part of their reference path lying in their domain.*

Corollary 3.1. *If $\gamma\colon J \to M$ is a regular C^1 path without self-intersections in C^∞ manifold endowed with C^∞ symmetric linear connection, then for every $s \in J$ exists a neighborhood U of $\gamma(s)$ and coordinates in it, in which the connection coefficients vanish on $\gamma(J) \cap U$.*

Remark 3.2. When proving the existence of the Fermi coordinates, we have used implicitly several times the absence of self-intersection points of γ. This is most evident in (3.3): if γ has self-intersection points, at them $x'^i(\beta_s))$ and $x^i(\beta_s))$ are not injective functions of s and, hence, f^1 can not be a diffeomorphism from J on $f^1(J)$. It is evident that the above considerations hold on any 'part' of γ without self-intersections, if any.

Remark 3.3. Our construction of the Fermi coordinates is essentially more precise and, correspondingly, longer than the one in [19, pp. 166–169]. In this reference a number of implicit conditions are presupposed. For example: γ is regular and without self-intersections, γ is contained in a single coordinate neighborhood (one and the same for $\{x^i\}$ and $\{x'^i\}$), $\dot\gamma^1(s) \neq 0$ for all $s \in J$, etc.

Remark 3.4. Often, especially in the physical literature, the Fermi coordinates are referred as ones in which the coefficients of a symmetric linear connection vanish along the *whole* path γ. This is true if γ is geodesic (or a path 'near' to geodesic), contained in a single coordinate neighborhood, but in the general case this is a wrong assertion. Generally along γ we can construct a family of Fermi coordinates such that $\gamma(J)$ is contained in the union of their domains and the connection coefficients in them vanish along the corresponding pieces of γ. Said in other words, as a rule there is not a single Fermi system of coordinates in whose domain $\gamma(J)$ is contained and in which the connection coefficients vanish on the whole set $\gamma(J)$.

Remark 3.5. If the connection is non-symmetric, in the constructed Fermi coordinates vanish the symmetric parts of some, but generally not all, of its coefficients – see (3.7) and (3.8). As the symmetrized connection coefficients are coefficients of a symmetric linear connection (see (I.3.9) and the comments before it), we can construct Fermi coordinates for this symmetric connection. In these particular coordinates, the symmetric parts of the initial connection coefficients vanish along their reference path.

For torsionless linear connection, in the Fermi coordinates, hold not only equations (3.7), but we also have (cf. (I.6.11))

$$\Gamma^i{}_{i_1\dots i_{n+1}}(p) = 0 \qquad n\in\mathbb{N},\ p\in\gamma(J'_U),\ i\geq 1,\ i_1,\dots,i_{n+1}\geq 2 \tag{3.10a}$$

$$\frac{\partial}{\partial x^{(i_1}}\Gamma^i{}_{i_2\dots i_{n+2})}(p) = 0 \qquad n\in\mathbb{N},\ p\in\gamma(J'_U),\ i\geq 1,\ i_1,\dots,i_{n+2}\geq 2 \tag{3.10b}$$

which equalities are obtained from (3.6) by repeated differentiation.

From the construction of the Fermi coordinates, a conclusion can be made that these coordinates are not the only ones (locally) normal along pieces of a given path. For instance, one can construct some analogue of the geodesic normal coordinates at a given point (see page 48) by replacing in (3.1) and in the next formulae $\Gamma^i{}_{i_1,\dots,i_n}$ by some constants $c^i{}_{i_1,\dots,i_n}$. We are not going to describe here such modifications because further a complete description of the coordinates normal along a path will be presented.

On the contrary to the Fermi coordinates along γ, which are essentially local, one can construct global frames along γ which are normal along the *whole* path γ. Actually, let $\gamma\colon J\to M$, J being open $\mathbb{R}$-interval, be regular C^1 path without self-intersections and $\{J_0^\alpha|\alpha\in A\}$, A being a non-empty set, be an open cover of J such that for every $p\in\gamma(J)$ the set $A_p := \{\alpha|\alpha\in A,\ \gamma(J_0^\alpha)\ni p\}$ consists of one or two elements and for each $\alpha\in N$ exist Fermi coordinates $\{x^i_\alpha\}$ with domain U_α such that $U_\alpha\cap\gamma(J)=\gamma(J_0^\alpha)$. Define along γ a frame $\{E_i\}$ such that, for every $p\in\gamma(J)$, $E_i|_p = \frac{\partial}{\partial x^i_\alpha}\big|_p$ if $\gamma(J_0^\alpha)\in p$ for a single $\alpha\in A$ and if the set A_p consists of two elements, we arbitrarily choose some $\beta\in A_p$ and set $E_i|_p = \frac{\partial}{\partial x^i_\beta}\big|_p$. The so-obtained frame $\{E_i\}$ is normal along γ but generally it is not smooth, even it may not be continuous, along the whole path γ. Hence the holonomicity problem for this frame is globally ill-posed.[12] All other frames normal along γ can be obtained from $\{E_i\}$ according to the recipe of Proposition I.5.2.

At the end, we want to mention one special type of Fermi coordinates in a neighborhood of (injective normal) geodesic in Riemannian manifold.

Example 3.1. Let $\gamma\colon J\to M$ be geodesic in a Riemannian manifold M with metric g. Let $\{E_i\}$ be a parallel frame along γ with $E_1=\dot\gamma$, i.e., $\nabla_{\dot\gamma}E_i=0$ and $E_1=\dot\gamma$; we can also say that $\{E_i|_{\gamma(s)}\}$, $s\in J$ is obtained from some fixed basis $\{e_i\}$ in $T_{\gamma(s_0)}(M)$, with $s_0\in J$ and $e_1=\dot\gamma(s_0)$, by a parallel transporting it from $\gamma(s_0)$ to the other points in $\gamma(J)$. For $(s,\boldsymbol{t}) = (s,t^2,\dots,t^{\dim_{\mathbb{R}}M})\in J\times\mathbb{R}^{\dim_{\mathbb{R}}M-1}$, we define $\imath\colon(s,\boldsymbol{t})\mapsto\sum_{i\geq 2}t^iE_i|_{\gamma(s)}\in T_{\gamma(s)}(M)$. If γ is injective, i.e., without self-intersections, and normal, i.e., $g(\dot\gamma,\dot\gamma)=1$, in [8, § 3.8 (ii)] it is proved that there

[12] With a little attention, one can see that locally $\{E_i\}$ is smooth and holonomic on the part of γ corresponding to a single interval J_0^α, but in the regions of intersections of such intervals the smoothness of the frame may be lost.

exists a neighborhood V_0 of the zero vector in $\mathbb{R}^{\dim_{\mathbb{R}} M-1}$ and a neighborhood U of $\gamma(J)$ in M such that the mapping $y := (\exp\circ\imath|_{J\times V_0})^{-1}\colon U \to J\times V_0 \subseteq \mathbb{R}^{\dim_{\mathbb{R}} M-1}$ is diffeomorphism. The associated to the chart (U, y) coordinates $\{y^i\}$ are called in [8, § 3.8 (ii)] Fermi coordinates. In the same reference it is proved that the coordinates $\{y^i\}$ so-defined are normal along γ, $\Gamma^i{}_{jk}\circ\gamma = 0$, and orthonormal, $g_{ij}(\gamma(s)) = \pm\delta_{ij}$, $s \in J$, at the same time. Thereof we conclude, the Fermi coordinates along injective (without self-intersections) normal geodesic are similar to the orthonormal Riemannian coordinates found by Birkhoff (see page 48), but when the latter are considered 'along paths'.

Exercise 3.1. Prove that the coordinates of Example 3.1 agree with our definition of 'Fermi coordinates'. (Hint: $\beta_s(t) = x^{-1}(s, \hat{\boldsymbol{t}})$ where one of the components of $\hat{\boldsymbol{t}} \in \mathbb{R}^{\dim_{\mathbb{R}} M-1}$ is equal to t, the others being constants.)

3.2. Complete description

As we said earlier, the second method for proving the existence of coordinates or frames normal along a path $\gamma\colon J \to M$ is via direct solving of the equation (I.5.4) along γ:

$$\big(\Gamma_k A + E_k(A)\big)\big|_{\gamma(J)} = 0 \tag{3.11}$$

or

$$\Gamma_k(\gamma(s))A(\gamma(s)) + \big(E_k(A)\big)\big|_{\gamma(s)} = 0. \tag{3.11$'$}$$

Let us recall the notation here. The coefficients of a linear connection ∇ are $\Gamma^i{}_{jk}$ and $\Gamma_k := \big[\Gamma^i{}_{jk}\big]_{i,j=1}^{\dim_{\mathbb{R}} M}$ are their matrices in an arbitrarily chosen frame $\{E_i\}$ in a neighborhood of $\gamma(J)$. We want to find a C^1 matrix-valued function $A := [A^j_i]$ such that the frame(s) $\{E'_i = A^j_i E_j\}$ is (are) normal along γ, i.e., on $\gamma(J)$. A necessary and sufficient condition for this is A to be a solution of (3.11).

Now we pose a problem: given a C^2 manifold M endowed with symmetric C^0 connection ∇ and a regular C^1 path $\gamma\colon J \to M$ without self-intersection, describe all frames and local coordinates normal respectively along γ and on (some part(s) of) $\gamma(J)$. Below we describe the general solution of this problem by adapting the method of [76, Section 3] to the particular situation just outlined; we also employ the initial idea underlying the construction of coordinates normal along a path in [25, pp. 428–431]. We shall see that the solution in terms of frames is easier and more simple than the one in terms of local coordinates.[13] This is not occasional: to a coordinate system $\{x^i\}$ corresponds the frame $\{\partial/\partial x^i\}$, but to obtain the local coordinates corresponding to a holonomic frame $\{A^j_i \partial/\partial x^j\}$, one has to solve a system of first-order partial differential equations. Said differently, the normal frames are obtained by solving a first-order system of partial differential equations

[13]The same situation will be met in Chapters III–V for much more general problems.

(see (I.5.4)) while the corresponding system for the normal coordinates is of second order (see (I.5.4′) on page 41).[14]

Below we are going to prove the existence of solutions of (3.11) and to find its general solution, which results in a complete description of all frames normal along a regular C^1 path. The main results are formulated below as Theorem 3.1 on page 97 to which theorem the reader can jump right now if he/she is not interested in the details of its proof.

To begin with, we shall construct local coordinates in a neighborhood of a point in $\gamma(J)$ such that a point in their domain has the path's parameter as its first coordinate.

Lemma 3.1. *Let $\gamma\colon J \to M$ be a regular C^1 injective path in a C^3 manifold M. For every $s_0 \in J$, there exists a chart (U_1, x) of M such that $\gamma(s_0) \in U_1$, $x\colon U_1 \to J_1 \times \mathbb{R}^{\dim_{\mathbb{R}} M-1}$ for some open subinterval $J_1 \subseteq J$, $s_0 \in J_1$ and $x(\gamma(s)) = (s, \boldsymbol{t}_0)$ for all $s \in J_1$ and some fixed $\boldsymbol{t}_0 \in \mathbb{R}^{\dim_{\mathbb{R}} M-1}$.*

Proof. Let $s_0 \in J$ be a point in J which is not an end point of J, if any, and (U, y) be a chart with $\gamma(s_0)$ in its domain, $U \ni \gamma(s_0)$, and $y\colon U \to \mathbb{R}^{\dim_{\mathbb{R}} M}$. From the regularity of γ, $\dot\gamma \neq 0$, follows that at least one of the numbers $\dot\gamma_y^1(s_0), \dots, \dot\gamma_y^{\dim_{\mathbb{R}} M}(s_0)$, where $\gamma_y^i := y^i \circ \gamma$, is non-zero. We, without lost of generality, choose this non-vanishing component to be $\dot\gamma_y^1(s_0)$.[15] Then, due to the continuity of $\dot\gamma$ (γ is of class C^1) and according to the implicit function Theorem [77, Chapter III, § 8], [7, Sections 1.37 and 1.38], [78, Chapter 10, Section 2], there exists an *open* subinterval $J_1 \subseteq J$ containing s_0, $J_1 \ni s_0$, and such that $\dot\gamma^1|_{J_1} \neq 0$ and the restricted mapping $\gamma_y^1|_{J_1}\colon J_1 \to \gamma_y^1(J_1)$ is a C^1 diffeomorphism on its image. Define a neighborhood

$$U_1 := \bigl\{p | p \in U,\ y^1(p) \in \gamma_y^1(J_1)\bigr\} = y^{-1}\bigl(\gamma_y^1(J_1) \times \mathbb{R}^{\dim_{\mathbb{R}} M-1}\bigr) \ni \gamma(s_0)$$

and a chart (U_1, x) with local coordinate functions

$$\begin{aligned} x^1 &:= \bigl(\gamma_y^1|_{J_1}\bigr)^{-1} \circ y^1 \\ x^k &:= y^k - \gamma_y^k \circ x^1 + t_0^k \qquad k = 2, \dots, \dim_{\mathbb{R}} M \end{aligned} \tag{3.12}$$

where $t_0^k \in \mathbb{R}$ are arbitrarily fixed constant numbers. Since $\frac{\partial x^1}{\partial y^j} = \frac{1}{\dot\gamma_y^1}\delta_j^1$, $\frac{\partial x^k}{\partial y^1} = -\frac{\dot\gamma_y^k}{\dot\gamma_y^1}$ for $k \geq 2$, and $\frac{\partial x^k}{\partial y^l} = \delta_l^k$ for $k, l \geq 2$, the Jacobian of the change $\{y^i\} \to \{x^i\}$ at $p \in U_1$ is $\frac{1}{\dot\gamma^1(p)} \neq 0, \infty$. Consequently $x\colon U_1 \to J_1 \times \mathbb{R}^{\dim_{\mathbb{R}} M-1}$ is really a coordinate homeomorphism with coordinate functions x^i.

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[14] This is realized implicitly in [25, p. 429] in the method developed in this work for finding a class of coordinates normal along a path.

[15] If it happens that $\dot\gamma_y^1(s_0) = 0$ and $\dot\gamma_y^{i_0}(s_0) \neq 0$ for some $i_0 \neq 1$, we have simply to renumber the local coordinates to get $\dot\gamma_y^1(s_0) \neq 0$. Practically this is a transition to new coordinates $\{y^i\} \to \{z^i\}$ with $z^1 = y^{i_0}$ and, for instance, $z^{i_0} = y^1$ and $z^i = y^i$ for $i \neq 1, i_0$, in which the first component of $\dot\gamma$ is non-zero. We suppose that, if required, this coordinate change is already done. If occasionally it happens that $\dot\gamma_y^{j_0}(s) \neq 0$ for all $s \in J$ and fixed j_0, it is extremely convenient to take this particular component of $\dot\gamma$ as $\dot\gamma_y^1$ – see the next sentence.

In the new chart (U_1, x), the coordinates of $\gamma(s)$, $s \in J_1$ are

$$\gamma^1(s) := (x^1 \circ \gamma)(s) = s, \quad \gamma^k(s) := (x^k \circ \gamma)(s) = t_0^k, \ k \geq 2, \tag{3.13}$$

i.e.,

$$x(\gamma(s)) = (s, \boldsymbol{t}_0)$$

for some $\boldsymbol{t}_0 = (t_0^2, \dots, t_0^{\dim_{\mathbb{R}} M}) \in \mathbb{R}^{\dim_{\mathbb{R}} M - 1}$ and all $s \in J_1$. □

Lemma 3.1 means that the chart (U_1, x) is so luckily chosen that the first coordinate in it of a point along γ coincides with the value of the corresponding path's parameter, the other coordinates being constant (along γ) numbers. Moreover, in U_1 the path γ can be considered as a representative of a family of paths $\eta(\,\cdot\,, \boldsymbol{t})\colon J_1 \to M$, $\boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M-1}$, defined by $\eta(s, \boldsymbol{t}) := x^{-1}(s, \boldsymbol{t})$ for $(s, \boldsymbol{t}) \in J_1 \times \mathbb{R}^{\dim_{\mathbb{R}} M-1}$; indeed, $\gamma = \eta(\,\cdot\,, \boldsymbol{t}_0)$ or $\gamma(s) = \eta(s, \boldsymbol{t}_0)$, $s \in J_1 \subseteq J$.

Remark 3.6. In this way we have obtained a natural foliation of U_1 along γ: putting $V_s := \{x^{-1}(s, \boldsymbol{t}) | \boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M-1}\}$, we see that $\bigcup_{s \in J_1} V_s = U_1$, $V_s \cap V_{s'} = \varnothing$ for $s \neq s'$, $s, s' \in J_1$, and $V_s \cap \gamma(J_1) = \gamma(s)$ for $s \in J_1$. Having this in mind, the main idea of the next considerations in this subsection is in each $(\dim_{\mathbb{R}} M - 1)$-dimensional submanifolds V_s (see p. 7) to be constructed frames or coordinates normal at the single point $\gamma(s)$ (as a point of V_s) and then to joint them smoothly along γ. We shall see in Section 5 below that this is possible due to the one-dimensionality of $\gamma(J_1)$ considered as a submanifold of M.

Remark 3.7. The vector field $\dot\gamma$ tangent to γ has in the coordinates $\{x^i\}$, associated with the chart (U_1, x) provided by Lemma 3.1, the components $(1, 0, \dots, 0)$. Consequently, if γ is a geodesic path (see Subsection I.3.4), the geodesic equation (I.3.23) reduces to

$$\Gamma^i{}_{11} \circ \gamma = 0. \tag{3.14}$$

Hence $\{x^i\}$ is generally not normal along a geodesic path, but along such a path the last equation holds.

Remark 3.8. We have met examples of concrete kinds of coordinates systems $\{x^i\}$ in Examples I.7.3, I.7.4 and I.7.5 in which they happen to be normal along the paths considered in them.

Now we are ready to solve equation (3.11) in the neighborhood U_1. Let us take an arbitrary point $p \in U_1$ with coordinates $(s, \boldsymbol{t}) \in J_1 \times \mathbb{R}^{\dim_{\mathbb{R}} M-1}$ in (U_1, x), i.e., $x(p) = (s, \boldsymbol{t})$ or $p = \eta(s, \boldsymbol{t}) = x^{-1}(s, \boldsymbol{t})$. The mapping $p \mapsto p_0 := \gamma(x^1(p)) \in \gamma(J_1)$ defines a useful for our purposes 'projection' of p on γ (or on $\gamma(J)$) along the family η; in coordinates, if $p = \eta(s, \boldsymbol{t})$, then $p_0 = \gamma(s) = \eta(s, \boldsymbol{t}_0)$, i.e., $x(p_0) = (s, \boldsymbol{t}_0)$ for $x(p) = (s, \boldsymbol{t})$.

Suppose A is of class C^3. Then [57, Section 4.10-5] there exist C^1 matrix-valued functions B_{kl} on U_1 such that they and their partial derivatives are bounded on $\gamma(J_1)$ and the following Taylor formula holds:

$$A(\eta(s,\boldsymbol{t})) = A(\eta(s,\boldsymbol{t}_0)) + \sum_{k=2}^{\dim_{\mathbb{R}} M} \frac{\partial A(\eta(s,\boldsymbol{t}))}{\partial t^k}\Big|_{\boldsymbol{t}=\boldsymbol{t}_0} (t^k - t_0^k) + \sum_{k,l=2}^{\dim_{\mathbb{R}} M} B_{kl}(\eta(s,\boldsymbol{t}))(t^k - t_0^k)(t^l - t_0^l). \quad (3.15)$$

In (3.11) we choose $E_j|_p = \frac{\partial}{\partial x^j}\big|_p$, $p \in U_1$, so that $E_1(A)|_{\gamma(s)} = \frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s}$ and $E_k(A)|_{\gamma(s)} = \frac{\partial A(\eta(s,\boldsymbol{t}))}{\partial t^k}\big|_{\boldsymbol{t}=\boldsymbol{t}_0}$, $k = \geq 2$ as $\gamma(s) = \eta(s,\boldsymbol{t}_0)$. Substituting (3.15) into (3.11′), we split (3.11′) into equivalent to it system of equations

$$\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s} = -\Gamma_1(\gamma(s))A(\gamma(s)), \quad (3.16a)$$

$$\frac{\partial A(\eta(s,\boldsymbol{t}))}{\partial t^k}\Big|_{\boldsymbol{t}=\boldsymbol{t}_0} = -\Gamma_k(\gamma(s))A(\gamma(s)), \qquad k \geq 2. \quad (3.16b)$$

Lemma 3.2. *Let J be $\mathbb{R}$-interval, $s_0 \in J$ be fixed, and Z be continuous, square matrix-valued, and non-zero function on J. On J exists a unique (square) non-degenerate matrix-valued function Y satisfying the initial-valued problem:*

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = ZY, \qquad Y|_{s=s_0} = \mathbb{1}, \quad Y = Y(s, s_0; Z) \quad (3.17)$$

where $s \in J$ and $\mathbb{1}$ is the identity (unit) matrix of the corresponding size.

Proof. See [34, Chapter IV, Section 1] or [82, pp.125–127].[16] □

Remark 3.9. This lemma and its multidimensional version (see Lemma 4.1) play a crucial role in our general approach for investigating normal frames. They are simple corollaries of the theorems of existence and uniqueness for normal systems of ordinary or partial differential equations of first order. By means of Y, we can express every solution of the matrix equation

$$\frac{\mathrm{d}X}{\mathrm{d}s} = ZX \quad (3.18)$$

with the C^1 matrix-valued function X being of the same size (order) as Z in (3.17). The general solution of (3.18) is

$$X = Y(s, s_0; Z) \quad (3.19)$$

with arbitrary $s_0 \in J$, or

$$X = Y(s, s_0; Z)X_0 \quad (3.20)$$

[16] The solution of (3.17) is called *matricant*.

with fixed $s_0 \in J$ and arbitrary X_0. The particular solution of (3.18) satisfying the initial condition

$$X|_{s=s_0} = X_0 \tag{3.21}$$

for some $s_0 \in J$ and X_0 is given by (3.20). At present, special properties of Y, such as $Y^{-1}(s, s_0; Z) = Y^{-1}(s_0, s; Z)$ or $Y(s, s_1; Z)Y(s_1, s_0; Z) = Y(s, s_0; Z)$, will not be needed.

Remark 3.10. If the order of the matrices in (3.17) is important, we write $\mathbb{1}_n$ for $\mathbb{1}$ with $\mathbb{1}_n$ being the $n \times n$, $n \in \mathbb{N}$, identity matrix; respective labels may be attached to Y and Z if required.

The general solution of equation (3.16a), which immediately follows from Lemma 3.2 and Remark 3.9, is

$$A(\gamma(s)) = Y(s, s_0; -\Gamma_1 \circ \gamma)B \tag{3.22}$$

where Y is defined by the initial-value problem (3.17), with $\mathbb{1} = \mathbb{1}_{\dim_{\mathbb{R}} M}$, and B is constant non-degenerate $\dim_{\mathbb{R}} M \times \dim_{\mathbb{R}} M$ matrix. Inserting (3.16b) in (3.15) and using (3.22), we find:

$$\begin{aligned} A(p) = \big\{\mathbb{1} - \Gamma_k(p_0)[x^k(p) - x^k(p_0)]\big\}Y(x^1(p), s_0; -\Gamma_1 \circ \gamma)B \\ + B_{kl}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] \end{aligned} \tag{3.23}$$

where we have used $p = \eta(s, \boldsymbol{t})$, $\eta = x^{-1}$, and $p_0 := \gamma(x^1(p)) = \gamma(s)$. Note, in (3.23) the terms corresponding to $k = 1$ and/or $l = 1$ vanish due to $x^1(p_0) = x^1(\gamma(x^1(p))) = (x^1 \circ \gamma)(x^1(p)) = x^1(p) = s$.[17]

In the above investigation we have supposed s_0 not to be an end-point of J, if any. This restriction is completely inessential for the final results, like (3.22) and (3.23). Actually, if J has end point(s), we, analogously to the scheme described on page 90, can proceed as follows. If s_0 is the left/right end of J, we define $\dot\gamma(s_0)$ such that $(\dot\gamma f)(s_0)$, $f \in \mathfrak{F}(\gamma(s_0))$ is the left/right derivative of $f \circ \gamma$ at $s = s_0$. Take an open interval $J' \supset J$ and let $\gamma' \colon J' \to M$ be such that $\gamma'|_J = \gamma$ and $\dot\gamma'|_J = \dot\gamma$. Applying all of the above considerations for γ' instead of γ and restricting the results to γ (or to $\gamma(J)$), we see that all of the above investigation remain valid with the only changes that now the interval J_1 (after the restriction) may turn to be closed from its left/right end and correspondingly U_1 may be neighborhood with boundary, i.e., a $(\dim_{\mathbb{R}} M)$-dimensional submanifold with boundary (see Remark I.2.1 and Footnote I.5 on page 7).

Let us explicitly formulate the results obtained.

Theorem 3.1. *Let $\gamma \colon J \to M$ be a C^1 regular path without self-intersections in C^3 manifold M endowed with (arbitrary) C^0 linear connection ∇. For every $s_0 \in J$, there exist a neighborhood U_1 of $\gamma(s_0)$ and a frame $\{E'_i\}$ on U_1 which is normal for*

[17] If we would have started with some $i_0 \neq 1$, for which $\dot\gamma^{i_0} \neq 0$, the result will be (3.22) and (3.23) with Γ_{i_0} instead of Γ_1. In the intermediary steps, e.g., starting from (3.12), the index 1 must be replace with i_0. For instance, p_0 will be defined by $p_0 := \gamma(x^{i_0}(p))$, so $x^{i_0}(p_0) = x^{i_0}(p)$ and the corresponding terms with $k = i_0$ and/or $l = i_0$ in (3.23) will vanish.

∇ *along* γ*, i.e., it is normal on* $U_1 \cap \gamma(J)$*. Moreover, in* U_1 *exist coordinates* $\{x^i\}$ *such that* $\gamma^1(s) = s$ *and* $\gamma^i(s) = \text{const}$*, for* $i \geq 2$ *and* $s \in J_1$*, with* $J_1 \subseteq J$ *being a subinterval of* J *and* $\gamma(J_1) \subset U_1$*. All such normal frames are* $\{E'_i\big|_p = A^j_i(p)\frac{\partial}{\partial x^j}\big|_p\}$*,* $p \in U_1$*, where the* C^3 *non-degenerate matrix-valued function* $A = [A^j_i]$ *is given via* (3.23)*.*

Remark 3.11. This result is considerably more general than we expected at the beginning: it concerns *all* linear connections, with or without torsion.

Proof. The first part of the theorem follows from the above-constructed solutions (3.22) for every $s_0 \in J$ and *arbitrary*, symmetric or not, connection ∇, as we did not use any assumption on the symmetry of its coefficients. The second assertion is a consequence of the fact that (3.22), with arbitrary B, is the general solution of (3.11) in the coordinates $\{x^i\}$ in U_1 defined by (3.12). □

Remark 3.12. If we replace the C^3 differentiability of A with C^1 differentiability, then (3.23) will be a solution of (3.11) but we cannot assert that it will be the general one. So, if M is of class C^2 and A is of class C^1, then Theorem 3.1 remains valid with an exception that there may exist matrices of class C^1 but not C^3 that are solutions of (3.11) and cannot be obtained by (3.23) for suitable choice of B and B_{kl}.

Said in a more free language, Theorem 3.1 asserts that along a path without self-intersections there exist frames locally normal for any linear connection. The assumption for the absence of points of self-intersections is important at the moment: at such points, if any, the coordinates $\{x^i\}$ given via (3.12) do not exist and, correspondingly, our proof fails. It is easily seen, this is a general feature, not one characterizing our particular proof: since the mapping $\gamma\colon J \to M$ is at least two-to-one at the points of self-intersections, there do not exist normal coordinates in their neighborhoods along γ. But, if we restrict γ to any subinterval of J on which it is injective, Theorem 3.1 is completely valid to the obtained path.

The constancy of B in (3.22) agrees with Proposition I.5.2. For, e.g., $B = \mathbb{1}$, we obtain a particular frame locally normal along γ, say $\{E_i\}$. Any other normal frame, say $\{E'_i\}$, is given by $E'_i = A^j_i E_j$ with $(E_i A)|_{\gamma(J)} = 0$, $A = [A^j_i]$, according to Proposition I.5.2. Combining this with (3.22) for some B, we get $A|_{\gamma(J)} = B$.

Let us turn now our attention to (3.23). Its meaning is that in U_1 we have constructed a frame $\{E'_i|_p = A^j_i(p)E\frac{\partial}{\partial x^j}\big|_p\}$ which is normal along γ, i.e., on $\gamma(J) \cap U_1$. By virtue of Proposition I.5.2, every frame on U_1 which is normal along γ is of the form $\{C^j_i E'_j\}$ where $C = [C^j_i]$ is C^1 non-degenerate matrix valued function on U_1 such that $E'_i(C)|_{\gamma(J)} = 0$. The holonomicity of these frames on $U_1 \backslash (\gamma(J) \cap U_1)$ may be completely arbitrary (see Remark I.5.1 on page 40), even it may change from point to point, but on $\gamma(J) \cap U_1$ part of them are holonomic if the connection is torsion free on this set (Corollary I.5.3);[18] one can check this directly, if required, by using (3.23).

[18]As $\gamma(J)$ is 1-dimensional submanifold of M, the condition in Corollary I.5.3 is identically satisfied.

Proposition 3.3. *Let $\gamma\colon J \to M$ be C^1 regular injective path in a C^2 manifold M endowed with C^1 linear connection ∇. There exist frames normal along the whole path γ for ∇.*

Proof. If it happens that $\gamma(J)$ lies entirely in U_1, the frame normal along γ constructed above is global, i.e., defined on the whole set $\gamma(J)$. But this may not be the case. Generally, we can only build an open cover $\{J_1^\alpha | \alpha \in A\}$ of J and neighborhoods $U_1^\alpha \supset \gamma(J_1^\alpha)$ on which frames $\{E_i^\alpha\}$ normal along γ exist. From these frames can be constructed global frames $\{E_i\}$, normal along γ, as follows. If for $s \in J$ exists a unique $\alpha \in A$ with $J_1^\alpha \ni s$, we put $E_i|_{\gamma(s)} = E_i^\alpha|_{\gamma(s)}$; if the set $P_s := \{\alpha | \alpha \in A,\ \gamma(s) \in \gamma(J_1^\alpha)\}$ consists of more than one element, we choose some $\beta \in P_s$ and put $E_i|_{\gamma(s)} = E_i^\beta|_{\gamma(s)}$. □

Generally the frames $\{E_i\}$ will not be differentiable, or even not continuous, in the regions of intersection of two or more neighborhoods U_1^α. Since on every U_1^α Theorem 3.1 gives a complete description of the frames (locally) normal along γ, we can combine this with the just described construction of a frame along the whole path γ to obtain a *full description of all frames globally normal along γ.*

At that moment, without any effort, we can make one generalization by partially omitting the assumption of the lack of self-intersection points of γ.

Theorem 3.2. *Let $\gamma\colon J \to M$ be locally injective C^1 regular path in a C^2 manifold M with C^0 linear connection ∇. There exists (global, on $\gamma(J)$) frames normal along the whole path γ for ∇. Locally these frames are of class C^1 but globally they need not to be such. The frames are (locally) holonomic iff ∇ is torsionless on $\gamma(J)$.*

Proof. Let $\gamma\colon J \to M$ be *locally injective path*, i.e., for every $s \in J$ there is a subinterval $J' \subseteq J$ such that the restricted path $\gamma|_{J'}\colon J' \to M$ is injective (without self-intersections). Let $J_I := \{s | s \in J,\ \gamma(s) \text{ is self-intersection point}\} \subset J$. From our assumption follows that J_I has no condensation points. Hence, if $J_I \neq \varnothing$, for every $s \in J$ there are three possibilities: (i) there exist $s', s'' \in J_I$, $s' < s''$ such that $s' < s \leq s''$; (ii) there is $s' \in J_I$ such that $s \leq s'$ and if $s'' \in J_I$, then $s'' \geq s'$; (iii) there is $s'' \in J_I$ such that $s > s''$ and if $s' \in J_I$, then $s' \leq s''$. Therefore, in any case, for $s \in J$ there exists a maximal (in J) interval $J_s \subseteq J$ containing s and such that $J_s \cap J_I$ is the empty set or consists of only one element; in the above notation, we, respectively, have $J_s = \{s_1 | s_1 \in J,\ s' < s_1 \leq s''\}$, $J_s = \{s_2 | s_2 \in J,\ s_2 \leq s'\}$, $J_s = \{s_3 | s_3 \in J,\ s_3 > s''\}$. Evidently, $J_s = J_t$ iff $s \in J_t$ or $t \in J_s$. Therefore J can be represented as $J = \bigcup_{\lambda \in \Lambda} J^\lambda$ where $\Lambda \neq \varnothing$ for $J_I \neq \varnothing$ and the intervals J^λ are such that $J^\lambda \cap J_I$ is the empty set or contains only one element and if $s \in J^\lambda$, then $J^\lambda = J_s$. Consequently, the above construction of a global frame normal along a path without self-intersections is applicable to every restricted path $\gamma|_{J^\lambda}$, viz., for every $\lambda \in \Lambda$, along $\gamma|_{J^\lambda}$ there is a (global) normal frame $\{E_i^\lambda\}$. At the end, for every $p \in \gamma(J)$ we define a basis $\{E_i|_p\}$ such that $E_i|_p = E_i^\lambda|_p$ if $p \in \gamma(J^\lambda)$ for some (unique) $\lambda \in \Lambda$. The bases $\{E_i|_p | p \in \gamma(J)\}$ form a frame along the whole path γ which is normal by construction. Obviously, any global frame normal along the whole path γ can be obtained in this way. □

Notice, all frames normal along γ can be constructed in the pointed way or from a single such frame according to the rule reflected in Proposition I.5.2.

Now we turn our attention to the coordinates normal along a given path which, as we know from Proposition I.5.4, could exist only for connections that are torsion free along γ. Our main result is formulated below as Theorem 3.3 on page 103 and the reader who is not interested in its proof and the construction of coordinates (locally) normal along γ can skip the text till it.

Let M be a C^3 manifold endowed with *symmetric* linear connection and $\gamma\colon J \to M$ be a regular C^1 path without self-intersections in it. The problem is to be found coordinates $\{z^i\}$ (locally) normal along γ. Such coordinates exist due to the symmetry of the connection and Corollary 3.1.

We start with the already employed chart (U_1, x) with coordinate functions x^i given by (3.12). Let (U_z, z) be a chart with $U_z \cap U_1 \supset \gamma(J_1)$. Since the transition functions $z^{-1} \circ x$ and $x^{-1} \circ z$ on $U_z \cap U_1$ are of class C^3, for $p \in U_z \cap U_1$, we can write the expansion (cf. (2.7))

$$z^i(p) = z^i(p_0) + a^i_j(p_0)[x^j(p) - x^j(p_0)] + a^i_{jk}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)]. \tag{3.24}$$

Here: $p = \eta(s, \boldsymbol{t})$, $p_0 = \gamma(x^1(p)) = \eta(s, \boldsymbol{t}_0) = \gamma(s)$, $a^i_j\colon \gamma(J_1) \to \mathbb{R}$ are C^1 functions, and $a^i_{jk}\colon U_z \cap U_1 \to \mathbb{R}$ are of class C^1 and together with their partial derivatives they are bounded. Now we invert the situation: we shall try to find a^i_j and a^i_{jk} such that the functions z^i given by (3.24) generate a normal frame $\{E'_i\}$,

$$\frac{\partial}{\partial z^i}\Big|_{\gamma(s)} = E'_i|_{\gamma(s)} := A^j_i(\gamma(s)) \frac{\partial}{\partial x^j}\Big|_{\gamma(s)}$$

with A given by (3.22).[19] So, $\frac{\partial z^i}{\partial x^j}\big|_{p_0} = \big(A^{-1}(p_0)\big)^i_j$ as, for $p = x^{-1}(s, \boldsymbol{t})$, we have $p_0 = \gamma(s)$. Inserting here (3.24), we get

$$\frac{\partial z^i(p_0)}{\partial x^1} = \frac{\partial z^i}{\partial x^1}\Big|_{p_0} = \frac{\partial z^i}{\partial x^1}\Big|_{\gamma(s)} = \big(A^{-1}(p_0)\big)^i_1 = \big(A^{-1}(\gamma(s))\big)^i_1$$

$$\iff z^i(p_0) = \int_{s_0}^{x^1(p_0)} \big(A^{-1}(\gamma(\sigma))\big)^i_1 \,\mathrm{d}\sigma + a^i$$

$$a^i_j(p_0) = \frac{\partial z^i(p)}{\partial x^j}\Big|_{p_0} = \frac{\partial z^i}{\partial x^j}\Big|_{\gamma(s)} = \frac{\partial z^i}{\partial t^j}\Big|_{\boldsymbol{t}=\boldsymbol{t}_0} = \big(A^{-1}(p_0)\big)^i_j, \qquad \text{for } j \geq 2$$

[19] The idea is to restrict a frame normal along γ *to* $\gamma(J)$ and then to extend the resulting frame, defined solely on $\gamma(J)$, in a neighborhood of $\gamma(J)$ in a holonomic way. (For a general result regarding such extensions – see Lemma III.10.1 on page 194.)

where $a^i \in \mathbb{R}$ are constants representing the coordinates of $\gamma(s_0)$ in $\{z^i\}$. Therefore, by virtue of (3.22) and (3.24), the looked for functions z^i have the form

$$
\begin{aligned}
z^i(p) = a^i + \int_{s_0}^{x^1(p)} \big(A^{-1}(\gamma(\sigma))\big)^i_1 \,\mathrm{d}\sigma + \big(A^{-1}(\gamma(x^1(p)))\big)^i_j [x^j(p) - x^j(p_0)] \\
+ a^i_{jk}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)], \quad A(\gamma(s)) = Y(s, s_0; -\Gamma_1 \circ \gamma)B,
\end{aligned}
\tag{3.25}
$$

where $p_0 = \gamma(x^1(p))$ and $a^i_{jk} \colon U_1 \to \mathbb{R}$ are C^1 functions which together with their partial derivatives are bounded on U_1. Notice, in (3.25) the terms with $j, k = 1$ vanish as $x^1(p) = x^1(p_0) = s$. Since the Jacobian of the change $\{x^i\} \to \{z^i\}$ along γ is $\det\big[\frac{\partial z^i}{\partial x^j}|_{\gamma(s)}\big] = \big(\det\big[\frac{\partial x^i}{\partial z^j}|_{\gamma(s)}\big]\big)^{-1} = \det\big(Y(s, s_0; -\Gamma_1 \circ \gamma)B\big)^{-1} \neq 0, \infty$ for every $s \in J_1$, there is a neighborhood $U_z \subseteq U_1$ of $\gamma(J)$, $U_z \supset \gamma(J)$, such that (U_z, z) is a chart of M with coordinate functions z^i.

At the moment we can not assert that the chart (U_z, z) is normal along γ: we proved that $\frac{\partial}{\partial z^i}\big|_{\gamma(J_1)} = E'_i|_{\gamma(J_1)}$ where $\{E'_i\}$ is normal on $\gamma(J_1)$ from where it generally does not follow that $\{\frac{\partial}{\partial z^i}\}$ is normal on $\gamma(J_1)$, but from here the conclusion can be made that every coordinate system $\{z^i\}$ normal along γ has the form (3.25). We shall try to choose the functions $a^i_{jk} \colon U_z \to \mathbb{R}$ so that to make $\{z^i\}$ normal coordinates system. For this purpose, the equation (I.5.4′) on page 41 with U_z for $U \cap V$ must be satisfied:

$$
\left(\frac{\partial^2 x^i}{\partial z^j \partial z^k} + \frac{\partial x^r}{\partial z^j} \frac{\partial x^n}{\partial z^k} \Gamma^i{}_{rn} \right)\bigg|_{\gamma(s)} = 0 \qquad s \in J_1. \tag{3.26}
$$

Multiplying this equality with $\frac{\partial z^j}{\partial x^l} \frac{\partial z^k}{\partial x^m}\big|_{\gamma(s)}$, summing over j and k, and using

$$
\frac{\partial^2 x^i}{\partial z^j z^k} \frac{\partial z^j}{\partial x^l} \frac{\partial z^k}{\partial x^m} + \frac{\partial x^i}{\partial z^j} \frac{\partial^2 z^j}{\partial x^l x^m} = 0,
$$

which can be obtained from $\frac{\partial x^i}{\partial z^k} \frac{\partial z^k}{\partial x^m} = \delta^i_m$ by differentiation with respect to x^l, we, after some simple algebra, find an equivalent to (3.26) equation

$$
\frac{\partial z^i}{\partial x^l}\bigg|_{\gamma(s)} \Gamma^l{}_{jk}(\gamma(s)) = \frac{\partial^2 z^i}{\partial x^j x^k}\bigg|_{\gamma(s)}, \qquad s \in J_1 \tag{3.26′}
$$

since $\frac{\partial^2 z^i}{\partial x^j x^k}$ is symmetric in j and k (the manifold is of class C^3). From here an immediate observation follows: *normal along γ coordinates* (*may*) *exist only for symmetric* (*torsionless*) *along γ connections.* This is in a full agreement with Corollary I.5.3. Correspondingly, below we deal with the symmetric case as until now the symmetry of the connection was not used.

From (3.25), the derivatives in (3.26′) can easily be calculated:

$$\frac{\partial z^i}{\partial x^l}\Big|_{\gamma(s)} = \big(A^{-1}(\gamma(s))\big)^i_l,$$
$$\frac{\partial^2 z^i}{\partial x^1 x^k}\Big|_{\gamma(s)} = \left(\frac{\mathrm{d}A^{-1}(\gamma(s))}{\mathrm{d}s}\right)^i_k, \qquad \frac{\partial^2 z^i}{\partial x^m x^n}\Big|_{\gamma(s)} = 2a^i_{(mn)}(\gamma(s)), \quad m,n \ge 2$$

with A given by (3.22). Since $\frac{\mathrm{d}}{\mathrm{d}s}A^{-1} = -A^{-1}\frac{\mathrm{d}A}{\mathrm{d}s}A^{-1} = A^{-1}\Gamma_1$ on $\gamma(J)$, where equation (3.16a) was used, the substitution of the above equalities into (3.26′) results in

$$\big(A^{-1}(\gamma(s))\big)^i_l \Gamma^l{}_{1m}(\gamma(s)) = \big(A^{-1}(\gamma(s))\Gamma_1(\gamma(s))\big)^i_m$$
$$\big(A^{-1}(\gamma(s))\big)^i_l \Gamma^l{}_{jk}(\gamma(s)) = 2a^i_{(jk)}(\gamma(s)), \qquad j,k \ge 2.$$

As $(\Gamma_1)^n_m := \Gamma^n{}_{m1} = \Gamma^n{}_{1m}$, the first of these equalities is an identity. This reflects the vanishment of the terms with $j,k = 1$ in (3.25). The second equation expresses $a^i_{(jk)}$ along γ through known quantities. For $p \in U_1 \backslash \gamma(J_1)$, the values of $a^i_{(jk)}(p)$ can not be determined from (3.26), so they are left completely arbitrary from the requirement $\{z^i\}$ to be normal along γ. Since we supposed M to be of class C^3, there are C^0 bounded functions $a^i_{jkl}(p)\colon U_z \to \mathbb{R}$ such that

$$a^i_{jk}(p) = a^i_{jk}(p_0) + a^i_{jkl}(p)[x^l(p) - x^l(p_0)]$$

where $p = x^{-1}(s,\boldsymbol{t}) \in U_z \subseteq U_1$ and $p_0 = \gamma(s) = x^{-1}(s,\boldsymbol{t}_0)$. Inserting this expansion in (3.25) and using the above ones for $a^i_{(jk)}$, $j,k \ge 2$, we finally get:

$$\begin{aligned} z^i(p) = a^i + \int_{s_0}^{x^1(p)} \big(A^{-1}(\gamma(\sigma))\big)^i_1 \,\mathrm{d}\sigma + \big(A^{-1}(\gamma(x^1(p)))\big)^i_j [x^j(p) - x^j(p_0)] \\ + \big(A^{-1}(\gamma(x^1(p)))\big)^i_l \Gamma^l{}_{jk}(\gamma(x^1(p)))[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)] \\ + a^i_{jkl}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} a^i \in \mathbb{R}, \quad A(\gamma(s)) = Y(s,s_0;-\Gamma_1\circ\gamma)B,\ s \in J_1, \\ p = x^{-1}(s,\boldsymbol{t}) \in U_1, \quad p_0 = x^{-1}(s,\boldsymbol{t}_0) = \gamma(s), \end{aligned} \tag{3.28}$$

B is constant non-degenerate matrix, and Y and x are defined via (3.17) and (3.12), respectively. Recall, the coordinates $\{x^i\}$ have been defined via (3.12).

In this way, after long and tedious computation, we have almost proved the following theorem.

Theorem 3.3. *Let $\gamma\colon J \to M$ be a regular C^1 path without self-intersections in a C^3 manifold endowed with C^0 symmetric linear connection. For every $s_0 \in J$, there exist an interval $J_1 \subseteq J$ containing s_0 and chart (U_z, z) such that $\gamma(J_1) \subset U_z$ and the associated with it local coordinates $\{z^i\}$ are normal along γ, or, more precisely, on $\gamma(J_1)$. All such local coordinates, which are normal along γ in U_z, are given via* (3.27).

Remark 3.13. The theorem remains true if γ is fixed and the connection is symmetric only on $\gamma(J)$, i.e., for a fixed path the symmetry on the whole manifold is not necessary, it is required only along γ.

Proof. Since in the derivation of (3.27) the point $s_0 \in J$ was completely arbitrary, the theorem is proved when J is an open $\mathbb{R}$-interval. If J is closed from one or both ends, one should proceed exactly as was pointed before the formulation of Theorem 3.1 (see page 97). Take some regular C^1 path $\gamma'\colon J' \to M$ without self-intersections such that $J' \supset J$ is open, $\gamma'|_J = \gamma$ and the tangent vectors to γ' and γ coincide on J (at the end point(s) $\dot\gamma$ is defined as one-sided derivative). Since for γ' the assertions of the theorem are valid, the restriction of them to γ, i.e., to $J \subset J'$, completes the proof. □

Remark 3.14. The assumption of absence of self-intersection points is important. If such points exist, at them the theorem fails as the mapping $\gamma\colon s \mapsto \gamma(s)$ is at least two-to-one at them and in no way the parameter $s \in J$ can be taken as one of the local coordinates in their neighborhood. Generally, normal frames can be introduced on any piece of γ without self-intersections, but at these points, if any, the normal coordinates can not be joint smoothly.

Remark 3.15. If we have started with some fixed $i_0 \neq 1$ for which $\dot\gamma_y^{i_0}(s_0) \neq 0$ instead of our choice $\dot\gamma_y^1(s_0) \neq 0$, the result will be given again via (3.27) and (3.28) in which the index 1 (not in the power -1 of A^{-1}!) must be replaced with i_0. Correspondingly, the terms with $j, k = i_0$ will vanish.

Corollary 3.2. *Let $\gamma\colon J \to M$ be regular C^1 path without self-intersections in a C^3 manifold endowed with C^0 symmetric linear connection. Let $\{E_i\}$ be a frame on U, with $U \cap \gamma(J) \neq \varnothing$, normal along $\gamma|_{J_1}$ for some subinterval $J_1 \subseteq J$. Then all frames $\{E'_i = A_i^j E_j\}$ normal along $\gamma|_{J_1}$ are such that*

$$A(p) = B + B_{kl}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] \tag{3.29}$$

for a constant matrix B and C^1 matrix-valued functions B_{kl} which together with their partial derivatives are bounded on U. Besides, the normal holonomic frames $\{E'_i\}$ are such that $E'_i = \frac{\partial}{\partial z^i}$, where the coordinates z^i are normal along $\gamma|_{J_1}$ and

$$\begin{aligned} z^i(p) = a^i + \delta_1^i[x^1(p) - s_0] + [x^i(p) - x^i(p_0)] \\ + a_{jkl}^i(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)]. \end{aligned} \tag{3.30}$$

Proof. Apply Theorems 3.1 and 3.3 for ${\Gamma^i}_{jk} \circ \gamma|_{J_1} = 0$ (and hence $Y(s, s_0; -\Gamma_1) = \mathbb{1}$) and $A \circ \gamma|_{J_1} = \mathbb{1}$ as $\{x^i\}$ and $\left\{E_i = \frac{\partial}{\partial x^i}\right\}$ are normal on $\gamma(J_1)$. □

Exercise 3.2. Prove Corollary 3.2 by applying only Theorem 3.1. Hint: use (3.23) and follow the scheme outlined in Remark 2.1.

Example 3.2. We want to emphasize once again that the coordinates normal along a path are local by their essence. Globally, i.e., on the whole path, they exist only as an exception. Such a rare case is when the path γ is a geodesic lying entirely in a normal neighborhood in which Riemannian coordinates exist and their origin lies on the given geodesic path. In this case, taking the Riemannian coordinates for $\{y^i\}$, the equation of γ is (see Section I.6) $\gamma^i(s) = (y^i \circ \gamma)(s) = X^i(s - s_0)$ with $X = \dot{\gamma}(s_0) \neq 0$. Letting $X^1 \neq 0$, we get $\dot{\gamma}^1(s) \neq 0$, $s \in J$ and $J_1 = J$ which results in $\gamma(J) \in U_1 = U$, and, at the end, that $\{z^i\}$ are defined in a neighborhood of $\gamma(J)$, i.e., $\{z^i\}$ are globally normal along γ.

Example 3.3. Now we can look back on the Fermi coordinates constructed in the previous subsection. At first, choose the initial coordinates $\{y^i\}$ as pointed in Remark 3.1 on page 87. Then, comparing (3.2) with (3.24) or (3.4) with (3.27), we see that these particular coordinates correspond along a given path to the choice $\beta_s(t) = \eta(s, \hat{\boldsymbol{t}}) = \exp_{\gamma(s),X}(t - t_0)$, where one of the components of $\hat{\boldsymbol{t}}$ is equal to $t \in J_X$, the others being constants, i.e., the paths $\beta_s \colon J_X \to M$ are geodesics. Besides, in the case of Fermi coordinates, by virtue of the assumption that all structures are of class C^∞, the appearing in (3.27) functions are expanded into power series with respect to $x^k(p) - x^k(p_0) = t^k - t_0^k = (t - t_0)\delta^k_{i_0}$ for some fixed i_0. (Note, the matrix A has completely different meanings in (3.27) and in (3.2) or (3.4)!)

4. The case in a neighborhood

The flat linear connections are usually associated with the path-independence of the parallel transport assigned to them (see p. 29 and especially Remark I.3.5) rather than with the normal coordinates or frames [8, § 2.6 (iii)], [11, Chapter II, § 9 and Chapter V, § 4], [1, Section 6.10].[1] Below we shall see that these are different sides of one and the same problem.

Theorem 4.1. *Let M be a C^3 manifold endowed with C^1 linear connection ∇ and U be $(\dim M)$-dimensional submanifold of M. Frames normal for ∇ on U exist if and only if ∇ is flat on U, $R|_U = 0$.*

Remark 4.1. Notice, the theorem covers equally well the cases without and with torsion.

Remark 4.2. In particular, the submanifold U can be a neighborhood, possibly coordinate one, or the whole manifold M.

[1] For the special case of Riemannian manifolds, see [12, p. 286 and p. 303] where, equivalently, a manifold is called (locally) flat if it is (locally) isometric to $\mathbb{R}^n$ with a metric $a, b \mapsto \sum_i \varepsilon_i a^i b^i$ for $a, b \in \mathbb{R}^n$ and $\varepsilon_i = \pm 1$.

Proof. Let $\{E_i\}$ be a frame on U. To save some writing, we introduce the matrices Γ_k and $\boldsymbol{R}_{kl}$ of, respectively, the coefficients and curvature of ∇:[2]

$$\Gamma_k := \left[\Gamma^i{}_{jk}\right]_{i,j=1}^{\dim M}, \qquad \boldsymbol{R}_{kl} := \left[R^i{}_{jkl}\right]_{i,j=1}^{\dim M}.$$

In this notation, equation (I.3.13) reads

$$\boldsymbol{R}_{kl} = -2\Gamma_{[k,l]} + 2\Gamma_{[k}\Gamma_{l]} - C^m_{kl}\Gamma_m \tag{4.1}$$

where $\Gamma_{k,l} := E_l(\Gamma_k)$ and C^m_{kl} are defined via (I.3.15).

NECESSITY. If ∇ admits a frame $\{E'_i\}$ normal on U, there is a non-degenerate C^1 matrix-valued function $A = [A^j_i]$ such that $E'_i = A^j_i E_j$ and (see (I.5.4))

$$\Gamma_k = -A_{,k}A^{-1} \quad \text{on } U, \qquad k = 1, \dots, \dim U = \dim M. \tag{4.2}$$

The substitution of this equality into (4.1) results in $R|_U = 0$, due to (I.3.15).

SUFFICIENCY. As we know from the considerations in Section I.5 (p. 39), a connection ∇ admits normal frames iff (I.5.4) has solutions with respect to the matrix A in a given frame $\{E_i\}$. Rewriting (I.5.4) in the form

$$E_k(A)|_p = -\Gamma_k(p)A(p), \qquad p \in U, \tag{4.3}$$

we shall find its general solution, under the condition $R|_U = 0$, by applying the following lemma.

Lemma 4.1. *Let N be a manifold and there be given C^1 matrix-valued functions $Z_a\colon N \to \mathrm{GL}(m,\mathbb{K})$, $a = 1, \dots, \dim N$, $\mathrm{GL}(m,\mathbb{K})$ being the group of $m \times m$ nondegenerate matrices on $\mathbb{K}$ for some $m \in \mathbb{N}$. Suppose $\{e_a | a = 1, \dots, \dim N\}$ is a (global) frame on N and consider the initial-value problem*

$$e_a(Y)|_q = Z_a(q)Y, \qquad q \in N,\ a = 1, \dots, \dim N, \tag{4.4a}$$
$$Y|_{q=q_0} = \mathbb{1} \tag{4.4b}$$

with respect to the $m \times m$ matrix-valued function Y on N. Here $q_0 \in N$ is fixed and $\mathbb{1} = \mathbb{1}_m$. Then:

(i) *The integrability conditions for* (4.4a) *are*

$$e_a(e_b(Y)) - e_b(e_a(Y)) = c^d_{ab}e_d(Y), \qquad a, b, d = 1, \dots, \dim N \tag{4.5}$$

where $[e_a, e_b]_- =: c^d_{ab}e_d$, or, equivalently,

$$R_{ab}(Z_1, \dots, Z_{\dim N}) := e_b(Z_a) - e_a(Z_b) + Z_aZ_b - Z_bZ_a + c^d_{ab}Z_d = 0, \tag{4.6}$$

i.e., (4.4a) *has solutions with respect to $Y\colon N \to \mathrm{GL}(m,\mathbb{K})$ under these conditions.*

[2] Since the Ricci tensor will not appear in this book, the quantities $\boldsymbol{R}_{kl}$ can not be confused with its components.

(ii) *The initial-value problem* (4.4) *has a solution, which is of class* C^2*, unique and smoothly depends on its arguments, if and only if the integrability conditions* (4.6) *are valid. This solution will be denoted by*

$$Y = Y(q, q_0; Z_1, \dots, Z_{\dim N}).$$

Remark 4.3. In this section, Lemma 4.1 will be used only for $N = U$, U being a neighborhood in M, resp. $\dim N = m = \dim M$. We formulate it in the above general form because we shall need this generalization further. The proofs in the both cases are practically identical.

Remark 4.4. The choice $N = J$, $\dim N = 1$, $q = s \in J$, $e_1 = \frac{\mathrm{d}}{\mathrm{d}s}$, and $q_0 = s_0 \in J$ returns us to Lemma 3.2. Moreover, for any N with $\dim N = 1$ the integrability conditions are identically valid, so the problem (4.4) always has a unique smooth solution in the one-dimensional case.

Remark 4.5. It is easily seen that the general solution of the matrix system of differential equations

$$e_a(X) = Z_a X \tag{4.7}$$

with $X \in \mathrm{GL}(m, \mathbb{K})$ is

$$X = Y(q, q_0; Z_1, \dots, Z_{\dim N}) \tag{4.8}$$

for arbitrary $q \in N$, or, equivalently,

$$X = Y(q, q_0; Z_1, \dots, Z_{\dim N}) X_0 \tag{4.9}$$

for fixed $q_0 \in N$ and arbitrary $X_0 \in \mathrm{GL}(m, \mathbb{K})$. Also, the particular solution satisfying the initial condition

$$X|_{q=q_0} = X_0 \tag{4.10}$$

for some $q_0 \in N$ and $X_0 \in \mathrm{GL}(m, \mathbb{K})$ is given by (4.9).

Proof of Lemma 4.1. Let (V, z) be a chart of N and $e_a|_V = D^b_a \frac{\partial}{\partial z^b}$ for some C^1 non-degenerate matrix-valued function $D = [D^b_a]$. In the coordinates $\{z_a\}$, the equation (4.4a) reads $\frac{\partial Y}{\partial z^a} = \left(B^{-1}\right)^b_a Z_b Y$. The integrability conditions for the last system are [34, Chapter VI, § 1] $\frac{\partial^2 Y}{\partial z^a \partial z^b} - \frac{\partial^2 Y}{\partial z^b \partial z^a} = 0$. Due to $e_a(e_b(Y)) - e_b(e_a(Y)) = D^{a'}_a D^{b'}_b \left(\frac{\partial^2 Y}{\partial z^{a'} \partial z^{b'}} - \frac{\partial^2 Y}{\partial z^{b'} \partial z^{a'}} \right) + c^d_{ab} e_d(Y)$, which is obtainable by direct calculation (cf. equations (I.8.1)–(I.8.3) in Section I.8), these conditions are equivalent to (4.5).

Expressing the derivatives $e_a(Y)$ from (4.4a) and inserting the results into equation (4.5), we get (4.6) (cf. [34, Chapter VI, equation (1.4)]). Reversing this process, we can derive (4.5) from (4.6). Hence (4.5) and (4.6) are equivalent (provided (4.4a) is valid, as we supposed).

The second part of the lemma's assertion is a corollary of its first part and the corresponding theorems of existence and uniqueness in the theory of differential equations; see, e.g., [34, Chapter VI, Theorem 6.1]. □

Let us return now to equation (4.3). It corresponds to (4.7) with $N = U$, $\dim N = m = \dim M$, $e_i = E_i$, and $Z_k = -\Gamma_k$. So, the integrability conditions for (4.3) are (4.6) with $Z_k = -\Gamma_k$:

$$0 = R_{kl}(-\Gamma_1, \dots, -\Gamma_{\dim M}) = -2\Gamma_{[k,l]} + 2\Gamma_{[k}\Gamma_{l]} - C^m_{kl}\Gamma_m = \boldsymbol{R}_{kl}$$

where (4.1) was used. These conditions are fulfilled due to the flatness of ∇ on U, $R|_U = 0$. Consequently, according to Lemma 4.1 (ii) and Remark 4.5, the general solution of (4.3) is

$$A(p) = Y(p, p_0; -\Gamma_1, \dots, -\Gamma_{\dim M})B \tag{4.11}$$

where $p \in U$, $p_0 \in U$ is fixed point, B is non-degenerate constant matrix in $\mathrm{GL}(\dim M, \mathbb{K})$, and Y is the unique solution of the initial-value problem

$$E_i(Y)|_p = -\Gamma_i(p)Y, \quad Y|_{p=p_0} = \mathbb{1}_{\dim M}. \tag{4.12}$$

Thereof, by construction, the frames $\{E'_i = A^j_i E_j\}$, with the matrices A given via (4.11), are normal on U. □

Corollary 4.1. *A C^3 manifold with C^1 linear connection admits a (global) normal frame on it iff it is flat.*

Proof. See Theorem 4.1. □

Theorem 4.2. *Given a C^1 linear connection on C^3 manifold M. If the connection is flat on a* $(\dim M)$*-dimensional submanifold U of M, then all frames normal on U for it are $\{E'_i = A^j_i E_j\}$ where $\{E_i\}$ is arbitrarily chosen frame on U and A is given by* (4.11) *in which $p \in U$, $p_0 \in U$ is fixed, Γ_k are the connection's matrices in $\{E_i\}$, and B is constant non-degenerate matrix.*

Proof. See the proof of the sufficiency of Theorem 4.1 in which is proved that (4.11) is the general solution of (I.5.4) in U. □

Remark 4.6. The constancy of B in (4.11) agrees with Proposition I.5.2 and Corollary I.5.1.

Corollary 4.2. *Let M be C^3 manifold endowed with a symmetric flat linear connection. On every* $(\dim M)$*-dimensional submanifold of M, in particular on an open set in M or on the whole M, exist normal frames whose general form is $\{E'_i = A^j_i E_j\}$ with $A = [A^j_i]$ given by* (4.11) *and $\{E_i\}$ being an arbitrary frame on it.*

Proof. See Theorem 4.2 □

Theorems 4.1 and 4.2 give a *complete* description of the frames normal on submanifolds of maximum dimensionality, in particular on neighborhoods or on the whole manifold. On their base an analogous description for the normal coordinates can be given.

Let M be C^3 manifold endowed with C^1 *symmetric* (torsionless) linear connection ∇ which is flat in the domain U of some local chart (U, x) of M, $R|_U = 0$. The problem is to be described all local coordinates normal on U for ∇. Its general solution is given by the following result.

Theorem 4.3. *Given a C^3 manifold M endowed with C^1 linear connection. If (U, x) is a chart of M on whose domain the connection is flat and torsionless, $R|_U = 0$ and $T|_U = 0$, then on U exist coordinates $\{x'^i\}$ normal for the connection given. All such coordinates are obtained from $\{x^i\}$ according to equation* (4.14) *below.*

Proof. According to Proposition I.5.3 and Theorem 4.1, coordinates $\{x'^i\}$ normal on U exist and they locally generate all frames normal on U. By Corollary I.5.2 and Theorem 4.2, *all* coordinates normal on U are such that $\frac{\partial}{\partial x'^i} = A_i^j E_j$ with $A = [A_i^j]$ given by (4.11) and $\{E_i\}$ being arbitrary frame on U. Below, for simplicity, we choose $E_i = \frac{\partial}{\partial x^i}$. So $\{x'^i\}$ must be such that $\frac{\partial}{\partial x'^i} = A_i^j \frac{\partial}{\partial x^j}$ with A given by (4.11). Consequently they must be solutions of the following system of partial differential equations

$$\frac{\partial x'^i}{\partial x^j} = \left(A^{-1}\right)^i_j, \qquad A(p) = Y(p, p_0; -\Gamma_1, \dots, -\Gamma_{\dim M})B, \tag{4.13}$$

where $p \in U$, $p_0 \in U$ is fixed, Γ_k are the connection coefficients' matrices in $\left\{\frac{\partial}{\partial x^i}\right\}$, and B is constant non-degenerate matrix. The integrability conditions for this system are [34, Chapter VI, §§ 1–6]

$$\begin{aligned} 0 &= \frac{\partial^2 x'^i}{\partial x^{[j}\partial x^{k]}} = \frac{\partial}{\partial x^{[j}}\left(A^{-1}\right)^i_{k]} = \left(\frac{\partial}{\partial x^{[j}} A^{-1}\right)^i_{k]} = \left(A^{-1}\frac{\partial A}{\partial x^{[j}} A^{-1}\right)^i_{k]} \\ &= \left(A^{-1}\Gamma_{[j}\right)^i_{k]} = \left(A^{-1}\right)^i_l \Gamma^l{}_{[kj]} \ , \end{aligned}$$

where (4.13) and (4.3) were used, which equations are identically satisfied, due to the symmetry of the connection. Hence the system (4.13) is completely integrable and defines x'^i as C^3 functions of x^i (as A is of class C^2).[3] The general solution of (4.13) can be written as

$$x'^i(p) = a^i + \int_{p_0}^{p} \left(B^{-1}Y^{-1}(q, p_0; -\Gamma_1, \dots, -\Gamma_{\dim M})\right)^i_k \, \mathrm{d}q^k \tag{4.14}$$

or

$$x'^i(p) = a^i + \int_{s_0}^{s} \left(B^{-1}Y^{-1}(\gamma(t), p_0; -\Gamma_1, \dots, -\Gamma_{\dim M})\right)^i_k \dot\gamma^k(t) \, \mathrm{d}t \tag{4.14$'$}$$

[3] The same result follows from the Poincaré Lemma [13, p. 121], [2, p. 55]: writing (4.13) as $\mathrm{d}x'^i = (A^{-1})^i_j \mathrm{d}x^j$, a necessary and sufficient condition for the existence of x'^i is $0 = \mathrm{d}[(A^{-1})^i_j \mathrm{d}x^j] = 2(A^{-1})^i_{[j,k]}\mathrm{d}x^j \wedge \mathrm{d}x^k$. For details concerning Pfaff systems, see, e.g., [12, Chapter IV, Section C].

Here: $p \in U$, $p_0 \in U$ is fixed, $a^i \in \mathbb{K}$ are constants representing the coordinates of p_0 in ${x'}^i$, Γ_k are the matrices of the connection coefficients in $\{\frac{\partial}{\partial x^i}\}$, the integrals are taken along paths lying entirely in U, $\gamma\colon J \to U$ is a C^1 path such that $\gamma(s_0) = p_0$ and $\gamma(s) = p$ for some $s_0, s \in J$, B is constant non-degenerate matrix, and Y is the solution of (4.4) with $N = U$, $m = \dim M$, and $e_k = \frac{\partial}{\partial x^k}$. i.e.,

$$\frac{\partial Y}{\partial x^k} = -\Gamma_k Y, \qquad Y|_{p=p_0} = \mathbb{1}, \quad Y = Y(p.p_0; -\Gamma_1, \dots, -\Gamma_{\dim M}). \tag{4.15}$$

Since the connection is supposed flat in U, $R|_U = 0$, it is a simple exercise to be shown that the integrals in (4.14) and (4.14′) are well-defined, i.e., independent of the concrete paths, like γ, lying in U along which the integration is performed. This ends the proof of Theorem 4.3. □

Exercise 4.1. Show that the choice $E_i = \frac{\partial}{\partial x^i}$ in the proof of Theorem 4.3 does not restrict the generality of our constructions. For the purpose, prove that the choice $E_i = D_i^j \frac{\partial}{\partial x^j}$, with a C^1 non-degenerate matrix-valued function $D = [D_i^j]$, does not influence the final results expressed by Theorem 4.3. (Hint: The matrix (4.11), which is a solution of (4.3), transforms $\{E_i\}$ into $\{E'_i\}$ and Γ_k, appearing in (4.3) and (4.11), are the connection's matrices in $\{E_i\}$; so, the particular form (and meaning) of A and Γ_k is different for arbitrary E_i and for $E_i = \frac{\partial}{\partial x^i}$.)

Corollary 4.3. *Let M be a C^3 manifold and ∇ be a C^1 flat torsionless linear connection on it. On every coordinate neighborhood of M exist coordinates normal for ∇ whose general form is given via* (4.14).

Proof. See Theorem 4.3. □

Theorem 4.3 and its Corollary 4.3 give a *complete* description of the normal coordinates on a C^3 manifold with C^1 (flat torsionless) linear connection.

If the manifold and the connection considered are of class C^∞, the integral in (4.14) (or (4.14′)) can be calculated explicitly in terms of the connection coefficients (in $\{\frac{\partial}{\partial x^i}\}$) as a power series in some neighborhood of a fixed point $p_0 \in U$. Indeed, in this case we can write

$$\begin{aligned} Y^{-1}(p, p_0; -\Gamma_1, \dots, -\Gamma_{\dim M}) &= Y^{-1}(p_0, p_0; -\Gamma_1, \dots, -\Gamma_{\dim M}) \\ &+ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=1}^{\dim M} \frac{\partial^n Y^{-1}(p, p_0; -\Gamma_1, \dots, -\Gamma_{\dim M})}{\partial p^{i_1} \cdots \partial p^{i_n}}\Big|_{p=p_0} \\ &\times [p^{i_1} - p_0^{i_1}] \cdots [p^{i_n} - p_0^{i_n}] \end{aligned}$$

where $p^k := x^k(p)$, $p_0^k := x^k(p_0)$, and the series is convergent in some neighborhood $V \subseteq U$ of p_0.

The values of Y^{-1} and its derivatives in the right-hand side of the last equation can be evaluated by means of (4.15) as it is equivalent to

$$\frac{\partial Y^{-1}}{\partial x^k} = Y^{-1}\Gamma_k \tag{4.16a}$$

$$Y^{-1}|_{p=p_0} = \mathbb{1}, \qquad Y^{-1} = \big(Y(p,p_0;-\Gamma_1,\dots,-\Gamma_{\dim M})\big)^{-1}. \tag{4.16b}$$

In fact, by successive differentiation of equation (4.16a), we get $\frac{\partial^n Y^{-1}}{\partial x^{(i_1}\cdots\partial x^{i_n)}} = Y^{-1}G_{i_1,\dots,i_n}$ where the matrices $G_{i_1,\dots,i_n}$ are symmetric in the subscripts and are defined through the relation

$$G_{i_1,\dots,i_{n+1}} := \Gamma_{(i_1}G_{i_2,\dots,i_{n+1})} + \frac{\partial}{\partial x^{(i_1}}G_{i_2,\dots,i_{n+1})}, \qquad n\in\mathbb{N}, \quad G_k := \Gamma_k. \tag{4.17}$$

Hence the expansion of Y^{-1} takes the form

$$Y^{-1}(p,p_0;-\Gamma_1,\dots,-\Gamma_{\dim M}) = \mathbb{1} + \Gamma_k(p_0)[x^k(p)-x^k(p_0)] + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1,\dots,i_n=1}^{\dim M} G_{i_1,\dots,i_n}(p_0)[x^{i_1}(p)-x_0^{i_1}(p_0)]\cdots[x^{i_n}(p)-x_0^{i_n}(p_0)].$$

Substituting this into (4.14) and taking into account that

$$\int_{p_0}^{p}[q^{i_1}-p_0^{i_1}]\cdots[q^{i_n}-p_0^{i_n}]\,\mathrm{d}q^j = \Big(1+\sum_{a=1}^{n}\delta^{i_a j}\Big)^{-1}[p^{i_1}-p_0^{i_1}]\cdots[p^{i_n}-p_0^{i_n}][p^j-p_0^j],$$

we get

$$\begin{aligned} x'^{\,i} = a^i + \big(B^{-1}\big)^i_j \Big\{ & x^j(p)-x^j(p_0) + \sum_{k,l=1}^{\dim M}\frac{1}{1+\delta^{kl}}\Gamma^j{}_{kl} \\ & \times [x^k(p)-x^k(p_0)][x^l(p)-x^l(p_0)] \\ & + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k,i_1,\dots,i_n=1}^{\dim M}\frac{1}{1+\sum_{a=1}^{n}\delta^{i_a k}}\big(G_{i_1,\dots,i_n}(p_0)\big)^j_k \\ & \times [x^{i_1}(p)-x^{i_1}(p_0)]\cdots[x^{i_n}(p)-x^{i_n}(p_0)][x^k(p)-x^k(p_0)]\Big\}. \end{aligned} \tag{4.18}$$

Since $\frac{\partial x'^{\,i}}{\partial x^j}\big|_{p_0} = \big(B^{-1}\big)^i_j$ and $B=[B^j_i]$ is non-degenerate, this series is convergent in some neighborhood $V\subseteq U$ of $p_0\in U$ and, consequently, the transition $\{x^i\}\to\{x'^{\,i}\}$ is (C^∞) well-defined and (C^∞) invertible.

Now we want to say a few words on the links between the normal frames/coordinates and parallel transport on manifolds with flat linear connection. (For simplicity and brevity, we consider only the global case below.)

It is a classical result that the parallel transport is (locally) path-independent iff the manifold M (or the connection) is (locally) flat [3,8,11]. Sometimes for such manifolds is said that they possess teleparallelism or absolute parallelism [19, p. 142], admit distant parallelism [55], or that they are parallelizable [28, p. 68]. A parallelization of M is called the choice of a covariantly-constant frame $\{E_i\}$ on M, i.e., $\nabla_X E_i = 0$ for all $X \in \mathfrak{X}(M)$. Hence $\{E_i\}$ is a parallelization of M iff $\nabla_{E_j} E_i = 0$ which is equivalent to $\Gamma^k{}_{ij} = 0$ (see (I.3.1)), i.e., iff $\{E_i\}$ is normal. Thereof, the parallelization is simply a concrete choice of some normal frame and the manifold is parallelizable iff it admits normal frames or, equivalently (see Theorem 4.1 and Corollary 4.1), iff it is flat. The following result, which agrees with Proposition I.5.5, expresses the same in another form.

Proposition 4.1. *A C^3 manifold endowed with C^1 linear connection admits (global) normal frames if and only if the parallel transport associated to the connection is path-independent.*

Proof. If M admits a normal frame $\{E_i\}$, in it the parallel transport is path-independent according to Proposition I.5.5. Conversely, let the parallel transport be path-independent and $\{E_i^0\}$ be a fixed basis at an arbitrarily chosen point $p \in M$. Define a frame $\{E_i\}$ on M such that $E_i^0|_p \to E_i|_q$ is the result of the parallel transport of E_i^0 from p to q for all $q \in M$. By construction, this frame is path-independent and satisfies the equation (see (I.3.18)) $\nabla_{\dot\gamma} E_i = 0$ for every path γ joining p and q. So, choosing γ such that at q the tangent vector $\dot\gamma$ coincides with $E_j|_q$, we get $0 = (\nabla_{E_j} E_i)|_q = (\Gamma^k{}_{ij} E_k)|_q$, i.e., $\Gamma^k{}_{ij}(q) = 0$, $q \in M$ which means that $\{E_i\}$ is normal on M. □

Remark 4.7. A trivial consequence of Corollary 4.1 and Proposition 4.1 is that the parallel transport is path-independent iff the generating it connection is flat. Conversely, if this result is proved by another way [3,11,28], from Proposition 4.1 immediately follows Corollary 4.1.

5. The case on arbitrary submanifolds

The problem for existence of coordinates normal on a submanifold of dimension higher than one was first posed by J. A. Schouten and D. J. Struik in 1935 [43, p. 106]. They showed that if N is n-dimensional submanifold of m-dimensional, $m \geq n$, manifold M endowed with symmetric linear connection, then on N exist (in our terminology) normal coordinates provided on N exist n linearly independent covariantly constant (with respect to every vector field) vector fields. In our notation this is translated as: on N exist normal coordinates if on it the connection admits normal frames. The same result is quoted in [19, p. 169] where a new problem is put (in our terminology): if on N, considered as a manifold, a (global) normal frame exists, are there coordinates on M which are normal on N considered as a submanifold of M? This problem was completely solved (for symmetric linear connections) by L. O'Raifeartaigh in 1958 [55].

5.1. Conventional method

In this subsection, we shall review (in our notation) and partially generalize the results of the paper [55]. Its idea [55, pp. 18 and 21, paragraphs 1 and 3] is quite simple and analogous to the one of the construction of Fermi coordinates (Subsection 3.1). If N is an n-dimensional submanifold of the m-dimensional, $m \geq n$, manifold M, in a neighborhood of N the manifold M can be represented (locally) as a direct sum of suitable $(m-n)$-dimensional submanifolds L_q, one for each $q \in N$, such that L_q is $(m-n)$-dimensional normal neighborhood of q (see Definition I.3.6) and for every p in the mentioned neighborhood of N there is unique $p_0 \in N$ such that $L_{p_0} \ni p$. If on N exist normal coordinates, such are the coordinates $(\varphi^1, \ldots, \varphi^n, \xi^1, \ldots, \xi^{m-n})$ where $\varphi^1(p), \ldots, \varphi^n(p)$ are the coordinates of p_0 in N ('considered as a flat space') and $\xi^1(p), \ldots, \xi^{m-n}(p)$ are the Riemannian normal coordinates of p with respect to p_0 in L_{p_0}. Special cases of this construction, corresponding to $n = 0, 1, m$, were investigated in Sections 2, 3, and 4, respectively. As we shall see, for $n \geq 2$ (if $m \geq 2$) coordinates normal on N exist iff some conditions, derived below, are satisfied.

Now the rigorous statement and solution of the problem outlined are in order.

Let M be C^3 manifold provided with C^1 torsionless linear connection and N be submanifold of M (see Subsection I.2.1, p. 7). The problem, we are going to investigate, is to find conditions under which on M exist local coordinates normal on (an open subset of) N and, when they are valid, to construct a particular example of such coordinates.

Lemma 5.1. *Let N be a submanifold of a C^3 manifold M endowed with C^1 linear connection ∇. The parallel transport with respect to ∇ along paths lying entirely in N is path-independent if and only if*

$$(R(X,Y))|_q = 0, \qquad X_q, Y_q \in T_q(N), \quad q \in N \tag{5.1}$$

where R is the curvature tensor of ∇ in M and X and Y are vector fields tangent to N (as a manifold), $X, Y \in \mathfrak{X}(N)$.

Remark 5.1. This lemma is true for arbitrary linear connections, without or with torsion,[1] and for $N = M$ it reduces to the wide-known result that the parallel transport is path-independent only for flat linear connections. Note, from (5.1) one can not conclude $R|_N = 0$ unless $\dim N = \dim M$. (In this context, see Remark IV.10.2 on page 276 concerning the same problem on vector bundles.)

[1] In [55] it is proved in the torsionless case.

Remark 5.2. The equation (5.1) does not concern separately M, N, or ∇. It involves simultaneously all of them: the manifold M is a carrier of ∇ and a set containing N and N is the set on which the parallel transport and the curvature of ∇ are restricted.

Proof. Let (U, x) be a chart of M with $\bar{U} := U \cap N \neq \varnothing$. By definition of a submanifold (page 7), the pair $(\bar{U}, \bar{x})$ with $\bar{x}(q) := (x^1(q), \dots, x^{\dim N}(q))$, $q \in N$, is a chart of N with $\{\bar{x}^a := x^a|_{\bar{U}} | a = 1, \dots, \dim N\}$ as associated coordinate system. So $\{\frac{\partial}{\partial \bar{x}^a} = \frac{\partial}{\partial x^a}|_{\bar{U}}\}$ is a frame on $\bar{U}$, i.e., $\{\frac{\partial}{\partial x^a}|_q\}$ is a basis in $T_q(N)$, $q \in N$.

Let $\gamma\colon J \to N$ and $Z \in \mathfrak{X}^1(N) \subseteq \mathfrak{X}^1(M)$ be C^1 vector field tangent to M over N and parallel along γ, viz. $\dot{\gamma}(s) \in T_{\gamma(s)}(N)$ and $Z_{\gamma(s)} \in T_{\gamma(s)}(M)$ for $s \in J$ and

$$0 = \nabla_{\dot{\gamma}} Z = \dot{\gamma}^j \Big(\frac{\partial Z^k}{\partial x^j} + \Gamma^k{}_{lj} Z^l \Big) \frac{\partial}{\partial x^k}.$$

Here and further in this proof, all components are with respect to $\{\frac{\partial}{\partial x^k}\}$ or $\{\frac{\partial}{\partial \bar{x}^k}\}$; in particular $\Gamma^k{}_{lj}$ are the coefficients of ∇ in the former frame. Since γ lies entirely in N, we have $\dot{\gamma}(s) \in T_{\gamma(s)}(N)$ and hence $\dot{\gamma}^j = 0$ for $j > \dim N$. The parallel transport in N is path-independent if[2] there is a vector field $Z \in \mathfrak{X}(N) \subseteq \mathfrak{X}(M)$ such that $\nabla_{\dot{\gamma}} Z = 0$ for every path $\gamma\colon J \to N$, which is equivalent to the existence of a solution of the system of differential equations

$$\frac{\partial Z^k}{\partial x^a} = -\Gamma^k{}_{la} Z^l, \qquad a = 1, \dots, \dim N, \quad k, l = 1, \dots, \dim M$$

with respect to Z^a (as $\dot{\gamma}^j = 0$ for $j > \dim N$). The integrability conditions for this system are [34, Chapter VI, § 1]

$$\begin{aligned} 0 = \frac{\partial^2 Z}{\partial x^{[a} \partial x^{b]}} = \frac{\partial}{\partial x^{[a}} \Big(-\Gamma^k{}_{|l|b]} Z^l \Big) &= \Big(-Z^l \frac{\partial}{\partial x^a} \Gamma^k{}_{lb} - \Gamma^k{}_{jb} \frac{\partial}{\partial x^a} Z^j \Big)_{[ab]} \\ &= \Big(-\frac{\partial}{\partial x^a} \Gamma^k{}_{lb} + \Gamma^k{}_{jb} \Gamma^j{}_{la} \Big)_{[ab]} Z^l = -R^k{}_{lab} Z^l \end{aligned}$$

where (I.3.13) was used (with $C^i_{jk} = 0$ as $\{\frac{\partial}{\partial x^i}\}$ is a coordinate frame). By virtue of the arbitrariness of (the initial value of) Z, these conditions are equivalent to

$$R^k{}_{lab}|_N = 0, \qquad k, l = 1, \dots, \dim M, \quad a, b = 1, \dots, \dim N \qquad (5.1')$$

in every chart (U, x) of M with $U \cap N \neq \varnothing$. If X and Y are vector fields tangent to N, then $X_q, Y_q \in T_q(N) \subseteq T_q(M)$, $q \in N$, so that $X^i = Y^i = 0$ for $i > \dim N$ and consequently (5.1′) is equivalent to (5.1). □

[2]However, see Remark I.3.5 on page 29.

Remark 5.3. Using the notation of (4.6), we can rewrite (5.1′) in more compact matrix form

$$R_{ab}(-\Gamma_1, \dots, -\Gamma_{\dim N})|_p = 0, \qquad a, b = 1, \dots, \dim N, \tag{5.1″}$$

which is valid in every chart of N.

Proposition 5.1. *Let N be a submanifold of a C^3 manifold M provided with C^1 linear connection ∇. If the connection ∇ admits frames normal on N, then* (5.1) *holds, i.e., equation* (5.1) *is a necessary condition for the existence of frames normal on N.*

Proof. If ∇ admits a normal frame on N, by Proposition I.5.5 the parallel transport in N along paths in N is path-independent and, hence, according to Lemma 5.1, equation (5.1) holds. □

We will prove in the next subsection (p. 120) that the condition (5.1) is also sufficient for the existence to frames normal on N for arbitrary, torsionless or not, linear connections. Below, following [55], we will prove only the existence of particular coordinates which are normal on N in the torsionless case provided (5.1) is valid.

Proposition 5.2. *Given a submanifold N of a C^3 manifold M endowed with C^1 torsionless linear connection ∇ for which equation* (5.1) *holds. Then ∇ admits frames and coordinates locally normal on N, i.e., for every $q \in N$ there exists a chart (U^0, y) of M with q in its domain, $U^0 \ni q$, such that the associated with it coordinate system $\{y^i\}$ is normal on $\bar{U} := U^0 \cap N$. A particular example of such coordinate is provided by equation* (5.5) *below.*

Proof. Let ∇ be torsion free, (U, x) be a chart of M such that $\bar{U} := U \cap N \neq \varnothing$ and $q_0 \in \bar{U}$. So $(\bar{U}, \bar{x})$ with $\bar{x}^a = x^a|_N$, $a = 1, \dots, \dim N$, is a chart of N in a neighborhood of q_0. Let $\{E_i^0\}$ be a basis in $T_{q_0}(M)$ such that $E_a^0 \in T_{q_0}(N)$, $a = 1, \dots, \dim N$. Consequently, the $(\dim M - \dim N)$-dimensional space spanned by $\{E_i^0 | i > \dim N\}$ does not contained vectors tangent to N except the zero vector.[3]

Suppose the equation (5.1) holds. Since the parallel transport in N along paths lying in N is path-independent (Lemma 5.1), the frame $\{E_i\}$ on N obtained from $\{E_i^0\}$ by parallel translation from q_0 to every point $q \in N$ along paths lying entirely in N is uniquely defined: $\nabla_X E_i = 0$, with X being tangent to N vector field, and $E_i|_{q_0} = E_i^0$.

By means of the frame $\{E_i\}$, we shall define the looked for coordinates. For the purpose, we write the expansion $E_i = D_i^j \frac{\partial}{\partial x^j}$. The non-degenerate matrix-valued function $D = [D_j^i]$ can be obtained from the parallel character of $\{E_i\}$ on N. Since $\nabla_X E_i = 0$ for all vector fields X tangent to N, for $X = \frac{\partial}{\partial x^a}$,

[3]For example, we can put $E_i^0 = \frac{\partial}{\partial x^i}\big|_{q_0}$; see the definition of a submanifold on page 7.

$a = 1, \dots, \dim N$, we get

$$0 = \nabla_{\partial/\partial x^a} E_j = \nabla_{\partial/\partial x^a} \Big(D^n_j \frac{\partial}{\partial x^n} \Big) = \Big(\frac{\partial}{\partial x^a} D^n_j + \Gamma^n{}_{ka} D^k_j \Big) \frac{\partial}{\partial x^n} \iff \frac{\partial D}{\partial x^a} = -\Gamma_a D.$$

As the integrability conditions for the last equation are (5.1″), which are equivalent to (5.1), in turn supposed to be valid, by Lemma 4.1 with $m = \dim M$, we have (see (4.4))

$$D(q) = Y(q, q_0; -\Gamma_1, \dots, -\Gamma_{\dim N}) D_0 \tag{5.2}$$

where the constant non-degenerate $\dim M \times \dim M$ matrix D_0 is such that $E_i|_{q_0} = E^0_i = D^{0j}_{\ i} \frac{\partial}{\partial x^j}|_{q_0}$, i.e., it represents the expansion of $\{E^0_i\}$ over $\{\frac{\partial}{\partial x^i}\big|_{q_0}\}$.[4]

Now we shall define new coordinates $\{y^i\}$ on the domain of the chart (U, x). At first we define them only on $\bar{U} := U \cap N$ (cf. (3.25)) by

$$y^i(q) := a^i + \int_{q_0}^{q} \big(D^{-1}(p)\big)^i_j \, \mathrm{d}p^j, \qquad q \in \bar{U} := U \cap N \subseteq N \tag{5.3}$$

where $p^j := x^j(p)$ and the integration is performed along some path connecting q_0 and q and lying entirely in N. The integral in this definition is path-independent as the corresponding integrability conditions are identically satisfied on $\bar{U}$:

$$\frac{\partial^2 y^i}{\partial \bar{x}^{[a} \bar{x}^{b]}} = \frac{\partial}{x^{[a}} (D^{-1})^i_{b]} = \Big(\frac{\partial D^{-1}}{\partial x^{[a}} \Big)^i_{b]} = \Big(-D^{-1} \frac{\partial D}{\partial x^{[a}} D^{-1} \Big)^i_{b]} = \big(D^{-1} \Gamma_{[a} \big)^i_{b]} = \big(D^{-1} \big)^i_k \Gamma^k{}_{[ba]} \equiv 0$$

where $\frac{\partial}{\partial \bar{x}^a}|_{\bar{U}} = \frac{\partial}{\partial x^a}|_{\bar{U}}$, $\frac{\partial D}{\partial x^a} = -\Gamma_a D$ (see above), and the symmetry of the connection coefficients were employed. The functions $y^1, \dots y^{\dim N}$ define a coordinate system in a neighborhood of q_0 in $\bar{U}$. Indeed, the Jacobian of the change $\{\bar{x}^a\} \to \{\bar{y}^a = y^a|_{\bar{U}}\}$ on $\bar{U}$ is $J(q) = \det\big[\big(D^{-1}(q)\big)^a_b\big]$. Since $D^{-1}(q_0) = D_0^{-1}$ and $E^0_a \in T_{q_0}(N)$, we have $J(q_0) = \det\big[\big(D_0^{-1}\big)^a_b\big] \neq 0, \infty$ as D is continuous (and also of class C^1). Hence there is a neighborhood $\bar{U}_{q_0} \subseteq \bar{U}$ of q_0 such that $J(q) \neq 0$ for $q \in \bar{U}_{q_0}$.[5] Therefore the functions y^i given via (5.3) define a coordinate system $\{\bar{y}^a\}$ on $\bar{U}_{q_0}$.

[4] The choice $E^0_i = \frac{\partial}{\partial x^i}\big|_{q_0}$ results in $D_0 = \mathbb{1}$; see Footnote3 on the preceding page.

[5] The choice $E^0_i = \frac{\partial}{\partial x^i}\big|_{q_0}$ results in $D_0 = \mathbb{1}$ and $J(q_0) = 1$. So, from the continuity of $J \colon \bar{U} \to \mathbb{K}$ follows that for every $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$, there is a neighborhood U_ε of q_0 such that $0 < 1 - \varepsilon < |J(q)| < 1 + \varepsilon$ for $q \in U_\varepsilon$. Thus $J(q) \neq 0$ for $q \in U_\varepsilon$.

From (5.3), we derive:

$$\frac{\partial}{\partial \bar{y}^a}\Big|_q = \frac{\partial}{\partial y^a}\Big|_q = \frac{\partial x^i}{\partial y^a}\Big|_q \frac{\partial}{\partial x^i}\Big|_q = \Big(\Big[\frac{\partial x^k}{\partial y^j}\Big|_q\Big]\Big)^i_a \frac{\partial}{\partial x^i}\Big|_q$$
$$= \Big(\Big[\frac{\partial y^k}{\partial y^j}\Big|_q\Big]^{-1}\Big)^i_a \frac{\partial}{\partial x^i}\Big|_q = \big((D(q)^{-1})^{-1}\big)^i_a \frac{\partial}{\partial x^i}\Big|_q = D^i_a(q)\frac{\partial}{\partial x^i}\Big|_q = E_a|_q$$

for $q \in \bar{U}_{q_0}$. So, $\{E_1, \dots, E_{\dim M}\}$ is a frame holonomic on $\bar{U}_{q_0}$ and it is generated by the coordinate system $\{\bar{y}^a\}$ of the chart $(\bar{U}_{q_0}, y)$ of N.

At this point we shall need the following lemma.

Lemma 5.2. *Let N be submanifold of a C^3 manifold M endowed with C^1 torsionless linear connection whose curvature tensor satisfies* (5.1). *Every parallel frame $\{E_i\}$ defined on N (resp. in a neighborhood of N) can locally be expanded (resp. be redefined) outside N in a holonomic way, i.e., if the vector fields E_i are parallel on N, for every point $p_0 \in N$, there is a chart (V, z) of M with $V \ni p_0$ such that $\bar{V} := V \cap N$ is a coordinate neighborhood in N and the associated with it coordinate system $\{z^i\}$ generates $\{E_i\}$ on $\bar{V}$: $E_i|_q = \frac{\partial}{\partial z^i}\big|_q$, $q \in \bar{V}$. If* $\dim N = 0$ *or if N is real and* $\dim N = 1$, *the assertion is valid for any frame, parallel or not, regardless of the existence of some connection on M, i.e., in the zero- and real one-dimensional cases any frame defined solely at a single point or along a given path can be (locally) generated by some local coordinates.*

Remark 5.4. From this lemma, generally, does not follow that any (parallel) frame is holonomic on N! This is valid if $\dim N = \dim M$. The meaning of the lemma is that any field of bases parallel on N, with respect to path-independent parallel transport generated by a linear connection, can be generated (locally) by some local coordinate system.

Remark 5.5. For $\dim N = 1$, when (5.1) is identically valid, and $\mathbb{K} = \mathbb{R}$ this lemma reduces to [76, Lemma 4.1]. See also Lemma III.10.1 on page 194.

Remark 5.6. The lemma means that any continuous frame defined solely on a submanifold and parallel on it can locally be extended to a holonomic frame on its neighborhood provided equation (5.1) holds. Of course, if required, such an extension can be done in an anholonomic way too (provided $\dim N < \dim M$). Consequently, the holonomicity problem for a (parallel) frame defined *only* on N (under the condition (5.1)) depends on the way this frame is extended outside the set N.

Proof of Lemma 5.2. Let $p_0 \in N$, (U, x) be a chart of M with $U \ni p_0$, and $(\bar{U}, \bar{x})$ be the associated chart of N, viz. $\bar{U} := U \cap N$, $\bar{x}(q) := (x^1(q), \dots, x^{\dim N}(q))$ for $x(q) := (x^1(q), \dots, x^{\dim N}(q), b^1, \dots, b^{\dim M - \dim N})$ where $q \in \bar{U}$ and the b's are constants in $\mathbb{K}$. Let $U_N := \{p | p \in U,\ x^a(p) = x^a(q),\ a = 1, \dots, \dim N,\ q \in \bar{U}\} \subseteq U$. This set is a neighborhood of $\bar{U}$ such that every $p \in U_N$ has a 'projection $p_0 \in \bar{U}$ along x': $p_0 := \bar{x}^{-1}(x^1(p), \dots, x^{\dim N}(p))$. Let the expansion $E_i|_q = D^j_i(q)\frac{\partial}{\partial x^j}\big|_q$ in

$\bar{U}$ be valid for some non-degenerate matrix-valued function $D := [D_i^j]$. We define $z^i : U_N \to \mathbb{K}$ by (cf. (5.3))

$$z^i(p) = a^i + \int_{q_0}^{p_0} \big(D^{-1}(p)\big)^i_j \, \mathrm{d}p^j + \big(D^{-1}(p_0)\big)^i_j \, [x^j(p) - x^j(p_0)] \\ + f^i_{jk}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)]. \quad (5.4)$$

Here: $p \in U_N$, $a^i \in \mathbb{K}$ are constants, $q_0 \in \bar{U}$ is fixed, p_0 is the above-defined 'projection' of p on $\bar{U}$, the integral is taken along some path lying entirely in $\bar{U}$, and $f^i_{jk} : U_N \to \mathbb{K}$ are of class C^1 and they and their first partial derivatives are bounded on $\bar{U} = U_N \cap N$. As we saw above (see (5.3) and the lines below it), the integral in (5.4) is path-independent due to the parallel character of $\{E_i\}$ on N and the symmetry of the connection. If N is real and $\dim N = 1$, this integral is well-defined for any initial frame $\{E_i\}$, regardless of the existence of some connection on M.

Exercise 5.1. Prove the last assertion! (Hint: in this case, N has locally the form $\gamma(J)$ where J is open $\mathbb{R}$-interval and $\gamma : J \to \gamma(J)$ is a diffeomorphism.)

For $\dim N = 0$, we take for p_0 some point in $\bar{U}$ and the above integral identically vanishes. The Jacobian of the transformation $\{x^i\} \to \{z^i\}$ at p_0 is $J(p_0) = \det\big(\frac{\partial z^i}{\partial x^j}\big|_{p_0}\big) = \det\big(D^{-1}(p_0)\big) = 1/\det(D(p_0)) \neq 0, \infty$ for every $p_0 \in \bar{U}$. Consequently $\{z^i\}$ is a coordinate system on some neighborhood $V \subseteq U_N$ of $\bar{U}$.

At last, we calculate the basic vectors of the frame $\big\{\frac{\partial}{\partial z^j}\big\}$, defined on V, at $p_0 \in \bar{U} = V \cap N = U_N \cap N = U \cap N$:

$$\frac{\partial}{\partial z^i}\Big|_{p_0} = \frac{\partial x^j}{\partial z^i}\Big|_{p_0} \frac{\partial}{\partial x^j}\Big|_{p_0} = \Big(\Big[\frac{\partial z^k}{\partial x^l}\Big|_{p_0}\Big]^{-1}\Big)^j_i \frac{\partial}{\partial x^j}\Big|_{p_0} \\ = \Big(\big(D^{-1}(p_0)\big)^{-1}\Big)^j_i \frac{\partial}{\partial x^j}\Big|_{p_0} = D^j_i(p_0) \frac{\partial}{\partial x^j}\Big|_{p_0} = E_i|_{p_0}.$$

Hence the chart (V, z) with coordinates functions z^i is the one we are looking for. This conclusion ends the proof of Lemma 5.2 □

Exercise 5.2. Prove that equation (5.4) defines the most general local coordinates generating locally a given (parallel) frame on a submanifold. From this view-point, the coincidence of $\{z^i\}$ on $\bar{U}$ with $\{y^i\}$ given via (5.3) (as $p = p_0$ for $p \in \bar{U}$) is evident as all such coordinate can differ only at points not lying in N.

Using the chart (V, z), $V \cap N = \bar{U}$, provided by Lemma 5.2, we can construct a foliation of V 'along' N: defining an $(\dim M - \dim N)$-dimensional manifolds V_q by

$$V_q := \{p | p \in V,\ z^a(p) = z^a(q),\ a = 1, \dots, \dim N\}, \qquad q \in \bar{U},$$

we see that

$$\bigcup_{q\in\bar{U}} V_q = V, \quad V_q \cap V_{q'} = \varnothing \text{ for } q \neq q',\ q, q' \in \bar{U}, \quad \text{and } V_q \cap \bar{U} = q \in \bar{U}.$$

Besides, by construction (see Lemma 5.2 and its proof), the tangent space $T_p(V_q)$, $p \in V_q$, does not contain vectors tangent to N; in particular, we have the direct decomposition $T_q(M) = T_q(V_q) \oplus T_q(N)$, $q \in \bar{U} \subseteq N$.

Consequently, for every $p \in V$, there is a unique point $\bar{p} \in \bar{U}$ such that $p \in V_{\bar{p}}$.[6]

Now in each V_q we take a $(\dim M - \dim N)$-dimensional normal neighborhood V_q^0 of q (see Definition I.3.6 on page 33). The union $V^0 := \bigcup_{q\in\bar{U}} V_q^0 \subseteq V$ forms a neighborhood of $\bar{U}$ such that, for every $p \in V^0$ in the submanifold $V_{\bar{p}}^0$, there exists a unique geodesic $\beta_p \colon J \to V_{\bar{p}}$, $J \ni 0$ joining $\bar{p}$ and p in $V_{\bar{p}}$, i.e., more precisely, $p = \exp_{\bar{p}}(tX_p) = \beta_p(t)$ where $\beta_p(0) = \bar{p}$, $\dot{\beta}_p(0) = X_p$ for some unique $X_p \in T_{\bar{p}}(V_{\bar{p}}) \subseteq \big\{\sum_{k=\dim N+1}^{\dim M} \alpha_k E_k|_{\bar{p}} : \alpha_{\dim N+1}, \dots, \alpha_{\dim M} \in \mathbb{K}\big\}$ and $t \in J$. Just to prove the existence of this unique X_p corresponding to every p in a neighborhood of $\bar{U}$ were needed all of the considerations after (5.3). With its help, we define the coordinates $\{y^i\}$ on V^0 (cf. (5.4)) by

$$y^i(p) = a^i + \int_{q_0}^{\bar{p}} \big(D^{-1}(p)\big)^i_j \,\mathrm{d}p^j + t\big(D^{-1}(\bar{p})\big)^i_j X_p^j. \tag{5.5}$$

Here: $p \in V^0$, $a^i \in \mathbb{K}$ are constants, $q_0 \in \bar{U}$ is fixed, $\bar{p}$ is the unique element in $\bar{U}$ for which $V_{\bar{p}} \ni p$, $p^i := x^i(p)$, the integral is taken along some path in $\bar{U}$, t is the increment of the parameter when passing from $\bar{p}$ to p along the unique geodesic connecting them in $V_{\bar{p}}$, and X_p is the tangent vector to this geodesic at $\bar{p}$ (corresponding to $t = 0$). The quantities ${}^yX_p^i := \big(D^{-1}(\bar{p})\big)^i_j X_p^j$ are the components of X_p in $\{E_i|_{\bar{p}}\}$ and, since $X_p \in T_{\bar{p}}(V_{\bar{p}})$, we have ${}^yX_p^a = 0$ for $a = 1, \dots, \dim N$.

Since (5.5) corresponds to (5.4) with $p_0 = \bar{p}$, $x^j(p) = x^j(p_0) + t\,{}^yX_p^j$, $f^i_{jk} = 0$, and $V = V^0$, from the considerations after (5.4) in the proof of Lemma 5.2, it follows that there exist a neighborhood $U^0 \subseteq V^0$ of $\bar{U}$ $(= U^0 \cap N = V^0 \cap N = \cdots)$ on which the transformation $\{x^i\} \to \{y^i\}$, given via (5.5), is a well-defined diffeomorphism.[7]

Ending these long considerations, we will prove that the chart (U^0, y) of M is normal on N, i.e., on $\bar{U} = U^0 \cap N$. Till now the chart (U, x) was complete arbitrary. Choosing $(U, x) = (U^0, y)$, we get $\frac{\partial}{\partial x^i}\big|_q = \frac{\partial}{\partial y^i}\big|_q = E_i|_q = D_i^j(q)\frac{\partial}{\partial x^j}\big|_q$, $q \in \bar{U}$. Thus $D|_{\bar{U}} = \mathbb{1}$ or $D_i^j(q) = \delta_i^j$ for $q \in \bar{U}$. From the parallel character of $\{E_i\}$ on N, we derive $0 = \nabla_{\partial/\partial x^a} E_j = \nabla_{\partial/\partial x^a} \frac{\partial}{\partial x^j} = \Gamma^i{}_{ja} E_i$ on $\bar{U}$, so that $\Gamma^i{}_{ja}\big|_{\bar{U}} = 0$

[6]In [55] this assertion is mentioned and used but its rigorous proof, which is not trivial, is only partially presented, the main details are missing.

[7]For another proof of this fact, see [55, pp. 21–22].

for $a = 1, \dots, \dim N$. Since $p \in U^0$ and $\bar{p} \in \bar{U}$ are situated on a geodesic, the coordinates (5.5) must satisfy the geodesic equation (I.3.23) with p for $\gamma(s)$, $y^j(p)$ for $\gamma^j(s)$, and t for s, which results in $\Gamma^i{}_{jk}(p) X^j_p X^k_p = 0$. Taking into account that p is arbitrary, $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$, and $X^a = 0$ for $a = 1, \dots, \dim N$ as $X_p \in T_{\bar{p}}(V_{\bar{p}})$ and $T_{\bar{p}}(V_{\bar{p}}) \cap T_{\bar{p}}(N) = \varnothing$, from here we find $\Gamma^i{}_{jk}\big|_{\bar{U}} = 0$ for $j, k > \dim N$. Combining this with $\Gamma^i{}_{ja}\big|_{\bar{U}} = 0$ for $a < \dim N$, we, finally obtain $\Gamma^i{}_{jk}\big|_{\bar{U}} = 0$ for all values of all indices.

Consequently the chart (U^0, y) is normal on $\bar{U}$, which ends the proof of Proposition 5.2 □

The coordinates (5.5) are direct analogue and multidimensional generalization of the Fermi coordinates along paths without self-intersections (see Subsection 3.1), to which they reduce in the one-dimensional case. For this reason, we propose, as in [55], to call *Fermi coordinates on a submanifold* this special kind of coordinates normal on a submanifold.

It should be emphasized, the Fermi coordinates on a submanifold are local by their nature. Generally global such coordinates, i.e., on the whole submanifold, do not exist. But, it is almost evident, we can construct a family of overlapping such coordinate systems whose domains form a neighborhood of the submanifold given.

From (5.5) (see also (5.4) for $x^i(p) = x^i(p_0) + \big(D^{-1}(p_0)\big)^i_j X^j_p$), one can derive that the basic vector fields of the normal frames associated to the Fermi coordinates are

$$\frac{\partial}{\partial y^i}\Big|_q = \frac{\partial x^j}{\partial y^i}\Big|_q \frac{\partial}{\partial x^j}\Big|_q = \left(\left[\frac{\partial x^k}{\partial y^l}\Big|_q\right]^{-1}\right)^j_i \frac{\partial}{\partial x^j}\Big|_q = D^j_i(q) \frac{\partial}{\partial x^j}\Big|_q = E_i|_q$$

for $q \in \bar{U} \subseteq N$. Consequently $\big\{\frac{\partial}{\partial y^j}\big\}$ on $\bar{U}$ coincides with the initial frame $\{E_i\}$ on $\bar{U}$. Since $\{E_i\}$ is parallel by definition, so is $\big\{\frac{\partial}{\partial y^j}\big\}$. This result completely agrees with Proposition I.5.6.

Combining Propositions 5.1 and 5.2, we obtain the following fundamental result.

Theorem 5.1. *Let N be a submanifold of a C^3 manifold M endowed with C^1 torsionless linear connection. There exist coordinates* (*in M*) *normal on N for the connection given if and only if the conditions* (5.1) *are valid. If* (5.1) *holds, the above-constructed Fermi coordinates are normal on the intersection of their domain with N.*

If M admits Fermi coordinates on its submanifold N, we can easily describe *all* frames normal on N. If (U^0, y) is a chart normal on N, i.e., on $\bar{U} = U^0 \cap N$, with $\{y^i\}$ being Fermi coordinates on U^0, the frame $\big\{\frac{\partial}{\partial y^j}\big\}$ is, by Definition I.5.2, normal on $\bar{U}$. According to Proposition I.5.2, in U^0 all frames normal on $\bar{U}$ have the form $\big\{E_i = A^j_i \frac{\partial}{\partial y^j}\big\}$ where the non-degenerate C^2 matrix-valued function $A := [A^j_i]$ on U^0 is such that $\frac{\partial A}{\partial y^j}\big|_{\bar{U}} = 0$.

Exercise 5.3. Prove that the general form of the matrix-valued function A is (cf. (5.11) in Subsection 5.2)

$$A(p) = B(p_0) - \frac{\partial B}{\partial y^j}\Big|_{p_0}[y^j(p) - y^j(p_0)] + B_{jk}(p)[y^j(p) - y^j(p_0)][y^k(p) - y^k(p_0)] \tag{5.6}$$

where $p \in U^0$, B is non-degenerate C^2 matrix-valued function on U^0, p_0 is the 'projection' of p on $\bar{U}$ as defined before (5.4), the matrix-valued functions B_{jk} on $\bar{U}$ are of class C^1, and B, $\frac{\partial B}{\partial y^i}$, and B_{jk} together with their partial derivatives are bounded on $\bar{U}$.

According to Theorem 5.1, between the normal frames exist holonomic ones. The *full local description* of all coordinates normal on a submanifold will be given in the next subsection. In the particular case considered, they are defined on U^0 and normal on $\bar{U} = U^0 \cap N$.

If we cover N with overlapping systems of Fermi coordinates (U^0_λ, y_λ), $\lambda \in \Lambda \neq \varnothing$, we can construct on N a global (on N) frame normal on the whole submanifold N. Let $\{E^\lambda_i\}$ be a frame defined on U^0_λ and normal on $\bar{U}_\lambda = U^0_\lambda \cap N$. If $q \in N$ is such that $q \in \bar{U}_\lambda$ for a single $\lambda \in \Lambda$, we put $E_i|_p := E^\lambda_i|_p$, $p \in U^0_\lambda$; if such $\lambda \in \Lambda$ are more than one, we arbitrarily choose some $\mu \in \Lambda$ and put $E_i|_p := E^\mu_i|_p$, $p \in U^0_\mu$. The so-obtained frame $\{E_i\}$ is defined in the neighborhood $\bigcup_{\lambda\in\Lambda} U^0_\lambda$ of N and is normal on N. Obviously, in this way can be constructed all frames normal on N. Generally these global on N normal frames are not smooth in the regions of overlapping of two or more local Fermi coordinates but locally, on each particular U^0_λ, they can be constructed in a smooth way.

5.2. Complete description

Below, following the main ideas of Subsection 3.2 and [83], we are going to give a full constructive description of the frames (resp. coordinates) normal on a submanifold of manifold with arbitrary linear connection, if such frames (resp. coordinates) exist.

Let M be C^2 manifold endowed with C^0 linear connection (of arbitrary torsion) and N be its submanifold. The problem is to find necessary and sufficient conditions for the existence of frames defined on a neighborhood of N and normal on N. If such frames exist, we want to give their complete description. From Proposition 5.1, we know that the conditions (5.1) are necessary for the existence of frames normal on N. Our first goal now is to prove the sufficiency of these conditions, which assertion is a part of the following theorem (cf. Theorem 3.1).

Theorem 5.2. *Let N be submanifold of C^3 manifold M provided with C^1 linear connection, with or without torsion. Then:*

(i) *The equation* (5.1) *is a necessary and sufficient condition for the existence of frames normal on N.*

(ii) *If* (5.1) *holds, every point* $q_0 \in N$ *has a neighborhood* U_N *(in* M*) such that on* U_N *exist* C^1 *frames normal on* N*, i.e., on* $\bar{U} = U_N \cap N \ni q_0$*. Moreover, on* U_N *exist coordinates* $\{x^i\}$ *such that every* $p = x^{-1}(\boldsymbol{s}, \boldsymbol{t}') \in U_N$*,* $(\boldsymbol{s}, \boldsymbol{t}') \in W_N \times W' \subseteq \mathbb{K}^{\dim N} \times \mathbb{K}^{\dim M - \dim N}$*, has a unique 'projection'* $p_0 = x^{-1}(\boldsymbol{s}, \boldsymbol{t}'_0) \in \bar{U} \subseteq N$ *for fixed* $\boldsymbol{t}'_0 \in W'$ *which depends on* x *but not on* $\boldsymbol{t}'$*. In these local coordinates, all frames defined on* U_N *and normal on* $\bar{U} = U_N \cap N$ *are* $\{E'_i = A^j_i \frac{\partial}{\partial x^j}\}$ *with* $A = [A^j_i]$ *given via equation* (5.11) *below in which* $\boldsymbol{s}_0 = \bar{x}(q_0)$ *are the coordinates of* q_0 *in the chart* $(\bar{U}_N, \bar{x})$ *of* N *induced by the chart* (U_N, x) *of* M*.*

Proof. Let $\{E_i\}$ be a frame defined in a neighborhood of N. From Section I.5, we know that all frames $\{E'_i\}$ defined on the same neighborhood and normal on N are $\{E'_i = A^j_i E_j\}$ where the C^1 non-degenerate matrix-valued function $A := [A^j_i]$ is a solution of the normal frame equation (I.5.4) with N for U, viz.

$$\Gamma_k(q)A(q) + (E_k(A))|_q = 0 \tag{5.7}$$

where $q \in N$ and $\Gamma_k := \left[\Gamma^i{}_{jk}\right]_{i,j=1}^{\dim M}$ are the matrices of the connection coefficients in $\{E_i\}$. Hence, the finding of the general solution of (5.7) with respect to A is equivalent to the complete description of all frames normal on N, if any.

Let $q \in N$ and (U, x) be a chart of M with $\bar{U} := U \cap N \ni q$. So $x\colon U \to W$ is homeomorphism, which actually is a C^1 diffeomorphism, for some open subset W of $\mathbb{K}^{\dim M}$. Representing W as a Cartesian product of two open sets $W'' \subseteq \mathbb{K}^{\dim N}$ and $W' \subseteq \mathbb{K}^{\dim M - \dim N}$, i.e., $W = W'' \times W'$, from the definition of a submanifold (p. 7) follows the existence of $\boldsymbol{t}'_0 = (t'^{\dim N+1}_0, \dots, t'^{\dim M}_0) \in W'$ such that

$$x(q) = \left(x^1(q), \dots, x^{\dim N}(q)\right) \times \boldsymbol{t}'_0 \qquad \text{for all } q \in N$$

and $(\bar{U}, \bar{x})$, with $\bar{x}(q) := (x^1(q), \dots, x^{\dim N}(q))$ for $q \in N$, provides a coordinate system $\{\bar{x}^i\}$ on $\bar{U}$. Let $W_N := \bar{x}(\bar{U}) \subseteq W''$, i.e., $\bar{x}\colon \bar{U} \to W_N$ is the local coordinate homeomorphism from $\bar{U}$ on the open subset W_N of $\mathbb{K}^{\dim N}$.

Consider the chart (U_N, x) of M with $U_N := x^{-1}(W_N \times W')$.[8] Evidently $U_N \cap N = \bar{U} \ni q$ and, for every $p \in U_N$, we have $x(p) = (\boldsymbol{s}, \boldsymbol{t}')$ for some unique $\boldsymbol{s} = (s^1, \dots, s^{\dim N}) \in W_N \subseteq \mathbb{K}^{\dim N}$ and $\boldsymbol{t}' = (t'^{(\dim N+1}, \dots, t'^{\dim M}) \in W' \subseteq \mathbb{K}^{\dim M - \dim N}$; in particular, if $q \in \bar{U}$, then $x(q) = (\boldsymbol{s}, \boldsymbol{t}'_0)$. These facts give us a possibility to define a natural 'projection' $\pi\colon U_N \to \bar{U}$ such that if $p = x^{-1}(\boldsymbol{s}, \boldsymbol{t}') \in U_N$, then $\pi\colon p \mapsto p_0 := \pi(p) := x^{-1}(\boldsymbol{s}, \boldsymbol{t}'_0)$, i.e., p_0 is the unique point in $\bar{U}$ with $\bar{x}(p_0) = \boldsymbol{s}$.[9]

Suppose the matrix-valued function A is of class C^3. There exist C^1 matrix-valued functions B_{kl} on U_N such that they and their partial derivatives are

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[8] Strictly speaking, x in (U_N, x) has to be replace by the restriction $x_N := x|_{U_N}$; we ignore this to save some writing.

[9] The triple $(U_N, \pi, \bar{U})$ is a fibre bundle with base $\bar{U}$, (total) bundle space U_N, and projection π (see further Section IV.2). From here the name 'projection' for π comes from.

bounded on $\bar{U}$ and (cf. (3.15))

$$A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}')\big) = A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}'_0)\big) + \sum_{k=\dim N+1}^{\dim M} \frac{\partial A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}')\big)}{\partial t'^{\,k}}\bigg|_{\boldsymbol{t}'=\boldsymbol{t}'_0} (t'^{\,k} - t'^{\,k}_0)$$
$$+ \sum_{k,l=\dim N+1}^{\dim M} B_{kl}\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}')\big)(t'^{\,k} - t'^{\,k}_0)(t'^{\,l} - t'^{\,l}_0) \quad (5.8)$$

Choosing $E_k|_p = \frac{\partial}{\partial x^k}\big|_p$ for $p \in U_N$, we get

$$E_a(A)|_{\pi(p)} = \frac{\partial A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}')\big)}{\partial s^a}\bigg|_{\boldsymbol{t}'=\boldsymbol{t}'_0}, \qquad E_k(A)|_{\pi(p)} = \frac{\partial A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}')\big)}{\partial t'^{\,k}}\bigg|_{\boldsymbol{t}'=\boldsymbol{t}'_0},$$

where $a = 1, \dots, \dim N$, $k = \dim N + 1, \dots, \dim M$, $p = x^{-1}(\boldsymbol{s},\boldsymbol{t}')$ for $(\boldsymbol{s},\boldsymbol{t}') \in W_N \times W'$. Taking this into account and substituting equation (5.8) into (5.7), we find the equivalent to (5.7) system of equations for A on $\bar{U}$:

$$\frac{\partial A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}'_0)\big)}{\partial s^a} = -\Gamma_a\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}'_0)\big)A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}'_0)\big) \quad (5.9a)$$
$$\frac{\partial A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}')\big)}{\partial t'^{\,k}}\bigg|_{\boldsymbol{t}'=\boldsymbol{t}'_0} = -\Gamma_k\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}'_0)\big)A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}'_0)\big), \quad (5.9b)$$

where $a = 1, \dots, \dim N$ and $k = \dim N + 1, \dots, \dim M$. Since equation (5.9a) corresponds to (4.4a) with W_N for N, $m = \dim M$, $e_a = \frac{\partial}{\partial s^a}$, $q = x^{-1}(\boldsymbol{s},\boldsymbol{t}'_0)$, and $Z_a = -\Gamma_a$, by Lemma 4.1, Assertion (i) it has solutions if and only if the conditions (5.1$''$) hold on U_N. But as equation (5.1$''$) is equivalent to (5.1) (see Remark 5.3 on page 114), this means that (5.7) has solutions on N with respect to A iff (5.1) is valid.

Now we suppose (5.1) to be true. Applying Lemma 4.1, Assertion (ii) and taking into account Remark 4.5 on page 106, we find the general solution of (5.9a) in the form (cf. (3.22))

$$A\big(x^{-1}(\boldsymbol{s},\boldsymbol{t}'_0)\big) = Y\big(\boldsymbol{s},\boldsymbol{s}_0; -\Gamma_1\circ(x^{-1}(\,\cdot\,,\boldsymbol{t}'_0)), \dots, -\Gamma_{\dim N}\circ(x^{-1}(\,\cdot\,,\boldsymbol{t}'_0))\big)B(\boldsymbol{t}'_0). \quad (5.10)$$

Here Y is the unique solution of (5.9a) with $A = Y$ satisfying the initial conditions

$$Y|_{\boldsymbol{s}=\boldsymbol{s}_0} = \mathbb{1}_{\dim M} \quad (5.9c)$$

for some fixed $\boldsymbol{s}_0 \in W_N \subseteq \mathbb{K}^{\dim N}$, and B is non-degenerate $\dim M \times \dim M$ matrix-valued function on W' of class C^1. Substituting (5.9b) into (5.8) and using (5.10), we get:

$$A(p) = \big\{\mathbb{1} - \Gamma_k(p_0)[x^k(p) - x^k(p_0)]\big\}$$
$$\times Y\big(\bar{x}(p_0), \boldsymbol{s}_0; -\Gamma_1\circ(x^{-1}(\,\cdot\,,\boldsymbol{t}'_0)), \dots, -\Gamma_{\dim N}\circ(x^{-1}(\,\cdot\,,\boldsymbol{t}'_0))\big)B(\boldsymbol{t}'_0)$$
$$+ B_{kl}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] \quad (5.11)$$

where $\boldsymbol{s}_0 \in W_N \subseteq \mathbb{K}^{\dim N}$ is fixed and we used $p = x^{-1}(\boldsymbol{s}, \boldsymbol{t}') \in U_N$, $p_0 = \pi(p) = x^{-1}(\boldsymbol{s}, \boldsymbol{t}'_0) \in \bar{U} = U_N \cap N$, and $\bar{x}(p_0) = \boldsymbol{s}$. Notice, in this equality all terms corresponding to $k, l = 1, \dots, \dim N$ are zero as $x^a(p) = x^a(p_0) = \bar{x}^a(p_0) = \boldsymbol{s}^a$ for $a = 1, \dots, \dim N$. Since (5.11) is the general solution of (5.7) on $\bar{U}$ by construction, the above results, together with Proposition 5.1 complete the proof of Theorem 5.2. □

Remark 5.7. Since $\boldsymbol{t}'_0$ in (5.8) depends only on the choice of the initial coordinates $\{x^i\}$, the matrix $B(\boldsymbol{t}'_0)$ is constant on N for every such particular choice. This agrees with Proposition I.5.2 and Corollary I.5.1.

Remark 5.8. Notice, equation (5.11) corresponds to (5.6) for $x^i = y^i$ ($\{y^i\}$ are Fermi coordinates) and $B(p) = Y(\cdots)$. The terms corresponding to the index values from 1 to $\dim N$ are zero in the both equations.

Remark 5.9. Theorem 5.2 is a first in a group of similar Theorems III.7.1, III.8.1, IV.6.1, and IV.9.1 generalizing it. (See also the comments after the proof of Theorem IV.9.1 on page 269.)

Corollary 5.1. *Let N be a submanifold of C^3 manifold with C^1 linear connection. On N (locally) exist normal frames if and only if the parallel transport in N along paths lying in N is path-independent.*[10]

Proof. See Theorem 5.2 and Lemma 5.1 on page 112 □

Corollary 5.2. *Let N be a submanifold of C^3 manifold with C^1 linear connection. On N (locally) exist normal frames iff on N exists a parallel frame with respect to the paths in N.*

Proof. If $\{E_i\}$ is frame normal on N, then it is also parallel on N, due to Proposition I.5.6. Conversely, let $\{E_i\}$ be parallel frame on N, i.e., $(\nabla_X E_i)|_N = 0$ for all $X \in \mathfrak{X}(N)$. The parallel transport of some $Y_0 = Y_0^i E_i|_{p_0} \in T_{p_0}(M)$, $p_0 \in N$, along $\gamma\colon J \to N$ from $p_0 = \gamma(s_0)$ to every $p = \gamma(s)$ for some $s_0, s \in J$ gives the vector $Y_p = Y_{\gamma(s)}$ where Y is such that (see Definition I.3.2) $(\nabla_{\dot\gamma} Y)|_{\gamma(s)} = 0$ and $Y_{p_0} = Y_0$. But $\nabla_{\dot\gamma} Y|_{\gamma(s)} = \big(\dot\gamma^k \nabla_{E_k}(Y^j E_j)\big)|_{\gamma(s)} = \dot\gamma^k(s)(\nabla_{E_k} Y^j)|_{\gamma(s)} E_j|_{\gamma(s)} = \dot\gamma(s)(Y^j_{\gamma(s)}) E_j|_{\gamma(s)}$, so that $\dot\gamma(s)(Y^j_{\gamma(s)}) = 0$, i.e., $Y^j_{\gamma(s)} = Y^j_0 = \text{const}$. Hence the parallel transport of Y_0 from p_0 to p along any path connecting them in N gives the vector $Y_0^j E_j|_p$ and, consequently, it is path-independent in N. Now Corollary 5.1 implies the existence of frames normal on N. □

Theorem 5.2 gives a *complete local description* of the frames normal on a submanifold. On this base a complete *global description* of all frames normal on a submanifold can be given. Indeed, let $\{\bar{U}^\lambda | \lambda \in \Lambda\}$, $\Lambda \neq \varnothing$, be an open cover of N such that for every $\lambda \in \Lambda$ there is a neighborhood U_N^λ in M such that $\bar{U}^\lambda = U_N^\lambda \cap N$ and on U_N^λ exist frames $\{E_i^\lambda\}$ normal on $\bar{U}^\lambda$, like the ones constructed above. In the neighborhood $V_N := \bigcup_{\lambda\in\Lambda} U_N^\lambda$ of N we define a frame $\{E_i\}$ by putting

[10] If M is multiply connected, see Remark I.3.5 on page 29.

$E_i|_p = E_i^\lambda|_p$, $p \in V_N$ for some $\lambda \in \Lambda$ for which $p \in U_N^\lambda$; if there are more than one such λ, we arbitrary choose some of them. By construction $\{E_i\}$ is a global frame which is normal on the whole submanifold N. Generally $\{E_i\}$ is not smooth, even not continuous, on the sets (neighborhoods) of intersection (overlapping) of two or more neighborhoods U_N^λ. Relying on Theorem 5.2 (ii), we conclude that in this way can be constructed *all* frames (globally) normal on any submanifold.

Now we have at our disposal the full machinery required for the *complete description of all local coordinates normal on a given submanifold.* Meanwhile we shall find a necessary and sufficient condition for the existence of coordinates normal on a submanifold.

The following theorem provides a *complete constructive description of all local coordinates normal on a submanifold.*

Theorem 5.3. *Let N be submanifold of C^3 manifold M endowed with C^1 torsion free linear connection for which the condition* (5.1) *holds. Then for every point $q \in N$ exists a chart (U_z, z) of M with $U_z \ni q$ which is normal on $U_z \cap N$. The coordinate functions $z^i : U_z \to \mathbb{K}$ of all such charts normal on N are given via equation* (5.16) *below.*

Proof. Suppose M is a C^3 manifold endowed with C^0 torsionless linear connection ∇ for which the (integrability) conditions (5.1) are valid on some submanifold N of M. Theorem 5.1 ensures the existence of coordinates normal on N, the Fermi coordinates providing an example. The problem is to find *all* such coordinates. For its solution we shall follow the scheme of Subsection 3.2 for the complete description of the local coordinates normal along a path (see p. 100 and further).

Consider the chart (U_N, x) of M, defined in the proof of Theorem 5.2, for which each point $p = x^{-1}(\boldsymbol{s}, \boldsymbol{t}') \in U_N$ has a unique projection $p_0 := \pi(p) = x^{-1}(\boldsymbol{s}, \boldsymbol{t}'_0)$ on N. By Theorem 5.2, Assertion (ii) on U_N all frames normal on $\bar{U} = U_N \cap N$ are $\{E'_i = A_i^j \frac{\partial}{\partial x^j}\}$ with $A = [A_i^j]$ given via (5.11). Therefore if local coordinates $\{z^i\}$, defined in a neighborhood U_z of $\bar{U}$ in U_N, $U_z \cap N = \bar{U}$ and normal on $\bar{U}$, exist, then they must satisfy the conditions

$$\frac{\partial}{\partial z^i}\Big|_q = E'_i|_q = A_i^j(q) \frac{\partial}{\partial x^j}\Big|_q \iff \frac{\partial z^i}{\partial x^j}\Big|_q = \left(A^{-1}(q)\right)^i_j. \tag{5.12}$$

Here $q = x^{-1}(\boldsymbol{s}, \boldsymbol{t}'_0) \in \bar{U}$ and A is some matrix-valued function given by equation (5.11). Since the manifold is of class C^3, the Taylor expansion

$$z^i(p) = z^i(p_0) + a^i_j(p_0)[x^j(p) - x^j(p_0)] + a^i_{jk}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)] \tag{5.13}$$

is valid for every $p \in U_z$, $p_0 = \pi(p)$. Here $a^i_j : \bar{U} \to \mathbb{K}$ are of class C^1 and the C^1 functions $a^i_{jk} : \bar{U} \to \mathbb{K}$ and their first partial derivatives are bounded on $\bar{U}$. Notice, due to $p_0 = \pi(p)$, we have $x^a(p) = x^a(p_0) = s^a$ for $a = 1, \dots, \dim N$; so in (5.13) the terms with $j, k = 1, \dots, \dim N$ are zero. From (5.12) and (5.13), we derive:

$$\frac{\partial z^i(p_0)}{\partial x^a} = \frac{\partial z^i}{\partial x^a}\Big|_{p_0} = \big(A^{-1}(p_0)\big)^i_a, \qquad a = 1, \dots, \dim N$$

$$\iff z^i(p_0) = a^i + \int_{q_0}^{p_0} \sum_{a=1}^{\dim N} \big(A^{-1}(q)\big)^i_a \,\mathrm{d}q^a$$

$$a^i_j(p_0) = \frac{\partial z^i(p)}{\partial x^j}\Big|_{p_0} = \frac{\partial z^i}{\partial x^j}\Big|_{p_0} = \frac{\partial z^i}{\partial t'^j}\Big|_{\boldsymbol{t}'=\boldsymbol{t}'_0} = \big(A^{-1}(p_0)\big)^i_j,$$

where $j = \dim N + 1, \dots, \dim M$, $a^i \in \mathbb{K}$ are constants representing the coordinates of p_0 in $\{z^i\}$ with respect to some fixed point $q_0 \in \bar{U}$, and the integral is taken along some path lying entirely in $\bar{U}$. The integral in the above equality is path-independent. This can be proved in the same way we proved the path-independence of the integral in (5.3). As $\{E_i = A^j_i \frac{\partial}{\partial x^j}\}$ is normal, by Proposition I.5.6 it is also parallel in U which, in particular, means that $\nabla_{\partial/\partial x^a} E_k = 0$. This equation is equivalent to $\frac{\partial A}{\partial x^a} = -\Gamma_a A$ or $\frac{\partial A^{-1}}{\partial x^a} = A^{-1}\Gamma_a$, by virtue of which and the torsionless of the connection the integrability conditions for the path-independence of the above integral are identically valid on $\bar{U}$:

$$\frac{\partial^2 z^i}{\partial x^{[a} x^{b]}} = \frac{\partial}{x^{[a}} \big(A^{-1}\big)^i_{b]} = \Big(\frac{\partial A^{-1}}{\partial x^{[a}}\Big)^i_{b]} = \big(A^{-1}\Gamma_{[a}\big)^i_{b]} = \big(A^{-1}\big)^i_k \Gamma^k{}_{[ba]} \equiv 0.$$

Substituting the values of $z^i(p_0)$ and $a^i_j(p_0)$ obtained above in (5.13), we derive the following general form of the coordinates $\{z^i\}$:

$$z^i(p) = a^i + \int_{q_0}^{p_0} \sum_{a=1}^{\dim N} \big(A^{-1}(q)\big)^i_a \,\mathrm{d}q^a + \big(A^{-1}(p_0)\big)^i_j \,[x^j(p) - x^j(p_0)]$$
$$+ a^i_{jk}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)], \quad (5.14)$$

with A given by (5.11). Since the Jacobian of the transformation $\{x^i\} \to \{z^i\}$ at $q \in \bar{U}$ is

$$\det\Big[\frac{\partial x^i}{\partial z^j}\Big|_q\Big] = \Big(\det\Big[\frac{\partial z^i}{\partial x^j}\Big|_q\Big]\Big)^{-1} = \big(\det(A^{-1}(q))\big)^{-1} = \det\big(A(q)\big) \neq 0, \infty,$$

there exists a neighborhood U_z of $\bar{U}$ in U_N such that $U_z \cap N = \bar{U}$ and (U_z, z) is a chart of M with coordinate functions z^i.

The charts (U_z, z) (locally) generate the restrictions to N of all frames normal on U_z but generally these charts are not normal on N, i.e., on $\bar{U}$. To make them such, we are going to choose the functions $a^i_{jk} \colon U_z \to \mathbb{K}$ in such a way as to satisfy equation (I.5.4′) on page 41 with U_z for $U \cap V$ and z for x. Repeating the arguments leading from (3.26) to (3.26′) with $q \in \bar{U}$ for $\gamma(s)$, we get[11]

$$\frac{\partial z^i}{\partial x^l}\Big|_q \Gamma^l{}_{jk}(q) = \frac{\partial^2 z^i}{\partial x^j x^k}\Big|_q, \qquad q \in \bar{U}. \tag{5.15}$$

The derivatives entering here can be calculated from (5.14):

$$\frac{\partial z^i}{\partial x^l}\Big|_q = \big(A^{-1}(q)\big)^i_l,$$
$$\frac{\partial^2 z^i}{\partial x^a x^k}\Big|_q = \left(\frac{\partial A^{-1}}{\partial x^a}\Big|_q\right)^i_k, \qquad \frac{\partial^2 z^i}{\partial x^m x^n}\Big|_q = 2a^i_{(mn)}(q)$$

where $q \in \bar{U}$, $a = 1, \dots, \dim N$, $m, n = \dim N + 1, \dots, \dim M$, and A is provided by (5.11). (Notice, in (5.14) the terms with $j, k = 1, \dots, \dim N$ vanish as $p_0 = \pi(p)$, so that $x^a(p) = \bar{x}^a(p) = \bar{x}^a(p_0)$ for $a = 1, \dots, \dim N$.) Substituting these expressions into (5.15) and using $\frac{\partial A^{-1}}{\partial x^a} = A^{-1}\Gamma_a$ (see the paragraph preceding the one containing equation (5.14)), we find:

$$\big(A^{-1}(q)\big)^i_l \Gamma^l{}_{ak}(q) = \big(A^{-1}(q)\Gamma_a(q)\big)^i_k, \qquad q \in \bar{U}, \quad a = 1, \dots, \dim N$$
$$\big(A^{-1}(q)\big)^i_l \Gamma^l{}_{mn}(q) = 2a^i_{(mn)}(q), \qquad m, n = \dim N + 1, \dots, \dim M.$$

The first of these equalities is an identity as $(\Gamma_a)^l_k := \Gamma^l{}_{ka} = \Gamma^l{}_{ak}$ (the connection is symmetric); this reflects the vanishment of the terms with $j, k = 1, \dots, \dim N$ in (5.14). The second equation determines $a^i_{(mn)}$ on $\bar{U}$; on $U_z \backslash \bar{U}$ the values of $a^i_{(mn)}$, as well as $a^i_{[mn]}$, are left arbitrary from the condition that $\{z^i\}$ is normal on $\bar{U}$. Since M is supposed to be of class C^3, there exist continuous bounded functions $a^i_{jkl} \colon \bar{U} \to \mathbb{K}$ such that

$$a^i_{jk}(p) = a^i_{jk}(p_0) + a^i_{jkl}(p)[x^l(p) - x^l(p_0)].$$

At last, inserting in this expansion the obtained expressions for $a^i_{jk}(p_0)$ and the result into (5.14), we, finally, get:

$$\begin{aligned} z^i(p) = a^i + \int_{q_0}^{p_0} \sum_{a=1}^{\dim N} \big(A^{-1}(q)\big)^i_a \,\mathrm{d}q^a + \big(A^{-1}(p_0)\big)^i_j [x^j(p) - x^j(p_0)] \\ + \big(A^{-1}(p_0)\big)^i_l \Gamma^l{}_{jk}(p_0)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)] \\ + a^i_{jkl}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)]. \end{aligned} \tag{5.16}$$

Here:

$$\begin{aligned} &p = x^{-1}(\boldsymbol{s}, \boldsymbol{t}') \in U_z, \quad p_0 = x^{-1}(\boldsymbol{s}, \boldsymbol{t}'_0) \in \bar{U} = U_z \cap N, \\ &a^i \in \mathbb{K}, \quad \bar{x}(p_0) = \boldsymbol{s}, \quad q_0 \in \bar{U} \text{ is fixed,} \end{aligned} \tag{5.17}$$

[11] From (5.15), once again, follows that normal coordinates exist only for torsion free linear connections.

and the matrix-valued C^1 function $A\colon U_z \to \mathrm{GL}(\dim M, \mathbb{K})$ is given by (5.11) in which B is $\dim M \times \dim M$ non-degenerate matrix-valued function, the matrix-valued functions B_{lk} on U_z are of class C^1 and they and their first partial derivatives are bounded on $\bar{U}$, and Y is the unique solution of the initial-value problem (4.4) with $\frac{\partial}{\partial s^a}$, $-\Gamma_a$, $\bar{x}(p_0)$, $s_0 = \bar{x}(q_0)$, and s for e_a, Z_a, p, p_0, and q_0, respectively. This ends the proof of Theorem 5.3. □

Remark 5.10. Theorem 5.3 remains valid if N is fixed and the connection is torsionless only on N (see Proposition 5.3).

Remark 5.11. Generally, under the conditions of Theorem 5.3, not all of the normal frames provided by Theorem 5.2, point (ii), are holonomic.

Proposition 5.3. *Let N be submanifold of a C^3 manifold endowed with C^1 linear connection ∇ admitting frames normal on N. Holonomic frames normal on N or, equivalently, coordinates normal on N for ∇ exist if and only if the torsion T of ∇ vanishes on N,*

$$(T(X, Y,))|_q = 0, \qquad X_q, Y_q \in T_q(N), \quad q \in N \tag{5.18}$$

where X and Y are vector fields on N, $X, Y \in \mathfrak{X}(N) \subseteq \mathfrak{X}(M)$.

Proof. The result follows from Proposition I.5.4 and the observation that in the proof of Theorem 5.3, as well as in the one of Theorem 5.1, only the torsionless on N of the connection was used. □

Now, we leave to the reader as exercises the next four problems:

Exercise 5.4. Show by explicit calculation that the normal frame $\{\frac{\partial}{\partial z^i}\}$ induced by $\{z^i\}$ (see (5.16)) is parallel on $\bar{U}$. (Hint: prove that $\frac{\partial}{\partial z^i} = A_i^j \frac{\partial}{\partial x^j}$ on $\bar{U}$.) This agrees with Proposition I.5.6.

Exercise 5.5. Give a description of the Fermi coordinates on N in the framework of Theorem 5.3. (Hint: compare (5.5) with (5.16) and construct the submanifolds V_q using the chart (U_z, z) provided by (5.16).)

Exercise 5.6. Obtain the results of Section 4 from the ones of the present subsection. (Hint: for $\dim N = \dim M$, we have $p = p_0 = \pi(p)$.)

Exercise 5.7. Find the explicit formula expressing the initial coordinates $\{x^i\}$ via the normal coordinates $\{z^i\}$ on N, i.e., invert equation (5.16) with respect to x^i. (Hint: use (5.16) or repeat the procedure of constructing $\{z^i\}$ but starting with expansion of $\{x^i\}$ with respect to $\{z^i\}$ analogous to (5.13).)

Once again, we would like to notice that the coordinates normal on a submanifold N of M are local by their essence, they can be global, i.e., defined on a neighborhood of the whole N, only as an exception. But, generally, we can construct a family of overlapping coordinates normal on N such that the union of their domains forms a neighborhood of N in M.

At the end, going rather ahead, we want to mention that the condition (5.1), ensuring the existence of frames normal on a submanifold N, has a quite natural interpretation in the theory of linear transports along paths (see Chapter IV, in particular Proposition IV.11.2 on page 282): it expresses the flatness on N of the parallel transport assigned to the linear connection under consideration.

6. Examples of normal frames and coordinates

Similarly to Section I.7, the present one contains different examples of frames and/or coordinates that are normal for some concrete connections on some sets. Their role is to illustrate the general theory presented in the preceding sections of this chapter.

Since the complete and explicit description of all coordinates normal at a point is expressible in a simple way via the coefficients of a symmetric linear connection (see Theorem 2.1), the reader can easily obtain (by means of (2.11′)) as exercise this description for the Riemannian connections considered in Examples I.7.3–I.7.11 in which the coefficients of the corresponding connections are explicitly written.

Unfortunately, a 'more realistic' example is, the more difficult are the concrete calculations; besides, the problems of finding explicitly frames/coordinates normal on (sub)sets different from a single point cannot be solved completely in closed form (in elementary functions or in radicals); for instance, to find a frame or coordinate system (locally) normal along a path, one needs the solution of the initial-value problem (3.17) (for concrete Z) and the coordinates constructed in Lemma 3.1.

