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Fuchsian Reduction

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To my parents

Preface

The nineteenth century saw the systematic study of new “special functions”, such as the hypergeometric, Legendre and elliptic functions, that were relevant in number theory and geometry, and at the same time useful in applications. To understand the properties of these functions, it became important to study their behavior near their singularities in the complex plane. For linear equations, two cases were distinguished: the Fuchsian case, in which all formal solutions converge, and the non-Fuchsian case. Linear systems of the form

$$z\frac{d\boldsymbol{u}}{dz} + A(z)\boldsymbol{u} = 0,$$

with A holomorphic around the origin, form the prototype of the Fuchsian class. The study of expansions for this class of equations forms the familiar “Fuchs–Frobenius theory,” developed at the end of the nineteenth century by Weierstrass’s school. The classification of singularity types of solutions of nonlinear equations was incomplete, and the Painlevé–Gambier classification, for second-order scalar equations of special form, left no hope of finding general abstract results.

The twentieth century saw, under the pressure of specific problems, the development of corresponding results for partial differential equations (PDEs): The Euler–Poisson–Darboux equation

$$u_{tt} + \frac{\lambda}{t}u_t - \Delta u = 0$$

and its elliptic counterpart arise in axisymmetric potential theory and in the method of spherical means; it also comes up in special reductions of Einstein’s equations. In particular, one realized that equations with different values of λ could be related to each other by transformations $u \mapsto t^m u$. Elliptic problems in corner domains and problems with double characteristics also led to further generalizations. This development was considered as fairly mature in the 1980s; it was realized that some problems required complicated expansions with logarithms and variable powers, beyond the scope of existing results, but it was assumed that this behavior was nongeneric. Nonlinear problems were practically ignored.

The word “Fuchsian” had come to stand for “equations for which all formal power series solutions are convergent.” Of course, Fuchsian ODEs have solutions involving logarithms, but by Frobenius’s trick, logarithms could be viewed as limiting cases of powers, and were therefore not thought of as generic.

However, in the 1980s difficulties arose when it became necessary to solve Fuchsian problems arising from other parts of mathematics, or other fields. The convergence of the “ambient metric” realizing the embedding of a Riemannian manifold in a Lorentz space with a homothety could not be proved in even dimensions. When, in the wake of the Hawking–Penrose singularity theorems, it became necessary to look for singular solutions of Einstein’s equations, existing results covered only very special cases, although the field equations appeared similar to the Euler–Poisson–Darboux equation. Numerical studies of such space-times led to spiky behavior: were these spikes artefacts? indications of chaotic behavior?

Other problems seemed unrelated to Fuchsian PDEs. For the blowup problem for nonlinear wave equations, again in the eighties, Hörmander, John, and their coworkers computed asymptotic estimates of the blowup time—which is not a Lorentz invariant. For elliptic problems $\Delta u = f(u)$ with monotone nonlinearities, solutions with infinite data dominate all solutions, and come up in several contexts; the boundary behavior of such solutions in bounded $C^{2+\alpha}$ domains is not a consequence of weighted Schauder estimates. Outside mathematics, we may mention laser collapse and the weak detonation problem. In astrophysics, stellar models raise similar difficulties; equations are singular at the center, and one would like to have an expansion of solutions near the singularity to start numerical integration. Also, the theory of solitons has provided, from 1982 on, a plethora of formal series solutions for completely integrable PDEs, of which one would like to know whether they represent actual solutions. Do these series have any relevance to nearly integrable problems?

The method of Fuchsian reduction, or reduction for short, has provided answers to the above questions. The upshot of reduction is a representation of the solution $\boldsymbol{u}$ of a nonlinear PDE in the typical form

$$\boldsymbol{u} = \mathbf{s} + T^m \boldsymbol{v},$$

where $\mathbf{s}$ is known in closed form, is singular for $T = 0$, and may involve a finite number of arbitrary functions. The function $\boldsymbol{v}$ determines the regular part of $\boldsymbol{u}$. This representation has the same advantages as an exact solution, because one can prove that the remainder $T^m \boldsymbol{v}$ is indeed negligible for T small. In particular, it is available where numerical computation fails; it enables one to compute which quantities become infinite and at what rate, and to determine which combinations of the solution and its derivatives remain finite at the singularity. From it, one can also decide the stability of the singularity under perturbations, and in particular how the singularity locus may be prescribed or modified.

Reduction consists in transforming a PDE $F[\boldsymbol{u}] = 0$, by changes of variables and unknowns, into an asymptotically scale-invariant PDE or system of PDEs

$$L\boldsymbol{v} = f[\boldsymbol{v}]$$

such that (i) one can introduce appropriate variables $(T, x_1, \dots)$ such that $T = 0$ is the singularity locus; (ii) L is scale-invariant in the T-direction; (iii) f is "small" as T tends to zero; (iv) bounded solutions $\boldsymbol{v}$ of the reduced equation determine singular $\boldsymbol{u}$ that are singular for $T = 0$. The right-hand side may involve derivatives of $\boldsymbol{v}$. After reduction to a first-order system, one is usually led to an equation of the general form

$$\left(T\frac{d}{dT} + A\right)\mathbf{w} = f[T, \mathbf{w}],$$

where the right-hand side vanishes for $T = 0$. PDEs of this form will be called "Fuchsian." The Fuchsian class is itself invariant under reduction under very general hypotheses on f and A. This justifies the name of the method.

Since $\boldsymbol{v}$ is typically obtained from $\boldsymbol{u}$ by subtracting its singularities and dividing by a power of T, $\boldsymbol{v}$ will be called the renormalized unknown. Typically, the reduced Fuchsian equations have nonsmooth coefficients, and logarithmic terms in particular are the rule rather than the exception. Since the coefficients and nonlinearities are not required to be analytic, it will even be possible to reduce certain equations with irregular singularities to Fuchsian form. Even though L is scale-invariant, $\mathbf{s}$ may not have power-like behavior. Also, in many cases, it is possible to give a geometric interpretation of the terms that make up $\mathbf{s}$.

The introduction, Chapter 1, outlines the main steps of the method in algorithmic form.

Part I describes a systematic strategy for achieving reduction. A few general principles that govern the search for a reduced form are given. The list of examples of equations amenable to reduction presented in this volume is not meant to be exhaustive. In fact, every new application of reduction so far has led to a new class of PDEs to which these ideas apply.

Part II develops variants of several existence results for hyperbolic and elliptic problems in order to solve the reduced Fuchsian problem, since the transformed problem is generally not amenable to classical results on singular PDEs.

Part III presents applications. It should be accessible after an upper-undergraduate course in analysis, and to nonmathematicians, provided they take for granted the proofs and the theorems from the other parts. Indeed, the discussion of ideas and applications has been clearly separated from statements of theorems and proofs, to enable the volume to be read at various levels.

Part IV collects general-purpose results, on Schauder theory and the distance function (Chapter 12), and on the Nash–Moser inverse function theorem (Chapter 13). Together with the computations worked out in the solutions to the problems, the volume is meant to be self-contained.

Most chapters contain a problem section. The solutions worked out at the end of the volume may be taken as further prototypes of application of reduction techniques.

A number of forerunners of reduction may be mentioned.

1. The Briot–Bouquet analysis of singularities of solutions of nonlinear ODEs of first order, continued by Painlevé and his school for equations of higher order. It has remained a part of complex analysis. In fact, the catalogue of possible singularities in this limited framework is still not complete in many respects. Most of the equations arising in applications are not covered by this analysis.
2. The regularization of collisions in the N-body problem. This line of thought has gradually waned, perhaps because of the smallness of the radius of convergence of the series in some cases, and again because the relevance to nonanalytic problems was not pursued systematically.
3. A number of special cases for simple ODEs have been rediscovered several times; a familiar example is the construction of radial solutions of nonlinear elliptic equations, which leads to Fuchsian ODEs with singularity at $r = 0$.

In retrospect, reduction techniques are the natural outgrowth of what is traditionally called the "Weierstrass viewpoint" in complex analysis, as opposed to the Cauchy and Riemann viewpoints. This viewpoint, from the present perspective, puts expansions at the main focus of interest; all relevant information is derived from them. For this approach to be relevant beyond complex analysis, it was necessary to understand which aspects of the Weierstrass viewpoint admit a generalization to nonanalytic problems with nonlinearities—and this generalization required a mature theory of nonlinear PDEs which was developed relatively recently. The development of reduction techniques in the early nineties seems to have been stimulated by the convergence of five factors:

1. The emergence of singularities as a legitimate field of study, as opposed to a pathology that merely indicates the failure of global existence or regularity.
2. The existence of a mature theory of elliptic and hyperbolic PDEs, which could be generalized to singular problems.
3. The failure of the search for a weak functional setting that would include blowup singularities for the simplest nonlinear wave equations.
4. The rediscovery of complex analysis stimulated by the emergence of soliton theory.
5. The availability of a beginning of a theory of Fuchsian PDEs, as opposed to ODEs, albeit developed for very different reasons, as we saw.

On a more personal note, a number of mathematicians have, directly or indirectly, helped the author in the emergence of reduction techniques: D. Aronson, C. Bardos, L. Boutet de Monvel, P. Garrett, P. D. Lax, W. Littman, L. Nirenberg, P. J. Olver, W. Strauss, D. H. Sattinger, A. Tannenbaum, E. Zeidler. In fact, my indebtedness extends to many other mathematicians whom I have met or read, including the anonymous referees. H. Brezis, whose mathematical influence may be felt in several of my works, deserves a special place. I am also grateful to him for welcoming this volume in this series, and to A. Kostant and A. Paranjpye at Birkhäuser, for their kind help with this project.

Paris
February 27, 2007

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1

Introduction

This introduction defines Fuchsian reduction, or reduction for short, illustrates it with a number of simple examples and outlines its main successes. The technical aspects of the theory are developed in the subsequent chapters. The upshot of reduction is a parameterized representation of solutions of nonlinear differential equations, in which singularity locus may be one of the parameters. We first show, on a very simple example, the advantages of such a representation. We then describe the main steps of the reduction process in general terms, and show in concrete situations how this reduction is achieved. We close this introduction with a survey of the impact of reduction on applications.

1.1 Singularity locus as parameter

Consider the ODE

$$\frac{du}{dt} = u^2.$$

The solution taking the value a for $t = t_0$ is

$$u(a, t_0, t) = \frac{a}{1 - a(t - t_0)}.$$

The set of solutions may be parameterized by two parameters (a, t_0). The procedure is quite similar to solving the algebraic equation $s^2 + t^2 = 1$ in the form $s = s(t) = \pm\sqrt{1 - t^2}$, in which t plays the role of a parameter, or local coordinate. But unlike the representation $s = s(t)$, the representation $u = u(a, t_0, t)$ is redundant: only one parameter suffices to describe the general solution. Indeed, let $b = a/(1 + at_0)$; we obtain

$$u(a, t_0, t) = \frac{b}{1 - bt}.$$

The parameter b gives the position of the (only) singular point of the solution, namely $1/b$.

Quite generally, finding the general solution of a differential equation amounts to finding a set of parameters that label all solutions close to a given one. The process is comparable to finding local coordinates on a manifold. Taking singularity locus as one of the parameters, one obtains a parameterization without redundancy, unlike the parameterization by t_0 and the Cauchy data at time t_0.

1.2 The main steps of reduction

A complete application of the reduction technique to a specific problem

$$F[\boldsymbol{u}] = 0$$

follows four steps, detailed below. The square brackets indicate that F may depend on $\boldsymbol{u}$ and its derivatives, as well as on independent variables.

- Leading-order analysis.
- First reduction and formal solutions.
- Second reduction and characterization of solutions.
- Invertibility and stability of solutions.

Let us briefly describe how these steps would be carried out for a typical class of problems: those for which the leading term is a power. Many other types of leading behavior arise in applications, including logarithms and variable powers. They will be discussed in due time.

The objective of leading-order analysis is to find a function T and a pair $(\boldsymbol{u}_0, \nu)$ such that $F[\boldsymbol{u}_0 T^\nu]$ vanishes to leading order. The hope is to find solutions such that

$$\boldsymbol{u} \sim \boldsymbol{u}_0 T^\nu. \tag{1.1}$$

The objective of the first reduction is to construct a formal solution of the typical form

$$\boldsymbol{u} = T^\nu \sum_{j=0}^{\infty} \sum_{p=0}^{j} u_{jp} T^j (\ln T)^p. \tag{1.2}$$

It is achieved by introducing a renormalized unknown $\boldsymbol{v}$, of which a typical definition has the form

$$\boldsymbol{u} = T^\nu (\boldsymbol{u}_0 + T^\varepsilon \boldsymbol{v}).$$

Change variables so that T is the first independent variable. Let $D = T\frac{\partial}{\partial T}$. If ε is small enough, $\boldsymbol{v}$ solves a system of the form

$$(D + A + \varepsilon)\boldsymbol{v} = T^\sigma f[T, \boldsymbol{v}].$$

One then chooses ε such that $\sigma > 0$. Such is the typical form of a Fuchsian first-order system for us. If it is possible to transform a problem into this form by a change of variables and unknowns, we say that it admits of reduction.

Chap. 3 gives general situations in which this reduction is possible, and further special cases are treated in the applications. General results from Chap. 2 give formal series solutions, and identify the terms containing arbitrary functions or parameters. The set of arbitrary functions, together with the equation of the singular set, form the *singularity data.* In some problems, the singular set is prescribed at the outset, and is not a free parameter; the singularity data consist then only of the arbitrary functions or parameters in the expansion.

Remark 1.1. In some cases, it is convenient to reduce first to a higher-order equation or system, of the form

$$P(D+\varepsilon)\boldsymbol{v} = T^{\sigma} f[T, \boldsymbol{v}].$$

The resonances are then defined as the roots of P.

The objective of the second reduction is to prove that the singularity data determine a unique solution of the equation $F[\boldsymbol{u}] = 0$. Introduce a new renormalized unknown $\mathbf{w}$ that satisfies

$$(D + A + m)\mathbf{w} = T^{\tau} g[T, \mathbf{w}]$$

with $\tau > 0$. It is typically defined by a relation of the form

$$\boldsymbol{v} = \varphi + T^{\mu}\mathbf{w},$$

where φ is known in closed form, and *what* contains all the arbitrary elements in the formal series solution. If μ is large enough, it turns out that m also is. One then chooses μ such that $A+m$ has no eigenvalue with negative real part. One then appeals to one of the general results of Chaps. 4, 5, or 6 to conclude that the equation for $\mathbf{w}$ has a unique solution that remains bounded as $T \to 0+$. It may be necessary to take some of the variables that enter the expansion, such as $t_0 = T$, $t_1 = T \ln T$, as new independent variables; this is essential for the convergence proof, and provides automatically a uniformization of solutions; see Chap. 4.

We now turn to the fourth step of the reduction process. Denoting by SD the singularity data, we have now constructed a mapping $\Phi : SD \mapsto \boldsymbol{u}$. If, on the other hand, we have another way of parameterizing solutions, we need to compare these two parameterizations. For instance, if we are dealing with a hyperbolic problem, we have a parameterization of solutions by Cauchy data, symbolically represented by a mapping Ψ : CD $\mapsto \boldsymbol{u}$. The objective of the "invertibility" step is to determine a map CD $\mapsto \boldsymbol{u} \mapsto SD$. This requires inverting Φ; hence the terminology. At this stage, we know how singularity data vary: perturbation of Cauchy data merely displaces the singular set or changes the arbitrary parameters in the expansion, or both.

Thus, the main technical point is the reduction to Fuchsian form and its exploitation. For this reason, we now give a few very simple illustrations of the process leading to Fuchsian form.

1.3 A few definitions

Let us define some terminology that will be used throughout the volume. Let T be one of the independent variables, and let

$$D = T\frac{\partial}{\partial T}.$$

A system is said to be Fuchsian if it has the form

$$(D + A)\boldsymbol{u} = F[T, \boldsymbol{u}], \tag{1.3}$$

where F vanishes with T, and A is linear. The eigenvalues of $-A$ for $T = 0$ are called resonances, or (Fuchs) indices; they determine the exponents λ such that the equation $(D+A)\boldsymbol{u} = 0$ may be expected to have a solution that behaves like T^λ for T small, real, and positive. Similarly, an equation will be called Fuchsian if it has the form

$$P(D)u = F[T, u],$$

where P is a polynomial, possibly with coefficients depending on variables other than T, and F vanishes with T. The roots of P are called resonances or Fuchs indices, and they are again associated to solutions with power leading behavior. Unlike Fuchs–Frobenius theory, the right-hand side and the solution may not have a continuation to a full complex neighborhood of the origin. The unknown may have several components, and if so, it is written in boldface; A is generally a matrix, but could be a differential operator—this makes no difference in the formal theory. Seemingly more general equations in which $A = A(T, \boldsymbol{u})$ may usually be reduced to the standard form (1.3); see Problem 2.7. Note that $T^2\partial/\partial T = s\partial/\partial s$ if $s = \exp(-1/T)$, so that equations with irregular singularities may be reduced to Fuchsian form by a nonanalytic change of variables. A treatment by reduction of some equations with irregular singular points is given in Problem 4.3. Similarly, higher-order equations may be reduced to first-order ones, by introducing a set of derivatives of the unknown as new unknowns, just as in the case of the Cauchy problem.

1.4 An algorithm in eight steps

Let us further subdivide the four basic steps into separate tasks. This yields an algorithm in eight steps:

- **Step A.** Choose the expansion variable T. Change variables so that $\boldsymbol{u} = \boldsymbol{u}(y, T)$, where y represents new coordinates.
- **Step B.** List all possible leading terms and choose one. This is generally achieved by writing

$$F[u_0 T^\nu] = \phi[u_0, \nu]T^\rho(1 + o(1))$$

and choosing u_0 and ν by the conditions

$$\phi[u_0, \nu] = 0 \text{ and } u_0 \not\equiv 0. \tag{1.4}$$

- **Step C.** Compute the first reduced equation. If convenient, convert the equation into a first-order system.
- **Step D.** Choose ε and determine the resonance equation.
- **Step E.** Determine the form of the solution. Determine in particular which coefficients are arbitrary.
- **Step F.** Compute the second reduced equation.
- **Step G.** Show that formal solutions are associated to actual solutions.
- **Step H.** Determine whether the solutions of step G are stable, by inverting the mapping from singularity data to solutions.

1.5 Simple examples of reduced Fuchsian equations

We show, on prototype situations, how a Fuchsian equation arises naturally, and how reduction techniques encompass familiar concepts: the Cauchy problem, stable manifolds, and the Dirichlet problem. We also work out completely a simple example of analysis of blowup, and outline another, which introduces the need for logarithmic terms.

1.5.1 The Cauchy problem as a special case of reduction

Consider, to fix ideas, the equation

$$u_{tt} = u^2,$$

and the solution of the Cauchy problem with data prescribed for $t = a$. Let $T = t - a$ and

$$u = u_0 + T(u_1 + v),$$

where $u_0 = u(a)$ and $u_1 = u'(a)$. We obtain

$$D(D+1)v = T(u_0 + Tu_1 + Tv).$$

This is a Fuchsian equation. We have taken $\nu = 0$, $\varepsilon = 1$, and the resonances are 0 and 1. The solution, which contains three parameters (u_0, u_1, a), is redundant, because the mapping $(u_0, u_1, a) \mapsto u$ is many-to-one, as in the example of Sect. 1.1.

1.5.2 Stable manifolds

Let us seek the (one-dimensional) manifold of solutions of

$$u_{tt} - u = 3u^2$$

that decay as $t \to +\infty$. Let $T = \exp(-t)$. Since $\partial_t = -T\partial_T = -D$, we obtain

$$(D+1)(D-1)u = 6u^2.$$

Letting $u = aT + T^2 v$, where a is a given constant, we obtain

$$(D+1)(D+3)v = 3(a+Tv)^2.$$

This is not yet a Fuchsian equation, because the right-hand side is not divisible by T. We therefore let $v = a^2 + w$, and obtain

$$(D+1)(D+3)w = 3[(a+Tv)^2 - a^2] = 3Tv(2a^2 + Tv).$$

We have achieved a Fuchsian reduction with $\nu = 1$ and $\varepsilon = 1$. The resonances are 0 and -2. The stable manifold is parameterized by a.

1.5.3 Dirichlet problem

Consider the problem

$$-\Delta u + f(u) = 0$$

on a smooth bounded domain Ω, with boundary condition $u = \varphi(x)$ on $\partial\Omega$. Let $d(x)$ denote the distance from x to the boundary of Ω; it is smooth in a neighborhood of the boundary; see Part IV. Let $u = \varphi + dz$ and $g(x,z) := df(\varphi + dz)$. Substituting into the equation and multiplying by d, we obtain

$$-d^2\Delta z - 2d\nabla d \cdot \nabla z + dg(x,z) = 0.$$

That z admits an expansion in powers of d is a consequence of Schauder theory. However, scaled Schauder estimates are not sufficient to handle equations of the form $-d^2\Delta z - ad\nabla d \cdot \nabla z + bz + dg(x,z) = 0$, for general values of a and b. Now, such operators arise naturally from the asymptotic analysis of geometric problems leading to boundary blowup. We develop an appropriate regularity theory in Chap. 6 to obtain an expansion of solutions in such cases.

1.5.4 Blowup for an ODE

Consider the equation

$$u_{tt} - 6u^2 - t = 0. \tag{1.5}$$

We illustrate with this example a practical method for organizing computations. We are interested in solutions that become singular for $t = a$.

Leading-order analysis

Let $T = t - a$. The equation becomes $u_{TT} - 6u^2 - T - a = 0$. We seek a possible leading behavior of the form $u \sim u_0 T^\nu$ with $u_0 \neq 0$. It is convenient to set up the table[1]

	u_{tt}	$-6u^2$	$-T$	$-a$
Exponent	$\nu - 2$	2ν	1	0
Coefficient	$\nu(\nu-1)u_0$	$-6u_0^2$	-1	$-a$

We now seek the smallest exponent in the "Exponent" line in this table.

If $\nu < 0$, the smallest is $\nu - 2$ or 2ν. If these two exponents are distinct, $\phi[u_0, \nu]$ is proportional to a power of u_0, and therefore may vanish only if $u_0 = 0$, which contradicts (1.4). Therefore, $\nu - 2 = 2\nu$, or $\nu = -2$. We then obtain $\phi[u_0, \nu] = 6(u_0 - u_0^2)$; hence $u_0 = 1$.

If $\nu \geq 0$, we obtain $2\nu > \nu - 2$. We are therefore left with the following cases:

- If $0 \leq \nu < 2$, $\phi[u_0, \nu] = \nu(\nu - 1)u_0$. Therefore, $\nu = 0$ or 1, corresponding to solutions such that $u \sim u_0$ and $u \equiv u_0 T$ respectively. These are special cases of the Cauchy problem in which the Cauchy data are nonzero.
- If $\nu = 2$, the terms u_{tt} and $-a$ balance each other, leading to $2u_0 = -a$, or $u \sim -\frac{1}{2}aT^2$. This is admissible if $a \neq 0$.
- If $\nu > 2$, the only possibility for a nontrivial balance is $a = 0$ and $\nu - 2 = 1$. This leads to $u \sim \frac{1}{6}T^3$.

The last two cases correspond to the Cauchy problem in which both Cauchy data vanish. Since these solutions may be investigated by standard means, we do not pursue their study any further.

First reduction

Consider first the case $u_0 = 1$, $\nu = -2$. Upon multiplication by T^2, equation (1.5) turns into

$$D(D-1)u - 6(Tu)^2 - T^3 - aT^2 = 0.$$

Let

$$u = \frac{1}{T^2}(1 + T^\varepsilon v(T)).$$

Substituting into the equation for u, and multiplying through by $T^{2-\varepsilon}$, we obtain

[1] For ODEs, it is possible to use a variant of Newton's diagram, as in Puiseux theory. For PDEs in which u_0 may be determined by a differential equation, rather than an algebraic equation, it is not convenient to do so. For this reason, we do not use Newton's diagram.

$$(D+\varepsilon-2)(D+\varepsilon-3)v-6T^{-\varepsilon}[(1+T^{\varepsilon}v)^2-1]-T^{2-\varepsilon}[T^3+aT^2]=0.$$

Since

$$(D+\varepsilon-2)(D+\varepsilon-3)-12=(D+\varepsilon+1)(D+\varepsilon-6),$$

this may be rearranged into

$$(D+\varepsilon+1)(D+\varepsilon-6)v=6T^{\varepsilon}v^2+T^{4-\varepsilon}[T+a].$$

The resonance polynomial is read off from the previous step: $P(X)=(X+1)(X-6)$. The resonances are -1 and 6. Since the right-hand side involves T^{ε} and $T^{4-\varepsilon}$, we may take ε in the range $[0,4]$. The best is to take ε as large as possible, namely $\varepsilon=4$. This gives

$$(D+5)(D-2)v=a+T+6T^4v^2.$$

Replacing v by $v-\frac{a}{10}$ leads to a Fuchsian equation with right-hand side divisible by T. The general results of Chap. 2 yield the formal series solution

$$u=\frac{1}{T^2}-\frac{a}{10}T^2-\frac{1}{6}T^3+T^4(b+\cdots),$$

where b is arbitrary. The first reduction is now complete.

Second reduction

The second reduction is carried out as follows: define w by

$$u=\frac{1}{T^2}-\frac{a}{10}T^2-\frac{1}{6}T^3+T^4w(T).$$

In other words, $v=-\frac{a}{10}-\frac{1}{6}T+T^2w$. One finds, by direct computation, that w solves

$$D(D+7)w=Tg(T,a,w),$$
$$w(0)=b,$$

where

$$g(T,a,w)=6T^2\left(-\frac{1}{10}-\frac{1}{6}T+T^2w\right)^2.$$

The general results of Chap. 5 show that this problem for w has a unique local solution. The second reduction is now complete: given any pair (a,b), there is a unique solution $u(t;a,b)$ of equation (1.5) of the form

$$u(t;a,b)=\frac{1}{(t-a)^2}-\frac{a}{10}(t-a)^2-\frac{1}{6}(t-a)^3+b(t-a)^4+\cdots,$$

where the expansion converges on some disk $|t-a|<2R$, where $R=R(a,b)$ depends smoothly on its arguments. The singularity data are (a,b); solutions have a double pole for $t=a$, and b is the coefficient of $(t-a)^4$ in the Laurent expansion of u.

Stability of singular behavior

To fix ideas, restrict a and b to a neighborhood of 0 in such a way that the series converges for $|t| < 3R(0,0)/2$. We may then compute u and its derivatives for $t = 0$, for $a \neq 0$. We obtain

$$u(0;a,b) = \frac{1}{a^2} + \cdots + ba^4 + \cdots \text{ and } u_t(0;a,b) = \frac{2}{a^3} + \cdots - 4ba^3 + \cdots .$$

It follows that

$$\left.\frac{\partial(u,u_t)}{\partial(a,b)}\right|_{t=0} = \begin{vmatrix} -2/a^3 + \cdots & a^4 + \cdots \\ -6/a^4 + \cdots & -4a^3 + \cdots \end{vmatrix} = 14 + \cdots \neq 0.$$

The map $(a,b) \mapsto (u(0), u_t(0))$ therefore satisfies the assumptions of the inverse function theorem near any (a,b) with $a \neq 0$ and b both small. To sum up, we have proved the following theorem:

Theorem 1.2. *Consider a solution $u = u(t;a,b)$ with $a \neq 0$. If a and b are small, and if v is a solution with Cauchy data close to $(u(0), u_t(0))$, then $v = u(t;\tilde{a},\tilde{b})$, with $(\tilde{a},\, \tilde{b})$ close to (a,b).*

If a becomes large, it is conceivable that the solution has another singularity between 0 and a. The appropriate stability statement, which involves setting up a correspondence between singularity data at the two singularities, is left to the reader.

1.5.5 Singular solutions of ODEs with logarithms

Let us seek singular solutions of

$$u'' = u^2 + e^t. \tag{1.6}$$

It is proved in [14, p. 166] that (1.6) has no solution of the form $u = T^{-2}(u_0 + u_1 T + \cdots)$, $T = t - a$, and that terms of the form $T^4 \ln T$ must be included. However, this leads to higher and higher powers of $\ln T$ if the computation is pushed further. We cope with this difficulty by expanding the solution in powers of T and $T \ln T$.

Theorem 1.3. *There is a family of solutions of (1.6) such that $u \sim 6/T^2$. This family is a local representation of the general solution: the parameters describing the asymptotics are smooth functions of the Cauchy data at a nearby regular point.*

Proof. The argument is similar to the one just given, and we merely indicate the differences. The formal solution now takes the form

$$u = \frac{6}{T^2} - \frac{A}{10}T^2 - \frac{A}{6}T^3 + \frac{A}{14}T^4 \ln T + T^4 v(T, T\ln T) \tag{1.7}$$

with $A = e^a$. Here, v is a power series in *two* variables T and $T \ln T$, entirely determined by its constant term. Since we have the form of the solution at hand, let us directly write the equation solved by v, which is the second reduced equation:

$$D(D+7)v = T^2 \left(-\frac{A}{10} - \frac{AT}{6} + T^2 w \right)^2 + A \left(\frac{T}{3!} + \frac{T^2}{4!} + \cdots \right). \tag{1.8}$$

It has the general form

$$D(D+7)v = Tf(T, T \ln T, v), \tag{1.9}$$

and Theorem 4.3 gives the existence and uniqueness of a local solution with $v(0)$ prescribed; it is the sum of a convergent power series in T and $T \ln T$. Let $b = v(0)$. The singularity data are (a, b). We conclude with the stability analysis. Since $A = e^a$, we have

$$\frac{\partial u}{\partial a} = \frac{12}{T^3} \frac{A}{5} T + A \left(-\frac{1}{10} + \frac{1}{3} \right) T^2 + O(T^3),$$
$$\frac{\partial u}{\partial b} = T^4 + o(T^4).$$

These relations can be differentiated with respect to t. We can therefore compute the Jacobian of the mapping $\Phi : (a, b) \mapsto (u(t_0), u'(t_0))$ if t_0 is close enough to zero. In fact, in that case, one may replace $\partial u / \partial a$ and $\partial u / \partial b$ by their equivalents. We can then invert the map Φ and conclude, as before, that we have achieved a local representation of the general solution. □

Even though v exhibits branching because of the logarithm, it is obtained from a single-valued function of two variables by performing a multivalued substitution. In other words, this representation is a uniformization of the solution.

1.5.6 Blowup for a PDE

We now move to the next level of difficulty: a PDE that requires logarithmic terms in the expansion of solutions. Since the manipulations involved are typical of those required for all the applications to nonlinear waves, we write out the computations in detail. In particular, all background definitions from Riemannian geometry are included, so that the treatment is self-contained.

Let us perform the first reduction for the hyperbolic equation:

$$\Box u = \exp u,$$

where $\Box = \partial_{tt} - \Delta$ is the wave operator in n space variables. This equation is the n-dimensional Liouville equation. In one space dimension, this equation is exactly solvable; see Sect. 10.6. It was this exact solution that suggested the introduction of Fuchsian reduction in the first place [120]. The objective is to show that near singularities, the equation is not governed by the wave operator, but by an operator for which the singular set is characteristic. This suggests that blowup singularities for nonlinear wave equations are not due to the focusing of rays for the wave operator, and that the correct results on the propagation of singularities must be based on this Fuchsian principal part rather than the wave operator. This statement will be substantiated in Chap. 10, where the other steps of reduction, including stability, will be carried out.

Leading-order analysis

Let us first define new independent variables:

$$X^0 = T = \phi(x,t) = t - \psi(x), \quad X^{i'} = x^i \text{ for } 1 \le i \le n. \tag{1.10}$$

Note that $\partial_i T = -\psi_i$ and $\partial_i g = \partial_{i'} g$ if $g = g(X)$, so that $\Delta g = \Delta' g$ in particular. It is convenient to put coordinate indices as exponents, and to use primed indices to denote derivatives with respect to the coordinates (X, T).

Lemma 1.4. *In these coordinates, the wave operator takes the form*

$$\Box = \gamma \partial_T^2 - \left(\sum_{i'} \partial_{i'}^2 - 2\psi_i \partial_{Ti'} \right) + (\Delta\psi)\partial_T, \tag{1.11}$$

where

$$\gamma = (1 - |\nabla\psi|^2). \tag{1.12}$$

Proof. For fixed i, ∂_{ii} can be expressed as follows (we write $\partial_{i'}$ for $\delta_i^{i'}\partial_{i'}$):

$$\begin{aligned}
(\partial_{i'} - \psi_i \partial_T)^2 &= (\partial_{i'} - \psi_i \partial_T)\partial_{i'} - \partial_i(\psi_i \partial_T) \\
&= (\partial_{i'} - \psi_i \partial_T)\partial_{i'} - (\Delta\psi)\partial_T - \psi_i \partial_{iT} \\
&= \partial_{i'}^2 - \psi_i \partial_{Ti'} - (\Delta\psi)\partial_T - \psi_i(\partial_{i'} - \psi_i \partial_T)\partial_T \\
&= \partial_{i'}^2 - 2\psi_i \partial_{Ti'} + |\nabla\psi|^2 \partial_T^2 - (\Delta\psi)\partial_T.
\end{aligned}$$

The result follows. □

By tabulating possible cases as before, we find that there is no consistent leading term for which u behaves like a power of T; therefore, we seek u with logarithmic behavior, and require $\exp u \sim u_0 T^\nu$. Substituting into the equation and balancing the most singular terms leads to $u_0 = 2$ and $\nu = -2$. The leading term is therefore $u \approx \ln(2/T^2)$.

First Reduction

We define a first renormalized unknown $v(X,T)$ by

$$u = \ln(\frac{2}{\phi^2}) + v(X,T)T^{\varepsilon}.$$

We leave it to the reader to compute the first reduced equation and check that $\varepsilon = 0$ leads to a Fuchsian PDE for v. We obtain the resonance polynomial $P(X) = (X+1)(X-2)$. The indices are therefore -1 and 2.

General results from Chap. 2 imply that no logarithms enter the solution until the term in T^2, since the smallest positive index is 2; furthermore, since it is simple, there is a formal solution in powers of T and $T\ln T$, which is entirely determined by the coefficient of T^2. To perform reduction, we need to compute the first few terms of the expansion. Inserting $v = v^{(0)}(X) + Tv^{(1)}(X) + \cdots$ into the first reduction and setting to zero the coefficients of T^{-2} and T^{-1} in it, we obtain

$$v^{(0)} = \ln\gamma, \quad v^{(1)} = -\gamma^{-1}\Delta\psi.$$

However, it is not possible to continue the expansion with a term $v^{(2)}T^2$: substitution into the equation shows that $v^{(2)}$ does not contribute any term of degree 0 to the equation. In fact, $v^{(2)}$ is arbitrary, and we must include a term in $R_1(X)T^2\ln T$ in the expansion.

Second reduction

Define the second renormalized unknown w by

$$u = \ln\frac{2}{T^2} + v^{(0)} + v^{(1)}T + R_1T^2\ln T + T^2w(X,T), \tag{1.13}$$

where R_1 will be determined below.

Lemma 1.5. *The second reduction leads to the Fuchsian PDE*

$$\begin{aligned}
&\gamma(T\partial_T)(T\partial_T+3)w \\
&\quad + T\Big[(\Delta\psi)(R_1+(T\partial_T)w) + 2\psi^i\delta_i^{i'}\partial_{i'}(R_1+(T\partial_T)w) \\
&\quad -T\ln T\Delta R_1 - T\Delta w\Big] \\
&\quad + 4\psi^i\delta_i^{i'}\partial_{i'}(R_1T\ln T+Tw) + 2(\Delta\psi)(R_1T\ln T+Tw) - T\Delta v^{(1)} \\
&= (1-|\nabla\psi|^2)\Big\{(v^{(1)}+R_1T\ln T+Tw)^2 - [v^{(1)}]^2 \\
&\qquad +T(v^{(1)}+R_1T\ln T+Tw)^3 \\
&\qquad \times\int_0^1(1-\sigma)^2\exp(T\sigma(v^{(1)}+R_1T\ln T+Tw))\,d\sigma\Big\}
\end{aligned} \tag{1.14}$$

for w.

This equation for w has the form

$$(1-|\nabla\psi|^2)D(D+3)w+\alpha(X)+3\gamma R_1=\mathcal{O}(T),$$

where $\mathcal{O}(T)$ refers to terms that all have a factor of T. We must therefore take $R_1=-\alpha/3\gamma$. The vanishing of α is the necessary and sufficient condition for the absence of logarithmic terms in the expansion of w; in that case, $\exp u$ does not involve logarithms at all. In the analytic case, the existence theorems from Chap. 4 imply that the equation $\Box u=e^u$ has a solution such that e^{-u} is holomorphic near the hypersurface of the equation $t=\psi(x)$. Since this regularity statement is independent of the representation of the hypersurface, we expect the vanishing of α to have a geometric meaning in terms of the geometry of the blowup surface in Minkowski space. In this case, one can say more:

Theorem 1.6. *The quantity R_1 equals $-2R/(3\gamma)$, where R is the scalar curvature of the blowup surface; furthermore, $\alpha=2R$.*

For the proof, see Problem 3.8.

Remark 1.7. For other nonlinearities, the no-logarithm condition may also involve the second fundamental form of the blowup surface. An example of such computations will be outlined in Chap. 10. Similar results are available in the elliptic case. Taking T to be the distance to the singular surface enables one to give a geometric interpretation for other elements of the expansion of the solution.

Stability is proved in Sect. 10.2.

1.6 Reduction and applications

We have seen that reduction arises naturally when one attempts to perform an asymptotic analysis of nonlinear PDEs near singularities. We now turn to the benefits of such information for applications.

1.6.1 Blowup pattern

Applications to nonlinear wave equations rest on the notion of a blowup pattern, which is not a wave. To describe the difference in intuitive terms, consider the following situation. Take a flashlight, and direct it toward a wall. One sees a spot of light. Now move your hand slightly, so that the spot of light moves on the wall. Clearly, the motion of the spot is not a wave propagation, because the spot does not move by itself, but merely because its source moves. Similarly, it is a familiar fact that nonlinear wave equations may have solutions that develop singularities in finite time. Due to the finite speed of propagation, singularities usually do not appear simultaneously at all points in space. The locus of singular points at any given time therefore defines a specific evolving pattern that forms spontaneously. This pattern is a collective result of the evolution of the solution as a whole, for one singularity is not necessarily causally related to nearby singularities in space-time. If this causal relation does hold, we may talk of wave propagation; but patterns are distinct from wave propagation, in which a definite physical quantity is being tracked as it propagates gradually and causally. In the problems considered here, the pattern at time t is given by an equation of the form $\psi(x) = t$. The considerations leading to this concept are further elaborated in Sect. 10.1.

If a singularity pattern is to be significant, it is necessary that it should be stable under perturbations of the initial data. If the equation itself is a model in which various effects have been neglected, we should also require stability of the pattern under perturbations of the equation: thus, if a singularity pattern is present for an exactly solvable model, it should also be present for its nearly integrable perturbations, as in Sect. 10.5. Reduction techniques investigate whether it is possible to embed a singular solution into a family of solutions with the maximum number of free functions or parameters; if this is the case, we say that the singular solution is stable.

Advantages of this viewpoint include the following: (i) the blowup time is obtained as the infimum of the equation ψ of the blowup surface; (ii) unlike self-similar estimates, one obtains precise information on the behavior of solutions in directions nearly tangential to the blowup surface; (iii) geometric information on the blowup set is obtained; (iv) continuation after blowup may be studied easily whenever it is relevant. In addition, unlike asymptotic methods, reduction provides a representation of solutions in a finite neighborhood of their singular set. Therefore, it represents large-amplitude waves accurately, a short time before blowup. This is appropriate since in many applications, the solution becomes large, but not actually infinite; in particular, reduction predicts the approximate shape of the set where the size of the solution exceeds a given quantity, and furnishes combinations of the solution and its derivatives that remain finite at blowup.

1.6.2 Laser collapse

We consider a model for laser collapse, which improves on the familiar NLS model in media with Kerr nonlinearity, by taking into account normal dispersion and lack of paraxiality. Modeling leads to a nonlinear hyperbolic equation with smooth data for $t = 0$, the solutions of which blow up on a hypersurface $t = \psi(x)$; see Chap. 10 for details. The main practical consequences of reduction are these:

- The rate of concentration of energy may be computed, and is related to the mean curvature of the singular locus.
- It is possible to compute solutions that blow up at two nearby points, possibly at different times; such solutions may account for "pulse-splitting."

- The solutions are stable in the sense that a small deformation of the blowup set and the asymptotics induces a small change in the solution, even though it is singular.
- Near singularities, solutions have the form $u = v + w$, where v is given in closed form and may therefore be treated as a substitute for an exact solution, which takes over when numerical computations break down.

From a mathematical standpoint:

- The local model near the singularity is not the wave equation, but a linear Fuchsian equation for which the blowup surface, which is spacelike, is characteristic.
- Reduction methods give in particular self-similar asymptotics, but it also provides information in a full neighborhood of the blowup set, in particular outside null cones or backward parabolas under the first blowup point.
- If we decompose $u = v + w$, where v contains the first few terms of the expansion of u, and write CD for the Cauchy data for $t = 0$, the map CD$\mapsto u(t, .)$ fails to be continuous in, say, the H^1 topology for t large, even if CD is very smooth; nevertheless, reduction shows that the map CD$\mapsto w(t, .)$ is well behaved, and that v is determined by two functions that are also well behaved.