Example 6.1 (Coordinates normal at a point in $\mathbb{S}^2$). Consider the 2-sphere $\mathbb{S}^2$, investigated in Example I.7.3, and a point $p \in \mathbb{S}^2$. Choose spherical coordinates (θ, φ) on $\mathbb{S}^2$ such that $\theta(p) = \frac{\pi}{2}$ and $\varphi(p) = 0$; so p will be in the great equatorial circle of $\mathbb{S}^2$. The coordinates (θ, φ) are normal at p (as well as at any point in the equatorial circle) for the Riemannian ∇ according to Example I.7.3 in which ∇ is defined. Taking into account Remark 2.1, we can assert that all coordinate systems $\{z^1, z^2\}$ on $\mathbb{S}^2$, which are normal at p, are given via the equations (see (2.13))

$$\begin{aligned} z^1(q) &= \frac{\pi}{2} + a_1^1\Big[\theta(q) - \frac{\pi}{2}\Big] + a_2^1\varphi(q) \\ &\quad + \sum_{j,k,l=1,2} b_{jkl}^1 [y^j(q) - y^j(p)][y^k(q) - y^k(p)][y^l(q) - y^l(p)]\Big|_{\substack{y^1=\theta \\ y^2=\varphi}} \\ z^2(q) &= a_1^2\Big[\theta(q) - \frac{\pi}{2}\Big] + a_2^2\varphi(q) \\ &\quad + \sum_{j,k,l=1,2} b_{jkl}^2 [y^j(q) - y^j(p)][y^k(q) - y^k(p)][y^l(q) - y^l(p)]\Big|_{\substack{y^1=\theta \\ y^2=\varphi}}, \end{aligned} \tag{6.1}$$

where the point q belongs to some open subset containing the point p.

Example 6.2 (Frames/coordinates normal along a great circle on $\mathbb{S}^2$). Let us suppose that the path $\gamma\colon [0,2\pi) \to \mathbb{S}^2$ is a great circle on the 2-sphere $\mathbb{S}^2$ (see Example I.7.3) and (θ,φ) are spherical coordinates on $\mathbb{S}^2$ such that γ lies on the equatorial plane, i.e., $\theta\circ\gamma = \frac{\pi}{2}$ and $\varphi\circ\gamma = \mathsf{id}_{[0,2\pi)}$. We know from Example I.7.3 that the coordinates system $\{\theta,\varphi\}$ is normal along the whole path γ for the Riemannian connection ∇ describe in the example mentioned.

Due to Corollary 3.2 on page 103, all frames $\{E'_1, E'_2\}$ normal for ∇ along γ are given by (see (3.29))

$$\begin{aligned}
E'_1 &= B^1_1\frac{\partial}{\partial\varphi} + B^2_1\frac{\partial}{\partial\theta} + \sum_{j,k,l=1,2} B^j_{1kl}\Big\{[y^k - y^k(p_0)][y^l - y^l(p_0)]\frac{\partial}{\partial y^j}\Big\}\Big|_{\substack{y^1=\theta\\ y^2=\varphi}}\\
E'_2 &= B^1_2\frac{\partial}{\partial\varphi} + B^2_2\frac{\partial}{\partial\theta} + \sum_{j,k,l=1,2} B^j_{2kl}\Big\{[y^k - y^k(p_0)][y^l - y^l(p_0)]\frac{\partial}{\partial y^j}\Big\}\Big|_{\substack{y^1=\theta\\ y^2=\varphi}},
\end{aligned}\tag{6.2}$$

where B^i_j are real constants, B^i_{jkl} are C^1 functions that together with their partial derivatives are bounded, and $p_0 \in \gamma([0,2\pi))$ is fixed and hence $\theta(p_0) = \frac{\pi}{2}$ and $\varphi(p_0)$ is a fixed number in $[0,2\pi)$. By the same corollary, all coordinate systems $\{z^1, z^2\}$ on $\mathbb{S}^2$, which are normal for ∇ along γ, have the representation (see (3.30))

$$\begin{aligned}
z^1 &= a^1 - s_0 - \varphi(p_0) + 2\varphi\\
&\quad + \sum_{j,k,l=1,2} a^1_{jkl}\{[y^j - y^j(p_0)][y^k - y^k(p_0)][y^l - y^l(p_0)]\}\Big|_{\substack{y^1=\theta\\ y^2=\varphi}}\\
z^2 &= a^2 - \theta(p_0) + \theta\\
&\quad + \sum_{j,k,l=1,2} a^2_{jkl}\{[y^j - y^j(p_0)][y^k - y^k(p_0)][y^l - y^l(p_0)]\}\Big|_{\substack{y^1=\theta\\ y^2=\varphi}},
\end{aligned}\tag{6.3}$$

where a^1 and a^2 are real constants and a^i_{jkl} are C^1 functions that together with their partial derivatives are bounded.

Example 6.3 (Weyl connections: general considerations). Recall (see Footnote I.5 on page 35), if M is a Riemannian manifold with metric g and ∇ is a Weyl connection on M characterized by a 1-form ω, the coefficients of ∇ in an arbitrary (local) frame $\{E_i\}$ are (see (I.4.7))

$$\begin{aligned}
\Gamma^i{}_{jk} &= \begin{Bmatrix} i\\ jk\end{Bmatrix} + \frac{1}{2}\big(g^{im}C^l_{mj}g_{lk} + g^{im}C^l_{mk}g_{lj} - C^i_{jk}\big) + \frac{1}{2}(g_{jk}g^{il}\omega_l - \delta^i_j\omega_k - \delta^i_k\omega_j)\\
&= {}^{\mathrm{R}}\Gamma^i{}_{jk} + W^i_{jk},
\end{aligned}\tag{6.4}$$

where the Christoffel symbols $\left\{{i \atop jk}\right\}$ are given by (I.4.14), the functions

$${}^{\mathrm{R}}\Gamma^i{}_{jk} = \begin{Bmatrix} i\\ jk\end{Bmatrix} + \frac{1}{2}\big(g^{im}C^l_{mj}g_{lk} + g^{im}C^l_{mk}g_{lj} - C^i_{jk}\big)\tag{6.5}$$

are the coefficients in $\{E_i\}$ of the Riemannian connection ∇^{R} induced by g, and the functions

$$W^i_{jk} = W^i_{kj} = \frac{1}{2}(g_{jk}g^{il}\omega_l - \delta^i_j\omega_k - \delta^i_k\omega_j) \tag{6.6}$$

are the components in $\{E_i\}$ of a tensor field W of type $(1,2)$, which field characterizes the concrete Weyl connection relative to the Riemannian one with coefficients (6.5).

Suppose the components g_{ij} of g in $\{E_i\}$ form a diagonal matrix

$$[g_{ij}] = \operatorname{diag}(g_1, \dots, g_{\dim M}). \tag{6.7}$$

Then the Christoffel symbols can be calculated via equation (I.4.15) and the components (6.6) of the tensor W reduce to (do not sum over i!)

$$W^i_{jk} = \frac{1}{2}\Big(\delta_{jk}\frac{g_k}{g_i}\omega_i - \delta^i_j\omega_k - \delta^i_k\omega_j\Big) \tag{6.8}$$

as $[g^{ij}] = [g_{ij}]^{-1} = \operatorname{diag}(1/g_1, \dots, 1/g_{\dim M})$. An elementary calculations shows that the non-vanishing components of W are

$$\begin{aligned} W^i_{ik} = W^i_{ki} &= -\frac{1}{2}\omega_k \quad \text{for all } i, k = 1, \dots, \dim M \\ W^i_{kk}|_{k\neq i} &= \frac{1}{2}\frac{g_k}{g_i}\omega_i. \end{aligned} \tag{6.9}$$

Applying the identity (do not sum over i, j and k!)

$$W^i_{jk} \equiv W^i_{jk}\delta_{ij} + W^i_{jk}(1-\delta_{ij})\delta_{jk} + W^i_{jk}(1-\delta_{ij})(1-\delta_{jk})$$

and (6.9), we get

$$W_k := [W^i_{jk}] = -\frac{1}{2}\omega_k\mathbb{1} + \frac{1}{2}g_k\Big[\frac{1}{g_i}\omega_i(1-\delta_{ij})\delta_{jk}\Big] - \frac{1}{2}[(1-\delta_{ij})(1-\delta_{jk})\delta_{ik}\omega_j]. \tag{6.10}$$

In particular, for $\dim M = 2$, the last equation reduces to

$$W_1 = [W^i_{j1}] = \frac{1}{2}\begin{pmatrix} -\omega_1 & -\omega_2 \\ \frac{g_1}{g_2}\omega_2 & -\omega_1 \end{pmatrix} \qquad W_2 = [W^i_{j2}] = \frac{1}{2}\begin{pmatrix} -\omega_2 & \frac{g_2}{g_1}\omega_1 \\ -\omega_1 & -\omega_2 \end{pmatrix}. \tag{6.11}$$

Below in this and in the next two examples, we shall consider some problems concerning frames normal for Weyl connections.

First of all, we notice that, if a frame $\{E_i\}$ on $V \subseteq M$ is normal on $U \subseteq V$ for the Riemannian connection ∇^{R} (with coefficients (6.5)), then it is normal for the Weyl connection ∇ on the set

$$U^W := \{p \in U : \omega_i|_p = 0 \text{ for } i = 1, \dots, \dim M\} = \{p \in U : \omega|_p = 0\} \tag{6.12}$$

which may be empty. This is a consequence from (6.9) and

$${\Gamma^i}_{jk}|_U = W^i_{jk}|_U \tag{6.13}$$

due to (6.4), (6.5) and ${}^{\mathrm{R}}\Gamma^{i}{}_{jk}|_U = 0$ as the frame $\{E_i\}$ is normal for ∇^{R} on U. As a result of the simplification (6.13), a frame like $\{E_i\}$ can be used as an initial one from which frames normal for ∇ on subset(s) U may be constructed.

Example 6.4 (Weyl connections on $\mathbb{S}^2$). Let ∇^{R} be the Riemannian connection on the 2-sphere $\mathbb{S}^2$ investigated in Example I.7.3. Its coefficients in spherical coordinates $(x^1 = \theta, x^2 = \varphi)$ are given by (I.7.11). The coordinate system $\{x^1, x^2\}$ is normal along the equatorial circle $\gamma\colon [0, 2\pi) \to \mathbb{S}^2$ with $x^1 \circ \gamma = \frac{\pi}{2}$ and $x^2 \circ \gamma = \mathsf{id}_{[0,2\pi)}$.

Suppose ∇ is a Weyl connection on $\mathbb{S}^2$ whose Riemannian part is ∇^{R}. Then, due to (6.13), we have $(i, j = 1, 2)$

$$\Gamma^{i}{}_{jk} \circ \gamma = W^{i}_{jk} \circ \gamma \tag{6.14}$$

with $W_k = [W^{i}_{jk}]$ given by (6.11). Thus, if there exist $p \in \gamma([0, 2\pi))$ such that $\omega|_p = 0$, then $\{x^1, x^2\}$ is normal for ∇ at p. From here immediately follows that, if $\omega|_p = 0$ for some $p \in \mathbb{S}^2$, then one can construct spherical coordinates (with p in their equatorial plane) which are normal at p for ∇.

In the general case, frames/coordinates normal for ∇ at an arbitrarily fixed point $p \in \mathbb{S}^2$ can be constructed as follows. Take spherical coordinates $(x^1 = \theta, x^2 = \varphi)$ on $\mathbb{S}^2$ such that p is in their equatorial plane, e.g., we can choose them in such a way that $\theta(p) = \frac{\pi}{2}$ and $\varphi(p) = 0$. Then $\{x^1, x^2\}$ is normal for ∇^{R} along the equatorial circle and, in particular, at p.

According to Theorem 2.3, all frames normal at p for ∇ (but not for ∇^{R}) are $\{E_i = A^{j}_{i} \frac{\partial}{\partial x^j}\}$ with $A = [A^{i}_{j}]$ given by (2.14) in which one should put $\Gamma_1(p) = W_1(p)$ and $\Gamma_2(p) = W_2(p)$ (see (6.13)), where W_1 and W_2 are given by (6.11) with $\omega_1 = \omega\big(\frac{\partial}{\partial x^1}\big)$ and $\omega_2 = \omega\big(\frac{\partial}{\partial x^2}\big)$, viz.

$$A(q) = \Bigg\{\mathbb{1} - \frac{1}{2}[x^1(q) - x^1(p)] \begin{pmatrix} -\omega_1 & -\omega_2 \\ \frac{g_1}{g_2}\omega_2 & -\omega_1 \end{pmatrix} - \frac{1}{2}[x^2(q) - x^2(p)] \begin{pmatrix} -\omega_2 & \frac{g_2}{g_1}\omega_1 \\ -\omega_1 & -\omega_2 \end{pmatrix}\Bigg\} A_0 \\ + A_{jk}(q)[x^j(q) - x^j(p)][x^k(q) - x^k(p)]. \tag{6.15}$$

Similarly, to obtain all coordinates normal at p, one has to substitute the equality $\Gamma^{i}{}_{(jk)}(p) = W^{i}_{jk}(p)$, with W^{i}_{jk} given by (6.11), in (2.11′). The reader may wish to write as an exercise the explicit form of the so-obtained coordinates normal at p.

Example 6.5 (Weyl connection on Minkowski spacetime). Let now ∇ be a Weyl connection on the Minkowski spacetime $\mathbb{R}^4_3$ and $\gamma\colon J \to \mathbb{R}^4_3$ be a straight line; e.g., it may be a trajectory of a photon, $\dot{\gamma} = 0$, and hence lying on the light cone (see Example I.7.12). We shall look for a coordinate system normal for ∇ along γ.

From the standard coordinate system $\{u^i : i = 1, \ldots, 4\}$, we can obtain via a pseudo-orthogonal transformation, which preserves the components of the metric e^4_3 of $\mathbb{R}^4_3$, a new coordinates system $\{x^i\}$ such that $x^1 \circ \gamma = \mathsf{id}_{\mathbb{R}}$, i.e., the x^1-axis to coincide with the line γ. [1] The coordinate system $\{x^i\}$ is of the type constructed in Lemma 3.1; it corresponds to $U_1 = \mathbb{R}^4_3$ and $\mathbf{t}_0 = (0, 0, 0)$.

[1] Here we shall not discuss the physical possibilities for realization of the coordinate system $\{x^i\}$.

Since in $\{x^i\}$ the components of e^4_3 form the matrix diag$(1,-1,-1,-1)$, in it the coefficients of ∇ are

$$\Gamma^i{}_{jk} = W^i_{jk}$$

with W^i_{jk} given by (6.9) (or (6.10)). Consequently, by Theorem 3.1 (resp. 3.3), all frames $\{E_i = A^j_i \frac{\partial}{\partial x^j}\}$ (resp. coordinate systems $\{z^i\}$) on $\mathbb{R}^4_3$ normal along the (whole) path γ are given via (3.23) (resp. (3.27)–(3.28)) in which the matrix Γ_1 has the form (see (6.10))

$$\Gamma_1 = W_1 := [W^i_{j1}] = -\frac{1}{2}\omega_1 \mathbb{1} + \frac{1}{2} g_1 \Big[\frac{1}{g_i}\omega_i(1-\delta_{ij})\delta_{j1}\Big] - \frac{1}{2}[(1-\delta_{ij})(1-\delta_{j1})\delta_{i1}\omega_j].$$

The matrix-valued function $Y(s,s_0;-\Gamma_1\circ\gamma)$ is now the solution of the initial-value problem $(s \in \mathbb{R})$

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = -(W_1\circ\gamma)Y \qquad Y|_{s=s_0} = \mathbb{1}.$$

Unfortunately, an explicit and finite expression for Y can be obtained only in some exceptional cases. For instance, if $\omega_i = \omega\big(\frac{\partial}{\partial x^i}\big)$ are constant functions, then W_1 is a constant matrix. This immediately implies

$$Y(s,s_0;-\Gamma_1\circ\gamma) = \mathrm{e}^{-(s-s_0)W_1\circ\gamma}.$$

Exercise 6.1. Generalize the above results for $\mathbb{R}^n_q$, $0 \le q \le n \in \mathbb{N}$, and when γ is a straight line in $\mathbb{R}^n_q$.

Example 6.6 (Flat Riemannian connections). Example I.7.1 will be re-considered below from the view-point of the results of Section 4.

Let M be a C^3 Riemannian manifold with C^2 metric g and ∇ be the Riemannian connection generated by g. Suppose ∇ is flat on an open set $U \subseteq M$. By Theorem 4.3, there exist normal coordinates x^i with domain $V \subseteq U$ for ∇. The coefficients of ∇ in these coordinates are $\Gamma^i{}_{jk} = \left\{ {i \atop ji} \right\} = 0$, which implies $g_{ij,k} = 0$ in them (see (I.4.16)). Consequently, the components g_{ij} of g in $\{x^i\}$ are constant on V. In this way we have inverted the implication/conclusions of Example I.7.1.

The coordinates x^i can be calculated by means of (4.14), but the finding of their explicit form may turn to be a difficult problem. However, if one finds a coordinate system $\{x^i\}$ on $V \subseteq U$ in which the metric components are constant, then it will be also normal for ∇ on V (see (I.4.16)). For instance, such are the standard coordinates on the Euclidean space $\mathbb{E}^n$ and on the pseudo-Euclidean space $\mathbb{R}^n_q$ (see Example I.7.1 and (I.7.16)). In particular, the standard (Cartesian) coordinates on the Minkowski spacetime $M_4 = \mathbb{R}^4_3$ (or $\mathbb{R}^4_1$) are normal on M_4 for the flat Riemannian connection on it induced by the metric e^4_3.

Example 6.7 (One-dimensional real manifolds). There are at lest two reasons for the global (local) existence of normal frames (coordinates) on 1-dimensional C^3 real manifolds endowed with a linear connection ∇, viz.:

(i) The curvature and torsion tensors identically vanish in the 1-dimensional case (see (I.3.13) and (I.3.14) with all indices taking the sole value 1). We can rephrase this by saying that any linear connection on 1-dimensional manifold is flat and torsionless. Consequently Theorems 4.1 and 4.3 imply the above result. Respectively, Theorem 4.2 and Theorem 4.3 give a complete description of all normal frames and coordinates.

(ii) A chart (U, x) on 1-dimensional manifold M consists of an open set $U \subseteq M$ and homomorphism $x: U \to J$ with J being an open real interval. Thus $x^{-1}: J \to M$ is a path for which the results of Section 3 are applicable. In particular, Theorems 3.1 and 3.2 (resp. Theorem 3.3) describe(s) all frames (resp. coordinates) normal along x^{-1}. Besides, since M can be represented as a union of curves like $x^{-1}(J)$, frames globally normal on M for ∇ can be constructed in a way similar to the one described in the proof of Theorem 3.2.

Let us now write some explicit formulae regarding coordinates/frames normal for a C^1 connection ∇ on a C^3 1-dimensional real manifold M. Let (U, x) be a chart of M with $x: U \to J \subseteq \mathbb{R}$ and $\{x^1\}$ be the corresponding coordinate system consisting of the sole coordinate function $x^1 = u^1 \circ x = \mathsf{id}_{\mathbb{R}} \circ x = x$. The connection ∇ has in $\{x^1\}$ a sole coefficient ${\Gamma^1}_{11}$ coinciding with the matrix $\Gamma_1 = [{\Gamma^1}_{11}] = {\Gamma^1}_{11}$.

To construct coordinates/frames normal for ∇, we need, according to the scheme describe in Section 4, the solution $Y(p, p_0; -\Gamma_1)$, for fixed $p_0 \in U$ and any $p \in U$, of the initial-value problem

$$\frac{\mathrm{d}Y}{\mathrm{d}x^1} = -\Gamma_1 Y \qquad Y_{p=p_0} = \mathbb{1}_1 = 1.$$

This solution is

$$Y(p, p_0; -\Gamma_1) = \exp\Big(-\int_{p_0}^{p} \Gamma_1(q)\,\mathrm{d}q\Big) = \exp\Big(-\int_{s_0}^{s} \Gamma_1(\gamma(s))\dot{\gamma}^1(s)\,\mathrm{d}s\Big) \tag{6.16}$$

where $\gamma: J' \to U$ is any path in U such that $\gamma(s_0) = p_0$ and $\gamma(s) = p$ for some $s_0, s \in J$; in particular, one can set $\gamma = x^{-1}$ and $J' = J$.

According to Theorem 4.3, all coordinate systems $\{x'^1\}$ on U which are normal for ∇ on U are such that

$$\begin{aligned} x'^1(p) &= a^1 + \frac{1}{B}\int_{p_0}^{p} Y^{-1}(q, p_0; -\Gamma_1)\,\mathrm{d}q^1 \\ &= a^1 + \frac{1}{B}\int_{p_0}^{p} \exp\Big(\int_{p_0}^{q} \Gamma_1(r)\,\mathrm{d}r\Big)\,\mathrm{d}q \\ &= a^1 + \frac{1}{B}\int_{s_0}^{s} \exp\Big(\int_{s_0}^{t} \Gamma_1(\gamma(u))\dot{\gamma}^1(u)\,\mathrm{d}u\Big)\dot{\gamma}^1(t)\,\mathrm{d}t \end{aligned} \tag{6.17}$$

where a^i are constant numbers (representing the coordinates of p_0 in $\{x^1\}$) and B is a non-zero constant.

Exercise 6.2. If ∇ is a Riemannian connection, derive the results of Example I.7.2 from (6.17). (Hint: use equation (I.7.5).)

Similarly, by Theorem 4.2, all frames $\{E'_1\}$ normal on U for ∇ are such that

$$\begin{aligned} E'_1|_p &= B\exp\Bigl(\int_{p_0}^{p}\Gamma_1(q)\,\mathrm{d}q\Bigr)\frac{\partial}{\partial x^1}\Big|_p \\ &= B\exp\Bigl(\int_{s_0}^{s}\Gamma_1(\gamma(t))\dot\gamma^1(t)\,\mathrm{d}t\Bigr)\frac{\partial}{\partial x^1}\Big|_p\ . \end{aligned} \tag{6.18}$$

Comparing (6.18) and (6.17), we get

$$E'_1 = \frac{\partial}{\partial x'^1} \tag{6.19}$$

which means that any normal frame is holonomic as it should be in 1-dimensional manifolds.[2]

Exercise 6.3. Derive equations (6.17)–(6.19) by applying the results of subsection 3.2. Hint: use Theorems 3.3 and 3.1.

Example 6.8 (Frames/coordinates normal along geodesic in 2-manifold). This example demonstrates that frames/coordinates normal along geodesic paths in a 2-dimensional manifold can always be found explicitly via quadratures.

Suppose M is a C^3 2-dimensional manifold, ∇ is a C^0 connection on M and $\gamma\colon J\to M$ is a geodesic. Due to Remark 3.7 on page 95, the equations ${\Gamma^1}_{11}\circ\gamma=0$ and ${\Gamma^2}_{11}\circ\gamma=0$ hold in the coordinates $\{x^1,x^2\}$ provided by Lemma 3.1. Therefore the matrix Γ_1 along γ is

$$\Gamma_1\circ\gamma = [{\Gamma^i}_{j1}\circ\gamma]_{i,j=1,2} = \begin{pmatrix} 0 & {\Gamma^1}_{21}\circ\gamma \\ 0 & {\Gamma^2}_{21}\circ\gamma \end{pmatrix}\ . \tag{6.20}$$

The matrix-valued function $Y(s,s_0;-\Gamma_1\circ\gamma)$, entering into the main results of Subsection 3.2, is the solution of the matrix initial-value problem

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = -(\Gamma_1\circ\gamma(s))Y \qquad Y_{s=s_0} = \mathbb{1}$$

[2] It is an elementary exercise to be proved that all C^0 frames on C^1 1-dimensional manifold are (locally) holonomic.

which in an expanded form reads (see (6.20); $Y = [Y_{ij}]$):

$$\begin{aligned}
\frac{\mathrm{d}Y_{11}(s)}{\mathrm{d}s} &= -\Gamma^1{}_{21}(\gamma(s))Y_{21} \qquad Y_{11}|_{s=s_0} = 1\\
\frac{\mathrm{d}Y_{12}(s)}{\mathrm{d}s} &= -\Gamma^1{}_{21}(\gamma(s))Y_{22} \qquad Y_{12}|_{s=s_0} = 0\\
\frac{\mathrm{d}Y_{21}(s)}{\mathrm{d}s} &= -\Gamma^2{}_{21}(\gamma(s))Y_{21} \qquad Y_{21}|_{s=s_0} = 0\\
\frac{\mathrm{d}Y_{22}(s)}{\mathrm{d}s} &= -\Gamma^2{}_{21}(\gamma(s))Y_{22} \qquad Y_{22}|_{s=s_0} = 1.
\end{aligned}$$

The solutions of the last two initial-value problems are

$$Y_{21} = 0 \qquad Y_{22} = \exp\Big(-\int_{s_0}^{s} \Gamma^2{}_{21}(\gamma(t))\,\mathrm{d}t\Big) .$$

Inserting these functions into the previous two initial-value problems, we get

$$Y_{11} = 1 \qquad Y_{12} = 0$$

and consequently

$$Y(s, s_0; -\Gamma_1 \circ \gamma) = \operatorname{diag}\Big(1, \exp\Big(-\int_{s_0}^{s} \Gamma^2{}_{21}(\gamma(t))\,\mathrm{d}t\Big)\Big) . \tag{6.21}$$

According to Theorem 3.1, all frames $\{E'_i\}$ normal for ∇ along the geodesic path γ are such that $E'_i(p) = A^j_i(p)\frac{\partial}{\partial x^j}\big|_p$, where $p \in U_1$ and the matrix-valued function $A = [A^j_i]$ is given by (3.23) and (6.21), i.e.,

$$\begin{aligned}
A(p) = \{\mathbb{1} - \sum_{k=1,2} \Gamma_k(p_0)[x^k(p) - x^k(p_0)]\} \operatorname{diag}\Big(1, \exp\Big(-\int_{s_0}^{x^1(p)} \Gamma^2{}_{21}(\gamma(t))\,\mathrm{d}t\Big)\Big)B\\
+ \sum_{k,l=1,2} B_{kl}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] \quad (6.22)
\end{aligned}$$

with $\Gamma_1(p_0)$ given via (6.20) (as $p_0 = \gamma(s)$).

Similarly, by Theorem 3.3, all coordinate systems $\{z^I : i = 1, 2\}$ normal for a symmetric connection ∇ on some subset $U_z \subseteq U_1$ are given via (3.27), (3.28), (6.20) and (6.21), that is we have the explicit expression ($i = 1, 2$).

$$
\begin{aligned}
z^i(p) &= a^i + (B^{-1})^i_1[x^1(p) - s_0] \\
&+ (B^{-1})^i_1[x^1(p) - x^1(p_0)] + (B^{-1})^i_2 \exp\Bigl(-\int_{s_0}^{x^1(p)} {\Gamma^2}_{21}(\gamma(t))\,\mathrm{d}t\Bigr)[x^2(p) - x^2(p_0)] \\
&+ (B^{-1})^i_1[x^2(p) - x^2(p_0)] \sum_{j=1,2} {\Gamma^1}_{j2}(\gamma(x^1(p)))[x^j(p) - x^j(p_0)] \\
&+ (B^{-1})^i_2[x^2(p) - x^2(p_0)] \exp\Bigl(-\int_{s_0}^{x^1(p)} {\Gamma^2}_{21}(\gamma(t))\,\mathrm{d}t\Bigr) \\
&\times \sum_{j=1,2} {\Gamma^2}_{j2}(\gamma(x^1(p)))[x^j(p) - x^j(p_0)] \\
&+ \sum_{j,k,l=1,2} a^i_{jkl}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)]. \qquad (6.23)
\end{aligned}
$$

The formula (6.22) (resp. (6.23)) gives a complete local description of all frames (resp. coordinates) normal along a geodesic path γ for arbitrary (resp. symmetric) linear connection ∇ on a 2-dimensional manifold. If one needs frames globally normal along γ, the scheme proposed in the proof of Theorem 3.2 can be applied for their construction and/or complete description.

Example 6.9 (Open Einstein-de Sitter Universe). The (open) Einstein-de Sitter Universe is a concrete type of non-static homogeneous cosmological model (see [58] and Example I.7.11). Its geometrical base is the Einstein-de Sitter manifold which is a 4-dimensional Riemannian manifold with Riemannian metric g which in a suitable coordinate system $\{x^1 = x, x^2 = y, x^3 = z, x^4 = ct\}$ has components g_{ij}, $i, j = 1, \dots, 4$, forming the diagonal matrix [58, § 164]

$$[g_{ij}] = \operatorname{diag}\bigl(-\mathrm{e}^{f(x^4)}, -\mathrm{e}^{f(x^4)}, -\mathrm{e}^{f(x^4)}, 1\bigr) \qquad (6.24)$$

for some C^1 function f; the range of x^i is $\mathbb{R}$. The metric g induces a Riemannian connection ∇ whose non-zero coefficients in $\{x^i\}$ are (see (I.4.13) and (I.4.15))

$$
\begin{aligned}
{\Gamma^i}_{i4} = {\Gamma^i}_{4i} &= \frac{1}{2} f'(x^4) \qquad \text{for } i = 1, 2, 3 \\
{\Gamma^4}_{kk} &= \frac{1}{2} f'(x^4)\mathrm{e}^{f(x^4)} \qquad \text{for } k = 1, 2, 3
\end{aligned}
\qquad (6.25)
$$

where $f'(x^4) = \frac{\mathrm{d}f(x^4)}{\mathrm{d}x^4}$.

These equations yield the following coefficients' matrices

$$\Gamma_k = [\Gamma^i{}_{jk}] = \frac{1}{2} f'(x^4) \begin{pmatrix} 0 & 0 & 0 & \delta_{1k} \\ 0 & 0 & 0 & \delta_{2k} \\ 0 & 0 & 0 & \delta_{3k} \\ \delta_{1k} e^{f(x^4)} & \delta_{2k} e^{f(x^4)} & \delta_{3k} e^{f(x^4)} & 0 \end{pmatrix} \qquad \text{for } k = 1,2,3$$

$$\Gamma_4 = [\Gamma^i{}_{j4}] = \frac{1}{2} f'(x^4) \operatorname{diag}(1,1,1,0). \tag{6.26}$$

Consequently, the coordinate system $\{x^i\}$ is normal on the set $\{p : f'(x^4(p)) = 0\}$, which may be empty if f' is nowhere vanishing function. However, if, for instance, there is a point p_0 such that $f'(x^4(p_0)) = 0$, then $\{x^i\}$ is normal on the 3-dimensional submanifold $\{p : x^4(p) = x^4(p_0)\}$.

Due to Theorem 2.1, all coordinates y^i normal for ∇ at an arbitrarily chosen point p are given by (2.11′). So, substituting (6.25) into (2.11′), we get their explicit form as:

$$\begin{aligned} y^i(q) = y^i(p) + b^i_j[x^j(q) - x^j(p)] + \frac{1}{2} f'(x^4)[x^4(q) - x^4(p)] \sum_{l=1}^{3} b^i_l [x^l(q) - x^l(p)] \\ + \frac{1}{2} f'(x^4) e^{f(x^4)} b^i_4 \sum_{l=1}^{3} [x^l(q) - x^l(p)]^2 \\ + b^i_{jkl}(q)[x^j(q) - x^j(p)][x^k(q) - x^k(p)][x^l(q) - x^l(p)]. \end{aligned} \tag{6.27}$$

Similarly, all frames normal at a point p are described via Theorem 2.3. Thus, to obtain an explicit expression for them, one simply has to substitute (6.26) into (2.14).

Exercise 6.4. Investigate frames/coordinates normal along a C^1 path in the Einstein-de Sitter spacetime. For the purpose, the coefficients of ∇ should be calculated in the coordinates provided by Lemma 3.1 (see (I.3.6), (3.12) and (6.25)) and then Theorems 3.1 and 3.3 should be applied.

7. Conclusion

The main subject of this chapter was the existence and (non-)uniqueness of normal frames and coordinates for linear connections on differentiable manifolds. The existing literature on the problem was reviewed (in modern notation). It deals only with normal coordinates which exist only for torsionless connections. The few works investigating the asymmetric case do not add nothing new as they treat the symmetric part of the connection, thus transferring the problem to the exploration of some other torsion free linear connection.

The major classical results concerning the normal coordinates for linear connections are summarized in the Table 7.1 on the next page.

Table 7.1: Main contributions in the theory of normal coordinates for torsionless linear connections.

Year	Person	Result and original reference
1854	B. Riemann	Existence and construction of ('Riemannian') coordinates in a Riemannian manifold which are normal at a single point. [48]
1922	O. Veblen	Existence and construction of ('Riemannian normal') coordinates in a manifold with torsionless linear connection which are normal at a single point. [72]
1922	E. Fermi	Existence of ('Fermi') coordinates in a Riemannian manifold which are normal along a path without self-intersections. [52]
1926	T. Levi-Civita	Explicit transformation to the Fermi coordinates along paths without self-intersections. [81]
1927	L.P. Eisenhart	Existence and construction of particular kind of ('Fermi') coordinates on a manifold with torsionless linear connection which are normal along a path without self-intersections. [53]
1958	L. O'Raifeartaigh	Necessary and sufficient conditions for existence of coordinates normal on submanifold of a manifold with torsionless linear connection. If such coordinates exist, a particular example of them ('Fermi coordinates') is constructed. [55]

Besides the detailed review of these results, we have presented a number of their generalizations and wide discussion of related topics and methods, most of which seem to be new ones and appearing in the present book for the first time.[1] They include mainly: (i) Description of *all* frames normal for symmetric connections; (ii) Complete description of *all* coordinates normal for torsionless linear connections (cases at a given point, along paths, and on submanifolds); (iii) Existence and complete constructive description of *all* frames normal at a single point, along (locally injective) paths, and on submanifolds for arbitrary linear connections, with or without torsion. Moreover, with respect to the references cited, we have made a number of improvements of the existing proofs of known results, a lot of facts are formulated more precisely, and some assertions are (partially) generalized. The reader interested in what exactly is new in Subsections 2.1, 3.1, and 5.1 and in Section 4 may wish to compare them with the references given at their beginnings and the ones of Sections 2–5. Subsections 2.2, 3.2, and 5.2, entitled 'Complete description' contain original material.

Here we want to emphasize on the following facts. First, normal coordinates (may) exist only for torsionless linear connections while normal frames (may) exist for arbitrary connections; the normal frames for connections with non-zero (resp.

[1] Preliminary (implicit) versions of some of these results are contained in the author's papers [80], [76, Section 6], and [83, Section 5].

zero) torsion are with necessity anholonomic (resp. holonomic). Second, at a single point or along a path normal coordinates (resp. frames) always exist for torsionless (resp. arbitrary) linear connections; on submanifolds of dimension greater than one they exist if and only if the parallel transport on it along paths lying entirely in it is path-independent; in particular on the whole manifold normal coordinates/frames exist iff it is flat. Third, the normal coordinates, if any, are essentially local; globally they exist only as an exception. If local normal frames exist, then global normal frames always exist but they are smooth generally only locally.

Looking over the Riemannian coordinates, Riemannian normal coordinates, and Fermi coordinates (along a path or on submanifold, if any in the last case), we see that they are constructed in a uniform common way by employing the (local) existence of geodesics. From this point of view, the above coordinates are realization of one special kind of normal coordinates on submanifolds of zero, one, or higher dimension respectively.

Chapter III

Normal Frames and Coordinates for Derivations on Differentiable Manifolds

The existence, uniqueness, and construction of frames and coordinates normal for derivations (along vector fields, fixed vector field, paths, and fixed path) of the tensor algebra over a manifold are explored in details. For arbitrary vector fields or paths, normal frames (resp. coordinates) exist always (resp. if the torsion vanishes); on other submanifolds or along more general mappings necessary and sufficient conditions for such existence are derived. For derivations along fixed vector field or path normal frames and coordinates exist always. With a few exceptions, a complete constructive description of the normal frames and coordinates, if any, is presented. Frames simultaneously normal for two derivations are studied. With respect to the normal frames, the unique role of the linear connections amongst the other derivations is pointed out.

1. Introduction

The aim of this chapter is the investigation of frames and coordinates normal for different kinds of derivations of the tensor algebra over a differentiable manifold. Since the linear connections are a particular example of such derivations, the presented here material is a direct continuation and generalization of the one in Chapter II. But, as we shall see, a number of problems concerning normal frames and charts for general derivations are 'locally' reduced to the same problems for linear connections and, consequently, their (local) solutions could be found, in more or less ready form, in Chapter II.

Some of the results in the present chapter are partially based on the ones in the series of works [76, 80, 83–87] and are completely revised and generalized their versions. But most of the material is new and original.

Section 2 has an introductory character. The concepts of derivations and derivations along vector fields of the tensor algebra over a manifold are introduced. Their components, coefficients (if they are linear), curvature, and torsion are defined. Next, in Section 3, the normal frames and coordinates are defined as ones in which the components of a derivation along vector fields vanish (on some set). The equations describing the transition to normal frames or coordinates are derived and the linearity of a derivation along vector fields is pointed as a necessary conditions for their existence.

In Section 4 (resp. Section 5) is proved that at a single point (resp. along a (locally injective) path) frames normal for a linear at it (resp. along it) derivation along vector fields always exist and their complete descriptions are given. Besides, if the derivation is torsionless, all normal coordinates are found. In Sections 6–8, the problems of existence, uniqueness, and complete description of frames and local charts (or coordinates) on neighborhoods, on submanifolds, and along (injective or locally injective) mappings, respectively, for derivations along vector fields are studied in details and solved.

To the problems concerning frames or coordinates normal for derivations along fixed vector field is devoted Section 9. The existence of normal frames and coordinates in this case is proved. A complete description of the frames normal at a single point, along a path, and on the whole manifold are presented. The local charts (or coordinates) normal at a point are completely described. Along a path the explicit system of differential equations, which the normal coordinates must satisfy and which always have (local) solutions, is derived. A method for obtaining the coordinates (locally) normal on the whole manifold is pointed in the C^∞ case.

Normal frames for derivations along paths are investigated in Section 10. After the introduction of the basic definitions and notation, it is proved that frames normal for a derivation along a given (fixed) path always exist and their general form is found. A (local) holonomic extension of such frames, as well as of any other frame defined only along a path, is constructed. For derivations along arbitrary paths is proved that they admit normal frames iff they are covariant derivatives along paths induced by linear connections for which normal frames exist. Since the normal frames for the derivations and connections turn to be identical, all problems for these frames are transferred to similar ones considered in Chapter II.

Section 11 deals with problems connected with frames simultaneously normal for two derivations along arbitrary/fixed vector field or path. Necessary and sufficient conditions for the existence of such frames are found. In particular, in the case of arbitrary vector field or path, they exist iff the two derivations coincide. Normal frames for mixed linear connections are explored. It is shown that this range of problems is completely and equivalently reduced to similar one for two, possibly identical, linear connections, the contra- and co-variant 'parts' of the initial mixed connection.

In Section 12 are collected and commented some results concerning linear connections obtained in the preceding sections of this chapter.

Section 13 illustrates the theory of the preceding sections with concrete examples.

Section 14 contains a discussion of some terminological problems linked to the normal frames or coordinates.

The chapter ends with certain general remarks in Section 15.

2. Derivations of the tensor algebra over a manifold

The idea of a derivation of the algebra $\boldsymbol{T}(M)$ of tensor fields over a manifold M is in $\boldsymbol{T}^1(M)$ to be introduced an operator analogous to the (ordinary or partial) derivative of scalar functions and 'compatible' with the tensor structure of $\boldsymbol{T}^1(M)$. The covariant derivative (along vector fields), introduced via Definition I.3.1 is, as we shall see below, a particular example of a derivation of the tensor algebra over a differentiable manifold. The analysis of the system of axioms in Definition I.3.1 reveals that not all of them are equally important for an abstract definition of derivation and leads to the following definition (see [11, Chapter I, § 3] and [12,88]).

Definition 2.1. A *derivation of the tensor algebra* $\boldsymbol{T}^1(U)$ (or of $\boldsymbol{T}(U)$), U being an open set in M, is a mapping $D\colon \boldsymbol{T}^1(U) \to \boldsymbol{T}^0(U)$ possessing the following properties:

(i) Linearity,

$$D(aK + bL) = aD(K) + bD(L), \qquad a, b \in \mathbb{K}, \quad K, L \in \mathfrak{T}^{r;1}_s(U).$$

(ii) Leibnitz rule (relative to tensor multiplication),

$$D(K \otimes L) = (D(K)) \otimes L + K \otimes D(L) \quad K, L \in \boldsymbol{T}^1(U).$$

(iii) Type preservation,

$$D\colon \mathfrak{T}^{r;1}_s(U) \to \mathfrak{T}^{r;0}_s(U).$$

(iv) Commutativity with every contraction operator,

$$[D, C]_{\_} := D \circ C - C \circ D = 0.$$

Example 2.1. The comparison of Definitions I.3.1 and 2.1 (see also comments I.3.1) shows that every covariant derivative ∇_X along $X \in \mathfrak{X}(U)$ is a derivation of $\boldsymbol{T}^1(U)$. Other examples of derivations are provided by the Lie derivative $\mathcal{L}_X$ along $X \in \mathfrak{X}(U)$ and by arbitrary tensor field $S \in \mathfrak{T}^1_1$ of type $(1,1)$ [11]. This is easily seen from the following local expansions in the tensor frame with basic fields (I.2.41) induced by a frame $\{E_i\}$ on U [11, 19, 89]:

$$\begin{aligned}(\mathcal{L}_X(K))^{i_1\dots i_r}_{j_1\dots j_s} = X\big(K^{i_1\dots i_r}_{j_1\dots j_s}\big) &+ \sum_{a=1}^{r}[-E_k(X^{i_a}) + C^{i_a}_{lk}X^l]K^{i_1\dots i_{a-1}ki_{a+1}\dots i_r}_{j_1\dots j_s}\\ &- \sum_{b=1}^{s}[-E_{j_b}(X^k) + C^k_{lj_b}X^l]K^{i_1\dots i_r}_{j_1\dots j_{b-1}kj_{b+1}\dots j_s},\end{aligned} \tag{2.1}$$

$$(S(K))^{i_1\dots i_r}_{j_1\dots j_s} = \sum_{a=1}^{r} S^{i_a}_k K^{i_1\dots i_{a-1}ki_{a+1}\dots i_r}_{j_1\dots j_s} - \sum_{b=1}^{s} S^k_{j_b} K^{i_1\dots i_r}_{j_1\dots j_{b-1}kj_{b+1}\dots j_s}, \tag{2.2}$$

where $K \in \mathfrak{T}^{r;1}_s(U)$ is arbitrary C^1 tensor field of type (r, s) on U.

The following proposition describes the general structures of the derivations of $\boldsymbol{T}^1(U)$ [11, Chapter I, Proposition 3.3].

Proposition 2.1. *For every derivation D of $\boldsymbol{T}^1(U)$, there exist unique vector field $X \in \mathfrak{X}^1(U)$ and tensor field $S_X \in \mathfrak{T}^1_1(U)$, generally depending on X, such that*

$$D = \mathcal{L}_X + S_X, \tag{2.3}$$

i.e., every derivation admits a unique decomposition (2.3).

Proof. See [11, Chapter I, proof of Proposition 3.3]. □

Example 2.2. In particular, a covariant derivative ∇_X along $X \in \mathfrak{X}(U)$ has a representation (2.3) with S_X given via

$$S_X(Y) = \Sigma_X(Y) := \nabla_X Y - [X, Y]_{\_} = \nabla_Y X + T(X, Y) \tag{2.4}$$

where $Y \in \mathfrak{X}(U)$ and T is the torsion of ∇ (see (I.3.12)).

Evidently, every pair (X, S) with $X \in \mathfrak{X}(U)$ and $S\colon \mathfrak{X}(U) \to \mathfrak{T}^1_1(U)$, $S\colon X \mapsto S_X$, defines a unique derivation of $\boldsymbol{T}^1(U)$ by (2.3). There is a one-to-one onto correspondence, given via (2.3), between the sets of derivations of $\boldsymbol{T}^1(U)$ and the set of pairs (X, S). If some derivation D has a decomposition (2.3), we shall write D^S_X instead of D; if the dependence on X is essential and the one on S is not, we reduce D^S_X to D_X. We say that D_X is a *derivation along* X.

Using (2.1)–(2.3), we find the local components of $D_X(K)$, $K \in \mathfrak{T}^{r;1}_s$, in some local frame as

$$
\begin{aligned}
(D_X(K))^{i_1\dots i_r}_{j_1\dots j_s} = X\big(K^{i_1\dots i_r}_{j_1\dots j_s}\big) &+ \sum_{a=1}^{r} {\Gamma_X}^{i_a}{}_{k} K^{i_1\dots i_{a-1}ki_{a+1}\dots i_r}_{j_1\dots j_s} \\
&- \sum_{b=1}^{s} {\Gamma_X}^{k}{}_{j_b} K^{i_1\dots i_r}_{j_1\dots j_{b-1}kj_{b+1}\dots j_s}.
\end{aligned}
\tag{2.5}
$$

Here the functions ${\Gamma_X}^i{}_j \in \mathfrak{F}(U)$ are defined via the expansion

$$D_X(E_k) =: {\Gamma_X}^i{}_k E_i \tag{2.6}$$

and their explicit form is

$${\Gamma_X}^i{}_j = (S_X)^i_j - E_j(X^i) + C^i_{kj}X^k \tag{2.7}$$

where $C^i_{jk} \in \mathfrak{F}(U)$ define the commutators of the basic vector fields according to (I.3.15). We call ${\Gamma_X}^i{}_j$ the (local) *components* of D_X in $\{E_i\}$; respectively, we call $\mathbf{\Gamma}_X := \big[{\Gamma_X}^i{}_j\big]$ *matrix* (of the components) or *components' matrix* of D_X.

An important result follows from the comparison of equations (2.6) and (I.3.1) (or (2.5) and (I.3.2) and taking into account (I.3.3)).

Proposition 2.2. *A derivation D_X of $\boldsymbol{T}^1(U)$ with local components ${\Gamma_X}^i{}_j$ in a frame $\{E_i\}$ is a covariant derivative along X iff*

$${\Gamma_X}^i{}_j = {\Gamma^i}_{jk}X^k \tag{2.8}$$

for some functions ${\Gamma^i}_{jk}\colon U \to \mathbb{K}$, which are coefficients of a linear connection on U.

Hence the covariant derivatives are derivations whose components depend linearly on the generating them via (2.3) vector field X.

Definition 2.2. A *derivation D_X* of $\boldsymbol{T}^1(M)$ is said to be *linear on (in) a set $U \subseteq M$ or along a mapping $\eta\colon Q \to M$* for a set $Q \neq \varnothing$, if in some frame (and hence in *all* frames) $\{E_i\}$ the relation

$$\mathbf{\Gamma}_X(p) = [{\Gamma_X}^i{}_j(p)] = \Gamma_k(p)X^k(p) \tag{2.9}$$

holds for some matrix-valued functions $\Gamma_1, \dots, \Gamma_{\dim M}$ on U or on $\eta(Q)$ and every $p \in U$ or $p \in \eta(Q)$, respectively.

According to Proposition 2.2, the covariant derivatives are derivations which are linear at every point.

If we transform the frame $\{E_i\}$ on U into a frame $\{E'_i = A^j_i E_j\}$ by means of a C^1 non-degenerate matrix-valued function $A = [A^j_i]$, we see from (2.6) and Definition 2.1 that the components ${\Gamma_X}^i{}_j$ of D_X transform as (cf. (I.3.5))

$${\Gamma_X}^i{}_j \mapsto {\Gamma'_X}^i{}_j = (A^{-1})^i_k[{\Gamma_X}^k{}_m A^m_j + X(A^k_j)]. \tag{2.10}$$

Introducing the *matrix* $\mathbf{\Gamma}_X := [\Gamma_X{}^i{}_j]_{i,j=1}^{\dim M}$ of the components of the derivation D_X, we rewrite (2.10) in the form (cf. (I.5.3))

$$\mathbf{\Gamma}_X \mapsto \mathbf{\Gamma}'_X = A^{-1}[\mathbf{\Gamma}_X A + X(A)] \tag{2.11}$$

where $X(A) := [X(A^i_j)]_{i,j=1}^{\dim M}$.

Definition 2.3. A *derivation D along vector fields of the tensor algebra $\boldsymbol{T}^1(U)$*, $U \subseteq M$, is a mapping assigning to every $X \in \mathfrak{X}(U)$ a derivation D_X of $\boldsymbol{T}^1(U)$ with decomposition $D_X = \mathcal{L}_X + S_X$ for some $S\colon \mathfrak{X}(U) \to \mathfrak{T}^1_1(U)$, $S\colon X \mapsto S_X$.

The concept of a derivation along vector fields[1] generalizes the one of a linear connection (see Definition I.3.1) to which it reduces if the derivative D_X along X is a covariant one. In a given frame $\{E_i\}$, the main difference between the linear connections and derivations along vector fields is expressed by equations (I.3.1) and (2.6). The former one defines the *coefficients* $\Gamma^i{}_{jk}\colon U \to \mathbb{K}$ of a linear connection ∇ irrespectively of some vector fields, while the latter one defines the *components* $\Gamma^i{}_j\colon U \to \mathbb{K}$ of a derivation D along vector fields only with respect to some vector field X and, generally, one can not get rid of the dependence on X unless D_X is linear in X.

Definition 2.4. A *derivation D along vector fields* is *linear on (in) a set $U \subseteq M$ or along a mapping $\eta\colon Q \to M$, $Q \neq \varnothing$*, if in some frame (and hence in all frames) $\{E_i\}$ the matrix $\mathbf{\Gamma}_X := \left[\Gamma^i{}_j\right]_{i,j=1}^{\dim M}$ of the components of D along every $X \in \mathfrak{X}(M)$ admits a decomposition (2.9) for some matrix-valued functions $\Gamma_1, \dots, \Gamma_{\dim M}$ on U or on $\eta(Q)$ and every $p \in U$ or $p \in \eta(Q)$, respectively.

Example 2.3. It is trivial to verify that the linear connections are derivation along vector fields linear at every point. Conversely, a derivation D along vector fields linear on U is a linear connection on U. In fact, equation (2.8) holds for the components of such a derivation[2] and, by Proposition 2.2, D_X is a covariant derivative along X. So, by Definition I.3.1, D is a linear connection on U.

The elements $\Gamma^i{}_{jk}$ of the matrices Γ_k, corresponding via (2.9) to a linear derivation D along vector fields, will be called *coefficients* of D on U. It is clear, the coefficients of D exist only on sets on which D is linear and they transform as coefficients of a linear connection by (I.3.5). The matrices $\Gamma_k = \left[\Gamma^i{}_{jk}\right]_{i,j=1}^{\dim M}$ will be called *matrices of the derivation coefficients*, or *derivation's matrices*, or *coefficients' matrices* of the derivation.

The definition of a C^r derivation along vector fields (on C^k manifold with $k \geq r$) is similar to the one of C^r linear connection (see Subsection I.3.2) and it is based on the transformation law (2.11). Given a class of frames connected via linear transformations whose matrices are $C^{r'}$, $r' \geq 1$, matrix-valued functions, a

[1] In the series of papers [76, 80, 83] the name S-derivation is used instead of derivation along vector fields.

[2] If (2.8) is valid, the equality (2.5) is equivalent on U to (I.3.2) and (I.3.3) with $\nabla = D$.

derivation along vector fields is said to be of class C^r, $r \le r'-1$ with respect to it if its local components in one (and hence in all) frame(s) in the above set of frames are of class C^r. In what follows, we shall suppose by default that a given class of $C^{r'}$, $r' \le k-1$, frames is fixed on a C^k, $k \ge 2$, manifold M and it consists of all frames associated to one (or all) systems of local coordinates on M or its open subsets and all frames obtainable from them by means of linear transformations with $C^{r'}$ matrices. For instance, by a C^1 derivation along vector fields we understand one on a C^3 manifold whose local components are C^1 functions in any frame which is coordinate or is obtainable from such by means of C^2 transformations.

Analogously to the case of linear connections (see the end of Subsection I.3.2), the concepts of curvature and torsion can be defined for derivations along vector fields.

The (operators of) *curvature* R^D and *torsion* T^D of a C^1 or arbitrary, respectively, derivation D along vector fields on C^3 or C^2 manifold M, resp., are mappings

$$R^D \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \boldsymbol{T}(M) \to \boldsymbol{T}(M)$$
$$T^D \colon \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \to \mathfrak{X}^0(M)$$

such that

$$R^D \colon (X,Y) \mapsto R^D(X,Y) := D_X \circ D_Y - D_Y \circ D_X - D_{[X,Y]_{\_}} \tag{2.12}$$
$$T^D \colon (X,Y) \mapsto T^D(X,Y) := D_X(Y) - D_Y(X) - [X,Y]_{\_} \tag{2.13}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. By abuse of the language, the quantities $R^D(X,Y)$ and $T^D(X,Y)$ will also be called (operators of) curvature and torsion, respectively, of D (along the pair (X,Y)). The mapping $R\colon D \mapsto R^D$ (resp. $T\colon D \mapsto T^D$), assigning to a derivation D along vector fields its curvature (resp. torsion), will be called curvature (resp. torsion) operator, or simply curvature (resp. torsion).

Using (2.5), it is a simple exercise to verify that $R^D(X,Y)$ is a tensor field of type $(1,1)$, which we consider as a derivation, while $T^D(X,Y)$ is a vector field and their components in a local frame $\{E_i\}$ are given respectively by

$$\left[\left(R^D(X,Y)\right)^i_{\;j}\right] = X(\mathbf{\Gamma}_Y) - Y(\mathbf{\Gamma}_X) + \mathbf{\Gamma}_X\mathbf{\Gamma}_Y - \mathbf{\Gamma}_Y\mathbf{\Gamma}_X - \mathbf{\Gamma}_{[X,Y]_{\_}}, \tag{2.14}$$
$$\left(T^D(X,Y)\right)^i = {\Gamma_X}^i_{\;j} Y^j - {\Gamma_Y}^i_{\;j} X^j - C^i_{jk} X^j Y^k \tag{2.15}$$

where $\mathbf{\Gamma}_X := \left[{\Gamma_X}^i_{\;j}\right]$ is the matrix of the components of the derivation D along a vector field X in $\{E_i\}$ (see (2.6)) and the functions C^i_{jk} are defined via $[E_i, E_j]_{\_} =: C^k_{ij} E_k$.

Since the equivalent equations (2.8) and (2.9) hold on U for a linear on $U \subseteq M$ derivation D along vector fields, the equations (2.14) and (2.15) take respectively the forms[3]

$$\big(R^D(X,Y)\big)^i_j = R^i_{jkl} X^k Y^l \tag{2.16}$$

$$\big(T^D(X,Y)\big)^i = T^i_{jk} X^j Y^k. \tag{2.17}$$

Here R^i_{jkl} and T^i_{jk} are given by (I.3.13) and (I.3.14), respectively, i.e., they are the components of the ordinary curvature and torsion tensors of a linear connection ∇ on U with coefficients coinciding with the ones of D on U.

A derivation along vector fields with vanishing curvature (resp. torsion) on a set $U \subseteq M$ will be called *flat*, or *curvature free*, or *integrable* (resp. *torsionless* or *torsion free*) on U. For $U = M$, we call it simply flat (resp. torsionless).

Details on part of the above material, as well as other general results concerning derivations of the tensor algebra over a manifold, the reader can find in [11].

The symmetry of the coefficients,

$$\Gamma^i{}_{jk} = \Gamma^i{}_{kj}, \tag{2.18}$$

of a torsion free derivation in coordinate frames on the set(s) on which it is linear is a trivial corollary from (2.15) and (2.9).

3. General overview

The heuristic arguments at the beginning of Section I.5, concerning linear connections, are completely applicable to the derivations along vector fields. According to them, one can expect the existence of (local) frames in which all terms except the first one in the right-hand side of (2.5) vanish. Obviously, this is equivalent to the vanishment of the local components of a given derivation along vector fields in some frame. Further in this chapter, we shall investigate at length when such special frames exist. Below in the present section, some general properties of these frames will be found provided they exist.

Definition 3.1. Given a manifold M, a subset $U \subseteq M$ and a derivation D along vector fields of the tensor algebra $\boldsymbol{T}^1(M)$. A *frame* $\{E_i\}$, defined on an open subset of M containing U or equal to it, is called *normal* on U for D if in it the components of D along every vector field $X \in \mathfrak{X}(M)$, i.e., of D_X, vanish everywhere on U. Respectively, if $g\colon Q \to M$, Q being non-empty set, a *frame* $\{E_i\}$, defined on an open subset of M containing $g(Q)$ or equal to it, is called *normal for D along g* if it is normal on $g(Q)$.

Similarly to (I.5.1), we can rewrite the first part of this definition as

$$\{E_i\} \quad \text{is } normal \text{ on } U \quad\iff\quad \Gamma_X{}^i{}_j|_U = 0 \quad \text{in } \{E_i\}. \tag{3.1}$$

[3]It should be emphasized, equations (2.16) and (2.17) are valid only on the set(s) on which the derivation D is linear.

Consequently, in a normal frame $\{E_i\}$, due to (2.5), the components of $D_X K$, $K \in \mathfrak{T}^{r;1}_s(U)$, are

$$(D_X(K))^{i_1\dots i_r}_{j_1\dots j_s} = X\Big(K^{i_1\dots i_r}_{j_1\dots j_s}\Big) = X^l E_l\Big(K^{i_1\dots i_r}_{j_1\dots j_s}\Big). \tag{3.2}$$

If $\{E_i\}$ is an arbitrary frame on (a neighborhood of) $U \subseteq M$, then on U exists a normal frame $\{E'_i\}$ if and only if there is a C^1 non-degenerate matrix-valued function $A = [A^i_j]$ transforming $\{E_i\}$ into $\{E'_i\}$, $E'_i = A^j_i E_j$, and such that, according to (3.1) and (2.11), satisfies the *normal frame equation*

$$(\mathbf{\Gamma}_X A + X(A))|_U = 0 \tag{3.3}$$

which in component form reads (see (2.10))

$$({\Gamma_X}^i{}_m A^m_j + X^m E_m(A^i_j))|_U = 0. \tag{3.3$'$}$$

Therefore a derivation D along vector fields admits normal frames on U iff (3.3) has solutions with respect to A in some frame $\{E_i\}$ and every vector field X. Practically all properties, including the existence, of the normal frames are, explicitly or implicitly, related to the equation (3.3). Below we shall make some simple general conclusions from it provided it has (non-degenerate) solution(s).

Proposition 3.1. *Let a derivation D along vector fields of $\boldsymbol{T}^1(M)$ admits a frame normal on $U \subseteq M$. Then the derivation D is linear on U, i.e., equation* (2.9) (*or* (2.8) *in component form*) *holds for every $p \in U$.*

Proof. Let $\{E'_i\}$ be a frame normal for D on U and $\{E_i\}$ be arbitrary frame on U. There is a C^1 non-degenerate matrix-valued function $A = [A^i_j]$ such that $E'_i = A^j_i E_j$ for which (3.3) is valid. Hence, we have $\mathbf{\Gamma}_X|_U = -[(X(A))A^{-1}]|_U = -[X^k(E_k(A))A^{-1}]|_U$, i.e., equation (2.9) holds for

$$\Gamma_k(p) = -[E_k(A)A^{-1}]|_p, \qquad p \in U, \tag{3.4}$$

and hence, by Definition 2.4, the derivation D is linear on U. □

Thus the linearity on U of a derivation D along vector fields is a necessary condition for the existence of frames normal for D on U. As we shall see in the next sections, this condition is generally not sufficient, exceptions being the zero- and one-dimensional cases.

Proposition 3.1 allows an essential simplification of equation (3.3) and its equivalent version (3.3$'$): the dependence on the arbitrary vector field X can be removed. Indeed, if D admits normal frame(s), then $\mathbf{\Gamma}_X = \Gamma_l X^l$ on U for some matrix-valued functions $\Gamma_1, \dots, \Gamma_{\dim M}$ which, when inserted into (3.3), reduces (3.3) to

$$(\Gamma_l A + E_l(A))|_U = 0 \tag{3.5}$$

due to the arbitrariness of $X^1, \dots, X^{\dim M}$. Correspondingly (3.3′) takes the form

$$\Gamma^i{}_{mk} A^m_j + E_k(A^i_j))|_U = 0. \tag{3.5'}$$

where $\Gamma^i{}_{jk} \colon U \to \mathbb{K}$ are the components of Γ_k, $\Gamma_k = [\Gamma^i{}_{jk}]_{i,j=1}^{\dim M}$, i.e., they are the coefficients of the initial derivation.

Equation (3.5) is identical with equation (I.5.4), as it should be: according to the above-said, the normal frames on U, if any, for a derivation D along vector fields and for a linear connection coinciding on U with D are identical. This explains why most of the properties of the normal frames for derivations along vector fields and linear connections are similar or identical. Regardless of this, below some results, concerning normal frames for derivations, will be proved independently, for others a reference to the case of linear connections will be given, and, at last, to some peculiarities of the case of derivations attention will be paid.

According to the aforesaid, if frames normal on U for D exist, they are uniquely defined by the coefficients $\Gamma^i{}_{jk} \colon U \to \mathbb{K}$ of D.

The following proposition and the first corollary of it describe the set of normal frames, in any, for a given derivation along vector fields.

Proposition 3.2. *Let a derivation D along vector fields admits a frame normal on $U \subseteq M$. The set of all frames normal for D on U consists of the frames that can be obtained from a fixed frame normal for D on U by means of linear transformations whose matrices vanish on U under the action of the basic vector fields of the normal frames.*

Proof. Suppose the frames $\{E_i\}$ and $\{E'_i\}$ are normal for D on U. Then, due to (3.1), $\mathbf{\Gamma}_X|_U = \mathbf{\Gamma}'_X|_U = 0$ for every vector field X, so equation (3.3) reduces to $(X(A))|_U = 0$, $A = [A^i_j]$. The choice $X = E_i$ implies $E_i(A)|_U = 0$. The converse assertion is almost evident: if $\{E_i\}$ is normal on U, i.e., $\mathbf{\Gamma}_X|_U = 0$, and $E'_i = A^j_i E_j$ with $E_i(A)|_U = 0$, then, from the transformation law (2.11), we get $\mathbf{\Gamma}'_X|_U = 0$, i.e., the frame $\{E'_i\}$ is normal for D on U. □

Corollary 3.1. *All frames normal on U for some derivation along vector fields, if any, are connected via linear transformations whose coefficients vanish on U under the action of the basic vector fields of the normal frames.*

Proof. See Proposition 3.2 or its proof. □

Corollary 3.2. *If a derivation D along vector fields admits a frame normal on $U \subseteq M$, then* (2.9) *holds in it with*

$$\Gamma_k|_U = 0. \tag{3.6}$$

Proof. See Proposition 3.1, equation (3.4), and Corollary 3.1. □

Remark 3.1. Here and below, we suppose the derivation D and the vector field X to be defined on the whole manifold M, i.e., $D\colon X \mapsto D_X$ with D_X being derivation of $\boldsymbol{T}^1(M)$ and $X \in \mathfrak{X}(M)$. If D and X are defined on a smaller set, e.g., on a submanifold N of M with $\dim N < \dim M$, then all of the above (and next) results are valid *mutatis mutandis* provided M is replaced with N; in particular, all frames will be defined on (subsets of) N and, correspondingly, the Latin indices $i, j, \dots$ should run from 1 to $\dim N$, not to $\dim M$.

Due to equation (3.6), Definition I.5.1 is a special case of Definition 3.1 when applied to linear connections. In fact, if ∇ is a linear connection, its components ${\Gamma_X}^i{}_j$ are defined by (2.6) with ∇ for D and satisfy (2.8) at every point in U and for every $X \in \mathfrak{X}(U)$. So, if ∇ admits frames normal on U, equation (3.6) implies that a frame normal for ∇ on U according to Definition 3.1 is normal for ∇ on U by Definition I.5.1. Conversely, if ∇ admits frames normal on U according to Definition I.5.1, by equation (2.8) they are also normal by Definition 3.1. Consequently, Definitions 3.1 and I.5.1 are equivalent when linear connections are concerned.

Now we shall turn our attention to the holonomicity of the normal frames, if such exist.

Proposition 3.3. *Let a derivation D along vector fields admits C^1 frames normal on a neighborhood $U \subseteq M$. All of these frames are either holonomic or anholonomic depending on is the torsion of D zero of non-zero on U, respectively.*

Proof. Let $\{E_i\}$ be a frame normal for D on U. In it, by virtue of (2.15), the torsion along the pair (E_i, E_j) is $(T^D(E_i, E_j))|_U = -(C^k_{ij} E_k)|_U = -([E_i, E_j]_\text{-})|_U$ and, consequently (see (2.15)), we have $(T^D(X, Y))|_U = -(X^i Y^j [E_i, E_j]_\text{-})|_U$ for every $X, Y \in \mathfrak{X}(U)$. Therefore $[E_i, E_j]_\text{-}|_U = 0$ iff $T^D|_U = 0$. $\square$

Remark 3.2. Remarks I.5.2 and I.5.3 are valid *mutatis mutandis* in a case of derivations along vector fields.

Recalling Remark I.5.1 on page 40, which is completely valid in a case of general derivations along vector fields, we have to emphasize that the (an)holonomicity of a frame $\{E_i\}$ normal on U does not imply any conclusions concerning the (an)holonomicity of the frame $\{E_i\}$ outside U if it is defined on a set larger than U. Moreover, if $\{E_i\}$ is normal on U but not outside it, the derivation's components in $\{E_i\}$ must vanish solely on U and generally are non-zero outside U.

Proposition 3.3 simply means that only the torsionless derivations along vector fields admit holonomic frames normal on a neighborhood. Moreover, for these derivations anholonomic normal frames do not exist.

Definition 3.2. Let D be a derivation along vector fields admitting normal frames. A *chart* (V, x) of M and the associated to it *coordinate system* $\{x^i\}$ are called *normal* (*for* D) *on a set* $U \subseteq V$ if the coordinate frame $\{\frac{\partial}{\partial x^i}\}$ is normal for D on U.

So, the normal coordinates are the ones for which the associated local frames are normal. They may exist only in the torsionless case, as stated in the following assertion.

Corollary 3.3. *Normal coordinates may exist on a neighborhood only for torsionless derivations along vector fields and they* (*locally*) *generate all frames normal for them, if any.*

Proof. See Definition 3.2 and Proposition 3.3. □

We can equivalently restate Corollary 3.3 by saying that derivations (along vector fields) with non-vanishing torsion do not admits holonomic frames normal on a neighborhood and if for them normal frames exist, they are anholonomic.

Now we want to turn our attention to coordinates normal on a subset $U \subseteq M$ which may not be a neighborhood.

The normal coordinates, if any, for a derivation (along vector fields) admitting normal frames can be found as solutions of (3.3′). If on $U \subseteq M$ normal frames exist and (V, x) is a chart with $V \cap U \neq \varnothing$ and associated coordinates $\{x^i\}$, the coordinates $x'^{\,i} = x'^{\,i}(x^1, \dots, x^{\dim M})$ defined on V are normal on $U \cap V$ if and only if (3.3′) holds for $E_i = \frac{\partial}{\partial x^i}$ and $A_i^j = \frac{\partial x^j}{\partial x'^{\,i}}$, i.e., iff

$$\left({\Gamma_X}^i{}_m \frac{\partial x^m}{\partial x'^{\,j}} + X^m \frac{\partial}{\partial x^m}\Big(\frac{\partial x^i}{\partial x'^{\,j}}\Big)\right)\bigg|_{U\cap V} = 0$$

which, by virtue of $\boldsymbol{\Gamma}_{\boldsymbol{X}} = \Gamma_k X^k$, is equivalent to[1]

$$\left(\frac{\partial^2 x^i}{\partial x'^{\,j} \partial x'^{\,k}} + \frac{\partial x^m}{\partial x'^{\,j}} \frac{\partial x^n}{\partial x'^{\,k}} \Gamma^i{}_{mn} \right)\bigg|_{U\cap V} = 0. \tag{3.7}$$

As one can expect, this equation is identical with the normal frame equation (I.5.4′) on page 41 defining the normal coordinates for linear connections.

Proposition 3.4. *If a derivation D along vector fields admits coordinates normal on a set $U \subseteq M$, then it is torsionless on U. In other words, if D admits a holonomic frame normal on U, it is torsion free on U.*

Proof. If $p \in U$ and $\{x'^{\,i}\}$ are coordinates in a neighborhood of p normal on U, then (3.7) is valid. Antisymmetrizing this equation with respect to j and k, we find $\Gamma^i{}_{[mn]}|_U = 0$. Combining the last result with (2.17) and (I.3.14), we get $T|_U = 0$ as in a holonomic frame, like $\{\frac{\partial}{\partial x^i}\}$ in (3.7), is fulfilled $C^i_{jk} = 0$. □

The last proposition immediately implies the following assertion.

Corollary 3.4. *If $U \subseteq M$, only the torsionless on U derivations along vector fields may admit frames which are normal and holonomic on U.*

Later it will be proved that the vanishment of the torsion on a submanifold is also a sufficient condition for the existence of normal coordinates for derivations along vector fields admitting frames normal on them.

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[1] The same result follows also from (3.5′) for $E_i = \partial/\partial x^i$ and $A_i^j = \partial x^j / \partial' x^i$.

If a derivation D along vector fields admits frames normal on $U \subseteq M$, it is a linear connection on U, as pointed above. To this connection corresponds a parallel transport (see Subsection I.3.3) for which *in ex.* are valid the results concerning parallel transports and normal frames of Section I.5 (see page 42 and further). In particular, Propositions I.5.5 and I.5.6 remain true for the parallel transport in U associate to D.

4. Frames and coordinates normal at a point

From the considerations in the previous section and Subsection II.2.2, we can expect the existence of frames normal for every derivation D along vector fields at every fixed point p in a C^2 manifold M provided D is linear at p, i.e.,

$$\mathbf{\Gamma}_X(p) = X^k(p)\Gamma_k \tag{4.1}$$

in every frame for some *constant* $\dim M \times \dim M$ matrices $\Gamma_1, \dots, \Gamma_{\dim M}$. Below we shall see that just this is the situation, which can be proved in a number of ways.

Theorem 4.1. *Let M be a C^2 manifold, p be a given point in M, and D be a derivation along vector fields of $\boldsymbol{T}^1(M)$. There exist frames normal for D at p if and only if D is linear at p. Moreover, if D is linear at p and (V,x) is a chart with $V \ni p$, then in V all frames normal for D at p are $\{E_i = A_i^j \frac{\partial}{\partial x^j}\}$ where $A := [A_i^j]\colon V \to \mathrm{GL}(\dim M, \mathbb{K})$ is of class C^1, non-degenerate, and its general form is*

$$A(q) = \{\mathbb{1} - \Gamma_j[x^j(q) - x^j(p)]\}A_0 \\ + A_{jk}(q)[x^j(q) - x^j(p)][x^k(q) - x^k(p)], \qquad q \in V \tag{4.2}$$

where $\Gamma_1, \dots, \Gamma_{\dim M}$ are the (constant) matrices of the coefficients of D given via (4.1), A_0 is a constant non-degenerate matrix, and A_{jk} are C^1 matrix-valued functions on U such that they and their first partial derivatives are bounded at p.

Proof. If D admits frames normal at p, by Proposition 3.1, it is linear at p, i.e., (4.1) holds. Conversely, if D is linear at p and (V,x) is a chart with p in its domain, a frame $\{E_i = A_i^j \frac{\partial}{\partial x^j}\}$ is normal for D at p iff $A = [A_i^j]$ satisfies (3.5) with $U = \{p\}$. In the proof of Theorem II.2.3 (p. 82), we proved that this equation always has solutions in V, the general one being given by (4.2) (see (II.2.14) with Γ_j for $\Gamma_j(p)$ and V for U). □

Theorem 4.1 provides a *complete* description of all frames normal at a single point for a given derivation (along vector fields) linear at this point. By Corollary 3.1, these frames are connected via linear transformations whose matrices vanish at the given point under the action of the basic vector fields of the normal frames.[1] According to Corollary 3.4 for $U = \{p\}$, the frames normal at a single point p and holonomic at p exist if the derivation is torsionless at p.

In the torsion free case, the linear derivations along vector fields admit normal coordinates at every fixed point p, which coordinates are solutions of (3.7) for $U = \{p\}$ with respect to x'^{i}.[2] Equivalently, in this case, coordinates $\{y^i\}$ normal at p can be found by transforming some local coordinates x^i, defined in a neighborhood of p, in such a way that the transformation $\{x^i\} \to \{y^i\}$ leads to the vanishment at p of the derivation's coefficients in $\{y^i\}$. Since they change according to (I.3.5), this problem is solved in the discussion preceding Proposition II.2.3 (see the paragraph containing equation (II.2.11) on page 80). So, the general form of the coordinates $\{y^i\}$ normal at p is given by equation (II.2.11′) on page 80 in which ${}^{x}\Gamma^{l}{}_{(ij)}(p)$ should be replace with the coefficients ${}^{x}\Gamma^{l}{}_{ij}$ of the derivation (along vector fields) at p. In this way we have proved the following theorem generalizing Theorem II.2.1.

Theorem 4.2. *Let M be a C^3 manifold, $p \in M$ and D be torsionless and linear at p derivation along vector fields. There exist coordinates $\{y^i\}$ normal at p for D whose general form is*

$$\begin{aligned} y^i(q) = y^i(p) &+ b^i_j[x^j(q) - x^j(p)] \\ &+ b^i_l\,{}^{x}\Gamma^{l}{}_{jk}[x^j(q) - x^j(p)][x^k(q) - x^k(p)] \\ &+ b^i_{jkl}(q)[x^j(q) - x^j(p)][x^k(q) - x^k(p)][x^l(q) - x^l(p)], \end{aligned} \tag{4.3}$$

where $y^i(p)$ are constant numbers, x^i are arbitrarily chosen local coordinates in some neighborhood V of p, $q \in V$, $[b^j_i]$ is constant non-degenerate matrix, ${}^{x}\Gamma^{l}{}_{jk}$ are the coefficients of D at p in $\{x^i\}$, and the C^3 functions $b^i_{jkl}\colon V \to \mathbb{K}$ together with their first partial derivatives are bounded at p.

This result gives a *complete* description of *all* coordinates normal at a single point for torsionless and linear at it derivations along vector fields.

Following the discussion before Proposition II.2.3, Theorem 4.2 can be slightly generalized: if D is arbitrary, with or without torsion, derivation linear at p, there exist local coordinates $\{y^i\}$ in which the symmetric part of the coefficients of D, i.e., ${}^{y}\Gamma^{l}{}_{(jk)} := \frac{1}{2}\big({}^{y}\Gamma^{l}{}_{jk} + {}^{y}\Gamma^{l}{}_{kj}\big)$, vanish at p and their general form is (4.3) (in which ${}^{x}\Gamma^{l}{}_{jk}$ may be replaced with ${}^{x}\Gamma^{l}{}_{(jk)}$). But these coordinates are not normal for D at p unless D is torsionless at p.

5. Frames and coordinates normal along paths

Relaying on the results of Section 3 and Subsection II.3.2, we may assume the existence of normal frames along a path $\gamma\colon J \to M$ for a derivation D along

[1] This result is also a consequence of (3.5) and (3.6) for $U = \{p\}$.

[2] The vanishment of the torsion plays a role of an integrability condition for (3.7) with $U = \{p\}$.

vector fields provided D is linear along γ, i.e., on the set $\gamma(J)$:

$$\boldsymbol{\Gamma}_X(\gamma(s)) = \Gamma_k(\gamma(s))X^k(\gamma(s)), \qquad s \in J \tag{5.1}$$

for some matrix-valued functions Γ_k on $\gamma(J)$, $\boldsymbol{\Gamma}_X$ being the matrix (of the components) of D in a frame $\{E_i\}$ defined in a neighborhood of $\gamma(J)$. (Notice, we suppose D to be linear on $\gamma(J)$ but outside $\gamma(J)$ it could not be such.) The truthfulness of this supposition is verified by the results presented below.

In this section, as in Section II.3 (see p. 85), the manifold M will be considered as real one, i.e., as real manifold of dimension $\dim_{\mathbb{R}} M$. (Recall, $\dim_{\mathbb{R}} M = \dim M$ if M is real and $\dim_{\mathbb{R}} M = 2\dim M = 2\dim_{\mathbb{C}} M$ if M is complex; see also p. 7.)

Theorem 5.1. *Let M be C^2-manifold, $\gamma\colon J \to M$ be C^1 regular path without self-intersections, and D be C^0, i.e., continuous, derivation along vector fields of $\boldsymbol{T}^1(M)$. There exist frames (locally) normal for D along γ if and only if D is linear along γ, that is on $\gamma(J)$. Moreover, if D is linear along γ, for every $s_0 \in J$, there exist neighborhood U_1 of $\gamma(s_0)$ and a frame $\{E'_i\}$ on U_1 which is normal for D along γ in U_1, i.e., on $U_1 \cap \gamma(J)$. Besides, there exist subinterval $J_1 \subseteq J$ with $\gamma(J_1) = U_1 \cap \gamma(J)$ and a chart (U_1, x) with associated coordinates $\{x^i\}$ in U_1 in which $\gamma^1(s) = s$, the other components of $\gamma(s)$ being constants, such that all of the mentioned normal frames are $\{E'_i|_p = A^j_i(p)\frac{\partial}{\partial x^j}\big|_p\}$, where $p \in U_1$ and the C^1 non-degenerate matrix-valued function $A = [A^j_i]\colon U_1 \to \mathrm{GL}(\dim_{\mathbb{R}} M, \mathbb{R})$ is given by*

$$\begin{aligned} A(p) = \big\{\mathbb{1} - \Gamma_k(p_0)[x^k(p) - x^k(p_0)]\big\}Y(x^1(p), s_0; -\Gamma_1 \circ \gamma)B \\ + B_{kl}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)]. \end{aligned} \tag{5.2}$$

Here $\mathbb{1}$ is the identity (unit) matrix, Γ_k are the coefficients' matrices of D (see equation (5.1)), $p = x^{-1}(s, \boldsymbol{t}) \in U_1$ with $s \in J_1$ and $\boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M - 1}$, $p_0 = \gamma(s) = x^{-1}(s, \boldsymbol{t}_0)$ for fixed $\boldsymbol{t}_0 \in \mathbb{R}^{\dim_{\mathbb{R}} M - 1}$, Y is the unique solution of the initial-value problem (II.3.17) with $s = x^1(p)$ and $Z = -\Gamma_1 \circ \gamma$, B is constant non-degenerate matrix, and the matrix valued functions B_{kl} on U_1 are of class C^1 and they and their first partial derivatives are bonded on $\gamma(J_1)$.

Proof. If frames (locally) normal for D along γ exist, by Proposition 3.1 the derivation D is linear along γ, i.e., (5.1) holds.

Suppose D is linear along $\gamma\colon J \to M$. Let $\{E_i\}$ be a frame defined on a neighborhood of $\gamma(J)$. As it was proved in Section 3, D admits frames $\{E'_i = A^j_i E_j\}$ normal along γ if there exist non-degenerate matrix-valued C^1 functions $A = [A^j_i]$ which satisfy equation (3.5) for $U = \gamma(J)$, i.e.,

$$\Gamma_k(\gamma(s))A(\gamma(s)) + E(A)|_{\gamma(s)} = 0 \tag{5.3}$$

for $s \in J$. So, D admits frames normal along γ iff this equation has solutions with respect to A for some $\{E_i\}$. Besides, these solutions, if any, describe all such normal frames. In Subsection II.3.2 (see the paragraph containing equation (II.3.12) on page 94), we proved that for every $s_0 \in J$ there exist a subinterval $J_1 \subseteq J$ with $J_1 \ni s_0$, neighborhood $U_1 \supset \gamma(J_1) = \gamma(J) \cap U_1$, and local chart (U_1, x) with coordinate functions (II.3.12) in which the components of $\gamma(s)$, $s \in J_1$, are $\gamma^1(s) = s$ and $\gamma^k(s) = t_0^k$ for some constant numbers t^k for $k \geq 2$ (see (II.3.13)) and every point $p \in U_1$ has coordinates of the form $x(p) = (s, \boldsymbol{t})$ for some $s \in J_1$ and $\boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M - 1}$.

Further, it was proved in Subsection II.3.2 that (5.3) for $E_k = \frac{\partial}{\partial x^k}$ reduces to the system (II.3.16) which always has solutions, the general one being given by (5.2) (see the discussion before (II.3.23)). □

Now the discussion between Theorems II.3.1 on page 97 and II.3.2 on page 99 can be repeated *in extenso* for derivations (along vector fields) linear along a given path γ. Below we summarize the main results of it.

The holonomicity of the (local) frames normal along γ for D depends on the torsion of D on $\gamma(J)$: if D is torsion free on $\gamma(J)$, i.e., along γ, these frames are holonomic and, correspondingly, along γ exist normal coordinates; otherwise, if D has non-vanishing torsion along γ, the mentioned frames are anholonomic. It should be emphasized that the holonomicity of the normal frames on $\gamma(J)$ does not imply some conclusions of their holonomicity outside the set $\gamma(J)$ where they can be holonomic or anholonomic depending on the free parameters B and B_{kl} in (5.2).

The normal frames provided by Theorem 5.1 are essentially *local*: they exist in the corresponding local coordinate neighborhoods and only as an exception such a neighborhood may contain the whole path γ. Covering the set $\gamma(J)$ with coordinate neighborhoods of this kind and taking a normal frame in each of them,[1] we can construct all frames *globally* normal along γ. These global normal frames may turn not to be smooth in the region(s) of intersection of the domains of the local coordinate neighborhoods.

The above concerns injective paths, i.e., ones without self-intersections. This restriction can be weakened considerably: the results are valid *mutatis mutandis* for locally injective paths. For this end, the interval J has to be presented as a union of subintervals J_λ, $\lambda \in \Lambda$, such that the restricted paths $\gamma|_{J_\lambda}$ are without self-intersections. So the afore-given construction of, possibly global, normal frames is applicable for $\gamma|_{J_\lambda}$. From the frames normal along $\gamma|_{J_\lambda}$ all frames globally normal along the whole path γ can be constructed. The details of this scheme are given before Theorem II.3.2 on page 99 whose version for derivations along vector fields reads:

Theorem 5.2. *Let M be C^2 manifold, $\gamma\colon J \to M$ be locally injective, C^1, and regular path, and D be C^0 derivation along vector fields. There exist frames (globally, i.e., on $\gamma(J)$) normal for D along the whole path γ iff D is linear along γ. Locally these frames, described via Theorem 5.1, are of class C^1 but globally they may not be such. Frames normal and holonomic along γ exist iff D is torsion free on $\gamma(J)$.*

[1] In the region(s) of intersection, we arbitrarily fix (choose) some frame normal in it (them).

As was proved above, a derivation D along vector fields which is linear and torsion free along regular locally injective path γ admits holonomic frames normal along γ. Since these frames coincide with the ones for a linear connection ∇ with the same coefficients along γ as those of D, the corresponding normal coordinates, locally generating the holonomic normal frames, for ∇ and D coincide. The normal coordinates for a symmetric C^0 linear connection are described by Theorem II.3.3 on page 103. Hence the normal coordinates for derivations along vector fields are given via the same theorem with the only change that the words "symmetric linear connection" must be replace with "derivation along vector fields which is linear and torsionless along γ".

6. Frames and coordinates normal in a neighborhood

The main result concerning the existence of frames normal for derivations along vector fields in a neighborhood, or submanifold of maximum dimensionality, is that the flat linear connections are the only derivations admitting normal frames on them.

Theorem 6.1. *Let M be C^3 manifold, D be C^1 derivation along vector fields on M, and U be $(\dim M)$-dimensional submanifold of M. There exist frames normal for D on U if and only if the restriction of D to U, i.e., to $\boldsymbol{T}^1(U)$, is a flat linear connection.*

Proof. Suppose there is a frame $\{E'_i\}$ normal for D on U. By Proposition 3.1 and its proof, D is linear on U and in an arbitrary frame $\{E_i = (A^{-1})^j_i E'_j\}$ its coefficients' matrices are given by (3.4). Hence D reduces on U to a linear connection ∇ with coefficients' matrices (3.4) in $\{E_i\}$ and whose curvature, according to (II.4.1) and (3.4), has in $\{E_i\}$ components $R^i{}_{jkl}$ such that on U is fulfilled

$$\begin{aligned} R_{kl} &:= [R^i{}_{jkl}] \\ &= \big\{2E_l[+(E_k(A))A^{-1}] + 2(E_k(A))A^{-1}(E_l(A))A^{-1} + C^m_{kl}(E_m(A))A^{-1}\big\}_{[kl]} \\ &\qquad\qquad = \big\{2(E_{[l}E_{k]})(A) + C^m_{kl}E_m(A)\big\}A^{-1} \equiv 0 \end{aligned}$$

where $[E_i, E_j]_\_ = 2E_{[i} \circ E_{j]} = C^k_{ij}E_k$ was used. Consequently ∇ (and also D) is flat on U.

Conversely, if D is a flat linear connection on U, by Theorem II.4.1, on U exist frames normal for this linear connection which, by virtue of (2.9), are also normal for D on U. □

Theorem 6.1 reduces all problems concerning normal frames, if any, for derivations (along vector fields) on submanifolds of maximum dimension to the same problems for (flat) linear connections. Since the last range of problems was completely investigated in Section II.4, we are not going to repeat them and the corresponding discussion here. We shall only summarize the main points of them as the following exercises.

Exercise 6.1. Prove that every C^1 flat linear derivation along vector fields on a C^3 manifold admits (global) normal frames as, in fact, it is a flat linear connection. (See Theorem 6.1 and Corollary II.4.1.)

Exercise 6.2. Prove that all frames $\{E'_i\}$ normal for a C^1 flat linear derivation on an $(\dim M)$-dimensional submanifold U are $\{E'_i = A^j_i E_j\}$, where $\{E_i\}$ is a frame on U and $A = [A^j_i]\colon U \to \mathrm{GL}(\dim M, \mathbb{K})$ is given by (II.4.11) in which $p \in U$, $p_0 \in U$ is fixed, $B \in \mathrm{GL}(\dim M, \mathbb{K})$, Γ_k are the coefficients' matrices in $\{E_i\}$, and Y is the solution of (II.4.4) with $q = p$, $q_0 = p_0$, $N = M$, $e_a = E_a$, and $Z_a = -\Gamma_a$ where $a = 1, \dots, \dim M$. (See Theorems 6.1 and II.4.2.)

Exercise 6.3. Prove that, if a C^1 derivation which is linear, flat and torsionless in the domain U of some local chart (U, x) of a C^3 manifold, then it admits in U normal coordinates which can be obtained from $\{x^i\}$ via (II.4.14) with $a^i \in \mathbb{K}$.

7. Frames and coordinates normal on submanifolds

As a consequence of the general constructions in Section 3, the equations describing frames normal on a submanifold U, if any, for derivations along vector fields coincide with the ones for a linear connection with the same coefficients on U. This allows an almost obvious direct transferring of the results obtained for linear connections to the general case of arbitrary derivations along vector fields. Here is a list of the main of them.

Theorem 7.1. *Let N be submanifold of C^3 manifold M and D be C^1 derivation along vector fields of $\boldsymbol{T}^1(M)$. Then:*

(i) *There exist frames normal for D on N if and only if it is linear on N and it's curvature vanishes on N along vector fields tangent to N as a manifold:*

$$\big(R^D(X,Y)\big)\big|_q = 0, \qquad X_q, Y_q \in T_q(N), \quad q \in N \subseteq M. \tag{7.1}$$

(ii) *If D admits frames normal on N, every point $q_0 \in N$ has a neighborhood U_N in M such that on U_N exist C^1 frames normal for D on N, i.e., on $\bar{U} = U_N \cap N \ni q_0$. Moreover, on U_N exist coordinates $\{x^i\}$ such that:*

(a) *to every $p = x^{-1}(\boldsymbol{s}, \boldsymbol{t}')$, $(\boldsymbol{s}, \boldsymbol{t}') \in W_N \times W' \subseteq \mathbb{K}^{\dim N} \times \mathbb{K}^{\dim M - \dim N}$ corresponds a unique point $p_0 = x^{-1}(\boldsymbol{s}, \boldsymbol{t}'_0) \in \bar{U}$ for some $\boldsymbol{t}'_0 \in W'$ independent of $\boldsymbol{t}'$ and*

(b) *all frames defined on* U_N *and normal on* $\bar{U}$ *are* $\{E'_i = A^j_i \frac{\partial}{\partial x^j}\}$ *with*

$$A(p) = [A^j_i] = \{\mathbb{1} - \Gamma_k(p_0)[x^k(p) - x^k(p_0)]\} \times Y\big(\bar{x}(p_0), \boldsymbol{s}_0; -\Gamma_1 \circ (x^{-1}(\,\cdot\,, \boldsymbol{t}'_0)), \dots, -\Gamma_{\dim N} \circ (x^{-1}(\,\cdot\,, \boldsymbol{t}'_0))\big) B(\boldsymbol{t}'_0) + B_{kl}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)]. \quad (7.2)$$

Here Γ_k *are the coefficients' matrices of* D*,* Y *is the unique solution of equation* (II.5.9a) *with* $A = Y$ *and initial condition* (II.5.9c) *on page* 122*,* $(\bar{U}_N, \bar{x})$ *is the chart of* N *induced by* (U_N, x)*,* $s_0 = \bar{x}(q_0)$*,* B *is bounded matrix-valued function, and the matrix-valued functions* B_{kl} *on* U_N *are of class* C^1 *and they and their first partial derivatives are bounded on* $\bar{U}_N$*.*

Proof. Let $\{E_i\}$ be a frame on (a neighborhood of) N. Suppose D admits a frame $\{E'_i = A^j_i E_j\}$ normal on N. By Proposition 3.1, the derivation D is linear on N and, according to the results after it, the matrix $A = [A^j_i]$ satisfies the normal frame equation (3.5) with $U = N$, viz.

$$\Gamma_k(p)A(p) + (E_k(A))|_p = 0 \qquad p \in N. \quad (7.3)$$

In Subsection II.5.2, we proved that this equation, written there as (II.5.7), has solutions iff the integrability conditions (II.5.1) hold with R being the curvature tensor of a linear connection on M whose coefficients, when restricted to N, coincide with the ones of D. Because of (2.16), the conditions (II.5.1) are equivalent to (7.1). Consequently, as A satisfies (II.5.7) with Γ_k being the coefficients' matrices of D, the equality (7.1) holds. Conversely, let the derivation D be linear with coefficients' matrices Γ_k and (7.1) be valid. Then, as we just said, the equation (7.3) admits solutions with respect to A. If A is such a solution, the frame $\{E'_i = A^j_i E_j\}$ is normal for D on N due to (2.11) and (2.9). This completes the first part of the theorem's proof.

Now suppose D admits frames normal on N. Then D is linear on N and equation (7.1) and its equivalent version (II.5.1) hold. Then Theorem II.5.2 is valid for a connection whose coefficients on N coincide with the ones of D. Hence the coordinates with the properties (a) exist and in them, i.e., for $E_i = \frac{\partial}{\partial x^i}$, the general solution of (7.3) is given via (7.2). So, the frames $\{E'_i = A^j_i E_j\}$ on U_N with $A = [A^j_i]$ given by (7.2) are normal on $\bar{U} = U_N \cap N$ for D and on U_N there are not other frames with this property. □

Combining Theorem 7.1 with Corollaries II.5.1 and II.5.2, we can formulate the following result.

Corollary 7.1. *Let* N *be a submanifold of a* C^3 *manifold* M *endowed with* C^1 *derivation* D *along vector fields which is linear on* N*. Suppose* ∇ *is a linear connection on* M *and the coefficients of* ∇ *on* N *coincide with the ones of* D*. Then on (a neighborhood of)* N *exist frames normal for* D *if and only if on* N *exists a frame parallel with respect to* ∇ *or, equivalently, iff the parallel transport generated by* ∇ *is path-independent on* N *along all paths lying entirely in* N*.*

If a derivation along vector fields admits frames normal on a submanifold N, Assertion (ii) of Theorem 7.1 gives a complete *local* description of all such frames.

Exercise 7.1. Using the explicit form (7.2) of the transformation matrix A, prove the validity of Proposition 3.2 and Corollary 3.1 on submanifolds; this is reflected in the constancy of the matrix $B(\boldsymbol{t}'_0)$ in (7.2) for a fixed choice of the local coordinates $\{x^i\}$.

Exercise 7.2. Construct a frame globally normal on the whole N from the frames locally normal on N. The idea is to take a system of overlapping (coordinate) neighborhoods, forming a neighborhood of N, on each of which C^1 normal frames exist and from these local normal frames to be constructed a global frame normal on N. In this way all frames globally normal on N can be obtained, the details of this construction were explained after the proof of Corollary II.5.2.

If a derivation D along vector fields admits frames normal on a submanifold N, there are holonomic frames normal on N iff D has zero torsion on N (see Corollary 3.4). Since the normal frames for derivations, linear on N for which (7.1) holds, and linear connections, with the same coefficients on N, coincide in the torsionless case, the corresponding normal coordinates for them coincide too. All of these normal coordinates $\{z^i\}$ can be found explicitly by integrating the equation $\frac{\partial z^i}{\partial x^j}\big|_q = \left(A^{-1}(q)\right)^i_j$, where $q \in \bar{U}$ and A is given by (7.2), with respect to z^i in some neighborhood U_z of $\bar{U}$ in U_N with $U_z \cap N = \bar{U}$ (see (II.5.12)); for the notation see Theorem 7.1. This system of equations was completely solved in Subsection II.5.2 (see pages 124–126) and its general solution is (II.5.16). This result, combined with the just said in this paragraph, completes the proof of the following theorem (cf. Theorem II.5.3 on page 124).

Theorem 7.2. *Let N be a submanifold of a C^3 manifold M and D be a C^1 derivation along vector fields such that on N it is linear, torsionless and* (7.1) *holds. For every point $q \in N$ exists a chart (U_z, z) of M, with q in its domain U_z, which is normal for D on $U_z \cap N$. Moreover, the coordinate functions $z^i \colon U_z \to \mathbb{K}$ of all such charts normal on N are given by*

$$\begin{aligned} z^i(p) = a^i + \int_{q_0}^{p_0} \sum_{a=1}^{\dim N} \left(A^{-1}(q)\right)^i_a \mathrm{d}q^a + \left(A^{-1}(p_0)\right)^i_j [x^j(p) - x^j(p_0)] \\ + \left(A^{-1}(p_0)\right)^i_l \Gamma^l{}_{jk}(p_0)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)] \\ + a^i_{jkl}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)], \quad (7.4) \end{aligned}$$

the notation being explained in Subsection II.5.2.

8. Frames and coordinates normal along mappings

In Section 5, we have investigated the problems of existence, uniqueness, and holonomicity of frames normal along a path $\gamma\colon J \to M$ for a derivation D along vector fields. The aim of the present section is a generalization of these results for frames normal along sufficiently general mappings $\gamma\colon J^n \to M$ or $f\colon N \to M$ where J^n is a neighborhood in $\mathbb{R}^n$ (or n-dimensional submanifold of $\mathbb{R}^n$) and N is n-dimensional manifold for some integer $n \in \{1, \dots, \dim M\}$. Obviously, the choice $n = 1$ and an open interval for J^1 returns us to Section 5.[1] If we suppose $\gamma(J^n)$ to be a single point or, more generally, an n-dimensional submanifold of M (and γ to be injective in the last case),[2] the results of Sections 4 and 7, respectively, should be reproduced. Relaying on these special cases, we can expect that for $n = 0, 1$ the linearity of the derivation should be a necessary and sufficient condition for the existence of frames normal along γ, while for $n \geq 2$ (if $\dim M \geq 2$) this might not be the general case.

Now the rigorous statements are in order.

Recall, according to Definition 3.1, we say that a *frames $\{E_i\}$ is normal along a mapping $g\colon Q \to M$, Q being non-empty set, for a derivation D along vector fields of $\boldsymbol{T}^1(M)$ if it is defined in a neighborhood of $g(Q)$ (or on M if $g(Q) = M$) and in it the components of D along every vector field vanish everywhere on the set $g(Q)$.*

By Proposition 3.1, a necessary condition for the existence of frames normal along a mapping $\gamma\colon J^n \to M$ is the linearity of D along γ, i.e.,

$$\boldsymbol{\Gamma}_X(\gamma(s)) = \Gamma_k(\gamma(s))X^k(\gamma(s)), \qquad s \in J^n \tag{8.1}$$

in an arbitrary frame $\{E_i\}$. Here $\boldsymbol{\Gamma}_X$ is the components' matrix of D in it, $X \in \mathfrak{X}(M)$, and Γ_k are matrix-valued functions on $\gamma(J^n)$ which are the coefficients' matrices of D.

From Section 3, we know that a linear derivation D admits normal frames along $\gamma\colon J^n \to M$ iff for some (and hence for every) frame $\{E_i\}$ there exists a non-degenerate matrix-valued function $A = [A^i_j]$ such that the normal frame equation (3.5) holds along γ, viz.

$$(\Gamma_k A + E_k(A))|_{\gamma(J^n)} = 0 \tag{8.2}$$

or

$$\Gamma_k(\gamma(s))A(\gamma(s)) + E_k(A))|_{\gamma(s)} = 0, \qquad s \in J^n. \tag{8.2$'$}$$

[1] In Section 5 the interval $J^1 \equiv J$ is of arbitrary type, i.e., it may be open or closed from one or both ends. To incorporate the last case, when J^1 is closed from the left or/and right, as a special case of the next considerations, we must admit J^n to be n-dimensional submanifold with boundary in $\mathbb{R}^n$. The following results can be modified to cover this more general case, which we leave to the reader as an exercise. (See Remark I.2.1 on page 6 for the notion of a manifold with boundary.)

[2] The set $\gamma(J^n)$ is a submanifold if, for instance, γ is a regular embedding [12, p. 230].

If (8.2) has solutions with respect to A, all frames $\{E'_i\}$ normal for D along γ are $\{E'_i = A^j_i E_j\}$ for some A satisfying (8.2).

Since the system of equations (8.2) is a multidimensional generalization of (II.3.11), corresponding to $n = 1$, we shall partially follow the scheme of Subsection II.3.2.

The problem is to find, if any, all frames normal along a mapping $\gamma\colon J^n \to M$ for a C^1 linear derivation D along vector fields on a C^3 manifold M. The next two subsections deal with the case when γ is respectively injective or locally injective while Subsection 8.3 investigates the case of a mapping $f\colon N \to M$ between manifolds instead of γ.

In the present section, to detour some technical problems,[3] we shall look on the manifold M as on a real one, i.e., if M is real, no changes are necessary and, if M is complex, we consider it as a real manifold of dimension $\dim_{\mathbb{R}} M = 2\dim M \equiv 2\dim_{\mathbb{C}} M$ (see p. 7). Below this will be formally reflected in writing $\dim_{\mathbb{R}} M$ instead of $\dim M$. (Recall, $\dim_{\mathbb{R}} M$ is equal to $\dim M$ if M is real or to $2\dim_{\mathbb{C}} M \equiv 2\dim M$ if M is complex.) Respectively, all Latin indices, with not explicitly specified range, run from 1 to $\dim_{\mathbb{R}} M$ and the ranges of the coordinate homeomorphisms lie in $\mathbb{R}^{\dim_{\mathbb{R}} M}$. In short, in the present section M is real manifold or complex manifold which we consider as real manifold of dimension $\dim_{\mathbb{R}} M$ and endowed with a real differentiable structure.

8.1. Injective mappings

At first we suppose the mapping $\gamma\colon J^n \to M$ to be injective, i.e., without self-intersections: if $s_1, s_2 \in J^n$ and $s_1 \neq s_2$, then $\gamma(s_1) \neq \gamma(s_2)$. This requires $n = \dim J^n \leq \dim_{\mathbb{R}} M$; otherwise γ can not be 1:1. Below we are going to study and solve equation (8.2′), if it has solutions, in a frame $\{E_i\}$ associated to a special kind of local coordinates. More precisely, we are going to show that $\gamma(J^n)$ is n-dimensional submanifold of M (in a sense of our definition on page 7) and then to apply the results of Subsection II.5.2 or of Section 7.

Let $\gamma\colon J^n \to M$ be C^1 regular injective mapping from a neighborhood $J^n \subseteq \mathbb{R}^n$ on a C^3 manifold M with $\dim_{\mathbb{R}} M \geq n$. Consider J^n as an n-dimensional manifold with a global chart (J^n, id_{J^n}) and standard coordinate functions r^a, $a = 1, \dots, n$, such that $r^a(s) = s^a$ for $s = (s^1, \dots, s^n) \in J^n$. If (U, y), $y\colon U \to \mathbb{R}^{\dim_{\mathbb{R}} M}$, is a chart of M with $U \cap \gamma(J^n) \neq \varnothing$ and associated coordinate functions y^i, the regularity of γ implies (see p. 15) that the Jacobi matrix of γ in the charts introduced, i.e., $\left[\frac{\partial(y^i\circ\gamma)}{\partial s^a}\Big|_s\right] = \left[\frac{\partial\gamma^i_y}{\partial s^a}\Big|_s\right]$ where $\gamma^i_y := y^i \circ \gamma\colon J^n \to \mathbb{R}$ and here and

[3]For instance, if γ is (locally) injective, C^1 and regular, the set $\gamma(J^n)$ is a submanifold of M and, roughly speaking, its real dimension is $\dim_{\mathbb{R}}(\gamma(J^n)) = n$ while its complex dimension, if M is complex, is $\dim_{\mathbb{C}}(\gamma(J^n)) = n/2$ which is not an integer for odd n. Besides, for complex M and odd n, the below-constructed chart (U_1, x), which plays a crucial role in proving most of the results in this section, requires a complicated redefinition if we do not look on M as on a real manifold. (If n is even, the almost only change in the complex case is to replace $\dim_{\mathbb{R}} M - n$ with $\dim M - n/2$ and to identify the neighborhoods J^n in $\mathbb{R}^n$ with some neighborhoods in $\mathbb{C}^{n/2}$.)

below $a, b, c = 1, \dots, n$ and $i, j, k, l = 1, \dots, \dim_{\mathbb{R}} M$ if the index ranges are not explicitly written, has a rank equal to $n (\leq \dim_{\mathbb{R}} M)$ whenever s is such that $\gamma(s) \in U \cap \gamma(J^n)$.

Lemma 8.1. *Let $n \in \mathbb{N}$, M be a C^3 manifold with $\dim M \geq n$, and $\gamma\colon J^n \to M$ be C^1 regular injective mapping. For every $s_0 \in J^n$, there exists a chart (U_1, x) of M such that $\gamma(s_0) \in U_1$, $x\colon U_1 \to J_1^n \times \mathbb{R}^{\dim_{\mathbb{R}} M - n}$ for some open subset $J_1^n \subseteq J^n$, $s_0 \in J_1^n$, and $x(\gamma(s)) = (s, \boldsymbol{t}_0)$ for some fixed $\boldsymbol{t_0} \in \mathbb{R}^{\dim_{\mathbb{R}} M - n}$ and all $s \in J_1^n$.*

Remark 8.1. This lemma is a multidimensional generalization of Lemma II.3.1 on page 94, which is its special case for $n = 1$.

Proof. Let us arbitrarily choose some $s_0 \in J^n$ and a chart (U, y) with $U \ni \gamma(s_0)$ and $y\colon U \to \mathbb{R}^{\dim_{\mathbb{R}} M}$. Since the regularity of γ at s_0 means that $\bigl[\frac{\partial \gamma_y^i}{\partial s^a}\big|_{s_0}\bigr]$ has maximal rank, equal to n, we, without loss of generality, can suppose the coordinates $\{y^i\}$ to be taken such that $\det\bigl[\frac{\partial \gamma_y^a}{\partial s^b}\big|_{s_0}\bigr] \neq 0, \infty$.[4] Then the implicit function Theorem [7, 77, 78] implies the existence of a subneighborhood $J_1^n \subseteq J^n$ with $J_1^n \ni s_0$ and such that the matrix $\bigl[\frac{\partial \gamma_y^a}{\partial s^b}\big|_{s}\bigr]$ is non-degenerate for $s \in J_1^n$ and the mapping

$$(\gamma_y^1, \dots, \gamma_y^n)|_{J_1^n} : J_1^n \to (\gamma_y^1(J_1^n), \dots, \gamma_y^n(J_1^n)) \subseteq \mathbb{R}^n,$$

with $(\gamma_y^1, \dots, \gamma_y^n)|_{J_1^n} : s \mapsto (\gamma_y^1(s), \dots, \gamma_y^n(s))$ for $s \in J_1^n$, is a C^1 diffeomorphism. Define a chart (U_1, x) of M with domain

$$\begin{aligned} U_1 :=& \{p | p \in U,\ y^a(p) \in \gamma_y^a(J_1^n),\ a = 1, \dots, n\} \\ =& y^{-1}\bigl((\gamma_y^1(J_1^n), \dots, \gamma_y^n(J_1^n)) \times \mathbb{R}^{\dim_{\mathbb{R}} M - n}\bigr) \ni \gamma(s_0) \end{aligned} \tag{8.3a}$$

and local coordinate functions x^i given via

$$\begin{aligned} y^a &=: \bigl(\gamma_y^a\big|_{J_1^n}\bigr) \circ (x^1, \dots, x^n), \qquad && a = 1, \dots, n \\ y^k &=: x^k + \bigl(\gamma_y^k\big|_{J_1^n}\bigr) \circ (x^1, \dots, x^n) - t_0^k, \qquad && k = n+1, \dots, \dim_{\mathbb{R}} M \end{aligned} \tag{8.3b}$$

where $(x^1, \dots, x^n)\colon p \mapsto (x^1(p), \dots, x^n(p))$, $p \in U_1$, and $t_0^k \in \mathbb{R}$ are arbitrarily fixed constant numbers. Since $\frac{\partial y^a}{\partial x^b} = \frac{\partial \gamma_y^a}{\partial s^b}$, $\frac{\partial y^a}{\partial x^k} = \delta_k^a$ for $k \geq n+1$, $\frac{\partial y^k}{\partial x^a} = \frac{\partial \gamma_y^k}{\partial s^a}$ for $k \geq n+1$, and $\frac{\partial y^k}{\partial x^l} = \delta_l^k$ for $k, l \geq n+1$, the Jacobian of the change $\{y^i\} \to \{x^i\}$ on U_1 is $\det\bigl[\frac{\partial x^i}{\partial y^j}\bigr] = \bigl(\det\bigl[\frac{\partial \gamma_y^a}{\partial s^b}\bigr]\bigr)^{-1} \neq 0, \infty$. Consequently x^i are really coordinate functions and $x\colon U_1 \to J_1^n \times \mathbb{R}^{\dim_{\mathbb{R}} M - n}$ is in fact coordinate homeomorphism.[5]

[4] If we start from a chart (U, z) for which the matrix $\bigl[\partial \gamma_z^a / \partial s^b\big|_{s_0}\bigr]$ is degenerate, we can make a coordinate change $\{z^i\} \to \{y^i\}$ with $y^i = z^{\alpha_i}$, where the integers $\alpha_1, \dots, \alpha_{\dim_{\mathbb{R}} M}$ form a permutation of $1, \dots, \dim_{\mathbb{R}} M$, such that $\bigl[\partial \gamma_y^a / \partial s^b\big|_{s_0}\bigr]$ is non-degenerate. (For the proof, see any book on matrices, e.g., [49, 90].) Further, we suppose that such a renumbering of the local coordinates is already done if required. (Cf. Footnote II.15 on page 94.)

[5] The so-constructed chart (U_1, x) is, obviously, a multidimensional generalization of a similar chart defined in Subsection II.3.2 – see the paragraph containing equation (II.3.12) on page 94.

The coordinates $\{x^i\}$ can be expressed through $\{y^i\}$ explicitly. Indeed, writing the first raw of (8.3b) as

$$(y^1,\dots,y^n) = (\gamma_y^1|_{J_1^n},\dots,\gamma_y^n|_{J_1^n})\circ(x^1,\dots,x^n) = (\gamma_y^1,\dots,\gamma_y^n)|_{J_1^n}\circ(x^1,\dots,x^n)$$

and using that $(\gamma_y^1,\dots,\gamma_y^n)|_{J_1^n}$ is a C^1 diffeomorphism and the second raw of (8.3b), we find (cf. (II.3.12))

$$\begin{aligned}&(x^1,\dots,x^n) = \big((\gamma_y^1,\dots,\gamma_y^n)|_{J_1^n}\big)^{-1}\circ(y^1,\dots,y^n)\\ &x^k = y^k - (\gamma_y^k|_{J_1^n})\circ\big((\gamma_y^1,\dots,\gamma_y^n)|_{J_1^n}\big)^{-1}\circ(y^1,\dots,y^n) + t_0^k, \qquad k\ge n+1.\end{aligned} \tag{8.3b$'$}$$

Using (8.3), we see that the local coordinates of $\gamma(s)$ for $s=(s^1,\dots,s^n)\in J_1^n$ in (U_1,x) are

$$\gamma^a(s) := x^a(\gamma(s)) = s^a, \quad \gamma^k(s) := x^k(\gamma(s)) = t_0^k, \qquad k\ge n+1, \tag{8.4}$$

i.e., $x(\gamma(s)) = (s,\boldsymbol{t}_0)$ for some fixed $\boldsymbol{t}_0 = (t_0^{n+1},\dots,t_0^{\dim_{\mathbb{R}} M})\in\mathbb{R}^{\dim_{\mathbb{R}} M-n}$. □

Thus, in the chart (U_1,x) or in the coordinates $\{x^i\}$ constructed above, the first n coordinates of a point lying in $\gamma(J^n)$, i.e., in $\gamma(J_1^n)$, coincide with the corresponding parameters $s^1,\dots,s^n$ of γ, the remaining coordinates, if any, being constant numbers. This conclusion allows locally, in U_1, the mapping γ to be considered as a representative of a family of mappings $\eta(\,\cdot\,,\boldsymbol{t})\colon J_1^n\to M$, $\boldsymbol{t}\in\mathbb{R}^{\dim_{\mathbb{R}} M-n}$, defined by $\eta(s,\boldsymbol{t}) := x^{-1}(s,\boldsymbol{t})$ for $(s,\boldsymbol{t})\in J_1^n\times\mathbb{R}^{\dim_{\mathbb{R}} M-n}$. In fact, we have $\gamma=\eta(\,\cdot\,,\boldsymbol{t}_0)$ or $\gamma(s)=\eta(s,\boldsymbol{t}_0)$.[6]

It is almost evident, for C^1 regular injective mapping $\gamma\colon J^n\to M$, the set $\gamma(J^n)$ is an n-dimensional submanifold of M (in a sense of the definition on page 7). Indeed, for every point $p=\gamma(s_0)\in\gamma(J^n)$, $s_0\in J^n$, we can construct the afore-described chart (U_1,x) which is such that (*vide supra*) $p\in U_1$ and

$$\begin{aligned}&\gamma(J^n)\cap U_1 = \gamma(J_1^n) \xrightarrow{x} J_1^n\times\{\boldsymbol{t}\},\\ &\qquad x\colon q\mapsto (x^1(q),\dots,x^n(q),t_0^{n+1},\dots,t_0^{\dim_{\mathbb{R}} M}) = s\times\boldsymbol{t}_0\end{aligned}$$

for every point $q=\gamma(s)\in\gamma(J^n)\cap U=\gamma(J_1^n)$, where $s=(s^1,\dots,s^n)$, $x^a(q)=s^a$, and some $\boldsymbol{t}_0=(t_0^{n+1},\dots,t_0^{\dim_{\mathbb{R}} M})\in\mathbb{R}^{\dim_{\mathbb{R}} M-n}$.

Let us return to our initial problem, the existence and description of normal frames and coordinates, if any, along the mapping γ. One can solve it by continuing the analogy with the material of Subsection II.3.2 and its multidimensional generalization. We leave this method to the reader as an exercise; in this way he/she can independently verify the below-presented assertions.[7] Alternatively, we are going to apply the already obtained results of Section 7 (or Section II.5).

[6] In [83] the existence of η is taken as a given fact without proof.

[7] If one follows this way, Lemma II.4.1 should be used instead of Lemma II.3.2.

Since $\gamma(J^n)$ is n-submanifold of M, if the C^1 regular mapping $\gamma\colon J^n \to M$ is injective, for the problems concerning frames normal along γ are completely valid and applicable the results of Section 7. We shall partially reproduce them for the mentioned range of problems utilizing the above-introduced special chart (U_1, x).

Theorem 8.1. *Let M be C^3 manifold endowed with C^1 derivation D along vector fields and the mapping $\gamma\colon J^n \to M$ be injective, C^1, and regular. Then:*

(i) *There exist frames normal for D along γ if and only if D is linear along γ, i.e., on $\gamma(J^n)$, and it is curvature free on $\gamma(J^n)$ as a manifold,*

$$\big(R^D(X,Y)\big)\big|_q = 0, \qquad X_q, Y_q \in T_q(\gamma(J^n)), \quad q \in \gamma(J^n), \tag{8.5}$$

the last condition being locally, in the chart (U_1, x) introduced above, equivalent to

$$[R_{ab}(-\Gamma_1 \circ \gamma, \dots, -\Gamma_n \circ \gamma)](s) = 0, \qquad a, b = 1, \dots, n \tag{8.6}$$

where $s = (s^1, \dots, s^n) \in J_1^n$ with $\gamma(J_1^n) = U_1 \cap \gamma(J^n)$ and

$$[R_{ab}(\Gamma_1 \circ \gamma, \dots, \Gamma_n \circ \gamma)]|_s := \Big(\frac{\partial(\Gamma_a \circ \gamma)}{\partial s^b} - \frac{\partial(\Gamma_b \circ \gamma)}{\partial s^a}\Big)\Big|_s + (\Gamma_a\Gamma_b - \Gamma_b\Gamma_a)|_{\gamma(s)} \tag{8.7}$$

with $\Gamma_1, \dots, \Gamma_{\dim_{\mathbb{R}} M}$ being the matrices of the coefficients of D in the frame $\{\frac{\partial}{\partial x^i}\}$.

(ii) *If D admits frames normal along γ, for every $s_1 \in J^n$, there exists a chart (U_1, x), the same as in Assertion* (i)*, with $U_1 \ni \gamma(s_1)$ such that:*

(a) *In the associated coordinates $\{x^i\}$, every $p \in U_1$ has coordinates $x(p) = (s, \boldsymbol{t})$ for some $s \in J_1^n$, where $J_1^n \subseteq J^n$ and $U_1 \cap \gamma(J^n) = \gamma(J_1^n)$, $\boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M - n}$, and the coordinates of $\gamma(s)$, $s \in J_1^n$, are $x(\gamma(s)) = (s, \boldsymbol{t}_0)$ for some constant (independent of s) vector $\boldsymbol{t}_0 = (t_0^{n+1}, \dots, t_0^{\dim_{\mathbb{R}} M}) \in \mathbb{R}^{\dim_{\mathbb{R}} M - n}$;*

(b) *All frames normal for D along γ in U_1, i.e., normal on $\gamma(J_1^n)$, are $\{E'_i|_p = A_i^j(p) E_j|_p\}$, $p = x^{-1}(s, \boldsymbol{t}) \in U_1$, where*

$$\begin{aligned} A(x^{-1}(s,\boldsymbol{t})) = [A_j^i(x^{-1}(s,\boldsymbol{t}))] &= \Big\{\mathbb{1} - \sum_{k=n+1}^{\dim_{\mathbb{R}} M} \Gamma_k(\gamma(s))(t^k - t_0^k)\Big\} \\ &\times Y(s, s_0; -\Gamma_1 \circ \gamma, \dots, -\Gamma_n \circ \gamma) B(\boldsymbol{t}_0) \\ &+ \sum_{k,l=n+1}^{\dim_{\mathbb{R}} M} B_{kl}(x^{-1}(s,\boldsymbol{t}))(t^k - t_0^k)(t^l - t_0^l). \end{aligned} \tag{8.8}$$

Here: $s_0 \in J_1^n$ is fixed, the non-degenerate matrix $B(\boldsymbol{t}_0) \in \mathrm{GL}(\dim_{\mathbb{R}} M, \mathbb{R})$ may depend only on $\boldsymbol{t}_0$, the matrix-valued functions B_{kl} on U_1 are of class C^1 and they *and their first partial derivatives are bounded when $\boldsymbol{t} \to \boldsymbol{t}_0$, and Y is the unique solution of the initial-value problem*

$$\frac{\partial Y}{\partial s^a}\Big|_s = -\Gamma_a(\gamma(s))Y, \quad a = 1, \dots, n, \quad Y|_{s=s_0} = \mathbb{1}. \tag{8.9}$$

Remark 8.2. The fact that in (8.8) the summations are only over the range $n+1, \dots, \dim_{\mathbb{R}} M$ and in Y enter only $\Gamma_1, \dots, \Gamma_n$ is a consequence of the special choice of the coordinates $\{y^i\}$ used in the construction of $\{x^i\}$: we have admitted $\det\big[\frac{\partial \gamma_y^a}{\partial s^b}\big|_{s_0}\big] \neq 0, \infty$. If we have chosen another non-degenerate nth-order submatrix of $\big[\frac{\partial \gamma_y^i}{\partial s^b}\big|_{s_0}\big]$, this will result in the corresponding changes in the summation ranges and in the indices of the Γ's in (8.8).

Proof. Since $\gamma(J^n)$ is n-dimensional submanifold of M, which was proved earlier, by Theorem 7.1 (i) the condition (8.5) and the linearity of D on $\gamma(J^n)$ are necessary and sufficient for the existence of frames normal for D along γ. By (2.16), equation (8.5) is equivalent to $R^i_{jkl}|_{\gamma(J^n)} = 0$ (cf. (II.5.1′) on page 113 with $N = \gamma(J^n)$) in any chart (U, y) of M with $U \cap \gamma(J^n) \neq \varnothing$. So, according to Remark II.5.3 on page 114, the condition (8.5) is equivalent to (II.5.1″) on page 114 with $N = \gamma(J^n)$, which in the particular chart (U_1, x) takes the form (8.6).

Suppose now D admits frames normal along γ; so it is linear along γ and (8.6) holds.

The existence and explicit construction of a chart (U_1, x) with the properties described in (ii), point (a), was presented in the proof of Lemma 8.1 and it is given by (8.3) in terms of some other chart (U, y). To prove the rest of the assertion, i.e., point (b) of (ii), we have to demonstrate that (8.8) is the general solution of (8.2′) with $E_i = \frac{\partial}{\partial x^i}$, $\{x^i\}$ being the associated to (U_1, x) local coordinates. It is almost evident, this proof is contained in the discussion preceding Theorem II.5.2 in Subsection II.5.2. Indeed, in it we proved that (II.5.11) is the general solution of (II.5.7) on a submanifold N of M in a chart (U_N, x) described there. If we put $N = \gamma(J^n)$, then equation (II.5.7) reduces to (8.2′) and the chart (U_1, x) introduced above is a concrete realization of (U_N, x) for $\boldsymbol{t}_0' = \boldsymbol{t}_0$. Therefore (II.5.11), with $p = x^{-1}(s, \boldsymbol{t})$, $p_0 = x^{-1}(s, \boldsymbol{t}_0) = \gamma(s)$, $\boldsymbol{t}_0' = \boldsymbol{t}_0$, and $\bar{x}(p_0) = s$, is the general solution of (8.2′). Taking into account (8.4) and $x(p) = (s, \boldsymbol{t}) = (s^1, \dots, s^n, t^{n+1}, \dots, t^{\dim_{\mathbb{R}} M})$, we conclude that in this case (8.8) is an equivalent representation of the matrix-valued function $A\colon U_1 \to \mathrm{GL}(\dim_{\mathbb{R}} M, \mathbb{R})$ given by (II.5.11). □

Remark 8.3. An independent proof of Theorem 8.1 is presented in [83].

Corollary 8.1. *Let $\gamma\colon J^n \to M$ be a C^1 regular injective mapping in C^3 manifold M endowed with linear on $\gamma(J^n)$ C^1 derivation D along vector fields and ∇ be a linear connection coinciding on $\gamma(J^n)$ with D. For D exist frames normal along γ iff there exists a frame parallel on $\gamma(J^n)$ with respect to ∇ or iff the parallel transport generated by ∇ is path-independent on $\gamma(J^n)$ along the paths lying entirely in $\gamma(J^n)$.*

Proof. Use that $\gamma(J^n)$ is a submanifold of M and see Corollary 7.1 for $N = \gamma(J^n)$ or Theorem 8.1 (ii) and Lemma II.5.1. □

If a derivation D along vector fields admits frames normal along regular C^1 injective mapping $\gamma\colon J^n \to M$, Theorem 8.1 (ii) provides a complete local description of all such frames. A frame $\{E_i^g\}$ globally normal along the whole mapping γ can be constructed by covering the set $\gamma(J^n)$ by a set $\{U^\lambda | \lambda \in \Lambda\}$ of overlapping coordinate neighborhoods, like U_1 above, in which exist frames $\{E_i^\lambda\}$ normal along the 'part' of γ in them and then letting $E_i^g|_p = E_i^\lambda|_p$ if p is in a single neighborhood U^λ and, if the sets U^λ with $U^\lambda \ni p$ are more than one, we arbitrary choose some of them. The obtained in this way frame globally normal along γ is defined in a neighborhood of $\gamma(J^n)$ equal to the union of the neighborhoods U^λ on which the local normal frames forming it are defined. In this way all frames globally normal along γ can be constructed. Notice, the frames globally normal along γ are generally not smooth on the sets of intersection of two or more neighborhoods U^λ.

If frames normal along $\gamma\colon J^n \to M$ exist for a derivation D along vector fields, then, according to Corollary 3.4, these frames can be (locally) holonomic if D is torsionless on $\gamma(J^n)$ in which case normal coordinates for it can exist.

Theorem 8.2. *Let M be C^3 manifold and $\gamma\colon J^n \to M$ be injective, C^1 and regular. Let D be C^1, linear along γ and torsionless derivation along vector fields for which* (8.5) *holds. Then for every $s_1 \in J^n$ there exists a subneighborhood $J_1^n \ni s_1$ of J^n and a chart (U_z, z) of M, with $U_z \subseteq U_1$ ((U_1, x) is the above-constructed chart, see* (8.3)*) and $U_z \cap \gamma(J^n) = \gamma(J_1^n)$, which is normal for D along $\gamma|_{J_1^n}$, i.e., on $\gamma(J_1^n)$. The coordinate functions $z^i\colon U_z \to \mathbb{R}^{\dim_{\mathbb{R}} M}$ of all such charts are given through the equation*

$$
\begin{aligned}
z^i(x^{-1}(s,\boldsymbol{t})) = a^i &+ \int_{s_0}^{s} \sum_{a,b=1}^{n} \big(A^{-1}(\gamma(\sigma_1,\dots,\sigma_n))\big)^i_a \frac{\mathrm{d}\gamma^a}{\mathrm{d}\sigma^b}\,\mathrm{d}\sigma^b \\
&+ \sum_{k=n+1}^{\dim_{\mathbb{R}} M} \big(A^{-1}(\gamma(s))\big)^i_k (t^k - t^k_0) + \sum_{l=1}^{\dim_{\mathbb{R}} M} \big(A^{-1}(\gamma(s))\big)^i_l \sum_{j,k=n+1}^{\dim_{\mathbb{R}} M} {\Gamma^l}_{jk}(\gamma(s))(t^j - t^j_0) \\
&\times (t^k - t^k_0) + \sum_{j,k,l=n+1}^{\dim_{\mathbb{R}} M} a^i_{jkl}(x^{-1}(s,\boldsymbol{t}))(t^j - t^j_0)(t^k - t^k_0)(t^l - t^l_0). \quad (8.10)
\end{aligned}
$$

Here $(s,\boldsymbol{t}) \in J_1^n \times \mathbb{R}^{\dim_{\mathbb{R}} M - n}$ are the coordinates of an arbitrary point $p \in U_z \subseteq U_1$ in the chart (U_1, x), i.e., $x(p) = (s,\boldsymbol{t})$, in which $\gamma = x^{-1}(\,\cdot\,,\boldsymbol{t}_0)$ for some fixed $\boldsymbol{t}_0 \in \mathbb{R}^{\dim_{\mathbb{R}} M - n}$, $s_0 \in J_1^n$ is fixed, a^i are constant numbers, A is given by (8.8)*, and $a^i_{jkl}\colon U_z \to \mathbb{R}$ are bounded functions.*

Proof. In fact, we have to solve the system $\frac{\partial z^i}{\partial x^j}\big|_q = (A^{-1}(q))^i_j$, $q \in U_z$ (cf. (II.5.12)) with respect to z^i in a neighborhood U_z of an arbitrarily chosen point in $\gamma(J^n)$. In

Subsection II.5.2 (see pp. 122–126) we solved this equation on arbitrary submanifold N of M for A given by (II.5.11), the general solution being (II.5.16). Since in the concluding part of the proof of Theorem 8.1 we proved that in the chart (U_1, x) the equality (II.5.11) reduces to (8.8) for $N = \gamma(J^n)$, $\boldsymbol{t}'_0 = \boldsymbol{t}_0$, $x(p) = (s, \boldsymbol{t})$, $p_0 = x^{-1}(s, \boldsymbol{t}_0) = \gamma(s)$, and $\bar{x}(p_0) = s$, we can assert that the general form of the looked for coordinates $\{z^i\}$ is (II.5.16) for the pointed special choice of the parameters in it. One can easily verify that, in the last case, equation (8.8) is simply an equivalent form of (II.5.16). □

Theorem 8.2 gives a full constructive description of all local coordinates normal along an injective mapping $\gamma\colon J^n \to M$ for a torsionless derivation along vector fields. Hence it completes the exploration of the normal frames and coordinates along such mappings.

8.2. Locally injective mappings

Relying on the proof of Theorem II.3.2 on page 99, one can expect that the requirement the mapping $\gamma\colon J^n \to M$ to be injective, imposed in the previous subsection, can be weakened. Indeed, this happens to be the case.

Call a mapping $f\colon X \to Y$, X and Y being sets, *locally injective* if for every $x \in X$ there exists a subset $X_x \subseteq X$ with $X_x \ni x$ such that the restricted mapping $f|_{X_x}\colon X_x \to Y$ is injective. If X has some structure, e.g., if it is n-manifold, we further admit that X_x possesses the same structure; e.g., X_x to be n-manifold too in the particular example. For instance, the mapping $\gamma\colon J^n \to M$, J^n being neighborhood of $\mathbb{R}^n$, is locally injective if for every $s \in J^n$ there is a subneighborhood $I^n_s \subseteq J^n$ with $I^n_s \ni s$ such that $\gamma|_{I^n_s}\colon I^n_s \to M$ is injective. Obviously, γ is locally injective if the set $J^n_I := \{s | s \in J^n,\ \gamma(s) \text{ is self-intersection point}\} \subset J^n$ has not condensation points. The maximum neighborhood of s on which γ is injective will be denoted by J^n_s. It can be represented as a union of all neighborhoods of s, like I^n_s, on which γ is injective.

Proposition 8.1. *Theorems* 8.1 *and* 8.2 *are valid if in them the condition on* γ *to be injective is replaced with the requirement it to be locally injective and* J^n_1 *and* (U_1, x) *are constructed for the restricted mapping* $\gamma|_{J^n_{s_0}}$ *instead for* γ.

Proof. Since $\gamma|_{J^n_{s_0}}$ is injective, for it Theorems 8.1 and 8.2 hold. Consequently, they are also valid for the whole mapping γ if the pointed changes are made in their formulations, as in the considered version they concern only the local properties of γ on subneighborhoods on which it is injective. □

Following the proof of Theorem II.3.2 and the lines after the proof of Corollary II.5.2, we are going to prove the following theorem.

Theorem 8.3. *Let* M *be* C^3 *manifold endowed with* C^1 *derivation* D *along vector fields and the mapping* $\gamma\colon J^n \to M$ *be locally injective,* C^1*, and regular. Suppose* D *is linear along* γ *and its curvature satisfies* (8.5)*. Then there exist frames globally* *normal for D along the whole mapping γ, i.e., normal on the whole set $\gamma(J^n)$. Locally holonomic such frames exist iff D is torsion free on $\gamma(J^n)$.*

Proof. By Proposition 8.1 and the discussion after the proof of Corollary 8.1, for every $I^n \subseteq J^n$ for which $\gamma|_{I^n}$ is injective there exist frames normal for D along $\gamma|_{I^n}$. Let $\{I^n_\lambda|\lambda \in \Lambda\}$ be an open cover of J^n such that the restricted mappings $\gamma|_{I^n_\lambda}$ are injective.[8] For each mapping $\gamma|_{J^n_\lambda}$, $\lambda \in \Lambda$, there exists a frame $\{E^\lambda_i\}$ globally normal for D, i.e., on $\gamma(J^n_\lambda)$ and all of them can be constructed in the way pointed in the mentioned discussion. From the frames $\{E^\lambda_i\}$, $\lambda \in \Lambda$, a frame $\{E_i\}$ globally normal along γ can be constructed: at every point $p \in \gamma(J^n)$, we put $E_i|p = E^\lambda_i|_p$ if there is a single $\lambda \in \Lambda$ for which $p \in \gamma(I^n_\lambda)$, otherwise, if the set $\{\mu|\mu \in \Lambda,\ \gamma(I^n_\mu) \ni p\}$ consists of at least two elements, we arbitrary choose some $\mu_0 \in \Lambda$ for which $\gamma(I^n_{\mu_0}) \ni p$ and define $E_i|_p = E^{\mu_0}_i|_p$. The domain of $\{E_i\}$ is a union of the neighborhoods on which the frames $\{E^\lambda_i\}$ forming it are defined. It is obvious that all frames globally normal along γ can be constructed by the just describe method.

The last assertion of the theorem is a consequence of Proposition 3.4, Theorem 8.2, and Proposition 8.1 □

Notice, the proof of Theorem 8.3 provides a complete *global* description of all frames normal along locally injective mappings, if such frames exist at all.

8.3. Mappings between manifolds

Since the differentiable manifolds are 'locally Euclidean' (Subsection I.2.1), the results obtained until now in the present section have natural generalization for mappings like $f\colon N \to M$, N being n-manifold. Below we briefly sketch this situation. Without loss of generality, N and M will be considered as real manifolds (see p. 7).

Let $f\colon N \to M$ be C^1 mapping from a C^1 manifold N on a C^3 manifold M with $\dim_{\mathbb{R}} M \geq \dim_{\mathbb{R}} N$ and D be C^1 derivation along vector fields of $\boldsymbol{T}^1(M)$. We want to explore the problems of existence, uniqueness, construction, etc. of frames (or, possibly, coordinates) normal for D along f. From the considerations in Section 3, we know that D admits frames normal along f iff it is linear along f, i.e., on $f(N) \subseteq M$, and its coefficients' matrices Γ_k in a frame $\{E_i\}$ defined on (a neighborhood of) $f(N)$ satisfy the normal frame equation

$$\Gamma_k(f(q))A(f(q)) + E_k(A)|_{f(q)} = 0, \qquad q \in N \tag{8.11}$$

for some non-degenerate matrix-valued function A whose domain is the same as the one of $\{E_i\}$. Moreover, all frames normal along f have the form $\{E'_i = A^j_i E_j\}$ where $A = [A^j_i]$ is a non-degenerate solution of (8.11), if such exists. Hence the above

[8]Let $J^n_I := \{s|s \in J^n,\ \exists s' \in J^n,\ s' \neq s,\ \gamma(s) = \gamma(s')\}$. If $J^n_I = \varnothing$, we put $\Lambda = \{1\}$ and $I^n_1 = J^n$. If $J^n_I \neq \varnothing$, we define $\Lambda = J^n_I$ and $I^n_s = J^n_s$, J^n_s being the maximal neighborhood of $s \in I^n_s$ for which $\gamma|_{I^n_s}$ is injective.

range of problems (for a linear along f derivation) is reduced to the investigation of the equation (8.11) in some frame $\{E_i\}$.

Locally (8.11) is equivalent to (8.2) for a suitable choice of the mapping $\gamma\colon J^n \to M$, J^n being neighborhood in $\mathbb{R}^n$ for $n = \dim_{\mathbb{R}} N$. To demonstrate this, for arbitrary $q_0 \in N$, we take a chart (V, φ) of N with $V \ni q_0$ and $\varphi\colon V \to J^n$ for some neighborhood J^n in $\mathbb{R}^n$. Since φ is, by definition, homeomorphism, for every $q \in V$ exists a unique $s = (s^1, \dots, s^n) \in J^n$ such that $s = \varphi(q)$ or $q = \varphi^{-1}(s)$. Therefore in (V, φ) equation (8.11) is equivalent to

$$\Gamma_k(f(\varphi^{-1}(s)))A(f(\varphi^{-1}(s))) + E_k(A)|_{f(\varphi^{-1}(s))} = 0, \qquad s \in J^n \tag{8.12}$$

which corresponds to (8.2) with

$$\gamma = f \circ \varphi^{-1}. \tag{8.13}$$

Consequently in each chart (V, φ) of N the problems connected to the frames normal along $f\colon N \to M$ are equivalent to the same problems for the frames normal along the mapping $\gamma = f \circ \varphi^{-1}\colon J^n \to M$ with $J^n = \varphi(V) \subseteq \mathbb{R}^n$ and $n = \dim_{\mathbb{R}} N$. Moreover, the normal frames, if any, for the both mappings, f and γ, are identical. Since γ is (resp. locally) injective iff f is (resp. locally) injective (as φ is bijection), the results in Subsection 8.1 and 8.2 have corresponding variants concerning mappings between manifolds.

Theorem 8.4 (cf. Theorem 8.1). *Let $f\colon N \to M$ be a mapping from a C^1 manifold N into a C^3 manifold M, both considered as real manifolds, if some of them is/are complex, with $\dim_{\mathbb{R}} N \le \dim_{\mathbb{R}} M$, and D be C^1 derivation along vector fields of $\boldsymbol{T}^1(M)$. Suppose f is locally injective, of class C^1, and regular. Then:*

(i) *There exist frames normal along f for D if and only if D is linear along f, i.e., on $f(N)$, and it is locally curvature free on $f(N)$ in a sense that for every $(\dim N)$-submanifold N' of N on which the restricted mapping $f|_{N'}\colon N' \to M$ is injective, the derivation D is curvature free on $f(N')$ as a manifold:*

$$(R^D(X,Y))|_q = 0, \qquad X_q, Y_q \in T_q(f(N')), \quad q \in f(N'). \tag{8.14}$$

For every chart (V, φ) of N', $\varphi\colon V \to J^n \subseteq \mathbb{R}$ with $n = \dim_{\mathbb{R}} N$, there exists a chart (U_1, x) of M with $U_1 \cap f(N') = f(V_1)$, $V_1 = \varphi^{-1}(J_1^n)$ being subneighborhood of V, such that in it (8.14) is equivalent to (8.6) with $\gamma = f \circ \varphi^{-1}$.

(ii) *If D admits frames normal along f, for every $p_1 \in N$, there exists a chart (U_1, x) of M, the same as in (i) above, with $U_1 \ni p_1$ and such that:*

(a) *The coordinates of every $p \in U_1$ in the associated coordinates $\{x^i\}$ are $x(p) = (s, \boldsymbol{t})$ for some $s \in J_1^n$, where J_1^n is a subneighborhood of J^n and $U_1 \cap V = (f \circ \varphi^{-1})(J_1^n)$ (for the notation, see (i)), and $\boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M - n}$, and the coordinates of $q \in U_1 \cap V$ are $x(q) = (\varphi(q), \boldsymbol{t}_0) = (s, \boldsymbol{t}_0)$ for some $s \in J_1^n$ and constant, independent of s, vector $\boldsymbol{t}_0 = (t_0^{n+1}, \dots, t_0^{\dim_{\mathbb{R}} M}) \in \mathbb{R}^{\dim_{\mathbb{R}} M - n}$;*

(b) *All frames normal for* D *along* f *in* U_1*, i.e., normal on* $U_1 \cap V = (f\circ\varphi^{-1})(J_1^n)$*, are* $\{E'_i|_p = A_i^j(p)\frac{\partial}{\partial x^j}\big|_p\}$, $p = x^{-1}(s,t) \in U_1$*, where* A *is given by* (8.8) *with* $\gamma = f\circ\varphi^{-1}$.

Remark 8.4. The formulation of the theorem implicitly uses the fact that $f(N')$ is n-dimensional, $n = \dim_{\mathbb{R}} N$, real submanifold of M. (Recall, all manifolds are considered as real ones in the present section.) This follows from the proved in Subsection 8.1 assertion that $\gamma(J^n)$ is n-submanifold of M for regular C^1 mapping $\gamma\colon J^n \to M$, J^n being neighborhood in $\mathbb{R}^n$. In fact, since in every chart (V,φ), $\varphi\colon V \to J^n$, of N we have $f(V) = (f\circ\varphi^{-1})(J^n)$ and, consequently, $f(N')$ is n-submanifold of M as $\gamma = f\circ\varphi^{-1}$ is locally injective (φ is bijection), the image $f(N')$ is n-submanifold of M as it can be represented as a union of n-submanifolds like $f(V)$.