1.6.3 The weak detonation problem

The mathematical issue is to analyze the solution of a nonlinear hyperbolic problem, with smooth data, that blows up, representing the onset of detonation. In addition to the above advantages (explicit formulas, geometric interpretation, substitute for numerics), reduction explains how to interpret rigorously the linearized solutions that are more singular than the solution of the detonation problem. The blowup surface is spacelike, reflecting the supersonic character of the detonation front.

As in the previous application, Reduction shows that blowup leads to the formation of a pattern—as opposed to a wave: the various points on the blowup surface are not causally related to one another, but nearby points on this surface have nearby domains of dependence. These two points are respectively reflected in two facts: (i) blowup singularities do not propagate along characteristic surfaces for the wave operator; (ii) the regularity of the blowup surface is related to the regularity of the Cauchy data. These facts will be ascertained by a direct procedure: construct the solution almost explicitly, and read off the desired information. The explicit character of reduction accounts for its practical usefulness.

1.6.4 Cosmology

As a further application, we turn to cosmology, referring to Chap. 8 for details. The big-bang model has been derived on the assumption that the large-scale structure of the universe is spatially isotropic and homogeneous. Since the universe is obviously not exactly homogeneous, and since the Friedmann–Lemaître–Robertson–Walker (FLRW) solution of Einstein's equation underlying the model does not seem to be stable under inhomogeneous perturbations, it is desirable to find cosmological solutions that allow inhomogeneities, and to investigate whether stable solutions are at least asymptotically isotropic and inhomogeneous. In the mid-seventies, the mathematical issue was clearly identified [54]: is there a mechanism whereby "space derivatives" dominate "time derivatives" near the singularity? If this is the case, the space-time is said to be "asymptotically velocity-dominated" (AVD). After many inconclusive attempts, numerics were tried; they were consistent with AVD behavior, except at certain places corresponding to spikes in the output of computation.

Reduction gave an explanation of AVD behavior: the relevant equations can be reduced to a Fuchsian form

$$tu_t + A(x)u = t^{\varepsilon} f(t, x, u, u_x),$$

with $\varepsilon > 0$, in which x-derivatives are indeed less important than time derivatives as $t \to 0$, because the time derivatives enter with an expression homogeneous of degree zero in t (the left-hand side), while the space derivatives are homogeneous of positive degree ε. Thus, any term in the expansion of u of the form t^k, with k constant, contributes terms of order t^k to the left hand side, but $t^{k+\varepsilon}$ to the right hand side. Since the exponent k must also be allowed to depend on x, a careful treatment is necessary, but the above justification remains in essentials. A detailed analysis of the expansion of the solutions in this case shows that it involves four arbitrary functions, and that the form of the expansion changes if the derivative of one of the arbitrary functions vanishes. Some spikes observed in computations are not numerical artefacts, but correspond precisely to the extrema of this arbitrary function. Other spikes, due to a poor choice of coordinates, may also be analyzed. This work has been extended to other types of matter terms. It seems to be the only practical and rigorous procedure for systematically constructing solutions of Einstein's equations with singularities containing arbitrary functions.

1.6.5 Conformal geometry

Conformal geometry is the geometry of a class of metrics related to one another by a multiplicative conformal factor $e^{2\sigma}$ that varies from point to point. Thus, angles between curves are well determined, but length scales may vary from point to point.

The first application of reduction in this context concerns the two-dimensional Liouville equation. It is one of the very first nonlinear PDEs to have been studied. The number of contexts in which it arises is extremely large, and contributions to its study span one and a half centuries, from Liouville's paper [136] onward. Our results pertain to the so-called conformal radius, defined in terms of conformal mapping.

Let us therefore recall some background information on conformal mapping. The Riemann mapping theorem states that any simply connected domain $\Omega \subset \mathbb{R}^2$, which we assume bounded for simplicity, may be mapped onto the unit disk by an analytic function of $z = x + iy$; this mapping is not isometric, but effects a conformal change of metric, with conformal factor $|f'(z)|^2$. This map, unique up to homographies, may be found as follows: fix any $z_0 = x + iy \in \Omega$, and consider the class of all analytic functions $f(z)$ defined on Ω such that $f(z_0) = 0$ and $f'(z_0) = 0$. For any such f, let $R(f, z_0)$ be the least upper bound of the numbers R such that $f(\Omega)$ is included in the disk of center 0 and radius R. Let

$$r(z_0) = \inf_f R(f, z_0),$$

where f varies in the class of analytic mappings defined on Ω. It turns out that there is a conformal mapping from Ω onto the disk of radius $r(z_0)$ about the origin; a rescaling furnishes a conformal mapping onto the unit disk. The consideration of minimizing sequences for R is the simplest strategy to prove the Riemann mapping theorem. The function $(x, y) \mapsto r(z_0)$ is the mapping radius function of Ω; it is also called conformal radius or hyperbolic radius because the metric $r^{-2}(dx^2 + dy^2)$, which blows up at the boundary, is a complete, conformally flat metric that generalizes Poincaré's hyperbolic metric on the unit disk or the half-plane. It is possible to recover a conformal mapping from Ω to the unit disk from the mapping radius function. The mapping radius was extensively studied in the twentieth century, and has several other applications that require understanding the boundary behavior of the mapping radius; see the review article [8].

It was conjectured in the mid-eighties that the mapping radius is a $C^{2+\alpha}$ function up to the boundary if Ω is of class $C^{2+\alpha}$. Reduction leads to a proof of this result, without assuming the domain to be simply connected. This improves the result of [36] to the effect that r is of class $C^{2+\beta}$ for some $\beta > 0$ if Ω is (convex and) of class $C^{4+\alpha}$.

To see how reduction enters the problem, which at first sight has no connection with singular solutions of PDEs, let us write $v(x, y) = r(x + iy)$. It turns out (Problem 9.1) that $u = -\ln v$ satisfies

$$-\Delta u + 4 \exp(2u) = 0 \tag{1.15}$$

in Ω; v solves $v\Delta v = |\nabla v|^2 - 4$. Equation (1.15) is known as the Liouville equation [136]. No boundary condition is imposed; u tends to $+\infty$ as (x, y) approaches $\partial\Omega$ and majorizes all solutions of this equation with smooth boundary values. The latter property holds in higher dimensions, and for large classes of superlinear monotone nonlinearities, in non-simply-connected domains as well. As a consequence, solutions to the Liouville equation satisfy an interior a priori bound involving only the distance to the boundary and not the boundary values at all. Keller and Rademacher also studied this equation in three dimensions, which is relevant to electrohydrodynamics. The minima of the radius function also occur as points of concentration of minimizing sequences in variational problems of recent interest. This and many other applications require a detailed knowledge of v [8]. Finally, the numerical computation of the radius function is effected by solving a Dirichlet problem in a slightly smaller domain, with Dirichlet data obtained from asymptotics of v. For these reasons, it is desirable to know the boundary behavior of v.

We will also give a similar result for the n-dimensional analogue of Liouville's' equation, introduced by Loewner and Nirenberg [137], who showed that some of the properties of the mapping radius may be generalized by considering the equation

$$-\Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0 \tag{1.16}$$

in an n-dimensional domain. Letting $v = u^{-n/(n-2)}$, one solves $v\Delta v = \frac{n}{2}(|\nabla v|^2 - 4)$. We seek to obtain $C^{2+\alpha}$ regularity of v to ensure that v is a classical solution of this equation. By contrast, u cannot be interpreted as a distribution solution of (1.16).

After these preliminaries, if $v = 2d + d^2 w$, w solves a Fuchsian PDE of the form

$$Lw := d^2\Delta w + (4-n)d\nabla d \cdot \nabla w + (2-2n) - 2\Delta d + dF(w, d\nabla w); \tag{1.17}$$

the Liouville equation leads to the same problem, with $n = 2$. In fact, it is not necessary that n should be equal to the dimension of Ω; inspection shows that all we need is that the parameter n in the equation should be larger than $1+\alpha$. For this reason, we now allow n to be a real parameter, unrelated to the space dimension. Therefore, the Liouville and Loewner–Nirenberg equations admit of reduction, and the regularity of the hyperbolic radius is equivalent to the extension of Schauder theory to the Fuchsian, degenerate elliptic equation (1.17).

Even though Δd is of class C^α, the modern form of the interior weighted Schauder estimates is insufficient to obtain the desired regularity, namely $d^2 w \in C^{2+\alpha}$. The reason is that weighted estimates estimate the scale-invariant ratio

$$\min(d(x), d(y))^{2+\alpha} \frac{|\nabla^2 w(P) - \nabla^2 w(Q)|^\alpha}{|P-Q|^\alpha}$$

(see Chap. 12). It is apparent that such an estimate cannot yield $d^2 w \in C^{2+\alpha}$, because the distances occur with the power $2+\alpha$ rather than 2. In fact, the result cannot be the sole consequence of ellipticity; simple examples show that the result is false if one does not take the form of lower-order terms into account. This issue is familiar in the theory of PDEs with degenerate quadratic form, such as the so-called Keldysh or Fichera problems, but the estimates we need do not follow from these L^p results. Also, the singularity of Green's function for the corresponding operator on the half-space—an operator similar to the Laplace–Beltrami operator on symmetric spaces—does not seem to be known. One example of a problem with linear degeneracy has been worked out [71], but the method does not apply to quadratic degeneracy, such as in our case.

The hyperbolic form of Liouville's equation—the one actually solved by Liouville [136]—can be solved completely in closed form.[2] Since the detailed study of this solution formula motivated the development of reduction, it will be considered in some detail in Chap. 10.

A second application concerns Fefferman's ambient metric construction. In 1936, Schouten and Haantjes suggested that it was possible to generalize the classical derivation of the conformal group of the two-sphere, by embedding it as the section $\{t = 1\}$ of the light cone in Minkowski space M^4, and letting the Lorentz group act on M^4. The problem is to embed an analytic manifold M of dimension n in a null hypersurface in a Lorentzian manifold G of dimension $n + 2$. This idea was taken up by Fefferman and Graham, who were interested in deriving conformal invariants of M from Riemannian invariants of G; they were also motivated by Fefferman's discovery, from a completely different perspective, of embeddings of this type in the context of complex geometry. The problem reduces to the construction of Ricci-flat metrics with a homothety, constrained to have a special form on a null hypersurface. There are no symmetry assumptions on the metric. They solved this problem for the case of n odd; we have solved the problem in full generality. This seems to be useful in the so-called holographic representation (Witten).

[2] In the elliptic case, and in simply connected domains, the solution of the equation in closed for depends on knowledge of the Riemann mapping.

Part I

Fuchsian Reduction

2

Formal Series

The purpose of this chapter is to construct formal solutions for equations or systems of the general form (1.3), which, we recall, reads

$$(D + A)\boldsymbol{u} = F[T, \boldsymbol{u}],$$

where F vanishes with T, and A is linear.

We are interested in finding solutions in a space $\mathcal{FS}$ of formal series of the form

$$\boldsymbol{u} = \sum_{\lambda \in \Lambda} \boldsymbol{u}_\lambda T^\lambda, \tag{2.1}$$

where the set Λ of possible exponents is countable and admits a total ordering such that (i) any set of the form $\{\mu \in \Lambda : \mu < \lambda\}$ is finite; (ii) $\boldsymbol{u}_\lambda$ may itself depend on T, but its form is restricted: it must belong to a suitable vector space E_λ; (iii) the difference between consecutive exponents is bounded below. The space and the exponents may be real or complex, depending on the examples. In the simplest situation, λ is an integer, and $\boldsymbol{u}_\lambda$ is a polynomial in $\ln T$, with coefficients depending on other "spatial" variables.

The space $\mathcal{FS}$ will be chosen so that the Fuchsian system may be solved recursively: formally, the $\boldsymbol{u}_\lambda$ are given by the equations

$$(D + \lambda + A)\boldsymbol{u}_\lambda = F_\lambda, \tag{2.2}$$

where F_λ is the coefficient of T^λ in the expansion of $F[T, \boldsymbol{u}]$. Now, if F vanishes with T, F_λ will depend only on the $\boldsymbol{u}_\mu$ with $\mu < \lambda$; as a consequence, (2.2) is a *recurrence relation* for computing the $\boldsymbol{u}_\lambda$. The spaces E_λ are determined by two requirements:

1. F should act on $\mathcal{FS}$: in practice, this requires $\mathcal{FS}$ to be closed under products and certain derivations;
2. one should be able to solve $(D + \lambda + A)\boldsymbol{u}_\lambda = F_\lambda$ in $\mathcal{FS}$ if $\boldsymbol{u}_\mu \in \mathcal{FS}$ for $\mu < \lambda$.

These requirements enable one in all practical situations to tailor the space $\mathcal{FS}$ to any one of them. The space $\mathcal{FS}$ should be taken large enough to automatically contain all solutions of (1.3) that remain bounded for T real, small, and positive. It is convenient to treat certain basic expressions in T, such as $T \ln T$, or T^x, as new independent variables; this leads to spaces with several "time variables," in which the operator D must be replaced by another first-order operator, which we call N. Systems of the form

$$(N + A)\boldsymbol{u} = F[T, \boldsymbol{u}],$$

with A and F as before, will be called generalized Fuchsian systems.

The main issue is therefore to invert $D + A$ or $N + A$. For this reason, we begin with properties of the operator $D = T\partial_T$, which are of constant use. We then discuss the main spaces of power series with constant exponents (independent of x), focusing on the space A_ℓ and its generalizations [124, 112]. We then introduce the operator N and discuss the mapping properties of $D+A$ and $N + A$ between these spaces. An example of a set of series with variable exponents is discussed next [16]; further examples are left to the exercises. Finally, the relation between A_ℓ and a representation of SL(2) is outlined.

2.1 The Operator *D* and its first properties

Consider two variables T and $L = \ln T$, which will have a purely formal meaning in this chapter. In applications, one may take $L = \ln |T|$ if one is interested in real solutions only, or any branch of the logarithm if one wishes to have complex-valued solutions. We are interested in expressions

$$u = \sum_{\lambda \in \Lambda}^{\infty} u_\lambda(L) T^\lambda,$$

where u_λ is a polynomial in L with scalar coefficients.

The following formal properties of operator $D = T\partial_T$ enable one to define Du

1. $DT^\lambda = \lambda T^\lambda$;
2. $DL = 1$;
3. $Du = 0$ if u is independent of T and L;
4. D satisfies Leibniz's rule ($D(uv) = uDv + vDu$ for scalar-valued u and v).

One extends D to functions with coefficients in a fixed vector space E that does not depend on λ by choosing a basis independent of T and working componentwise: $D(u_1, \dots, u_k) = (Du_1, \dots, Du_k)$. Two properties are of constant use:

$$\text{For any } \lambda \in \mathbb{C}, \quad D(T^\lambda u) = T^\lambda (D + \lambda) u, \tag{2.2a}$$

$$\text{For any } k = 1, 2 \dots, \quad T^k \partial_T^k = D(D - 1) \cdots (D - k + 1). \tag{2.2b}$$

As a consequence, for any polynomial P,

$$P(D)(T^\lambda u) = T^\lambda P(D+\lambda)u. \tag{2.3}$$

Similarly, if $\boldsymbol{u}$ is vector-valued and A is an operator that commutes with multiplication by T^λ, then

$$(D+A)(T^\lambda \boldsymbol{u}) = T^\lambda (D+A+\lambda)\boldsymbol{u}. \tag{2.4}$$

Finally, for any polynomial Q,

$$D[Q(L)] = Q'(L), \tag{2.5}$$

where Q' is the derivative of the polynomial Q.

Definition 2.1. *The* order *of a polynomial P at $\lambda \in \mathbb{C}$ is the smallest power that occurs in $P - P(\lambda)$ with a nonzero coefficient. It is the order of vanishing of P at λ. It is written* $\operatorname{ord}(P,\lambda)$. *Similarly, the order of a matrix A at λ is the maximal size of the Jordan blocks for the eigenvalue λ of A;*[1] *it is zero if λ is not an eigenvalue of A. It is written* $\operatorname{ord}(A,\lambda)$.[2] *We write* $\operatorname{ord} P$ *for* $\operatorname{ord}(P,0)$, *and similarly for matrices.*

Theorem 2.2. *Let q_0 be constant. The equation*

$$P(D)u = q_0 T^\lambda L^m$$

admits solutions of the form $q(L)T^\lambda$, with q polynomial in L, with

$$\deg q \le m + \operatorname{ord}(P,\lambda).$$

Proof. Writing $u = T^\lambda v$, we are reduced to the case $\lambda = 0$. We may write $P(D) = D^m R(D)$, where $m = \operatorname{ord}(P,0)$, and $R(0) \neq 0$. The action of $R(D)$ on the space of polynomials in L of degree at most m is therefore represented by a nonsingular triangular matrix. It follows that there is a polynomial Q_1 such that $R(D)Q_1(L) = q_0 L^m$, and $\deg Q_1 \le m$. But the degree of Q_1 cannot be less than m. Therefore, $\deg Q_1 = m$. If $q^{(m)} = Q_1$, we obtain $P(D)q(L) = R(D)Q_1(L) = q_0 L^m$, as desired. □

Since $D^m L^k = 0$ for $k < m$, $q(L)$ is determined up to the addition of a polynomial of degree less than m. We record this observation in the form of a theorem.

---

[1] It is possible to work with the decomposition $A = S + N$ of A into a diagonalizable and nilpotent part, instead of the Jordan decomposition [88]. Both are of course closely related.

[2] It is at most equal to the multiplicity of λ as an eigenvalue of A. If λ is an eigenvalue, $\operatorname{ord}(A,\lambda)$ is the smallest s such that $(A-\lambda)^s$ vanishes on the generalized eigenspace for eigenvalue λ. If p_A is the minimal polynomial of A, $\operatorname{ord}(A,\lambda) = \operatorname{ord}(p_A,\lambda)$.

Theorem 2.3. *If* $P(D)u = 0$, *then*

$$u = \sum_{\lambda} T^{\lambda} Q_{\lambda}(L),$$

where the sum extends over the roots λ *of* P, *and* $\deg Q_{\lambda} \le \mathrm{ord}(P, \lambda) - 1$.

As for systems, we have the following theorem.

Theorem 2.4. *Let* A *be a matrix and* $\mathbf{q}$ *a vector, both independent of* T *and* L; *let* m *be a nonnegative integer. Then the equation*

$$(D + A)\mathbf{u} = \mathbf{q} L^m T^{\lambda} \tag{2.6}$$

has a solution of the form $Q(L)T^{\lambda}$, *where* Q *is a polynomial of degree at most* $m + \mathrm{ord}(A, -\lambda)$.

Proof. Writing $\mathbf{u} = T^{\lambda}\mathbf{v}$, we may reduce the problem to the case $\lambda = 0$. If A is invertible, for any polynomial Q, $\varphi = \sum_{j=0}^{m} (-1)^j A^{-j-1} Q^{(j)}(L)$ solves $(D + A)\varphi = Q(L)$. The result follows in this case. If A is singular, we decompose the space on which A acts into the direct sum of a space on which it is invertible and one on which it is nilpotent.[3] Since we have already treated the case of invertible A, it suffices to determine the component of the solution in the latter space. We therefore assume that A is nilpotent, and seek a solution of the form

$$u = \sum_{j=0}^{m+s} \mathbf{u}_j \frac{L^j}{j!},$$

where s is to be determined. We obtain

$$(D + A)\mathbf{u} = \sum_{j=0}^{m+s-1} (A\mathbf{u}_j + \mathbf{u}_{j+1}) \frac{L^j}{j!} + A\mathbf{u}_{m+s} \frac{L^{m+s}}{(m+s)!}.$$

It follows that

$$\mathbf{u}_1 = -A\mathbf{u}_0, \ldots, \mathbf{u}_m = (-1)^m A^m \mathbf{u}_0,$$
$$\mathbf{u}_{m+1} = m!\mathbf{q} - A\mathbf{u}_m, \quad \mathbf{u}_{m+2} = -A\mathbf{u}_{m+1}, \ldots, \quad A\mathbf{u}_{m+s} = 0.$$

Therefore, all the $\mathbf{u}_k$ for $k < m + s$ are uniquely determined by u_0. The second line now gives $0 = A\mathbf{u}_{m+s} = \cdots = (-1)^{s-1} A^s \mathbf{u}_{m+1}$. We therefore need $A^s[\mathbf{q} - A\mathbf{u}_m/m!] = 0$, with $\mathbf{u}_m = (-1)^m A^m \mathbf{u}_0$. Choose s and u_0 such that

$$A^s \mathbf{q} \in \mathrm{Ran}(A^{m+1+s}), \tag{2.7}$$

where Ran denotes the range. Since A is nilpotent of order $\mathrm{ord}(A)$, this is certainly possible with $s \le \mathrm{ord}(A)$. This completes the proof. □

[3] This follows from the Jordan decomposition theorem. Recall that A is nilpotent if some power of A vanishes.

2.2 The space A_ℓ and its variants

This section collects solvability results for nonlinear Fuchsian equations in spaces adapted to series in integral powers of T and L; they cover the most common applications. Still larger spaces of series are considered in the problems.

2.2.1 The space A_ℓ

Definition 2.5. *For $\ell \geq 0$, A_ℓ is the vector space of formal expressions of the form*

$$u = \sum_{0\leq p\leq \ell j} u_{jp} T^j L^p,$$

where the coefficients u_{jp} may depend on additional variables x. The same notation will be used for vector-valued spaces of series if no confusion ensues.

Remark 2.6. If $\ell = 0$, we recover the usual space of formal series in T. If ℓ is a positive integer and $S^\ell = T$, we may write any element of A_ℓ as a series in S and SL. If $1/\ell = m$ is a positive integer, any element of A_ℓ may be written as a series in T and $T^m L$. The real number ℓ is not necessarily an integer. The restriction $p \leq \ell j$ also occurs in the definition of Ecalle's "seriable functions" [56]; the latter are, as a rule, represented by divergent series.

It is convenient to treat the variables $t_0 = T$, $t_1 = TL$, $t_2 = TL^2$,... as new independent variables; for this reason, we introduce a second space of series.

Definition 2.7. *For any integer $\ell \geq 0$, B_ℓ is the vector space of formal series in $\ell + 1$ indeterminates $\boldsymbol{t} = (t_0, \ldots, t_\ell)$. An element of B_ℓ will be written*

$$u(\boldsymbol{t}) = \sum_a u_a \boldsymbol{t}^a,$$

where $a = (a_0, \ldots, a_\ell)$ and $\boldsymbol{t}^a = \Pi_j t_j^{a_j}$.

Remark 2.8. More formally, one could introduce a $\mathcal{D}$-module structure on the space of series of the form $\sum_{q\leq lp} a_{pq} T^p L^q$, by letting operators of the form $\sum_k b_k(T, TL, \ldots, TL^l) D^k$ act on it with the rules $DT = T$, $DL = 1$.

The space B_ℓ is a graded algebra, with the grading given by total degree.[4] The spaces A_ℓ and B_ℓ are related through the map

$$\begin{aligned}\varphi : B_\ell &\to A_\ell,\\ t_k &\mapsto TL^k,\\ 1 &\mapsto 1.\end{aligned}$$

[4] In other words, any element of B_ℓ may be written as a sum of homogeneous components of increasing degrees, and the product of homogeneous polynomials of degrees m and n is homogeneous of degree $m + n$.

Definition 2.9. *Elements of* $\operatorname{Ker}\varphi$ *will be called inessential. Thus, a polynomial or power series* $P(\boldsymbol{t})$ *is inessential if*

$$P(t, t\ln t, \ldots, t(\ln t)^{\ell}) \equiv 0.$$

We now identify A_ℓ with $B_\ell/\operatorname{Ker}\varphi$, and define on B_ℓ a counterpart N of the operator D:

Theorem 2.10. *(a) The map* φ *admits a degree-preserving right inverse, and is onto.*
(b) $\operatorname{Ker}\varphi$ *coincides with the ideal* I *generated by the polynomials* $t_k t_j - t_{k-1}t_{j+1}$ *with* $0 \le j \le k-2$*. The space* A_ℓ *is isomorphic to* B_ℓ/I*.*
(c) There is a unique derivation N *on* B_ℓ *such that* $\varphi\circ N = D\circ\varphi$*.*

Proof. (a) We must find ψ such that $\varphi\circ\psi$ is the identity; that φ is onto will follow. First, let $\psi(1) = 1$. Next, for any monomial T^pL^q with $p > 0$, let k be the smallest integer such that $q \le kp$. It is easy to check that $\varphi(t_k^a t_{k-1}^b) = T^pL^q$ if and only if $a = q-(k-1)p$ and $b = kp-q$. We therefore let

$$\psi(r^p(\ln r)^q) = t_k^{q-(k-1)p}t_{k-1}^{kp-q}.$$

The map ψ has the desired properties. In fact, φ is an isomorphism for $\ell = 1$; it is also invertible on polynomials of degree zero or one, for any ℓ.

(b) First of all, it is readily verified that $I \subset \operatorname{Ker}\varphi$. Next, let $u \in B_\ell$. Consider a typical monomial $t_0^{a_0}\cdots t_\ell^{a_\ell}$ in u. If $a_\ell = 0$, it already belongs to $B_{\ell-1}$. If $a_k > 0$ for some $k \le \ell-2$, we may subtract a multiple of $t_\ell t_k - t_{\ell-1}t_{k+1}$ from u, and thereby reduce a_ℓ. In finitely many steps, we are left, at most, with a monomial of the form $t_\ell^a t_{\ell-1}^b$. One repeats this operation for every monomial occurring in u. This generates a decomposition

$$u = u_1 + u_2 + w, \tag{2.8}$$

where $u_1 \in B_{\ell-1}$, $u_2 \in I$, and $\psi\circ\varphi(w) = w$. In fact, w is a linear combination of terms of the form $t_\ell^a t_{\ell-1}^b$ with $a > 0$; it follows that $\varphi(w) \in A_{k-1}$ if and only if $w = 0$.

Let us now assume in addition that $u \in \operatorname{Ker}\varphi$. Since $\varphi(w) = -\varphi(u_1)$ and $u_1 \in B_{\ell-1}$, we obtain $\varphi(w) \in A_{\ell-1}$. Therefore, $w = 0$. We have therefore proved that $\operatorname{Ker}\varphi \subset B_{\ell-1} + I$. Since φ is injective on B_1, we find, by induction on ℓ, that $\operatorname{Ker}\varphi \subset I$. This concludes the proof.

(c) Since φ is injective on polynomials of degree one, we must have $Nt_k = \psi(D(TL^k)) = t_k + kt_{k-1}$. Similarly, N must annihilate terms that do not contain $\boldsymbol{t}$. Since the action of a derivation on polynomials of degrees zero and one determines it completely, we obtain

$$N = \sum_{k=0}^{\ell}(t_k + kt_{k-1})\frac{\partial}{\partial t_k}, \tag{2.9}$$

with the convention $t_{-1} = 0$. This operator has the desired properties. The result amounts to the identity

$$D[w(T, T\ln T, \dots, T(\ln T)^\ell)] = (Nw)(T, T\ln T, \dots, T(\ln T)^\ell).$$

2.2.2 Operations on A_ℓ and B_ℓ

One can define $f(u)$ for $u \in A_\ell$ or B_ℓ, provided that the Taylor series of f converges at the constant term of u. We will need a more precise result. Write $\boldsymbol{t}^{\,a}$ for $t_0^{a_0} \cdots t_\ell^{a_\ell}$ and $|a| = a_0 + \cdots + a_\ell$ for the length of the multi-index a.

Lemma 2.11. *Let $f(u) = \sum_k t_k f_k(u)$ and $u = v + \sum_{|a|=\mu} w_a \boldsymbol{t}^{\,a}$, where v and the coefficients w_a belong to B_ℓ. Then, if f is analytic at the constant term of u, we may write*

$$f(u) = f(v) + \sum_{|a|=\mu} \boldsymbol{t}^{\,a} \left[\sum_k t_k \phi_{ak}[w] \right],$$

where w denotes the collection of the coefficients w_a.

Proof. It suffices to expand f in a Taylor series around v: writing $z = \sum_{|a|=\mu} w_a \boldsymbol{t}^{\,a}$, we have

$$\begin{aligned} f_k(v+z) &= f_k(v) + f_k'(v) \cdot \left(\sum_a w_a \boldsymbol{t}^{\,a} \right) + \cdots \\ &= f_k(v) + \sum_a \boldsymbol{t}^{\,a} \left(f_k'(v) \cdot w_a + \frac{1}{2} f_k''(v) \cdot (w_a, z) + \dots \right). \end{aligned}$$

□

Remark 2.12. The decomposition of the lemma is not unique in general. Also, if f involves Du or x-derivatives of u, the ϕ_{ak} will involve the Dw_a as well as spatial derivatives of w_a; finally, if the dependence of $f(u)$ on spatial derivatives is linear, then ϕ_{ak} will be linear in the corresponding derivatives of w.

Lemma 2.13. *If u is as in the preceding lemma, then*

$$N\left(\sum_{|a|=\mu} w_a \boldsymbol{t}^{\,a} \right) = \sum_{|a|=\mu} \boldsymbol{t}^{\,a} [(N + \mu + B)w]_a,$$

where Bw_a is a linear combination of the $w_{a'}$ for which

$$(a'_\ell, a'_{\ell-1}, \dots, a'_0) < (a_\ell, a_{\ell-1}, \dots, a_0),$$

in the sense of lexicographical ordering. In particular, B is nilpotent.

Proof. One obtains

$$N(w_a \boldsymbol{t}^{\,a}) = \boldsymbol{t}^{\,a}(N+\mu)w_a + \sum_{a,k} k a_k w_a t_0^{a_0} \cdots t_{k-1}^{a_{k-1}+1} t_k^{a_k-1} \cdots t_\ell^{a_\ell},$$

which has the desired form. □

The solution of equations in A_ℓ rests on Theorems 2.2 and 2.4. From them, we can determine ℓ such that a given equation admits solutions in A_ℓ. We turn to such results.

2.2.3 Solving Fuchsian equations in A_ℓ and B_ℓ

Consider first

$$P(D)u = T^a L^b f[u], \tag{2.10}$$

where P is a polynomial, f maps A_ℓ to A_ℓ, $a > 0$, and $b \geq 0$. The function f could also involve spatial derivatives, since such operators leave A_ℓ invariant.

Theorem 2.14. *Equation (2.10) admits a formal solution in A_ℓ, provided that*

$$a\ell \geq b + \max_{j\in\mathbb{N}\setminus\{0\}} \operatorname{ord}(P, j).$$

Proof. Since $a > 0$, one may seek $u = \sum_{j\geq 0} T^j u_j$, where the u_j are polynomials in L. They can be computed recursively using Theorem 2.2. We must show that the degree of u_j does not exceed $j\ell$. It suffices to check that for every integer $j \geq 0$,

$$(j-a)\ell + b + \operatorname{ord}(P, j) \leq j\ell.$$

This is equivalent to the condition given in the theorem.

Remark 2.15. If we replace (2.10) by a system of the same form, we must replace $\operatorname{ord}(P, j)$ by an integer M_j such that $P(D)u = T^j Q(L)$ admits a solution of the form $T^j R(L)$, with $\deg R \leq \deg Q + M_j$.

We record a simple special case.

Corollary 2.16. *If $f : A_\ell \to A_\ell$, P is a polynomial, then the equation*

$$P(D)u = Tf(u)$$

is solvable in A_ℓ if $\ell \geq \max_{j>0} \operatorname{ord}(P, j)$.

For generalizations of these spaces to other sets of exponents, see Problem 2.2.

We turn to the main solvability result relative to the operator N.

Lemma 2.17. *If k is an integer, q is independent of $\boldsymbol{t}$, and A is a constant matrix, there is an integer ℓ_0 that depends on k such that if $\ell \geq \ell_0$, the equation*

$$(N + A)\boldsymbol{u} = qt_k \tag{2.11}$$

has a solution of the form

$$\sum_{j=0}^{\ell} t_j v_j,$$

where the v_j are independent of $\boldsymbol{t}$.

Proof. First, we observe that

$$(N + A)\sum_{j=0}^{\ell} t_j v_j = \sum_{j=0}^{\ell} t_j\{(N + A + 1)v_j + (j + 1)v_{j+1}\},$$

where $v_{\ell+1}$ is taken to be zero. We therefore need to solve

$$(N + A + 1)v_j + (j + 1)v_{j+1} = \delta_{jk}q.$$

This leads to the system

$$(A + 1)v_j + (j + 1)v_{j+1} = \delta_{jk}q.$$

We may decompose the v_j along two complementary subspaces, on which $A + 1$ is invertible and nilpotent respectively. The invertible part is solved immediately by taking $v_0 = (A+1)^{-1}q$ and all the other $v_j = 0$. We therefore assume that $(A + 1)$ is nilpotent. We may then take $v_0 = 0$ and solve for the other v_j recursively. We have $v_{k+1} = q/(k + 1)$ and

$$v_j = \frac{[-(A + 1)]^{j-k-1}v_{k+1}}{j(j - 1)\cdots(k + 2)},$$

for $j > k+1$. To find a solution with $\ell+1$ variables, we need to choose ℓ such that one can satisfy the last equation of this system, namely

$$(A + 1)^{\ell-k}q = 0,$$

which holds for ℓ large enough if A is nilpotent.

We close this section with a few results on still more general spaces of series. They allow us in particular to solve equations in spaces in which ℓ may be less than one, corresponding to a more economical representation of the solution.

Definition 2.18. *For any nondecreasing sequence $\mathcal{L} = (\ell_0, \ell_1, \dots)$, $A_{\mathcal{L}}$ is the space of formal series of the form*

$$u = \sum_{0\leq p\leq j\ell_j} u_{jp}T^jL^p,$$

where the coefficients u_{jp} are functions of "space" variables $(x_1, \dots, x_n)$, and $A_{\mathcal{L},s}$ is the space of series of the form

$$\sum_{j\geq 0} q_j(x, L)T^j,$$

where $\deg q_j \leq \ell_j(j+s)$ *and* $s \geq 0$*. For any of the above series, we write* $\{u\}_j$ *for the coefficient of* T^j*; it is a polynomial in* L*.*

Thus, $T^s A_{\mathcal{L},s} \subset A_{\mathcal{L}}$. The following is easily checked.

Proposition 2.19. *If* $j\ell_j + k\ell_k \leq (j+k)\ell_{j+k}$ *for every* j *and* k*,* $A_{\mathcal{L}}$ *is closed under products.*

We first analyze the action of nonlinear functions on $A_{\mathcal{L},s}$.

Lemma 2.20. *If* $u \in A_{\mathcal{L},s}$*,* $j \geq r \geq 1$*, and* p *and* r *are integers, then* $\{T^r u^p\}_j$ *has degree in* L *equal to at most* $\ell_{j-r}(j + ps - r)$*.*

Proof. Writing $u = \sum_{j\geq 0} q_j(L)T^j$, and suppressing the x dependence, we need to estimate the degree of any expression $q_{j_1} \cdots q_{j_p}$ in which $j_1 + \cdots + j_p = j - r$ and, for every k, $\deg q_{j_k} \leq \ell_{j_k}(j_k + s)$. Since the sequence $\mathcal{L}$ is nondecreasing, this degree is at most

$$\sum_{k=1}^{p} \ell_{j_k}(j_k + s) \leq \ell_{j-r}(j - r + ps),$$

as announced. □

We now turn to solvability results for Fuchsian equations in $A_{\mathcal{L},s}$. The unknown u may have several components.

Lemma 2.21. *Let* p *and* r *be integers equal to at least* 1*. Let* g *be a polynomial of total degree in* u *at most* p*. Then the equation* $P(D)u = T^r g(u)$ *is solvable in* $A_{\mathcal{L},s}$ *if* $(p-1)s \leq r$ *and, for* $j \geq r$*,*

$$\ell_j - \ell_{j-r} \geq \frac{\operatorname{ord}(P, j)}{j + s}.$$

Proof. By assumption, $T^r g$ is a sum of expressions to which Lemma 2.20 applies. By Theorem 2.2, we need to have, for $j \geq r$,

$$\ell_j(j + s) \geq \ell_{j-r}(j + ps - r) + \operatorname{ord}(P, j).$$

In other words, we need $(\ell_j - \ell_{j-r})(j+s) \geq \ell_{j-r}[(p-1)s - r] + \operatorname{ord}(P, j)$. If we require s to satisfy $(p-1)s - r \geq 0$, the statements of the lemma follow. □

We may now state the main result.

Theorem 2.22. *If g is a polynomial in the components of u, g' its linearization, and $f(u) = [g(Tu) - g(0) - g'(0)Tu]/T$, the equation*

$$P(D)u = f(u)$$

is solvable in $A_{\mathcal{L},1}$ if for every $j \geq 0$, $\ell_j - \ell_{j-1} \geq \operatorname{ord}(P, j)/(j+1)$.

Proof. The assumption on f implies that it is a sum of terms of the form $T^{p-1}u^p$ with $p \geq 2$. We therefore take $r = p - 1$ and $s = 1$ in Lemma 2.21; the theorem follows. □

Remark 2.23. More generally, we may allow g to be a formal power series in u, $A_1 u, \ldots$, where $A_1, \ldots$ are linear.

2.2.4 Two tools for the second reduction

We present two general results that simplify the determination of the second reduced equation. The first shows how to transform the first reduced equation so as to increase the eigenvalues of A. The second shows how to perform a second reduction directly if a formal solution of high order is already at hand.

How to increase the eigenvalues of A

Since the existence theorems require the eigenvalues of A to have nonnegative real parts, it is useful to be able to transform the equation at hand so as to achieve this. The Fuchsian form, thanks to Lemma 2.13, is convenient for this purpose. Consider a *generalized Fuchsian system* with several time variables

$$(N + A)\boldsymbol{u} = \sum_{q \leq k_0} t_q f_q(t_0, \ldots, t_\ell, x, \boldsymbol{u}, \partial_x \boldsymbol{u}). \tag{2.12}$$

Theorem 2.24. *Given a Fuchsian system (2.12), one can, after increasing ℓ if necessary, produce another system of the same form, the solutions of which generate solutions of (2.12), in which the eigenvalues of A have been raised by one.*

Proof. We write ∂ for ∂_x. We seek u in the form

$$\boldsymbol{u} = u_0(x) + t \cdot v(x, t) = u_0 + t_0 v_0 + \cdots + t_\ell v_\ell. \tag{2.13}$$

We must choose u_0 in the null space of A. The new unknown $\boldsymbol{v} = (v_0, \ldots, v_\ell)$ has $(\ell + 1)$ times as many components as $\boldsymbol{u}$. Substituting, we find as before that

$$(N + A)\sum_{j=0}^{\ell} t_j v_j = \sum_{j=0}^{\ell} t_j \{(N + A + 1)v_j + (j+1)v_{j+1}\},$$

where $v_{\ell+1}$ is taken to be zero, and

$$t_q f_q(t, x, u_0 + t \cdot v, \partial(u_0 + t \cdot v)) = t_q(f_q(0, x, u_0, \partial u_0) + \sum_{j=0}^{\ell} t_j g_{qj}(t, x, v, \partial v)).$$

We therefore require v to solve the system

$$(N+A+1)v_j+(j+1)v_{j+1} = \sum_q \delta_{jq} f_q(0, x, u_0, \partial u_0)+t_q g_{qj}(t, x, v, \partial v), \quad (2.14)$$

where δ_{jq} is the Kronecker symbol. Any solution of (2.14) generates a solution of (2.12) via (2.13). We now need to absorb $\delta_{jq} f_q(0, x, u_0, \partial u_0)$ into $\boldsymbol{v}$. *This is where the value of ℓ may need to be changed.* By Lemma 2.17 it is possible to find $\mathbf{w} = (w_j)$ such that

$$(N + A + 1)v_j + (j + 1)w_{j+1} = \delta_{jq} f_q(0, x, u_0, \partial u_0)$$

if ℓ is chosen large enough. Therefore, $\tilde{\boldsymbol{u}} := \boldsymbol{v} - \mathbf{w}$ solves a new Fuchsian system of the form

$$(N + A_1)\tilde{\boldsymbol{u}} = \sum_{q \le \ell} t_q g_q(t_0, \ldots, t_\ell, x, \tilde{\boldsymbol{u}}, \partial_x \tilde{\boldsymbol{u}}),$$

in which the eigenvalues of A_1 have the form $\lambda + 1$, where λ runs through the eigenvalues of A. Thus, we have replaced (2.12) by a Fuchsian problem of the same form, but in which the eigenvalues of A have been raised by 1. □

How to make use of an approximate solution to high order

Given the existence of an approximate solution to a very high order, one can directly obtain a second reduced equation; in addition to clarifying the structure of the argument, this technique is useful when it is not convenient to write out the details of the formal solution. Consider a nonlinear generalized Fuchsian system

$$(N + A)u = \sum_{0 \le j \le \ell} t_j f_j. \quad (2.15)$$

For definiteness, we work with systems in which no derivative higher than the first occurs in the right-hand side.

Definition 2.25. *An expression $\boldsymbol{v} = \sum_{j \le g} v_g$ is a formal solution of (2.15)* up to order g *if*

$$(N + A)\boldsymbol{v} = \sum t_j f_j[\boldsymbol{v}] + \sum_{|a|=g+1} \boldsymbol{t}^a \phi_a(\boldsymbol{t}, x), \quad (2.16)$$

where $\phi_a \in B_\ell$. The remainder may be written in the form

$$\sum_{|a|=g+1} \boldsymbol{t}^a \phi_a(\boldsymbol{t}, x) = \sum_{j=0}^{\ell} \sum_{|b|=g} \boldsymbol{t}^b \phi_{bj}(\boldsymbol{t}, x).$$

Theorem 2.26. *If (2.15) has a solution up to order g, there is a system of the form*

$$((N + A')\mathbf{w})_a = \sum_j t_j g_{j,a}[\mathbf{w}],$$

the solutions $\mathbf{w} = (w_a)$ of which generate solutions of (2.15) via the substitution

$$\boldsymbol{u} = \boldsymbol{v} + \sum_{|a|=g} t^a w_a.$$

The eigenvalues of A' have the form $\lambda + g$, where λ runs through the eigenvalues of A.

Proof. We compute the result of the substitution to find the desired system for $\mathbf{w}$: by Lemma 2.13, there is a nilpotent matrix B such that

$$(N + A) \sum_{|a|=g} \boldsymbol{t}^a w_a = \sum_{|a|=g} \boldsymbol{t}^a [(N + A + g + B)\mathbf{w}]_a.$$

As for the nonlinear terms, there are functions $h_{j,a}$ such that

$$f_j(x, t, \boldsymbol{v} + \sum_a t^a w_a) = f_j[\boldsymbol{v}] + \sum_a t^a h_{j,a}(x, \mathbf{w}, \partial \mathbf{w}).$$

Since $\boldsymbol{v}$ is a formal solution up to order g, (2.16) holds. Letting $A' = A + g + B$, we are led to the system

$$(N + A')w_a = \sum_j t_j [h_{j,a}(x, \mathbf{w}, \partial \mathbf{w}) + \phi_{aj}],$$

which has the desired property. □

2.3 Formal series with variable exponents

We now study a space of series adapted to variable powers, such as T^x. Situations of this type are relevant for the applications in Chap. 8. After introducing the basic variables and studying the action of D and ∂_x on them, we turn to the spaces of series on which the equations $Du = f$ and $(D - x)u = f$ will be solved. The space of power series in x with positive radius of convergence is denoted by $\mathbb{C}\{x\}$.

2.3.1 Basic variables and their properties

We start from the three indeterminates t, t^x, and $\ln t$, and the operator $D = t\partial_t$. They satisfy the formal rules

$$Dt = t, \quad D \ln t = 1, \quad Dt^x = xt^x, \quad \partial_x t^x = t^x \ln t.$$

All the formal results in this section are derived from them. We cannot work with power series in t and t^x because the operator D is not invertible on this space. We need to enlarge our space of series to include solutions of equations $D^k u = 1$ and $D^k u = t^x$ for every $k \geq 0$. Let us define indeterminates τ_k and θ_k by $\tau_k = (\ln t)^k/k!$, $\theta_0 = t^x$, and

$$\theta_k = x^{-k}\left(t^x - \sum_{j=0}^{k-1} \frac{(x\ln t)^j}{j!}\right) = x^{-k}\left(t^x - \sum_{j=0}^{k-1} x^j \tau_j\right). \tag{2.17}$$

It is convenient to make the convention $\tau_{-1} = 0$. These new variables are not independent, since we have

$$x\theta_{k+1} = \theta_k - \tau_k \text{ for } k \geq 0. \tag{2.18}$$

However, it is convenient to keep τ_k as an independent variable, because it is annihilated by ∂_x.

Remark 2.27. If we consider θ_k as a holomorphic function for, say, $\operatorname{Re} x > 0$ and $\operatorname{Re} t > 0$, we have

$$\theta_k = \sum_{j=k}^{\infty} x^{j-k}\tau_j = k\tau_k \int_0^1 (1-\sigma)^{k-1} t^{\sigma x} d\sigma, \tag{2.19}$$

using Taylor's formula with integral remainder.

These new variables are adapted to the solution of equations involving D and $D - x$ because of the following results.

Proposition 2.28. *For $k \geq 0$, we have*

$$D\tau_k = \tau_{k-1}, \; D\theta_{k+1} = \theta_k, \tag{2.20}$$

$$(D-x)\theta_{k+1} = \tau_k, \; D\theta_0 = x\theta_0. \tag{2.21}$$

Proof. The first and fourth relations are readily verified. Using (2.17) and (2.18), the second and third relations follow. □

Proposition 2.29. *For $k \geq 1$ and $q \geq 0$, we have*

$$D(\theta_k\tau_{q+1}) = \theta_k\tau_q + \theta_{k-1}\tau_q, \; D(\theta_{k+1}\tau_0) = \theta_k,$$
$$D(\theta_0\tau_{q+1}) = \theta_0\tau_q + x\theta_0\tau_{q+1}, \; D(\theta_0\tau_0) = x\theta_0.$$

Proof. This is a direct consequence of Proposition 2.28. □

Proposition 2.30. *For $k \geq 1$ and $q \geq 0$, we have*

$$(D-x)\tau_k = \tau_{k-1} - x\tau_k, \qquad (D-x)(\theta_0\tau_k) = \theta_0 D\tau_k = \theta_0\tau_{k-1},$$

and

$$(D-x)\left(\theta_k\tau_{q+1} - \binom{k+q}{q+1}\theta_{k+q+1}\right) = \theta_k\tau_q.$$

Proof. The first two relations follow from Proposition 2.28. We have, using (2.20),

$$(D-x)(\theta_k \tau_{q+1}) = \tau_{k-1}\tau_{q+1} + \theta_k \tau_q = \theta_k \tau_q + \frac{(k+q)!}{(k-1)!(q+1)!}\tau_{k+q}.$$

The result follows. □

Proposition 2.31. *For $k \geq 0$, $\partial_x \theta_k = \theta_k \ln t - k\theta_{k+1}$.*

Proof. The statement is true if $k=0$. If $k \geq 1$, we have, using the relation $j\tau_j = \tau_{j-1}\ln t$,

$$\begin{aligned}
\partial_x \theta_k = & -kx^{-1-k}\left(t^x - \sum_0^k x^j \tau_j + x^k \tau_k\right) + x^{-k}\left(t^x - \sum_0^{k-2} x^j \tau_j\right)\ln t \\
= & -k\theta_{k+1} + \left(t^x - \sum_0^{k-2} x^j \tau_j - x^{k-1}\tau_{k-1}\right) x^{-k}\ln t, \\
= & \, \theta_k \ln t - k\theta_{k+1}.
\end{aligned}$$

□

Since $\ln t = \theta_1 - x\theta_2$, the variables τ_q may be expressed in terms of the θ_k if we allow x-dependent coefficients.

Proposition 2.32. *The variables τ_q belong to the algebra over $\mathbb{C}\{x\}$ generated by the θ_k. This algebra is invariant under the operators D and ∂_x.*

2.3.2 Spaces of series

We now define three families of finite-dimensional spaces: for $M \geq 0$,

$$\begin{aligned}
E_M &:= \left\{ u(x,t) : u = \sum_{k+q=M} a_{kq}(x)\theta_k \tau_q \right\}, \\
F_M &:= \left\{ u(x,t) : u = \sum_{k+q=M} a_{kq}(x)\theta_k \tau_q + b_0(x)\tau_M \right\}, \\
G_M &:= \bigoplus_{p \leq M} F_p,
\end{aligned}$$

where the coefficients are holomorphic near $x=0$. The main result of this section is the inversion of D, $D-x$, and $D-\lambda(x)$ (with λ holomorphic and nonzero at the origin) in these spaces.

Theorem 2.33. *Let $\lambda(0) \neq 0$ and $M \geq 0$. Then*

1. *The equation $Du = f$ has a solution in F_{M+1} if $f \in F_M$.*
2. *$(D - x) : E_{M+1} \to F_M$ is an isomorphism. On E_0, $(D - x)$ vanishes.*
3. *$(D - \lambda(x)) : G_M \to G_M$ is an isomorphism.*
4. *If f involves only the variables τ_k, there is a solution of $Du = f$ that does not involve the θ_k. The same holds for the equation $(D - \lambda(x))u = f$.*
5. *∂_x maps F_M to F_{M+1}.*

Proof. (1) Let $k + q = M$. We obtain

$$D\left(\sum_{j=0}^{q}(-1)^j \theta_{k+j+1}\tau_{q-j}\right) = \theta_k \tau_q.$$

Furthermore, $D\tau_{M+1} = \tau_M$. The result follows. Note that DG_{M+1} is not included in G_M since $D(\theta_0\tau_{M+1}) = \theta_0(\tau_M + x\tau_{M+1})$.

(2) Proposition 2.30 shows that

$$(D - x)(\theta_k \tau_{M-k+1} - \binom{M}{M-k+1}\theta_{M+1}) = \theta_k \tau_{M-k}$$

if $1 \leq k \leq M$. Also, Proposition 2.28 gives

$$(D - x)(\theta_0 \tau_{M+1}) = \theta_0 \tau_M,$$
$$(D - x)\theta_{M+1} = \tau_M.$$

The result follows.

(3) G_M admits the basis $\{\theta_k\tau_q, \tau_q\}_{k+q \leq M}$. Define an operator D_0 that satisfies $D_0(\theta_0\tau_q) = x\theta_0\tau_q$ and sends the other basis elements to zero. By Proposition 2.29, D restricted to G_M is the sum of a nilpotent operator and the operator D_0. Now, if $\lambda(0) \neq 0$, $D_0 - \lambda(x)$ is an invertible diagonal operator that differs from $D - \lambda(x)$ by a nilpotent operator. The result follows.

(4) Since $D\tau_k = \tau_{k-1}$, the space generated by the τ_k is invariant under D.

(5) This follows from Proposition 2.31. □

We now solve Fuchsian PDEs with variable indices [16] in spaces of series of the form $\sum_{j \geq 0} u_j(t, x)t^j$ with $u_j \in G_{M(j)}$, where $M(j)$ depends on the problem at hand. Unlike the spaces A_ℓ, such spaces are not algebras. However, they are included in the algebra generated by t, x, and the θ_k.

2.3.3 Formal solutions of linear Fuchsian PDEs

Consider Fuchsian PDEs of the form

$$Lu = f(x, t), \tag{2.22}$$

where

$$L = (D - c)(D - x - d) - t^2 P_2(x, \partial_x),$$

P_2 is a second-order operator with analytic coefficients, and f is analytic, both near the origin. The parameters c and d are real. The goal is to find formal solutions of the form

$$u = \sum_{j \geq 0} u_j(t, x) t^j, \tag{2.23}$$

where $u_j \in G_{M(j)}$, and $M(j)$ will be specified below.

Theorem 2.34. *(a) If neither c nor d is a nonnegative integer, equation (2.22) has a unique formal solution in integral powers of t:*

$$U_3(x, t) = \sum_{j=0}^{\infty} u_j(x) t^j.$$

(b) If c or d is a nonnegative integer, equation (2.22) has formal solutions

$$U_3(x, t) = \sum_{j=0}^{\infty} u_j(x, t) t^j,$$

where $u_j \in G_{j+2}$.

Proof. We prove (a) and (b) together. Write $f(x, t) = \sum_j f_j(x) t^j$. Let us seek solutions of (2.22) of the form $u = \sum_j u_j(x, t) t^j$. Substitution into the equation yields

$$(D + j - c)(D + j - x - d) u_j = P_2 u_{j-2} + f_j,$$

with the convention that $f_j = 0$ if j is negative. From the results of Sect. 2.3.2, if $u_{j-2} \in G_{M(j-2)}$, one can find $u_j \in G_{M(j-2)+k(j)}$, where $k(j) = 2$ if $j - c$ and $j - d$ are both nonzero, $k(j) = 3$ if precisely one of them vanishes, and $k(j) = 4$ if both vanish. If neither c nor d is a nonnegative integer, we see by induction that there is a solution with u_j independent of t: u is a solution in pure powers of t. If precisely one of them is a nonnegative integer, $k(j)$ equals 2 except for precisely one value j_0 of j. It follows that $M(j) = j$ for $j < j_0$, and $j + 1$ for $j \geq j_0$.

If both are nonnegative integers, but $c \neq d$, then $k(j)$ equals 2 for $j \neq c$, d, and 3 otherwise. Therefore, $M(j) \leq j + 2$.

If $c = d$ is a nonnegative integer, $k(j)$ equals 2 for $j \neq c$, and 4 otherwise. Therefore, we again have $M(j) \leq j + 2$. □

2.4 Relation of A_ℓ to the invariant theory of binary forms

We relate Fuchsian analysis in the space A_ℓ to the Lie algebra of the group SL(2) of complex 2×2 matrices with unit determinant. This group therefore serves as a symmetry for singularity analysis of very general classes of PDEs without nontrivial point symmetries. This is the only section of the book that requires some knowledge of representation theory, and will not be needed in the sequel. For further information on invariant theory, see [81].

A binary form is an expression

$$p(x, y) := \sum_{k=0}^{\ell} \binom{\ell}{k} t_k x^k y^{\ell-k}.$$

The group SL(2) acts on the coefficients of p in the following way: if $x = ax'+by'$, $y = cx'+dy'$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)$, we define another binary form, p', in x' and y', such that $p(x, y) = p'(x', y')$. If $(t'_0, \ldots, t'_\ell)$ are the coefficients of p', the SL(2) action is expressed by the mapping $(t_0, \ldots, t_\ell) \mapsto (t'_0, \ldots, t'_\ell)$. An invariant is a function of the coefficients $(t_0, \ldots, t_\ell)$ that remains unchanged under this transformation; a covariant has the same property, but is allowed to have homogeneous dependence on x and y. We consider polynomial invariants and covariants. Thus, invariants are elements of the algebra B_ℓ.

At the infinitesimal level, the invariance of u under the subgroups $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$ is expressed by $Mu = 0$ and $M'u = 0$ respectively, where

$$M = \sum_{k=0}^{\ell} k t_{k-1} \partial/\partial t_k$$

and

$$M' = \sum_{k=0}^{\ell} (\ell - k) t_{k+1} \partial/\partial t_k,$$

with the convention that $t_{-1} = t_{\ell+1} = 0$. We also define $G = \sum_k t_k \partial_k$ and $P = \sum_k k t_k \partial_k$, where $\partial_k = \partial/\partial t_k$. Thus, $G + M$ may be identified with the operator N of Sect. 2.2. One can also show that the coefficients of the top power of x in covariants coincide with the solutions of $Mu = 0$; these coefficients are called *semi-invariants.* These operators are related to SL(2) in the following way.

Theorem 2.35. *G commutes with P, M, and M'. In addition, $\{W = lG - 2P, M, M'\}$ satisfy $[W, M] = -2M$, $[W, M'] = 2M'$, and $[M, M'] = W$. The triplet $(-W, -M', -M)$ gives the standard presentation of the Lie algebra $\mathfrak{sl}(2)$.*

Proof. For any monomial $\boldsymbol{t}^a$, we have

$$G\boldsymbol{t}^a = g(a)\boldsymbol{t}^a \text{ and } \quad P\boldsymbol{t}^a = p(a)\boldsymbol{t}^a,$$

where $g(a) = \sum_j a_j$ and $p(a) = \sum_j j a_j$; they are respectively called the *degree* and the *weight* of the monomial $\boldsymbol{t}^a$.

It suffices to check the statements of the theorem on monomials. First, for any monomial u of degree g and weight p, the polynomials Mu and $M'u$ are homogeneous of the same degree, but their respective weights are $p-1$ and $p+1$. Therefore, relations $[G,M]=[G,M']=0$, $[P,M]=-M$ and $[P,M']=M'$ hold on all monomials. Also, by direct calculation, $[M,M']=(\ell G-2P)$. The other commutation relations follow easily from these. □

Theorem 2.36. *Let u be a sum of monomials of the same degree g and weight p, in the variables $(t_0,\dots,t_\ell)$, and $k\ge 1$.*

1. *If $M^k v=u$ and u is inessential in the sense of Definition 2.9, then v is the sum of a linear combination of $t_0^g,\dots,t_0^{g-k+1}t_1^{k-1}$, and an inessential polynomial. Conversely, if u is inessential, so is Mu.*
2. *If u is a monomial with $\ell g-2p>0$, then u is in the range of M_ℓ. In particular, any monomial in $(t_0,\dots,t_k)$ is in the range of M_ℓ if $\ell>2k$.*
3. *Assume $u=\sum_{q\le k_0} t_q u_q'(t_0,\dots,t_{\ell'})$ and $k_0\le\ell'\le\ell$. Then, modulo inessential polynomials, the equation $M^k v=u$ has a polynomial solution that depends only on $(t_0,\dots,t_{\ell'})$, provided that $k+k_0\le\ell'$.*

Proof. (1) The statement is clear if $g=0$. Let us therefore assume $g>0$. Let $s(t)=v(t,t\ln t,\dots)$ and assume $k=1$. We have, since $N-g=M$ on polynomials of degree g,

$$t\frac{ds}{dt}-gs=(Mv)(t,t\ln t,\dots)=u(t,t\ln t,\dots)=0.$$

Since $s(0)=0$, $s\equiv ct^g$, so $u-ct_0^g$ is inessential. This settles the case $k=1$. The other cases, as well as the converse, are proved similarly.

(2) The statement follows from a general property of representations of $\mathfrak{sl}(2)$: the irreducible representations contained in the present one act on a chain

$$(v_k,v_{k-2},\dots,v_{-k})$$

of polynomials of degree g, where for every j, v_j is an eigenvector of $\ell G-2P$. Now, M maps every v_j to a nonzero multiple of v_{j+2} if $j<k$ respectively 0 if $j=k$; therefore, any v_j with $j>0$ must lie in the range of M. But these polynomials span precisely the sum of the eigenspaces of $\ell G-2P$ with positive eigenvalues, as desired. In particular, if $u=u(t_0,\dots,t_k)$, we have at any rate $p\le kg$, and $\ell>2k$ is certainly sufficient.

(3) It suffices to consider monomials. Let $u(\boldsymbol{t})=\boldsymbol{t}^a=t_0^{a_0}t_1^{a_1}\dots$, and $u(t,t\ln t,\dots)=t^g(\ln t)^{p(a)}$. We know that $g=a_0+\dots+a_{l'}$, and that there is an index $q\le k_0$ such that $a_q>0$. Write $v(t,t\ln t,\dots)=r(t)$; we want

$$\left(t\frac{d}{dt}-g\right)^k r=t^g(\ln t)^{p(a)},$$

so that $r=\sum_{h<k}c_h t^g(\ln t)^h+t^gR(\ln t)$, where R is a polynomial of degree $p(a)+k$, and the c_h are arbitrary. Now, since $a_q>0$, one can always write any expression $t^g(\ln t)^{p(a)+k}$ in the form

$$[t(\ln t)^q]^{a_q-1}[t(\ln t)^{k+q}]\prod_{j\neq q}[t(\ln t)^j]^{a_j}.$$

If $k+k_0 \le \ell'$, replacing t, $t\ln t, \ldots$ by t_0, $t_1, \ldots$ in the above expressions, we obtain a polynomial v' in $(t_0, \ldots, t_{\ell'})$ such that $M^k v' - u$ is inessential. This is the desired result. □

Problems

2.1. For each of the following equations, find a space of formal series, as large as possible, in which one can find at least one solution, and specify the arbitrary coefficients in it. Reduce it to a first-order system for $\boldsymbol{u} = (u, Du, \ldots)$:

1. $D(D-2)(D-3)^2 u = T\sin u$.
2. $(D^2-16)u = T^4 \ln T(u^2+1)$.
3. $(D^2-16)(D-2)^3 u = T^4 \ln T(u^2+1)$.
4. $(D^2-4D+5)u = \exp(Tu)-1$.
5. $(D+2)(D-3)u = T + T^3(\ln T)^3 + (\ln T)/T^2 + T^6u^2$ (first let $u = v + q(\ln T)/T^2$, for a suitable polynomial q).

2.2. (a) Discuss the formal solutions of $(D+A)u = Tf(u)$ representing solutions bounded as $T \to 0+$ along the real (positive) axis, when A is diagonal with real nonzero eigenvalues and f is a polynomial.

(b) Same question if A is real, but may have complex eigenvalues with nonzero real part. Study separately the case of real and complex solutions. In particular, show that in the real case, the appropriate space of formal series should contain all expressions of the form $q(L, C_1, C_2, \ldots)T^\lambda$, where λ runs through all linear combinations, with nonnegative integral coefficients, of the real parts of the nonnegative eigenvalues, and the C_k represent $\cos(T\operatorname{Im}\lambda)$ and $\sin(T\operatorname{Im}\lambda)$, where λ runs through the nonreal eigenvalues of A.

(c) Examples: write out the spaces of formal series if A is 2×2 with eigenvalues (i) 1 and 5/3; (ii) 1 and $\sqrt{2}$; (iii) $\sqrt{2}+i\sqrt{3}$ and $\sqrt{2}-i\sqrt{3}$.

(d) For 2×2 systems, interpret the results in terms of phase plane analysis. Generalize to $n \times n$ systems.
Remark: In a somewhat more modern form, this is how Poincaré studied the behavior of trajectories near a fixed point, around 1900.

2.3. Consider a problem of the form

$$(D+1)(D-4)u = (f(Tu) - f(0) - Tf'(0))/T,$$

where f is an analytic function of one variable u. Show that there are nontrivial formal solutions such that $Tu \in A_{1/4}$ [34]. Extend the result to more general equations.

2.4. Build an algebra of formal series invariant under multiplication by t, $t^{k(x)}$, $t^{1-k(x)}$, and the operators $D = t\partial_t$, $t\partial_x$, assuming that k is smooth and satisfies $a \le k(x) < b$, where a and b are constants in $(0, 1)$. This question is relevant for the Gowdy problem; see Chap. 8.

2.5. (a) Show that $|\boldsymbol{u}| = r^{-\operatorname{ord}(\boldsymbol{u})}$, where r is a fixed number in $(0, 1)$ defines an ultrametric absolute value[5] on A_ℓ, which makes it complete. Henceforth, A_ℓ is endowed with the associated topology.

(b) Show that polynomials act continuously on A_ℓ. What about analytic functions?

(c) Show that D and spatial differentiations also act continuously on A_ℓ.

(d) Show that $(A - \lambda)$ is continuously invertible on A_ℓ if for any integer k, $\lambda - k$ is not an eigenvalue of A.

(e) Show that if F satisfies the conditions in (b), the mapping $\boldsymbol{u} \mapsto TF[T, \boldsymbol{u}]$ is a contraction. Conclude that the equation $(D+A)\boldsymbol{u} = TF[T, \boldsymbol{u}]$ admits a unique solution in A_ℓ if all the eigenvalues of A have positive real parts, by applying the contraction mapping theorem.

(f) Generalize this approach to other examples of spaces of formal series in this chapter.

2.6. Find solutions to

$$(t\partial_t - 3)u = t^2 - 3u^{1/3}$$

such that $u = 1 + o(1)$ near $t = 0$.

2.7. Consider a Fuchsian PDE of the form

$$(D + A(T, \boldsymbol{u}))\boldsymbol{u} = F[T, \boldsymbol{u}],$$

and a space $\mathcal{FS}$ of formal series. Assume, as usual, that $F[T, \boldsymbol{u}]$ has no constant term if $\boldsymbol{u} \in \mathcal{FS}$. Let $\boldsymbol{u} = \boldsymbol{u}_0 + T^\varepsilon \boldsymbol{v}(T)$. Give sufficient conditions on A, F, and the space $\mathcal{FS}$ for $\boldsymbol{v}$ to satisfy a Fuchsian system in standard form (1.3).

[5] In other words, this defines a distance for which $|\boldsymbol{u} + \boldsymbol{v}| \le \max(|\boldsymbol{u}|, |\boldsymbol{v}|)$. This is stronger than the triangle inequality.

3

General Reduction Methods

We describe two general situations in which Fuchsian reduction with integer indices is possible. They are adapted to applications to nonlinear waves, and cover in particular all the applications to soliton theory. We limit ourselves to fairly common situations, and do not strive for the best hypotheses on the nonlinearities. Results are taken from [120] and [124].

3.1 Reduction of a single equation

We consider first a scalar equation with polynomial dependence on the unknown and its derivatives, written

$$F[u] := F(t, x_1, \dots, x_n, u, \partial_t u, \partial_{x_1} u, \dots) = 0. \tag{3.1}$$

We assume that variables have been chosen so that the singular set has equation $t = 0$. Let m be the order of the equation, which will also be assumed to be the order of the highest time derivative. Thus, the singularity surface is assumed to be noncharacteristic. All considerations are local, near $(x, t) = (0, 0)$.

3.1.1 Preliminaries

To represent and manipulate in a compact form nonlinear combinations of the derivatives of a function u of $n+1$ variables $(x^1, \dots, x^n, t)$, it is convenient to introduce a multi-index notation: any spatial derivative will be labeled by a multi-index $I = (i_1, \dots, i_n)$:

$$\partial_x^I u := \prod_{q=1}^{n} \left(\frac{\partial}{\partial x^q} \right)^{i_q}.$$

The total order of derivation in $\partial_x^I u$ is the *length* of the multi-index I, written $|I| = i_1 + \dots + i_n$. Similarly, any derivative with respect to space and time variables has the form $\partial_t^j \partial_x^I u$. Any product of u and its derivatives is labeled by a list of multi-indices $a = (a_I)$, where each a_I has the form $a_I = (a_{1,I}, \dots, a_{m,I})$:

$$u^a := \prod_{j,I} \left(\partial_t^j \partial_x^I u\right)^{a_{j,I}}. \tag{3.2}$$

The *degree* $g(a)$ and *weight* $p(a)$ of the multi-index a are defined by

$$g(a) = \sum_{j,I} a_{j,I}, \qquad p(a) = \sum_{j,I} j a_{j,I}. \tag{3.3}$$

It is helpful to introduce a special notation for monomials that do not contain space derivatives:

$$u^A = u^{A_0}(u_t^{A_1}) \cdots (\partial_t^m u)^{A_m}.$$

They correspond to monomials (3.2) with $I = (0, \dots, 0)$. To minimize technicalities, we assume that F is polynomial in u and its derivatives, so that the sum in the definition of F is finite. The most general PDE with polynomial nonlinearity may now be written

$$F[u] := \sum_{a=(a_I)} f_a(x,t) u^a. \tag{3.4}$$

We shall assume that

$$f_a = \sum_{b \geq 0} f_{ab}(x) t^{\mu(a)+b}.$$

3.1.2 Leading-order analysis

We seek solutions of the form $u = u_0 t^\nu + \text{h.o.t.}$, where h.o.t. refers to higher-order terms in t; $u_0 t^\nu$ represents a balance of the top-order time derivatives and some nonlinear terms. Assume

$$u = [u_0(x) + v(x,t)t^\varepsilon]t^\nu,$$

where ν and ε are constant. We need the notation

$$[\nu]_j := \nu(\nu-1)\cdots(\nu+j-1), \tag{3.5}$$

and the relation

$$\partial_t^j u = ([\nu]_j + t^\varepsilon [D+\nu+\varepsilon]_j v)\, t^{\nu-j},$$

where $[D]_j = D(D-1)\cdots(D-j+1)$. We have $\partial_t^j(t^\nu v) = t^{\nu-j}[D+\nu]_j v$. We prove in this section and the next that

$$F[u] = t^\rho[P(u_0) + t^\varepsilon Q(D+\varepsilon)v + o(t^\varepsilon)].$$

Definition 3.1. *Any pair* (u_0, ν) *such that*

1. $P[u_0] = 0$ *and*
2. $u_0 \not\equiv 0$

is said to determine a leading balance $u \sim u_0 t^\nu$*. Since* Q *does not involve derivatives with respect to* x*,* Q *is a polynomial, and its roots, which maydepend on* x*, are the* resonances*. Solutions corresponding to the same* *value of u_0 and ν, and differing only by the choice of arbitrary coefficients, are said to belong to the same* branch.

If $u = u_0(x)t^\nu + \text{h.o.t.}$, we have

$$\partial_t^j \partial_x^I u = [\nu]_j t^{\nu-j} \partial_x^I u_0 + \text{h.o.t.}$$

Therefore

$$\prod_j (\partial_t^j \partial_x^I u)^{a_{j,I}} = c(\nu, a_I) t^{\nu g(a_I) - p(a_I)} \prod_j (\partial_x^I u_0)^{a_{j,I}} + \text{h.o.t.},$$

where

$$c_{a_I}(\nu) = \prod_j [\nu]_j^{a_{j,I}}.$$

It follows that

$$f_a(x,t) \prod_{j,I} (\partial_t^j \partial_x^I u)^{a_I} = t^{\mu(a)+\nu g(a)-p(a)} f_{a0}(x) c(\nu, a) \prod_{j,I} (\partial_x^I u_0)^{a_{j,I}} + \text{h.o.t.} \tag{3.6}$$

The assumption that the leading balance does not involve space derivatives implies that the most singular terms one obtains upon substitution of the leading behavior into the equation *never contain any space derivatives*, and that the top-order time derivatives enter only into the most singular terms. This enables us to write

$$F[u_0(x)t^\nu + \text{h.o.t.}] = t^\rho (P(u_0) + \text{h.o.t.}),$$

where

$$\rho = \min_A \{\nu g(A) - p(A) + \mu(A)\}, \tag{3.7}$$

and

$$P(u_0, \nu) := \sum_{\nu g(A) - p(A) + \mu(A) = \rho} f_{A0}(x) c(\nu, A) u_0^{g(A)}. \tag{3.8}$$

Thus, the leading term is determined by an algebraic equation ($P(u_0) = 0$), rather than a differential equation. Let us summarize the results:

Theorem 3.2. *Leading-order terms $u \sim u_0 t^\nu$ are determined by those solutions of $P(u_0, \nu) = 0$ with $u_0 \neq 0$, where P is given by (3.8).*

Remark 3.3. In practice, one determines ν by requiring that the minimum in (3.7) be attained for two values of A, the corresponding monomials in F balancing each other.

3.1.3 First reduction

We now perform the first reduction, with $\varepsilon = 1$ for simplicity; this creates some complication if 1 is a resonance, but avoids having to introduce new variables in the expansion. We further require that the second most singular terms also not involve space derivatives.

Remark 3.4. A still more general reduction would be obtained by taking $t^{1/p}$ as expansion variable, where p is a large integer. This is necessary if there are resonances between 0 and 1; one should then take ε less than the smallest of them; otherwise, arbitrary functions associated with these indices will be missed. However, many examples do not require this more complete treatment.

We therefore take

$$t^{-\nu}u = u_0 + \sum_{q \leq k_0} h_q(x)t(\ln t)^q + tw(x,t), \tag{3.9}$$

where the integer k_0 will be determined later. Fix u_0 among the roots of P. Assume that we are not in the case of the Cauchy problem, so that $\nu(\nu - 1)\cdots(\nu - m + 1) \neq 0$. We prove that the substitution (3.9) leads to a Fuchsian equation for w. In fact, we will establish that

$$\begin{aligned} &F[t^\nu(u_0 + \sum_{q \leq k_0} h_q(x)t(\ln t)^q + tw(x,t)] \\ &= t^\rho \left[P(u_0) + t \left\{ Q(x, D+1)w - \sum_{q \leq 2k_0} t(\ln t)^q G_q \right\} \right] \end{aligned}$$

if the h_q and k_0 are chosen suitably.

Theorem 3.5. *(a) After performing substitution (3.9), (3.1) is equivalent to an equation of the form*

$$\begin{aligned} Q(x, t\partial_t + 1)w = \varphi(x) + \sum_{q \leq l_0} t(\ln t)^q G_q(t, t\ln t, \\ \cdots, t(\ln t)^{l_0}, x, w, \ldots, D^{m-1}w, \{tD^k \partial_x^J w\}_{k+|J| \leq m,\ k<m}), \end{aligned} \tag{3.10}$$

where $D = t\partial_t$, for a suitable integer l_0.

(b) Q is given by

$$\begin{aligned} Q(x,r) = \sum_{\nu g - p + \mu = \rho} & c(\nu, A) f_{A0} \\ & \times \left[A_0 + \frac{\nu + r}{\nu} A_1 + \cdots + \frac{[\nu + r]_j}{[\nu]_j} A_j \right] u_0^{g(A)-1}. \end{aligned} \tag{3.11}$$

(c) If $Q(x, D+1) = D^s R(x, D)$ with $R(x,0) \neq 0$, one can choose k_0 and the functions h_q so that $l_0 = 2k_0 = 2s$ and $\varphi = 0$. In particular, if $Q(x,0) \neq 0$, no logarithm is required on the right-hand side.

Remark 3.6. The equation $Q(x,r) = 0$ is the resonance equation. Resonances could vary with x. However, in many cases, even if u_0 is not constant, the resonances are.

Remark 3.7. It will follow from the proof that $k_0 = (l_0/2)$ equals the multiplicity of 1 as a resonance.

Proof. **Step 1: First change of unknown.** Let $u = t^\nu v(x,t)$ and $D = t\partial_t$. Since $\partial_t^j u = t^{\nu-j}[D+\nu]_j v$,

$$u^a = \prod_{j,I} \left(t^{\nu-j}[D+\nu]_j \partial_x^I v\right)^{a_{j,I}} = t^{\nu g(a)-p(a)} \prod_{j,I} \left[[D+\nu]_j \partial_x^I v\right]^{a_{j,I}}.$$

Substituting into the equation and setting $t = 0$, one recovers the relation $P(u_0) = 0$.

Step 2: Introduction of logarithms and second change of unknown. Fixing u_0 among the roots of P, we now let

$$v = u_0 + \sum_{q \le k_0} h_q(x) t_q + t w(x,t),$$

where $t_q = t(\ln t)^q$, and the h_q, as well as the integer k_0, will be determined below. We obtain

$$[D+\nu]_j \partial_x^J v = [\nu]_j \partial_x^J u_0 + t_0 [D+\nu+1]_j \left[\partial_x^J w + \sum_q \partial_x^J h_q (\ln t)^q\right].$$

This expression is a first-degree polynomial in $(t_0, \ldots, t_{k_0})$, with coefficients involving functions of x, and derivatives of w of the form $tD^k \partial_x^J w$.

Step 3: Substitution into (3.4). Upon substitution, one obtains a series in the t_q, in which the most singular term is $t^\rho P(u_0)$. From Step 2, we obtain

$$u^a = t^{\nu g(a)-p(a)} \Phi_a(x, t, \{t_q \partial h_q\}, \{tD^j w\}_{j \le m}, \{tD^j \partial^J w\}),$$

where ∂ stands for all space derivatives. We now substitute this result into (3.4), which produces an expression of the form $t^\rho P(u_0) + \mathcal{O}(t^{\rho+1}(\ln t)^{2k_0})$. We divide this by $t^{\rho+1}$, since the sum of the terms in t^ρ vanishes by the choice of u_0. Consider each term $f_a u^a$ separately. Each such term contributes terms of degree $\nu g(a) - p(a) + \mu(a)$ or higher. We also know that $\nu g(a) - p(a) + \mu(a) \ge \rho$, and that this sum equals ρ or $\rho + 1$ only for terms that do not contain spatial derivatives. Now, the terms such that $\nu g(a) - p(a) + \mu(a) \ge \rho + 2$ still have a factor of t left after division by $t^{\rho+1}$, and therefore already have the desired form. For the others, use the Taylor expansion of u^a up to second order to extract the contributions in t^ρ and $t^{\rho+1}$. We therefore need to consider only two types of terms:

1. Monomials with $\nu g(a) - p(a) + \mu(a) = \rho$; they contribute

$$t^\rho \left[P(u_0) + t(Q(x, D+1) \left[w + \sum_q h_q (\ln t)^q \right] + \varphi_1(x)) \right.$$
$$\left. + t \sum_{q \le 2k_0} t(\ln t)^q \Psi_{1q}(x, \{t_q\}, \{D^j w\}_{j \le m}) \right].$$

   By inspection, the operator Q has the announced form; φ_1 is a function of x, the expression of which is not needed.
2. Monomials with $\nu g(a) - p(a) + \mu(a) = \rho + 1$; they contribute

$$t^{\rho+1} \left[\varphi_2(x) + t \sum_{q \le 2k_0} t(\ln t)^q \Psi_{2q}(x, \{t_q\}, \{D^j w\}_{j \le m}) \right].$$

   The function φ_2 depends only on x.

Combining these equations, we reach the desired assertion.

Step 4: Choice of k_0 and (h_q). We now finish the proof by showing that one can choose k_0 and (h_q) to eliminate $\varphi(x)$. We have to solve

$$D^s R(x, D) \sum_{q \le k_0} h_q (\ln t)^q + \varphi = 0,$$

where φ is independent of t. Therefore, $R(x, D) \sum_{q \le k_0} h_q (\ln t)^q + \frac{(\ln t)^s}{s!} \varphi$ must be a polynomial of degree less than s in $\ln t$. The arguments of Sect. 2.2 give a φ with these properties if $R(x, 0) \neq 0$ and $k \ge s$; it contains s arbitrary constants. The theorem is proved, with $l_0 = 2k_0 = 2s$, as announced.

3.1.4 Is −1 a resonance?

We give a necessary and sufficient condition for $Q(-1)$ to be equal to zero. It holds for translation-invariant PDEs. Recall that $P(u_0) = 0$, where P is given by (3.8).

Theorem 3.8. *Assume that $\nu \neq 0, 1, \ldots, m-1$. Then, $Q(-1) = 0$ if and only if*

$$\sum_{\nu g - p + \mu = \rho} c(\nu, A) f_{A0} \mu(A) u_0^{g(A)} = 0. \tag{3.12}$$

This holds in particular if $\mu(A)$ is independent of A, or if balancing terms do not involve t.

Proof: We compute $Q(-1)$:

$$\begin{aligned}
u_0 Q(-1) = \quad & \sum c(\nu, A) f_{A0} u_0^{g(A)} \\
& \times \left[A_0 + \frac{\nu - 1}{\nu} A_1 + \frac{(\nu-1)(\nu-2)}{\nu(\nu-1)} A_2 + \cdots \right] \\
= & \sum c(\nu, A) f_{A0} u_0^{g(A)} \\
& \times \left[A_0 + (1 - \frac{1}{\nu}) A_1 + (1 - \frac{2}{\nu}) A_2 + \cdots \right] \\
= & \sum_{\nu g(A) - p(A) + \mu(A) = \rho} c(\nu, A) f_{A0} u_0^{g(A)} [g(A) - p(A)/\nu] \\
= & \sum c(\nu, A) f_{A0} u_0^{g(A)} [\rho - \mu(A)]/\nu \\
= & \frac{1}{\nu} \left[P(u_0) - \sum c(\nu, A) f_{A0} \mu(A) u_0^{g(A)} \right].
\end{aligned}$$

Since $P(u_0) = 0$, the result follows.

3.2 Introduction of several time variables and second reduction

We now view w as a function of the independent variables $(x, t_0, \dots, t_\ell)$; w solves the generalized Fuchsian equation

$$Q(N+1)w = \sum_q t_q G_q[w, Nw, \dots]. \tag{3.13}$$

We begin by proving that (3.13) may be replaced by a first-order Fuchsian system of the form

$$(N + A)u = \sum_q t_q G_q.$$

Introduce the new unknown $(w, \dots, D^{m-1}w, \{tD^k \partial_x^J w\}_{k+|J|<m})$, where m is the order of the equation. Compute the action of D on each of the new unknowns, taking (3.10) into account. Let $w_k = D^k w$ and $tD^k \partial_x^J w = w_{k,J}$. We have

$$Dw_k = w_{k+1} \tag{3.14}$$

for $k + 1 < m$. On the other hand, let $\partial^J = \partial_{j_1} \partial_{j_2} \cdots = \partial_{j_1} \partial_x^{J'}$, with $j_1 \le j_2, \dots$. If $k + 1 + |J| < m$,

$$Dw_{k,J} = w_{k,J} + w_{k+1,J}. \tag{3.15}$$

If $k + 1 + |J| = m$,

$$Dw_{k,J} = w_{k,J} + tD^{k+1} \partial_x^J w = w_{k,J} + t\partial_{j_1} w_{k+1,J'}. \tag{3.16}$$

For the last derivative, namely $D(D^{m-1}w)$, use (3.10). Next, any $tD^k\partial_x^J w$ with $k+|J|=m$ and $k<m$ may be expressed as a first-order spatial derivative of one of our unknowns. We then write Q as

$$Q(x, D+1) = D^m + Q_1 D^{m-1} + \cdots ,$$

and obtain

$$Dw_{m-1} + \sum_{j>0} Q_j w_{m-j} = \sum_q t_q G_q(x, t_0, \ldots, w, \{\partial_j w_{k,J}\}). \tag{3.17}$$

Equations (3.14)–(3.17) now form a Fuchsian system, where A may depend on x. In practice, u_0 and the f_{A0} are constant, and so are the coefficients Q_j. Using Theorem 2.24 m times, one may, after a further reduction, replace A by $A+m$; in other words, we may assume that the eigenvalues of A have positive real parts. This completes the second reduction.

3.3 Semilinear systems

We now show how to cast rather general semilinear systems in the form (2.12), given a set of assumptions that encapsulate the result of leading-order analysis. Again, for simplicity, we limit ourselves to a simple setup with algebraic leading behavior and constant resonances. Consider a system of the form

$$u_t = \sum_{j=1}^{n} a^j \partial_j u + b(u), \tag{3.18}$$

where $a^j = a^j(x,t) = \sum_{k\geq 0} a_k^j(x) t^k$, and t is one-dimensional. All considerations are local near $x=0$, $t=0$. We are interested in solutions that blow up on Σ defined by $t=\psi(x)$; we seek $u \sim (t-\psi(x))^{-p/q} v_0(x)$ for integers p and q as below. Four assumptions are now described and motivated.