Proof. The first part of Assertion (i) is a trivial corollary of Theorem 7.1 (i) and the fact that the frames globally normal along f can be constructed from the frames normal along the restricted mappings $f|_{N'}$ in a way analogous to the one described in the proof of Theorem 8.3 (with N for J^n and f for γ). To prove the equivalence of (8.14) and (8.6) for $\gamma = f\circ\varphi^{-1}$, we notice that for every chart (V,φ) of N', $\varphi\colon V \to J^n \subseteq \mathbb{R}^n$, Theorem 8.1 (i) is valid for the mapping $\gamma = f\circ\varphi^{-1}\colon J^n \to M$. So, equation (8.14) in (V,φ) is equivalent to (8.6) with $\gamma = f\circ\varphi^{-1}$ and there is a chart (U_1, x) of M, described in Subsection 8.1, in which (8.5) is equivalent to (8.6) for $\gamma = f\circ\varphi^{-1}$ and $U_1\cap\gamma(J^n) = U_1\cap f(N') = (f\circ\varphi^{-1})(J_1^n) = f(V_1)$.

The proof of Assertion (ii) rests on Theorem 8.1 (ii). Let $p_1 \in N$ and N' be $(\dim N)$-submanifold of N with $N' \ni p_1$ and such that $f|_{N'}$ is injective. Let (V,φ), $\varphi\colon V \to J^n \subseteq \mathbb{R}^n$, be a chart of N' and $s_1 = \varphi(p_1) \in J^n$. Now it is evident that the second part of the theorem is a simple restatement of the second part of Theorem 8.1 for the mapping $\gamma = f\circ\varphi^{-1}\colon J^n \to M$. □

Corollary 8.2 (cf. Corollary 8.1). *Let the manifolds* N *and* M*, regarded as real ones, if some of them is/are complex, be of classes* C^1 *and* C^3*, respectively, and the mapping* $f\colon N \to M$ *be locally injective, of class* C^1*, and regular. Let a derivation* D *along vector fields of* $\boldsymbol{T}^1(M)$ *be of class* C^1 *and linear along* f *and* ∇ *be a* C^1 *linear connection on* M *coinciding on* $f(N)$ *with* D*. For* D *exist frames normal along* f *iff there is a frame locally parallel on* N *with respect to* ∇ *or iff the parallel transport generated by* ∇ *is locally path-independent on* N *along paths locally lying entirely in* N*. Here by 'locally' is understood on any* $(\dim N)$*-dimensional submanifold* N' *of* N *such that* $f|_{N'}$ *is injective.*

Proof. See Corollary 8.1 for $\gamma = f\circ\varphi^{-1}$, where φ is the coordinate homeomorphism of a chart (V,φ) of N', and note that N can be covered by the domains of charts like (V,φ). □

If the frames normal along a mapping $f\colon N \to M$ exist, the ones provided by Theorem 8.4 (ii) are local in a sense that they are defined only on some neighborhood (in M) of every point in $f(N)$. From these neighborhoods one can construct a neighborhood of $f(N)$ in M on which a frame globally normal along f can be defined by identifying it in each coordinate neighborhood with the corresponding local normal frame; in a case of overlapping of two or more neighborhoods, one arbitrary can fix some of the local frames to represent the global one. (For details, see the paragraph after the proof of Corollary 8.1.) It is clear, all frames globally normal along f can be constructed in this way.

If a derivation is torsionless, some of the frames normal for it, if any, are holonomic and hence locally they are generated by (normal) local coordinates. Of course, this is true for the frames normal along mappings between manifolds as the following result states explicitly.

Theorem 8.5 (cf. Theorem 8.2). *Let N and M be manifolds, considered as real ones if some of them is/are complex, of classes C^1 and C^3, respectively, and $f\colon N \to M$ be locally injective, C^1, and regular. Let a derivation D along vector fields of $\boldsymbol{T}^1(M)$ be torsionless, of class C^1, linear along f, and its curvature to satisfies equation* (8.14) *on every* $(\dim N)$*-dimensional submanifold N' of N on which $f|_{N'}$ is injective. Then for every point $p_1 \in N$ there exist a chart (V_1, φ) of N, with $\varphi\colon V_1 \to J_1^n \subseteq \mathbb{R}^n$, $V_1 \ni p_1$, $n = \dim_{\mathbb{R}} N$ and $f|_{V_1}$ injective, and a chart (U_z, z) of M, with $U_z \subseteq U_1$ ((U_1, x) is the chart provided by Theorem* 8.4 (ii)) *and $U_z \cap N = f(V_1) = (f \circ \varphi^{-1})(J_1^n)$, which is normal for D along f in $f(V_1)$, i.e., along $f|_{V_1}$ or on $f(V_1)$. The associated coordinates functions $z^i\colon U_z \to \mathbb{R}^{\dim_{\mathbb{R}} M}$ of all such normal charts are given via equation* (8.10) *for $\gamma = f \circ \varphi^{-1}$.*

Proof. By Theorem 8.4 and Corollary 3.3, the derivation D admits normal coordinates. Let $p_1 \in N$ and (V, φ) and (U_1, x) be the charts described in the Theorem 8.4 with $U_1 \ni f(p_1) \in V$. In the notation of Theorem 8.4, define $V_1 = U_1 \cap V = (f \circ \varphi^{-1})(J_1^n)$ and $\gamma = f \circ \varphi^{-1}\colon J_1^n \to M$. Now the rest of the proof follows from the remark that Theorem 8.2 is valid for the mapping $\gamma = f \circ \varphi^{-1}$. □

9. Normal frames and coordinates for derivations along a fixed vector field

In Sections 3–8, we have explored problems connected with frames normal for derivations along vector fields of the tensor algebra over a manifold (see Definition 2.3). In the present section, we want briefly to pay attention to the same range of problems for derivations along a fixed vector field or, equivalently, for a given derivation of the same algebra (see Definition 2.1).

Throughout this section X will denote a *fixed* vector field over a manifold M, $X \in \mathfrak{X}(M)$. Respectively, D_X means a derivation along this concrete X of $\boldsymbol{T}^1(M)$ or, in other words, D_X is a derivation of $\boldsymbol{T}^1(M)$ with decomposition like (2.3) with fixed X and arbitrary S.

Definition 9.1 (cf. Definition 3.1). Let U be a subset of a manifold M and D_X be a derivation along some fixed vector field $X \in \mathfrak{X}(M)$ of $\boldsymbol{T}^1(M)$. A *frame* $\{E_i\}$, defined on an open subset of M containing U or equal to it, is called *normal on U for D_X* if the components of D_X in $\{E_i\}$ vanish on U (for the fixed X).

Proposition 9.1. *The only derivation $D_{\mathbf{0}}$ along the zero vector field for which frames normal on $U \subseteq M$ exist is the zero derivation, assigning to a C^1 tensor field the zero tensor field of the same type. Every frame defined on (a neighborhood of) U is normal for $D_{\mathbf{0}}$ on its domain.*

Proof. If $D_{\mathbf{0}}$ admits a frame $\{E_i\}$ normal on U, $\Gamma_{\mathbf{0}}{}^i{}_j|_U = 0$, equation (2.5) implies $D_{\mathbf{0}}K = 0$ for every tensor field K, so $D_{\mathbf{0}}$ is the zero derivation, with decomposition (2.3) with $X = \mathbf{0}$ and $S_{\mathbf{0}} = 0$. If $D_{\mathbf{0}}$ is the zero derivation, from (2.6) follows that every frames $\{E_i\}$ is normal for it. □

Proposition 9.1 has a slightly more general local version:

Proposition 9.2. *Let M be manifold, $U \subseteq M$, $X \in \mathfrak{X}(M)$, $X|_U = 0$, and D_X be derivation along X. Frames normal on U for D_X exist if and only if D_X is the zero derivation on U, $D_X\big|_{\boldsymbol{T}^1(U)} = D_{\mathbf{0}}|_{\boldsymbol{T}^1(U)}$, assigning to every tensor field on U the zero tensor field of the same type on U. If on U the derivation D_X is the zero one, then every frame defined on U or on a larger set is normal for D_X on U.*

Proof. If $\{E_i\}$ is a frame normal for D_X on U, i.e., $\boldsymbol{\Gamma}_X|_U = 0$, from (2.5) we get $(D_X K)|_U = 0$ for every tensor field K as $X|_U = 0$. Conversely, if D_X is the zero derivation on U, from (2.6), we derive $\boldsymbol{\Gamma}_X|_U = 0$ in every frame $\{E_i\}$. □

Nevertheless the above two results are completely trivial, they give a full description of the frames, if any, normal for derivations along vector fields on the sets on which these vector fields vanish.[1] For this reason, further in this section we assume X to be a non-zero vector field on U, $X|_U \neq 0$, but there could exist points in U at which X vanishes. Notice, from (2.6) follows that if D_X is the zero derivation along X on $U \subseteq M$, then its components vanish identically in any frame defined on U. Combining this with (2.11), we get $X|_U = 0$. Hence, the zero derivation on U along fixed vector field can be such if this vector field vanished on U. So, assuming $X|_U \neq 0$, we automatically exclude the case when D_X is the zero derivation on U from further considerations.

One may call a derivation D_X along a fixed vector field X linear on $U \subseteq M$ if in some (and hence in any) frame its components' matrix has the form $\boldsymbol{\Gamma}_X(p) = \Gamma_k(p)X^k(p)$, $p \in U$, for some matrix-valued functions Γ_k on U. Since the components' matrix of *every* D_X for *every fixed* X possess (generally infinitely many) such representations,[2] this concept is useless for exploring the normal frames for D_X. Paraphrasing, we may say that every derivation along a fixed vector field is linear on every set.

[1] Evidently, if D_X is the zero derivation on U and $\{E_i\}$ is a frame defined on V with $V \cap U \neq \varnothing$, the frame $\{E_i\}$ is normal on $V \cap U$ for D_X. Consequently, every chart or coordinate system is normal for the zero derivation on the intersection of their domains.

[2] For example, if $X^1(p) \neq 0$, we put $\Gamma_1(p) = \boldsymbol{\Gamma}_X(p)/X^1(p)$ and $\Gamma_k(p) = 0$ for $k \geq 2$.

Let $\{E_i\}$ be a frame defined on a (neighborhood of) $U \subseteq M$. A frame $\{E'_i = A^j_i E_i\}$, where $A = [A^j_i]$ is a C^1 non-degenerate matrix valued-function, is normal for D_X on U iff A is a solution of the normal frame equation (see (2.11), cf. (3.3))

$$(\mathbf{\Gamma}_X A + X(A))|_U = 0 \qquad \text{for fixed } X \in \mathfrak{X}(M). \tag{9.1}$$

So, D_X admits frames normal on U iff this equation (for fixed X) has solutions; if they exist, all such frames are $\{E'_i = A^j_i E_j\}$ with A being a solution of (9.1). Below we shall prove that equation (9.1) (for fixed X) always has (non-degenerate) solutions (in a neighborhood of the non-singular points of X) which results in the general conclusion that frames normal for a derivation along a fixed vector field exist on arbitrary subsets of M, in particular on the whole manifold M. The set of frames normal for a derivation along fixed vector field is described by the following result.

Proposition 9.3. *Let D_X be derivation along some fixed $X \in \mathfrak{X}(M)$ and $U \subseteq M$. The frames normal on U for D_X (if any) are connected by linear transformations whose matrices vanish under the action of X on U. Equivalently, the set of all frames normal for D_X (if any) consists of frames that can be obtained from a fixed normal frame for D_X by transformations of the described type.*

Proof. If $\{E_i\}$ and $\{E'_i = A^j_i E_j\}$ are normal for D_X on U, then $\mathbf{\Gamma}_X = \mathbf{\Gamma}'_X = 0$, so (9.1) (or (2.11)) implies $X(A)|_U = 0$, $A = [A^j_i]$. □

Since the concepts of curvature and torsion can not be introduced for derivations along a fixed vector fields,[3] for them there are no analogues of Proposition 3.3 and Corollary 3.3. Regardless of that, they always admit normal coordinates (in a neighborhood of the non-singular points of X) which are defined as follows.

Definition 9.2 (cf. Definition 3.2)**.** A *chart* (V, x) of M and the associated to it *coordinate functions or system* are called *normal on $U \subseteq V$ for a derivation D_X along fixed $X \in \mathfrak{X}(M)$* if the coordinate frame $\left\{\frac{\partial}{\partial x^i}\right\}$ is normal for D_X on U.

If $U \subseteq M$ and (V, x) is a chart of M such that $V \cap U \neq \varnothing$, the coordinates $x'^i = x'^i(x^1, \dots, x^{\dim M})$ are normal on $U \cap V$ iff (9.1) holds for $A^j_i = \frac{\partial x^j}{\partial x'^i}$, i.e., iff they are solutions of the normal frame/coordinates equation

$$\Big({\Gamma_X}^i{}_m \frac{\partial x^m}{\partial x'^j} + X^m \frac{\partial}{\partial x^m}\Big(\frac{\partial x^i}{\partial x'^j}\Big)\Big)\Big|_{U\cap V} = 0. \tag{9.2}$$

Using the equalities

$$\frac{\partial x^i}{\partial x'^l}\frac{\partial x'^l}{\partial x^k} = \delta^i_k, \qquad \frac{\partial}{\partial x^j}\Big(\frac{\partial x^i}{\partial x'^k}\Big) = -\frac{\partial x^i}{\partial x'^l}\frac{\partial^2 x'^l}{\partial x^j \partial x^m}\frac{\partial x^m}{\partial x'^k},$$

[3] In some sense the curvature and torsion for every D_X are identically zero as seen from (2.12) and (2.13) for $Y = X$.

the second of which is a consequence of the first one and its derivative with respect to x^j, we rewrite (9.2) as

$$\Big(X^m \frac{\partial^2 x'^i}{\partial x^m \partial x^k} - \frac{\partial x'^i}{\partial x^m} \Gamma_X{}^m{}_k\Big)\Big|_{U\cap V} = 0. \tag{9.2$'$}$$

This equation is more suitable than (9.2) for finding x'^i if x^i are known.

Now we turn our attention to the problems of existence and construction of normal frames and coordinates for derivations along fixed vector fields. Formally they are equivalent to the solutions of equations (9.1) and (9.2), respectively, under certain conditions.

9.1. The case at a single point

Let $p_0 \in M$ be a given point in a manifold M and D_X be a derivation along some fixed vector field $X \in \mathfrak{X}(U)$ of $\boldsymbol{T}^1(M)$ with components' matrix $\mathbf{\Gamma}_X$ in a frame $\{E_i\}$ defined on a neighborhood of p_0. The frames normal for D_X at p_0, if any, are $\{E'_i = A^j_i E_j\}$, where $A = [A^j_i]$ is a non-degenerate solution of (9.1) for $U = \{p_0\}$:

$$\mathbf{\Gamma}_X(p_0)A(p_0) + (X(A))|_{p_0} = 0. \tag{9.3}$$

Theorem 9.1. *Let M be C^1 manifold, $p_0 \in M$ and D_X be a derivation along fixed $X \in \mathfrak{X}(M)$. If $X_{p_0} = 0$, then either every frame in a neighborhood of p_0 is normal at p_0 for D_X (if D_X is the zero derivation at p_0) or such frames do not exist (if D_X is not the zero derivation at p_0). If $X_{p_0} \neq 0$, then D_X always admits frames $\{E'_i\}$ normal at p_0 and, if X is continuous, all such frames are $\big\{E'_i|_p = A^j_i(p)E_j|_p\big\}$, where $A\colon U_1 \to \mathrm{GL}(\dim M, \mathbb{K})$ is given by equation* (9.8) *below and $p \in U_1$, with (U_1, x) being the special chart constricted in Subsection* II.3.2 *for $U_1 \ni p_0$ and γ being the integral path for X through p_0.*[4]

Proof. If $X_{p_0} = 0$, then equation (9.3) has solution(s) with respect to A iff $\mathbf{\Gamma}_X(p_0) = 0$ in which case D_X is the zero derivation at p and every non-generate matrix-valued function A is its solution. This trivial conclusion, which follows also from Proposition 9.2 for $U = \{p\}$, means that for $X_{p_0} = 0$ either every frame in a neighborhood of p_0 is normal for D_X at p_0 or frames normal for D_X at p_0 do not exist. To check the case, one should take some frame and calculate the matrix of the components of D_X in it. If it happens to be non-vanishing at p_0, frames normal at p_0 do not exist, otherwise every frame is normal for D_X at p_0.

Suppose now $X_{p_0} \neq 0$. Since (9.3) implies a single condition at p_0 on A and its directional derivative along X, one can expect that it admits infinitely many solutions which will be proved below.

[4]Equivalently, in an arbitrary chart (U, y) with $U \ni p_0$, the matrix-valued function A has the form (9.4) under the condition (9.5). Formula (9.7) provides a class of such A which is its general form iff $\det(\mathbf{\Gamma}_X(p_0)) \neq 0$.

Let (U, y) be a chart of M with $U \ni p_0$, $E_i = \frac{\partial}{\partial y^i}$, and $A\colon U \to \mathrm{GL}(\dim M, \mathbb{K})$ be of class C^2. In U is valid the Taylor expansion

$$A(p) = A_0 + A_k[y^k(p) - y^k(p_0)] + A_{kl}(p)[y^k(p) - y^k(p_0)][y^l(p) - y^l(p_0)] \quad (9.4)$$

where $A_0 \in \mathrm{GL}(\dim M, \mathbb{K})$, A_k are constant matrices and the matrix-valued functions A_{kl} on U are of class C^1 and they and their first partial derivatives are bounded at p_0. Inserting (9.4) into (9.3), we get

$$\mathbf{\Gamma}_X(p_0)A_0 + X^i_y(p_0)A_i = 0, \qquad \det A_0 \neq 0 \quad (9.5)$$

with X^i_y being the components of X in $\{\frac{\partial}{\partial y^i}\}$.

A simple verification shows that

$$A_0 = X^i_y(p_0)B_i, \qquad A_k = -\mathbf{\Gamma}_X(p_0)B_k, \quad (9.6)$$

with arbitrary constant matrices $B_1, \ldots, B_{\dim M}$ for which $\det(X^i_y(p_0)B_i) \neq 0$, is a solution of (9.5) to which corresponds a solution of (9.3) of the form[5]

$$A(p) = X^i_y(p_0)B_i - \mathbf{\Gamma}_X(p_0)B_k[y^k(p) - y^k(p_0)] \\ + A_{kl}(p)[y^k(p) - y^k(p_0)][y^l(p) - y^l(p_0)]. \quad (9.7)$$

If $\mathbf{\Gamma}_X(p_0)$ is non-degenerate, then, evidently, (9.6) is the general solution of (9.5) and, consequently, (9.7) is the general solution of (9.3). A method for obtaining the general solution of (9.5), and hence of (9.3), for arbitrary $\mathbf{\Gamma}_X(p_0)$ and continuous X is the following one.[6]

Suppose X is of class C^0. Since $X_{p_0} \neq 0$, there is a unique integral path $\gamma\colon J \to M$ through p_0, i.e., (see equation (I.2.19) on page 13) $\dot\gamma(s) = X_{\gamma(s)}$, $s \in J$, and $\gamma(s_0) = p_0$ for some fixed $s_0 \in J$. In Subsection II.3.2, we saw that there is a chart (U_1, x) with $p_0 \in U_1 \subseteq U$ and coordinates functions (II.3.12) in which the coordinates of $\gamma(s)$ are given via (II.3.13). Therefore in (U_1, x) we have $X^k_{\gamma(s)} = \dot\gamma^k(s) = \delta^k_1$ for $s \in J_1$, where $J_1 \subseteq J$ is such that $J_1 \ni s_0$ and $\gamma(J_1) = \gamma(J) \cap U_1$. Hence

$$X_{\gamma(s)} = \dot\gamma(s) = \frac{\partial}{\partial x^1}\Big|_{\gamma(s)},$$

and, in particular, $X_{p_0} = \frac{\partial}{\partial x^1}\big|_{p_0}$.

Repeating the aforesaid with $(U_1.x)$ for (U, y), we get

$$\mathbf{\Gamma}_X(p_0)A^{(x)}_0 + A^{(x)}_1 = 0 \quad (9.5')$$

---

[5]This solution was first derived in [84, Section IV].

[6]If X is not of class C^0 in a neighborhood of p_0 and $\det(\mathbf{\Gamma}_X(p_0)) = 0$, the methods of matrix algebra should be used for explicit solving of (9.5). Equivalently, one can take any C^1 vector field Y with $Y_{p_0} = X_{p_0}$ and $\mathbf{\Gamma}_Y(p_0) = \mathbf{\Gamma}_X(p_0)$ for X in the next lines.

instead of (9.5), with the index (x) indicating that the corresponding quantities are computed in $\{\frac{\partial}{\partial x^i}\}$. So $A_1^{(x)} = -\mathbf{\Gamma}_X(p_0)A_0^{(x)}$, with arbitrary non-degenerate $A_0^{(x)}$, is the general solution of (9.5). Consequently, the general solution of (9.3) in (U_1, x) is (see (9.4))

$$A^{(x)}(p) = A_0^{(x)} - \mathbf{\Gamma}_X(p_0)A_0^{(x)}[x^1(p) - x^1(p_0)] \\ + \sum_{k=2}^{\dim M} A_k^{(x)}[x^k(p) - x^k(p_0)] + A_{kl}^{(x)}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] \quad (9.8)$$

where $A_0^{(x)}, A_2^{(x)}, \dots, A_{\dim M}^{(x)}$ are constant matrices, the matrix $A_0^{(x)}$ is non-degenerate, and the matrix-valued functions A_{kl} on U_1 are of class C^1 and they and their first partial derivatives are bounded at p_0. □

The problem for existence and description of charts (or coordinates) normal at a single point p_0 can be attacked by two independent but equivalent ways. The first one is to select the free parameters in (9.8) so that there to exist local coordinates $\{z^i\}$ with $\frac{\partial}{\partial z^i} = E_i' = \left(A^{(x)}(p)\right)_i^j \frac{\partial}{\partial x^j}$. The second one, which we will developed below, is to solve directly the normal frame equation (9.2′) for $U = \{p_0\}$, i.e.,

$$X^m(p_0)\frac{\partial^2 z^i}{\partial y^m \partial y^k}\Big|_{p_0} - \frac{\partial z^i}{\partial y^m}\Big|_{p_0} {\Gamma_X}^m{}_k(p_0) = 0 \quad (9.9)$$

where $\{y^i\}$ are some given local coordinates in a neighborhood of p_0, the quantities ${\Gamma_X}^k{}_m(p_0)$ and $X^m(p_0)$ are computed in $\{\frac{\partial}{\partial y^i}\}$, and $\{z^i\}$ are the looked for normal coordinates, if any.

Theorem 9.2. *Let M be a C^3 manifold, $p_0 \in M$ and D_X be some derivation along fixed $X \in \mathfrak{X}(M)$. If p_0 is a singular point for X, i.e., $X_{p_0} = 0$, then either every chart with p_0 in its domain is normal for D_X (in a case D_X is the zero derivation at p_0) or charts normal at p_0 for D_X do not exist (in a case D_X is not the zero derivation at p_0). If $X_{p_0} \neq 0$, then for D_X always exist charts (U_z, z), with $U_z \ni p_0$, normal at p_0 and the general form of their coordinate functions z^i is given via equation (9.13) below in which: $a^i, a^i_j, a^i_{jk} \in \mathbb{K}$, $\det[a^i_j] \neq 0$, $a^i_{jkl} \colon U_z \cap V \to \mathbb{K}$ are of class C^1 and they and their first partial derivatives are bounded at p_0, (U_1, x), with $U_1 \supseteq U_z$, is the special chart constructed above in this subsection, and $\mathbf{\Gamma}_X$ is the matrix of D_X in $\{\frac{\partial}{\partial x^i}\}$.*

Proof. If $X_{p_0} = 0$, then (9.9) holds iff $\mathbf{\Gamma}_X(p_0) = 0$ (see also (9.2) for $U \cap V = \{p_0\}$). Hence if $X_{p_0} = 0$, then charts normal at p_0 either do not exist or every chart with p_0 in its domain is normal. (Recall, in the last case D_X must be the zero derivation at p_0; see Proposition 9.2.)

Let M be C^3 manifold, $p_0 \in M$, $X_{p_0} \neq 0$, and (U, y) be a given chart in M with $U \ni p_0$. We search for a chart (U_z, z) with $U_z \ni p_0$ which is normal at p_0 for

D_X. It is valid the Taylor expansion

$$z^i(p) = a^i + a^i_j[y^j(p) - y^j(p_0)] + a^i_{jk}[y^j(p) - y^j(p_0)][y^k(p) - y^k(p_0)] \\ + a^i_{jkl}(p)[y^j(p) - y^j(p_0)][y^k(p) - y^k(p_0)][y^l(p) - y^l(p_0)], \quad (9.10)$$

where $a^i, a^i_j, a^i_{jk} \in \mathbb{K}$ with $\det[a^i_j] \neq 0$ and $a^i_{jkl} \colon U \cap V \to \mathbb{K}$ are of class C^1 and they and their first partial derivatives are bounded at p_0 (see the lines preceding equation (II.3.24) on page 100).

From (9.10), we get $\frac{\partial z^i}{\partial y^j}\big|_{p_0} = a^i_j$ and $\frac{\partial^2 z^i}{\partial y^j \partial y^k}\big|_{p_0} = 2a^i_{(jk)}$. Substituting these equalities into (9.9), we find

$$a^i_j {\Gamma_X}^j{}_k(p_0) - X^m(p_0) 2a^i_{(mk)} = 0, \quad \det[a^i_j] \neq 0. \quad (9.11)$$

This equation always has solutions, a class of which for $\mathbf{\Gamma}_X(p_0) \neq 0$ is

$$a^i_j = c^i b_j X^m(p_0)\big(b_l {\Gamma_X}^l{}_m(p_0)\big), \quad 2a^i_{(mk)} = c^i\big(b_l {\Gamma_X}^l{}_m(p_0)\big)\big(b_n {\Gamma_X}^n{}_k(p_0)\big) \quad (9.12)$$

where $b_i, c^i \in \mathbb{K}$ are such that $\det[a^i_j] \neq 0$. The general solution of (9.11) can be found analogously to the one of (9.5) above. For this end, we take for (U, y) the afore-constructed chart (U_1, x) in which $X = \frac{\partial}{\partial x^1}$ along the integral path of X through p_0 in U_1. In it (9.11) reduces to $2a^i_{(1k)} = a^i_l {\Gamma_X}^l{}_k(p_0)$, $\det[a^i_j] \neq 0$, the remaining quantities being left arbitrary. Consequently, due to (9.10), the general form of the looked for coordinates $\{z^i\}$ normal at p_0 is

$$z^i(p) = a^i + a^i_j[x^j(p) - x^j(p_0)] + a^i_l {\Gamma_X}^l{}_k(p_0)[x^k(p) - x^k(p_0)][x^1(p) - x^1(p_0)] \\ + \sum_{j,k=2}^{\dim M} a^i_{jk}[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)] \\ + a^i_{jkl}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)]. \quad (9.13)$$

Since the Jacobian of the transformation $\{x^i\} \to \{z^i\}$ at p_0 is $\det\big[\frac{\partial z^i}{\partial x^j}\big|_{p_0}\big] = \det[a^i_j] \neq 0$, there is a subneighborhood $U_z \subseteq U_1$ with $U_z \ni p_0$ in which x^i are well-defined coordinate functions of a chart (U_z, z) which is normal by construction.[7]

□

9.2. The case along paths

The first step for generalizing the results form the previous subsection is to investigate the problems concerning frames or coordinates normal along paths for derivations along fixed vector field.

[7] The constructed in [80, proof of Proposition 4.1] chart normal at a single point corresponds to the choices $a^i_j = \delta^i_j$, $a^i_{jk} = 0$ for $j, k \geq 2$, and $a^i_{jkl} \equiv 0$.

In this section, as in Sections II.3, 5 and 8, we shall look on the manifold M as on a real one, i.e., if M is real, no changes are necessary and, if M is complex, we consider it as a real manifold of dimension $\dim_{\mathbb{R}} M = 2\dim M \equiv 2\dim_{\mathbb{C}} M$ (see p. 7). We shall formally reflected this in writing $\dim_{\mathbb{R}} M$ instead of $\dim M$. Respectively, the Latin indices with not explicitly specified range will run from 1 to $\dim_{\mathbb{R}} M$ and the ranges of the coordinate homeomorphisms will lie in $\mathbb{R}^{\dim_{\mathbb{R}} M}$.

Suppose $\gamma\colon J \to M$ is a locally injective C^1 regular path in a C^2 manifold M endowed with C^0 derivation D_X along a fixed non-zero vector field X of class C^0, i.e., $X \in \mathfrak{X}^0(M)$ with $X \neq 0$. The problem is to study the equation (9.1) for $U = \gamma(J)$, i.e.,

$$\mathbf{\Gamma}_X(\gamma(s))A(\gamma(s)) + X(A)|_{\gamma(s)} = 0 \tag{9.14}$$

for $s \in J$. The idea for investigating (9.14) is to be constructed a chart along γ such that the first (or the first two) coordinate(s) of a point in its domain to be equal to some value of the parameter of γ (or to it and a function of the parameter of an integral path of X) and in which X coincides with or is proportional to certain basic coordinate vector.

Theorem 9.3. *Let M be C^3 manifold* (*which should be considered as a real one of dimension* $\dim_{\mathbb{R}} M$ *if it is complex*), *$\gamma\colon J \to M$ be locally injective C^1 and regular, $s_0 \in J$, and D_X be a C^1 derivation along a fixed C^0 vector field X on M. If $\gamma(s_0)$ is a singular point for X, $X|_{\gamma(s_0)} = 0$, then either every frame in a neighborhood of $\gamma(s_0)$ is normal at $\gamma(s_0)$ for D_X* (*in which case D_X is the zero derivation at $\gamma(s_0)$*) *or such frames do not exist* (*in which case D_X is not the zero derivation at $\gamma(s_0)$*)*. If $\gamma(s_0)$ is not a singular point for X, $X|_{\gamma(s_0)} \neq 0$, then there exist a subinterval $J_1 \ni s_0$ of J and a frame $\{E'_i\}$ such that $\{E'_i\}$ is normal for D_X along $\gamma|_{J_1}$, i.e., on $\gamma(J_1)$. Moreover, there is a neighborhood V_1 of $\gamma(J_1)$ on which all frames normal for D_X are $\{E'_i = A^j_i \frac{\partial}{\partial x^j}\}$, where $A = [A^j_i]\colon V_1 \to \mathrm{GL}(\dim_{\mathbb{R}} M, \mathbb{R})$, $V_1 \subseteq U_1$, and (U_1, x) is a chart of M with associated coordinates $\{x^i\}$. The chart (U_1, x), transformation matrix A, and neighborhood V_1 are constructed in different ways depending on is the condition* (9.15) *below* (*with $J^0 \supseteq J_1$*) *valid or not:*

(a) *If* (9.15) *holds, (U_1, x) is the special chart described in Subsection* II.3.2 *for the path $\gamma|_{J_1}$, A is given by equation* (9.19) *below, and $V_1 = x^{-1}(J_1, W)$ where W is a neighborhood of $\boldsymbol{t}_0$ in $\mathbb{R}^{\dim_{\mathbb{R}} M - 1}$ on which A_0 is non-degenerate;*

(b) *If* (9.15) *is not valid, (U_1, x) is the special chart constructed at the beginning of Subsection* 8.1 *for the mapping α, introduced below on page* 181 *in the proof of this theorem, A is given by equation* (9.22) *below, and $V_1 = x^{-1}(J_1, J'_0, \mathbb{R}^{\dim_{\mathbb{R}} M - 2})$ where J'_0 is such that $A|_{\gamma(J'_0)}$ is non-degenerate.*

Proof. If $X|_{\gamma(s)} = 0$ for some $s \in J$, then (9.14) implies $\mathbf{\Gamma}_X(\gamma(s)) = 0$ as A is non-degenerate (cf. a similar situation explored in Subsection 9.1). Hence at the singular points of X along γ, if any, either every frame is normal (D_X is the zero derivation at them) or frames normal at them do not exist (D_X is not the zero derivation at them). In particular, if X is singular along γ, $X|_{\gamma(J)} = 0$, then either every frame along γ is normal for D_X or such frames do not exist at all.

Below we shall suppose the existence of $s_0 \in J$ such that $X_{\gamma(s_0)} \neq 0$, i.e., that X is not singular along the whole path γ. Since X is of class C^0, there is a neighborhood U_0 of $\gamma(s_0)$ in which X is non-singular, $X|_{U_0} \neq 0$.

Let $\beta_p \colon J'_p \to M$, $p \in \gamma(J)$ be the integral path of X through p (see p. 13), i.e., $\beta_p(\sigma_{p,0}) = p \in \gamma(J)$ for some $\sigma_{p,0} \in J'_p$ and $\dot\beta_p(\sigma_p) = X_{\beta_p(\sigma_p)}$, $\sigma_p \in J'_p$. For the path $\beta_{\gamma(s_0)}$ (in a neighborhood of $\gamma(s_0)$) there are two possibilities: it (locally) intersects γ only at $\gamma(s_0) = \beta_{\gamma(s_0)}(\sigma_{\gamma(s_0),0})$ for fixed $\sigma_{\gamma(s_0),0} \in J'_{\gamma(s_0)}$ or it is (locally) obtainable from γ by a possible change of its parameter. We shall write that in terms of X. The second possibility means the existence of a subinterval J^0 of J with $J^0 \ni s_0$ and function $g\colon \gamma(J^0) \to \mathbb{R}\backslash\{0\}$ such that $\gamma(J^0) \subset U_0$, $\gamma|_{J^0}$ is injective (recall that γ is supposed locally injective), and on $\gamma(J^0)$ the field X is proportional to the vector field $\dot\gamma$ tangent to γ with factor g,

$$X|_{\gamma(J^0)} = g\dot\gamma, \tag{9.15}$$

while the first possibility is equivalent to the nonexistence of such connection between X and $\dot\gamma$. (Notice, (9.15) with $g = 1$ means that γ is, possibly locally, an integral path for X.)

Let the relation (9.15) hold. Then, by virtue of equation (I.2.3), we have $X(A)|_{\gamma(s)} = g(\gamma(s))[\dot\gamma(s)(A)] = g(\gamma(s))\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s}$, so (9.14) reduces to

$$\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s} + \frac{1}{g(\gamma(s))}\mathbf{\Gamma}_X(\gamma(s))A(\gamma(s)) = 0, \qquad s \in J^0. \tag{9.16}$$

By Remark II.3.9 on page 96, the general solution of this equation, defining A only on $\gamma(J^0)$, is

$$A(\gamma(s)) = Y\Big(s, s_0; -\frac{1}{g\circ\gamma}\mathbf{\Gamma}_X \circ \gamma\Big)A_0 \tag{9.17}$$

where $A_0 \in \mathrm{GL}(\dim_{\mathbb{R}} M, \mathbb{R})$, $s \in J^0$, $s_0 \in J^0$ is fixed as above, and Y is the unique solution of the initial-value problem

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = -\frac{1}{g(\gamma(s))}\mathbf{\Gamma}_X(\gamma(s))Y, \quad Y|_{s=s_0} = \mathbb{1}. \tag{9.18}$$

If $\dim_{\mathbb{R}} M = 1$, which is possible only if M is real, then (9.15) always holds[8] and (9.17) gives the general solution of (9.14) with respect to A in $\gamma(J^0) \ni \gamma(s_0)$.[9] For $\dim_{\mathbb{R}} M \geq 2$, the general form of A on some neighborhood of $\gamma(s_0)$ could be found as follows. Since the integral paths are without self-intersections, i.e.,

[8]For $\dim_{\mathbb{R}} M = 1$, the vector $\dot\gamma(s) \neq 0$, $s \in J^0$, can be taken as a basis of the one-dimensional tangent space $T_{\gamma(s)}(M)$, respectively $\{\dot\gamma\}$ is a frame on $\gamma(J^0)$. Therefore (9.15) is simply the expansion of X over $\{\dot\gamma\}$ in this case.

[9]For $\dim_{\mathbb{R}} M = 1$, the 1×1 matrix $\mathbf{\Gamma}_X(\gamma(s))$ is simply a number, so that the explicit form of Y is $Y\big(s, s_0; -\frac{1}{g\circ\gamma}\mathbf{\Gamma}_X\circ\gamma\big) = \exp\Big(-\int_{s_0}^{s} \frac{\mathbf{\Gamma}_X(\gamma(\sigma))}{g(\gamma(\sigma))}\,\mathrm{d}\sigma\Big)$.

injective [9, 11], such is the restricted path $\gamma|_{J^0}$. Construct for $\gamma|_{J^0}$ the special chart (U_1, x) described in Subsection II.3.2 (see p. 94 with $\gamma|_{J^0}\colon J^0 \to M$ for γ). Then, for $p \in U_1$, we have $x(p) = (s, \boldsymbol{t})$ for some $s \in J_1 \subseteq J^0$ (recall that $\gamma(J_1) = U_1 \cap \gamma(J^0)$) and $\boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M-1}$ and $x(\gamma(s)) = (s, \boldsymbol{t_0})$ for a fixed $\boldsymbol{t}_0 \in \mathbb{R}^{\dim_{\mathbb{R}} M-1}$. Expanding A as

$$A(x^{-1}(s,\boldsymbol{t})) = A(\gamma(s)) + \sum_{k=2}^{\dim_{\mathbb{R}} M} A_k(x^{-1}(s,\boldsymbol{t}))(t^k - t_0^k)$$

and using (9.17), we, finally, get the general solution of (9.14) in U_1 as

$$A(x^{-1}(s,\boldsymbol{t})) = Y\big(s, s_0; -\frac{1}{g\circ\gamma}\mathbf{\Gamma}_X \circ \gamma\big)A_0 + \sum_{k=2}^{\dim_{\mathbb{R}} M} A_k(x^{-1}(s,\boldsymbol{t}))(t^k - t_0^k), \quad (9.19)$$

where A_0 is constant non-degenerate matrix and the matrix-valued functions $A_2, \dots, A_{\dim_{\mathbb{R}} M}$ on U_1 are of class C^1 and they and their first derivatives are bounded when $\boldsymbol{t} \to \boldsymbol{t}_0$, i.e., on $\gamma(J_1)$. Since A is continuous (it is supposed of class C^1), there is a neighborhood of $\boldsymbol{t}_0$ in $\mathbb{R}^{\dim_{\mathbb{R}} M-1}$ in which $A(x^{-1}(s,\boldsymbol{t}))$ is non-degenerate.

Let us now suppose $\dim_{\mathbb{R}} M \geq 2$ and the integral path $\beta_{\gamma(s_0)}\colon J'_{\gamma(s_0)} \to M$ to intersect $\gamma\colon J \to M$ at the single point $\gamma(s_0) = \beta_{\gamma(s_0)}(\sigma_{\gamma(s_0),0})$ for some unique $\sigma_{\gamma(s_0),0} \in J'_{\gamma(s_0)}$.[10] Since γ is locally injective, there is a subinterval $J_3 \subseteq J$ with $J_3 \ni s_0$ such that $\gamma|_{J_3}\colon J_3 \to M$ is without self-intersections. Let $J_2 := \{s | s \in J_3,\ \gamma(s) \in U_0\}$. Then $\gamma|_{J_2}$ is injective path and along it X does not vanish. At last, since X and γ are continuous, there exists a subinterval $J_0 \subseteq J_2$ with $J_0 \ni s_0$ such that the integral path $\beta_{\gamma(s)}$, $s \in J_2$, intersects $\gamma|_{J_0}$ only once (at the point $\gamma(s) = \beta_{\gamma(s)}(\sigma_{\gamma(s),0})$). A 'local' solution of (9.14) for the restricted path $\gamma|_{J_0}$ can be found as follows.

Let, for every $p \in \gamma(J)$, there be fixed (chosen) some C^1 bijective mapping (C^1 diffeomorphism) $\tau_p\colon J' \to J'_p$ from some open $\mathbb{R}$-interval J' to the domain of the integral path β_p such that $\tau_p(\sigma_0) = \sigma_{p,0}$ for fixed $\sigma_0 \in J'$.[11] Consider a mapping $\alpha\colon J_0 \times J' \to M$ with $\alpha(s,\sigma) := \beta_{\gamma(s)}(\tau_{\gamma(s)}(\sigma))$ for $(s,\sigma) \in J_0 \times J'$. Due to the aforesaid, α is injective, of class C^1 and regular. Consequently, for it are applicable the results at the beginning of Subsection 8.1 with α, 2, and $J_0 \times J'$ for γ, n, and J^n respectively. Therefore for α can be constructed a chart (U_1, x), explicitly given by (8.3) for $n = 2$, $\gamma = \alpha$, and $J_1^2 \subseteq J^2 = J_0 \times J'$, with $U_1 \ni \alpha(s_0, \sigma_0) = \gamma(s_0)$ and $U_1 \subseteq U_0$, such that there is a subneighborhood $J_1^2 := J_1 \times {J'}^0 \subseteq J_0 \times J'$ with $\alpha(J_1^2) = U_1 \cap \gamma(J_2)$, the coordinates of $p \in U_1$ are

[10] This is only a local property if (9.15) does not hold. But in this way we do not loose generality since if $\sigma_- \leq \sigma_{\gamma(s_0),0}$ and $\sigma_+ \geq \sigma_{\gamma(s_0),0}$ are respectively the maximal and minimal elements of the set $\{\sigma | \sigma \in J'_{\gamma(s_0)},\ \beta_{\gamma(s_0)}(\sigma_{\gamma(s_0)}) = \gamma(s) \text{ for some } s \in J\} \ni \sigma_{\gamma(s_0),0}$ and $\sigma_- \neq \sigma_+$, then we can take the interval $(\sigma_-, \sigma_+) \ni \sigma_{\gamma(s_0),0}$ for $J'_{\gamma(s_0)}$.

[11] Since all open real intervals are (C^∞-) diffeomorphic, mappings like τ_p always exist.

$x(p) = (s, \sigma, \boldsymbol{t})$ for some $(s, \sigma) \in J_1^2$ and $\boldsymbol{t} \in \mathbb{K}^{\dim_{\mathbb{R}} M-2}$, and $x(\alpha(s, \sigma)) = (s, \sigma, \boldsymbol{t}_0)$ for a fixed $\boldsymbol{t}_0 \in \mathbb{K}^{\dim_{\mathbb{R}} M-2}$ (see (8.4)).

Since in the special chart (U_1, x) we have

$$x(\beta_{\gamma(s)}(\sigma_{\gamma(s)})) = x(\beta_{\gamma(s)}(\tau_{\gamma(s)}(\sigma))) = x(\alpha(s, \sigma)) = (s, \sigma, \boldsymbol{t}_0)$$

for $\sigma_{\gamma(s)} = \tau_{\gamma(s)}(\sigma) \in J'_s$, $\sigma \in J'$, the components of $X_{\beta_{\gamma(s)}(\sigma_{\gamma(s)})}$ in the frame $\{\frac{\partial}{\partial x^i}\}$ are $X_{\beta_{\gamma(s)}(\sigma_{\gamma(s)})} = \frac{\partial(x^k(\beta_{\gamma(s)}(\sigma_{\gamma(s)})))}{\partial \sigma_{\gamma(s)}} = \frac{\partial(\delta_2^k \sigma)}{\partial \sigma_{\gamma(s)}} = \delta_2^k \big(\frac{\mathrm{d}\tau_{\gamma(s)}(\sigma)}{\mathrm{d}\sigma}\big)^{-1}$. Hence

$$X_{\beta_{\gamma(s)}(\sigma_{\gamma(s)})} = \frac{1}{\frac{\mathrm{d}\tau_{\gamma(s)}(\sigma)}{\mathrm{d}\sigma}} \frac{\partial}{\partial x^2}\Big|_{\beta_{\gamma(s)}(\sigma_{\gamma(s)})}. \tag{9.20}$$

So, in (U_1, x), i.e., for $s \in J_1$ and $E_i = \frac{\partial}{\partial x^i}$, the basic equation (9.14) reads

$$\frac{\partial A(x^{-1}(s, \sigma, \boldsymbol{t}))}{\partial \sigma}\Big|_{\substack{\sigma=\sigma_0 \\ \boldsymbol{t}=\boldsymbol{t}_0}} + \frac{\mathrm{d}\tau_{\gamma(s)}(\sigma)}{\mathrm{d}\sigma}\Big|_{\sigma=\sigma_0} \boldsymbol{\Gamma}_X(\gamma(s)) A(\gamma(s)) = 0. \tag{9.21}$$

The general solution of this equation with respect the matrix-valued function $A\colon U_1 \to \mathrm{GL}(\dim_{\mathbb{R}} M, \mathbb{R})$ can be found by expanding A with respect to $(\sigma - \sigma_0)$ up to second order terms. In this way, we find

$$\begin{aligned} A(x^{-1}(s, \sigma, \boldsymbol{t})) &= A_0(x^{-1}(s, \sigma_0, \boldsymbol{t})) \\ &- \frac{\mathrm{d}\tau_{\gamma(s)}(\sigma)}{\mathrm{d}\sigma}\Big|_{\sigma=\sigma_0} \boldsymbol{\Gamma}_X(\gamma(s)) A_0(\gamma(s))(\sigma - \sigma_0) + B(x^{-1}(s, \sigma, \boldsymbol{t}))(\sigma - \sigma_0)^2 \end{aligned} \tag{9.22}$$

where $A_0\colon x^{-1}(J_1, \sigma_0, \mathbb{R}^{\dim M-2}) \to \mathrm{GL}(\dim_{\mathbb{R}} M, \mathbb{R})$ and the matrix-valued function B on U_1 is of class C^1 and it and its first partial derivatives are bounded on $\gamma(J_1)$, i.e., when $\sigma \to \sigma_0$ and $\boldsymbol{t} \to \boldsymbol{t}_0$. Since we have $\det A(x^{-1}(s, \sigma_0, \boldsymbol{t})) = \det A_0(x^{-1}(s, \sigma_0, \boldsymbol{t})) \neq 0, \infty$ and A is continuous (as it is of class C^1), there is a subinterval $J'_0 \subseteq J'$ containing σ_0 such that $A\colon x^{-1}(J_1, J'_0, \mathbb{R}^{\dim M-2}) \to \mathrm{GL}(\dim_{\mathbb{R}} M, \mathbb{R})$. This last result completes the proof of Theorem 9.3. □

The Theorem 9.3 provides a complete local description of the frames normal along a path $\gamma\colon J \to M$ for a derivation D_X along fixed vector field $X \in \mathfrak{X}(M)$ at the points in which X does not vanish. In particular, if $X_{\gamma(J)} \neq 0$, then every point along γ has a neighborhood on which frames normal along γ exist and from these frames all frames globally normal along γ can be constructed.

In a neighborhood of every point $\gamma(s_0)$, $s_0 \in J$, for which $X_{\gamma(s_0)} \neq 0$ coordinates $\{z^i\}$ normal along γ exist. They can be found by integrating the system $\frac{\partial}{\partial z^i} = A_i^j \frac{\partial}{\partial x^j}$ for some A given by (9.19) or by (9.22) (with corresponding $\{x^i\}$ as specified above) or via direct solving of the normal frame equation (9.2′) along γ, viz. of

$$X^m(\gamma(s)) \frac{\partial^2 z^i}{\partial x^m \partial x^k}\Big|_{\gamma(s)} + \frac{\partial z^i}{\partial x^m}\Big|_{\gamma(s)} \Gamma_X{}^m{}_k(\gamma(s)) = 0. \tag{9.23}$$

Since the results obtained considerably simplify the second method, we shall follow it.

Proposition 9.4. *Let M be C^3 manifold, $\gamma\colon J\to M$ be locally injective, C^1 and regular path, and D_X be C^1 derivation along a fixed $X\in\mathfrak{X}^0(M)$. For every $s_0\in J$ for which $X|_{\gamma(s_0)}\neq 0$, there exist local coordinates in a neighborhood of $\gamma(s_0)$ which are normal along γ. Their general form is given by equation* (9.24) *below under the conditions* (9.26) (*resp.* (9.27)) *below if* (9.15) *holds* (*resp. does not hold*).

Proof. Let $s_0\in J$, $X_{\gamma(s_0)}\neq 0$, and (U_1,x) with $U_1\ni\gamma(s_0)$ be the special chart constructed above with reference mapping $\gamma|_{J_0}$ or α if (9.15) holds or not, respectively. We shall look for a chart (U_z,z) with $\gamma(s_0)\in U_z\subseteq U_1$ in which (9.23) is valid. Since M is supposed of class C^3, we can write the expansion

$$z^i(p)=a^i(p_0)+a^i_j(p_0)[x^j(p)-x^j(p_0)]+a^i_{jk}(p)[x^j(p)-x^j(p_0)][x^k(p)-x^k(p_0)]. \tag{9.24}$$

Here: $p\in U_z$, p_0 is the 'projection' of p along γ or α (i.e., $p_0=x^{-1}(s,\boldsymbol{t}_0)=\gamma(s)$ or $p_0=x^{-1}(s,\sigma,\boldsymbol{t}_0)=\beta_{\gamma(s)}(\tau_{\gamma(s)}(\sigma))$ if $p=x^{-1}(s,\boldsymbol{t})$ or $p=x^{-1}(s,\sigma,\boldsymbol{t})$ respectively), a^i and a^i_j are C^1 functions of p_0 and $a^i_{jk}\colon U_z\to\mathbb{R}$ are C^1 and they and their first partial derivatives are bounded when $p\to p_0$.

In the chart (U_1,x), the fixed vector field X has the expansion $X|_{\gamma(s)}=g(\gamma(s))\frac{\partial}{\partial x^1}|_{\gamma(s)}$ or (9.20) if (9.15) holds or not, respectively. So, in these two cases, equation (9.23) takes respectively the form (cf. (9.16) and (9.21))

$$\frac{\partial^2 z^i}{\partial x^1\partial x^k}\Big|_{\gamma(s)}-\frac{1}{g(\gamma(s))}\frac{\partial z^i}{\partial x^j}\Big|_{\gamma(s)}{\Gamma_X}^j{}_k(\gamma(s))=0 \tag{9.25a}$$

$$\frac{\partial^2 z^i}{\partial x^2\partial x^k}\Big|_{\substack{p=p_0\\ \sigma=\sigma_0}}-\frac{\mathrm{d}\tau_{\gamma(s)}(\sigma)}{\mathrm{d}\sigma}\Big|_{\sigma=\sigma_0}\frac{\partial z^i}{\partial x^j}\Big|_{\substack{p=p_0\\ \sigma=\sigma_0}}{\Gamma_X}^j{}_k(\gamma(s))=0. \tag{9.25b}$$

The entering here derivatives of the z's with respect to the x's can be expressed through the unknown functions a^i and a^i_j by employing (9.24):

$$\frac{\partial z^i}{\partial x^j}\Big|_{\gamma(s)}=\begin{cases}\frac{\mathrm{d}a^i(\gamma(s))}{\mathrm{d}s} & \text{for } j=1\\ a^i_j(\gamma(s)) & \text{for } j\geq 2\end{cases}\qquad \frac{\partial^2 z^i}{\partial x^1\partial x^k}\Big|_{\gamma(s)}=\begin{cases}\frac{\mathrm{d}^2a^i(\gamma(s))}{\mathrm{d}s^2} & \text{for } k=1\\ \frac{\mathrm{d}a^i_k(\gamma(s))}{\mathrm{d}s} & \text{for } k\geq 2\end{cases}$$

$$\frac{\partial z^i}{\partial x^j}\Big|_{p=p_0}=\begin{cases}\frac{\partial a^i}{\partial s}\big|_{p=p_0} & \text{for } j=1\\ \frac{\partial a^i}{\partial \sigma}\big|_{p=p_0} & \text{for } j=2\\ a^i_j(p_0) & \text{for } j\geq 3\end{cases}\qquad \frac{\partial^2 z^i}{\partial x^2\partial x^k}\Big|_{p=p_0}=\begin{cases}\frac{\partial^2 a^i}{\partial s\partial\sigma}\big|_{p=p_0} & \text{for } k=1\\ \frac{\partial^2 a^i}{\partial \sigma^2}\big|_{p=p_0} & \text{for } k=2\\ \frac{\partial a^i_k}{\partial\sigma}\big|_{p=p_0} & \text{for } k\geq 3\end{cases}$$

where the first (resp. second) raw is valid when (9.15) holds (resp. does not hold) and we have used that $p_0=x^{-1}(s,\boldsymbol{t}_0)=\gamma(s)$ (resp. $p_0=x^{-1}(s,\sigma,\boldsymbol{t}_0)=\beta_{\gamma(s)}(\tau_{\gamma(s)}(\sigma)))$ if $p=x^{-1}(s,\boldsymbol{t})$ (resp. $p=x^{-1}(s,\sigma,\boldsymbol{t})$). The substitution of these equations into (9.25) results in the following two systems of linear differential equations for the functions a^i and a^i_j:

$$\frac{1}{g(\gamma(s))}\Big[\frac{\mathrm{d}a^i(\gamma(s))}{\mathrm{d}s}{\Gamma_X}^1{}_k(\gamma(s)) + \sum_{j=2}^{\dim_{\mathbb{R}} M} a^i_j(\gamma(s)){\Gamma_X}^j{}_k(\gamma(s))\Big] = \begin{cases} \frac{\mathrm{d}^2a^i(\gamma(s))}{\mathrm{d}s^2} & \text{for } k=1 \\ \frac{\mathrm{d}a^i_k(\gamma(s))}{\mathrm{d}s} & \text{for } k\ge 2 \end{cases} \qquad (9.26)$$

$$\begin{aligned}\frac{\mathrm{d}\tau_{\gamma(s)}(\sigma)}{\mathrm{d}\sigma}\Big|_{\sigma=\sigma_0}\Big[&\frac{\partial a^i(\beta_{\gamma(s)}(\tau_{\gamma(s)}(\sigma)))}{\partial s}\Big|_{\sigma=\sigma_0}{\Gamma_X}^1{}_k(\gamma(s)) \\ &+ \frac{\partial a^i(\beta_{\gamma(s)}(\tau_{\gamma(s)}(\sigma)))}{\partial \sigma}\Big|_{\sigma=\sigma_0}{\Gamma_X}^2{}_k(\gamma(s)) \\ &+ \sum_{j=3}^{\dim_{\mathbb{R}} M} a^i_j(\gamma(s)){\Gamma_X}^j{}_k(\gamma(s))\Big] = \begin{cases} \frac{\partial^2 a^i(\beta_{\gamma(s)}(\tau_{\gamma(s)}(\sigma)))}{\partial s\partial\sigma}\Big|_{\sigma=\sigma_0} & \text{for } k=1 \\ \frac{\partial^2 a^i(\beta_{\gamma(s)}(\tau_{\gamma(s)}(\sigma)))}{\partial \sigma^2}\Big|_{\sigma=\sigma_0} & \text{for } k=2 \\ \frac{\partial a^i_k(\beta_{\gamma(s)}(\tau_{\gamma(s)}(\sigma)))}{\partial\sigma}\Big|_{\sigma=\sigma_0} & \text{for } k\ge 3 \end{cases}. \end{aligned} \qquad (9.27)$$

We are not going to solve these systems of differential equations, but we must note that they always have solutions according to the general theorems of the theory of differential equations [34]. The substitution of these solutions into (9.24) gives the corresponding coordinates normal along the path γ. The domain U_z of the normal chart (U_z, z) should be chosen such that in it the Jacobian $\det\big[\frac{\partial z^i}{\partial x^j}\big]$ be non-vanishing. □

Problem 9.1. Explore the system (9.26)–(9.27) of differential equations and find its general solution.

9.3. The case on the whole manifold

The purpose of this section is to be shown that frames normal for a derivation along fixed vector field exist on the whole manifold. The general form of these frames will be found too.

Let M be C^1 manifold, $X \in \mathfrak{X}^0(M)$, $X \neq 0$, $\{E_i\}$ be a frame on M, and D_X be a C^0 derivation along X. As we know, a frame $\{E'_i = A^j_i E_j\}$ is normal for D_X (on M) iff (9.1) holds for $U = M$,

$$\mathbf{\Gamma}_X A + X(A) = 0 \qquad (9.28)$$

where $\mathbf{\Gamma}_X$ is the matrix of D_X in $\{E_i\}$. We intend to demonstrate that this equation always has (local) non-degenerate solutions (at the non-singular points for X) in contrast to the similar (system of) equation(s) for linear connections (Section II.4) or for derivations along vector fields (Section 6).

Theorem 9.4. *Let M be C^1 manifold, X be continuous vector field on M, X be not identically zero on M, and D_X be a C^0 derivation along X. For every $p_0 \in M$ for which $X_{p_0} \neq 0$, there exists a chart (U, x) with associated coordinate system $\{x^i\}$ such that: $U \ni p_0$, $X|_U = \frac{\partial}{\partial x^1}\big|_U$, there exist frames normal on U for D_X and all such frames are $\{E'_i = A^j_i \frac{\partial}{\partial x^j}\}$ where $A = [A^j_i] \colon U \to \mathrm{GL}(\dim M, \mathbb{R})$ is given by equation (9.31) below in which Y is defined by (9.32) below and A_0 is a non-degenerate matrix-valued function. If $p_0 \in M$ and $X_{p_0} = 0$, at p_0 exist frames normal for D_X iff D_X is the zero derivation at p_0, $(D_X K)|_{p_0} = 0$ for every tensor field K, in which case every frame (in a neighborhood of p_0) is normal at p_0 for D_X.*

Proof. Let us choose some $p_0 \in M$ such that $X_{p_0} \neq 0$. According to [7, Proposition 1.53], there is a chart (U, x) of M with $U \ni p_0$ and local coordinates $\{x^i\}$ for which[12]

$$X|_U = \frac{\partial}{\partial x^1}\Big|_U \neq 0. \tag{9.29}$$

So, in (U, x) the normal frame equation (9.28) (on U) with $E_i = \frac{\partial}{\partial x^i}$ reads

$$\frac{\partial A}{\partial x^1}\Big|_p = -\mathbf{\Gamma}_X(p) A(p). \tag{9.30}$$

Its general solution on U is

$$A(p) = Y\big(x^1(p), x^1(p_0); -\mathbf{\Gamma}_X(x^{-1}(\,\cdot\,, x^2(p), \dots, x^{\dim M}(p)))\big) \\ \times A_0\big(x^2(p), \dots, x^{\dim M}(p)\big), \tag{9.31}$$

where $p \in U$, the matrix-valued function $Y = Y(x^1(p), x^1(p_0); Z)$, with continuous matrix-valued function Z on $x^1(U)$, is the unique solution of the initial-value problem (see Lemma II.3.2 on page 96 or Lemma II.4.1 on page 105)

$$\frac{\partial Y}{\partial x^1}\Big|_p = ZY, \qquad Y|_{x^1(p) = x^1(p_0)} = \mathbb{1} \tag{9.32}$$

and A_0 is non-degenerate matrix-valued function of its arguments.

This ends the proof of the first part of the theorem. The second one is a consequence of Proposition 9.2 (for $U = \{p_0\}$). □

The Theorem 9.4 gives a complete local description of all frames normal on M. If X is non-zero everywhere on M or if D_X is the zero derivation at the points at which X vanishes, all frames globally normal on the whole manifold M for D_X can be constructed from the local normal frames provided by Theorem 9.4. For this end, a method similar to the one described in Subsection II.5.2 should be applied (see page 123).

[12] This is a special case of the theorem for straightening (of integral paths); see [91, p. 121].

If (U, x) is the chart provided by Theorem 9.4, the coordinate functions z^i of every normal chart (V, z) with $p \in V \subseteq U$ are solutions of (9.2′) with $x'^i = z^i$ and $X^m = \delta_1^m$:

$$\Big(\frac{\partial^2 z^i}{\partial x^1 \partial x^k} - \frac{\partial z^i}{\partial x^j}{\Gamma_X}^j{}_k\Big)\Big|_V = 0. \tag{9.33}$$

According to the theorems of the theory of (linear partial) differential equations [34], this equation always has (local) solutions with respect to z^i. The general solution of (9.33) can not be expressed in a closed form for arbitrary $\mathbf{\Gamma}_X$. But if M and D_X are of class C^∞, it can be found by expanding $z^i = z^i(x^1, \dots, x^{\dim M})$ and ${\Gamma_X}^j{}_k$ in power series with respect to $[x^i(p) - x^i(p_0)]$,

$$z^i(p) = a^i + \sum_{n=1}^{\infty} a^i_{i_1 \dots i_n}[x^{i_1}(p) - x^{i_1}(p_0)] \cdots [x^{i_n}(p) - x^{i_n}(p_0)] \tag{9.34a}$$

$${\Gamma_X}^j{}_k(p) = \gamma^j_k + \sum_{n=1}^{\infty} \gamma^j_{k i_1 \dots i_n}[x^{i_1}(p) - x^{i_1}(p_0)] \cdots [x^{i_n}(p) - x^{i_n}(p_0)] \tag{9.34b}$$

where the a's and γ's are constant numbers symmetric in $i_1 \dots i_n$ and $\det[a^j_i] \neq 0$. Indeed, substituting these expansions into (9.33), we get the following infinite system of recurrent equations

$$\begin{aligned} a^i_{1k} &= a^i_j \gamma^j_k \\ a^i_{1ki_2} &= a^i_{ji_2}\gamma^j_k + a^i_j \gamma^j_{ki_2} \\ a^i_{1ki_2i_3} &= \big(a^i_{ji_2i_3}\gamma^j_k + a^i_{ji_2}\gamma^j_{ki_3} + a^i_j\gamma^j_{ki_2i_3}\big)_{(i_2i_3)} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{9.35}$$

From these equalities and (9.34a), the general solution of (9.33) can be found. Since $\frac{\partial z^i}{\partial x^j} = a^i_j$ and $[a^i_j]$ is non-degenerate, the obtained series for z^i is convergent in some neighborhood $V \subseteq U$ of p_0 and the change $\{x^i\} \to \{z^i\}$ is well-defined in V, i.e., it is with non-zero Jacobian in V. In this way all charts (V, z) normal for the initial derivation D_X can be found in a neighborhood of every point $p_0 \in M$ at which $X_{p_0} \neq 0$. (Recall, if $X_{p_0} = 0$ and D_X is the zero derivation at p_0, then every chart in a neighborhood of p_0 is normal at p_0 for D_X, but if $X_{p_0} = 0$ and D_X is not the zero derivation at p_0, then charts (and frames) normal at p_0 for D_X do not exist.)

9.4. Other cases

Following the methods of Subsections 9.1–9.3 and the ideas of Sections 4–8, one can try to attack the problems for existence and construction of frames and coordinates normal on submanifolds or along (locally injective) mappings for derivations along fixed vector field. The main obstacles on this way are the complicated systems of partial differential equations one gets, a typical example of which are (9.26) and (9.27).[13] For this reason, we are not going to study these situations here and want only to make some comments on them.

Proposition 9.5. *Let M be C^1 manifold, U be any subset of M, X be continuous vector field on M, and D_X be derivation along X. If $p \in U$ and $X_p \neq 0$, then there is a subset U_p of U containing p on which frames normal for D_X exist. If $p \in U$ and $X_p = 0$, then at p exist frames normal for D_X iff D_X is the zero derivation at p and if this is the case, every frame defined in a neighborhood of p is normal at p for D_X.*

Proof. If $p \in U \subset M$ and $X_p \neq 0$, by Theorem 9.4 exist frames normal on some neighborhood U_0 of p in M (and all of them are described by this theorem). If $\{E_i\}$ is such a frame, then, obviously, it is normal for D_X on $U_p := U_0 \cap U \ni p$ as $U_p \subseteq U$, which completes the proof of the first part of the theorem. The second part of its assertion is a consequence of Proposition 9.2. □

Corollary 9.1. *If N is a submanifold of a C^1 manifold M, $X \in \mathfrak{X}^0(M)$, $X \neq 0$, and D_X is a derivation along X, then for every $p \in N$ for which $X_p \neq 0$ exists a neighborhood U_p^N of p in N on which frames normal for D_X exist.*

Proof. See Proposition 9.5 or its proof for $U = N$ and put $U_p^N = U_p$. □

Corollary 9.2. *Let $f\colon N \to M$ be locally injective mapping between C^1 manifolds N and M, $X \in \mathfrak{X}^0(M)$, $X \neq 0$, and D_X be derivation along X. For every* $(\dim N)$-*dimensional submanifold N' of N on which f is injective and $X|_{f(N')} \neq 0$ there exist frames normal along $f|_{N'}$ for D_X.*

Proof. See Proposition 9.5 and construct, if required, a frame globally normal along $f_{N'}$ from the local frames normal on the constituents of an open cover of $f(N')$ on each of which frames normal along f exist. □

Applying some freedom of the language, we may paraphrase Corollaries 9.1 and 9.2 by saying that on any C^1 submanifold N of a C^1 manifold M or along a locally injective mapping f with range in M, frames locally normal on N or along f exist for every derivation along a fixed vector field X provided X does not vanish on N or on the range of f, respectively.

An important remark in connection with Proposition 9.5 and Corollaries 9.1 and 9.2 should be made. We proved the existence of frames (locally) normal on $U \subseteq M$ by restricting to (subsets of) U the frames locally normal on the *whole* manifold M. All frames locally normal on M are described by Theorem 9.4 and their restrictions to (subsets of) U are, of course locally normal on U. The essential moment is that if $U \neq M$, then, generally, exist frames locally normal on U which are *not* (locally) normal on M. In other words, the set of frames locally normal on

[13]In the general case, the analogues of (9.26) (resp. (9.27)) correspond to the cases when the fixed vector field X lies (resp. does not lie) in the tangent space of the corresponding submanifold considered as a manifold.

U is larger than the one consisting of locally normal frames obtained by frames (locally) normal on M via restriction to (subsets of) U.

An example of these more or less intuitively clear assertions is provided by a comparison of Theorems 9.4 and 9.1, the latter corresponding to $U = \{p_0\}$, with $p_0 \in M$ and $X_{p_0} \neq 0$ in the above notation. Since in the both theorems x^i are coordinates in which $X = \frac{\partial}{\partial x^1}$ (on M or at p_0 resp.), one can compare the matrices (9.31) and (9.8) by means of which is achieved the transition to normal frames (on M or at p_0 resp.). Take, for simplicity, p_0 in (9.31) and (9.8) to be one and the same point. Expanding the right-hand side of (9.31) with respect to $x^k(p) - x^k(p_0)$ and taking into account (9.30), we get

$$\begin{aligned}
A(p) = &\left\{\mathbb{1} - \mathbf{\Gamma}_X(p_0)[x^1(p) - x^1(p_0)] + B(x^1(p))[x^1(p) - x^1(p_0)]^2\right\} \\
&\times\Big\{B_0 + \sum_{k=2}^{\dim M} B_k[x^k(p) - x^k(p_0)] \\
&+ \sum_{k,l=2}^{\dim M} B_{kl}(x^2(p), \dots, x^{\dim M}(p))[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)]\Big\} \\
= &B_0 - \mathbf{\Gamma}_X(p_0)B_0[x^1(p) - x^1(p_0)] + \sum_{k=2}^{\dim M} B_k[x^k(p) - x^k(p_0)] \\
&+ B(x^1(p))B_0[x^1(p) - x^1(p_0)]^2 \\
&- \mathbf{\Gamma}_X(p_0)[x^1(p) - x^1(p_0)] \sum_{k=2}^{\dim M} B_k[x^k(p) - x^k(p_0)] \\
&+\sum_{k,l=2}^{\dim M} B_{kl}(x^2(p), \dots, x^{\dim M}(p))[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] + \cdots .
\end{aligned}$$

Here B_0 and B_k, $k \geq 2$ are arbitrary constant matrices, $\det B_0 \neq 0$, B_{kl} with $k, l \geq 2$ are arbitrary matrix-valued functions (in both cases the arbitrariness comes from the arbitrariness of A_0 in (9.31)), and $B(x^1(p))$ is a *fixed* matrix-valued function whose form depends entirely on $\mathbf{\Gamma}_X$ (B comes from the expansion of Y in (9.31)). Comparing the last expression with (9.8), we see that the both expressions coincide up to notation if in (9.8) we put

$$\begin{aligned}
A_{kl}^{(x)}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] = &B(x^1(p))B_0[x^1(p) - x^1(p_0)]^2 \\
&- \mathbf{\Gamma}_X(p_0)[x^1(p) - x^1(p_0)] \sum_{k=2}^{\dim M} B_k[x^k(p) - x^k(p_0)] \\
+ \sum_{k,l=2}^{\dim M} B_{kl}(x^2(p), \dots, x^{\dim M}(p))[x^k(p) - x^k(p_0)]&[x^l(p) - x^l(p_0)] + \cdots . \qquad (9.36)
\end{aligned}$$

Hence, up to third order terms, (9.8) corresponds to (9.31) with

$$A^{(x)}_{11}(p) = B(x^1(p))B_0, \quad A^{(x)}_{1k}(p) = -\mathbf{\Gamma}_X(p_0)B_k \text{ for } k \geq 2,$$
$$A^{(x)}_{kl}(p) = B_{kl}(x^2(p), \dots, x^{\dim M}(p)) \text{ for } k, l \geq 2.$$

Exercise 9.1. Find the exact expressions for $A^{(x)}_{kl}$ for which (9.8) reduces to (9.31). (Hint: in the both sides of (9.36), take into account all quantities up to fifth order with respect to $[x^k(p) - x^k(p_0)]$.)

The general idea and conclusion from the above discussion is: on every $U \subseteq M$ frames normal on U for a given derivation along some fixed vector field always exist but the 'smaller' U is, the 'larger' the variety of these normal frames is; this variety is 'maximal' at a single point and it is 'minimal' for the whole manifold M.

The aforesaid, concerning normal frames, can, evidently, *mutatis mutandis* be transferred to the description of frames normal on subsets of a manifold (or along locally injective mappings with range in it) for derivations along fixed vector field in it.

10. Normal frames and coordinates for derivations along paths

As it is well known (see p. 11), to every injective C^1 path $\gamma\colon J \to M$ corresponds a vector field $\dot\gamma$ along γ, i.e., on $\gamma(J)$, called tangent to γ, assigning to every point $\gamma(s)$, $s \in J$, the vector $\dot\gamma(s)$ tangent to γ at $\gamma(s)$ and defined via (I.2.3).[1] The vector field $\dot\gamma$ can be extended to a vector field X on a neighborhood of $\gamma(J)$, $X_{\gamma(J)} = \dot\gamma$, in infinitely many ways. Consider some derivation D_X along X of $\boldsymbol{T}^1(M)$. Since outside $\gamma(J)$ the field X is completely arbitrary, such is D_X too. One can get real profit of D_X if it can be made to depend only on γ. Since it is clear that outside $\gamma(J)$ this is impossible without some additional very restrictive assumptions on X, let us see what happens with D_X when its action is restricted to $\gamma(J)$. Suppose K is a C^1 tensor field of type (r, q) defined on a neighborhood of $\gamma(J)$ (or on $\gamma(J)$ if $\dim_{\mathbb{R}} M = 1$ and $\gamma(J) = M$). If $\{E_i\}$ is a frame on this neighborhood, the components of D_X in the tensor frame induced by it are (2.5). Hereof, applying $X_{\gamma(J)} = \dot\gamma$ and (I.2.3), we obtain

$$\big((D_X(K))^{i_1\dots i_r}_{j_1\dots j_q}\big)\big|_{\gamma(s)} = \frac{\mathrm{d}\big(K^{i_1\dots i_r}_{j_1\dots j_q}(\gamma(s))\big)}{\mathrm{d}s} + \sum_{a=1}^{r} \Gamma_X{}^{i_a}{}_k(\gamma(s)) K^{i_1\dots i_{a-1}ki_{a+1}\dots i_r}_{j_1\dots j_q}(\gamma(s))$$
$$- \sum_{b=1}^{q} \Gamma_X{}^{k}{}_{j_b}(\gamma(s)) K^{i_1\dots i_r}_{j_1\dots j_{b-1}kj_{b+1}\dots j_q}(\gamma(s)).$$

[1] If γ is not injective, the correspondence $\gamma(s) \to \dot\gamma(s)$ is generally multiple-valued, while the mapping $s \to \dot\gamma(s)$ is always single-valued.

From here an important conclusion follows: if D_X is a derivation along X and the restriction of its matrix $\mathbf{\Gamma}_X$ to $\gamma(J)$ depends only on the path γ,

$$\mathbf{\Gamma}_X(\gamma(s))|_{X_{\gamma(s)}=\dot\gamma(s)} = \mathbf{\Gamma}(s;\gamma), \qquad s\in J, \tag{10.1}$$

then the restriction to $\gamma(J)$ of the action of D_X on any tensor field is independent of the particular definition of X and depends only on γ. Indeed, combining (10.1) with the equation preceding it, we find

$$\begin{aligned}\bigl((D_X(K))^{i_1\dots i_r}_{j_1\dots j_q}\bigr)\big|_{\gamma(s)} = & \frac{\mathrm{d}\bigl(K^{i_1\dots i_r}_{j_1\dots j_q}(\gamma(s))\bigr)}{\mathrm{d}s} \\ & + \sum_{a=1}^{r}\Gamma^{i_a}{}_{k}(s;\gamma)K^{i_1\dots i_{a-1}ki_{a+1}\dots i_r}_{j_1\dots j_q}(\gamma(s)) \\ & - \sum_{b=1}^{q}\Gamma^{k}{}_{j_b}(s;\gamma)K^{i_1\dots i_r}_{j_1\dots j_{b-1}kj_{b+1}\dots j_q}(\gamma(s))\end{aligned} \tag{10.2}$$

for $X|_{\gamma(J)} = \dot\gamma$. One should have in mind the invariant character of the condition (10.1): if it holds (for some or every path γ) in some frame $\{E_i\}$, then it is valid in any other frame $\{E'_i = A^j_i E_j\}$ as (2.11) implies

$$\mathbf{\Gamma}(s;\gamma)\mapsto\mathbf{\Gamma}'(s;\gamma) = A^{-1}(\gamma(s))\Bigl[\mathbf{\Gamma}(s;\gamma)A(\gamma(s)) + \frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s}\Bigr] \tag{10.3}$$

due to $X|_{\gamma(J)} = \dot\gamma$.

Example 10.1. An important example of a derivation along vector fields possessing the property (10.1) is provided by the linear connections which, as we know from Section 2, are derivations linear at every point of the manifold. In fact, if ∇ is a linear connection with coefficients' matrices $\Gamma_k = [\Gamma^i{}_{jk}]$, $\Gamma^i{}_{jk}$ being its coefficients, then, by Proposition 2.2 (see also Definition 2.4), we have

$$\mathbf{\Gamma}_X(\gamma(s))|_{X_{\gamma(s)}=\dot\gamma(s)} = \Gamma_k(\gamma(s))\dot\gamma^k(s) \tag{10.4}$$

as $X|_{\gamma(J)} = \dot\gamma$. The mapping assigning to a C^1 tensor field K defined in a neighborhood of $\gamma(J)$ the tensor field $(\nabla_X K)|_{\gamma(J)}$, where $X\in\mathfrak{X}(M)$ and $X|_{\gamma(J)} = \dot\gamma$, is called *covariant derivative along* γ *generated by* (*associated to*) ∇. It is denoted by $\nabla_{\dot\gamma}$ (and sometimes by $\frac{D}{\mathrm{d}s}$ or $\frac{D}{\mathrm{d}s}\big|_\gamma$).[2] According to (10.2) and (10.4), the com-

[2] A far more logical is the covariant derivative along γ to be denoted by ∇^γ (see below Definition 10.1) instead by $\nabla_{\dot\gamma}$, but we shall follow the established tradition.

ponents of $\nabla_{\dot\gamma} K$ at $\gamma(s)$ are

$$\begin{aligned}\bigl((\nabla_{\dot\gamma}(K))^{i_1\dots i_r}_{j_1\dots j_q}\bigr)\big|_{\gamma(s)} = & \frac{\mathrm{d}\bigl(K^{i_1\dots i_r}_{j_1\dots j_q}(\gamma(s))\bigr)}{\mathrm{d}s} \\ & + \dot\gamma^l(s)\sum_{a=1}^{r} \Gamma^{i_a}{}_{kl}(\gamma(s)) K^{i_1\dots i_{a-1}ki_{a+1}\dots i_r}_{j_1\dots j_q}(\gamma(s)) \\ & - \dot\gamma^l(s)\sum_{b=1}^{q} \Gamma^{k}{}_{j_b l}(\gamma(s)) K^{i_1\dots i_r}_{j_1\dots j_{b-1}kj_{b+1}\dots j_q}(\gamma(s)).\end{aligned} \tag{10.5}$$

where $\Gamma^i{}_{jk}(\gamma(s))$ are the coefficients of ∇ at $\gamma(s)$. It should be mentioned the different meaning of the symbols ∇_X and $\nabla_{\dot\gamma}$: in the former X is a vector field defined on the manifold M while in the latter $\dot\gamma$ is a vector field defined only along a path $\gamma\colon J \to M$ and not outside the set $\gamma(J)$. That is why an expansion like $\nabla_{\dot\gamma} = \dot\gamma^k \nabla_{E_k}$ for $\dot\gamma(s) = \dot\gamma^k(s) E_k|_{\gamma(s)}$ is not quite correct; one should write $\nabla_{\dot\gamma}\big|_{\boldsymbol{T}^1(\gamma(J))} = \dot\gamma^k \bigl(\nabla_{E_k}\big|_{\boldsymbol{T}^1(\gamma(J))}\bigr)$ which, in more free terms, can be reduced to $\nabla_{\dot\gamma} = \dot\gamma^k(\nabla_{E_k}|_{\gamma(J)})$ or to $\nabla_{\dot\gamma}|_{\gamma(s)} = \dot\gamma^k(s)(\nabla_{E_k}|_{\gamma(s)})$, $s \in J$.

The above considerations lead to the concept of a derivation along paths (of the tensor algebra over a manifold) assigning to every C^1 injective path a derivation along it with properties like (10.1) and (10.2).

Definition 10.1. A *derivation* D *along paths of the tensor algebra over a* C^1 *manifold* M, is a mapping assigning to every C^1 path $\gamma\colon J \to M$ without self-intersections a mapping D^γ, called *derivation along* γ (of the algebra of C^1 tensor fields along γ), from the algebra of C^1 tensor fields along γ into the one of C^0 tensor fields,

$$\mathsf{D}^\gamma\colon \boldsymbol{T}^1(\gamma(J)) \to \boldsymbol{T}^0(\gamma(J)),$$

such that:

- **(i)** $\mathsf{D}^\gamma(K + L) = \mathsf{D}^\gamma K + \mathsf{D}^\gamma L, \quad K, L \in \mathfrak{T}^{r;1}_s(\gamma(J))$;
- **(ii)** $\mathsf{D}^\gamma(K \otimes L) = (\mathsf{D}^\gamma K) \otimes L + K \otimes (\mathsf{D}^\gamma L), \quad K, L \in \boldsymbol{T}^1(\gamma(J))$;
- **(iii a)** $\mathsf{D}^\gamma\colon \mathfrak{F}^1(\gamma(J)) \to \mathfrak{F}^0(\gamma(J))$;
- **(iii b)** $\mathsf{D}^\gamma\colon \mathfrak{T}^{1;1}_0(\gamma(J)) \to \mathfrak{T}^{1;0}_0(\gamma(J))$;
- **(iii c)** $\mathsf{D}^\gamma\colon \mathfrak{T}^{0;1}_1(\gamma(J)) \to \mathfrak{T}^{0;0}_1(\gamma(J))$;
- **(iv)** $\mathsf{D}^\gamma(g) = \dot\gamma(g), \quad g \in \mathfrak{F}^1(\gamma(J))$;
- **(v)** $\mathsf{D}^\gamma(\omega(Z)) = C^1_1\bigl(\mathsf{D}^\gamma(\omega \otimes Z)\bigr)$, $\omega \in \mathfrak{T}^{0;1}_1(\gamma(J)), \quad Z \in \mathfrak{T}^{1;1}_0(\gamma(J))$.

Comments 10.1. This definition is similar to the Definition I.3.1 of a linear connection: the fist two properties of the latter one are removed and all covariant derivatives along a vector field X are replaced with derivatives along γ. Hence the points (2)–(6) of comments I.3.1 can be repeated *mutatis mutandis*. Hereof we can say that D^γ: is $\mathbb{K}$-linear, satisfies the Leibnitz rule (with respect to the tensor multiplication), preserves the types of the tensor fields, commutes with all contraction operators, and on scalar functions reduces to the vector field $\dot\gamma$ tangent to γ.

Remark 10.1. In connection with the theory of derivations (along paths) in vector bundles (see Subsection IV.2.3), the so-defined derivations along paths should be called section- or tensor-derivations along paths as the tensor fields are, in fact, sections of the corresponding tensor bundles over a manifold (see Subsection IV.2.4). At the moment, this more complicated name is not required and, respectively, it will not be employed in the present chapter. (For details, see the definition of a section-derivation along paths on page 221 and Subsection IV.13.1.)

Suppose D is a derivation along paths, $\gamma\colon J \to M$ is injective and C^1, and $\{E_i\}$ is a C^1 frame on $\gamma(J)$ which may be defined and outside $\gamma(J)$. By condition (iii b) of Definition 10.1, there exist unique functions ${\Gamma_\gamma}^i{}_j$ for which the expansion

$$\mathsf{D}^\gamma(E_i) =: {\Gamma_\gamma}^j{}_i E_j \tag{10.6}$$

holds. They are called *(local) components of* D^γ *or of* D *along* γ. We shall write $\Gamma^i{}_j(s;\gamma)$ for the value of ${\Gamma_\gamma}^i{}_j$ at $\gamma(s)$, ${\Gamma_\gamma}^i{}_j\colon \gamma(s) \mapsto \Gamma^i{}_j(s;\gamma)$ which is correct as γ is supposed to be injective. The components of D^γ uniquely describe it in $\{E_i\}$ as from Definition 10.1 follows that the components of $\mathsf{D}^\gamma(K)$ for $K \in \mathfrak{T}^{r;1}_q(\gamma(J))$ in the tensor frame induced by $\{E_i\}$ are

$$\begin{aligned}\big((\mathsf{D}^\gamma(K))^{i_1\dots i_r}_{j_1\dots j_q}\big)\big|_{\gamma(s)} = & \frac{\mathrm{d}\big(K^{i_1\dots i_r}_{j_1\dots j_q}(\gamma(s))\big)}{\mathrm{d}s} \\ & + \sum_{a=1}^{r} \Gamma^{i_a}{}_k(s;\gamma) K^{i_1\dots i_{a-1}k i_{a+1}\dots i_r}_{j_1\dots j_q}(\gamma(s)) \\ & - \sum_{b=1}^{q} \Gamma^k{}_{j_b}(s;\gamma) K^{i_1\dots i_r}_{j_1\dots j_{b-1}k j_{b+1}\dots j_q}(\gamma(s)).\end{aligned} \tag{10.7}$$

which can be verified by direct calculation.

Comparing (10.7) with (10.2), we conclude that to every derivation D along vector fields, whose components have the property (10.1), there corresponds a derivation along paths D such that $\mathsf{D}\colon \gamma \mapsto D_X\big|_{\boldsymbol{T}^1(\gamma(J))}$ for $X \in \mathfrak{X}(M)$ with $X|_{\gamma(J)} = \dot\gamma$. A particular example is provided by the covariant derivative along paths associating to γ the covariant derivative $\nabla_{\dot\gamma}$ along γ with ∇ being a linear connection.

If the frame $\{E_i\}$ is change to $\{E'_i = A^j_i E_j\}$ (along γ) by means of a non-degenerate matrix $A = [A^j_i]$, equation (10.6) implies the change

$${\Gamma_\gamma}^i{}_j \mapsto {\Gamma'_\gamma}^i{}_j = \big(A^{-1}\big)^i_l \big[{\Gamma_\gamma}^l{}_k A^k_j + \dot\gamma(A^l_j)\big]. \tag{10.8}$$

Introducing the *matrix* $\boldsymbol{\Gamma}_\gamma := \big[{\Gamma_\gamma}^i{}_j\big]^{\dim M}_{i,j=1}$ of the components of D^γ, we can rewrite this as (cf. (2.11))

$$\boldsymbol{\Gamma}_\gamma \mapsto \boldsymbol{\Gamma}'_\gamma = A^{-1}[\boldsymbol{\Gamma}_\gamma A + \dot\gamma(A)]. \tag{10.9}$$

Notice, the main difference between the pairs of transformations (2.10) and (2.11), on one hand, and (10.8) and (10.9), on another hand, is that the former ones are defined on an arbitrary neighborhood of M, while the latter ones have a sense only along γ, i.e., on $\gamma(J)$. If we put $\mathbf{\Gamma}(s;\gamma) = \mathbf{\Gamma}_\gamma(\gamma(s))$, $s \in J$, which is the matrix of D^γ (in $\{E_i\}$) at the point $\gamma(s)$, the change (10.9) takes the form (10.3), as one can expect. If one is interested in holonomic frames, like $\{E_i = \frac{\partial}{\partial x^i}\}$ with $\{x^i\}$ being coordinates in $U \subseteq M$ and $U \cap \gamma(J) \neq \varnothing$, the coordinates change $\{x^i\} \to \{x'^i\}$ will imply, e.g., the change (10.8) with $A^i_j = \frac{\partial x^i}{\partial x'^j}$ and all quantities should be restricted on $U \cap \gamma(J)$.

Example 10.2. It is easily seen that, by virtue of (10.4), the component's matrix of a covariant derivative $\nabla_{\dot\gamma}$ along γ is

$$\mathbf{\Gamma}_\gamma = \Gamma_k \dot\gamma^k \tag{10.10}$$

where $\Gamma_k := \left[\Gamma^i{}_{jk}\right]$ are the coefficients' matrices of a linear connection ∇.

A derivation along paths is called *differentiable of class C^k* or simply of class C^k, if its matrix is of class C^k with respect to all coordinate frames or the ones obtainable from them by means of C^{k+1} transformations (cf. similar convention concerning derivations along vector fields on page 146).

Now the problem for the existence and properties of a special kind of frames, called normal, in which the components of a derivation along paths vanish can be posed. It has two sides: local, along particular path, and global, along every path. We shall start with the former case.

Definition 10.2. Let D be a derivation along paths and $\gamma\colon J \to M$ be injective C^1 path. A *frame* $\{E'_i\}$ defined on $\gamma(J)$ or on its subset is called *normal for* D^γ (or *for* D *along* γ) if in it the components of D^γ (of D along γ) vanish. A *frame* defined on a neighborhood (of part) of $\gamma(J)$ is called *normal for* D^γ if its restriction to $\gamma(J)$ is normal for D^γ.

Frames $\{E'_i\}$ normal along particular path γ for a C^1 derivation D along paths always exist and can easily be found on the base of the transformation law (10.9) of the components of D along γ.

Theorem 10.1. *If D is a C^0 derivation along paths in a C^2 manifold M and $\gamma\colon J \to M$ is injective and C^1, then all frames (globally) normal for D along γ, i.e., for D^γ, are $\{E'_i = A^j_i E_j\}$, where $\{E_i\}$ is arbitrary frame along γ and $A = [A^i_j]$ is given by*

$$A(\gamma(s)) = Y(s, s_0; -\mathbf{\Gamma}_\gamma \circ \gamma)A_0, \tag{10.11}$$

in which Y is the unique solution of (II.3.17)*, $s \in J$, $s_0 \in J$ is fixed, $\mathbf{\Gamma}_\gamma$ is the matrix of D in $\{E_i\}$, and A_0 is constant non-degenerate matrix.*

Proof. If $\{E_i\}$ is an arbitrary frame on $\gamma(J)$, a frame $\{E'_i = A^j_i E_j\}$ is normal along γ, $\mathbf{\Gamma}'_\gamma = 0$, iff the normal frame equation

$$\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s} + \mathbf{\Gamma}(s;\gamma)A(\gamma(s)) = 0 \tag{10.12}$$

holds along γ, where $A := [A^i_j] \colon \gamma(J) \to \mathrm{GL}(\dim M, \mathbb{K})$ is of class C^1 and (10.3) was used. By Remark II.3.9 on page 96, the general solution of the last equation with respect to A is given by (10.11). □

Since the provided by Theorem 10.1 frames normal for a derivation D^γ along a concrete path $\gamma \colon J \to M$ are defined only on $\gamma(J)$, the problem for their holonomicity is ill-posed. The correct question is: can these frames be (locally) extended in a holonomic or anholonomic way on a neighborhood with non-empty intersection with $\gamma(J)$? As the existence of anholonomic extensions is clear and beyond doubt, we shall show the existence of holonomic extensions on which base, if required, anholonomic ones can be constructed. Looking on M as on a real manifold (of dimension $\dim_{\mathbb{R}} M = 2 \dim M$ if M is complex) and admitting γ to be regular, the needed result is an evident corollary of the following lemma.

Lemma 10.1. *Let $\gamma \colon J \to M$ be injective C^1 regular path in a C^1 real manifold M and $\{E_i\}$ be a frame defined on $\gamma(J)$. For every $s_0 \in J$, there exists a chart (U, y) with $U \ni \gamma(s_0)$ such that the coordinate frame $\{\frac{\partial}{\partial y^i}\}$ reduces along γ to $\{E_i\}$, $\frac{\partial}{\partial y^i}\big|_{U \cap \gamma(J)} = E_i\big|_{U \cap \gamma(J)}$, i.e., in a neighborhood of every point in $\gamma(J)$ the initial frame $\{E_i\}$ can be extended in a holonomic way.*

Remark 10.2. Since now $\gamma(J)$ is one-dimensional (real) submanifold of M, this lemma is a corollary of Lemma II.5.2 on page 116. Because of the importance of this result, we present here its independent proof following the one of [76, Lemma 4.1].

Proof. Since γ is C^1, regular, and without self-intersections, for every $s_0 \in J$ the special chart (U_1, x), with $U_1 \ni \gamma(s_0)$ and described in Subsection II.3.2 on page 94, can be constructed. Recall, (see (II.3.12) and (II.3.13)), for $p \in U_1$, we have $x(p) = (s, \boldsymbol{t})$ for some $s \in J_1 := \{\sigma | \sigma \in J,\ \gamma(\sigma) \in U_1\} \subseteq J$ and $\boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M - 1}$ and $x(\gamma(s)) = (s, \boldsymbol{t}_0)$, $s \in J_1$, for fixed $\boldsymbol{t}_0 \in \mathbb{R}^{\dim_{\mathbb{R}} M - 1}$. Let the non-degenerate matrix $B = [B^i_j(s)]$, $s \in J_1$, be defined via the expansion

$$E_i|_{\gamma(s)} = B^j_i(s) \frac{\partial}{\partial x^j}\Big|_{\gamma(s)}. \tag{10.13}$$

Define the functions $y^i \colon U_1 \to \mathbb{R}$ by

$$y^i(x^{-1}(s, \boldsymbol{t})) := b^i + \int_{s_0}^{s} \big(B^{-1}(\sigma)\big)^i_1 \,\mathrm{d}\sigma + \sum_{k=2}^{\dim_{\mathbb{R}} M} \big(B^{-1}(\sigma)\big)^i_k (t^k - t^k_0) \\ + \sum_{k,l=2}^{\dim_{\mathbb{R}} M} b^i_{kl}(x^{-1}(s, \boldsymbol{t}))(t^k - t^k_0)(t^l - t^l_0) \tag{10.14}$$

where $b^i \in \mathbb{R}$ and $b^i_{kl} \colon U_1 \to \mathbb{R}$ are C^1 and together with their derivatives are bounded on $\gamma(J_1)$. Since

$$\frac{\partial y^i}{\partial x^j}\Big|_{\gamma(s)} = \frac{\partial y^i}{\partial x^j}\Big|_{x^{-1}(s, \boldsymbol{t}_0)} = \big(B^{-1}(s)\big)^i_j$$

and $B(s)$ is non-degenerate, the transformation $\{x^i\} \to \{y^i\}$ is well-defined C^1 diffeomorphism on $U := x^{-1}(J_1, V) \subseteq U_1$ for some neighborhood V of $\boldsymbol{t_0}$ in $\mathbb{R}^{\dim_{\mathbb{R}} M-1}$. Hence (U, y) is a chart of M with $U \ni \gamma(s_0)$ and the coordinate frame generated by the associated with it coordinates $\{y^i\}$ has the following basic vectors along γ, i.e., on $\gamma(J_1) \subseteq \gamma(J)$:

$$\frac{\partial}{\partial y^i}\Big|_{\gamma(s)} = \frac{\partial x^j}{\partial y^i}\Big|_{\gamma(s)} \frac{\partial}{\partial x^j}\Big|_{\gamma(s)} = \Big(\Big[\frac{\partial y^k}{\partial x^l}\Big|_{\gamma(s)}\Big]^{-1}\Big)^j_i \frac{\partial}{\partial x^j}\Big|_{\gamma(s)} = B^j_i(s)\frac{\partial}{\partial x^j}\Big|_{\gamma(s)}.$$

Therefore $\frac{\partial}{\partial y^i}\big|_{\gamma(s)} = E_i|_{\gamma(s)}$ and, consequently, $\big\{\frac{\partial}{\partial y^i}\big\}$ is a holonomic extension of $\{E_i\}$ in the neighborhood U of $\gamma(s_0)$. □

The next proposition reveals the complete (local) consistency between the normal frames provided by Theorem 10.1 for the covariant derivatives along paths (generated by a linear connection ∇) and the ones given via Theorem II.3.1 on page 97.

Proposition 10.1. *Let $\gamma\colon J \to M$ be a C^1 regular injective path in a C^1 manifold M endowed with a C^0 linear connection ∇. The frames on $\gamma(J)$ normal along γ for ∇ and the ones normal for the covariant derivative $\nabla_{\dot\gamma}$ along γ are identical.*

Proof. If a frame $\{E'_i\}$ is normal along γ for ∇, then, due to (10.10), the components' matrix of $\nabla_{\dot\gamma}$ in $\{E'_i|_{\gamma(J)}\}$ is $\boldsymbol{\Gamma}'_\gamma = \Gamma'_k\dot\gamma^k = \sum_k 0 \times \dot\gamma^k = 0$ as, by definition, in a frame normal for ∇ the coefficients' matrices Γ'_k of ∇ vanish.

To prove the converse assertion, we shall use the special chart (U_1, x), in a neighborhood of some point $\gamma(s_0) \in \gamma(J)$, constructed at the beginning of Subsection II.3.2. Since in the frame $\big\{E_i = \frac{\partial}{\partial x^i}\big|_{U_1}\big\}$ we have $\dot\gamma^k = \delta^k_1$, in it (10.10) gives the matrix $\boldsymbol{\Gamma}_\gamma$ of $\nabla_{\dot\gamma}$ as $\boldsymbol{\Gamma}_\gamma = -\Gamma_1$. Hereof, applying Theorem 10.1, we see that a frame $\{E'_i\}$ on $\gamma(J)$ normal for $\nabla_{\dot\gamma}$ has on $\gamma(J) \cap U_1$ basic vectors $E'_i|_{\gamma(s)} = A^j_i(\gamma(s))\frac{\partial}{\partial x^j}\big|_{\gamma(s)}$ for $s \in J_1 := \{\sigma | \sigma \in J,\ \gamma(\sigma) \in U_1\} \ni s_0$ with

$$A(\gamma(s)) = Y(s, s_0; -\Gamma_1 \circ \gamma)A_0 \tag{10.15}$$

which, up to notation, coincides with the matrix (II.3.22) on page 97 transforming $\big\{\frac{\partial}{\partial x^i}\big|_{\gamma(J)}\big\}$ into a frame on $\gamma(J)$ normal for ∇. So, by Theorem II.3.1, locally, i.e., on $\gamma(J_1)$, every frame $\{E'_i\}$ normal for $\nabla_{\dot\gamma}$ is a restriction along γ (more precisely, to $\gamma(J_1)$) of a frame normal for ∇ along γ. □

In short, admitting some abuse of the language, we can say that the frames normal along a path for a linear connection are normal for the generated by it covariant derivative along paths and *vice versa*.

One should have in mind a significant difference between the normal frames provided by Theorem 10.1 for $\nabla_{\dot\gamma}$ and the ones provided by Theorem II.3.1 along γ for ∇: the former normal frames are *global*, i.e., defined along the whole path γ, on the set $\gamma(J)$, while the latter ones are *local*, i.e., defined only on a neighborhood in $\gamma(J)$ of each point in $\gamma(J)$.

Now we turn our attention to the 'global' case.

Definition 10.3. A *frame* $\{E_i\}$ defined on a subset $U \subseteq M$ or on a larger set (if $U \neq M$) is called *normal on U for a derivation* D *along paths* if for every C^1 injective path $\gamma\colon J \to U$ lying entirely in U the restricted frame $\{E_i|_{\gamma(J)}\}$ is normal along γ for the derivation D^γ, $\mathsf{D}\colon \gamma \to \mathsf{D}^\gamma$. In particular, a *frame* normal on M for D is called simply *normal for* D.

Proposition 10.2. *A derivation along paths admits frames normal on $U \subseteq M$ if and only if on U it is a covariant derivative along paths induced by some linear connection for which frames normal on U exist.*

Proof. Suppose, a derivation D along paths admits a frame $\{E_i'\}$ normal on U. Let $\{E_i\}$ be a frame on U and $A = [A_i^j]\colon U \to \mathrm{GL}(\dim M, \mathbb{K})$ be defined via the expansion $E_i'|_{\gamma(s)} = A_i^j(\gamma(s))E_j|_{\gamma(s)}$. By Theorem 10.1, the matrix $\mathbf{\Gamma}(s;\gamma)$ of D along *arbitrary* path $\gamma\colon J \to U$ in U and the matrix-valued function A are connected by (10.11) or, equivalently, satisfy (10.12). Hence, from (10.12), we get

$$\mathbf{\Gamma}(s;\gamma) = -\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s}A^{-1}(\gamma(s)) = -E_k(A)|_{\gamma(s)}\dot\gamma^k(s)A^{-1}(\gamma(s))$$

where $\dot\gamma^k$ are the components of $\dot\gamma$ in $\{E_k\}$. Therefore the matrix $\mathbf{\Gamma}_\gamma$ of D^γ in $\{E_k|_{\gamma(J)}\}$ is (10.10) with

$$\Gamma_k\colon p \mapsto \Gamma_k(p) = -(E_k(A))|_p A^{-1}(p). \tag{10.16}$$

Since a simple verification shows that these matrices Γ_k transform according to (I.5.3), they are matrices of the coefficients of a linear connection ∇ which, due to $\mathbf{\Gamma}(s;\gamma) = \Gamma_k(\gamma(s))\dot\gamma^k(s)$, is such that $\mathsf{D}^\gamma = \nabla_{\dot\gamma}$, i.e., the covariant derivative along paths generated by ∇ coincides with the initial derivation D along paths. As $\{E_i'\}$ is normal for D, in it is fulfilled $0 = \mathbf{\Gamma}'(s;\gamma) = \Gamma_k'(\gamma(s))\dot\gamma'^k(s)$ for *arbitrary* $\gamma\colon J \to U$. Hereof $\Gamma_k'(p)\dot\gamma'^k(s_0) = 0$ for every path $\gamma\colon J \to U$ with $\gamma(s_0) = p \in U$ for some $s_0 \in J$ which implies $\Gamma_k'|_U = 0$ and, consequently, $\{E_i'\}$ is normal on U for ∇.

Conversely, suppose $\mathsf{D}^\gamma = \nabla_{\dot\gamma}$ for a linear connection ∇ for which a frame $\{E_i'\}$ normal on U exist. Then the matrix $\mathbf{\Gamma}_\gamma$ of D^γ is given by (10.10) for all frames $\{E_i\}$ on U and all paths $\gamma\colon J \to U$ in U. In particular, for $E_i = E_i'|_{\gamma(J)}$, we have $\mathbf{\Gamma}_\gamma'\colon \gamma(s) \mapsto \mathbf{\Gamma}'(s;\gamma) = \mathbf{\Gamma}_k'(\gamma(s))\dot\gamma^k(s) = \sum_k 0 \times \dot\gamma^k(s) = 0$ as $\mathbf{\Gamma}_k' = 0$ in a frame $\{E_i'\}$ normal for ∇. Therefore $\mathbf{\Gamma}_\gamma = 0$ in $\{E_i'\}$ for every path γ lying entirely in U and, by Definition 10.3, the frame $\{E_i'\}$ is normal for $\mathsf{D}^\gamma = \nabla_{\dot\gamma}$. ☐

The meaning of Proposition 10.2 is that the condition on a derivation D along paths to admit frames normal on $U \subseteq M$ is so strong that it reduces such a derivation to a covariant derivative along paths admitting frames normal on U. Since, by Proposition 10.1, such a covariant derivative along paths is always generated by a linear connection ∇ admitting frames normal on U, all problems concerning frames normal for D are reduced to similar problems for ∇, which we have investigated earlier at length.

11. On frames simultaneously normal for two derivations

The problems concerning frames simultaneously normal for two derivations (along vector fields, fixed vector field, paths, fixed path) are new and not investigated. They arise naturally in some physical theories of gravity in which as a space-time model is taken a differentiable manifold endowed with two, generally or initially independent, linear connections. Such are the metric-affine theories [36, 92], the ones with metric and background affine connection [93, 94] in which there are two connections, one of them being a Riemannian connection induced by some Riemannian metric, as well as the theories with covariant and contravariant affine connection [35, 95, 96] in which two different linear connections in the tangent and cotangent bundles are used.[1] The importance of the normal frames in these theories comes from the fact that the normal frames are the mathematical tool for description of the physical concept 'inertial frame' [97]. So, the equivalence principle selects only linear connections possessing normal frames for the description of the pure gravity. Hereof, if two linear connections admit common normal frame(s), they can serve together as a mathematical base for a gravitational theory, otherwise only one of them should be selected for the same purpose.

Below in this section, we shall present some quite simple results whose main moral is: frames simultaneously normal for two arbitrary and different derivations (along vector fields, fixed vector field, paths, fixed path), generally, do not exist; however, there are some exceptions. Physically, this can be interpreted as: in a gravity theory with two linear connections only one of them can describe the gravity directly, the other one may be connected with it but, generally, it should be primary related to an object different from the gravity. However, there exist exceptions from this conclusion which will be pointed.

The order of the rigorous results comes now.

Proposition 11.1. *Let $D^{(1)}$ and $D^{(2)}$ be derivations along vector fields admitting frames normal on $U \subseteq M$. There exist frames normal on U for $D^{(1)}$ and $D^{(2)}$ simultaneously iff $D^{(1)}|_U = D^{(2)}|_U$.*

Proof. Let $\{E_i\}$ be a frame on (a neighborhood of) U and $\Gamma_k^{(a)}$ be the coefficients' matrices of $D^{(a)}$, $a = 1, 2$. (Recall, by Proposition 3.1, the derivations are linear on U.) A frame $\{E'_i = A_i^j E_j\}$ is normal for $D^{(1)}$ and $D^{(2)}$ iff (3.5) holds for $\Gamma_k^{(a)}$ instead of Γ_k: $(\Gamma_k^{(a)} A + E_k(A))|_U = 0$, $a = 1, 2$, $A = [A_i^j]$ which is possible iff $\Gamma_k^{(1)}|_U = \Gamma_k^{(2)}|_U$ as A is non-degenerate which, in its turn, means $D^{(1)}|_U = D^{(2)}|_U$. □

[1] In the last case, the two connections are equivalent to a single operator ∇ satisfying all conditions in Definition I.3.1 but the condition (vii); see the definition of a mixed linear connection on page 199.

Proposition 11.2. *Let $D^{(1)}_{X_{(1)}}$ and $D^{(2)}_{X_{(2)}}$ be derivations along fixed vector fields $X_{(1)}$ and $X_{(2)}$, respectively, $U \subseteq M$, $\{E_i\}$ be a frame on (a neighborhood of) U, and $D^{(1)}_{X_{(1)}}$ and $D^{(2)}_{X_{(2)}}$ admit frames normal on U. Then a frame $\{E'_i = A^j_i E_j\}$ normal on U for $D^{(1)}_{X_{(1)}}$ is also normal for $D^{(2)}_{X_{(2)}}$ if and only if in $\{E_i\}$ the components' matrix of $D^{(2)}_{X_{(2)}}$ on U is*

$$\Gamma^{(2)}_{X_{(2)}}|_U = -\big[(X_{(2)}(A))A^{-1}\big]\big|_U, \qquad A = [A^j_i].$$

Proof. The frame $\{E'_i = A^j_i E_j\}$ is normal for $D^{(2)}_{X_{(2)}}$ if its matrix $\Gamma^{(2)}_{X_{(2)}}$ and A are connected via (9.1) with $X_{(2)}$ and $\Gamma^{(2)}_{X_{(2)}}$ for X and Γ_X, respectively, from where the assertion follows. □

Proposition 11.3. *Let $U \subseteq M$, $X \in \mathfrak{X}(M)$ be fixed, $D^{(1)}$ and $D^{(2)}_X$ be derivations along vector fields and X, respectively, for which frames normal on U exist. Frames normal on U for $D^{(1)}$ and $D^{(2)}_X$ simultaneously exist if and only if $D^{(2)}_X|_U = D^{(1)}_X|_U$, where $D^{(1)}\colon X \mapsto D^{(1)}_X$.*

Proof. Let $\{E_i\}$ be a frame on (a neighborhood of) U and $\{E'_i = A^j_i E_j\}$ be a frame normal on U for $D^{(1)}$. Hereof $A = [A^j_i]$ satisfies (3.5), with the coefficients' matrices $\Gamma^{(1)}_l$ of $D^{(1)}$ for Γ_l, which implies $(\Gamma^{(1)}_l X^l A + X(A))|_U = 0$, $X = X^k E_k$. The frame $\{E'_i\}$ is also normal for $D^{(2)}_X$ iff (9.1) holds, with the matrix $\mathbf{\Gamma}^{(2)}_X$ of $D^{(2)}_X$ for $\mathbf{\Gamma}_X$, i.e., $(\mathbf{\Gamma}^{(2)}_X A + X(A))|_U = 0$, which is equivalent to $\mathbf{\Gamma}^{(2)}_X|_U = (\Gamma^{(1)}_l X^l)|_U$, due to the previous equality, which, in its turn, expresses $D^{(2)}_X|_U = D^{(1)}_X|_U$ in $\{E_i\}$. □

Proposition 11.4. *Let the derivations ${}^{(1)}\mathsf{D}^\gamma$ and ${}^{(2)}\mathsf{D}^\gamma$ along one and the same C^1 injective path $\gamma\colon J \to M$ admit normal frames (along γ). There is a frame simultaneously normal for ${}^{(1)}\mathsf{D}^\gamma$ and ${}^{(2)}\mathsf{D}^\gamma$ if and only if ${}^{(1)}\mathsf{D}^\gamma = {}^{(2)}\mathsf{D}^\gamma$.*

Proof. If $\{E_i\}$ is a frame on $\gamma(J)$ and the frame $\{E'_i = A^j_i E_j\}$ is normal for ${}^{(1)}\mathsf{D}^\gamma$, i.e., (10.12) holds for the matrices $\mathbf{\Gamma}^{(1)}_\gamma$ of ${}^{(1)}\mathsf{D}^\gamma$ in $\{E_i\}$ for $\mathbf{\Gamma}_\gamma$, $\mathbf{\Gamma}_\gamma = \mathbf{\Gamma}^{(1)}_\gamma$, then $\{E'_i\}$ is also normal for ${}^{(2)}\mathsf{D}^\gamma$ iff $\mathbf{\Gamma}^{(2)}(s;\gamma)$ satisfies (10.12) with $\mathbf{\Gamma}_\gamma = \mathbf{\Gamma}^{(2)}_\gamma$ which, due to the previous equality, is equivalent to $\mathbf{\Gamma}^{(1)}_\gamma = \mathbf{\Gamma}^{(2)}_\gamma$, i.e., to ${}^{(1)}\mathsf{D}^\gamma = {}^{(2)}\mathsf{D}^\gamma$. □

Proposition 11.5. *Let ${}^{(1)}\mathsf{D}$ and ${}^{(2)}\mathsf{D}$ be derivations along paths which admit frames normal on $U \subseteq M$. Then frames simultaneously normal for ${}^{(1)}\mathsf{D}$ and ${}^{(2)}\mathsf{D}$ exist iff ${}^{(1)}\mathsf{D}|_U = {}^{(2)}\mathsf{D}|_U$.*

Proof. The result follows from Proposition 11.4 for arbitrary path $\gamma\colon J \to M$. The assertion is also a consequence of Propositions 10.2 and 11.1 as any linear connection is a (linear) derivation along vector fields. □

Proposition 11.6. *Frames simultaneously normal along $\gamma\colon J \to M$ for derivations ${}^{(1)}\mathsf{D}$ and ${}^{(2)}\mathsf{D}^\gamma$ along paths and along γ, respectively, exist iff ${}^{(2)}\mathsf{D}^\gamma = {}^{(1)}\mathsf{D}^\gamma$, with ${}^{(1)}\mathsf{D}\colon \gamma \mapsto {}^{(1)}\mathsf{D}^\gamma$.*

Proof. See Proposition 11.4. □

Now we want to say a few words on the physical theories with covariant and contravariant affine (linear) connections. As it was mentioned in Footnote 1 on page 197, their mathematical base are the so-called *linear connections of mixed type* [41, § 37], or, simply, *mixed linear connections* which are a generalization of the linear connections. A mapping ∇ is called *mixed linear connection (of type* $(1,1)$*)* if it satisfies all conditions of Definition I.3.1 on page 21 except conditions (vii) in it. If $\{E_i\}$ and $\{E^i\}$ are frames in $\mathfrak{T}^1_0(M)$ and $\mathfrak{T}^0_1(M)$ respectively,[2] the *contravariant* $\Gamma^i{}_{jk}$ and the *covariant* $\Gamma_j{}^i{}_k$ *coefficients* of ∇ in these frames are defined via the expansions

$$\nabla_{E_k} E_j =: \Gamma^i{}_{jk} E_i, \quad \nabla_{E_k} E^j =: \Gamma_i{}^j{}_k E^i. \tag{11.1}$$

If $K = K^{i_1\dots i_r}_{j_1\dots j_s} E_{i_1} \otimes \dots \otimes E_{i_r} \otimes E^{j_1} \otimes \dots \otimes E^{j_s}$ and $X = X^k E_k \in \mathfrak{X}(M)$, then

$$\nabla_X K = X^k \nabla_{E_k} K \tag{11.2}$$

and the components of $\nabla_{E_k} K$ are (cf. (I.3.3))

$$\begin{aligned}\left(\nabla_{E_k} K\right)^{i_1\dots i_r}_{j_1\dots j_s} := E_k\left(K^{i_1\dots i_r}_{j_1\dots j_s}\right) &+ \sum_{a=1}^{r} \Gamma^{i_a}{}_{lk} K^{i_1\dots i_{a-1} l i_{a+1}\dots i_r}_{j_1\dots j_s} \\ &+ \sum_{b=1}^{s} \Gamma_{j_b}{}^l{}_k\, K^{i_1\dots i_r}_{j_1\dots j_{b-1} l j_{b+1}\dots j_s}.\end{aligned} \tag{11.3}$$

If we change the frame and coframe by means of non-degenerate matrix-valued functions $A = [A^j_i]$ and $B = [B^j_i]$,

$$E_i \mapsto E'_i = A^j_i E_j, \quad E^i \mapsto E'^{\,i} = B^i_j E^j, \tag{11.4}$$

the definitions (11.1) imply the following transformations of the coefficients of ∇ (cf. (I.3.5)):

$$\Gamma^i{}_{jk} \mapsto \Gamma'^{\,i}{}_{jk} = \left(A^{-1}\right)^i_l A^m_j A^n_k \Gamma^l{}_{mn} + \left(A^{-1}\right)^i_l E'_k\left(A^l_j\right) \tag{11.5}$$

$$\Gamma_i{}^j{}_k \mapsto \Gamma'_i{}^j{}_k = \left(B^{-1}\right)^l_i B^j_m A^n_k \Gamma_l{}^m{}_n + \left(B^{-1}\right)^l_i E'_k\left(B^j_l\right) \tag{11.6}$$

which, if we introduce the *coefficients' matrices* $\Gamma_k := \left[\Gamma^i{}_{jk}\right]^{\dim M}_{i,j=1}$ and $\overline{\Gamma}_k := \left[\Gamma_j{}^i{}_k\right]^{\dim M}_{i,j=1}$ of ∇, in a matrix form read (see the convention on the matrix indices

[2]The coframe $\{E^i\}$ is not necessary to be dual to $\{E_i\}$.

on page xii)

$$\Gamma_k \mapsto \Gamma'_k = A^l_k A^{-1}\big[\Gamma_l A + E_l(A)\big] \tag{11.5$'$}$$

$$\overline{\Gamma}_k \mapsto \overline{\Gamma}'_k = A^l_k\big[B\overline{\Gamma}_l + E_l(B)\big]B^{-1} = A^l_k B\big[\overline{\Gamma}_l B^{-1} - E_l(B^{-1})\big]. \tag{11.6$'$}$$

Hence $\Gamma^i{}_{jk}$ are coefficients of (ordinary) linear connection which we shall denote by ${}^{(1)}\nabla$.

If $B = A^{-1}$, which is equivalent to the requirement the action of the basic vectors of the coframe $\{E^i\}$ on the ones of the frame $\{E_i\}$ to be invariant under the change (11.4), i.e., $E^i(E_j) = E'^i(E'_j)$,[3] then (11.6′) reduces to

$$\overline{\Gamma}_k \mapsto \overline{\Gamma}'_k = A^l_k A^{-1}\big[\overline{\Gamma}_l A - E_l(A)\big] \tag{11.7}$$

which implies that

$${}^{(2)}\Gamma^i{}_{jk} := -\Gamma_j{}^i{}_k, \qquad {}^{(2)}\Gamma_k := -\overline{\Gamma}_k \tag{11.8}$$

are, respectively, coefficients and coefficients' matrices of a (usual) linear connection which we denote by ${}^{(2)}\nabla$.[4]

It is almost evident, ∇ is ordinary linear connection if and only if one, and hence any, of the following equivalent equations holds:

$${}^{(1)}\nabla = \nabla, \quad {}^{(1)}\nabla = {}^{(2)}\nabla, \quad {}^{(2)}\nabla = \nabla, \quad \Gamma_j{}^i{}_k + \Gamma^i{}_{jk} = 0, \tag{11.9}$$

the last of which is valid in, e.g., a pair of dual frame and coframe. Consequently, any one of these equalities is equivalent to the conditions (vii) in Definition I.3.1.

Definition 11.1. A *pair* $\big(\{E_i\},\{E^i\}\big)$ *of a frame and coframe* is called *normal on* $U \subseteq M$ *for a mixed linear connection* ∇ if in it all of the coefficients, contravariant and covariant ones, of ∇ vanish on U.

Proposition 11.7. *If only changes preserving the action of the coframe's basic vectors on the frame's ones are considered,*[5] *a mixed linear connection* ∇ *admits a pair of such frame and coframe normal on* $U \subseteq M$*, if and only if on* U ∇ *reduces to ordinary linear connection for which frames normal on* U *exist.*

Proof. If $\big(\{E_i\},\{E^i\}\big)$, with $E^i(E_j)$ constant under transformations of the frame and coframe, is normal on U for ∇, then $\{E_i\}$ is normal on U for ${}^{(1)}\nabla$ and ${}^{(2)}\nabla$ simultaneously and, by Proposition 11.1, ${}^{(1)}\nabla|_U = {}^{(1)}\nabla|_U$ which, as we said above, implies $\nabla|_U = {}^{(1)}\nabla|_U$ and $\{E_i\}$ is normal on U for $\nabla|_U$. Conversely, if ∇ is ordinary linear connection and $\{E_i\}$ is a frame normal on U for ∇, then $\nabla = {}^{(1)}\nabla = {}^{(2)}\nabla$ and $\big(\{E_i\},\{E^i\}\big)$, with invariant $E^i(E_j)$, is normal on U for ∇ considered as a mixed linear connection. □

[3] In particular such are all pairs of a frame and the coframe dual to it.

[4] In some sense [95, 96] ${}^{(1)}\nabla$ (resp. ${}^{(2)}\nabla$) is the 'contravariant' (resp. covariant) component of ∇.

[5] In particular, such is the set of all dual frames and coframes.

Consider now the general case, when the frame $\{E_i\}$ and coframe $\{E^i\}$ are completely arbitrary, $B \neq A^{-1}$. By virtue of (11.5′) and (11.6′), the pair $\big(\{E'_i = A^j_i E_j\}, \{E'^i = B^i_j E^j\}\big)$ is normal on $U \subseteq M$ for a mixed linear connection ∇ iff the *system of normal frame equations*

$$(\Gamma_k A + E_k(A))|_U = 0 \tag{11.10a}$$

$$(-\overline{\Gamma}_k B^{-1} + E_k(B^{-1}))|_U = 0 \tag{11.10b}$$

holds. These equations are equivalent to the assertions that, respectively, the frame $\{E'_i = A^j_i E_j\}$ is normal on U for the linear connection ${}^{(1)}\nabla$ and the frame $\{E''_i = (B^{-1})^j_i E_j\}$ is normal on U for the linear connection ${}^{(2)}\nabla$.[6] These very simple results imply the following theorem.

Theorem 11.1. *A mixed linear connection ∇ admits frames normal on $U \subseteq M$ if and only if the associated to it linear connections ${}^{(1)}\nabla$ and ${}^{(2)}\nabla$ admit frames normal on U. Moreover, if frames normal on U for ${}^{(1)}\nabla$ and ${}^{(2)}\nabla$ exist and $\{E_i\}$ and $\{E^i\}$ are respectively frame and coframe on (a neighborhood of) U, then all pairs of frames and coframes normal on U for ∇ are $\big(\{E'_i = A^j_i E_j\}, \{E'^i = B^i_j E^j\}\big)$ where $\{E'_i\}$ is a frame normal on U for ${}^{(1)}\nabla$ and $\{E''_i = (B^{-1})^j_i E_j\}$ is a frame normal on U for ${}^{(2)}\nabla$.*

Remark 11.1. For ${}^{(1)}\nabla \neq {}^{(2)}\nabla$, the second part of the theorem does not contradict to Propositions 11.7 and 11.1 because nowhere is imposed the additional condition ${}^{(1)}\nabla$ and ${}^{(2)}\nabla$ to have a common frame normal on U, which could happen iff (11.10a) and (11.10b) have a common solution with respect to A and B^{-1} respectively.

From the above discussion the following conclusions can be made.

If pairs of frames $\{E_i\}$ and coframes $\{E^i\}$ for which the functions $f^j_i := E^j(E_i)$ are invariant under the change (11.4) are considered, a mixed linear connection does not admit normal frames unless it is an ordinary linear connection for which normal frames exist; in particular, this is true if only pairs of dual frames and coframes, for which $f^j_i = \delta^j_i$, are taken into account.

If the considered frames and coframes are completely arbitrary, with no additional conditions on them imposed, then all problems concerning normal frames for a mixed linear connection ∇ are reduced to the same problems but for the linear connections ${}^{(1)}\nabla$ and ${}^{(2)}\nabla$.

Consequently, in the both cases, any problem concerning normal frames for mixed linear connections can equivalently be (re)formulated in terms of similar

---

[6] From here immediately follows an independent proof of Proposition 11.7: If $E'^i(E'_j) = E^i(E_j)$, then $B^{-1} = A$ is a solution of (11.10b), so $E''_i = E'_i$ and hence $\{E'_i\}$ is a common normal frame for ${}^{(1)}\nabla$ and ${}^{(2)}\nabla$ which, by Proposition 11.1, implies ${}^{(1)}\nabla = {}^{(2)}\nabla$. The last result yields $\nabla = {}^{(1)}\nabla = {}^{(2)}\nabla$.

problem for ordinary linear connections. Since the last range of problems is explored at length in this book, we shall stop here with the investigation of the mixed linear connections from the view-point of normal frames.

Exercise 11.1. Reformulate *mutatis mutandis* the definitions and results of this section for the case of (normal) local coordinates or (normal) charts.

12. Normal frames for linear connections (review)

Since the linear connections find a far more numerous applications than the other derivations (along vector fields), in the present section are summarized the main ideas and are discussed the results concerning normal frames for linear connections which are spread over the preceding material in this chapter.

First of all, recall that a derivation along vector fields, which is linear on the whole manifold M, is a linear connection on M. More generally, a derivation D along vector fields which is linear on some subset(s) of M coincides on it (them) with a linear connection whose coefficients on this (these) set(s) coincide with the ones of D. This fact allows the results on normal frames for derivations along vector fields to be applied directly for 'local' linear connections, i.e., ones that are defined on a proper subset of the whole manifold M.

In Chapter II, a complete description of the frames and coordinates normal on submanifolds of M and along paths in M for linear connections defined on the whole manifold M was presented. In the present chapter (Sections 3–7) analogous results for derivations D along vector fields were proved. Frames normal on $U \subseteq M$ for D exist iff $D|_U = \nabla|_U$ where ∇ is linear connection on U or on a larger set (if $U \neq M$) and ∇ admits frames normal on U; moreover, if such ∇ exists, all frames normal on U for D are normal on U for ∇ and *vice versa*. Reversing the situation, we have

Assertion 12.1. If ∇ is a linear connection defined on U, then all problems concerning the frames normal on $U' \subseteq U$ for ∇, if any, are equivalent to the same problems for a derivation D along vector fields such that $D|_U = \nabla|_U$ and D is not linear everywhere on $M \backslash U$, i.e., outside U (if $U \neq M$).

From here an essential conclusion follows.

Conclusion 12.1. The linear connections defined on M (resp. on subsets of M) are globally (resp. locally) the only derivations along vector fields for which frames normal on M (resp. on the corresponding subsets of M) could exist.

As we proved, normal frames always exist at a single point or along given (locally injective) path; on submanifolds of dimension at least two, normal frames exist as an exception iff some additional conditions, derived here, are satisfied.

In Section 8, we studied problems devoted to frames normal along (locally injective) mappings like $f\colon N \to M$, N and M being manifolds, for derivations

D along vector fields. The main results are expressed by Theorem 8.4, the first assertion of which can be restated as (see also Theorems 7.1 and II.5.2): A derivation D along vector fields admits frames normal along locally injective mapping $f\colon N \to M$ if and only if for every $(\dim N)$-dimensional submanifold N' of N such that $f|_{N'}$ is injective, the restriction of D to $f(N')$ (i.e., to $\boldsymbol{T}^1(f(N'))$) coincides with some linear connection (possibly restricted to $f(N')$) for which frames normal on $f(N')$ exist. Therefore the exploration of frames normal along f for D can equivalently be reformulated in terms of linear connections. Reversing, ones again, the situation, we get

Assertion 12.2. If ∇ is a linear connection, all problems regarding frames normal along f for ∇ are equivalent to the same problems for a derivation D along vector fields such that $D|_{f(N')} = \nabla|_{f(N')}$ and D is not linear on $M\backslash f(N')$.

Consequently a conclusion similar to the above one can be made.

Conclusion 12.2. Locally the linear connections are the only derivations along vector fields for which frames normal along locally injective mappings exist. If $\dim_{\mathbb{R}} N = 0, 1$, then such frames always exist, otherwise some additional conditions must be satisfied for their existence.

The only case, when the linear connections are not 'selected' from the viewpoint of normal frames or coordinates, is the one for derivations along fixed vector field (Section 9). This is not surprising, on the opposite such a result is quite natural as the linear connections are special type of derivations along (arbitrary) vector fields, not along fixed vector field. Of course, all results of Section 9 are completely valid for every covariant derivative ∇_X along fixed vector field X.[1] We hope that they can be useful in theoretical physics where derivatives with respect to a fixed vector X find applications. The role of X can be played by a number of vectorial physical quantities, for instance it could be the velocity field of a fluid, some force field, the (four-)potential of an electromagnetic field, etc.

Since the derivations along a fixed path $\gamma\colon J \to M$ can be considered as a suitable restriction of special kind of derivations along a fixed vector field X with $X_{\gamma(J)} = \dot\gamma$, the just said for such derivations is valid *mutatis mutandis* for derivations along a fixed path. The derivations along a fixed path always admit frames normal along that path (Theorem 10.1), in particular this is true for the covariant derivative $\nabla_{\dot\gamma}$ along γ generated by some linear connection ∇. An interesting result is the coincidence of the frames normal along γ for $\nabla_{\dot\gamma}$ and for ∇ (Proposition 10.1). As derivatives like $\nabla_{\dot\gamma}$ often arise in physics, where $\dot\gamma$ is interpreted as a velocity of a (point) particle moving along γ, it is possible that the results of Section 10 could find suitable physical applications.

If derivations along arbitrary paths are considered, then, as one can expected, such derivations admit normal frames iff they are covariant derivatives generated

[1] Following the terminology of Section 2 (see, in particular, Definition 2.3), one can call ∇_X a linear connection along X: a linear connection ∇ is a derivation along vector fields which is linear at every point and ∇_X is the value of ∇ at X (see Definition I.3.1).

by linear connections for which normal frames exist (Proposition 10.2). This reveals, once again, the privileged role of the linear connections between the other derivations along vector fields.

If one wants frames simultaneously normal for two derivations along vector fields (resp. along paths) to exist, then these derivations must (locally) coincide (according to Proposition 11.1 (resp. 11.5)) and, consequently, they are globally or locally identical linear connections (resp. identical covariant derivatives along paths) (see Proposition 10.2) induced by some linear connection (see also Proposition 10.1).

By Theorem 11.1, all problems regarding pairs of a frame and coframe normal for a mixed linear connection are equivalent to similar ones for a pair of linear connections (its 'contravariant' and 'covariant' components). If these linear connections admit normal frames, the pairs of a frame and coframe normal for the mixed linear connections exist. It is important to be noted, such pairs of a frame and coframe for a mixed linear connection, which is not (ordinary) linear connection, may exist if no additional conditions on the employed frames and coframes are imposed, which is a highly unusual situation. Practically everywhere, the considered frames $\{E_i\}$ and coframes $\{E^i\}$ are supposed to be dual, $E^i(E_j) = \delta^i_j$, or such that the functions $f^i_j := E^i(E_j)$ to be invariant under changes of the frames and coframes. If this is the case, then pairs of a frame and coframe normal for a mixed linear connection may exist if it is an ordinary linear connection (Proposition 11.7).

13. Examples

In the previous sections, we have demonstrated that different problems concerning frames normal for derivations along (arbitrary) vector fields and paths are more or less reduced to similar problems for suitable linear connections. For that reason, the reader is referred to Sections I.7 and II.6 for instances of normal frames of this kind for linear connections.

Below will be presented several examples of frames normal for derivations along fixed vector field or fixed path. As it was mentioned earlier, these cases have no analogues in the theory of frames normal for linear connections (along arbitrary vector fields).

Example 13.1 (Covariant derivatives). If ∇ is a linear connection and X is a fixed vector field on a manifold M, then ∇_X is a derivation along X whose coefficients in a frame $\{E_i\}$ are ${\Gamma_X}^i{}_j = \Gamma^i{}_{jk}X^k$ (see Proposition 2.2), where $\Gamma^i{}_{jk}$ are the coefficients of ∇ in $\{E_i\}$.

Let M be of class C^1 and $p_0 \in M$. If $X_{p_0} = 0$, then all frames/coordinates in a neighborhood of p_0 are normal for ∇_X at p_0 as ${\Gamma_X}^i{}_j(p_0) = 0$. Suppose $X_{p_0} \neq 0$. Then, in the notation and suppositions of Theorems 9.1 or 9.2, all respectively frames $\{E'_i = A^j_i E_j\}$ and coordinate systems $\{z^i\}$ defined on a neighborhood of

p_0 and normal at p_0 are such that (see (9.8) and (9.13))

$$A^{(x)}(p) = A_0^{(x)} - \sum_k X_{p_0}^k \big[{\Gamma^i}_{jk}(p_0)\big]_{i,j=1}^{\dim M} A_0^{(x)}[x^1(p) - x^1(p_0)]$$
$$+ \sum_{k=2}^{\dim M} A_k^{(x)}[x^k(p) - x^k(p_0)] + A_{kl}^{(x)}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] \quad (13.1)$$

$$z^i(p) = a^i + a_j^i[x^j(p) - x^j(p_0)] + a_l^i X_{p_0}^j {\Gamma^l}_{kj}(p_0)[x^k(p) - x^k(p_0)][x^1(p) - x^1(p_0)]$$
$$+ \sum_{j,k=2}^{\dim M} a_{jk}^i[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)]$$
$$+ a_{jkl}^i(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)]. \quad (13.2)$$

Similarly, applying Theorem 9.3 or Proposition 9.4, one can describe the frames or coordinates, respectively, (locally) normal for ∇ along a fixed path.

Example 13.2 (Lie derivatives). The Lie derivative $\mathcal{L}$ (see Example 2.1 and Subsection IV.13.3 below) is a fundamental derivation along vector fields on any C^1 manifold [11]. If $X \in \mathfrak{X}^1(M)$, the coefficients of the Lie derivative $\mathcal{L}_X$ along (with respect to) X are defined by

$$\mathcal{L}_X E_i =: {\Gamma_X}^j{}_i E_j$$

in a frame $\{E_i\}$ and, due to (2.1), are ($X = X^i E_i$)

$$ {\Gamma_X}^i{}_j = -E_j(X^i) + C_{kj}^i X^k. \quad (13.3)$$

Consequently, $\mathcal{L}$ is nowhere linear derivation, precisely the only set $U \subseteq M$ on which ${\Gamma_X}^i{}_j = X^k {\Gamma^i}_{jk}$ for some functions ${\Gamma^i}_{jk} \colon U \to \mathbb{K}$ and all X, is the empty set, $U = \varnothing$. Thus Proposition 3.1 implies the non-existence of frames normal for $\mathcal{L}$ on any subset $U \subseteq M$.

Let now X be a fixed vector field. Suppose $U \subseteq M$ and $X|_U = 0$. If $E_j(X^i)|_U = 0$ (which is equivalent $\mathcal{L}_X$ to be the zero derivation on U), then any frame or coordinate system defined on (a neighborhood of) U is normal for $\mathcal{L}_X$ on U. Otherwise, i.e., if $E_j(X^i)|_p \neq 0$ for some $p \in U$, frames/coordinates normal on U for $\mathcal{L}_X$ do not exist.

Let $p_0 \in U$ and $X_{p_0} \neq 0$. Then, in the notation and suppositions of Theorem 9.1 or 9.2, all respectively frames $\{E'_i = A_i^j E_j\}$ and coordinate systems $\{z^i\}$ defined on a neighborhood of p_0 and normal at p_0 are such that (see (9.8) and (9.13))

$$A^{(x)}(p) = A_0^{(x)} - \big[(-E_j(X^i) + C_{mj}^i X^m)|_{p_0}\big]_{i,j=1}^{\dim M} A_0^{(x)}[x^1(p) - x^1(p_0)]$$
$$+ \sum_{k=2}^{\dim M} A_k^{(x)}[x^k(p) - x^k(p_0)] + A_{kl}^{(x)}(p)[x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)] \quad (13.4)$$

$$
\begin{aligned}
z^i(p) = a^i + a^i_j[x^j(p) - x^j(p_0)] \\
+ a^i_l(-E_k(X^l) + C^l_{mk}X^m)[x^k(p) - x^k(p_0)][x^1(p) - x^1(p_0)] \\
+ \sum_{j,k=2}^{\dim M} a^i_{jk}[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)] \\
+ a^i_{jkl}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)][x^l(p) - x^l(p_0)]. \quad (13.5)
\end{aligned}
$$

Similarly, applying Theorem 9.3 or Proposition 9.4, one can describe the frames or coordinates, respectively, (locally) normal for $\mathcal{L}$ along a fixed path. For instance, if the condition (9.15) is not valid for a fixed (regular, C^1, and locally injective) path $\gamma\colon J \to M$, then all frames locally normal along γ are, in the notation of Theorem 9.3 and its proof, $\{E'_i = A^j_i \frac{\partial}{\partial x^i}\}$, where the matrix-valued function $A = [A^i_j]$ is given by (9.22), which in the particular case reads (see (13.3))

$$
\begin{aligned}
A(x^{-1}(s,\sigma,\boldsymbol{t})) = A_0(x^{-1}(s,\sigma_0,\boldsymbol{t})) \\
- \frac{\mathrm{d}\tau_{\gamma(s)}(\sigma)}{\mathrm{d}\sigma}\Big|_{\sigma=\sigma_0} \big[(-E_j(X^i) + C^i_{kj}X^k)|_{\gamma(s)}\big]_{i,j=1}^{\dim M} A_0(\gamma(s))(\sigma - \sigma_0) \\
+ B(x^{-1}(s,\sigma,\boldsymbol{t}))(\sigma - \sigma_0)^2. \quad (13.6)
\end{aligned}
$$

Example 13.3 (Tensor fields of type $(1,1)$**).** As we said in Example 2.1, a tensor field S of type $(1,1)$ on a manifold M can be considered as a derivation. Hence $S\colon \mathfrak{X}(M) \ni X \mapsto S \in \mathfrak{T}^1_1(M)$ is a derivation along (relative to) vector fields; it can be called (generalized) Frenet-Serret derivative due to the considerations in Example 13.6 below. If X is a vector field and $\{E_i\}$ is a frame, then (2.2) yields the following coefficients of S (considered as a derivation):

$$
{\Gamma_X}^i{}_j = S^i_j. \quad (13.7)
$$

Since the coefficients of S along X are independent of X, Proposition 3.1 implies a non-existence of frames/coordinates normal for S on any subset (unless $S = 0$).

However, if we consider X as an arbitrarily fixed vector field, frames or coordinates normal for S exist on any subset, according to the results of Section 9. For instance, Theorems 9.1 and 9.2 (resp. 9.3 and Proposition 9.4) with respectively $\Gamma_X = [S^i_j]$ and ${\Gamma_X}^i{}_j = S^i_j$ give the complete (local) description of all frames and coordinates, respectively, normal at a single point (resp. along a path).

If $p_0 \in M$, X is fixed and $X_{p_0} \neq 0$, then Theorem 9.4 says that there is a neighborhood U of p_0 on which a frame $\{E'_i = A^j_i \frac{\partial}{\partial x^j}\}$ is normal for S iff (see the notation in Theorem 9.4)

$$
\begin{aligned}
A(p) = Y\big(x^1(p), x^1(p_0); -\big[S^i_j(x^{-1}(\,\cdot\,, x^2(p), \dots, x^{\dim M}(p)))\big]_{i,j=1}^{\dim M}\big) \\
\times A_0\big(x^2(p), \dots, x^{\dim M}(p)\big), \quad (13.8)
\end{aligned}
$$

where Y is the solution of the initial-value problem

$$\frac{\partial Y}{\partial x^1}\Big|_p = -\big[S^i_j(x^{-1}(\,\cdot\,, x^2(p), \dots, x^{\dim M}(p)))\big]_{i,j=1}^{\dim M} Y, \qquad Y|_{x^1(p)=x^1(p_0)} = \mathbb{1}. \tag{13.9}$$

Example 13.4 (General derivations and particular cases). As we know from Definition 2.3, any derivation D along vector fields is a sum of the Lie derivative and a mapping $S\colon \mathfrak{X}(M) \to \mathfrak{T}^1_1(M)$ assigning a tensor field S_X of type $(1,1)$ to a vector field X; the choice $S_X = \Sigma_X$, with Σ_X defined by (2.4), corresponds to a linear connection ∇. According to (2.7), the coefficients of D and their matrix (at X) are

$$\Gamma_X{}^i{}_j = (S_X)^i_j - E_j(X^i) + C^i_{kj}X^k \tag{13.10}$$

$$\boldsymbol{\Gamma}_X = \big[(S_X)^i_j - E_j(X^i) + C^i_{kj}X^k\big]_{i,j=1}^{\dim M}. \tag{13.11}$$

Consequently, if X is fixed, frames and/or coordinates normal for D on some subset can be found by applying the results of Section 9, precisely by substituting in them the equations (13.10) and (13.11). We are not going to write the corresponding explicit expressions as the mentioned substitution is a trivial technical task.

Suppose on a manifold M is given a derivation D_X along a fixed vector field X and a linear connection ∇. As a result of (2.3) and Example 2.2, we can write the representations

$$D_X = \mathcal{L}_X + S_X \tag{13.12a}$$

$$D_X = \nabla_X + S_X - \Sigma_X. \tag{13.12b}$$

The former equality (which is independent of the existence of ∇) is suitable for general mathematical considerations, while the latter one finds applications in some physical applications.

Here is a list of four derivations along fixed vector field X based on the decompositions (13.12):

$$D^{\text{F-W}} = \nabla_X - 2Q_X \qquad \text{(Fermi-Walker derivative)} \tag{13.13a}$$

$$D^{\text{F}} = \nabla^{\{\}}_X - 2\bar{Q}_X \qquad \text{(Fermi derivative)} \tag{13.13b}$$

$$D^{\text{T}} = \mathcal{L}_X + \theta\delta \qquad \text{(Truesdell derivative)} \tag{13.13c}$$

$$D^{\text{J}} = \nabla^{\{\}}_X - \omega \qquad \text{(Jaumann derivative),} \tag{13.13d}$$

where δ is the unit tensor field, the tensor fields Q_X, $\bar{Q}_X$ and ω are type $(1,1)$ and they, as well as the scalar function θ, are defined below in Examples IV.13.3–IV.13.6 to which the reader is referred for further details.

Exercise 13.1. Calculate the coefficients of the derivations (13.13) and find all frames/coordinates normal for them, e.g., at a given point.

Example 13.5 (Fermi derivative along a path). Let D^{F} be Fermi derivative, with a representation (13.13b), on a C^3 Riemannian manifold M with metric g. Its restriction to a C^2 path $\gamma\colon J \to M$ results into the Fermi derivative D^{F}_{γ} along γ,

$$D^{\mathrm{F}}_{\gamma} = \nabla^{\{\}}_{\dot\gamma} - 2\bar{Q}_{\gamma}, \tag{13.14}$$

where, according to Example IV.13.4 on page 297,

$$\begin{aligned}\bigl(\bar{Q}_{\gamma}\bigr)^i_j &= \dot\gamma^i(s) g_{jk}(\gamma(s)) (\nabla^{\{\}}_{\dot\gamma}\dot\gamma)^k\big|_{\gamma(s)} \\ &= \dot\gamma^i(s) g_{jk}(\gamma(s)) \Bigl(\frac{\mathrm{d}\dot\gamma^k(s)}{\mathrm{d}s} + \begin{Bmatrix} k \\ lm \end{Bmatrix}\Big|_{\gamma(s)} \dot\gamma^l(s)\dot\gamma^m(s)\Bigr)\end{aligned} \tag{13.15}$$

with the Christoffel symbols $\left\{ {i \atop jk} \right\}$ being defined by (I.4.14).

In particular, if γ is a geodesic, then $\bar{Q}_{\gamma} = 0$, by virtue of the equation of geodesics (I.4.17), and hence

$$D^{\mathrm{F}}_{\gamma} = \nabla^{\{\}}_{\dot\gamma} \qquad \text{for a geodesic path } \gamma. \tag{13.16}$$

Combining this result with Proposition 10.1, we can assert that all frames and/or coordinates normal for the Fermi derivative along a (fixed) geodesic path γ coincide with the ones for the Riemannian connection $\nabla^{\{\}}$ along the path γ. For further details on frames normal along a (geodesic) path for a linear connection, the reader is referred to Section II.3; for a concrete instance of this kind, see Example II.6.8.

In the general case, the components of the Fermi derivative (13.14) are (see (10.6))

$$\Gamma^i_j(s;\gamma) = \begin{Bmatrix} i \\ jk \end{Bmatrix}\Big|_{\gamma(s)} - 2\dot\gamma^i(s) g_{jk}(\gamma(s)) \Bigl(\frac{\mathrm{d}\dot\gamma^k(s)}{\mathrm{d}s} + \begin{Bmatrix} k \\ lm \end{Bmatrix}\Big|_{\gamma(s)} \dot\gamma^l(s)\dot\gamma^m(s)\Bigr). \tag{13.17}$$

Theorem 10.1, with $\mathbf{\Gamma}_{\gamma} \circ \gamma = [\Gamma_{\gamma}{}^i_j \circ \gamma]$ defined via (13.17), gives a complete description of all frames on $\gamma(J)$ which are normal for D^{F}_{γ}. If one needs holonomic frames defined on a neighborhood of $\gamma(J)$ and normal on $\gamma(J)$, the results of Lemma 10.1 should be applied.

Example 13.6 (Generalized Frenet-Serret derivative along path). The generalized Frenet-Serret derivative $D^{\text{F-S}}_{\gamma}$ along a C^1 path γ is a derivation along γ whose coefficients in a frame $\{E_i\}$ (along γ) are

$$\Gamma^i_j(s;\gamma) = S^i_j(\gamma(s)) \tag{13.18}$$

where $S^i_j(\gamma(s))$ are the components at $\gamma(s)$ in $\{E_i\}$ of a tensor field S of type $(1,1)$.[1] Comparing (13.18) and (13.7), we see that the latter reduces to the former one for

[1] The (usual) Frenet-Serret derivative is obtainable from (13.18) for a particular choice of $S^i_j(\gamma(s))$ as a function of the curvature(s) and torsion(s) of the path γ – see [98, eq. (6)].

$X = \dot{\gamma}(s)$ at the point $\gamma(s)$. In this sense, the derivation determined by a tensor field of type $(1,1)$ can be called (generalized) Frenet-Serret derivation along vector fields; respectively, the restriction to a path of this derivative along the vector field tangent to the path results into the generalized Frenet-Serret derivative along that path.

According to Theorem 10.1, all frames $\{E'_i = A^j_i E_j\}$ defined on $\gamma(J)$ and normal for $D^{\text{F-S}}_\gamma$ are such that

$$A(\gamma(s)) = Y(s, s_0; -[S^i_j \circ \gamma])A_0 \tag{13.19}$$

with A_0 being a constant non-degenerate matrix and Y being the solution of

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = -[S^i_j(\gamma(s))]Y \qquad Y|_{s=s_0} = \mathbb{1};$$

in particular, if $S^i_j(\gamma(s))$ is independent of s, then $Y = \exp\bigl(S^i_j(\gamma(s))(s - s_0)\bigr)$. Holonomic frames $\bigl\{\frac{\partial}{\partial y^i}\bigr\}$ defined on a neighborhood of $\gamma(J)$ and normal on $\gamma(J)$ for $D^{\text{F-S}}_\gamma$ can be constructed as pointed in the proof of Lemma 10.1, viz. one should define B^i_j via (10.13) with E'_i for E_i, then the coordinates y^i are given by (10.14).

14. Terminology 2: Normal and geodesic frames

We have called normal some special kinds of local bases, frames, charts, and coordinates investigated in the present book. This needs some explanations which are given below.

For symmetric linear connections, the local coordinates in which their components vanish at a given point are called normal in [70, Chapter V, Section 3] or in [20, § 11.6]. In [11, Chapter III, § 8] and in [71, p. 278] the local coordinates normal at a point, introduced there via the exponential mapping (see Subsection II.2.3) for linear connection (symmetric or not), are defined as such for which the symmetric part of the connection's components vanish at that point (see Proposition II.2.3). Evidently, the latter definition includes the former one as a special case. Note that the both definitions originate from the consideration of the equation (I.3.22) of geodesic paths [11, 70, 71]. This is the primary reason for calling these local coordinates geodesic (or Riemannian, or normal Riemannian [20, § 11.5]) in the special case of a Riemannian manifold [51, § 42, p. 201], where they are (sometimes) equivalently introduced via the condition that in them the partial derivatives of the metric's components vanish at a given point [51, § 42] (cf. Proposition I.6.1).

The case of a symmetric linear connection is investigated in [19, Chapter III, § 7, pp. 156–158] (see the references therein too). There is made a distinction between geodesic and normal at a point local coordinates. Geodesic coordinates are called the ones in which the connection's coefficients vanish at that point and normal coordinates are called the geodesic ones satisfying at the given point equation (7.23) of [19, Chapter III, § 7] which, in particular, implies the vanishment at that point of the connection's coefficients together with their symmetrized partial derivatives.[1] (Note that the possibility for the existence of the last type of coordinates is ensured by our (non-)uniqueness result expressed by Proposition I.5.2 with which is compatible the mentioned equation; see also equations (I.6.11) and (II.3.10) and Proposition II.2.3.) Analogous opinion is shared in [99, pp. 13–14].

It is known that the symmetric parts of the connection coefficients of arbitrary linear connection ∇ are directly linked with the equation (I.3.22) of geodesic lines (curves, paths) and uniquely determine them [11, 19]. By our opinion, this suggests the following convenient convention. Call normal or respectively geodesic on a set U a local coordinate system (chart, frame), defined in a neighborhood of U, in which the local components of ∇ or respectively their symmetric parts vanish on U. Thus, in the torsion free case, the concepts of normal and geodesic coordinate system coincide. Generally a normal frame is geodesic, the converse being not valid. In this sense, the normal coordinates described in [19, p. 158] are a special type of (our) normal coordinates, specified by the additional conditions described in this reference. These conditions are consistent with Proposition I.5.2. Note that the proposed definition is in accordance with the special one used in [100].

If one adopts the suggested convention, then the generalization from linear connections to arbitrary derivations D (along vector fields, fixed vector field, paths, fixed path) of the tensor algebra over a manifold is evident: only the concept of a normal frame is applicable because, generally, of some symmetry properties of the components of D cannot be spoken about.[2] This explains the terminology accepted in the present monograph.

Let us mention that the so-defined normal frames on U for D have a connection with a kind of generalized geodesic paths corresponding to D (cf. [101]) which will be discussed briefly in Section IV.15.

15. Conclusion

As the title of the present chapter indicates, it deals with different kinds of problems concerning frames (coordinates, charts) normal for derivations (along vector fields, fixed vector field, paths, fixed path) of the tensor algebra over (vector fields on) a differentiable manifold. Excluding the cases along fixed vector field or fixed path, the general conclusion is that a derivation admits frames normal on some set if and only if on it the derivation coincides with some linear connection (with the

[1] Cf. the derived here similar equations (I.6.11) and (II.3.10) for respectively the Riemannian coordinates and Fermi coordinates.

[2] If D is derivation along arbitrary vector fields or paths and it admits frames normal on U, then one can speak of the symmetry properties of the coefficients of D on U as D is linear on U (Proposition 3.1).

same coefficients as the derivation on the set given) for which normal frames exist. This fact singles out the linear connections, defined locally or globally, as the only derivations which admit normal frames in the above sense. The last conclusion is important for the theoretical physics and explains to a great extent why the linear connections find so many applications in it. The reason is, from certain positions, quite simple: due to the equivalence principle, the 'normal frames' are the mathematical concept for describing the physical notion of 'inertial frame' [97].

Most of the material in this chapter is new and written for the present book. Exceptions are some results in Sections 4–7 which are taken from [76, 80, 83],[1] but the proofs are completely revised or new.

The main results obtained can be summarized as follows: Necessary and sufficient conditions for the existence of frames or coordinates normal at a single point, along locally injective paths or, more generally, mappings, and on submanifolds for derivations along either arbitrary or fixed vector field or path are found. Frames normal at a single point or along a path always exist. On other sets normal frames exist iff some additional conditions are satisfied, exception being the case of a derivation along a fixed vector field, when normal frames exist on every subset of the manifold or, in particular, on the whole manifold. When normal frames or coordinates exist, their complete constructive description is given.[2]

[1] These papers are revised journal versions of the earlier works [84–86], respectively.

[2] Exceptions are some cases of a derivation along fixed vector field, in which complicated systems of partial differential equations arise.

Chapter IV

Normal Frames in Vector Bundles

The theory of linear transports along paths in vector bundles, generalizing the parallel transports generated by linear connections, is developed. The normal frames for them are defined as ones in which their matrices are the identity one. A number of results, including theorems of existence and uniqueness, concerning normal frames are derived. Special attention is paid to the case when the bundle's base is a manifold. The normal frames are defined and investigated also for derivations along paths and along tangent vector fields in the last case. Frames normal at a single point or along a given path always exist. On other subsets normal frames exist only in the curvature free case. The privileged role of the parallel transports is pointed out in this context.

♡

1. Introduction

The analysis of Corollary II.5.1 on page 123 reveals that the properties of the parallel transport assigned to a linear connection, not directly the ones of the connection itself, are responsible for the existence of frames normal on a submanifold for the connection.[1] This observation forms the groundwork of the idea the 'normal' frames to be defined directly for (parallel) transports without referring to the concept of a (linear) connection (or some other derivation along vector fields). The main obstacle for the realization of such an approach to 'normal frames' is that, ordinary, the concept of a parallel transport is a secondary one, it is introduced on the base of the concept of a (linear) connection. To the solution of the last problem and the development of the mentioned approach to normal frames (in finite-dimensional vector bundles) is devoted the present chapter of the book. As we shall demonstrate below, the consistent realization of the above idea leads to a completely new look on the 'normal frames', which is self-contained and incorporates as special cases all of the results of the preceding chapters.

The material in Sections 3–6 and 8 is based on the work [102] and the one after them is practically new and written especially for the present book.[2]

In the present chapter is studied a wide range of problems concerning frames normal for linear transports and derivations along paths in vector bundles and for derivations along tangent vector fields in the case when the bundle's base is a differentiable manifold. In the last case, when tangent bundles are concerned, the only general result, known to the author and regarding normal frames, is [23, p. 102, Theorem 2.106].

The structure of this chapter is as follows.

Section 2 introduces some basic concepts from the theory of (fibre) bundles, in particular of the one of vector bundles, required for the investigations in this chapter. After the concepts of bundle, section, and vector bundle are fixed, a special attention to the ones of liftings of paths and derivations along paths, which will play an important role further, is paid. The tensor bundles over a manifold are pointed as particular examples of vector bundles. Details on these and many other concepts regarding (fibre) bundles, the reader can fined in the monographs [7, 11, 106–110].

Section 3 is devoted to the general theory of linear transports along paths in vector fibre bundles which is a far reaching generalization of the theory of parallel transports generated by linear connections.[3] The general form and other properties of these transports are studied. A bijective correspondence between them and derivations along paths is established. In Section 4, the normal frames are de-

[1] Here the situation is similar to the one described in the second paragraph of Section II.1 on page 74: the properties of the Christoffel symbols, not directly the ones of the Riemannian metric generating them, are fully responsible for the existence of coordinates normal at a single point in a Riemannian manifold.

[2] Although, some initial ideas and results are borrowed from the papers [103–105].

[3] This section is based on the early works [101, 105, 111–115] of the author. For some more general results, see Chapter V.

fined as ones in which the matrix of a linear transport along paths is the identity (unit) one or, equivalently, in which its coefficients, as defined in Section 3, vanish 'locally'. A number of properties of the normal frames are found. In Section 5 is explored the problem of existence of normal frames. Several necessary and sufficient conditions for such existence are proved and the explicit construction of normal frames, if any, is presented.

Section 6 concentrates on, possibly, the most important special case of frames normal for linear transports or derivations along smooth paths in vector bundles with a differentiable manifold as a base. A specific necessary and sufficient condition for existence of normal frames in that case is proved. In particular, normal frames may exist only for those linear transports or derivations along paths whose (2–index) coefficients linearly depend on the vector tangent to the path along which they act. Obviously, this is a generalization of the derivation along curves assigned to a linear connection. Section 7 examplifies the theory in a case of line bundles. Section 8 is devoted to problems concerning frames normal for derivations along tangent vector fields in a bundle with a manifold as a base. Necessary and sufficient conditions for the existence of these frames are derived. The conclusion is made that there is a one-to-one onto correspondence between the sets of linear transports along paths, derivations along paths, and derivations along tangent vector fields all of which admit normal frames.

In the first part of Section 9, based on [103], the concept of a curvature of a linear transport along paths is introduced and some its properties are explored. In its second part, relations between the curvature of a linear transports along paths and the frames normal for them are studied. The main result is that only the curvature free transports admit normal frames. The concept of a torsion of a linear transport along paths in the tangent bundle over a manifold is introduced in Section 10 (cf. the early paper [103]). Links between the torsion and holonomic normal frames are investigated. The vanishment of the torsion is pointed as a necessary and sufficient condition for existence of normal coordinates on submanifolds. If such coordinates exist, their complete description is given.

Section 11 deals with parallel transports in the tangent bundles over manifolds and frames normal for these transports. It is shown that the parallel transport assigned to a linear connection is a special kind of a linear transport in tangent bundles. As a side result, an axiomatic definition of a parallel transport is obtained, on the base of which a new definition of a linear connection, equivalent to the usual one, is given. The flat parallel transports are pointed as the only linear transports along paths in tangent bundles which transports admit normal frames. The coordinates normal for flat and torsionless parallel transports are explicitly presented.

Section 12 concerns a special type of normal frames in which the 3-index coefficients, if any, of a linear transport along paths vanish.

Section 13 is similar to Section 11, but it deals with the interrelations between different types of derivations along vector fields over a manifold and the linear transports along paths in the tangent bundle over it. As examples, particular derivations or transports, such as Fermi-Walker, Jaumann, etc., are considered.

The aim of Section 14 is twofold. On one hand (Subsections 14.1–14.3), the rigorous relations between the theory of linear transports along paths in vector bundles and the one of parallel transports and connections in these bundles are investigated. On the base of the axiomatic approach to the theory of parallel transports, as presented in [23], we show how it (and hence the one of connections) is incorporated as a special case in the general theory of linear transports along paths. On another hand (Subsections 14.4 and 14.5), we demonstrate how the results concerning normal frames and derived for linear connections on manifolds and linear transports along paths are almost *in extenso* applicable to the theory of parallel transports and connections on vector bundles.

In Section 15 is introduced the notion of autoparallel paths in manifolds whose tangent bundle is endowed with a linear transport along paths. If this transport is a parallel one, it is proved that the autoparallels coincide with the geodesics of the linear connection generating the transport.

Section 16 gives an idea of the role of the linear transports along paths and normal frames in a fibre bundle formulation of quantum mechanics.

The chapter ends with some notes in Section 17.

All fibre bundles in this chapter are vectorial ones. The base and total bundle space of such bundles can be general topological spaces. However, if some kind of differentiation in one/both of these spaces is needed to be introduced (considered), it/they should possess a smooth structure; if this is the case, we require it/they to be smooth, of class C^1, differentiable manifold(s). Starting from Section 6, the base and total bundle space are supposed to be C^1 manifolds. Sections 3–5 do not depend on the existence of a smoothness structure in the bundle's base. Smoothness of the bundle space is partially required in Sections 2–5.[4]

2. Vector bundles

The purpose of the present section is the introduction of some simple basic concepts of the theory of (fibre) bundles, in particular of vector bundles, on which this chapter rests.

The most general intuitive idea of a bundle is the one of an object consisting of a set (space) to each point of which is 'attached' some other set (space). Hence a bundle should be thought as a mapping whose values are some sets, the attached sets. On the sets forming a bundle may be imposed a large number of 'reasonable' conditions whose goal is particular bundles to be obtained for the exploration of more or less concrete problems; for instance, the sets can be topological spaces, in

[4] The bundle space is required to be a C^1 manifold in Section 2 (starting from Definition 2.1), in Definition 4.1′, in Proposition 4.1–4.2, if (4.1c) and (4.1d) are taken into account, in Theorem 5.2, and in Proposition 5.6.

particular manifolds. Often the attached sets are vector (linear) spaces, in which case the bundle is called vector bundle. For quite a big class of problems, the notion of a 'smooth' bundle is required: the idea is when one 'moves' in the initial set, the corresponding attached sets (or selected points in them) to change smoothly.

A particular example of a smooth vector bundle is the tangent bundle to a manifold M: its basic set (space) is the manifold M itself and the set attached at $p \in M$ is the space $T_p(M)$ tangent to M at p. Similar structure have the tensor bundles of different types over M. If $f\colon M \to \mathbb{K}$ is a smooth function, a completely trivial smooth bundle structure over M arises by assigning to each $p \in M$ the value $f(p)$ of f at p.

2.1. Basic definitions

A *bundle* is a triple (E, π, B) of sets E and B, called (total) *bundle space* and *base* (space) respectively, and (generally) surjective mapping $\pi\colon E \to B$, called *projection.*[1] For every $b \in B$, the set $\pi^{-1}(b)$ is called the *fibre over b*.

If $X \subseteq B$, the bundle $(E, \pi, B)|_X := (\pi^{-1}(X), \pi|_{\pi^{-1}(X)}, X)$ is called the *restriction to* X of a bundle (E, π, B).

A *section* of the bundle (E, π, B) is a mapping $\sigma\colon B \to E$ such that $\pi \circ \sigma = \mathsf{id}_B$, i.e., $\sigma\colon b \mapsto \sigma(b) \in \pi^{-1}(b)$. The set of sections of (E, π, B) is denoted by $\mathrm{Sec}(E, \pi, B)$.

If E and B are topological spaces, which is the most widely considered case, the bundle (E, π, B) is called *topological.* In this case in the definition of a bundle is included the *bundle property*: there exists a (topological) space $\mathcal{E}$ such that, for each $b \in B$, there is an open set ('directory space') $W \ni b$ in B and homeomorphism ('decomposition function') $\phi_W\colon W \times \mathcal{E} \to \pi^{-1}(W)$ of $W \times \mathcal{E}$ onto $\pi^{-1}(W)$ satisfying the condition $(\pi \circ \phi_W)(w, e) = w$ for $w \in W$ and $e \in \mathcal{E}$. Besides, if the restriction $\phi_W|_b\colon \{b\} \times \mathcal{E} \to \pi^{-1}(b)$, $b \in B$, is homeomorphism, the bundle property is called *local triviality*, $\mathcal{E}$ is called (*typical, standard*) *fibre of the bundle*, and the fibre $\pi^{-1}(b)$ is homeomorphic to $\mathcal{E}$ for every $b \in B$.

A *vector bundle* is a locally trivial bundle (E, π, B) such that: (i) the fibres $\pi^{-1}(b)$, $b \in B$, and the standard fibre $\mathcal{E}$ are (linearly) isomorphic vector spaces and (ii) the decomposition mappings ϕ_W and their restrictions $\phi_W|_b$ are (linear) isomorphisms between vector spaces. The dimension of $\mathcal{E}$, $\dim \mathcal{E} = \dim \pi^{-1}(b)$ for all $b \in B$, is called the (fibre) dimension of the vector bundle, resp. it is called $(\dim \mathcal{E})$-dimensional. Here the vector spaces are considered over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

All bundles in this chapter are supposed to be vectorial.

[1] The notation (E, π, B) for a bundle comes from the one $E \xrightarrow{\ \pi\ } B$ for a mapping π from E on B.

2.2. Liftings of paths

A *lifting*[2] (in a vector bundle (E,π,B)) of a mapping $g\colon X\to B$ to E, X being a set, is a mapping $\overline{g}\colon X\to E$ such that $\pi\circ\overline{g}=g$. The set of all liftings of a mapping $g\colon X\to B$ will be denoted by $\mathrm{Lift}_g(E,\pi,B):=\{\overline{g}\colon X\to E|\pi\circ\overline{g}=g\}$. In particular, the liftings of the identity mapping id_B of the base B are called *sections* and their set is $\mathrm{Sec}(E,\pi,B):=\mathrm{Lift}_{\mathsf{id}_B}(E,\pi,B)=\{\sigma|\sigma\colon B\to E,\ \pi\circ\sigma=\mathsf{id}_B\}$ (see Subsection 2.1).

Let $\mathrm{P}(A):=\{\gamma|\gamma\colon J\to A\}$ and $\mathrm{PLift}(E,\pi,B):=\{\lambda|\lambda\colon \mathrm{P}(B)\to\mathrm{P}(E),\ (\pi\circ\lambda)(\gamma)=\gamma \text{ for } \gamma\in\mathrm{P}(B)\}$ be respectively the *set of paths* in a set A and the *set of liftings of paths* from B to E.[3] The set $\mathrm{PLift}(E,\pi,B)$ is: (i) A natural $\mathbb{K}$-vector space if we put $(a\lambda+b\mu)\colon\gamma\mapsto a\lambda_\gamma+b\mu_\gamma$ for $a,b\in\mathbb{K}$, $\lambda,\mu\in\mathrm{PLift}(E,\pi,B)$, and $\gamma\in\mathrm{P}(B)$, where, for brevity, we write λ_γ for $\lambda(\gamma)$, $\lambda\colon\gamma\mapsto\lambda_\gamma$; (ii) A natural left module with respect to $\mathbb{K}$-valued functions on B: if $f,g\colon B\to\mathbb{K}$, we define $(f\lambda+g\mu)\colon\gamma\mapsto(f\lambda)_\gamma+(g\mu)_\gamma$ with $(f\lambda)_\gamma(s):=f(\gamma(s))\lambda_\gamma(s)$ for $\gamma\colon J\to B$ and $s\in J$; (iii) A left module with respect to the set $\mathrm{PF}(B):=\{\varphi|\varphi\colon\gamma\mapsto\varphi_\gamma,\ \gamma\colon J\to B,\ \varphi_\gamma\colon J\to\mathbb{K}\}$ of *functions along paths* in the base B: for $\varphi,\psi\in\mathrm{PF}(B)$, we set $(\varphi\lambda+\psi\mu)\colon\gamma\mapsto(\varphi\lambda)_\gamma+(\psi\mu)_\gamma$ where $(\varphi\lambda)_\gamma(s):=(\varphi_\gamma\lambda_\gamma)(s):=\varphi_\gamma(s)\lambda_\gamma(s)$.

If we consider $\mathrm{PLift}(E,\pi,B)$ as a $\mathbb{K}$-vector space, its dimension is equal to infinity. If we regard $\mathrm{PLift}(E,\pi,B)$ as a left $\mathrm{PF}(B)$-module, its rank is equal to the dimension of (E,π,B) (i.e., to the dimension of the fibre(s) of (E,π,B)). In the last case, a basis in $\mathrm{PLift}(E,\pi,B)$ can be constructed as follows (cf. [116, Proposition 2.1.14] in a case of sections of a vector bundle).

For every path $\gamma\colon J\to B$ and a point $s\in J$, choose a basis $\{e_i(s;\gamma)|i=1,\dots,\dim\pi^{-1}(\gamma(s))\}$ in the fibre $\pi^{-1}(\gamma(s))$; if the total space E is a C^r manifold, we suppose $e_i(s;\gamma)$ to have a C^r dependence on s. Define liftings along paths $\hat{e}_i\in\mathrm{PLift}(E,\pi,B)$ by $\hat{e}_i\colon\gamma\mapsto\hat{e}_i|_\gamma:=e_i(\,\cdot\,;\gamma)$, i.e., $\hat{e}_i|_\gamma\colon s\mapsto\hat{e}_i|_\gamma(s):=e_i(s;\gamma)$. The set $\{\hat{e}_i\}$ is a basis in $\mathrm{PLift}(E,\pi,B)$, i.e., for every $\lambda\in\mathrm{PLift}(E,\pi,B)$ there are $\lambda^i\in\mathrm{PF}(B)$ such that $\lambda=\sum_i\lambda^i\hat{e}_i$ and $\{\hat{e}_i\}$ are $\mathrm{PF}(B)$-linearly independent. Actually, for any path $\gamma\colon J\to B$ and number $s\in J$, we have $\lambda_\gamma(s)\in\pi^{-1}(\gamma(s))$, so there exists numbers $\lambda^i_\gamma(s)\in\mathbb{K}$ such that $\lambda_\gamma(s)=\sum_i\lambda^i_\gamma(s)e_i(s;\gamma)$. Defining $\lambda^i\in\mathrm{PF}(B)$ by $\lambda^i\colon\gamma\mapsto\lambda^i_\gamma$ with $\lambda^i_\gamma\colon s\mapsto\lambda^i_\gamma(s)$, we get $\lambda=\sum_i\lambda^i\hat{e}_i$; if $e_i(\,\cdot\,;\gamma)$ is of class C^r, so are λ^i_γ. The $\mathrm{PF}(B)$-linear independence of $\{\hat{e}_i\}$ is an evident corollary of the $\mathbb{K}$-linear independence of $\{e_i(s;\gamma)\}$. As we notice above, if E is C^r manifold, we choose $\hat{e}_i$, i.e., $\hat{e}_i|_\gamma$, to be C^r and, consequently, the components λ^i, i.e., λ^i_γ, are of class C^r too.

Let (E,π,B) be a vector bundle of dimension $n\in\mathbb{N}$, $n<\infty$, $U\subseteq B$, and $g\colon Q\to B$ with $Q\neq\varnothing$. A *frame* on (over) U (resp. along g) is a set $\{e_i|i=1,\dots,n\}$ of n $F(U)$-linearly (resp. $F(g(Q))$-linearly) independent sections of $(E,\pi,B)|_U$ (resp. of $(E,\pi,B)|_{g(Q)}$), where $F(U):=\{f:f\colon U\to\mathbb{K}\}$ is the

[2] For detail see, e.g., [108].

[3] Every linear transport L along paths (*vide infra* Section 3, in particular Definition 3.3) provides a lifting of paths: for every $\gamma\colon J\to B$ fix some $s\in J$ and $u\in\pi^{-1}(\gamma(s))$, the mapping $\gamma\mapsto\overline{\gamma}_{s;u}$ with $\overline{\gamma}_{s;u}(t):=L^\gamma_{s\to t}u$, $t\in J$ is a lifting of paths from B to E.

set of functions on U. Said differently, $\{e_i\}$ is a frame on U (resp. along g) if $e_i \in \operatorname{Sec}\big((E,\pi,B)|_U\big)$ (resp. $e_i \in \operatorname{Sec}\big((E,\pi,B)|_{g(Q)}\big)$) and, for every $p \in U$ (resp. $p \in g(Q)$), the set of vectors $\{e_i|_p\}$ is a basis in $\pi^{-1}(p)$.

2.3. Derivations along paths

Let (E,π,B) be a vector bundle whose bundle space E is C^r, $r \in \mathbb{N}$, manifold. A lifting $\lambda \in \mathrm{PLift}(E,\pi,B)$ is said to be of class C^k, $k = 0,1,\dots,r$, if in some (and hence in any) C^k frame in $\mathrm{PLift}(E,\pi,B)$ its components are of class C^k along paths in some set of paths, i.e., λ is of class C^k if λ_γ is a C^k path for a path γ in that set. Obviously, not every path in B has a C^k lifting in E; for instance, all liftings of a discontinuous path in B are discontinuous paths in E. The set of paths in B having C^k liftings in E is $\pi \circ \mathrm{P}^k(E) := \{\pi \circ \overline{\gamma} | \overline{\gamma} \in \mathrm{P}^k(E)\}$, with $\mathrm{P}^k(E)$ being the set of C^k paths in E. Therefore, when talking of C^k liftings, we shall implicitly assume that they are acting on paths in $\pi \circ \mathrm{P}^k(E) \subset \mathrm{P}(B)$. The discontinuous paths in B are, of course, not in $\pi \circ \mathrm{P}^k(E)$, so that they are excluded from the considerations below.

Analogously, $\varphi \in \mathrm{PF}(B)$ is of class C^k if φ_γ is of class C^k for a path γ in some set of paths. Denote by $\mathrm{PLift}^k(E,\pi,B)$, $k = 0,1,\dots,r$, the set of C^k liftings of paths from B to E and by $\mathrm{PF}^k(B)$, $k = 0,1,\dots,r$, the set of C^k functions along paths in B.

If the base B is $C^{r'}$, $r' \in \mathbb{N}$, manifold, we denote by $\operatorname{Sec}^k(E,\pi,B)$, $k = 0,1,\dots,r'$, the set of C^k sections of the bundle (E,π,B).

Definition 2.1. A *derivation along paths in* (E,π,B) or a *derivation of liftings of paths* in (E,π,B) is a mapping

$$D\colon\ \mathrm{PLift}^1(E,\pi,B) \to \mathrm{PLift}^0(E,\pi,B) \tag{2.1a}$$

which is $\mathbb{K}$-linear,

$$D(a\lambda + b\mu) = aD(\lambda) + bD(\mu) \tag{2.2a}$$

for $a,b \in \mathbb{K}$ and $\lambda,\mu \in \mathrm{PLift}^1(E,\pi,B)$, and the mapping

$$D_s^\gamma\colon\ \mathrm{PLift}^1(E,\pi,B) \to \pi^{-1}(\gamma(s)), \tag{2.1b}$$

defined via $D_s^\gamma(\lambda) := \big((D(\lambda))(\gamma)\big)(s) = (D\lambda)_\gamma(s)$ and called *derivation along* $\gamma\colon J \to B$ *at* $s \in J$, satisfies the 'Leibnitz rule':

$$D_s^\gamma(f\lambda) = \frac{\mathrm{d}f_\gamma(s)}{\mathrm{d}s}\lambda_\gamma(s) + f_\gamma(s)D_s^\gamma(\lambda) \tag{2.2b}$$

for every $f \in \mathrm{PF}^1(B)$. The mapping

$$D^\gamma\colon\ \mathrm{PLift}^1(E,\pi,B) \to \mathrm{P}\big(\pi^{-1}(\gamma(J))\big), \tag{2.1c}$$

defined by $D^\gamma(\lambda) := (D(\lambda))|_\gamma = (D\lambda)_\gamma$, is called a *derivation along* γ.

Let $\{e_i\}$ be a *frame along paths* γ, i.e., for any $\gamma\colon J \to B$ and $s \in J$, $e_i\colon (s;\gamma) \mapsto e_i(s;\gamma)$ and $\{e_i(s;\gamma)\}$ is a basis in $\pi^{-1}(\gamma(s))$ that can depend on s and γ. The *components*[4] $\Gamma^i{}_j(s;\gamma)$ of a derivation D along paths (along γ at s in $\{e_i\}$) are defined via the expansion

$$D^\gamma_s \hat{e}_j =: \Gamma^i{}_j(s;\gamma) e_i(s;\gamma), \tag{2.3}$$

where $\hat{e}_i\colon \gamma \mapsto e_i(\,\cdot\,;\gamma)$ is a lifting of paths, generated by e_i according to (2.8) below, and the dimension of (E,π,B) is assumed to be finite. If $\lambda \in \mathrm{PLift}^1(E,\pi,B)$ has an expansion $\lambda = \lambda^i \hat{e}_i$, then the properties (2.2) imply the explicit local expansion

$$D^\gamma_s \lambda = \sum_i \Big(\frac{\mathrm{d}\lambda^i_\gamma(s)}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)\lambda^j_\gamma(s) \Big) e_j(s;\gamma). \tag{2.4}$$

Hence the components of D in a given frame uniquely define it.

From the last equation, we get the explicit expansion for $D\lambda$:

$$D(\lambda) = (\dot{\lambda}^i + \Gamma^i{}_j \lambda^j)\hat{e}_i \tag{2.5}$$

where $\dot{\lambda}^i, \Gamma^i{}_j \in \mathrm{PF}(B)$ and $\hat{e}_i \in \mathrm{PLift}(E,\pi,B)$ are such that

$$\dot{\lambda}^i\colon \gamma \mapsto \dot{\lambda}^i_\gamma\colon s \mapsto \frac{\mathrm{d}\lambda^i_\gamma(s)}{\mathrm{d}s} \tag{2.6}$$

$$\Gamma^i{}_j\colon \gamma \mapsto \Gamma^i{}_j(\,\cdot\,;\gamma)\colon s \mapsto \Gamma^i{}_j(s;\gamma) \tag{2.7}$$

$$\hat{e}_i\colon \gamma \mapsto e_i(\,\cdot\,;\gamma)\colon s \mapsto e_i(s;\gamma). \tag{2.8}$$

Similarly, the derivation of λ along γ is

$$D^\gamma(\lambda) := (D\lambda)(\gamma) = (\dot{\lambda}^i_\gamma + \Gamma^i{}_j(\,\cdot\,;\gamma)\lambda^j_\gamma) e_i(\,\cdot\,;\gamma). \tag{2.9}$$

If the frame $\{e_i\}$ is changed by means of a non-degenerate matrix-valued function $A = [A^j_i]$, $\{e_i(s;\gamma)\} \mapsto \{e'_i(s;\gamma) = A^j_i(s;\gamma)e_j(s;\gamma)\}$, from (2.3) and (2.2), we see that the *matrix* $\mathbf{\Gamma}(s;\gamma) := [\Gamma^i{}_j(s;\gamma)]$ of D transform into

$$\mathbf{\Gamma}'(s;\gamma) := [\Gamma'^i{}_j(s;\gamma)] = A^{-1}(s;\gamma)\mathbf{\Gamma}(s;\gamma)A(s;\gamma) + A^{-1}(s;\gamma)\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s}. \tag{2.10}$$

Applying (2.3), one can easily prove that every system $\Gamma^i{}_j \in \mathrm{PF}(B)$ of functions along paths, $\Gamma^i{}_j\colon \gamma \mapsto \Gamma^i{}_j(\,\cdot\,;\gamma)\colon s \mapsto \Gamma^i{}_j(s;\gamma)$, which transform according to (2.10), defines via (2.4) a derivation D along paths in (E,π,B). Combining this result with the above one, we can assert the existence of a bijective correspondence

[4] In connection with the theory of normal frames (see Section 4 and further), it is convenient to call $\Gamma^i{}_j(s;\gamma)$ also *(2-index) coefficients* of D^γ. This agrees with the fact that $\Gamma^i{}_j$ are coefficients of some linear transport along paths (see below).

between the set of functions $\{\Gamma^i{}_j\}$ along paths with transformation law (2.10) and the set of derivations along paths of a vector bundle (E,π,B).

The set $\mathrm{PSec}(E,\pi,B)$ of *sections along paths* of a bundle (E,π,B) consists of mappings $\boldsymbol{\sigma}\colon \gamma \mapsto \boldsymbol{\sigma}_\gamma$ assigning to every path $\gamma\colon J \to B$ a section $\boldsymbol{\sigma}_\gamma \in \mathrm{Sec}\big((E,\pi,B)|_{\gamma(J)}\big)$ of the bundle restricted to $\gamma(J)$. Every (ordinary) section $\sigma \in \mathrm{Sec}(E,\pi,B)$ generates a section $\boldsymbol{\sigma}$ along paths via $\boldsymbol{\sigma}\colon \gamma \mapsto \boldsymbol{\sigma}_\gamma := \sigma|_{\gamma(J)}$, i.e., $\boldsymbol{\sigma}_\gamma$ is simply the restriction of σ to $\gamma(J)$; hence $\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}_\gamma$ for every path $\alpha\colon J_\alpha \to B$ with $\alpha(J_\alpha) = \gamma(J)$. Every $\boldsymbol{\sigma} \in \mathrm{PSec}(E,\pi,B)$ generates a lifting $\hat{\boldsymbol{\sigma}} \in \mathrm{PLift}(E,\pi,B)$ by $\hat{\boldsymbol{\sigma}}\colon \gamma \mapsto \hat{\boldsymbol{\sigma}}_\gamma := \boldsymbol{\sigma}_\gamma \circ \gamma$; in particular, the lifting $\hat{\sigma}$ associated to $\sigma \in \mathrm{Sec}(E,\pi,B)$ is given via $\hat{\sigma}\colon \gamma \mapsto \hat{\sigma}_\gamma = \sigma|_{\gamma(J)} \circ \gamma$.

If B is a manifold, every derivation D along paths generates a mapping

$$\overline{D}\colon\ \mathrm{PSec}^1(E,\pi,B) \to \mathrm{PLift}^0(E,\pi,B)$$

such that, if $\boldsymbol{\sigma} \in \mathrm{PSec}^1(E,\pi,B)$, then $\overline{D}\colon \boldsymbol{\sigma} \mapsto \overline{D}\boldsymbol{\sigma} = \overline{D}(\boldsymbol{\sigma})$ where $\overline{D}\boldsymbol{\sigma}\colon \gamma \mapsto \overline{D}^\gamma \boldsymbol{\sigma}$ is a lifting of paths defined by $\overline{D}^\gamma \boldsymbol{\sigma}\colon s \mapsto (\overline{D}^\gamma \boldsymbol{\sigma})(s) := D^\gamma_s \hat{\boldsymbol{\sigma}}$ with $\hat{\boldsymbol{\sigma}}$ being the lifting generated by $\boldsymbol{\sigma}$, i.e., $\gamma \mapsto \hat{\boldsymbol{\sigma}}_\gamma := \boldsymbol{\sigma}_\gamma \circ \gamma$. The mapping $\overline{D}$ may be called a *derivation of C^1 sections along paths*. Notice, if $\gamma\colon J \to B$ has intersection points and $x_0 \in \gamma(J)$ is such a point, the mapping $\gamma(J) \to \pi^{-1}(\gamma(J))$ given by $x \mapsto \{D^\gamma_s(\hat{\boldsymbol{\sigma}}) | \gamma(s) = x,\ s \in J\}$, $x \in \gamma(J)$, is generally multiple-valued at x_0 and, consequently, is not a section of $(E,\pi,B)|_{\gamma(J)}$.

If B is a C^1 manifold and for some $\gamma\colon J \to B$ there exists a subinterval $J' \subseteq J$ on which the restricted path $\gamma|J\colon J' \to B$ is without self-intersections, i.e., $\gamma(s) \neq \gamma(t)$ for $s,t \in J'$ and $s \neq t$, we can define the *derivation along γ of sections* over $\gamma(J')$ as a mapping

$$\mathsf{D}^\gamma\colon\ \mathrm{Sec}^1\big((E,\pi,B)|_{\gamma(J')}\big) \to \mathrm{Sec}^0\big((E,\pi,B)|_{\gamma(J')}\big) \tag{2.11}$$

such that

$$(\mathsf{D}^\gamma \sigma)(x) := D^\gamma_s \hat{\sigma} \qquad \text{for } x = \gamma(s) \tag{2.12}$$

where $s \in J'$ is unique for a given $x \in \gamma(J')$ and $\hat{\sigma} \in \mathrm{PLift}\big((E,\pi,B)|_{\gamma(J')}\big)$ is given by $\hat{\sigma} = \sigma|_{\gamma(J')} \circ \gamma|_{J'}$. Generally the mapping (2.11) defined by (2.12) is multiple-valued at the points of self-intersections of γ, if any, as $(\mathsf{D}^\gamma\sigma)(x) := \{D^\gamma_s \hat{\sigma} : s \in J,\ \gamma(s) = x\}$. The so-defined mapping $\mathsf{D}\colon \gamma \mapsto \mathsf{D}^\gamma$ is called a *section-derivation* along paths.[5] As we said, it is single-valued only along paths without self-intersections.

Let D be a derivation along paths in a vector bundle (E,π,B), $\gamma\colon J \to B$ be injective, D be the generated by D section-derivation along paths, and $\sigma \in \mathrm{Sec}^1(E,\pi,B)|_{\gamma(J)}$. As a consequence of (2.12) and (2.4), we have

$$(\mathsf{D}^\gamma \sigma)|_{\gamma(s)} = \sum_i \Big[\frac{\mathrm{d}\sigma^i(\gamma(s))}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)\sigma^j(\gamma(s))\Big] e_i(s;\gamma) \tag{2.13}$$

[5] A particular example of a section-derivation along injective C^1 paths is the derivation along paths provided by Definition III.10.1. For details – see Remark III.10.1 and Subsection 13.1.

where $\{e_i(s;\gamma)\}$ is a basis in $\pi^{-1}(\gamma(s))$, $\sigma(\gamma(s)) =: \sigma^i(\gamma(s))e_i(s;\gamma)$, and $\Gamma^i{}_j(s;\gamma)$ are the components of D (or of D) in $\{e_i\}$.

Generally a section along paths or lifting of paths does not define a (single-valued) section of the bundle as well as to a lifting along paths there does not correspond some (single-valued) section along paths. The last case admits one important special exception, viz. if a lifting λ is such that the lifted path λ_γ is an 'exact topological copy' of the underlying path $\gamma\colon J \to B$, i.e., if there exist $s,t \in J$, $s \neq t$ for which $\gamma(s) = \gamma(t)$, then $\lambda_\gamma(s) = \lambda_\gamma(t)$, which means that if γ has intersection points, then the lifting λ_γ also possesses such points and they are in the fibres over the corresponding intersection points of γ. Such a lifting λ generates a section $\overline{\lambda} \in \mathrm{PSec}(E,\pi,B)$ along paths given by $\overline{\lambda}\colon \gamma \mapsto \overline{\lambda}_\gamma$ with $\overline{\lambda}\colon \gamma(s) \mapsto \lambda_\gamma(s)$. In the general case, the mapping $\gamma(s) \mapsto \lambda_\gamma(s)$ for a lifting λ of paths is multiple-valued at the points of self-intersection of $\gamma\colon J \to B$, if any; for injective path γ, this mapping is a section of $(E,\pi,B)|_{\gamma(J)}$. Such mappings will be called *multiple-valued sections along paths*.

2.4. Tensor bundles

Over every differentiable manifold M exists a collection of natural bundles, viz. the different tensor bundles with M as a base, uniquely related to the differentiable structure of M.

The *tangent bundle* $(T(M),\pi,M)$, called also *bundle tangent to* M, is a well explored example of a tensor bundle over a manifold M. Its base is the manifold M itself, the bundle space $T(M)$ is the disjoint union of the spaces tangent to M (see Subsection I.2.3), i.e., $T(M) := \bigcup_{p\in M} T_p(M)$, and the projection $\pi\colon T(M) \to M$ is defined by $\pi(X_p) := p$ if $X_p \in T_p(M)$. So the fibre of $(T(M),\pi,M)$ over $p \in M$ is exactly the space tangent to M at p, $\pi^{-1}(p) = T_p(M)$. It is a trivial exercise to be verified the equivalence

$$X \in \mathfrak{X}(M) \iff X \in \mathrm{Sec}(T(M),\pi,M),$$

i.e., the vector fields over a manifold are simply sections of its tangent bundle, and, consequently,

$$\mathfrak{X}(M) = \mathrm{Sec}(T(M),\pi,M).$$

Respectively, the sections along paths of the tangent bundle are mappings assigning to a path $\gamma\colon J \to M$ a vector field over $\gamma(J)$ etc.

Since each space $T_p(M)$ is $(\dim M)$-dimensional vector (linear) space, the tangent bundle $(T(M),\pi,M)$ is $(\dim M)$-dimensional vector bundle for which $\mathbb{K}^{\dim M}$ can be taken as a (standard) fibre.

Similar to the tangent bundle is the *cotangent bundle* $(T^*(M),\pi^*,M)$ whose bundle space is $T^*(M) := \bigcup_{p\in M} T^*_p(M)$ and the projection π^* is such that $\pi^*(\omega_p) := p$ if $\omega_p \in T^*_p(M)$ where $T^*_p(M)$ is the cotangent space to M at p (see Subsection I.2.4).

The *tensor bundle* $(T^r_s(M), \pi^r_s, M)$ *of type* (r, s), $r, s \in \mathbb{N} \cup \{0\}$ is a natural generalization of the tangent and cotangent bundles. Its bundle space is the disjoint union of the tensor spaces of type (r, s) over M,

$$T^r_s(M) := \bigcup_{p\in M} T_p{}^r_s(M)$$

with $T_p{}^r_s(M)$ being the tensor space of type (r, s) at $p \in M$ (see Subsection I.2.5). The projection $\pi^r_s \colon T^r_s(M) \to M$ is defined by

$$\pi^r_s(K_p) := p \quad \text{for } K_p \in T_p{}^r_s(M).$$

Therefore the dimension of $(T^r_s(M), \pi^r_s, M)$ is $(\dim M)^{r+s} = \dim T_p{}^r_s(M)$ and $\mathbb{K}^{(\dim M)^{r+s}}$ can be taken as its standard fibre. The choices $(r, s) = (1, 0)$ and $(r, s) = (0, 1)$ correspond to the tangent and cotangent bundles, respectively. Since $T_p{}^0_0 \equiv \mathbb{K}$, the tensor bundle of type $(0, 0)$ is $(M \times \mathbb{K}, \pi^0_0, M)$ with $\pi^0_0(p, a) := p$ for $(p, a) \in M \times \mathbb{K}$.

Obviously, the tensor fields over M are sections of the corresponding tensor bundles and *vice versa*,

$$\mathfrak{T}^r_s(M) = \operatorname{Sec}\bigl(T^r_s(M), \pi^r_s, M\bigr).$$

The *algebraic tensor bundle* $(\mathbf{T}(M), \boldsymbol{\pi}, M)$ on (over) a manifold M is also an example of a bundle structure over manifolds. By definition, its bundle space is the disjoint union of the tensor algebras over M, $\mathbf{T} := \bigcup_{p\in M} \boldsymbol{T}_p(M)$, and its projection is such that $\boldsymbol{\pi}(K_p) := p$ if $K_p \in \boldsymbol{T}_p(M)$ for $p \in M$, i.e., the fibre over p is the tensor algebra at p, $\pi^{-1}(p) = \boldsymbol{T}_p(M)$. All tensor fields are specific (of type (r, s)) sections of the algebraic tensor bundle.

One can expect the existence of a link between the derivations along vector fields of the tensor algebra over M (see Definition III.10.1 on page 191) and the derivations along paths of the tensor bundles over M. Since the former derivations are (uniquely) defined only along injective paths, the mentioned connection could be expressed via the section-derivations along paths of the tensor bundles over the manifold M (see (2.11) and (2.12)).

Let D be a derivation along paths of $\boldsymbol{T}(M)$, $\gamma \colon J \to M$ be injective, and K be a C^1 tensor field(=section) of type (r, q) along γ. Locally, in some frame along γ, the derivative $D^\gamma(K)$ of K along γ is given by (III.10.7) in which $\Gamma^i{}_j(s; \gamma)$ are the components of D along γ at $s \in J$.

Let now D be a section-derivation along paths, generated by a derivation along paths, in the tensor bundle $(T^r_q(M), \pi^r_q, M)$. If $K \in \mathfrak{T}^r_q(M)$, then, in the tensor frame (I.2.41) induced by a frame $\{E_i\}$, equation (2.13), with K for σ and this frame for $\{e_i\}$, reads[6]

$$(\mathsf{D}^\gamma K)(\gamma(s)) = \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_q}} \Big[\frac{\mathrm{d}K^{i_1,\dots,i_r}_{j_1,\dots,j_q}(\gamma(s))}{\mathrm{d}s} + \Gamma^{i_1,\dots,i_r,k_1,\dots,k_q}_{j_1,\dots,i_q,l_1,\dots,l_r}(s;\gamma) K^{l_1,\dots,l_r}_{k_1,\dots,k_q}(\gamma(s)) \Big] E^{j_1,\dots,j_q}_{i_1,\dots,i_r}|_{\gamma(s)}. \quad (2.14)$$

This expression is identical for every K with the one corresponding to the right-hand side of (III.10.7) if and only if the components of the section-derivation D have the form

$$\Gamma^{i_1,\dots,i_r,k_1,\dots,k_q}_{j_1,\dots,i_q,l_1,\dots,l_r}(s;\gamma) = \Big(\sum_{a=1}^{r} \delta^{i_1}_{l_1} \cdots \delta^{i_{a-1}}_{l_{a-1}} \Gamma^{i_a}{}_{l_a}(s;\gamma) \delta^{i_{a+1}}_{l_{a+1}} \cdots \delta^{i_r}_{l_r} \Big) \prod_{b=1}^{q} \delta^{k_b}_{j_b} - \Big(\prod_{a=1}^{q} \delta^{i_a}_{l_a} \Big) \sum_{b=1}^{q} \delta^{k_1}_{j_1} \cdots \delta^{k_{b-1}}_{j_{b-1}} \Gamma^{k_b}{}_{i_b}(s;\gamma) \delta^{k_{b+1}}_{j_{b+1}} \cdots \delta^{k_q}_{j_q} \quad (2.15)$$

where $\Gamma^i{}_l$ are the components of a (section-)derivation along paths in the tangent bundle $(T(M),\pi,M)$.

Thus we have proved that a derivation along vector fields of the tensor algebra over a manifold M is equivalent to a set of (section-)derivations along paths, one in each tensor bundle $(T^r_q(M),\pi^r_q,M)$ with $q,r\in\mathbb{N}\cup\{0\}$, whose components have in some (and hence in every) frame the form (2.15); the corresponding (section-) derivation along paths in $(T^r_q(M),\pi^r_q,M)$ being equal to the restriction of the initial derivation along paths of $\boldsymbol{T}(M)$ to the bundle space $T^r_q(M)$ of the tensor bundle of type (r,q).

Example 2.1. A particular example of the above derivations is provided by the one generated by a linear connection ∇ over M. As we said earlier in Section III.10, any linear connection ∇ generates along every C^1 path γ a derivation $\nabla_{\dot\gamma}$ along γ (of the tensor algebra along γ), locally given via (III.10.5). Taking into account the afore-said, we can assert that in the tensor bundle of type (r,q) every connection ∇ generates a derivation along paths with local coefficients (2.15) in which

$$\Gamma^i{}_j(s;\gamma) = \Gamma^i{}_{jk}(\gamma(s))\dot\gamma^k(s) \quad (2.16)$$

where $\Gamma^i{}_{jk}$ are the coefficients of ∇.

[6]Notice, for $r=q=0$ equation (2.14) implies $(\mathsf{D}^\gamma K)(\gamma(s)) = \frac{\mathrm{d}K(\gamma(s))}{\mathrm{d}s} = \dot\gamma(s)(K)$, i.e., on scalar functions D^γ acts as ordinary directional derivative, as it should be.

3. Linear transports along paths in vector bundles

In the majority of the mathematical literature, the concept of a connection (in locally trivial differentiable bundles of class C^1) is taken as a basic one and the concept of a parallel transport is introduced by its means (see, e.g., [6, 10–13, 16, 28, 106, 117]). However, the opposite approach is also known [17, 23, 30–33]: in it the 'parallel transport' is axiomatically defined and on its base the concept of a 'connection' is introduced as a secondary one.[1] It seems that the earliest written accounts on this approach are the ones due to Ü.G. Lumiste [30, Section 2.2] and C. Teleman [17, Chapter IV, Section B.3] (both published in 1964), the next essential steps being made by P. Dombrowski [31, § 1] and W. Poor [23].[2] The comparison and the features of the both approaches to the theory of connections and parallel transports is not a subject of the present book,[3] but, for completeness, it is partially investigated in Section V.8.

In the present section, the concept of a 'linear transport along paths in vector bundles' is axiomatically defined and some its simple properties are derived. This concept is a straightforward generalization of the one of a parallel transport assigned to a linear connection (see Section 11) and some other transports defined on its base (see Section 13). As we shall see later (see Definitions 11.2 and 11.3 and the comments after them), on its base independent, but equivalent, definition of a parallel transport and linear connection can be given. Moreover, in Subsection 14.3 will be shown that the parallel transports in general vector bundles are a special type of linear transports along paths, as a consequence of which the whole theory of connections and covariant derivatives can be deduced from the theory of linear transports in these bundles.

3.1. Definition and general form

Let (E, π, B) be a $\mathbb{K}$-vector bundle[4] with bundle (total) space E, base B, projection $\pi\colon E \to B$, and homeomorphic fibres $\pi^{-1}(p)$, $p \in B$. Whenever some kind of differentiation in E is considered, the bundle space E will be required to be a C^1 differentiable manifold. The base B is supposed to be a general topological space in Sections 3–5 and from Section 6 onwards is required to be a C^1 differentiable manifold. By J and $\gamma\colon J \to B$ are denoted a real interval and path in B,

[1] A summary of the realization of this approach in [23] is presented in Subsection 14.1 below; see also [118].

[2] The author of [31] states that his paper is based on unpublished lectures of prof. W. Rinow in 1949. See also [23, p. 46] where the author claims that the first axiomatical definition of a parallel transport in the tangent bundle case is given by prof. W. Rinow in his lectures at the Humboldt University in 1949. Some heuristic comments on the axiomatic approach to parallel transport theory can be found in [8, Section 2.1] too.

[3] For some details, see [115, 118] and the references given therein.

[4] Most of our results are valid for vector bundles over more general fields too but this is inessential for the following.

respectively. The paths considered are generally *not* supposed to be continuous or differentiable unless their differentiability class is stated explicitly.

Definition 3.1. A *linear transport along paths* in the bundle (E,π,B) is a mapping L assigning to every path γ a mapping L^γ, *transport along* γ, such that $L^\gamma\colon (s,t)\mapsto L^\gamma_{s\to t}$ where the mapping

$$L^\gamma_{s\to t}\colon \pi^{-1}(\gamma(s))\to\pi^{-1}(\gamma(t)) \qquad s,t\in J, \tag{3.1}$$

called *transport along* γ *from* s *to* t, has the properties:

$$L^\gamma_{s\to t}\circ L^\gamma_{r\to s} = L^\gamma_{r\to t}, \qquad r,s,t\in J, \tag{3.2}$$

$$L^\gamma_{s\to s} = \mathsf{id}_{\pi^{-1}(\gamma(s))}, \qquad s\in J, \tag{3.3}$$

$$L^\gamma_{s\to t}(\lambda u+\mu v) = \lambda L^\gamma_{s\to t}u + \mu L^\gamma_{s\to t}v, \qquad \lambda,\mu\in\mathbb{K}, \quad u,v\in\pi^{-1}(\gamma(s)). \tag{3.4}$$

Remark 3.1. Equations (3.2) and (3.3) mean that L is a *transport along paths* in the bundle (E,π,B) (see Definition V.8.1 below or [114, Definition 2.1]), which may be an arbitrary topological bundle, not only a vector one in the general case,[5] while (3.4) specifies that it is *linear* [114, equation (2.8)]. In the present book, with an exception of Section V.8, only linear transports in vector bundles will be explored.

Remark 3.2. Definition 3.1 is a generalization of the concept of 'linear connection' given, e.g., in [31, Section 1.2] (see especially [31, p. 138, axiom (L_1)]) which practically defines the covariant derivative in terms of linear transports along paths (see (3.29) below which is equivalent to [31, p. 138, axiom (L_3)]). Our definition is much weaker; e.g., we completely drop [31, p. 138, axiom (L_3)] and use, if required, weaker smoothness conditions. An excellent introduction to the theory of vector bundles and the parallel transports in them can be found in the book [23]. In particular, in this reference is proved the equivalence of the concepts parallel transport, connection and covariant derivative operator in vector bundles (as defined there). Analogous results concerning linear transports along paths will be presented below. The detailed comparison of Definition 3.1 with analogous ones in the literature is not a subject of this work.

Further, in Subsection 13.3 (see also Definition 3.1, [105, Definition 2.1] and [105, Proposition 4.1]), it will be proved that special types of linear trans-

[5] The definition of a connection in a *topological* bundle (E,π,B) in [17, Chapter IV, Section B.3] is, in fact, an axiomatic definition of a parallel transport. If we neglect the continuity condition in this definition, it defines a connection in (E,π,B) as a mapping $C\colon(\gamma,q)\mapsto C(\gamma,q)$ assigning to any continuous path $\gamma\colon[0,1]\to B$ and a point $q\in\pi^{-1}(\gamma(0))$ a path $C(\gamma,q)\colon[0,1]\to E$ such that $C(\gamma,q)|_0=q$ and $\pi\circ C(\gamma,q)=\gamma$. If I is a transport along paths in (E,π,B), then $C\colon(\gamma,q)\mapsto C(\gamma,q)\colon t\mapsto C(\gamma,q)|_t = I^\gamma_{0\to t}(q)$ defines a connection C in (E,π,B) in the sense mentioned. Moreover, if this definition is broadened by replacing $[0,1]$ with an arbitrary and not fixed closed interval $[a,b]$, with $a,b\in\mathbb{R}$ and $a\le b$, then the converse is also true, i.e., $C(\gamma,q)|_t = I^\gamma_{a\to t}(q)$, $t\in[a,b]$, for some transport I. However, the proof of this statement is not trivial; cf. similar considerations in Subsections 3.2 and 3.3 below, which deal with analogous problems in vector bundles (with other definition of a parallel transport).

ports along paths are: the parallel transport assigned to a linear connection (covariant derivative) of the tensor algebra of a manifold [11, 19],[6] Fermi-Walker transport [62, 119], Fermi transport [62], Truesdell transport [120, 121], Jaumann transport [122], Lie transport [19, 119], the modified Fermi-Walker and Frenet-Serret transports [98], etc. Consequently, Definition 3.1 is general enough to cover a list of important transports used in theoretical physics and mathematics. Thus studying the properties of the linear transports along paths, we can make corresponding conclusions for any one of the transports mentioned.[7]

From (3.2) and (3.3), we get that $L^{\gamma}_{s\to t}$ are invertible mappings and

$$(L^{\gamma}_{s\to t})^{-1} = L^{\gamma}_{t\to s}, \qquad s,t \in J. \tag{3.5}$$

Hence the linear transports along paths are in fact linear isomorphisms of the fibres over the path along which they act.

The following two propositions establish the general structure of linear transports along paths.[8]

Proposition 3.1. *A mapping* (3.1) *is a linear transport along* γ *from* s *to* t *for every* $s,t \in J$ *if and only if there exist a vector space* V*, isomorphic with* $\pi^{-1}(x)$ *for all* $x \in B$*, and a family* $\{F(s;\gamma)\colon \pi^{-1}(\gamma(s)) \to V,\ s \in J\}$ *of linear isomorphisms such that*

$$L^{\gamma}_{s\to t} = F^{-1}(t;\gamma) \circ F(s;\gamma), \qquad s,t \in J. \tag{3.6}$$

Proof. If (3.1) is a linear transport along γ from s to t, then fixing some $s_0 \in J$ and using (3.3) and (3.5), we get $L^{\gamma}_{s\to t} = L^{\gamma}_{s_0\to t} \circ L^{\gamma}_{s\to s_0} = \left(L^{\gamma}_{t\to s_0}\right)^{-1} \circ L^{\gamma}_{s\to s_0}$. So (3.6) holds for $V = \pi^{-1}(\gamma(s_0))$ and $F(s;\gamma) = L^{\gamma}_{s\to s_0}$. Conversely, if (3.6) is valid for some linear isomorphisms $F(s;\gamma)$, then a straightforward calculation shows that it converts (3.2) and (3.3) into identities and (3.4) holds due to the linearity of $F(s;\gamma)$. □

Proposition 3.2. *Let a representation* (3.6) *for some vector space* V *and linear isomorphisms* $F(s;\gamma)\colon \pi^{-1}(\gamma(s)) \to V,\ s \in J$ *be given for a linear transport along paths in the bundle* (E,π,B)*. For a vector space* ${}^{\star}V$*, there exist linear isomorphisms* ${}^{\star}F(s;\gamma)\colon \pi^{-1}(\gamma(s)) \to {}^{\star}V,\ s \in J$*, for which*

$$L^{\gamma}_{s\to t} = {}^{\star}F^{-1}(t;\gamma) \circ {}^{\star}F(s;\gamma), \qquad s,t \in J, \tag{3.7}$$

[6] For the proof, see Proposition 11.1 on page 282 below.

[7] The concept of linear transport along paths in vector bundles can be generalized to the transports along paths in arbitrary bundles [114] and to transports along mappings in bundles [123]. An interesting consideration of the concept of (parallel) 'transport' (along closed paths) in connection with homotopy theory and the classification problem of bundles can be found in [124]. These generalizations are out of the scope of the present book.

[8] Particular examples of Proposition 3.1 are known for parallel transports in vector bundles. For instance, Proposition 1 in [125, p. 240] realizes it for parallel transport in a bundle associated to a principal one and induced by a connection in the latter case; see also the proof of the lemma in the proof of Proposition 1.1 in [11, Chapter III, § 1], where a similar result is obtained implicitly.

iff there exists a linear isomorphism $D(\gamma)\colon V \to {}^\star V$ *such that*

$$ {}^\star F(s;\gamma) = D(\gamma)\circ F(s;\gamma), \qquad s\in J. \tag{3.8}$$

Proof. If equation (3.8) holds, the substitution of $F(s;\gamma) = D^{-1}(\gamma)\circ {}^\star F(s;\gamma)$ into (3.6) yields (3.7). Vice versa, if (3.7) is valid, then from its comparison with (3.6) follows that $D(\gamma) = {}^\star F(t;\gamma)\circ \bigl(F(t;\gamma)\bigr)^{-1} = {}^\star F(s;\gamma)\circ \bigl(F(s;\gamma)\bigr)^{-1}$ is the required (independent of $s,t\in J$) isomorphism. □

Starting from this point, we shall investigate further only the finite-dimensional case, $\dim \pi^{-1}(x) = \dim\pi^{-1}(y) < \infty$ for all $x,y\in B$. In this way we shall avoid a great number of specific problems arising when the fibres have infinite dimension (see, e.g., [126] for details). A lot of our results are valid, possibly *mutatis mutandis*, in the infinite-dimensional treatment too. One way for transferring results from finite to infinite-dimensional spaces is the direct limit from the first to the second ones. Then, for instance, if the bundle's dimension is countably or uncountably infinite, the corresponding sums must be replaced by series or integrals whose convergence, however, requires special exploration [126]. Linear transports along paths in infinite-dimensional vector bundles naturally arise, e.g., in the fibre bundle formulation of quantum mechanics [127–131]. Generally, there are many difficulties with the infinite-dimensional problem which deserves a separate investigation.

3.2. Representations in frames along paths

Let $\{e_i(s;\gamma)\}$ be a basis in $\pi^{-1}(\gamma(s))$, $s\in J$.[9] So, along $\gamma\colon J\to B$ we have a set $\{e_i\}$ of bases on $\pi^{-1}(\gamma(J))$, i.e., $\{e_i\}$ is a frame along γ.[10] The dependence of $e_i(s;\gamma)$ on s is inessential if we are interested only in the *algebraic* properties of the linear transports along paths. The mapping $s\mapsto e_i(s;\gamma)$ will be required to be of class C^1 if some kind of differentiation of liftings of paths will be considered.

The *matrix* $\boldsymbol{L}(t,s;\gamma) := \bigl[L^i{}_j(t,s;\gamma)\bigr]$ *(along* γ *at* (s,t) *in* $\{e_i\}$*) of a linear transport* L along γ from s to t is defined via the expansion[11]

$$ L^\gamma_{s\to t}\bigl(e_i(s;\gamma)\bigr) =: L^j{}_i(t,s;\gamma)e_j(t;\gamma) \qquad s,t\in J. \tag{3.9}$$

We call $\boldsymbol{L}\colon (t,s;\gamma)\to \boldsymbol{L}(t,s;\gamma)$ the *matrix (function)* of L; respectively $L^j{}_i$ are its *matrix elements* or *components* in the given frame.

It is almost evident that

$$ L^j{}_i(t,s;\gamma)e_j(t;\gamma)\otimes e^i(s;\gamma) \in \pi^{-1}(\gamma(t))\otimes \bigl(\pi^{-1}(\gamma(s))\bigr)^* \tag{3.10}$$

[9] Here and henceforth the Latin indices run from 1 to $\dim\pi^{-1}(p)$, $p\in B$.

[10] Regardless of the fact that $\{e_i(s;\gamma)\}$ is a basis in $\pi^{-1}(\gamma(s))$, we write $e_i(s;\gamma)$, not $e_i(\gamma(s))$, as we consider $e_i\colon \gamma\mapsto e_i(\,\cdot\,;\gamma)$ as a lifting of paths; hence $e_i(s;\gamma)$ generally depends on s and γ, not only on the combination $\gamma(s)$. If γ has self-intersections, then the mapping $p\mapsto e_i(s;\gamma)$, with $p\in\gamma(J)$ and $s\in J$ such that $p=\gamma(s)$, is, generally, multiple-valued at these points.

[11] Notice the different positions of the arguments s and t in $L^\gamma_{s\to t}$ and in $\boldsymbol{L}(t,s;\gamma)$.

where the asterisk ($*$) denotes dual objects and $e^i(s;\gamma) := (e_i(s;\gamma))^*$. Hence the change of the bases $\{e_i(s;\gamma)\} \mapsto \{e'_i(s;\gamma) := A^j_i(s;\gamma)e_j(s;\gamma)\}$, $s \in J$, by means of a non-degenerate matrix $A(s;\gamma) := \left[A^j_i(s;\gamma)\right]$, implies

$$\boldsymbol{L}(t,s;\gamma) \mapsto \boldsymbol{L}'(t,s;\gamma) = A^{-1}(t;\gamma)\boldsymbol{L}(t,s;\gamma)A(s;\gamma) \tag{3.11}$$

or in component form

$$L'^j{}_i(t,s;\gamma) = \left(A^{-1}(t;\gamma)\right)^j_k L^k{}_l(t,s;\gamma)A^l_i(s;\gamma). \tag{3.11$'$}$$

Evidently, for $u = u^i(s;\gamma)e_i(s;\gamma) \in \pi^{-1}(\gamma(s))$, due to (3.4), we have

$$L^\gamma_{s\to t}u = \left(L^j{}_i(t,s;\gamma)u^i(s;\gamma)\right)e_j(t;\gamma). \tag{3.12}$$

In terms of the matrix $\boldsymbol{L}$ of L, the basic equations (3.2) and (3.3) read respectively

$$\boldsymbol{L}(t,s;\gamma)\boldsymbol{L}(s,r;\gamma) = \boldsymbol{L}(t,r;\gamma) \qquad r,s,t \in J, \tag{3.13}$$

$$\boldsymbol{L}(s,s;\gamma) = \mathbb{1} \qquad s \in J \tag{3.14}$$

with $\mathbb{1}$ being the identity (unit) matrix of corresponding size. From these equalities immediately follows that $\boldsymbol{L}$ is always non-degenerate.

Proposition 3.3. *A linear mapping* (3.1) *is a linear transport along γ from s to t iff its matrix, defined via* (3.9)*, satisfies* (3.13) *and* (3.14)*.*

Proof. The necessity was already proved. The sufficiency is trivial: a simple checking proves that (3.13) and (3.14) convert respectively (3.2) and (3.3) into identities. □

Proposition 3.4. *A non-degenerate matrix-valued function $\boldsymbol{L}\colon (t,s;\gamma) \mapsto \boldsymbol{L}(t,s;\gamma)$ is a matrix of some linear transport along paths L (in a given field $\{e_i\}$ of bases along γ) iff*

$$\boldsymbol{L}(t,s;\gamma) = \boldsymbol{F}^{-1}(t;\gamma)\boldsymbol{F}(s;\gamma) \tag{3.15}$$

where $\boldsymbol{F}\colon (t;\gamma) \mapsto \boldsymbol{F}(t;\gamma)$ is a non-degenerate matrix-valued function.

Proof. This proposition is simply a matrix form of Proposition 3.1. If $\{f_i\}$ is a basis in V and $F(s;\gamma)e_i(s;\gamma) = F^j{}_i(s;\gamma)f_j$, then (3.15) with $\boldsymbol{F}(s;\gamma) = \left[F^j{}_i(s;\gamma)\right]$ is equivalent to (3.6). □

Proposition 3.5. *If the matrix $\boldsymbol{L}$ of a linear transport L along paths has a representation*

$$\boldsymbol{L}(t,s;\gamma) = {}^\star\boldsymbol{F}^{-1}(t;\gamma)\,{}^\star\boldsymbol{F}(s;\gamma) \tag{3.16}$$

for some matrix-valued function ${}^\star\boldsymbol{F}(s;\gamma)$, then all matrix-valued functions $\boldsymbol{F}$ representing $\boldsymbol{L}$ via (3.15) *are given by*

$$\boldsymbol{F}(s;\gamma) = \boldsymbol{D}^{-1}(\gamma)\,{}^\star\boldsymbol{F}(s;\gamma) \tag{3.17}$$

where $\boldsymbol{D}(\gamma)$ is a non-degenerate matrix depending only on γ.

Proof. In fact, this propositions is a matrix variant of Proposition 3.2; $\boldsymbol{D}(\gamma)$ is simply the matrix of the mapping $D(\gamma)$ in some bases. □

If $\boldsymbol{F}(s;\gamma)$ and $\boldsymbol{F}'(s;\gamma)$ are two matrix-valued functions, representing the matrix of L via (3.15) in two bases $\{e_i\}$ and $\{e'_i\}$ respectively, then, as a consequence of (3.11), the relation

$$\boldsymbol{F}'(s;\gamma) = C(\gamma)\boldsymbol{F}(s;\gamma)A(s;\gamma) \tag{3.18}$$

holds for some non-degenerate matrix-valued function C of γ.

3.3. Linear transports and derivations along paths

Below we want to consider some properties of the linear transports along paths connected with their 'differentiability'; in particular, we shall establish a bijective correspondence between them and the derivations along paths. For the purpose is required a smooth, of class at least C^1, transition from fibre to fibre when moving along a path in the base. Rigorously this is achieved by exploring transports in bundles whose *bundle space is a* C^1 *differentiable manifold* which will be supposed from now on in the present chapter.

Let (E,π,B) be a vector bundle whose bundle space E is a C^r, $r\in\mathbb{N}\cup\{0,\infty,\omega\}$, differentiable manifold. A linear transport L^γ along $\gamma\colon J\to B$ is called *differentiable of class* C^k, $k=0,1,\dots,r$, or simply C^k transport, if for arbitrary $s\in J$ and $u\in\pi^{-1}(\gamma(s))$, the path $\overline{\gamma}_{s;u}\colon J\to E$ with $\overline{\gamma}_{s;u}(t) := L^\gamma_{s\to t}u\in\pi^{-1}(\gamma(t))$, $t\in J$, is a C^k mapping in the bundle space E. If a C^k linear transport has a representation (3.6), the mapping $s\mapsto F(s;\gamma)$ is of class C^k. So, the transport L^γ is of class C^k iff $L^\gamma_{s\to t}$ has C^k dependence on s and t simultaneously. If $\{e_i(\,\cdot\,;\gamma)\}$ is a C^k frame along γ, i.e., $\{e_i(s;\gamma)\}$ is a basis in $\pi^{-1}(\gamma(s))$ and the mapping $s\mapsto e_i(s;\gamma)$ is of class C^k for all i, from (3.12) follows that L^γ is of class C^k iff its matrix $\boldsymbol{L}(t,s;\gamma)$ has C^k dependence on s and t.

Let E be a C^1 manifold and S a set of paths in B, $S\subseteq \mathrm{P}(B)=\{\gamma\colon J\to B\}$. A transport L along paths in (E,π,B), E being C^r manifold, is said to be of class C^k, $k=0,1,\dots,r$, on S if the corresponding transport L^γ along γ is of class C^k for all $\gamma\in S$. A transport along paths may turn to be of class C^k on some set S of paths in B and not to be of class C^k on other set S' of paths in B. Below, through Section 6, the set S will not be specialized and written explicitly; correspondingly, we shall speak simply of C^k transports implicitly assuming that they are such on some set S. Starting from Section 6, we shall suppose B to be a C^1 manifold and the set S to be the one of C^1 paths in B. Further we consider only C^1 linear transports along paths whose matrices will be referred to smooth frames along paths.

Definition 3.2. The *derivation D along paths generated by a C^1 linear transport L along paths* in (E,π,B), E being a C^1 manifold, is a mapping of type (2.1a) such that for every path $\gamma\colon J \to B$, we have $D^\gamma\colon \lambda \mapsto (D\lambda)_\gamma$ with $D^\gamma\lambda\colon s \mapsto D^\gamma_s\lambda$, $s \in J$, where D^γ_s is a mapping (2.1b) given via

$$D^\gamma_s(\lambda) := \lim_{\varepsilon\to 0}\Big\{\frac{1}{\varepsilon}\big[L^\gamma_{s+\varepsilon\to s}\lambda_\gamma(s+\varepsilon) - \lambda_\gamma(s)\big]\Big\} \tag{3.19}$$

for every lifting $\lambda \in \mathrm{PLift}^1(E,\pi,B)$ with $\lambda\colon \gamma \mapsto \lambda_\gamma$. The mapping D^γ (resp. D^γ_s) will be called a *derivation along γ generated by L* (resp. a *derivation along γ at s assigned to L*).

Remark 3.3. The operator D^γ_s is an analogue of the covariant derivative along paths assigned to a linear connection; cf., e.g., [31, p. 139, equation (12)].

Remark 3.4. Notice, if γ has self-intersections and $x_0 \in \gamma(J)$ is such a point, the mapping $x \mapsto \pi^{-1}(x)$, $x \in \gamma(J)$, given by $x \mapsto \{D^\gamma_s(\lambda) | \gamma(s) = x,\ s \in J\}$ is, generally, multiple-valued at x_0.

Let L be a linear transport along paths in (E,π,B). For every path $\gamma\colon J \to B$, choose some $s_0 \in J$ and $u_0 \in \pi^{-1}(\gamma(s_0))$. The mapping

$$\overline{L}\colon \gamma \mapsto \overline{L}^\gamma_{s_0,u_0}, \quad \overline{L}^\gamma_{s_0,u_0}\colon J \to E, \quad \overline{L}^\gamma_{s_0,u_0}\colon t \mapsto \overline{L}^\gamma_{s_0,u_0}(t) := L^\gamma_{s_0\to t}u_0 \tag{3.20}$$

is, evidently, a lifting of paths.

Definition 3.3. The lifting of paths $\overline{L}$ from B to E in (E,π,B), defined via (3.20), is called *lifting* (*of paths*) *generated by the* (*linear*) *transport* L.

Equations (3.2) and (3.4), combined with (3.19), immediately imply

$$D^\gamma_t(\overline{L}) \equiv 0, \qquad t \in J, \tag{3.21}$$

$$D^\gamma_s(a\lambda + b\mu) = aD^\gamma_s\lambda + bD^\gamma_s\mu, \qquad a,b \in \mathbb{K}, \quad \lambda,\mu \in \mathrm{PLift}^1(E,\pi,B), \tag{3.22}$$

where $s_0 \in J$ and $u_0 \in \pi^{-1}(\gamma(s_0))$ are fixed. In other words, equation (3.21) means that the lifting $\overline{L}$ is constant along every path γ with respect to D.

Let $\{e_i(s;\gamma)\}$ be a smooth field of bases along $\gamma\colon J \to B$, $s \in J$, i.e., $\{e_i\}$ to be a frame along γ. Combining (3.12) and (3.19), we find the explicit local action of D^γ_s as[12]

$$D^\gamma_s\lambda = \left[\frac{\mathrm{d}\lambda^i_\gamma(s)}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)\lambda^j_\gamma(s)\right]e_i(s;\gamma). \tag{3.23}$$

Here the *(2-index) coefficients* $\Gamma^i{}_j$ of the linear transport L are defined by

$$\Gamma^i{}_j(s;\gamma) := \left.\frac{\partial L^i{}_j(s,t;\gamma)}{\partial t}\right|_{t=s} = -\left.\frac{\partial L^i{}_j(s,t;\gamma)}{\partial s}\right|_{t=s} \tag{3.24}$$

and, evidently, uniquely determine the derivation D generated by L.

[12] The existence of derivatives like $\mathrm{d}\lambda^i_\gamma(s)/\mathrm{d}s$, viz. that $\lambda^i_\gamma\colon J \to \mathbb{K}$ are C^1 mappings, follows from $\lambda \in \mathrm{PLift}^1(E,\pi,B)$.

Exercise 3.1. Prove that the derivation along paths generated by a linear transport is actually a derivation along paths (see Definition 2.1). (Hint: use (3.22) and (3.23).)

Below, we shall prove that, freely speaking, a linear transport along path(s) can locally, in a given field of local bases, be described equivalently by the set of its local coefficients (with transformation law (3.26) written below).

If the transport's matrix $\boldsymbol{L}$ has a representation (3.15), from (3.24) we get

$$\boldsymbol{\Gamma}(s;\gamma) := \big[\Gamma^i{}_j(s;\gamma)\big] = \frac{\partial \boldsymbol{L}(s,t;\gamma)}{\partial t}\bigg|_{t=s} = \boldsymbol{F}^{-1}(s;\gamma)\frac{\mathrm{d}\boldsymbol{F}(s;\gamma)}{\mathrm{d}s}. \tag{3.25}$$

From here, (3.11), and (3.14), we see that the change $\{e_i\} \to \{e'_i = A^j_i e_i\}$ of the bases along γ with a non-degenerate C^1 matrix-valued function $A(s;\gamma) := \big[A^j_i(s;\gamma)\big]$ implies

$$\boldsymbol{\Gamma}(s;\gamma) = \big[\Gamma^i{}_j(s;\gamma)\big] \mapsto \boldsymbol{\Gamma}'(s;\gamma) = \big[\Gamma'^i{}_j(s;\gamma)\big]$$

with

$$\boldsymbol{\Gamma}'(s;\gamma) = A^{-1}(s;\gamma)\boldsymbol{\Gamma}(s;\gamma)A(s;\gamma) + A^{-1}(s;\gamma)\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s}. \tag{3.26}$$

Proposition 3.6. *Let along every (resp. given) path $\gamma\colon J \to B$ be given a geometrical object Γ whose local components $\Gamma^i{}_j$ in a frame $\{e_i\}$ along γ change according to* (3.26) *with $\boldsymbol{\Gamma}(s;\gamma) = \big[\Gamma^i{}_j(s;\gamma)\big]$. There exists a unique linear transport L along paths (resp. along γ) the matrix of whose coefficients is exactly $\boldsymbol{\Gamma}(s;\gamma)$ in $\{e_i\}$ along γ. Moreover, the matrix of the components of L in $\{e_i\}$ is*

$$\boldsymbol{L}(t,s;\gamma) = Y(t,s_0;-\boldsymbol{\Gamma}(\,\cdot\,;\gamma))Y^{-1}(s,s_0;-\boldsymbol{\Gamma}(\,\cdot\,;\gamma)), \qquad s,t \in J \tag{3.27}$$

where $s_0 \in J$ is arbitrarily fixed and the matrix $Y(s,s_0;Z)$, for a C^0 matrix-valued function $Z\colon s \mapsto Z(s)$, is the unique solution of the initial-valued problem

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = Z(s)Y, \qquad Y = Y(s,s_0;Z), \quad s \in J, \tag{3.28a}$$

$$Y(s_0,s_0;Z) = \mathbb{1}. \tag{3.28b}$$

Proof. At the beginning, we note that the existence and uniqueness of the solution of (3.28) is ensured by Lemma II.3.2 on page 96.

Given a linear transport L with a matrix (3.15). Suppose its components are exactly $\Gamma^i{}_j(s;\gamma)$ in a frame $\{e_i\}$. Solving (3.25) with respect to $\mathrm{d}\boldsymbol{F}^{-1}/\mathrm{d}s$, we obtain $\mathrm{d}\boldsymbol{F}^{-1}(s;\gamma)/\mathrm{d}s = -\boldsymbol{\Gamma}(s;\gamma)\boldsymbol{F}^{-1}(s;\gamma)$ and, consequently, $\boldsymbol{F}^{-1}(s;\gamma) = Y(s,s_0;-\boldsymbol{\Gamma}(\,\cdot\,;\gamma))\boldsymbol{F}^{-1}(s_0;\gamma)$. So, due to equation (3.15), the matrix of L is (3.27). Because of [34, Chapter IV, equation (1.10)], the expression

$$Y(t,s;Z) = Y(t,s_0;Z)Y(s_0,s;Z) = Y(t,s_0;Z)Y^{-1}(s,s_0;Z)$$

is independent of s_0. Besides, as a consequence of (3.26), the matrix (3.27) transforms according to (3.11) when the local bases are changed. Hence equation (3.3) holds and, due to (3.12), the linear mapping L with a matrix (3.27) in $\{e_i\}$ is a linear transport along γ. In this way we have proved two things: On one hand, a linear mapping with a matrix (3.27) in $\{e_i\}$ is a linear transport with local coefficients $\Gamma^i{}_j(s;\gamma)$ in $\{e_i\}$ along γ and, on the other hand, any linear transport with local coefficients $\Gamma^i{}_j(s;\gamma)$ in $\{e_i\}$ has a matrix (3.27) in $\{e_i\}$. □

Now we are ready to prove a fundamental result: **there exists a bijective mapping between the sets of C^1-linear transports along paths and derivations along paths**. *The explicit correspondence between linear transports along paths and derivations along paths is through the equality of their local coefficients and components, respectively, in a given field of bases.* After the prove of this result, we shall illustrate it in a case of linear connections on a manifold.

Proposition 3.7. *A mapping* (2.1a) (*resp.* (2.1c)) *is a derivation along paths* (*resp. along* γ) *iff there exists a unique linear transport along paths* (*resp. along* γ) *generating it via* (3.19). *Besides, the components and coefficients of corresponding in this way, respectively, derivation and linear transport along paths are equal.*

Proof. Let $\{e_i(s;\gamma)\}$ be a frame along γ and D (resp. D^γ) be a derivation along paths (resp. along γ). Define the *components* $\Gamma^i{}_j(s;\gamma)$ of D^γ in $\{e_i\}$ by the expansion (2.3) in which we consider $\hat{e}_i\colon \gamma \mapsto e_i(\,\cdot\,;\gamma)$ as a lifting of paths. They uniquely define D^γ as (2.2) implies (3.23) (see the identical equality (2.4)). Besides, it is trivial to verify the transformation law (3.26) for them. So, by Proposition 3.6, there is a unique linear transport along paths (resp. along γ) with the same local coefficients.

Conversely, as we already proved, to any linear transport L along paths (resp. along γ) there corresponds a derivation D^γ along γ given via (3.19) whose components coincide with the coefficients of L^γ and transform according to (3.26). □

Example 3.1. Let ∇ be a linear connection (see Definition I.3.1) on a C^1 differentiable manifold M and $\Gamma^i{}_{jk}(p)$, $i,j,k = 1,\dots,\dim M$, $p \in M$, be its local coefficients in a field $\{E_i(p)\}$ of bases in the tangent bundle over M, i.e., $\nabla_{E_i} E_j = \Gamma^k{}_{ji} E_k$. If γ is a C^1 path in M, the covariant derivative $\nabla_{\dot\gamma} = \dot\gamma^i \nabla_{E_i}$ along γ (see Section III.10, in particular equations (III.10.4) and (III.10.5)), $\dot\gamma$ being the vector field tangent to γ, is a derivation along γ in the tensor bundles over M and its local components are

$$\Gamma^i{}_j(s;\gamma) = \Gamma^i{}_{jk}(\gamma(s))\dot\gamma^k(s). \tag{3.29}$$

Exercise 3.2. Verify that *the unique linear transport along paths, corresponding, in accordance with Proposition* 3.7, *to the derivation with local components given by* (3.29), *along* $\gamma\colon J \to M$ *from* s *to* t, *with* $s,t \in J$ *and* $s \le t$, *is exactly the* *parallel transport along* $\gamma|_{[s,t]}$ *generated via the initial connection* ∇. (See Definition I.3.2 and, for some details, Section 11.)

Exercise 3.3. Prove that, given on the tangent bundle over M a derivation D along paths with local components (3.29), then $\Gamma^i{}_{jk}$ are the coefficients of a linear connection ∇, which, when restricted to the tangent bundle, is such that $(\nabla_V U)|_p = D^\gamma_s(U \circ \gamma)$, where $\gamma\colon J \to M$ is any C^1 path with $\dot\gamma(s) = V_p$, $\gamma(s) = p \in M$ for some $s \in J$, and U, V are vector fields on M.

Example 3.2. Consider a concrete kind of a linear transports L in the trivial line bundle $(B\times\mathbb{R}, \mathrm{pr}_1, B)$, where B is a topological space, which in particular can be a C^0 manifold, $\times$ is the Cartesian product sign, and $\mathrm{pr}_1\colon B\times\mathbb{R} \to B$ is the projection on B. An element of $B\times\mathbb{R}$ is of the form $u = (b, y)$ for some $b \in B$ and $y \in \mathbb{R}$ and the fibre over $c \in B$ is $\mathrm{pr}_1^{-1}(c) = \{c\} \times \mathbb{R} = \{(c, z) : z \in \mathbb{R}\}$; the linear structure of $\mathrm{pr}_1^{-1}(c)$ is given by $\lambda_1(c, z_1) + \lambda_2(c, z_2) = (c, \lambda_1 z_1 + \lambda_2 z_2)$ for $\lambda_1, \lambda_2, z_1, z_2 \in \mathbb{R}$. The bundle $(B \times \mathbb{R}, \mathrm{pr}_1, B)$ admits a global frame field $\{e_1\}$ consisting of a single section $e_1 \in \mathrm{Sec}(B \times \mathbb{R}, \mathrm{pr}_1, B)$ such that $e_1\colon B \ni b \mapsto e_1(b) = (b, 1) \in \mathrm{pr}_1^{-1}(b)$. For $\gamma\colon J \to B$ and $s, t \in J$, define $L\colon \gamma \mapsto L^\gamma\colon (s,t) \mapsto L^\gamma_{s\to t}\colon \mathrm{pr}_1^{-1}(\gamma(s)) \to \mathrm{pr}_1^{-1}(\gamma(t))$ by

$$L^\gamma_{s\to t}(u) = \Bigl(\gamma(t), \frac{f(\gamma(s))}{f(\gamma(t))}y\Bigr) \qquad \text{for } u = (\gamma(s), y) \in \mathrm{pr}_1^{-1}(\gamma(s)), \tag{3.30}$$

where $f\colon \gamma(J) \to \mathbb{R}\setminus\{0\}$ is a non-vanishing function on $\gamma(J)$. The verification of (3.2)-(3.4) is trivial and hence L is a linear transport along paths. Its matrix in the frame $\{e_1\}$ is $\boldsymbol{L}(t, s; \gamma) = L^1{}_1(t, s; \gamma) = \frac{f(\gamma(s))}{f(\gamma(t))}$, in conformity with (3.15). If $f\circ\gamma\colon J \to \mathbb{R}\setminus\{0\}$ is of class C^1, the single coefficient of L is (see (3.24)) $\Gamma^1{}_1(s;\gamma) = \frac{\mathrm{d}}{\mathrm{d}s}\ln(f(\gamma(s)))$; however, this coefficient is a useful quantity if $B\times\mathbb{R}$ (and hence B) is a C^1 manifold – see (3.23). Going some pages ahead (see Proposition 5.2 and Definition 4.4 below), we see that the transport L satisfies equation (5.2) below and therefore admits normal frames; in particular the frame $\{f_1\}$ such that (see (5.3) below)

$$f_1|_{\gamma(s)} = L^\gamma_{s_0\to s}\bigl(e_1|_{\gamma(s_0)}\bigr) = \Bigl(\gamma(s), \frac{f(\gamma(s_0))}{f(\gamma(s))}\Bigr)$$

for a fixed $s_0 \in J$ and any $s \in J$ is normal along γ, i.e., the matrix of L in $\{f_1\}$ is the identity matrix (the number one in the particular case).

4. Normal frames for linear transports

The parallel transport in a Euclidean space $\mathbb{E}^n$ (or in $\mathbb{R}^n$) has the property that, in Cartesian coordinates, it preserves the components of the vectors that are transported, changing only their initial points [3]. This evident observation, which can be taken even as a definition for parallel transport in $\mathbb{E}^n$, is of fundamental importance when one tries to generalize the situation.

Let L be a linear transport along paths in a vector bundle (E, π, B), $U \subseteq B$ be an arbitrary subset in B, and $\gamma\colon J \to U$ be a path in U.

Definition 4.1. A *frame* (*field of bases*) in $\pi^{-1}(\gamma(J))$ is called *normal along* γ *for* L if the matrix of L in it is the identity matrix along the given path γ.

Definition 4.2. A *frame field* (*of bases*) defined on U is called *normal on* U *for* L if it is normal along every path $\gamma\colon J \to U$ in U. The *frame* is called *normal for* L if $U = B$.

Notice that 'normal' refers to a 'normal form' as opposed to orthogonal to tangential.

In the context of the present book, we pose the following problem. Given a linear transport along paths, is it possible to find a local basis or a field of bases (frame) in which its matrix is the identity one? Below we shall rigorously formulate and investigate this problem. If frames with this property exist, we call them *normal* (for the transport given). According to (3.12), the linear transports do not change vectors' components in such a frame and, conversely, a frame with the last property is normal. Hence the normal frames are a straightforward generalization of the Cartesian coordinates in Euclidean space.[1] Because of this and following the established terminology with respect to metrics [11, 12], we call *Euclidean* a linear transport admitting normal frame(s).

Since a frame field on an arbitrary set U is actually a basis in the set $\mathrm{Sec}\big((E, \pi, B)|_U\big) = \mathrm{Sec}(\pi^{-1}(U), \pi|_U, U)$, we call such a basis *normal* if the corresponding field of bases is normal on U.

Remark 4.1. It should be emphasized that a frame normal on U must be defined on U but outside U (if $U \neq M$), i.e., at points in $M \backslash U$, it may not be defined. In this aspect the frames normal for linear transports differ from the ones for derivations along vector fields, in particular for linear connections, in which case they must be defined on an open subset of M containing or equal to U. This difference comes from the fact that in the definitions of the normal frames in the latter case are involved implicitly derivations with respect to some frames or local coordinates which derivations are well defined only on neighborhoods, while in the former case appear only derivations along paths.

Definition 4.3. A *linear transport along paths* (or *along a path* γ) is called *Euclidean along some* (*or the given*) *path* γ if it admits a frame normal along γ.

Definition 4.4. A *linear transport along paths* is called *Euclidean on* U if it admits frame(s) normal on U. It is called *Euclidean* if $U = B$.

[1] According to the argument presented, it is more natural to call Cartesian the special kind of local bases (or frames) we are talking about. But, in our opinion and for historical reasons, it is better to use the already established terminology for linear connections and derivations of the tensor algebra over a differentiable manifold (see below and [97, appendix A] or [83]).

We want to note that the name "Euclidean transport" is connected with the fact that if we put $B = \mathbb{R}^n$ and $\pi^{-1}(p) = T_p(\mathbb{R}^n)$ (the tangent space to $\mathbb{R}^n$ at p) and identify $T_p(\mathbb{R}^n)$ with $\mathbb{R}^n$, then in an orthonormal frame, i.e., in Cartesian coordinates, the Euclidean transport coincides with the standard parallel transport in $\mathbb{R}^n$ (leaving the vectors' components unchanged).

Example 4.1. Euclidean transports exist always in a case of a trivial bundle $(B \times V, \mathrm{pr}_1, B)$, with V being a vector space and $\mathrm{pr}_1 : B \times V \to B$ being the projection on B; cf. Example 3.2. For instance, the mapping $L^\gamma_{s\to t}(\gamma(s), v) = (\gamma(t), v)$, for $v \in V$, defines a Euclidean transport which is similar to the parallel one in $\mathbb{R}^n$. Indeed, if $\{f_i : i = 1, \dots, , \dim V\}$ is a basis of V and $v = v^i f_i$, then $e_i : p \mapsto e_i|_p := (p, f_i)$, $p \in B$, is a (global) frame on B if we put $v^i e_i|_p = (p, v^i f_i) = (p, v)$ and therefore $L^\gamma_{s\to t}(e_i|_{\gamma(s)}) = e_i|_{\gamma(t)}$, which means that $L : \gamma \mapsto L^\gamma : (s,t) \mapsto L^\gamma_{s\to t}$ is a Euclidean transport and $\{e_i\}$ is a normal frame for it (see Corollary 4.1 below).

Below we present some general results concerning normal frames leaving the problem of their existence for the next section.

The importance of normal frames is established by the following result.

Proposition 4.1. *The following statements are equivalent in a given frame $\{e_i\}$ over $U \subseteq B$:*

(i) *The matrix of L is the identity (unit) matrix on U, i.e., along every path γ in U:*

$$\boldsymbol{L}(t, s; \gamma) = \mathbb{1}. \tag{4.1a}$$

(ii) *The matrix of L along every $\gamma : J \to U$ depends only on γ, i.e., it is independent of the points at which it is calculated:*

$$\boldsymbol{L}(t, s; \gamma) = C(\gamma) \tag{4.1b}$$

where C is a matrix-valued function of γ.

(iii) *If E is a C^1 manifold, the coefficients $\Gamma^i{}_j(s;\gamma)$ of L vanish on U, i.e., along every path γ in U*

$$\boldsymbol{\Gamma}(s;\gamma) = 0. \tag{4.1c}$$

(iv) *The explicit local action of the derivation D along paths generated by L reduces on U to differentiation of the components of the liftings with respect to the path's parameter if the path lies entirely in U:*

$$D^\gamma_s \lambda = \frac{\mathrm{d}\lambda^i_\gamma(s)}{\mathrm{d}s}\, e_i(s;\gamma) \tag{4.1d}$$

where $\lambda = \lambda^i e_i \in \mathrm{PLift}^1\big((E,\pi,B)|_U\big)$, with E being a C^1 manifold, and $\lambda : \gamma \mapsto \lambda_\gamma$.

(v) *The transport L leaves the vectors' components unchanged along any path in U:*

$$L^{\gamma}_{s\to t}\big(u^i e_i(s;\gamma)\big) = u^i e_i(t;\gamma) \tag{4.1e}$$

where $u^i \in \mathbb{K}$.

(vi) *The basic vector fields are L-transported along any path $\gamma\colon J \to U$:*

$$L^{\gamma}_{s\to t}\big(e_i(s;\gamma)\big) = e_i(t;\gamma). \tag{4.1f}$$

Proof. We have to prove the equivalences

$$\begin{aligned}
\boldsymbol{L}(t,s;\gamma) = C(\gamma) \iff \boldsymbol{L}(t,s;\gamma) = \mathbb{1} \iff \boldsymbol{\Gamma}(s;\gamma) = 0 \\
\iff D^{\gamma}_{s}\lambda = \frac{\mathrm{d}\lambda^i_{\gamma}(s)}{\mathrm{d}s} e_i(s;\gamma) \iff L^{\gamma}_{s\to t}\big(u^i e_i(s;\gamma)\big) = u^i e_i(t;\gamma) \\
\iff L^{\gamma}_{s\to t}\big(e_i(s;\gamma)\big) = e_i(t;\gamma).
\end{aligned} \tag{4.2}$$

If $\boldsymbol{L}(t,s;\gamma) = C(\gamma)$, then, using the representation (3.15), we get $\boldsymbol{F}(t;\gamma) = \boldsymbol{F}(s;\gamma)C(\gamma) = \boldsymbol{F}(s_0;\gamma)$ for some fixed $s_0 \in J$ as the points s and t are arbitrary, so $\boldsymbol{L}(t,s_0;\gamma) = \boldsymbol{F}^{-1}(s_0;\gamma)\boldsymbol{F}(s_0;\gamma) = \mathbb{1}$. The inverse implication is trivial. The second equivalence is a consequence of (3.25) and (3.15) since $\boldsymbol{\Gamma} = 0$ implies $\boldsymbol{F}(s;\gamma) = \boldsymbol{F}(\gamma)$, while the third one is a corollary of (3.23). The validity of the last but one equivalence is a consequence of $\boldsymbol{L}(t,s;\gamma) = \mathbb{1} \iff L^{\gamma}_{s\to t}\big(u^i e_i(s;\gamma)\big) = u^i e_i(t;\gamma)$ which follows from (3.12). The last equivalence is a corollary of the linearity of L and the arbitrariness of u^i. □

Remark 4.2. An evident corollary of the last proof is

$$\boldsymbol{L}(t,s;\gamma) = \mathbb{1} \iff \boldsymbol{F}(s;\gamma) = \boldsymbol{B}(\gamma) \tag{4.3}$$

with $\boldsymbol{B}$ being a matrix-valued function of the path γ only. According to Proposition 3.5, this dependence is inessential and, consequently, in a normal frame, we can always choose representation (3.15) with

$$\boldsymbol{F}(s;\gamma) = \mathbb{1}. \tag{4.4}$$

Corollary 4.1. *Any one of the equalities* (4.1a)–(4.1f) *express a necessary and sufficient condition for a frame to be normal for L in U.*

Proof. This result is a direct consequence of Definition 4.2 and Proposition 4.1. □

Proposition 4.2. *The equations* (4.1a)–(4.1f) *are equivalent in a given frame $\{e_i\}$ along a* (*fixed*) *path $\gamma\colon J \to B$.*

Proof. This proof is identical with the one of Proposition 4.1 for $U = \gamma(J)$. □

Corollary 4.2. *A frame is normal along γ for L if and only if in that frame one* (*and hence all*) *of the equalities* (4.1a)–(4.1f) *is* (*are*) *valid.*

Proof. The result follows from Definition 4.1 and Proposition 4.2. □

In particular, a frame is normal for L along γ iff in the frame the coefficients of L vanish along γ, i.e., iff (4.1c) holds.

A lifting of paths $\lambda \in \mathrm{PLift}(E, \pi, B)$ is called L-transported along $\gamma\colon J \to B$, if for every $s, t \in J$ is fulfilled

$$\lambda_\gamma(t) = L^\gamma_{s\to t}\lambda_\gamma(s)$$

with $\lambda\colon \gamma \mapsto \lambda_\gamma$. Hence a frame $\{e_i(s, \gamma)\}$ along γ is L-transported along γ if the liftings $\hat{e}_1, \dots, \hat{e}_{\dim B}$, defined via (2.8), are L-transported along γ.

Therefore a frame is normal for L along γ iff it is L-transported along γ, i.e., if, by definition, its basic vectors $e_i(s; \gamma)$ satisfy (4.1f). As we shall see below (see Proposition 4.4), this allows a convenient and useful way for constructing normal frames, if any.

For the above reasons, sometimes, it is convenient for the Definition 4.1 to be replaced, equivalently, by the next ones.

Definition 4.1′. If E is a C^1 manifold, a *frame* (*or frame field*) over $\gamma(J)$ is called *normal along* $\gamma\colon J \to B$ for a linear transport L along paths if the coefficients of L along γ vanish in that frame.

Definition 4.1″. A *frame* over $\gamma(J)$ is called *normal along* $\gamma\colon J \to B$ for a linear transport L along paths if it is L-transported along γ.

The last definition of a normal frame is, in a sense, the 'most invariant (basis-free)' one.

The next proposition describes the class of normal frames, if any, along a given path.

Proposition 4.3. *All frames normal for some linear transport along paths which is Euclidean along a certain* (*fixed*) *path are connected by linear transformations whose matrices may depend only on the given path but not on the point at which the bases are defined.*

Proof. Let $\{e_i\}$ and $\{e'_i := A^j_i e_j\}$ be frames normal along $\gamma\colon J \to B$ for a linear transport L along paths and $\boldsymbol{L}$ and $\boldsymbol{L}'$ be the matrices of L in them respectively. As, by definition $\boldsymbol{L} = \boldsymbol{L}' = \mathbb{1}$, from (3.11), we get $A(s; \gamma) = A(t; \gamma)$ for any $s, t \in J$, i.e., $A(s; \gamma)$ depends only on γ and not on s.

If E is a C^1 manifold and $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}'$ are the matrices of the coefficients of L in $\{e_i\}$ and $\{e'_i\}$, respectively, by Proposition 4.1 we have $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}' = 0$, so the transformation law (3.26) implies $\mathrm{d}A(s; \gamma)/\mathrm{d}s = 0$, $A(s; \gamma) := \left[A^j_i\left(s; \gamma\right)\right]$. □

Corollary 4.3. *All frames normal for a Euclidean transport along a given path are obtained from one of them via linear transformations whose matrices may depend only on the path given but not on the point at which the bases are defined.*

Proof. See Proposition 4.3 or its proof. □

The following two results describe the class of all frames normal on an arbitrary set U, if such frames exist.

Corollary 4.4. *If a linear transport along paths admits frames normal on a set U, then all of them are connected via linear transformations with constant (on U) matrices.*

Proof. Let $\{e_i\}$ and $\{e'_i := A^j_i e_i\}$ be frames normal on U and $p \in U$. By Proposition 4.3 (see also Definition 4.2) for any paths β and γ in U passing though p, we have $A(p) := \left[A^j_i\right] = \boldsymbol{B}(\beta) = \boldsymbol{B}(\gamma)$ for some matrix-valued function $\boldsymbol{B}$ on the set $\mathrm{P}(U)$ of the paths in U. Hence $A(x) = \mathrm{const}$ on U, due to the arbitrariness of β and γ. □

Corollary 4.5. *If a linear transport along paths admits a frame normal on a set U, then all such frames on U for it are obtained from that frame by linear transformations with constant (on U) coefficients.*

Proof. The result immediately follows from Corollary 4.4 □

We end this section with a simple but important result which shows how the normal frames, if any, can be constructed along a given path.

Proposition 4.4. *If L is Euclidean transport along $\gamma\colon J \to B$ and $\{e^0_i\}$ is a basis in $\pi^{-1}(\gamma(s_0))$ for some $s_0 \in J$, then the frame $\{e_i\}$ along γ defined by*

$$e_i(s;\gamma) = L^{\gamma}_{s_0\to s}\left(e^0_i\right), \qquad s \in J \tag{4.5}$$

is normal for L along γ.

Proof. Due to (3.2) and (4.5), the frame $\{e_i\}$ satisfies (4.1f) along γ. Hence, by Corollary 4.2, it is normal for L along γ. □

An analogous result on a set $U \subseteq B$ will be presented in the next section (see Proposition 5.5 below).

5. On the existence of normal frames

In the previous section there were derived a number of properties of the normal frames, but the problem of their existence was neglected. This is the subject of the present section.

Following the ideas of the previous chapters, one may attack the problem of existence of frames normal for a linear transport L along paths as follows. Suppose $\{e_i\}$ is a frame on $U \subseteq B$ (resp. along γ) and $\boldsymbol{\Gamma}(s;\gamma)$ is the matrix of the coefficients of L along $\gamma\colon J \to U$ (resp. along the given path γ). Then L admits a frame $\{e'_i\}$ normal on U (resp. along γ) iff there is a matrix-valued function $A :=$

$[A^i_j]\colon (s,\gamma) \mapsto A(s;\gamma) = A(\gamma(s))$ (resp. $A\colon (s,\gamma) \mapsto A(s;\gamma)$) transforming $\{e_i\}$ into $\{e'_i\}$, $e'_i(s;\gamma) = A^j_i(s;\gamma)e_j(s;\gamma)$, and satisfying, by virtue of (4.1c) and (3.26), the equation

$$\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s} + \mathbf{\Gamma}(s;\gamma)A(s;\gamma) = 0 \tag{5.1}$$

for every $\gamma\colon J \to U$ and $s \in J$ (resp. for the given path $\gamma\colon J \to B$ and $s \in J$). This assertion is also an obvious corollary of the transformation law (3.26) and Definition 4.1′ (or Proposition 4.1, points (i) and (iii)). Therefore all problems concerning the properties and existence of frames normal on U (resp. along γ) can be reduced to the investigation of the equation (5.1) with $A(s;\gamma) = A(\gamma(s))$, $\gamma\colon J \to U$ (resp. with A depending, generally, separately on s and γ); for this, we call (5.1) the *normal frame equation* along γ for L. We shall comment on this approach in Subsection 13.4. Below we shall follow other, more direct, methods which are primary related to the ('global') general properties of the linear transports along paths.

At a given point $p \in B$ the following result is valid.

Proposition 5.1. *A linear transport L^γ along $\gamma\colon J \to B$ such that $\gamma(J) = \{p\}$ for a given point $p \in B$ admits normal frame(s) iff it is the identity mapping of the fibre over p, i.e., $L^\gamma_{s\to t} = \mathsf{id}_{\pi^{-1}(p)}$ for every $s, t \in J$.*

Proof. The sufficiency is trivial (see Definition 3.1). If $\{e_i\}$ is normal for L^γ (at p), then $L^\gamma_{s\to t}(u^i e_i|_p) = u^i L^\gamma_{s\to t} e_i|_p = u^i e_i|_p$, $u^i \in \mathbb{K}$, due to $\gamma(s) = \gamma(t) = p$ and Proposition 4.1, point (iv). Therefore $L^\gamma_{s\to t} = \mathsf{id}_{\pi^{-1}(p)}$. □

Thus, for a degenerate path $\gamma\colon J \to \{p\} \subset B$ for some $p \in B$, the identity mapping of the fibre over p is the only realization of a Euclidean transport along paths. Evidently, for such a transport every basis of that fibre is a frame normal at p for it.

Proposition 5.2. *A linear transport L along paths admits frame(s) normal along a given path $\gamma\colon J \to B$ iff*

$$L^\gamma_{s\to t} = \mathsf{id}_{\pi^{-1}(\gamma(s))} \qquad \text{for every } s, t \in J \text{ such that } \gamma(s) = \gamma(t), \tag{5.2}$$

i.e., if γ contains loops, the L-transport along each of them reduces to the identity mapping of the fibre over the initial/final point of the transportation.

Remark 5.1. For $s = t$ the equation (5.2) is identically satisfied due to (3.3). But for $s \neq t$, if such s and t exist, this is highly non-trivial restriction: it means that the result of L-transportation along γ of a vector $u \in \pi^{-1}(x_0)$ for some $x_0 \in \gamma(J)$ from x_0 to a point $x \in \gamma(J)$ is independent of how long the vector has 'travelled' along γ or, more precisely, if $x_0, x \in \gamma(J)$ are fixed and, for each $y \in \gamma(J)$, $J_y := \{r \in J : \gamma(r) = y\}$, then the vector $L^\gamma_{s_0\to s}(u)$ is independent of the choice of the points $s_0 \in J_{x_0}$ and $s \in J_x$ (if some of the sets J_{x_0} and/or J_x contain more than one point). This is trivial if γ is without self-intersections (see (3.2)). If γ has self-intersections, e.g., if γ intersects itself one time at $\gamma(s)$, i.e., if $\gamma(s) = \gamma(t)$ for some $s, t \in J,\ s \neq t$, then the result of L-transportation of $u \in \pi^{-1}(\gamma(s_0))$ from $p_0 = \gamma(s_0)$ to $p = \gamma(s) = \gamma(t)$ along γ is $u_s = L^{\gamma}_{s_0 \to s} u$ or $u_t = L^{\gamma}_{s_0 \to t} u$. We have $u_s = u_t$ iff (5.2) holds. Rewording, if we fix some $u_0 \in \pi^{-1}(\gamma(s_0))$, the bundle-valued function $u\colon \gamma(J) \to E$ given by $u\colon \gamma(s) \to u_s = L^{\gamma}_{s_0 \to s} u_0 \in \pi^{-1}(\gamma(s))$ for $s \in J$ is single-valued iff (5.2) is valid.[1] Notice, since $\pi \circ u_s \equiv \gamma(s)$ (see (3.1)), the mapping u is (a single-valued) lifting of γ in E through u_0 irrespectively of the validity of (5.2).

Prima facie the above may be reformulated in terms of the concept of holonomy in vector bundles [23, pp. 51–54]. But a rigorous analysis reveals that this is impossible in the general case without imposing further restrictions, like equation (5.6) below, on the transports involved. For instance, without requiring equation (5.6) below to be valid, one cannot introduce the concept of a holonomy group.

Proof. If L is Euclidean along γ, then (5.2) follows from equation (4.1e) as it holds for every $u^i \in \mathbb{K}$ in some normal frame $\{e_i\}$. Conversely, let (5.2) be valid. Put

$$e_i|_{\gamma(s)} := L^{\gamma}_{s_0 \to s}(e_i^0) \tag{5.3}$$

where $\{e_i^0\}$ is a fixed basis in $\pi^{-1}(\gamma(s_0))$ for a fixed $s_0 \in J$. Due to the nondegeneracy of L, $\{e_i\}$ is a basis at $\gamma(s)$ for every s. According to (5.2), the so-defined field of bases $\{e_i\}$ along γ is single-valued. By means of (3.2), we easily verify that (4.1f) holds for $\{e_i\}$. Hence $\{e_i\}$ is normal for L along γ. □

Remark 5.2. Regardless of the validity of (5.2), equation (5.3) defines a field of, generally multiple-valued, normal frames in the set of sections along γ of (E, π, B). Such a multi-valued property can be avoid if γ is supposed to be injective ($\Leftrightarrow$ without self-intersections). *Prima facie* one may think that this solves the multi-valued problem in the general case by decomposing γ into a union of injective paths. However, this is not the most general situation because a transport along a composition of paths does not generally equal to the composition of the transports along its constituent sub-paths (see equation (5.6) below); besides, since equation (5.10) below does not hold generally, the absents of a natural/canonical definition of composition (product) of paths introduces an additional indefiniteness.

Corollary 5.1. *Every linear transport along paths is Euclidean along every fixed path without self-intersections.*

Proof. For a path $\gamma\colon J \to B$ without self-intersections, the equality $\gamma(s) = \gamma(t)$, $s, t \in J$ is equivalent to $s = t$. So, according to (3.3), the condition (5.2) is identically satisfied. □

Now we shall establish an important necessary and sufficient condition for the existence of frames normal on an arbitrary subset $U \subseteq B$.

[1] The so-defined mapping u is a multiple-valued section along γ (see Subsection 2.3).

Theorem 5.1. *A linear transport along paths admits frames normal on some set (resp. along a given path) if and only if its action along every path in this set (resp. along the given path) depends only on the initial and final points of the transportation but not on the particular path connecting these points. In other words, a transport is Euclidean on $U \subseteq B$ iff it is path-independent on U.*

Proof. Let a linear transport L admit a frame $\{e_i\}$ normal on $U \subseteq B$. By (3.12) and Definitions 4.1 and 4.2, this implies $L^{\gamma}_{s\to t} u^i(\gamma(s))\big(e_i|_{\gamma(s)}\big) = u^i(\gamma(s)) e_i|_{\gamma(t)}$ for $\gamma\colon J \to U$ and $u(p) \in \pi^{-1}(p)$, $p \in B$. Conversely, let $L^{\gamma}_{s\to t} u(\gamma(s))$ depend only on $\gamma(s)$ and $\gamma(t)$ but not on γ and $\{e_i\}$ be a field of bases on U (resp. on $\gamma(J)$). Then, due to (3.12), the matrix $\boldsymbol{L}$ of L in $\{e_i\}$ has the form $\boldsymbol{L}(t,s;\gamma) = \boldsymbol{B}(\gamma(t),\gamma(s))$ for some matrix-valued function $\boldsymbol{B}$ on $U \times U$. Combining this result with Propositions 3.4 and 3.5, we see that $\boldsymbol{L}$ admits a representation

$$\boldsymbol{L}(t,s;\gamma) = \boldsymbol{F}_0^{-1}(\gamma(t)) \boldsymbol{F}_0(\gamma(s)), \qquad s,t \in J \tag{5.4}$$

for some non-degenerate matrix-valued function $\boldsymbol{F}_0$ on the set U. Putting $e'_i|_p = \big(\boldsymbol{F}_0^{-1}(p)\big)^j_{\ i} e_j|_p$, $p \in U$, we obtain from (3.11) that the matrix of L in $\{e'_i\}$ is $\boldsymbol{L}'(t,s;\gamma) = \mathbb{1}$, i.e., the frame $\{e'_i\}$ is normal for L on U. □

An evident corollary of Theorem 5.1 is the following assertion. Let a linear transport L be Euclidean on $U \subseteq B$ and $h_a\colon J \to U$, $a \in [0,1]$, be a homotopy of paths passing through two fixed points $p, q \in U$, i.e., $h_a(s_0) = p$ and $h_a(t_0) = q$ for some $s_0, t_0 \in J$ and any $a \in [0,1]$. Then $L^{h_a}_{s_0\to t_0}$ is independent of $a \in [0,1]$. In particular, we have $L^{h_a}_{s_0\to t_0}\big|_{q=p} = \mathsf{id}_{\pi^{-1}(p)}$ owing to Proposition 5.2.

Equation (5.4) and the part of the proof of Theorem 5.1 after it are a hint for the formulation of the following result.

Theorem 5.2. *A linear transport L along paths in a vector bundle, with C^1 manifold as a bundle space, is Euclidean on U (resp. along γ) iff for some, and hence for every, frame $\{e_i\}$ on U (resp. on $\gamma(J)$) there exists a matrix-valued function $\boldsymbol{F}_0$ on U such that the matrix $\boldsymbol{L}$ of L in $\{e_i\}$ is given by* (5.4) *for every $\gamma\colon J \to U$ (resp. for the given γ) or, equivalently, iff the matrix $\boldsymbol{\Gamma}$ of the coefficients of L in $\{e_i\}$ is*

$$\boldsymbol{\Gamma}(s;\gamma) = \boldsymbol{F}_0^{-1}(\gamma(s)) \frac{\mathrm{d}\boldsymbol{F}_0(\gamma(s))}{\mathrm{d}s}. \tag{5.4$'$}$$

Proof. Suppose a linear transport L is Euclidean. There is a frame $\{e^0_i\}$ normal for L on U (resp. along γ). Define a matrix $\boldsymbol{F}_0(p)$ via the expansion $e_i|_p = \big(\boldsymbol{F}_0(p)\big)^j_{\ i} e^0_j\big|_p$, $p \in U$. Since, by definition, the matrix of L in $\{e^0_i\}$ is the unit (identity) matrix on U, the matrix of L in $\{e_i\}$ is given via (5.4) due to (3.11). Conversely, if (5.4) holds in $\{e_i\}$ on U, the frame $\{e'_i|_p = \big(\boldsymbol{F}_0^{-1}(p)\big)^j_{\ i} e^0_j\big|_p\}$ is normal for L on U (resp. along γ), as we saw at the end of the proof of Theorem 5.1. The equivalence of (5.4′) and (5.4) is a consequence of (3.24) (cf. (3.25), (3.26), and (4.2)). □

Remark 5.3. An alternative proof of theorem follows directly from equation (5.1): we have $A(s;\gamma) = A(\gamma(s))$ and (5.4′) is valid for $\boldsymbol{F}_0 = A^{-1}$.

The proof of Theorem 5.2 suggest a way for generating Euclidean transports along paths by 'inverting' the definition of normal frames: take a given field of bases over $U \subseteq B$ and define a linear transport by requiring its matrix to be the identity matrix in the given field of bases. More precisely, we have in mind the following. Let $\{e_i\}$ be a fixed frame on U, $\{e'_i = A^j_i e_j\}$, with $A = \left[A^j_i\right]$ being non-degenerate, be arbitrary frame on U, and $\gamma\colon J \to U$ be a path in U. Define a linear mapping (3.1) by its matrix in $\{e'_i\}$ (see (3.12)):

$$\boldsymbol{L}(t,s;\gamma) = A(\gamma(t))A^{-1}(\gamma(s)), \qquad A(p) := \left[A^j_i(p)\right], \quad p \in U. \tag{5.5}$$

According to Proposition 3.4, the mapping $L\colon \gamma \mapsto L^\gamma\colon (s,t) \mapsto L^\gamma_{s\to t}$ is a linear transport along paths. By Theorem 5.1, this transport is Euclidean. Moreover, from (3.11), we see that the matrix of L in $\{e_i\}$ is unit on U, i.e., $\{e_i\}$ is a frame normal for L on U. We call this Euclidean transport *generated by (or assigned to)* the given initial frame, which is normal for it.

Proposition 5.3. *All frames normal for a Euclidean linear transport along paths in U generate one and the same Euclidean transport along paths in U coinciding with the initial one.*

Proof. The result is an almost evident consequence of (5.5) and Corollary 4.5. □

Proposition 5.4. *Two or more frames on U generate one and the same Euclidean transport along paths iff they are connected via linear transformations with constant* (*on U*) *coefficients.*

Proof. If $\{e_i\}$ and $\{e'_i\}$ generate L, then they are normal for it (Proposition 5.3) and, by Corollary 4.4, they are connected in the way pointed. The converse is a trivial corollary of (5.5) for $A(p) = \text{const}$ with $p \in U$. □

In this way we have established a bijective correspondence between the set of Euclidean linear transports along paths in U and the class of sets of frames on U connected by linear transformations with constant coefficients.

The comparison of Proposition 5.2 with Theorem 5.1 suggests that a transport is Euclidean in $U \subseteq B$ iff (5.2) holds for every $\gamma\colon J \to U$. But this is not exactly the case. The right result is the following one.

Theorem 5.3. *A linear transport L along paths is Euclidean on some path-connected set*[2] $U \subseteq B$ *iff the next three conditions are valid:*

(i) *Equation* (5.2) *holds for every continuous path* $\gamma\colon J \to U$*;*

[2]A set is path-connected if every two its points can be connected by a continuous path lying entirely in it. Sometimes such sets are called linearly connected or arc-connected.

(ii) *The transport along a product of paths is equal to the composition of the transports long the paths of the product, i.e.,*

$$L^{\gamma_1\gamma_2} = L^{\gamma_2} \circ L^{\gamma_1} \tag{5.6}$$

where γ_1 *and* γ_2 *are paths in* U *such that the end of* γ_1 *coincides with the beginning of* γ_2 *and* $\gamma_1\gamma_2$ *is the product of these paths;*

(iii) *For any subinterval* $J' \subseteq J$ *the locality condition*

$$L^{\gamma|J'}_{s\to t} = L^{\gamma}_{s\to t}, \qquad s,t \in J' \subseteq J, \tag{5.7}$$

with $\gamma|J'$ *being the restriction of* $\gamma\colon J \to U$ *to* J'*, is valid.*

Remark 5.4. Here and below we do not present and use a particular definition of the product of paths. There are slightly different versions of that definition; for details see [108,132] or [114, Section 3]. Our results are independent of any concrete such definition because the transports, we are considering here, are independent of the particular path they are acting along (see Theorem 5.1).

Proof. If L is Euclidean, then, by Definition 4.4, it admits normal frame(s) along every $\gamma\colon J \to U$ and, consequently, according to Proposition 5.2, the condition (5.2) is valid along every $\gamma\colon J \to U$. By Theorem 5.1, the transport $L^{\gamma}_{s\to t}$, $s,t \in J$ depends only on the points $p = \gamma(s)$ and $q = \gamma(t)$ but not on the particular path γ connecting $p, q \in U$. Equations (5.6) and (5.7) follow from here.

Conversely, let (5.2), (5.6), and (5.7) be true for all paths γ, γ_1, and γ_2 in U, the end of γ_1 coinciding with the beginning of γ_2, and subinterval $J' \subseteq J$. Meanwhile, we notice the equality

$$L^{\gamma^{-1}} = \bigl(L^{\gamma}\bigr)^{-1}, \tag{5.8}$$

γ^{-1} being the path inverse to γ,[3] which is a consequence of (5.2) and (5.6).

Let p_0 be arbitrarily chosen fixed point in U and $\{e_i^0\}$ an arbitrarily fixed basis in the fibre $\pi^{-1}(p_0)$ over it. In the fibre $\pi^{-1}(p)$ over $p \in U$, we define a basis $\{e_i|_p\}$ via (cf. (5.3))

$$e_i|_p := L^{\gamma_{p_0,p}}_{s_0\to s}\bigl(e_i^0\bigr) \tag{5.9}$$

where $\gamma_{p_0,p}\colon J \to U$ is an arbitrary continuous path through p_0 and p, i.e., for some $s_0, s \in J$, we have $\gamma_{p_0,p}(s_0) = p_0$ and $\gamma_{p_0,p}(s) = p$. Below we shall prove that the field $\{e_i\}$ of bases over U is normal for L on U.

At first, we shall prove the independence of $e_i|_p$ from the particular continuous path $\gamma_{p_0,p}$. Let $\beta_a\colon J_a \to U$, $a = 1, 2$ and $\beta_a(s_a) = p_0$ and $\beta_a(t_a) = p$ for some $s_a, t_a \in J_a$, $a = 1, 2$. For definiteness, we assume $s_a \le t_a$. (The other combinations

[3]Here we do not need a particular definition of γ^{-1} (cf. Remark 5.4). More precisely, if $\gamma\colon [s,t] \to U$, and $\gamma^{-1}\colon [s',t'] \to U$, for $s,t,s',t' \in \mathbb{R}$, $s < t$, $s' < t'$, and $\gamma^{-1}(s') = \gamma(t)$, we shall apply (5.8) in the form $L^{\gamma^{-1}}_{s'\to t'} = \bigl(L^{\gamma}_{s\to t}\bigr)^{-1} = L^{\gamma}_{t\to s}$.

of ordering between s_1, t_1, s_2, and t_2 can be considered analogously.) Defining $\beta'_a := \beta_a|[s_a, t_a]$, $a = 1, 2$ and using (5.7), (5.8), (5.6), and (5.2), we get

$$L^{\beta_2}_{s_2\to t_2} \circ L^{\beta_1}_{t_1\to s_1} = L^{\beta'_2}_{s_2\to t_2} \circ L^{\beta'_1}_{t_1\to s_1} = L^{\beta'_2}_{s_2\to t_2} \circ L^{(\beta'_1)^{-1}}_{s_1\to t_1} = L^{(\beta'_1)^{-1}\beta'_2}_{s_0\to t_0} = \mathsf{id}_{\pi^{-1}(p)},$$

where $\left(\beta'_1\right)^{-1}\beta'_2\colon [s_0, t_0] \to U$ is the product of $\left(\beta'_1\right)^{-1}$ and β'_2 and we have used that, from the definition of $\left(\beta'_1\right)^{-1}$ and β'_2, is clear that $\left((\beta'_1)^{-1}\beta'_2\right)(s_0) = \left((\beta'_1)^{-1}\beta'_2\right)(t_0) = p$, i.e., $\left(\beta'_1\right)^{-1}\beta'_2$ is a closed path passing through p. Applying the last result, (3.2), and (3.3), we obtain:

$$L^{\beta_2}_{s_2\to t_2} e^0_i = \left(L^{\beta_2}_{s_2\to t_2} \circ L^{\beta_1}_{t_1\to s_1}\right) \circ \left(L^{\beta_1}_{s_1\to t_1}\right) e^0_i = L^{\beta_1}_{s_1\to t_1} e^0_i.$$

Since β_1 and β_2 are arbitrary, from here we conclude that the frame $\{e_i\}$, defined via (5.9) on U, is independent from the particular path used in (5.9).

Now we shall prove that $\{e_i\}$ is normal for L on U, which will complete this proof.

From the proof of Proposition 5.2 (compare (5.9) and (5.3)) follows that $\{e_i\}$ is normal for L along any path in U passing through p_0. Let $\gamma\colon J \to U$ be such a path, $s_0 \in J$ be fixed, and $\beta\colon [0, 1] \to U$ be such that $\beta(0) = p$ and $\beta(1) = \gamma(s_0) =: p_0$. Defining $\gamma_\pm := \gamma|J_\pm$ for $J_\pm := \{s \in J,\ \pm s \geq \pm s_0\}$, we conclude that $\{e_i\}$ is normal for L along $\beta\gamma_+$ and $\beta\gamma_-^{-1}$. Take, for example, the path $\beta\gamma_+$. If for some $s'_0, s', s^* \in \mathbb{R}$ is fulfilled $(\beta\gamma_+)(s'_0) = p$, $(\beta\gamma_+)(s') = \gamma(s)$, and $(\beta\gamma_+)(s^*) = p_0$, then, applying (5.9), (5.6), and (5.7), we find for $s \geq s_0$:

$$\begin{aligned} e_i|_{\gamma(s)} &= L^{\beta\gamma_+}_{s'_0\to s'}\left(e_i|_p\right) = L^{\beta\gamma_+}_{s'_0\to s'} \circ L^{\beta\gamma_+}_{s^*\to s'_0}\left(e_i|_{p_0}\right) = L^{\beta\gamma_+}_{s^*\to s'}\left(e_i\big|_{p_0}\right) \\ &= L^{\gamma_+}_{s_0\to s} \circ L^{\beta}_{0\to 1}\left(e_i|_{p_0}\right) = L^{\gamma_+}_{s_0\to s}\left(e_i|_p\right) = L^{\gamma}_{s_0\to s}\left(e_i|_p\right). \end{aligned}$$

Analogously one can prove that $e_i|_{\gamma(s)} = L^{\gamma}_{s_0\to s}\left(e_i|_p\right)$ for $s \leq s_0$ by using $\beta\gamma_-^{-1}$ instead of $\beta\gamma_+$. So, due to (3.2), the frame $\{e_i\}$ satisfies (4.1f) along γ. Consequently, by Corollary 4.1, the frame so-constructed is normal for L along γ. □

Remark 5.5. According to [114, Proposition 3.4], the equality (5.6) is a consequence of (5.7) and the reparametrization condition

$$L^{\gamma\circ\tau}_{s\to t} = L^{\gamma}_{\tau(s)\to\tau(t)}, \qquad s, t \in J'' \tag{5.10}$$

where J'' is an $\mathbb{R}$-interval and $\tau\colon J'' \to J$ is bijection. Hence in the formulation of Theorem 5.3 we can replace the condition (5.6) with (5.10). So, we have:

Theorem 5.3′. *A transport L is Euclidean on a path-connected set $U \subseteq B$ if* (5.2), (5.7), *and* (5.10) *are valid for every continuous path $\gamma\colon J \to U$.*

Remark 5.6. A transport along paths satisfying (5.7) and (5.10) is in fact a parallel transport (along paths). For details, see Section V.8.

The next result is analogous to Proposition 4.4. According to it, a frame normal for L on $U \subseteq B$, if any, can be obtained by L-transportation of a fixed basis over some point in U to the other points in U.

Proposition 5.5. *If L is a Euclidean transport on a path-connected set $U \subseteq B$ and $\{e_i^0\}$ is a given basis in $\pi^{-1}(p_0)$ for a fixed $p_0 \in U$, then the frame $\{e_i\}$ over U defined via*

$$e_i|_p = L^\gamma_{s_0 \to s}\big(e_i^0\big), \tag{5.11}$$

where $\gamma\colon J \to U$ is such that $\gamma(s_0) = p_0$ and $\gamma(s) = p$ for some $s_0, s \in J$, is normal for L on U.

Proof. By Theorem 5.1, the basis $\{e_i|_p\}$ is independent of the particular path γ used in (5.11). According to Theorem 5.3, the conditions (5.2), (5.6), and (5.7) hold for L. Further, repeating step-by-step the last paragraph of the proof of Theorem 5.3, we verify that $\{e_i\}$ is normal for L on U.

Alternatively, the assertion is a consequence of (3.21) and Proposition 5.6 presented a few lines below. □

A simple way to check whether a given frame is normal along some path is provided by the following proposition.

Proposition 5.6. *A frame $\{e_i\}$ along $\gamma\colon J \to B$ is normal for a linear transport L along paths in (E, π, B), E being a C^1 manifold, if and only if the liftings $\hat{e}_i\colon \gamma \mapsto e_i(\,\cdot\,, \gamma)$ are constant (along γ) with respect to the derivation D generated by L:*

$$D^\gamma \hat{e}_i = 0. \tag{5.12}$$

Proof. If $\{e_i\}$ is normal for L along γ, equation (4.1f) is valid (see Corollary 4.2), so (5.12) follows from (3.21). If (5.12) holds, by virtue of (3.21), its solution is[4] $e_i|_{\gamma(s)} = L^\gamma_{s_0 \to s}\big(e_i|_{\gamma(s_0)}\big)$ and consequently, by Proposition 4.4, the frame $\{e_i\}$ is normal along γ. □

Recall (see the remark preceding Definition 2.1), the path γ in Proposition 5.6 cannot be an arbitrary continuous path in B as it must be in the set $\pi \circ \mathrm{P}^k(E)$, with $\mathrm{P}^k(E)$, $k = 0, 1$, being the set of C^k paths in E. Notice, the derivative in (5.12) does not require B to be a manifold.

Of course, it is true that if (5.12) holds in a frame $\{e_i\}$ along every path γ in U, the frame $\{e_i\}$ is normal for L on U. But it is more natural to find a 'global' version of (5.12) concerning the whole set U, not the paths in it. The corresponding result is formulated below as Theorem 9.1 (see also Theorem 6.1).

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[4]Equation (5.12) is an ordinary differential equation of first order with respect to the local components of e_i (see (3.23)).

6. The case of a manifold as a base

Starting from this section, we consider some peculiarities of frames normal for linear transports along paths in a vector bundle (E, π, M) whose base M is a C^1 differentiable manifold. Besides, the bundle space E will be required to be a C^1 manifold too. This will allow links to be made with the general results of Chapter III concerning frames normal for derivations of the tensor algebra of the vector space of vector fields over a manifold which, in particular, can be linear connections.

The local coordinates of $p \in M$ will be denoted by p^μ, $p^\mu := x^\mu(p)$. Here and below the Greek indices $\alpha, \beta, \dots, \mu, \nu, \dots$ run from 1 to $\dim M$ and, as usual, a summation from 1 to $\dim M$ on such indices repeated on different levels will be assumed. The below-considered paths, like $\gamma\colon J \to M$, are supposed to be of class C^1 and by $\dot\gamma(s)$ is denoted the vector tangent to γ at $\gamma(s)$, $s \in J$, (more precisely at s) i.e., $\dot\gamma$ is the vector field tangent to γ provided γ is injective. By $\{E_\mu\}$ will be denoted a frame along γ in the bundle space tangent to M, i.e., for every $s \in J$ the vectors $E_1|_{\gamma(s)}, \dots, E_{\dim M}|_{\gamma(s)}$ form a basis in the space $T_{\gamma(s)}(M)$ tangent to M at $\gamma(s)$. In particular, the frame $\{E_\mu\}$ can be a coordinate one, $E_\mu|_p = \frac{\partial}{\partial x^\mu}\big|_p$, in some neighborhood of $p \in \gamma(J)$. The transports along paths investigated below are supposed to be of class C^1 on the set of C^1 paths in M.

Proposition 6.1. *Let L be a linear transport along paths in (E, π, M), E and M being C^1 manifolds, and L be Euclidean on $U \subseteq M$ (resp. along a C^1 path $\gamma\colon J \to M$). Then the matrix $\mathbf{\Gamma}$ of its coefficients has the representation*

$$\mathbf{\Gamma}(s;\gamma) = \sum_{\mu=1}^{\dim M} \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) \equiv \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) \tag{6.1}$$

in any frame $\{e_i\}$ along every (resp. the given) C^1 path $\gamma\colon J \to U$, where $\Gamma_\mu = \big[\Gamma^i{}_{j\mu}\big]_{i,j=1}^{\dim \pi^{-1}(p)}$ are some matrix-valued functions, defined on an open set V containing U (resp. $\gamma(J)$) or equal to it, and $\dot\gamma^\mu$ are the components of $\dot\gamma$ in some frame $\{E_\mu\}$ along γ in the bundle space $T(M)$ tangent to M, $\dot\gamma = \dot\gamma^\mu E_\mu$.

Proof. By Theorem 5.2, the representation (5.4′) is valid in $\{e_i\}$ for some matrix-valued function $\boldsymbol{F}_0$ on U. Hence, if U is a neighborhood, equation (6.1) holds for

$$\Gamma_\mu(p) = \boldsymbol{F}_0^{-1}(p)\big(E_\mu(\boldsymbol{F}_0)|_p\big) \tag{6.2}$$

with $p \in U$. In the general case, e.g., if U is a submanifold of M of dimension less than the one of M, the terms $E_\mu(\boldsymbol{F}_0)|_U$, $\mu = 1, \dots, \dim M$, in the last equality may turn to be undefined as the matrix-valued function $\boldsymbol{F}_0$ is defined only on U. To overcome this possible problem, let us take some C^1 matrix-valued function $\boldsymbol{F}$, defined on an open set V containing U (resp. $\gamma(J)$) or equal to it, such that $\boldsymbol{F}|_U = \boldsymbol{F}_0$. Since (5.4) and (5.4′) depend only on the values of $\boldsymbol{F}_0$, i.e., on the ones of $\boldsymbol{F}$ on U, these equations hold also if we replace $\boldsymbol{F}_0$ in them with $\boldsymbol{F}$. From the so-modified equality (5.4′), with $\boldsymbol{F}$ for $\boldsymbol{F}_0$, we see that (6.1) is valid for

$$\Gamma_\mu(p) = \boldsymbol{F}^{-1}(p)(E_\mu(\boldsymbol{F}))|_p \tag{6.3}$$

with $p \in V$. □

Consider now the transformation properties of the matrices Γ_μ in (6.1). Let U be an open set, e.g., $U = M$. If we change the frame $\{E_\mu\}$ in the bundle space tangent to M, $\{E_\mu\} \mapsto \{E'_\mu = B^\nu_\mu E_\nu\}$ with $B = \big[B^\nu_\mu\big]^{\dim M}_{\mu,\nu=1}$ being non-degenerate matrix-valued function, and simultaneously the bases in the fibres $\pi^{-1}(p)$, $p \in M$, $\{e_i|_p\} \mapsto \{e'_i|_p = A^j_i(p) e_j|_p\}$, then, from (3.26) and (6.1), we see that Γ_μ transforms into Γ'_μ such that

$$\Gamma'_\mu = B^\nu_\mu A^{-1}\Gamma_\nu A + A^{-1}E'_\mu(A) = B^\nu_\mu A^{-1}\big(\Gamma_\nu A + E_\nu(A)\big) \tag{6.4}$$

where $A := \big[A^j_i\big]^{\dim\pi^{-1}(p)}_{i,j=1}$ is non-degenerate and of class C^1.

Note 6.1. While deriving (6.4), we supposed (6.1) to be valid on M, i.e., for $U = M$. If $U \neq M$, equation (6.1) holds only on U, i.e., for $\gamma\colon J \to U$. Therefore the result (6.4) is true only on U, but in this case the frames $\{e_i\}$ and $\{e'_i\}$ *must* be defined on an open subset of M containing or equal to U. This follows from (3.26) in which the derivative $\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s} = \frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s}$ enters. To derive (6.4), we have expressed $\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s}$ as $(E_\mu(A))|_{\gamma(s)}\dot\gamma^\mu(s)$ which is meaningful iff A is defined on a neighborhood of each point in U. Consequently A, as well as $\{e_i\}$ and $\{e'_i\}$, must be defined on an open set $V \supseteq U$. For this reason, below, when derivatives like $E_\mu(A)$ appear, we admit the employed frames in the bundle space E to be defined always on some neighborhood in M containing or equal to the set U on which some normal frames are investigated.

The above considerations, when applied to parallel transports assigned to linear connections (see below Section 11) in the tangent bundle $(T(M), \pi, M)$, are the cause for which in the definition of frames normal for a linear connection (see Definition I.5.1 on page 37) is included the requirement they to be defined on a neighborhood, possibly coinciding with U if U is a neighborhood.

Denoting by $\Gamma^i{}_{j\mu}$ the components of Γ_μ, we can rewrite (6.4) as

$$\Gamma'^i{}_{j\mu} = \sum_{\nu=1}^{\dim M}\sum_{k,l=1}^{\dim\pi^{-1}(p)} B^\nu_\mu \big(A^{-1}\big)^i_k A^l_j \Gamma^k{}_{l\nu} + \sum_{\nu=1}^{\dim M}\sum_{k=1}^{\dim\pi^{-1}(p)} B^\nu_\mu \big(A^{-1}\big)^i_k E_\nu(A^k_j). \tag{6.5}$$

Thus, we observe that the functions $\Gamma^i{}_{j\mu}$ are very similar to the coefficients of a linear connection.[1] Below, in Subsection 13.4, we shall see that this is not accidental (compare (6.1) with (3.29)). These functions are also called coefficients of

[1] Compare (6.4) with (I.5.3) or (6.5) with (I.3.5). The equations in these pairs are identical for $\dim M = \dim\pi^{-1}(p)$, $p \in M$, and $B = A$.

the transport L. To make a distinction between $\Gamma^i{}_j$ and $\Gamma^i{}_{j\mu}$, we call the former ones *2-index coefficients* of L and the latter ones *3-index coefficients* of L when there is a risk of ambiguities. It should be emphasized on the fact that the 2-index coefficients are defined with respect to a *single* frame $\{e_i\}$ along γ in the vector bundle (E, π, M), while the 3-index coefficients are defined with respect to a *pair* $(\{e_i\}, \{E_\mu\})$ of frames, $\{e_i\}$ in the bundle space E of (E, π, M) and $\{E_\mu\}$ in the bundle space $T(M)$ of the bundle $(T(M), \pi, M)$ tangent to M. Besides, if (6.1) holds for every $\gamma\colon J \to U$ for a transport L, then, in the general case, there are (infinitely) many such representations unless U an open set. For instance, if (6.1) is valid for some Γ_μ, it is also true if we replace in it Γ_μ with $\Gamma_\mu + G_\mu$ where the matrix-valued functions G_μ are such that $G_\mu \dot\gamma^\mu = 0$ for every $\gamma\colon J \to U$. Hence, generally, the 3-index coefficients of L depend also on U and are not unique due to, e.g., the nonuniqueness of $\boldsymbol{F}$ in (6.3).