1. Ensure that the blowup surface Σ is noncharacteristic:

$$Q(x) = \left(1 + \sum_j a_0^j \partial_j \psi\right) \text{ is invertible.} \tag{3.19}$$

2. Require power growth for the nonlinearity $b(u)$: assume that there are integers p and q, with $q>0$, such that $\tau^{p+q} b(\tau^{-p}\xi)$ is analytic in $\tau \in \mathbb{C}$ and $\xi \in \mathbb{C}^m$, near $\tau=0$, $\xi=0$. We write

$$\tau^{p+q} b(\tau^{-p}\xi) = c(\tau,\xi) := \sum_{j\geq 0} c_j(\xi)\tau^j. \tag{3.20}$$

3. Express that the leading term balances the derivatives with the nonlinearity: substitution of the leading behavior into the equation leads to

$$-pv_0 = qQ(x)^{-1}c_0(v_0), \tag{3.21}$$

which we assume has a nontrivial solution v_0.
4. Ensure that the resonances are constant: there exists a matrix-valued function $P(x)$ such that

$$P^{-1}Q^{-1}c_0'(v_0)P \quad \text{is constant.} \tag{3.22}$$

Here, c_0' is the matrix of derivatives of c_0 with respect to the components of u.

Theorem 3.9. *Under assumptions (3.19)–(3.22), system (3.18) admits of reduction to a generalized Fuchsian system.*

Proof. Introduce the new time variable $T = t - \psi(x)$, and write the equation as

$$Qu_T = a(\partial u) + b(u) + (a_0 - a)(\partial\psi)u_T,$$

where $a(\partial u) = \sum_j a^j \partial_j u$ and $a_0(\partial u) = \sum_j a_0^j \partial_j u$; ∂u stands for all the first-order spatial derivatives of u. Note that $(a_0 - a) = \mathcal{O}(T)$. Next, since we expect u to behave like $T^{-p/q}$, we let $T = \tau^q$ and $u = v\tau^{-p}$; using the assumption on $b(u)$, we obtain

$$Q(\tau v_\tau - pv)/q = \tau^q a(\partial v) + c(\tau, v) + (a_0 - a)(\partial\psi)(\tau v_\tau - pv)/q.$$

Since by (3.19), Q^{-1} exists, we may write $(Q-(a_0-a)(\partial\psi))^{-1} = Q^{-1}+\mathcal{O}(T) = Q^{-1} + \tau^q R$, and we obtain

$$\tau v_\tau - pv = q(Q^{-1} + \tau^q R)[\tau^q a(\partial v) + c(\tau, v)]. \tag{3.23}$$

We now substitute

$$v = v_0 + \boldsymbol{\tau} \cdot w := v_0 + \tau_0 w_0 + \cdots + \tau_l w_l,$$

where $\tau_j = \tau(\ln \tau)^j$ and $\boldsymbol{\tau} = (\tau_0, \ldots, \tau_l)$. We obtain, using (3.20),

$$\begin{aligned} c(\boldsymbol{\tau}, v) &= c(\tau_0, v_0 + \boldsymbol{\tau} \cdot w) \\ &= c_0(v_0) + c_0'(v_0)[\boldsymbol{\tau} \cdot w] + \tau_0 c_1(v_0) + \sum_k \tau_k \boldsymbol{\tau} \cdot h_k(\boldsymbol{\tau}, x, w, \partial w). \end{aligned}$$

It will be convenient to write $\boldsymbol{\tau} \cdot [c_0'(v_0)w]$ for $c_0'(v_0)[\boldsymbol{\tau} \cdot w]$, where

$$c_0'(v_0)w = (c_0'(v_0)w_0, \ldots, c_0'(v_0)w_l).$$

Since $\tau\partial_\tau v = N(\boldsymbol{\tau} \cdot w)$, where $N = \sum_k (\tau_k + k\tau_{k-1})\partial/\partial\tau_k$, we have

$$(N-p)(\boldsymbol{\tau}\cdot w)=\sum_{j=0}^{l}\tau_j\{(N-p)w_j+w_j+(j+1)w_{j+1}\}.$$

Since $q\geq 1$, there exist φ_1 and h_1 such that

$$q(Q^{-1}+\tau_0^q R)(\tau_0^q a(\partial v_0))=\boldsymbol{\tau}\cdot\left(\delta_{j0}\varphi_1(x)+\sum_k \tau_k h_{1k}(\boldsymbol{\tau},x,w,\partial w)\right),$$

while there exist φ_2 and h_2 such that

$$\begin{aligned}&q(Q^{-1}+\tau_0^q R)\left[\tau_0^q a(\boldsymbol{\tau}\cdot\partial w)+c_0(v_0)+\boldsymbol{\tau}\cdot[c_0'(v_0)w]+\sum_k\tau_k\boldsymbol{\tau}\cdot h_k\right]\\&=qQ^{-1}c_0(v_0)+\boldsymbol{\tau}\cdot\left\{qQ^{-1}[c_0'(v_0)w]+\delta_{j0}\varphi_2(x)+\sum_k\tau_k h_{2k}(\boldsymbol{\tau},x,w,\partial w)\right\}.\end{aligned}$$

We are ready to write (3.23), which now becomes

$$\begin{aligned}&(N-p)(\boldsymbol{\tau}\cdot w)-pv_0\\&=q(Q^{-1}+\tau_0^q R)\Bigg[\tau_0^q(a(\partial v_0)+\boldsymbol{\tau}\cdot a(\partial w))+c_0(v_0)+\boldsymbol{\tau}\cdot[c_0'(v_0)w]\\&\qquad+\tau_0 c_1(v_0)+\sum_k\tau_k\boldsymbol{\tau}\cdot h_k(\boldsymbol{\tau},x,w,\partial w)\Bigg]\\&=qQ^{-1}c_0(v_0)+\boldsymbol{\tau}\cdot[qQ^{-1}[c_0'(v_0)w]+(\varphi_1+\varphi_2)\delta_{j0}+\sum_k\tau_k(h_{1k}+h_{2k}+h_k)].\end{aligned}$$

Letting $\varphi=\varphi_1+\varphi_2$ and $g=h_1+h_2+h$, it is now natural to consider the system

$$(N-p-qQ^{-1}c_0'(v_0))w_j+w_j+(j+1)w_{j+1}=\varphi\delta_{j0}+\sum_k\tau_k g_{kj},$$

where g_{kj} is the jth component of g_k. Letting $w_j=Pz_j$, to take advantage of (3.21), we obtain a system of Fuchsian form. It remains to eliminate φ by introducing more variables as necessary, as in Theorem 2.24. This completes the reduction. □

3.4 Structure of the formal series with several time variables

We revert to a single equation, of the form

$$Q(t\partial_t)u=\sum_{q\leq k_0}t(\ln t)^q G_q[t,t\ln t,\dots,t(\ln t)^{\ell_0},u,Du,tD\partial_x u,\dots].\tag{3.24}$$

Recall that $D = t\partial_t$. There is an integer ℓ such that solutions in powers of $t(\ln t)^j$, $j \le \ell$, exist. We give here an estimate of the optimal (i.e., smallest) value of ℓ that enters in the solution. This estimate will be called ℓ'. It is useful in practice to take ℓ' as small as possible, to minimize the amount of computation. We begin by viewing the solution as a function of $(x, t_0, \dots, t_\ell)$; we therefore replace D by N. The basic observation is that we may perform all computations modulo inessentials in the sense of Definition 2.9. Thus, we may replace (3.13) by

$$Q(N)u = \sum_q t_q(G_q + I_q), \tag{3.25}$$

where I_q is any inessential polynomial. An appropriate choice of I_q will enable us to considerably lower the value of ℓ. We now state the results:

Theorem 3.10. *Let ℓ' be the the sum of (i) twice the multiplicity of 1 as a resonance, or $\ell_0 = 2k_0$ if it is greater, and (ii) the maximum multiplicity of any other positive resonance. Then there are inessential polynomials I_0, $\dots$, I_{ℓ_0} such that all formal formal solutions of (3.13) have the form $u = u(t, \dots, t(\ln t)^{\ell'})$, where $u(t_0, \dots, t_{\ell'})$ solves*

$$Q(N)u = \sum_q t_q(G_q + I_q(t_0, \dots, t_{\ell'})).$$

The number of arbitrary functions in the resulting solution equals the sum of the multiplicities of the positive integer resonances.

Specializing to the case of simple resonances, we obtain the following corollary:

Corollary 3.11. *If all resonances are simple and greater than 1, one may take $\ell = \ell' = 1$. More precisely, there is a formal solution of (3.13) of the form $u = u(t, t\ln t)$, with as many arbitrary functions as there are positive resonances.*

Proof. To say that u is a solution of (3.25) means that

$$Qu - \sum_q t_q G_q[u]$$

is inessential, and therefore may be written $\sum_q t_q J_q(\boldsymbol{t})$. We therefore consider the most general series solution of this equation and show that its essential part is independent of J_q. We then compute the formal solution to some high order, and introduce the I_q. Let us substitute

$$u = \sum_g u_g,$$

where u_g is a homogeneous polynomial in $(t_0, \dots, t_\ell)$, of degree g, into equation (3.13). We first prove, by induction on g, that u_g is the sum of an essential and an inessential part, the former depending on $(t_0, \dots, t_{\ell'})$, where ℓ' is defined below. We then show that one may introduce inessential polynomials I_q into the equation in such a way that the resulting equation will have a solution in which the inessential part is identically zero.

Step 1: Case $Q(g) \neq 0$. The u_g must be determined recursively from equations of the form

$$Q(N)u_g = \sum_q t_q \{G_q + J_q\}_{g-1},$$

where $\{\quad\}_g$ indicates that one takes the homogeneous part of degree g only. We argue by induction on g. On polynomials of degree g, $N = g + M$, with $M = \sum_k kt_{k-1}\partial/\partial t_k$. Therefore, writing $Q(N) = M^k R(N)$ with $R(g) \neq 0$, the recurrence relation reduces to

$$M^k R(N)u_g = \sum_q t_q \{G_q + J_q\}_{g-1},$$

where k is the multiplicity of g as a resonance. The operator M has been studied in Sect. 2.4. Since $Q(g) \neq 0$, $k = 0$. We merely need to check that the right-hand side has the desired form, since u_g will be uniquely determined. Indeed, $R(N)$ is invertible on the space of polynomials in $(t_0, \dots, t_{l'})$. Since $\{G_q\}_{g-1}$ involves only $u_0, \dots, u_{g-1}$, we may use the induction hypothesis and write $\sum_{j<g} u_j = v(t_0, \dots, t_{\ell'}) + w$, where w is inessential. It follows that

$$\begin{aligned} G_q &= G_q(t_0, \dots, t_{\ell_0}, v + w, \dots) \\ &= G_q(t_0, \dots, t_{\ell_0}, v, \dots) \\ &\quad + \int_0^1 [F_u(t, v + sw, Nv, \dots)w + F_{Du}(t, v, Nv + sNw, \dots)Nw + \cdots]ds. \end{aligned}$$

Now, w, Nw,..., and all their derivatives are inessential. Since inessential functions are stable by product with other functions (i.e., they form an ideal), we see that $\{G_q + J_q\}_{g-1}$ is the sum of a polynomial in $(t_0, \dots, t_{\ell'})$ and an inessential polynomial.

Step 2: General case. We now assume $k > 0$. The earlier results about the form of G_q still hold. Using Theorem 2.36 of Sect. 2.4, the general solution has the form

$$u_g = G(t_0, \dots, t_{k+\ell_0}) + \text{inessential}.$$

Therefore, we need $\ell' \geq k + \ell_0$. Also, the essential part of u_g involves k arbitrary functions of x, because case (1) of Theorem 2.36 ensures that u_g is determined, modulo inessentials, up to a combination of $t_0^g, \dots, t_0^{g-k+1} t_1^{k-1}$. Since the solutions of $M^k v = t_q J_q$ for different J_q's differ by inessential polynomials, the essential part of u does not depend on J_q. In practice, we have to solve at each resonance an equation of the form $M^k u =$ known, and we can make use of the special form of the right-hand side to further reduce the value of ℓ.

Step 3: Introduction of I_q. We now fix g very large, and let v_g be the essential part of the formal solution we just computed, truncated at order g. Define I_q of degree g so that v_g is a formal solution, up to order g, of

$$Q(N)v_g = \sum_q (G_q[v_g] + I_q).$$

We may now apply Theorem 2.24 to conclude. Note that v_g contains arbitrary functions of x corresponding to each resonance. This completes the proof of Theorem 3.10.

Step 4: Proof of Corollary 3.11. If all resonances are simple and greater than 1, or if 1 is a simple and compatible resonance (i.e., if it does not introduce logarithms), an important simplification is that $\ell_0 = k_0 = 0$: no logarithms appear in the first step of the reduction. If we assume that g is a simple resonance, and that for $j < g$, $u_g = u_g(t_0, t_1)$, we must, in order to find u_g, solve an equation of the form

$$MR(N)u_g = t_0 F_g(t_0, t_1),$$

where F_g is a polynomial of degree $g - 1$. The operator $R(N)$ is invertible on the space of such polynomials. By case 3 of Theorem 2.36, we may find a solution that depends only on t_0 and t_1. The argument is now finished as in the general case. This completes the proof. □

Remark 3.12. If there is a single simple resonance $r > 1$, the solution is given by a series in t_0 and $t_0^{r-1}t_1$ (i.e., t and $t^r \ln t$). Indeed, since 1 is not a resonance, we have $k_0 = \ell_0 = 0$, and the formal solution $u = \sum_j u_j$ is computed by solving recursively an equation of the form

$$Q(N)u_j = t_0 R_j(t_0, t_1),$$

where R_j and u_j are independent of t_1 if $j < r$. Now N (and therefore $Q(N)$) leaves invariant the space of polynomials in t_0 and $t_0^{r-1}t_1$ of total degree $j \neq r$; $Q(N)$ is invertible on this space. On the other hand, the right-hand side R_j must involve t_1 linearly if $j < 2r$. Assume by induction that the u_k for $k < j$ contain only monomials of the form $t_0^b t_1^c$ where $c \leq [k/r]$. Then R_j is a combination of polynomials

$$u_{j_1} \cdots u_{j_q}$$

such that $j_1 + \cdots + j_q + 1 = j$. Each of the u_{j_s} contains only monomials $t_0^{b_s} t_1^{c_s}$ with $c_s \leq [j_s/r]$. It follows that $t_0 R_j$, and therefore u_j, contains only monomials $t_0^b t_1^c$ with

$$c = \sum_s c_s \le \sum_s [j_s/r] \le \sum_s \frac{j_s}{r} = (j-1)/r,$$

as announced. This property may fail if 1 is a resonance.

3.5 Resonances, instability, and group invariance

3.5.1 Practical determination of resonances

The determination of resonances may be greatly simplified using shortcuts given below. The resonances are obtained by the following procedure: determine the leading term, say $u_0 T^\nu$, linearize about it and seek solutions of the linearized equation with leading order $w(x)T^{\nu+r}$. The values of r leading to a nontrivial solution are the resonances. While it is easy to prove this result in the general situations described in the previous chapter, it is as easy to check it afresh in new situations. Therefore, the practical procedure is, with the notation of Sect. 3.1.2, to compute the coefficient of $T^{\rho+r}$ in $F[u_0 T^\nu + w(x)T^{\nu+r}]$, and set it equal to zero.

If the problem admits a group action, we may obtain solutions of the linearization from any solution that is not invariant by the group. Indeed, the existence of a one-parameter group action, with parameter ε, means that there is a one-parameter family $\varepsilon \mapsto u(\varepsilon)$ such that whenever $F[u(0)] = 0$, necessarily $F[u(\varepsilon)] = 0$. It follows, by differentiating this equation with respect to ε, that

$$\frac{\partial u(\varepsilon)}{\partial \varepsilon}$$

always solves the linearized equation. For instance, assume $u(x,t) \sim u_0(t - \psi(x))^\nu$, with $u_0 \neq 0$, solves an autonomous equation (*i.e.*, one in which x and t do not enter explicitly). It follows that for every ε, $u(x, t+\varepsilon)$ is also a solution. If it is permissible to differentiate the equation, we find that if $\nu \neq 0$,

$$\frac{\partial u(\varepsilon)}{\partial \varepsilon} \sim \nu u_0 (t - \psi(x))^{\nu-1}$$

solves the linearized equation; indeed, the term involving the $\partial_x u_0$ is only of order ν and does not enter at leading order. It follows that

-1 is necessarily a resonance.

This result, which is a special case of Theorem 3.8, is often useful as a double check. Intuitively, the presence of the resonance -1 in a translation-invariant problem is related to the possibility of perturbing a singular solution by shifting the singularity locus. Since this operation does not affect the type of singularity, it is compatible with stability of the singularity. If the only resonance with negative real part is -1, we may expect stability.

3.5.2 Instability: a case study

We discuss in this section one of the few examples for which an analysis of instability can be made complete [108]. Some information is also available for homogeneous systems of ODEs [5]; a complete classification seems out of reach at the present time. A simpler example is provided by Problem 3.4(2).

The Chazy equation

$$y''' - 2yy'' + 3y'^2 = 0 \tag{3.26}$$

possesses the exact solution $-6/x$, for which the resonances are -1, -2, and -3. The negative resonances express that all nonzero solutions of the linearized equation are more singular than the reference solution $-6/x$; however, this does not suffice to indicate instability, since the negative resonance -1 is compatible with stability, as we just saw.

The objective of this section is to prove that there are families of solutions $y(x;\varepsilon,r)$ such that

$$y(x;0,r) = -6/x \quad \text{and} \quad \frac{dy}{d\varepsilon}(x;0,r) = x^{-1+r}$$

for $r = -1$, -2, and -3. This will account for the three "negative resonances," by giving the singular behavior of solutions close to the reference solution. It will be apparent from the proofs that $y(x;\varepsilon;r)$ may in fact be expanded to higher order, and that the coefficients of the higher-order terms are increasingly more singular in x.

Background

The Chazy equation came up in the course of Chazy's extension of Painlevé's program to third-order equations [41, 42, 43]. It is one of the "class XII" equations

$$y''' - 2yy'' + 3y'^2 = E(6y' - y^2)^2, \tag{3.27}$$

where $E = 4/(36 - k^2)$ with $k \geq 0$, omitting the "complementary" terms. This equation has the same special solution $-6/x$, but the general solution is completely different. It is possible for the simple pole to split into two or more poles by perturbation. This behavior cannot be captured adequately in a series representation in powers of x: take a solution with two poles at $x = 0$ and $x = \alpha$. The radius of convergence of a pole expansion around $x = 0$ cannot be greater than $|\alpha|$, and therefore tends to zero if there is a confluence of the two singularities ($\alpha \to 0$). A perturbative approach can correctly describe what happens at a fixed location away from the singularities, but its asymptotics as $x \to 0$ do not describe the confluence correctly.

Equation (3.26) is also closely related to a system considered by Halphen, and its general solution can be parameterized using the solutions of a hypergeometric equation if $k > 0$, and the Airy equation if $k = 0$; see [45, 1].

This equation has attracted recent interest as a reduction of self-dual Yang–Mills equations [3, 1, 175]. It also arises in connection with one of the special reductions of Einstein's equations in a Bianchi IX space-time. A particular solution is

$$\frac{1}{2}\frac{d}{dx}\ln \Delta(x),$$

where $\Delta(x)$ is the discriminant modular form [175, 32, 33, 43, 126]; it has the real axis as a natural boundary.

The Chazy equation admits an SL(2) action: if $y(x)$ is a solution, so is

$$\frac{ad-bc}{(cx+d)^2}y\left(\frac{ax+b}{cx+d}\right)-\frac{6c}{cx+d} \tag{3.28}$$

for any choice of the complex parameters a, b, c, and d, subject to $ad-bc\neq 0$. There are effectively only three parameters, since scaling the parameters by a common factor does not generate a new solution. The transformed solution has, in general, a circular natural boundary.

Results

The results below actually apply to any equation that admits the transformation formula (3.28) and for which any uniform limit of analytic solutions is also a solution. This assumption is clear for ODEs; it allows one to extend (3.28) to some cases in which the transformation $(ax+b)/(cx+d)$ is noninvertible, by viewing it as a limit of invertible transformations.

Fix a solution $y(x)$ analytic near $x=0$. The construction below depends on the possibility of prescribing y, y', y'' arbitrarily at one point. This construction precisely fails for the nongeneric solutions $-6/x+A/x^2$. The resonance structure will be derived on the sole basis of the representation formula.

Consider the family

$$y(x;\varepsilon)=-\frac{6}{x-\eta}+\frac{\mu}{(x-\eta)^2}y\left(\frac{-\mu}{x-\eta}\right), \tag{3.29}$$

where η and μ depend on ε, and are assumed to be small as $\varepsilon\to 0$. This is a special case of the transformation (3.28). Our results are as follows.

Theorem 3.13. *If $\mu y(0)-6\eta=\varepsilon$, and μ and η are proportional to ε, then*

$$y(x;0)=\frac{-6}{x} \quad \textit{and} \quad \frac{dy}{d\varepsilon}(x;0)=\frac{1}{x^2}.$$

Theorem 3.14. *Assume $\mu y(0)-6\eta=0$, but $6y'(0)-y(0)^2\neq 0$. Then, if μ and η are both proportional to $\varepsilon^{1/2}$, we have*

$$y(x;0)=\frac{-6}{x} \quad \textit{and} \quad \frac{dy}{d\varepsilon}(x;0)=\frac{c}{x^3},$$

where $c\neq 0$.

Theorem 3.15. *Assume* $\mu y(0) - 6\eta = 0$ *and* $6y'(0) - y(0)^2 = 0$, *but* $y'' - yy' + y^3/9 \neq 0$ *when* $x = 0$. *Then, if* μ *and* η *are both proportional to* $\varepsilon^{1/3}$, *we have*

$$y(x;0) = \frac{-6}{x} \quad \text{and} \quad \frac{dy}{d\varepsilon}(x;0) = \frac{c}{x^4},$$

where $c \neq 0$.

Theorem 3.16. *If an equation of order three or higher to which the Cauchy existence theorem applies admits the special solution* $y = -6/x$, *and if the linearization of the equation at this solution does not have* $1/x^2$, $1/x^3$, *and* $1/x^4$ *among its solutions, then the given equation cannot admit the* SL(2) *action (3.28).*

The restriction that the solution $y(x)$ be analytic is essential. In fact, we can obtain quite different results if y admits branching:

Theorem 3.17. *If there is a solution of the form* $y(x) = x^{-1}h(x^k)$, *where* h *is analytic,* $k > 0$, *and* $h(0) = (k-6)/2$, *there exist two families of solutions,* $y_1(x;\varepsilon)$ *and* $y_2(x;\varepsilon)$, *such that*

$$y_1(x;0) = y_2(x;0) = -\frac{k+6}{2x}$$

and

$$\frac{dy_1}{d\varepsilon}(x;0) = \frac{c}{x^2} \quad \text{and} \quad \frac{dy_2}{d\varepsilon}(x;0) = \frac{c}{x^{k+1}}.$$

Remarks

If we take $y = -6/(x-x_0) + A/(x-x_0)^2$, then Theorem 3.14 applies, but Theorem 3.15 does not. Indeed, in this case,

$$y' - y^2/6 = \frac{-A^2(x-x_0)^{-4}}{6}, \tag{3.30}$$

and

$$y'' - yy' + \frac{y^3}{9} = \left(\frac{d}{dx} - \frac{2}{3}y\right)\left(y' - \frac{y^2}{6}\right) = A^3\frac{(x-x_0)^{-6}}{9}. \tag{3.31}$$

It is therefore not possible to make (3.30) vanish without having $A = 0$, in which case (3.31) vanishes as well. One can rephrase the assumption in Theorem 3.15 as follows: $y' = y^2/6$ but $y'' \neq y^3/18$, for $x = 0$. Theorems 3.13, 3.14, and 3.15 all apply, for example, when there is a solution for every choice of $y(0)$, $y'(0)$, and $y''(0)$. These results therefore hold for any third-order autonomous equation, hence for both equations (3.26) and (3.27), which have completely different singularity structures. It follows from Theorem 3.19 below that Theorem 3.17 applies to equations (3.27). A result similar to Theorem 3.16 could of course be stated for this situation. An illustration of Theorem 3.16 is the equation

$$y''' = 2yy'' - 3y'^2 + cy'(6y' - y^2),$$

which has the solution $-6/x$, but whose resonance equation is $(r+1)[(r+2)(r+3) - 36c] = 0$. Therefore -2 and -3 are both resonances only if $c = 0$. We conclude, without computing the symmetry group of the equation, that this equation does not admit the transformation law (3.28) if $c \neq 0$.

Proofs

Any solution $y(x)$ generates the one-parameter family of solutions

$$y(x;\varepsilon) = -\frac{6}{x - \eta(\varepsilon)} + \frac{\mu(\varepsilon)}{(x - \eta(\varepsilon))^2} y\left(-\frac{\mu(\varepsilon)}{x - \eta(\varepsilon)}\right).$$

If x is fixed and nonzero, and if μ and η are small as $\varepsilon \to 0$, we can expand this solution in the form

$$\begin{aligned} y(x;\varepsilon) = & \frac{-6}{x} + \frac{\mu y - 6\eta}{x^2} + \frac{\mu(2\eta y - \mu y') - 6\eta^2}{x^3} \\ & + x^{-4}[-6\eta^3 + \mu(3\eta^2 y - 3\eta\mu y' + \tfrac{\mu^2}{2} y'')] \\ & + \mathcal{O}(\eta^4, \eta^3\mu, \eta^2\mu^2, \eta\mu^3, \mu^4), \end{aligned}$$

where y, y', ... stand for $y(0)$, $y'(0)$, ...

Any such family has the property that $y(x;0) = -6/x$. Furthermore, it is clear that the above expansion could be pushed to all orders, and that the coefficients of the higher-order terms contain higher and higher powers of $1/x$.

Proof of Theorem 3.13 : If we take μ and η proportional to ε in such a way that $\mu y(0) - 6\eta \sim \varepsilon$, we have $\partial y/\partial\varepsilon = 1/x^2$ for $\varepsilon = 0$.

Proof of Theorem 3.14 : If we take μ and η proportional to $\varepsilon^{1/2}$ in such a way that $\mu y(0) - 6\eta = 0$, and if y is such that $6y'(0) \neq y(0)^2$, we have $\partial y/\partial\varepsilon = \text{const}/x^3$ for $\varepsilon = 0$.

Proof of Theorem 3.15 : If we take μ and η proportional to ε in such a way that $\mu y(0) - 6\eta = 0$, and assume that $6y'(0) - y(0)^2 = 0$ but $y'' - yy' + y^3/9 \neq 0$ for $x = 0$, we find that $\partial y/\partial\varepsilon = \text{const}/x^4$ for $\varepsilon = 0$.

This proves Theorem 3.13, 3.14, and 3.15.

Proof of Theorem 3.16: Consider an equation $F[u] = 0$ of order three or higher with such a group action. Solving the Cauchy problem, we can construct solutions to which each of Theorems 3.13, 3.14, and 3.15 applies. Consequently, there are differentiable families of solutions $y(x;\varepsilon)$ as in these theorems. Since $F[y(x;\varepsilon)]$ is identically zero, we have

$$0 = \frac{d}{d\varepsilon} F[y(x;\varepsilon)]\bigg|_{\varepsilon=0} = F'[-6/x]\left(\frac{dy}{d\varepsilon}(x;0)\right),$$

where F' denotes the linearization of F. We conclude that this linearized equation must admit the three solutions $1/x^m$, $m = 2$, 3, and 4. If these three functions do not solve the linearization, there can be no such group action, QED. The specific coefficients of the group action are not essential to the result: only the existence of an expansion of families of solutions matters.

Proof of Theorem 3.17 : The solution $y(x)$ in the statement of the theorem is constructed in Theorem 3.19. From y, we construct the one-parameter family

$$y_2(x;\varepsilon) = -\frac{6}{x} - \frac{1}{x}h\left(\frac{\varepsilon}{x^k}\right),$$

using (3.28) for the inversion $x \mapsto \varepsilon^{1/k}/x$. Letting $\varepsilon \to 0$, we find that $-(6+h(0))/x = -(k+6)/2x$ must be a solution. We now define

$$y_1(x;\varepsilon) = -\frac{k+6}{x-\varepsilon}.$$

The properties listed in the theorem are now readily verified.

Instability of isolated poles

Even though the construction of the perturbation expansion of solutions close to $-6/x$ can be made solely on the basis of the group action on solutions, the singularities that arise by perturbation of simple poles are different for (3.26) and (3.27). We know that perturbation series near a single pole do not allow an analytical description of confluence phenomena. However, even though a function such as $(x-a)^{-1} + (x-b)^{-1}$ is not jointly analytic in x, a and b small, it is the logarithmic derivative of $(x-a)(x-b)$, which is perfectly well behaved. More generally, we show that a Cole–Hopf transformation provides an analytical description of confluence phenomena in the Chazy equation.

Theorem 3.18. *For any constant a, equation (3.26) has precisely one solution of the form $y(x) = u'/2u$ with*

$$u(x) = e^x(1 + e^x w(e^x)),$$

where w is analytic when its argument is small, and $w(0) = a$. Using transformations (3.28), this solution generates a one-parameter family of perturbations of $-6/x$, with a natural boundary shrinking to a point as the parameter vanishes.

For equation (3.27), we have the following result.

Theorem 3.19. *Let a be a constant. For $k \neq 0$ or 1, equation (3.27) has a unique solution of the form*

$$y(x) = x^{-1}h(x^k),$$

where h is analytic when its argument is small, $h(0) = (k-6)/2$, and $h'(0) = a$. If $k = 2$, 3, 4, or 5, this solution is rational. Using transformations (3.28), this solution generates a one-parameter family of perturbations of $-6/x$, where all poles, except possibly one, cluster at the origin as the parameter vanishes.

Thus, $-6/x$ is unstable; the next result shows that $(k-6)/2x$ is stable, but $-(k+6)/2x$ is unstable:

Theorem 3.20. *For $k = 2, 3, 4, 5$, there is a three-parameter family of solutions of (3.27) that contains the solution $(k-6)/2x$. These parameters are in correspondence with the Cauchy data at a nearby regular point. Solutions with leading term $-(k+6)/2x$, on the other hand, are unstable under perturbation: they arise from the confluence of all singularities save one.*

These three theorems are proved in Problem 3.3. We close this chapter with a general result on stability of solutions.

3.6 Stability and parameter dependence

We prove a stability result for pole singularities of an equation with the maximum number of coefficients in their pole expansions. This result makes rigorous the idea that a series with as many free parameters as there are Cauchy data must represent the general solution locally. It is the simplest case in which Step H of the general program can be carried out. To prove the result, one must show that these parameters are not redundant. We achieve this by a reduction to the implicit function theorem.[1]

To fix ideas, let us consider an autonomous equation of the form

$$(d/dx)^m u = f(u, \ldots, u^{(m-1)}), \tag{3.33}$$

where f is a polynomial. Let $u(x - x_0, c_1, \ldots, c_{m-1})$ be a family of solutions that depends analytically on $(x_0, c_1, \ldots, c_{m-1})$ for $|x_0|$ and $|c_k| < a$ and $0 < |x - x_0| < b$, for some positive a and b. We have the following result:

Theorem 3.21. *Assume that $\partial u/\partial x_0$, $\partial u/\partial c_1, \ldots, \partial u/\partial c_{m-1}$ form a linearly independent set of solutions of the linearization of (3.33). Then*

$$u(x - x_0, c_1, \ldots, c_{m-1})$$

is a local representation of the general solution.

Proof. Consider the reference solution $U = u(x, 0, \ldots, 0)$ for definiteness. Given any point x_1 with $0 < |x_1| < b$, consider

$$\varphi : (x_0, c_1, \ldots, c_{m-1}) \mapsto \left(u(x_1), u'(x_1), \ldots, u^{(m-1)}(x_1)\right),$$

[1] An example of a redundant representation is the two-parameter family of series

$$u(x; \varepsilon, \eta) = \sum_{j \geq 0} \frac{\eta^j}{(x-\varepsilon)^{j+1}}. \tag{3.32}$$

The parameters ε and η are redundant, because $u(x; \varepsilon, \eta) = 1/(x - \varepsilon - \eta)$: the pairs (ε, η) with the same value of $\varepsilon + \eta$ define the same function.

where $u(x_1) = u(x_1 - x_0, c_1, \ldots, c_{m-1})$, and similarly for the derivatives of u. Applying the inverse function theorem to this map near $(x_0, 0, \ldots, 0)$, we conclude that any set of Cauchy data close to the data of U at x_1 coincides with the Cauchy data set of a member of our family. □

Remark 3.22. The assumption usually holds for any pole expansion with the maximum number of parameters if $\nu \neq 0$. Indeed, it suffices to consider the family $u(x - x_0, c_1, \ldots, c_{m-1})$, where the c_l's are the arbitrary coefficients in the expansion of u. The functions $\partial u/\partial x_0$, $\partial u/\partial c_l$ are derivatives of families of solutions, and are therefore solutions of the linearized equation. It is easy to check that these derivatives all have different leading behaviors at $x = x_0$, and are therefore linearly independent.

Remark 3.23. If $\nu = 0$, we are in the case of the Cauchy problem, and the series for the solution contains $m + 1$ parameters, namely the location of the initial point and the m Cauchy data. These data are clearly redundant. In the case of (3.32), the representation is redundant because $\partial u/\partial \varepsilon = \partial u/\partial \eta$.

Problems

3.1. Prove that the space of homogeneous inessential polynomials of degree n in $\ell + 1$ variables, as defined in Sect. 3.4, is a vector space over $\mathbb{R}$. Find a basis of this space for small values of n and ℓ.

3.2. (a) Show that -1 is not a resonance for the Cauchy problem $u_{TT} = f(u)$, $u = 1 + \mathcal{O}(T^2)$ if f is, say, a polynomial.

(b) Study solutions of the ODE [46]

$$u_T u_{TTT} - 2u_{TT}^2 + 18u_T = Tuu_T^2$$

such that $u = a_0 + \mathcal{O}(T^2)$. Is -1 a resonance?

3.3. Apply reduction to the ODE $u_t = 2\sqrt{u}$, and construct solutions such that $u \sim a(t - t_0)^2$ for $t > 0$. Show that even though the Cauchy problem with initial condition $u(0) = 0$ has infinitely many solutions near t_0, there is only one solution with the properties $u(0) = 0$ and $u_{tt} \neq 0$.

3.4. For each of the following equations, choose an expansion variable and determine (a) the possible leading balances; (b) the first reduction; (c) the resonances in each case; (d) the most general formal solution for each of them:

1. $u_t = u^2 + 1$.
2. $u_{tt} + 3uu_t + u^3 = 0$. What happens if one introduces a new unknown via $u = v_t/v$?
3. $u_{tt} = e^u - e^{-2u}$.
4. $u_{tt} = u^2 + cu_t$, with c constant.

5. $u_{tt} = u^4$.
6. $u_{ttt} + uu_{tt} = 0$.
7. $u_{ttt} - u^p u_t = 0$ with $p > 0$.

See Problem 10.1 for further examples.

3.5. Prove Theorems 3.18 to 3.20.

3.6. Reduce the following equations to a first-order Fuchsian system:
(a) The Euler–Poisson–Darboux (EPD) equation

$$u_{tt} - \frac{\lambda - 1}{t} u_t = \Delta u,$$

where Δ is the Laplace operator in n variables.
(b) $(t\partial_t)^2 u - tu_{xx} = 0$.

3.7. Perform reduction for the Lotka–Volterra system

$$dx/dt = ax + cxy,$$
$$dy/dt = bx + dxy,$$

where a, b, c, d are nonzero constants.

3.8. Prove Theorem 1.6.

3.9. In Sect. 3.4, can one lower ℓ' by allowing it to be fractional?

3.10. Let $D_k = \partial_{rr} + (k/r)\partial_r$. Let u solve the EPD equation $D_k u = \Delta u$.
(a) Show that $r^{k-1} D_k u = D_{2-k}(r^{k-1}u)$.
(b) Show that $t^{k-1}u$ solves another EPD equation. Show that $r^{-1}u_r$ also solves an EPD equation [76, 184]. The transformation $\rho = r^{k-1}$, $\xi = x/(k-1)$ for $k \neq 1$ transforms the EPD equation into a generalized Tricomi equation.
(c) Reduce the generalized Stokes–Beltrami system $y^3\phi_x = \psi_y$, $y^3\phi_y = -\psi_x$ to (the elliptic analogue of) an EPD equation for ϕ [183].

3.11. Find all solutions of

$$x^2(a + bx^n)\frac{d^2u}{dx^2} + x(c + ex^n)\frac{du}{dx} + (f + gx^n) = 0,$$

where a, b, c, e, f, g are constants, $a \neq 0$, and n is a positive integer (Euler [58]).

Part II

Theory of Fuchsian Partial Differential Equations

4

Convergent Series Solutions of Fuchsian Initial-Value Problems

This section presents the two basic existence theorems for complex solutions of Fuchsian PDEs. Thanks to them, as soon as the second reduction has been achieved, one can immediately conclude that the singularity data determine a unique singular solution, and that the formal asymptotics of Part I are valid. The first result, Theorem 4.3, yields solutions that are continuous in time and analytic in space; the second, Theorem 4.5, yields convergent series solutions in both space and generalized time variables $T, \dots, T(\ln T)^\ell$. The method of proof is based on an iteration in a Banach space, as in the modern approach to the Cauchy–Kovalevskaya theorem. We begin with an overview of the classical Fuchs–Frobenius theory for ODEs in the complex domain.

4.1 Theory of linear Fuchsian ODEs

Consider a first-order linear system of ODEs

$$\boldsymbol{u}' = A(z)\boldsymbol{u},$$

where the prime denotes the derivative with respect to z. We review some basic facts, without proofs [50, 23]. If A is analytic and single-valued in a pointed neighborhood of the origin, $\{0 < |z| < a\}$, any fundamental matrix $\Phi(z)$ of solutions[1] may be continued analytically about the singularity at the origin. Therefore $\Phi(e^{2i\pi}z)$ must be of the form $\Phi(z)C$, with C constant. Writing $C = e^{2i\pi P}$, we find that $\Phi(z)z^{-P} = S(z)$ is single-valued.

Now let us assume in addition that $A(z)$ has at most a pole of first order at z_0; the system is then said to be Fuchsian, or of Fuchs type at z_0. The point z_0 is said to be a singular point of the first kind. The point at infinity is said to be a Fuchsian singularity if the change of variables $z \mapsto z' = 1/z$ leads to an equation with a Fuchsian singularity for $z' = 0$. For the rest of

[1] $\Phi(z)$ is a fundamental matrix if its columns form a basis of the space of solutions.

this section, we assume that the origin is a Fuchsian singularity; Problem 4.3 shows that certain matrices A with stronger singularities may be converted into Fuchsian ODEs with nonanalytic coefficients.

Next, for any Fuchsian singularity, S has at most power growth as $z \to 0$. The converse is not true (see Problem 4.4). One often calls the origin a regular singular point if there is a fundamental solution of the form $S(z)z^P$, where S has at most a pole at the origin, and P is constant; non-Fuchsian ODEs may admit the origin as a regular singular point.[2] For a regular singularity, if $\Psi(z)$ is a formal solution in powers and logarithms of z and finitely many negative powers, it must always converge: because solutions have power growth, the matrix $S(z)$ must be a Laurent series with finitely many negative powers; since $\Psi^{-1}\Phi$ must be formally equal to the identity, and Φ and Ψ are both series of the same form, they must be equal, so that Φ converges.

A more direct approach is to construct a fundamental matrix. Let R be the residue of A at the origin, so that $A(z) - R/z$ is analytic at the origin. The Fuchs indices are, by definition, the eigenvalues of R. This tallies with the developments of Chap. 2. If no two eigenvalues of R differ by an integer, one can find a fundamental matrix of the form $\Phi = Sz^R$, where S is given by a series in nonnegative powers of z, the constant term of which is the identity matrix. The proof is achieved by direct recursive computation of the coefficients of the expansion of S. In Chap. 2, we directly computed the expansion of each of these solutions instead of dealing with the fundamental matrix; this approach is preferable if one is interested only in solutions that tend to zero.

If two eigenvalues do differ by an integer, the natural route, from our point of view, would be to introduce new logarithmic variables and seek a solution of the form $z^\nu \boldsymbol{v}$, where the components of $\boldsymbol{v}$ belong to a space A_ℓ, with ℓ sufficiently large. The classical procedure is different: it seeks to reduce the difference between the eigenvalues as follows. Performing a linear change of unknown, it is always possible to assume that R has been reduced to a block-diagonal form

$$\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

where R_1 is the Jordan block corresponding to the eigenvalue λ_1, and R_2 does not have λ_1 as an eigenvalue. Assume that R_1 has size $p \times p$, while u has n components. Perform the change of unknown[3]

$$u = Mv, \text{ where } M = \begin{pmatrix} zI_p & 0 \\ 0 & I_{n-p} \end{pmatrix}.$$

Write $A = z^{-1}R + A_0 + A_1 z + \cdots$, where

[2] However, there is a tendency in the literature to use the phrase "regular singularity" for "Fuchsian singularity," and we shall follow this usage.