Prima facie one may think that the converse of Proposition 6.1 is true, but this is not the general case, i.e., if in some frame equation (6.1) holds, then $\{e_i\}$ is generally not normal unless some conditions are fulfilled. Thus (6.1) is only a necessary, but, generally, not sufficient condition for a frame to be normal.

Theorem 6.1. *A C^2 linear transport L along paths is Euclidean on a neighborhood $U \subseteq M$ if and only if in every frame the matrix $\boldsymbol{\Gamma}$ of its coefficients has a representation* (6.1) *along every C^1 path γ in U in which the matrix-valued functions Γ_μ, defined on an open set containing U or equal to it, satisfy the equalities*

$$\big(R_{\mu\nu}(-\Gamma_1, \dots, -\Gamma_{\dim M})\big)(p) = 0 \tag{6.6}$$

where $p \in U$ and

$$R_{\mu\nu}(-\Gamma_1, \dots, -\Gamma_{\dim M}) := -E_\nu(\Gamma_\mu) + E_\mu(\Gamma_\nu) + \Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu - C^{\varkappa}_{\mu\nu}\Gamma_\varkappa \tag{6.7}$$

in a frame $\{E_\mu\}$ over U in the bundle space tangent to M, with $C^{\varkappa}_{\mu\nu}$ being the structure functions of $\{E_\mu\}$, $[E_\mu, E_\nu]_{\_} =: C^{\varkappa}_{\mu\nu}E_\varkappa$.

Remark 6.1. This result is a direct analogue of Theorem III.6.1 on page 157 in the theory considered here.

Remark 6.2. If we change the frame $\{e_i\}$, $\{e_i\} \mapsto \{e'_i = A^j_i e_j\}$, over U and simultaneously the frame $\{E_\mu\}$, $\{E_\mu\} \mapsto \{E'_\mu = B^\nu_\mu E_\nu\}$, in the tangent bundle space over U, from (6.7), (I.8.4), and (6.4), we get that the matrices $R_{\mu\nu}$ transform into

$$R'_{\mu\nu} = B^{\varkappa}_\mu B^\lambda_\nu A^{-1} R_{\varkappa\lambda} A. \tag{6.8}$$

Therefore the conditions (6.6) have an invariant character, i.e., they are independent of the particular choice of the frames (or coordinates) involved in them. The last result also shows that the quantities $R_{\mu\nu}$ depend on the frames $\{e_i\}$ and $\{E_\mu\}$ on U and are completely independent of the their values outside U (if $U \neq M$) in case they are defined there (see Note 6.1).

Remark 6.3 *(Important!).* This theorem has a stronger version: the requirement the representation (6.1) to hold can be dropped as it is a consequence of (6.6). We skip this detail till Subsection 9.2 (see the proof of Theorem 9.1 and the comments after it on page 269). Corollary 9.1 on page 269 is a formulation of the mentioned stronger variant of Theorem 6.1 employing the concept of a 'curvature'. The same remarks are valid with respect to Theorem 6.2 on page 253 below.

Proof. NECESSITY. For a transport L Euclidean on U is valid (6.1) due to Proposition 6.1. Moreover, we know from the proof of this proposition that Γ_μ admit representation (6.3) for some C^1 non-degenerate matrix-valued function $\boldsymbol{F}$. The proof of the necessity is completed by the following lemma.

Lemma 6.1. *A set of matrix-valued functions* $\{\Gamma_\mu : \mu = 1, \dots, \dim M\}$*, of class* C^1 *and defined on a neighborhood* V*, admits a representation* (6.3) *iff the conditions* (6.6) *are fulfilled for* $p \in V$*.*

Proof of Lemma 6.1*.* A representation (6.3) exists iff it, considered as a matrix linear partial differential equation of first order, has a solution with respect to $\boldsymbol{F}$. Rewriting (6.3) as

$$E_\mu(\boldsymbol{F}^{-1})|_p = -\Gamma_\mu(p)\boldsymbol{F}^{-1}(p), \qquad p \in V,$$

from Lemma II.4.1 on page 105 with $N = M$ and E_μ for e_μ, we conclude that the solutions of this equation with respect to $\boldsymbol{F}^{-1}$ exist iff (6.6) holds. In fact, fixing some initial value $\boldsymbol{F}^{-1}(p_0) = f_0$, we get (see Remark II.4.5)

$$\boldsymbol{F}(p) = f_0^{-1} Y^{-1}(p, p_0; -\Gamma_1, \dots, -\Gamma_{\dim M}) \tag{6.9}$$

where $Y(p, p_0; Z_1, \dots, Z_{\dim M})$ is the solution of the initial-value problem

$$E_\mu(Y)|_p = Z_\mu(p) Y|_p, \qquad Y|_{p=p_0} = \mathbb{1}. \tag{6.10}$$

Here $Z_1, \dots, Z_{\dim M}$ are continuous matrix-valued functions and $\mathbb{1}$ is the identity (unit) matrix of the corresponding size. According to Lemma II.4.1, the problem (6.10) with $Z_\mu = -\Gamma_\mu$ has (a unique) solution (of class C^2) iff the (integrability) conditions (6.6) are valid. □

SUFFICIENCY. Let (6.1) and (6.6) be valid. As a consequence of Lemma 6.1, there is a representation (6.3) for $\boldsymbol{\Gamma}$ with some $\boldsymbol{F}$. Substituting (6.3) into (6.1), we get

$$\boldsymbol{\Gamma}(s;\gamma) = \boldsymbol{F}^{-1}(\gamma(s)) E_\mu(\boldsymbol{F})|_{\gamma(s)} \dot\gamma^\mu(s) = \boldsymbol{F}^{-1}(\gamma(s)) \frac{\mathrm{d}\boldsymbol{F}(\gamma(s))}{\mathrm{d}s}.$$

So, by Theorem 5.2 (see (5.4′) for $\boldsymbol{F}_0 = \boldsymbol{F}|_U$), the considered transport L along paths is Euclidean. □

The just-proved Theorem 6.1 expresses a very important practical necessary and sufficient condition for existence of frames normal on neighborhoods because the conditions (6.1) and (6.6) are easy to check for a given linear transport along paths in bundles with a differentiable manifold as a base.

Now, combining (4.1c) and (6.1), applying Corollaries 4.1 and 4.2, and using the arbitrariness of γ, we can formulate the following essential result.

Proposition 6.2. *A necessary and sufficient condition for a frame to be normal on a neighborhood $U \subseteq M$ for a Euclidean on U linear transport along paths in (E, π, M) is the vanishment of its 3-index coefficients, i.e.,*

$$\Gamma_\mu(p) := \left[\Gamma^i{}_{j\mu}\right]_{i,j=1}^{\dim \pi^{-1}(p)} = 0 \qquad (6.11)$$

for every $p \in U$, where $\Gamma_\mu(p)$ define the (2*-index*) *coefficients of the transport via* (6.1).

Remark 6.4. The assumption in this proposition for U to be a neighborhood is an essential one. From it and the arbitrariness of γ follows that $\dot\gamma(s)$ is an arbitrary vector in $T_{\gamma(s)}(M)$ which, together with (6.1) and $\mathbf{\Gamma}(s;\gamma) = 0$, implies (6.11). If U is not a neighborhood, then, generally, such a conclusion can not be made. For instance, if U is a submanifold of M and $\dim U < \dim M$, then $\dot\gamma(s) \in T_{\gamma(s)}(U)$ but $\dot\gamma(s) \notin T_{\gamma(s)}(M) \backslash T_{\gamma(s)}(U)$ as $\gamma\colon J \to U$ is a path *in* U. Therefore, in this example, only $\dim U$ of the matrices $\Gamma_1, \dots, \Gamma_{\dim M}$ must vanish in a (suitable) frame normal on U, the remaining $\dim M - \dim U$ of them need not to be zeros on U. Obviously, this assertion can be inverted, i.e., frames normal on a submanifold U exist iff in them $\dim U$ of the matrices of the transport's 3-index coefficients vanish on U (in suitable coordinates or frames). More details for normal frames in which the 3-index coefficients of a linear transport vanish are presented in Section 12.

Because of Propositions 6.1 and 6.2, as we said above, the functions $\Gamma^i{}_{j\mu}$ are convenient to be called also coefficients of the transport, like $\Gamma^i{}_j$. If there is a risk of ambiguity whether we have in mind $\Gamma^i{}_j$ or $\Gamma^i{}_{j\mu}$, when speaking about transport's (or corresponding derivation's) coefficients, we shall call them respectively 2-*index* and 3-*index* coefficients. Note that any linear transport has 2-index coefficients while 3-index ones exist only for some of them; in particular such are the Euclidean transports (see Proposition 6.1 and Theorem 6.2). Through the 3-index coefficients can be defined concrete classes of, possibly Euclidean on some sets, linear transports along paths in a given vector bundle with a manifold as a base. For the purpose one should define a transport by the matrix (6.1) of its 2-index coefficients in which Γ_μ are *fixed* matrix-valued functions over the whole base M. In particular, if Γ_μ are the matrices of the coefficients of a linear connection in the tangent bundle over M, we obtain in this way the class of parallel transports generated by such connections in this bundle (see (3.29) and the assertion after it).

It should be emphasized, the 3-index coefficients $\Gamma^i{}_{j\mu}$ of a given linear transport L admitting them are defined *uniquely* on $U \subseteq M$ by (6.3) or (6.2) if (and only if) U is an open subset of M, e.g., if $U = M$; the same is valid for the transformed coefficients (6.5). If U is not such, the 3-index coefficients of L contain an arbitrariness connected with the one of the matrix-valued function $\boldsymbol{F}$, appearing in (6.3), which is subjected only on the condition $\boldsymbol{F}|_U = \boldsymbol{F}_0$. Besides, if (6.1) holds for some Γ_μ, it remains true if we replace in its right-hand side Γ_μ with $\Gamma_\mu + G_\mu$ where the matrix-valued functions G_μ are such that $G_\mu(\gamma(s))\dot{\gamma}^\mu(s) = 0$ for all C^1 paths $\gamma\colon J \to U$ and $s \in J$. The mentioned arbitrariness will be described below when U is a submanifold of M.

Now we are going to find an analogue of Theorem 6.1 when the neighborhood $U \subseteq M$ in it is replaced with a submanifold of the base M.

Let N be a submanifold of M and L a linear transport along paths in (E, π, M) which is Euclidean on N. Let the C^1 matrix-valued function $\boldsymbol{F}_0$ determines the coefficients' matrix of L via (5.4′). Suppose $p_0 \in N$ and (V, x) is a chart of M such that $V \ni p_0$ and the local coordinates of every $p \in N \cap V$ are $x(p) = (x^1(p), \dots, x^{\dim N}(p), t_0^{\dim N+1}, \dots, t_0^{\dim M})$, where t_0^ρ, $\rho = \dim N + 1, \dots, \dim M$, are constant numbers (see the definition of a submanifold on page 7).

In the chart (V, x), we have $\frac{\mathrm{d}\boldsymbol{F}_0(\gamma(s))}{\mathrm{d}s} = \sum_{\alpha=1}^{\dim N} \frac{\partial \boldsymbol{F}_0}{\partial x^\alpha}\Big|_{\gamma(s)} \dot{\gamma}^\alpha(s)$, with $\gamma^\mu := x^\mu \circ \gamma$, for every C^1 path $\gamma\colon J \to N$ and $s \in J$. From here and (5.4′), it follows that (6.1) holds for

$$\Gamma_\alpha(p) = \boldsymbol{F}_0^{-1}(p) \frac{\partial \boldsymbol{F}_0}{\partial x^\alpha}\Big|_p, \qquad \alpha = 1, \dots, \dim N \tag{6.12}$$

and *arbitrary* $\Gamma_{\dim N+1}, \dots, \Gamma_{\dim M}$ since in the coordinates $\{x^\mu\}$ is fulfilled $\gamma^\rho(s) = t_0^\rho = \text{const}$ and hence

$$\dot{\gamma}^{\dim N+1} = \dots = \dot{\gamma}^{\dim M} \equiv 0. \tag{6.13}$$

Comparing (6.12) with (6.2) for $E_\mu = \frac{\partial}{\partial x^\mu}$, we conclude that Γ_α, given via (6.12), are exactly the first $\dim N$ of the matrices $\Gamma_\mu = [\Gamma^i{}_{j\mu}]$ of the 3-index coefficients of the transport L in the pair of frames $\big(\{e_i\}, \{\frac{\partial}{\partial x^\mu}\}\big)$. As we said, the rest of the 3-index coefficients of L (on N) are completely arbitrary. In particular, one can choose them according to (6.3),

$$\Gamma_\rho(p) = \boldsymbol{F}^{-1}(p) \frac{\partial \boldsymbol{F}}{\partial x^\rho}, \qquad \rho = \dim N + 1, \dots, \dim M, \quad \boldsymbol{F}|_N = \boldsymbol{F}_0, \tag{6.14}$$

which leads to the validity of (6.3) in every frame, or, if the representation (6.1) holds for every $\gamma\colon J \to M$ (this does not mean that L is Euclidean on M!), the matrices Γ_ρ can be identified with the ones appearing in (6.1) in the frame $\{\frac{\partial}{\partial x^\mu}\}$.

If $\{x'^\mu\}$ are other coordinates on V like $\{x^\mu\}$, i.e., $x'^\rho(p) = \text{const}$ for $p \in N \cap V$ and $\rho = \dim N + 1, \dots, \dim M$, the change $\{x^\mu\} \mapsto \{x'^\mu\}$, combined with $\{e_i\} \mapsto \{e'_i = A_i^j e_j\}$ leads to

$$\Gamma_\alpha \mapsto \Gamma'_\alpha = B_\alpha^\beta A^{-1} \Gamma_\beta A + A^{-1} \frac{\partial A}{\partial x'^\alpha}, \quad B_\alpha^\beta := \frac{\partial x^\beta}{\partial x'^\alpha}, \quad \alpha, \beta = 1, \dots, \dim N \tag{6.15}$$

on $N \cap V$. So, equation (6.4) remains valid only for frames $\{E_\mu\}$ normal on N. But using the arbitrariness of Γ_ρ, we can force (6.4) to hold on N for arbitrary frames defined on a neighborhood of N.

The above discussion implies that the condition (6.6) in Theorem 6.1, when applied on a submanifold N, imposes restrictions on the transport L as well as ones on the 'inessential' 3-index coefficients of L, like Γ_ρ above, or on the matrix-valued function $\boldsymbol{F}$ entering in (6.3) or in (6.14). Since the restrictions of the last type are not connected with the transport L, below we shall 'repair' Theorem 6.1 on submanifolds in such a way as to exclude them from the final results.

Theorem 6.2. *A linear transport L along paths is Euclidean on a submanifold N of M if and only if in every frame $\{e_i\}$, in the bundle space over N, the matrix of its coefficients has a representation* (6.1) *along every C^1 path in N and, for every $p_0 \in N$ and a chart (V, x) of M such that $V \ni p_0$ and $x(p) = (x^1(p), \dots, x^{\dim N}(p), t_0^{\dim N+1}, \dots, t_0^{\dim M})$ for every $p \in N \cap V$ and constant numbers $t_0^{\dim N+1}, \dots, t_0^{\dim M}$, the equalities*

$$\big(R^N_{\alpha\beta}(-\Gamma_1, \dots, -\Gamma_{\dim N})\big)(p) = 0, \qquad \alpha, \beta = 1, \dots, \dim N \tag{6.16}$$

hold for all $p \in N \cap V$ and

$$R^N_{\alpha\beta}(-\Gamma_1, \dots, -\Gamma_{\dim N}) := R_{\alpha\beta}(-\Gamma_1, \dots, -\Gamma_{\dim M}) = -\frac{\partial \Gamma_\alpha}{\partial x^\beta} - \frac{\partial \Gamma_\beta}{\partial x^\alpha} + \Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha. \tag{6.17}$$

Here $\Gamma_1, \dots, \Gamma_{\dim N}$ are first $\dim N$ of the matrices of the 3-index coefficients of L in the coordinate frame $\{\frac{\partial}{\partial x^\mu}\}$ in the tangent bundle space over $N \cap V$. They are uniquely defined via (6.12).

Remark 6.5. In the theory considered here, this result is a direct analogue of Theorem III.8.1, point (i), on page 165.

Remark 6.6. It is intuitively clear, generally not all of the equations (6.16) are independent. One can expect only $(\dim N)[(\dim N)-1]/2$ of them to be independent because of $R_{\mu\nu} = -R_{\nu\mu}$, due to (6.7).

Remark 6.7. This theorem is, in fact, a special case of Theorem 6.1: if in the latter theorem we put $U = N$, restrict the transport L to the bundle $(\pi^{-1}(N), \pi|_N, N)$, replace M with N, and notice that $\{x^1, \dots, x^{\dim N}\}$ provide an internal coordinate system on N, we get the former one with $E_\alpha = \partial/\partial x^\alpha$. Because of the importance of the result obtained, we call it 'theorem' and present below its independent proof.

Proof. If L is Euclidean on N, equation (6.1) holds in every frame on N (Proposition 6.1); in particular it is valid in the frame $\{\frac{\partial}{\partial x^\mu}\}$, induced by the chart (V, x), in which, as was proved above, equation (6.12) is satisfied. The substitution of (6.12) into (6.17) results in (6.16). Conversely, let (6.1) for $\gamma\colon J \to N$ and (6.16) be valid. By Lemma 6.1 with N for M, from (6.16) follows the existence of a representation (6.12) for some matrix-valued function $\boldsymbol{F}_0$ on N. Substituting (6.12) into (6.1) and using that γ is a path in N and (6.13) is valid, in the frame $\{\frac{\partial}{\partial x^\mu}\}$, we obtain:

$$\begin{aligned}\mathbf{\Gamma}(s;\gamma) = \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) = \sum_{\alpha=1}^{\dim N} \Gamma_\alpha(\gamma(s))\dot\gamma^\alpha(s) \\ = \boldsymbol{F}_0^{-1}(\gamma(s)) \sum_{\alpha=1}^{\dim N} \frac{\partial \boldsymbol{F}_0}{\partial x^\alpha}\Big|_{\gamma(s)} \dot\gamma^\alpha(s) = \boldsymbol{F}_0^{-1}(\gamma(s)) \frac{\partial \boldsymbol{F}}{\partial x^\mu}\Big|_{\gamma(s)} \dot\gamma^\mu(s) \\ = \boldsymbol{F}_0^{-1}(\gamma(s)) \frac{\mathrm{d}\boldsymbol{F}(\gamma(s))}{\mathrm{d}s} = \boldsymbol{F}_0^{-1}(\gamma(s)) \frac{\mathrm{d}\boldsymbol{F}_0(\gamma(s))}{\mathrm{d}s}\end{aligned}$$

where $\boldsymbol{F}$ is a C^1 matrix-valued function defined on an open set containing N or equal to it and such that $\boldsymbol{F}|_N = \boldsymbol{F}_0$. Thus, by Theorem 5.2, the transport L is Euclidean on N. □

Corollary 6.1. *Every linear transport along paths in a vector bundle whose base and bundle spaces are C^1 manifolds, is Euclidean at every single point or along every path without self-intersections.*

Proof. See Theorem 6.2 for $\dim N = 0, 1$, in which cases $R^N_{\alpha\beta} \equiv 0$. □

It should be noted, the last result agrees completely with Proposition 5.1 and Corollary 5.1.

7. Linear transports and normal frames in line bundles

The purpose of this section is to exemplify the above general considerations concerning linear transports and normal frames on *line vector bundles*, i.e., ones with one-dimensional fibres.

Let (E, π, M) be one-dimensional vector bundle over a C^1 manifold M. Thus the (typical) fibre of (E, π, M) can be identified with $\mathbb{K}$ and then the fibre $\pi^{-1}(x)$ over $x \in M$ will be an isomorphic image of $\mathbb{K}$. Let $\gamma\colon J \to M$ be of class C^1 and L be a linear transport along paths in (E, π, M). A frame $\{e\}$ along γ consists of a single non-zero vector field $e\colon (s;\gamma) \to e(s;\gamma) \in \pi^{-1}(\gamma(s))\backslash\{0\}$, $s \in J$, and in it the matrix of L^γ at $(t,s) \in J \times J$ is simply a number $\boldsymbol{L}(t,s;\gamma) \in \mathbb{K}$, $L^\gamma_{s\to t}(ue(s;\gamma)) = u\boldsymbol{L}(t,s;\gamma)e(t;\gamma)$ for $u \in \mathbb{K}$ and $s,t \in J$. By Proposition 3.4, the general form of $\boldsymbol{L}$ is

$$\boldsymbol{L}(t,s;\gamma) = \frac{f(s;\gamma)}{f(t;\gamma)} \tag{7.1}$$

where $f\colon (s;\gamma) \mapsto f(s;\gamma) \in \mathbb{K}\backslash\{0\}$ is defined up to (left) multiplication with a function of γ (Proposition 3.5). Respectively, due to (3.25), the matrix of the coefficient(s) of L is

$$\mathbf{\Gamma}(s;\gamma) = \frac{\partial \boldsymbol{L}(t,s;\gamma)}{\partial s}\Big|_{t=s} = \frac{1}{f(s;\gamma)}\frac{\mathrm{d}f(s;\gamma)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\big[\ln(f(s;\gamma)\big] \tag{7.2}$$

and (3.27) takes the form

$$\boldsymbol{L}(t,s;\gamma)=\exp\Bigl(-\int_s^t \boldsymbol{\Gamma}(\sigma;\gamma)\,\mathrm{d}\sigma\Bigr). \tag{7.3}$$

A change $e(s;\gamma)\mapsto e'(s;\gamma)=a(s;\gamma)e(s;\gamma)$, with $a(s;\gamma)\in\mathbb{K}\backslash\{0\}$, of the frame $\{e\}$ implies (see (3.11) and (3.26))

$$\boldsymbol{L}(t,s;\gamma)\mapsto\boldsymbol{L}'(t,s;\gamma)=\frac{a(s;\gamma)}{a(t;\gamma)}\boldsymbol{L}(t,s;\gamma) \tag{7.4a}$$

$$\boldsymbol{\Gamma}(s;\gamma)\mapsto\boldsymbol{\Gamma}'(s;\gamma)=\boldsymbol{\Gamma}(s;\gamma)+\frac{\mathrm{d}}{\mathrm{d}s}\bigl[\ln(a(s;\gamma)\bigr]. \tag{7.4b}$$

The explicit local action of the derivation D along paths generated by L is

$$D_s^\gamma\lambda=\Bigl(\frac{\mathrm{d}\lambda_\gamma(s)}{\mathrm{d}s}+\boldsymbol{\Gamma}(s;\gamma)\lambda_\gamma(s)\Bigr)e(s;\gamma) \tag{7.5}$$

where $\lambda\in\mathrm{PLift}^1(E,\pi,M)$ and (3.23) was used. Let us now look on the normal frames on one-dimensional vector bundles.

A frame $\{e\}$ is normal for L along γ (resp. on U) iff in that frame (7.1) holds with

$$f(s;\gamma)=f_0(\gamma) \tag{7.6}$$

where $\gamma\colon J\to M$ (resp. $\gamma\colon J\to U$) and $f_0\colon\gamma\mapsto f_0(\gamma)\in\mathbb{K}\backslash\{0\}$ (see Remark 4.2 and Proposition 4.1). Since, in a frame normal along γ (resp. on U), it is fulfilled

$$\boldsymbol{L}(t,s;\gamma)=\mathbb{1},\quad \boldsymbol{\Gamma}(s;\gamma)=0 \tag{7.7}$$

for the given path γ (resp. every path in U), in every frame $\{e'=ae\}$, we have

$$\boldsymbol{L}'(t,s;\gamma)=\frac{a(s;\gamma)}{a(t;\gamma)},\quad \boldsymbol{\Gamma}'(s;\gamma)=\frac{\mathrm{d}}{\mathrm{d}s}\bigl[\ln(a(s;\gamma)\bigr]. \tag{7.8}$$

In addition, for Euclidean on $U\subseteq M$ transport L, the representation

$$\boldsymbol{\Gamma}'(s;\gamma)=\Gamma'_\mu(\gamma(s))\dot\gamma'^{\mu}(s) \tag{7.9}$$

holds for every C^1 path $\gamma\colon J\to U$ and some $\Gamma'_\mu\colon V\to\mathbb{K}$ with V being an open set such that $V\supseteq U$ (Proposition 6.1). This means (see Theorems 5.1 and 5.2) that (7.8) holds for

$$a(s;\gamma)=a_0(\gamma(s)), \tag{7.10}$$

where $a_0\colon U\to\mathbb{K}\backslash\{0\}$, and, consequently, the equality (7.9) can be satisfied if we choose

$$\Gamma'_\mu=E_\mu(a) \tag{7.11}$$

with $a\colon V \to \mathbb{K}$, $a|_U = a_0$ and $\{E_\mu\}$ being a frame in the bundle space tangent to M which, in particular, can be a coordinate one, $E_\mu = \frac{\partial}{\partial x^\mu}$. Of course, if U is not an open set, this choice of Γ'_μ is not necessary (see Section 6); for example, the equality (7.9) will be preserved, if to the right-hand side of (7.11) is added a function G'_μ such that $G'_\mu \dot{\gamma}'^{\,\mu} = 0$.

By virtue of (6.4), the functions Γ_μ and Γ'_μ in two arbitrary pairs of frames $(\{e\}, \{E_\mu\})$ and $(\{e' = ae\}, \{E'_\mu = B^\nu_\mu E_\nu\})$, respectively, are connected via

$$\Gamma'_\mu = B^\nu_\mu \Gamma_\nu + \frac{1}{a} E'_\mu(a) = B^\nu_\mu \big(\Gamma_\nu + E_\nu(\ln a)\big) \tag{7.12}$$

and, consequently, with respect to changes of the frames in the tangent bundle space over M, when $a = 1$, they behave like the components of a covariant vector field (one-form). Therefore, on an open set U, e.g., $U = M$, the quantity

$$\omega = \Gamma_\mu E^\mu, \tag{7.13}$$

where $\{E^\mu\}$ is the coframe dual to $\{E_\mu\}$ (in local coordinates: $E_\mu = \frac{\partial}{\partial s^\mu}$ and $E^\mu = \mathrm{d}x^\mu$), is a 1-form over M (with respect to changes of the local coordinates on M or of the frames in the (co)tangent bundle space over M). However, it depends on the choice of the frame $\{e\}$ in the bundle space E and a change $e \mapsto e' = ae$ implies

$$\omega \mapsto \omega' = \omega + (E_\nu(\ln a))E^\nu = \omega + (E'_\nu(\ln a))E'^{\,\nu}. \tag{7.14}$$

Using the 1-form (7.13), we see that

$$\mathbf{\Gamma}(s;\gamma) = \omega|_{\gamma(s)}(\dot{\gamma}(s)) \tag{7.15}$$

and (7.3) can be rewritten as

$$\boldsymbol{L}(t,s;\gamma) = \exp\Big(-\int_{\gamma(s)}^{\gamma(t)} \omega\Big) \tag{7.16}$$

where the integration is along some path in U (on which the transport L is Euclidean). Hence $\boldsymbol{L}$ (or L) depends only on the points $\gamma(s)$ and $\gamma(t)$, not on the particular path connecting them, as it should be (Theorem 5.1). The self-consistency of our results is confirmed by the equation

$$R_{\mu\nu}|_U = 0 \tag{7.17}$$

which is a consequence of (7.11) and (6.7) and which is a necessary and sufficient condition for the existence of frames normal on an open set U (Theorem 6.1).

We end this section with the remark that frames normal along injective paths always exist (Corollary 5.1), but on an arbitrary submanifold $N \subseteq M$ they exist iff the functions Γ_μ satisfy the conditions (6.16) with $x \in N$ in the coordinates described in Theorem 6.2.

8. Normal frames for derivations in vector bundles with a manifold as a base

For a general bundle (E, π, B) whose bundle space E is C^1 manifold, we call a frame $\{e_i\}$ *normal on* $U \subseteq B$ (resp. along $\gamma\colon J \to M$) for a derivation D along paths (resp. D^γ along γ) (see Definition 2.1 on page 219) if $\{e_i\}$ is normal on U (resp. along γ) for the linear transport L along paths generating it by (3.19) (see Proposition 3.7). We can also, equivalently, define a frame normal for D (resp. D^γ) as one in which the components of D (resp. D^γ) vanish (see the proof of Proposition 3.7, Proposition 4.2, and Corollary 4.2). A derivation admitting normal frame(s) is called *Euclidean*.

In connection with concrete physical applications, far more interesting case is the case of a bundle (E, π, M) with a differentiable manifold M as a base. The cause for this is the existence of natural structures over M, e.g., the different tensor bundles and the tensor algebra over it. Below we concentrate on this particular case.

Definition 8.1. A *derivation* over an open set $V \subseteq M$ or in $(E, \pi, M)|_V$ *along tangent vector fields* is a mapping $\mathcal{D}$ assigning to every tangent vector field X over V a $\mathbb{K}$-linear mapping

$$\mathcal{D}_X\colon \operatorname{Sec}^1\big((E, \pi, M)|_V\big) \to \operatorname{Sec}^0\big((E, \pi, M)|_V\big), \tag{8.1}$$

called a *derivation along* X, such that

$$\mathcal{D}_X(f \cdot \sigma) = X(f) \cdot \sigma + f \cdot \mathcal{D}_X(\sigma) \tag{8.2}$$

for every C^1 section σ over V and every C^1 function $f\colon V \to \mathbb{K}$.

Example 8.1. The mapping $X \mapsto \nabla_X|_{\mathfrak{T}^{r;1}_s(M)}$, where ∇ is a linear connection and $\mathfrak{T}^{r;1}_s(M) = \operatorname{Sec}^1(T^r_s(M), \pi^r_s, M)$ is the set of C^1 tensor fields of type (r, s) over M, is a derivation along tangent vector fields in the tensor bundle $(T^r_s(M), \pi^r_s, M)$ of type (r, s).

Obviously (see Definition 2.1), if $\gamma\colon J \to V$ is a C^1 path, the mapping $\overline{D}\colon \hat{\sigma} \mapsto \overline{D}\hat{\sigma}$, with $\overline{D}\hat{\sigma}\colon \gamma \mapsto \overline{D}^\gamma\hat{\sigma}$, where $\overline{D}^\gamma\hat{\sigma}\colon s \mapsto \overline{D}^\gamma_s\hat{\sigma}$ is defined via

$$\overline{D}^\gamma_s(\hat{\sigma}) = \big((\mathcal{D}_X\sigma)|_{X=\dot{\gamma}}\big)(\gamma(s)), \qquad \hat{\sigma}\colon \gamma \mapsto \sigma \circ \gamma, \tag{8.3}$$

is a derivation along paths on the set of C^1 liftings generated by sections of $(E, \pi, M)|_V$. From Section 2, we know that along paths without self-intersections every derivation along paths generates a derivation of the sections of (E, π, M) (see (2.11) and (2.12)). Thus to any derivation $\mathcal{D}$ along (tangent) vector fields on V there corresponds, via (8.3), a natural derivation D along the paths in V on the set of liftings generated by sections. These facts are a hint for the possibility to introduce 'normal' frames for $\mathcal{D}$. This can be done as follows.

Let $\{e_i\}$ be a C^1 frame in $\pi^{-1}(V)$. We define the *components* or *(2-index) coefficients* ${\Gamma_X}^i{}_j \colon V \to \mathbb{K}$ of $\mathcal{D}_X$ by the expansion (cf. (2.3))

$$\mathcal{D}_X e_i = {\Gamma_X}^j{}_i \, e_j. \tag{8.4}$$

So $\mathbf{\Gamma}_X := \big[{\Gamma_X}^j{}_i\big]$ is the *matrix* of $\mathcal{D}_X$ in $\{e_i\}$.

Applying (8.2) to $\sigma = \sigma^i e_i$ and using the linearity of $\mathcal{D}_X$, we get the explicit expression (cf. (3.23))

$$\mathcal{D}_X(\sigma) = \big(X(\sigma^i) + {\Gamma_X}^i{}_j \, \sigma^j\big) e_i. \tag{8.5}$$

A simple verification proves that the change $\{e_i\} \mapsto \{e'_i = A^j_i e_j\}$, with a non-degenerate C^1 matrix-valued function $A = \big[A^j_i\big]$, leads to (cf. (3.26))

$$\mathbf{\Gamma}_X := \big[{\Gamma_X}^i{}_j\big] \mapsto \mathbf{\Gamma}'_X := \big[{\Gamma'_X}^i{}_j\big] = A^{-1}\mathbf{\Gamma}_X A + A^{-1}X(A), \tag{8.6}$$

where $X(A) := \big[X(A^j_i)\big]$ and the frames $\{e_i\}$ and $\{e'_i\}$ (as well as A) are supposed to be defined on an open subset of M containing or equal to U (see Note 6.1). Conversely, if a geometrical object with components ${\Gamma_X}^i{}_j$ is given in a frame $\{e_i\}$ and a change $\{e_i\} \mapsto \{e'_i = A^j_i e_j\}$ implies the transformation (8.6), then there exists a unique derivation along X, defined via (8.5), whose components in $\{e_i\}$ are exactly ${\Gamma_X}^i{}_j$ (cf. Proposition 3.6).

Below, for the sake of simplicity, we take $V = M$, i.e., the derivations are over the whole base M.

Definition 8.2. A *frame* $\{e_i\}$, defined on an open subset of M containing or equal to a set U, is called *normal for a derivation $\mathcal{D}$ along tangent vector fields* (*resp. for $\mathcal{D}_X$ along a given tangent vector field X*) *on U* if in $\{e_i\}$ the components of $\mathcal{D}$ (resp. $\mathcal{D}_X$) vanish on U for every (resp. the given) tangent vector field X.

If $\mathcal{D}$ (resp. $\mathcal{D}_X$) admits frames normal on $U \subseteq M$, we call it *Euclidean* on U. A number of results, analogous to those of Sections 4–6, can be proved for such derivations. Here we shall mention only a few of them.

Proposition 8.1 (cf. Theorem 5.2)**.** *A derivation $\mathcal{D}$ along tangent vector fields admits frame(s) normal on $U \subseteq M$ iff in every frame its matrix on U has the form*

$$\Gamma_X|_U = \big(F^{-1}X(F)\big)\big|_U \tag{8.7}$$

where F is a C^1 non-degenerate matrix-valued function defined on an open set containing U.

Proof. If $\{e'_i\}$ is normal on U for $\mathcal{D}$, then (8.7) with $F = A^{-1}$ follows from (8.6) with $\mathbf{\Gamma}'_X|_U = 0$. Conversely, if (8.7) holds, then (8.6) with $A = F^{-1}$ yields $\mathbf{\Gamma}'_X|_U = 0$. □

Proposition 8.2 (cf. Corollary 4.5)**.** *The frames normal on $U \subseteq M$ for a Euclidean derivation along vector fields* (*resp. given vector field X*) *are connected by linear transformations whose matrices A are constant* (*resp. $X(A) = 0$*) *on U.*

Proof. The result is a consequence of (8.6) for $\mathbf{\Gamma}_X = \mathbf{\Gamma}'_X = 0$. □

Definition 8.3. A *derivation $\mathcal{D}$ along* (*tangent*) *vector fields* is called *linear on U* if in some (and hence in any) frame its components admit the representation

$$\Gamma_X{}^i{}_j(p) = \Gamma^i{}_{j\mu}(p) X^\mu(p) \qquad \text{or} \quad \mathbf{\Gamma}_X = \Gamma_\mu X^\mu \tag{8.8}$$

where $p \in U$, $\Gamma_\mu = \left[\Gamma^i{}_{j\mu}(p)\right]_{i,j=1}^{\dim \pi^{-1}(p)}$ are matrix-valued functions on (a neighborhood of) U, and X^μ are the local components of a vector field X in some frame $\{E_\mu\}$ in the bundle space $T(M)$ of the tangent bundle, $X = X^\mu E_\mu$.

Remark 8.1*.* The invariant definition of a derivation linear on U is via the equation

$$\mathcal{D}_{fX+gY} = f\mathcal{D}_X + g\mathcal{D}_Y \tag{8.9}$$

where $f, g \colon U \to \mathbb{K}$ and X and Y are tangent vector fields over U. But for the purposes of this work the above definition is more suitable. Comparing Definitions 8.1 and 8.3 (see also (8.9)) with [23, p. 74, Definition 2.51] or with Definition 14.7 on page 305, we see that a derivation along tangent vector fields is linear iff it is a covariant derivative operator (covariant derivative) in $(E, \pi, B)|_U$. Therefore the concepts linear derivation along tangent vector fields and covariant derivative operator (covariant derivative) coincide.

We call $\Gamma^i{}_{j\mu}$ 3*-index coefficients* of $\mathcal{D}$ or simply coefficients if there is no risk of misunderstanding. It is trivial to check that under changes of the frames they transform according to (6.5). It is easy to verify that to every linear derivation $\mathcal{D}$ there corresponds a unique derivation along paths or linear transport along paths whose 2-index coefficients are given via (6.1) with $\Gamma_\mu := \left[\Gamma^i{}_{j\mu}\right]$ being the matrices of the 3-index coefficients of $\mathcal{D}$.[1] Conversely, to any such transport or derivation along paths there corresponds a unique linear derivation along tangent vector fields with components ((2-index) coefficients) given by (8.8), i.e., with the same 3-index coefficients. So, *there is a bijective correspondence between the sets of linear derivations along tangent vector fields and derivations* (*or linear transports*) *along paths whose* (2*-index*) *coefficients admit the representation* (6.1). It should be emphasized, if the above discussion is restricted to a subset U, i.e., only for paths lying entirely in U, it remains valid iff U is an open set in M. Otherwise, if U is not an open set, the correspondence between derivations along tangent vector fields and derivations or linear transports along paths via their 3-index coefficients is surjective in

[1] One can verify that the action of the derivation along paths induced by $\mathcal{D}$ on the liftings generated by sections is given by (8.3).

the right direction.[2] In the opposite direction it is injective, if the 3-index coefficients of the derivations along paths are fixed, or/and to a single derivation along paths may correspond different derivations along tangent vector fields, if the arbitrariness of the 3-index coefficients of the former derivations is taken into account. This remark is important when the normal frames in the both cases are compared.

Proposition 8.3. *A derivation along tangent vector fields is Euclidean on U iff it is linear on U and, in every frame $\{e_i\}$ over U in the bundle space and every frame $\{E_\mu\}$ over U in the tangent bundle space over U, the matrices Γ_μ of its 2-index coefficients have the form* (6.3) *for some non-degenerate C^1 matrix-valued function $\boldsymbol{F}$ on U.*

Proof. The result is a corollary from Proposition 8.1 as $X = X^\mu E_\mu$ and (8.7) imply (8.8) with $\Gamma_\mu := [\Gamma^i{}_{j\mu}] = F^{-1}E_\mu(F)$. □

Theorem 8.1 (cf. Theorem 6.1). *Frames normal on a neighborhood U for a derivation $\mathcal{D}$ along vector fields exist iff it is linear on U and its 3-index coefficients satisfy the conditions* (6.6) *on U.*

Proof. By Proposition 8.3, a derivation $\mathcal{D}$ along vector fields is Euclidean iff (6.3) holds for some $\boldsymbol{F}$ which, according to Lemma 6.1, is equivalent to (6.6). □

Proposition 8.4 (cf. Proposition 6.2). *A frame is normal on $U \subseteq M$ for some linear derivation along tangent vector fields iff the derivation's* 3*-index coefficients vanish on U.*

Proof. This result is a corollary of Definition 8.2, equation (8.8) and the arbitrariness of X in it. □

Remark 8.2. The arbitrariness of U in this Proposition does not contradict Remark 6.4 as the restriction of a derivation along tangent vector fields to (a neighborhood of) U is along vector fields X on U tangent to M, i.e., $X_p \in T_p(M)$ with $p \in U$.

In this way we have proved the *existence of a bijective mappings between the sets of Euclidean derivations along paths and Euclidean linear transports along paths.* It is given via the (local) coincidence of their 3-index coefficients in some (local) frames. Moreover, the normal frames for the corresponding objects of these sets coincide. What concerns the frames normal for Euclidean derivations along tangent vector fields, in them, by Proposition 8.4, vanish not only their 2-index coefficients, but also the 3-index ones. Hence the set of these frames is, generally, a subset of the one of frames normal for derivations or linear transports along paths. More details on frames in which the 3-index coefficients of a derivations or transport vanish will be presented in Section 12. These observations, combined with Remark 8.1, are quite important because they allow *in extenso* transferring

[2]If Γ_μ and $\overline{\Gamma}_\mu$ are the matrices of the 3-index coefficients of $\mathcal{D}$ and $\overline{\mathcal{D}}$, then they define a single derivation along paths in U iff $(\overline{\Gamma}_\mu - \Gamma_\mu)\dot{\gamma}^\mu = 0$ for all C^1 paths $\gamma\colon J \to U$.

of the results obtained for linear transports and derivations (along paths or along tangent vector fields) to the theory of connections, parallel transports, and covariant derivatives in general vector bundles. We shall return to this range of problems in Section 14.

9. Curvature and normal frames

Until now the concept of a 'curvature', in connection with normal frames, appeared several times in similar contexts. For the first time, we met it in Section II.4: the vanishment of the curvature (I.3.11) of a linear connection is a necessary and sufficient condition for the existence of frames normal on a neighborhood for the linear connection on it (Theorem II.4.1 on page 104). Next, the curvature of a linear connection appeared in Theorem II.5.2, point (i), on page 120 for analogous result on submanifolds. Similar results for arbitrary derivations along vector fields on a manifold are expressed by Theorems III.6.1, III.7.1, III.8.1, III.8.3, and III.8.4 in which the curvature (III.2.12) of the derivation plays an essential role. One would probably ask for the origin of the condition for ('partial') vanishment of the curvature in all of these results. From algebraic view-point, this condition is nothing else but the integrability condition for a system of partial differential equations of first order, the rigorous result being expressed by Lemma II.4.1 on page 105 (see, in particular, equation (II.4.6) in it).[1] Alternatively, from geometrical point of view, this means the path-independence of the parallel transport, generated by a (suitable) linear connection, along paths in some set (see Lemma II.5.1 on page 112 and equations (II.5.1′) on page 113 and (II.5.1″) on page 114).

All of the above-mentioned results have natural analogues in the theory of frames normal for linear transports along paths (or derivations along paths or along tangent vector fields). Practically they were derived, in algebraic terms, in Sections 5 and 6. To the reformulation of a part of the corresponding assertions, employing the concept of a 'curvature', is devoted the present section.

9.1. Curvature of linear transport or derivation along paths

Since there is a bijective correspondence between linear transports along paths and derivations along paths (see Section 3), two equivalent ways for the introduction of curvature can be pointed out. The algebraic approach is based on the consideration of a 'commutator' of two derivations along two different paths passing through one and the same point. Alternatively, in the geometrical approach, a vector is transported along a closed path and the result of this transportation is investigated when the path is contracted (tends) to a single point. Below we shall realize the former approach as it is somewhat more direct and concise.

[1] From here the adjective 'integrable' as a synonym for 'flat' in connection with vanishing curvature comes from.

The direct transferring of the concept of a 'curvature' of a derivation along vector fields (of the tensor algebra over a manifold), introduced via (III.2.12) (see also (I.3.11) in the particular case of linear connections), to derivations along paths is impossible as, due to (2.1c), the derivative along a path of a lifting is not a lifting. For the purpose is required a kind of derivation along paths which preserves the type of the objects on which it acts. Without going into details, we notice that the only derivation of this kind, introduce until now and having the property mentioned, is the section-derivation along paths generated by a derivation along paths and defined via (2.11) and (2.12). Alternatively, the 'commutator' of two derivations along paths should be suitable defined. Below we briefly sketch these approaches to the concept of a curvature of linear transports along paths.

Let D be a C^1 derivation along paths in a vector bundle (E, π, B), E being C^1 manifold, and D be the section-derivation generated by it (see page 221). Let $\eta\colon J \times J' \to B$ be an *injective* mapping, i.e., for every $(s, t) \in J \times J'$ there does not exist $(s', t') \in J \times J'$ such that $(s', t') \neq (s, t)$ and $\eta(s', t') = \eta(s, t)$.

The *(section-)curvature (operator)* along η at $(s, t) \in J \times J'$ of a C^1 section-derivation D along paths, generated by a C^1 derivation D along paths, is a mapping

$$\mathsf{R}^\eta(s,t)\colon\ \mathrm{Sec}^2\big((E,\pi,B)|_{\eta(J,J')}\big) \to \pi^{-1}(\eta(s,t)) \tag{9.1}$$

defined as follows. For every section $\sigma \in \mathrm{Sec}^2\big((E,\pi,B)|_{\eta(J,J')}\big)$, define sections $\sigma_1, \sigma_2 \in \mathrm{Sec}^1\big((E,\pi,B)|_{\eta(J,J')}\big)$ such that, for every $(s,t) \in J \times J'$,

$$\begin{aligned}
\sigma_1(\eta(s,t)) &:= \big(\mathsf{D}^{\eta(s,\,\cdot\,)}(\sigma|_{\eta(s,J')})\big)(\eta(s,t)) = D_t^{\eta(s,\,\cdot\,)}(\hat\sigma) \in \pi^{-1}(\eta(s,t))\\
\sigma_2(\eta(s,t)) &:= \big(\mathsf{D}^{\eta(\,\cdot\,,t)}(\sigma|_{\eta(J,t)})\big)(\eta(s,t)) = D_s^{\eta(\,\cdot\,,t)}(\hat\sigma) \in \pi^{-1}(\eta(s,t))
\end{aligned}$$

where the lifting $\hat\sigma \in \mathrm{PLift}(E,\pi,B)$ is such that $\hat\sigma\colon \gamma \mapsto \hat\sigma_\gamma := \sigma \circ \gamma$ for every path γ in $\eta(J, J')$.[2]

By definition, the action of the mapping (9.1) on σ is

$$\begin{aligned}
\big(\mathsf{R}^\eta(s,t)\big)(\sigma) &:= \big(\mathsf{D}^{\eta(\,\cdot\,,t)}(\sigma_1|_{\eta(J,t)})\big)(\eta(s,t)) - \big(\mathsf{D}^{\eta(s,\,\cdot\,)}(\sigma_2|_{\eta(s,J')})\big)(\eta(s,t))\\
&= D_s^{\eta(\,\cdot\,,t)}(\hat\sigma_1) - D_t^{\eta(s,\,\cdot\,)}(\hat\sigma_2)
\end{aligned} \tag{9.2}$$

where $\hat\sigma_1, \hat\sigma_2 \in \mathrm{PLift}(E,\pi,B)|_{\eta(J,J')}$ and $\hat\sigma_a\colon \gamma \mapsto (\hat\sigma_a)_\gamma := \sigma_a \circ \gamma$, $a = 1, 2$ for every path γ in $\eta(J, J')$.

Symbolically, by abuse of the notation, one may write

$$\mathsf{R}^\eta(s,t) = \mathsf{D}^{\eta(\,\cdot\,,t)} \circ \mathsf{D}^{\eta(s,\,\cdot\,)} - \mathsf{D}^{\eta(s,\,\cdot\,)} \circ \mathsf{D}^{\eta(\,\cdot\,,t)}$$

but, as a consequence of (2.11), compositions like $\mathsf{D}^{\eta(\,\cdot\,,t)} \circ \mathsf{D}^{\eta(s,\,\cdot\,)}$ are not quite correctly defined since the action of this expression on σ must be defined as $\mathsf{D}^{\eta(\,\cdot\,,t)}(\sigma_1|_{\eta(J,t)})$.

[2]The index 1 (resp. 2) in σ_1 (resp. σ_1) reflects that in the right-hand side of its definition the first (resp. second) argument, i.e., s (resp. t), is fixed, i.e., considered as a parameter.

Exercise 9.1. Suppose $\{e_i(p)\}$, $p \in \eta(J, J')$, is a basis in the fibre $\pi^{-1}(p)$. In the frame $\{\hat{e}_i\}$, with $\hat{e}_i \colon \gamma \mapsto (\hat{e}_i)_\gamma \colon r \mapsto e_i(r;\gamma) = e_i(\gamma(r))$, $r \in J''$, for a path $\gamma \colon J'' \to \eta(J, J')$, the expansion (2.4) with $s = r$ holds for every lifting $\lambda \in \mathrm{PLift}^1(E, \pi, B)|_{\eta(J,J')}$. Applying this formula, find explicit expressions for the derivatives $D_t^{\eta(s,\,\cdot\,)}\hat{\sigma}$, $D_s^{\eta(\,\cdot\,,t)}\hat{\sigma}$, $D_t^{\eta(s,\,\cdot\,)}\hat{\sigma}_2$, and $D_s^{\eta(\,\cdot\,,t)}\hat{\sigma}_1$, entering in (9.2). Substituting the so-obtained expressions into (9.2), prove that the (section-)curvature operator (9.1) is *linear*,

$$(\mathsf{R}^\eta(s,t))(\sigma) = \big((\mathsf{R}^\eta(s,t))\big)^i{}_j\, \sigma^j(\eta(s,t)) e_i(\eta(s,t)), \tag{9.3}$$

and that its components in a frame $\{e_i\}$ are

$$\begin{aligned}\big((\mathsf{R}^\eta(s,t))\big)^i{}_j = {}& \frac{\partial \Gamma^i{}_j(t;\eta(s,\,\cdot\,))}{\partial s} - \frac{\partial \Gamma^i{}_j(s;\eta(\,\cdot\,,t))}{\partial t} \\ & + \Gamma^i{}_k(s;\eta(\,\cdot\,,t))\Gamma^k{}_j(t;\eta(s,\,\cdot\,)) - \Gamma^i{}_k(t;\eta(s,\,\cdot\,))\Gamma^k{}_j(s;\eta(\,\cdot\,,t))\end{aligned} \tag{9.4}$$

where $\Gamma^i{}_j(\cdots)$ are the components of the section-derivation D, or of the derivation D generating it, in the frame $\{\hat{e}_i\}$ (see (2.3)).

Having in mind the above introduction of the curvature of a section-derivation, now we shall define the curvature (operator) of arbitrary derivation along paths.

Let D be a C^1 derivation along paths in a vector bundle (E, π, B), E being C^1 manifold, and $\eta \colon J \times J' \to B$ be *arbitrary*, injective or not, mapping. Let $\bar{\eta} \colon J \times J' \to E$ be of class C^2 and a lifting of η, $\pi \circ \bar{\eta} := \eta$.

The *curvature (operator)* along η at $(s,t) \in J \times J'$ of the derivation D is a mapping

$$R^\eta(s,t) \colon \bar{\eta} \mapsto (R^\eta(s,t))(\bar{\eta}) \in \pi^{-1}(\eta(s,t)) \tag{9.5}$$

defined in the following way.

Let $\hat{\eta}_1, \hat{\eta}_2 \in \mathrm{PLift}^2(E, \pi, B)$ and $\check{\eta}_1, \check{\eta}_2 \colon J \times J' \to E$ be such that

$$\begin{aligned}\hat{\eta}_1(\eta(s,\,\cdot\,)) &:= \bar{\eta}(s,\,\cdot\,), \quad \check{\eta}_1(s,t) := \big(D^{\eta(s,\,\cdot\,)}(\hat{\eta}_1)\big)(t) \\ \hat{\eta}_2(\eta(\,\cdot\,,t)) &:= \bar{\eta}(\,\cdot\,,t), \quad \check{\eta}_2(s,t) := \big(D^{\eta(\,\cdot\,,t)}(\hat{\eta}_2)\big)(s).\end{aligned}$$

By definition, the action of the curvature $R^\eta(s,t)$ (along η at (s,t) of D) on $\bar{\eta}$ is

$$(R^\eta(s,t))(\bar{\eta}) := \big(D^{\eta(\,\cdot\,,t)}(\hat{\hat{\eta}}_2)\big)(s) - \big(D^{\eta(s,\,\cdot\,)}(\hat{\hat{\eta}}_1)\big)(t) \tag{9.6}$$

where the liftings $\hat{\hat{\eta}}_1, \hat{\hat{\eta}}_2 \in \mathrm{PLift}^1(E, \pi, B)$ are such that[3]

$$\hat{\hat{\eta}}_1(\eta(s,\,\cdot\,)) := \check{\eta}_2(s,\,\cdot\,), \quad \hat{\hat{\eta}}_2(\eta(\,\cdot\,,t)) := \check{\eta}_1(\,\cdot\,,t).$$

[3]Notice the positions of the subscripts 1 and 2 in the next definitions.

The explicit 'local' expression for the curvature $R^\eta(s,t)$ can be found by means of the expansion (2.4). Let $\{\hat{e}_i(p)\}$, $p \in \eta(J, J')$, be a basis in $\pi^{-1}(p)$. In the frame $\{e_i\}$, with $\hat{e}_i \colon \gamma \mapsto (\hat{e}_i)_\gamma \colon r \mapsto e_i(r;\gamma) = e_i(\gamma(r))$ for a path $\gamma \colon J'' \to \eta(J, J')$ and $r \in J''$, we have (see (2.9))

$$\begin{aligned} D^{\eta(s,\,\cdot\,)}(\hat{\eta}_1) &= \big[\dot{\hat{\eta}}{}^i_1(\eta(s,\,\cdot\,)) + \Gamma^i{}_j(\,\cdot\,;\eta(s,\,\cdot\,))\hat{\eta}^j_1(s,\,\cdot\,)\big] e_i(\,\cdot\,;\eta(s,\,\cdot\,)) \\ D^{\eta(\,\cdot\,,t)}(\hat{\eta}_2) &= \big[\dot{\hat{\eta}}{}^i_2(\eta(\,\cdot\,,t)) + \Gamma^i{}_j(\,\cdot\,;\eta(\,\cdot\,,t))\hat{\eta}^j_2(\,\cdot\,,t)\big] e_i(\,\cdot\,;\eta(\,\cdot\,,t)) \end{aligned}$$

where $\Gamma^i{}_j(s;\gamma)$ are the components of D in $\{\hat{e}_i\}$ and (see (2.6))

$$\dot{\hat{\eta}}{}^i_1(\eta(s,\,\cdot\,)) \colon t \mapsto \frac{\partial\big(\hat{\eta}^i_1(\eta(s,\,\cdot\,))(t)\big)}{\partial t} = \frac{\partial \bar{\eta}^i(s,t)}{\partial t}, \quad \dot{\hat{\eta}}{}^i_2(\eta(\,\cdot\,,t)) \colon s \mapsto \frac{\partial \bar{\eta}^i(s,t)}{\partial s}.$$

Using these results, (2.4), and the definitions of $\check{\eta}_1$ and $\check{\eta}_2$, we find:

$$\begin{aligned} \check{\eta}_1(s,t) &= \Big[\frac{\partial \bar{\eta}^i(s,t)}{\partial t} + \Gamma^i{}_j(t;\eta(s,\,\cdot\,))\bar{\eta}^j(s,t)\Big] e_i(t;\eta(s,\,\cdot\,)) \\ \check{\eta}_2(s,t) &= \Big[\frac{\partial \bar{\eta}^i(s,t)}{\partial s} + \Gamma^i{}_j(s;\eta(\,\cdot\,,t))\bar{\eta}^j(s,t)\Big] e_i(s;\eta(\,\cdot\,,t)). \end{aligned}$$

Finally, calculating the derivatives $\big(D^{\eta(\,\cdot\,,t)}(\hat{\check{\eta}}_2)\big)(s)$ and $\big(D^{\eta(s,\,\cdot\,)}(\hat{\check{\eta}}_1)(t)\big)$ by means of equation (2.4), using the relations (see (2.6))

$$\dot{\hat{\check{\eta}}}{}^i_1(\eta(s,\,\cdot\,)) \colon t \mapsto \frac{\partial\big(\hat{\check{\eta}}{}^i_1(\eta(s,\,\cdot\,))(t)\big)}{\partial t} = \frac{\partial \check{\eta}^i_2(s,t)}{\partial t}, \quad \dot{\hat{\check{\eta}}}{}^i_2(\eta(\,\cdot\,,t)) \colon s \mapsto \frac{\partial \check{\eta}^i_1(s,t)}{\partial s},$$

and the formulae just-obtained, from (9.6), we get

$$(R^\eta(s,t))(\bar{\eta}) = \big(R^\eta(s,t)\big)^i{}_j\, \bar{\eta}^j e_i(\eta(s,t)), \tag{9.7}$$

where $e_i(s;\eta(\,\cdot\,,t)) = e_i(t;\eta(s,\,\cdot\,)) = e_i(\eta(s,t))$ was taken into account, and

$$\begin{aligned} \big(R^\eta(s,t)\big)^i{}_j = {}& \frac{\partial \Gamma^i{}_j(t;\eta(s,\,\cdot\,))}{\partial s} - \frac{\partial \Gamma^i{}_j(s;\eta(\,\cdot\,,t))}{\partial t} \\ & + \Gamma^i{}_k(s;\eta(\,\cdot\,,t))\Gamma^k{}_j(t;\eta(s,\,\cdot\,)) - \Gamma^i{}_k(t;\eta(s,\,\cdot\,))\Gamma^k{}_j(s;\eta(\,\cdot\,,t)) \end{aligned} \tag{9.8}$$

which in a matrix form reads

$$\begin{aligned} \boldsymbol{R}^\eta(s,t) := \big[\big(R^\eta(s,t)\big)^i{}_j\big] = {}& \frac{\partial \boldsymbol{\Gamma}(t;\eta(s,\,\cdot\,))}{\partial s} - \frac{\partial \boldsymbol{\Gamma}(s;\eta(\,\cdot\,,t))}{\partial t} \\ & + \boldsymbol{\Gamma}(s;\eta(\,\cdot\,,t))\boldsymbol{\Gamma}(t;\eta(s,\,\cdot\,)) - \boldsymbol{\Gamma}(t;\eta(s,\,\cdot\,))\boldsymbol{\Gamma}(s;\eta(\,\cdot\,,t)). \end{aligned} \tag{9.9}$$

Consequently the curvature $R^\eta(s,t)$ of a derivation D along paths is a *linear* operator whose *matrix elements (components)* are (9.8) in $\{\hat{e}_i\}$ and which coincide, for injective η, with the ones of the section-derivation D along paths generated by D (see (9.4)),

$$\big(R^{\eta}(s,t)\big)^{i}_{\ j} = \big(\mathsf{R}^{\eta}(s,t)\big)^{i}_{\ j}. \tag{9.10}$$

This equality is not accidental: if η is injective and one puts $\bar{\eta} = \sigma \circ \eta$ for a C^1 section σ over $\eta(J, J')$, then, making the corresponding changes, one can transform (9.7) into (9.3) with $\big(\mathsf{R}^{\eta}(s,t)\big)^{i}_{\ j} = \big(R^{\eta}(s,t)\big)^{i}_{\ j}$.

Definition 9.1. The *curvature* (*operator*) *along* $\eta\colon J \times J' \to B$ *at* $(s,t) \in J \times J'$ *of a* C^1 *linear transport along paths* is called the curvature (operator) along $\eta\colon J \times J' \to B$ at $(s,t) \in J \times J'$ of the generated by it via (3.19) derivation along paths.

Definition 9.2. The *curvature* (*operator*) *of linear transports or derivations along paths* is a mapping R assigning to each transport or derivation and every $\eta\colon J \times J' \to B$ and $(s,t) \in J \times J'$ the corresponding curvature $R^{\eta}(s,t)$ along η at (s,t).

The following result is important in connection with the theory of normal frames (see Proposition 6.1 on page 247), as well as for making a link with the concept of a curvature of the linear connections.

Proposition 9.1. *Let* (E, π, M) *be a vector bundle whose bundle space* E *and base* M *are* C^1 *manifolds,* $U \subseteq M$*, and the matrix of the* 2*-index coefficients of a linear transport* L *or derivations* D *along paths has on* U *the form*

$$\mathbf{\Gamma}(s;\gamma) = \sum_{\mu=1}^{\dim M} \Gamma_{\mu}(\gamma(s))\dot{\gamma}^{\mu}(s) \tag{9.11}$$

in every frame $\{e_i\}$ *on* U*, where* $\Gamma_{\mu} = \big[\Gamma^{i}_{\ j\mu}\big]$ *are some matrix-valued functions on an open set containing* U *or equal to it,* $\gamma\colon J'' \to U$ *is of class* C^1*,* $s \in J''$*, and* $\dot{\gamma}^{\mu}$ *are the components of the tangent vector field* $\dot{\gamma}$ *in some frame* $\{E_{\mu}\}$ *over* U *in the tangent bundle space over* M*,* $\dot{\gamma} = \dot{\gamma}^{\mu}E_{\mu}$*. Then, for every* C^1 *mapping* $\eta\colon J \times J' \to U$*, the components of the curvature of* L *or* D *in* $\{e_i\}$ *along* η *at* $(s,t) \in J \times J'$ *are*

$$\big(R^{\eta}(s,t)\big)^{i}_{\ j} = R^{i}_{\ j\alpha\beta}(\eta(s,t))\,(\eta'(s,t))^{\alpha}\,(\eta''(s,t))^{\beta}, \tag{9.12}$$

where

$$R^{i}_{\ j\alpha\beta} := -\Gamma^{i}_{\ j\alpha,\beta} + \Gamma^{i}_{\ j\beta,\alpha} - \Gamma^{k}_{\ j\alpha}\Gamma^{i}_{\ k\beta} + \Gamma^{k}_{\ j\beta}\Gamma^{i}_{\ k\alpha} - \Gamma^{i}_{\ j\mu}C^{\mu}_{\alpha\beta}\colon U \to \mathbb{K} \tag{9.13}$$

with $[E_{\alpha}, E_{\beta}]_{\_} =: C^{\mu}_{\alpha\beta}E_{\mu}$ *and* $f_{,\mu} := E_{\mu}(f)$ *for a* C^1 *function* f *on* U*. Here* $\eta'(\,\cdot\,,t)$ *and* $\eta''(s,\,\cdot\,)$ *denote the tangent vectors to the paths* $\eta(\,\cdot\,,t)$ *and* $\eta(s,\,\cdot\,)$*, respectively.*

Proof. One should substituted (9.11) for the corresponding paths into (9.8), then to use the equalities $\frac{\partial}{\partial s}\big|_{\eta(s,t)} = (\eta'(s,t))^\alpha E_\alpha|_{\eta(s,t)} = \eta'(s,t)$ and $\frac{\partial}{\partial t}\big|_{\eta(s,t)} = (\eta''(s,t))^\beta E_\beta|_{\eta(s,t)} = \eta''(s,t)$, and, at last, to apply (I.8.1).[4] □

For future purposes, we shall rewrite (9.13) in a matrix form:

$$\boldsymbol{R}_{\alpha\beta} := \left[R^i{}_{j\alpha\beta}\right]_{i,j=1}^{\dim\pi^{-1}(p)} = -E_\beta(\Gamma_\alpha) + E_\alpha(\Gamma_\beta) - \Gamma_\beta\Gamma_\alpha + \Gamma_\alpha\Gamma_\beta - C^\mu_{\alpha\beta}\Gamma_\mu \quad (9.14)$$

for some point p in M (recall, $\dim\pi^{-1}(p)$ is simply the (fibre) dimension of the bundle).

We shall call the functions $R^i{}_{j\alpha\beta}$, given via (9.13), *4-index components* of the curvature with respect to the pair of frames $(\{e_i\}, \{E_\mu\})$. In this connection, the matrix elements $(R^\eta(s,t)^i{}_j)$ are natural to be called *2-index components* of the curvature with respect to the frame $\{e_i\}$. Evidently, the curvature of every linear transport has 2-index components, while only the ones of transports with coefficients' matrix (9.11) have 4-index components. A straightforward calculation shows that a change $\{e_i\} \mapsto \{e' = A^j_i e_j\}$ implies

$$\boldsymbol{R}^\eta(s,t) \mapsto \boldsymbol{R}'^\eta(s,t) = A^{-1}(\eta(s,t)\boldsymbol{R}^\eta(s,t)A(\eta(s,t), \quad (9.15)$$

while its combination with $\{E_\alpha\} \mapsto \{E'_\alpha = B^\beta_\alpha E_\beta\}$ leads to

$$\boldsymbol{R}_{\alpha\beta} \mapsto \boldsymbol{R}'_{\alpha\beta} = B^\mu_\alpha B^\nu_\beta A^{-1}\boldsymbol{R}_{\mu\nu}A. \quad (9.16)$$

So, the 4-index components of the curvature of a transport/derivation for which (9.11) holds, of which kind are the Euclidean transports or derivations, are independent of the particular mapping η by means of which they are defined, while the 2-index components depend on the point at which they are calculated and on the tangent vectors to η at it.

Proposition 9.2. *Let (E,π,M), E and M being C^1 manifolds, be a vector bundle and L be a linear transport along paths in it for which* (9.11) *holds for every path $\gamma\colon J \to U$, $U \subseteq M$. If* $\dim M = 0, 1$, *then the curvature of L identically vanishes on every $U \subseteq M$.*

Proof. See (9.13) for $\alpha = \beta = 1$ (if $\dim M = 1$) or remove the subscripts α and β (and put $C^\mu = 0$) in it (if $\dim M = 0$). □

Example 9.1. A simple look over equations (9.13) on the preceding page and (I.3.13) on page 26 reveals a striking similarity between them. Of course, this is not accidental: if we take the tangent bundle $(T(M),\pi,M)$ for (E,π,M) and the covariant derivative along paths induced by some linear connection ∇ (see

[4]The computation is somewhat simpler if one proves (9.13) for a coordinate frame $\{E_\alpha = \frac{\partial}{\partial x^\alpha}\}$ and then transforms the result, by means of (6.5) with $A = \mathbb{1}$, to an arbitrary frame $E_\alpha = B^\beta_\alpha \frac{\partial}{\partial x^\beta}$ and, at last, applies (I.8.3).

Section III.10) for D, then (9.11) holds for the coefficients' matrices Γ_μ of ∇ and, consequently, the right hand sides of (9.13) and (I.3.13) coincide as now all indices run from 1 to $\dim M$ and ${\Gamma^i}_{jk}$ are the coefficients of ∇. We can express this result by saying that the components of the curvature tensor of a linear connection ∇ are identical with the (4-index) components of the curvature of the parallel transport or covariant derivative along paths generated by ∇.

Example 9.2. The result of Example 9.1 can be proved in a basis-free way as follows. Let ∇ be a linear connection on a manifold M, D be a derivation along paths in $(T(M), \pi, M)$ with matrix (9.11) in some (every) frame in which Γ_μ are the matrices of the coefficients of ∇, and $A, B \in \mathfrak{X}^2(M)$. Suppose $p \in M$ and a C^2 mapping $\eta\colon J \times J' \to M$ be defined by the equations $\eta(s_0, t_0) = p$, $\eta'(s_0, t_0) = A_p$, and $\eta''(s_0, t_0) = B_p$ for a fixed $(s_0, t_0) \in J \times J'$. By means of (2.12), one can easily prove that the section-derivation D generated by D is identical with the covariant derivative along paths induced by ∇, $\mathsf{D}\colon \gamma \mapsto \mathsf{D}^\gamma = \nabla_{\dot\gamma}$ for every injective path γ. Applying this result and (9.2), we find

$$(\mathsf{R}^\eta(s_0, t_0))(X) = [(R^\nabla(A, B))(X)]|_p, \qquad X \in \mathfrak{X}(M) \tag{9.17}$$

where R^∇ is the curvature (operator) of ∇, given by the right-hand side of (I.3.11) (see also (III.2.12) for $D = \nabla$).

Exercise 9.2. Prove that, if $\mathbf{D}$ is a *linear* derivation along vector fields of $\boldsymbol{T}^1(M)$ (see Definition III.2.2 on page 145) and D is a derivation along paths in the tangent bundle $(T(M), \pi, M)$ for which (9.11) is valid with Γ_μ given via (III.2.9) (with $k = \mu$), then $\mathsf{D}\colon \gamma \mapsto \mathsf{D}\gamma = \mathbf{D}_{\dot\gamma}$ and

$$(\mathsf{R}^\eta(s_0, t_0))(X) = [(R^{\mathbf{D}}(A, B))(X)]|_p, \qquad X \in \mathfrak{X}(M),\ p = \eta(s_0, t_0), \tag{9.18}$$

where $R^{\mathbf{D}}$ is the curvature (operator) of $\mathbf{D}$ defined via (III.2.12) and whose local components are given by (III.2.14).[5]

Definition 9.3. A linear transport or derivation along paths is called *flat*, or *curvature free*, or *integrable* (resp. on U) if its curvature vanishes on M (resp. on a set $U \subseteq M$), i.e., $R^\eta(s, t) = 0$ for every $\eta\colon J \times J' \to M$ (resp. for every $\eta\colon J \times J' \to U$).

Remark 9.1. It should be emphasized on a principal difference between the flatness on $U \subseteq B$ of a linear transport along paths in a bundle (E, π, B) and of a linear connection or derivation along tangent vector fields on a manifold M. In the former case, the curvature is zero on U along mappings η *in* U, i.e., $\eta\colon J \times J' \to U$, while in the latter one it vanishes only *along vectors tangent to* U, i.e., from the tangent space $T_p(M)$, $p \in U$. If $(E, \pi, B) = (T(M), \pi, M)$ and U is a submanifold of M, the first case means that $\eta'(s,t), \eta''(s,t) \in T_{\eta(s,t)}(U) \subseteq T_{\eta(s,t)}(M)$, while in the second one, we have $\eta'(s,t), \eta''(s,t) \in T_{\eta(s,t)}(M)$, i.e., η' and η'' are not generally in the tangent bundle space over U of $(T(U), \pi, U)$.

[5]For further generalization of these results, see Proposition 14.15.

9.2. On the curvature of Euclidean transports along paths

The main result expressing the relation between curvature of and the frames normal for a linear transport along paths is the following one.

Theorem 9.1. *Let (E, π, B), E being C^1 manifold, be a vector bundle, L be a linear transport along paths in it, and $U \subseteq B$. The transport L is Euclidean on U if and only if it is flat on U, i.e., frames normal on U for L exist iff the curvature of L along mappings in U is identically zero.*

Remark 9.2. In some sense, this theorem is the 'peak' or 'summit' of the series of similar Theorems II.5.2, III.7.1, III.8.1, and 6.1 which are its special cases.

Proof. NECESSITY. If L is Euclidean on U, by Definition 4.1′ or Proposition 4.1, its coefficients vanish in every frame normal on U, $\Gamma^i{}_j(r;\gamma) \equiv 0$ for every $\gamma\colon J'' \to U$ and $r \in J''$. Substituting this into equation (9.8) with $\eta\colon J \times J'' \to U$, we get $R^\eta(s,t) \equiv 0$.

SUFFICIENCY. According to the remark at the beginning of Section 5 (see the paragraph containing equation (5.1) on page 240), we have to prove the existence of solution of the normal frame equation

$$\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s} + \mathbf{\Gamma}(s;\gamma)A(\gamma(s)) = 0 \tag{9.19}$$

in a frame $\{e_i\}$ on U with respect to $A\colon U \to \mathrm{GL}(\dim \pi^{-1}(p), \mathbb{K})$, $p \in B$, for a flat on U linear transport along every path $\gamma\colon J \to U$.

Let $V := J \times \cdots \times J$ ($\dim \pi^{-1}(p)$-times), $\chi\colon V \to U$, $y \in V$, and the mappings $\chi^y_{ij}\colon J \times J \to U$ be defined by $\chi^y_{ij}(s,t) := \chi(y)\big|_{y^i=s,y^j=t}$, $s,t \in J$, for $j \neq i$ and $\chi^y_{ij}(s,t) := \chi(y)|_{y^i=s}$ for $j = i$.

The auxiliary system of partial differential equations

$$\frac{\partial A(\chi(y))}{\partial y^j} = -\mathbf{\Gamma}(y^j; \chi^y_{jl}(\,\cdot\,, y^l))A(\chi(y)) \tag{9.19$*$}$$

(the concrete value of the index l is insignificant: $\chi^y_{jk}(\,\cdot\,,y^k) = \chi^y_{jl}(\,\cdot\,,y^l)$), obtained from (9.19) for $\gamma = \chi^y_{jk}(\,\cdot\,,y^k)$ and $s = y^j$, always has (non-zero) solutions due to the flatness of the transport L on U. Indeed, the integrability conditions for it [34] are identically satisfied:

$$\begin{aligned}
&2\frac{\partial^2}{\partial y^{[j}\partial y^{k]}}A(\chi(y)) \\
&= \frac{\partial}{\partial y^j}[-\mathbf{\Gamma}(y^k;\chi^y_{kj}(\,\cdot\,,y^j))A(\chi(y))] - \frac{\partial}{\partial y^k}[-\mathbf{\Gamma}(y^j;\chi^y_{jk}(\,\cdot\,,y^k))A(\chi(y))]
\end{aligned}$$

$$= \Big[-\frac{\partial \mathbf{\Gamma}(y^k; \chi^y_{kj}(\,\cdot\,, y^j))}{\partial y^j} + \mathbf{\Gamma}(y^k; \chi^y_{kj}(\,\cdot\,, y^j))\mathbf{\Gamma}(y^j; \chi^y_{jk}(\,\cdot\,, y^k))$$
$$+ \frac{\partial \mathbf{\Gamma}(y^j; \chi^y_{jk}(\,\cdot\,, y^k))}{\partial y^k} - \mathbf{\Gamma}(y^j; \chi^y_{jk}(\,\cdot\,, y^k))\mathbf{\Gamma}(y^k; \chi^y_{kj}(\,\cdot\,, y^j))\Big] A(\chi(y))$$
$$= -R^{\chi^y_{jk}}(y^j, y^k) A(\chi(y)) \equiv 0$$

where we have used the initial system (9.19∗) (for different l) several times, $\chi^y_{kj}(\cdot, y^j) = \chi^y_{jk}(\cdot, y^k)$, (9.9) for $\eta = \chi^y_{jk}$, $s = y^j$, and $t = y^k$, and the flatness of L on U.

Let now A be a solution of the system (9.19∗) for y, χ, and χ^y_{jk} such that $y^{i_0} = s \in J$ for some fixed i_0, $\chi(y_0) = \gamma(s)$ for some $y_0 \in V$ with $y_0^{i_0} = s$, and $\chi^{y_0}_{i_0 l}(\,\cdot\,, y_0^l) = \gamma$.[6] Then the restriction of (9.19∗) at y_0 for $j = i_0$ is identical with (9.19) and, consequently, the frame $\{e'_i\}$ with $e'_i|_{\gamma(s)} := A^j_i(\gamma(s)) e_j|_{\gamma(s)}$ is normal along $\gamma\colon J \to U$. □

Now we have at our disposal the machinery required to look on Theorems 6.1 on page 249 and 6.2 on page 253 from the view-point of transports' curvature. Let (E, π, M) be a vector bundle whose bundle space and base are C^1 manifolds and $U \subseteq M$. By Proposition 9.1, if the coefficients' matrix of a linear transport along paths has the form (9.11), the matrices of the 4-index components of the curvature are (9.14) and consequently

$$R_{\mu\nu}(-\Gamma_1, \dots, -\Gamma_{\dim M}) = \boldsymbol{R}_{\mu\nu} \tag{9.20}$$

where $R_{\mu\nu}(\cdots)$ are defined by (6.7).[7] From this equality, the conclusion can be made that in Theorem 6.1 the condition (6.6) can, equivalently, be replace by the flatness of the transport on U. Moreover, in this theorem the validity of the representation (6.1) can be dropped: it follows either from the Euclidness of the transport (Proposition 6.1) or from the flatness (as by Theorem 9.1 the flatness implies the Euclidness). This completes the proof of the following assertion which is a stronger version of Theorem 6.1.

Corollary 9.1. *A linear transport along paths in vector bundle (E, π, M), E and M being C^1 manifolds, is Euclidean on $U \subseteq M$ iff it is flat on U or, equivalently, iff the conditions* (6.6) *hold for $p \in U$.*

Corollary 9.2. *Let (E, π, M), E and M being C^1 manifolds, be a vector bundle and L be a linear transport along paths in it for which* (9.11) *holds for every path $\gamma\colon J \to U$, $U \subseteq M$. If* $\dim M = 0, 1$, *then L is Euclidean on every set $U \subseteq M$.*

Proof. See Proposition 9.2 and Theorem 9.1. □

[6] Such a choice of $\chi^{y_0}_{i_0 l}$ is always (locally) possible; for the case when B is a C^1 manifold, see the paragraph containing equations (II.3.12) and (II.3.13).

[7] In the equality (9.20), we implicitly suppose that the matrix-valued functions Γ_μ, entering in (9.11) and defining $\boldsymbol{R}_{\mu\nu}$ via (9.14), are identical with the matrices of the 3-index coefficients of the transport which appear in (6.1). If U is not a $(\dim M)$-dimensional submanifold of M, in this way we introduce in the definition of $\boldsymbol{R}_{\mu\nu}$ an arbitrariness which is a consequence of the one in the definition of Γ_μ; for some details, see the discussion after Proposition 6.2.

The combination of the above results with the ones of Section 6 leads to a necessary and sufficient condition for the flatness on $U \subseteq M$ of a linear transport whose coefficient matrix has a representation (9.11) on U. At first sight, one may suppose L is flat on U iff

$$R_{\mu\nu} = 0 \tag{9.21}$$

on U. But, as it is clear from (9.12), these equalities imply the flatness on U but the opposite is generally not true unless U is a neighborhood, e.g., if $U = M$, or if on Γ_μ are imposed some additional conditions. Besides, as we know from Section 6, the equations (9.21) do not concern solely the transport under consideration, but they also imply some restrictions on the arbitrariness in the definition of the transport's 3-index coefficients. The elimination of the last restrictions on a submanifold of M from (9.21) leads to the following result.

Proposition 9.3. *A linear transport L along paths in (E, π, M) is flat on a submanifold N of M if and only if for every $p_0 \in N$ and a chart (V, x) of M such that $V \ni p_0$ and $x(p) = (x^1(p), \dots, x^{\dim N}(p), t_0^{\dim N+1}, \dots, t_0^{\dim M})$ for every $p \in N \cap V$ and constant numbers $t_0^{\dim N+1}, \dots, t_0^{\dim M}$, the equalities* (6.16) *hold for $p \in N \cap V$ with $R^N_{\alpha\beta}$ given via* (6.17) *in which the matrices $\Gamma_1, \dots, \Gamma_{\dim N}$, given via* (6.12), *are the first* $\dim N$ *of the matrices of the 3-index coefficients of L in the coordinate frame $\left\{\frac{\partial}{\partial x^\mu}\right\}$ in the tangent bundle space over $N \cap V$.*

Proof. See Theorem 6.2 on page 253 and Theorem 9.1 or Corollary 9.1. □

The essence of Proposition 9.3 is that it provides a criterion for a flatness on a submanifold of a linear transport along paths in terms of the 4-index components of its curvature, if they exist.

An important consequence of Theorem 9.1 is that the combination of it and any one of the necessary and sufficient conditions or assertions for the existence of normal frames derived in Sections 5 and 6 results in a corresponding criterion or assertion, respectively, for flatness of a linear transport or derivation along paths. For example, a transport is curvature free iff its action depends only on the initial and final points of the transportation, not on the particular path connecting them (cf. Theorem 5.1); along every fixed path every transport is flat (see Corollary 5.1), etc.

A simple corollary of Theorems 9.1 and 5.1 is the following important criterion for path-independence of a linear transport along paths.

Corollary 9.3. *Let L be a linear transport along paths in a vector bundle (E, π, B) and $U \subseteq B$. The transport L is path-independent over U, i.e., along paths in U, if and only if L is flat on U.*

Proof. If L is path-independent over U, it is Euclidean on U(Theorem 5.1) and so it is flat on U (Theorem 9.1). Conversely, if L is flat on U, it is Euclidean on U (Theorem 9.1) and, consequently, it is path-independent over U (Theorem 5.1). □

In short, we can summarize the content of this section by stating that the sets of Euclidean, flat, and path-independent linear transports along paths coincide.

10. Torsion and normal coordinates

In the preceding chapters, we have had a number of chances to see that, if frames normal for a linear connection or derivation of the tensor algebra on a manifold exist, then normal coordinates exist iff the torsion of the connection or derivation, respectively, vanishes. *Prima facie* one can expect similar results for linear transports or derivations along paths in vector bundles. But a simple look on the corresponding definitions (I.3.12) and (III.2.13) of the torsion reveals a peculiarity that can not be generalized to arbitrary vector bundles: they include the commutator of two tangent vector fields and the action of a derivation along one of them on the another one. These operations are primary related to the bundle $(T(M), \pi, M)$ tangent to a manifold M and at present are not known suitable generalizations of them for arbitrary vector bundles.[1] Since it seems that the tangent bundles are the only sufficiently general and frequently used bundles for which the concept of a 'torsion' can be introduced, they will be considered below in the present section.

10.1. Torsion of linear transport or derivation along paths in the tangent bundle over a manifold

Let M be a C^1 manifold and $(T(M), \pi, M)$ the bundle tangent to it. Suppose $\eta\colon J\times J' \to M$ is a C^2 *injective* mapping. Denote by $\eta'(\,\cdot\,, t)$ and $\eta''(s, \,\cdot\,)$ for $(s,t) \in J \times J'$ the vector fields tangent to the paths $\eta(\,\cdot\,, t)\colon J \to M$ and $\eta(s, \,\cdot\,)\colon J' \to M$, respectively. We consider η' and η'' as vector fields on $\eta(J, J')$, $\eta', \eta'' \in \mathfrak{X}(\eta(J, J')) = \mathrm{Sec}((T(M), \pi, M)|_{\eta(J,J')})$, such that

$$\eta'\colon \eta(s,t) \mapsto \eta'_{\eta(s,t)} := \eta'(s,t) \quad \text{and} \quad \eta''\colon \eta(s,t) \mapsto \eta''_{\eta(s,t)} := \eta''(s,t)$$

for every $(s,t) \in J \times J'$.[2]

At last, if $X \in \mathfrak{X}(\eta(J, J'))$, we denote with $\hat{X}$ a lifting of paths $\hat{X} \in \mathrm{PLift}((T(M), \pi, M)|_{\eta(J,J')})$ such that for every path $\gamma\colon J'' \to \eta(J, J')$, we have $\hat{X}\colon \gamma \mapsto \hat{X}_\gamma \in \mathrm{P}(\pi^{-1}(\gamma(J'')))$ with $\hat{X}_\gamma\colon r \mapsto \hat{X}_\gamma(r) := X|_{\gamma(r)}$, $r \in J''$. In particular, the equalities $\hat{\eta}'_{\eta(s,\,\cdot\,)}(t) = \eta'(s,t)$ and $\hat{\eta}''_{\eta(\,\cdot\,,t)}(s) = \eta''(s,t)$ are valid.

The *torsion* (*vector*) *along* $\eta\colon J \times J' \to M$ *at* $(s,t) \in J \times J'$ of a derivation D along paths in the tangent bundle $(T(M), \pi, M)$ is called the vector

$$T^\eta(s,t) := \big(D^{\eta(\,\cdot\,,t)}(\hat{\eta}'')\big)(s) - \big(D^{\eta(s,\,\cdot\,)}(\hat{\eta}')\big)(t) \in \pi^{-1}(\eta(s,t)). \tag{10.1}$$

[1] The analogue of a derivation along vector fields in a general vector bundle is the concept of a derivation along paths but it acts on liftings of paths, (version: on sections (possibly along paths)), not on paths or on the elements of the total bundle space. If the elements of the bundle space are mappings (operators), under certain conditions, their commutator can be defined, but this is an exceptional case.

[2] This is possible as η is supposed injective.

Using the operator D_s^γ (see Definition 2.1) and the one of the section-derivation D generated by D (see (2.12)), we can write

$$\begin{aligned} T^\eta(s,t) &= D_s^{\eta(\,\cdot\,,t)}(\hat{\eta}'') - D_t^{\eta(s,\,\cdot\,)}(\hat{\eta}') \\ &= \big(\mathsf{D}^{\eta(\,\cdot\,,t)}(\eta'')\big)(\eta(s,t)) - \big(\mathsf{D}^{\eta(s,\,\cdot\,)}(\eta')\big)(\eta(s,t)). \end{aligned} \tag{10.2}$$

Introducing a frame $\{E_i\}$ on $\eta(J,J')$, applying (2.4), and using the relation

$$\frac{\partial \eta'^{\,i}(s,t)}{\partial t} - \frac{\partial \eta''^{\,i}(s,t)}{\partial s} = C^i_{kl}(\eta(s,t))\eta'^{\,k}(s,t)\eta''^{\,l}(s,t),$$

which is a consequence of the definitions of η' and η'' and the C^2 differentiability of η,[3] from (10.2), we find the local expression[4]

$$T^\eta(s,t) = \big(T^\eta(s,t)\big)^i E_i|_{\eta(s,t)} \tag{10.3}$$

where the *components* of the torsion in $\{E_i\}$ are

$$\begin{aligned} \big(T^\eta(s,t)\big)^i := \Gamma^i{}_j(s;\eta(\,\cdot\,,t))\eta''^{\,j}(s,t) - \Gamma^i{}_j(t;\eta(s,\,\cdot\,))\eta'^{\,j}(s,t) \\ - C^i_{jk}(\eta(s,t))\eta'^{\,j}(s,t)\eta''^{\,k}(s,t), \end{aligned} \tag{10.4}$$

with $\Gamma^i{}_j$ being the components of D in $\{E_i\}$ and $[E_j,E_k]_{\_} =: C^i_{jk}E_i$.

Definition 10.1. The *torsion* (*vector*) *along* $\eta\colon J\times J'\to B$ *at* $(s,t)\in J\times J'$ *of a linear transport along paths in the tangent bundle over a manifold* is called the torsion (vector) along $\eta\colon J\times J'\to B$ at $(s,t)\in J\times J'$ of the generated by it via (3.19) derivation along paths.

Definition 10.2. The *torsion* (*operator*) *of linear transports or derivations along paths* is a mapping T assigning to each transport or derivation and every $\eta\colon J\times J'\to B$ and $(s,t)\in J\times J'$ the corresponding torsion $T^\eta(s,t)$ along η at (s,t).

Proposition 10.1. *If the coefficients of a linear transport along paths in the tangent bundle* $(T(M),\pi,M)$ *have on* $U\subseteq M$ *the representation* (9.11) *along every path* γ *in* U *in some* (*every*) *frame* $\{E_i\}$ *on* U*, then the components of its torsion vector along* $\eta\colon J\times J'\to U$ *at* $(s,t)\in J\times J'$ *are*

$$\big(T^\eta(s,t)\big)^i = T^i_{jk}(\eta(s,t))\eta'^{\,j}(s,t)\eta''^{\,k}(s,t) \tag{10.5}$$

with

$$T^i_{jk} := -\Gamma^i{}_{jk} + \Gamma^i{}_{kj} - C^i_{jk}. \tag{10.6}$$

Proof. Substitute (9.11) for $\gamma = \eta(\,\cdot\,,t), \eta(s,\,\cdot\,)$ into (10.4). □

[3] To derive this equality, calculate the commutator $[\eta',\eta'']_{\_}|_{\eta(s,t)}$ in $\{E_i\}$ and use $[\eta',\eta'']_{\_}=0$ as $(\eta'(s,t))(f) = \frac{\partial f(\eta(s,t))}{\partial s}$ and $(\eta''(s,t))(f) = \frac{\partial f(\eta(s,t))}{\partial t}$ for a C^1 function f on $\eta(J,J')$.

[4] One can at first derive (10.4) in a coordinate frame and then, applying the corresponding transformation laws, the result can be transformed into an arbitrary frame.

Lemma 10.1. *The representation* (9.11) *on page* 265 *holds on* $U \subseteq M$*, i.e., for paths lying entirely in* U*, for some linear transport* L *along paths in the tangent bundle* $(T(M), \pi, M)$ *if and only if* L *along paths in* U *coincides with the parallel transport generated by a linear connection.*

Proof. If (9.11) holds, the matrices Γ_μ transform according to (6.4) with $B = A$ (as now $(E, \pi, M) = (T(M), \pi, M)$) and, consequently (see (I.5.3)), they are coefficients' matrices of a linear connection ∇. Therefore the derivation D along paths generated by L coincides with the covariant derivative along paths generated by ∇. Since the linear transport along paths generated by the last derivation is the parallel transport assigned to ∇ (see Definition I.3.2 on page 27 and Proposition 3.7), the last transport coincides with L.

Conversely, from definitions (I.3.18) and (I.3.17) follows that, for every parallel transport generated by a linear connection ∇, the representation (9.11) is valid if in it Γ_μ are the coefficients' matrices of ∇. □

Lemma 10.1 explains the reason why in equation (10.5) appear the components T^i_{jk} of the torsion tensor of some linear connection with local coefficients ${\Gamma^i}_{jk}$ (see (I.3.14)).

Example 10.1. The torsion tensor T^∇ of a linear connection ∇ can be introduced by means of the torsion operator T of the assigned to it parallel transport or covariant derivative along paths. Indeed, defining the mapping $\eta\colon J \times J' \to M$ as we did before equation (9.17), after a simple calculation, we find

$$T^{\eta_p}(s_0, t_0) = \big(T^\nabla(A, B)\big)\big|_p \tag{10.7}$$

where T^∇ is exactly the torsion tensor of ∇ defined by (I.3.12) and with local components (I.3.14), i.e., (10.6).

Example 10.2. If $\mathbf{D}$ is a linear derivation along vector fields (see Definition III.2.2) and D is a derivation along paths in the bundle $(T(M), \pi, M)$ tangent to M for which (9.11) holds for the coefficients' matrices Γ_μ of $\mathbf{D}$, then the torsion of D along η_p at (s_0, t_0) is

$$T^{\eta_p}(s_0, t_0) = \big(T^{\mathbf{D}}(A, B)\big)\big|_p \tag{10.8}$$

with $T^{\mathbf{D}}$ being the torsion of $\mathbf{D}$ given via (III.2.13) and with local components given by (III.2.15).

Definition 10.3. A linear transport or derivation along paths in $(T(M), \pi, M)$ is called *torsionless* or *torsion free* (resp. on $U \subseteq M$) if its torsion (operator) vanishes on M (resp. on U).

Remark 10.1 *(Cf. Remark* 9.1 *on page* 267*).* It should be emphasized on a principal difference between the torsionless on $U \subseteq B$ of a linear transport along paths in the bundle $(T(M), \pi, M)$ and of a linear connection or derivation along tangent vector fields on a manifold M. In the former case, the torsion is zero on U along mappings

η *in* U, i.e., $\eta\colon J\times J'\to U$, while in the latter one it vanishes only *along vectors tangent to* U, i.e., from the tangent space $T_p(M)$, $p\in U$. If U is a submanifold of M, the first case means that $\eta'(s,t),\eta''(s,t)\in T_{\eta(s,t)}(U)\subseteq T_{\eta(s,t)}(M)$, while in the second one, we have $\eta'(s,t),\eta''(s,t)\in T_{\eta(s,t)}(M)$, i.e., η' and η'' are not generally in the tangent bundle space over U in $(T(U),\pi,U)$.

Proposition 10.2. *A linear transport* (*resp. section-derivation*) *along paths in the tangent bundle* $(T(M),\pi,M)$ *is torsionless over a submanifold* N *of* M, *i.e., along paths in* N, *if and only if its restriction to the bundle space* $T(N):=\bigcup_{p\in N}T_p(N)$ (*resp. to* $\operatorname{Sec}^1(T(N),\pi|_N,N)$) *coincides with the restriction to the same set of the parallel transport* (*resp. covariant derivative*) *along paths in* N *generated by a linear connection and which transport* (*resp. covariant derivative*) *is torsion free on the mentioned set.*

Proof. If D (resp. L) is a torsion free covariant derivative (resp. parallel transport) along paths generated by a linear connection ∇, then, due to Lemma 10.1 (resp. Definition I.3.2 and (I.3.17)), its components (resp. coefficients) have locally the representation (9.11) in which Γ_μ are the coefficients' matrices of ∇. Therefore the torsion of D (resp. L) is given via (10.5) which vanishes on N, by virtue of (10.6), (10.7), and the torsionless of D (resp. L).

Conversely, let D (resp. L) be a section-derivation (resp. linear transport) along paths which is torsion free on the tangent bundle $(T(N),\pi,N)$ over N.

At first, we consider the case $\dim N=\dim M$.

Let $\eta\colon J\times J'\to N$, $(s,t)\in J\times J'$, $\{E_i\}$ be a frame on N, and $\Gamma^i{}_j$ be the components of D (resp. the coefficients of L) in $\{E_i\}$. From (10.3) and (10.4) follows that the torsionless is equivalent to

$$\Gamma^i{}_j(s;\eta(\,\cdot\,,t))\eta''^{\,j}(s,t)=\Gamma^i{}_j(t;\eta(s,\,\cdot\,))\eta'^{\,j}(s,t)+C^i_{jk}(\eta(s,t))\eta'^{\,j}(s,t)\eta''^{\,k}(s,t).$$

Since η is arbitrary, so are η' and η'' and, consequently, the last equation implies the expansion[5, 6]

$$\Gamma^i{}_j(r;\gamma)=\Gamma^i{}_{jk}(r;\gamma)\dot\gamma^k(r),\qquad r=s,t\quad\text{and resp. }\gamma=\eta(\,\cdot\,,t),\eta(s,\,\cdot\,)$$

for some functions $\Gamma^i{}_{jk}$ (generally non-symmetric in j and k) which do not depend on $\dot\gamma$. Inserting this representation for $\Gamma^i{}_j$ into the last equation, we get

$$\Gamma^i{}_{kj}(s;\eta(\,\cdot\,,t))=\Gamma^i{}_{jk}(t;\eta(s,\,\cdot\,))+C^i_{jk}(\eta(s,t))$$

as η' and η'' are arbitrary. Using, once again, the arbitrariness of η, hereof follows that $\Gamma^i{}_{jk}(r;\gamma)$ must depend on the combination $\gamma(r)$, not on r and γ separately,

[5]From the differentiation of the last equality with respect to $\eta'^{\,i}(s,t)$ or $\eta''^{\,i}(s,t)$ follows the below-written expansion for $\Gamma^i{}_j$ and the second derivative of the this equality with respect to $\eta'^{\,i}(s,t)$ and $\eta''^{\,i}(s,t)$ results in the equation presented after the expansion.

[6]Here and below the assumption that N is a neighborhood is essentially used as otherwise, generally, from equality like $A_i(t;\eta(s,\,\cdot\,))\eta'^{\,i}(s,t)=0$ does not follow $A_i(t;\eta(s,\,\cdot\,))=0$.

i.e., ${\Gamma^i}_{jk}(r;\gamma) = {\Gamma^i}_{jk}(\gamma(r))$. Therefore, we have:

$$\begin{aligned} {\Gamma^i}_j(r;\gamma) &= {\Gamma^i}_{jk}(\gamma(r))\dot\gamma^k(r) \\ {\Gamma^i}_{kj}(\gamma(r)) &= {\Gamma^i}_{jk}(\gamma(r)) + C^i_{jk}(\gamma(r)) \end{aligned}$$

for $r = s, t$ and respectively $\gamma = \eta(\,\cdot\,, t), \eta(s, \,\cdot\,)$. Since $\eta\colon J \times J' \to N$ is arbitrary, the last two equations must hold for *every* path $\gamma\colon J'' \to N$. The former of these equations, combined with Lemma 10.1, implies that D (resp. L) is on N a covariant derivative (resp. parallel transport) along paths assigned to a linear connection ∇ (with local coefficients ${\Gamma^i}_{jk}$) on N. From the latter equation, when substituted into (10.6) or (I.3.14), follows that ∇ and the assigned to it parallel transport are torsionless on N.

Let now N be a submanifold of M of dimension $\dim N < \dim M$.

Note 10.1. The proof of the theorem can be completed by noticing that N is a manifold by itself (as it is a submanifold of M – see page 7) and, consequently, to this case is valid the already proved theorem for $\dim N = \dim M$ with N for M. The reformulation of this observation in terms of submanifolds yields the required result. The below-presented alternative and independent proof is considerably longer but it reveals some features of the methods employed and parts of it will be used later. If, at this point, this is not interesting for the reader, he/she can skip the text after the present note till the end of this proof.

Suppose $\{E_i = \frac{\partial}{\partial x^i}\}$ is the coordinate frame associated to a local coordinate system $\{x^i\}$ in a neighborhood of $\eta(s,t)$ in M such that $x^i(q) = a^i = \text{const} \in \mathbb{K}$ for every $i > \dim N$ and q in a neighborhood of $\eta(s,t)$ in N (see page 7). Then in the above proof of the case with $\dim N = \dim M$ one should put ${\eta'}^i = {\eta''}^i = 0$ for $i > \dim N$ as the range of η is in N. Taking this into account and repeating *mutatis mutandis* the above lines, concerning the case $\dim N = \dim M$, we conclude that, for every path $\gamma\colon J'' \to N$, is fulfilled

$$\begin{aligned} {\Gamma^i}_j(r;\gamma) &= {\Gamma^i}_{jk}(\gamma(r))\dot\gamma^k(r), && j \le \dim N,\ i = 1,\dots,\dim M \\ {\Gamma^i}_{kj}(p) &= {\Gamma^i}_{jk}(p) + C^i_{jk}(p), && j,k \le \dim N,\ i = 1,\dots,\dim M,\ p \in N \end{aligned}$$

and for $j,k > \dim N$ the functions ${\Gamma^i}_{jk}\colon N \to \mathbb{K}$ are completely arbitrary. (Notice, since γ is a path in N, we have $\dot\gamma^k = 0$ for $k > \dim N$; besides, $C^i_{jk} = 0$ as $E_i = \frac{\partial}{\partial x^i}$.)

Let ∇ be a linear connection on (a neighborhood of) N whose coefficients on N in $\{E_i\}$ are equal to the above functions ${\Gamma^i}_{jk}$ which for $j,k > \dim N$ are arbitrarily fixed. Consider the restriction to $\operatorname{Sec}^1(T(N), \pi|_N, N)$ of the covariant derivative (resp. to $T(N)$ of the parallel transport) along paths in N assigned to ∇. According to (I.3.17) (resp. Definition I.3.2), it depends only on ${\Gamma^i}_j(r;\gamma) = {\Gamma^i}_j(\gamma(r))$ with $j \le \dim N$ when applied on tensor fields (resp. tensors) over N as a manifold, i.e., sections of the tangent bundle $(T(N), \pi|_N, N)$ over N (resp. in the bundle space $T(N)$). Hence, by construction, this restriction coincides with the one to the same set of the initial section-derivation D (resp. linear transport L) along paths. As a consequence of the last result, it is torsionless. □

Proposition 10.2 has an equivalent version which is more suitable for the most applications.

Proposition 10.2′. *Let L (resp. D) be a linear transport (resp. section-derivation) along paths in the tangent bundle $(T(M), \pi, M)$ over a C^1 manifold M and N be a submanifold of M. Then L (resp. D) is torsionless on N if and only if it coincides on N, i.e., along paths in N, with the torsionless on N parallel transport (resp. the torsionless on N covariant derivative) along paths assigned to some linear connection (on (a neighborhood of) N).*

Proof. This proof is identical with the one of Proposition 10.2 till, but *not* including, its last paragraph. It should be replace by the following text.

Let ∇ be a linear connection on (a neighborhood of) N whose coefficients $\overline{\Gamma}^i{}_{jk}$ on N in $\{E_i\}$ are such that $\overline{\Gamma}^i{}_{jk} = \Gamma^i{}_{jk}$ for $j, k \leq \dim N$ ($\Gamma^i{}_{jk}$ are the same as above) and $\overline{\Gamma}^i{}_{jk} = 0$ for $j, k > \dim N$.[7] Then the parallel transport (resp. covariant derivative) along paths in N depends only on $\overline{\Gamma}^i{}_{jk}$ with $j, k \leq \dim N$ (see Definition I.3.2 (resp. (I.3.17))) and, by construction, it coincides with the initial transport L (resp. section-derivation D), due to which it is torsion free on N. □

Remark 10.2. Having in mind Remarks 9.1 and 10.1, we want to emphasize on the fact that the vanishment on $U \subseteq M$ of the curvature/torsion of the parallel transport assigned to a linear connection ∇, generally, does not imply that ∇ is flat/torsionless. An exception is the case when U is a $(\dim M)$-dimensional submanifold of M.[8]

10.2. Holonomic normal frames in the tangent bundle

The main purpose of this subsection is the proof of the below-presented theorem whose meaning is that, in the tangent bundle over a manifold, the flat torsionless parallel transports (resp. covariant derivatives) along paths generated by linear connections are the only linear transports (resp. section-derivations) along paths admitting normal coordinates, i.e., holonomic normal frames. Notice, here the flatness is required for the existence of the normal frames (Theorem 9.1).

Theorem 10.1. *Let L be an arbitrary linear transport along paths in the tangent bundle $(T(M), \pi, M)$ over a C^3 manifold M and L be flat over a neighborhood*

---

[7] The condition $\overline{\Gamma}^i{}_{jk} = 0$ for $k > \dim N$ is not necessary as $\overline{\Gamma}^i{}_{jk}$ appear only in the combination $\overline{\Gamma}^i{}_{jk}\dot{\gamma}^k$ with γ in U, i.e., $\dot{\gamma}^k = 0$ for $k > \dim N$ in the coordinates used.

[8] These assertions are clearly seen from Corollary 11.1 and Remark 11.3 below in a case U is a submanifold of M.

$U \subseteq M$. Holonomic on U frames normal for L on U exist if and only if L is torsionless over U or, equivalently, iff L coincides over U with the (flat and) torsionless parallel transport assigned to a (flat and torsionless) linear connection on U.

Proof. **(i)** It follows from Proposition 10.2′ and (9.17) that L is torsionless and flat over (a submanifold, in particular neighborhood) U iff over U it coincides with the parallel transport generated by a linear connection and which transport is flat and torsion free on U.

(ii) If L admits a frame $\{E_i\}$ which is holonomic and normal on U, i.e., such that $C^i_{jk} = 0$ on (a neighborhood of) U and $\Gamma^i{}_j = 0$ on U, equation (10.4) yields $T^\eta(s,t) = 0$ for every $\eta\colon J \times J' \to U$. Hence, by definition, L is torsionless on U.

(iii a) At last, let on $T(U)$ the transport L coincides with the flat torsionless parallel transport generated by a linear connection ∇ (on U).[9] Then (see Lemma 10.1) the coefficients of L are $\Gamma^i{}_j(s;\gamma) = \Gamma^i{}_{jk}(\gamma(s))\dot\gamma^k(s)$, $s \in J$, for every path $\gamma\colon J \to U$ with $\Gamma^i{}_{jk}$ being the coefficients of the mentioned linear connection. Therefore the curvature (resp. torsion) of L is given via (9.12) and (9.13) (resp. (10.5) and (10.6)) with $i, j, \alpha, \beta = 1, \dots, \dim M$ and the values of η in U.

(iii b) Since L is flat on U, it is Euclidean on U (Theorem 9.1) and, by Theorem 5.2, its coefficients' matrix on U in a frame $\{E_i\}$ on U has the form $\mathbf{\Gamma}(s;\gamma) = \boldsymbol{F}_0^{-1}(\gamma(s))\frac{\mathrm{d}\boldsymbol{F}_0(\gamma(s))}{\mathrm{d}s}$ for some non-degenerate matrix-valued function $\boldsymbol{F}_0\colon U \to \mathrm{GL}(\mathbb{K}, \dim M)$ of class C^1 and every path $\gamma\colon J \to U$. On U, we define a frame $\{E'_i\}$ by $E'_i = \big(\boldsymbol{F}_0^{-1}\big)^j_i E_j$.

Our intention is to prove that the frame $\{E'_i\}$ is holonomic and normal on U, i.e., $[E'_j, E'_k]_{\_} =: C'^i_{jk} E'_i$ with $C'^i_{jk} = 0$ and $\Gamma'^i{}_j = 0$ on U. The functions C'^i_{jk} are expressible via the analogous ones C^i_{jk} for $\{E_i\}$, $[E_j, E_k]_{\_} =: C^i_{jk} E_i$, through equation (I.8.4) on page 69 with $A = \boldsymbol{F}_0^{-1}$:

$$C'^i_{jk} = (\boldsymbol{F}_0)^i_l \big[\big(\boldsymbol{F}_0^{-1}\big)^m_j \big(\boldsymbol{F}_0^{-1}\big)^n_k C^l_{mn} + 2E'_{[j}\big(\boldsymbol{F}_0^{-1}\big)^l_{k]}\big].$$

After some simple algebraic manipulations, from here we get

$$C'^i_{jk} = (\boldsymbol{F}_0)^i_l \big(\boldsymbol{F}_0^{-1}\big)^m_j \big(\boldsymbol{F}_0^{-1}\big)^n_k T^l_{mn} = -T'^i_{jk} \tag{10.9}$$

where T^i_{jk} are the components in $\{E_i\}$ of the torsion tensor of ∇ given by (10.6) and T'^i_{jk} are the same ones in $\{E'_i\}$.[10] Since the parallel transport generated by the

[9]Since U is a neighborhood, from here follows (see (10.7), (9.17), and Definition I.3.2 on page 27) that ∇ is also flat and torsion free on U. Therefore, by Theorem II.4.3 on page 108, ∇ admits holonomic frames normal on U. As the 3-index coefficients of ∇ and of the parallel transport assigned to it coincide, this transport is also Euclidean on U and the (holonomic) frames normal for it and ∇ are identical. This completes the proof of the theorem. If the reader is not interested from alternative proof and some other details, he/she can skip the rest of the proof of Theorem 10.1, parts (iii b)–(iii d) below.

[10]From (10.6) follows that (10.9) is valid in a frame $\{E''_i\}$ iff $\Gamma''^i{}_{jk} = \Gamma''^i{}_{kj}$, in particular this is true in every frame normal for ∇ on U.

linear connection ∇ is supposed torsionless on U, the equality

$$C'^{\,i}_{jk}(\eta(s,t))(\eta'(s,t))'^{\,j}(\eta''(s,t))'^{\,k} = 0 \tag{10.10}$$

is valid for every $\eta\colon J\times J' \to U$ as a consequence of (10.9), (10.5), and $T^{\eta}(s,t) = 0$ on U. Here $(\eta')'^{\,i}$ and $(\eta'')'^{\,i}$ are the components of η' and η'' in $\{E'_i\}$.

(iii c) Taking into account that U is a neighborhood in M (and hence it is $(\dim M)$-dimensional submanifold of M) and the arbitrariness of η (which means that $\eta'(s,t)$ and $\eta''(s,t)$ are arbitrary vectors in $T_{\eta(s,t)}(M)$), from equation (10.10), we obtain $C'^{\,i}_{jk}|_U = 0$, i.e., $\{E'_i\}$ is holonomic on U.

(iii d) The proof is completed by the observation that the frame $\{E'_i = (\boldsymbol{F}_0^{-1})^j_i E_j\}$ is normal on U. In fact, inserting the representation (5.4′) into the transformation law (3.26) with $A = \boldsymbol{F}_0^{-1}$, we find $\boldsymbol{\Gamma}'(s;\gamma) = 0$ for every $\gamma\colon J \to U$. □

In the above proof of Theorem 10.1, we intentionally have singled out its several parts, (i)–(iii d), and the intermediate equations (10.9) and (10.10). A superficial look on them reveals that parts (ii)–(iii b) and (iii d) are valid for arbitrary subset $U \subseteq M$ as the initial assumption on U to be a neighborhood in M was not used in them. On the contrary, the conclusions in part (iii c) rely entirely on the supposition that U is a neighborhood in M as otherwise from (10.10) one can not, generally, infer the holonomicity of the frame $\{E'_i\}$, $C'^{\,i}_{jk} = 0$ on some open set containing or equal to U: if U is not a neighborhood in M, then, generally, $\eta'(s,t)$ and $\eta''(s,t)$ are in some *proper* subset of $T_{\eta(s,t)}(M)$ (see, e.g., Proposition 11.3 below). The origin of this situation comes from the fact that the torsionlessness on $U \subseteq M$ of the parallel transport generated by a linear connection ∇ is, generally, weaker condition than the torsionlessness of ∇: the latter one implies the former but the former, generally, does not lead to the latter one. This conclusion is completely similar to an analogous result concerning the curvature on a submanifold N of M of linear connections which is expressed by Lemma II.5.1 on page 112: the path-independence on N in it is equivalent to the flatness of the transport on U, so it is equivalent to (II.5.1), not to the flatness of the connection on N (cf. Remark II.5.1). At the end, the conclusions in part (i) rest on Proposition 10.2′ which is valid if U is a submanifold of M, not only a neighborhood in it, but it may not hold on an arbitrary its subset.

On the base of the above comments, we can formulate the next two propositions which are stronger variants of different parts of Theorem 10.1.

Proposition 10.3. *A linear transport along paths in the tangent bundle* $(T(M),\pi,M)$ *is flat and torsionless on a submanifold* N *of* M *if and only if it coincides over* N *with the parallel transport generated by a linear connection on* (*a neighborhood of*) N *which transport is flat and torsionless over* N.

Exercise 10.1. Prove, independently of (10.9), that the frame $\{E'_i = (\boldsymbol{F}_0^{-1})^j_i E_j\}$ is normal for ∇ on U. (Notice, here is important the supposition U to be a neighborhood.)

Proof. See Proposition 10.2′ and (9.17) □

Proposition 10.4. *If a linear transport along paths in $(T(M), \pi, M)$ admits a frame which is normal and holonomic on a set $U \subseteq M$, then it is flat and torsion free on U.*

Proof. See part (ii) of the proof of Theorem 10.1. □

Proposition 10.5. *Let L be a linear transport along paths in the tangent bundle $(T(M), \pi, M)$. Let L be flat and torsionless over a neighborhood U in the manifold M and the C^1 non-degenerate matrix-valued function $\boldsymbol{F}_0 \colon U \to \mathrm{GL}(\dim M, \mathbb{K})$ generates the coefficients of L via (5.4′) in a frame $\{E_i\}$ on U. Then:*

(i) *The frame $\{E'_i\}$ with*

$$E'_i = \left(\boldsymbol{F}_0^{-1}\right)_i^j E_j \tag{10.11}$$

is holonomic and normal for L on U.

(ii) *All frames normal for L on U are holonomic.*

(iii) *All frames which are holonomic and normal for L on U are obtainable from $\{E'_i\}$ by linear transformations with coefficients constant on U.*

Proof. (i) See parts (iii b)–(iii d) of the proof of Theorem 10.1

(ii) According to Corollary 4.5 on page 239, all frames normal for L on U are $\{E''_i = B_i^j E'_j\}$ with $B_j^i|_U = \text{const}$. Hence these frames are holonomic on U, $[E''_j, E''_k]_- = 0$, as the frame $\{E'_i\}$ is such, $[E'_j, E'_k]_- = 0$.

(iii) This part of the assertion follows from part (ii) and Corollary 4.5. □

Remark 10.3. The non-degenerate matrix-valued function $\boldsymbol{F}_0$, appearing in equation (10.11), is not arbitrary: since the transport L is assumed torsionless on U and U is a neighborhood, from (6.3) (see Theorem 5.2) and (10.6)) follows that the matrix elements $(\boldsymbol{F}_0)_j^i$ of $\boldsymbol{F}_0$ must be solutions of the system of equations

$$-(\boldsymbol{F}_0^{-1})_m^i E_k (\boldsymbol{F}_0)_j^m + (\boldsymbol{F}_0^{-1})_m^i E_j (\boldsymbol{F}_0)_k^m - C'^{\,i}_{jk} = 0 \tag{10.12}$$

on U, where $C'^{\,i}_{jk}$ are given by (I.8.4) with $A = \boldsymbol{F}_0^{-1}$. The constant solutions of (10.12) correspond to frames holonomic and normal for the linear connection generating the parallel transport with which L coincides on U.

Definition 10.4. Let L (resp. D) be a linear transport (resp. derivation) along paths in the tangent bundle $(T(M), \pi, M)$ over a C^1 manifold M. A *chart* (V, x) of M and the associated to it *coordinate system* $\{x^i\}$ are called *normal* (*for* L (*resp.* D)) *on a set* $W \subseteq V$ if the coordinate frame $\left\{\frac{\partial}{\partial x^i}\right\}$ is normal for L (resp. D) on W.

Proposition 10.6. *Let L be a linear transport along paths in the tangent bundle $(T(M), \pi, M)$. Let L be flat and torsion free on a neighborhood U in M and $\boldsymbol{F}_0 \colon U \to \mathrm{GL}(\dim, \mathbb{K})$ be of class C^1 and generates on V the coefficients of L*

via (5.4′) *in the frame* $\left\{\frac{\partial}{\partial y^i}\right\}$ *associate to a local chart* (V, y) *of* M *such that* $V \subseteq U$. *Then all coordinates* $\{x^i\}$ *normal for* L *on* V *are given by*

$$x^i(p) = x_0^i + \int_{p_0}^{p} (\boldsymbol{F}_0 B^{-1})^i_j \,\mathrm{d}y^j = x_0^i + (B^{-1})^k_j \int_{p_0}^{p} (\boldsymbol{F}_0)^i_k \,\mathrm{d}y^j \tag{10.13}$$

where $p \in V$, $p_0 \in V$ *is fixed,* $x_0^i \in \mathbb{K}$ *are constants, and* $B = [B^i_j]$ *is constant non-degenerate matrix.*

Proof. From the proof of Proposition 10.5 follows that $\{x^i\}$ is normal iff $\frac{\partial}{\partial x^i} = B^j_i (\boldsymbol{F}_0^{-1})^k_j \frac{\partial}{\partial y^k}$ for non-degenerate $B = [B^j_i] \in \mathrm{GL}(\dim M, \mathbb{K})$, which is equivalent to (10.13). □

Exercise 10.2. Prove that the path-independence of the integral in (10.13) is a consequence of the torsionless of L on $U \supseteq V$. (Cf. the proof of the path-independence of the integrals in (II.5.14), or (II.5.3), or (11.13). For a rigorous proof, see (11.13) below and the proof after it for $A = B\boldsymbol{F}_0^{-1}$.)

Proposition 10.7. *If* L *is torsionless linear transport along paths in the tangent bundle* $(T(M), \pi, M)$ *over a* C^1 *real manifold* M, *then along every fixed injective path* $\gamma\colon J \to M$ *exist holonomic frames* (*local coordinates*) *normal for* L *along* γ.

Proof. By Corollary 5.1 and Proposition 5.5, the transport L admits frames defined solely along γ and normal along γ. All of them, by Lemma III.10.1 on page 194, can be extended in a holonomic way outside $\gamma(J)$. □

Here we shall stop the investigation of the holonomic frames normal for linear transports along paths in the tangent bundle over a differentiable manifold. The reason is that these problems, as well as any ones concerning such transports on submanifolds, can be reduced to similar problems for parallel transports (assigned to linear connections) or to linear connections over the same manifold (see Proposition 10.3). Besides, the problems for existence and description of the holonomic frames are solved by Proposition 10.5. The explicit description of the coordinates normal on a neighborhood is provided by Proposition 10.6. A similar result on arbitrary submanifolds will be derived in Subsection 11.2.

11. Parallel transports in tangent bundles

The (flat and/or torsionless) parallel transports assigned to a linear connections appeared several times in Section 10, when the theory of normal frames was concerned. In the present section, we shall investigate the parallel transports (assigned to linear connections) from the view-point of the general theory of linear transports along paths. In particular, we shall show that the parallel transports are uniquely selected among the other transports along paths when normal frames are involved.

11.1. The parallel transport as a transport along paths

The concept of a 'parallel transport', assigned to a linear connection over a manifold M, was introduced by Definition I.3.2 on page 27 in Subsection I.3.3 where some essential its properties were mentioned. They are enough to be proved that every parallel transport, when restricted to tensors of a fixed type, is a special kind of a linear transport along paths in the corresponding tensor bundle over M. To save some writing, below we shall investigate only the *case of the tangent bundle* $(T(M),\pi,M)$ over a C^1 manifold M but, *mutatis mutandis* (by adding additional indices, e.g., replacing $T(M)$ with $T^r_q(M)$), the below-presented material can be transferred on a tensor bundle $(T^r_q(M),\pi^r_q,M)$ of arbitrary type (r,q) over M.

Let M be a C^1 manifold, $(T(M),\pi,M)$ be the tangent bundle over it, $\gamma\colon J\to M$ be a C^1 path, and P be the parallel transport assigned to a C^0 linear connection ∇ on M. As we know (see Definition I.3.2), if $a,b\in J$ and $a\le b$, then $\mathsf{P}^{\gamma|[a,b]}\colon T_{\gamma(a)}(M)\to T_{\gamma(b)}(M)$ and $\mathsf{P}^{\gamma|[a,b]}(X_0)=X_{\gamma(b)}$ for every $X_0\in T_{\gamma(a)}(M)$ and a vector field X (over $\gamma(J)$) which is the solution of $\nabla_{\dot\gamma}X=0$ with $X_{\gamma(a)}=X_0$. A simple analysis reveals that the only place, where the ordering of the real numbers a and b, $a\le b$, is used, is in the interval $[a,b]$ in which, by definition, its left end must be less than or equal to its right one. This restriction in the definition of a parallel transport is, by our opinion, completely unnecessary and can be removed by a suitable generalization or redefinition of the concept of a 'parallel transport'.

Definition 11.1. A *parallel transport* (*along paths*)*, assigned to a linear connection* ∇*, in the tangent bundle* $(T(M),\pi,M)$ is a mapping $P\colon\gamma\mapsto P^\gamma$, $\gamma\colon J\to M$ being a C^1 path, where P^γ, called *parallel transport along* γ, maps every pair $(a,b)\in J\times J$ into a mapping

$$P^\gamma_{a\to b}\colon T_{\gamma(a)}(M)\to T_{\gamma(b)}(M), \tag{11.1}$$

called *parallel transport along* γ *from* a *to* b, such that if $X_0\in T_{\gamma(a)}(M)$, then

$$P^\gamma_{a\to b}(X_0)=X_{\gamma(b)} \tag{11.2}$$

where X is a vector field over $\gamma(J)$ defined as the unique solution of the initial-value problem

$$\nabla_{\dot\gamma}X=0 \tag{11.3a}$$

$$X_{\gamma(a)}=X_0. \tag{11.3b}$$

Remark 11.1. Recall (see Remark I.3.3 on page 28), the vector field X defined via (11.3) is generally multiple-valued at the points of self-intersection of γ, if any. To restore the single-valueness, one should consider X as a lifting of γ in $(T(M),\pi,M)$, i.e., $X\colon J\to T(M)$ with $\pi\circ X=\gamma$; in this case it is better to write X_γ for X and $X_\gamma(s)$ for $X_{\gamma(s)}$, $s\in J$ and the action of $\nabla_{\dot\gamma}$ on X_γ is given by (I.3.19) which is in a full agreement with (2.4) and (2.16) (see also (III.10.5) and Section 13).

Comparing Definitions I.3.2 and 11.1, we see that the parallel transports P and P, introduced by them, are equivalent in a sense that each of them can be expressed through the other one:

$$P^{\gamma}_{a\to b} = \begin{cases} \mathsf{P}^{\gamma|[a,b]} & \text{for } a \le b \\ \big(\mathsf{P}^{\gamma|[b,a]}\big)^{-1} & \text{for } a \ge b \end{cases} \tag{11.4a}$$

$$\mathsf{P}^{\beta} = P^{\beta}_{a\to b} \qquad \text{for } \beta\colon [a,b] \to M,\ a \le b. \tag{11.4b}$$

Proposition 11.1. *The parallel transport P, as defined by Definition* 11.1*, is a linear transport along paths in the tangent bundle* $(T(M), \pi, M)$*.*

Proof. Due to Definition 3.1 and (11.1), we have to check (3.2)–(3.4) with P for L, which is almost trivial: the equality (3.2) is a simple consequence of the uniqueness of the solutions of the initial-value problem (11.3) along γ written in terms of P, equation (3.3) follows from (11.2) with $b = a$ and (11.3b) as X_0 is arbitrary, and the linearity condition (3.4) is a result of the linearity of equation (11.3a) as $\nabla_{\dot\gamma}$ is a linear operator (see Definition I.3.1, points (ii) and (iii), or (I.3.17)). □

This quite simple result has two consequences which are important for us. On one hand, the parallel transport (along paths and assigned to a linear connection) provides a concrete and essential example of a linear transport along paths and, on another hand, all results derived until now in the present chapter are valid for the parallel transports (in the tensor bundles over a manifold).

Below, when talking about parallel transports, we shall have in mind the ones introduced by Definition 11.1, *not* by Definition I.3.2. If there is a risk of misunderstanding, we shall call such transports *parallel transports along paths.*[1]

It is well known, a parallel transport on a neighborhood U, i.e., along paths in U, is path-independent iff the generating it linear connection is flat on U. Analogous result on arbitrary submanifold N of M is expressed by Lemma II.5.1 on page 112. Proposition 11.1 ensures the formulation of this important assertion in internal terms, i.e., ones involving only parallel transports.

Proposition 11.2. *Let N be a submanifold of a C^3 manifold M. A C^1 parallel transport is path-independent over N, i.e., along C^1 paths in N, if and only if it* (*considered as a linear transport along paths in* $(T(M), \pi, M)$) *is flat on N .*

Proof. The result follows from Lemma II.5.1, Remark II.5.3 after its proof, and equations (9.20) and (9.12). □

A little below (see Corollary 11.2), it will be proved the validity of Proposition 11.2 for arbitrary set $U \subseteq M$, not only for submanifolds of M.

Proposition 11.3. *The parallel transports in the tangent bundle* $(T(M), \pi, M)$ *are the only linear transports along C^1 paths in this bundle whose coefficients have the*

[1] For some general relations between parallel transports, connections, and transports along paths in differentiable bundles, the reader is referred to Section V.8.

representation

$$\mathbf{\Gamma}(s;\gamma) = \Gamma_k(\gamma(s))\dot{\gamma}^k(s) \tag{11.5}$$

in any frame.

Proof. This is a reformulation of Lemma 10.1 on page 273. □

Proposition 11.4. *If P is a parallel transport along paths in $(T(M),\pi,M)$ assigned to a linear connection ∇, then the generated by it derivation along C^1 paths D, defined via* (3.19), *is such that*

$$\big(D^\gamma(\hat{X})\big)(s) = (\nabla_{\dot{\gamma}}X)|_{\gamma(s)} \tag{11.6}$$

where $\gamma\colon J\to M$ is a C^1 path, $X\in\mathfrak{X}(M)$, and $\hat{X}\in \mathrm{PLift}^1(T(M),\pi,M)$ is given by $\hat{X}\colon\gamma\mapsto\hat{X}_\gamma\colon s\mapsto\hat{X}_\gamma(s) := X_{\gamma(s)}$, $s\in J$. The section-derivation D corresponding to D coincides with the covariant derivative along injective C^1 paths generated by ∇:

$$\mathsf{D}^\gamma = \nabla_{\dot{\gamma}} \tag{11.7}$$

for every C^1 path γ without self-intersections.

Proof. (Cf. the first part of the proof of Lemma 10.1.) Since P is a parallel transport, the equality (11.5) holds in a frame $\{E_i\}$ (Proposition 11.3) and in any other frame $\{E'_i = A^j_i E_j\}$ the matrices Γ'_k are given by (6.4) with $B = A := [A^j_i]$ and $\mu,\nu = 1,\dots,\dim M$ as now $(E,\pi,B) = (T(M),\pi,M)$. Hence Γ_k transform according to (I.5.3), or (I.3.5) in components, which means that they are coefficients' matrices of a linear connection ∇. Since the covariant derivative assigned to ∇ acts according to (I.3.17), in which exactly the components of (11.5) (see (11.5′) below) appear, and the explicit action of D^γ_s is given by (3.23) with $\{E_i\}$ for $\{e_i\}$, we derive that $D^\gamma_s(\hat{X}) = (\nabla_{\dot{\gamma}}X)|_{\gamma(s)}$, i.e., (11.6) holds.

The equality (11.7) is a consequence of (11.6) and (2.12) for injective path γ. □

At this point, we want to mention a corollary from Propositions 11.3 and 11.4 which is quite important in the theory of (linear) connections and parallel transports: on the base of these propositions, *the concept of a parallel transport can be defined axiomatically, independently of the connection theory, and by its means the notion of a* (*linear*) *connection can be introduced.* For the purpose, Definitions 11.1 (or I.3.2) and I.3.1 should be replaced, respectively, by the following ones.

Definition 11.2. A *parallel transport P along paths in $(T(M),\pi,M)$ for a C^1 manifold M* is a linear transport along paths in $(T(M),\pi,M)$ the coefficients' matrix of which has the representation (11.5) in some (and hence in any) frame.

Definition 11.3. A *linear connection* (*resp. assigned to a parallel transport P*) *on a manifold M* (or in $(T(M),\pi,M)$) is a mapping ∇ assigning to every $X\in\mathfrak{X}(M)$ a mapping $\nabla_X\colon\mathfrak{X}^1(M)\to\mathfrak{X}^0(M)$ such that

$$(\nabla_X(Y))|_p := D^\gamma_s(\hat{Y}), \qquad Y\in\mathfrak{X}^1(M), \quad p\in M \tag{11.8}$$

where D is the derivation along paths corresponding via (3.19) to some parallel transport in $(T(M), \pi, M)$ (resp. to P), $\gamma \colon J \to M$ is the integral path of X through p, $\gamma(s) = p$ for some $s \in J$, and $\hat{Y} \colon \gamma \mapsto \hat{Y}_\gamma := Y \circ \gamma$.

Exercise 11.1. Prove that the mapping ∇, introduced via Definition 11.3, is in fact a linear connection (restricted to the algebra of vector fields over M) according to Definition I.3.1.

Exercise 11.2. Prove that the parallel transport assigned to a connection ∇ via Definition 11.1 (or I.3.2) coincides with the one generating ∇ via (11.8).

Therefore we conclude that a linear connection (resp. parallel transport) according to Definition I.3.1 (resp. I.3.2) is such with respect to Definition 11.3 (resp. 11.2) and *vice versa*, i.e., both definitions are equivalent. Thus we have at our disposal two equivalent approaches to the theory of linear connections and parallel transports in which one of these objects is taken (and axiomatically defined) as a primitive one, while the other one is introduced on its base.

Proposition 11.5. *In the representation* (11.5) *for the coefficient matrix of a parallel transport, the matrices Γ_k are the coefficients' matrices of the generating it linear connection ∇, i.e., in components, we have*

$$\Gamma^i{}_j(s;\gamma) = \Gamma^i{}_{jk}(\gamma(s))\dot{\gamma}^k(s) \tag{11.5$'$}$$

with $\Gamma^i{}_{jk}$ being the local coefficients of ∇.

Proof. In the proof of Proposition 11.4, we have demonstrated that Γ_k are coefficients' matrices of a linear connection ∇. According to (3.23) and (2.3), the coefficients of the parallel transport P assigned to ∇ coincide with the components of the derivation D along paths generated by P. Combining this result with (11.6) for P and (I.3.17), we see that the coefficients of P coincide with the ones given by (11.5′) for the initial parallel transport. Hence both transports are identical. □

Remark 11.2. This statement does not contradict to the arbitrariness in the 3-index coefficients of a linear transport along paths discussed in Section 6 because the 2-index coefficients of a parallel transport have the form (11.5′) on any subset $U \subseteq M$, i.e., for $\gamma \colon J \to U$, with $\Gamma^i{}_{jk}$ being the coefficients of ∇.

Proposition 11.6. *In an arbitrary frame $\{E_i\}$, the curvature and torsion of a parallel transport along paths are*

$$\mathsf{R}^\eta(s,t) = [(R(\eta', \eta'')E_j) \otimes E^j]|_{\eta(s,t)} \tag{11.9}$$

$$T^\eta(s,t) = [T(\eta', \eta'')]|_{\eta(s,t)} \tag{11.10}$$

where $\eta \colon J \times J' \to M$, $(s,t) \in J \times J'$, $E^j := (E_j)^*$ is the dual of E_j, and R and T, given via (I.3.11) *and* (I.3.12)*, are respectively the curvature and torsion of the linear connection generating the parallel transport.*

Proof. See (9.17) and (10.7), or Propositions 9.1 and 10.1 and use Proposition 11.5, (I.3.13) and (I.3.14). □

Corollary 11.1. *A parallel transport is flat or torsionless on a submanifold N of M iff, respectively,*

$$R(X,Y)=0 \qquad \text{for } X,Y \in \operatorname{Sec}(T(N),\pi|_N,N) \tag{11.11}$$

$$T(X,Y)=0 \qquad \text{for } X,Y \in \operatorname{Sec}(T(N),\pi|_N,N) \tag{11.12}$$

with R and T being respectively the curvature and torsion of the linear connection generating the initial parallel transport.

Proof. See Proposition 11.6 for $\eta\colon J \times J' \to N$ and use the arbitrariness of η. □

Remark 11.3 *(Cf. Remarks* 9.1 *and* 10.1*).* If the dimension of the submanifold N is less than the one of M, $\dim N < \dim M$, then from $R(X,Y)=0$ or $T(X,Y)=0$ one can not conclude that $R=0$ or $T=0$, respectively, as, for vector fields A, B in $[\operatorname{Sec}(T(M),\pi,M)]\backslash[\operatorname{Sec}(T(N),\pi|_N,N)] \neq \varnothing$, the equality $R(A,B)=0$ or $T(A,B)=0$ may not hold. Therefore one should make a clear distinction between flatness/torsionless of a parallel transport and of the linear connection generating it: the latter implies the former one, but the opposite is, generally, not true.

11.2. Normal frames for parallel transports along paths

First of all, applying Theorems 9.1 and 5.1 for a parallel transport in the tangent bundle $(T(M),\pi,M)$ over a C^1 manifold M, we obtain, respectively, the following two propositions.

Proposition 11.7. *A parallel transport is Euclidean on a set $U \subseteq M$, i.e., admits frames normal along C^1 paths in U, if and only if it is flat on U.*

Proposition 11.8. *A parallel transport is Euclidean on a set $U \subseteq M$ iff it is path-independent on U, i.e., its action along paths in U depends only on the initial and final points of the transportation, not on the particular C^1 path in U connecting them.*

Combining these assertions, we derive the following generalization of Proposition 11.2.

Corollary 11.2. *A parallel transport is path-independent over a set $U \subseteq M$ iff it is flat on U.*

Alternatively, this result follows from Corollary 9.3 with $(T(M),\pi,M)$ for (E,π,B) and a parallel transport for L.

Proposition 11.9. *A parallel transport P is Euclidean on a submanifold N of a C^1 manifold M if and only if the generating it linear connection ∇ is Euclidean on N.*

Remark 11.4. This is untrivial result as (11.5) for $\mathbf{\Gamma}(s;\gamma)=0$ with $\gamma\colon J\to N$ does not imply $\Gamma_k=0$ if $\dim N<\dim M$.

Proof. The transport P is Euclidean on N iff it is path-independent on N (Proposition 11.8) a criterion for which is the fulfillment of (II.5.1) with R being the curvature of ∇ (Lemma II.5.1 on page 112 or Corollary 11.1). By Theorem II.5.2, point (i), on page 120, the last condition is a necessary and sufficient one for the existence of frames normal for ∇ on N. □

Proposition 11.10. *If a linear connection ∇ admits frames normal on a set U, then all frames normal on U for ∇ are also normal on U for the parallel transport P assigned to ∇. If U is a neighborhood, the opposite statement is true too, i.e., a frame normal on a neighborhood U for P is normal on U for ∇, and hence on a neighborhood the frames normal for P and ∇ are identical.*

Proof. The first part of the assertion follows from Propositions 11.3 and 11.4 and Definitions 4.1, 4.2 and I.5.1. The second part is a consequence of the same propositions and definitions combined with the remark that, if U is a neighborhood, in (11.5) the vector $\dot\gamma(s)$ is an arbitrary vector in $T_{\gamma(s)}(M)$ and hence $\mathbf{\Gamma}|_U=0$ is equivalent to $\Gamma_k|_U=0$. □

The statement opposite to the first part of Proposition 11.10 is, generally not true, i.e., in the general case not all, but only some, of the frames normal for a parallel transport are also normal for the linear connection generating it. This is clearly seen from (11.5) as well as from the (un)uniqueness Proposition I.5.2 and Corollary 4.5. In fact, from these results follows that, if $\{E_i\}$ and $\{E'_i\}$ are frames normal on $U\subseteq M$ for ∇ or P resp., then $E'_i=A^j_iE_j$ where the matrix-valued function $A:=[A^j_i]$, defined on an open set containing U or equal to it or on U resp., is such that $A|_U=\text{const}$ or $(E_i(A))|_U=0$ resp. Generally, the restriction $A|_U=\text{const}$ is stronger than $(E_i(A))|_U=0$ and, consequently, the set of the solutions of the equation $A|_U=\text{const}$ with respect to A is not greater than the one of the $(E_i(A))|_U=0$.

Proposition 11.11. *A linear transport along paths in $(T(M),\pi,M)$ for a C^1 manifold M admits frames normal for it on $U\subseteq M$ if and only if it coincides on U with some parallel transport which is flat on U.*

Proof. The sufficiency of the assertion is an obvious corollary of Proposition 11.7.

If a transport L in $(T(M),\pi,M)$ is Euclidean on U, it is flat on U (Theorem 9.1) and the matrix of its coefficients has the form (9.11) on U in every frame over U (Proposition 6.1). Therefore, by virtue of Lemma 10.1, the transport L coincides over U with some parallel transport which is flat on U as L is such. □

The last proposition demonstrates the 'privileged' place of the parallel transports amongst all linear transports along paths in the tangent bundle over a manifold: they are the only ones which may admit normal frames, a necessary and sufficient condition for this being their flatness. Before going ahead with the normal frames, we want to prove an auxiliary statement which is a generalization of Lemma II.5.2 on page 116 for transports along path in the tangent bundle $(T(M), \pi, M)$, which, in fact, are parallel transports in it.

Lemma 11.1. *Let L be a linear transport along paths in $(T(M), \pi, M)$ for a C^3 manifold M, N be a submanifold of M, and L be flat and torsionless on N. Every frame $\{E_i\}$ defined on (resp. a neighborhood of) N and L-transported on N, i.e., $E_i|_{\gamma(t)} = L^{\gamma}_{s\to t}(E_i|_{\gamma(s)})$ for $\gamma\colon J \to N$ and $s, t \in J$, can be expanded (resp. redefined) outside of N in a holonomic way, i.e., for every $p_0 \in N$ there is a chart (V, z) of M with $V \ni p_0$ such that $E_i|_{\bar V} = \frac{\partial}{\partial z^i}\big|_{\bar V}$, $\bar V := V \cap N$. If $\dim N = 0$ or if N is real and $\dim N = 1$, the condition $\{E_i\}$ to be L-transported is not necessary.*

Remark 11.5. According to Proposition 4.1, points (i) and (vi), any frame L-transported on a set U is normal on U for L and *vice versa* (see also Definition 4.1″).

Proof. Since the proof of the assertion is practically identical with the one of Lemma II.5.2 on page 116, we shall point below only the new place requiring additional study in order to be obtained the needed proof from the one of Lemma II.5.2. The notation below is the same as in the proof of Lemma II.5.2.

To prove the lemma, define $A := [A^j_i]\colon U_N \to \mathrm{GL}(\dim M, \mathbb{K})$ by the expansion $E_i|_q =: A^j_i(q) \frac{\partial}{\partial x^j}\big|_q$, $q \in U_N$, and simply repeat the proof of Lemma II.5.2 with A for D.

The only new problem, arising in this way, is to be proved the path-independence of the integral appearing in the definition of the looked for coordinate functions z^i, i.e., the integral in (see (II.5.4))

$$z^i(p) = a^i + \int_{q_0}^{p_0} \big(A^{-1}(p)\big)^i_j \,\mathrm{d}p^j + \big(A^{-1}(p_0)\big)^i_j\, [x^j(p) - x^j(p_0)] + f^i_{jk}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)]. \quad (11.13)$$

The conditions for this independence are $\frac{\partial^2 z^i}{\partial x^{[a}\partial x^{b]}}\big|_{U_N} = 0$ with a and b running from 1 to $\dim N$ as the integration is performed along some path in N. To verify these equalities, we shall proceed as follows.

Since $\{E_i\}$ is normal on N for L (Remark 11.5), every vector field E_i, considered as a lifting of paths according to (2.8), i.e., $\hat E_i\colon \gamma \mapsto (\hat E_i)_\gamma\colon s \mapsto E_i|_{\gamma(s)}$, satisfies the equation $D^\gamma \hat E_i = 0$ with D being the derivation along paths generated by L (Proposition 5.6). Writing this equation in the coordinate frame $\big\{\widehat{\frac{\partial}{\partial x^i}}\big\}$, where $\widehat{\frac{\partial}{\partial x^i}}$ are liftings of paths, analogous to $\hat E_i$ above, and using (3.23), we get

$$\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s} = -\mathbf{\Gamma}(s;\gamma)A(\gamma(s))$$

as $E_i = A_i^j \frac{\partial}{\partial x^j}$. [2] Here $\mathbf{\Gamma}$ is the matrix of (2-index) coefficients of L in $\{\frac{\partial}{\partial x^i}\}$. Since γ is a path in N, in $\{\frac{\partial}{\partial x^i}\}$ we have $\dot\gamma^k = 0$ for $k > \dim N$. Therefore in $\{\frac{\partial}{\partial x^i}\}$ is fulfilled $\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s} = \frac{\partial A}{\partial x^k}\big|_{\gamma(s)}\dot\gamma^k(s) = \frac{\partial A}{\partial x^a}\big|_{\gamma(s)}\dot\gamma^a(s)$ and $\mathbf{\Gamma}(s;\gamma) = \Gamma_k(\gamma(s))\dot\gamma^k(s) = \Gamma_a(\gamma(s))\dot\gamma^a(s)$, where Γ_k are the matrices of the 3-index coefficients of L in $\{\frac{\partial}{\partial x^i}\}$ (Proposition 6.1). Due to these equalities, the last displayed equation is equivalent to

$$\frac{\partial A}{\partial x^a} = -\Gamma_a A, \qquad a = 1, \dots, \dim N.$$

Now, repeating the calculation after (II.5.3) and using the last equation, we find

$$\frac{\partial^2 y^i}{\partial x^{[a}\partial x^{b]}} = \frac{\partial}{\partial x^{[a}}\big(A^{-1}\big)^i_{b]} = \dots = \big(A^{-1}\Gamma_{[a}\big)^i_{b]} = \big(A^{-1}\big)^i_k \Gamma^k{}_{[ba]}.$$

The proof is completed by the observation that $\Gamma^k{}_{[ba]} \equiv 0$ is equivalent to the torsionless of L in the coordinate frame $\{\frac{\partial}{\partial x^i}\}$. In fact, by Corollary 11.1, the torsionless is equivalent to (11.12), which in $\{\frac{\partial}{\partial x^i}\}$ reduces to $T^i_{ab} = 0$ as $X^k = Y^k = 0$ for $k > \dim N$ and, consequently, (see (10.6)) $-\Gamma^i{}_{ab} + \Gamma^i{}_{ba} = 0$ since the frame $\{\frac{\partial}{\partial x^i}\}$ is holonomic, $C^i_{jk} = 0$. □

Proposition 11.12. *Let $U \subseteq M$, L be a linear transport along paths in the tangent bundle $(T(M), \pi, M)$, and L be flat on U. Then:*

(i) *If some frame normal for L on U is holonomic, then the transport L is torsionless on U and coincides over U with some parallel transport which is flat and torsionless on U.*

(ii) *If U is a submanifold of M and L is torsion free on U, then all frames normal on U for L are holonomic or can be redefined outside U (if $U \neq M$) in such a way that the redefined frames turn to be holonomic on U.*

Proof. (i) The result follows from Propositions 10.4 and 11.11.

(ii) Since all frames normal on U for L are L-transported on U (Remark 11.5), by Lemma 11.1 they are holonomic on U or can be redefined outside U so that the redefined frames are holonomic on U. □

Corollary 11.3. *A linear transport along paths in $(T(M), \pi, M)$ admits holonomic frame(s) normal on a submanifold N of M if and only if it coincides over N with some parallel transport which is flat and torsionless on N.*

Proof. See Proposition 11.12. □

Thus the moral is: *the flat and torsion free parallel transports are the only linear transports along paths in $(T(M), \pi, M)$ admitting holonomic normal frames.*

[2] If one does not want to use liftings of paths, the same result follows from $\mathsf{D}^\gamma E_i = 0$ with E_i considered as a basic vector field and D^γ being the section-derivation corresponding to D along injective path γ.

Besides, all frames normal for such transports are holonomic or can be redefined to be such.

Since all frames normal for a given flat linear transport are described via Propositions 4.3 and Corollary 4.4, in the torsionless case, the explicit description of all local coordinates normal on a submanifold is given via (see (11.13) and the proof of Lemma 11.1)

$$z^i(p) = a^i + \int_{q_0}^{p_0} \big(B^{-1}A^{-1}(p)\big)^i_j \,\mathrm{d}p^j + \big(B^{-1}A^{-1}(p_0)\big)^i_j \,[x^j(p) - x^j(p_0)] \\ + f^i_{jk}(p)[x^j(p) - x^j(p_0)][x^k(p) - x^k(p_0)]. \quad (11.14)$$

Here $B \in \mathrm{GL}(\dim M, \mathbb{K})$ is a constant matrix, $\{E_i = A^j_i \frac{\partial}{\partial x^j}\}$ is a frame normal on the submanifold, and the other notation being explained in the proofs of lemmas II.5.2 and 11.1.

12. Strong normal frames

Let M be a manifold and $(T(M), \pi, M)$ the tangent bundle over it. Let ∇ and P be, respectively, a linear connection on M and the parallel transport along paths in $(T(M), \pi, M)$ generated by ∇ (see (3.29) and the statement after it). Suppose ∇ and P admit frames normal on a set $U \subseteq M$. Here a natural question arises: what are the links between both types of normal frames, the ones normal for ∇ on U and the ones normal for P on U?

Recall (see Definitions I.5.1 and 4.1′ and equation (11.5′)), if $\Gamma^i{}_{jk}$ are the coefficients of ∇ in a frame $\{E_i\}$, the frame $\{E_i\}$ is normal on $U \subseteq M$ for ∇ or P iff respectively

$$\Gamma^i{}_{jk}(p) = 0 \quad (12.1)$$

$$\Gamma^i{}_j(s;\gamma) = \Gamma^i{}_{jk}(\gamma(s))\dot\gamma^k(s) = 0 \quad (12.2)$$

for all $p \in U$, $\gamma\colon J \to U$, and $s \in J$.

Two simple but quite important conclusions can be made from these equalities: (i) The frames normal for ∇ are normal for P, the converse being generally not valid, which is the main contents of Proposition 11.10, and (ii) in a frame normal for ∇ vanish the 2-index as well as the 3-index coefficients of P.

Definition 12.1. Let P be a parallel transport in $(T(M), \pi, M)$ and $U \subseteq M$. A *frame* $\{E_i\}$, defined on an open set containing U, is called *strong normal on* U *for* P if the 3-index coefficients of P in $\{E_i\}$ vanish on U. Respectively, $\{E_i\}$ is *strong normal along* $g\colon Q \to M$ if it is strong normal on $g(Q)$.

Obviously, the set of frames strong normal on U for a parallel transport P coincides with the set of frames normal for the linear connection ∇ generating P.

The above considerations can be generalized directly to linear transports for which 3-index coefficients exist and are fixed.

Definition 12.2. Let E and M be C^1 manifolds, $U \subseteq M$, and (E, π, M) be a vector bundle over M. Let L (resp. D) be a linear transport (resp. derivation) along paths in (E, π, M) admitting 3-index coefficients on U which are supposed to be fixed, i.e., its coefficient matrix is of the form

$$\mathbf{\Gamma}(s;\gamma) = \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) \tag{12.3}$$

in every pair of frames $\{e_i\}$ in E and $\{E_\mu\}$ in $T(M)$ defined on an open set containing U or equal to it, where $\gamma\colon J \to U$ is of class C^1 and $\Gamma_\mu := [\Gamma^i{}_{j\mu}]$ are the (fixed) matrices of the 3-index coefficients of L. A *frame* $\{e_i\}$, defined on an open set containing U or equal to it, is called *strong normal on U for L* (resp. D), if in the pair $(\{e_i\}, \{E_\mu\})$ for some (and hence any) $\{E_\mu\}$ the 3-index coefficients of L vanish on U. Respectively, $\{e_i\}$ is *strong normal along $g\colon Q \to M$* if it is strong normal on $g(Q)$.

So, a frame $\{e_i\}$ is strong normal or normal on U if (cf. (12.1) and (12.2)) respectively

$$\Gamma_\mu(p) = 0 \tag{12.4}$$
$$\mathbf{\Gamma}(s;\gamma) = \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) = 0 \tag{12.5}$$

for all $p \in U$, $\gamma\colon J \to U$, and $s \in J$. From these equations, is evident, a strong normal frame is a normal one, the opposite being valid as an exception, e.g., if U is a neighborhood. This situation is identical with the one for parallel transports in $(T(M), \pi, M)$ which is a consequence of the fact that Definition 12.2 incorporates Definition 12.1 as its obvious special case.

The main difference between the cases of parallel transports and arbitrary linear transports along paths is that for the former the condition (12.3) holds globally, i.e., for every path $\gamma\colon J \to M$, for some uniquely fixed Γ_μ, while for the latter (12.3) is valid, generally, locally, i.e., for $\gamma\colon J \to U$ with $U \subseteq M$, and in it Γ_μ are fixed but are not uniquely defined by the transport and may depend on U (see Sections 6 and 11). The cause for this is that for a parallel transport, equation (12.3) on M with uniquely defined Γ_μ follows from its definition, while if for a given linear transport L this equation holds on U for some Γ_μ, it is also true if we replace Γ_μ with $\Gamma_\mu + G_\mu$ where the matrix-valued functions G_μ are subjected to the condition $G_\mu\dot\gamma^\mu = 0$ for every path γ in U. If U is an open set, then $\dot\gamma(s)$ is an arbitrary vector in $T_{\gamma(s)}(M)$, which implies $G_\mu|_U = 0$, i.e., in this case the 3-index coefficients of L are unique; just this is the case with a parallel transport when $U = M$ and its 3-index coefficients are fixed and, by definition, are equal to the coefficients of the linear connection generating it.

If in Definition 12.2 one replaces D with a derivation $\mathcal{D}$ along tangent vector fields and (12.4) with (8.8), the definition of a frame strong normal on U for $\mathcal{D}$

will be obtained. But, by Proposition 8.4, every frame normal on U for $\mathcal{D}$ is strong normal on U for $\mathcal{D}$ and *vice versa.* Therefore the concepts of a 'normal frame' and 'strong normal frame', when applied to derivations along tangent vector fields, are identical. Returning to the considerations in Section 8, we see that frames (strong) normal for a derivation along tangent vector fields are *strong* normal for some derivation or linear transport along paths and *vice versa.* For this reason, below only strong normal frames for the latter objects will be investigated.

To make the situation easier and clearer, below the following problem will be studied. Let (E,π,M) be a vector bundle over a C^1 manifold M, $V\subseteq M$ be an *open* subset, $U\subseteq V$, and L be a linear transport along paths in (E,π,M) whose coefficient matrix has the form (12.3) on V, i.e., for every C^1 path $\gamma\colon J\to V$.[1] The problem to be investigated frames strong normal on U for L.

Let $\{e_i\}$ be a frame over V in E and $\{E_\mu\}$ a frame over V in $T(M)$. A frame $\{e'_i=A^j_ie_j\}$ over V in E is strong normal on $U\subseteq V$ if for some frame $\{E'_\mu\}$ over V in $T(M)$ is fulfilled $\Gamma'_\mu|_U=0$ with Γ'_μ given by (6.4). Hence $\{e'_i\}$ is strong normal on U iff the matrix-valued function $A=[A^j_i]$ satisfies the (*strong*) *normal frame equation*

$$(\Gamma_\mu A+E_\mu(A))|_U=0 \tag{12.6}$$

where Γ_μ are the 3-index coefficients' matrices of L in $(\{e_i\},\{E_\mu\})$. This equation, describing the matrix $A=[A^j_i]$ which provides a transition from an arbitrary to a strong normal frame(s), is also a consequence of (5.1), (12.3), and the arbitrariness of $\gamma\colon J\to V\supseteq U$.

If on U exists a frame $\{e_i\}$ strong normal for L, then all frames $\{e'_i=A^j_ie_j\}$ which are normal or strong normal on U can easily be described: for the normal frames, the matrix $A=[A^j_i]$ must be constant on U (Corollary 4.5), $A|_U=0$, while for the strong normal frames it must be such that $E_\mu(A)|_U=0$ for some (every) frame $\{E_\mu\}$ over U in $T(M)$ (see Corollary III.3.1 or (12.6) with $\Gamma_\mu|_U=0$).[2]

Comparing equations (12.6) and (III.3.5) or (I.5.4), we see that they are identical with the only difference that the size of the square matrices $\Gamma_1,\dots,\Gamma_{\dim M}$, and A in (III.3.5) is $\dim M\times\dim M$ while in (12.6) it is $v\times v$, where v is the dimension of the vector bundle (E,π,M), i.e., $v=\dim\pi^{-1}(p)$, $p\in M$, which is generally not equal to $\dim M$. But this difference is completely insignificant from the view-point of solving these equations (in a matrix form) or with respect to the integrability conditions for them. Therefore all of the results of Chapter III (or II) concerning the solution of the matrix differential equation (12.6) are (*mutatis mutandis*) applicable to the investigation of the frames strong normal on a set $U\subseteq M$.

The transferring of results from Chapter III is so trivial that their explicit reformulations makes a sense if one really needs the corresponding rigorous as-

[1] From here follows the existence of unique 3-index coefficients of L on V which, under a change of frames, transform into (6.5). We suppose the 3-index coefficients of L on U to be fixed and equal to the ones on V when restricted to U.

[2] This conclusions agree with the discussion after the proof of Proposition 11.10.

sertions for some concrete purpose. For this reason, we describe below briefly the general situation and one its corollary.

The only peculiarity one must have in mind, when such transferring is carried out, consist in the observation that in this way can be obtained, generally, only *part* of the frames normal for some linear transport, viz. the frames strong normal for it. But such a state of affairs is not a trouble as we need a single normal frame to construct all of them by means of Corollary 4.5.

Let $n \in \mathbb{N}$, J^n be a neighborhood in $\mathbb{R}^n$, and $\gamma_n \colon J^n \to M$ be a C^1 locally injective mapping. Then, from Theorem III.8.1 and Proposition III.8.1, we derive the following theorem.

Theorem 12.1. *A necessary and sufficient condition for the existence of frame(s) strong normal on $\gamma_n(J^n)$ for some linear transport along paths or derivation along paths or along vector fields tangent to M, is its (3-index) coefficients to satisfy, in some neighborhood (in $\mathbb{R}^n$) of every $s \in J^n$, the equations*

$$\big(R_{\mu\nu}(-\Gamma_1 \circ \gamma_n, \dots, -\Gamma_{\dim M} \circ \gamma_n)\big)(s) = 0, \qquad \mu, \nu = 1, \dots, n \tag{12.7}$$

where $R_{\mu\nu}$ are given via (6.7) for $x^\mu = s^\mu$, $\mu, \nu = 1, \dots, n$ with $\{s^\mu\}$ being Cartesian coordinates in $\mathbb{R}^n$.

It is almost evident, in the coordinates used, equation (12.7) is identical with (6.16) for $N = \gamma_n(J^n)$ and $p = \gamma_n(s)$. Thus, on a submanifold or along locally injective mappings, the existence of normal frames (for linear transports of the considered type) implies the existence of strong normal frames.

From (12.7), an immediate observation follows: strong normal frames always exist at every point ($n = 0$) or/and along every locally injective path ($n = 1$). Besides, these are the *only cases* when strong normal frames *always exist* because for them (12.7) is identically valid. On submanifolds with dimension greater than or equal to two normal frames exist only as an exception if (and only if) (12.7) holds. Notice, equations (12.7) express the flatness of the corresponding linear transports (see Section 9, equation (9.14), or Corollary 9.1) or, if $n = \dim M$, derivations (see Section III.2).

13. Linear transports assigned to derivations in tangent bundles

Since the linear connections are a special kind of derivations along vector fields (see Section III.2), one can expect a possible existence of linear transports along paths whose relation to the derivations is similar to the one between parallel transports and linear connections (see Subsection 11.1). Below we shall see that such an expectation has a firm background. The aim of the present section is to be clarified the links between the different derivations of the algebra of vector fields over a manifold M and the linear transports along paths in the tangent bundle

$(T(M), \pi, M)$ over M. The next ideas and results have a natural generalization in the case of arbitrary tensor bundles (or the algebraic tensor bundle) over M which we leave to the reader.

13.1. Derivations along paths

Since the Definition 11.1 on page 281 of a parallel transport employs only the covariant derivative along paths assigned to a linear connection, not directly the connection, it can *mutatis mutandis* be transferred to arbitrary derivations along paths. On this base most of the material in Section 11 could be suitably generalized. Here are some details of this procedure.

Let M be a C^1 manifold, D be a derivation along (injective C^1) paths of the tensor algebra over M (see Definition III.10.1 on page 191), and $\gamma\colon J \to M$ be injective and of class C^1. Define a mapping L on the set $\mathrm{P}^1(M)$ of C^1 paths in M, $L\colon \gamma \mapsto L^\gamma$, such that

$$L^\gamma\colon (s,t) \mapsto L^\gamma_{s\to t}\colon T_{\gamma(s)}(M) \to T_{\gamma(t)}(M) \tag{13.1}$$

for $s, t \in J$ and

$$L^\gamma_{s\to t}(X_0) = X_{\gamma(t)}, \qquad X_0 \in T_{\gamma(s)}(M) \tag{13.2}$$

where the vector field X over $\gamma(J)$ is the unique solution of the initial-value problem

$$\mathsf{D}^\gamma X = 0 \tag{13.3a}$$

$$X_{\gamma(s)} = X_0. \tag{13.3b}$$

Proposition 13.1. *The mapping $L\colon \gamma \to L^\gamma$, defined via* (13.1)–(13.3)*, is a linear transport along paths in the tangent bundle* $(T(M), \pi, M)$ *over the manifold M.*

Proof. Take into account (III.10.7) and repeat the proof of Proposition 11.1 with L for P and D^γ for $\nabla_{\dot\gamma}$. □

Definition 13.1. The linear transport L along paths in $(T(M), \pi, M)$, defined via equations (13.1)–(13.3), will be called *transport assigned to or generated by the derivation* D *along paths* (of $\boldsymbol{T}^1(M)$ or of $\boldsymbol{T}(M)$).

On the base of (13.1)–(13.3), one can prove a number of properties of the transport L assigned to D, e.g., Proposition 11.4 holds with L for P, D for the linear connection ∇, and D^γ for $\nabla_{\dot\gamma}$. But such efforts, for deriving the properties of L using only (13.1)–(13.3), are practically completely needless and the result of them will be only new ways for derivation or proving the properties of the general linear transports in $(T(M), \pi, M)$. The cause for this situation is that the afore-presented definitions of a linear transport (along injective C^1 paths) assigned to a derivation along paths is simply an equivalent way for defining what a linear transport (along injective C^1 paths) in the tangent bundle is.

The above assertion can be proved, most easily, in some local frame $\{E_i\}$ on (a neighborhood of) $\gamma(J)$ for an arbitrary injective C^1 path $\gamma\colon J \to M$.

Comparing the local expansions (III.10.7) on page 192 for $(r,q) = (1,0)$ and (2.13) on page 221 for $e_i(s;\gamma) = E_i|_{\gamma(s)}$ and $\sigma = K$, we see that they are identical for every vector field K defined on $\gamma(J)$ (or on a larger set). Therefore a derivation along paths according to Definition III.10.1, when restricted to vector fields, is in fact a section-derivation along paths in the tangent bundle, according to the definition of the latter one on page 221, assigned to a derivation D along paths in $(T(M),\pi,M)$ (see Definition 2.1) whose local components $\Gamma^i{}_j$ along γ coincide with the functions $\Gamma^i{}_j(s;\gamma)$ appearing in (III.10.7). For this reason, further the derivations along paths of the algebra of C^1 vector fields over M will be called *vector-derivations along paths.* Thus we have proved the following result.

Proposition 13.2. *For every vector-derivation* D *along paths of the algebra of vector fields over a manifold M, there exists a unique derivation D along paths in the tangent bundle* $(T(M),\pi,M)$ *over M which generates it via* (2.12)*. In a local frame, D is explicitly defined via its local components which coincide with the ones of* D*.*

Remark 13.1*.* The components of D are defined along injective paths, but this is insignificant for D which is defined along arbitrary, with or without self-intersections, paths.

Denoting, in some frame $\{E_i\}$, by $\boldsymbol{L}(t,s;\gamma)$ the matrix of the transport L assigned to a vector-derivation D along paths, from equations (13.2), (13.3), and (III.10.7), we derive that it is the unique solution of the initial-value problem

$$\frac{\partial \boldsymbol{L}(t,s;\gamma)}{\partial t} = -\boldsymbol{\Gamma}(t;\gamma)\boldsymbol{L}(t,s;\gamma) \tag{13.4a}$$

$$\boldsymbol{L}(s,s;\gamma) = \mathbb{1} \tag{13.4b}$$

where $s,t \in J$ and $\boldsymbol{\Gamma} := [\Gamma^i{}_j]$ is the matrix of the components of D (or of D) in $\{E_i\}$. So the explicit form of $\boldsymbol{L}$ is

$$\boldsymbol{L}(t,s;\gamma) = Y(t,s_0;-\boldsymbol{\Gamma}(\,\cdot\,;\gamma))Y^{-1}(s,s_0;-\boldsymbol{\Gamma}(\,\cdot\,;\gamma)) \tag{13.5}$$

where Y is defined via (3.28) and $s_0 \in J$ is fixed.

At last, noticing that (13.5) is identical with (3.27), we conclude from Proposition 3.6 that L is identical with the linear transport along paths generating D via (3.19) and whose coefficients coincide with the components of D (or of D). This result completes the proof of our statement that a linear transport along paths assigned to a vector-derivation is simply a linear transport along injective C^1 paths in the tangent bundle and *vice versa*.

13.2. Derivations along vector fields

From Section III.10, we know that to every derivation along vector fields whose matrix in some (and hence in every – see (III.2.11)) frame $\{E_i\}$ satisfies the condition

$$\mathbf{\Gamma}_X(\gamma(s))|_{X_{\gamma(s)}=\dot{\gamma}(s)} = \mathbf{\Gamma}(s;\gamma) \tag{13.6}$$

for every (injective) C^1 path $\gamma\colon J \to M$ and $s \in J$, there corresponds a vector-derivation D along paths in $(T(M), \pi, M)$ such that (see (III.10.2))

$$(\mathsf{D}^\gamma V)(\gamma(s)) = \Big(\frac{\mathrm{d}V^i(\gamma(s))}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)V^j_{\gamma(s)}\Big)E_i|_{\gamma(s)} \tag{13.7}$$

for $V \in \mathfrak{X}^1(M)$.

Example 13.1. A kind of vector-derivation is the covariant derivative assigned to a linear connection ∇ for which

$$\mathbf{\Gamma}_X(p) = \Gamma_k(p)X^k_p, \tag{13.8}$$

where $p \in M$ and Γ_k are the coefficients' matrices of ∇. In this case, obviously, equation (13.6) holds with

$$\mathbf{\Gamma}(s;\gamma) = \Gamma_k(\gamma(s))\dot{\gamma}^k(s). \tag{13.9}$$

Since to a vector-derivation along paths in the tangent bundle $(T(M), \pi, M)$ there corresponds unique linear transport along paths in the same bundle (see Subsection 13.1), to every derivation D along vector fields of $\boldsymbol{T}(M)$, whose components' matrix satisfies the condition (13.6), there corresponds a linear transport along paths generating the assigned to D vector-derivation D via (3.19) and (2.12).

So, the general scheme for generating a linear transport from a derivation along vector fields is

derivation along vector fields satisfying (13.6)
→ vector-derivation along paths
→ linear transport along paths. (13.10)

A typical example, realizing this procedure, is the parallel transport assigned to a linear connection. A modification of the scheme (13.10) for derivations along fixed vector fields will be developed in the next subsection .

13.3. Derivations along fixed vector field

In the present subsection, we want to apply (possibly a modification of) the scheme (13.10) to a derivation D_X along a *fixed* vector field X of the tensor algebra over a manifold M. Of course, such a derivation can not define a linear transport along arbitrary (C^1) paths in $(T(M), \pi, M)$ since X is *fixed*, not arbitrary, vector field and hence (13.6) cannot hold for arbitrary path γ.[1] As one could expect, the transports corresponding to D_X are along the integral paths of X. Except this peculiarity, the other general considerations in Subsection 13.2 preceding (13.10) remain unchanged. Here are some details of the procedure assigning a linear transport to a derivation along a fixed vector field.

Let $X \in \mathfrak{X}^1(M)$ be fixed and non-singular, $X \neq 0$. Let $\gamma_p \colon J \to M$ be the integral path of X through $p \in M$, i.e., $\gamma_p(s_0) = p$ for some $s_0 \in J$ and $\dot{\gamma}_p(s) = X_{\gamma_p(s)}$ for all $s \in J$. Suppose D_X is a derivation along X such that the matrix of its components $\mathbf{\Gamma}_X$ (in some and hence in any frame) satisfies the condition (13.6) with $\gamma = \gamma_p$ for every $p \in M$. Then, for every integral path γ of X, it is defined a unique derivation D^γ along γ, explicitly given via (13.7). (Note, D^γ is defined only along integral paths γ of X!) Repeating the proof of Proposition 13.1 with γ being an integral path of X, we see that (13.1)–(13.3) define a unique linear transport L along the integral paths of X in $(T(M), \pi, M)$. In a local frame, the matrix of L satisfies the initial-value problem (13.4) and its explicit form is (13.5).

Thus the conclusion is: to every derivation along a fixed vector field X (which is C^1 and non-singular) of the tensor algebra over a manifold M, there corresponds a unique linear transport along the integral paths of X in the tangent bundle over M (provided (13.6) holds with an integral path of X for γ). Therefore, we have the following modification of (13.10):

derivation along fixed vector field X
$\quad\to$ vector-derivation along the integral paths of X
$\quad\quad\to$ linear transport along the integral paths of X. (13.11)

In this scheme, we intentionally dropped the condition (13.6). The reason is that it works independently of (13.6): if (13.6) is not valid, the resulting derivation and linear transport along the integral paths of X could depend also on the vector field X in a neighborhood of the integral paths of X, not only on the values of X on them. In other words, if (13.6) does not hold, the corresponding derivation and linear transport along a particular integral path may depend not only on it but also on a congruence of the integral paths of X in a neighborhood of the one along which the derivation and transport act. This situation is not an exceptional one, on the opposite, it is often met in concrete applications, first of all in the theoretical physics.[2] Below we shall present a list of important examples of this kind.

[1] For a fixed X, the equality $X_{\gamma(s)} = \dot{\gamma}(s)$ cannot hold for arbitrary γ as it defines the integral paths of X – see (I.2.19).

[2] In the physically-oriented literature, the situation is 'inverse' with respect to X: there is given a congruence of (injective, C^1, and regular) paths and X is defined as the vector field tangent to the paths of that congruence.

Recall (see Definition III.2.3), for every vector field X and derivation D along vector fields, the decomposition $D_X = \mathcal{L}_X + S_X$, with $D\colon X \to D_X$ and $\mathcal{L}_X$ being the Lie derivative along X, holds for some tensor field S_X of type $(1,1)$. Since for a linear connection ∇ this equality holds for

$$S_X = \Sigma_X, \quad \Sigma_X(Y) := \nabla_X Y - [X,Y]_{\_} = \nabla_Y X + T(X,Y), \quad Y \in \mathfrak{X}(M), \tag{13.12}$$

with T being the torsion of ∇, for every derivation D along vector fields is fulfilled

$$D_X = \nabla_X + S_X - \Sigma_X \tag{13.13}$$

for (arbitrarily chosen) linear connection ∇ and some $S_X \in \mathfrak{T}^1_1(M)$. Applying the last decomposition and the explicit local expression (III.2.2), one can prove the following statements.

Example 13.2. Let M be a manifold endowed with a linear connection ∇. To ∇ corresponds a decomposition (13.13) with

$$S_X = \Sigma_X \tag{13.14a}$$

where X can be fixed, as well as arbitrary, and the condition (13.6) is valid.

Example 13.3. Let M be Einstein-Cartan manifold, i.e., one endowed with (generally said independent) Riemannian metric g and linear connection ∇. The Fermi–Walker derivative [62, 119] is a one for which (13.13) holds with

$$S_X = \Sigma_X - 2Q_X. \tag{13.14b}$$

Here X is fixed, (time-like,) C^1, and unit vector field[3] and $Q_X \in \mathfrak{T}^1_1(M)$ with

$$g_{im}(Q_X)^m_j := \left\{ h^k_i h^l_j (g(\,\cdot\,, \nabla_{E_k} X))_l + h^m_i g_{ml} T^l_{jk} X^k + (g(\,\cdot\,, \nabla_{E_j} X))_i \right\}_{[ij]}$$

where $h^k_i := \delta^k_i - X^k (g(\,\cdot\,, X))_i / g(X,X)$ and T^i_{jk} are the components of the torsion tensor of ∇.

Example 13.4. On a Riemannian manifold, the Fermi derivative assigned to its metric g [62] corresponds to the choice

$$S_X = \Sigma^{\{\}}_X - 2\overline{Q}_X \tag{13.14c}$$

where X is a fixed, (time-like,) C^1 and unit vector field, the index $\{\}$ means that the connection is with respect to the Christoffel symbols (I.4.14) formed from g, $\overline{Q}_X \in \mathfrak{T}^1_1(M)$, and $(\overline{Q}_X)^i_j := X^i (g(\,\cdot\,, \nabla^{\{\}}_X X))_j$.

[3]The choice $g(X,X) = +1$ or $g(X,X) = -1$ depends on the accepted signature of g: the former (resp. latter) one corresponds to a positive (resp. negative) ‘time’ eigenvalue of g in a diagonal form.

Example 13.5. On a Riemannian manifold, the Truesdell derivative assigned to its metric g [120, 121] is obtained from (13.13) for[4]

$$S_X = \theta \cdot \delta. \tag{13.14d}$$

Here X is a fixed, C^1 and null or unit vector field (which could be time-like or space-like), $\theta := \sum_i (\nabla_{E_i} X)^i$ is the expansion of X, and δ is the unit tensor with local components δ_i^j (in any frame).

Example 13.6. On a Riemannian manifold, the Jaumann derivative assigned to its metric g [122] is described via (13.13) with

$$S_X = \Sigma_X^{\{\}} - \omega. \tag{13.14e}$$

Here X is a fixed C^1 and null or unit vector field (which could be time-like or space-like), $\omega \in \mathfrak{T}_1^1(M)$ and[5]

$$g_{ik}\omega_j^k := \big\{ (g(\,\cdot\,, \nabla_{E_j}^{\{\}} X))_i - (g(\,\cdot\,, \nabla_{\nabla_{E_j}^{\{\}} X}^{\{\}} X))_i \big\}_{[ij]}.$$

The above list of particular derivations (along, possibly, fixed vector fields) obtained via (13.13) can easily be completed by the Lie derivative [19, 89, 119], modified Fermi-Walker and Frenet-Serret derivatives [98], etc.

Excluding the case of linear connections, all of the above-mentioned derivatives are along a fixed vector field and, generally, the condition (13.6) is not satisfied for them. Consequently, the derivations and linear transports along the integral paths of X corresponding to them depend, generally, on a congruence of integral paths, not only on the particular integral path along which they act. Regardless of this, these derivations and the corresponding transports find a number of important applications.

Exercise 13.1. Following the implications in (13.10) (or in (13.11)), prove that from (13.13) and (13.14a)–(13.14e) can be obtained respectively: the parallel transport assigned to a linear connection (covariant derivative) of the tensor algebra of a manifold [11, 19], Fermi-Walker transport [62, 119], Fermi transport [62], Truesdell transport [120, 121], and Jaumann transport [122]. This list of transports can be completed with the Lie transport [19, 119], modified Fermi-Walker and Frenet-Serret transports [98], etc.

13.4. Normal frames

In Subsections 13.1–13.3, it was demonstrated how to different derivations of the tensor algebra over a manifold can be assigned linear transports along paths in

[4] The choice of ∇ is insignificant as $\nabla_X - \Sigma_X = \mathcal{L}_X$.

[5] The sign before the second term in the right-hand side of the next equality must be opposite to the one of the eigenvalue of the 'time' term of g in a diagonal form.

the tangent bundle over it. Hence to every such derivation can be assigned two kinds of normal frames: the ones corresponding to it as a derivation and the ones corresponding to the linear transports generated by the derivations. These frames were introduced and investigated respectively in Chapter III and in the present one. Since a simple observation reveals that the derivations' components coincide with the coefficients of the linear transports generated by the derivations, one can expect the existence of some links between both sorts of normal frames. More precisely, relying on the considerations in Section 12, we can assert that the frames, normal for derivations along paths or (arbitrary) vector fields, are strong normal for the linear transports generated by these derivations. What concerns the frames normal for derivations along a fixed vector field, it is easy to be proved that they are also normal for the linear transports (along the integral paths of the vector field) generated by this kind of derivations and *vice versa*. The opposite link is partially valid in a sense that if a linear transport generated by a derivation admits normal frames, then some (or all) of them, viz. the strong normal ones, are normal for the corresponding derivation generating the initial transport.

In this way, we have came to the following important

Conclusion 13.1. All of the results, concerning normal frames and obtained in Chapter III for different derivations of the tensor algebra over a manifold, are *mutatis mutandis* valid for the frames (strong) normal for the (corresponding) linear transports along paths in the tangent bundle over the same manifold.

The afore-presented conclusions can also be confirmed by the next considerations some of which are valid for general vector bundles with a manifold as a base, not only for the tangent bundle over a manifold.

From Proposition 6.1 and Theorem 6.2, we know that only linear transports/derivations along paths with (2-index) coefficients given by (6.1) admit normal frames. Besides, from equations (6.1) and (6.4), it follows that frames normal on a subset $U \subseteq M$ for such transports/derivations along paths exist if and only if the matrix differential equation (5.1), or, equivalently,

$$\left[\dot\gamma^{\mu}(\Gamma_{\mu}A + E_{\mu}(A))\right]\Big|_U = 0, \tag{13.15}$$

has a solution for every $\gamma\colon J \to U$ with respect to A.[6] In fact, the equations (6.16) are the integrability conditions for (13.15).[7] Evidently, the same is the situation with derivations along tangent vector fields (see Section 8) when, due to (8.6), such a derivation admits frames normal on U iff the equation

$$\left(\mathbf{\Gamma}_X A + X(A)\right)\Big|_U = 0, \tag{13.16}$$

[6]If such A exist in a frame $\{e_i\}$, then the frame $\{e'_i = A^j_i e_j\}$ is normal on U and *vice versa*; see (6.4).

[7]If (6.6) hold and U is a neighborhood, then $A = Y(p, p_0; -\Gamma_1, \dots, -\Gamma_{\dim M})A_0$, A_0 being non-degenerate matrix.

$\mathbf{\Gamma}_X$ being the derivation's matrix along a vector field X, has a solution with respect to A. As we proved in Section 8, if X is arbitrary and tangent to the paths in U, this equation is equivalent to (13.15) with Γ_μ being the matrices of the 3-index coefficients of the derivation; if X is completely arbitrary, (13.16) is equivalent to equation (13.17) below.

Now it is time to recall that, from mathematical view-point, the material of Chapter III is actually devoted precisely to the solution of the normal frame equation

$$\left(\Gamma_\mu A + E_\mu(A)\right)\big|_U = 0 \qquad (13.17)$$

which is equivalent to (13.15) if U is a neighborhood. The fact that in Chapter III are studied frames normal for derivations of the tensor algebra over a manifold M is inessential because the *equations describing the matrices* by means of which is performed the transformation from an arbitrary frame to a (strong) normal one are the same in this chapter and in Chapter III. The only difference is *what objects* are transformed by means of the matrices satisfying (13.16): in the present chapter these are the frames in the restricted bundle space $\pi^{-1}(U) \subseteq E$, while in Chapter III they are the tensor bases over U, in particular the ones in the bundle tangent to M. In Chapter III, the only explicit use of the derivations of the tensor algebra over M was to define their components (2-index coefficients) and the transformation law for the latter. Since this law (see (III.2.11) and (III.10.9)) is identical with (8.6),[8] all results concerning the 2- and 3-index coefficients of derivations of the tensor algebra over M and the ones of derivations along tangent vectors in vector bundle (E, π, M) *coincide.*

Thus, we have came to the following very important conclusion.

Conclusion 13.2. All of the results of Chapter III, concerning derivations along vector fields, their components, and frames normal for them, are *mutatis mutandis* valid (as investigated in the present chapter) for linear transports along paths, derivations along paths or along tangent vector fields, their coefficients (or components), and the frames (strong) normal for them in vector bundles with a differentiable manifold as a base.

The only change, if required, to transfer the results is to replace the term 'derivation along vector fields' with 'derivation along tangent vector fields', or 'derivation along paths', or 'linear transport along paths' and, possibly, the term 'normal frame' with 'strong normal frame'.

Because of the widespread usage of covariant derivatives (linear connections), we want to mention them separately regardless of the fact that this case was completely covered in Chapters II and III. As a consequence of (3.29), the covariant derivatives are derivations linear on the whole base M (as well as on any its subset). Thus for them the condition (6.1) is identically satisfied. Therefore, by

[8] The transformation laws (3.26) and (6.4) can be considered, under certain conditions, as special cases of (8.6).

Theorem 6.2, a covariant derivative (or the corresponding parallel transport) admits normal frames on a submanifold $U \subseteq M$ iff (6.16) holds on U. Consequently, every covariant derivative admits normal frames at every point or along given (smooth locally injective) path. However, only the flat covariant derivatives on U admit frames normal on U if U is a neighborhood ($\dim U = \dim M$).

In the theoretical physics, we find applications of a number of linear transports along paths: parallel [11, 19], Fermi-Walker [62, 119], Fermi [62], Truesdell [120, 121], Jaumann [122], Lie [19, 119], modified Fermi-Walker and Frenet-Serret [98], etc. Our results are fully applicable to all of them, in particular for all of them there exist frames normal at a given point or/and along (smooth locally injective) paths.

14. Links with the theory of connections and parallel transports

The goal of this section is twofold: on one hand, we would like to clarify the relations between the linear transports along paths and parallel transports in general vector bundles and, on another hand, to transfer the results obtained for connections and parallel transports in vector bundles, thus generalizing the material from Sections 11–13 concerning mainly the tangent bundle over a manifold. A part of these problems will be generalized further in Chapter V from the view-point of general connection theory on differentiable bundles.

The book [23] will be used consistently for reference purposes below. This is due to the fact that in it is elaborated an axiomatical approach to the concept of a parallel transport in vector bundles (whose bundle and base spaces are C^∞ manifolds) and on this base the connection theory is studied. Such an exposition perfectly matches our aims, as pointed at the beginning of Section 3.

In the present section (E, π, M) will denote a vector bundle whose bundle and base spaces, E and M, are C^∞ manifolds, N stands for a C^∞ manifold, and $g\colon N \to M$ is supposed to be of class C^∞. In general, for the consistency with [23], all manifolds and mappings between them are supposed to be of class C^∞ in this section.[1]

14.1. Parallelism structures, connections and covariant derivatives

In this subsection the (axiomatical) definition of a parallel transport and its links with the concepts of connection and covariant derivative from [23] are reproduced

[1] Such a supposition is too strong. For the most of the material that follows, smoothness of class C^2 (and sometimes C^1) is sufficient. (Smoothness of class C^3 is required if normal coordinates, if any, are concerned.)

and briefly reviewed. We have chosen the pointed definition from [23] for two reasons: (i) in the sense of [23, Theorems 2.28 and 2.33] it provides an equivalent description of the notion of a connection on vector bundles, and (ii) in the available to the author literature, the book [23] provides the most advanced and well-developed axiomatical approach to the notion of a parallel transport (in vector bundles).

Definition 14.1 (cf. [23, Definition 2.7])**.** Let (E, π, M), E and M being C^∞ manifolds, be a vector bundle and $\beta\colon [a, b] \to M$, $a, b \in \mathbb{R}$ with $a \le b$, be a C^∞ path in M. A mapping $\mathbb{P}$ associating to each pair (u, β), $u \in \pi^{-1}(\beta(a))$, a (unique) C^∞ lifting of β from M to E is called a *parallelism structure* (or a *system of parallel transport*) in (E, π, M) if the following five conditions are satisfied:

(i) Existence. The value $\mathbb{P}^\beta_u$ of $\mathbb{P}$ at (u, β), $\mathbb{P}\colon (u, \beta) \mapsto \mathbb{P}^\beta_u$, is such that

$$\mathbb{P}^\beta_u(a) = u \tag{14.1}$$

(and, by definition, $\mathbb{P}_u\colon \beta \mapsto \mathbb{P}^\beta_u\colon [a, b] \to E$ with $\pi \circ \mathbb{P}^\beta_u = \beta$), i.e., $\mathbb{P}^\beta_u$ is a lifting of β through u starting exactly from the point $u \in \pi^{-1}(\beta(a))$.

(ii) Linearity and invertability. The mapping

$$\mathsf{P}^\beta\colon \pi^{-1}(\beta(a)) \to \pi^{-1}(\beta(b)) \tag{14.2}$$

defined by

$$\mathsf{P}^\beta(u) := \mathbb{P}^\beta_u(b) \tag{14.3}$$

for every $u \in \pi^{-1}(\beta(a))$ is a vector space isomorphism. It is called *parallel transport along* $\beta\colon [a, b] \to M$ (assigned to $\mathbb{P}$). The parallel transport along the inverse path $\beta^-\colon [a, b] \to M$, $\beta^-\colon s \mapsto \beta^-(s) := \beta(a + b - s)$ for each $s \in [a, b]$, is

$$\mathsf{P}^{\beta^-} = \left(\mathsf{P}^\beta\right)^{-1}. \tag{14.4}$$

(iii) Parameterization independence. Let $\varphi\colon [c, d] \to [a, b]$ be C^∞ function such that $\varphi(c) = a$ and $\varphi(d) = b$. Then

$$\mathbb{P}^{\beta\circ\varphi}_u = \mathbb{P}^\beta_u \circ \varphi. \tag{14.5}$$

(iv) C^∞ dependence on initial conditions. For every open set $U \subseteq M$ and each C^∞ mapping $f\colon T(U) \to M$ such that $f(0_p) = p$, $p \in U$, with 0_p being the zero vector in $T_p(U) = T_p(M)$, the mapping

$$\tilde{f}\colon T(U) \times \pi^{-1}(U) \to E, \quad \tilde{f}\colon (X, u) \mapsto \mathbb{P}^\alpha_u(1), \qquad \alpha(s) := f(sX) \tag{14.6}$$

where $(X, u) \in T(U) \times \pi^{-1}(U)$ and $s \in [0, 1]$, is of class C^∞.

(v) Initial uniqueness. If $p \in M$ and the C^∞ paths $\alpha, \beta\colon [0, 1] \to M$ are such that $\alpha(0) = \beta(0) = p$ and[2] $\dot\alpha(0) = \dot\beta(0)$, then for each $u \in \pi^{-1}(p)$ the tangent vectors to the paths $\mathbb{P}^\alpha_u, \mathbb{P}^\beta_u\colon [0, 1] \to E$ at the point $s = 0$ coincide:

$$\dot{\widehat{\mathbb{P}^\alpha_u}}(0) = \dot{\widehat{\mathbb{P}^\beta_u}}(0). \tag{14.7}$$

[2]The tangent vectors $\dot\alpha(0)$ and $\dot\beta(0)$ are considered as defined by one-sided derivative operators – see (I.2.3).

The parallelism structures are an equivalent way for describing general connections on vector bundles. To formulate this statement in a rigorous way (see Theorem 14.1 below), we have to present some preliminary material.

Definition 14.2. A *lifting* (or lift) $\bar{\beta}\colon [a,b] \to E$ of a C^∞ path $\beta\colon [a,b] \to M$, $\pi \circ \bar{\beta} = \beta$, is said to be *parallel with respect to a parallelism structure* $\mathbb{P}$ in a vector bundle (E,π,M) if $\bar{\beta} = \mathbb{P}^{\beta}_{\bar{\beta}(a)}$; respectively, the *paths* $\mathbb{P}^{\beta}_{u}$, with $u \in \pi^{-1}(\beta(a))$ and $\beta\colon [a,b] \to M$, in E are called *parallel* (*with respect to* $\mathbb{P}$). A *section* $\sigma \in \operatorname{Sec}(E,\pi,M)$ is *parallel along* β (*with respect to* $\mathbb{P}$) if the lifting $\hat{\beta} = \sigma \circ \beta$ is parallel (with respect to $\mathbb{P}$).

Definition 14.3 (cf. [23, Definition 2.26])**.** A *connection on vector bundle* (E,π,M) is a $(\dim M)$-dimensional distribution $T^h(E)\colon E \to T(E)$ such that:

(i) For each $u \in E$ the value $T^h_u(E)$ of $T^h(E)$ at u, $T^h(E)\colon u \mapsto T^h_u(E)$, is in the tangent space $T_u(E)$ and is (direct) complement to its vertical component $T^v_u(E) := T_u\big(\pi^{-1}(\pi(u))\big)$, i.e., to the space tangent to the fibre $\pi^{-1}(\pi(u))$ through u:

$$T_u(E) =: T^h_u(E) \oplus T^v_u(E). \tag{14.8}$$

(ii) The spaces $T^h_u(E)$ are homogeneous in a sense that

$$\mu_{c*}\big(T^h_u(E)\big) = T^h_{\mu_c(u)}(E) = T^h_{cu}(E), \qquad c \in \mathbb{K} \tag{14.9}$$

where $\mu_c\colon E \to E$ with $\mu_c\colon u \mapsto cu$ and $\mu_{c*} := (\mu_c)_*$ is the differential of μ_c.

Let us set $T^v(E) := \cup_{u\in E} T^v_u(E)$. By abuse of the notation, we put $T^h(E)$ to denote also the union $\cup_{u\in E} T^h_u(E)$. Thus, we have the direct decomposition

$$T(E) = T^h(E) \oplus T^v(E). \tag{14.10}$$

Remark 14.1. A connection on the tangent bundle $(T(M),\pi_T,M)$ over M is called (linear) connection on M. For some details, see below Proposition 14.5 and Remark 14.3.

Definition 14.4. The *spaces* $T^h_u(E)$ and $T^v_u(E)$ are called *horizontal* and *vertical*, respectively, space (tangent to E at u); the vectors of these spaces are called respectively *horizontal and vertical vectors.*

Definition 14.5 (cf. [23, Definition 2.27])**.** Let $T^h(E)$ be a connection on (E,π,M). A *lifting* $\bar{f}\colon N \to E$ of a C^∞ mapping $f\colon N \to M$, N being a manifold, is *horizontal with respect to* $T^h(E)$ (*or is* $T^h(E)$*-horizontal*) if $\bar{f}_*(T_p(N))) \subseteq T^h_{\bar{f}(p)}(E)$ for every $p \in N$, i.e., the induced tangent mapping $\bar{f}_*$ of $\bar{f}$ sends the vectors tangent to N at p to horizontal vectors (tangent) at $\bar{f}(p)$. In particular, a *path* $\bar{\beta}$ in E, which is a lifting of the path $\pi \circ \bar{\beta}$ in M, is *horizontal* if its tangent vector field is a path in $T^h(E)$.

Definition 14.6. If $T^h(E)$ is a connection on (E, π, M), the *($T^h(E)$-)horizontal lifting* of a vector field $X \in \mathfrak{X}(M) = \mathrm{Sec}(T(M), \pi_T, M)$ is the unique vector field $\bar{X} \in \mathfrak{X}(E) = \mathrm{Sec}(E, \pi, M)$ such that $\bar{X}_u \in T^h_u(E)$ and $\bar{X}$ and X are π-related, i.e., $\pi_* \circ \bar{X} = X \circ \pi$.

Proposition 14.1. *Let $T^h(E)$ be a connection on some vector bundle (E, π, M) and $\beta\colon [a, b] \to M$ be a C^∞ path in M. For every $t_0 \in [a, b]$ and $u \in \pi^{-1}(\beta(t_0))$, there exists a unique $T^h(E)$-horizontal lifting $\bar{\beta}$ of β to E such that $\bar{\beta}(t_0) = u$.*

Proof. See [23, Proposition 2.32 on p. 59]. □

Theorem 14.1 (cf. [23, Theorems 2.28 and 2.33]**).** *Each given parallelism structure $\mathbb{P}$ in a vector bundle (E, π, M) determines a connection $T^h(E)$ on (E, π, M) such that a path $\bar{\beta}$ in E is a parallel lifting of the path $\beta = \pi \circ \bar{\beta}$ in M with respect to $\mathbb{P}$ if and only if $\bar{\beta}$ is horizontal with respect to $T^h(E)$, i.e., iff $\dot{\bar{\beta}}$ is a path in $T^h(E)$. Conversely, if $T^h(E)$ is a connection on a vector bundle (E, π, M), the system of $T^h(E)$-horizontal liftings to E of the C^∞ paths in M is a parallelism structure in (E, π, M) and, besides, the connection on (E, π, M) determined by it according to the previous assertion is just $T^h(E)$.*

Proof. See [23, pp. 55–61]. □

The last theorem is a rigorous expression of the assertion that a parallelism structure in a vector bundle is equivalent to a connection on the same bundle. By its means one can easily prove some important properties of the parallelism structures.

Proposition 14.2. *Let $\mathbb{P}$ be a parallelism structure in vector bundle (E, π, M) and $\beta\colon [a, b] \to M$ be a C^∞ path. For each $c \in [a, b]$ and $u \in \pi^{-1}(\beta(a))$,*

$$\mathbb{P}^\beta_u(\lambda) = \mathbb{P}^{\beta|[a,c]}_u(\lambda), \qquad \lambda \in [a, c]. \tag{14.11}$$

Proof. Let $T^h(E)$ be the connection on (E, π, M) determined by $\mathbb{P}$ as in Theorem 14.1. Since $\mathbb{P}^\beta_u$ and $\mathbb{P}^{\beta|[a,c]}_u$ are parallel liftings of respectively β and $\beta|_{[a,c]}$ through u, they are the *unique* $T^h(E)$-horizontal liftings of respectively β and $\beta|_{[a,c]}$ through u. Consider the restricted lifting $\mathbb{P}^\beta_u\big|_{[a,c]}$. As $\pi \circ \big(\mathbb{P}^\beta_u\big|_{[a,c]}\big) = \beta|_{[a,c]}$ and $\mathbb{P}^\beta_u\big|_{[a,c]}(a) = \mathbb{P}^\beta_u(a) = u$, it is a $T^h(E)$-horizontal lifting through u and, due to the uniqueness of such liftings (see Proposition 14.1), we have

$$\mathbb{P}^\beta_u\big|_{[a,c]} = \mathbb{P}^{\beta|[a,c]}_u \tag{ 14.11$'$}$$

which is equivalent to (14.11). □

Proposition 14.3. *Let $\mathbb{P}$ be a parallelism structure in (E, π, M), the path $\beta\colon [a, b] \to M$ be C^∞, and $u \in \pi^{-1}(\beta(a))$. For each $c \in [a, b]$,*

$$\mathbb{P}^\beta_u(\lambda) = \mathbb{P}^{\beta|[c,b]}_{\mathbb{P}^{\beta|[a,c]}_u(c)}(\lambda), \qquad \lambda \in [c, b]. \tag{14.12}$$

Proof. Let $T^h(E)$ be the connection on (E, π, M) determined by $\mathbb{P}$ according to Theorem 14.1. Since $\mathbb{P}^\beta_u$ and $\mathbb{P}^{\beta|[c,b]}_{\mathbb{P}^{\beta|[a,c]}_u(c)}$ are parallel liftings of β through u and of $\beta|[c,b]$ through $\mathbb{P}^{\beta|[a,c]}_u(c)$, respectively, they are the *unique* $T^h(E)$-horizontal liftings of β through u and of $\beta|[c,b]$ through $\mathbb{P}^{\beta|[a,c]}_u(c)$, respectively. Evidently $\pi \circ \big(\mathbb{P}^\beta_u|[c,b]\big) = \beta|[c,b]$ and $\mathbb{P}^\beta_u(c) = \mathbb{P}^{\beta|[a,c]}_u(c)$ (see (14.11) with $\lambda = c$). Hence $\mathbb{P}^\beta_u|[c,b]$ is a $T^h(E)$-horizontal lifting of $\beta|_{[c,b]}$ through $\mathbb{P}^{\beta|[a,c]}_u(c)$ and, by Proposition 14.1, it coincides with $\mathbb{P}^{\beta|[c,b]}_{\mathbb{P}^{\beta|[a,c]}_u(c)}$, i.e.,

$$\mathbb{P}^\beta_u\big|_{[c,b]} = \mathbb{P}^{\beta|[c,b]}_{\mathbb{P}^{\beta|[a,c]}_u(c)} \tag{14.12$'$}$$

which is tantamount to (14.12). □

Now we shall consider briefly the description of parallelism structures (and hence connections) on vector bundles in terms of covariant derivatives.

Definition 14.7. Let (E, π, M), E and M being C^∞ manifolds, be a $\mathbb{K}$-vector bundle and $g\colon N \to M$, N being C^∞ manifold, be of class C^∞. A mapping $\nabla\colon \mathfrak{X}(N) \times \mathrm{Lift}_g(E, \pi, M) \to \mathrm{Lift}_g(E, \pi, M)$ is called a *covariant derivative* (or *covariant derivative operator*) in (E, π, M) along g if for every $U, V \in \mathfrak{X}(N)$ and $X, Y \in \mathrm{Lift}_g(E, \pi, M)$ the mapping $\nabla\colon (U, X) \mapsto \nabla_U X$ has the properties:

(i) $\nabla_{U+V} X = \nabla_U X + \nabla_V X$;
(ii) $\nabla_{fU} X = f \nabla_U X, \quad f \in \mathfrak{F}(N)$;
(iii) $\nabla_U (X + Y) = \nabla_U X + \nabla_U Y$;
(iv) $\nabla_U (hX) = U(h) X + h \nabla_U X, \quad h \in \mathfrak{F}^1(N)$.

The mapping $\nabla X\colon \mathfrak{X}(N) \to \mathrm{Lift}_g(E, \pi, M)$, associated to a covariant derivative ∇ and such that $\nabla X\colon U \mapsto (\nabla X)(U) := \nabla_U X$, is called covariant differential of X (along g) and the lifting $\nabla_U X$ of g is called covariant derivative (along g) of X with respect (or along) U. If $u \in T_p(N)$, $p \in N$, we set $\nabla_u X := (\nabla_U X)_p$ with $U \in \mathfrak{X}(N)$ such that $U_p = u$, which is well-defined due to (i) and (ii).

Two important special cases are worth singling out. If $N = M$ and $g = \mathsf{id}_M$, then ∇ is called covariant derivative in (E, π, M). Besides, if in addition $E = T(M)$, i.e., if ∇ is a covariant derivative in the tangent bundle over M, then ∇ is called covariant derivative *on* M.

To formulate the second main result of this subsection, Theorem 14.2 below, we shall briefly present some preliminary material.[3]

Put $\pi^* E := \pi^*(E) := \{(\zeta, \xi) | \zeta, \xi \in E,\ \pi(\zeta) = \pi(\xi)\}$.[4] Evidently, $\pi^* E$ is a vector space. We assert that $\pi^* E$ is isomorphic to the vertical component $T^v(E) := \cup_{u \in E} T^v_u(E) = \cup_{u \in E} T_u\big(\pi^{-1}(\pi(u))\big)$ of the bundle space $T(E)$. To prove this, define

[3] For details, see [23]. A reader familiar with [23] or equivalent material can skip the text from this point to Theorem 14.2.

[4] The set $\pi^*(E)$ is the bundle space of the pullback of (E, π, M) along π [23, p. 8].

$\mathfrak{g}\colon \pi^*E \to T^v(E)$ by $\mathfrak{g}\colon (\zeta,\xi) \mapsto \mathfrak{g}_\zeta\xi := \dot\alpha(0)$ where $(\zeta,\xi)\in\pi^*$ and α is a path in E given via $\alpha\colon t\mapsto \alpha(t) := \zeta + t\xi$, $t\in\mathbb{R}$. The mapping $\mathfrak{g}$ is a vector space isomorphism and its inverse is $\mathfrak{g}^{-1}\colon w\mapsto(\zeta,\xi)$, $w\in T^v(E)\subset T(E)$, where $\zeta := \pi_{T(E)}(w)\in E$, $\pi_{T(E)}$ being the projection of the tangent bundle $(T(E),\pi_{T(E)},E)$ over E, and ξ is a point in $\pi^{-1}(\pi(\zeta))$, considered as a manifold, such that its coordinates in some fixed local coordinates $\{x^i\}$ are equal to the local coordinates of w (as an element of $T^v_\zeta(E)\subset T_\zeta(E)$) in the associated frame $\{\frac{\partial}{\partial x^i}\}$ at ζ.

Denote by $\mathrm{pr}_2\colon T^v(E)\to E$ a mapping assigning to each $w\in T^v(E)$ the "second component" of the element of $\pi^*(E)$ corresponding to w via $\mathfrak{g}$, i.e., $\mathrm{pr}_2(w) := \xi$ if $w = \mathfrak{g}_\zeta\xi$ for some (unique) $(\zeta,\xi)\in\pi^*(E)$.

If $T^h(E)$ is a connection on a vector bundle (E,π,M), every vector $w\in T(E)$ has, according to (14.8), a unique decomposition $w = w^h + w^v$, where, if $w\in T_u(E)$ for some $u\in E$, $w^h\in T^h_u(E)$ and $w^v\in T^v_u(E)$ are, respectively, the horizontal and vertical components of w. The *connection mapping* $\varkappa\colon T(E)\to E$ of $T^h(E)$ is defined by $\varkappa\colon w\mapsto\varkappa(w) := \mathrm{pr}_2(w^v) = \mathrm{pr}_2(w - w^h)$ for each $w\in T(E)$.

Theorem 14.2. *Let $T^h(E)$ be a connection on some vector bundle (E,π,M), the mapping $\varkappa\colon T(E)\to E$ be its connection mapping, and $g\colon N\to M$, where N is C^∞ manifold, be of class C^∞. The mapping $\nabla\colon \mathfrak{X}(N)\times \mathrm{Lift}_g(E,\pi,M)\to \mathrm{Lift}_g(E,\pi,M)$ given by $\nabla\colon (U,X)\mapsto \nabla_U X := \varkappa(X_*(U))$ for every $(U,X)\in \mathfrak{X}(N)\times\mathrm{Lift}_g(E,\pi,M)$ is a covariant derivative in (E,π,M) along g. Besides, a lifting $X\in\mathrm{Lift}_g(E,\pi,M)$ is parallel (with respect to the parallelism structure assigned to $T^h(E)$ via 14.1) if and only if $\nabla_U X = 0$ for all $U\in\mathfrak{X}(N)$ and if $h\colon L\to N$ for a C^∞ manifold L, then $\nabla(X\circ h) = (\nabla X)\circ h_*$ where the first (resp. second) ∇ is along $g\circ h$ (resp. g). Conversely, if ∇ is a covariant derivative in (E,π,M) (along id_M), then there exists a connection $T^h(E)$ on (E,π,M) such that $X_*(v)\in T^h(E)$ for given $X\in\mathrm{Sec}(E,\pi,M)$ and $v\in T(M)$ if and only if $\nabla_v X = 0$.*

Proof. See [23, pp. 74–77]. □

Corollary 14.1 (cf. [23, Corollary 2.59]**).** *A covariant derivative ∇ in a vector bundle (E,π,M) determines a parallelism structure $\mathbb{P}$ in it such that a lifting $X\in\mathrm{Lift}_\beta(E,\pi,M)$ of a C^∞ locally injective path $\beta\colon[a,b]\to M$ is parallel (along β with respect to $\mathbb{P}$) iff*

$$\nabla_{\dot\beta} X = 0. \tag{14.13}$$

Remark 14.2. Regardless of the appearance of equation (14.13) in the book [23, Corollary 2.59], its left-hand side is not well defined without special explanations which are not given in [23], where implicitly is assumed β to be injective, as in our text until now. The problem is that the mapping ∇_V, $V\in\mathfrak{X}(M)$, in particular ∇_v with $v\in T(M)$ or $\nabla_{\dot\beta(s)}$, $s\in[a,b]$, *acts on sections* of (E,π,M), which are liftings of $g = \mathsf{id}_M$ to E, *not on liftings of paths or on a lifting of a particular path* such as $X\in\mathrm{Lift}_\beta(E,\pi,M)$. Besides, the symbol $\nabla_{\dot\beta}$ is not defined as $\dot\beta$ is a lifting of β to $T(M)$ and, if β has self-intersections, it cannot be considered as a vector field on $\beta([a,b])$. For these reasons, the symbol $\nabla_{\dot\beta} X$ is incorrect. For injective path β, the symbol $\nabla_{\dot\beta} X$ may be defined as a section of $(E,\pi,M)|_{\beta([a,b])}$ such that $\nabla_{\dot\beta} X\colon \beta(s)\mapsto \nabla_{\dot\beta(s)} X := (\nabla_V \check X)_{\beta(s)}$, $s\in[a,b]$, where $V\in\mathfrak{X}(M)$ satisfies $V_{\beta(s)} := \dot\beta(s)$ and $\check X\in \operatorname{Sec}(E,\pi,M)$ is such that $\check X\colon \beta(s)\mapsto \check X_{\beta(s)} := X(s)$. If β is not injective, i.e., if it has self-intersections, then $\nabla_{\dot\beta} X$, with $X\in \operatorname{Lift}_\beta(E,\pi,M)$, should be regarded as a lifting of β to E, $\nabla_{\dot\beta} X\in \operatorname{Lift}_\beta(E,\pi,M)$, such that

$$(\nabla_{\dot\beta} X)\colon s\mapsto (\nabla_{\dot\beta} X)(s) := \Big(\frac{\mathrm{d}X^i(s)}{\mathrm{d}s} + {\Gamma^i}_{j\mu}(\beta(s))X^j(s)\dot\beta^\mu(s)\Big)e_i(\beta(s)) \qquad (14.14)$$

where the notation is explained in the paragraphs containing equations (14.18)–(14.21) below. In a case of injective path β both definitions agree in a sense that $\nabla_{\dot\beta(s)}\check X = (\nabla_{\dot\beta} X)(s)$, i.e., the value of $\nabla_{\dot\beta} X$, as a section over $\beta([a,b])$, at $\beta(s)$ is equal to the value of $\nabla_{\dot\beta} X$, as a lifting of β, at $s\in[a,b]$.

Proof. The operator ∇, by the second part of Theorem 14.2, determines a connection $T^h(E)$ in (E,π,M) which in turn, by Theorem 14.1, determines a parallelism structure $\mathbb{P}$ in (E,π,M) (via the $T^h(E)$-horizontal liftings of the C^∞ paths in M). Since a path in E is parallel with respect to $\mathbb{P}$ iff it is $T^h(E)$-horizontal, X is parallel iff $\dot X_{X(s)}\in T^h_{X(s)}(E)$ for each $s\in[a,b]$. Denoting by $D := \frac{\mathrm{d}}{\mathrm{d}t}\in\mathfrak{X}(\mathbb{R})$ the tangent vector field to $\mathbb{R}$, we can write [7, Section 1.23 (e)] $\dot X_{X(s)} = X_*|_s(D|_s)$. So, X is parallel iff

$$\dot X_{X(s)} = X_*|_s(D|_s)\in T^h_{X(s)}(E). \qquad (14.15)$$

Suppose β is injective and denote by $\beta^{-1}\colon \beta([a,b])\to[a,b]$ the mapping inverse to β on the image $\beta([a,b])$. Since $\dot\beta(s) = \beta_*|_{\beta(s)}(D|_s)$, we have[5] $D|_s = \big(\beta_*|_{\beta(s)}\big)^{-1}(\dot\beta(s)) = \big((\beta^{-1})_*|_{\beta(s)}\big)(\dot\beta(s))$. Substituting this in $\dot X_{X(s)} = X_*|_s(D|_s)$, we get

$$\dot X_{X(s)} = [(X\circ\beta^{-1})_*|_{\beta(s)}](\dot\beta(s)) \qquad (14.16)$$

for every injective path $\beta\colon[a,b]\to M$ and $s\in[a,b]$. Let $\check X\in\operatorname{Sec}(E,\pi,M)$ be such that $\check X|_{\beta([a,b])} := X\circ\beta^{-1}$.[6] Now the condition (14.15) is tantamount to $\check X_*|_{\beta(s)}(\dot\beta(s))\in T^h_{X(s)}(E)$ which, by the last assertion of Theorem 14.2, is equivalent to

$$\nabla_{\dot\beta(s)}\check X := (\nabla_V\check X)|_{\beta(s)} = 0 \qquad (14.17)$$

where $V\in\mathfrak{X}(M)$ and $V|_{\beta(s)} := \dot\beta(s)$. At the end, if we identify $\check X$ along β with X, as $\check X\colon \beta(s)\mapsto (X\circ\beta^{-1})(\beta(s))\equiv X(s)$, the last equality can be rewritten as (14.13).

Let now $\beta\colon[a,b]\to M$ be locally injective and, for each $s\in[a,b]$, the symbol J_s denotes the maximal subinterval $J_s\subseteq[a,b]$ containing s, $J_s\ni s$, such that

[5] Here and below we apply the formula $(\varphi\circ\psi)_*|_p = \varphi_*|_{\psi(p)}\circ\psi_*|_p$, where $\psi\colon L\to M$, $\varphi\colon M\to N$, L, M, and N are C^1 manifolds and $p\in L$; see [7, Section 1.25 (c)].

[6] The existence of a C^∞ section $\check X$ with the property mentioned can be easily proved by applying [15, Lemma 5 on p. 28] to the components of $\check X$ in some frame over $\beta([a,b])$.

$\beta_s := \beta|_{J_s}$ is injective. Repeating the above procedure, we see that $X_s := X|_{J_s}$ is parallel along β_s iff $\nabla_{\dot\beta_s(t)}\check X_s = 0$, $t \in J_s$, with $\check X_s \in \operatorname{Sec}(E,\pi,M)$ such that $\check X_s|_{\beta_s(J_s)} = \check X_s|_{\beta(J_s)} := X\circ\beta_s^{-1}$. From the derived further local expression (14.20) follows that $\nabla_{\dot\beta_{s_1}(t)}\check X_{s_1} = \nabla_{\dot\beta_{s_2}(t)}\check X_{s_2}$ for $s_1, s_2 \in [a,b]$, $J_{s_1}\cap J_{s_2} \neq \varnothing$, and $t \in J_{s_1}\cap J_{s_2}$ and, moreover, $\nabla_{\dot\beta_s(t)}\check X_s = \hat X(t)$, where the lifting $\hat X \in \operatorname{Lift}_\beta(E,\pi,M)$ is locally given by the right-hand side of (14.14). At the end, if, as above, we identify $\check X_s$ with $X|_{J_s}$, we can set $\nabla_{\dot\beta_s(t)}X_s = \hat X|_{J_s}$ which is a local version of (14.14) and, consequently, (14.13) is a global variant of $\nabla_{\dot\beta_s}\check X_s = 0$, $s \in [a,b]$. □

Theorems 14.1 and 14.2 and Corollary 14.1 imply

Conclusion 14.1. The concepts of parallelism structure, connection, and covariant derivative respectively in, on, and in a vector bundle are equivalent: given any one of them, the other two can appropriately be assigned to it.

By abuse of the language but without risk of ambiguity, the covariant derivatives are called connections (as in Definition I.3.1 on page 21), the reason being the last conclusion.

Before closing this long introductory subsection, we would like to write some local expressions and decompositions which will essentially be used further.

Let ∇^g be a covariant derivative in a vector bundle (E,π,M) along a C^∞ mapping $g\colon N \to M$, where E, M, and N are C^∞ manifolds. Let U be an open subset in N. Suppose $\{E_\mu | \mu = 1,\dots,\dim N\}$ is a frame in $T(N)$ over U and $\{e_i | i = 1,\dots,\pi^{-1}(p),\ p \in M\}$ is a frame, possibly depending on g, along $g|_U$ in E, i.e., for each $q \in N$, $\{E_\mu|q\}$ is a basis in $T_q(N)$ and $\{e_i(q;g)\}$ is a basis in $\pi^{-1}(g(q))$ (along g). Notice, E_μ are vector fields over U in $T(N)$, while $e_i(\,\cdot\,;g)\colon q \mapsto e_i(q;g)$ are liftings of g.

For every $V \in \mathfrak{X}(U)$ and $X_g \in \operatorname{Lift}_g((E,\pi,M)|_U)$, we write the expansions ($p \in M$)

$$V = V^\mu E_\mu \equiv \sum_{\mu=1}^{\dim N} V^\mu E_\mu, \quad X_g = X_g^i e_i(\,\cdot\,;g) \equiv \sum_{i=1}^{\dim \pi^{-1}(p)} X_g^i e_i(\,\cdot\,;g)$$

for some C^∞ functions $V^\mu, X_g^i\colon U \to \mathbb{K}$. Applying the properties (i)–(iv) of Definition 14.7 and using that $\nabla_V^g X_g \in \operatorname{Lift}_g((E,\pi,M)|_U)$ (here and below g is not a summation index!), we get

$$\nabla_V^g(X_g) = V^\mu\bigl[E_\mu(X_g^i) + \Gamma^i{}_{j\mu}(\,\cdot\,;g)X_g^j\bigr]e_i(\,\cdot\,;g) \tag{14.18}$$

where the summation over $\mu(,\nu,\dots)$ is from 1 to $\dim N$ and over $i(,j,\dots)$ is from 1 to $\dim\pi^{-1}(p)$, $p \in M$. Here the functions $\Gamma^i{}_{j\mu}(\,\cdot\,;g)\colon U \to \mathbb{K}$ are defined by (cf. (I.3.1))

$$\nabla_{E_\mu}(e_j(\,\cdot\,;g)) =: \Gamma^i{}_{j\mu}(\,\cdot\,;g)e_i(\,\cdot\,;g) \tag{14.19}$$

and are called *local coefficients* of the covariant derivative ∇^g with respect to the pair of frames $(\{E_\mu\},\{e_i\})$.

If ∇ is a covariant derivative in (E,π,M), $\nabla = \nabla^{\mathsf{id}_M}$, then the argument $g(=\mathsf{id}_M)$ above will be omitted, i.e.,

$$\nabla_V(X) = V^\mu(E_\mu(X^i) + \Gamma^i{}_{j\mu}X^j)e_i \tag{14.20}$$

where $X \in \mathrm{Sec}((E,\pi,M)|_U)$, e_i are considered as vector fields, and the functions $\Gamma^i{}_{j\mu}\colon U \to \mathbb{K}$, $U \subseteq M$, are given via (cf. (I.3.1))

$$\nabla_{E_\mu}(e_j) =: \Gamma^i{}_{j\mu}e_i. \tag{14.21}$$

Proposition 14.4. *Every derivation D^γ in (E,π,M) along a fixed C^1 path $\gamma\colon J \to M$ with coefficients* (*cf.* (2.9) *and* (6.1))

$$\Gamma^i{}_j(\,\cdot\,;\gamma) = \Gamma^i{}_{j\mu}(\,\cdot\,;\gamma)\dot\gamma^\mu, \tag{14.22}$$

where $\Gamma^i{}_{j\mu}(\,\cdot\,;\gamma)\colon J \to \mathbb{K}$, is a covariant derivative in (E,π,M) along γ with respect to $\dot\gamma$.

Proof. Compare (14.18) with (2.9) or check the properties described in Definition 14.7 with γ and D^γ for g and ∇, respectively, invoking the definition of D^γ (see Definition 2.1 on page 219). □

Proposition 14.5. *A linear connection ∇ in the tangent bundle over a manifold M is a covariant derivative in this bundle* (*along the identity mapping* id_M).

Proof. Compare (14.18) with (I.3.2) and (I.3.3) for $r = s = 1$ or restrict Definition I.3.1 on page 21 to the tangent bundle over M and check the properties described in Definition 14.7. □

Remark 14.3. If one starts with the general theory of connections on vector bundles, Proposition 14.5 should be converted into a definition, i.e., in such an approach a linear connection on a manifold is defined as a connection on its tangent bundle and, by abuse of the language, the associated covariant derivative is also called linear connection on the same manifold. For details, see [23, p. 93 ff].

Suppose the frames $\{E_\mu\}$ and $\{e_i\}$ are subjected simultaneously to the following changes

$$E_\mu|_q \mapsto E'_\mu|_q = B^\nu_\mu(q)E_\nu|_q, \quad e_i(q;g) \mapsto e'_i(q;g) = A^j_i(q;g)e_j(q;g) \tag{14.23}$$

where $q \in N$, and $B := [B^\nu_\mu]$ and $A(\,\cdot\,;g) := [A^j_i(\,\cdot\,;g)]$ are non-degenerate matrix-valued functions of class C^∞. (In what follows, it is sufficient $A(\,\cdot\,;g)$ to be of class C^1 or C^2 and B to be arbitrary of class at most C^2.) A straightforward calculation, based on (14.19), reveals that the change (14.23) leads to the transformation

${\Gamma^i}_{j\mu} \mapsto {\Gamma'^i}_{j\mu}$ with

$$\begin{aligned}{\Gamma'^i}_{j\mu}(\,\cdot\,;g) = \sum_{\nu=1}^{\dim N} \sum_{k,l=1}^{\dim \pi^{-1}(p)} B^\nu_\mu \big(A^{-1}(\,\cdot\,;g)\big)^i_k A^l_j(\,\cdot\,;g){\Gamma^k}_{l\nu}(\,\cdot\,;g) \\ + \sum_{\nu=1}^{\dim N} \sum_{k=1}^{\dim \pi^{-1}(p)} B^\nu_\mu \big(A^{-1}(\,\cdot\,;g)\big)^i_k E_\nu(A^k_j(\,\cdot\,;g)). \quad (14.24)\end{aligned}$$

This means that the functions ${\Gamma'^i}_{j\mu}(\,\cdot\,;g)$ are the coefficients of ∇^g in the new pair of frames $(\{E'_\mu\},\{e'_i\})$, $\nabla^g_{E'_\mu}(e'_j(\,\cdot\,;g)) = {\Gamma'^i}_{j\mu}(\,\cdot\,;g)e'_i(\,\cdot\,;g)$. Introducing the *matrices of the coefficients* of ∇^g (the *coefficients' matrices* of ∇^g) $\Gamma_\mu(\,\cdot\,;g) := \big[{\Gamma^i}_{j\mu}(\,\cdot\,;g)\big]_{i,j=1}^{\dim \pi^{-1}(p)}$, $p \in M$, we can rewrite (14.24) in a more compact form:

$$\Gamma'_\mu(\,\cdot\,;g) = B^\nu_\mu[A^{-1}(\,\cdot\,;g)\Gamma_\nu(\,\cdot\,;g)A(\,\cdot\,;g) + A^{-1}(\,\cdot\,;g)E_\nu(A(\,\cdot\,;g))]. \quad (14.25)$$

The reader may have notice that, up to notation and meaning of the symbols, (14.24) and (14.25) are identical with (6.5) and (6.4), respectively. We shall comment on this fact later in subsection 14.3.

If one is dealing with a connection in a bundle (E,π,M) (along id_M), the argument $g(=\mathrm{d}_M)$ should be deleted in equations (14.23)–(14.25) in accordance with (14.20) and (14.21).

Now we shall derive the parallel transport equation in a given vector bundle (E,π,M) (cf. [23, p. 77, Exercise 1]).

Let ∇ be a covariant derivative in (E,π,M) and $\mathbb{P}$ be the determined by it parallelism structure according to Corollary 14.1. By the same corollary, a section $X \in \mathrm{Sec}(E,\pi,M)$ is parallel along $\beta\colon [a,b] \to M$ iff $\nabla_{\dot\beta} X = 0$. Writing this equation in a pair of frames $(\{E_\mu\},\{e_i\})$, $\{E_\mu\}$ in $T(M)$ and $\{e_i\}$ in E, and applying (14.20), we get

$$\dot\beta(X^i) + {\Gamma^i}_{j\mu}X^j\dot\beta^\mu = 0 \quad (14.26)$$

along β, or

$$\frac{\mathrm{d}X^i(\beta(s))}{\mathrm{d}s} + {\Gamma^i}_{j\mu}(\beta(s))X^j(\beta(s))\dot\beta^\mu(s) = 0 \quad (14.26')$$

with $s \in [a,b]$, $\dot\beta(s) = \dot\beta^\mu(s)E_\mu|_{\beta(s)} \in T_{\beta(s)}(M)$, and $X = X^i e_i$.

Relying on the results obtained, we can partially generalize Definition 14.2 as follows.[7]

Definition 14.8. Let ∇^g be a covariant derivative in a vector bundle (E,π,M) along a mapping $g\colon N \to M$. A *lifting* $X \in \mathrm{Lift}_g(E,\pi,M)$ is *called parallel* (*with respect to* ∇^g) if $\nabla^g_V X = 0$ for every $V \in \mathfrak{X}(N)$. A *lifting* X of g to E is called *parallel* (*with respect to* ∇^g) *along a path* $\beta\colon [a,b] \to N$ in N (not in M!) if $\nabla^g_{\dot\beta} X = 0$.

[7]For the rest of the generalization of Definition 14.2, *vide infra* Definition 14.9 on the facing page.

Exercise 14.1. Verify the consistency of Definitions 14.8 and 14.2.

Exercise 14.2. By repeating *mutatis mutandis* the procedure leading to equations (14.26) and (14.26′), prove that a lifting $X_g \in \text{Lift}_g(E, \pi, M)$ is parallel along β iff in any pair of frames $(\{E_\mu\}, \{e_i\})$, with $\{E_\mu\}$ in $T(N)$ (not in $T(M)$) and $\{e_i\}$ in E, is valid

$$\dot\beta(X^i_g) + \Gamma^i{}_{j\mu}(\,\cdot\,; g) X^j_g \dot\beta^\mu = 0 \tag{14.27}$$

along $\beta\colon [a, b] \to N$, or

$$\frac{\mathrm{d}X^i_g(\beta(s))}{\mathrm{d}s} + \Gamma^i{}_{j\mu}(\beta(s); g) X^j_g(\beta(s))\dot\beta^\mu(s) = 0 \tag{14.27′}$$

were $s \in [a, b]$, $\dot\beta(s) = \dot\beta^\mu(s) E_\mu|_{\beta(s)} \in T_{\beta(s)}(N)$, and $X_g = X^i_g e_i(\,\cdot\,; g)$.

14.2. Parallel transports in vector bundles

This subsection is devoted to some problems connected with the axiomatical definition of parallel transports in vector bundles.

Definition 14.9. The *parallel transport* P *in a vector bundle* (E, π, M) *corresponding (assigned) to a parallelism structure* $\mathbb{P}$ in (E, π, M) is a mapping $\mathsf{P}\colon \beta \mapsto \mathsf{P}^\beta$ assigning to every C^∞ path $\beta\colon [a, b] \to M$ in M the parallel transport along β defined by $\mathbb{P}$ via (14.2) and (14.3).

Proposition 14.6. *The parallel transport is reparameterization invariant in a sense that, if* $\beta\colon [a, b] \to M$ *and* $\varphi\colon [c, d] \to [a, b]$ *is* C^∞ *and such that* $\varphi(c) = a$ *and* $\varphi(d) = b$*, then*

$$\mathsf{P}^{\beta\circ\varphi} = \mathsf{P}^\beta \tag{14.28}$$

Proof. See (14.3) and (14.5). □

Define the *product* (or the *product path* $\alpha\beta$) of two paths $\alpha\colon [a, b] \to M$ and $\beta\colon [b, c] \to M$ with $\alpha(b) = \beta(b)$ by[8]

$$\alpha \cdot \beta := \alpha\beta\colon [a, c] \to M, \quad \alpha\beta(s) := \begin{cases} \alpha(s) & \text{for } s \in [a, b] \\ \beta(s) & \text{for } s \in [b, c] \end{cases}. \tag{14.29}$$

The product of paths is associative but not commutative.

Definition 14.10. Suppose $\alpha\colon [a, b] \to M$ and $\beta\colon [b, c] \to M$ are of class C^∞ and $\alpha(b) = \beta(b)$. If the product path $\alpha\beta$ is not C^∞ at the point $s = b$, the *parallel transport along* $\alpha\beta$ is defined to be composition of the parallel transports along the constituent paths:

$$\mathsf{P}^{\alpha\beta} = \mathsf{P}^\beta \circ \mathsf{P}^\alpha. \tag{14.30}$$

[8] In [23, p. 51] the product $\alpha\beta$ is denoted by $\beta * \alpha$; this is a matter of convention. See also the comments in [23, Section 2.16].

More generally, if $\alpha_i\colon [a_i, a_{i+1}] \to M$, $i = 1, \dots, k$ with $k \in \mathbb{N}\backslash\{1\}$, are C^∞ paths, $\alpha_i(a_{i+1}) = \alpha_{i+1}(a_{i+1})$ for $i = 1, \dots, k-1$, and the product path $\alpha_1 \cdots \alpha_k$ is not C^∞ at the points $a_2, \dots, a_{k-1}$, then by definition

$$\mathsf{P}^{\alpha_1 \cdots \alpha_k} = \mathsf{P}^{\alpha_k} \circ \cdots \circ \mathsf{P}^{\alpha_1}. \tag{14.31}$$

Remark 14.4. In (14.30) and (14.31) we use the equality sign (=) instead of the equal by definition one (:=) because they are valid also in the case when, respectively, $\alpha\beta$ and $\alpha_1 \cdots \alpha_k$ are C^∞ paths (not only C^∞ piecewise as assumed above) which will be proved later; but the validity of, e.g., (14.30) for a C^∞ path $\alpha\beta$ is a theorem, not a definition (see Proposition 14.8).

Remark 14.5. Suppose $\alpha\colon [a, b] \to M$ is (finite) C^∞ piecewise path, i.e., there are $k \in \mathbb{N}\backslash\{1\}$ and numbers $a_1, \dots, a_{k+1} \in [a, b]$ such that $a = a_1 < a_2 < \cdots < a_{k+1} = b$, the paths $\alpha_i := \alpha|_{[a_i, a_{i+1}]}$, $i = 1, \dots, k$ are C^∞, and α is not C^∞ at the points $a_2, \dots, a_{k-1}$. Since

$$\beta = (\beta|_{[b_1, b_2]}) \cdots (\beta|_{[b_k, b_{k+1}]}) \tag{14.32}$$

for every $\beta\colon [c, d] \to M$ and $b_1, \dots, b_{k+1} \in [c, d]$ such that $c = b_1 < b_2 < \cdots < b_{k+1} = d$, we have $\alpha = \alpha_1 \cdots \alpha_k$ and, by (14.31),

$$\mathsf{P}^{\alpha} = \mathsf{P}^{\alpha_k} \circ \cdots \circ \mathsf{P}^{\alpha_1}. \tag{14.33}$$

Therefore the parallel transport along a piecewise C^∞ path is equal to the composition (in the corresponding order) of the parallel transports along the C^∞ parts of the initial path.

Proposition 14.7. *If P is a parallel transport assigned to a parallelism structure $\mathbb{P}$ in a vector bundle, $\beta\colon [a, b] \to M$ is C^∞, and $c \in [a, b]$, then*

$$\mathsf{P}^{\beta} = \mathsf{P}^{\beta|[c,b]} \circ \mathsf{P}^{\beta[a,c]} \tag{14.34}$$

Proof. The equality (14.34) is a consequence of (14.12) with $\lambda = b$ and equation (14.3). □

Proposition 14.8. *The equalities* (14.30) *and* (14.31) *are valid for C^∞* (*and, by definition, for piecewise C^∞*) *paths $\alpha\beta$ and $\alpha_1 \cdots \alpha_k$ respectively.*

Proof. Apply (14.34) with $\alpha\beta$ for β and use (14.29). □

Let us recall, in the approach of [23], which was followed in the previous subsection and until now in the present one, the concept of a parallel transport is a secondary one as it was defined on the base of the one of a parallelism structure (or, equivalently, connection or covariant derivative). Now we want to revert the situation by showing that the parallel transport can be defined in an independent axiomatic way as a consequence of which the parallelism structures (and, hence, connections and covariant derivatives) can be introduced on its ground.[9] Before giving the corresponding definition, we would like to mention that there are three groups of axioms when one tries to define a parallel transport axiomatically. First, there are axioms of a "functional type" describing the parallel transport as a mapping on the set of paths, which we assume to be of class C^∞ for the consistency with [23] but, generally, this is a too strong restriction. These axioms are well known and appear in practically identical form in different works. Below they are written as (14.35)–(14.38). The second group describes the "smoothness properties" of the parallel transport. The concrete axioms in it vary significantly in the different works depending on their particular aims. Here we choose them in a way consistent with the properties of the parallelism structures (see (14.39) and (14.40) below). The last group includes axioms describing the compatibility of the parallel transport with other structures, if any, such as metric, (almost) complex structure, etc. This group is not a subject of this investigation, so we admit it to be empty. As an example of such a type of axiom, we point the condition (I.4.2) on page 35 for a linear connection which means that the parallel transport does not change the scalar products of the vectors when they are transported by its means.

Definition 14.11. Let (E, π, M) be a vector bundle. A *parallel transport in* the bundle (E, π, M) is a mapping P assigning to each path $\beta\colon [a, b] \to M$ a mapping

$$\mathsf{P}^\beta\colon \pi^{-1}(\beta(a)) \to \pi^{-1}(\beta(b)) \tag{14.35}$$

such that:

(i) P^β is a vector space isomorphism;

(ii) if $\varphi\colon [c, d] \to [a, b]$ is C^∞ and $\varphi(c) = a$ and $\varphi(d) = b$, then

$$\mathsf{P}^{\beta\circ\varphi} = \mathsf{P}^\beta; \tag{14.36}$$

(iii) if β^- is the path inverse to β,

$$\mathsf{P}^{\beta^-} = \left(\mathsf{P}^\beta\right)^{-1}; \tag{14.37}$$

(iv) if $\alpha\colon [c, a] \to M$, $\alpha(a) = \beta(a)$ and $\alpha\beta$ is the product of α and β,

$$\mathsf{P}^{\alpha\beta} = \mathsf{P}^\beta \circ \mathsf{P}^\alpha; \tag{14.38}$$

(v) under the hypotheses preceding equation (14.6) in condition (iv) of Definition 14.1, the mapping

$$\tilde{f}\colon T(U) \times \pi^{-1}(U) \to E, \quad \tilde{f}\colon (X, u) \mapsto \mathsf{P}^\alpha(u), \qquad \alpha(s) := f(sX), \tag{14.39}$$

where $(X, u) \in T(U) \times \pi^{-1}(U)$ and $s \in [0, 1]$, is of class C^∞;

[9] A reader interested in the earliest essential published account on this problem should read the paper [31].

(vi) if $\alpha, \beta\colon [0,1] \to M$, $\alpha(0) = \beta(0)$, $\dot\alpha(0) = \dot\beta(0)$, $\bar\alpha_u(t) := \mathsf{P}^{\alpha|[0,t]}(u)$, and $\bar\beta_u(t) := \mathsf{P}^{\beta|[0,t]}(u)$ for $u \in \pi^{-1}(\alpha(0))$ and $t \in [0,1]$, then

$$\dot{\bar\alpha}(0) = \dot{\bar\beta}(0). \tag{14.40}$$

The following theorem says that, when talking of a parallel transport, it is insignificant whether we have in mind Definition 14.11 or Definition 14.9.

Theorem 14.3. *The parallel transport* P *corresponding to a parallelism structure* $\mathbb{P}$ *according to Definition* 14.9 *is a parallel transport in a sense of Definition* 14.11*. Conversely, if* P *is a parallel transport according to Definition* 14.11*, then the mapping* $\mathbb{P}\colon (u,\beta) \mapsto \mathbb{P}_u^\beta$*,* $\beta\colon [a,b] \to M$*,* $u \in \pi^{-1}(\beta(a))$*, where the mapping* $\mathbb{P}_u^\beta\colon [a,b] \to E$ *is given by*

$$\mathbb{P}_u^\beta(t) = \mathsf{P}^{\beta|[a,t]}, \tag{14.41}$$

is a parallelism structure such that the assigned to it parallel transport according to Definition 14.9 *coincides with* P*, i.e., given* P*, there is* $\mathbb{P}$ *generating it via* (14.3)*.*

Proof. Suppose $\mathbb{P}$ is a parallelism structure and P is the assigned to it parallel transport. The properties (14.36) and (14.38) ware already established in Propositions 14.6 and 14.8, while (14.35), (14.37), and point (i) of Definition 14.11 are valid due to Definition 14.1, condition (ii). The conditions (v) and (vi) of Definition 14.11 are simple reformulation of, respectively, conditions (iv) and (v) of Definition 14.1.

Conversely, it is almost completely trivial to be verified that the mapping $\mathbb{P}$ defined via (14.41), in which P is a parallel transport according to Definition 14.11, satisfies all of the conditions mentioned in Definition 14.1. For instance, (14.5) follows from

$$\mathbb{P}_u^{\beta\circ\varphi}(s) = \mathsf{P}^{(\beta\circ\varphi)|[a,s]}(u) = \mathsf{P}^{\beta|[\varphi(a),\varphi(s)]} = \mathbb{P}_u^\beta(\varphi(s)), \qquad s \in [a,b],$$

while $\mathbb{P}_u^\beta(a) = \mathsf{P}^{\beta|[a,a]}(u) = u$ is a result of

$$\mathsf{P}^{\beta_p} = \mathsf{id}_{\pi^{-1}(p)}, \qquad \beta_p\colon [a,a] = \{a\} \to \{p\} \in M \tag{14.42}$$

which is a consequence of $\mathsf{P}^{\beta_p} = \mathsf{P}^{\beta_p\beta_p^-} = \mathsf{P}^{\beta_p^-} \circ \mathsf{P}^{\beta_p} = (\mathsf{P}^{\beta_p})^{-1} \circ \mathsf{P}^{\beta_p} = \mathsf{id}_{\pi^{-1}(p)}$ as $\beta_p\beta_p^- = \beta_p = \beta_p^-$. We leave to the reader to check the other properties. At last, by (14.3), the parallel transport $\bar{\mathsf{P}}$ corresponding to $\mathbb{P}$ is given via $\bar{\mathsf{P}}^\beta(u) := \mathbb{P}_u^\beta(b) = \mathsf{P}^{\beta|[a,b]}(u) = \mathsf{P}^\beta(u)$ for every $u \in \pi^{-1}(\beta(a))$ and $\beta\colon [a,b] \to M$, which means $\bar{\mathsf{P}} = \mathsf{P}$. □

The fundamental meaning of the last theorem is that the concept of a parallel transport in a vector bundle is equivalent to the one of a parallelism structure in it and, consequently, to the ones of connection on it or a covariant derivative in it.

In the present, as well as in the previous, subsection all mappings (manifolds, and other structures) were supposed to be of classes C^∞. This is sufficient, but not necessary for proving the results. Most of them remain valid when C^1 or C^2 (and rarely C^3) smoothness is supposed. To simplify the formulation of some results, as well as to keep in touch with [23], further in this section we also suppose C^∞ smoothness. But a careful reader can reformulate the material by counting the necessary class of smoothness really required.

14.3. Parallel transports and linear transports along paths

The main aim of this subsection is to be shown that the parallel transports can be considered as linear transports along paths. Once this is proved, one can freely apply the results, concerning normal frames for linear transports along paths, to arbitrary parallel transports and, as a consequence of the results obtained, to parallelism structures, connections and covariant derivatives in/on vector bundles.

Proposition 14.9. *Let (E, π, M) be a vector bundle and P be a parallel transport in it. If $\gamma\colon J \to M$, with J being $\mathbb{R}$-interval, the mapping $P\colon \gamma \mapsto P^\gamma\colon (s,t) \mapsto P^\gamma_{s\to t}$, $s, t \in J$, such that*

$$P^\gamma_{s\to t} = \begin{cases} \mathsf{P}^{\gamma|[s,t]} & \textit{for } s \le t \\ \left(\mathsf{P}^{\gamma|[t,s]}\right)^{-1} & \textit{for } s \ge t \end{cases}, \tag{14.43}$$

is a linear transport along paths in (E, π, M).

Note 14.1. Compare this result with Proposition 11.1, as well as equation (14.43) with (11.4a).

Proof. We have to check the conditions in Definition 3.1 with P for L: equation (3.1) follows from (14.35), (3.3) is a consequence of (14.42), while (3.4) is true as $\mathsf{P}^{\gamma|[a,b]}$, $a, b \in J$, $a \le b$, is a vector space isomorphism between $\pi^{-1}(\gamma(a))$ and $\pi^{-1}(\gamma(b))$. At last, (3.2) is a corollary of (14.38) as, for every path $\gamma\colon J \to M$, $\gamma|[a,c] = (\gamma|[a,c])|_{[a,b]}(\gamma|[a,c])|_{[b,c]} = \gamma|_{[a,b]}\gamma|_{[b,c]}$ for all $a, b, c \in J$ such that $a \le b \le c$ (see (14.32)). □

Exercise 14.3. Prove that (14.43) is equivalent to

$$P^\gamma_{s\to t} = F^{-1}(t,\gamma) \circ F(s,\gamma) \tag{14.43$'$}$$

where

$$F(r,\gamma) := \begin{cases} \mathsf{P}^{\gamma|[r,a]} & \text{for } r \le a \\ \left(\mathsf{P}^{\gamma|[a,r]}\right)^{-1} & \text{for } r \ge a \end{cases}, \qquad r = s, t$$

for some fixed $a \in [\min(s,t), \max(s,t)]$. (Hint: use the equality (14.32) and invoke equation (14.38).) Then Proposition 14.9 follows from Proposition 3.1.

It is almost evident, (14.43) can be inverted, i.e., P can be expressed via P (cf. (11.4b)):

$$\mathsf{P}^\beta = P^\beta_{a\to b} \qquad \text{for } \beta\colon [a,b] \to M,\ a \le b. \tag{14.44}$$

Therefore the mapping P provides an equivalent description of the parallel transport P, the only difference being that P is defined along arbitrary paths $\gamma\colon J \to M$. We shall call P *parallel transport along paths* if we want to emphasize its difference from P but, if it is not important and there is not a risk of ambiguities, P will simply be called parallel transport as P.

At this point a natural problem arises: when a linear transport along paths is a parallel transport (along paths)? Since this problem is aside from our main topic and its solution will not be used further in the present work, we shall consider it quite briefly.[10] For the purpose one should, using (14.43) or (14.44), express all of the properties of a parallel transport P mentioned in Definition 14.11 through the parallel transport P along paths corresponding to P.

Lemma 14.1. *If P is a linear transport along paths in a vector bundle (E, π, M) and it satisfies the conditions*

$$P^{\gamma\circ\varphi}_{s\to t} = P^{\gamma}_{\varphi(s)\to\varphi(t)}, \qquad s,t \in J'', \tag{14.45}$$

where $\gamma\colon J \to M$ and $\varphi\colon J'' \to J$ is orientation preserving diffeomorphism, and

$$P^{\gamma|J'}_{s\to t} = P^{\gamma}_{s\to t}, \qquad s,t \in J', \tag{14.46}$$

where $J' \subseteq J$ is a subinterval, then the mapping $\mathsf{P}\colon \beta \mapsto \mathsf{P}^{\beta}$, $\beta\colon [a,b] \to M$, given via (14.44) *satisfies the conditions* (14.35) *and* (i)–(iv) *of Definition* 14.11*. Conversely, given a mapping P satisfying the last mentions conditions, the mapping $P\colon \gamma \mapsto P^{\gamma}\colon (s,t) \mapsto P^{\gamma}_{s\to t}$, $\gamma\colon J \to M$, $s,t \in J$, defined via* (14.43) *is a linear transport along paths possessing the properties* (14.45) *and* (14.46)*.*

Proof. Necessity: (14.35) follows from (3.1), (i) is a consequence of equations (3.4) and (3.5), (14.36) is a corollary of (14.45), (14.37) is a result of (14.45) with $\varphi(t) = a + b - t$, $t \in [a,b]$ and (3.5), and, at last, in the proof of (14.38) the conditions (14.46), (14.45) and (3.4) are involved: $\mathsf{P}^{\alpha\beta} = P^{\alpha\beta}_{c\to b} = P^{\alpha\beta}_{a\to b} \circ P^{\alpha\beta}_{c\to a} = P^{(\alpha\beta)|[a,b]}_{a\to b} \circ P^{(\alpha\beta)|[c,a]}_{c\to a} = P^{\beta}_{a\to b} \circ P^{\alpha}_{c\to a} = \mathsf{P}^{\beta} \circ \mathsf{P}^{\alpha}$.

Sufficiency: The fact that P is a linear transport along paths was established in Proposition 14.9 in which proof the conditions (v) and (vi) of Definition 14.11 were not used. Taking for definiteness $s \le t$, the proof of (14.45) and (14.46) is respectively:

$$P^{\gamma\circ\varphi}_{s\to t} = \mathsf{P}^{(\gamma\circ\varphi)|[s,t]} = \mathsf{P}^{\gamma|[\varphi(s),\varphi(t)]} = P^{\gamma}_{\varphi(s)\to\varphi(t)}$$
$$P^{\gamma|J'}_{s\to t}, = \mathsf{P}^{(\gamma|J')|[s,t]} = \mathsf{P}^{\gamma|[s,t]} = P^{\gamma}_{s\to t}.$$

The case $s \ge t$ can be proved similarly. □

[10] For some details and a more general discussion of the situation, see [115] and Section V.8 below.

Theorem 14.4. *If* P *is a parallel transport, then the mapping* $P\colon \gamma \mapsto P^{\gamma}\colon (s,t) \mapsto P^{\gamma}_{s\to t}$, $\gamma\colon J \to M$, $s,t \in J$, *defined by* (14.43) *is a linear transport along paths satisfying* (14.45), (14.46) *and the conditions:*

(a) *under the hypotheses preceding* (14.6) *in condition* (iv) *of Definition* 14.1, *the mapping*

$$\tilde{f}\colon T(U)\times\pi^{-1}(U) \to E, \quad \tilde{f}\colon (X,u) \mapsto P^{\alpha}_{0\to 1}(u), \qquad \alpha(s) := f(sX), \tag{14.47}$$

where $(X,u) \in T(U)\times\pi^{-1}(U)$ *and* $s \in [0,1]$, *is of class* C^{∞};

(b) *if* $\alpha,\beta\colon [0,1] \to M$, $\alpha(0) = \beta(0)$, $\dot\alpha(0) = \dot\beta(0)$, $\bar\alpha_u(t) := P^{\alpha}_{0\to t}(u)$, *and* $\bar\beta_u(t) := P^{\beta}_{0\to t}(u)$ *for* $u \in \pi^{-1}(\alpha(0))$ *and* $t \in [0,1]$, *then*

$$\dot{\bar\alpha}(0) = \dot{\bar\beta}(0). \tag{14.48}$$

Besides, the corresponding to P via (14.44) *mapping is a parallel transport coinciding with* P. *Conversely, if P is a linear transport along paths satisfying the conditions* (14.45)–(14.48), *the mapping* $\mathsf{P}\colon \beta \mapsto \mathsf{P}^{\beta}$ *given via* (14.43) *is a parallel transport along paths; moreover, the corresponding to* P *via* (14.43) *linear transport along paths coincides with P.*

Proof. See Lemma 14.1 and reformulate conditions (v) and (vi) of Definition 14.11 in terms of the mapping P. ☐

Thus we can make two important conclusions:

Conclusion 14.2. The parallel transports along paths are the only linear transports along paths having the special properties (14.45)–(14.48).

Therefore, we can give the following definition of a parallel transport along paths which, by Theorem 14.4 is equivalent to the one presented above.

Definition 14.12. A *parallel transport along paths in a vector bundle* (E,π,M) is a linear transport along paths in it satisfying the conditions (14.45)–(14.48).

Conclusion 14.3. The parallel transports along paths provide an equivalent description of the parallel transports and, consequently, of the parallelism structures, connections, and covariant derivatives. Therefore any result concerning linear transports along paths, in general, or parallel transports along pats, in particular, can *mutatis mutandis* be formulated equivalently in terms of parallel transports, parallelism structures, connections, or covariant derivatives.

Let ∇^{g} be a covariant derivative in a vector bundle (E,π,M) along $g\colon N \to M$ and $\Gamma_{\mu}(\,\cdot\,;g) = \left[\Gamma^{i}{}_{j\mu}(\,\cdot\,;g)\right]$ be the matrix of its coefficients in a pair of frames $(\{E_{\mu}\},\{e_i\})$, over a set $U\times g(U) \subseteq N\times M$, $\{E_{\mu}\}$ being a frame in $T(N)$ over $U\subseteq N$ and $\{e_i\}$ being a frame over $g(U)$ in E.

Comparing the transformation equations (14.25) on page 310 and (6.4) on page 248, or equivalently (14.24) and (6.5), one can notice from the first sight a striking similarity between them: the only formal difference being that in equation (14.25) the indices $\mu, \nu, \dots$ run from 1 to $\dim N$ and the matrix-valued function $A(\,\cdot\,; g)\colon U \to \mathrm{GL}(\dim(\pi^{-1}(p)), \mathbb{K})$, $U \subseteq N$, $p \in M$, stands for $A\colon W \to \mathrm{GL}(\dim(\pi^{-1}(p)), \mathbb{K})$, $W \subseteq M$. This analogy is a hint to be considered the quantities (cf. (6.1))

$$\begin{aligned}\mathbf{\Gamma}(s;\gamma,g) &:= \left[\Gamma^{i}{}_{j\mu}(\gamma(s);g)\dot{\gamma}^{\mu}(s)\right]_{i,j=1}^{\dim \pi^{-1}(p)} = \mathbf{\Gamma}(\gamma(s);g)\\ &:= \sum_{\mu=1}^{\dim N} \Gamma_{\mu}(\gamma(s);g)\dot{\gamma}^{\mu}(s) \equiv \Gamma_{\mu}(\gamma(s);g)\dot{\gamma}^{\mu}(s),\end{aligned} \tag{14.49}$$

where $\gamma\colon J \to N$ is a C^1 path and $s \in J$, which under a change $(\{E_\mu\},\{e_i\}) \mapsto (\{E'_\mu = B^\nu_\mu E_\nu\},\{e'_i = A^j_i e_j\})$ transform into (see (14.25) and cf. (3.26))

$$\mathbf{\Gamma}'(s;\gamma,g) = A^{-1}(\gamma(s);g)\mathbf{\Gamma}(\gamma(s);g)A(\gamma(s);g) + A^{-1}(\gamma(s);g)\frac{\partial A(\gamma(s);g)}{\partial s}. \tag{14.50}$$

Prima facie one may think, relying on the Proposition 3.6 on page 232, that there is a (unique) linear transport along paths in some bundle over N whose 2-index coefficient matrix is $\mathbf{\Gamma}(s;\gamma,g)$. To clarify the situation, it must be emphasized on the fact that the path γ is in N (not in M) while the 'transport' should act between some of the fibres of (E,π,M), viz. the ones over $g(N)$. Hence the bundle in which the hypothetical transport should act must have N as a base while its bundle space must be a subspace of E, viz. $\pi^{-1}(g(N))$. Besides, the fibre over $q \in N$ must be $\pi^{-1}(g(q))$. This is admissible iff g is injective, $g(q_1) \neq g(q_2)$ for each $q_1, q_2 \in N$ such that $q_1 \neq q_2$, in which case g is bijective on the image $g(N)$ and *vice versa*, i.e., if $g\colon N \to M$ is C^∞ and $g|_N\colon N \to g(N)$ is bijection.

The results obtained in the above analysis can be summarized in the following assertion.

Proposition 14.10. *Let ∇^g be a covariant derivative in a vector bundle (E,π,M) along an injective mapping $g\colon N \to M$ and $g^{-1}\colon g(N) \to N$ be the inverse of g on the image $g(N) \subseteq M$. If $\Gamma_\mu(\,\cdot\,,g)$ are the coefficients' matrices of ∇^g in a pair of frames $(\{E_\mu\},\{e_i\})$, then there exists a unique linear transport ${}^gP$ along paths in the vector bundle $(\pi^{-1}(g(N)),\pi_N,N)$ with $\pi_N := g^{-1}\circ(\pi|_{\pi^{-1}(g(N))})$ such that in $\{e_i\}$ the matrix of its $(2$-index$)$ coefficients is given by (14.49) for every C^1 path $\gamma\colon J \to N$ or, equivalently, such that in $(\{E_\mu\},\{e_i\})$ its 3-index coefficients coincide with the coefficients of ∇^g. Conversely, to every linear transport along paths in $(\pi^{-1}(g(N)),\pi_N,N)$ with given 3-index coefficients $\Gamma^i{}_{j\mu}(\,\cdot\,;g)$, there corresponds a unique covariant derivative ∇^g in (E,π,M) with coefficients $\Gamma^i{}_{j\mu}(\,\cdot\,;g)$.*

Proof. See the above discussion, Proposition 3.6 on page 232, and the definition of the 3-index coefficients of a linear transport along paths in Section 6. ☐

Remark 14.6. Notice, in the case $N = M$ and $g = \mathsf{id}_M$, i.e., when a covariant derivative in (E, π, M) is considered, the bundle $(\pi^{-1}(g(N)), \pi_N, N)$, in which the transport ${}^gP$ determined by Proposition 14.9 acts, coincides with the initial bundle (E, π, M).

Proposition 14.11 (cf. Proposition 11.4). *Let ${}^gD$ be the derivation along paths corresponding via* (3.19) *to the linear transport ${}^gP$ along paths described in Proposition* 14.10. *If X_g is a lifting of g to E and $\hat{X}_g$ is a lifting of the paths in N to $\pi^{-1}(N)$ in the bundle $(\pi^{-1}(g(N)), \pi_N, N)$ such that $\hat{X}_g \colon \gamma \mapsto X_g \circ \gamma$ for every C^1 path $\gamma\colon J \to N$, then, for each $s \in J$,*

$$\big({}^gD^\gamma(\hat{X}_g)\big)(s) := {}^gD^\gamma_s(\hat{X}_g) = \big(\nabla^g_{\dot\gamma}(X_g)\big)(\gamma(s)). \tag{14.51}$$

Proof. Invoke (3.23), (14.18), and (14.49). □

Remark 14.7. In a case of a covariant derivative ∇ in (E, π, M), i.e., for $N = M$ and $g = \mathsf{id}_M$, equation (14.51) reads $D^\gamma_s(\hat{X}) = (\nabla_{\dot\gamma}(X))|_{\gamma(s)}$ for every $X \in \operatorname{Sec}(E, \pi, M)$ and $\hat{X} \in \operatorname{PLift}(E, \pi, M)$ such that $\hat{X}\colon \gamma \mapsto \hat{X}_\gamma\colon s \mapsto X_{\gamma(s)}$. In the tangent bundle case, $(E, \pi, M) = (T(M), \pi, M)$, the last equality is identical with (11.6).

The above result, combined with Proposition 11.4 on page 283 is a tip for the existence of some connection between ${}^gP$ and the parallel transport determined by ∇^g.

Definition 14.13 (see Definitions 14.2 and 14.8)**.** Suppose the symbols (E, π, M), N, g, ∇^g, and ${}^gP$ have the same meaning as in Proposition 14.10. A *path* $\bar\gamma\colon J \to \pi^{-1}(g(N)) \subseteq E$ is called *parallel (along $\gamma := \pi_N \circ \bar\gamma\colon J \to N$) with respect to* ∇^g if

$$\bar\gamma(t) = {}^gP^\gamma_{s\to t}\bar\gamma(s) \tag{14.52}$$

for some (and hence for any) $s \in J$ and all $t \in J$; respectively, in this case $\bar\gamma$ is called a parallel (along γ with respect to ∇^g) lifting of γ. A *lifting* λ of the paths in N to paths in $\pi^{-1}(g(N))$ is called *parallel with respect to* ∇^g if for any $\gamma\colon J \to N$, the path λ_γ, $\lambda\colon \gamma \mapsto \lambda_\gamma$, is parallel along γ with respect to ∇^g. A *section* $\sigma \in \operatorname{Sec}(\pi^{-1}(g(N)), \pi_N, N)$ is *parallel along* $\gamma\colon J \to N$ (resp. on $U \subseteq N$) with respect to ∇^g, if the lifting $\hat\sigma\colon \gamma \mapsto \hat\sigma_\gamma = \sigma \circ \gamma$ is parallel along the given path γ (resp. along every path $\gamma\colon J \to U$) with respect to ∇^g.

Proposition 14.12. *Let ∇^g and ${}^gP$ be as in Proposition* 14.10. *A lifting $X_g \in \operatorname{Lift}_g(E, \pi, M)$ is parallel* (*resp. along $\gamma\colon J \to N$*) *with respect to ∇^g if and only if the lifting of paths $\hat{X}_g \in \operatorname{PLift}(\pi^{-1}(g(N)), \pi_N, N)$ with $\hat{X}_g\colon \gamma \mapsto X_g \circ \gamma$, $\gamma\colon J \to N$, is parallel on N* (*resp. along the given path γ*) *with respect to ∇^g.*

Proof. See Definitions 14.8 and 14.13 and use (3.21), on page 231, and (14.51). □

So, allowing some freedom of the language, we can say that equation (14.52) describes, besides the parallel paths in $\pi^{-1}(g(N))$, also the parallel sections of $(\pi^{-1}(g(N)), \pi_N, N)$, and the parallel liftings of g to $E \supseteq \pi^{-1}(g(N))$.

Proposition 14.13. *Let* ∇ *be a covariant derivative in a vector bundle* (E, π, M) *and* $\mathbb{P}$ *be the determined by it parallelism stricture according to Corollary* 14.1 *on page* 306. *The linear transport* ${}^{\mathsf{id}_M}P$ *along paths in* (E, π, M) *determined by* ∇ *according to Proposition* 14.10 (*with* $N = M$ *and* $g = \mathsf{id}_M$) *coincides with the parallel transport* P *along paths corresponding to* $\mathbb{P}$ *via* (14.3) *and* (14.43), *i.e.,* $P = {}^{\mathsf{id}_M}P$.

Proof. Let $\gamma\colon J \to M$ be a C^1 path, $s_0 \in J$, and $u_0 \in \pi^{-1}(\gamma(s_0))$. We have to show that the liftings $\bar\gamma$ and $\bar\gamma\,'\colon J \to E$ of γ, given via

$$\bar\gamma(t) := P^{\gamma}_{s_0\to t}(u_0), \quad \bar\gamma\,'(t) := {}^{\mathsf{id}_M}P^{\gamma}_{s_0\to t}(u_0), \qquad t \in J$$

are identical, $\bar\gamma = \bar\gamma\,'$. It is trivial to see, the path $\bar\gamma$ is parallel with respect to $\mathbb{P}$ (see Definition 14.2). Therefore, by Corollary 14.1, $\nabla_{\dot\gamma}\bar\gamma = 0$ or, in component form (see (14.14) or (14.26′))

$$\frac{\mathrm{d}\bar\gamma^i(t)}{\mathrm{d}t} + \Gamma^i{}_{j\mu}(\gamma(t))\dot\gamma^\mu(t) = 0 \tag{14.53}$$

where $\Gamma^i{}_{j\mu}$ are simultaneously 3-index coefficients of P and coefficients of ∇ (see Proposition 14.10). According to (3.20) and (3.21), the path $\bar\gamma\,'$ satisfies the equation $D^\gamma_t\bar\gamma\,' = 0$, D being the derivation along paths generated by ${}^{\mathsf{id}_M}P$ (see Definition 3.2 on page 230), which, due to (3.23) and (6.1), coincides with (14.53) with $\bar\gamma\,'$ for $\bar\gamma$. Hence (the components of) $\bar\gamma$ and $\bar\gamma\,'$ are solutions of identical first order ordinary differential equations passing through one and the same point $u_0 = \bar\gamma(s_0) = \bar\gamma\,'(s_0)$. Consequently, by the uniqueness of the solutions of such equations, $\bar\gamma = \bar\gamma\,'$, which, due to the arbitrariness of γ, s_0, and u_0, implies $P = {}^{\mathsf{id}_M}P$. □

14.4. Normal frames for parallel transports, connections and covariant derivatives

The conclusions made in the previous subsection enable all of the general results and definitions related to linear transports along paths in vector bundles to be transferred, with appropriate changes, to the theory of connections, parallelism structures, parallel transports, and covariant derivatives in/on these bundles. In particular, such a procedure can be realized with respect to the theory of normal frames (and, possibly, coordinates) for linear transports along paths. Below we turn our attention to its brief description. To save some space and writing, we shall perform it with respect to the parallel transports (along paths) and covariant derivatives, which are most suitable for our purposes, but, applying Theorems 14.1, 14.2, 14.3, and 14.4, one can reformulate the material in terms of parallelism structures or connections.

Definition 14.14 (see Definitions 4.1–4.4**).** Let P be a parallel transport along paths in a vector bundle (E, π, M). A *frame* normal for P along $\gamma\colon J \to M$

(resp. on $U \subseteq M$) is called *normal along* γ (*resp. on* U) *for the parallel transport* P*, parallelism structure* $\mathbb{P}$*, or/and connection* $T^h(E)$ determined by it. If such *frames* exist, P, P, $\mathbb{P}$, and/ or $T^h(E)$ are called *Euclidean along* γ (*resp. on* U). A *frame* strong normal for P (on an open set containing or equal to U) is called *strong normal for* P, $\mathbb{P}$, and/or $T^h(E)$.

Definition 14.15 (cf. Definition I.5.1)**.** Let (E, π, M) be a vector bundle, $g\colon N \to M$, ∇^g be a covariant derivative in (E, π, M) along g, and $U \subseteq N$, where E, M, and N are manifolds. A *frame* $\{e_i\}$ in E, defined over an open set containing or equal to $g(U)$, is called *normal for* ∇^g *on* U, if for some (and hence for any – see (14.25) or Conclusion 14.5 below) frame $\{E_\mu\}$ in $T(N)$ over U, the coefficients of ∇^g in the pair of frames $(\{E_\mu\}, \{e_i\})$ vanish everywhere on U. Respectively, a *frame* $\{e_i\}$ is *normal for* ∇^g *along* $\varphi\colon Q \to N$, $Q \neq \varnothing$, if it is normal on $\varphi(Q) \subseteq N$.

Obviously, it is valid

Conclusion 14.4. Definition 14.14 transfers all problems concerning frames normal for parallelism structures, parallel transports, and connections to similar problems for a special type of linear transports along paths, viz. to the parallel transports along paths, to which are applicable all of the results from Sections 3–9.

As a result of this conclusion, we are not going to consider the mentioned in it type of problems here. Of course, the additional conditions (14.45)–(14.48), singling out the parallel transports along paths from the other linear transports along paths, lead to some specific properties of the corresponding normal frames. But their concrete description is out of the main subject of the present book. However, below some of these special properties will be described (implicitly) when frames normal for covariant derivatives will be explored.

Let us now turn our attention to the frames normal for covariant derivatives. We distinguish these frames from the ones described by Definition 14.14 because the class of these frames is, generally, larger than the one of the letter. As we shall see below, a frame normal for a covariant derivative ∇ in a vector bundle is strongly normal for the corresponding to ∇ (or determining ∇) parallel transport P along paths. The relations between the frames normal for ∇ and P is similar to the ones described in Section 12 in the case of linear connections on a manifold.

Suppose (E, π, M) is a vector bundle, $g\colon N \to M$ is of class C^∞, ∇^g is a covariant derivative in (E, π, M) along g, $U \subseteq N$, $\{E_\mu | \mu = 1, \dots, \dim N\}$ is a frame over U in $T(N)$, and $\{e_i | i = 1, \dots, \dim \pi^{-1}(p),\ p \in M\}$ is a frame over an open set V containing or equal to $g(U) \subseteq M$ in E. Let $B = [B^\nu_\mu]^{\dim N}_{\mu,\nu=1}$ and $A = [A^j_i]^{\dim \pi^{-1}(p)}_{i,j=1}$, be non-degenerate matrix-valued functions on V and $g(V)$, respectively, and A be of class C^1. The frame $\{e'_i = A^j_i e_j\}$ over $V \supseteq g(U)$ in E is normal for ∇^g on U iff there can be found A and B such that in the pair of frames $(\{E_\mu\}, \{e_i\})$ the coefficients ${\Gamma'}^i{}_{j\mu}$ of ∇^g vanish on U. According to the transformation law (14.25), the frame $\{e'_i\}$ is normal for ∇^g on U iff the normal frame equation (cf. (I.5.4) on page 39 or (12.6) on page 291)

$$(\Gamma_\mu(\,\cdot\,;g)A + E_\mu(A))|_U = 0 \tag{14.54}$$

holds. From here two quite fundamental conclusions can be made:

Conclusion 14.5. The choice of the frame $\{E_\mu\}$ over U in $T(N)$ involved in the definition of a frame normal for a covariant derivative along a mapping is completely insignificant (see (14.54) and (14.25)): a frame normal for one such choice is normal for any other choice. From general positions, this choice may be important only when one tries to solve equation (14.54) with respect to A as it may lead to a simplification as well as to a complication of the concrete form of the equation. So, it is a matter of convenience to select a particular frame $\{E_\mu\}$ when exploring frames normal for covariant derivatives along mappings.

Conclusion 14.6. The more important result is that (14.54) is identical with equation (12.6) (with $\mu = 1, \dots, \dim N$ while in Section 12 is supposed $\mu = 1, \dots, \dim M$) describing the frames strongly normal on U for linear transports along paths with fixed 3-index coefficients. Therefore the discussion after (12.6) in Section 12 can be repeated word by word if we replace in it the base manifold M with the manifold N (which is the domain of the mapping $g\colon N \to M$ along which is the covariant derivative ∇^g) and the phrase "frame(s) (strong) normal on … for …" with "frame(s) normal for ∇^g". In short, this results in the conclusion that all of the results of Chapter III (or II) concerning the solution of the matrix equation (14.54) are *mutatis mutandis* valid with respect to frames normal for covariant derivatives along mappings.

In particular, we can assert the existence of frames normal at any fixed point $q \in N$, i.e., for $U = \{q\}$, or along any C^1 locally injective path $\gamma\colon J \to N$, i.e., for $U = \gamma(J)$; also the frames normal on U, if any, are connected via linear transformations such that their matrices A satisfy the equation $E_\mu(A)|_U = 0$ for some (and hence any) frame $\{E_\mu\}$ in $T(N)$ over an open set containing or equal to U, etc.

Thus the general, and possibly unexpected, inference is encoded in

Conclusion 14.7. The methods developed for the investigation of frames normal for linear connections on manifolds and the ones for linear transports along paths in vector bundles are completely applicable for the exploration of frames normal for covariant derivatives along mappings and frames strongly normal for linear transports along paths; moreover, all kinds of problems concerning the last two types of frames can be solved by the methods mentioned.

This conclusion is in full agreement with the ones at the end of the last paragraph of Section 8 and with Remark 8.1 on page 259.

Now it is time to formulate and solve the problem: what are the links between frames normal for a covariant derivative and the determined by (or determining it) parallel transport?

Proposition 14.14. *Let* ∇^g *and* ${}^gP$ *be as in Proposition* 14.9 *on page* 315*. The frames normal on* $U \subseteq N$ *for* ∇^g *are strong normal on* U *for* ${}^gP$ *and* vice versa*.*

Proof. See Definitions 14.15 and 12.2 and invoke Proposition 14.9. □

14.5. On the role of the curvature

This subsection shows that the curvature of a connection (or parallel transport or covariant derivative) plays with respect to the normal frames the same role as the curvature of a linear connection (transport) with respect to this kind of frames.

The *curvature tensor field* or simply *curvature* of a connection $T^h(E)$ on a vector bundle (E, π, M) is a mapping [23, Section 2.43]

$$R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times E \to E$$

such that, given $U, V \in \mathfrak{X}(M)$ and $\xi \in E$,

$$R\colon (U, V, \xi) \mapsto R(U, V)\xi := -\mathrm{pr}_2\big(([\bar{U}, \bar{V}]_{\text{-}})^v_\xi\big). \tag{14.55}$$

Here: $\mathrm{pr}_2\colon T^v(E) \to E$ was defined on page 306, $\bar{U}$ and $\bar{V}$ are the horizontal liftings to E of U and V respectively (see Definition 14.6), $([\bar{U}, \bar{V}]_{\text{-}})_\xi$ is the commutator of $\bar{U}$ and $\bar{V}$ at ξ, and the superscript v denotes the vertical component of a vector in $T(E)$ with respect to $T^h(E)$. If $X \in \mathrm{Sec}(E, \pi, M)$, we set $R(U, V)X \in \mathrm{Sec}(E, \pi, M)$ with $(R(U, V)X)(p) := R(U, V)(X(p))$, $p \in M$.

For us will be important that R is tensorial in all its three arguments, it is skewsymmetric in the first two arguments, and, if ∇ is a covariant derivative in (E, π, M) which determines $T^h(E)$, then[11]

$$R(U, V)X = \big(R^\nabla(U, V)\big)(X) \equiv R^\nabla(U, V)X \tag{14.56}$$

where $X \in \mathrm{Sec}(E, \pi, M)$ and $R^\nabla\colon \mathrm{Sec}(E, \pi, M) \to \mathrm{Sec}(E, \pi, M)$ is a mapping given by

$$R^\nabla(U, V) := [\nabla_U, \nabla_V]_{\text{-}} - \nabla_{[U,V]_{\text{-}}} = \nabla_U \circ \nabla_V - \nabla_V \circ \nabla_U - \nabla_{[U,V]_{\text{-}}} \tag{14.57}$$

and it is called *curvature* (*operator*) *of the covariant derivative* ∇. Notice, in the tangent bundle case, $(E, \pi, M) = (T(M), \pi, M)$, the last formula reduces to the Definition (I.3.11) on page 25, as one can expect.

Let $\{E_\alpha | \alpha = 1, \dots, \dim M\}$ be a frame in $T(M)$ over a set $W \subseteq M$ and $\{e_i | i = 1, \dots, \dim \pi^{-1}(p),\ p \in M\}$ be a frame in E over W. The *components* $R^i{}_{j\alpha\beta}$ and $(R^\nabla)^i{}_{j\alpha\beta}$ of the curvatures R and R^∇, respectively, are defined by

$$R(E_\alpha, E_\beta)e_j =: R^i{}_{j\alpha\beta} e_i, \qquad R^\nabla(E_\alpha, E_\beta)e_j =: (R^\nabla)^i{}_{j\alpha\beta} e_i.$$

Obviously, due to (14.56), if ∇ induces $T^h(E)$, then $R^i{}_{j\alpha\beta} = (R^\nabla)^i{}_{j\alpha\beta}$

[11] For the proof of these assertions, as well as for other properties of the curvature, see [23, p. 67 ff and p. 82 ff].

Exercise 14.4. Let R^{∇} be the curvature of a covariant derivative in a vector bundle (E, π, M). Applying (14.56), (14.20), and (14.21), prove that

$$(R^{\nabla})^{i}{}_{j\alpha\beta} := -\Gamma^{i}{}_{j\alpha,\beta} + \Gamma^{i}{}_{j\beta,\alpha} - \Gamma^{k}{}_{j\alpha}\Gamma^{i}{}_{k\beta} + \Gamma^{k}{}_{j\beta}\Gamma^{i}{}_{k\alpha} - \Gamma^{i}{}_{j\mu}C^{\mu}_{\alpha\beta} \colon W \to \mathbb{K} \quad (14.58)$$

where $[E_\alpha, E_\beta]_{\_} =: C^{\mu}_{\alpha\beta} E_\mu$ and $f_{,\mu} := E_\mu(f)$ for a C^1 function f on U.

Proposition 14.15. *Let ∇ be a covariant derivative in a vector bundle (E, π, M) and P be the corresponding to it unique linear transport along paths according to Propositions* 14.10 (*which coincides with the parallel transport assigned to the parallelism structure $\mathbb{P}$ described in Corollary* 14.1). *The components $(R^{\nabla})^{i}{}_{j\alpha\beta}$ of the curvature R^{∇} of ∇ coincide with the 4-index components $R^{i}{}_{j\alpha\beta}$ of the curvature of P* (*defined by* (9.13) *in Proposition* 9.1 *on page* 265)*:*

$$(R^{\nabla})^{i}{}_{j\alpha\beta} = R^{i}{}_{j\alpha\beta}. \quad (14.59)$$

Moreover, if ${}^{s}\mathsf{R}$ is the curvature of the section-derivation corresponding to the derivation along paths generated by P (*see Subsection* 9.1), *then*

$$\big[(R^{\nabla}(U,V))(X)\big]\big|_{p} = \big({}^{s}\mathsf{R}^{\eta}(s_0,t_0)\big)(X), \quad (14.60)$$

where: $U, V \in \mathfrak{X}(M)$, $X \in \operatorname{Sec}(E,\pi,M)$, $p \in M$, $\eta\colon J \times J' \to M$ is of class C^2, $\eta(s_0,t_0) = p$ for some $(s_0,t_0) \in J \times J'$, $\eta'(s_0,t_0) = U_p$ and $\eta''(s_0,t_0) = V_p$ with $\eta'(\,\cdot\,,t_0)$ and $\eta''(s_0,\,\cdot\,)$ being the tangent vectors to $\eta(\,\cdot\,,t_0)$ and $\eta(s_0,\,\cdot\,)$, respectively.

Proof. Equation (14.59) follows from (14.58), (9.13), and the coincidence of the coefficients of ∇ and the 3-index coefficients of P (by Proposition 14.10). To verify the relation (14.60), one should simply repeat the derivation of equality (9.17) on page 267 with the only difference that now ∇ is a covariant derivative in (E,π,M) and (E,π,M) should stand for the tangent bundle $(T(M),\pi,M)$. □

Call a connection, covariant derivative, or parallel transport (along paths) *flat* (on $W \subseteq M$) if their curvatures vanish (on W). If a connection, covariant derivative, and parallel transport (along paths) are connected as above supposed, then the results obtained demonstrate that the flatness of one of these three quantities implies the flatness of the remaining two of them.

Of course, all general results of Section 9, in particular the ones connecting curvature and normal frames, are completely applicable to the parallel transports along paths in vector bundles as they are a special type of linear transports along paths (Theorem 14.4 on page 317). But Proposition 14.15, combined with the results of Subsection 14.4, permits something more:

Conclusion 14.8. The material of Section 11, excluding the one concerning torsion and normal coordinates, can *mutatis mutandis*, practically *in extenso*, be generalized to the case of parallel transports (along paths) in general vector bundles. To this end one should replace the tangent bundle $(T(M), \pi, M)$ with a vector bundle (E, π, M), the linear connection ∇ on M with a connection ∇ in (E, π, M), and to make some other, not so important changes.

This transferring of results is so trivial that the explicit formulation of the corresponding assertions makes sense if they are really required for some purpose.

Exercise 14.5. Reformulate some important inferences of Section 11, such as the analogues of Propositions 11.1–11.4 and 11.9–11.11, for parallel transports (along paths) in general vector bundles.

A partial generalization to arbitrary vector bundles admits the material of Section 13, but this is almost out of the subject of this book.

15. Autoparallel paths

The geodesics in a C^2 manifold M endowed with a C^0 linear connection ∇ are C^1 paths whose tangent vector undergoes parallel transport along them by means of the parallel transport P assigned to ∇ (see Definition I.3.5 on page 30). If P is the parallel transport along paths corresponding to P via (11.4a), then a C^1 path $\gamma\colon J \to M$ is geodesic iff

$$\dot{\gamma}(s) = P^{\gamma}_{s_0 \to s}\dot{\gamma}(s_0) \tag{15.1}$$

for some (and hence any) $s_0 \in J$ and all $s \in J$, by virtue of (I.3.20). Obviously, this equality can be taken as a base for a new, but equivalent, definition of a geodesic path.

Denote by $\mathcal{P} \in \mathrm{PLift}(T(M), \pi, M)$ the lifting of paths generated by P via equation (3.20) with $u_0 = \dot{\gamma}(s_0)$ and P for L, viz., for some $s_0 \in J$,

$$\mathcal{P}\colon \gamma \to \mathcal{P}_\gamma\colon s \to \mathcal{P}_\gamma(s) := P^{\gamma}_{s_0 \to s}\dot{\gamma}(s_0). \tag{15.2}$$

Evidently, the equality (15.1) is equivalent to

$$\mathcal{P}_\gamma = \dot{\gamma}, \tag{15.3}$$

where $\dot{\gamma}\colon s \mapsto \dot{\gamma}(s)$ is considered as a lifting of γ to $T(M)$.

If $\tau \in \mathrm{PLift}(T(M), \pi, M)$ denotes the lifting of paths

$$\tau\colon \gamma \mapsto \tau_\gamma := \dot{\gamma}, \tag{15.4}$$

which may be called *tangent lifting of paths*, the equality (15.3) is tantamount to

$${}^{\nabla}D^{\gamma}\tau = 0 \tag{15.5}$$

as a result of the equality (3.21) on page 231. Here ${}^{\nabla}D$ is the derivation along paths in $(T(M), \pi, M)$ generated by P via (3.19). Its components $\Gamma^{i}{}_{j}$ coincide with the coefficients of P (Proposition 3.7 on page 233) and along $\gamma\colon J \to M$ in a frame $\{E_i\}$ in $T(M)$ over $\gamma(J)$ are

$$\Gamma^i{}_j(s;\gamma) = \Gamma^i{}_{jk}(\gamma(s))\dot\gamma^k(s), \qquad s \in J \tag{15.6}$$

where $\Gamma^i{}_{jk}$ are the coefficients of ∇ in $\{E_i\}$ (see, e.g., Propositions 11.3 and 11.4).

The above conclusions can be summarized in the following assertion.

Proposition 15.1. *A C^1 path in a C^2 manifold endowed with C^0 linear connection is geodesic if and only if it satisfies any one* (*and hence all*) *of the equivalent equations* (15.1), (15.3) *and* (15.5).

Remark 15.1. Applying equation (3.23) on page 231 one can prove that, due to (15.4) and (15.6), the equation (15.5) is an equivalent invariant form of the equation (I.3.22) of geodesic paths (see also equations (15.11) and (15.12) below). Thus (15.1), (15.3) and (15.5) are equivalent versions of the *geodesic equation*. Also the equation (15.5) is a rigorous version of (I.3.21) on page 30 for arbitrary, with or without self-intersections, paths (see Remark I.3.8 on page 31): if γ is not injective, one should *define* the symbol $\nabla_{\dot\gamma}\dot\gamma$ as ${}^{\nabla}D^{\gamma}\tau$, $\nabla_{\dot\gamma}\dot\gamma := {}^{\nabla}D^{\gamma}\tau$, otherwise, for an injective path γ, the lifting $\dot\gamma$ of γ is equivalent to a vector field $\gamma(s) \mapsto \dot\gamma(s)$ on $\gamma(J)$ and $\nabla_{\dot\gamma}\dot\gamma$ has its usual meaning if we identify this field with $\dot\gamma$.

One should observe, in (15.1), (15.3), and (15.5) are involved only the parallel transport P along paths and the generated by it derivation along paths ${}^{\nabla}D$, not directly the linear connection ∇. This observation hints to the existence of a generalization of the concept of geodesic paths to more general ones when P is replace with arbitrary linear transport along paths in the tangent bundle $(T(M),\pi,M)$ over M. The following definition is a rigorous realization of such an idea.

Definition 15.1. Let M be a C^2 manifold and L be a C^0 linear transport along paths in its tangent bundle $(T(M),\pi,M)$. Call a C^1 *path* $\gamma\colon J \to M$ *autoparallel with respect to* L if

$$\dot\gamma(s) = L^{\gamma}_{s_o \to s}(\dot\gamma(s_0)) \tag{15.7}$$

for some (and hence any) $s_0 \in J$ and all $s \in J$.

Remark 15.2. For other meaning of the term 'autoparallel path', see Remark I.3.7 on page 30.

Proposition 15.2. *The autoparallel paths with respect to a parallel transport along paths generated by a linear connection coincide with the geodesics of this connection.*

Proof. Compare (15.1) with (15.7) and invoke Proposition 15.1. □

Similarly to the above considerations, we can rewrite (15.7) as

$$\mathcal{L}_\gamma = \dot\gamma \tag{15.8}$$

or

$$D^{\gamma}\tau = 0 \tag{15.9}$$

where

$$\mathcal{L}\colon \gamma \mapsto \mathcal{L}_\gamma\colon s \mapsto \mathcal{L}_\gamma(s) := L^\gamma_{s_0\to s}\dot\gamma(s_0), \tag{15.10}$$

τ is the tangent lifting of paths given via (15.4), and D is the derivation along paths generated by L via (3.19). The derivation of these equations is in fact the proof of the following proposition (cf. Proposition 15.1).

Proposition 15.3. *A C^1 path γ in C^2 manifold is autoparallel with respect to a C^0 linear transport along paths in the tangent bundle over that manifold if and only if for it any one (and hence both) of the equivalent conditions* (15.8) *and* (15.9) *is (are) valid.*

From practical view-point, *the equation of autoparallels* (15.9) is quite suitable for the study of the (local) properties of the autoparallels. In particular this concerns its representation in a frame $\{E_i\}$ along γ in $T(M)$. As a consequence of (3.23) on page 231 with $\lambda = \tau$ and (15.4), in $\{E_i\}$, the equation (15.9) is equivalent to

$$\frac{\mathrm{d}\dot\gamma^i(s)}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)\dot\gamma^j(s) = 0 \tag{15.11}$$

where $\Gamma^i{}_j$ are the (local) coefficients of the transport L in $\{E_i\}$. In the special case when $\{E_i\}$ is the frame induced by some local coordinates $\{x^i\}$, $E_i = \frac{\partial}{\partial x^i}$, the last equation transforms into

$$\frac{\mathrm{d}^2\gamma^i(s)}{\mathrm{d}s^2} + \Gamma^i{}_j(s;\gamma)\frac{\mathrm{d}\gamma^j(s)}{\mathrm{d}s} = 0 \tag{15.12}$$

where $s \in J$ must be such that $\gamma(s)$ is in the domain of $\{x^i\}$ and $\gamma^i := x^i \circ \gamma$.

In the case of a parallel transport, $L = P$ when (15.6) holds, equations (15.11) and (15.12) reduce, respectively, to the equations (I.3.22) and (I.3.23) of the geodesics. Therefore they, as well as (15.9), should be called *equation(s) of the autoparallel paths.* It is clear, the equation of the autoparallels is a system of ordinary differential equations with respect to the local coordinates of the paths (in some local coordinates). The concrete order and type of this system depends on the transport L through its coefficients $\Gamma^i{}_j$. The investigation of the autoparallels' equation is out of the subject of our work. Below we present only one (almost trivial) result that generalizes the well know theorem that the geodesics are described via linear equations in normal coordinates (see the paragraph containing equation (II.2.15) on page 83 or, e.g., [11, Chapter III, Proposition 8.3]).

Proposition 15.4. *Let M be C^2 manifold and L be a C^0 linear transport along paths in its tangent bundle $(T(M),\pi,M)$. Suppose L admits an autoparallel path $\gamma\colon J \to M$ such that $L^\gamma_{s\to t} = \mathsf{id}_{\pi^{-1}(\gamma(s))}$ for all $s,t \in J$ with $\gamma(s) = \gamma(t)$, i.e., the mapping $\gamma(s) \mapsto \mathcal{L}_\gamma(s) = \dot\gamma(s)$ (see* (15.10) *or* (15.9)*) is a single-valued vector field over $\gamma(J)$. Then, in any frame $\{E_i\}$ normal along γ for L, the components $\dot\gamma^i(s)$ of $\dot\gamma(s) \in T_{\gamma(s)}(M)$ are constant, i.e., we have*

$$\dot\gamma(s) = a^i E_i \tag{15.13}$$

for some numbers $a^i \in \mathbb{K}$. Besides, if L is torsionless, γ is injective and (V, x) is a chart normal along γ for L, then

$$x^i(\gamma(s)) = a^i s + b^i, \quad \dot{\gamma}(s) = a^i \frac{\partial}{\partial x^i} \tag{15.14}$$

where $\{x^i\}$ is the associated to (V, x) normal coordinate system, $s \in J$ is such that $\gamma(s) \in V$, and $a^i, b^i \in \mathbb{K}$ are constant numbers.

Remark 15.3. The condition $L^{\gamma}_{s \to t} = \mathsf{id}_{\pi^{-1}(\gamma(s))}$ for all $s, t \in J$ with $\gamma(s) = \gamma(t)$, by Proposition 5.2 on page 240, is a criterion for the existence of frames normal along γ while, by Proposition 10.7, the torsionless of L and the injectiveness of γ ensure the existence of coordinates normal along γ.

Proof. Recall, by definition, the coefficients $\Gamma^i{}_j$ of L vanish in a normal frame or in normal coordinates. So, evidently, (15.13) (resp. (15.14)) is the general solution of the equations of autoparallels (15.11) (resp. (15.12)) in a normal frame (resp. normal coordinate system $\{x^i\}$). □

Consequently, in the sense of Proposition 15.4, the autoparallel paths in a manifold with a transport in its tangent bundle are the analogues of the straight lines (resp. geodesics) in a Euclidean space $\mathbb{E}^n$ (resp. manifold with a linear connection).