[3] I_p denotes the $p \times p$ identity matrix.

$$A_0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Direct computation shows that v solves a Fuchsian equation in which R has been replaced by

$$\begin{pmatrix} R_1 - I_p & A_{12} \\ 0 & R_2 \end{pmatrix}.$$

This operation therefore has the net effect of replacing λ_1 by $\lambda_1 - 1$. Iterating this operation, one reduces the problem in finitely many steps to a problem in which R has been replaced by a matrix $\tilde{R}$ of which no two eigenvalues differ by an integer. It follows that there is a fundamental system of solutions of the form $\tilde{S}(z)z^{\tilde{R}}$, where $\tilde{S}(z)$ is analytic near the origin.

Let us review some corresponding facts for nth-order ODEs

$$Lw := w^{(n)} + a_{n-1}(z)w^{(n-1)} + \cdots + a_0(z)w = 0.$$

The origin is said to be a Fuchsian singularity if for every k between 0 and $n-1$, $b_k := z^k a_k$ is analytic at the origin. Taking $u := (w, zw', \ldots, z^{n-1}(d/dz)^{n-1}w)$ as the new unknown, one then obtain an equation of the form

$$zu' = B(z)u,$$

with B analytic. The Fuchs indices may be found directly from the equation: they are the roots of the polynomial

$$P(\lambda) := [\lambda]_n + \sum_{k<n} b_k(0)[\lambda]_k = 0$$

using the notation introduced in (3.5). Frobenius's trick consists in noticing that if one can find a family of solutions u_ε satisfying

$$Lu_\varepsilon = f(\varepsilon)u_\varepsilon,$$

where $f(0) = f'(0) = 0$, then $(d/d\varepsilon)u_\varepsilon|_{\varepsilon=0}$ solves the equation $Lw = 0$. This trick, and its obvious generalizations to higher-order derivatives, is useful for finding several solutions in the case in which P has multiple roots. It does not seem to be applicable to nonlinear situations and is therefore not further discussed.

4.2 Initial-value problem for Fuchsian PDEs with analytic data

Consider now a Fuchsian system for a "vector" unknown $\boldsymbol{u}(x,t)$ of the form

$$(D + A(x))\boldsymbol{u} = F[\boldsymbol{u}] := tf(t, x, \boldsymbol{u}, \boldsymbol{u}_x). \tag{4.1}$$

We assume that A is an analytic matrix near $x = 0$, where $x = (x_1, \ldots, x_n)$, such that $\|\sigma^A\| \leq C$ for $0 < \sigma < 1$. The unknown $\boldsymbol{u}$ depends on complex x, but only on real $t > 0$.

Remark 4.1. The seemingly more general system $(D + A)\boldsymbol{u} = t^{\varepsilon} f$ may be reduced to the form (4.1) by taking t^{ε} as time variable, and by modifying A accordingly. Similarly, if A depends on t, and may be written $A = A_0(x) + tA_1(x, t, \boldsymbol{u})$, the system may be written

$$(D + A_0)\boldsymbol{u} = t(f - A_1\boldsymbol{u}).$$

For these reasons, we limit ourselves to the case $A = A(x)$.

Remark 4.2. If the equation contains parameters, it is often useful to view them as new space variables, whose derivatives do not enter into f. In this way, analytic dependence with respect to parameters is obtained as a by-product of the proof.

The nonlinearity f is assumed to preserve analyticity in space and continuity in time, and to be Lipschitz in $\boldsymbol{u}$ and $\boldsymbol{u}_x$ whenever $\boldsymbol{u}$ is bounded. That is, we assume that f depends linearly on $\boldsymbol{u}_x$; fully nonlinear equations may be reduced to this case by differentiation; see Problem 4.1. To fix ideas, we assume that f is a sum of products of analytic functions of x, $\boldsymbol{u}$, and $\boldsymbol{u}_x$ by powers of t, $t^{k(x)}$, and $\ln t$; this suffices to ensure estimate (4.2) below.

Theorem 4.3. *The system (4.1) has exactly one solution near* $x = 0$ *and* $t = 0$*, which is analytic in* x*, continuous in* t*, and tends to zero as* $t \downarrow 0$*.*

Proof. Let

$$H[v] = \int_0^1 \sigma^{A(x)-1} \boldsymbol{v}(\sigma t)\, d\sigma.$$

This provides the solution of

$$(D + A)\boldsymbol{u} = \boldsymbol{v},$$

with $\boldsymbol{u}(0) = 0$, provided that $\boldsymbol{v} = O(t)$ near $t = 0$. The proof will consist in showing that the operator $\boldsymbol{v} \mapsto G[\boldsymbol{v}] := F[H[\boldsymbol{v}]]$ is a contraction for a suitable norm. Its fixed point generates a solution $\boldsymbol{u} = H[\boldsymbol{v}]$ to our problem. We are ultimately interested in real values of x in some open set Ω. We therefore work in a small complex neighborhood of the real set Ω. We also define two norms. The s-norm of a function of x is given by

$$\|\boldsymbol{u}\|_s = \sup\{|\boldsymbol{u}(x)| : x \in \mathbb{C}^n \text{ and } d(x, \Omega) < s\}.$$

The a-norm of a function of x and t is defined by

$$|u|_a = \sup\left\{\frac{s_0 - s}{t}\|u(t)\|_s\sqrt{1 - \frac{t}{a(s_0 - s)}} : t < a(s_0 - s)\right\}.$$

The objective is to prove that the iteration $u_0 = 0$, $u_{n+1} = G[u_n]$ is well defined and converges to a fixed point of G, which gives us the desired solution. This will be achieved by exhibiting a set of functions that contains zero and on which G is contractive in the a-norm. Since a contraction has a unique fixed point, we also obtain uniqueness.

Remark 4.4. The a-norm allows functions to become unbounded when $t = a(s_0 - s)$. Thus, the solution will be bounded on a domain that shrinks with time; this is consistent with the behavior of the domain of dependence of solutions of the Cauchy problem. The iteration would not be well defined if we had worked simply with the supremum of the s-norm over some time interval, because the derivative of a holomorphic function is not bounded by its supremum on the same domain, but on a slightly larger domain.

We choose $R > 0$ and s_0 such that $\|F[0](t)\|_{s_0} \leq Rt$. This can always be achieved since we are allowed to take R very large.

Step 1: Estimating *H*

Using the definition of $|u|_a$, we obtain, after the change of variables $\rho = \sigma t/a(s_0 - s)$,

$$\begin{aligned}\|H[u](t)\|_s &\leq \frac{|u|_a}{s_0 - s}\int_0^1 |\sigma^A|\frac{\sigma t}{\sigma}\left(1 - \frac{\sigma t}{a(s_0 - s)}\right)^{-1/2} d\sigma \\ &= \frac{C|u|_a}{s_0 - s}\int_0^{t/a(s_0-s)} \frac{a(s_0 - s)\, d\rho}{\sqrt{1-\rho}} \\ &\leq C_0 a|u|_a.\end{aligned}$$

Step 2: Estimating *F*

We claim that there is a constant C_1 such that

$$\|F[p] - F[q]\|_s(t) \leq \frac{C_1 t}{s' - s}\|p - q\|_{s'} \tag{4.2}$$

if $s' > s$ and $\|p\|_s$ and $\|q\|_s$ are both less than R; this constraint will be ensured in Step 3 thanks to the argument of Step 1.

Proof. $F[p]$ is the product of t by a linear expression in the gradient of p, with coefficients that are Lipschitz functions of p. The bound on the s-norm ensures that all the partial derivatives of F with respect to p and $\nabla_x p$ are bounded by some constant C. Therefore, we have

$$|F[p] - F[q]| \leq Ct(|p - q| + |\nabla_x p - \nabla_x q|).$$

We now estimate the supremum of this expression as x varies so as to satisfy $\mathrm{dist}(x, \Omega) < s$. The first term is bounded by $\|p - q\|_s$, and is *a fortiori* no larger than $\|p - q\|_{s'}$. The second is estimated by Cauchy's inequality on each variable. Thus, to estimate $\partial_1(p - q)$, we write

$$p(x,t) - q(x,t) = \frac{1}{2\pi i}\int_{|z-x_1|=s'-s} \frac{(p(z, x_2, \ldots, t) - q(z, x_2, \ldots, t))dz}{z - x_1}.$$

Differentiating with respect to x_1, we obtain

$$\begin{aligned}|\partial_1(p-q)| &= \left|\frac{1}{2\pi i}\int_{|z-x_1|=s'-s}\frac{(p(z,x_2,\dots,t)-q(z,x_2,\dots,t))dz}{(z-x_1)^2}\right| \\ &\le \frac{1}{2\pi}\int_{|z-x_1|=s'-s}\frac{|p(z,x_2,\dots,t)-q(z,x_2,\dots,t)||dz|}{(s'-s)^2} \\ &\le \frac{1}{2\pi}\|p-q\|_{s'}\frac{2\pi(s'-s)}{(s'-s)^2},\end{aligned}$$

which provides the desired estimate for the second term as well. □

Step 3: *G* is a contraction

Assume that $|u|_a$ and $|v|_a$ are both less than $R/2C_0a$. We claim that

$$|G[u]-G[v]|_a \le C_2 a|u-v|_a.$$

Proof. First write

$$G[u]-G[v] = \sum_{j=1}^{n} F[w_j]-F[w_{j-1}],$$

where

$$w_j = \int_0^{j/n}\sigma^{A-1}u(\sigma t)\,d\sigma + \int_{j/n}^{1}\sigma^{A-1}v(\sigma t)\,d\sigma.$$

By Step 1, we have $\|w_j\|_s < R/2$ for $t < a(s_0-s)$. We therefore have, using Step 2 with $p = w_j$ and $q = w_{j-1}$,

$$\|G[u]-G[v]\|_s(t) \le \sum_{j=1}^{n}\frac{Ct}{s_j-s}\|w_j-w_{j-1}\|_{s_j}.$$

We now choose $s_j = s(j/n)$, where

$$s(\sigma) = \frac{1}{2}\left(s+s_0-\frac{\sigma t}{a}\right).$$

We then obtain

$$\begin{aligned}\|w_j-w_{j-1}\|_{s_j} &= \left\|\textstyle\int_{(j-1)/n}^{j/n}\sigma^{A-1}[u(\sigma t)-v(\sigma t)]\,d\sigma\right\|_{s_j} \\ &\le \textstyle\int_{(j-1)/n}^{j/n} C\|u-v\|_{s(\sigma)}(\sigma t)\,d\sigma/\sigma \\ &\le \int_{(j-1)/n}^{j/n}\frac{Ct}{s_0-s(\sigma)}\frac{|u-v|_a d\sigma}{\sqrt{1-\sigma t/a(s_0-s(\sigma))}}.\end{aligned}$$

Letting n tend to infinity, we obtain

$$\|G[u] - G[v]\|_s(t) \leq \int_0^1 C \frac{t^2|u-v|_a}{(s(\sigma)-s)(s_0 - s(\sigma))} \frac{d\sigma}{\sqrt{1 - \sigma t/a(s_0 - s(\sigma))}}.$$

Perform the change of variables $\rho = \sigma t/a(s_0 - s)$. Since

$$(s(\sigma)-s)(s_0 - s(\sigma)) = \frac{(s_0 - s)^2}{4}(1-\rho^2), \quad 1 - \frac{\sigma t}{a(s_0 - s(\sigma))} = \frac{1-\rho}{1+\rho},$$

we obtain

$$\begin{aligned} \|G[u] - G[v]\|_s(t) &\leq \frac{Cat|u-v|_a}{s_0 - s} \int_0^{t/a(s_0-s)} \frac{d\rho}{(1-\rho)^{3/2}}. \\ &\leq \frac{Cat|u-v|_a}{s_0 - s} \Big(1 - \frac{t}{a(s_0-s)}\Big)^{-1/2} \end{aligned}$$

Using the definition of the a-norm, we obtain the desired estimate. □

Step 4: End of proof

Let $u_0 = 0$ and define inductively u_n by $u_{n+1} = G[u_n]$. We show that this sequence converges in the a-norm if a is small.

Since $\|u_1\|_{s_0} \leq Rt$, $|u_1|_a < R/4C_0 a$ if a is small. We may assume $C_2 a < \frac{1}{2}$. It follows by induction that $|u_{n+1} - u_n|_a \leq 2^{-n}|u_1|_a$ and $|u_{n+1}|_a < R/2C_0 a$, hence $\|Hu_n\|_s < R/2$. Therefore, all the iterates are well defined and lie in the domain in which G is a contraction. As a result, the iteration converges, as desired. □

4.3 Generalized Fuchsian systems

Consider the problem

$$Nz + Az = f(\boldsymbol{t}, x, z, Dz), \tag{4.3}$$

where $N = N_l = \sum_{k=0}^{l}(t_k + kt_{k-1})\partial/\partial t_k$, A is constant, and f is analytic near $(0,0,0,0)$ without a constant term in $\boldsymbol{t}'$. The unknown z has m components, and $\boldsymbol{t}' = (t_0, \ldots, t_l)$. All functions are analytic in their arguments unless otherwise specified. One may, by introducing new dependent variables, assume, as we will, that f is linear in Dz. We prove the following result.

Theorem 4.5. *If A has no eigenvalue with negative real part, (4.3) has, near the origin, exactly one analytic solution that vanishes for $\boldsymbol{t} = 0$.*

Remark 4.6. Instead of requiring A to be constant, one may require that there exist a matrix-valued function $P(x)$ such that $P(x)^{-1}AP(x)$ is constant, since the latter case reduces to the former by a redefinition of u. Also, N may be replaced by a more general first-order operator, with similar proofs, but the present setup is sufficient for most applications. Since there are several "time" variables, this result is not a consequence of the previous one; however, the techniques are very similar, and the main differences are outlined at the end of this chapter. The special case ($k = 0$) of the equation $t\boldsymbol{u}_t + A\boldsymbol{u} = tf(t, x, \boldsymbol{u}, D\boldsymbol{u})$, where $\boldsymbol{u}$ and f are vector-valued, with m components, could also be handled by Theorem 4.3.

Proof. We first treat the case of two time variables ($l = 1$), and indicate the modifications in the general case afterward. We let $(t_0, t_1) = (T, Y)$. The problem therefore reads:

$$\begin{aligned}(N + A)z &= f(T, Y, x, z, \nabla z),\\ z(0, 0, x) &= z_0(x) \in \operatorname{Ker} A,\end{aligned} \tag{4.4}$$

where $\nabla = \nabla_x$ and $f \equiv 0$ for $T = Y = 0$. Replacing z by $z - z_0(x)$, we may assume, since $Az_0(x) = 0$,

$$z_0(x) = 0.$$

The solution of (4.4) therefore depends on the choice of one function $z_0(x)$ which has been incorporated into the right-hand side. We define

$$F[z] := f(T, Y, x, z, \nabla z).$$

The argument is in five steps.

Step 1

For given $k = k(T, Y)$, the problem

$$\begin{aligned}(N + A)\, z(T, Y) &= k(T, Y),\\ z(0, 0) &= 0,\end{aligned} \tag{4.5}$$

where k is analytic, independent of x, and vanishes for $T = Y = 0$, has a unique analytic solution, given by

$$z(T, Y) = H[k] := \int_0^1 \sigma^{A-1} k(\sigma T, \sigma(T \ln \sigma + Y))\, d\sigma. \tag{4.6}$$

Indeed, let $g(\sigma) = z(\sigma T, \sigma(T \ln \sigma + Y))$ for $0 < \sigma < 1$. We obtain

$$\frac{d}{d\sigma}(\sigma^A g(\sigma)) = \sigma^{A-1} k(\sigma T, \sigma(T \ln \sigma + Y)).$$

Since $g(\sigma)$ must tend to zero as σ goes to zero, while σ^A remains bounded, we have

$$\sigma^A g(\sigma) = \int_0^\sigma \tau^{A-1} k(\tau T, \tau(T \ln \tau + Y))\, d\tau.$$

Since k vanishes at the origin, the contribution from k to the integral is $O(\tau^{1-\varepsilon})$ for any $\varepsilon > 0$, and the integral converges. Equation (4.6) follows by letting $\sigma = 1$ in the last equation. One checks directly, using the Cauchy–Riemann equations, that this does provide an analytic solution to the problem. Let $u = F[z]$, and write (4.4) as the integral equation

$$u = G[u] := F[H[u]]. \tag{4.7}$$

It will be solved by a fixed-point argument. The desired solution will then be given by $z = H[u]$.

Step 2

We define two norms. Assume that f is analytic for $x \in \mathbb{C}^n$ and $d(x, \Omega) < 2s_0$, and $u \in \mathbb{C}^m$ with $|u| < 2R$, for some positive constants s_0 and R, where Ω is a bounded open neighborhood of 0. For any function $u = u(x)$ we define the s-norm

$$\|u\|_s := \sup\{|u(x)| : d(x, \Omega) < s\}. \tag{4.8}$$

For any function $u = u(T, Y, x)$, and a a sufficiently small positive number, to be chosen later, we define the a-norm

$$|u|_a := \sup_{\delta_0(T,Y)<a(s_0-s)0\leq s<s_0} \left\{ \delta_0^{-1}\|u\|_s(T,Y)(s_0-s)\sqrt{1-\frac{\delta_0}{a(s_0-s)}} \right\}, \tag{4.9}$$

where $\delta_0 = \delta_0(T, Y) := |T|+\theta|Y|$, and $0 < \theta < 1$ is fixed. We write $\|u\|_s(T, Y)$ for the s-norm of $u(., T, Y)$. We also let $\delta(\sigma) = \delta_0\sigma(1 - \theta \ln \sigma)$. The main properties of $\delta(\sigma)$ are

1. $\delta(\sigma)$ increases strictly from 0 to δ_0,
2. $\delta_0(\sigma T, \sigma(Y + T \ln \sigma)) \leq \delta(\sigma)$ if $0 < \sigma < 1$,
3. $d\delta(\sigma)/d\sigma \geq \delta(\sigma)/(C_0\sigma)$. One can take $C_0 = 1 - \theta$.

In particular, if $|u|_a < \infty$,

$$\|u\|_s(\sigma T, \sigma(T \ln \sigma + Y)) \leq \frac{\delta(\sigma)|u|_a}{s_0 - s}\left(1 - \frac{\delta(\sigma)}{a(s_0-s)}\right)^{-1/2}.$$

Step 3

We estimate the s-norm of Hu in terms of the a-norm of u. From the definitions of the various norms, it follows that

$$\|Hu\|_s(T, Y) \leq \frac{|u|_a}{s_0 - s}\int_0^1 |\sigma^A|\frac{\delta(\sigma)}{\sigma}\left\{1 - \frac{\delta(\sigma)}{a(s_0-s)}\right\}^{-1/2} d\sigma \tag{4.10}$$

if $\delta_0 < a(s_0 - s)$. We estimate σ^A by a constant C_1. Let us define

$$\rho = \frac{\delta(\sigma)}{a(s_0 - s)}$$

so that by property 3 above,

$$\frac{d\sigma}{d\rho} \leq C_0 \frac{\sigma}{\delta(\sigma)} a(s_0 - s).$$

It follows that

$$\|Hu\|_s(T, Y) \leq \frac{|u|_a}{s_0 - s} \int_0^1 C_0 C_1 \frac{a(s_0 - s)\, d\rho}{\sqrt{1 - \rho}} = 2C_0 C_1 a |u|_a,$$

or

$$\|Hu\|_s(T, Y) \leq C_2 a |u|_a. \tag{4.11}$$

Step 4

Since f is linear in the spatial derivatives, Cauchy's inequality gives

$$\|F[u] - F[v]\|_{s'}(T, Y) \leq \frac{C_3 \delta_0(T, Y)}{s - s'} \|u - v\|_s \tag{4.12}$$

for $0 < s' < s < s_0$, if $\|u\|_s \leq R$ and $\|v\|_s \leq R$.

Step 5

Assume now $|u|_a$, $|v|_a < R/(2C_2 a)$. Let $G[u] = F[Hu]$. We prove

$$|G[u] - G[v]|_a \leq C_4 a |u - v|_a \tag{4.13}$$

for some constant C_4. To this end, let $\sigma_j = j/n$, for $0 \leq j \leq n$, and

$$w_j = \int_0^{\sigma_j} \sigma^{A-1} u(\sigma T, \sigma(T \ln \sigma + Y))\, d\sigma - \int_{\sigma_j}^1 \sigma^{A-1} v(\sigma T, \sigma(T \ln \sigma + Y))\, d\sigma,$$

and observe that

$$G[u] - G[v] = \sum_{j=1}^n F[w_j] - F[w_{j-1}]. \tag{4.14}$$

Using the argument of Step 3, we obtain $\|w_j\|_s \leq R$ for $\delta_0(T, Y) < a(s_0 - s)$, so that $F[w_j]$ is indeed defined. If $s_j \in (s, s_0 - \delta_0(T, Y)/a)$ for every j, we find from Step 4 that

$$\|F[w_j] - F[w_{j-1}]\|_s \leq \frac{C_3 \delta_0}{s_j - s} \|w_j - w_{j-1}\|_{s_j}.$$

On the other hand,

$$\|w_j - w_{j-1}\|_{s_j} \le \int_{\sigma_{j-1}}^{\sigma_j} |\sigma^{A-1}| \, \|u - v\|_{s_j}(\sigma T, \sigma(T \ln \sigma + Y)) \, d\sigma.$$

This suggests the choice $s_j = \min\{s(\sigma) : \sigma_{j-1} \le \sigma \le \sigma_j\}$, where

$$s(\sigma) = \frac{1}{2}\left[s + s_0 - \frac{\delta(\sigma)}{a}\right].$$

Observe next that $\sum_j s_j \chi_{[\sigma_{j-1},\sigma_j)} \to s(\sigma)$ as j tends to infinity, uniformly and *from below* on $(0,1)$, if $\delta_0(T,Y) < a(s_0 - s)$. Furthermore, if $\sigma_{j-1} \le \sigma \le \sigma_j$,

$$\begin{aligned}\|u - v\|_{s_j}(\sigma T, \sigma(T \ln \sigma + Y)) &\le \|u - v\|_{s(\sigma)}(\sigma T, \sigma(T \ln \sigma + Y)) \\ &\le \frac{\delta(\sigma)|u - v|_a}{s_0 - s(\sigma)} \left(1 - \frac{\delta(\sigma)}{a(s_0 - s(\sigma))}\right)^{-1/2}.\end{aligned}$$

We therefore obtain, since $|\sigma^{A-1}| \le C_1/\sigma$, letting $j \to \infty$,

$$\begin{aligned}&\|G[u] - G[v]\|_s(T,Y) \\ &\quad\le C_3\delta_0 \int_0^1 \frac{\delta(\sigma)|u - v|_a}{(s(\sigma) - s)(s_0 - s(\sigma))} \left(1 - \frac{\delta(\sigma)}{a(s_0 - s(\sigma))}\right)^{-1/2} C_1 \frac{d\sigma}{\sigma}.\end{aligned}$$

Since

$$s(\sigma) - s = \frac{s_0 - s}{2}\left(1 - \frac{\delta(\sigma)}{a(s_0 - s)}\right)$$

and

$$s_0 - s(\sigma) = \frac{s_0 - s}{2}\left(1 + \frac{\delta(\sigma)}{a(s_0 - s)}\right),$$

we again let $\rho = \delta(\sigma)/[a(s_0 - s)]$. As σ varies from 0 to 1, ρ varies from 0 to $\delta_0/[a(s_0 - s)]$ (which is always less than 1); also,

$$1 - \frac{\delta(\sigma)}{a(s_0 - s(\sigma))} = \frac{1 - \rho}{1 + \rho}.$$

Performing this change of variables, we obtain, using $d\sigma/d\rho \le C_0\sigma/\rho$,

$$\begin{aligned}&\|G[u] - G[v]\|_s(T,Y) \\ &\le C_1C_3\delta_0 \int_0^{\delta_0/[a(s_0-s)]} C_0\delta(\sigma)\frac{4|u - v|_a}{(s_0 - s)^2}(1 - \rho^2)^{-1}\sqrt{\frac{1+\rho}{1-\rho}}\,\frac{a(s_0 - s)\,d\rho}{\delta(\sigma)} \\ &= 4a(s_0 - s)^{-1}C_0C_1C_3\delta_0|u - v|_a \int_0^{\delta_0/[a(s_0-s)]} d\rho/(1-\rho)^{3/2} \\ &= C_4\delta_0 a(s_0 - s)^{-1}|u - v|_a \left(1 - \frac{\delta_0}{a(s_0 - s)}\right)^{-1/2}.\end{aligned}$$

Therefore

$$|G[u] - G[v]|_a \le C_4 a|u - v|_a.$$

End of proof ($l = 1$)

Let us define $u_0 = 0$ and $u_1 = G[u_0]$. There is a constant R_0 such that

$$\|u_1\|_{s_0} \le R_0 \delta_0(T, Y)$$

if $|T| + \theta|Y| = \delta_0$. Assume that a is chosen so small that

$$C_4 a < 1/2 \quad \text{and} \quad R_0 s_0 < R/(4C_2 a).$$

In particular, $|u_1|_a \le R_0 s_0 < R/(4C_2 a)$. The mapping G is a contraction in the a-norm on the set $\{|u|_a \le R/(2C_2 a)\}$. The existence and uniqueness of the desired solution now follow from the contraction mapping principle. This completes the proof of the theorem in this case.

End of proof (general case)

The proof parallels the one given for (4.4), and we therefore only indicate the differences. Write

$$N = \sum_j m_{ij} t_j \partial_i$$

for suitable coefficients m_{ij} forming a matrix M, with only one eigenvalue. One must then replace $H[k]$ by

$$\int_0^1 \sigma^{A-1} k(\sigma^M \boldsymbol{t})\, d\sigma,$$

and use $\delta_0(t_0, \dots, t_l) = \sum_{k=0}^l \theta^k |t_k|$, where $\theta \in (0, 1/l)$ is fixed. We then use for $\delta(\sigma)$ the quantity $\delta_0 \sigma (1 - \theta \ln \sigma)^l$.

We need to check the three properties of δ in Step 2. The first follows from the assumption $0 < \theta < 1/l$. The second is checked as follows: Since $\boldsymbol{t}(\sigma) := \sigma^M \boldsymbol{t}$ solves $\sigma\, d\boldsymbol{t}/d\sigma = M\boldsymbol{t}$, $\boldsymbol{t}(1) = \boldsymbol{t}$, we obtain, using the expression of M,

$$(\sigma^M \boldsymbol{t})_k = \sum_{j=0}^k t_j \binom{k}{j} \sigma (\ln \sigma)^{k-j}.$$

We then compute

$$\begin{aligned}
\delta_0(\sigma^M \boldsymbol{t}) = \sum_k \theta^k |t_k(\sigma)| &\le \sigma \sum_{0 \le j \le k \le l} t_j \theta^k \binom{k}{j} |\ln \sigma|^{k-j} \\
&= \sigma \sum_{0 \le j \le k \le l} \theta^j |t_j| \binom{k}{j} (-\theta \ln \sigma)^{k-j}
\end{aligned}$$

$$
\begin{aligned}
&= \sigma \sum_{0\le j\le k\le l} (-\theta \ln \sigma)^j \binom{k}{j} \theta^{k-j} |t_{k-j}| \\
&\le \sigma \sum_{j=0}^{l} (-\theta \ln \sigma)^j \binom{l}{j} \sum_{k\ge j} \theta^{k-j} |t_{k-j}| \\
&\le \sigma \delta_0(\boldsymbol{t}) \sum_{j=0}^{l} (-\theta \ln \sigma)^j \binom{l}{j} \\
&= \sigma \delta_0(\boldsymbol{t})(1-\theta \ln \sigma)^l,
\end{aligned}
$$

which is the desired result. The third property follows from

$$\frac{\delta'(\sigma)}{\delta(\sigma)} \ge \frac{1-l\theta}{\sigma}.$$

The rest of the proof proceeds verbatim. We obtain existence on domains of the form

$$\{\delta(\boldsymbol{t}) < a(s_0 - s); \quad d(x, \Omega) < s\}.$$

This completes the proof of the existence result.

We include a closely related result that could be used to prove an existence result for generalized Fuchsian systems in which the matrix M is still more general; see Sect. 5.3.

Lemma 4.7. *For any matrix M with positive eigenvalues, there is a function $\delta(t)$ such that*

1. *δ is continuous for all $\boldsymbol{t}$, and C^∞ for $t \neq 0$,*
2. *$\delta(\sigma^M \boldsymbol{t})$ increases from 0 to $\delta(\boldsymbol{t})$ as σ increases from 0 to 1.*
3. *$\delta(\sigma^M \boldsymbol{t}) \le \delta(t)\sigma^m(1-\theta \ln \sigma)^q$ for some positive m and q, and some $\theta \in (0,1)$.*

Proof. Assume that a linear change of variables has been performed to put M in block diagonal form, with blocks of the form

$$\begin{pmatrix} \lambda_r & & & & \\ 1 & \lambda_r & & & \\ & 2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & l_r & \lambda_r \end{pmatrix},$$

with $\lambda_r > 0$. The integer l_r may be zero, in the case of a 1×1 block. (It is convenient not to take the off-diagonal elements equal to 1.) Let us label $(t_{r0}, t_{r1}, \dots, t_{rl_r})$ the components of $\boldsymbol{t}$ corresponding to the rth block. Define

$$\delta(t) = \sum_r \left(\sum_{k=0}^{l_r} \theta^k |t_{rk}| \right).$$

We show that one can choose $\theta > 0$ so that $\delta(\boldsymbol{t})$ has the required properties. First, by solving the equation $\sigma \partial \boldsymbol{t} / \partial \sigma = M\boldsymbol{t}$, one shows that

$$(\sigma^M \boldsymbol{t})_{rk} = \sum_{j=0}^{k} t_{rj} \binom{k}{j} \sigma^{\lambda_r} (\ln \sigma)^{k-j}$$

for $k \leq l_r$. We now compute, for $\sigma \in (0, 1)$,

$$\begin{aligned}
\delta(\sigma^M \boldsymbol{t}) &= \sum_{k,r} \theta^k |(\sigma^M \boldsymbol{t})_{rk}| \\
&\leq \sum_r \sum_{0 \leq j \leq k \leq l_r} \sigma^{\lambda_r} |t_{rj}| \theta^k \binom{k}{j} (\ln \sigma)^{k-j} \\
&\leq \sum_{0 \leq j \leq k} \sigma^{\lambda_r} \binom{k}{j} [-\theta \ln \sigma]^{k-j} \theta^j |t_{rj}| \\
&= \sum_{0 \leq j \leq k} \sigma^{\lambda_r} \binom{k}{j} [-\theta \ln \sigma]^{j} \theta^{k-j} |t_{r,k-j}| \\
&\leq \sum_r \sum_{j=0}^{l_r} \binom{l_r}{j} [-\theta \ln \sigma]^j \sum_{k \geq j} \theta^{k-j} |t_{r,k-j}| \sigma^{\lambda_r} \\
&\leq \sum_r \sigma^{\lambda_r} \delta(\boldsymbol{t}) (1 - \theta \ln \sigma)^{l_r}. \qquad (4.15)
\end{aligned}$$

(We used the inequality $\binom{k}{j} \leq \binom{l}{j}$ if $k \leq l$.) If we choose m such that $0 < m \leq \lambda_r$ for all r, and estimate l_r by, say, q, we find that

$$\delta(\sigma^M \boldsymbol{t}) \leq \delta(\boldsymbol{t}) \sigma^m (1 - \theta \ln \sigma)^q.$$

We finally choose $\theta \in (0, m/q)$, to ensure that $\sigma^m (1 - \theta \ln \sigma)^q$ is increasing for $\sigma \in (0, 1)$. All the desired properties follow.

4.4 Notes

The classical Fuchs–Frobenius theory is treated in most treatises on ODEs in the complex domain, such as [50]. An abstract convergence result for linear ODEs may be found in [99]. For classical applications, see for instance [23]. For ODEs with non analytic coefficients, the classical equivalence "Fuchsian $\equiv$ all formal solutions converge" is either false, or meaningless; indeed, the coefficients might themselves be given by divergent series. The results for PDEs have a long history. It seems that Nagumo [146] first realized in 1942 that it was possible to drop the analyticity condition in time in the Cauchy–Kovalevskaya theorem: continuity in time and analyticity in space suffice. It was also realized gradually that the Cauchy–Kovalevskaya theorem could be proved by an iteration procedure, at first in scales of Banach spaces; the most famous contributions in this direction are due to Ovsjannikov [154], and to Nirenberg [148], who stressed the parallel with the Nash–Moser inverse function theorem. In fact, it is possible to prove the result using the contraction mapping principle, but as the reader can see, the choice of norm is non trivial, motivated as it is by the proofs based on iteration in scales of Banach spaces. The modern interest in Fuchsian PDEs seems to go back to [77, 11], although the stage had been prepared to some extent by work on the Euler–Poisson–Darboux equation; see [135, 184]. The techniques of the convergence proofs are essentially modifications of those from [11]; see also [178, 120, 122, 161]. Generalized Fuchsian problems are necessary to deal with in order to take logarithmic terms into account [120]. As far as reduction is concerned, the iteration used in the proof of the existence theorem generates automatically the expansion of the solution: it is not necessary to know beforehand the form of this expansion. On the contrary, it is generated automatically; this is important for applications in relativity, where the form of the expansion is difficult to determine beforehand. The results on generalized Fuchsian systems prove that if the expansion contains powers and logarithms alone, there is a unique expansion that converges; it has been uniformized by the introduction of the higher "time variables." These results are taken from [120, 104, 124], with minor extensions suggested by [103].

Problems

4.1. Recover the Cauchy–Kovalevskaya theorem for equations linear in the derivatives from the results of this chapter. Reduce the general "fully nonlinear" equation to this case, by adjoining to the equation all its derivatives, and adding to the list of unknowns all the first-order derivatives of the unknown.

4.2. What type of equation would lead to an initial-value problem involving the operator $N\sum_j m_{ij}t_j\partial_i$ with a matrix (m_{ij}) of the type considered in Sect. 4.3?

4.3. This problem shows how to recover asymptotics of solutions of linear equations with irregular, or non-Fuchsian, singularities by Fuchsian reduction. Question (a) shows how to simplify such an ODE by a change of variables that itself solves a nonlinear Fuchsian equation; (b) shows how to construct solutions of the simplified equation, and (c–d) give examples. Results are taken from [91].

(a) Consider

$$a(y) = a_{-1}y^{-1} + a_0 + a_\varepsilon y^\varepsilon + y^{\varepsilon+\nu}\alpha(y),$$
$$b(y) = b_{-2}y^{-2} + b_{-1}y^{-1} + b_{\varepsilon-1}y^{\varepsilon-1} + b_0 + y^\sigma\beta(y),$$

with $a_{-1},\ a_0,\ a_\varepsilon,\ b_{-2},\ b_{-1},\ b_0,\ b_{\varepsilon-1}$ complex constants, $b_{-2} \neq 0$, $\varepsilon > 0$, $\sigma > 0$, $\varepsilon + \nu > 1$, and where $\alpha(y)$ and $\beta(y)$ are continuous for $y \geq 0$ and small. Find functions $\psi_1(y)$, $\psi_2(s)$, and a change of variables $y = \theta(s)$, with $\theta(s) = s + as^2 \ln s + 0(s^{2+\mu})$, such that the function

$$f(s) = v(y)\exp(-\psi_1(y) - \psi_2(s))$$

satisfies

$$D_s^2 f(s) + [2\gamma s^{-1} + 1 + s^\mu\alpha_1(s)]D_s f(s) + s^{\mu-1}\alpha_0(s)f(s) = 0, \tag{4.16}$$

for some positive $\mu < \min(1, \varepsilon, \varepsilon + \nu - 1, \sigma)$, with $D_s = s\,d/ds$, and α_0, α_1 continuous for $s \geq 0$ and small.

(b) Using the results of Sect. 5.1, show that equation (4.16) has precisely one solution such that $f(s) = 1 + o(1)$ as $s \to 0$.

(c) Show that the modified Mathieu equation

$$-u''(x) + c^2\sinh^2(x)u(x) = \lambda u(x),$$

with real parameters c, λ, where $c > 0$, has a solution characterized by the asymptotic behavior

$$u(x) = (\sinh(x))^{-1/2}e^{-c/\sinh(x)}[1 + o(1)]$$

for the eigenfunction with fastest decay.

(d) Show that

$$\Delta u + \left(C|x|^{-1} + D|x|^{-1-\varepsilon}\right)u = \lambda u$$

has a unique radial solution such that if $\lambda = -k^2 \neq 0$ (hence $\gamma = -ik$),

$$u(x) \sim |x|^{(-n+1)/2}\exp(i(k|x| + C(2k)^{-1}\ln|x|)), \quad |x| \to \infty.$$

4.4. Give an example of a non-Fuchsian system with a regular singular point.

5

Fuchsian Initial-Value Problems in Sobolev Spaces

This chapter is devoted to the main techniques for solving the initial-value problem for Fuchsian ODEs and PDEs without analyticity assumptions. The results are adapted to application to Fuchsian equations obtained by the second reduction, see Sect. 5.5, in which the matrix A has no eigenvalue with negative real part. Recall that Sect. 2.2.4 gave general methods for reducing the situation to this case.

The simplest result, which parallels the Cauchy–Lipschitz theorem, is easy to state and to prove. In any Banach space X, the problem

$$tu_t + Au = t^\varepsilon F(t, u),$$

where F is locally Lipschitz in u and continuous in both variables, and $\varepsilon > 0$, has precisely one solution defined for small t that tends to zero as $t \to 0+$, as soon as the operator norm of σ^A is bounded for $\sigma \in (0, 1]$. To prove it, one casts the equation into the integral equation

$$u(t) = t^\varepsilon \int_0^1 \sigma^{A-1+\varepsilon} F[t\sigma, u(t\sigma)] d\sigma,$$

and one applies the Banach fixed point theorem in the space $C([0, \tau]; X)$ of continuous X-valued functions defined for $0 \le t \le \tau$, with τ sufficiently small. The corresponding result for generalized Fuchsian ODEs is given in Theorem 5.7. We begin this chapter with a more precise theorem, Theorem 5.1, which will be needed in Chap. 7; it allows the right-hand side to behave like an inverse power of $\ln t$ rather than t^ε. We then turn to the case of PDEs, in which we allow several Fuchsian variables, to cope with the type of equations that may arise in applications. We first assume that the Fuchs indices are constant, then indicate the simple modifications for the case in which the indices depend on the space variables. The development is modeled on the theory of symmetric-hyperbolic systems [116, 176], which it contains as a special case.

5.1 Singular systems of ODEs in weighted spaces

5.1.1 Setup and assumptions

We are interested in problems of the form

$$\begin{aligned} t\tfrac{dV}{dt}(t) + AV(t) &= f(\lambda, t, V(t)), \\ V(t) &= \mathcal{O}(\zeta(t)) \text{ as } t \to 0+, \end{aligned} \tag{5.1}$$

for $0 < t < T$, where ζ is a given nonnegative weight function, and V has p real components. The nonlinearity f depends smoothly on a (vector) parameter λ. All estimates and assumptions below are assumed to hold uniformly for λ in a fixed ball $\{|\lambda| \le \lambda_0\}$. Denote by $\|A\|$ the matrix norm of A. We let

$$E_{t,\zeta} = \{V \in C([0,T]; \mathbb{R}^p) : V/\zeta \text{ is bounded on } (0,\tau)\}$$

and

$$|V|_{\tau,\zeta} := \sup_{t\in(0,\tau)} |V(t)/\zeta(t)|.$$

If no confusion is possible, we write $|V|_\zeta$ for $|V|_{\tau,\zeta}$. We assume $\tau < 1$ since the only difficulty arises from the degeneracy for $t = 0$. In all assumptions, m is a fixed constant.

(H1) A is a constant $p \times p$ matrix, $0 < \zeta \le 1$, and

$$\zeta(t) \to 0 \text{ and } \zeta(t)\|t^A\| \to 0 \text{ as } t \to 0+\,.$$

(H2) The equation

$$t\frac{dV}{dt}(t) + AV(t) = f(\lambda, t, 0) \tag{5.2}$$

has a unique solution in $E_{\tau,\zeta}$, with norm $|V|_{\tau,\zeta} \le m$.

(H3) f is continuous in (λ, t, V) for $|\lambda| \le \lambda_0$, $0 \le t \le \tau$, and $|V| \le 2m$.

(H4) For $|\lambda| \le \lambda_0$, $|V| \le 2m$, $|W| \le 2m$, and $0 < t < \tau$,

$$|f(\lambda, t, V) - f(\lambda, t, W)| \le \psi(t)|V - W|, \tag{5.3}$$

where

$$K(t) := \frac{1}{\zeta(t)} \int_0^1 \|\sigma^A\|\zeta(\sigma t)\psi(\sigma t)\frac{d\sigma}{\sigma}$$

is bounded on $[0, \tau]$ and tends to zero as $t \to 0+$.

(H5) f is continuously differentiable with respect to λ and V. In addition, $|f_\lambda|$ is $\mathcal{O}(\psi_0(t))$, where

$$\frac{1}{\zeta(t)} \int_0^1 \|\sigma^A\|\psi_0(\sigma t)\frac{d\sigma}{\sigma} \le m. \tag{5.4}$$

Furthermore, if $\lambda_n \to \lambda$ and $V_n \to V$ in the $E_{\tau,\zeta}$ norm, with $|V_n|_{\tau,\zeta} \le 2m$, then the sequence of functions $f_\lambda(\lambda_n, t, V_n(t))$ tends to $f_\lambda(\lambda, t, V(t))$ in E_{τ,ψ_0}. The same holds for the partial derivatives f_V of f with respect to the components of V.

Examples of weight functions satisfying these conditions are given in Sect. 5.1.3. Given (H1) and (H3), hypothesis (H2) is equivalent to assuming that

$$t^{-A}\int_0^t s^{A-1}f(\lambda,s,0)\,ds$$

belongs to $E_{\tau,\zeta}$ and is bounded by $m\zeta(t)$ on $[0,\tau]$. Indeed, (5.2) is equivalent to

$$(t^A V)_t = t^{A-1}f(\lambda,t,0),$$

and (H1) enables one to integrate the equation from 0 to τ for $V \in E_{\tau,\zeta}$, since it implies $t^A V(t) \to 0$ as $t \to 0$. Furthermore, if $|f| \le \psi_0(t)$ with ψ_0 as in (H5), we are led to assumption (H2).

5.1.2 General results

Theorem 5.1. *Under assumptions (H1)–(H4), problem (5.1) has a unique solution in $E_{\tau,\zeta}$ for τ small enough.*

Proof. Any solution of (5.1) with $|V|_{\tau,\zeta} \le 2m$ satisfies $|V| \le 2m$ pointwise and

$$(t^A V)_t = t^{A-1}f(\lambda,t,V).$$

For $V \in E_{\tau,\zeta}$, (H1) implies

$$V = G(V,\lambda) \tag{5.5}$$

with

$$G(V,\lambda) = t^{-A}\int_0^t s^{A-1}f(\lambda,s,V(s))\,ds. \tag{5.6}$$

We prove that G is a contraction on the ball of radius $2m$ in $E_{\tau,\zeta}$. (H2) implies

$$|G(0,\lambda)|_{\tau,\zeta} \le m$$

and (H4) that for any V, W,

$$|G(V,\lambda)-G(W,\lambda)|_{\tau,\zeta} \le \sup_{[0,\tau]} K(t)\;|V-W|_{\tau,\zeta}$$

if $|V| \le 2m$ and $|W| \le 2m$ on $[0,\tau]$. It follows that G is well defined and satisfies

$$|G(V,\lambda)|_{\tau,\zeta} \le m + \sup_{[0,\tau]} K(t)\;|V|_{\tau,\zeta}.$$

Let us choose τ so small that $\sup_{[0,\tau]} K(t) \le 1/2$. If $|V|_{\tau,\zeta} \le 2m$, then

$$\sup_{[0,\tau]}|V(t)| \le |V|_{\tau,\zeta}\sup_{[0,\tau]}\zeta(t) \le 2m$$

and

$$|G(V,\lambda)|_{\tau,\zeta} \le 2m.$$

Furthermore, if $|V|_{\tau,\zeta} \le 2m$ and $|W|_{\tau,\zeta} \le 2m$, then

$$|G(V,\lambda) - G(W,\lambda)|_{\tau,\zeta} \le \frac{1}{2}|V - W|_{\tau,\zeta}.$$

Theorem 5.1 follows. □

If assumption (H5) holds, we have the following Theorem.

Theorem 5.2. *Under assumptions (H1)–(H5), and for τ small enough, (5.1) has a unique solution $V(t,\lambda) \in E_{\tau,\zeta}$. Furthermore, for fixed $t \in (0,\tau)$, $V(t,\lambda)$ is continuously differentiable with respect to λ, for small $|\lambda|$, and its differential may be computed by differentiating the equation.*

Proof. The assumptions ensure that the map G is well defined. (H4) implies $|f_V| \le \psi(t)$, so that by dominated convergence, one finds that G is Gâteaux differentiable in (λ, V), and its differentials, with respect to V or λ, are given by

$$G_V(\lambda,t,V)\cdot W = t^{-A}\int_0^t s^{A-1} f_V(\lambda,s,V(s))\cdot W(s)\,ds$$

and

$$G_\lambda(\lambda,t,V) = t^{-A}\int_0^t s^{A-1} f_\lambda(\lambda,s,V(s))\,ds.$$

The continuity assumptions on f_λ and f_V ensure that the differential of $G(\lambda,V)$ is continuous for the topology of $\mathbb{R}\times E_{\tau,\zeta}$. Therefore $(\lambda,V) \mapsto V - G(\lambda,V)$ satisfies the assumptions of the inverse function theorem for small τ. The result follows. □

5.1.3 Simple special cases

We now turn to two simple cases in which (H1)–(H4) hold if f is continuous in its arguments.

Theorem 5.3. *Assume that the eigenvalues of A have nonnegative real parts, and that the eigenvalues with zero real part have Jordan blocks of maximal size M. Assume further that (5.3) holds with*

$$|f(\lambda,t,0)| \le C_1|\ln t|^{-M}\zeta(t),$$
$$0 \le \zeta(t) \sim C_2|\ln t|^{-\alpha},$$
$$0 \le \psi(t) \sim C_3|\ln t|^{-a},$$

where $a > M$, $\alpha > M-1$, $\alpha > 0$, and C_1, C_2, and C_3 are positive. Then (H1), (H2), and (H4) hold. They also hold if $|t^{-\mu}f(\lambda,t,0)|$ is bounded and ζ and ψ are equivalent to C_4t^μ with $\mu > 0$.

The proof is given in Problem 5.2. It shows that we may take ζ equal to $|\ln t|^{-\alpha}$ $(\ln|\ln t|)^{-\beta}$ and $\psi = |\ln t|^{-a}(\ln|\ln t|)^{-b}$ for small t, with β, b nonnegative.

Remark 5.4. The assumption on A implies that $\|\sigma^A\| = 0(|\ln\sigma|^{M-1})$.

A simple corollary is the following:

Theorem 5.5. *Take A as in Theorem 5.3. If $f = f(\lambda, t, V, 1/\ln t)$ is a smooth function of its four arguments near $(0,0,0,0)$ such that f, f_λ, and f_V are all $\mathcal{O}(|\ln t|^{-a})$ with $a > 2M$, then (5.1) has a unique solution near $\lambda = 0$ for τ small, which is $\mathcal{O}(|\ln t|^{-a/2})$ and continuously differentiable with respect to λ.*

Proof. The assumptions of Theorem 5.2 are satisfied with $\zeta = |\ln t|^{-a/2}$, $\psi(t) = \psi_0(t) = |\ln t|^{-a}$. The results follow. □

Remark 5.6. The optimality of Theorem 5.3, hence of Theorems 5.1 and 5.2, is shown by the following counter examples. First, take $a = M = 1$ and $\alpha > M - 1 (= 0)$. For $0 < \alpha < 1$, and $\tau < 1$,

$$t\frac{dv}{dt} = \frac{v(t)}{|\ln t|} + \frac{\alpha - 1}{|\ln t|^{\alpha+1}}$$

has infinitely many solutions, all of which belong to $E_{\tau,\zeta}$:

$$v(t) = \frac{c}{|\ln t|} + \frac{1}{|\ln t|^\alpha},$$

where c is arbitrary. Second, take $a > M = 1$ and $\alpha = M - 1 = 0$. Then

$$t\frac{dv}{dt} = \frac{v(t)}{|\ln t|^2} - \frac{\exp(-1/\ln t)}{|\ln t|}$$

has the general solution

$$v(t) = (\ln|\ln t| + c)\exp(-1/\ln t),$$

where c is arbitrary. None of these solutions is $\mathcal{O}(1)$.

5.2 A generalized Fuchsian ODE

We now turn to a simple existence theorem for generalized Fuchsian ODEs with $\ell + 1$ time variables $t_0, \ldots, t_\ell$; we use the notation of Sect. 4.3. Let E be a Banach space. Let f be a Lipschitz map from $\mathbb{R}^{\ell+1} \times E$ to E, with Lipschitz constant L. We seek solutions of

$$(N + A)u(\boldsymbol{t}) = \boldsymbol{t} \cdot f(\boldsymbol{t}, u) \tag{5.7}$$

that remain bounded as $t \to 0$.

Theorem 5.7. *Let A be a bounded operator and f as above. Equation (5.7) has a unique continuous solution, defined for small $\boldsymbol{t}$, that vanishes at the origin. If f is infinitely differentiable, so is u.*

Variants are suggested in the problems.

Proof. The problem is to solve the integral equation

$$u = T(u),$$

where

$$T[u] := \int_0^1 \sigma^A \sigma^M \boldsymbol{t} \cdot f\left(\sigma^M \boldsymbol{t}, u(\sigma^M \boldsymbol{t})\right) \frac{d\sigma}{\sigma}.$$

We let $c = \int_0^1 |\sigma^A| \sigma^{m-1}(1 - \theta \ln \sigma)^q \, d\sigma$. Then T is a contraction on $C(\delta(\boldsymbol{t}) \leq \delta_0; E)$, provided that $\delta_0 < 1/(Lc)$. The existence of continuous solutions with values in E follows immediately. For higher $\boldsymbol{t}$-derivatives, one defines a sequence $\{u_k\}$ by $u_0 = 0$, $u_{k+1} = T[u_k]$. The differential T'_u of T with respect to u at $u = u_k$ is a contraction on $C(\delta(\boldsymbol{t}) \leq \delta_0; E)$, so that the sequence of first-order derivatives $\{\nabla_t u_k\}$ satisfies a recurrence relation

$$\nabla_{\boldsymbol{t}} u_{k+1} = T'_u \nabla_{\boldsymbol{t}} u_k + \int_0^1 \sigma^A [\nabla_{\boldsymbol{t}}(\sigma^M \boldsymbol{t}) \cdot f(\sigma^M \boldsymbol{t}, u_k(\sigma^M \boldsymbol{t})) + \sigma^M \boldsymbol{t} \cdot \partial f/\partial \boldsymbol{t}] \frac{d\sigma}{\sigma},$$

which has the form

$$\nabla_{\boldsymbol{t}} u_{k+1} = T'_u \nabla_{\boldsymbol{t}} u_k + \varphi[u_k].$$

Since we know that $\{u_k\}$ converges at an exponential rate in $C(\delta(\boldsymbol{t}) \leq \delta_0; E)$, we have, all norms being taken in this space,

$$\|\nabla_{\boldsymbol{t}} u_{k+1} - \nabla_{\boldsymbol{t}} u_k\| \leq \varepsilon \|\nabla_{\boldsymbol{t}} u_k - \nabla_{\boldsymbol{t}} u_{k-1}\| + C\alpha^k,$$

for some ε and α in $(0, 1)$. It follows by induction that one can choose a and κ such that $\|\nabla_{\boldsymbol{t}} u_k - \nabla_{\boldsymbol{t}} u_{k-1}\| \leq a(\kappa\alpha)^k$, while $0 < \kappa\alpha < 1$. The existence of $\boldsymbol{t}$-derivatives and their continuity follows. For higher derivatives, an iteration of the same argument proves their existence and continuity. □

5.3 Fuchsian PDEs: abstract results

We solve Fuchsian hyperbolic systems of the form

$$Q(N + A)\boldsymbol{u} = \sum_{k=0}^{\ell} t_k (B_k + f_k(t, \boldsymbol{u})) := \boldsymbol{t} \cdot (\mathcal{B}\boldsymbol{u} + f), \tag{5.8}$$

where $\boldsymbol{t} = (t_0, \ldots, t_\ell)$, u is vector-valued, the $B_k = \sum_{j=1}^n A_{jk} \partial_j$ are first-order differential operators, and $N = \sum_{i,j=0}^{\ell} m_{ij} t_j \partial/\partial t_i$, $M = (m_{ij})$ being a matrix with real, positive eigenvalues. For $\ell = 0$, we recover

$$Q(t\partial_t + A)\boldsymbol{u} = t(B + f(t, \boldsymbol{u})), \tag{5.9}$$

with one time variable. Such problems arise from nonlinear wave equations by second reduction.

Assumptions

The assumptions on the coefficients and the nonlinearity spelled out below follow closely the needs of the blowup problem, and are therefore not claimed to be optimal; see Sect. 5.4. We first deal with the case of constant A, and then give the modifications of the proofs for the case of variable A. We seek solutions for which u is, for $\boldsymbol{t} = 0$, a prescribed element of the kernel of A. By redefining $\boldsymbol{u}$, we may, and will, assume that $\boldsymbol{u}(0) = 0$. We denote by (u, v) both the Euclidean scalar product on $\mathbb{R}^{n+2}$ and the associated L^2 scalar product.

Our assumptions on Q, A, M, $\mathcal{B}$, and f are as follows. For constant A we require the following conditions:

(A1) Multiplication by Q, Q^{-1}, and A_{jk} are bounded operators in H^s; all the eigenvalues of A have nonnegative real parts.
(A2) The function f is C^∞ in u and defines a map from $\mathbb{R}^{l+1} \times H^s$ to H^s; furthermore, $f \equiv 0$ if $\|\boldsymbol{u}\|_{L^\infty}$ or $|\boldsymbol{t}|$ is large enough.
(A3) There is a positive-definite matrix-valued function V, which commutes with Q and A_{jk}, and such that $(\boldsymbol{u}, VQA\boldsymbol{u}) \geq 0$, and $(\boldsymbol{u}, VQ\boldsymbol{u})$ is equivalent to the L^2 norm. In addition, $VA_{jk} = A_{jk}$, and multiplication by V is a bounded operator in H^s.
(A4) The eigenvalues of M are real and positive, and $M + M^T$ is positive definite (M^T being the transpose of M).

Remark 5.8. Condition (A2) serves to establish that the H^s norm of f grows at most linearly with the H^s norm of $\boldsymbol{u}$, given an L^∞ bound on $\boldsymbol{u}$ ("Moser-type estimate" [144, 143]). In practice, (A2) can be satisfied by truncating nonlinearities for large arguments if $s > n/2 + 1$. Since we are interested in solutions that vanish initially, this truncation is reasonable.

Remark 5.9. The introduction of V should not be confused with the change of scalar product commonly encountered in the theory of symmetric systems [116]: it is due here to the fact that (Au, u) may change sign, even if all the eigenvalues of A are nonnegative. The second part of (A4) is used to prove the estimates on the time derivatives of the solution. If M is lower triangular with positive eigenvalues, we may ensure that $M + M^T$ is positive definite by replacing t_k by $\varepsilon^{-k} t_k$, which replaces m_{jk} by $\varepsilon^{j-k} m_{jk}$: taking ε small, we may arrange so that the off-diagonal elements become arbitrarily small, while the diagonal elements remain the same. Since the latter are positive, we see that $M + M^T$ can be assumed to be positive definite by redefining $\boldsymbol{t}$ if need be.

Remark 5.10. It is important that the equation should contain $Q(N + A)$ rather than $QN + A = Q(N + Q^{-1}A)$: indeed, one can find Q diagonal and positive definite, and A with eigenvalues in the right half-plane, for which $Q^{-1}A$ has some eigenvalues with *negative* real parts. In such a case, there may be nontrivial solutions of $(QN + A)\boldsymbol{u} = 0$ with zero initial data, violating uniqueness.

If A is variable, we replace assumption (A3) by

(A3′) Let $A_S = A - [A, S]S^{-1}$, where $S = (1 - \Delta)^{s/2}$. There is a positive-definite matrix-valued function V that commutes with Q and A_{jk} such that $(\boldsymbol{u}, VQA_S\boldsymbol{u}) \geq 0$ and (u, VQu) is equivalent to the L^2 norm. In addition, $VA_{jk} = A_{jk}$, and multiplication by V is a bounded operator in H^s.

Remark 5.11. In practice, one may perform a further reduction to satisfy (A3′): Sect. 2.2.4 shows that as soon as we have a formal solution $\tilde{\boldsymbol{u}}$ of (5.9) of order $m + 1$, say, the change of unknown $\boldsymbol{u} = \tilde{\boldsymbol{u}} + t^m v(x, t)$ leads to a Fuchsian system for v, of the form (5.9), but with A replaced by $A + m$. As a consequence, A_s is replaced by $A_s + m$. (A3′) therefore certainly holds for m sufficiently large.

It is convenient to measure the size of $\boldsymbol{t}$ in terms of a norm invariant under the characteristic flow of M; such a norm is given by Lemma 4.7. We may now state the result. Let s be an integer.

Theorem 5.12. *Under assumptions (A1) through (A4) (respectively (A1), (A2), (A3′), (A4)), if $s > n/2 + 1$, equation (5.9) has a unique local solution, continuous with values in H^s.*

Remark 5.13. In case all coefficients are C^∞, one could derive H^s estimates on higher-order derivatives, for all s, by a similar argument.

We first establish *a priori* estimates, and then apply them to approximate equations in which $\mathcal{B}$ is replaced by a bounded approximation. One then passes to the limit. We seek continuous solutions defined on sets of the form $\{\delta(\boldsymbol{t}) \leq \delta_0\}$, rather than $\{|\boldsymbol{t}| \leq \delta_0\}$, because the latter do not form a basis of neighborhoods of the origin invariant under the characteristic flow of M.

Estimates

We consider solutions of

$$Q(N + A)\boldsymbol{u} = \boldsymbol{t} \cdot (B\boldsymbol{u} + f(\boldsymbol{u})),$$

where Q, N, f, A are as in (A1)–(A4), but B is subject instead to the conditions of the next theorem.

We assume $\boldsymbol{u}(0) = 0$, and prove L^2 and H^s *a priori* bounds. They will be applied to regularized equations, where B will be a smooth approximation to $\mathcal{B}$. We recall that $S = (1 - \Delta)^{s/2}$. An operator P is said to be bounded above if $(P\boldsymbol{u}, \boldsymbol{u}) \leq C(\boldsymbol{u}, \boldsymbol{u})$. It is equivalent to require $P + P^*$ to be bounded above.

Theorem 5.14. *(1) If $VB + (VB)^* = C_1$ is bounded above, then $\|\boldsymbol{u}(\boldsymbol{t})\|_{L^2} \leq C\delta(t)$ for small $\boldsymbol{t}$;*

(2) If $SQ^{-1}BS^{-1} - Q^{-1}B = C_2$ is bounded above, then $\|u(t)\|_{H^s} \le C\delta(t)$ for small $\boldsymbol{t}$.

The constants in these estimates depend only on Q, N, f, A, and the bounds on C_1 and C_2.

Proof. (1) We have

$$N(\boldsymbol{u}, VQ\boldsymbol{u}) + (\boldsymbol{u}, VQA\boldsymbol{u}) = \boldsymbol{t} \cdot (\boldsymbol{u}, VB\boldsymbol{u} + Vf(\boldsymbol{u})).$$

Now, $(\boldsymbol{u}, VQA\boldsymbol{u}) \ge 0$ and $(\boldsymbol{u}, VB\boldsymbol{u}) = (\boldsymbol{u}, C_1\boldsymbol{u})/2 \le C(\boldsymbol{u}, VQ\boldsymbol{u})$. Therefore $e(\sigma) := (\boldsymbol{u}(\sigma^M \boldsymbol{t}), VQ\boldsymbol{u}(\sigma^M \boldsymbol{t}))$, which is equivalent to the square of the L^2 norm of $\boldsymbol{u}(\sigma^M \boldsymbol{t})$, satisfies $e(0) = 0$ and

$$\sigma\partial_\sigma e \le C\delta(\boldsymbol{t})\sigma^\beta[1 + e],$$

where $\beta > -1$. Integrating, we obtain $\ln(1+e(1)) \le C\delta(\boldsymbol{t})$. The result follows, for small $\boldsymbol{t}$.

(2) Let $\boldsymbol{v} = S\boldsymbol{u}$. We seek an L^2 estimate on v. Now, if A is constant, $\boldsymbol{v}$ solves

$$(N + A)\boldsymbol{v} = \boldsymbol{t} \cdot (SQ^{-1}BS^{-1}\boldsymbol{v} + SQ^{-1}f(S^{-1}\boldsymbol{v})).$$

Multiplying by Q, we obtain

$$Q(N + A)\boldsymbol{v} = \boldsymbol{t} \cdot (Bv + QC_2\boldsymbol{v} + QSQ^{-1}f(S^{-1}\boldsymbol{v})).$$

Since the nonlinear term is bounded and sublinear on L^2, we may apply the procedure of (1) to derive an L^2 estimate of v. If A is not constant, but satisfies (A3′), we have

$$(N + A_S)\boldsymbol{v} = \boldsymbol{t} \cdot (SQ^{-1}BS^{-1}\boldsymbol{v} + SQ^{-1}f(S^{-1}\boldsymbol{v})).$$

The argument continues as before. □

Approximate equation

The strategy consists in approximating $\mathcal{B}$ by bounded operators. We use the Yosida regularization

$$B_{i\lambda} = \lambda(\lambda - B_i)^{-1}B_i,$$

and let $\mathcal{B}_\lambda = (B_{0\lambda}, \ldots, B_{l\lambda})$, which is semibounded. Since $s > n/2 + 1$ and $\mathcal{B}$ is a first-order operator with coefficients in H^s, we know that $\mathcal{B} + \mathcal{B}^*$ is bounded on L^2, and that therefore $\mathcal{B}_\lambda$ exists for λ real and large enough. We consider the approximate equation

$$Q(N + A)\boldsymbol{u}_\lambda = \boldsymbol{t} \cdot (\mathcal{B}_\lambda\boldsymbol{u}_\lambda + f(\boldsymbol{u}_\lambda)), \tag{5.10}$$

with $\boldsymbol{u}_\lambda(0) = 0$. The parameter λ is large and positive, and will eventually tend to infinity. The existence and differentiability of H^s solutions to this equation follow from Theorem 5.7. We establish here *a priori* estimates. We check the assumptions of Theorem 5.14 for $\mathcal{B}_\lambda$, taking care that the operators C_1 and C_2 be bounded above uniformly in λ.

First of all, since V and A_{jk} commute, we have

$$(V\mathcal{B} + (V\mathcal{B})^*)_k = -\sum_j \partial_j (VA_{jk}),$$

which is a bounded function if $s > n/2 + 1$. To obtain information on $\mathcal{B}_\lambda$, we prove the following lemmas; throughout the rest of the argument, we write B for any of the B_i's, and let $R_\lambda = (\lambda - B)^{-1}$, $B_\lambda = \lambda B R_\lambda$.

Lemma 5.15. *Assume* $|\lambda R_\lambda| \le C$ *for* $\lambda > \lambda_0$*, and* $(B\boldsymbol{u}, \boldsymbol{u}) \le C(\boldsymbol{u}, \boldsymbol{u})$ *for* u *in the domain of* B*. Then* $(\lambda R_\lambda B\boldsymbol{u}, \boldsymbol{u}) \le C'(\boldsymbol{u}, \boldsymbol{u})$*, where* C' *is independent of* $\lambda > \lambda_0$*.*

Proof. We write

$$\begin{aligned}\lambda R_\lambda B + (\lambda R_\lambda B)^* &= \lambda B R_\lambda + \lambda R_\lambda^* B^* \\ &= \lambda R_\lambda^* (B + B^* - 2B^* B/\lambda)\lambda R_\lambda,\end{aligned}$$

so that since $(R_\lambda^* B^* B R_\lambda \boldsymbol{u}, \boldsymbol{u}) = (BR_\lambda \boldsymbol{u}, BR_\lambda \boldsymbol{u}) \ge 0$,

$$2(\lambda R_\lambda B\boldsymbol{u}, \boldsymbol{u}) \le ((B + B^*)\lambda R_\lambda \boldsymbol{u}, \lambda R_\lambda \boldsymbol{u}) \le C(\boldsymbol{u}, \boldsymbol{u}),$$

which is the desired result. □

Lemma 5.16. $VB_\lambda + (VB_\lambda)^*$ *is bounded above on* L^2*.*

Proof. We know that $VB = B$. We show that $VB_\lambda = B_\lambda$, which, using Lemma 5.15, will give the desired result. If $B_\lambda x = y$, we have $Bx = (\lambda - B)y$, and therefore $Bx = VBx = \lambda Vy - VBy = \lambda Vy - \lambda y + Bx$, or $\lambda(y - Vy) = 0$. Since λ is positive, the result follows. □

Lemma 5.17. *There are bounded operators* C *and* C' *such that*

$$SQ^{-1}BS^{-1} = Q^{-1}B + C$$

and

$$SBS^{-1} = B + C'.$$

Proof. Both B and $Q^{-1}B$ are sums of terms of the form $a_j(x)\partial_j$, with $a_j \in H^s$ or a_j constant. By a well-known theorem,

$$\|[S, a_j]f\|_{L^2} \le C(\|a_j\|_{\mathrm{Lip}} \|f\|_{H^{s-1}} + \|a_j\|_{H^s} \|f\|_{L^\infty}). \tag{5.11}$$

In particular, for any u,

$$\|[S, a_j]\partial_j S^{-1} u\|_{L^2} \le C\|u\|_{L^2}.$$

Now

$$S(a_j \partial_j)S^{-1} - a_j \partial_j = [S, a_j]\partial_j S^{-1},$$

and we just saw that the right-hand side is a bounded operator if $a_j \in H^s$, the case of constant a_j being trivial. The lemma is therefore proved. □

We also recall for later use the following classical result (see [26]):

Lemma 5.18. *If $B + B^*$ is bounded, then λR_λ and $BR_\lambda = \lambda R_\lambda - 1$ are uniformly bounded for λ large enough.*

Lemma 5.19. *$SQ^{-1}B_\lambda S^{-1}$ is bounded on L^2.*

Proof. Using Lemma 5.17, we have

$$\begin{aligned} SQ^{-1}B_\lambda S^{-1} &= \lambda SQ^{-1}B(\lambda - B)^{-1}S^{-1} \\ &= \lambda SQ^{-1}BS^{-1}S(\lambda - B)^{-1}S^{-1} \\ &= \lambda(Q^{-1}B + C)S(\lambda - B)^{-1}S^{-1}. \end{aligned}$$

Now [155, pp. 123 and 125],

$$S(\lambda - B)^{-1}S^{-1} = (\lambda - SBS^{-1})^{-1} = (\lambda - B - C')^{-1},$$

and since

$$(\lambda - B - C')^{-1} = R_\lambda(I - C'R_\lambda)^{-1},$$

we conclude that

$$\begin{aligned} SQ^{-1}B_\lambda S^{-1} &= (Q^{-1}B + C)\lambda R_\lambda(I - C'R_\lambda)^{-1} \\ &= Q^{-1}B_\lambda + C\lambda R_\lambda(I - C'R_\lambda)^{-1} + Q^{-1}B\lambda R_\lambda C'R_\lambda(I - C'R_\lambda)^{-1}. \end{aligned}$$

Since C, C', and λR_λ are bounded, it follows that $C\lambda R_\lambda(I - C'R_\lambda)^{-1}$ is bounded for λ large enough.

As for $Q^{-1}B\lambda R_\lambda C'R_\lambda(I - C'R_\lambda)^{-1}$, it is bounded as well because $B\lambda R_\lambda C'R_\lambda$ $(= (\lambda R_\lambda - 1)C'\lambda R_\lambda)$ is. This concludes the proof. □

By application of Theorem 5.14, we find that the approximate solutions satisfy

$$\|\boldsymbol{u}_\lambda(\boldsymbol{t})\|_{H^s} \le C\delta(\boldsymbol{t}).$$

Estimating the time derivatives

We turn to energy-type estimates on the time derivatives $z_\lambda := \nabla_{\boldsymbol{t}}\boldsymbol{u}_\lambda$. They form a vector of length $(l + 1)(n + 2)$. In this paragraph only, we use the following notation: for any matrix or operator, such as A, we write A' for the matrix consisting of $(l + 1)$ diagonal blocks equal to A:

$$A'z = (Az_0, \ldots, Az_l).$$

In a similar way, Q' and $\mathcal{B}'_\lambda$ are related to Q and $\mathcal{B}_\lambda$. Differentiating (5.10), we obtain

$$Q'(N + \tilde{A})z_\lambda = \boldsymbol{t} \cdot [\mathcal{B}'_\lambda z_\lambda + f_u z_\lambda + \nabla_{\boldsymbol{t}} f] + \mathcal{B}_\lambda \boldsymbol{u}_\lambda + f(\boldsymbol{u}_\lambda), \qquad (5.12)$$

where $\tilde{A}$ is the block matrix

$$\tilde{A} = \begin{pmatrix} A + m_{00} & m_{10}I & \ldots \\ m_{01}I & A + m_{11}I & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}.$$

Lemma 5.20. *The eigenvalues of $\tilde{A}$ consist of the sums $\lambda_A + \lambda_M$, where λ_A and λ_M are eigenvalues of A and M respectively.*

Proof. Assume that $P^{-1}MP$ is in upper-triangular Jordan form. Let $P = (p_{ij})$ and

$$\tilde{P} = \begin{pmatrix} p_{00}I & p_{10}I & \dots \\ p_{01}I & p_{11}I & \dots \\ \dots & \dots & \dots \end{pmatrix}.$$

Then $\tilde{P}^{-1}\tilde{A}\tilde{P}$ is block-diagonal with blocks of the form

$$\begin{pmatrix} A+\lambda_M & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & A+\lambda_M \end{pmatrix}.$$

The lemma follows. □

In particular, $\tilde{A}$ is invertible, and we may define uniquely $z_\lambda(0)$, which is bounded in H^s (set $\boldsymbol{t} = 0$ in (5.12), and use $\boldsymbol{u}_\lambda(0) = 0$). Subtracting it from z_λ and calling again the difference z_λ for convenience, we obtain a system of the form (5.12), but with coefficients bounded in H^{s-1}: indeed, since we already have $\|\boldsymbol{u}_\lambda(\boldsymbol{t})\|_{H^s} \le C\delta(\boldsymbol{t})$, we know that

$$\|\mathcal{B}_\lambda \boldsymbol{u}_\lambda + f(\boldsymbol{u}_\lambda) - f(0)\|_{H^s} \le C\delta(\boldsymbol{t})$$

as well, and we are therefore assured that the right-hand side of the equation for z_λ is $\mathcal{O}(\boldsymbol{t})$. This will play the role of (A2) in the sequel.

To prove energy estimates on z_λ, we check the rest of (A1)–(A4) for (5.12). Only (A3) requires a separate argument. Let $(\tilde{A}_{qr})$ be the block decomposition of $\tilde{A}$:

$$\tilde{A}_{qr} = A + m_{rq}I.$$

Recall that $(u, VQAu) \ge 0$ for any u. We estimate from below

$$\begin{aligned}(V'Q'\tilde{A}z, z) &= \sum_{q,r}(z_q, VQ\tilde{A}_{qr}z_r) \\ &= \sum_{q,r}(z_q, VQAz_q + VQm_{rq}z_r) \\ &\ge \sum_{q,r}\frac{1}{2}[m_{rq} + m_{rq}]((VQ)^{1/2}z_q, (VQ)^{1/2}z_r).\end{aligned}$$

The result follows, since $M + M^T$ is positive definite. Applying the procedure of Sect. 5.2, we find that

$$\|\nabla_{\boldsymbol{t}} \boldsymbol{u}_\lambda\|_{H^{s-1}} \le C\delta(\boldsymbol{t})$$

uniformly in λ.

Existence result

Proof. We now pass to the limit $\lambda \to \infty$. From the bounds of $\boldsymbol{u}_\lambda$ in $C(\delta(\boldsymbol{t}) \leq \delta_0; H^s) \cap C^1(\delta(\boldsymbol{t}) \leq \delta_0; H^{s-1})$, we obtain a solution in $C(\delta(\boldsymbol{t}) \leq \delta_0; H^{s-1})$, by application of Ascoli's theorem. The *a priori* estimate in H^s also implies that the solution is continuous at $t = 0$, in the H^s topology, and is uniformly bounded in H^s. □

Let us now turn to systems (5.9). We now have a solution w of (5.8) defined for small $(t_0, \dots, t_l)$, such that $(x, t, \dots, t(\ln t)^l)$ belongs to H^s, is continuous in t, with values in H^{s-1}, and which is bounded, and tends to zero as $t \to 0$, in the H^s topology. Now, we may substitute back $t_0 = t$, $t_1 = t \ln t, \dots$ into (5.8), which gives a symmetric-hyperbolic system similar to (5.9) for $t \neq 0$. For such a system, standard results [116] show that solutions that start in H^s are continuous in time, with values in H^s, and therefore $w(x, t, \dots, t(\ln t)^l)$ is continuous at every $t \neq 0$, with values in H^s. Combining this with the estimates near $t = 0$, we conclude that the solution is continuous for all t with values in H^s. From the definition of w, it follows that v has the same continuity properties. We have therefore proved Theorem 5.12.

5.4 Optimal regularity for Fuchsian PDEs

The question of optimal regularity for Fuchsian PDEs is somewhat more complicated than its counterpart for the Cauchy problem. We outline two issues: (i) expressions involving the solution, such as $tu(x, t)$, may be more regular than u; (ii) the optimal regularity may not be obtained by taking the arbitrary functions to be smooth. These issues are illustrated on examples; no general statement seems to be available at this time.

The first issue will be illustrated; by the wave equation

$$u_{tt} - \Delta u = f(x, t).$$

If the Cauchy data u_0 and u_1 are respectively in H^s and H^{s-1}, and if f is polynomial in time and H^{s-1} in space, the solution u at time t is of class H^s in the space variables x, and this is best possible. However, if we perform reduction by letting $u = u_0(x) + tu_1(x) + t^2 v(x, t)$, we obtain

$$(t\partial_t + 1)(t\partial_t + 2)v - t^2 \Delta v = t^2 (f - \Delta u_0 - t\Delta u_1).$$

Since the right-hand side is now of class H^{s-3}, it appears at face value that $v \in H^{s-2}$ for fixed t.

The second issue pertains to the regularity of the arbitrary functions. For the wave equation, the arbitrary functions are u_0 and u_1, and for a right-hand side in H^{s-1}, we should take them in H^s and H^{s-1} respectively to obtain the optimal regularity of solutions. By contrast, for Fuchsian equations, the correct regularity of the arbitrary functions may require going beyond Sobolev spaces. We analyze in detail the situation for the one-dimensional model problem

$$t(u_t - u_x) = u - u_0; \quad u(0) = u_0, \tag{5.13}$$

where $u_0 \in H^s$ is given. Here, u_0 occurs both as an initial condition, and in the right-hand side. The structure of solutions is given by the following theorem.

Theorem 5.21. *There are infinitely many solutions that are H^s with respect to x, for every t. Furthermore, if u_0 is analytic,*

$$u = u_0 + \sum_{j \geq 1} (a_j(x) + b_j(x) \ln t) t^j, \tag{5.14}$$

where a_1 is arbitrary, and the other coefficients can be found inductively; in particular, $b_1 = u_{0x} := \partial_x u_0$. If u_0 is not analytic, the expansion remains valid as an asymptotic expansion. Solutions are uniquely determined by the choice of a_1.

Before giving the proof, let us state the main point: taking a_1 very smooth does not generate the H^s solutions in the theorem:

Theorem 5.22. *For any $u_0 \in H^s$, the solutions in H^s are those in which*

$$a_1(x) - k\left(\frac{\partial}{\partial x}\right) u_0 \in H^s,$$

where $k(i\xi) = i\xi(\gamma + \ln(i\xi))$, and γ is Euler's constant.

We now turn to the proof of these results.

Proof of Theorem 5.21

Let us Fourier transform in x. We obtain

$$t(\hat{u}_t - i\xi\hat{u}) = \hat{u} - \hat{u}_0,$$

or

$$(t^{-1}e^{-i\xi t}\hat{u})_t = -t^{-2}e^{-i\xi t}\hat{u}_0.$$

The general solution has the form

$$\hat{u}(\xi, t) = te^{i\xi t}\hat{\alpha}(\xi) + \left(t \int_t^{\infty} s^{-2} e^{i\xi(t-s)}\, ds\right) \hat{u}_0(\xi)$$

for $t > 0$, where $\alpha(\xi)$ is arbitrary. This can be rewritten

$$\hat{u} = (t\alpha(x+t))\hat{\ } + g(t)\hat{u}_0(\xi), \tag{5.15}$$

where

$$g(t,\xi) = \int_0^\infty \frac{t}{(t+\tau)^2} e^{-i\xi\tau}\, d\tau = \int_1^\infty \sigma^{-2} e^{i\xi t(1-\sigma)}\, d\sigma. \tag{5.16}$$

From the second form of g, it is apparent that our problem has infinitely many solutions that are in H^s for each $t > 0$, if $u_0 \in H^s$; more precisely,

$$\|u(t)\|_s \le \|u_0\|_s + t\|\alpha\|_s. \tag{5.17}$$

This proves the first part of Theorem 5.21. For the second, we let $v = (u - u_0)/t$ so that v, too, solves a Fuchsian equation

$$t(v_t - v_x) = u_{0x}, \tag{5.18}$$

where $u_{0x} = \partial_x u_0$. This equation has no solution of class C^1 unless u_0 is constant. On the other hand, from the study of the analytic case, if u_0 is analytic, then tv is an analytic function of x, t, and $t \ln t$. Indeed, one finds, by substitution, that (5.18) has infinitely many *formal* solutions of the form

$$\sum_{j\ge 1} \frac{1}{t}(a_j + b_j \ln t)t^j.$$

The coefficients a_j and b_j are found recursively if u_0 and a_1 are given. One obtains In particular, $b_1 = u_{0x}$. The convergence follows from Theorem 4.5.

Uniqueness follows from the above argument. It may also be seen by interpreting the coefficient a_1 as an initial value for the Fuchsian system

$$\begin{aligned} t\lambda_t + \mu &= t\lambda_x + u_{0x}, \\ t\mu_t &= t\mu_x. \end{aligned} \tag{5.19}$$

The following lemma states this more precisely.

Lemma 5.23. *If (λ, μ) is a solution of (5.19) with*

$$\lambda(x,0) = a_1(x) \quad \text{and} \quad \mu(x,0) = u_{0x},$$

then $u = u_0 + t(\lambda + \mu \ln t)$ solves (5.13).

The result is proved by direct substitution. □

This completes the proof of Theorem 5.21. □

We now turn to the estimation of the regularity of a_1 for solutions in H^s.

Optimal regularity

We show that the solution will not have optimal regularity, namely H^s in the space variable, if a_1 is C^∞; one must instead take $a \in H^{s-2}$ of a very particular form, as given in Theorem 5.22. To see this, we first evaluate $g(t)$ more precisely. The Laplace transform of $(t+\tau)^{-2}$ with respect ot τ is, for $t > 0$,

$$p \mapsto \frac{1}{t} - pe^{pt}E_1(pt),$$

where $E_1(z) = -\gamma - \ln z - \sum_{n\geq 1}(-1)^n z^n/(n(n!))$ is the exponential-integral function (also equal to $\int_z^\infty t^{-1}e^{-t}\,dt$; γ is Euler's constant, and we are taking $|\arg z| < \pi$ with the principal determination of the logarithm). Therefore, we obtain

$$g(t) = 1 + i\xi t e^{i\xi t}\left(\gamma + \ln(i\xi t) + \sum_{n\geq 1}\frac{(-i\xi t)^n}{n(n!)}\right). \tag{5.20}$$

Since $t > 0$, we may expand this as

$$g(t) = 1 + t\{i\xi \ln t + i\xi(\gamma + \ln(i\xi)) + \mathcal{O}(t\ln t)\}.$$

This provides an expression for the most general solution in H^s. Let us now assume $u = u_0 + t(\lambda + \mu \ln t)$, where (λ, μ) solve (5.19) with initial data (a_1, u_{0x}). From the equation for μ, we obtain

$$\mu(x,t) = u_{0x}(x+t).$$

Therefore $\hat{\mu} = i\xi\hat{u}_0 e^{-it\xi} = i\xi\hat{u}_0(1 + \mathcal{O}(t))$ as $t \to 0$ for fixed ξ. We can now compute a_1 for this solution: as $t \to 0$,

$$\frac{\hat{u} - \hat{u}_0}{t} = e^{i\xi t}\hat{\alpha}(\xi) + \frac{g(t) - 1}{t}\hat{u}_0 = i\xi\hat{u}_0 \ln t + \hat{\alpha} + k(i\xi)\hat{u}_0 + o(1),$$

where $k(i\xi) = i\xi(\gamma + \ln(i\xi))$. Observe that k is not a classical symbol. Using the expansion of $\hat{\mu}$, we compute $\hat{\lambda}$ and conclude that $\hat{a}_1 = \hat{\alpha} + k(i\xi)\hat{u}_0$. Since α must be in H^s for estimate (5.17) to hold, we see that a_1 cannot be in H^s. Rather,

$$a_1 - k(\partial/\partial x)u_0 \in H^s,$$

as claimed in Theorem 5.22. □

The restriction on the arbitrary function occurring in the general solution is missed by the formal calculation in powers of t and $t\ln t$. This restriction is of a global nature as in the so-called connection problem, in which one seeks relations between representations of solutions of ODEs at zero and infinity. Set $\rho = \xi t$. We obtain

$$\rho(\hat{u}_\rho - i\hat{u}) = \hat{u} - \hat{u}_0(\xi).$$

All solutions of this equation are bounded near $\rho = 0$. They depend on one parameter. Any two solutions differ by a multiple of $\rho e^{i\rho}$; thus, at most one solution can remain bounded for all ρ. More precisely, let us require that

$$|\hat{u}| \leq C|\hat{u}_0|$$

for all ρ. This singles out the solution

$$\int_1^\infty \sigma^{-2}e^{i\rho(1-\sigma)}\,d\sigma\, u_0(\xi).$$

If we expand this solution in the form (5.14), there is no reason why the coefficient a_1 should turn out to be zero. In fact, is it easy to see that this coefficient contains precisely the expression $k(\partial_x)u_0$ of Theorem 5.22.

5.5 Reduction to a symmetric system

To apply the abstract results of this chapter, it is often necessary to convert a hyperbolic problem into a symmetric first-order system. We give a typical example of this reduction.