16. On a fibre bundle view at quantum mechanics

The purpose of this section is to be given an idea of the fibre bundle formulation of nonrelativistic quantum mechanics which essentially employs the theory of linear transports along paths in vector bundles; the reader can find in [127–131] a detailed exposition of this approach to quantum mechanics. In particular, it will be demonstrated below that the bundle Schrödinger and Heisenberg pictures of motion are (locally) identical in a suitable normal frame

The reader should be aware that some of the considerations below will not be quite rigorous from mathematical view-point. This is a result of the fact that Hilbert bundles of generally infinite fibre dimension will be used. Correspondingly, we will be dealing with infinite matrices applying the rules known from the theory of finite matrices, which is a common practice in quantum mechanics [68, 133–135] but such an approach requires additional investigation and even may turn to be wrong sometimes [126] (see also [136] and [135, Chapter 7, Section 18]).

Before going on, we shall make a technical remark. The (fibre) indices below can take discrete and/or continuous values depending on the spectrum of the corresponding operators (the Hamiltonian in the particular case). For that reason, sums like $\lambda^i e_i(x)$ must be understood as $\sum_{i \in \Lambda_d} \lambda^i e_i(x) + \int_{i \in \Lambda_c} \lambda^i e_i(x) \, \mathrm{d}i$, where Λ_d (resp. Λ_c) is the set of discrete (resp. continuous) values the index i can take. For more details on this item, see [128, p. 4921].

In the bundle approach to quantum mechanics, to a quantum system is assigned a unique C^1 Hilbert bundle (F,π,M), M being a C^1 manifold, whose (standard) fibre is identified with the usual Hilbert space of states $\mathcal{F}$ of the quantum system. The system state is described via a lifting of paths $\Psi \in \operatorname{PLift}(F,\pi,M)$ such that, if $\gamma\colon J \to M$ is a C^1 path,

$$\Psi\colon \gamma \mapsto \Psi_\gamma\colon t \in J \mapsto \Psi_\gamma(t) = l^{-1}_{\gamma(t)}(\psi(t)) \in \pi^{-1}(\gamma(t)), \tag{16.1}$$

where $l^{-1}_x\colon \mathcal{F} \to \pi^{-1}(x)$, $x \in M$, are fixed isomorphisms and $\psi(t) \in \mathcal{F}$ is the conventional state vector of the system at a moment $t \in J$.

The time evolution of the system (or of its state lifting Ψ) is governed by the equation

$$\Psi_\eta(t) = U_\gamma(t,s)(\Psi_\gamma(s)) \qquad s,t \in J, \tag{16.2}$$

where the *evolution transport*

$$U\colon \gamma \mapsto U_\gamma\colon (s,t) \mapsto U_\gamma(t,s)\colon \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t))$$

is a (Hermitian and isometric) linear transport along paths in (F,π,M) such that

$$U_\gamma(t,s) = l^{-1}_{\gamma(t)} \circ \mathcal{U}(t,s) \circ l_{\gamma(s)}\colon \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \tag{16.3}$$

with $\mathcal{U}(t,s)$ being the system's evolution operator in $\mathcal{F}$. It is a remarkable fact that the matrix $\boldsymbol{\Gamma}(s;\gamma)$ of the coefficients of U is

$$\boldsymbol{\Gamma}(s;\gamma) = -\frac{1}{\mathrm{i}\hbar}\boldsymbol{H}^{\mathrm{m}}_\gamma(t) \tag{16.4}$$

where $\mathrm{i} = +\sqrt{-1}$ is the imaginary unit, $\hbar$ is the Planck constant (deviled by 2π) and the matrix-bundle Hamiltonian $\boldsymbol{H}^{\mathrm{m}}_\gamma(t)$ is

$$\boldsymbol{H}^{\mathrm{m}}_\gamma(t) = \boldsymbol{l}^{-1}_{\gamma(t)}(t)\boldsymbol{\mathcal{H}}(t)\boldsymbol{l}_{\gamma(t)}(t) - \mathrm{i}\hbar\boldsymbol{l}^{-1}_{\gamma(t)}(t)\Big(\frac{\mathrm{d}\boldsymbol{l}_{\gamma(t)}(t)}{\mathrm{d}t} + \boldsymbol{E}(t)\boldsymbol{l}_{\gamma(t)}(t)\Big). \tag{16.5}$$

Here $\boldsymbol{\mathcal{H}}(t)$ is the matrix of the system's Hamiltonian in a basis $\{f_i(t)\}$ of $\mathcal{F}$, $\boldsymbol{l}_{\gamma(t)}(t)$ is the matrix of $l_{\gamma(t)}$ in this basis and a frame $\{e_i\}$ over M in F, and the matrix $\boldsymbol{E}(t) := [E^j_i(t)]$ is defined via the expansion $\frac{\mathrm{d}f_i(t)}{\mathrm{d}t} = E^j_i(t) f_j(t)$.

If D is the derivation along paths generated by U (see Definition 3.2 on page 230), then the state lifting Ψ is a solution of the *bundle Schrödinger equation*

$$D(\Psi) = 0 \tag{16.6}$$

satisfying some initial condition, like $\Psi_\gamma(t_0) = \Psi^0_\gamma$ for fixed $t_0 \in J$ and $\Psi^0_\gamma \in \pi^{-1}(\gamma(t_0))$.

Let $\mathcal{A}(t)\colon \mathcal{F} \to \mathcal{F}$ be the linear Hermitian (self-adjoint) operator corresponding to a dynamical variable $\mathbb{A}$ in the standard quantum mechanics. In the bundle approach, to $\mathbb{A}$ corresponds a lifting A of paths in the bundle of restricted M-morphisms of (F, π, M) such that

$$A_\gamma(t) = l^{-1}_{\gamma(t)} \circ \mathcal{A}(t) \circ l_{\gamma(t)} \colon \pi^{-1}(\gamma(t)) \to \pi^{-1}(\gamma(t)). \tag{16.7}$$

The description of a quantum system via liftings like Ψ and A is known as the bundle Schrödinger picture of motion. The shift to the bundle Heisenberg picture of motion is via the changes

$$\Psi \mapsto \Psi^{\mathrm{H}} \colon (t, t_0; \gamma) \mapsto \Psi^{\mathrm{H}}_{\gamma,t}(t_0) \qquad A \mapsto A^{\mathrm{H}} \colon (t, t_0; \gamma) \mapsto A^{\mathrm{H}}_{\gamma,t}(t_0), \tag{16.8}$$

where $t, t_0 \in J$ and

$$\Psi^{\mathrm{H}}_{\gamma,t}(t_0) = \Psi_\gamma(t_0) = l^{-1}_{\gamma(t_0)}(\psi(t_0)) = U_\gamma(t_0, t)\Psi_\gamma(t) \tag{16.9a}$$

$$A^{\mathrm{H}}_{\gamma,t}(t_0) = U_\gamma(t, t_0)^{-1} \circ A_\gamma(t) \circ U_\gamma(t, t_0). \tag{16.9b}$$

(The mapping $A \mapsto A^{\mathrm{H}}$ defines a linear transport in the bundle of point restricted morphisms – see [127, p. 4903] and [129, p. 4941].) Thus, in the bundle Heisenberg picture, the time dependence is entirely transferred to the observables liftings of paths. The Schrödinger and Heisenberg pictures of motion are completely equivalents from physical point of view as the mean values of the dynamical variables in them are identical.

Let $\{e_i\}$ be a frame in F (locally) normal along γ for the evolution transport U.[1] By Definition 4.1 this means that the matrix $\boldsymbol{U}$ of U in $\{e_i\}$ is the (corresponding infinite) identity matrix, $\boldsymbol{U}(t, t_0) = \mathbb{1}$. Consequently equations (16.9) take the following form in the normal frame $\{e_i\}$:

$$\boldsymbol{\Psi}^{\mathrm{H}}_{\gamma,t}(t_0) = \boldsymbol{\Psi}_\gamma(t_0) = \boldsymbol{\Psi}_\gamma(t) \tag{16.10a}$$

$$\boldsymbol{A}^{\mathrm{H}}_{\gamma,t}(t_0) = \boldsymbol{A}_\gamma(t). \tag{16.10b}$$

In this way we have proved the following result.

Proposition 16.1. *The bundle Schrödinger and Heisenberg pictures of motion are identical in a frame* (*locally*) *normal along the reference path* γ.

17. Conclusion

The following three main topics have found their place in this chapter. First, the grounds of the theory of linear transports along paths in vector bundles are set and its relations with the one of parallel transports and connections in these bundles are revealed. Second, the theory of normal frames for such transports was developed. And third, it was demonstrated that the obtained formalism naturally

[1] Here we suppose that the condition (5.2) with U for L in Proposition 5.1 is fulfilled.

agrees with the already existing theory of frames (and coordinates) normal for (linear) connections and other derivations of the tensor algebra over a differentiable manifold.

It should be noticed, from the view-point of linear transports, a lot of the earlier-obtained results concerning linear connections (or other derivations) look considerably simpler and more natural. Here are two typical examples. (i) A linear transport admits frames normal on some set iff it is flat on it; a linear connection admits frames normal on a neighborhood iff it is flat on it. (ii) If a linear transport admits frames normal on some set, such a frame can be obtained via transportation by means of the transport of a basis over a fixed point in this set to the other points in it along paths lying entirely in that set; previously this result was known only on neighborhoods for linear connections and the parallel transports generated by them: a frame normal on a neighborhood for a flat linear connection can be obtained by parallelly transporting a basis over a fixed point in this neighborhood to the other points in it along paths lying in the neighborhoods.

Separately should be mentioned the case of linear transports along paths in the tangent bundle over a manifold. In it, the flat parallel transports generated by linear connections are the only ones admitting normal frames. This conclusion is similar to the one derived in Chapter III concerning frames normal for derivations along vector fields and linear connections. Besides, the torsionless flat parallel transports are the only ones admitting normal coordinates.

At the end, we would like to express the hope that the material in the present chapter could serve as a groundwork for a systematic exploration of frames (and, possibly, coordinates) normal for different kinds of derivations or transports along paths in infinitely dimensional vector bundles, in particular for linear connections in infinitely dimensional manifolds. Partially or *mutatis mutandis* some of our results remain valid in the infinite-dimensional case, but, rigorously speaking, this case is not studied until now and it is open for further research.

Chapter V

Normal Frames for Connections on Differentiable Fibre Bundles

The general connection theory on differentiable fibre bundles, with emphasis on the vector ones, is partially considered. The theory of frames normal for general connections on these bundles is developed. Links with the theory of frames normal for linear connections in vector bundles are revealed. Existence of bundle coordinates normal at a given point and/or along injective horizontal path is proved and a necessary and sufficient condition of existence of bundle coordinates normal along injective horizontal mappings is proved. The concept of a transport along paths in differentiable bundles is introduced. Different links between connections, parallel transports (along paths) and transports along paths are investigated.

1. Introduction

All connections considered until now, on manifolds and on vector bundles, were linear. It is well known that there exist non-linear connections on vector bundles as well as on non-vector ones. Can normal frames (and/or coordinates) be introduced for such more general connections? The positive solution of that problem is the main goal of the present chapter of this book. For the purpose and for a comparison with the definitions and results already obtained is required some preliminary material on general connection theory on differentiable bundles, which is collected in Sections 2–5. On its base, the normal frames for connections on such bundles are studied in Sections 6 and 7.

Sections 2–5 follow the work [137],[1] Sections 6 and 7 are a slightly revised version of [139], and Section 8 reproduces in a modified form the paper [118]

The work is organized as follows.

In Section 2 is collected some introductory material, like the notion of Lie derivatives and distributions on manifolds, needed for our exposition. Here some of our notation is fixed too.

Section 3 is devoted to the general connection theory on bundles whose base and bundles spaces are differentiable manifolds. From different view-points, this theory can be found in many works, like [6, 7, 10–13, 16, 28, 60, 98, 106, 107, 117, 138, 140–146]. In Subsection 3.1 are reviewed some coordinates and frames/bases on the bundle space which are compatible with the fibre structure of a bundle. Subsection 3.2 deals with the general connection theory. A connection on a bundle is defined as a distribution on its bundle space which is complimentary to the vertical distribution on it. The notions of parallel transport generated by connection and of specialized frame are introduced. The fibre coefficients and fibre components of the curvature of a connection are defined via part of the components of the anholonomicity object of a specialized frame. Frames adapted to local bundle coordinates are introduced and the local (2-index) coefficients in them of a connection are defined; their transformation law is derived and it is proved that a geometrical object with such transformation law uniquely defines a connection.

In Section 4, the general connection theory from Section 3 is specified on vector bundles. The most important structures in/on them are the ones that are consistent/compatible with the vector space structure of their fibres. The vertical lifts of sections of a vector bundle and the horizontal lifts of vector fields on its base are investigated in more details in Subsection 4.1. Subsection 4.2 is devoted to linear connections on vector bundles, i.e., connections such that the assigned to them parallel transport is a linear mapping. It is proved that the 2-index coefficients of a linear connection are linear in the fibre coordinates, which leads to the introduction of the (3-index) coefficients of the connection; the latter coefficients

[1] The presentation of the material in Sections 2–4 is according to some of the main ideas of [138, Chapters 1 and 2], but their realization here is quite different and follows the modern trends in differential geometry.

being defined on the base space. The transformations of different objects under changes of vector bundle coordinates are explored. The covariant derivatives are introduced and investigated in Subsection 4.3. They are defined via the Lie derivatives [138] and a mapping realizing an isomorphism between the vertical vector fields on the bundle space and the sections of the bundle. The equivalence of that definition with the widespread one, defining them as mappings on the module of sections of the bundle with suitable properties, is proved. In Subsection 4.4, the affine connections on vector bundles are considered briefly.

In Section 5, some of the results of the previous sections are generalized when frames more general than the ones generated by local coordinates on the bundle space are employed. The most general such frames, compatible with the fibre structure, and the frames adapted to them are investigated. The main differential-geometric objects, introduced in the previous sections, are considered in such general frames. Particular attention is paid on the case of a vector bundle. In vector bundles, a bijective correspondence between the mentioned general frames and pairs of bases, in the vector fields over the base and in the sections of the bundle, is proved. The (3-index) coefficients of a connection in such pairs of frames and their transformation laws are considered. The covariant derivatives are also mentioned on that context.

The theory of normal frames for connections on bundles is considered in Section 6. Subsection 6.1 deals with the general case. Loosely said, an adapted frame is called normal if the 2-index coefficients of a connection vanish in it (on some set). It happens that a frame is normal if and only if it coincides with the frame it is adapted to. The set of these frames is completely described in the most general case. The problems of existence, uniqueness, etc. of normal frames adapted to holonomic frames, i.e., adapted to local coordinates, are discussed in Subsection 6.2. If such frames exist, their general form is described. The existence of frames normal at a given point and/or along an injective horizontal path is proved. The flatness of a connection on an open set is pointed as a necessary condition of existence of (locally) holonomic frames normal on that set. Some links between the general theory of normal frames and the one of normal frames in vector bundles, presented in Chapter IV, are given in Subsection 6.3. It is proved that a frame is normal on a vector bundle with linear connection if and only if in it vanish the 3-index coefficients of the connection. The equivalence of the both theories on vector bundles is established.

In Section 7 is formulated and proved a necessary and sufficient condition for existence of coordinates normal along injective mappings with non-vanishing horizontal component, in particular along injective horizontal mappings.

Section 8 is devoted to some aspects of the axiomatical approach to parallel transport theory [17,23,30–33,91,147–150] and its relations to connection theory; it is based on the paper [118]. It starts with a definition of a transport along paths in a bundle and a result stating that, under some assumptions, it defines a connection. The most important properties of the parallel transports generated by connections are used to be (axiomatically) defined the concept of a parallel transport (irrespectively to some connection on a bundle). In a series of results are constructed bijective mappings between the sets of transports along paths satisfying some additional conditions, connections, and parallel transports. In this way, two different, but equivalent, systems of axioms defining the concept "parallel transport" will be established.

The chapter ends with some remarks and conclusions in Section 9.

2. Preliminaries

This section contains an introductory material, notation etc. that will be needed for our exposition. The reader is referred for details to Chapter I, Section III.2, and standard books on differential geometry, like [7, 23, 151].

A differentiable finite-dimensional manifold over a field $\mathbb{K}$ will be denoted typically by M. Here $\mathbb{K}$ stands for the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The manifolds we consider are supposed to be smooth of class C^2.[1] The set of vector fields, realized as first-order differential operators, and of differential k-forms, $k \in \mathbb{N}$, over M will be denoted by $\mathcal{X}(M)$ and $\Lambda^k(M)$, respectively. The space tangent (resp. cotangent) to M at $p \in M$ is $T_p(M)$ (resp. $T_p^*(M)$) and $(T(M), \pi_T, M)$ (resp. $(T^*(M), \pi_{T^*}, M)$) will stand for the tangent (resp. cotangent) bundle over M. The value of $X \in \mathcal{X}(M)$ at $p \in M$ is $X_p \in T_p(M)$ and the action of X on a C^1 function $\varphi\colon M \to \mathbb{K}$ is a function $X(\varphi)\colon M \to \mathbb{K}$ such that $X(\varphi)|_p := X_p(\varphi) \in \mathbb{K}$.

If M and $\bar{M}$ are manifolds and $f\colon \bar{M} \to M$ is a C^1 mapping, then $f_* := \mathrm{d}f\colon T(\bar{M}) \to T(M)$ denotes the induced tangent mapping (or differential) of f such that, for $p \in \bar{M}$, $f_*|_p := \mathrm{d}f|_p := T_p(f)\colon T_p(\bar{M}) \to T_{f(p)}(M)$ and, for a C^1 function g on M, $(f_*(X))(g) := X(g \circ f)\colon p \mapsto f_*|_p(g) = X_p(g \circ f)$, with $\circ$ being the composition of mappings sign. Respectively, the induced cotangent mapping is $f^*\colon T^*(M) \to T^*(\bar{M})$. If $h\colon N \to \bar{M}$, N being a manifold, we have the chain rule $\mathrm{d}(f \circ h) = \mathrm{d}f \circ \mathrm{d}h$, which is an abbreviation for $\mathrm{d}(f \circ h)_q = (\mathrm{d}f)_{f(q)} \circ (\mathrm{d}h)_q$ for $q \in N$.

By $J \subseteq \mathbb{R}$ will be denoted an arbitrary real interval that can be open or closed at one or both its ends. The notation $\gamma\colon J \to M$ represents an arbitrary path in M. For a C^1 path $\gamma\colon J \to M$, the vector tangent to γ at $s \in J$ will be denoted by $\dot{\gamma}(s) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s}(\gamma(t)) = \gamma_*\big(\frac{\mathrm{d}}{\mathrm{d}r}\big|_s\big) \in T_{\gamma(s)}(M)$, where r in $\frac{\mathrm{d}}{\mathrm{d}r}\big|_s$ is the standard coordinate function on $\mathbb{R}$, i.e., $r\colon \mathbb{R} \to \mathbb{R}$ with $r(s) := s$ for all $s \in \mathbb{R}$ and hence $r = \mathsf{id}_{\mathbb{R}}$ is the identity mapping of $\mathbb{R}$. If $s_0 \in J$ is an end point of J and J

[1] Some of our definitions or/and results are valid also for C^1 or even C^0 manifolds, but we do not want to overload the material with continuous counting of the required degree of differentiability of the manifolds involved. Some parts of the text admit generalizations on more general spaces, like the topological ones, but this is out of the subject of the present work.

is closed at s_0, the derivative in the definition of $\dot{\gamma}(s_0)$ is regarded as a one-sided derivative at s_0.

The Lie derivative relative to $X \in \mathcal{X}(M)$ will be denoted by $\mathcal{L}_X$. It is defined on arbitrary geometrical objects on M [89], but below we shall be interested in its action on tensor fields [11, Chapter I, § 2] (see also [152]). If f, Y, and θ are C^1 respectively function, vector field and 1-form on M, then

$$\mathcal{L}_X(f) = X(f) \tag{2.1a}$$

$$\mathcal{L}_X(Y) = [X, Y]_{\_} \tag{2.1b}$$

$$(\mathcal{L}_X(\theta))(Y) = X(\theta(Y)) - \theta([X, Y]_{\_}) = (\mathrm{d}\theta)(X, Y) + Y(\theta(X)), \tag{2.1c}$$

where $[A, B]_{\_} = A \circ B - B \circ A$ is the commutator of operators A and B (with common domain) and d denotes the exterior derivative operator.

Since $\mathcal{L}_X$ is a derivation of the tensor algebra over the vector fields on M, for a tensor field $S \colon \Lambda^1(M) \times \cdots \times \Lambda^1(M) \times \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathfrak{F}(M)$, we have

$$\begin{aligned}(\mathcal{L}_X S)(\theta, \ldots; Y, \ldots) \\ = X(S(\theta, \ldots; Y, \ldots)) - S(\mathcal{L}_X\theta, \ldots; Y, \ldots) - \cdots - S(\theta, \ldots; \mathcal{L}_X Y, \ldots)) - \cdots ,\end{aligned} \tag{2.2}$$

which defines $\mathcal{L}_X S$ explicitly, due to (2.1).

Let the Greek indices $\lambda, \mu, \nu, \ldots$ run over the range $1, \ldots, \dim M$ and $\{E_\mu\}$ be a C^1 frame in $T(M)$, i.e., $E_\mu \in \mathcal{X}(M)$ be of class C^1 and, for each $p \in M$, the set $\{E_\mu|_p\}$ be a basis of the vector space $T_p(M)$.[2] Let $\{E^\mu\}$ be the coframe dual to $\{E_\mu\}$, i.e., $E^\mu \in \Lambda^1(M)$, $\{E^\mu|_p\}$ be a basis in $T^*_p(M)$, and $E^\mu(E_\nu) = \delta^\mu_\nu$ with δ^ν_μ being the Kronecker deltas ($\delta^\nu_\mu = 1$ for $\mu = \nu$ and $\delta^\nu_\mu = 0$ for $\mu \neq \nu$). Assuming the Einstein's summation convention (summation on indices repeated on different levels over the whole range of their values), we define the *components* $(\Gamma_X)^\mu{}_\nu$ of $\mathcal{L}_X$ in (relative to) $\{E_\mu\}$ via the expansion

$$\mathcal{L}_X E_\mu =: (\Gamma_X)^\nu{}_\mu E_\nu \tag{2.3}$$

which is equivalent to

$$\mathcal{L}_X E^\mu = -(\Gamma_X)^\mu{}_\nu E^\nu \tag{2.3$'$}$$

by virtue of $E^\mu(E_\nu) = \delta^\mu_\nu$ and the commutativity of the Lie derivatives and contraction operators.[3] From (2.3) and (2.1b), we get

$$(\Gamma_X)^\nu{}_\mu = -E_\mu(X^\nu) - C^\nu_{\mu\lambda} X^\lambda, \tag{2.4}$$

[2]There are manifolds, like the even-dimensional spheres $\mathbb{S}^{2k}$, $k \in \mathbb{N}$, which do not admit global, continuous (and moreover C^k for $k \geq 1$), and nowhere vanishing vector fields [153]. If this is the case, the considerations must be localized over an open subset of M on which such fields exist. We shall not overload our exposition with such details.

[3]The sign before $(\Gamma_X)^\mu{}_\nu$ in (2.3) or (2.3$'$) is conventional and we have chosen it in a way similar to the accepted convention for the components of a covariant derivative (or, equivalently, the coefficients of a linear connection – see Section 4).

in $\{E_\mu\}$, where $X = X^\mu E_\mu$ and the functions $C^\nu_{\mu\lambda}$, known as the *components of the anholonomicity object of* $\{E_\mu\}$, are defined by

$$[E_\mu, E_\nu]_- =: C^\lambda_{\mu\nu} E_\lambda \tag{2.5}$$

or, equivalently, by its dual (see (2.1c))

$$dE^\lambda = -\frac{1}{2} C^\lambda_{\mu\nu} E^\mu \wedge E^\nu, \tag{2.5$'$}$$

with $\wedge$ being the exterior (wedge) product sign.[4] Equation (2.4) is a special case of (III.2.7) for $S_X = 0$. The explicit local action of $\mathcal{L}_X$ on a general C^1 tensor field is given by (III.2.1).

A frame $\{E_\mu\}$ or its dual coframe $\{E^\mu\}$ is called *holonomic* (*anholonomic*) if $C^\lambda_{\mu\nu} = 0$ ($C^\lambda_{\mu\nu} \neq 0$) for all (some) values of the indices μ, ν, and λ. For a holonomic frame there always exist local coordinates $\{x^\mu\}$ on M such that *locally* $E_\mu = \frac{\partial}{\partial x^\mu}$ and $E^\mu = \mathrm{d}x^\mu$. Conversely, if $\{x^\mu\}$ are local coordinates on M, then the local frame $\{\frac{\partial}{\partial x^\mu}\}$ and local coframe $\{\mathrm{d}x^\mu\}$ are well defined and holonomic on the domain of $\{x^\mu\}$. For more information concerning (an)holonomic frames, see Section I.8.

A straightforward calculation by means of (2.5) reveals that a change

$$\{E_\mu\} \to \{\bar{E}_\mu = B^\nu_\mu E_\nu\} \tag{2.6}$$

of the frame $\{E_\mu\}$, where $B = [B^\nu_\mu]$ is a non-degenerate matrix-valued function, entails the transformation

$$C^\lambda_{\mu\nu} \mapsto \bar{C}^\lambda_{\mu\nu} = (B^{-1})^\lambda_\varrho \big(B^\sigma_\mu E_\sigma(B^\varrho_\nu) - B^\sigma_\nu E_\sigma(E^\varrho_\mu) + B^\sigma_\mu B^\tau_\nu C^\varrho_{\sigma\tau} \big). \tag{2.7}$$

Besides, from (2.4) and (2.7), we see that the quantities $(\Gamma_X)^\nu{}_\mu$ undergo the change (cf. equation (III.2.10))

$$(\Gamma_X)^\nu{}_\mu \mapsto (\bar{\Gamma}_X)^\nu{}_\mu = (B^{-1})^\mu_\varrho \big((\Gamma_X)^\varrho{}_\sigma B^\sigma_\nu + X(B^\sigma_\nu) \big) \tag{2.8}$$

when (2.6) takes place. Setting $\Gamma_X := [(\Gamma_X)^\nu{}_\mu]$ and $\bar{\Gamma}_X := [(\bar{\Gamma}_X)^\nu{}_\mu]$, we can rewrite (2.8) in a more compact matrix form as (cf. (III.2.11))

$$\Gamma_X \mapsto \bar{\Gamma}_X = B^{-1} \cdot (\Gamma_X \cdot B + X(B)). \tag{2.9}$$

If $n \in \mathbb{N}$ and $n \leq \dim M$, an n-dimensional *distribution* Δ on M is defined as a mapping $\Delta\colon p \mapsto \Delta_p$ assigning to each $p \in M$ an n-dimensional subspace Δ_p of the tangent space $T_p(M)$ of M at p, $\Delta_p \subseteq T_p(M)$. A distribution is *integrable* if there is a submersion $\psi\colon M \to N$ such that $\operatorname{Ker} \psi_* = \Delta$; a necessary and locally

[4] If M is a Lie group and $\{E_\mu\}$ is a basis of its Lie algebra (defined as the set of left invariant vector fields in on M), then $C^\lambda_{\mu\nu}$ are constants, called structure constants of M, and (2.5) and (2.5$'$) are known as the structure equations of M.

sufficient condition for the integrability of Δ is the commutator of every two vector fields in Δ to be in Δ. We say that a vector field $X \in \mathcal{X}(M)$ is in Δ and write $X \in \Delta$, if $X_p \in \Delta_p$ for all $p \in M$. A *basis on* $U \subseteq M$ *for* Δ is a set $\{X_1, \dots, X_n\}$ of n linearly independent (relative to functions $U \to \mathbb{K}$) vector fields in $\Delta|_U$, i.e., $\{X_1|_p, \dots, X_n|_p\}$ is a basis for Δ_p for all $p \in U$.

A distribution is convenient to be described in terms of (global) frames or/and coframes over M. In fact, if $p \in M$ and $\varrho = 1, \dots, n$, in each $\Delta_p \subseteq T_p(M)$, we can choose a basis $\{X_\varrho|_p\}$ and hence a frame $\{X_\varrho\}$, $X_\varrho \colon p \mapsto X_\varrho|_p$, in $\{\Delta_p : p \in M\} \subseteq T(M)$; we say that $\{X_\rho\}$ is a basis for/in Δ. Conversely, any collection of n linearly independent (relative to functions $M \to \mathbb{K}$) vector fields X_ϱ on M defines a distribution $p \mapsto \left\{\sum_{\varrho=1}^{n} f^\varrho X_\varrho|_p : f^\varrho \in \mathbb{K}\right\}$. Consequently, a frame in $T(M)$ can be formed by adding to a basis for Δ a set of $(\dim M - n)$ new linearly independent vector fields (forming a frame in $T(M) \setminus \{\Delta_p : p \in M\}$) and v.v., by selecting n linearly independent vector fields on M, we can define a distribution Δ on M. Equivalently, one can use $\dim M - n$ linearly independent 1-forms ω^a, $a = n+1, \dots, \dim M$, which are annihilators for it, $\omega^a|_{\Delta_p} = 0$ for all $p \in M$. For instance, if $\{X_\mu : \mu = 1, \dots, \dim M\}$ is a frame in $T(M)$ and $\{X_\varrho : \varrho = 1, \dots, n\}$ is a basis for Δ, then one can define ω^a to be elements in the coframe $\{\omega^\mu\}$ dual to $\{X_\mu\}$. We call $\{\omega^a\}$ a *cobasis* for Δ.

3. Connections on bundles

Before presenting the general connection theory in Subsection 3.2, we shall fix first some notation and concepts concerning fibre bundles in Subsection 3.1.The reader is referred for more details to Subsection IV.2.1 and to the literature cited in Section IV.1.

3.1. Frames and coframes on the bundle space

Let (E, π, M) be a bundle with bundle space E, projection $\pi \colon E \to M$, and base space M. Suppose that the spaces M and E are manifolds of finite dimensions $n \in \mathbb{N}$ and $n+r$, for some $r \in \mathbb{N}$, respectively; so the dimension of the fibre $\pi^{-1}(x)$, with $x \in M$, i.e., the fibre dimension of (E, π, M), is r. Besides, let these manifolds be C^2 differentiable, if the opposite is not stated explicitly.[1]

Let the Greek indices $\lambda, \mu, \nu, \dots$ run from 1 to $n = \dim M$, the Latin indices $a, b, c, \dots$ take the values from $n+1$ to $n+r = \dim E$, and the uppercase Latin

[1]Most of our considerations are valid also if C^1 differentiability is assumed and even some of them hold on C^0 manifolds. By assuming C^2 differentiability, we skip the problem of counting the required differentiability class of the whole material that follows. Sometimes, the C^2 differentiability is required explicitly, which is a hint that a statement or definition is not valid otherwise. If we want to emphasize that some text is valid under a C^1 differentiability assumption, we indicate that fact explicitly. However, the proofs of Proposition 6.5 and all assertions in Section 7 require C^3 differentiability, which will be indicated explicitly.

indices $I, J, K, \dots$ take values in the whole set $\{1, \dots, n+r\}$. One may call these types of indices respectively base, fibre, and bundle indices.

Suppose $\{u^I\} = \{u^\mu, u^a\} = \{u^1, \dots, u^{n+r}\}$ are local *bundle coordinates* on an open set $W \subseteq E$, i.e., on the set $\pi(W) \subseteq M$ there are local coordinates $\{x^\mu\}$ such that $u^\mu = x^\mu \circ \pi$;[2] the coordinates $\{u^\mu\}$ (resp. $\{u^a\}$) are called *basic* (resp. *fibre*) *coordinates* [23].[3]

Further only coordinate changes

$$\{u^\mu, u^a\} \mapsto \{\tilde{u}^\mu, \tilde{u}^a\} \tag{3.1a}$$

on E between bundle coordinates will be considered. This means that

$$\begin{aligned} \tilde{u}^\mu(p) &= f^\mu(u^1(p), \dots, u^n(p)) \\ \tilde{u}^a(p) &= f^a(u^1(p), \dots, u^n(p), u^{n+1}(p), \dots, u^{n+r}(p)) \end{aligned} \tag{3.1b}$$

for $p \in U \cap \tilde{U}$, with U being the domain of the coordinates $\{\tilde{u}^I\}$ and some functions f^I. The bundle coordinates $\{u^\mu, u^a\}$ induce the (local) frame $\big\{\partial_\mu := \frac{\partial}{\partial u^\mu}, \partial_a := \frac{\partial}{\partial u^a}\big\}$ and coframe $\{\mathrm{d}u^\mu, \mathrm{d}u^a\}$ over W in respectively the tangent $T(E)$ and cotangent $T^*(E)$ bundle spaces of the tangent and cotangent bundles over the bundle space E. Since a change (3.1) of the coordinates on E implies $\partial_I \mapsto \tilde{\partial}_I := \frac{\partial}{\partial \tilde{u}^I} = \frac{\partial u^J}{\partial \tilde{u}^I}\partial_J$ and $\mathrm{d}u^I \mapsto \mathrm{d}\tilde{u}^I = \frac{\partial \tilde{u}^I}{\partial u^J}\mathrm{d}u^J$, the transformation (3.1) leads to

$$(\partial_\mu, \partial_a) \mapsto (\tilde{\partial}_\mu, \tilde{\partial}_a) = (\partial_\nu, \partial_b) \cdot A \tag{3.2a}$$

$$(\mathrm{d}u^\mu, \mathrm{d}u^a)^\top \mapsto (\mathrm{d}\tilde{u}^\mu, \mathrm{d}\tilde{u}^a)^\top = A^{-1} \cdot (\mathrm{d}u^\nu, \mathrm{d}u^b)^\top. \tag{3.2b}$$

Here and below expressions like $(\partial_\mu, \partial_a)$ are shortcuts for ordered $(n+r)$-tuples like $(\partial_1, \dots, \partial_{n+r}) = \big([\partial_\mu]_{\mu=1}^n, [\partial_a]_{a=n+1}^{n+r}\big)$, $\top$ is the matrix transpositions sign, the centered dot $\cdot$ stands for the matrix multiplication, and the transformation matrix A is

$$A := \Big[\frac{\partial u^I}{\partial \tilde{u}^J}\Big]_{I,J=1}^{n+r} = \begin{pmatrix} \big[\frac{\partial u^\nu}{\partial \tilde{u}^\mu}\big] & 0_{n\times r} \\ \big[\frac{\partial u^b}{\partial \tilde{u}^\mu}\big] & \big[\frac{\partial u^b}{\partial \tilde{u}^a}\big] \end{pmatrix} =: \begin{bmatrix} \frac{\partial u^\nu}{\partial \tilde{u}^\mu} & 0 \\ \frac{\partial u^b}{\partial \tilde{u}^\mu} & \frac{\partial u^b}{\partial \tilde{u}^a} \end{bmatrix}, \tag{3.3}$$

where $0_{n\times r}$ is the $n \times r$ zero matrix. Besides, in expressions of the form $\partial_I a^I$, like the one in the right-hand side of (3.2a), the summation excludes differentiation, i.e., $\partial_I a^I := a^I \partial_I = \sum_I a^I \partial_I$; if a differentiation really takes place, we write $\partial_I(a^I) := \sum_I \partial_I(a^I)$. This rule allows a lot of formulae to be written in a compact matrix form, like (3.2a). The explicit form of the matrix inverse to (3.3) is $A^{-1} = \big[\frac{\partial \tilde{u}^I}{\partial u^J}\big] = \dots$ and it is obtained from (3.3) via the change $u \leftrightarrow \tilde{u}$.

---

[2]On a bundle or fibred manifold, these coordinates are known also as adapted coordinates [116, Definition 1.1.5].

[3]If (W, v) is a bundle chart, with $v\colon W \to \mathbb{K}^n \times \mathbb{K}^r$ and $e^a\colon \mathbb{K}^r \to \mathbb{K}$ are such that $e^a(c_1, \dots, c_r) = c_a \in \mathbb{K}$, then one can put $u^a = e^a \circ \mathrm{pr}_2 \circ v$, where $\mathrm{pr}_2\colon \mathbb{K}^n \times \mathbb{K}^r \to \mathbb{K}^r$ is the projection on the second multiplier $\mathbb{K}^r$.

The formula (3.2a) can be generalized for arbitrary frames $\{e_I\} = \{e_\mu, e_a\}$ and $\{\tilde{e}_I\} = \{\tilde{e}_\mu, \tilde{e}_a\}$ in $T(E)$ and their dual coframes $\{e^I\} = \{e^\mu, e^a\}$ and $\{\tilde{e}^I\} = \{\tilde{e}^\mu, \tilde{e}^a\}$ in $T^*(E)$ whose *admissible changes* are given by

$$(e_I) = (e_\mu, e_a) \mapsto (\tilde{e}_I) = (\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot A \tag{3.4a}$$

$$\begin{pmatrix} e^\mu \\ e^a \end{pmatrix} \mapsto \begin{pmatrix} \tilde{e}^\mu \\ \tilde{e}^a \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} e^\nu \\ e^b \end{pmatrix}. \tag{3.4b}$$

Here $A = [A^I_J]$ is a nondegenerate matrix-valued function with a block structure similar to (3.3), viz.

$$A = \begin{pmatrix} [A^\nu_\mu]^n_{\mu,\nu=1} & 0_{n\times r} \\ \left[A^b_\mu\right]_{\substack{\mu=1,\dots,n \\ b=n+1,\dots,n+r}} & [A^b_a]^{n+r}_{a,b=n+1} \end{pmatrix} =: \begin{bmatrix} A^\nu_\mu & 0 \\ A^b_\mu & A^b_a \end{bmatrix} \tag{3.5a}$$

with inverse matrix

$$A^{-1} = \begin{pmatrix} [A^\nu_\mu]^{-1} & 0 \\ -[A^a_b]^{-1} \cdot [A^a_\mu] \cdot [A^\nu_\mu]^{-1} & [A^a_b]^{-1} \end{pmatrix}. \tag{3.5b}$$

Here $A^a_\mu \colon W \to \mathbb{K}$ and $[A^\nu_\mu]$ and $[A^a_b]$ are non-degenerate matrix-valued functions on W such that $[A^\nu_\mu]$ is constant on the fibres of E, i.e., for $p \in U$, $A^\nu_\mu(p)$ depends only on $\pi(p) \in M$, which is equivalent to any one of the equations

$$A^\nu_\mu = B^\nu_\mu \circ \pi \qquad \frac{\partial A^\nu_\mu}{\partial u^a} = 0, \tag{3.6}$$

with $[B^\nu_\mu]$ being a nondegenerate matrix-valued function on $\pi(W) \subseteq M$. Obviously, (3.2) corresponds to (3.4) with $e_I = \frac{\partial}{\partial u^I}$, $\tilde{e}_I = \frac{\partial}{\partial \tilde{u}^I}$, and $A^J_I = \frac{\partial u^J}{\partial \tilde{u}^I}$.

All frames $\{\tilde{e}_I\}$ on E connected via (3.4)–(3.5), which are (locally) obtainable from holonomic ones $\{e_I\}$, induced by bundle coordinates, via admissible changes, will be referred as *bundle frames*. Only such frames will be employed in the present chapter.

If we deal with a *vector* bundle (E, π, M) endowed with vector bundle coordinates $\{u^I\}$ [23], then the new fibre coordinates $\{\tilde{u}^a\}$ in (3.1) must be *linear and homogeneous* in the old ones $\{u^a\}$, i.e.,

$$\tilde{u}^a = (B^a_b \circ \pi) \cdot u^b \text{ and } u^a = ((B^{-1})^a_b \circ \pi) \cdot \tilde{u}^b, \tag{3.7}$$

with $B = [B^a_b]$ being a non-degenerate matrix-valued function on $\pi(W) \subseteq M$. In that case, the matrix (3.3) and its inverse take the form

$$A = \begin{bmatrix} \frac{\partial u^\mu}{\partial \tilde{u}^\nu} & 0 \\ \left(\frac{\partial (B^{-1})^a_b}{\partial \tilde{x}^\nu} \circ \pi\right) \cdot \tilde{u}^b & (B^{-1})^b_a \circ \pi \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \frac{\partial \tilde{u}^\nu}{\partial u^\mu} & 0 \\ \left(\frac{\partial B^b_a}{\partial x^\mu} \circ \pi\right) \cdot u^a & B^b_a \circ \pi \end{bmatrix}. \tag{3.8}$$

More generally, in the vector bundle case, admissible are transformations (3.4) with matrices like

$$A = \begin{pmatrix} [A^\mu_\nu] & 0 \\ [A^b_{c\mu}\tilde{u}^c] & [A^a_b] \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} [A^\mu_\nu]^{-1} & 0 \\ -[A^a_b]^{-1}\cdot[A^b_{c\mu}\tilde{u}^c]\cdot[A^\mu_\nu]^{-1} & [A^a_b]^{-1} \end{pmatrix} \tag{3.9}$$

with $A^a_{b\mu}\colon W \to \mathbb{K}$ being functions on W which are constant on the fibres of E,

$$A^a_{b\mu} = B^a_{b\mu}\circ\pi \qquad \frac{\partial A^a_{b\mu}}{\partial u^c} = 0 \tag{3.10}$$

for some functions $B^a_{b\mu}\colon \pi(W) \to \mathbb{K}$. Obviously, (3.9) corresponds to (3.5) with $A^b_\mu = A^b_{c\mu}\tilde{u}^c$ and the setting $A^J_I = \frac{\partial u^J}{\partial \tilde{u}^I}$ reduces (3.9) to (3.8) due to (3.7).

3.2. Connection theory

From a number of equivalent definitions of a connection on differentiable manifold [146, Sections 2.1 and 2.2], we shall use the following one.

Definition 3.1 (cf. Definition IV.14.3 on page 303)**.** A *connection on differentiable bundle* (E,π,M) is an $n = \dim M$-dimensional distribution Δ^h on E such that, for each $p\in E$ and the *vertical distribution* Δ^v defined by

$$\Delta^v\colon p\mapsto \Delta^v_p := T_{\imath(p)}\big(\pi^{-1}(\pi(p))\big) \cong T_p\big(\pi^{-1}(\pi(p))\big), \tag{3.11}$$

with $\imath\colon \pi^{-1}(\pi(p)) \to E$ being the inclusion mapping, is fulfilled

$$\Delta^v_p \oplus \Delta^h_p = T_p(E), \tag{3.12}$$

where $\Delta^h\colon p\mapsto \Delta^h_p \subseteq T_p(E)$ and $\oplus$ is the direct sum sign. The distribution Δ^h is called *horizontal* and symbolically we write $\Delta^v\oplus\Delta^h = T(E)$.

A *vector* at a point $p\in E$ (resp. a *vector field* on E) is said to be *vertical* or *horizontal* if it (resp. its value at p) belongs to Δ^v_p or Δ^h_p, respectively, for the given (resp. any) point p. A vector $Y_p\in T_p(E)$ (resp. vector field $Y\in\mathcal{X}(E)$) is called a *horizontal lift of a vector* $X_{\pi(p)}\in T_{\pi(p)}(M)$ (resp. *vector field* $X\in\mathcal{X}(M)$ on $M=\pi(E)$) if $\pi_*(Y_p) = X_{\pi(p)}$ for the given (resp. any) point $p\in E$. Since $\pi_*|_{\Delta^h_p}\colon \Delta^h_p \to T_{\pi(p)}(M)$ is a vector space isomorphism for all $p\in E$ [23, Section 1.24], any vector in $T_{\pi(p)}(M)$ (resp. vector field in $\mathcal{X}(M)$) has a unique horizontal lift in $T_p(E)$ (resp. $\mathcal{X}(E)$).

As a result of (3.12), any vector $Y_p\in T_p(E)$ (resp. vector field $Y\in\mathcal{X}(E)$) admits a unique representation $Y_p = Y^v_p\oplus Y^h_p$ (resp. $Y = Y^v\oplus Y^h$) with $Y^v_p\in\Delta^v_p$ and $Y^h_p\in\Delta^h_p$ (resp. $Y^v\in\Delta^v$ and $Y^h\in\Delta^h$). If the distribution $p\mapsto\Delta^h_p$ is differentiable of class C^m, $m\in\mathbb{N}\cup\{0,\infty,\omega\}$, it is said that the *connection* Δ^h *is (differentiable) of class* C^m. A connection Δ^h is of class C^m if and only if, for every C^m vector field Y on E, the vertical Y^v and horizontal Y^h vector fields are of class C^m.

A C^1 *path* $\beta\colon J \to E$ is called *horizontal* (*vertical*) if its tangent vector $\dot\beta$ is horizontal (vertical) vector along β, i.e., $\dot\beta(s) \in \Delta^h_{\beta(s)}$ ($\dot\beta(s) \in \Delta^v_{\beta(s)}$) for all $s \in J$. A *lifting* $\bar\gamma\colon J \to E$ of a path $\gamma\colon J \to M$, i.e., $\pi \circ \bar\gamma = \gamma$, is called *horizontal* if $\bar\gamma$ is a horizontal path, i.e., when the vector field $\dot{\bar\gamma}$ tangent to $\bar\gamma$ is horizontal or, equivalently, if $\dot{\bar\gamma}$ is a horizontal lift of $\dot\gamma$. Since $\pi^{-1}(\gamma(J))$ is an $(r+1)$-dimensional submanifold of E, the distribution $p \mapsto \Delta^h_p \cap T_p(\pi^{-1}(\gamma(J)))$ is one-dimensional and, consequently, is integrable for arbitrary C^1 path γ. The integral paths of that distribution are horizontal lifts of γ and, *for each* $p \in \pi^{-1}(\gamma(J))$, *there is (locally) a unique horizontal lift* $\bar\gamma_p$ *of* γ *passing through* p.[4]

Definition 3.2. Let $\gamma\colon [\sigma,\tau] \to M$, with $\sigma,\tau \in \mathbb{R}$ and $\sigma \le \tau$, and $\bar\gamma_p$ be the unique horizontal lift of γ in E passing through $p \in \pi^{-1}(\gamma([\sigma,\tau]))$. The *parallel transport (translation, displacement)* generated by (assigned to, defined by) a connection Δ^h is a mapping $\mathsf{P}\colon \gamma \mapsto \mathsf{P}^\gamma$, assigning to the path γ a mapping

$$\mathsf{P}^\gamma\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma,\tau] \to M \tag{3.13}$$

such that, for each $p \in \pi^{-1}(\gamma(\sigma))$,

$$\mathsf{P}^\gamma(p) := \bar\gamma_p(\tau). \tag{3.14}$$

In vector bundles are important the *linear* connections for which is required the parallel transport assigned to them to be linear in a sense that the mapping (3.13) is linear for every path γ (see Subsection 4.2 below).

Let us now look on the connections Δ^h on a bundle (E,π,M) from a view point of (local) frames and their dual coframes on E. Let $\{e_\mu\}$ be a basis for Δ^h, i.e., $e_\mu \in \Delta^h$ and $\{e_\mu|_p\}$ is a basis for Δ^h_p for all $p \in E$, and $\{e^a\}$ be the coframe for Δ^h, i.e., a collection of 1-forms e^a, $a = n+1,\dots,n+r$, which are linearly independent (relative to functions $E \to \mathbb{K}$) and such that $e^a(X) = 0$ if $X \in \Delta^h$.

Definition 3.3. A frame $\{e_I\}$ in $T(E)$ over E is called *specialized* for a connection Δ^h if the first $n = \dim M$ of its vector fields form a basis $\{e_\mu\}$ for the horizontal distribution Δ^h and its last $r = \dim \pi^{-1}(x)$, $x \in M$, vector fields form a basis $\{e_a\}$ for the vertical distribution Δ^v. Respectively, a coframe $\{e^I\}$ on E is called *specialized* if $\{e^a\}$ is a cobasis for Δ^h and $\{e^\mu\}$ is a cobasis for Δ^v.

The horizontal lifts of vector fields and 1-forms can easily be described in specialized (co)frames. Indeed, let $\{e_I\}$ and $\{e^I\}$ be respectively a specialized frame and its dual coframe. Define a frame $\{E_\mu\}$ and its dual coframe $\{E^\mu\}$ on M which are π-related to $\{e_I\}$ and $\{e^I\}$, i.e., $E_\mu := \pi_*(e_\mu)$ and $e^\mu := \pi^*(E^\mu) = E^\mu \circ \pi_*$.[5]

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[4]In this sense, a connection Δ^h is an Ehresmann connection [16, p. 314] and *vice versa* [116, pp. 85–89].

[5]Recall, $\pi_*|_{\Delta^h_p}\colon \Delta^h_p \to T_{\pi(p)}(M)$ is a vector space isomorphism.

If $Y = Y^\mu E_\mu \in \mathcal{X}(M)$ and $\phi = \phi_\mu e^\mu \in \Lambda^1(M)$, then their horizontal lifts (from M to E) respectively are

$$\bar{Y} = (Y^\mu \circ \pi) e_\mu \qquad \bar{\phi} = (\phi_\mu \circ \pi) e^\mu. \tag{3.15}$$

The specialized (co)frames transform into each other according to the general rules (3.4) in which the transformation matrix and its inverse have the following block structure:

$$A = \begin{pmatrix} [A_\mu^\nu] & 0_{n\times r} \\ 0_{r\times n} & [A_a^b] \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} [A_\mu^\nu]^{-1} & 0_{n\times r} \\ 0_{r\times n} & [A_a^b]^{-1} \end{pmatrix}, \tag{3.16}$$

where $A_\mu^\nu, A_a^b \colon E \to \mathbb{K}$ and the functions A_μ^ν are constant on the fibres of the bundle (E, π, M), that is, we have

$$A_\mu^\nu = B_\mu^\nu \circ \pi \quad \text{or} \quad \frac{\partial A_\mu^\nu}{\partial u^a} = 0 \tag{3.17}$$

for some nondegenerate matrix-valued function $[B_\mu^\nu]$ on M. Besides, in a case of vector bundle, the functions A_b^a are also constant on the fibres of the bundle (E, π, M), i.e.,

$$A_a^b = B_a^b \circ \pi \quad \text{or} \quad \frac{\partial A_a^b}{\partial u^a} = 0 \tag{3.18}$$

for some nondegenerate matrix-valued function $B = [B_a^b]$ on M. Changes like (3.4), with A given by (3.16), respect the fibre as well as the connection structure of the bundle.

Let E be a C^2 manifold and Δ^h a C^1 connection on (E, π, M). The components C_{IJ}^K of the anholonomicity object of a *specialized* frame $\{e_I\}$ are (local) functions on E defined by (see (2.5))

$$[e_I, e_J]_{\_} =: C_{IJ}^K e_K \tag{3.19}$$

and are naturally divided into the following six groups (cf. [138, p. 21]):

$$\{C_{\mu\nu}^\lambda\}, \quad \{C_{\mu\nu}^a\}, \quad \{C_{\mu b}^\lambda = 0\}, \quad \{C_{ab}^\lambda = 0\}, \quad \{C_{\mu b}^c\}, \quad \{C_{ab}^c\}. \tag{3.20}$$

The functions $C_{\mu\nu}^\lambda$ are constant on the fibres of (E, π, M), precisely $C_{\mu\nu}^\lambda = f_{\mu\nu}^\lambda \circ \pi$ where $f_{\mu\nu}^\lambda$ are the components of the anholonomicity object for the π-related frame $\{\pi_*(e_\mu)\}$ on M, as the commutators of π-related vector fields are π-related [7, Section 1.55]. Besides, since the vertical distribution Δ^v is integrable (the space Δ_p^v is the space tangent to the fibre through $p \in E$ at p), we have

$$[e_a, e_b]_{\_} = C_{ab}^c e_c \tag{3.21}$$

(so that $C_{ab}^\lambda = 0$), due to which C_{ab}^c are called components of the vertical anholonomicity object. To prove that $C_{\mu b}^\lambda = 0$, one should expand $\{e_I\}$ along $\left\{\partial_I = \frac{\partial}{\partial u^I}\right\}$, with $\{u^I\}$ being some bundle coordinates, viz. $e_\mu = e^\nu_\mu \partial_\nu + e^b_\mu \partial_b$ and $e_a = e^b_a \partial_b$, with some functions e^ν_μ, e^b_μ and e^b_a, and to notice that e^ν_μ are constant on the fibres, i.e., $\partial_a(e^\nu_\mu) = 0$.

The non-trivial mixed "vertical-horizontal" components between (3.20), viz. $C^a_{\mu\nu}$ and $C^a_{\mu b}$, are important characteristics of the connection Δ^h. The functions

$$^\circ\Gamma^a_{b\mu} := +C^a_{b\mu} = -C^a_{\mu b} \tag{3.22a}$$

$$R^a_{\mu\nu} := +C^a_{\mu\nu} = -C^a_{\nu\mu}, \tag{3.22b}$$

which enter into the commutators

$$\mathcal{L}_{e_\mu} e_b = [e_\mu, e_b]_{\_} = {}^\circ\Gamma^a_{b\mu} e_a \tag{3.23a}$$

$$\mathcal{L}_{e_\mu} e_\nu = [e_\mu, e_\nu]_{\_} = R^a_{\mu\nu} e_a + C^\lambda_{\mu\nu} e_\lambda, \tag{3.23b}$$

are called respectively the *fibre coefficients of* Δ^h (or *components of the connection object of* Δ^h) and *fibre components of the curvature of* Δ^h (or *components of the curvature (object) of* Δ^h) in $\{e_I\}$. Under a change (3.4), with a matrix (3.16), of the specialized frame, the functions (3.22) transform into respectively

$$^\circ\tilde{\Gamma}^a_{b\mu} = A^\nu_\mu \big([A^f_e]^{-1}\big)^a_d \big({}^\circ\Gamma^d_{c\nu} A^c_b + e_\nu(A^d_b)\big) \tag{3.24a}$$

$$\tilde{R}^a_{\mu\nu} = \big([A^f_e]^{-1}\big)^a_b A^\lambda_\mu A^\varrho_\nu R^b_{\lambda\varrho}, \tag{3.24b}$$

which formulae are direct consequences of (3.23). If we put $\bar{A} := [A^b_a]$, ${}^\circ\Gamma_\nu := [{}^\circ\Gamma^d_{c\nu}]$, and ${}^\circ\tilde{\Gamma}_\nu := [{}^\circ\tilde{\Gamma}^d_{c\nu}]$, then (3.24a) is tantamount to

$$\begin{aligned} {}^\circ\tilde{\Gamma}_\mu &= A^\nu_\mu \bar{A}^{-1} \cdot ({}^\circ\Gamma_\nu \cdot \bar{A} + e_\nu(\bar{A})) \\ &= A^\nu_\mu (\bar{A}^{-1} \cdot {}^\circ\Gamma_\nu - e_\nu(\bar{A}^{-1})) \cdot \bar{A}. \end{aligned} \tag{3.25}$$

Up to a meaning of the matrices $[A^\nu_\mu]$ and $\bar{A}$ and the size of the matrices ${}^\circ\Gamma_\nu$ and $\bar{A}$, the last equation is identical with the one expressing the transformed matrices of the coefficients of a linear connection (covariant derivative operator) in a vector bundle (see (IV.14.25) for $N = M$ and $g = \mathsf{id}_M$ or [87, eq. (3.5)]) on which we shall return later in this chapter (see Section 4, in particular equation (4.23′) in it). Equation (3.24b) indicates that $R^a_{\mu\nu}$ are components of a tensor, viz.

$$\Omega := \frac{1}{2} R^a_{\mu\nu} e_a \otimes e^\mu \wedge e^\nu, \tag{3.26}$$

called *curvature tensor* of the connection Δ^h. By (3.23a), the horizontal distribution Δ^h is (locally) integrable iff its curvature tensor vanishes, $\Omega = 0$.

Definition 3.4. A connection with vanishing curvature tensor is called *flat*, or *integrable*, or *curvature free*.

Proposition 3.1. *The flat connections are the only ones that may admit holonomic specialized frames.*

Proof. See Definition 3.4 and (3.23b). □

The above considerations of specialized (co)frames for a connection Δ^h on a bundle (E, π, M) were *global* as we supposed that these (co)frames are defined on the whole bundle space E, which is always possible if no smoothness conditions on Δ^h are imposed. Below we shall show how *local* specialized (co)frames can be defined via local bundle coordinates on E.

Let $\{u^I\}$ be local bundle coordinates on an open set $W \subseteq E$. They define on $T(W) \subseteq T(E)$ the local basis $\{\partial_I := \frac{\partial}{\partial u^I}\}$, so that any vector can be expended along its vectors. In particular, we can do so with any basic vector field e_I^W of a *specialized* frame $\{e_I\}$ restricted to W, $e_I^W := e_I|_W$. Since $\{\partial_a|_p\}$, with $p \in W$, is a basis for Δ_p^v, we can write

$$(e_\mu^W, e_a^W) = (A_\mu^\nu \partial_\mu + A_\mu^a \partial_a, A_a^b \partial_b) = (\partial_\nu, \partial_b) \cdot \begin{pmatrix} [A_\mu^\nu] & 0 \\ [A_\mu^b] & [A_a^b] \end{pmatrix}, \tag{3.27}$$

where $[A_\mu^\nu]$ and $[A_a^b]$ are non-degenerate matrix-valued functions on W.[6]

Definition 3.5. A frame $\{X_I\}$ over W in $T(W)$ is called *adapted (to the coordinates* $\{u^I\}$ *in* W*)* for a connection Δ^h if it is the specialized frame obtained from (3.27) via admissible transformation (3.4) with the matrix $A = \begin{pmatrix} [A_\mu^\nu]^{-1} & 0 \\ 0 & [A_a^b]^{-1} \end{pmatrix}$.

Exercise 3.1. An arbitrary *specialized* frame $\{e_I^W\}$ in $T(E)$ over W enters in the definition of a frame $\{X_I\}$ adapted to bundle coordinates $\{u^I\}$ on W. Prove that $\{X_I\}$ is independent of the particular choice of the frame $\{e_I^W\}$. (Hint: apply Definition 3.5 and (3.4a) with A given by (3.16).) The below-derived equality (3.34) is an indirect proof of that fact too.

According to (3.4) and Definition 3.5, the adapted frame $\{X_I\}$ and the corresponding to it *adapted coframe* $\{\omega^I\}$ are given by

$$X_\mu = \partial_\mu + \Gamma_\mu^a \partial_a \qquad X_a = \partial_a \tag{3.28a}$$

$$\omega^\mu = \mathrm{d}u^\mu \qquad \omega^a = \mathrm{d}u^a - \Gamma_\mu^a \mathrm{d}u^\mu. \tag{3.28b}$$

Here the functions $\Gamma_\mu^a \colon W \to \mathbb{K}$ are defined via

$$[\Gamma_\mu^a] = +[A_\nu^a] \cdot [A_\mu^\nu]^{-1} \tag{3.29}$$

and are called (2-*index*) *coefficients* of Δ^h. In a matrix form, the equations (3.28) can be written as

$$(X_\mu, X_a) = (\partial_\nu, \partial_b) \cdot \begin{bmatrix} \delta_\mu^\nu & 0 \\ +\Gamma_\mu^b & \delta_a^b \end{bmatrix} \quad \begin{pmatrix} \omega^\mu \\ \omega^a \end{pmatrix} = \begin{bmatrix} \delta_\nu^\mu & 0 \\ -\Gamma_\nu^a & \delta_b^a \end{bmatrix} \cdot \begin{pmatrix} \mathrm{d}u^\nu \\ \mathrm{d}u^b \end{pmatrix}. \tag{3.30}$$

[6]The non-degeneracy of $[A_\mu^\nu]$ follows from the fact that the vector fields $\pi_*|_{\Delta^h}(e_\mu^W) = A_\mu^\nu \pi_*(\frac{\partial}{\partial u^\mu})$ form a basis for $\mathcal{X}(\pi(W)) \subseteq \mathcal{X}(M)$.

The operators $X_\mu = \partial_\mu + \Gamma^a_\mu \partial_a$ are known as *covariant derivatives on* $T(W)$ and the plus sing in (3.28a) before Γ^a_μ (hence in the right-hand side of (3.29)) is conventional.

If $\{u^I\}$ and $\{\tilde{u}^I\}$ are local coordinates on open sets $W \subseteq E$ and $\tilde{W} \subseteq E$, respectively, and $W \cap \tilde{W} \neq \varnothing$, then, on the overlapping set $W \cap \tilde{W}$, a problem arises: how are connected the adapted frames corresponding to these coordinates? Let us mark with a tilde all quantities that refer to the coordinates $\{\tilde{u}^I\}$. Since the adapted frames are, by definitions, specialized ones, we can write (see (3.4))

$$(\tilde{X}_\mu, \tilde{X}_a) = (X_\nu, X_b) \cdot A \qquad \begin{pmatrix} \tilde{\omega}^\mu \\ \tilde{\omega}^a \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} \omega^\nu \\ \omega^b \end{pmatrix}, \tag{3.31a}$$

where the transformation matrix A and its inverse have the form (3.16). Recalling (3.2) and (3.3), from these equalities, we get

$$A = \operatorname{diag}\Big(\Big[\frac{\partial u^\nu}{\partial \tilde{u}^\mu}\Big], \Big[\frac{\partial u^b}{\partial \tilde{u}^a}\Big]\Big) = \begin{pmatrix} \big[\frac{\partial u^\nu}{\partial \tilde{u}^\mu}\big] & 0 \\ 0 & \big[\frac{\partial u^b}{\partial \tilde{u}^a}\big] \end{pmatrix} . \tag{3.31b}$$

Combining (3.29) and (3.31), one can easily prove

Proposition 3.2. *A change $\{u^I\} \mapsto \{\tilde{u}^I\}$ of the local bundle coordinates implies the following transformation of the 2-index coefficients of the connection:*

$$\Gamma^a_\mu \mapsto \tilde{\Gamma}^a_\mu = \Big(\frac{\partial \tilde{u}^a}{\partial u^b}\Gamma^b_\nu + \frac{\partial \tilde{u}^a}{\partial u^\nu}\Big)\frac{\partial u^\nu}{\partial \tilde{u}^\mu}. \tag{3.32}$$

It is obvious, a connection Δ^h is of class C^m, $m \in \mathbb{N} \cup \{0\}$, if and only if its coefficients Γ^a_μ are C^m functions on W, provided ∂_I are C^m vector fields on W (which is the case when E is a C^{m+1} manifold). By virtue of (3.32), the C^{m+1} changes of the local bundle coordinates preserve the C^m differentiability of Γ^a_μ. Thus the C^{m+1} differentiability of the base M and bundle E spaces is a necessary condition for existence of C^m connections on (E, π, M); as we assumed $m = 1$ in this chapter, the connections considered here can be at most of differentiability class C^1.

The next proposition states that a connection on a bundle is locally equivalent to a geometric object whose components transform like (3.32).

Proposition 3.3. *To any connection Δ^h in a bundle (E, π, M) can be assigned, according to the procedure described above, a geometrical object on E whose components Γ^a_μ in bundle coordinates $\{u^I\}$ on E transform according to* (3.32) *under a change $\{u^I\} \mapsto \{\tilde{u}^I\}$ of the bundle coordinates on the intersection of the domains of $\{u^I\}$ and $\{\tilde{u}^I\}$. Conversely, given a geometrical object on E with local transformation law* (3.32)*, there is a unique connection Δ^h in (E, π, M) which generates the components of that object as described above.*

Proof. The first part of the statement was proved above, when we constructed the adapted frame (3.28a) and derived (3.32). To prove the second part, choose a point $p \in E$ and some local coordinates $\{u^I\}$ on an open set W in E containing p in which the geometrical object mentioned has local components Γ^a_μ. Define a local frame $\{X_I\} = \{X_\mu, X_a\}$ on W by (3.28a). The required connection is then $\Delta^h \colon q \mapsto \Delta^h_q := \{r^\mu X_\mu|_q : r^\mu \in \mathbb{K}\}$ for any $q \in W$, which means that Δ^h_q is the linear cover of $\{X_\mu|_q\}$. The transformation law (3.32) insures the independence of Δ^h from the local coordinates employed in its definition. □

From the construction of an adapted frame $\{X_I\}$, as well as from the proof of Proposition 3.3, follows that the set of vectors $\{X_\mu\}$ is a basis for the horizontal distribution Δ^h and the set $\{X_a\}$ is a basis for the vertical distribution Δ^v. The matrix of the restricted tangent projection $\pi_*|_{\Delta^h}$ in bundle coordinate system $\{u^\mu = x^\mu \circ \pi, u^a\}$ on E, where x^μ are local coordinates on M, is the identity matrix as $(\pi_*|_{\Delta^h_p})^\nu_\mu = \frac{\partial(x^\mu \circ \pi)}{\partial u^\mu}\Big|_p = \delta^\nu_\mu$ for any point p in the domain of $\{u^I\}$. Hereof we get

$$\pi_*|_{\Delta^h}(X_\mu) = \frac{\partial}{\partial x^\mu} \qquad \Big(\iff \pi_*|_{\Delta^h_p}(X_\mu|_p) = \frac{\partial}{\partial x^\mu}\Big|_{\pi(p)} \Big). \tag{3.33}$$

In particular, from here follows that $\pi_*|_{\Delta^h_p} \colon \Delta^h_p \to T_{\pi(p)}(M)$ is a vector space isomorphism. The inverse to equation (3.33), viz.

$$X_\mu = (\pi_*|_{\Delta^h})^{-1}\Big(\frac{\partial}{\partial x^\mu}\Big) = (\pi_*|_{\Delta^h})^{-1} \circ \pi_*\Big(\frac{\partial}{\partial u^\mu}\Big), \tag{3.34}$$

can be used in an equivalent definition of a frame $\{X_I\}$ adapted to local coordinates $\{u^I\}$, namely, this is the frame $\big\{(\pi_*|_{\Delta^h})^{-1} \circ \pi_*\big(\frac{\partial}{\partial u^\mu}\big), \frac{\partial}{\partial u^a}\big\}$. If one accepts such a definition of an adapted frame for Δ^h, the (2-index) coefficients of Δ^h have to be defined via the expansion (3.28a); the only changes this may entail are in the proofs of some results, like (3.31) and (3.32).

It is useful to be recorded also the simple fact that, by construction, we have

$$\pi_*(X_a) = 0. \tag{3.35}$$

Let E be a C^2 manifold and Δ^h be a C^1 connection. The *adapted frames are generally anholonomic* as the commutators between the basic vector fields of the adapted frame (3.28a) are (cf. (3.20) and (3.22))

$$[X_\mu, X_\nu]_\_ = R^a_{\mu\nu} X_a \quad [X_\mu, X_b]_\_ = {}^\circ\Gamma^a_{b\mu} X_a \quad [X_a, X_b]_\_ = 0, \tag{3.36}$$

with

$$R^a_{\mu\nu} = \partial_\mu(\Gamma^a_\nu) - \partial_\nu(\Gamma^a_\mu) + \Gamma^b_\mu \partial_b(\Gamma^a_\nu) - \Gamma^b_\nu \partial_b(\Gamma^a_\mu) = X_\mu(\Gamma^a_\nu) - X_\nu(\Gamma^a_\mu) \tag{3.37a}$$

$${}^\circ\Gamma^a_{b\mu} = -\partial_b(\Gamma^a_\mu) = -X_b(\Gamma^a_\mu) \tag{3.37b}$$

being the fibre components of the curvature and fibre coefficients of the connection.

An obvious corollary from (3.36) is

Proposition 3.4. *An adapted frame is holonomic if and only if*

$$R^a_{\mu\nu} = 0 \quad (\iff \quad \Omega = 0) \quad {}^\circ\Gamma^a_{b\mu} = 0. \tag{3.38}$$

Therefore only the flat (integrable) C^1 connections, for which $\Omega = 0$, may admit holonomic adapted frames (cf. Proposition 3.1). Besides, as a consequence of (3.37b) and (3.38), such connections admit holonomic adapted frames on $W \subseteq E$ if and only if there are local coordinates on W in which the coefficients Γ^a_μ are constant on the fibres passing through W, i.e., iff $\Gamma^a_\mu = G^a_\mu \circ \pi$ for some functions $G^a_\mu \colon \pi(W) \to \mathbb{K}$, which is equivalent to $\partial_b(\Gamma^a_\mu) = 0$.

Example 3.1 (horizontal lifting of a path). Recall, the horizontal lift of a C^1 path $\gamma\colon J \to M$ passing through a point $p \in \pi^{-1}(\gamma(t_0))$ for some $t_0 \in J$ is the unique path $\bar\gamma_p\colon J \to E$ such that $\pi \circ \bar\gamma_p = \gamma$, $\bar\gamma_p(t_0) = p$, and $\dot{\bar\gamma}_p(t) \in \Delta^h_{\bar\gamma_p(t)}$ for all $t \in J$. As in a specialized frame $\{e_I\}$ the relation $X_p \in \Delta^h_p$ is equivalent to $e^a(X) = 0$ for any $X \in \mathcal{X}(M)$, in an adapted coframe, given by (3.28b), the horizontal lift $\bar\gamma_p$ of γ is the unique solution of the initial value problem

$$\omega^a(\dot{\bar\gamma}_p) = 0 \tag{3.39a}$$

$$\bar\gamma_p(t_0) = p \tag{3.39b}$$

which is tantamount to any one of the initial-value problems ($t \in J$)

$$\dot{\bar\gamma}^a_p(t) - \Gamma^a_\mu(\bar\gamma_p(t))\dot{\bar\gamma}^\mu_p(t) = 0 \tag{3.39′a}$$

$$\bar\gamma^I_p(t_0) = p^I := u^I(p) \tag{3.39′b}$$

$$\frac{\mathrm{d}(u^a \circ \bar\gamma_p(t))}{\mathrm{d}t} - \Gamma^a_\mu(\bar\gamma_p(t))\frac{\mathrm{d}(x^\mu \circ \gamma(t))}{\mathrm{d}t} = 0 \tag{3.39″a}$$

$$u^I(\bar\gamma_p(t_0)) = u^I(p), \tag{3.39″b}$$

where x^μ are the local coordinates in the base space that induce the basic coordinates u^μ on the bundle space, $u^\mu = x^\mu \circ \pi$. (Note that the quantities $\frac{\mathrm{d}(x^\mu \circ \gamma(t))}{\mathrm{d}t}$, entering into (3.39″a), are the components of the vector $\dot\gamma$ tangent to γ at parameter value t.) One may call (3.39a), or any one of its versions (3.39′a) or (3.39″a), the *parallel transport equation* in an adapted frame as it uniquely determines the parallel transport along the restriction of γ to any closed subinterval in J (see Definition 3.2).

Example 3.2 (the equation of geodesic paths). A connection Δ^h on the tangent bundle $(T(M), \pi_T, M)$ of a manifold M is called a *connection on* M. In this case, equation (3.39) defines also the geodesics (relative to Δ^h) in M. A C^2 path $\gamma\colon J \to M$ in a C^2 manifold M is called a *geodesic path* if its tangent vector field $\dot\gamma$ undergoes parallel transport along the same path γ, i.e., $\mathsf{P}^{\gamma|[\sigma,\tau]}(\dot\gamma(\sigma)) = \dot\gamma(\tau)$

for all $\sigma, \tau \in J$, which means that the lifted path $\dot{\gamma}\colon J \to T(M)$ is a *horizontal* lift of γ (relative to Δ^h). So, if x^μ are local coordinates on $\pi(W) \in M$ and the bundle coordinates on $W \subseteq E$ are such that [7, Section 1.25] $u^\mu = x^\mu \circ \pi$ and $u^{n+\mu} = \mathrm{d}x^\mu$ ($\mu, \nu, \dots = 1, \dots, n = \dim M$), then (3.39″a) transforms into the *geodesic equation* (on M)

$$\frac{\mathrm{d}^2(x^\mu \circ \gamma(t))}{\mathrm{d}t^2} - \Gamma_\nu^{n+\mu}(\dot{\gamma}(t)) \frac{\mathrm{d}(x^\nu \circ \gamma(t))}{\mathrm{d}t} = 0 \qquad t \in J, \tag{3.40}$$

which (locally) defines all geodesics in M. Of course, a particular geodesic is specified by fixing some initial values for $\gamma(t_0)$ and $\dot{\gamma}(t_0)$ for some $t_o \in J$. Notice, equation (3.40) is an equation for a path γ in M, while (3.39″a) is an equation for the lifted path $\bar{\gamma}$ in $T(M)$ provided the path γ in M is known; for a geodesic path, evidently, we have $\bar{\gamma} = \dot{\gamma}$. With obvious renumbering of the indices, one usually writes Γ_ν^μ for $\Gamma_\nu^{n+\mu}$, so then (3.40) coincides with the classical geodesic equation (I.3.23); for other point of view on this equation, see Section IV.15.

4. Connections on vector bundles

In this section, by (E, π, M) we shall denote an arbitrary *vector* bundle (see Section IV.2 or, e.g., [23]). A peculiarity of such bundles is that their fibres are isomorphic vector spaces, which leads to a natural description of the vertical distribution Δ^v on their fibre spaces, as well as to existence of a kind of differentiation of their sections (known as covariant differentiation – see Subsection IV.14.1).

In the vector bundles are used, as we shall do in this section, the so-called vector bundle coordinates whose fibre coordinates are linear on their fibres and are constructed as follows (cf. [116, p. 30].

Let $\{e_a\}$ be a frame in E over a subset $W_M \subseteq M$, i.e., $\{e_a(x)\}$ to be a basis in $\pi^{-1}(x)$ for all $x \in W_M$. Then, for each $p \in \pi^{-1}(W_M)$, we have a unique expansion $p = p^a e_a(\pi(p))$ for some numbers $p^a \in \mathbb{K}$. The *vector fibre coordinates* $\{u^a\}$ on $\pi^{-1}(W_M)$ induced (generated) by the frame $\{e_a\}$ are defined via $u^a(p) := p^a$ and hence can be identified with the elements of the coframe $\{e^a\}$ dual to $\{e_a\}$, i.e., $u^a = e^a$. Conversely, if $\{u^I\}$ are coordinates on $\pi^{-1}(W_M)$ for some $W_M \subseteq M$ which are linear on the fibres over W_M, then there is a unique frame $\{e_a\}$ in $\pi^{-1}(W_M)$ which generates $\{u^a\}$ as just described; indeed, considering $u^{n+1}, \dots, u^{n+r}$ as 1-forms on $\pi^{-1}(W_M)$, one should define the frame $\{e_a\}$ required as a one whose dual is $\{u^a\}$, i.e., via the conditions $u^a(e_b) = \delta_b^a$.

A collection $\{u^I\}$ of basic coordinates u^μ and vector fibre coordinates u^a on $\pi^{-1}(W_M)$ is called *vector bundle coordinate system* on $\pi^{-1}(W_M)$. Only such coordinates on E will be employed in this section.

4.1. Vertical lifts

The idea of describing the vertical distribution Δ^v on a vector bundle is that, if L is a vector space, then to any $Y \in L$ there corresponds a 'vertical' vector field

$Y^v \in \mathcal{X}(L) = \operatorname{Sec}(T(L), \pi_T, L)$ whose value at $X \in L$ is the vector tangent to the path $t \mapsto X + tY \in L$, with $t \in \mathbb{R}$, at $t = 0$, i.e., $Y^v|_X := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(X + tY)$. Here, as usual, with $\operatorname{Sec}(E, \pi, M)$ (resp. $\operatorname{Sec}^m(E, \pi, M)$ with $m \in \mathbb{N} \cup \{0\}$) we denote the module of sections (resp. C^m sections) of a bundle (E, π, M) (resp. of a C^{m+1} bundle (E, π, M)).

Let (E, π, M) be a vector bundle and Δ^v the vertical distribution on it, viz., for each $p \in E$, $\Delta^v\colon p \mapsto \Delta^v_p := T_p(\pi^{-1}(\pi(p)))$. To every $Y \in \operatorname{Sec}(E, \pi, M)$, we assign a *vertical* vector field $Y^v \in \Delta^v$ on E such that, for $p \in E$,

$$Y^v_p := Y^v|_p := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(p + tY|_{\pi(p)}). \tag{4.1}$$

(The mapping $(p, Y_{\pi(p)}) \mapsto Y^v_p$ defines an isomorphism from the pullback bundle π^*E into the vertical bundle $\mathcal{V}(E)$ – see [23, Sections 1.27 and 1.28] and also [116, p. 41, Exercises 2.2.1 and 2.2.2].)

Lemma 4.1. *Let $\{u^a\}$ be vector fibre coordinates generated by a frame $\{e_a\}$ on M. If $Y \in \operatorname{Sec}(E, \pi, M)$ and $Y = Y^a e_a$, then*

$$Y^v = (Y^a \circ \pi)\frac{\partial}{\partial u^a}. \tag{4.2}$$

Proof. Using the Definition (4.1), we get for $p \in E$:

$$\begin{aligned} Y^v|_p &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(p + tY|_{\pi(p)}) = \frac{\mathrm{d}(u^a(p + tY|_{\pi(p)}))}{\mathrm{d}t}\Big|_{t=0}\frac{\partial}{\partial u^a}\Big|_p \\ &= \frac{\mathrm{d}(p^a + tY^a(\pi(p)))}{\mathrm{d}t}\Big|_{t=0}\frac{\partial}{\partial u^a}\Big|_p = Y^a(\pi(p))\frac{\partial}{\partial u^a}\Big|_p = \Big((Y^a \circ \pi)\cdot\frac{\partial}{\partial u^a}\Big)\Big|_p, \end{aligned}$$

where equation (I.2.7) was applied. □

If $Y \in \operatorname{Sec}(E, \pi, M)$, the vector field $Y^v := v(Y) \in \Delta^v$, defined via (4.1), is called the *vertical lift of the section* Y. It is (locally) given by (4.2) in vector bundle coordinates.

Proposition 4.1. *The mapping*

$$\begin{aligned} &v\colon \operatorname{Sec}(E, \pi, M) \to \{\text{vector fields in } \Delta^v\} \\ &v\colon Y \mapsto Y^v\colon p \mapsto Y^v|_p := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(p + tY_{\pi(p)}) \end{aligned} \tag{4.3}$$

is a bijection and it and its inverse are linear mappings.

Proof. The linearity and injectivity of v follow directly from (4.1). Now we shall prove that, for each $Z \in \Delta^v$, there is a $Y \in \operatorname{Sec}(E, \pi, M)$ such that $Y^v = Z$, i.e., v is also surjective. Let $Z = Z^a\frac{\partial}{\partial u^a}$, with $\{u^I\}$ being (local) vector bundle coordinates on E and the functions Z^a being constant on the fibres of E, that is $Z^I = z^I \circ \pi$

for some functions z^I on M. Define $Y = z^a e_a$ with $\{e_a\}$ being the frame in E over M generating $\{u^a\}$. By Lemma 4.1, we have $Y^v = (z^a \circ \pi) \frac{\partial}{\partial u^a} = Z^a \frac{\partial}{\partial u^a} = Z$. The linearity of v^{-1} follows from here too. □

Consider a section ω of the bundle dual to (E, π, M) [23]. Its *vertical lift* ω_v is a 1-form on Δ^v such that, for $Z \in \Delta^v$ and $p \in E$, $\omega_v(Z)|_p = \omega(Y)|_{\pi(p)}$ for the unique section $Y \in \mathrm{Sec}(E, \pi, M)$ with $Y^v = Z$ (see Proposition 4.1), i.e., we have $\omega_v(\cdot)|_p = (\omega \circ v^{-1}(\cdot))|_{\pi(p)}$ which means that

$$\omega_v(Z) = (\omega \circ v^{-1}(Z)) \circ \pi \quad \text{or} \quad \omega_v(Y^v)|_p = \omega(Y)|_{\pi(p)} \quad (= \omega_{\pi(p)}(Y_{\pi(p)})). \tag{4.4}$$

If $\{e^a = u^a\}$ is the coframe dual to $\{e_a\}$, and $\omega = \omega_a e^a$, then in the coframe $\{\mathrm{d}u^a\}$ dual to $\{\frac{\partial}{\partial u^a}\}$, we can write (cf. (4.2))

$$\omega_v = (\omega_a \circ \pi) \mathrm{d}u^a. \tag{4.5}$$

It should be mentioned, 'vertical' lifts of vector fields or 1-forms over the base space M are generally not defined unless $E = T(M)$ or $E = T^*(M)$, respectively.[1]

A *section* Y *of* (E, π, M) and *section* ω *of the bundle dual to* (E, π, M) can be lifted *vertically* via the mappings

$$v \colon Y \mapsto Y^v \in \Delta^v \tag{4.6a}$$
$$\omega \mapsto \omega_v \tag{4.6b}$$

respectively given by (4.3) and (4.4) (see also (4.2) and (4.5)). These mappings do not require a connection and arise only from the fibre structure of the bundle space induced from the projection $\pi \colon E \to M$.

If a connection Δ^h on (E, π, M) is given, it generates *horizontal lifts of the vector fields on the base space* M and of the *one-forms on the same base space* M into respectively vector fields in Δ^h and linear mappings on the vector fields in Δ^h. Precisely, if $F \in \mathcal{X}(M)$ and $\phi \in \Lambda^1(M)$, their *horizontal lifts* are defined by the mappings[2]

$$F \mapsto F^h \in \Delta^h \quad \text{with} \quad F^h \colon p \mapsto F^h_p := (\pi_*|_{\Delta^h_p})^{-1}(F_{\pi(p)}) \qquad p \in E \tag{4.7a}$$
$$\phi \mapsto \phi_h \quad \text{with} \quad \phi_h := \phi \circ \pi_*|_{\Delta^h} \colon p \mapsto \phi_h|_p = \phi|_{\pi(p)} \circ (\pi_*|_{\Delta^h_p}). \tag{4.7b}$$

The horizontal lift ϕ_h of ϕ can also be defined alternatively via

$$\phi_h(F^h)|_p = \phi(F)|_{\pi(p)} \tag{4.8}$$

which equation is tantamount to (4.7b).

[1]Since $\pi_*(\Delta^v_p) = 0_{\pi(p)} \in T_{\pi(p)}(M)$, $p \in E$, we can say that only the zero vector field over M has vertical lifts relative to π and any vector field in Δ^v is its vertical lift. This conclusion is independent of the existence of a connection on (E, π, M) and depends only on the fibre structure of E induced by π.

[2]Alternatively, one may define $\phi'_h = \phi \circ \pi_* = \pi^*(\phi)$, which expands the domain of ϕ_h, defined by (4.7b), on the whole space $\mathcal{X}(E)$. Obviously, $\phi'_h(Z) = \phi_h(Z)$ for $Z \in \Delta^h \subseteq \mathcal{X}(E)$ and $\phi'_h(Z) = 0$ for $Z \in \mathcal{X}(E) \setminus \{X \in \Delta^h\}$.

Let $\{u^\mu = x^\mu \circ \pi, u^a\}$ be vector bundle coordinate system and $\{X_I\}$ (resp. $\{\omega^I\}$) be the adapted to it frame (resp. coframe) constructed from it according to (3.28). If $Y = Y^a e_a$, $\omega = \omega_a e^a$, $F = F^\mu \frac{\partial}{\partial x^\mu} \in \mathcal{X}(M)$, and $\phi = \phi_\mu \mathrm{d}x^\mu \in \Lambda^1(M)$, the equations (4.2) and (4.5) imply

$$Y^v = (Y^a \circ \pi) X_a \quad \omega_v = (\omega_a \circ \pi)\omega^a, \tag{4.9}$$

while from (4.7) and (3.33), one gets

$$F^h = (F^\mu \circ \pi) X_\mu \quad \phi_h = (\phi_\mu \circ \pi)\omega^\mu, \tag{4.10}$$

which agree with (3.15).

Exercise 4.1. Consider the vertical and horizontal lifts of vector fields and 1-forms in a case of a connection Δ^h on the tangent bundle $(T(M), \pi_T, M)$.

Exercise 4.2. In a case of the tangent bundle $(T(M), \pi_T, M)$ (resp. the cotangent bundle $(T^*(M), \pi_T^*, M)$) over a manifold M, any coordinate system $\{x^\mu\}$ on an open set $W_M \subseteq M$ induces natural vector bundle coordinates in the bundle space [7, Section 1.25] (see also [116, pp. 8, 43]). For the purpose, we put $e_\mu = \frac{\partial}{\partial x^\mu}$, so that $e^\mu = \mathrm{d}x^\mu$ and we get ($\lambda, \mu, \ldots = 1, \ldots, \dim M$ and $a, b = \dim M + 1, \ldots, 2\dim M$)

$$\{u^I\} = \{x^\mu \circ \pi_T, \mathrm{d}x^\nu\} \quad \text{i.e.,} \quad u^\mu = x^\mu \circ \pi_T \quad u^a = \mathrm{d}x^{a-\dim M} \tag{4.11a}$$

on $\pi_T^{-1}(W_M)$, in the tangent bundle case, and

$$\{u^I\} = \Big\{x^\mu \circ \pi_{T^*}, (\cdot)\Big(\frac{\partial}{\partial x^\nu}\Big)\Big\} \quad \text{i.e.,} \quad u^\mu = x^\mu \circ \pi_{T^*} \quad u^{\dim M + \nu} \colon \xi \mapsto \xi\Big(\frac{\partial}{\partial x^\nu}\Big) \tag{4.11b}$$

on $\pi_{T^*}^{-1}(W_M) \ni \xi$, in the cotangent bundle case. In connection with the higher order (co)tangent bundles, it is convenient the vector fibre coordinates to be denoted also as $u_1^\mu := \dot{x}^\mu := \mathrm{d}x^\mu$ in $T(M)$ and by $u_\mu^1(\cdot) = (\cdot)\Big(\frac{\partial}{\partial x^\mu}\Big)$ in $T^*(M)$. Find in the coordinates (4.11) the adapted (co)vector fields and the vertical/horizontal lifts of vector fields or 1-forms in the (co)tangent bundle case.

4.2. Linear connections on vector bundles

The most valued structures in/on vector bundles are the ones which are compatible/consistent with the linear structure of the fibres of these bundles. Since a distribution $\Delta\colon p \mapsto \Delta_p \subseteq T_p(E)$, $p \in E$, on the bundle space E of a (vector) bundle (E, π, M) cannot be considered as a linear mapping without additional hypotheses, the concept of a linear connection arises from the one of the parallel transport assigned to a connection (see Definition 3.2). (For an alternative approach, see [146, p. 42].)

Definition 4.1. A connection on a vector bundle is called *linear* if the assigned to it parallel transport is a linear mapping along every path in the base space, i.e., if the mapping (3.13) is linear for all paths $\gamma\colon [\sigma,\tau] \to M$ in the base, viz.

$$\mathsf{P}^\gamma(\rho X) = \rho \mathsf{P}^\gamma(X) \tag{4.12a}$$

$$\mathsf{P}^\gamma(X+Y) = \mathsf{P}^\gamma(X) + \mathsf{P}^\gamma(Y), \tag{4.12b}$$

where $\rho \in \mathbb{K}$ and $X, Y \in \pi^{-1}(\gamma(\sigma))$.

Remark 4.1. The equivalence between Definitions 4.1 and IV.14.3 will be established below in Subsection 4.3.

The restriction on a connection to be linear is quite severe and is described locally by

Theorem 4.1 (cf. [138, Section 5.2])**.** *Let (E, π, M) be a vector bundle, $\{u^I\}$ be vector bundle coordinate system on an open set $W \subseteq E$, and Δ^h be a connection on it described in the frame $\{X_I\}$, adapted to $\{u^I\}$, by its 2-index coefficients Γ^a_μ (see* (3.27)–(3.29)). *The connection Δ^h is linear if and only if, for each $p \in W$,*

$$\Gamma^a_\mu(p) = -\Gamma^a_{b\mu}(\pi(p))u^b(p) = -\big((\Gamma^a_{b\mu}\circ\pi)\cdot u^b\big)(p), \tag{4.13}$$

where $\Gamma^a_{b\mu}\colon \pi(W) \to \mathbb{K}$ are some functions on the set $\pi(W) \subseteq M$ and the minus sign before $\Gamma^a_{b\mu}$ in (4.13) *is conventional.*

Proof. Let $\gamma\colon [\sigma,\tau] \to \pi(W)$ be a C^1 path. Consider the parallel transport equation (3.39″a), viz.

$$\frac{\mathrm{d}\bar\gamma^a_p(t)}{\mathrm{d}t} = \Gamma^a_\mu(\bar\gamma_p(t))\dot\gamma^\mu(t), \tag{4.14}$$

where $\bar\gamma_p\colon [\sigma,\tau] \to W$ is the horizontal lift of γ through $p \in \pi^{-1}(\gamma(\sigma))$, $\bar\gamma^a := u^a \circ \bar\gamma$, and $\dot\gamma^\mu(t) = \frac{\mathrm{d}(x^\mu\circ\gamma(t))}{\mathrm{d}t} = \frac{\mathrm{d}(u^\mu\circ\bar\gamma(t))}{\mathrm{d}t}$ as $u^\mu = x^\mu \circ \pi$ for some coordinate system $\{x^\mu\}$ on $\pi(W)$.

Sufficiency. If (4.13) holds, (4.14) is transformed into

$$\frac{\mathrm{d}\bar\gamma^a_p(t)}{\mathrm{d}t} = -\Gamma^a_{b\mu}(\gamma(t))\bar\gamma^b_p(t)\dot\gamma^\mu(t), \tag{4.15}$$

which is a system of r linear first order ordinary differential equations for the r functions $\bar\gamma^{n+1}_p, \dots, \bar\gamma^{n+r}_p$. According to the general theorems of existence and uniqueness of the solutions of such systems [34], it has a unique solution

$$\bar\gamma^a_p(t) = Y^a_b(t)p^b \tag{4.16}$$

satisfying the initial condition $\bar\gamma^a_p(\sigma) = u^a(p) =: p^a$, where $Y = [Y^a_b]$ is the fundamental solution of (4.15), i.e.,

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = -[\Gamma^a_{b\mu}(\gamma(t))\dot\gamma^\mu(t)]^{n+r}_{a,b=n+1}\cdot Y(t) \qquad Y(\sigma) = \mathbb{1}_{r\times r} = [\delta^a_b]. \tag{4.17}$$

The linearity of (3.14) with respect to p follows from (4.16) for $t = \tau$.

NECESSITY. Suppose (3.13) is linear in p for all paths $\gamma\colon [\sigma,\tau] \to \pi(W)$. Then $\bar\gamma_p(t) := \mathsf{P}^{\gamma|[\sigma,t]}(p)$ is the horizontal lift of $\gamma|[\sigma,t]$ through p and (cf. (4.16)) $\bar\gamma_p^a(t) = A_b^a(\gamma(t))p^b$ for some C^1 functions $A_b^a\colon \pi(W) \to \mathbb{K}$. The substitution of this equation in (4.14) results into

$$\frac{\partial A_b^a(x)}{\partial x^\mu}\Big|_{x=\gamma(t)=\pi(\bar\gamma_p(t))} \cdot \dot\gamma^\mu(t)p^b = \Gamma_\mu^a(\bar\gamma_p(t))\dot\gamma^\mu(t).$$

Since $\gamma\colon [\sigma,\tau] \to M$, we get equation (4.13) from here, for $t=\sigma$, with $\Gamma^a_{b\mu}(x) = -\frac{\partial A_b^a(x)}{\partial x^\mu}$ for $x \in \pi(U)$. □

The functions $\Gamma^a_{b\mu}\colon \pi(W) \to \mathbb{K}$ will be referred as the (*local*) 3-*index coefficients* of the linear connection Δ^h in the adapted frame $\{X_I\}$. If there is no risk to confuse them with the 2-index coefficients $\Gamma^a_\mu\colon W \to \mathbb{K}$, they will be called simply coefficients of Δ^h. Note, the 2-index coefficients of a linear connections are defined on (a subset of) the bundle space E, while the 3-index ones are define on (a subset of) the base space M. The equation (4.15) is simply the *parallel transport equation* for the linear connection considered.

Example 4.1. Since u^a is replaced by $u_1^\mu = \mathrm{d}x^\mu$ in the tangent bundle case (see Exercise 4.4), the linear connections in $(T(M),\pi_T,M)$ have 2-index coefficients of the form

$$\Gamma^\nu_\mu = -(\Gamma^\nu_{\lambda\mu}\circ\pi_T)\cdot u_1^\lambda = -(\Gamma^\nu_{\lambda\mu}\circ\pi_T)\cdot \mathrm{d}x^\lambda \tag{4.18}$$

and, consequently, they can be regarded as 1-forms on M.

Consider a linear connection Δ^h on a vector bundle (E,π,M). Let Γ^a_μ and $\Gamma^a_{b\mu}$ be its 2- and 3-index coefficients, respectively, in a frame $\{X_I\}$ adapted to vector bundle coordinates $\{u^I\}$.

Corollary 4.1. *The* 3-*index coefficients* $\Gamma^a_{b\mu}$ *of a linear connection* Δ^h *uniquely define the fibre coefficients of* Δ^h *in* $\{X_I\}$ *by*

$$^\circ\Gamma^a_{b\mu} = \Gamma^a_{b\mu}\circ\pi = \pi^*(\Gamma^a_{b\mu}), \tag{4.19}$$

that is the fibre coefficients of a linear connection are equal to the 3-*index ones lifted by the projection* π.

Proof. Since (3.28a) and (4.13) imply

$$[X_\mu, X_b]_\_ = (\Gamma^a_{b\mu}\circ\pi)X_a, \tag{4.20}$$

the equation (4.19) follows from (3.22a) and (3.23a) or (3.37b) and (4.13). □

As the vector bundle coordinates u^I are, by definition, linear on the fibres of the bundle, the general change of such coordinates is

$$\{u^\mu, u^a\} \mapsto \{\tilde u^\mu = \tilde x^\mu\circ\pi, \tilde u^a = (B^a_b\circ\pi)\cdot u^b\}, \tag{4.21}$$

with $B = [B^a_b]$ being a non-degenerate matrix-valued function on $\pi(W)$. The change (4.21) entails the following transformation of the corresponding adapted frames

$$\{X_\mu, X_a\} \mapsto \{\tilde{X}_\mu = (B^\nu_\mu \circ \pi) \cdot X_\nu, \tilde{X}_a = (B^b_a \circ \pi) \cdot X_b\}, \tag{4.22}$$

where $[B^\nu_\mu] = \left[\frac{\partial x^\nu}{\partial \tilde{x}^\mu}\right]$ is a non-degenerate matrix-valued function on the intersection of the domains of $\{x^\mu\}$ and $\{\tilde{x}^\mu\}$. (In (4.22) we have used that $\frac{\partial u^\nu}{\partial \tilde{u}^\mu}\big|_p = \frac{\partial(x^\nu \circ \pi)}{\partial(\tilde{x}^\mu \circ \pi)}\big|_p = \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\big|_{\pi(p)}$.)

Proposition 4.2. *The change* (4.21) *implies the following transformations of the* 3*-index coefficients of the linear connection:*

$$\Gamma^a_{b\mu} \mapsto \tilde{\Gamma}^a_{b\mu} = B^\nu_\mu \Big(B^a_d \Gamma^d_{c\nu} - \frac{\partial B^a_c}{\partial x^\nu} \Big) (B^{-1})^c_b, \tag{4.23}$$

Proof. Apply (4.22), (3.32) and (4.13). Alternatively, the same transformation law follows also from equations (3.24a) and (4.19). □

If we introduce the matrix-valued functions $\Gamma_\mu := [\Gamma^a_{b\mu}]$ and $\tilde{\Gamma}_\mu := [\tilde{\Gamma}^a_{b\mu}]$ on M, we can rewrite (4.23) as (cf. (IV.14.25) with $N = M$, $g = \mathsf{id}_M$, and $A = B^{-1}$)

$$\begin{aligned} \Gamma_\mu \mapsto \tilde{\Gamma}_\mu &= B^\nu_\mu \left(B \cdot \Gamma_\nu - \frac{\partial B}{\partial x^\nu} \right) \cdot B^{-1} \\ &= B^\nu_\mu B \cdot \left(\Gamma_\nu \cdot B^{-1} + \frac{\partial B^{-1}}{\partial x^\nu} \right). \end{aligned} \tag{4.23'}$$

This relation corresponds to (3.25) with $[A^a_b] = B^{-1} \circ \pi$ (see also (4.19)) as the frame $\{e_a \colon M \to E\}$, relative to which the vector fibre coordinate system $\{u^a\}$ is defined ($E \ni p \mapsto u^a(p)$ with $p = u^a(p) e_a(\pi(p))$), transforms via the matrix inverse to $B \circ \pi$.

Let E be a C^2 manifold and Δ^h a C^1 connection on (E, π, M). Substituting (4.13) into (3.37a), we get the fibre components of the curvature of a linear connection as

$$R^a_{\mu\nu} = -(R^a_{b\mu\nu} \circ \pi) \cdot u^b \tag{4.24}$$

where

$$R^a_{b\mu\nu} := \frac{\partial}{\partial x^\mu}(\Gamma^a_{b\nu}) - \frac{\partial}{\partial x^\nu}(\Gamma^a_{b\mu}) - \Gamma^c_{b\mu}\Gamma^a_{c\nu} + \Gamma^c_{b\nu}\Gamma^a_{c\mu}, \tag{4.25}$$

or in a matrix form

$$R_{\mu\nu} := [R^a_{b\mu\nu}] = \frac{\partial \Gamma_\nu}{\partial x^\mu} - \frac{\partial \Gamma_\mu}{\partial x^\nu} - \Gamma_\nu \cdot \Gamma_\mu + \Gamma_\mu \cdot \Gamma_\nu, \tag{4.25'}$$

are the *components of the curvature operator* (see below (4.42)). As a result of (3.22b) and (4.24), the transformation (4.21) entails the change

$$R^a_{b\mu\nu} \mapsto \tilde{R}^a_{b\mu\nu} = B^\lambda_\mu B^\varrho_\nu (B^{-1})^a_c B^d_b R^c_{d\lambda\varrho}, \tag{4.26}$$

or in a matrix form

$$R_{\mu\nu} \mapsto \tilde{R}_{\mu\nu} = B^\lambda_\mu B^\varrho_\nu B^{-1} \cdot R_{\lambda\varrho} \cdot B, \tag{4.26$'$}$$

which corresponds to (3.24b) with $A = B^{-1} \circ \pi$ (see also (4.24)).

4.3. Covariant derivatives in vector bundles

A possibility for introduction of differentiation in vector bundles, endowed with connection, comes from the vector space structure of their fibres. This operation can be defined in many independent ways, leading to identical results. In one of them is involved the parallel transport induced by the connection: the idea is the values of sections to be parallel transported (along paths in the base) into a single fibre (over the paths), where one can work with the 'transported' sections as with functions with values in a vector space. Other method uses the existence of natural vertical lifts of sections of the bundle and horizontal lifts of the vector fields on the base space; since the both lifts are vector fields on the bundle space, their commutator (or Lie derivative relative to each other) is well defined and can be used as a prototype of some sort of differentiation. We shall realize below the second method mentioned, which seems is first introduced in a rudimentary form in [138, p. 31].[3,4] The first way, as well as the axiomatic approach, for introduction of covariant derivatives will be obtained as theorems in what follows.

Let (E, π, M) be a vector bundle on which a *linear* connection Δ^h is defined. Suppose $\{E_a\}$ is a frame in E to which vector fibre coordinates u^a are associated and $\{u^I\}$ is the corresponding vector bundle coordinate system. The frame adapted to $\{u^I\}$ will be denoted by $\{X_I\}$ and $\{\omega^I\}$ will be its dual coframe, both defined by (3.28) through the (2-index) coefficients Γ^a_μ of Δ^h.

Let $\hat{Z} = \hat{Z}^a X_a \in \Delta^v$ and $\bar{Z} = \bar{Z}^\mu X_\mu \in \Delta^h$ be respectively vertical and horizontal vector fields on E. Define a mapping $\hat{\nabla}\colon \Delta^v \oplus \Delta^h = T(E) \to \mathcal{X}(E)$ such that[5]

$$\hat{\nabla}\colon (\hat{Z}, \bar{Z}) \mapsto \hat{\nabla}_{\bar{Z}}(\hat{Z}) := \Pi(\mathcal{L}_{\bar{Z}} \hat{Z}) \in \mathcal{X}(E), \tag{4.27}$$

---

[3] In [138, p. 31] is proved that, for $F = \frac{\partial}{\partial x^\mu}$ and in our notation, the ath component of the right hand sides of (4.35) and of (4.36) coincide in a frame $\{E_a\}$ in E.

[4] An equivalent alternative approach is realized in [23, Sections 2.49–2.52].

[5] The idea of the construction (4.27) is to drag the vertical vector field $\hat{Z}$ along the horizontal one $\bar{Z}$, which will give a vector field in $\mathcal{X}(E)$, and then to project the result onto the *vertical* distribution Δ^v by means of the invariant projection operator $\Pi = X_a \otimes \omega^a \colon \mathcal{X}(E) \to \mathcal{X}(E)$. Evidently $\Pi^2 = \Pi \circ \Pi = \Pi$ and Π is the unit (identity) tensor in the tensor product of vector fields and 1-forms on E.

where the (1,1) tensor field

$$\Pi := \sum_a X_a \otimes \omega^a \tag{4.28}$$

is considered as a operator on the set of vector fields on E. Since (see (2.1b) and (III.2.1))

$$\mathcal{L}_{\bar{Z}}\hat{Z} = \bar{Z}(\hat{Z}^a)X_a + \bar{Z}^\mu \hat{Z}^a [X_\mu, X_a]_{\_}$$

and $\omega^a(X_\mu) = \delta^a_\mu = 0$, from (3.36), (3.37b) and (4.27), we obtain

$$\hat{\nabla}_{\bar{Z}}\hat{Z} = \bar{Z}^\mu\{X_\mu(\hat{Z}^a) - \hat{Z}^b \partial_b(\Gamma^a_\mu)\}X_a, \tag{4.29}$$

from where one can prove, via direct calculation, the independence of $\hat{\nabla}_{\bar{Z}}\hat{Z}$ of the particular (co)frame used. For any particular point $p \in E$, the value of the vector field (4.29) at p is a vertical vector, $(\hat{\nabla}_{\bar{Z}}\hat{Z})|_p \in \Delta^v_p$, but generally $\hat{\nabla}_{\bar{Z}}\hat{Z}$ is not a vertical vector field. The reason is that a vertical vector field on E is a mapping $V\colon p \mapsto V_p \in \Delta^v_p := T_p(\pi^{-1}(\pi(p)) := T_{\imath(p)}(\pi^{-1}(\pi(p)) = (\pi_*|_p)^{-1}(0_{\pi(p)})$, with $\imath\colon \pi^{-1}(p) \to E$ being the inclusion mapping and $0_{\pi(p)} \in T_{\pi(p)}(M)$ being the zero vector, due to which V_p, and hence its components, must depend only on $\pi(p) \in M$. Therefore, we have

$$\hat{\nabla}_{\bar{Z}}\hat{Z} \in \Delta^v \iff \partial_b(\Gamma^a_\mu) = -\Gamma^a_{b\mu}\circ\pi \iff \Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot u^b + G^a_\mu \circ \pi, \tag{4.30}$$

for some functions $\Gamma^a_{b\mu}, G^a_\mu\colon M \to \mathbb{K}$. Thus $\hat{\nabla}_{\bar{Z}}\hat{Z}$ is a vertical vector field if and only if the 2-index coefficients Γ^a_μ in $\{X_I\}$ of the connection Δ^h are of the form

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot u^b + G^a_\mu\circ\pi. \tag{4.31}$$

This equality selects the set of *affine connections* among all connections (see Subsection 4.4 below);[6] in particular, of this type are the linear connections for which $G^a_\mu = 0$ and $\Gamma^a_{b\mu}$ are their 3-index coefficients (see (4.13)). For connections with 2-index coefficients (4.31), equation (4.29) reduces to

$$\hat{\nabla}_{\bar{Z}}\hat{Z} = \bar{Z}^\mu\{X_\mu(\hat{Z}^a) + \hat{Z}^b(\Gamma^a_{b\mu}\circ\pi)\}X_a \in \Delta^v. \tag{4.32}$$

Now the idea of introduction of a covariant derivative of a section $Y \in \operatorname{Sec}(E,\pi,M)$ along a vector field $F \in \mathcal{X}(M)$ is to 'lower' the operator $\hat{\nabla}$ from $T(E)$ to $T(M)$.

Definition 4.2. A *covariant derivative* or *covariant derivative operator*, associated to a linear (or affine) connection Δ^h on a vector bundle (E,π,M), is a mapping

$$\begin{aligned}\nabla\colon & \mathcal{X}(M)\times \operatorname{Sec}^1(E,\pi,M) \to \operatorname{Sec}^0(E,\pi,M)\\ \nabla\colon & (F,Y) \mapsto \nabla_F Y\end{aligned} \tag{4.33}$$

[6] Usually the affine connections are defined on affine bundles [11, 146].

such that, for $F \in \mathcal{X}(M)$ and $Y \in \operatorname{Sec}^1(E,\pi,M)$, $\nabla_F Y$ is the unique section of (E,π,M) whose vertical lift is $\hat{\nabla}_{F^h} Y^v$, with $\hat{\nabla}$ defined by (4.27) (or (4.32)), viz.

$$(\nabla_F Y)^v := \hat{\nabla}_{F^h} Y^v \tag{4.34}$$

or

$$\nabla_F Y = v^{-1} \circ \hat{\nabla}_{(\pi_*|_{\Delta^h})^{-1}(F)} (v(Y)) = (v^{-1} \circ \hat{\nabla}_{(\pi_*|_{\Delta^h})^{-1}(F)} \circ v)(Y), \tag{4.35}$$

where $F^h \in \Delta^h$ and $Y^v \in \Delta^v$ are respectively the horizontal and vertical lifts of F and Y.

Remark 4.2. Definition 4.2 and the rest of this subsection are valid also for affine connections for which (4.31) holds, not only for the linear ones. For some details, see Subsection 4.4.

Proposition 4.3. *Let $\{E_a\}$ be a frame in E and $\{x^\mu\}$ local coordinate system on M. If $Y = Y^a E_a \in \operatorname{Sec}^1(E,\pi,M)$ and $F = F^\mu \frac{\partial}{\partial x^\mu} \in \mathcal{X}(M)$, then we have the explicit local expression*

$$\nabla_F Y = F^\mu \Big(\frac{\partial Y^a}{\partial x^\mu} + \Gamma^a_{b\mu} Y^b \Big) E_a. \tag{4.36}$$

Proof. Apply (4.34), (4.9), (4.10), (4.32), and (4.2). □

Proposition 4.4. *Let Δ^h be a linear connection on (E,π,M) and P be the generated by it parallel transport. Let $x \in M$, $\gamma\colon [\sigma,\tau] \to M$, $\gamma(t_0) = x$ for some $t_0 \in [\sigma,\tau]$, and $\dot\gamma(t_0) = F_x$, i.e., γ to be the integral path of $F \in \mathfrak{X}(M)$ through x. Then*

$$(\nabla_F Y)|_x = \lim_{s\to t_0} \frac{P^\gamma_{s\to t_0}(Y_{\gamma(s)}) - Y_{\gamma(t_0)}}{s - t_0} = \lim_{\varepsilon\to 0} \frac{P^\gamma_{t_0+\varepsilon\to t_0}(Y_{\gamma(t_0+\varepsilon)}) - Y_{\gamma(t_0)}}{\varepsilon}, \tag{4.37}$$

where $Y \in \operatorname{Sec}^1(E,\pi,M)$ and

$$P^\gamma_{s\to t} := \begin{cases} \mathsf{P}^{\gamma|[s,t]} & \text{for } s \le t \\ \big(\mathsf{P}^{\gamma|[t,s]}\big)^{-1} & \text{for } s \ge t. \end{cases} \tag{4.38}$$

Proof. Use Definition 3.2 and apply the parallel transport equation (4.15) with initial value $\bar\gamma_{Y_{\gamma(s)}}(s) = Y_{\gamma(s)}$ at the point $t = s \in [\sigma,\tau]$. □

Remark 4.3. The mapping $P\colon \gamma \mapsto P^\gamma\colon (s,t) \mapsto P^\gamma_{s\to t}$ is a (parallel) transport along paths – see Proposition 8.6 on page 395.

By Proposition 4.4, the equation (4.37) can be used as an equivalent definition of a covariant derivative associated with a linear connection.

Proposition 4.5. *Let $F, G \in \mathcal{X}(M)$, $Y, Z \in \operatorname{Sec}^1(E,\pi,M)$, and $f\colon M \to \mathbb{K}$ be a C^1 function. Then:*

$$\nabla_{F+G} Y = \nabla_F Y + \nabla_G Y \tag{4.39a}$$
$$\nabla_{fF} Y = f \nabla_F Y \tag{4.39b}$$
$$\nabla_F (Y+Z) = \nabla_F Y + \nabla_F Z \tag{4.39c}$$
$$\nabla_F (fY) = F(f)\cdot Y + f\cdot \nabla_F Y. \tag{4.39d}$$

Proof. Apply (4.36). □

Proposition 4.6. *If a mapping* (4.33) *satisfies* (4.39), *there exists a unique linear connection Δ^h, the assigned to which covariant derivative is exactly ∇.*

Proof. Define local functions $\Gamma^a_{b\mu}$ on M, called *components* of ∇, by the decomposition

$$\nabla_{\frac{\partial}{\partial x^\mu}} E_b =: \Gamma^a_{b\mu} E_a. \tag{4.40}$$

A simple verification proves that they transform according to (4.23) and hence the quantities (4.13) transform by (3.32). Proposition 3.3 ensures the existence of a unique linear connection whose 2-index (3-index) coefficients are Γ^a_μ ($\Gamma^a_{b\mu}$). Thus the covariant derivative of $Y \in \operatorname{Sec}(E,\pi,M)$ relative to $F \in \mathcal{X}(M)$ is given by the right-hand side of (4.36). On another hand, (4.39) entail (4.36), with $\Gamma^a_{b\mu}$ defined by (4.40), so that ∇ is exactly the covariant derivative operator assigned to the connection with 3-index coefficients $\Gamma^a_{b\mu}$. □

Consequently, equations (4.39) and (4.40) provide a third equivalent definition of a covariant derivative (covariant derivative operator). Moreover, since Proposition 4.6 establishes a bijective correspondence between linear connections and operators (4.33) satisfying (4.39), quite often such operators are called linear connections.[7] As it is clear from the proof of Proposition 4.6, the bijection between linear connections and covariant derivative operators is locally given by the coincidence of their (3-index) coefficients and components, respectively.

The four equations (4.39) are identical with conditions (i)–(iv) in Definition IV.14.7 for $N = M$ and $g = \mathsf{id}_M$. This fact means that a covariant derivative according to Definition 4.2 is a covariant derivative in (E,π,M) according to Definition IV.14.7 and *vice versa*, so that the both definitions are equivalent (for $N = M$ and $g = \mathsf{id}_M$ in the latter one). Combining that result with Conclusion IV.14.1, we see that the Definitions 4.1 and IV.14.3 of a linear connection on a vector bundle are equivalent (see also Definition 3.1).

Exercise 4.3. A C^1 section $\omega = \omega_a E^a$ of the bundle dual to (E,π,M) can be differentiated covariantly similarly as the sections of (E,π,M). Show that the corresponding operator, say ∇^*, can equivalently be defined by (the 'Leibnitz rule')

$$(\nabla^*_F \omega)(Y) = F(\omega(Y)) - \omega(\nabla_F Y) \tag{4.41}$$

[7] See also [23, Sections 2.15 and 2.52].

and locally is valid the equation

$$\nabla^*_F \omega = F^\mu \Big(\frac{\partial \omega_a}{\partial x^\mu} - \Gamma^b_{a\mu} \omega_b \Big) E^a.$$

Equipped with the covariant derivative ∇ assigned to a C^1 linear connection Δ^h, we define the *curvature operator* of Δ^h (or ∇) by

$$\begin{aligned} &R\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \operatorname{End}(\operatorname{Sec}(E,\pi,M)) \\ &R\colon (F,G) \mapsto R(F,G) := \nabla_F \circ \nabla_G - \nabla_G \circ \nabla_F - \nabla_{[F,G]_-}, \end{aligned} \tag{4.42}$$

with End(...) denoting the set of endomorphisms of (...).

Exercise 4.4. Prove that locally

$$(R(F,G))(Y) = (R^a_{b\mu\nu} Y^b F^\mu G^\nu) E_a, \tag{4.43}$$

where the functions $R^a_{b\mu\nu}\colon M \to \mathbb{K}$, called the *components* of the curvature operator R in the pair of frames $\big(\{\frac{\partial}{\partial x^\mu}\}, \{E_a\}\big)$, are defined by

$$R\Big(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \Big)(E_b) =: R^a_{b\mu\nu} E_a \tag{4.44}$$

and are explicitly expressed through the coefficients of ∇(= 3-index coefficients of Δ^h) via (4.25).

A linear connection or covariant derivative operator is called *flat* or *curvature free* if

$$R = 0 \qquad (\iff R^a_{b\mu\nu} = 0). \tag{4.45}$$

Obviously, the flatness of Δ^h or ∇ is a necessary and sufficient condition for the (local) integrability of the horizontal distribution $\Delta^h\colon p \mapsto \Delta^h_p \subseteq T_p(E)$, $p \in E$ (see (3.23b) and (4.24)).

4.4. Affine connections

In Subsection 4.3, we met a class of connections on a vector bundle whose local 2-index coefficients have the form (see (4.31))

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi) \cdot u^b + G^a_\mu \circ \pi \tag{4.46}$$

in the frame $\{X_I\}$ adapted to a vector bundle coordinate system $\{u^I\}$. From $\partial_b \Gamma^a_\mu = -\Gamma^a_{b\mu}$ and (3.32), one derives that the functions $\Gamma^a_{b\mu}$ in (4.46) transform according to (4.23), viz.

$$\Gamma^a_{b\mu} \mapsto \tilde{\Gamma}^a_{b\mu} = B^\nu_\mu \Big(B^a_d \Gamma^d_{c\nu} - \frac{\partial B^a_c}{\partial x^\nu} \Big) (B^{-1})^c_b \tag{4.47}$$

when the vector bundle coordinates or adapted frames undergo the change (4.21) or (4.22), respectively. Thus, combining (3.32), (4.47) and (4.46), we see that (4.21) or (4.22) implies the transition

$$G^a_\mu \mapsto \tilde{G}^a_\mu = B^a_b G^b_\nu B^\nu_\mu. \tag{4.48}$$

Consequently, the functions $\Gamma^a_{b\mu}$ in (4.46) are 3-index coefficients of a linear connection, while G^a_μ in it are the components of a linear mapping $G\colon \mathcal{X}(M) \to \operatorname{End}(\operatorname{Sec}((E,\pi,M)^*))$ such that $G\colon F \mapsto G(F)\colon \omega \mapsto (G(F))(\omega)$, for $F \in \mathcal{X}(M)$ and a section ω of the bundle $(E,\pi,M)^*$ dual to (E,π,M), and $\big(G\big(\frac{\partial}{\partial x^\mu}\big)\big)(E^a) = G^a_\mu$. The invariant description of the connections with local 2-index coefficients of the type (4.46) is as follows.

Definition 4.3. A connection on a vector bundle is termed *affine connection* if the assigned to it parallel transport P is an affine mapping along all paths $\gamma\colon [\sigma,\tau] \to M$ in the base space, i.e.,

$$\mathsf{P}^\gamma(\rho X) = \rho \mathsf{P}^\gamma(X) + (1-\rho)\mathsf{P}^\gamma(\mathbf{0}) \tag{4.49a}$$

$$\mathsf{P}^\gamma(X+Y) = \mathsf{P}^\gamma(X) + \mathsf{P}^\gamma(Y) - \mathsf{P}^\gamma(\mathbf{0}), \tag{4.49b}$$

where $\rho \in \mathbb{K}$, $X, Y \in \pi^{-1}(\gamma(\sigma))$, and $\mathbf{0}$ is the zero vector in the fibre $\pi^{-1}(\gamma(\sigma))$, which is a $\mathbb{K}$-vector space.

An affine connection for which $\mathsf{P}^\gamma(\mathbf{0})$ is the zero vector in $\pi^{-1}(\gamma(\tau))$ is a linear connection and *vice versa* – see Definitions 4.1 and 4.3.

Theorem 4.2. *Let (E,π,M) be a vector bundle, $\{u^I\}$ be vector bundle coordinate system over an open set $U \subseteq E$, and Δ^h be a connection on it with* 2*-index coefficients Γ^a_μ in the frame $\{X_I\}$ adapted to $\{u^I\}$. The connection Δ^h is an affine connection if and only if equation* (4.46) *holds for some functions $\Gamma^a_{b\mu}, G^a_\mu\colon \pi(U) \to \mathbb{K}$.*

Proof (*cf. the proof of Theorem* 4.1). Take a C^1 path $\gamma\colon [\sigma,\tau] \to \pi(U)$ and consider the parallel transport equation (3.39″a), viz.

$$\frac{\mathrm{d}\bar{\gamma}^a_p(t)}{\mathrm{d}t} = \Gamma^a_\mu(\bar{\gamma}_p(t))\dot{\gamma}^\mu(t), \tag{4.50}$$

where $\bar{\gamma}_p\colon [\sigma,\tau] \to U$ is the horizontal lift of γ through $p \in \pi^{-1}(\gamma(\sigma))$, $\bar{\gamma}^a := u^a \circ \bar{\gamma}$, and $\dot{\gamma}^\mu(t) = \frac{\mathrm{d}(x^\mu \circ \gamma(t))}{\mathrm{d}t} = \frac{\mathrm{d}(u^\mu \circ \bar{\gamma}(t))}{\mathrm{d}t}$ as $u^\mu = x^\mu \circ \pi$ for some coordinates $\{x^\mu\}$ on $\pi(U)$.

SUFFICIENCY. If (4.46) holds, equation (4.50) can be transformed into

$$\frac{\mathrm{d}\bar{\gamma}^a_p(t)}{\mathrm{d}t} = -\Gamma^a_{b\mu}(\gamma(t))\bar{\gamma}^b_p(t)\dot{\gamma}^\mu(t) + G^a_\mu(\gamma(t))\dot{\gamma}^\mu(t), \tag{4.51}$$

which is a system of r linear inhomogeneous first order ordinary differential equations for the r functions $\bar{\gamma}^{n+1}_p, \dots, \bar{\gamma}^{n+r}_p$. According to the general theorems of existence and uniqueness of the solutions of such systems [34], it has a unique solution

$$\bar{\gamma}_p^a(t) = Y_b^a(t)p^b + y^a(t) \tag{4.52}$$

satisfying the initial condition $\bar{\gamma}_p^a(\sigma) = u^a(p) =: p^a$, where $Y = [Y_b^a]$ is the fundamental solution of (4.15) (see (4.17)) and $y^a(t)$ is the solution of (4.51) with $y^a(t)$ for $\bar{\gamma}_p^a(t)$ satisfying the initial condition $y^a(\sigma) = 0$. The affinity of (3.13) in p, i.e., (4.49), follows from (4.52) for $t = \tau$.

NECESSITY. Suppose (3.13) is affine in p for all paths $\gamma\colon [\sigma, \tau] \to \pi(U)$. Then $\bar{\gamma}_p(t) := \mathsf{P}^{\gamma|[\sigma,t]}(p)$ is the horizontal lift of $\gamma|[\sigma, t]$ through p and (cf. (4.52)) $\bar{\gamma}_p^a(t) = A_b^a(\gamma(t))p^b + A^a(\gamma(t))$ for some C^1 functions $A_b^a, A^a\colon \pi(U) \to \mathbb{K}$. The substitution of this equation in (4.50) results into

$$\frac{\partial A_b^a(x)}{\partial x^\mu}\bigg|_{x=\gamma(t)=\pi(\bar{\gamma}_p(t))} \cdot \dot{\gamma}^\mu(t)p^b + \frac{\partial A^a(x)}{\partial x^\mu}\bigg|_{x=\gamma(t)=\pi(\bar{\gamma}_p(t))} \cdot \dot{\gamma}^\mu(t). = \Gamma_\mu^a(\bar{\gamma}_p(t))\dot{\gamma}^\mu(t).$$

Since $\gamma\colon [\sigma, \tau] \to M$, we get equation (4.46) from here, for $t = \sigma$, with $\Gamma_{b\mu}^a(x) = -\frac{\partial A_b^a(x)}{\partial x^\mu}$ and $G_\mu^a(x) = \frac{\partial A^a(x)}{\partial x^\mu}$ for $x \in \pi(U)$. □

Proposition 4.7. *There is a bijective mapping α between the sets of affine connections and of pairs (∇, G) of a linear connection ∇ and a linear mapping $G\colon \mathcal{X}(M) \to \mathrm{End}(\mathrm{Sec}((E, \pi, M)^*))$.*

Proof. If ${}^A\Delta^h$ is an affine connection with 2-index coefficients given by (4.46) (see Theorem 4.2), then (see the discussion after equation (4.46)) to it corresponds the pair $\alpha({}^A\Delta^h) := ({}^L\Delta^h, G)$ of a linear connection, with 3-index coefficients $\Gamma_{b\mu}^a$ and linear mapping $G\colon \mathcal{X}(M) \to \mathrm{End}(\mathrm{Sec}((E, \pi, M)^*))$, with components G_μ^a. Conversely, to a pair $({}^L\Delta^h, G)$, locally described via the 3-index coefficients $\Gamma_{b\mu}^a$ of ${}^L\Delta^h$ and components G_μ^a of G, there corresponds an affine connection ${}^A\Delta^h = \alpha^{-1}({}^L\Delta^h, G)$ with 2-index coefficients given by (4.46). □

In Subsection 4.3, it was demonstrated that covariant derivatives can be introduced for affine connections, not only for linear ones.

Proposition 4.8. *The covariant derivative for an affine connection ${}^A\Delta^h$ coincides with the one for the linear connection ${}^L\Delta^h$ given via $\alpha({}^A\Delta^h) = ({}^L\Delta^h, G)$ with α defined in the proof of Proposition* 4.7.

Proof. Apply (4.29)–(4.36). □

If a linear connection ${}^L\Delta^h$ and an affine one ${}^A\Delta^h$ are connected by $\alpha({}^A\Delta^h) = ({}^L\Delta^h, G)$ for some G, then some of their characteristics coincide; e.g., such are their fibre coefficients (see (3.37b), (4.46) and (4.13)) and all quantities expressed via the corresponding to them (identical) covariant derivatives. However, quantities, containing (depending on) partial derivatives relative to the basic coordinates u^μ, are generally different for those connections. For instance, if ${}^A R_{\mu\nu}^a$ and ${}^L R_{\mu\nu}^a$

are the fibre components of the curvatures of ${}^{A}\Delta^{h}$ and ${}^{L}\Delta^{h}$, respectively, then, by (3.37a) and (4.46), we have

$$ {}^{A}R^{a}_{\mu\nu} = -({}^{L}R^{a}_{b\mu\nu}\circ\pi)\cdot u^{b} - T^{a}_{\mu\nu}\circ\pi \tag{4.53} $$

$$ {}^{L}R^{a}_{\mu\nu} = -({}^{L}R^{a}_{b\mu\nu}\circ\pi)\cdot u^{b} \tag{4.54} $$

where (see (4.25))

$$ {}^{L}R^{a}_{b\mu\nu} := \frac{\partial}{\partial x^{\mu}}(\Gamma^{a}_{b\nu}) - \frac{\partial}{\partial x^{\nu}}(\Gamma^{a}_{b\mu}) - \Gamma^{c}_{b\mu}\Gamma^{a}_{c\nu} + \Gamma^{c}_{b\nu}\Gamma^{a}_{c\mu}, \tag{4.55} $$

$$ T^{a}_{\mu\nu} := -\frac{\partial}{\partial x^{\mu}}(G^{a}_{\nu}) + \frac{\partial}{\partial x^{\nu}}(G^{a}_{\mu}) + \Gamma^{a}_{c\nu}G^{c}_{\mu} - \Gamma^{a}_{c\mu}G^{c}_{\nu} \tag{4.56} $$

and the functions $T^{a}_{\mu\nu}$ have a sense of components of the torsion of ${}^{L}\Delta^{h}$ relative to G [146, pp. 42, 46].

Thus, in general, the affine connections and linear connections are essentially different. However, they imply identical theories of covariant derivatives.

If, for some reason, the linear mapping G is fixed, then the set of linear connections $\{{}^{L}\Delta^{h}\}$ can be identified with the subset $\{\alpha^{-1}({}^{L}\Delta^{h}, G)\}$ of the set of affine connections $\{{}^{A}\Delta^{h}\}$.

Example 4.2. We shall exemplify the above material in a case of the tangent bundle $(T(M), \pi_T, M)$ over a manifold M. Using the base indices $\mu, \nu, \ldots$ for the fibre ones $a, b, \ldots$ according to the rule $a \mapsto \mu = a - \dim M$, we rewrite (4.46) as

$$ \Gamma^{\mu}_{\nu} = -(\Gamma^{\mu}_{\lambda\nu}\circ\pi_T)\cdot u^{\lambda}_{1} + G^{\mu}_{\nu}\circ\pi_T. \tag{4.57} $$

Now the affine connections on $(T(M), \pi_T, M)$ are the *generalized affine connections on M* [11, Chapter III, § 3]. The choice of G via

$$ G^{\mu}_{\nu}\colon M \to \delta^{\mu}_{\nu}, \tag{4.58} $$

which corresponds to the identical transformation of the spaces tangent to M, singles out the set of *affine connections on M*– see [11, Chapter III, § 3] or [23, pp. 103–105] – (known also as *Cartan connections on M* [146, p. 46]) whose 2-index coefficients have the form (see (4.57), (4.11a) and (4.58))

$$ \Gamma^{\mu}_{\nu} = -(\Gamma^{\mu}_{\lambda\nu}\circ\pi_T)\cdot \mathrm{d}x^{\lambda} + \delta^{\mu}_{\nu}. \tag{4.59} $$

Combining this example with Proposition 4.7, we derive

Proposition 4.9 (cf. [11, Chapter III, § 3, Theorem 3.3])**.** *There is a bijective correspondence between the sets of linear connections and of affine ones on a manifold.*

Often the terms “linear connection” and “affine connection” on a manifold are used as synonyms, due to the last result.

5. General (co)frames

Until now two special kinds of local (co)frames in the (co)tangent bundle to the bundle space of a bundle were employed, viz. the natural holonomic ones, induced by some local coordinates, and the adapted (co)frames determined by local coordinates and a connection on the bundle. The present section is devoted to (re)formulation of some important results and formulae in arbitrary (co)frames, which in particular can be natural or adapted (if a connection is presented) ones.

Let (E, π, M) be a C^2 bundle and $\{e_I\}$ a (local) frame in $T(E)$. The components C^K_{IJ} of the anholonomicity object of $\{e_I\}$ are defined by (3.19) and a change

$$\{e_I\} \mapsto \{\bar{e}_I = B^J_I e_J\} \tag{5.1}$$

with a non-degenerate matrix-valued function $B = [B^J_I]^{n+r}_{I,J=1}$ entails (see (2.7))

$$C^K_{IJ} \mapsto \bar{C}^K_{IJ} = (B^{-1})^K_L \big(B^M_I e_M(B^L_J) - B^M_J e_M(B^L_I) + B^M_I B^N_J C^L_{MN} \big). \tag{5.2}$$

Let a connection Δ^h on (E, π, M) be given. If $\{e_I\}$ is a specialized frame for Δ^h (see Subsection 3.2), then the set $\{C^K_{IJ}\}$ is naturally divided into the six groups (3.20). The value of that division is in its invariance with respect to the class of specialized frames, which means that, if $\{\bar{e}_I\}$ is also a specialized frame, then the transformed components of the elements of each group are functions only in the elements of the non-transformed components of the same group – see (3.24), (3.21), and (2.7).

Exercise 5.1. By means of (5.1), prove that, if a division like (3.20) holds in a frame $\{e_I\}$, then it holds in $\{\bar{e}_I\}$ if and only if the matrix-valued function B is of the form (3.16).

In particular, we cannot talk about fibre coefficients of Δ^h and of fibre components of the curvature of Δ^h in frames more general than the specialized ones as in that case the transformation (5.1), with $\{e_I\}$ (resp. $\{\bar{e}_I\}$) being a specialized (resp. non-specialized) frame, will mix, for instance, the fibre coefficients and the curvature's fibre components of Δ^h in $\{\bar{e}_I\}$ – see (5.2).

It is a simple, but important, fact that the specialized frames are (up to renumbering) the most general ones which respect the splitting of $T(E)$ into vertical and horizontal components. Suppose $\{e_I\}$ is a specialized frame. Then the general element of the set of all specialized frames is (see (3.4a) and (3.16))

$$(\bar{e}_\mu, \bar{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} A^\nu_\mu & 0 \\ 0 & A^b_a \end{bmatrix} = (A^\nu_\mu e_\nu, A^b_a e_b), \tag{5.3a}$$

where $[A^\nu_\mu]^n_{\mu,\nu=1}$ and $[A^b_a]^{n+r}_{a,b=n+1}$ are non-degenerate matrix-valued functions on E, which are constant on the fibres of (E, π, M), i.e., we can set $A^\nu_\mu = B^\nu_\mu \circ \pi$ and $A^b_a = B^b_a \circ \pi$ for some non-degenerate matrix-valued functions $[B^\nu_\mu]$ and $[B^b_a]$

on M. Respectively, the general specialized coframe dual to $\{\bar{e}_I\}$ is (see (3.4b) and (3.16))

$$\begin{pmatrix} \bar{e}^\mu \\ \bar{e}^a \end{pmatrix} = \begin{bmatrix} [A^\lambda_\rho]^{-1} & 0 \\ 0 & [A^c_d]^{-1} \end{bmatrix} \cdot \begin{pmatrix} e^\nu \\ e^b \end{pmatrix} = \begin{bmatrix} \big([A^\lambda_\rho]^{-1}\big)^\mu_\nu e^\nu \\ \big([A^c_d]^{-1}\big)^a_b e^b \end{bmatrix}, \tag{5.3b}$$

where $\{e^I\}$ is the specialized coframe dual to $\{e_I\}$, $e^I(e_J) = \delta^I_J$.

Since $\pi_*|_{\Delta^h} \colon \{X \in \Delta^h\} \to \mathcal{X}(M)$ is an isomorphism, any basis $\{\varepsilon_\mu\}$ *for* Δ^h defines a basis $\{E_\mu\}$ of $\mathcal{X}(M)$ such that

$$E_\mu = \pi_*|_{\Delta^h}(\varepsilon_\mu) \tag{5.4}$$

and v.v., a basis $\{E_\mu\}$ for $\mathcal{X}(M)$ induces a basis $\{\varepsilon_\mu\}$ for Δ^h via

$$\varepsilon_\mu = (\pi_*|_{\Delta^h})^{-1}(E_\mu). \tag{5.5}$$

Similarly, there is a bijection $\{\varepsilon^\mu\} \mapsto \{E^\mu\}$ between the 'horizontal' coframes $\{\varepsilon^\mu\}$ and the coframes $\{E^\mu\}$ dual to the frames in $T(M)$ ($E^\mu \in \Lambda^1(M)$, $E^\mu(E_\nu) = \delta^\mu_\nu$). Thus a 'horizontal' change

$$\varepsilon_\mu \mapsto \bar{\varepsilon}_\mu = (B^\nu_\mu \circ \pi)\varepsilon_\nu, \tag{5.6}$$

which is independent of a 'vertical' one given by

$$\varepsilon_a \mapsto \bar{\varepsilon}_a = (B^b_a \circ \pi)\varepsilon_b \tag{5.7}$$

with $\{\varepsilon_a\}$ being a basis for Δ^v, is equivalent to the transformation

$$E_\mu \mapsto \bar{E}_\mu = B^\nu_\mu E_\nu \tag{5.8}$$

of the basis $\{E_\mu\}$ for $\mathcal{X}(M)$, related via (5.4) to the basis $\{\varepsilon_\mu\}$ for Δ^h. Here $[B^\nu_\mu]$ and $[B^b_a]$ are non-degenerate matrix-valued functions on M.

As $\pi_*(\varepsilon_a) = 0 \in \mathcal{X}(M)$, the 'vertical' transformations (5.7) do not admit interpretation analogous to the 'horizontal' ones (5.6). However, in a case of a *vector* bundle (E, π, M), they are tantamount to changes of frames in the bundle space E, i.e., of the bases for $\mathrm{Sec}(E, \pi, M)$. Indeed, if v is the mapping defined by (4.3), the sections

$$E_a = v^{-1}(\varepsilon_a) \tag{5.9}$$

form a basis for $\mathrm{Sec}(E, \pi, M)$ as the vertical vector fields ε_a form a basis for Δ^v. Conversely, any basis $\{E_a\}$ for the sections of (E, π, M) induces a basis $\{\varepsilon_a\}$ for Δ^v such that

$$\varepsilon_a = v(E_a). \tag{5.10}$$

As v and v^{-1} are linear, the change (5.7) is equivalent to the transformation

$$E_a \mapsto \bar{E}_a = B^b_a E_b \tag{5.11}$$

of the frame $\{E_a\}$ in E related to $\{\varepsilon_a\}$ via (5.9). In this way, we have proved the following result.

Proposition 5.1. *There is a bijective correspondence between the set of specialized frames $\{\varepsilon_I\} = \{\varepsilon_\mu, \varepsilon_a\}$ on a vector bundle (E, π, M) and the set of pairs $(\{E_\mu\}, \{E_a\})$ of frames $\{E_\mu\}$ on $T(M)$ over M and $\{E_a\}$ on E over M.*[1]

Since conceptually the frames in $T(M)$ and E are easier to be understood and in some cases have a direct physical interpretation, one often works with the pair $(\{E_\mu = \pi_*|_{\Delta^h}(\varepsilon_\mu)\}, \{E_a = v^{-1}(\varepsilon_a)\})$ of frames instead with a specialized frame $\{\varepsilon_I\} = \{\varepsilon_\mu, \varepsilon_a\}$; for instance $\{E_\mu\}$ and $\{E_a\}$ can be completely arbitrary frames in $T(M)$ and E, respectively, while the specialized frames represent only a particular class of frames in $T(E)$.

One can *mutatis mutandis* localize the above considerations when M is replaced with an open subset U_M in M and E is replaced with $W = \pi^{-1}(U_M)$. Such a localization is important when the bases/frames considered are connected with some local coordinates or when they should be smooth.[2]

Let us turn now our attention to frames adapted to local coordinate system $\{u^I\}$ on an open set $W \subseteq E$ for a given connection Δ^h on a general C^1 bundle (E, π, M) (see (3.27)–(3.30)). Since in their definition the local coordinates u^I enter only via the vector fields $\partial_I := \frac{\partial}{\partial u^I} \in \mathcal{X}(E)$, we can generalize this definition by replacing $\{\partial_I\}$ with an arbitrary frame $\{e_I\}$ defined in $T(E)$ over an open set $W \subseteq E$ and such that $\{e_a|_p\}$ is a *basis for the space* $T_p(\pi^{-1}(\pi(p)))$ *tangent to the fibre through* $p \in W$. So, using $\{e_I\}$ for $\{\partial_I\}$, we have

$$(e_\mu^W, e_a^W) = (D_\mu^\nu e_\nu + D_\mu^a e_a, D_a^b e_b) = (e_\nu, e_b) \cdot \begin{pmatrix} [D_\mu^\nu] & 0 \\ [D_\mu^b] & [D_a^b] \end{pmatrix}, \tag{5.12}$$

where $\{e_I^W\}$ is a *specialized* frame in $T(W)$, $[D_\mu^\nu]$ and $[D_a^b]$ are non-degenerate matrix-valued functions on W, and $D_\mu^a \colon W \to \mathbb{K}$.

Definition 5.1. The specialized frame $\{X_I\}$ over W in $T(W)$, obtained from (5.12) via an admissible transformation (3.4a) with matrix $A = \begin{pmatrix} [D_\nu^\mu]^{-1} & 0 \\ 0 & [D_b^a]^{-1} \end{pmatrix}$, is called *adapted to the frame* $\{e_I\}$ *for* Δ^h.[3]

Exercise 5.2. Using (3.4) and (3.16), verify that the adapted frame $\{X_I\}$ and coframe $\{\omega^I\}$ dual to it are independent of the particular specialized frame $\{e_I^W\}$ entering into their definitions via (5.12). The equalities (5.13a) and (5.21) derived below are indirect proof of that fact too.

---

[1] It should be mentioned the evident fact that a frame $\{E_\mu\}$ in $T(M)$ over M is also a basis for the module $\mathfrak{X}(M)$ of vector fields over M and hence is a basis for the set $\mathrm{Sec}(T(M), \pi_T, M)$ of section of the bundle tangent to M, due to $\mathfrak{X}(M) = \mathrm{Sec}(T(M), \pi_T, M)$. Similarly, a frame $\{E_a\}$ on E over M is a basis for the set $\mathrm{Sec}(E, \pi, M)$ of sections of the vector bundle (E, π, M).

[2] Recall, not every manifold admits a *global* nowhere vanishing C^m, $m \geq 0$, vector field (see [153] or [1, Section 4.24]); e.g., such are the even-dimensional spheres $\mathbb{S}^{2k}$, $k \in \mathbb{N}$, in Euclidean space.

[3] Recall, here and below the adapted frames are defined only with respect to frames $\{e_I\} = \{e_\mu, e_a\}$ such that $\{e_a\}$ is a basis for the vertical distribution Δ^v over W, i.e., $\{e_a|_p\}$ is a basis for Δ_p^v for all $p \in W$. Since Δ^v is integrable, the relation $e_a \in \Delta^v$ for all $a = n+1, \dots, n+r$ implies $[e_a, e_b]_\_ \in \Delta^v$ for all $a, b = n+1, \dots, n+r$.