Consider the solution u of the n-dimensional Liouville equation considered in Sect. 1.5.6; the notation of this section will be used. We define three vector-valued functions $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, the components of which are defined in terms of u, its derivatives, and ψ. The first component of $\mathbf{u}$ is the function u, while the first component of $\mathbf{w}$ coincides with the function w defined by (1.13). Each of these functions solves a first-order system. Furthermore, $\mathbf{v}$ can be computed from $\mathbf{w}$, and $\mathbf{u}$ from $\mathbf{w}$. The system for $\mathbf{u}$ is simply the symmetric-hyperbolic system associated with the Liouville equation. The intuitive idea is this: if one expands u up to order m to define a renormalized unknown, it is advisable to expand first-order derivatives of u to order $m-1$ if one wants to respect the natural scaling of derivatives.

System for u

Let

$$\mathbf{u} := (u, u_0, u_i),$$

where i runs from 1 to n. The following system implies (10.2) if the u_i are, for $T = 0$, the components of the spatial gradient of u:

$$(Q\partial_T - A^i\partial_i)\mathbf{u} = \varphi(X, T, u) := \begin{pmatrix} u_0 \\ e^u - (\Delta\psi)u_0 \\ 0 \end{pmatrix}.$$

Here, the diagonal matrix Q of size $(n+2)\times(n+2)$ is defined by

$$Q(x) = \begin{pmatrix} 1 & & \\ & \gamma & \\ & & I_n \end{pmatrix}.$$

The symmetric matrix A^i has only three nonzero entries, namely

$$(A^i)_{2,2} = -2\psi_i; \quad (A^i)_{2,i+2} = (A^i)_{i+2,2} = 1.$$

Fuchsian system for v

We subtract from **u** a few terms from its expansion, and obtain a Fuchsian equation: Define **v** by

$$\begin{aligned} u &= \ln(2/T^2) + v^{(0)} + v^{(1)}T + vT^2, \\ u_0 &= -(2/T) + v^{(1)} + v_0 T, \\ u_i &= v_i^{(0)} + v_i T, \end{aligned} \tag{5.21}$$

where $\mathbf{v} = (v, v_0, v_i)$ is a new unknown, and

$$\exp(v^{(0)}) = \gamma; \qquad v^{(1)}\gamma + \Delta\psi = 0; \qquad v_i^{(0)} = \partial_i v^{(0)}.$$

The system for **u** now becomes

$$Q[T\partial_T + A]\mathbf{v} = \tilde{\varphi}(X) + TA^i\partial_i\mathbf{v} + TF(X, \mathbf{v}), \tag{5.22}$$

where

$$\tilde{\varphi} = \begin{pmatrix} 0 \\ -2R(X) \\ \partial_i v^{(1)} \end{pmatrix}; \qquad F = \begin{pmatrix} 0 \\ b_0 \\ 0 \end{pmatrix},$$

R is defined in (10.4), and

$$A = \begin{pmatrix} 2 & -1 & \\ -2 & 1 & \\ & & I_n \end{pmatrix}.$$

This matrix has eigenvalues 0, 3, and 1, with multiplicities $1, 1, n$. Its null space is generated by $(1, 2, 0, \ldots, 0)^T$. The function b_0 is given by

$$b_0(X, T, \mathbf{v}) = -v_0\Delta\psi + \gamma(2v^{(1)}v + Tv^2) + \gamma h(T, v^{(1)} + Tv),$$

where $h(T, z) = z^3 \int_0^1 (1-\sigma)^2 \exp[\sigma T z]\, d\sigma$. Since we are interested only in small **v**, we will truncate the nonlinear part of b_0, namely $\gamma[h(T, v^{(1)} + Tv) + Tv^2]$, so that it is smooth, identically zero for $|v| > 2$, and given by this expression for $|v| < 1$.

Remark 5.24. The principal part of (5.22) is equal to the principal part of the equation for **u** multiplied by T.

Fuchsian system for w

Since **v** is not free from logarithmic terms unless $R = 0$, we now view it as a function of the variables X, t_0, and t_1, where $t_0 = T$; $t_1 = T \ln T$. Introduce a second renormalized unknown $\mathbf{w} = (w, w_0, w_i)$ by the formulas

$$\begin{aligned} u &= \ln(2/t_0^2) + v^{(0)} + v^{(1)}t_0 + R_1 t_0 t_1 + w(t_0, t_1, X)t_0^2, \\ u_0 &= -(2/t_0) + v^{(1)} + (t_0 + 2t_1)R_1 + w_0 t_0, \\ u_i &= v_i^{(0)} + v_i^{(1)} t_0 + w_i t_0, \end{aligned} \tag{5.23}$$

where

$$R_1 = -\frac{2R}{3\gamma}.$$

In other words, $\mathbf{w}$ is defined by

$$\begin{aligned} v &= w + R_1 t_1/t_0, \\ v_0 &= w_0 + R_1(1 + 2t_1/t_0), \\ v_i &= w_i + v_i^{(1)}. \end{aligned}$$

This definition of w is consistent with (1.13). Equation (5.22) now takes the form

$$Q(N + A)\mathbf{w} = t_0 A^i \partial_i \mathbf{w} + t_0 g_0(X, t_0, t_1, \mathbf{w}) + t_1 g_1(X, t_0, t_1, \mathbf{w}), \tag{5.24}$$

where $N = t_0\partial/\partial t_0 + (t_0 + t_1)\partial/\partial t_1$,

$$g_0 = \begin{pmatrix} 0 \\ -(w_0 + R_1)\Delta\psi + \Delta v^{(1)} - \sum_i 2\psi_i \partial_i R_1 \\ \qquad + \gamma[h(v^{(1)} + t_0 w + t_1 R_1) + (2v^{(1)} + t_0 w + t_1 R_1)w] \\ \partial_i R_1 \end{pmatrix},$$

and

$$g_1 = \begin{pmatrix} 0 \\ \gamma(2v^{(1)} + t_0 w + t_1 R_1)R_1 - 2R_1\Delta\psi - 4\sum_i \psi_i \partial_i R_1 \\ -2v_i^{(1)} \end{pmatrix}.$$

Since the kernel of A is one-dimensional, the solutions of (5.24) are determined entirely by one function, namely the value of the first component of $\mathbf{w}$ for $t_0 = t_1 = 0$. We let

$$\mathbf{w}^{(0)} = w(t_0 = t_1 = 0),$$

and replace $\mathbf{w}$ by $\mathbf{w} - \mathbf{w}^{(0)}$. This doesn't affect the form of the equation, but yields a Fuchsian system with vanishing initial values.

We conclude by showing that the function w does determine a solution of the Liouville equation for u. Since the equations for $\mathbf{w}$ include those deduced from the relations $u_0 = \partial_T u$ and $\partial_T u_i = \partial_i u_0$ (hence $\partial_T(\partial_i u - u_i) = 0$), if $\mathbf{w}$ solves (5.24), $\mathbf{w}(T, T\ln T, X)$ satisfies

$$w_0 = 2w + T\partial_T w,$$

and

$$\partial_T[w_i T - T^2 \partial_i w - (T^2 \ln T)\partial_i R_1] = 0.$$

It follows that

$$w_i = T\partial_i w + (T \ln T)\partial_i R_1.$$

Thus, we have proved that w determines precisely one solution of (5.24), with the property that the u_i are the components of the gradient of u, so that this construction does produce solutions of $\Box u = e^u$. This is noteworthy, since not all solutions of the equation for $\mathbf{u}$ correspond to solutions of (10.2).

Problems

5.1. Can one replace continuity with respect to t in assumption (H3), Sect. 5.1.1, by Carathéodory-type conditions?

5.2. Prove Theorem 5.7.

5.3. Improve Theorem 5.7 in the following directions:
(a) Under what assumptions can one find a solution such that $u(0)$ is a given element of the null space of A?
(b) Under what assumptions may one expect solutions in L^p, $p > 1$?

5.4. Convert a linear Fuchsian PDE with smooth coefficents, in one unknown u, into a first-order system by introducing as new variables $\Lambda^k u$, where $\Lambda = (1 - \Delta)^{1/2}$ [25].

6

Solution of Fuchsian Elliptic Boundary-Value Problems

This chapter is devoted to a special class of elliptic boundary-value problems with quadratic boundary degeneracy. We briefly review the standard L^p setup for such problems, and give recent estimates of Schauder type.

Consider a domain Ω in $\mathbb{R}^n$, and the distance function $d(x)$, the properties of which are recalled in Sect. 12.1. We consider elliptic equations of the form

$$Au = f,$$

where A is strictly elliptic in Ω, but degenerates in a quadratic manner as one approaches the boundary. To be specific, we introduce two classes:

Definition 6.1. *An operator A is said to be* of type (I) *on Ω if it can be written*

$$A = \partial_i(d^2 a^{ij} \partial_j) + db^i \partial_i + c,$$

with (a^{ij}) uniformly elliptic and of class C^α, and b^i, c bounded.

An operator is said to be of type (II) *if it can be written*

$$A = d^2 a^{ij} \partial_{ij} + db^i \partial_i + c,$$

with (a^{ij}) uniformly elliptic and a^{ij}, b^i, c of class C^α.

Remark 6.2. Types (I) and (II) are invariant under changes of coordinates of class $C^{2+\alpha}$. If λ is a scale factor, we may think of derivatives of first order, identified with frames, as homogeneous of degree -1, and the distance as homogeneous of degree 1; thus Fuchsian operators are made up of scale-invariant combinations of d and the operators ∂_j. To check that an operator is of type (I) or (II), we may work indifferently in coordinates x or (T, Y) defined in Sect. 12.1. All proofs will be performed in the (T, Y) coordinates; an operator is of type (II) if and only if it has the above form with d replaced by T, and the coefficients a^{ij}, b^i, c are of class C^α as functions of T and Y; a similar statement holds for type (I). Problems with scale-invariance with respect to one variable arise in the study of domains with wedges and corners; see [127, 73]. Many intermediate situations can also be considered.

The Fuchs indices are defined in the usual way, by taking $T = d$ as first coordinate, and by writing the equation in the form

$$P(D)u = T^\varepsilon F(x, u, T\nabla u, T^2\nabla^2 u),$$

where ∇ stands for all derivatives; the indices are the roots of P. The operator $d^2\Delta$ has indices 0 and 1.

Problems with linear degeneracy such as

$$da^{ij}\partial_{ij}u + b^i\partial_i u = f$$

are in fact, after multiplication of the equation by d, reduced to the special case $c = 0$ of Fuchsian elliptic operators with quadratic degeneracy.

The main new qualitative feature of Fuchsian elliptic PDEs as compared with standard elliptic theory is that boundary conditions have the form $u \sim \varphi(x)d^\sigma$, where σ is a Fuchs index. If σ is negative, and the other index is positive, requiring $\varphi = 0$ amounts to looking for bounded solutions. In such cases, there is typically only one bounded solution, which does not require the prescription of a function on the boundary. The Legendre equation is the prototype of such behavior. The Laplace equation corresponds to the operator $d^2\Delta$, which has indices 0 and 1. In the Dirichlet problem, one seeks a solution such that $u \sim \varphi(x)$ near the boundary; for the Neumann problem, one would like $u \sim \varphi(x)d(x) + \psi(x)$, where φ is prescribed and ψ is unknown. Thus, in elliptic Fuchsian problems, unlike the hyperbolic case, one is allowed to prescribe only some of the arbitrary functions in the formal expansion of solutions.

Most of the chapter is devoted to analogues of Schauder estimates for such PDEs. Schauder-type estimates for Fuchsian elliptic PDEs differ from other weighted estimates in three respects: (i) scaled Schauder estimates do not yield the optimal regularity properties for Fuchsian operators, because they do not take the value of the Fuchs indices into account; (ii) equations generated by reduction generally have integral indices, so that even solutions with smooth data have (logarithmic) singularities; (iii) the Laplace operator is not the correct local model for regularity in the Fuchsian case. We begin with some classical results, adapted mostly to L^p solutions of equations with linear degeneracy; for background results, see [69, 150].

6.1 Basic L^p results for equations with degenerate characteristic form

Historically, linear operators with linear degeneracy were the first ones to be studied, for two different reasons. First of all, the hypergeometric equation

$$z(1-z)u_{zz} + [c - (a+b+1)z]u_z - abu = 0,$$

which contains many of the special functions of mathematical physics,[1] has linear degeneracy, and the behavior of solutions near the origin depends on the value of c, since the Fuchs indices at 0 are 0 and $1-c$. Therefore, the loss of ellipticity forces one to take into account the lower-order terms. The special case of Legendre polynomials[2] shows that degenerate PDEs lead to expansions in orthogonal polynomials, generalizing Fourier series. Recall that the Legendre polynomials are defined as bounded solutions of $[(1-x^2)P_n']'+n(n+1)P_n = 0$, normalized by $P_n(1) = 1$, for $n = 0, 1, \ldots$. The factor $(1 - x^2)$ degenerates linearly at the boundary. No boundary condition is imposed—apart from the boundedness condition. A second motivation comes from axisymmetric potential theory, which leads, in three dimensions, to

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = f(r, z),$$

or equivalently

$$r(u_{rr} + u_{zz}) + u_r = rf(r, z).$$

Any semilinear problem involving the Laplace operator will reduce to such a Fuchsian problem if one assumes axial symmetry. Here again, no boundary condition is required at $r = 0$ if we are interested in bounded solutions. Some applications lead to equations mathematically equivalent to an axisymmetric Laplace or Poisson in a fictitious space of higher dimension [184]. These early studies led to the realization that the lower-order terms are essential in understanding the solvability and regularity properties of the solutions.

When an elliptic problem degenerates at the boundary, it is possible to obtain an existence theory by the usual variational approach, but the regularity properties of the solution are delicate, because smooth data do not lead to smooth solutions, as in the nondegenerate case: regularity depends on the value of the indices. We briefly recall the some classical existence results in L^p spaces.

Consider an operator

$$L = a^{ij}(x)\partial_{ik} + b^k(x)\partial_k + c(x),$$

in which summation over repeated indices in different positions is understood. We assume that (a^{ij}) is symmetric and positive definite, that the coefficients are twice differentiable in Ω, and that the boundary is C^2. As will be clear from the proofs, these regularity assumptions can be weakened, but at the expense of complication in the statements of results.

[1] Around 1900, all the major special functions of use in physics and engineering had been related to the most general ODE of Fuchsian type with at most five regular singularities in the complex plane, and the equations deduced from it by confluence of singularities.

[2] Murphy's formula gives $P_n(\cos\theta) = (-1)^n F(n + 1, -n; 1; \cos^2 \frac{1}{2}\theta)$ where $F(a, b; c; z)$ denotes the hypergeometric function with $(a, b, c) = (n + 1, -n, 1)$.

Definition 6.3. *Let (n_k) denote the inward normal to Ω. Let*

$$l^k = b^k - \partial_j a^{jk}; \qquad b = l^k n_k.$$

Define the Fichera sets:

$$\Sigma_3 = \{x \in \partial\Omega : a^{ij} n_i n_j > 0\}, \tag{6.1a}$$
$$\Sigma_0 = \{x \in \partial\Omega \setminus \Sigma_3 : b = 0\}, \tag{6.1b}$$
$$\Sigma_1 = \{x \in \partial\Omega \setminus \Sigma_3 : b > 0\}, \tag{6.1c}$$
$$\Sigma_2 = \{x \in \partial\Omega \setminus \Sigma_3 : b < 0\}, \tag{6.1d}$$

so that $\partial\Omega = \cup_{j=0}^{3} \Sigma_j$.

Remark 6.4. The simplest example for which this theory is relevant is the heat operator: $Lu = u_{xx} - u_t$, $\Omega = (-1, 1) \times (0, t_0)$, say, considered as a degenerate elliptic operator in two variables, x and t. For an operator with linear degeneracy, b generally does not vanish identically; it has been generalized more recently as the subprincipal symbol. For an operator with quadratic degeneracy of type (II), b is identically zero.

Consider the boundary-value problem

$$Lu = f; \qquad u = 0 \text{ on } \Sigma_2 \cup \Sigma_3. \tag{6.2}$$

This problem can be solved by the usual procedure based on Green's identity:

$$\int_\Omega (vLu - uL^*v)dx = -\int_{\Sigma_3} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right) d\sigma - \int_{\partial\Omega} buv \, d\sigma, \tag{6.3}$$

where $\partial u/\partial\nu = a^{ij} n_i \partial_j u$, and

$$L^*v = \partial_j(a^{ij}\partial_i v) - \partial_k(b^k v) + cv = a^{ij}\partial_{ij} v + b^{*k}\partial_k v + c^* v,$$

with

$$b^{*k} = 2\partial_j a^{jk} - b^k; \qquad c^* = \partial_{ij} a^{ij} - \partial_k b^k + c = c - \partial_k l^k.$$

Observe also that

$$-\int_\Omega uLu \, dx = \int_\Omega \left[a^{ij}\partial_i u \partial_j u + \frac{1}{2}(\partial_k b^k - \partial_{ij} a^{ij} - 2c)u^2 \right] dx + \frac{1}{2}\int_{\partial\Omega} u^2 b \, d\sigma.$$

A variant of the usual argument based on the Lax–Milgram theorem gives the existence of a solution for any $f \in L^2$ provided that

$$\partial_k b^k - \partial_{ij} a^{ij} - 2c \geq c_0,$$

where c_0 is a positive constant.

It is possible to obtain L^p a priori bounds under a different set of hypotheses.

Theorem 6.5. *Assume $u \in C^2(\overline{\Omega})$, $u = 0$ on $\Sigma_2 \cup \Sigma_3$, $1 \le p < \infty$, and that $w \in C^2(\overline{\Omega})$ satisfies*

$$w \le 0 \text{ and } L^* w + (p-1)cw > 0 \text{ in } \overline{\Omega}.$$

Then

$$\|u\|_{L^p(\Omega)} \le \frac{\min_{\overline{\Omega}} p|w|}{\min_{\overline{\Omega}}[L^* w + (p-1)cw]} \|Lu\|_{L^p(\Omega)}.$$

The principle of the proof is to apply Green's formula with u and v replaced by $(u^2 + \delta)^{p/2}$ and w, and to let δ tend to zero. This estimate is the basis of many existence results, such as the following.

Theorem 6.6. *Assume that $\partial\Omega$ is defined by the equation $F = 0$, where $F > 0$ in Ω, and $LF \le 0$ at interior points of $\Sigma_0 \cup \Sigma_2$. Then, if $c < 0$ and $c^* < 0$ in $\overline{\Omega}$, the boundary-value problem (6.2) has a solution in L^p for every $f \in L^p$. It satisfies the above estimate with $w = -1$:*

$$\|u\|_{L^p(\Omega)} \le \frac{p}{\min_{\overline{\Omega}}[-c^* + (1-p)c]} \|f\|_{L^p(\Omega)}.$$

Let us see how these results apply to Fuchsian operators. To take a specific example, assume that near the boundary,

$$a^{ij} = d^2 \delta^{ij}; \quad b^k = (1 - \alpha - \beta) d d^k; \quad c = \alpha\beta.$$

This is a Fuchsian operator with indices α and β. We obtain

$$c^* = (\alpha + 1)(\beta + 1) + (1 + \alpha + \beta) d \Delta d.$$

Since $l^k = -(1 + \alpha + \beta) d^k$, we obtain $b = 0$, so that $\partial\Omega = \Sigma_0$. The conditions $c < 0$ and $c^* < 0$ require that the indices have opposite signs, and that $\alpha + 1$ and $\beta + 1$ also have opposite signs. If $\alpha > 0$, for instance, this yields the condition $\beta < -1$. Intuitively, if $-1 < \beta < \alpha$, we expect that $Lu = 0$ admits solutions near the boundary that behave like d^α and d^β respectively, and both of these will be in L^p for p sufficiently close to 1. Therefore, it is to be expected that the homogeneous problem has a nontrivial solution, and that the equation $Lu = f$ is not solvable for arbitrary f.

We leave to the reader the consideration of the case $c = 0$, where one of the indices is zero; in that case, if f/d is in some L^p space, we may apply the same analysis to the equation $d^{-1} Lu = f/d$, which now has linear degeneracy.

We now turn to recent regularity results, which are not consequences of these classical results, and were in fact motivated by difficulties in the application of reduction to elliptic problems.

6.2 Schauder regularity for Fuchsian problems

The Hölder and weighted Hölder spaces and their properties are recalled in Sect. 12.2.1, and the usual Schauder estimates in Sect. 12.3; for further details, see [117]. Let us introduce a $C^{2+\alpha}$ domain $\Omega' \subset \Omega$ on which d is of class $C^{2+\alpha}$ and does not exceed $\frac{1}{2}$. We let $\Omega_\delta = \Omega \cap \{x : d(x) \le \delta\}$. The basic results for type (I) operators are as follows:

Theorem 6.7. *If $Ag = f$, where f and g are bounded and A is of type (I) on Ω', then $d\nabla g$ is bounded, and dg and $d^2\nabla g$ belong to $C^\alpha(\Omega' \cup \partial\Omega)$.*

Theorem 6.8. *If $Ag = df$, where f and g are bounded, $g = \mathcal{O}(d^\alpha)$, and A is of type (I) on Ω', then $g \in C^\alpha(\Omega' \cup \partial\Omega)$ and $dg \in C^{1+\alpha}(\Omega' \cup \partial\Omega)$.*

These two results are proved in the next subsection. The main result for type (II) operators is the following:

Theorem 6.9. *If $Ag = df$, where $f \in C^\alpha(\Omega' \cup \partial\Omega)$, $g = \mathcal{O}(d^\alpha)$, and A is of type (II) on Ω', then d^2g belongs to $C^{2+\alpha}(\Omega' \cup \partial\Omega)$.*

Proof. The assumptions ensure that $a^{ij}\partial_{ij}(d^2 f)$ is Hölder continuous and that f is bounded; $d^2 f$ therefore solves a Dirichlet problem to which the Schauder estimates apply near $\partial\Omega$. Therefore $d^2 f$ is of class $C^{2+\alpha}$ up to the boundary. Since we already know that $f \in C^\alpha(\overline{\Omega}_\delta)$ and df is of class $C^{1+\alpha}(\overline{\Omega}_\delta)$, we have indeed f of class $C^{2+\alpha}_\#(\overline{\Omega}_{\delta'})$ for $\delta' < \delta$.

Let $\rho > 0$. Throughout the proofs, $t \le \frac{1}{2}$, and we shall use the sets

$$
\begin{aligned}
Q &= \{(T, Y) : 0 \le T \le 2 \text{ and } |y| \le 3\rho\},\\
Q_1 &= \left\{(T, Y) : \frac{1}{4} \le T \le 2 \text{ and } |y| \le 2\rho\right\},\\
Q_2 &= \left\{(T, Y) : \frac{1}{2} \le T \le 1 \text{ and } |y| \le \rho/2\right\},\\
Q_3 &= \left\{(T, Y) : 0 \le T \le \frac{1}{2} \text{ and } |y| \le \rho/2\right\}.
\end{aligned}
$$

We may assume, by scaling coordinates, that $Q \subset \Omega'$. It suffices to prove the announced regularity on Q_3.

6.2.1 First "type (I)" result

We prove Theorem 6.7. Let $Af = g$, with A, f, g satisfying the assumptions of the theorem over Q, and let y_0 be such that $|y_0| \le \rho$. For $0 < \varepsilon \le 1$ and $(T, Y) \in Q_1$, let

$$f_\varepsilon(T, Y) = f(\varepsilon T, y_0 + \varepsilon Y),$$

and similarly for g and other functions. We have $f_\varepsilon = (Ag)_\varepsilon = A_\varepsilon f_\varepsilon$, where

$$A_\varepsilon = \partial_i(T^2 a^{ij}_\varepsilon \partial_j) + T b^i_\varepsilon \partial_i + c_\varepsilon$$

is also of type (I), with coefficient norms independent of ε and y_0, and is uniformly elliptic in Q_1. Interior estimates give

$$\|g_\varepsilon\|_{C^{1+\alpha}(Q_2)} \le M_1 := C_1(\|f_\varepsilon\|_{L^\infty(Q_1)} + \|g_\varepsilon\|_{L^\infty(Q_1)}). \tag{6.4}$$

The assumptions of the theorem imply that M_1 is independent of ε and y_0. We therefore obtain

$$|\varepsilon\nabla g(\varepsilon T, y_0 + \varepsilon Y)| \le M_1, \tag{6.5}$$

$$\varepsilon|\nabla g(\varepsilon T, y_0 + \varepsilon Y) - \nabla g(\varepsilon T', y_0)| \le M_1(|T - T'| + |Y|)^\alpha, \tag{6.6}$$

if $\frac{1}{2} \le T, T' \le 1$ and $|Y| \le \rho/2$. It follows in particular, taking $Y = 0$, $\varepsilon = t \le 1$, $T = 1$, and recalling that $|y_0| \le \rho$, that

$$|t\nabla g(t, y)| \le M_1 \text{ if } |y| \le \rho, t \le 1. \tag{6.7}$$

This proves the first statement of the theorem.

Taking $\varepsilon = 2t \le 1$, $T = \frac{1}{2}$, and letting $y = y_0 + \varepsilon Y$, $t' = \varepsilon T'$, we have

$$2t|\nabla g(t, y) - \nabla g(t', y_0)| \le M_1(|t - t'| + |y - y_0|)^\alpha (2t)^{-\alpha}$$

for $|y - y_0| \le \rho t$ and $t \le t' \le 2t \le 1$.

Let us prove that

$$|t^2\nabla g(t, y) - t'^2\nabla g(t', y_0)| \le M_2(|t - t'| + |y - y_0|)^\alpha \tag{6.8}$$

for $|y|, |y_0| \le \rho$ and $0 \le t \le t' \le \frac{1}{2}$, which will prove

$$t^2\nabla g \in C^\alpha(Q_3).$$

It suffices to prove this estimate in the two cases: (i) $t = t'$ and (ii) $y = y_0$; the result then follows from the triangle inequality. We distinguish three cases.

1. If $t = t'$, we need only consider the case $|y - y_0| \ge \rho t$. We then obtain

$$t^2|\nabla g(t, y) - \nabla g(t, y_0)| \le 2M_1 t \le 2M_1|y - y_0|/\rho.$$

2. If $y = y_0$ and $t \le t' \le 2t \le 1$, we have $t + t' \le 2t'$, hence

$$\begin{aligned} &|t^2\nabla g(t, y_0) - t'^2\nabla g(t', y_0)| \\ &\qquad \le t^2|\nabla g(t, y_0) - \nabla g(t', y_0)| + |t - t'|(t + t')|\nabla g(t', y_0)| \\ &\qquad \le M_1 2^{-1-\alpha} t^{1-\alpha}|t - t'|^\alpha + 2M_1|t - t'| \\ &\qquad \le M_2|t - t'|^\alpha. \end{aligned}$$

3. If $y = y_0$, and $2t \le t' \le \frac{1}{2}$, we have $t + t' \le 3(t' - t)$, and

$$|t^2\nabla g(t, y_0) - t'^2\nabla g(t', y_0)| \le M_1(t + t') \le 3M_1|t - t'|.$$

This proves estimate (6.8). On the other hand, since g and $T\nabla g$ are bounded over Q_3,

$$Tg \in \mathrm{Lip}(Q_3) \subset C^\alpha(Q_3).$$

This completes the proof of Theorem 6.7. □

6.2.2 Second “type (I)” result

We prove Theorem 6.8. The argument is similar to the previous proof, except that M_1 is now replaced by $M_3\varepsilon^\alpha$, with M_3 independent of ε and y_0. It follows that

$$|t\nabla g(t,y)| \le M_3 t^\alpha \quad \text{if } |y| \le \rho \text{ and } t \le 1. \tag{6.9}$$

Taking $\varepsilon = 2t \le 1$, $T = \frac{1}{2}$, letting $y = y_0 + \varepsilon Y$, $t' = \varepsilon T'$ and noting that $\varepsilon^\alpha(|T - T'| + |Y|)^\alpha = (|t - t'| + |y - y_0|)^\alpha$, we obtain

$$2t|\nabla g(t,y) - \nabla g(t',y_0)| \le M_3(|t-t'| + |y-y_0|)^\alpha$$

for $|y - y_0| \le \rho t$ and $t \le t' \le 2t \le 1$. Let us prove that

$$|t\nabla g(t,y) - t'\nabla g(t',y_0)| \le M_4(|t-t'| + |y-y_0|)^\alpha \tag{6.10}$$

for $|y|, |y_0| \le \rho$ and $0 \le t \le t' \le \frac{1}{2}$, which will prove

$$T\nabla g \in C^\alpha(Q_3).$$

We again distinguish three cases:

1. If $t = t'$, $|y - y_0| \ge \rho t$,
$$t|\nabla g(t,y) - \nabla g(t,y_0)| \le 2M_3 t^\alpha \le 2M_3(|y-y_0|/\rho)^\alpha.$$
2. If $y = y_0$ and $t \le t' \le 2t \le 1$, we have $|t - t'| \le t \le t'$; hence
$$\begin{aligned}|t\nabla g(t,y_0) - t'\nabla g(t',y_0)| &\le \frac{1}{2}M_3|t-t'|^\alpha + |t-t'||\nabla g(t',y_0)| \\ &\le M_3|t-t'|^\alpha\left(\frac{1}{2} + t'^{1-\alpha}t'^{\alpha-1}\right) \le 2M_3|t-t'|^\alpha.\end{aligned}$$
3. If $y = y_0$, and $2t \le t' \le \frac{1}{2}$, we have $t \le t' \le 3(t' - t)$ and
$$|t\nabla g(t,y_0) - t'\nabla g(t',y_0)| \le M_3(t^\alpha + t'^\alpha) \le 2M_3(3|t-t'|)^\alpha.$$

Estimate (6.10) therefore holds.

The same type of argument shows that

$$g \in C^\alpha(Q_3).$$

In fact, we have, with $\varepsilon = 2t$ again, $\|g_\varepsilon\|_{C^\alpha(Q_2)} \le M_5\varepsilon^\alpha$, where M_5 depends on the r.h.s. and the uniform bound assumed on f. This implies

$$|g(t,y) - g(t',y_0)| \le M_5(|t-t'| + |y-y_0|)^\alpha$$

if $t \le t' \le 2t \le 1$ and $|y - y_0| \le \rho t$. The assumptions of the theorem yield in particular

$$|g(t,y)| \le M_5 t^\alpha,$$

for $t \leq \frac{1}{2}$ and $|y| \leq \rho$.

If $\rho t \leq |y - y_0| \leq \rho$, and $t \leq \frac{1}{2}$, we have

$$|g(t,y) - g(t,y_0)| \leq 2M_5 t^\alpha \leq 2M_5 \left(\frac{|y-y_0|}{\rho}\right)^\alpha .$$

If $2t \leq t' \leq \frac{1}{2}$ and $y = y_0$,

$$|g(t,y_0) - g(t',y_0)| \leq M_5(t^\alpha + t'^\alpha) \leq 2M_5(3|t-t'|)^\alpha .$$

If $t \leq t' \leq 2t \leq \frac{1}{2}$, we already have

$$|g(t,y_0) - g(t',y_0)| \leq M_5|t-t'|^\alpha .$$

The Hölder continuity of g follows. Combining these pieces of information, we conclude that

$$g \in C^{1+\alpha}_{\#}(Q_3),$$

QED. □

6.3 Solution of a model Fuchsian operator

Consider the operator

$$L := d^2 \Delta + (4-n) d \nabla d \cdot \nabla + (2-2n),$$

where d denotes the distance to the boundary of a $C^{2+\alpha}$ domain. Choose a coordinate system in which $T = d$ is the first variable; for properties of this coordinate system, see Chap. 12. The goal of this section is the following result.

Theorem 6.10. *If δ is sufficiently small, there is a $w_0 \in C^{2+\alpha}_{\#}(\overline{\Omega}_\delta)$ such that*

$$L w_0 + 2\Delta d = 0 \tag{6.11}$$

in Ω_δ. Furthermore, on $\partial\Omega$,

$$w_0 = -H, \tag{6.12}$$

where $H = -(\Delta d)/(n-1)$ is the mean curvature of the boundary.

Theorem 6.10 is proved in three steps: first, one decomposes L into a sum $L_0 + L_1$ in a coordinate system adapted to the boundary, where L_0 is the analogue of L in a half-space; next, one solves $Lf = k + \mathcal{O}(d^\alpha)$ in this coordinate system for any function k of class C^α, such as $-2\Delta d$, by inverting a model operator closely related to L_0; finally, we patch together the results to obtain a function w_0 such that $Lw_0 = g$.

Decomposition of L

We use the notation of Chap. 12. The operator L splits as follows:

$$L = L_0 + L_1,$$

where

$$L_0 w = (D+2)(D+1-n)w + T^2 \Delta' w$$

and

$$L_1 w = (4-n)\tilde{\nabla} d \cdot \nabla'(Tw) + 2T\tilde{\nabla} d \cdot \nabla'(Dw) + T(Dw)\Delta d,$$

with $D = T\partial_T$. We now solve approximately the equation $Lf = k$ by solving exactly a model problem, related to the operator L_0.

Solution of $Lf = k + \mathcal{O}(d^\alpha)$

Let C^α_{per} denote the space of functions $k(Y,T) \in C^\alpha(0 \le T \le \theta)$ such that $k(Y_j + 2\theta, T) = k(Y_j, T)$ for $1 \le j \le n-1$. We prove the following theorem.

Theorem 6.11. *Let $\theta > 0$, and $k(Y,T)$ be of class C^α_{per}. There is a function f such that*

1. $L_0 f = k + \mathcal{O}(d^\alpha)$,
2. *f is of class* $C^{2+\alpha}_\#(0 \le T \le \theta)$,
3. $f(Y,0) = k(Y,0)/(2-2n)$, *and*
4. $L_1 f = \mathcal{O}(d^\alpha)$.

Proof. Let

$$L_0' = (D+2)(D-1) + T^2\Delta' = L_0 + (n-2)(D+2).$$

We first solve the equation $L_0' f_0 = k$.

Lemma 6.12. *There is a bounded linear operator*

$$G : C^\alpha_{per} \longrightarrow C^{2+\alpha}_\#(0 \le T \le \theta)$$

such that $f_0 := G[k]$ satisfies

1. $L_0' f_0 = k$,
2. *f_0 is of class* $C^{2+\alpha}_\#(0 \le T \le \theta)$,
3. $f_0(Y,0) + k(Y,0)/2 = 0$, $Df_0(Y,0) = 0$, *and*
4. $L_1 f_0 = \mathcal{O}(d^\alpha)$.

Proof. First, construct $\tilde{k}$ such that $(D-1)\tilde{k} = -k$, and $\tilde{k}$ and $D\tilde{k}$ are both C^α up to $T = 0$. One may take

$$\tilde{k} = \int_1^\infty F_1[k](Y, T\sigma)\frac{d\sigma}{\sigma^2},$$

where F_1 is an extension operator, so that $F_1[k] = k$ for $T \leq \theta$. One checks that $\tilde{k} = k$ for $T = 0$.

Next, solve $(\partial_{TT} + \Delta')h + \tilde{k} = 0$ with periodic boundary conditions, of period 2θ, in each of the Y_j variables, and $h(Y, 0) = h_T(Y, \theta) = 0$; this yields

$$h \text{ is of class } C^{2+\alpha}(0 \leq T \leq \theta)$$

by the Schauder estimates. In particular, h_T is continuous up to $T = 0$, and $Dh = 0$ for $T = 0$ and $T = \theta$. Since $h = 0$ for $T = 0$, we also have $\Delta' h = 0$ for $T = 0$. The equation for h therefore gives

$$h_{TT} = -\tilde{k} = -k \text{ for } T = 0.$$

In addition,

$$(\partial_{TT} + \Delta')Dh = D(\partial_{TT} + \Delta')h + 2h_{TT} = k - \tilde{k} + 2h_{TT},$$

which is C^α. Since, on the other hand, Dh is of class C^1 and $Dh = 0$ for $T = 0$ and $T = \theta$, we conclude, using again the Schauder estimates, that

$$Dh \text{ is of class } C^{2+\alpha}(0 \leq T \leq \theta).$$

We now define f_0 by

$$f_0 := T^{-2}(D-1)h = \partial_T \left(\frac{h}{T} \right) = \int_0^1 \sigma h_{TT}(Y, T\sigma)\, d\sigma. \tag{6.13}$$

Since f_0 is itself uniquely determined by h, itself defined in terms of k, we define a map G by

$$G[k] = f_0.$$

A direct computation yields $L'_0 f_0 = k$:

$$\begin{aligned} L'_0 f_0 &= (D+2)(D-1)T^{-2}(D-1)h + (D-1)\partial_Y^2 h \\ &= T^{-2}D(D-3)(D-1)h + (D-1)\left\{-T^{-2}D(D-1)h - \tilde{k}\right\} \\ &= T^{-2}D(D-1)(D-3)h - T^{-2}(D-3)D(D-1)h - (D-1)\tilde{k} \\ &= k. \end{aligned}$$

Let us now consider the regularity of f_0 up to $\partial\Omega$, and the values of f_0 and its derivatives on $\partial\Omega$. Consider $g_0 := T^2 f_0$. Since $g_0 = (D-1)h$ belongs to $C^{2+\alpha}(0 \leq T \leq \theta)$ and vanishes for $T = 0$, we have $g_0 = \int_0^1 g_{0T}(Y, T\sigma) T d\sigma$. It follows that

$$T f_0(Y, T) = \int_0^1 g_{0T}(Y, T\sigma) d\sigma \in C^{1+\alpha}(0 \leq T \leq \theta).$$

Since, on the other hand, $G[k] = \int_0^1 \sigma h_{TT}(Y, T\sigma)\, d\sigma$, we find that $f_0 \in C^\alpha(0 \leq T \leq \theta)$ and

$$f_0(Y,0) = \frac{1}{2}h_{TT}(Y,0) = -\frac{1}{2}k(Y,0).$$

We therefore have

$$f_0 \text{ is of class } C^{2+\alpha}_{\#}(0 \le T \le \theta).$$

Since

$$(D+2)f_0 = T^{-2}D(D-1)h = h_{TT},$$

$Df_0(Y,0) = h_{TT}(Y,0) - 2f_0(Y,0) = 0$. By differentiation with respect to the Y variables, we obtain that $\tilde{\nabla} d \cdot \nabla'(Tf_0)$ is of class C^α and vanishes for $T = 0$. The same is true of $T(Df_0)\Delta d$. Similarly,

$$2T\tilde{\nabla} d \cdot \nabla' Df_0 = 2\tilde{\nabla} d \cdot \nabla'[\partial_T(T^2 f_0) - 2Tf_0]$$

is of class C^α, and vanishes for $T = 0$ because this is already the case for TDf_0. It follows that $L_1 f_0$ is a C^α function that vanishes for $T = 0$; it is therefore $\mathcal{O}(d^\alpha)$; as desired.

We are now ready to complete the proof of Theorem 6.11. Let a be a constant and $f = G[ak]$. We have $L_0' f = ak$ and for $T = 0$, $f = -\frac{1}{2}ak$. Since $L_1 f \in C^\alpha$, and $L_1 f$ and Df both vanish for $T = 0$, it follows that for $T = 0$,

$$Lf - k = (L_0' - (n-2)(D+2) + L_1)f - k = [a + (n-2)a - 1]k.$$

Taking $a = 1/(n-1)$, we find that f has the announced properties. □

Solution of $Lw_0 = g$

Let us now consider a function g of class $C^\alpha(\overline{\Omega}_\delta)$. Recall that there is a positive $r_0 < \delta$ such that any ball of radius r_0 centered at a point of the boundary is contained in a domain in which we have a system of coordinates of the type (Y, T). Let us cover (a neighborhood of) $\partial\Omega$ by a finite number of balls $(V_\lambda)_{\lambda\in\Lambda}$ of radius $r_1 < r_0$ and centers on $\partial\Omega$, and consider the balls $(U_\lambda)_{\lambda\in\Lambda}$ of radius r_0 with the same centers. We may assume that every U_λ is associated with a coordinate system (Y_λ, T_λ) of the type considered in Sect. 12.1; taking r_1 smaller if necessary, we may also assume that $\overline{V}_\lambda \subset Q_\lambda \subset U_\lambda$, where Q_λ has the form

$$Q_\lambda := \{(Y_{\lambda,1}, \dots, Y_{\lambda,n-1}, T_\lambda) : 0 \le Y_{\lambda,j} \le \theta \text{ for every } j, \text{ and } 0 < T_\lambda < \theta\}.$$

Consider a smooth partition of unity (φ_λ) and smooth functions (Φ_λ), such that

1. $\sum_{\lambda\in\Lambda} \varphi_\lambda = 1$ near $\partial\Omega$;
2. $\operatorname{supp} \varphi_\lambda \subset V_\lambda$;
3. $\operatorname{supp} \Phi_\lambda \subset U_\lambda \cap \{T < \theta\}$;
4. $\Phi_\lambda = 1$ on V_λ.