According to (3.4), the adapted frame $\{X_I\} = \{X_\mu, X_a\}$ and the coframe $\{\omega^I\} = \{\omega^\mu, \omega^a\}$ dual to it are given by the equations

$$(X_\mu, X_a) = (e_\nu, e_b) \cdot \begin{bmatrix} \delta^\nu_\mu & 0 \\ +\Gamma^b_\mu & \delta^b_a \end{bmatrix} = (e_\mu + \Gamma^b_\mu e_b, e_a) \tag{5.13a}$$

$$\begin{pmatrix} \omega^\mu \\ \omega^a \end{pmatrix} = \begin{bmatrix} \delta^\mu_\nu & 0 \\ -\Gamma^a_\nu & \delta^a_b \end{bmatrix} \cdot \begin{pmatrix} e^\nu \\ e^b \end{pmatrix} = \begin{pmatrix} e^\mu \\ e^a - \Gamma^a_\nu e^\nu \end{pmatrix}, \tag{5.13b}$$

where $\{e^I\}$ is the coframe dual to $\{e_I\}$, $e^I(e_J) = \delta^I_J$, and the functions $\Gamma^a_\mu \colon W \to \mathbb{K}$, called (2-*index*) *coefficients of* Δ^h in $\{X_I\}$, are defined by

$$[\Gamma^a_\mu] := +[D^a_\nu] \cdot [D^\nu_\mu]^{-1}. \tag{5.14}$$

Proposition 5.2. *A change* $\{e_I\} \mapsto \{\tilde{e}_I\}$ *with*

$$(\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{pmatrix} [A^\nu_\mu] & 0 \\ [A^b_\mu] & [A^b_a] \end{pmatrix} = (A^\nu_\mu e_\nu + A^b_\mu e_b, A^b_a e_b), \tag{5.15}$$

where $[A^\nu_\mu]$ *and* $[A^b_a]$ *are non-degenerate matrix-valued functions on* W*, which are constant on the fibres of* (E, π, M)*, and* $A^b_\mu \colon W \to \mathbb{K}$*, entails the transformations*

$$(X_\mu, X_a) \mapsto (\tilde{X}_\mu, \tilde{X}_a) = (\tilde{e}_\mu + \tilde{\Gamma}^b_\mu \tilde{e}_b, \tilde{e}_a) = (A^\nu_\mu X_\nu, A^b_a X_b) = (X_\nu, X_b) \cdot \begin{bmatrix} A^\nu_\mu & 0 \\ 0 & A^b_a \end{bmatrix} \tag{5.16}$$

$$\Gamma^a_\mu \mapsto \tilde{\Gamma}^a_\mu = \big([A^c_d]^{-1}\big)^a_b (\Gamma^b_\nu A^\nu_\mu - A^b_\mu) \tag{5.17}$$

of the frame $\{X_I\}$ *adapted to* $\{e_I\}$ *and of the coefficients* Γ^a_μ *of* Δ^h *in* $\{X_I\}$*, i.e.,* $\{\tilde{X}_I\}$ *is the frame adapted to* $\{\tilde{e}_I\}$ *and* $\tilde{\Gamma}^a_\mu$ *are the coefficients of* Δ^h *in* $\{\tilde{X}_I\}$*.*

Proof. Apply (5.12)–(5.14). □

Note 5.1. If $\{e_I\}$ and $\{\tilde{e}_I\}$ are adapted, then $A^b_\mu = 0$. If $\{Y_I\}$ is a specialized frame, it is adapted to any frame $\{e_\mu = A^\nu_\mu Y_\nu, e_a = A^b_a Y_b\}$ and hence any specialized frame can be considered as an adapted one; in particular, any specialized frame is a frame adapted to itself. Obviously (see (5.14)), the coefficients of a connection identically vanish in a given specialized frame considered as an adapted one. This leads to the concept of a *normal frame* to which is devoted Section 6 below. Besides, from the above observation follows that the set of adapted frames coincides with the one of specialized frames.

Exercise 5.3. Verify that the formulae dual to (5.15) and (5.16) are (see (3.4b) and (3.5b))

$$\begin{pmatrix}\tilde e^{\mu}\\ \tilde e^{a}\end{pmatrix} = \begin{pmatrix}[A^{\varrho}_{\tau}]^{-1} & 0\\ -[A^{c}_{d}]^{-1}[A^{c}_{\tau}][A^{\varrho}_{\tau}]^{-1} & [A^{c}_{d}]^{-1}\end{pmatrix}\cdot\begin{pmatrix}e^{\nu}\\ e^{b}\end{pmatrix}$$
$$= \begin{pmatrix}([A^{\varrho}_{\tau}]^{-1})^{\mu}_{\nu}\, e^{\nu}\\ ([A^{c}_{d}]^{-1})^{a}_{b}\, e^{b} - \bigl([A^{c}_{d}]^{-1}[A^{c}_{\tau}][A^{\varrho}_{\tau}]^{-1}\bigr)^{a}_{\nu}\, e^{\nu}\end{pmatrix} \tag{5.18}$$

$$\begin{pmatrix}\omega^{\mu}\\ \omega^{a}\end{pmatrix} \mapsto \begin{pmatrix}\tilde\omega^{\mu}\\ \tilde\omega^{a}\end{pmatrix} = \begin{pmatrix}([A^{\varrho}_{\tau}]^{-1}))^{\mu}_{\nu}e^{\nu}\\ ([A^{c}_{d}]^{-1})^{a}_{b}e^{b}\end{pmatrix}. \tag{5.19}$$

Example 5.1. If $\{e_I\}$ and $\{\tilde e_I\}$ are the frames generated by local coordinates $\{u^I\}$ and $\{\tilde u^I\}$, viz. $e_I = \frac{\partial}{\partial u^I}$ and $\tilde e_I = \frac{\partial}{\partial \tilde u^I}$, the changes (5.16) and (5.17) reduce to (3.31) and (3.32), respectively. The choice $e_I = \frac{\partial}{\partial u^I}$ also reduces Definition 5.1 to Definition 3.5.

A result similar to Proposition 3.3 is valid too provided in its formulation equation (3.32) is replaced with (5.17).

If e_μ has an expansion $e_\mu = e^{\nu}_{\mu}\frac{\partial}{\partial u^{\nu}} + e^{b}_{\mu}\frac{\partial}{\partial u^{b}}$ in the domain W of $\{u^I\} = \{u^{\mu} = x^{\mu}\circ\pi, u^{a}\}$, where $e^{b}_{\mu}\colon W\to\mathbb{K}$ and $e^{\nu}_{\mu} = x^{\nu}_{\mu}\circ\pi$ for some $x^{\nu}_{\mu}\colon \pi(W)\to\mathbb{K}$ such that $\det[x^{\nu}_{\mu}]\neq 0,\infty$, and we define a frame $\{x_\mu\}$ in $T(\pi(W))\subseteq T(M)$ by $\{x_\mu := x^{\nu}_{\mu}\frac{\partial}{\partial x^{\nu}}\}$, then

$$\pi_*(X_\mu) = x_\mu, \tag{5.20}$$

by virtue of (3.33) and (3.35). Thus, we have (cf. (3.34))

$$X_\mu = (\pi_*|_{\Delta^h})^{-1}(x_\mu) = (\pi_*|_{\Delta^h})^{-1}\circ\pi_*(e_\mu) \tag{5.21}$$

which can be used in an equivalent definition of a frame $\{X_I\}$ adapted to $\{e_I\}$ (with $\{e_a\}$ being a basis for Δ^v): X_μ should be defined by (5.21) and $X_a = e_a$. If one accepts such a definition of an adapted frame, the 2-index coefficients of a connection should be defined via the equation (5.13a), not by (5.14), and the proofs of some results, like (5.16) and (5.17), should be modified.

Proposition 5.3. *If $\{X_I\}$ is a frame adapted to a frame $\{e_I\}$, with $\{e_a\}$ being a basis for Δ^v, for a C^1 connection Δ^h, then (cf. (3.36))*

$$[X_\mu, X_\nu]_{\_} = R^{a}_{\mu\nu}X_a + S^{\lambda}_{\mu\nu}X_\lambda \tag{5.22a}$$
$$[X_\mu, X_b]_{\_} = {}^{\circ}\Gamma^{a}_{b\mu}X_a + C^{\lambda}_{\mu b}X_\lambda \tag{5.22b}$$
$$[X_a, X_b]_{\_} = C^{d}_{ab}X_d, \tag{5.22c}$$

where (*cf.* (3.37))

$$\left.\begin{aligned} R^a_{\mu\nu} &:= X_\mu(\Gamma^a_\nu) - X_\nu(\Gamma^a_\mu) - C^a_{\mu\nu} - \Gamma^b_\mu C^a_{\nu b} + \Gamma^b_\nu C^a_{\mu b} \\ &\quad + \Gamma^a_\lambda(-C^\lambda_{\mu\nu} + \Gamma^b_\mu C^\lambda_{\nu b} - \Gamma^b_\nu C^\lambda_{\mu b}) + \Gamma^b_\mu \Gamma^d_\nu C^a_{bd} \\ S^\lambda_{\mu\nu} &:= C^\lambda_{\mu\nu} + \Gamma^b_\mu C^\lambda_{\nu b} - \Gamma^b_\nu C^\lambda_{\mu b} \end{aligned}\right\} \tag{5.23a}$$

$$^\circ\Gamma^a_{b\mu} := -X_b(\Gamma^a_\mu) - C^a_{\mu b} + \Gamma^d_\mu C^a_{db} - \Gamma^a_\lambda C^\lambda_{\mu b} \tag{5.23b}$$

$$[e_I, e_J]_{\_} =: C^K_{IJ} e_K = C^a_{IJ} e_a + C^\lambda_{IJ} e_\lambda. \tag{5.23c}$$

Proof. Insert equation (5.13a) into the corresponding commutators, use the definition (5.23c) of the components of the anholonomicity object of $\{e_I\}$, and apply (5.13a). Notice, as $\{e_a\}$ is a basis for the integrable distribution Δ^v, we have $[e_a, e_b]_{\_} \in \Delta^v$ and consequently $C^\lambda_{ab} \equiv 0$. □

The functions $R^a_{\mu\nu}$ are the *fibre components of the curvature* of Δ^h and $^\circ\Gamma^a_{b\mu}$ are the *fibre coefficients* of Δ^h in the adapted frame $\{X_I\}$; if $e_I = \frac{\partial}{\partial u^I}$ for some bundle coordinates $\{u^I\}$ on E, they reduce to (3.37a) and (3.37b), respectively. From (5.22), we immediately derive

Corollary 5.1. *A connection Δ^h is integrable iff in some (and hence any) adapted frame*

$$R^a_{\mu\nu} = 0. \tag{5.24}$$

Corollary 5.2. *An adapted frame $\{X_I\}$ is (locally) holonomic iff in it*

$$R^a_{\mu\nu} = {}^\circ\Gamma^a_{b\mu} = S^\lambda_{\mu\nu} = C^d_{ab} = C^\lambda_{\mu b} = 0. \tag{5.25}$$

If the initial frame $\{e_I\}$ is changed into (5.15), then the transformation laws of the quantities (5.23) follow from (5.22) and (5.16); in particular, the curvature components transform according to the tensor equation (3.24b).

Let us now pay attention to the case when (E, π, M) is a *vector* bundle endowed with a connection Δ^h.

According to the above-said in this section, any *adapted* frame $\{X_I\} = \{X_\mu, X_a\}$ in $T(E)$ is equivalent to a pair of frames in $T(M)$ and E according to

$$\{X_\mu, X_a\} \leftrightarrow (\{E_\mu = \pi_*|_{\Delta^h}(X_\mu)\}, \{E_a = v^{-1}(X_a)\}). \tag{5.26}$$

Therefore the vertical and horizontal lifts are given by (cf. Lemma 4.1, (4.7a) and (4.10))

$$\mathrm{Sec}(E, \pi, M) \ni Y = Y^a E_a \xrightarrow{v} v(Y) := Y^v = (Y^a \circ \pi) X_a \in \Delta^v \tag{5.27a}$$

$$\mathcal{X}(M) \ni F = F^\mu E_\mu \xrightarrow{h} h(F) := F^h = (F^\mu \circ \pi) X_\mu \in \Delta^h. \tag{5.27b}$$

Thus, we have the linear isomorphism

$$\begin{aligned}(h,v)\colon \mathcal{X}(M)\times \mathrm{Sec}(E,\pi,M) &\to \mathcal{X}(E)\\ (h,v)\colon (F,Y) &\mapsto (F^h,Y^v)\end{aligned} \tag{5.28}$$

which explains why the covariant derivatives (see Definition 4.2) represent an equivalent description of the linear connections in vector bundles. Since any vector field $\xi = (\xi^I\circ\pi)X_I \in \mathcal{X}(E)$ has a unique decomposition $\xi = \xi^h \oplus \xi^v$, with $\xi^h = (\xi^\mu\circ\pi)X_\mu$ and $\xi^v = (\xi^a\circ\pi)X_a$, we have

$$(h,v)^{-1}(\xi) = (\pi_*|_{\Delta^h}(\xi^h), v^{-1}(\xi^v)) = (\xi^\mu E_\mu, \xi^a E_a). \tag{5.29}$$

Suppose $\{X_I\}$ and $\{\tilde{X}_I\}$ are two adapted frames. Then they are connected via (cf. (5.3a) and (5.16))

$$\tilde{X}_\mu = (B^\nu_\mu\circ\pi)X_\nu \quad \tilde{X}_a = (B^b_a\circ\pi)X_b, \tag{5.30}$$

where $[B^\nu_\mu]$ and $[B^b_a]$ are some non-degenerate matrix-valued functions on M. The pairs of frames corresponding to them, in accordance with (5.26), are related via

$$\tilde{E}_\mu = B^\nu_\mu E_\nu \quad \tilde{E}_a = B^b_a E_b \tag{5.31}$$

and *vice versa.*

Proposition 5.4. *Let Δ^h be a* linear *connection on a* vector *bundle (E,π,M) and $\{X_I\}$ be the frame adapted for Δ^h to a frame $\{e_I\}$ such that $\{e_a\}$ is a basis for Δ^v. Let $\{u^I\} = \{u^\mu, u^a\}$ be vector bundle coordinate system on $U\subseteq E$. Suppose that the frame $\{e_I\}$, to which $\{X_I\}$ is adapted to, is such that*

$$\begin{aligned}(e_\mu,e_a)|_U &= (\partial_\nu,\partial_b)\cdot \begin{bmatrix} B^\nu_\mu\circ\pi & 0\\ (B^b_{c\mu}\circ\pi)\cdot u^c & B^b_a\circ\pi\end{bmatrix}\\ &= \big((B^\nu_\mu\circ\pi)\partial_\nu + ((B^b_{c\mu}\circ\pi)\cdot u^c)\partial_b, (B^b_a\circ\pi)\partial_b\big),\end{aligned} \tag{5.32}$$

where $\partial_I := \frac{\partial}{\partial u^I}$, $[B^\nu_\mu]$ and $[B^b_a]$ are non-degenerate matrix-valued functions on $\pi(U)$, and $B^b_{c\mu}\colon \pi(U)\to\mathbb{K}$. Then the 2-index coefficients Γ^a_μ of Δ^h in $\{X_I\}$ have the representation

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot u^b \tag{5.33}$$

on U for some functions $\Gamma^a_{b\mu}\colon \pi(U)\to\mathbb{K}$, called 3-index coefficients of Δ^h *in $\{X_I\}$.*

Remark 5.1. The representation (5.33) is not valid for frames more general than the ones given by (5.32). Precisely, equation (5.33) is valid if and only if (5.32) holds for some local coordinates $\{u^I\}$ on W – see (5.17).

Remark 5.2. Since the vector fibre coordinates u^a are 1-forms on U, the 2-index coefficients (5.33) of a linear connection are also 1-forms on the bundle space.

Proof. Writing (5.17) for the transformation $\{\partial_I\} \mapsto \{e_I\}$, with the frame $\{e_I\}$ given by (5.32), we get (5.33) with

$$\Gamma^a_{b\mu} = ([B^e_d]^{-1})^a_c ({}^{\partial}\Gamma^c_{b\nu} B^\nu_\mu + B^c_{b\mu}),$$

where ${}^{\partial}\Gamma^a_{b\nu}$ are the 3-index coefficients of Δ^h in the frame adapted to the coordinates $\{u^I\}$ (see (4.13)). □

Let $\{X_I\}$ and $\{\tilde{X}_I\}$ be frames adapted to $\{e_I\}$ and $\{\tilde{e}_I\}$, respectively, such that (cf. (5.32))

$$(\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ (B^b_{c\mu} \circ \pi) \cdot u^c & B^b_a \circ \pi \end{bmatrix}, \tag{5.34}$$

and Δ^h admits 3-index coefficients in $\{X_I\}$ and $\{\tilde{X}_I\}$, which means that $\{e_I\}$ and $\{\tilde{e}_I\}$ are obtainable from the frames $\{\frac{\partial}{\partial u^I}\}$ and $\{\frac{\partial}{\partial \tilde{u}^I}\}$, respectively, for some bundle coordinate systems $\{u^I\}$ and $\{\tilde{u}^I\}$ via equations like (5.32) (with $\tilde{e}_I$ for e_I and $\tilde{\partial}_I$ for ∂_I in the letter case) in which the B's need not be the same as in (5.34).[4] Then, due to (5.17) and (5.33), the 3-index coefficients $\Gamma^a_{b\mu}$ and $\tilde{\Gamma}^a_{b\mu}$ of Δ^h in respectively $\{X_I\}$ and $\{\tilde{X}_I\}$ are connected by (see also footnote 4 and cf. (4.23))

$$\tilde{\Gamma}^a_{b\mu} = \big([B^e_f]^{-1}\big)^a_c (\Gamma^c_{d\nu} B^\nu_\mu + B^c_{d\mu}) B^d_b. \tag{5.35}$$

Exercise 5.4. Prove that the transformation $\{e_I\} \mapsto \{\tilde{e}_I\}$, with the frame $\{\tilde{e}_I\}$ given by (5.34), is the most general one that preserves the existence of 3-index coefficients of Δ^h provided they exist in $\{e_I\}$ in a sense that, if $\{e_I\}$ is given by (5.32) (which leads to (5.33)) and $\{\tilde{e}^I\}$ is given by (5.34), then there exist vector bundle coordinates $\{\tilde{u}^I\}$ which generate $\{\tilde{e}^I\}$ according to (5.32) with $\tilde{e}_I$ for e_I, $\tilde{\partial}_I$ for ∂_I and some B's, which leads to (5.33) with $\tilde{\Gamma}$ for Γ and $\tilde{u}$ for u.

Introducing the matrices $\Gamma_\mu := [\Gamma^a_{b\mu}]^{n+r}_{a,b=n+1}$, $\tilde{\Gamma}_\mu := [\tilde{\Gamma}^a_{b\mu}]^{n+r}_{a,b=n+1}$, $B := [B^a_b]$, and $B_\mu := [B^a_{b\mu}]$, we rewrite (5.35) as (cf. (4.23′))

$$\tilde{\Gamma}_\mu = B^{-1} \cdot (\Gamma_\nu B^\nu_\mu + B_\mu) \cdot B. \tag{5.35′}$$

A little below (see the text after equation (5.37)), we shall prove that the compatibility of the developed formalism with the theory of covariant derivatives requires further restrictions on the general transformed frames (5.15) to the ones given by (5.34) with

$$B_\mu = \tilde{E}_\mu(B) \cdot B^{-1} = B^\nu_\mu E_\nu(B) \cdot B^{-1}, \tag{5.36}$$

where $\tilde{E}_\mu := \pi_*|_{\Delta^h}(\tilde{X}_\mu) = \pi_*|_{\Delta^h}((B^\nu_\mu \circ \pi) X_\nu) = B^\nu_\mu E_\nu$. In this case, (5.35′) reduces to (cf. (4.23′))

$$\tilde{\Gamma}_\mu = B^\nu_\mu B^{-1} \cdot (\Gamma_\nu \cdot B + E_\nu(B)) = B^\nu_\mu (B^{-1} \cdot \Gamma_\nu - E_\nu(B^{-1})) \cdot B. \tag{5.37}$$

[4] Notice, from (5.34) follows that the vector fibre coordinate systems $\{u^a\}$ and $\{\tilde{u}^a\}$ are connected via $u^a = (B^a_b \circ \pi) \cdot \tilde{u}^b$.

At last, a few words on the covariant derivative operators ∇ are in order. Without lost of generality, we define such an operator (4.33) via the equations (4.39). Suppose $\{E_\mu\}$ is a basis for $\mathcal{X}(M)$ and $\{E_a\}$ is a one for $\mathrm{Sec}^1(E,\pi,M)$. Define the *components* $\Gamma^a_{b\mu}\colon M\to\mathbb{K}$ of ∇ in the pair of frames $(\{E_\mu\},\{E_a\})$ by (cf. (4.40))

$$\nabla_{E_\mu}(E_b)=\Gamma^a_{b\mu}E_a. \tag{5.38}$$

Then (4.39) imply (cf. (4.36))

$$\nabla_F Y = F^\mu(E_\mu(Y^a)+\Gamma^a_{b\mu}Y^b)E_a$$

for all $F=F^\mu E_\mu\in\mathcal{X}(M)$ and $Y=Y^aE_a\in\mathrm{Sec}^1(E,\pi,M)$. A frame change $(\{E_\mu\},\{E_a\})\mapsto(\{\tilde{E}_\mu\},\{\tilde{E}_a\})$, given via (5.31), entails

$$\Gamma^a_{b\mu}\mapsto\tilde{\Gamma}^a_{b\mu}=B^\nu_\mu\big([B^e_f]^{-1}\big)^a_c(\Gamma^c_{d\nu}B^d_b+E_\nu(B^c_b)), \tag{5.39}$$

as a result of (5.38). In a more compact matrix form, the last result reads

$$\tilde{\Gamma}_\mu = B^\nu_\mu B^{-1}\cdot(\Gamma_\nu\cdot B+E_\nu(B)) \tag{5.39$'$}$$

with $\Gamma_\mu := [\Gamma^a_{b\mu}]$, $\tilde{\Gamma}_\mu := [\tilde{\Gamma}^a_{b\mu}]$, and $B:=[B^a_b]$.

Thus, if we identify the 3-index coefficients of Δ^h, defined by (5.33), with the components of ∇, defined by (5.38),[5] then the quantities (5.35$'$) and (5.39$'$) must coincide, which immediately leads to the equality (5.36). Therefore

$$(e_\mu,e_a)\mapsto(\tilde{e}_\mu,\tilde{e}_a)=(e_\nu,e_b)\cdot\begin{bmatrix} B^\nu_\mu\circ\pi & 0\\ \big((B^\nu_\mu E_\nu(B^b_d)(B^{-1})^d_c)\circ\pi\big)u^c & B^b_a\circ\pi\end{bmatrix}\Bigg|_{B=[B^b_a]} \tag{5.40}$$

is the most general transformation between frames in $T(E)$ such that the frames adapted to them are compatible with the linear connection and the covariant derivative corresponding to it. In particular, such are all frames $\big\{\frac{\partial}{\partial u^I}\big\}$ in $T(E)$ induced by vector bundle coordinates $\{u^I\}$ on E – see (4.21) and (3.1)–(3.3); the rest members of the class of frames mentioned are obtained from them via (5.40) with $e_I=\frac{\partial}{\partial u^I}$ and non-degenerate matrix-valued functions $[B^\nu_\mu]$ and B.

If $\{X_I\}$ (resp. $\{\tilde{X}_I\}$) is the frame adapted to a frame $\{e_I\}$ (resp. $\{\tilde{e}_I\}$), then the change $\{e_I\}\mapsto\{\tilde{e}_I\}$, given by (5.40), entails $\{X_I\}\mapsto\{\tilde{X}_I\}$ with $\{\tilde{X}_I\}$ given by (5.30) (see (5.15) and (5.16)). Since the last transformation is tantamount to the change

$$(\{E_\mu\},\{E_a\})\mapsto(\{\tilde{E}_\mu\},\{\tilde{E}_a\}) \tag{5.41}$$

of the basis of $\mathcal{X}(M)\times\mathrm{Sec}(E,\pi,M)$ corresponding to $\{X_I\}$ via the isomorphism (5.28) (see (5.26), (5.30), and (5.31)), we can say that the transition (5.41)

[5]Such an identification is justified by the definition of ∇ via the parallel transport assigned to Δ^h (see Proposition 4.4) or via a projection, generated by Δ^h, of a suitable Lie derivative on $\mathfrak{X}(E)$ (see Definition 4.2).

induces the change (5.39) of the 3-index coefficients of the connection Δ^h. Exactly the same is the situation one meets in the literature [7, 11, 23] when covariant derivatives are considered (and identified with connections).

Regardless that the change (5.40) of the frames in $T(E)$ looks quite special, it is the most general one that, through (5.16) and (5.26), is equivalent to an arbitrary change (5.41) of a basis in $\mathcal{X}(M) \times \mathrm{Sec}(E, \pi, M)$, i.e., of a pair of frames in $T(M)$ and E.

We would like to remark that, in the general case, equation (4.43) also holds with $F = F^\mu E_\mu$, $G = G^\mu E_\mu$, and

$$(R(E_\mu, E_\nu))(E_b) = R^a_{b\mu\nu} E_a, \tag{5.42}$$

so that

$$R^a_{b\mu\nu} = E_\mu(\Gamma^a_{b\nu}) - E_\nu(\Gamma^a_{b\mu}) - \Gamma^c_{b\mu}\Gamma^a_{c\nu} + \Gamma^c_{b\nu}\Gamma^a_{c\mu} - \Gamma^a_{b\lambda}C^\lambda_{\mu\nu}, \tag{5.43}$$

where the functions $C^\lambda_{\mu\nu}$ define the anholonomicity object of $\{E_\mu\}$ via $[E_\mu, E_\nu]_- =: C^\lambda_{\mu\nu} E_\lambda$.

The above results, concerning linear connections on vector bundles, can be generalized for affine connections on vector bundles. For instance, the analogue of Propositions 5.4 reads.

Proposition 5.5. *Let Δ^h be an* affine *connection on a* vector *bundle (E, π, M) and $\{X_\mu\}$ be the frame adapted for Δ^h to a frame $\{e_I\}$ such that $\{e_a\}$ is a basis for Δ^v and*

$$\begin{aligned}(e_\mu, e_a)|_U &= (\partial_\nu, \partial_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ (B^b_{c\mu} \circ \pi) \cdot u^c & B^b_a \circ \pi \end{bmatrix} \\ &= \big((B^\nu_\mu \circ \pi)\partial_\nu + ((B^b_{c\mu} \circ \pi) \cdot u^c)\partial_b, (B^b_a \circ \pi)\partial_b\big),\end{aligned} \tag{5.44}$$

where $\partial_I := \frac{\partial}{\partial u^I}$ for some local bundle coordinate system $\{u^I\} = \{u^\mu = x^\mu \circ \pi, u^b = E^b\}$ on $U \subseteq E$, $[B^\nu_\mu]$ and $[B^b_a]$ are non-degenerate matrix-valued functions on U, and $B^b_{c\mu} \colon U \to \mathbb{K}$. Then the 2-index coefficients Γ^a_μ of Δ^h in $\{X_I\}$ have the representation (*cf.* (4.46))

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi) \cdot u^b + G^a_\mu \circ \pi \tag{5.45}$$

on U for some functions $\Gamma^a_{b\mu}, G^a_\mu \colon U \to \mathbb{K}$.

Remark 5.3. The representation (5.45) is not valid for frames more general than the ones given by (5.44). Precisely, equation (5.45) is valid if and only if (5.44) holds for some local coordinate system $\{u^I\}$ on U – see (5.17).

Proof. Writing the equation (5.17) for the transformation $\{\partial_I\} \mapsto \{e_I\}$, with $\{e_I\}$ given by (5.44), we get (5.45) with

$$\Gamma^a_{b\mu} = ([B^e_d]^{-1})^a_c ({}^\partial\Gamma^c_{b\nu} B^\nu_\mu + B^c_{b\mu}) \quad G^a_\mu = ([B^e_d]^{-1})^a_b\, {}^\partial G^b_\nu B^\nu_\mu$$

where ${}^{\partial}\Gamma^a_{b\nu}$ and ${}^{\partial}G^b_{\nu}$ are defined via the 2-index coefficients ${}^{\partial}\Gamma^a_{\mu}$ of Δ^h in the frame adapted to the coordinates $\{u^I\}$ via ${}^{\partial}\Gamma^a_{\mu} = -({}^{\partial}\Gamma^a_{b\mu} \circ \pi) \cdot E^b + {}^{\partial}G^a_{\mu} \circ \pi$ (see Theorem 4.2). □

Let $\{X_I\}$ and $\{\tilde{X}_I\}$ be frames adapted to $\{e_I\}$ and $\{\tilde{e}_I\}$, respectively, with (cf. (5.44))

$$(\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ (B^b_{c\mu} \circ \pi) \cdot u^c & B^b_a \circ \pi \end{bmatrix}, \tag{5.46}$$

in which (5.45) holds for Δ^h. Then, due to (5.17) and (5.45), the pairs $(\Gamma^a_{b\mu}, G^a_\mu)$ and $(\tilde{\Gamma}^a_{b\mu}, \tilde{G}^a_\mu)$ for Δ^h in respectively $\{X_I\}$ and $\{\tilde{X}_I\}$ are connected by (cf. (4.47) and (4.48))

$$\tilde{\Gamma}^a_{b\mu} = \big([B^e_f]^{-1}\big)^a_c (\Gamma^c_{d\nu} B^\nu_\mu + B^c_{d\mu}) B^d_b \tag{5.47a}$$

$$\tilde{G}^a_\mu = \big([B^e_f]^{-1}\big)^a_b G^b_\nu B^\nu_\mu. \tag{5.47b}$$

Exercise 5.5. Prove that the transformation $\{e_I\} \mapsto \{\tilde{e}_I\}$, with the frame $\{\tilde{e}_I\}$ given by (5.46), is the most general one that preserves the existence of the relation (5.45) for Δ^h provided it holds in $\{e_I\}$.

Further one can repeat *mutatis mutandis* the text after Exercise 5.4 to the paragraph containing equation (5.41) including.

6. Normal frames

In the theory of linear connections on a manifold, the normal frames are defined as frames in the tangent bundle space in which the connections' (3-index) coefficients vanish on some subset of the manifold (see Definition I.5.1 on page 37). The definition of normal frames for a connection on a vector bundle is practically the same, the only difference being that these frames are in the bundle space, not in the tangent bundle space over the base space (see Sections IV.8 and IV.14.4). The present section is devoted to the introduction of normal frames for general connections on fibre bundles and some their properties.

6.1. The general case

To save some space and for brevity, in what follows we shall not indicate explicitly that the frames $\{e_I\} = \{e_\mu, e_a\}$, with respect to which the adapted frames are defined, are such that $\{e_a\}$ is a (local) basis for the vertical distribution Δ^v on the bundle considered.

Definition 6.1. Given a connection Δ^h on a bundle (E, π, M) and a subset $W \subseteq E$. A *frame* $\{X_I\}$ *in* $T(E)$ *adapted to a frame* $\{e_I\}$ *in* $T(E)$ and defined over an open subset V of E containing or equal to W, $V \supseteq W$, is called *normal for* Δ^h *over/on*

W (*relative to* $\{e_I\}$) if all (2-index) coefficients Γ^a_μ of Δ^h vanish in it everywhere on W. Respectively, $\{X_I\}$ is *normal for* Δ^h *along a mapping* $g\colon Q \to E$, $Q \neq \varnothing$, if $\{X_I\}$ is normal for Δ^h over the set $g(Q)$.

Let $\{X_I\}$ be the frame in $T(E)$ adapted to a frame $\{e_I\}$ in $T(E)$ over an open subset $V \subseteq E$. Then the frame $\{\tilde{X}_I\}$ in $T(E)$ adapted to a frame $\{\tilde{e}_I\}$, given by (5.15), in $T(E)$ over the same subset V is normal for Δ^h over $W \subseteq V$ if and only if

$$(A^\nu_\mu \Gamma^b_\nu - A^b_\mu)|_W = 0, \tag{6.1}$$

due to (5.16) and (5.17). Since Γ^b_μ depend only on Δ^h and $\{e_I\}$, the existence of solutions of (6.1), relative to A^ν_μ and A^b_μ, and their properties are completely responsible for the existence and the properties of frames normal for Δ^h over W. For that reason, we call (6.1) the (*system of*) *equation*(*s*) *of the normal frames for* Δ^h *over* W or simply the *normal frame* (*system of*) *equation*(*s*) (for Δ^h over W).

In the most general case, when no additional restrictions on the frames considered are imposed, the normal frame equation (6.1) is a system of *nr linear algebraic equations* for $nr + n^2$ variables and, consequently, it has a solution depending on n^2 independent parameters. In particular, if we choose the functions $A^\nu_\mu\colon W \to \mathbb{K}$ (with $\det[A^\nu_\mu] \neq 0, \infty$) as such parameters, we can write the general solution of (6.1) as

$$(\{A^\nu_\mu\}, \{A^b_\mu\})|_W = (\{A^\nu_\mu\}, \{\Gamma^b_\nu A^\nu_\mu\})|_W. \tag{6.2}$$

It should be noted, equation (6.1) or its general solution (6.2) defines the frame $\{\tilde{e}_I\}$ and the frame $\{\tilde{X}_I\}$ adapted to $\{\tilde{e}_I\}$ only on W and leaves them completely arbitrary on $V \setminus W$, if it is not empty.

Proposition 6.1. *Let* $\{X_I\}$ *be the frame adapted to a frame* $\{e_I\}$ *in* $T(V) \subseteq T(E)$ *defined over an open set* $V \subseteq E$ *and* Γ^a_μ *be the coefficients of a connection* Δ^h *in* $\{X_I\}$. *Then all frames* $\{\tilde{X}_I\}$, *normal on* $W \subseteq V$ *for the connection* Δ^h, *are adapted to frames* $\{\tilde{e}_I\}$ *given on* W *by*

$$\tilde{e}_\mu|_W = (A^\nu_\mu(e_\nu + \Gamma^b_\nu e_b))|_W \quad \tilde{e}_a|_W = (A^b_a e_b)|_W. \tag{6.3}$$

where $[A^\nu_\mu]$ *and* $[A^b_a]$ *are non-degenerate matrix-valued functions on* V *which are constant on the fibres of* (E, π, M). *Moreover, the frame* $\{\tilde{X}_I\}$ *adapted on* V *to* $\{\tilde{e}_I\}$, *given by* (6.3) (*and hence normal on* W), *is such that*

$$\tilde{X}_\mu|_W = (A^\nu_\mu X_\nu)|_W = \tilde{e}_\mu|_W \quad \tilde{X}_a|_W = (A^b_a X_b)|_W = \tilde{e}_a|_W, \tag{6.4}$$

Proof. Apply (5.16), (5.15), and (5.13a) for the choice (6.2). □

The equations (6.4) are not accidental as it is stated by the following assertion.

Proposition 6.2. *The frame* $\{\tilde{X}_I\}$ *in* $T(E)$ *adapted to a frame* $\{\tilde{e}_I\}$ *in* $T(E)$ *and defined over an open set* $V \subseteq E$ *is normal on* $W \subseteq V$ *if and only if on* W *is fulfilled*

$$\tilde{X}_I|_W = \tilde{e}_I|_W. \tag{6.5}$$

Proof. Apply (5.13a) or (5.16) and Definition 6.1. □

Thus one can equivalently *define the normal frames as adapted frames that coincide on some set with the frames they are adapted to* or as frames (in the tangent bundles space over the bundle space) that coincide on some set with the frames adapted to them.

Since any specialized frame is adapted to itself (see Definition 5.1 and (5.12), with $D_I^J = \delta_I^J$), *the sets of normal, specialized, and adapted frames are identical.*

As we see from Proposition 6.1, which gives a complete description of the normal frames, the theory of normal frames in the most general setting is trivial. It becomes more interesting and richer if the class of frames $\{e_I\}$, with respect to which are defined the adapted frames, is restricted in one or other way. To the theory of normal frames, adapted to such restricted classes of frames in $T(E)$, is devoted the rest of this section.

6.2. Normal frames adapted to holonomic frames

The class of holonomic frames induced by local coordinates on E (see Subsection 3.2) is the most natural class of frames in $T(E)$ relative to which the adapted, in particular normal, frames are defined. To specify the consideration of the previous subsection to normal frames adapted to local coordinates on E, we set $e_I = \frac{\partial}{\partial u^I}$ and $\tilde{e}_I = \frac{\partial}{\partial \tilde{u}^I}$, where $\{u^I\}$ and $\{\tilde{u}^I\}$ are local coordinate systems on E whose domains have a non-empty intersection V and $W \subseteq V$. Then the matrix $[A_I^J]$ in (6.1) is given by (3.3) (as $\{e_I\} \mapsto \{\tilde{e}_I\}$ reduces to (3.2a)), so that the normal frame equation (6.1) reduces to the *normal coordinates equation* (see also (3.32))

$$\Big(\frac{\partial \tilde{u}^a}{\partial u^b}\Gamma_\mu^b + \frac{\partial \tilde{u}^a}{\partial u^\mu}\Big)\Big|_W = 0, \tag{6.6}$$

due to (3.1), which is a first-order system of *nr linear partial differential equations* on W relative to the r unknown functions $\{\tilde{u}^{n+1}, \dots, \tilde{u}^{n+r}\}$.

Since the connection Δ^h is supposed given and fixed, such are its coefficients Γ_μ^b in $\{\frac{\partial}{\partial u^I}\}$. Therefore the existence, uniqueness and other properties of the solutions of (6.6) strongly depend on the set W (which is in the intersection of the domains of the local coordinates $\{u^I\}$ and $\{\tilde{u}^I\}$ on E).

Proposition 6.3. *If the normal frame equation* (6.6) *has solutions, then all frames* $\{\tilde{X}_I\}$ *normal on* $W \subseteq E$ *and adapted to local coordinates, defined on an open set* $V \subseteq E$ *such that* $V \supseteq W$, *are described by*

$$\tilde{X}_\mu|_W = (A_\mu^\nu X_\nu)|_W = \frac{\partial}{\partial \tilde{u}^\mu}\Big|_W \quad \tilde{X}_a|_W = (A_a^b X_b)|_W = \frac{\partial}{\partial \tilde{u}^a}\Big|_W, \tag{6.7}$$

where $\{X_I\}$ *is the frame adapted to some arbitrarily fixed local coordinates* $\{u^I\}$*, defined on an open set containing or equal to* V*,* $\{\tilde{u}^I\}$ *are local coordinates with domain* V *and such that* $\tilde{u}^a$ *are solutions of* (6.6)*, and* $A_I^J = \frac{\partial u^J}{\partial \tilde{u}^I}$ *on the intersection of the domains of* $\{u^I\}$ *and* $\{\tilde{u}^I\}$*.*

Proof. Apply Proposition 6.1 for $e_I = \frac{\partial}{\partial u^I}$ and $\tilde{e}_I = \frac{\partial}{\partial \tilde{u}^I}$ and then use (3.2a) and (3.3). $\square$

This simple result gives a complete description of all normal frames, if any, adapted to (local) holonomic frames. It should be understood clearly, normal on W is the frame $\{\tilde{X}_I\}$, adapted to $\{\frac{\partial}{\partial \tilde{u}^I}\}$ and coinciding with it on W, but not the frame $\{\frac{\partial}{\partial \tilde{u}^I}\}$; in particular, the frame $\{\frac{\partial}{\partial \tilde{u}^I}\}$ is holonomic while the frame $\{\tilde{X}_I\}$ need not to be holonomic, even on W, if the connection considered does not satisfies some additional conditions, like the vanishment of its curvature on W.

Consider now briefly the existence problem for the solutions of (6.6). To begin with, we emphasize that in (6.6) enter only the fibre coordinates $\tilde{u}^a$, so that it leaves the basic ones $\tilde{u}^\mu$ completely arbitrary.

Proposition 6.4. *If* E *is of class* C^2*,* $p \in E$ *is fixed and* $W = \{p\}$*, then the general solution of* (6.6) *is*

$$\tilde{u}^a(q) = g^a + g_b^a\{-\Gamma_\mu^b(p)(q^\mu - p^\mu) + (q^b - p^b)\} + f_{IJ}^a(q)(q^I - p^I)(q^J - p^J), \quad (6.8)$$

where g^a *and* g_b^a *are constants in* $\mathbb{K} = \mathbb{R}, \mathbb{C}$*,* $\det[g_b^a] \neq 0, \infty$*, the point* q *is in the domain* V *of* $\{u^I\}$*,* $q^I := u^I(q)$*,* $p^I := u^I(p)$*, and* f_{IJ}^a *are* C^2 *functions on* V *such that they and their first partial derivatives are bounded when* $q^I \to p^I$*.*

Proof. Expand $\tilde{u}^a(q) = f^a(u^1(q), \ldots, u^n(q), \ldots, u^{n+r}(q)) = f^a(q^1, \ldots, q^{n+r})$ into a Taylor first order polynomial with remainder term quadratic in $(q^I - p^I)$ and insert the result into (6.6). In this way one gets (6.8) with $g^a = \tilde{u}^a(p)$ and $g_b^a = \frac{\partial \tilde{u}^a}{\partial u^b}\big|_p$. $\square$

Now we would like to investigate the existence of solutions of (6.6) along paths $\beta\colon J \to E$, i.e., for $W = \beta(J)$. The main result is formulated below as Proposition 6.5 and for its proof we shall essentially use Lemma II.3.1.

Proposition 6.5. *Let* Δ^h *be a* C^1 *connection on a real* C^3 *bundle* (E, π, M)*,* $n = \dim M \geq 1$*, and* $r = \dim \pi^{-1}(x) \geq 1$ *for* $x \in M$*. Let* $\beta\colon J \to E$ *be an injective regular* C^1 *path such that its tangent vector* $\dot{\beta}(s)$ *at* s *is* not *a vertical vector for all* $s \in J$*,* $\dot{\beta}(s) \notin \Delta_{\beta(s)}^v$*; in particular, the path* β *can be horizontal, i.e.,* $\dot{\beta}(s) \in \Delta_{\beta(s)}^h$ *for all* $s \in J$*, but generally the vector* $\dot{\beta}(s)$ *can have also and a vertical component for some or all* $s \in J$*. Then, for every* $s_0 \in J$*, there exist a neighborhood* W_1 *of the point* $\beta(s_0)$ *in* E *and bundle coordinates* $\tilde{u}^I$ *on* W_1 *which are solutions of* (6.6) *for* $W = W_1 \cap \beta(J) = \beta(J_1)$*, with* $J_1 := \{s \in J : \beta(s) \in W_1\}$*, i.e., along the restricted path* $\beta|_{J_1}$*. All such bundle coordinates* $\tilde{u}^I$ *are given via the equation* (6.9) *below.*

Proof. Consider the chart (W_1, u) with $W_1 \ni \beta(s_0)$ provided by Lemma II.3.1 for E and β instead of M and γ, respectively. For any $p \in W_1$, there is a unique $(s, \boldsymbol{t}) \in J_1 \times \mathbb{R}^{\dim_{\mathbb{R}} E-1}$ such that $p = u^{-1}(s, \boldsymbol{t})$, i.e., in the coordinate system $\{u^I\}$ associated to u, the coordinates of p are $u^1(p) = s$ and $u^I(p) = t^I \in \mathbb{R}$ for $I \geq 2$. Besides, we have $u(\beta(s)) = (s, \boldsymbol{t_0})$ for all $s \in J_1$ and some fixed $\boldsymbol{t_0} \in \mathbb{R}^{\dim_{\mathbb{R}} E-1}$.

Since $\dot\beta(s)$ is not a vertical vector for all $s \in J$, the coordinate system $\{u^I\}$ can be chosen to be *bundle* coordinate system. For the purpose, in the proof of Lemma II.3.1 one must choose $\{y^I\}$ as bundle coordinate system and to take for $\dot\beta^1_y(s_0)$ any non-vanishing component between $\dot\beta^1_y(s_0), \dots, \dot\beta^n_y(s_0)$, viz. if $\dot\beta^1_y(s_0) \neq 0$ the proof goes as it is written and, if $\dot\beta^1_y(s_0) = 0$, choose some μ_0 such that $\dot\beta^{\mu_0}_y(s_0) \neq 0$ and make, e.g., the change $\dot\beta^1_y(s_0) \leftrightarrow \dot\beta^{\mu_0}_y(s_0)$. This, together with (II.3.12), with u^I for x^k, ensures that $\{y^I\} \mapsto \{u^I\}$ is an admissible change, so that $\{u^I\}$ are bundle coordinates if the initial coordinates $\{y^I\}$ are such ones.

Let $\{u^I\}$ be so constructed bundle coordinates and $\eta := u^{-1}$, so that $\beta(s) = \eta(s, \boldsymbol{t}_0)$. Expanding $\tilde u^a(\eta(s, \boldsymbol{t}))$ into a first-order Taylor polynomial at the point $\boldsymbol{t}_0 \in K$, we find the general solution of (6.6), with $W = \beta(J_1) = W_1 \cap \beta(J)$, in the form

$$\begin{aligned}\tilde u^a(\eta(s,\boldsymbol{t})) &= B^a(s)\\ &\quad + B^a_b(s)\{-\Gamma^b_\mu(\beta(s))[u^\mu(\eta(s,\boldsymbol{t})) - u^\mu(\beta(s))] + [u^b(\eta(s,\boldsymbol{t})) - u^b(\beta(s))]\}\\ &\qquad + B^a_{IJ}(s,\boldsymbol{t};\eta)[u^I(\eta(s,\boldsymbol{t})) - u^I(\beta(s))][u^J(\eta(s,\boldsymbol{t})) - u^J(\beta(s))], \quad (6.9)\end{aligned}$$

where $B^a, B^a_b \colon J_1 \to \mathbb{K} = \mathbb{R}$, $\det[B^a_b] \neq 0, \infty$, and the C^1 functions B^a_{IJ} and their first partial derivatives are bounded when $\boldsymbol{t} \to \boldsymbol{t}_0$. (Notice, the terms with $\mu = 1$ and/or $I = 1$ and/or $J = 1$ do not contribute in (6.9) as $u^1(\eta(s, \boldsymbol{t})) \equiv s$ and, besides, the functions B^a_{IJ} can be taken symmetric in I and J, $B^a_{IJ} = B^a_{JI}$.) □

Remark 6.1. If there is $s_0 \in J$ for which $\dot\beta(s_0)$ is a vertical vector, $\dot\beta(s_0) \in \Delta^v_{\beta(s_0)}$, then Proposition 6.5 remains true with the only correction that the coordinates $\tilde u^I$ will *not be bundle* coordinates. If this is the case, the constructed coordinates $\tilde u^I$ will be solutions of (6.6), but we cannot assert that they are bundle coordinates which are (locally) normal along β in a neighborhood of the point $\beta(s_0)$.

Proposition 6.5 can be generalized by requiring β to be locally injective instead of injective, i.e., for each $s \in J$ to exist a subinterval $J_s \subseteq J$ such that $J_s \ni s$ and the restricted path $\beta|_{J_s}$ to be injective. Besides, if one needs a version of the above results for complex bundles, they should be considered as real ones (with doubled dimension of the manifolds) for which are applicable the above considerations.

Corollary 6.1. *At any arbitrarily fixed point in E and/or along a given injective regular C^1 path in E, whose tangent vector is not vertical, there exist* (*possibly local, in the latter case*) *normal frames.*

Proof. See Definition 6.1, Propositions 6.4 and 6.5, and equation (6.6). If the path is not contained in a single coordinate neighborhood, one should cover its image in the bundle space with such neighborhoods and, then, to apply Proposition 6.5; in the intersection of the coordinate domains, the uniqueness (and, possibly, continuity or differentiability) of the normal frames may be lost. □

Definition 6.2. Local bundle coordinates $\{\tilde{u}^I\}$, defined on an open set $V \subseteq E$, will be called *normal* on $W \subseteq V$ for a connection Δ^h if the frame $\{\tilde{X}_I\}$ in $T(E)$ adapted to $\{\frac{\partial}{\partial \tilde{u}^I}\}$ over V is normal for Δ^h on W.

Corollary 6.1 implies the existence of coordinates normal at a given point or (locally) along a given injective path whose tangent vector is not vertical; in particular, there exist coordinates normal along an injective horizontal path. However, normal coordinates generally do not exist on more general subsets of the bundle space E. A criterion for existence of coordinates normal on sufficiently general subsets $W \subseteq E$, e.g on 'horizontal' submanifolds, is given by Theorem 7.1 in Section 7 below. In particular, we have the following result.

Proposition 6.6. *If Δ^h is a C^1 connection, W is an open set in E, and normal frames for Δ^h on W exist, then there are holonomic such frames if Δ^h is flat on W. Said otherwise, the system of equations* (6.6) *may admit solutions on an open set W if*

$$R^a_{\mu\nu}|_W = 0 \tag{6.10}$$

where $R^a_{\mu\nu}$ are the fibre components of the curvature of Δ^h in some frame on E, defined by (3.37a) *in the frame $\{X_I\}$ adapted to a holonomic one.*

Proof. Since the normal frames are also adapted ones (see Definition 6.1) and $\Gamma^a_\mu|_W = 0$ in a frame normal on W, the statement is a corollary from Proposition 3.4. □

Remark 6.2. However, in the general case the flatness of a connection on an open set is only a necessary, but not sufficient, condition for the existence of coordinates normal on that set – see Theorem 7.1 in Section 7 below. Exceptions are the linear connections on vector bundles – see Remark 7.1 in Section 7.

Exercise 6.1. Show that part of the integrability conditions for (6.6) for an open set W are

$$0 = \frac{\partial^2 \tilde{u}^a}{\partial u^\nu \partial u^\mu} - \frac{\partial^2 \tilde{u}^a}{\partial u^\mu \partial u^\nu} \equiv \frac{\partial \tilde{u}^a}{\partial u^b} R^b_{\mu\nu} \tag{6.11}$$

from where Proposition 6.6 immediately follows. However, the flatness of the connection on W generally does not imply the rest of the integrability conditions, viz. $\frac{\partial^2 \tilde{u}^a}{\partial u^b \partial u^\mu} - \frac{\partial^2 \tilde{u}^a}{\partial u^\mu \partial u^b} = 0$ and $\frac{\partial^2 \tilde{u}^a}{\partial u^b \partial u^c} - \frac{\partial^2 \tilde{u}^a}{\partial u^c \partial u^b} = 0$.

The combination of Propositions 6.6 and 6.2 implies the non-existence of coordinates normal on an open set for non-flat (non-integrable) connections.

6.3. Normal frames on vector bundles

The normal frames in vector bundles for covariant derivative operators (linear connections), other derivations, and linear transports along paths were investigated in Chapter IV. The goal of the present subsection is to be made a link between them and the general theory of Subsection 6.1.

Consider a linear connection Δ^h on a vector bundle (E, π, M) (see Definition 4.1). Let the frame $\{e_I\}$ in $T(E)$ be given by (5.32) and $\{X_I\}$ be the frame adapted to $\{e_I\}$ for Δ^h. Then, by Proposition 5.4, the 2- and 3-index coefficients of Δ^h are connected via (5.33) in which $\{u^a\}$ is vector fibre coordinate system.

Proposition 6.7. *A frame $\{X_I\}$ is normal on $W \subseteq E$ for a linear connection Δ^h if and only if in it vanish the 3-index coefficients of Δ^h on $\pi(W) \subseteq M$,*

$$\Gamma^a_\mu|_W = 0 \iff \Gamma^a_{b\mu}|_{\pi(W)} = 0. \tag{6.12}$$

Proof. Since $u^{n+1}, \dots, u^{n+r}$ are 1-forms which are linearly independent for all $p \in W$, the assertion follows from equation (5.33). □

Combining Proposition 6.7 with (5.35), we see that the normal frame equation (6.1) in vector bundle is equivalent to

$$(B^\nu_\mu \Gamma^a_{b\nu} + B^a_{b\mu})|_{\pi(W)} = 0 \tag{6.13}$$

or to its matrix variant (see also (5.35′); $\Gamma_\nu := [\Gamma^a_{b\nu}]$, $B_\mu := [B^a_{b\mu}]$)

$$(B^\nu_\mu \Gamma_\nu + B_\mu)|_{\pi(W)} = 0. \tag{6.13′}$$

Taking into account (6.13) and (5.34), we can assert that the frame $\{\tilde{X}_I\}$ adapted to the frame

$$(\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ -((B^\lambda_\mu \Gamma^b_{c\lambda}) \circ \pi) \cdot u^c & B^b_a \circ \pi \end{bmatrix}, \tag{6.14}$$

where $[B^\nu_\mu]$ and $[B^a_b]$ are non-degenerate matrix-valued functions, is normal on W for Δ^h and hence $\tilde{X}_I = \tilde{e}_I$, by virtue of Proposition 6.2. Recall (see (5.15), (5.16), and (5.31)), the change $\{e_I\} \mapsto \{\tilde{e}_I\}$, given by (6.14), entails $\{X_I\} \mapsto \{\tilde{X}_I\}$, where

$$\tilde{X}_\mu = (B^\nu_\mu \circ \pi) X_\nu \quad \tilde{X}_a = (B^b_a \circ \pi) X_b, \tag{6.15}$$

which is equivalent to $\{E_I\} \mapsto \{\tilde{E}_I\}$ with

$$\tilde{E}_\mu = B^\nu_\mu E_\nu \quad \tilde{E}_a = B^b_a E_b. \tag{6.16}$$

Here (see (5.26)) $\{E_\mu = \pi_*|_{\Delta^h}(X_\mu)\}$ is a frame in $T(M)$ and $\{E_a = v^{-1}(X_a)\}$ is a frame in E.

Thus, if additional restriction are not imposed, the theory of normal frames in vector bundles is rather trivial, which reflects a similar situation in general bundles, considered in Subsection 6.1. However, the really interesting and sensible case is when one considers frames compatible with the covariant derivatives (see Section 5, the paragraphs including equations (5.38)–(5.40)). As we know (see (5.36)), it corresponds to arbitrary non-degenerate matrix-valued functions $[B_\mu^\nu]$ and $B = [B_b^a]$ and a matrix-valued functions $B_\mu = [B_{b\mu}^\nu]$ given by

$$B_\mu = \tilde{E}_\mu(B) \cdot B^{-1} = B_\mu^\nu E_\nu(B) \cdot B^{-1}. \tag{6.17}$$

In particular, such are all holonomic frames in $T(E)$, locally induced by local coordinates on E, as discussed in Section 4. Now the normal frames equation (6.13) (or (6.1)) reduces to

$$(\Gamma_\mu \cdot B + E_\mu(B))|_{\pi(W)} = 0 \tag{6.18}$$

which is exactly the equation (IV.14.54) for $N = M$, $g = \mathsf{id}_M$ and $U = \pi(W)$, i.e., the normal frame equation for a linear connection in a vector bundle. This equation leaves the frame $\{\tilde{E}_\mu = \pi_*|_{\Delta^h}(X_\mu)\}$ in $T(M)$ completely arbitrary and imposes restriction on the frame $\{\tilde{E}_a = v^{-1}(X_a) = B_a^b E_b\}$ in E. This conclusion justifies the following definition.

Definition 6.3. Given a *linear* connection Δ^h on a *vector* bundle (E, π, M) and a subset $W_M \subseteq M$. A *frame* $\{E_a\}$ *in* E, defined over an open set V_M containing W_M or equal to it, $V_M \supseteq W_M$, is called *normal for* Δ^h *over/on* W_M if their is a frame $\{X_I\}$ in $T(E)$, defined over an open set $V_E \subseteq E$, which is normal for Δ^h over a subset $W_E \subseteq E$ and such that $\pi(W_E) = W_M$, $\pi(V_E) = V_M$, and $E_a = v^{-1}(X_a)$, with the mapping v defined by (4.3). Respectively, $\{E_a\}$ is normal for Δ^h along a mapping $g\colon Q_M \to M$, $Q_M \neq \varnothing$, if $\{E_a\}$ is normal for Δ^h over $g(Q_M)$.

Taking into account Definition 6.1, we see that the so-defined normal frames in the bundle space E are just the ones used in the theory of frames normal for linear connections in vector bundles – see Definition IV.14.15 for $N = M$ and $g = \mathsf{id}_M$.

It is quite clear, to any frame $\{X_I\}$ in $T(E)$ normal over $W \subseteq E$, there corresponds a unique frame $\{E_a = v^{-1}(X_a)\}$ in E normal over $\pi(W) \subseteq M$. But, to a frame $\{E_a\}$ in E normal over $\pi(W)$, there correspond infinitely many frames $\{X_I\} = \{(\pi_*|_{\Delta^h})^{-1}(E_\mu), v(E_a)\}$ in $T(E)$ normal over W, where $\{E_\mu\}$ is an arbitrary frame in $T(M)$ over $\pi(W)$. Thus the problems of existence and (un)uniqueness of normal frames in $T(E)$ is completely reduced to the same problems for normal frames in E. The last kind of problems, as we noted at the beginning of the present section, are known and were investigated in Chapter IV.

We emphasize that a normal frame $\{E_a\}$ in E, as well as the basis $\{v(E_a)\}$ for Δ^v, can be holonomic as well as anholonomic (see Chapter IV); at the same time, a normal frame $\{X_I\}$ in $T(E)$ is anholonomic unless the conditions (5.25) hold (Corollary 5.2), a necessary condition being the flatness (integrability) of the horizontal distribution Δ^h.

Ending this section, let us say some words regarding frames normal for affine connections on vector bundles.

Proposition 6.8 (cf. Proposition 6.7). *A frame $\{X_I\}$ is normal on $U \subseteq E$ for an affine connection Δ^h, with 2-index coefficients* (5.45) *on U, if and only if in it is fulfilled*

$$\Gamma^a_{b\mu}|_{\pi(U)} = 0 \tag{6.19a}$$

$$G^a_{\mu}|_{\pi(U)} = 0. \tag{6.19b}$$

Proof. The assertion follows from Definition 6.1, equation (5.45), and the linear independence of the vector fibre coordinates $u^{n+1}, \dots, u^{n+r}$, considered as 1-forms. □

Corollary 6.2. *A necessary condition for existence of frames normal on $U \subseteq E$ for an affine connection is*

$$G^a_{\mu}|_{\pi(U)} = 0 \tag{6.20}$$

in all adapted frames on U; in particular, (6.20) *is equivalent to $G|_{(\pi(U))} = 0$ if U is an open set.*

Proof. Use (6.19b) and (5.47b). □

Corollary 6.3. *A necessary condition for existence of frames normal on $U \subseteq E$ for an affine connection is*

$$\Gamma^a_{\mu}|_U = -\{(\Gamma^a_{b\mu} \circ \pi) \cdot u^b\}|_U \tag{6.21}$$

in all adapted frames on U; in particular, if U is an open set, then (6.21) *means that the restriction of the connection considered on $(U, \pi|_U, \pi(U))$ is a linear connection.*

Proof. Apply Proposition 6.8 and Corollary 6.2. □

Corollary 6.4. *If an affine connection admits frames normal on $U \subseteq E$, then all of them are normal on U for the linear connection, corresponding to it via the mapping α constructed in the proof of Proposition* 4.7*, and* vice versa.

Proof. Use Corollary 6.3 and Proposition 6.7. □

Thus, if the condition (6.20) is satisfied, the above results completely reduce the problems of existence, (un)uniqueness and the properties of frames normal for affine connections to the same problems for linear connections (that correspond to them).

7. Coordinates normal along injective mappings with non-vanishing horizontal component

The purpose of this section is a multi-dimensional generalization of Proposition 6.5 in the real case, $\mathbb{K} = \mathbb{R}$. It is formulated below as Theorem 7.1. For its proof we shall essentially need Lemma III.8.1 on page 163.

Let (E, π, M) be a C^3 bundle endowed with C^1 connection Δ^h. Let $k \in \mathbb{N}$, $k \leq \dim M$, and J^k be an open set in $\mathbb{R}^k$. Consider a C^2 regular injective mapping $\beta \colon J^k \to E$ such that the vector fields $\dot{\beta}_\alpha \colon s \mapsto \dot{\beta}_\alpha(s) := \frac{\partial \beta^I(s)}{\partial s^\alpha} \frac{\partial}{\partial u^I}\big|_{\beta(s)}$, with $s := (s^1, \dots, s^k) \in J^k$ and $\alpha = 1, \dots, k$, do *not* belong to the vertical distribution Δ^v, $\dot{\beta}_\alpha(s) \notin \Delta^v_{\beta(s)}$ for all $s \in J^k$; in particular, the mapping β can be a horizontal mapping in a sense that $\dot{\beta}_\alpha(s) \in \Delta^h_{\beta(s)}$ for all $s \in J^k$, but generally these vectors can have a vertical component too. Our aim is to find the integrability conditions for the normal frame/coordinates equation (6.6) and its solutions, if any, when $U = \beta(J^k_1)$ for some subset $J^k_1 \subseteq J^k$.

Let us take some $s_0 \in J^k$ and construct the chart (U_1, u) with $U_1 \ni \beta(s_0)$ provided by Lemma III.8.1 with E for M and β for γ. If $J^k_1 := \{s \in J^k : \beta(s) \in U_1\}$ and $p \in U_1$, then there is a unique $(s, \boldsymbol{t}) \in J^k_1 \times \mathbb{R}^{\dim_\mathbb{R} E - k}$ such that $p = \eta(s, \boldsymbol{t})$ with $\eta := u^{-1}$, i.e., $u^I(p) = s^I$ for $I = 1, \dots, k$ and $u^I(p) = t^I$ for $I = k+1, \dots, n+r$. Besides, we have $u(\beta(s)) = (s, \boldsymbol{t}_0)$ for all $s \in J^k_1$ and some fixed $\boldsymbol{t}_0 \in \mathbb{R}^{\dim_\mathbb{R} E - k}$. Since the vector fields $\dot{\beta}_\alpha$, $\alpha = 1, \dots, k$, are not vertical, we can construct the coordinate system $\{u^I\}$, associated to the chart (U_1, u), so that they it be *bundle* coordinate system on U_1 (see the proof of Lemma III.8.1). Thus on U_1 we have bundle coordinates u^I such that

$$\begin{gathered}\big(u^1(\eta(s,\boldsymbol{t})), \dots, u^{n+r}(\eta(s,\boldsymbol{t}))\big) := (s, \boldsymbol{t}) \in \mathbb{R}^{n+r} \\ s = (s^1, \dots, s^k) \in J^k_1 \quad \boldsymbol{t} = (t^{k+1}, \dots, t^{n+r}) \in \mathbb{R}^{n+r-k}.\end{gathered} \tag{7.1}$$

Let the indices α and β run from 1 to k and the indices σ and τ take the values form $k+1$ to n; we set $\sigma = \tau = \varnothing$ if $k = n$. Thus, we have $u^\alpha(\eta(s,\boldsymbol{t})) = s^\alpha$, $u^\sigma(\eta(s,\boldsymbol{t})) = t^\sigma$, and $u^a(\eta(s,\boldsymbol{t})) = t^a$.

Proposition 7.1. *Under the hypotheses made above, the normal frame/coordinates equation* (6.6) *with* $U = \beta(J^k_1) = \beta(J^k) \cap U_1$ *has solutions if and only if the system of equations*

$$\Big(\frac{\partial \Gamma^b_\alpha}{\partial u^\beta} - \frac{\partial \Gamma^b_\beta}{\partial u^\alpha}\Big)\Big|_{\beta(s)} B^a_b(s) + \Gamma^b_\alpha(\beta(s)) \frac{\partial B^a_b(s)}{\partial s^\beta} - \Gamma^b_\beta(\beta(s)) \frac{\partial B^a_b(s)}{\partial s^\alpha} = 0, \tag{7.2}$$

where Γ^a_μ *are the 2-index coefficients of* Δ^h *in* $\{u^I\}$, *has solutions* $B^a_b \colon J^k_1 \to \mathbb{R}$ *with* $\det[B^a_b] \neq 0, \infty$. *Besides, if such solutions exist, then all solutions of* (6.6) *are given on* U_1 *by the formula*

$$\tilde{u}^a(\eta(s,\boldsymbol{t})) = -\int_{s_1}^{s} B_b^a(s)\Gamma_\alpha^b(\beta(s))\,\mathrm{d}s^\alpha$$
$$- B_b^a(s)\Gamma_\mu^b(\beta(s))[u^\mu(\eta(s,\boldsymbol{t})) - u^\mu(\beta(s))] + B_b^a(s)[u^b(\eta(s,\boldsymbol{t})) - u^b(\beta(s))]$$
$$+ f_{\mu\nu}^a(s;\boldsymbol{t};\eta)[u^\mu(\eta(s,\boldsymbol{t})) - u^\mu(\beta(s))][u^\nu(\eta(s,\boldsymbol{t})) - u^\nu(\beta(s))], \quad (7.3)$$

where $s_1 \in J_1^k$ *is arbitrarily fixed,* B_b^a*, with* $\det[B_b^a] \neq 0, \infty$*, are solutions of* (7.2)*, and the functions* $f_{\mu\nu}^a$ *and their first partial derivatives are bounded when* $\boldsymbol{t} \to \boldsymbol{t}_0$*.*

Remark 7.1. As $u^\alpha(\eta(s,\boldsymbol{t})) = u^\alpha(\beta(s)) \equiv s^\alpha$ for all $\alpha = 1,\dots,k$, the terms with $\mu,\nu = 1,\dots,k$ in (7.3) have vanishing contribution.

Remark 7.2. For $k = 1$, we have $\alpha = \beta = 1$, due to which the equations (7.2) are identically valid and Proposition 7.1 reduces to Proposition 6.5.

Proof. To begin with, we rewrite (6.6) as

$$\frac{\partial \tilde{u}^a}{\partial s^\alpha}\Big|_{\beta(s)} = -\frac{\partial \tilde{u}^a}{\partial t^b}\Big|_{\beta(s)}\Gamma_\alpha^b(\beta(s)) \quad \frac{\partial \tilde{u}^a}{\partial t^\sigma}\Big|_{\beta(s)} = -\frac{\partial \tilde{u}^a}{\partial t^b}\Big|_{\beta(s)}\Gamma_\sigma^b(\beta(s)).$$

Introducing a non-degenerate matrix-valued function $[B_b^a]$ on J_1^k by

$$B_b^a(s) = \frac{\partial \tilde{u}^a}{\partial t^b}\Big|_{\beta(s)} = \frac{\partial \tilde{u}^a(s,\boldsymbol{t})}{\partial t^b}\Big|_{\boldsymbol{t}=\boldsymbol{t}_0}, \quad (7.4)$$

we see that (6.6) is equivalent to

$$\frac{\partial \tilde{u}^a}{\partial s^\alpha}\Big|_{\beta(s)} = -B_b^a(s)\Gamma_\alpha^b(\beta(s)) \quad \alpha = 1,\dots,k \quad (7.5a)$$
$$\frac{\partial \tilde{u}^a}{\partial t^\sigma}\Big|_{\beta(s)} = -B_b^a(s)\Gamma_\sigma^b(\beta(s)) \quad \sigma = k+1,\dots,n. \quad (7.5b)$$

Expanding $\tilde{u}^a(\eta(s,\boldsymbol{t}))$ into a Taylor polynomial up to second-order terms relative to $(\boldsymbol{t} - \boldsymbol{t}_0)$ about the point $\boldsymbol{t}_0$ and using (7.4) and (7.5), we get:

$$\tilde{u}^a(\eta(s,\boldsymbol{t})) = f^a(s) - B_b^a(s)\Gamma_\sigma^b(\beta(s))[t^\sigma - t_0^\sigma] + B_b^a(s)[t^b - t_0^b]$$
$$+ f_{\sigma\tau}^a(s;\boldsymbol{t};\eta)[t^\sigma - t_0^\sigma][t^\tau - t_0^\tau]$$
$$= f^a(s) - B_b^a(s)\Gamma_\mu^b(\beta(s))[u^\mu(\eta(s,\boldsymbol{t})) - u^\mu(\beta(s))] + B_b^a(s)[u^b(\eta(s,\boldsymbol{t})) - u^b(\beta(s))]$$
$$+ f_{\mu\nu}^a(s;\boldsymbol{t};\eta)[u^\mu(\eta(s,\boldsymbol{t})) - u^\mu(\beta(s))][u^\nu(\eta(s,\boldsymbol{t})) - u^\nu(\beta(s))], \quad (7.6)$$

where f^a and $f_{\mu\nu}^a$ are C^1 functions and $f_{\mu\nu}^a$ and their first partial derivatives are bounded when $\boldsymbol{t} \to \boldsymbol{t}_0$. The equation (7.5a) is the only condition that puts some restrictions on f^a and B^a_b (besides $\det[B^a_b] \neq 0, \infty$). Inserting (7.6) into (7.5a) and using that $\beta(s) = \eta(s, \boldsymbol{t}_0)$, we obtain

$$\frac{\partial f^a(s)}{\partial s^\alpha} = -B^a_b(s)\Gamma^b_\alpha(\beta(s)). \tag{7.7}$$

Thus the initial normal coordinates equation (6.6), with $U = \beta(J^k_1)$, has solutions if and only if there exist solutions of (7.7) relative to f^a and/or B^a_b. The integrability conditions for (7.7) are [34]

$$0 = \frac{\partial^2 f^a}{\partial s^\beta \partial s^\alpha} - \frac{\partial^2 f^a}{\partial s^\alpha \partial s^\beta} = -\frac{\partial}{\partial s^\beta}\big(B^A_b(s)\Gamma^b_\alpha(s))\big) + \frac{\partial}{\partial s^\alpha}\big(B^A_b(s)\Gamma^b_\beta(s))\big) = \cdots$$

and coincide with (7.2), by virtue of $u^\alpha(\beta(s)) = s^\alpha$. This result concludes the proof of the first part of the proposition.

If (7.2) admits solutions B^a_b with $\det[B^a_b] \neq 0, \infty$, then the general solution of (7.7) is $f^a(s) = -\int_{s_1}^{s} B^a_b(s)\Gamma^b_\alpha(\beta(s))\,\mathrm{d}s^\alpha$ for some $s_1 \in J^k_1$ and this solution is independent of the integration path in J^k_1, due to (7.2). □

Lemma 7.1. *Let (E, π, M) be a C^3 bundle endowed with C^2 connection with coefficients Γ^a_μ in the frame adapted to local coordinate system $\{u^i\}$, defined before Proposition* 7.1. *There exist solutions B^a_b with $\det[B^a_b] \neq 0, \infty$ of the system of equations* (7.2) *if and only if the coefficients Γ^a_μ satisfy the equations*

$$R^a_{\alpha\beta}(\beta(s)) = 0 \qquad s \in J^k_1 \tag{7.8a}$$

$$\Big(\Gamma^d_\alpha \frac{\partial^2 \Gamma^c_\beta}{\partial u^b \partial u^d} - \Gamma^d_\beta \frac{\partial^2 \Gamma^c_\alpha}{\partial u^b \partial u^d}\Big)\Big|_{\beta(s)} = 0 \qquad s \in J^k_1 \tag{7.8b}$$

in which $R^a_{\mu\nu}$ are the (fibre) components in $\{u^I\}$ of the curvature of Δ^h, defined by (3.37a)*. If the conditions* (7.8) *are valid, the set of the solutions of* (7.2) *coincides with the set of solutions of the system*

$$\frac{\partial B^a_b(s)}{\partial s^\alpha} = -B^a_c(s)\frac{\partial \Gamma^c_\alpha}{\partial u^b}\Big|_{\beta(s)} + \frac{\partial D^a_b(s)}{\partial s^\alpha} \tag{7.9}$$

relative to B^a_b, where D^a_b are solutions of

$$\Big(\Gamma^b_\alpha(\beta(s))\frac{\partial}{\partial s^\beta} - \Gamma^b_\beta(\beta(s))\frac{\partial}{\partial s^\alpha}\Big)D^a_b(s) = 0. \tag{7.10}$$

Proof. Consider the integrability condition (7.2) for (6.6) in more details. Define functions $D^a_{b\alpha}\colon J^k_1 \to \mathbb{K} = \mathbb{R}$ via the equation

$$\frac{\partial B^a_b(s)}{\partial s^\alpha} = -B^a_c(s)\frac{\partial \Gamma^c_\alpha}{\partial u^b}\Big|_{\beta(s)} + D^a_{b\alpha}(s). \tag{7.11}$$

The substitution of this equality into (7.2) results in

$$R^b_{\beta\alpha}(\beta(s))B^a_b(s) - \Gamma^b_\alpha(\beta(s))D^a_{b\beta}(s) + \Gamma^b_\beta(\beta(s))D^a_{b\alpha}(s) = 0,$$

where the functions $R^a_{\alpha\beta}$ are the (fibre) components in $\{u^I\}$ of the curvature of Δ^h, defined by (3.37a). The simple observation that $\tilde{u}^\alpha$ and $\tilde{u}^a$, if they exist as solutions of (6.6), are normal coordinates on the whole bundle space of the restricted bundle $(U, \pi|_U, \pi(U))$ with $U = \beta(J^k_1)$ leads to

$$R^a_{\alpha\beta}(\beta(s)) = 0 \qquad s \in J^k_1, \tag{7.12}$$

by virtue of Proposition 6.6. Therefore the previous equation reduces to

$$\Gamma^b_\alpha(\beta(s))D^a_{b\beta}(s) - \Gamma^b_\beta(\beta(s))D^a_{b\alpha}(s) = 0. \tag{7.13a}$$

It is clear that (7.11)–(7.13a) are equivalent to (7.2). Consequently, the quantities $D^a_{b\alpha}$ must be solutions of (7.13a) while the C^1 functions B^a_b have to be solutions of (7.11). The integrability conditions $\big(\frac{\partial^2}{\partial s^\beta \partial s^\alpha} - \frac{\partial^2}{\partial s^\alpha \partial s^\beta}\big)B^a_b(s) = 0$ for (7.11) can be written as[1]

$$\Big(-\frac{\partial^2\Gamma^c_\alpha}{\partial u^\beta \partial u^b} + \frac{\partial^2\Gamma^c_\beta}{\partial u^\alpha \partial u^b} + \frac{\partial\Gamma^d_\alpha}{\partial u^b}\frac{\partial\Gamma^c_\beta}{\partial u^d} - \frac{\partial\Gamma^d_\beta}{\partial u^b}\frac{\partial\Gamma^c_\alpha}{\partial u^d}\Big)\Big|_{\beta(s)} B^a_c(s) + \frac{\partial D^a_{b\alpha}(s)}{\partial s^\beta} - \frac{\partial D^a_{b\beta}(s)}{\partial s^\alpha} = 0$$

which conditions split into

$$0 = \frac{\partial D^a_{b\alpha}(s)}{\partial s^\beta} - \frac{\partial D^a_{b\beta}(s)}{\partial s^\alpha} \tag{7.13b}$$

$$\begin{aligned} 0 &= \Big(-\frac{\partial^2\Gamma^c_\alpha}{\partial u^\beta \partial u^b} + \frac{\partial^2\Gamma^c_\beta}{\partial u^\alpha \partial u^b} + \frac{\partial\Gamma^d_\alpha}{\partial u^b}\frac{\partial\Gamma^c_\beta}{\partial u^d} - \frac{\partial\Gamma^d_\beta}{\partial u^b}\frac{\partial\Gamma^c_\alpha}{\partial u^d}\Big)\Big|_{\beta(s)} \\ &= \Big(-\Gamma^d_\alpha\frac{\partial^2\Gamma^c_\beta}{\partial u^b \partial u^d} + \Gamma^d_\beta\frac{\partial^2\Gamma^c_\alpha}{\partial u^b \partial u^d}\Big)\Big|_{\beta(s)}, \end{aligned} \tag{7.14}$$

where (7.12) and (3.37a) were applied in the derivation of the second equality in (7.14).

Since the system of equations (7.13) always has solutions, e.g., $D^b_{b\alpha}(s) = 0$, we can assert that (7.12) and (7.14) are the integrability conditions for (7.2) and, if (7.12) and (7.14) hold, every solution of (7.11), with $D^a_{b\alpha}$ satisfying (7.13), is a solution of (7.2) and *vice versa.*

At the end, the only unproved assertion is that $D^a_{b\alpha}$ in (7.11) equals to $\partial_\alpha(D^a_b)$ with D^a_b satisfying (7.10). Indeed, since J^k_1 is an open set and hence is contractible

[1] At this point one should require Δ^h to be of class C^2 which is possible if the manifolds E and M are of class C^3.

one, the Poincaré's lemma (see [6, Section 6.3] or [5, pp. 21, 106]) implies the existence of functions D^a_b on J^k_1 such that $D^b_{b\alpha}(s) = \partial_\alpha(D^a_b)(s)$, due to (7.13b); inserting this result into (7.13a), we get (7.10). □

Remark 7.3. Regardless that the conditions (7.8b) look quite special, they are identically valid for connections with

$$\Gamma^a_\alpha = -(\Gamma^a_{b\alpha} \circ \pi) \cdot u^b + G^a_\alpha \circ \pi, \tag{7.15}$$

where $\Gamma^a_{b\alpha}$ and G^a_α are C^2 functions on $\pi(\beta(J^k))$, i.e., for affine connections (see Subsection 4.4). In particular, of this kind are the linear connections on vector bundles – see Proposition 5.4 and 5.5.

At last, we shall formulate the main result of the above considerations as a combination of Proposition 7.1 and Lemma 7.1.

Theorem 7.1. *Let* (E, π, M) *be a* C^3 *bundle endowed with a* C^2 *connection. Under the hypotheses made and notation introduced before Proposition* 7.1*, there exist solutions of the normal frame/coordinates equation* (6.6) *if and only if the connection's coefficients satisfy the equations* (7.8)*. If these equations hold, all coordinates normal on* $\beta(J^k_1)$ *are given on* U_1 *by* (7.3)*, where* B^a_b *are solutions of* (7.9)*, with* D^a_b *being solutions of* (7.10)*.*

Remark 7.4. If there are $s_0 \in J^k$ and $\alpha \in \{1, \dots, k\}$ such that the vector $\dot\beta_\alpha(s_0)$ is a vertical vector, $\dot\beta_\alpha(s_0) \in \Delta^v_{\beta(s_0)}$, then Theorem 7.1 remains true with the only correction that the coordinate system $\{u^I\}$ will *not be bundle* coordinate system. If this is the case, the constructed coordinates $\tilde u^I$ will be solutions of (6.6), but we cannot assert that they are bundle coordinates which are (locally) normal along β in a neighborhood of the point $\beta(s_0)$.

Theorem 7.1 provides a necessary and sufficient condition for the existence of local coordinates in a neighborhood of $\beta(s_0)$ for any $s_0 \in J^k$ which are locally normal along β, i.e., on $\beta(J^k_1)$ for some open subset $J^k_1 \subseteq J^k$ containing s_0. Moreover, if this condition is valid, the theorem describes locally all coordinates normal along β.

Exercise 7.1. Prove that Theorem 7.1 remains valid by requiring β to be locally injective instead of injective, i.e., for each $s \in J$ to exist subset $J^k_s \subseteq J^k$ such that $J^k_s \ni s$ and the restricted mapping $\beta|_{J^k_s}$ to be injective.

If one needs a version of the above results for complex bundles, they should be considered as real ones (with doubled dimension of the manifolds) for which are applicable the above considerations.

8. Links between connections and transports along paths in fibre bundles

As the title of this section indicates, its content is outside of the main topic of the present book. It generalizes part of Section IV.14 and investigates relations between some axiomatic approaches to the general theories of connections, parallel transports, and transports along paths. We hope that the material below will clarify some problems that may have arisen in Chapter IV and will be useful for readers interested in the axiomatization of the concept of a 'parallel transport'.

The widespread approach to the concept of a "parallel transport" is it to be considered as a secondary one and defined on the basis of the connection theory [6,7,10–13,16,28,60,98,106,107,117,141–145]. However, the opposite approach, in which the parallel transport is axiomatically defined and from it the connection theory is constructed, is also known [17,23,30–33,91,147–150] and goes back to 1949[1]; e.g., it is systematically realized in [23], where the connection theory on vector bundles is investigated. In [114] the concept of a "parallel transport" was generalize to the one of "transport along paths". The relations between both concepts were analyzed in [115]; in particular, Theorem 3.1 of [115, p. 13] (see Theorem 8.1 below) contains a necessary and sufficient condition for a transport along paths to be (axiomatically defined) parallel transport. The aim of the present section is to be investigated some links between general connections on fibre bundles and transport along paths in them. Recall that similar problems, but in the linear case in vector bundles, were explored in Section IV.14.

The bundle and base spaces of the bundles in this section are supposed to be of differentiable of class C^1; however, some parts of the text below, like Definition 8.1, are valid in more general situations.

Definition 8.1. A *transport along paths* in bundle (E, π, B) is a mapping I assigning to every path $\gamma\colon J \to M$ a mapping I^γ, termed *transport along* γ, such that $I^\gamma\colon (s,t) \mapsto I^\gamma_{s\to t}$ where the mapping

$$I^\gamma_{s\to t}\colon \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \qquad s,t \in J, \tag{8.1}$$

called *transport along* γ *from* s *to* t, has the properties:

$$I^\gamma_{s\to t} \circ I^\gamma_{r\to s} = I^\gamma_{r\to t} \qquad r,s,t \in J \tag{8.2}$$

$$I^\gamma_{s\to s} = \mathsf{id}_{\pi^{-1}(\gamma(s))} \qquad s \in J, \tag{8.3}$$

[1] It seems that the earliest written accounts on this approach are the ones due to Ü.G. Lumiste [30, Section 2.2] and C. Teleman [17, Chapter IV, Section B.3] (both published in 1964), the next essential steps being made by P. Dombrowski [31, § 1] and W. Poor [23]. Besides, the author of [31] states that his paper is based on unpublished lectures of prof. Willi Rinow (1907–1979) in 1949; see also [23, p. 46] where the author claims that the first axiomatical definition of a parallel transport in the tangent bundle case is given by prof. W. Rinow in his lectures at the Humboldt University in 1949. Some heuristic comments on the axiomatic approach to parallel transport theory can be found in [8, Section 2.1] too.

where $\circ$ denotes composition of mappings and id_X is the identity mapping of a set X.

Remark 8.1. If $(E, \pi, M))$ is a vector bundle and the mappings (8.1) are linear, Definition 8.1 reduces to Definition IV.3.1.

An analysis and various comments on this definition can be found in [87,102, 114,115]; see also Section IV.3.

As we shall see below, an important special class of transports along paths is selected by the conditions

$$I^{\gamma|J'}_{s\to t} = I^{\gamma}_{s\to t} \qquad s,t\in J' \tag{8.4}$$
$$I^{\gamma\circ\chi}_{s\to t} = I^{\gamma}_{\chi(s)\to\chi(t)} \qquad s,t\in J'', \tag{8.5}$$

where $J' \subseteq J$ is a subinterval, $\gamma|J'$ is the restriction of γ to J', and $\chi\colon J'' \to J$ is a bijection of a real interval J'' onto J.

Putting $r = t$ in (8.2) and using (8.3), we see that the mappings (8.1) are invertible and

$$(I^{\gamma}_{s\to t})^{-1} = I^{\gamma}_{t\to s}. \tag{8.6}$$

The following result describes how a transport along paths generates a connection.

Proposition 8.1. *Let I be a transport along paths in a bundle (E, π, M). Let $\gamma\colon J \to M$ be a path and, for any $s_0 \in J$ and $p \in \pi^{-1}(\gamma(s_0))$, the lift $\bar\gamma_{s_0,p}\colon J \to E$ of γ be defined by*

$$\bar\gamma_{s_0,p}(t) = I^{\gamma}_{s_0\to t}(p) \qquad t\in J. \tag{8.7}$$

Suppose that I is such that $\bar\gamma_{s_0,p}$ is a C^1 path for every s_0 and p and the conditions (8.4) *and* (8.5) *hold for all paths $\gamma\colon J \to M$. Then the distribution*

$$\Delta^I : p \mapsto \Delta^I_p := \Big\{ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s_0} (\bar\gamma_{s_0,p}(t)) \\ : \gamma\colon J\to M \text{ is } C^1 \text{ and injective, } s_0\in J,\ \gamma(s_0) = \pi(p) \Big\} \subseteq T_p(E), \tag{8.8}$$

with $p \in E$, is a connection on (E, π, M), i.e.,

$$\Delta^v_p \oplus \Delta^I_p = T_p(E) \qquad p\in E, \tag{8.9}$$

with Δ^v being the vertical distribution on E, $\Delta^v_p = T_p(\pi^{-1}(\pi(p)))$.

Proof. Let $\{u^\mu = x^\mu\circ\pi, u^a\}$ be bundle coordinate system on E, p be a point in its domain, and $\bar\gamma\colon J\to E$ be a lift of $\gamma\colon J \to M$, $\pi\circ\bar\gamma = \gamma$. Since $\pi_*\big(\frac{\partial}{\partial u^I}\big|_p\big) = \frac{\partial(x^\mu\circ\pi)}{\partial u^I}\big|_p \frac{\partial}{\partial x^\mu}\big|_{\pi(p)}$ and $\dot{\bar\gamma}^\mu(t) = \frac{\mathrm{d}(u^\mu\circ\bar\gamma(t))}{\mathrm{d}t} = \frac{\mathrm{d}(x^\mu\circ\gamma(t))}{\mathrm{d}t} = \dot\gamma^\mu(t)$ for all $t\in J$, we have

$$\pi_*(\dot{\bar\gamma}(t)) = \dot\gamma(t) \tag{8.10}$$

for any lift $\bar{\gamma}$ in E of a path γ in M. Therefore, from (8.8) and (8.10), we get

$$\pi_*(\Delta_p^I) = \{\dot{\gamma}(s_0) : \gamma\colon J \to M \text{ is } C^1 \text{ and injective}, s_0 \in J,\ \gamma(s_0) = \pi(p)\} \\ = T_{\pi(p)}(M)$$

as $\dot{\gamma}(s_0)$ is an arbitrary vector in $T_{\pi(p)}(M) = T_{\gamma(s_0)}(M)$. Thus the mapping $\pi_*|_{\Delta_p^I}\colon \Delta_p^I \to T_{\pi(p)}(M)$ is surjective. It is also linear, due to the definition of a tangent mapping [7, Section 1.22]. At last, we shall prove that $\pi_*|_{\Delta_p^I}$ is injective, from where it follows that $\pi_*|_{\Delta_p^I}$ is a vector space isomorphism for every $p \in E$ which, in its turn, implies (8.9) as $\pi_*(\Delta_p^v) = 0_{\pi(p)} \in T_{\pi(p)}(M)$.

If $G_i \in \Delta_p^I$, $i = 1, 2$, then there exist paths $\gamma_i\colon J_i \to M$ such that $\gamma_i(s_i) = \pi(p)$ for some $s_i \in J_i$ and $G_i = \dot{\bar{\gamma}}_{i;s_i,p}$, with $i = 1, 2$ and the lifted paths in the right-hand side being given by (8.7). Then $\pi_*(G_i) = \dot{\gamma}_i(s_i)$, due to (8.10). Suppose that $\dot{\gamma}_1(s_1) = \dot{\gamma}_2(s_2)$. Since $\gamma_1(s_1) = \gamma_2(s_2) = \pi(p)$, the last equality entails $s_1 = s_2$, $J_1 \cap J_2 \neq \varnothing$, and the existence of an interval $J' \subseteq J_1 \cap J_2$ such that $s_1 = s_2 \in J'$ and $\gamma_1|J' = \gamma_2|J'$.[2] Combining this result with (8.4), we get $G_1 = G_2$, which means that $\pi_*|_{\Delta_p^I}\colon \Delta_p^I \to T_{\pi(p)}(M)$ is injective. □

Remark 8.2. The condition (8.5) was not used explicitly in the proof of Proposition 8.1, but it is important in the definition (8.8) of the connection Δ^I. Namely, it ensures that, if $X \in \Delta_p^I$, then $\varkappa X \in \Delta_p^I$ for all $\varkappa \in \mathbb{K}$. Indeed, if $X = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=s_0}(\bar{\gamma}_{s_0,p}(t))$ for some path $\gamma\colon J \to M$ and $\chi\colon J' \to J$ is a C^1 diffeomorphism, the vector tangent to the path $\bar{\beta}_{s'_0,p}(t')$ at $s'_0 = \chi^{-1}(s_0)$, with $\beta = \gamma \circ \chi$, is $\chi(s'_0)X$, by virtue of (8.7) and (8.5). So that $\chi(s'_0)X \in \Delta_p^I$ and the arbitrariness of χ leads to $\varkappa X \in \Delta_p^I$ for all $\varkappa \in \mathbb{K}$.

Remark 8.3. The role of the transport I along paths in Proposition 8.1 is on its base to be constructed a lifting of the paths in M to paths in E with appropriate properties. Namely, such a lifting should assign to a path $\gamma\colon J \to M$ a unique path $\bar{\gamma}_{s_0,p}\colon J \to E$ passing through a given point $p \in \pi^{-1}(\gamma(s_0))$, for some $s_0 \in J$, and such that $\pi \circ \bar{\gamma}_{s_0,p} = \gamma$ and, if $q \in \bar{\gamma}_{s_0,p}(t_0)$ for some $t_0 \in J$, then $\bar{\gamma}_{t_0,q} = \bar{\gamma}_{s_0,p}$. On this ground one can generalize Proposition 8.1 as well as some of the next considerations and results.

Definition 8.2. The connection Δ^I, defined in Proposition 8.1, will be called *assigned to* (*defined by, generated by*) the transport I along paths.

For the further exploration of the relations between transports along paths and connections (or parallel transports generated by them), we shall need the notion of an inverse path and of a product of paths. There are not 'natural' definitions of these concepts, but this is not important for us as the (parallel) transports we shall consider below are parametrization invariant in some sense, like (8.5).

[2] Here we use the local existence of a unique C^1 path passing trough a given point and having a fixed tangent vector at it.

For that reason, the concepts mentioned will be defined only for *canonical paths* $[0,1] \to M$, whose domain is the real interval $[0,1] := \{r \in \mathbb{R} : 0 \le r \le 1\}$. The path *inverse* to $\gamma\colon [0,1] \to M$ is $\gamma_{\text{-}} := \gamma \circ \tau_{\text{-}}\colon [0,1] \to M$, with $\tau_{\text{-}}\colon [0,1] \to [0,1]$ being given by $\tau_{\text{-}}(t) := 1 - t$ for $t \in [0,1]$. If $\gamma_1, \gamma_2\colon [0,1] \to M$ and $\gamma_1(1) = \gamma_2(0)$, the *product* $\gamma_1\gamma_2$ of γ_1 and γ_2 is a canonical path $\gamma_1\gamma_2\colon [0,1] \to M$ such that $(\gamma_1\gamma_2)(t) := \gamma_1(2t)$ for $t \in [0,1/2]$ and $(\gamma_1\gamma_2)(t) := \gamma_2(2t-1)$ for $t \in [1/2,1]$. For more details on this item, see [108, 132].

Recall now the basic properties of the parallel transports generated by connections.

Proposition 8.2. *Let*

$$\mathsf{P}\colon \gamma \mapsto \mathsf{P}^{\gamma}\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma,\tau] \to M \tag{8.11}$$

be the parallel transport generated by a connection on some bundle (E,π,M). *The mapping* P *has the following properties:*

(i) *The parallel transport* P *is invariant under orientation preserving changes of the paths' parameters. Precisely, if* $\gamma\colon [\sigma,\tau] \to M$ *and* $\chi\colon [\sigma',\tau'] \to [\sigma,\tau]$ *is an orientation preserving* C^1 *diffeomorphism, then*

$$\mathsf{P}^{\gamma\circ\chi} = \mathsf{P}^{\gamma}. \tag{8.12}$$

(ii) *If* $\gamma\colon [0,1] \to M$ *and* $\gamma_{\text{-}}\colon [0,1] \to M$ *is its canonical inverse,* $\gamma_{\text{-}}(t) = \gamma(1-t)$ *for* $t \in [0,1]$, *then*

$$\mathsf{P}^{\gamma_{\text{-}}} = \left(\mathsf{P}^{\gamma}\right)^{-1}. \tag{8.13}$$

(iii) *If* $\gamma_1, \gamma_2\colon [0,1] \to M$, $\gamma_1(1) = \gamma_2(0)$, *and* $\gamma_1\gamma_2\colon [0,1] \to M$ *is their canonical product, then*

$$\mathsf{P}^{\gamma_1\gamma_2} = \mathsf{P}^{\gamma_2} \circ \mathsf{P}^{\gamma_1}. \tag{8.14}$$

(iv) *If* $\gamma_{r,x}\colon \{r\} = [r,r] \to \{x\}$ *for some given* $r \in \mathbb{R}$ *and* $x \in M$, *then*

$$\mathsf{P}^{\gamma_{r,x}} = \mathsf{id}_{\pi^{-1}(x)}. \tag{8.15}$$

Remark 8.4. As a result of (8.12), some properties of the parallel transports generated by connections, like (8.13) and (8.14), are sufficient to be formulated/proved only for canonical paths $[0,1] \to M$.

Proof. The proofs of (8.12)–(8.15) can be found in a number of works, for example in [3, 4, 6, 11, 30, 32, 33, 148, 149, 154]. Alternatively, the reader can prove them by applying the definitions given in this book. (See also Subsections IV.14.1 and IV.14.2 in a case of a vector bundle.) □

Definition 8.3 (cf. Definition IV.14.11 on page 313)**.** A mapping (8.11) satisfying the equalities (8.12)–(8.15) will be called (*axiomatically defined*) *parallel transport.*

Proposition 8.3. *Let* P *be the parallel transport assigned to a* C^m, *with* $m \in \mathbb{N}\cup\{0\}$, *connection on a* smooth, *of class* C^{m+1}, *bundle* (E, π, M). *Then* P *is smooth, of class* C^m, *in a sense that, if* $\gamma\colon [\sigma, \tau] \to M$ *is a* C^1 *path, then* P^γ *is in the set of* C^m *diffeomorphisms between the fibres* $\pi^{-1}(\gamma(\sigma))$ *and* $\pi^{-1}(\gamma(\tau))$,

$$\mathsf{P}\colon \gamma \mapsto \mathsf{P}^\gamma \in \mathrm{Diff}^m\bigl(\pi^{-1}(\gamma(\sigma)), \pi^{-1}(\gamma(\tau))\bigr) \qquad \gamma\colon [\sigma, \tau] \to M. \tag{8.16}$$

Proof. See [16, 106, 107]. □

The axiomatic approach to parallel transport was developed mainly on the ground on the properties (8.12)–(8.16) of the parallel transports assigned to connections. However, this topic is out of the range of the present monograph and the reader is referred to the literature cited at the beginning of the present section.

Ending with the results we take for granted, we reproduce below a slightly modified version of [115, p. 13, Theorem 3.1].

Theorem 8.1. *Let* I *be a transport along paths in bundle* (E, π, M) *and* $\gamma\colon [\sigma, \tau] \to M$. *If* I *satisfies the conditions* (8.4) *and* (8.5), *then the mapping*

$$\mathsf{I}\colon \gamma \mapsto \mathsf{I}^\gamma := I^\gamma_{\sigma\to\tau}\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma, \tau] \to M \tag{8.17}$$

is a parallel transport, i.e., it possess the properties (8.12)–(8.15), *with* I *for* P. *Besides, if* I *is smooth in a sense that*

$$I^\beta_{s\to t} \in \mathrm{Diff}^m\bigl(\pi^{-1}(\beta(s)), \pi^{-1}(\beta(t))\bigr) \qquad \beta\colon J \to M \quad s, t \in J \tag{8.18}$$

for some $m \in \mathbb{N} \cup \{0\}$, *then the mapping* (8.17) *satisfies* (8.16), *with* I *for* P.

Conversely, suppose the mapping

$$\mathsf{P}\colon \gamma \mapsto \mathsf{P}^\gamma\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma, \tau] \to M \tag{8.19}$$

is a parallel transport, i.e., satisfies (8.12)–(8.15), *and define the mapping*

$$P\colon \beta \mapsto P^\beta\colon (s, t) \mapsto P^\beta_{s\to t} = \mathsf{P}^{(\beta|[\sigma,\tau])\circ\chi_t^{[\sigma,\tau]}} \circ \Bigl(\mathsf{P}^{(\beta|[\sigma,\tau])\circ\chi_s^{[\sigma,\tau]}}\Bigr)^{-1} \qquad \beta\colon J \to M, \tag{8.20}$$

where $s, t \in J$, $\sigma, \tau \in J$ *are such that* $\sigma \le \tau$ *and* $[\sigma, \tau] \ni s, t$,[3] *and* $\chi_s^{[\sigma,\tau]}\colon [\sigma, \tau] \to [\sigma, s]$ *are for* $s > \sigma$ *arbitrary orientation preserving* C^1 *diffeomorphisms* (*depending on* β *via the interval* $[\sigma, \tau]$). *Then the mapping* (8.20) *is a transport along paths in* (E, π, M), *which transport satisfies the conditions* (8.4) *and* (8.5), *with* P *for* I. *Besides, under the same assumptions, the condition* (8.16) *for* P *implies* (8.18), *with* P *for* I, *where* P *is given by* (8.20).

---

[3]In particular, one can set $\sigma = \min(s, t)$ and $\tau = \max(s, t)$ or, if J is a closed interval, define σ and τ as the end points of J, i.e., $J = [\sigma, \tau]$.

Remark 8.5. Instead by (8.20), the transport P along paths generated by a parallel transport P can be defined equivalently as follows. For a path $\gamma\colon [\sigma,\tau] \to M$ and $s,t \in [\sigma,\tau]$, we put.

$$P\colon \gamma \mapsto P^\gamma\colon (s,t) \mapsto P^\gamma_{s\to t} = \mathsf{P}^{\gamma\circ\chi_t^{[\sigma,\tau]}} \circ \Bigl(\mathsf{P}^{\gamma\circ\chi_s^{[\sigma,\tau]}}\Bigr)^{-1} = \mathsf{P}^{\gamma|[\sigma,t]} \circ \bigl(\mathsf{P}^{\gamma|[\sigma,s]}\bigr)^{-1}$$
$$\gamma\colon [\sigma,\tau] \to M \tag{8.21a}$$

Now, for an arbitrary path $\beta\colon J \to M$, with J being closed or open at one or both its ends, we set

$$P\colon \beta \mapsto P^\beta\colon (s,t) \mapsto P^\beta_{s\to t} = \begin{cases} P^{\beta|[s,t]}_{s\to t} & \text{for } s \le t \\ \bigl(P^{\beta|[t,s]}_{t\to s}\bigr)^{-1} & \text{for } s \ge t \end{cases} \qquad \beta\colon J \to M \quad s,t \in J. \tag{8.21b}$$

It can easily be verified that (8.21) are tantamount to

$$P\colon \beta \mapsto P^\beta\colon (s,t) \mapsto P^\beta_{s\to t} = \begin{cases} \mathsf{P}^{\beta|[s,t]} & \text{for } s \le t \\ \bigl(\mathsf{P}^{\beta|[t,s]}\bigr)^{-1} & \text{for } s \ge t \end{cases} \qquad \beta\colon J \to M \quad s,t \in J. \tag{8.22}$$

Definition 8.4. A transport I along paths which has the properties (8.4) and (8.5) will be called *parallel* transport along paths.

Theorem 8.1 simply says that there is a bijective correspondence between the parallel transports along paths and the parallel transports.

Definition 8.5. If I is a parallel transport along paths, then we say that the parallel transport (8.17) is *generated by* (*defined by, assigned to*) I. Respectively, if P is a parallel transport, then we say that the (parallel) transport along paths (8.20) is *generated by* (*defined by, assigned to*) P.

Let us now return to the connection Δ^I generated by a transport I along paths, introduced in Proposition 8.1.

Proposition 8.4. *If $\gamma\colon J \to M$ is an injective path and $p \in \pi^{-1}(\gamma(s_0))$ for some $s_0 \in J$, then there is a unique Δ^I-horizontal lift of γ (relative to Δ^I) in E through p and it is exactly the path $\bar\gamma_{s_0,p}$ defined by* (8.7).

Proof. Simply apply the definitions (8.7) and (8.8) and use the properties (8.1) and (8.2) of the transports along paths ($t_0 \in J$):

$$\dot{\bar\gamma}_{s_0,p}(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \bigl(I^\gamma_{s_0\to t}(p)\bigr) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \bigl(I^\gamma_{t_0\to t}(I^\gamma_{s_0\to t_0}(p))\bigr) \in \Delta^I_{\bar\gamma(t_0)}. \qquad \square$$

Remark 8.6. If γ is not injective and $\gamma(s_0) = \gamma(t_0)$ for some $s_0, t_0 \in J$ such that $s_0 \ne t_0$, then the paths $\bar\gamma_{s_0,p}, \bar\gamma_{t_0,p}\colon J \to E$ need not to coincide as $\bar\gamma_{s_0,p} =$

$I_{t_0\to s_0}\circ\bar{\gamma}_{t_0,p}$, due to (8.7) and (8.2). Therefore the Δ^I-horizontal lift of a non-injective path through a point in E lying in a fibre over a self-intersection point of the path, if any, may not be unique. Similar is the situation for an arbitrary connection. (This range of problems is connected with the so-called holonomy groups.)

Proposition 8.5. *The parallel transport* I *generated by the connection, defined by a transport along paths* I*, is such that*

$$\mathsf{I}^\gamma = I^\gamma_{\sigma\to\tau} \qquad \textit{for } \gamma\colon[\sigma,\tau]\to M. \tag{8.23}$$

Proof. According to Definition 3.2 and (8.7), we have:

$$\mathsf{I}\colon \gamma\mapsto \mathsf{I}^\gamma\colon \pi^{-1}(\gamma(\sigma))\to\pi^{-1}(\gamma(\tau)) \qquad \gamma\colon[\sigma,\tau]\to M$$
$$\mathsf{I}^\gamma\colon p\mapsto \mathsf{I}^\gamma(p)=\bar{\gamma}_{\sigma,p}(\tau)=I^\gamma_{\sigma\to\tau}(p) \qquad p\in\pi^{-1}(\gamma(\sigma)).$$ □

Corollary 8.1. *The parallel transport* I *generated by the connection* Δ^I*, assigned to a transport* I *along paths, is a parallel transport, i.e., it satisfies* (8.12)–(8.15) *with* I *for* P*.*

Proof. This result is a particular case of Proposition 8.2. An alternative proof can be carried out by using (8.1)–(8.7), (8.23), and the definitions of inverse path and product of paths. The assertion is also a consequence of (8.23) and Theorem 8.1. □

Until this point, we have studied how a transport along paths generates a connection (Proposition 8.1) and parallel transport (Proposition 8.5). Besides, Theorem 8.1 establishes a bijective correspondence between particular class of transports along paths and mappings having (some of) the main properties of the parallel transports generated by connections. Below we shall pay attention, in a sense, to the opposite links, starting from a connection on a bundle.

Proposition 8.6. *Let* P *be the parallel transport assigned to a connection* Δ^h *on a bundle* (E,π,M)*. The mapping*

$$P\colon \gamma\mapsto P^\gamma\colon (s,t)\mapsto P^\gamma_{s\to t} \qquad \gamma\colon J\to M \tag{8.24a}$$

defined by

$$P^\gamma_{s\to t} = \begin{cases} \mathsf{P}^{\gamma|[s,t]} & \textit{for } s\le t\\ \left(\mathsf{P}^{\gamma|[t,s]}\right)^{-1} & \textit{for } s\ge t\end{cases} \tag{8.24b}$$

is a transport along paths in (E,π,M)*. Moreover,* P *is parallel transport along paths, i.e., it satisfies the equations* (8.4) *and* (8.5) *with* P *for* I*.*

Proof. One should check the conditions (8.1)–(8.5) with P for I. The relations (8.1) and (8.2) follow directly from Definition 3.2 of a parallel transport generated by a connection. The rest conditions are consequences of (8.24) and a simple, but tedious, application of the properties (8.12)–(8.14) of the parallel transports. Alternatively, this proposition is a consequence of the second part of Theorem 8.1 and Remark 8.5. □

Remark 8.7. Applying (8.12)–(8.14), the reader can verify that

$$P^{\gamma}_{s\to t} = F^{-1}(t;\gamma)\circ F(s;\gamma) \tag{8.25}$$

with

$$F(r;\gamma) = \begin{cases} \mathsf{P}^{\gamma|[r,w]} & \text{for } r\le w \\ \big(\mathsf{P}^{\gamma|[w,r]}\big)^{-1} & \text{for } r\ge w \end{cases} \qquad r=s,t \tag{8.26}$$

for any (arbitrarily) fixed $w\in J$. This result is a special case of the general structure of the transports along paths [114, Theorem 3.1].

Definition 8.6. The parallel transport along paths, defined by a connection Δ^h on a bundle through Proposition 8.6, will be called parallel transport along paths *assigned to* (*defined by, generated by*) the connection Δ^h.

Corollary 8.2. *Let Δ^I be the connection generated by a parallel transport I along paths according to Proposition* 8.1*. If I is the parallel transport assigned to Δ^I, then the transport along paths assigned to I (or Δ^I), as described in Proposition* 8.6*, coincides with the initial transport I along paths.*

Proof. Substitute (8.23) into (8.24), with I for P and I for P. □

Corollary 8.3. *Let P be the transport along paths assigned to a connection Δ^h (via its parallel transport P) according to Proposition* 8.6*. The connection Δ^P generated by P, as described in Proposition* 8.1*, coincides with the initial connection Δ^h, $\Delta^P=\Delta^h$.*

Proof. On one hand, if $p\in E$, the space Δ^P_p consists of the vectors tangent at s_0 to the paths $\bar\gamma_{s_0,p}\colon t\mapsto P^{\gamma}_{s_0\to t}(p)$, with $\gamma\colon J\to M$, $s_0\in J$, and $\pi(p)=\gamma(s_0)$, due to Proposition 8.1. On another hand, Δ^h_p consists of the vectors tangent at s_0 to the paths $\tilde\gamma_{s_0,p}\colon t\mapsto \mathsf{P}^{\gamma|[s_0,t]}(p)$, by virtue of Definition 3.2. Equation (8.24b) says that both types of paths coincide, $\tilde\gamma_{s_0,p}=\bar\gamma_{s_0,p}$, so that their tangent vectors at $t=s_0$ are identical and, consequently, Δ^P_p and Δ^h_p are equal as sets, $\Delta^P_p=\Delta^h_p$, for all $p\in E$. □

Roughly speaking, the above series of results says that a connection Δ^h is equivalent to a mapping P (the assigned to it parallel transport) satisfying (8.11)–(8.15) or to a mapping P (the assigned to it parallel transport along paths) satisfying (8.1)–(8.5) (with P for I). Besides, the smoothness of Δ^h is equivalent to the one of P or P. Let us summarize these results as follows.

Theorem 8.2. *Given a connection Δ^h on a bundle (E,π,M), there exists a unique parallel transport I along paths in (E,π,M) which generates Δ^h via* (8.8)*, i.e., $\Delta^I=\Delta^h$. Besides, the parallel transport P defined by Δ^h is given by* (8.17)*, i.e., $\mathsf{P}=\mathsf{I}$.*

Proof. See Propositions 8.1 and 8.5 and Theorem 8.1. □

Theorem 8.3. *Given a parallel transport I along paths in a bundle (E, π, M), then there exists a unique connection Δ^h on (E, π, M) such that the parallel transport P along paths assigned to Δ^h coincides with I, $P = I$. Besides, the connection Δ^I generated by I is identical with Δ^h, $\Delta^I = \Delta^h$.*

Proof. Apply Proposition 8.1 and corollaries 8.2 and 8.3. □

Theorem 8.4. *Given a parallel transport along paths in a bundle, there is a unique (axiomatically defined) parallel transport generating it. Conversely, given a parallel transport, there is a unique parallel transport along paths generating it.*

Proof. This statement is a reformulation of Theorem 8.1. □

Theorem 8.5. *Given a parallel transport P, there exists a unique connection Δ^h generating it. Besides, the parallel transport assigned to Δ^h coincides with P.*

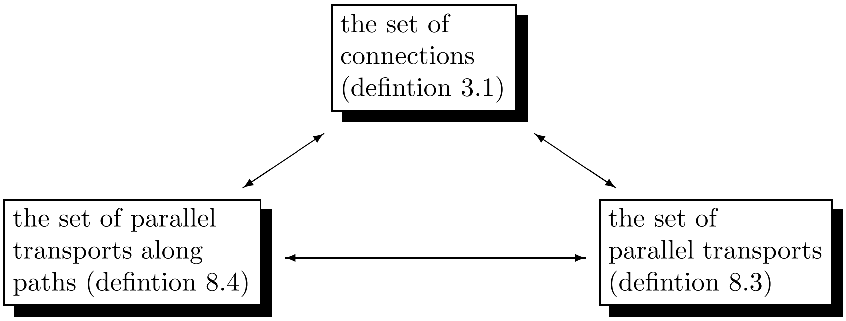
Proof. See Theorems 8.4 and 8.3 and Definitions 3.2 and 8.5. □

Theorem 8.6. *Given a connection Δ^h, there is a unique parallel transport P such that the defined by it parallel transport P along paths generates Δ^h, $\Delta^P = \Delta^h$. Besides, P coincides with the parallel transport assigned to Δ^h.*

Proof. Apply Theorems 8.2 and 8.4. □

The above results can be summarized in the commutative diagram shown on figure 8.1, the mappings in which are described via Theorems 8.2–8.6. Besides, if

Figure 8.1: Mappings between the sets of parallel transports, connections and parallel transports along paths



one of these objects is smooth, so are the other ones corresponding to it via the bijections constructed in the present section.

We end with the main moral of this section. The concepts "connection," "(axiomatically defined) parallel transport" and "parallel transport along path" (i.e., one that satisfies equations (8.4) and (8.5)) are equivalent in a sense that there are bijective mappings between the sets of these objects. Besides, if one of these objects is smooth, so are the other ones corresponding to it via the bijections constructed in the present section.

9. Conclusion

In this chapter we have presented a short (and partial) review of (one of the approaches to) the connection theory on bundles whose base and bundle spaces are (C^2) differentiable manifolds. Special attention was paid to connections, in particular linear ones, on vector bundles, which find wide applications in physics [146,155]. However, many other approaches, generalizations, alternative descriptions, particular methods, etc. were not mentioned at all. In particular, these include: connections on more general (e.g., topological) bundles, connections on principal bundles (which are important in the gauge field theories), holonomy groups, flat connections, Riemannian connections, etc., etc. The surveys [32, 156] contain essential information on these and many other items. Consistent and self-contained exposition of such problems can be found in [23, 110, 151, 157].

If additional geometric structures are added to the theory considered in Section 3, there will become important connections compatible with these structures. In this way arise many theories of particular connections; we have demonstrated that on the example of linear connections on vector bundles (Section 4).

The consideration of arbitrary (co)frames in Section 5 may seem slightly artificial as the general theory can be developed without them. However, this is not the generic case when one starts to apply the connection theory for investigation of particular problems. It may happen that some problem has solutions in general (co)frames but it does not possess solutions when (co)frames generated directly by local coordinates are involved. For example, local coordinates (holonomic frames) normal at a given point for a covariant derivative operator (linear connection) ∇ on a manifold exist if ∇ is torsionless at that point, but anholonomic frames normal at a given point for ∇ exist in a case of non-vanishing torsion.

In Section 6, we saw that the theory of normal frames in the most general case is quite trivial. This reflects the understanding that the more general a concept is, the less particular properties it has, but the more concrete applications it can find if it is restricted somehow. This situation was demonstrate when holonomic normal frames were considered; e.g., they exist at a given point or along an injective horizontal path, but on an open set they may exist only in the flat case. A feature of a vector bundle (E, π, M) is that the frames in $T(E)$ over E are in bijective correspondence with pairs of frames in E over M and in $T(M)$ over M. This result allows the normal frames in $T(E)$, if any, to be 'lowered' to ones in E. From here a conclusion was made that the theory of frames in $T(E)$ normal for linear connections on a vector bundle is equivalent to the existing one of frames in E normal for covariant derivatives in (E, π, M).

The concept “parallel transport” precedes historically the one of a “connection” and was first clearly formulated in the work [29] of Levi Civita on a parallel transport of a vector in Riemannian geometry. The connection theory was formulated approximately during the period 1920–1949 in a series of works on particular connections and their subsequent generalizations and has obtained an almost complete form in 1950–1955 together with the clear formulation of the concepts “manifold” and “fibre bundle” [32]; the concept of connection on general fibre bundle was established at about 1970 [158]. During that period, with a few exceptional works, the ‘parallel transport’ was considered as a secondary concept, defined by means of the one of a ‘connection’. Later, as we pointed at the beginning of Section 8, there appear several attempts for axiomatizing the concept of a parallel transport and by its means the connection theory to be constructed; e.g., this approach is developed deeply in vector bundles in [23]. The major classical results on axiomatization of parallel transport theory are presented in Table 9.1.

Table 9.1: Main contributions in axiomatizing the concept of “parallel transport”.

Year	Person	Result and original reference
1917	T. Levi-Civita	Definition of a parallel transport of a vector in Riemannian geometry. [29]
1949	Willi Rinow	An axiomatic definition of parallel transport in tangent bundle is introduced in unpublished lectures at Humboldt university. (See [31] and [23, p. 46].)
1964	Ü.G. Lumiste	Definition of a connection in principal bundle (with homogeneous fibres) as a parallel transport along canonical paths $\alpha\colon [0,1] \to M$ in its base M. The parallel transport is defined as a mapping from the fibre over $\alpha(0)$ into the one over $\alpha(1)$ satisfying some axioms. [30, Section 2.2]
1964	C. Teleman	Definition of a connection in topological bundle as a parallel transport along canonical paths $\alpha\colon [0,1] \to M$ in its base M.. The parallel transport is defined as a lifting of these paths through a point in the fibre over their initial points $\alpha(0)$. [17, Chapter IV, Section B.3]
1968	P. Dombrowski	Definition of a linear connection in vector bundle as a parallel transport along paths $\beta\colon [a,b] \to M$, with $a, b \in \mathbb{R}$ and $a \leq b$, in its base M. The parallel transport is defined as a mapping from the fibre over $\alpha(a)$ into the one over $\alpha(b)$ satisfying certain axioms. The theory of covariant derivatives is constructed on that base. [31, § 1]
1981	Walter Poor	A detailed axiomatic definition of a parallel transport in vector bundles. The whole theory of linear connections in such bundles is deduced on that ground. [23]

The main merit from Section 8 is a necessary and sufficient conditions when an axiomatically defined parallel transport (or a parallel transport along paths) defines a unique connection (with suitable properties) and *vice versa*. Moreover, the concepts “connection,” “(axiomatically defined) parallel transport,” and “parallel transport along path” (i.e., one that satisfies equations (8.4) and (8.5)) are equivalent in a sense that there are bijective mappings between the sets of these objects. However, the concept of a parallel transport admits a generalization to the one of a transport along paths and there exist transports along paths that cannot be generated by connections or axiomatically defined parallel transports.

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Subject List of Symbols

The following table contains a list of the main symbols which consistently refer to the some concept and frequently are employed. The page numbers denote the page where the symbols first appear. A more or less alphabetically ordered list of the same symbols is presented in the Notation index, beginning on page 423.

Symbol	Explanation	Page
Set theory		
$\varnothing$	empty set	5
$\in$	belongs to, included in	4
$\ni$	which contains	7
$\{x : P\}$	set whose elements have the property P	5
$\{x \vert P\}$	the same as $\{x : P\}$	5
$\{a, b, c, \ldots\}$	set with elements $a, b, c, \ldots$	4
$\subset$	contained in, proper subset	8
$\supset$	proper superset (overset)	27
$\subseteq$	contained in or equal, proper subset or equal	5
$\supseteq$	proper superset (overset) or equal	177
$\cup$	union (cup) sing	5
$\cap$	intersection (cap) sign	5
$=$	equals to, equality sign	4
$:=$ or $=:$	equals by definition: a quantity sitting from the side of the colon being defined	4
$\setminus$	subtraction of sets; $A\setminus B := \{a \vert a \in A,\ a \notin B\}$	27
$\vert_A$ or $\vert A$	restriction to a set A	8
$\times$	Cartesian (direct) product	4

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Symbol	Explanation	Page
$\times^k A$	$A\times\cdots\times A$ where the set A is taken k times, $k \in \mathbb{N}\cup\{0\}$; $\times^0 A := \varnothing$	4
$\iff$ or iff	if and only if, logical equivalence	34
$\to$	mapping sign	5
$f\colon A \to B$	mapping f with domain A and range B	5
$\mapsto$	maps to sign	6
$f\colon x \mapsto y$	mapping f assigning to x the value y	6
f^{-1}	inverse mapping to a mapping f	5
$\circ$	sign of the composition of mappings; $(f \circ g)(p) := f(g(p))$	5
id_A	identity mapping of set A; $\mathsf{id}_A : a \mapsto a$ for all $a \in A$	5

Linear/vector spaces

Symbol	Explanation	Page
$\mathbb{N}$	natural numbers, integers	4
$\mathbb{R}$	real numbers, (field of) reals, real line	4
$\mathbb{C}$	complex numbers, (field of) complex, complex plane	4
$\mathbb{K}$	$\mathbb{R}$ or $\mathbb{C}$	4
$\mathbb{E}$	1-dimensional Euclidean space	34
$\mathbb{R}^n$	n-th Cartesian power of $\mathbb{R}$, $n \in \mathbb{N}$	4
$\mathbb{C}^n$	n-th Cartesian power of $\mathbb{C}$, $n \in \mathbb{N}$	4
$\mathbb{K}^n$	n-th Cartesian power of $\mathbb{K}$, $n \in \mathbb{N}$; $\mathbb{R}^n$ or $\mathbb{C}^n$	4
$\mathbb{E}^n$	n-th Cartesian power of $\mathbb{E}$, $n \in \mathbb{N}$	34
J	real interval of arbitrary type (open or closed from one or both ends); the same is the meaning of J with some attached to it indices if the opposite is not stated explicitly	6
$[a, b]$	close real interval with left (resp. right) end a (resp. b), with $a, b \in \mathbb{R}$ and $a \le b$	27
$(a, b]$	closed from right and opened from left real interval with left (resp. right) end a (resp. b), with $a, b \in \mathbb{R}$ and $a < b$	27
$[a, b)$	closed from left and opened from right real interval with left (resp. right) end a (resp. b), with $a, b \in \mathbb{R}$ and $a < b$	27

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Symbol	Explanation	Page
(a, b)	open real interval with left (resp. right) end a (resp. b), with $a, b \in \mathbb{R}$ and $a < b$	27
J^n	neighborhood in $\mathbb{R}^n$, $n \in \mathbb{N}$	161
$\otimes$	tensor product sign	17
$\otimes^k V$	$V \otimes \cdots \otimes V$ where the vector space V over $\mathbb{K}$ is taken k times, $k \in \mathbb{N} \cup \{0\}$; $\otimes^0 V := \mathbb{K}$	17
A^*	object dual to an object A, A being a vector or vector space	15
$i, j, k, \ldots$	indices running from 1 to the dimension of a vector space	5
$\{e_i\}$	basis in a vector space	83
$A_{(ij)}$	symmetrization over the indices i and j; equals to $\frac{1}{2}(A_{ij} + A_{ji})$ (see the list of conventions, page xii)	24
$A_{[ij]}$	antisymmetrization over the indices i and j; equals to $\frac{1}{2}(A_{ij} - A_{ji})$ (see the list of conventions, page xii)	24
$[A, B]_{-}$	commutator of operators A and B on a linear space; equals to $A \circ B - B \circ A$	13
Matrices		
$\mathbb{1}$	identity (unit) matrix of the corresponding size	89
$\mathbb{1}_n$	the identity (unit) $n \times n$, $n \in \mathbb{N}$, matrix	97
$\delta_{ij} = \delta^{ij} = \delta_i^j$	Kronecker delta symbol(s), $\delta_i^j = 1$ for $i = j$ and $\delta_i^j = 0$ for $i \neq j$	15
$A = [A_i^j]$	square matrix A with elements A_i^j	11
$A = [A_i^j]_{i,j=1}^n$	the same, but the range of the indexes i and j is written explicitly	11
$\operatorname{diag}(a_1, \ldots, a_n)$	diagonal matrix with diagonal elements $a_1, \ldots, a_n$; equals to $[\delta_{ij} a_j]_{i,j=1}^n$	36
$\det(A) \equiv \det A$	determinant of square matrix A	36
A^{-1}	inverse matrix of non-degenerate square matrix A	11
$A^\top$	transposed matrix of a matrix $A = [A_{ij}]$, $(A^\top)_{ij} := A_{ji}$; for $A = [A^j{}_i]$, we put $(A^\top)_j{}^k := A^j{}_k$, i.e., the superscript is considered as 'first' index, which numbers the rows of a matrix	47

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Symbol	Explanation	Page
$\mathrm{GL}(n, \mathbb{K})$	the set of $n \times n$, $n \in \mathbb{N}$, invertible matrices on a field $\mathbb{K}$	82

Manifolds

Symbol	Explanation	Page
M	differentiable manifold	4
$\dim M \equiv \dim(M)$	dimension of manifold M	5
$\dim_{\mathbb{C}} M$	dimension of a complex manifold M; $\dim_{\mathbb{C}} M := \dim M$	5
$\dim_{\mathbb{R}} M$	the real dimension of manifold M: $\dim_{\mathbb{R}} M = \dim M$ if M is real and $\dim_{\mathbb{R}} M = 2\dim M = 2\dim_{\mathbb{C}} M$ if M is complex.	5
U	subset of M, often an open set (neighborhood)	5
N	submanifold of M	7
$T_p(M)$	space tangent to M at $p \in M$	9
$T_p^*(M)$	space cotangent to M at $p \in M$	15
$T(M)$	bundle space tangent to M; $T(M) := \bigcup_{p\in M} T_p(M)$	11
$T^*(M)$	bundle space cotangent to M; $T^*(M) := \bigcup_{p\in M} T_p^*(M)$	70
${T_p}_s^r(M)$	tensor space of type (r, s), $r, s \in \mathbb{N} \cup \{0\}$, over $p \in M$	17
$T_s^r(M)$	tensor bundle space of type (r, s), $T_s^r(M) := \bigcup_{p\in M} {T_p}_s^r(M)$	223
$\lambda, \mu, \nu, \ldots$	indices running from 1 to the dimension of a differentiable manifold (see the list of conventions, page xi)	247
$[\,\cdot\,,\,\cdot\,]_-$	commutator (of vector fields); $[X, Y]_- := X \circ Y - Y \circ X$	13
$\{r^i\}$	standard Cartesian coordinates on $\mathbb{K}^{\dim M}$	5
$\{E_i\|_p\}$	basis in $T_p(M)$	11
$\{E^i\|_p\}$	basis in $T_p^*(M)$ dual to $\{E_i\|_p\}$; $E^i\|_p := (E_i\|_p)^*$	15
$\{E_i\}$	basis in $T(M)$; frame on $U \subseteq M$	12
$\{E^i\}$	basis in $T^*(M)$ dual to $\{E_i\}$; coframe on $U \subseteq M$, $E^i := (E_i)^*$	16
C^i_{jk}	structure functions of a frame $\{E_i\}$; $[E_j, E_k]_- =: C^i_{jk} E_i$	26
(U, φ)	(local) chart with domain U and homomorphism φ	5
$\{\varphi_i\}$	(local) coordinate system assigned to a chart (U, φ)	5
$\{\frac{\partial}{\partial x^i}\big\|_p\}$	coordinate basis in $T_p(M)$, $p \in M$	10

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Symbol	Explanation	Page
$\left\{\frac{\partial}{\partial x^i}\right\}$	coordinate frame on $U \subseteq M$	12
$\left\{\mathrm{d}x^i\|_p\right\}$	coordinate (co)basis in $T_p^*(M)$, $p \in M$ dual to $\left\{\frac{\partial}{\partial x^i}\Big\|_p\right\}$	15
$\left\{\mathrm{d}x^i\right\}$	coordinate coframe on $U \subseteq M$ dual to $\left\{\frac{\partial}{\partial x^i}\right\}$	16
C^k mapping	mapping having continuous derivative(s) up to k-th order including, $k \in \mathbb{N} \cup \{0, \infty, \omega\}$	7
f_*	differential (induced tangent) mapping of a differentiable mapping f	13
$\mathfrak{F}(U)$	the algebra of functions on $U \subseteq M$	8
$\mathfrak{F}^r(U)$	the algebra of C^r functions on $U \subseteq M$	8
$\mathfrak{X}(U)$	the $\mathfrak{F}(U)$-module of vector fields on $U \subseteq M$	11
$\mathfrak{X}^r(U)$	the $\mathfrak{F}^r(U)$-module of C^r vector fields on $U \subseteq M$	13
$\mathfrak{T}_s^r(U)$	the $\mathfrak{F}(U)$-module of tensor fields of type (r, s) on $U \subseteq M$	18
$\mathfrak{T}_s^{r;k}(U)$	the $\mathfrak{F}^k(U)$-module of C^k tensor fields of type (r, s) on $U \subseteq M$	19
$\boldsymbol{T}_p(M)$	the tensor algebra at $p \in M$	17
$\boldsymbol{T}(U)$	the algebra of tensor fields on $U \subseteq M$	19
$\boldsymbol{T}^r(U)$	the algebra of C^r tensor fields on $U \subseteq M$	19
C_s^r	contraction operator of type (r, s), acting on the r^{th} superscript and s^{th} subscript	19
C	contraction operator of arbitrary type	19
$f_{,i}$	the action of a basic vector field E_i on $f \in \mathfrak{F}^1(M)$; equals to $E_i(f)$ (in coordinate frame coincides with the i^{th} partial derivative of f)	26
g	Riemannian metric	34
$\gamma\colon J \to M$	path in M	9
$\dot\gamma(s)$	the vector tangent to a C^1 path γ at parameter value $s \in J$, $\dot\gamma(s) \in T_{\gamma(s)}(M)$	9
$\dot\gamma$	the vector field tangent to C^1 injective path γ, i.e., $\dot\gamma\colon \gamma(s) \mapsto \dot\gamma(s)$; the tangent lifting of arbitrary C^1 path γ, i.e., $\dot\gamma\colon s \mapsto \dot\gamma(s)$	11
Δ	distribution on manifold	338
Δ_p	the value of Δ at $p \in M$; $\Delta_p \subseteq T_p(M)$	338
$\mathcal{L}_X$	Lie derivative along vector field X	144

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Symbol	Explanation	Page
Linear connections on manifolds		
∇	linear connection; covariant (absolute) differential	21
∇_X	covariant derivative along vector field X	21
T	torsion, torsion tensor field (operator)	25
R	curvature, curvature tensor field (operator)	25
$\Gamma^i{}_{jk}$	(local) coefficients of linear connection in a (local) frame $\{E_i\}$	22
$\Gamma_k := \left[\Gamma^i{}_{jk}\right]_{i,j=1}^{\dim M}$	matrices of the coefficients of a linear connection; connection matrices, coefficients' matrices	38
$T^i{}_{jk}$	(local) components of the torsion in a (local) frame $\{E_i\}$	26
$R^i{}_{jkl}$	(local) components of the curvature in a (local) frame $\{E_i\}$	26
$\boldsymbol{R}_{kl}$	matrices of the curvature components, curvature matrices; $\boldsymbol{R}_{kl} := \left[R^i{}_{jkl}\right]_{i,j=1}^{\dim M}$	105
$\left\{ {i \atop jk} \right\}$	Christoffel symbols (of a metric)	36
Derivations on manifolds		
D	derivation or derivation along vector fields	143
D_X	derivation along (tangent) vector field X	144
$\Gamma_X{}^i{}_j$	components of derivation along vector field X in a (local) frame $\{E_i\}$	147
$\boldsymbol{\Gamma}_X = \left[\Gamma_X{}^i{}_j\right]_{i,j=1}^{\dim M}$	the matrix of the components of a derivation along a vector field X	147
R	curvature, curvature operator	147
T	torsion, torsion operator	147
R^D	curvature of a derivation D along vector fields	147
T^D	torsion of a derivation D along vector fields	147
$\Gamma^i{}_{jk}$	coefficients of a linear derivation along vector fields	146
$\Gamma_k = \left[\Gamma^i{}_{jk}\right]_{i,j=1}^{\dim M}$	matrices of the coefficients of a linear derivation along vector fields; coefficients' matrices	145
D	(section-, vector-, tensor-)derivation along paths	191
D^γ	derivation along path γ	191

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Symbol	Explanation	Page
${\Gamma_\gamma}^i{}_j$	components of derivation along $\gamma\colon J \to M$	192
$\Gamma^i{}_j(s;\gamma)$	the value of ${\Gamma_\gamma}^i{}_j$ at $s \in J$	192
$\mathbf{\Gamma}_\gamma := \left[{\Gamma_\gamma}^i{}_j\right]$	the matrix of derivation along γ	192
$\mathbf{\Gamma}(s;\gamma)$	the value of $\mathbf{\Gamma}_\gamma$ at s	190

Fibre bundles

Symbol	Explanation	Page
(E,π,B)	(fibre) bundle with (total) bundle space E, projection $\pi\colon E \to B$, and base (space) B	217
$\gamma\colon J \to B$	path in B	218
$\mathrm{Sec}(\xi)$	the set of sections of bundle ξ	217
$\mathrm{Sec}^k(\xi)$	the set of C^k sections of bundle ξ, $k \in \mathbb{N} \cup \{0,\infty,\omega\}$	219
$\mathrm{P}(A)$	the set of paths in a set A	218
$\mathrm{PLift}(\xi)$	the set of liftings of paths in a bundle ξ	218
$\mathrm{PLift}^k(\xi)$	the set of C^k liftings of paths in a bundle ξ	219
$\mathrm{PF}(B)$	the set of functions along paths in bundle (E,π,B)	218
$\mathrm{PF}^k(B)$	the set of C^k functions along paths in bundle (E,π,B)	219
$\mathrm{Lift}_g(E,\pi,B)$	the set of liftings of a mapping $g\colon X \to B$ to E in bundle (E,π,M)	218
σ	section of a bundle	217
$\hat\sigma$	lifting of paths assigned to a section σ of a bundle, $\hat\sigma\colon \gamma \mapsto \hat\sigma_\gamma := \sigma \circ \gamma$	221
D	derivation along paths in a bundle	219
D^γ	derivation along a path $\gamma\colon J \to B$ in bundle (E,π,B)	219
$\overline{D}$	derivation along paths of sections in a bundle	221
D	section-derivation along paths in a bundle	221
$i,j,k,\dots$	indices running from 1 to the (fibre) dimension of a vector bundle	218
$\alpha,\beta,\dots,\mu,\nu,\dots$	indices running from 1 to the dimension of the base of a bundle with a manifold as a base space	247
$I,J,K,\dots$	indices running from 1 to the dimension $\dim E$ of the bundle space of a bundle	340
$a,b,c,\dots$	indices running from $1+\dim M$, $\dim M$ being the dimension of the base space, to the dimension $\dim E$ of the bundle space of a bundle	339

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Symbol	Explanation	Page
$\{e_i\}$	frame in the bundle space of a vector bundle	218
$\{\hat{e}_i\}$	bases for liftings of paths	218
$\Gamma^i{}_j$	components (2-index coefficients) of derivation	220
$\mathbf{\Gamma} := [\Gamma^i{}_j]$	matrix (of the components) of derivation	220
$\Gamma^i{}_{j\mu}$	(3-index) coefficients of derivation	248
$\mathbf{\Gamma}_\mu := [\Gamma^i{}_{j\mu}]$	coefficients' matrices of derivation	247
$(T(M), \pi, M)$	tangent bundle over manifold M	222
$(T^*(M), \pi^*, M)$	cotangent bundle over manifold M	222
$(T^r_s(M), \pi^r_s, M)$	tensor bundle of type (r, s) over manifold M	223
$(\mathbf{T}(M), \boldsymbol{\pi}, M)$	algebraic tensor bundle over manifold M	223
Δ^h	connection on differentiable bundle; horizontal distribution on that bundle	342
Δ^v	vertical distribution on differentiable bundle	342

Linear transports and derivations along paths

Symbol	Explanation	Page
L	linear transport along paths in a vector bundle (E, π, B)	226
L^γ	linear transport along path $\gamma\colon J \to B$	226
$L^\gamma_{s\to t}$	linear transport along γ from s to t	226
D	derivation along paths generated by linear transport along paths	230
D^γ	derivation along paths generated by linear transport along a path $\gamma\colon J \to B$	231
D^γ_s	derivation along paths generated by linear transport along a path $\gamma\colon J \to B$ at s, $s \in J$	231
$\mathcal{D}$	derivation along tangent vector fields	257
$\mathcal{D}_X$	derivation along a tangent vector field X	257
$\Gamma^j{}_i$	(2-index) coefficients of transport/derivation along paths	231
$\Gamma^j{}_{i\mu}$	(3-index) coefficients of transport/derivation along paths	248
$\Gamma := [\Gamma^j{}_i]$	matrix of the (2-index) coefficients of transport or derivation along paths	232

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Symbol	Explanation	Page
$\Gamma_{\mu} := [\Gamma^{j}{}_{i\mu}]_{i,j=1}^{n}$	matrices of the 3-index coefficients of transport or derivation along paths	247
R	curvature (operator) of linear transport or derivations along paths	265
$R^{\eta}(s,t)$	curvature along $\eta\colon J\times J'\to B$ at $(s,t)\in J\times J'$	262
$\big(R^{\eta}(s,t)\big)_{j}^{i}$	(2-index) components of $R^{\eta}(s,t)$ in a frame $\{e_i\}$	263
$R^{i}_{j\alpha\beta}$	4-index components of $R^{\eta}(s,t)$ in a pair of frames $\{e_i\}$ in the bundle space E and $\{E_{\mu}\}$ in the bundle space $T(M)$ of the tangent bundle	265
$\boldsymbol{R}^{\eta}(s,t)$	matrix of $R^{\eta}(s,t)$ in a frame $\{e_i\}$, $\boldsymbol{R}^{\eta}(s,t) := \big[(R^{\eta}(s,t))_{j}^{i}\big]$	264
$\boldsymbol{R}_{\alpha\beta} := \big[R^{i}_{j\alpha\beta}\big]$	matrices of $R^{\eta}(s,t)$ in a pair of frames $\{e_i\}$ and $\{E_{\mu}\}$	266
T	torsion (vector, operator) of a linear transport or derivation along paths in the tangent bundle over a manifold	272
$T^{\eta}(s,t)$	torsion along $\eta\colon J\times J'\to M$ at $(s,t)\in J\times J'$ (in the tangent bundle over a manifold)	272
$(T^{\eta}(s,t))^{i}$	components of $T^{\eta}(s,t)$ in a frame $\{E_i\}$	272
$T^{i}{}_{jk}$	components of the torsion tensor in $\{E_i\}$	272
P	parallel transport in $(T(M),\pi,M)$ considered as a linear transport along paths	281

Parallel transports, connections, covariant derivatives

(E,π,M)	vector bundle, E and M being C^{∞} manifolds	301
N	C^{∞} manifold	301
$\mathbb{P}$	parallelism structure	302
P	parallel transport	311
P^{β}	parallel transport along a path $\beta\colon[a,b]\to M$	302
P	parallel transport along paths	315
$P^{\beta}_{s\to t}$	parallel transport along $\beta\colon[a,b]\to M$ from s to t, $s,t\in[a,b]$	315
$T^{h}(E)$	connection on a vector bundle (E,π,M), horizontal subspace of $T(E)$	303

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Symbol	Explanation	Page
$T_u^h(E)$	horizontal subspace of $T_u(E)$, $u \in E$	303
$T^v(E)$	the vertical subspace of $T(E)$	303
$T_u^v(E)$	the vertical subspace of $T_u(E)$, $u \in E$	303
∇^g	covariant derivative in (E, π, M) along a C^∞ mapping $g\colon N \to M$	308
∇	the same as ∇^g if g is insignificant or a covariant derivative in (E, π, M); $\nabla = \nabla^{\mathsf{id}_M}$	224
$i, j, \ldots$	indices running from 1 to the (fibre) dimension of (E, π, M)	308
$\mu, \nu, \ldots$	indices running from 1 to the dimension of N (or M if $N = M$)	308
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$\Gamma^i{}_{j\mu}$	coefficients of ∇^{id_M}; $\Gamma^i{}_{j\mu} = \Gamma^i{}_{j\mu}(\,\cdot\,; \mathsf{id}_M)$	309
$\Gamma_\mu(\,\cdot\,; g)$	matrices of the coefficients of ∇^g, i.e., $\left[\Gamma^i{}_{j\mu}(\,\cdot\,; g)\right]$	310
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Δ^v	vertical distribution on a differentiable manifold (E, π, M)	342
$I, J, K, \ldots$	indices running from 1 to the dimension $\dim E$ of the bundle space of a bundle	340
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Notation Index

The frequently used symbols with a fixed meaning are listen below. They are arranged in more or less alphabetical order. Since the sorting of mathematical symbols is not unique, they are sorted according to up to three criterions: the symbol's name or pronunciations (if any), the name of symbol's kernel (root) letter, and the meaning of the symbol as a whole. For some symbols such a classification is not unique, due to which they are listen more than ones; e.g., ∇ can be found under the letters "C", standing for Connection or Covariant derivative (sorting by meaning), and "N", standing for Nabla (sorting by name). Besides, a multiple appearance of a symbol under different sorting letters may mean that it has different meanings in different but similar contexts. The number(s) standing to the right of the symbols denote the page(s) where the symbols first appear or are defined.

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