In particular, $\Phi_\lambda\varphi_\lambda = \varphi_\lambda$. The function $g\varphi_\lambda$ is of class $C^\alpha(\overline{Q}_\lambda)$; it may be extended by successive reflections to an element of C^α_{per}, with period 2θ in the Y_λ variables; this extension will be denoted by the same symbol for simplicity. Let us apply Theorem 6.11, and consider, for every λ, the function $w_\lambda := G[g\varphi_\lambda/(n-1)]$. We have

$$Lw_\lambda = g\varphi_\lambda + R_\lambda,$$

in $U_\lambda \cap \{T < \theta\}$, where R_λ is Hölder continuous for $T \le \theta$, and vanishes on $\partial\Omega$; as a consequence, $R_\lambda = \mathcal{O}(d^\alpha)$. The function $\Phi_\lambda w_\lambda$ is compactly supported in U_λ, and may be extended, by zero, to all of Ω; it is of class $C^{2+\alpha}_\#(\overline{\Omega})$. We may therefore consider

$$w_1 := \sum_{\lambda\in\Lambda} \Phi_\lambda w_\lambda,$$

which is supported near $\partial\Omega$. Now, near $\partial\Omega$,

$$\begin{aligned}\sum_\lambda L(\Phi_\lambda w_\lambda) &= \sum_\lambda \Phi_\lambda L(w_\lambda) + 2d^2\nabla\Phi_\lambda\cdot\nabla w_\lambda + d^2 w_\lambda \Delta\Phi_\lambda \\ &\qquad\qquad + (4-n)w_\lambda d\nabla d\cdot\nabla\Phi_\lambda \\ &= \sum_\lambda [g\Phi_\lambda\varphi_\lambda + R'_\lambda] = g + f,\end{aligned}$$

where $f = \sum_\lambda R'_\lambda$ has the same properties as R_λ. It therefore suffices to solve $Lw_2 = f$ when f is a Hölder continuous function vanishing on the boundary.

Lemma 6.13. *For any $f \in C^\alpha(\overline{\Omega})$, there is, for δ small enough, an element $w_2 \in C^{2+\alpha}_\#(\overline{\Omega}_\delta)$ such that*

$$Lw_2 = f \text{ and } w_2 = \mathcal{O}(d^\alpha) \text{ near } \partial\Omega.$$

Proof. Consider the solution w_ε of the Dirichlet problem $Lw_\varepsilon = f$ on a domain of the form $\{\varepsilon < d(x) < \delta\}$, with zero boundary data. As before, δ is taken small enough to ensure that $d \in C^{2+\alpha}(\overline{\Omega}_\delta)$. Schauder theory gives $w_\varepsilon \in C^{2+\alpha}(\{\varepsilon \le d(x) \le \delta\})$. By assumption, $|f| \le ad^\alpha$ for some constant a. Let $A > (\alpha+2)(n-1-\alpha)$. Since

$$-L(d^\alpha) = d^\alpha[(\alpha+2)(n-1-\alpha) - \alpha d\Delta d],$$

$Ad(x)^\alpha$ is a supersolution if δ is small, and the maximum principle gives us a uniform bound on w_ε/d^α. By interior regularity, we obtain that for a sequence $\varepsilon_n \to 0$, the w_{ε_n} converge in C^2, in every compact set away from the boundary, to a solution w_2 of $Lw_2 = f$ with $w_2 = \mathcal{O}(d^\alpha)$. Since the right-hand side f is also $\mathcal{O}(d^\alpha)$, we obtain, by the "type (I)" Theorem 6.8, that w_2 is of class $C^{1+\alpha}_\#(\overline{\Omega}_\delta)$. Theorem 6.9 now ensures that w_2 is in fact of class $C^{2+\alpha}_\#(\overline{\Omega}_\delta)$, QED. □

It now suffices to take $g = -2\Delta d$ and let

$$w_0 = w_1 - w_2.$$

By construction, $Lw_0 + 2\Delta d = 0$ near the boundary, and w_0 is of class $C^{2+\alpha}_{\#}(\overline{\Omega}_\delta)$ if δ is small. In addition, we know from Theorem 6.11 that on $\partial\Omega$, $w_1 = (2\Delta d)/(2n-2)$, which is equal to $-H$ on $\partial\Omega$. Lemma 6.13 gives us $w_2 = \mathcal{O}(d^\alpha)$. We conclude that $w_0 = -H$ on the boundary.

This completes the proof of Theorem 6.10. □

Problems

6.1. State and prove regularity results corresponding to type (I) operators, assuming that A also contains terms of the form $\partial_i(b'^i u)$ in Au, with b'^i of class C^α.

6.2. Let Ω be a bounded domain of class $C^{2+\alpha}$. For $\sigma > 0$, convert the problem $\Delta(ud^{-\sigma}) = f \in C^\alpha$ into an equation of type (II) near the boundary. Same question for $\operatorname{div}(d^{-\sigma}\nabla u) = f$.

Part III

Applications

7

Applications in Astronomy

Stellar modeling leads to a system of equations exhibiting a singularity at the center of the star, which is an obstacle to direct integration of the equations of the model. We illustrate the use of reduction to overcome this difficulty in two typical cases: the polytropic and the point-source model.

7.1 Notions on stellar modeling

The information we have on stars is exclusively derived from light received from them. We thus obtain their composition, and indirect information about mass, distance, or relative velocities. The inference of such information relies on stellar modeling. Stars have been classified into a relatively small number of categories, such as white dwarfs or red giants, which have been arranged into evolutionary sequences. These sequences have also been inferred from the study of stellar models, in which stars are considered approximately in equilibrium for long times, until a change of composition, or some other event, makes the star evolve from one stage of a sequence to the next. The great majority of current work is concerned with spherically symmetric stars, a case in which many issues are still unresolved.

The modeling process consists in relating assumptions on stellar composition, and on the mechanisms of generation of radiant energy, to the possible values of mass, size, temperature, and luminosity of the star. The mathematical difficulty comes from the fact that, as we shall see, the equations are singular at the center of the star, which makes numerical methods break down at the center. It is not possible to determine the solution by integrating from the outside in, because there are too few boundary conditions at the surface. A possibility consists in giving arbitrary values to the missing data on the surface, and adjusting them by trial and error, but again, the singularity at the center causes the computation to break down close to the center. Reduction, by giving the correct behavior at the center, and by relating the coefficients of the asymptotics at the center to the behavior at small but positive distance from the center, helps to push the computation through.

We describe a general class of models of gaseous stars in Newtonian mechanics, and explain the upshot of the reduction process in sample cases. The star, of mass M_s, is assumed to lie in the ball of radius R about the origin. We take $R < \infty$ for simplicity. Let S_r denote the sphere of radius $r \leq R$. The unknowns are (i) the mass $M(r)$ contained within S_r; (ii) the pressure $P(r)$, density $\rho(r)$, and temperature $T(r)$ on S_r; (iii) the luminosity $L(r)$, representing the net outward flow of energy per second through S_r. The density is related to M by

$$\frac{dM}{dr} = 4\pi r^2 \rho. \tag{7.1}$$

Boundary conditions are (i) at the surface, $r = R$, $M = M_s$ (total mass), $P = 0$, $T = 0$, $L = L_s$ (total luminosity of the star); (ii) at the center, $r = 0$, $M = 0$, $L = 0$, and P and T have two unknown values P_c and T_c. Note that M_s and L_s are not known a priori either.

It is clear that M is an increasing function of r; one may therefore view either (M, P, T, L) as functions of r, or (r, P, T, L) as functions of M.

The condition of hydrostatic equilibrium expresses the equation $\rho\mathbf{g} = \nabla P$, where $\mathbf{g} = -g(r)\mathbf{r}/r$ is the acceleration of gravity, which points inward, with magnitude $g(r)$. Since $\operatorname{div}\mathbf{g} = -4\pi G\rho$ (Poisson's equation), where G is constant, we find, after integration by parts on the ball of radius r, that $4\pi r^2 g(r) = 4\pi G\rho(r)$. This expresses the fact that the gravitational field on S_r is the same as if all of the mass within S_r were concentrated at the center. We therefore have

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}. \tag{7.2}$$

The pressure is the sum of gas and radiation pressure:

$$P = P_{\text{gas}} + P_{\text{rad}}.$$

One takes

$$P_{\text{rad}} = \frac{1}{4}aT^4,$$

where $a = 7.56 \cdot 10^{-15}$ cgs. It is negligible for the Sun, but may be appreciable in other stars. The gas pressure is given, in the case of a perfect gas, by

$$P_{\text{gas}} = \frac{RT}{V} = nkT = \frac{\rho}{\mu}RT,$$

where one mole of molecules has a mass of μ grams and occupies a volume V. $R = 8.32 \cdot 10^7$ cgs $= k/m_H$, where $m_H = 1.66 \cdot 10^{-24}$ g. Composition determines μ by an averaging process. Thus, assume that one gram of the star contains X grams of hydrogen, Y of helium, and Z of other elements, with $X + Y + Z = 1$. Assuming total ionization, each hydrogen atom contributes two particles (one proton and one electron), and each helium atom contributes three (one nucleus and two electrons). Let N_A be Avogadro's number and neglect Z. A mole of gas contains XN_A hydrogen atoms, and $\frac{1}{4}YN_A$ helium atoms, because the helium nucleus is (about) four times heavier than the hydrogen nucleus. Therefore, the total number of particles is $(2X + \frac{3}{4}Y)N_A$. It follows that $\mu^{-1} = (2X + \frac{3}{4}Y) = (\frac{3}{4} + \frac{5}{4}X)$, taking the relation $X + Y = 1$ into account.

Energy balance is expressed in two stages. First, let $L(r)$ denote the net flow of radiation across the surface S_r. The difference between $L(r+dr)$ and $L(r)$ corresponds to the energy balance emitted in the layer bounded by S_r and S_{r+dr}. Letting $\varepsilon(r)$ denote the total rate of energy outflow per second and per gram in this layer, we have $dL = \varepsilon dM$, or

$$\frac{dL}{dr} = 4\pi r^2 \varepsilon \rho. \tag{7.3}$$

A possible choice is $\varepsilon \propto \rho T^{\nu}$.

Finally, the temperature may change as a result of the absorption of radiant energy by the layers of the star; this is generally written

$$\frac{dT}{dm} = -\frac{GM\rho T}{r^2 P}\nabla^*, \tag{7.4}$$

where ∇^* equals $d\ln T/d\ln P$ in convective regions (and is therefore determined by the equation of state).[1] If radiation is responsible for energy transport, ∇^* is given by

$$\frac{3}{16\pi acG}\frac{\kappa LP}{MT^4},$$

where the opacity κ may be given by expressions of the form $\kappa_0\rho^{\alpha}T^{-\beta}$, with α and β positive. A possible choice is $T \propto P^{2/5}$ for convection, and $\kappa = \kappa_0\rho^{2-\alpha}T^{-7/2}$ with $\alpha = 0$ or $\frac{1}{4}$, and $\kappa_0 = 2.4 \cdot 10^{23}(1+X)^{3/4}$.

The problem is to integrate the set (7.1–4) of four equations in four unknowns for (M, P, T, L). Four special cases may be mentioned, of which we study two: (i) polytropic case: ρ is proportional to a power of T, or equivalently, P is proportional to a power of ρ; (ii) point-source model: $\varepsilon = 0$, κ constant; (iii) isothermal case: $P = K_0\rho + K_1$ with K_0 and K_1 constant, and T is constant; (iv) radiative envelopes: $\varepsilon = 0$ and $r > r_0$.

7.2 Polytropic model

Write $\rho = \lambda T^n$, so that, since P is proportional to ρT, we find $P = KT^{n+1}$. Substituting into equation (7.2) and solving for $M(r)$, we find that $M(r)$ is

[1] This quantity is often denoted by ∇ in the astrophysics literature; we write it ∇^*, to distinguish it from the gradient.

proportional to $r^2 dT/dr$. Equation (7.1) now takes the form of the *Lane–Emden–Fowler* equation

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} + CT^n = 0,$$

see Problem 7.1.

7.3 Point-source model

We assume the source of radiant energy of the star is concentrated at its center, so that $L = L_s$ is constant, and that κ is constant. According to [40, pp. 354–355], this is appropriate if the source of radiant energy is limited to a region of radius less than 17% of the radius of the star. The equations reduce to (7.1), (7.2), and (7.4); the latter takes the form

$$\frac{dP_{\text{rad}}}{dr} = -\frac{\kappa L_s}{4\pi c r^2}\rho,$$

with $P_{\text{rad}} = \frac{1}{3}aT^4$. From this equation and (7.2), we obtain

$$M(r) = \frac{\kappa L_s}{4\pi c G}\left(\frac{dP}{dP_{\text{rad}}} + 1\right).$$

Differentiating once more with respect to P_{rad}, we obtain

$$\frac{\kappa L_s}{4\pi c G}\frac{d^2P}{dP_{\text{rad}}^2} = 4\pi r^2 \rho \frac{dr}{dP_{\text{rad}}} = -\frac{16\pi^2 c}{\kappa L_s} r^4.$$

Similarly, using the relations $P = (k\rho/\mu m_H)T$ and the definition of P_{rad}, we obtain

$$\frac{d}{dP_{\text{rad}}}(1/r) = -\frac{1}{r^2}\frac{dr}{dP_{\text{rad}}} = \frac{4\pi c}{\kappa L_s}\frac{k}{\mu m_H}\frac{1}{P}(3P_{\text{rad}}/a)^{1/4}.$$

Define new unknowns ξ, z, and t by $r = A\xi$, $P = \Pi z$, $P_{\text{rad}} = \Pi_{\text{rad}} t$; choosing the constants A, Π, and Π_{rad} appropriately, we obtain the equations

$$\frac{d^2z}{dt^2} = -\xi^4; \quad \frac{d}{dt}(1/\xi) = z^{-1}t^{1/4}.$$

Setting $x = 1/\xi$, we are led to the following mathematical problem. Consider the system

$$dx/dt = z^{-1}t^{1/4}, \qquad d^2z/dt^2 = -x^{-4}, \tag{7.5}$$

where $z(t)$ and $x(t)$ represent respectively dimensionless pressure and inverse distance to the center of a star, as a function of radiation pressure t, which is proportional to the fourth power of temperature T; see [40]. Thus, solutions in which x becomes infinite as $t \to b$ represent the behavior at the center of the star.

The problem is to investigate solutions defined for $0 \le a < t < b \le \infty$, for given a and b. We prove the stability of a family of solutions in which x has logarithmic blowup as $t \uparrow b < \infty$, thus justifying the computations sketched in [40]: locally, all solutions are contained in this family, parameterized by b and two other parameters c and d that determine the leading asymptotics. Our results also give rigorous bounds on the error made if one keeps only the first few terms. As a result, the formal asymptotics may be used as exact solutions; they provide quantitative information near singularities—a place where numerical computations become difficult. In Chandrasekhar's interpretation of this regime, "this asymptotic behavior corresponds to the case of the density falling off exponentially as $[r = 1/x(t)] \to 0$, while the temperature very slowly attains its maximum; the central regions will be practically isothermal." Indeed, $b - t$ varies like the $T_b^4 - T^4$, where T_b is the central temperature. We may add, as a result of our analysis, that even though the difference between T^4 and T_b^4, where T_b is the central temperature, decays exponentially as $r \to 0$, this exponential is multiplied by a coefficient (namely, the exponential of the parameter d) that may take any value. Our stability results make it possible in principle to compute it in some cases, since d is given by an implicit function theorem. It would be interesting to implement such a method numerically.

Let $T = b - t$. The discussion in [40] suggests that one look for solutions in which z' has a negative limit at $T = 0$, while x behaves logarithmically. Our results are as follows [91].

Theorem 7.1. *For any b, c, d with b and c positive:*

- *There is a unique solution such that*

$$x = b^{1/4}c^{-1}|\ln T| + d + \mathcal{O}(|\ln T|^{-3}), \tag{7.6}$$

$$z = cT + \mathcal{O}(T^2|\ln T|^{-3}). \tag{7.7}$$

- *These solutions are stable: if $b - t_0$ is small, the solution with initial data prescribed at t_0 close to $(x(t_0), z(t_0), z'(t_0))$ has the same asymptotics as above, with slightly different values of b, c, and d.*

Remark 7.2. For a more complete study, one should also investigate other possible singular behaviors; for instance, one may seek x and z equivalent to powers of $(t - b)^{1/3}$. A similar theorem could be stated and proved for them. The argument leads to a Fuchsian system without logarithmic terms. However, these solutions are positive for $t > b$ rather than $t < b$ and are therefore not considered here.

Proof. For the existence statement, we define renormalized unknowns $X(T)$ and $Z(T)$ by

$$x = b^{1/4}c^{-1}|\ln T| + d + X(T), \tag{7.8}$$

$$z = cT + T^2 Z(T). \tag{7.9}$$

Let $D = T\,d/dT$. We obtain

$$DX = Tf(T, Z), \tag{7.10}$$

$$(D+1)(D+2)Z = |\ln T|^{-4} g(X, 1/\ln T), \tag{7.11}$$

where f and g are smooth near the origin. It follows that (X, Z, DZ) solves a Fuchsian system to which Theorem 5.3 applies, with $\lambda = (b, c, d)$, $M = 1$, $\alpha = 3$, and $a = 4$. Existence follows. The bounds in the theorem follow from the expression of x and z in terms of X and Z. It is easy to see from the equation for X that $X = \mathcal{O}(T)$. Theorem 5.5 ensures that the solution is smooth in b, c, and d.

To prove stability (step H of the algorithm of Chapter 1), one must prove that these three parameters are nonredundant. This is achieved by considering the map

$$(b, c, d) \mapsto (x(t_0), z(t_0), z'(t_0)). \tag{7.12}$$

We now prove that this map can be locally inverted using the inverse function theorem. This will complete the proof. From now on, (b, c, d) will be assumed to lie in a small neighborhood of some reference value (b_0, c_0, d_0) with b_0 and c_0 positive, $T = b - t$ small. Now, (X, Z, DZ) solves a Fuchsian system with right-hand side depending smoothly on (b, c, d), and the linearized system satisfies the assumptions of Theorem 5.3; the partial derivatives of the solution with respect to these parameters are computed by solving the linearization, and are therefore $\mathcal{O}(|\ln T|^{-3})$ for T small, uniformly in (b, c, d). Furthermore, $D^2 Z$ also satisfies a similar bound, using equation (7.11).

We may now differentiate the equations

$$x(t; b, c, d) = b^{1/4} c^{-1} |\ln(b-t)| + d + X(b-t; b, c, d), \tag{7.13}$$

$$z(t; b, c, d) = c(b-t) + (b-t)^2 Z(b-t; b, c, d), \tag{7.14}$$

$$z'(t; b, c, d) = -c - (b-t)[(D+2)Z](b-t; b, c, d). \tag{7.15}$$

The Jacobian determinant of (x, z, z') with respect to (b, c, d) now becomes, using the information we have on X, Z, and their derivatives,

$$\begin{vmatrix} b^{1/4}(cT)^{-1} + \mathcal{O}(|\ln T|) & -b^{1/4}c^{-2}|\ln T| + \mathcal{O}(1) & 1 + o(1) \\ c + \mathcal{O}(T) & T + \mathcal{O}(T^2) & \mathcal{O}(T^2) \\ \mathcal{O}(|\ln T|^{-3}) & -1 + \mathcal{O}(T) & \mathcal{O}(T) \end{vmatrix} = -c + o(1)$$

as T tends to zero; the estimates on remainders ($o(1)$ etc.) are uniform in the parameters in the range under consideration. This shows that the Jacobian of the map (7.12) is nonzero for t_0 close to b, and completes the proof.

Problems

7.1. Apply reduction to the search of radial solutions of $-\Delta u + f(u) = 0$ for $r > 0$, small, in N space dimensions. Study in particular the following cases:

1. $u = a + \mathcal{O}(r)$ as $r \to 0$, f smooth.
2. $u \sim br^{\nu}$ as $r \to 0$, $f(u) = u^q$. Show in particular that there are two types of singular solutions; those of the first solve $-\Delta u + f(u) = c\delta$, the others are more singular; the latter are generally called "very singular solutions." Are these solutions stable?

8

Applications in General Relativity

This chapter presents the first two applications of reduction in relativity. At the present time, reduction appears to be the only systematic procedure for finding solutions of Einstein's equations that at the same time (i) contains arbitrary functions as opposed to constants; (ii) enables explicit description of singular behavior, in principle to all orders; (iii) is not a "group-theoretic generation technique": it does not use in any essential way the algebraic structure of the symmetry group. Both applications deal with cosmological models, that is, models of the space-time at large scale; they were chosen because they are the first and simplest illustration of the impact of reduction techniques on general relativity; other applications to more complicated settings may be found in the literature. For background information on general relativity, see Chap. 6 of [104], the notation of which is followed here.

We focus on the mathematical issues addressed by reduction methods in applications to general relativity. In particular, reduction techniques give a mathematically rigorous explanation for "asymptotically velocity-dominated[1] behavior," in which time derivatives become more important than space derivatives near the cosmological singularity. The following is by no means a complete review of cosmological models current today among physicists, nor a suggestion that this particular class of cosmological models is better or worse than others.

8.1 The big-bang singularity and AVD behavior

The universe at large scale appears to be isotropic and homogeneous. The standard Friedmann–Lemaître–Roberston–Walker (FLRW) model uses a global time function with respect to which the spatial metric expands at a constant rate (Hubble effect), leading in particular to a contribution to the redshift of distant objects. The "big bang" is obtained by extrapolating

[1] Some authors speak of "asymptotically velocity-term dominated" solutions.

this behavior to the distant past. However, since the universe is not exactly isotropic and homogeneous, it is necessary to embed the FLRW solution into a family of nearly homogeneous and isotropic solutions of Einstein's equations. This would establish the stability of the big bang. It was suggested in the 1960s [13] that stable singularities of Einstein's equations do not behave like the FLRW solution in the distant past, but rather like one of the Kasner solutions of the vacuum equations:

$$-dt^2 + \sum_{j=1}^{3} t^{2p_j} (dx^j)^2,$$

where $\sum_j p_j = \sum_j p_j^2 = 1$.[2] The p_j are called Kasner exponents. In view of difficulties in this program, the alternative view that one should look for a succession of "epochs" during each of which the metric should be close to a different Kasner solution was also suggested (see [83, 181, 13, 182]).[3] Homogeneous solutions of this type may be constructed by ODE methods. The question whether singularities could be destroyed by perturbation remained unanswered.

The situation changed with the Hawking–Penrose singularity theorems [78], which assert that some kind of singular behavior (geodesic incompleteness) must occur in cosmological solutions under a sign condition on the Ricci tensor. However, the singularity theorems, by their very generality, give no analytic information on what actually happens at this singularity.

The next step was taken in 1972 [54] with the suggestion that solutions should be "asymptotically velocity-term dominated" (AVD): one still assumes that a cosmic time coordinate t exists and requires that the metric tensor $g_{ab}(\mathbf{x}, t)$ evolve in such a way that an observer with given $\mathbf{x}_0$ moving toward the singularity sees the dynamics of $g_{ab}(\mathbf{x}_0, t)$ asymptotically approach that of a Kasner space-time, with possibly a different Kasner limit for each different $\mathbf{x}_0$ (see [54, 13, 89, 122]).

AVD behavior in a family of inhomogeneous solutions was first studied on the case of the so-called polarized Gowdy space-times; see [49]. In this case, the field equations reduce to the linear Euler–Poisson–Darboux equation. For this reason, the techniques developed in proving that result have not been readily extended to more general families. Instead, most of the recent evidence for AVD and oscillatory behavior in cosmological space-times has been based on numerical work: Berger and Moncrief [18] provide strong numerical

[2] In this line of thought, it is widely held that there is an initial singularity, near which the right-hand side of Einstein's field equations become negligible compared to the terms on the left-hand side: the latter are therefore expected to balance each other rather than the r.h.s. This is expressed by saying that "matter does not matter."

[3] In fact, the history of the universe accessible to observation seems to cover a relatively small number of those epochs, so that such oscillatory behavior seems difficult to check against observed data.

evidence for AVD behavior in general Gowdy space-times, but find that the Kasner exponents should satisfy some inequalities in generic solutions (the solutions should be "low-velocity"). A special feature of Einstein's equations is the presence of "constraint" equations that are propagated as a consequence of the other field equations; the situation may be compared with the equation $\operatorname{div} B = 0$ in Maxwell's theory, which does not contain time derivatives. We meet a similar difficulty in Sect. 9.4. It is not always easy to be sure, in numerical computations, that the constraint equations do hold. Now, the computations lead to "spiky" behavior in some of the components, and it was not clear whether one should interpret these spikes as indicative of chaotic or AVD behavior.

The first mathematical proof of AVD behavior for solutions without closed-form solutions was one of the first successes of reduction techniques [122, 119] by providing solutions with explicit asymptotics that can take over precisely when numerical computation fails. In particular, one can identify the location of the spikes, after discounting those related to a bad choice of coordinates, with the critical points of the function $X_0(x)$; see Sect. 8.2.3. We now present these results.

8.2 Gowdy space-times

Gowdy spacetimes [70] are Lorentzian manifolds with spacelike slices homeomorphic to the three-torus, on which a two-dimensional abelian isometry group acts without fixed points. Gowdy $\mathbb{T}^3$ space-times [70] have been extensively studied over the years [89, 48, 47, 18, 122]. It is convenient to take as time coordinate the area t of the orbits of this two-dimensional group; the space-time corresponds to the region $t > 0$. The metric then takes the form

$$ds^2 = e^{\lambda/2} t^{-1/2}(-dt^2 + dx^2) + t[e^{-Z}(dy + X\,dz)^2 + e^Z dz^2],$$

where λ, X, and Z are functions of t and x only, and are periodic of period 2π with respect to x. A related metric occurs in the framework of axial symmetry, where a radial variable r plays the role of t; our results apply to this situation with straightforward modifications. The system is equivalent to the vanishing of the Ricci tensor of this metric.

8.2.1 The field equations

The mathematical problem is the analysis of the behavior, as $t \to 0+$, of the following system in two unknowns X and Z, which depend on two real variables $t > 0$ and x:

$$D^2 X - t^2 X_{xx} = 2(DX\,DZ - t^2 X_x Z_x), \tag{8.1}$$

$$D^2 Z - t^2 Z_{xx} = -e^{-2Z}((DX)^2 - t^2 X_x^2), \tag{8.2}$$

$$\lambda_x = 2(Z_x DZ + e^{-2Z} X_x DX), \tag{8.3}$$

$$D\lambda = (DZ)^2 + t^2 Z_x^2 + e^{-2Z}((DX)^2 + t^2 X_x^2), \tag{8.4}$$

where subscripts denote derivatives, and $D = t\partial_t$. The last two equations arise respectively from the momentum and Hamiltonian constraints. It suffices to solve the first two equations, and we therefore focus on them from now on. One should also ensure that the integral of λ_x from 0 to 2π vanishes.

The equations for X and Z are often interpreted as expressing that (X, Z) generates a "harmonic-like" map from 1+1 Minkowski space with values in hyperbolic space with the metric

$$dZ^2 + e^{-2Z} dX^2.$$

The usual Cartesian coordinates on the Poincaré model of hyperbolic space are X and $Y = e^Z$, so that the metric coincides with the familiar expression $(dX^2 + dY^2)/Y^2$. It is occasionally useful to use polar coordinates (w, ϕ) on the hyperbolic space, so that the metric on the target space is

$$dw^2 + \sinh^2 w \, d\phi^2.$$

The equations for w and ϕ then take the form

$$D^2 w - t^2 \partial_{xx} w = \frac{1}{2} \sinh 2w[(Dw)^2 - t^2 w_x^2]; \tag{8.5}$$

$$D^2 \phi - t^2 \partial_{xx} \phi = -2 \coth w [Dw D\phi - t^2 w_x \phi_x]. \tag{8.6}$$

For fixed t, the solution represents a *loop* in hyperbolic space. For extensive references on Gowdy space-times, see [47, 49, 18, 74].

8.2.2 Exact solutions

If $X = Z = 0$, we recover a metric equivalent to the Kasner solution with exponents $(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3})$. Other Kasner solutions are recovered for $X = 0$ and $Z = k \ln t$; the corresponding Kasner exponents are $(k^2 - 1)/(k^2 + 3)$, $2(1 - k)/(k^2 + 3)$, and $2(1 + k)/(k^2 + 3)$.

If X vanishes identically, the equation for Z reduces to a linear Euler–Poisson–Darboux equation, which can be solved explicitly. This provides a family of solutions involving two arbitrary functions that satisfy

$$Z \sim k \ln t + \mathcal{O}(1),$$

where k now depends on x and can be arbitrary. Such spaces are called "polarized" Gowdy space-times.

Both sets of equations (8.1–8.4) and (8.5–8.6) can be solved exactly if we seek solutions independent of x. In terms of the (X, Z) variables for example, these solutions have leading behavior of the form

$$Z \sim k \ln t + \mathcal{O}(1), \qquad X = \mathcal{O}(1),$$

where k is a positive constant and represent solutions in which the loop degenerates to a point that follows a geodesic and tends to a point at infinity in hyperbolic space.

Numerical computations suggest more complicated behavior in the full nonlinear system for X and Z [18]. Indeed, if one monitors the "velocity" $v(x,t) = \sqrt{(DZ)^2 + \exp(-2Z)(DX)^2}$, which should tend to $|k(x)|$, one finds that it is not possible to find solutions such that $v > 1$ on any interval as $t \to 0$. Even if one starts out with $v > 1$ and solves toward $t = 0$, the parameter v dwindles to values less than 1, except for some sharp spikes located near places where $X_x = 0$, which eventually disappear at any fixed resolution. They may persist longer at higher resolutions. Solutions such that $v < 1$ are said to be "low-velocity," and others are called "high-velocity"; see [74]. Solutions with k positive and negative are qualitatively quite different, even though they would have the same value for v. We show below that solutions with negative k may be obtained from those with positive k by a suitable transformation.

The problem can be summarized as follows: If the geodesic loop approximation is valid, v approaches $|k|$. We therefore need a mechanism that forces $|k| < 1$, but if v must be smaller than 1, how do we account for the polarized solutions? Also, should we restrict ourselves to $k > 0$, given that numerical computations do not give information on the sign of k? The Gowdy system is not Fuchsian, despite the form of its left-hand side, because the right-hand side is not zero for $t = 0$. We therefore need to perform a reduction.

8.2.3 Results

The reduction of a PDE to Fuchsian form explains why solutions should become AVD, i.e., how the spatial derivative terms can become less important than the temporal derivatives near singularities, even though the solution is genuinely inhomogeneous. The results below also account for the various types of behavior observed on numerical and special solutions by exhibiting a solution with the maximum number of "degrees of freedom," and which, under specialization, reproduces the main features listed above.

Leading-order analysis depends on the value of k; to obtain Kasner-like behavior, it is natural to take $Z \sim k(x)\ln t$, $X \sim X_0(x)$. However, it is necessary, in order to obtain a reduction, to include one more term in the expansion of Z and X: $Z \approx k(x)\ln t + \varphi(x)$ and $X \approx X_0(x) + t^{2k(x)}\psi(x)$. We have four arbitrary functions in these asymptotics, which is reasonable for a set of two equations of second order. We call such asymptotics "generic" for this reason. The singularity data are k, X_0, φ, and ψ.

The second task is the definition of the renormalized unknowns $u(x,t)$ and $v(x,t)$. Consider first the case $0 < k < 1$, and define u and v by the relations

$$Z(x,t) = k(x)\ln t + \varphi(x) + t^{\varepsilon}u(x,t), \tag{8.7}$$

$$X(x,t) = X_0(x) + t^{2k(x)}(\psi(x) + v(x,t)), \tag{8.8}$$

where ε is a small positive constant to be chosen later. We prove in the next section that u and v solve a Fuchsian system. The equations will involve variable powers of t. Thus, we achieved the desired reduction; it turns out that we directly obtain the second reduced equation in this manner. If $0 < k < 1$, the periodicity condition $\oint \lambda_x dx = 0$ is equivalent to

$$\int_0^{2\pi} k(\varphi_x + 2X_{0x}\psi e^{-2\varphi})dx = 0, \tag{8.9}$$

which we assume from now on. If $k > 1$, we will require in addition that $X_{0x} \equiv 0$, for reasons that will be clear after we derive the reduced system. In both cases, we find $\lambda = k^2 \ln t + \mathcal{O}(1)$ as $t \to 0$.

The results, Theorems 8.1 and 8.2, are summarized as follows:

- If k lies strictly between 0 and 1, we obtain a "generic" solution involving four arbitrary functions of x, namely k, X_0, φ, and ψ.
- If $k > 0$ and X_0 is independent of x, we obtain a solution involving only three functions of x and one constant. It includes both the x-independent solutions and the polarized solutions; this explains why these cases do not lead to a restriction on k.

High velocity is allowed when X_0 is constant; this is consistent with the numerical results that show spikes when $X_x = 0$.

If k is negative, one can proceed in a similar manner, except that one should start with

$$Z = k(x)\ln t + \varphi(x) + t^{\varepsilon}u(x,t), \tag{8.10}$$
$$X = X_0(x) + t^{\varepsilon}v(x,t), \tag{8.11}$$

where k, φ, and X_0 are arbitrary functions. In fact, one can generate solutions with negative k from solutions with positive k. Indeed, if (X, Z) is any solution of the Gowdy equations, so is $(\tilde{X}, \tilde{Z})$, where

$$\tilde{X} = \frac{X}{X^2 + Y^2}, \quad \tilde{Z} = \ln \frac{Y}{X^2 + Y^2},$$

with $Y = e^Z$ as before. This corresponds to an inversion in the Poincaré half-plane.

Our existence results may be applied in two different ways to the problem. One is to assume the arbitrary functions to be analytic and 2π-periodic, and to produce solutions that are periodic in x. One can also use the results to produce solutions that are defined only near some value of x. This is useful when the solution is not conveniently represented in the (X, Z) coordinates, in which one of the points at infinity in hyperbolic space plays a distinguished role. In such cases, one can patch local solutions obtained from several local charts in hyperbolic space.

8.2.4 Reduced system

In this section, we first reduce the Gowdy equations to a second-order system for u and v, which is then converted to a first-order Fuchsian system. The equations now become

$$\begin{aligned}(D+\varepsilon)^2 u = t^{2-\varepsilon}&[k_{xx}\ln t + \varphi_{xx} + t^\varepsilon u_{xx}] \\ &- \exp(-2\varphi - 2t^\varepsilon u)\big\{t^{2k-\varepsilon}((D+2k)(v+\psi))^2 \\ &- t^{2-2k-\varepsilon}[X_{0x} + t^{2k}(v_x+\psi_x+2k_x(v+\psi)\ln t)]^2\big\}, \qquad (8.12)\\ D(D+2k)v = t^{2-2k}&X_{0xx} + 2t^\varepsilon(D+\varepsilon)u(D+2k)(v+\psi) \\ &+ t^2[(v+\psi)_{xx} + 4k_x(v_x+\psi_x)\ln t \\ &+ (2k_{xx}\ln t + 4k_x^2(\ln t)^2)(v+\psi)] \\ &- 2t^{2-2k}[X_{0x} + t^{2k}(v_x+\psi_x+k_x(v+\psi)\ln t)] \\ &\times [k_x \ln t + \varphi_x + t^\varepsilon u_x]. \qquad (8.13)\end{aligned}$$

This second-order system will now be reduced to a first-order system. To this end, let us introduce the new variables

$$\mathbf{u} = (u_0, u_1, u_2, v_0, v_1, v_2) = (u, Du, tu_x, v, Dv, tv_x).$$

We then obtain

$$\begin{aligned}Du_0 &= u_1;\\ Du_1 &= -2\varepsilon u_1 - \varepsilon^2 u_0 + t^{2-\varepsilon}(k_{xx}\ln t + \varphi_{xx}) + t\partial_x u_2\\ &\quad - \exp(-2\varphi - 2t^\varepsilon u_0)\big\{t^{2k-\varepsilon}(v_1+2kv_0+2k\psi)^2 - t^{2-2k-\varepsilon}X_{0x}^2\\ &\quad - 2t^{1-\varepsilon}X_{0x}(v_2+t\psi_x+k_x(v_0+\psi)t\ln t)\\ &\quad - t^{2k-\varepsilon}(v_2+t\psi_x+2k_x(v_0+\psi)t\ln t)^2\big\};\\ Du_2 &= t\partial_x(u_0+u_1);\\ Dv_0 &= v_1;\\ Dv_1 &= -2kv_1 + t^{2-2k}X_{0xx} + t\partial_x(v_2+t\psi_x) + 4k_x(v_2+t\psi_x)t\ln t\\ &\quad + (v_0+\psi)[2k_{xx}t^2\ln t + 4(k_x t\ln t)^2]\\ &\quad + 2t^\varepsilon(v_1+2kv_0+2k\psi)(u_1+\varepsilon u_0)\\ &\quad - 2X_{0x}t^{2-2k}(k_x\ln t+\varphi_x+t^\varepsilon\partial_x u_0)\\ &\quad - 2t(\partial_x(v_0+\psi)+2k_x(v_0+\psi)\ln t)(k_x t\ln t + t\varphi_x + t^\varepsilon u_2);\\ Dv_2 &= t\partial_x(v_0+v_1).\end{aligned}$$

This system therefore has the form

$$(D+A)\mathbf{u} = g(t,x,\mathbf{u},\mathbf{u}_x), \qquad (8.14)$$

where the right-hand side g involves various powers of t, possibly multiplied by logarithms. We choose ε so that all of these terms tend to zero as t goes to zero. The low-velocity case is precisely the one in which it is possible to achieve this without making any assumptions on the singularity data, namely k, X_0, φ, and ψ. The high- and low-velocity cases are now distinguished by the absence or presence of the terms involving t^{2-2k} (and $t^{2-2k-\varepsilon}$). As is clear from the above equations, these terms disappear precisely if X_0 is a constant (i.e., $X_{0x} = 0$). The matrix A has eigenvalues ε, 0, and $2k$, and there is a constant C such that $|\sigma^A| \leq C$ for any $\sigma \in (0, 1)$ if $\varepsilon > 0$.

We obtain solutions of (8.14) that satisfy $\mathbf{u} = 0$ for $t = 0$. Let us check that these solutions generate solutions u_0 and v_0 of the original Gowdy system. Since the second and fifth equations of the system satisfied by $\mathbf{u}$ are obtained directly from the second-order system, it suffices to check that $u_1 = Du_0$, $v_1 = Dv_0$, $u_2 = tu_{0x}$, and $v_2 = tv_{0x}$. The first two statements are identical with the first and fourth equations respectively. As for the last two, we note that the first and third equations imply

$$D(u_2 - t\partial_x u_0) = t\partial_x(u_0 + u_1 - Du_0 - u_0) = 0.$$

Since $u_2 - t\partial_x u_0$ tends to zero as $t \to 0$, it must be identically zero for all time, as desired. The same argument applies to v. The computations for the case $k < 0$ are entirely analogous, and are therefore omitted. We now study the low- and high-velocity cases separately.

8.2.5 Low-velocity case

Assume that k lies between zero and one:

Theorem 8.1. *Let $k(x)$, $X_0(x)$, $\phi(x)$, and $\psi(x)$ be real analytic, and assume $0 < k(x) < 1$ for $0 \leq x \leq 2\pi$. Then there exists a unique solution of the form (8.7–8.8), where u and v tend to zero as $t \to 0$.*

Proof. By inspection, the vector $\mathbf{u}$ satisfies a system of the form (8.14), where g can be written $t^\alpha f$, provided that we take α and ε to be small enough. Letting $t = s^m$, we obtain a new system of the same form, but with α replaced by $m\alpha$. By taking α large enough, we may therefore assume that we have a system to which Theorem 4.3 applies. The result follows. □

8.2.6 High-velocity case

We now assume that k is positive, and may take values greater than one. If k is less than one, we recover the solutions obtained above, with $X_{0x} = 0$:

Theorem 8.2. *Let $k(x)$, $\phi(x)$, and $\psi(x)$ be real analytic, and assume $X_{0x} = 0$ and $k(x) > 0$ for $0 \leq x \leq 2\pi$. Then there exists a unique solution of the form (8.7–8.8), where u and v tend to zero as $t \to 0$.*

Proof. Since X_{0x} is now zero, we find that $\mathbf{u}$ satisfies, if $\varepsilon > 0$, a Fuchsian system of the form (8.14), where g can be written $t^\alpha f$, provided again that we take α and ε to be small enough. Letting as before $t = s^m$, we obtain a new system of the same form, but with α replaced by $m\alpha$. By taking m large enough, we may therefore assume that we have a system to which Theorem 4.3 applies. The result follows. □

8.3 Space-times with twist

We now turn to a second example, in which the constraint equations do not decouple from the evolution equations, resulting in a considerably more complicated PDE system than what obtains in the case of Gowdy space-times [48, 17, 159]. These spaces admit an abelian isometry group with spacelike generators, in which, unlike the Gowdy case, the Killing vectors have a nonvanishing twist. The difficulty is overcome by abandoning the separation of constraint and evolution equations. Combining some of the constraints with some of the "evolution" equations, one can form a system that determines the metric. One then proves that the remaining constraints hold everywhere if they hold asymptotically at the singularity. This latter condition can be expressed explicitly in terms of the "singularity data."

The Gowdy subfamily is characterized by the requirement that the Killing fields X and Y that generate the isometry group have vanishing twist constants $\kappa_X := \varepsilon^{abcd} X_a Y_b \nabla_c X_d$ and $\kappa_Y := \varepsilon^{abcd} X_a Y_b \nabla_c Y_d$, where ε^{abcd} is the Levi-Civita tensor. If one chooses the orbital area as time variable ("Gowdy time") [70], the constraint equations decouple from the evolution equations in the Gowdy case. If, however, either κ_X or κ_Y is nonzero, then no such decoupling occurs.

8.3.1 Field equations

The general form of the metric and the field equations for the $\mathbb{T}^2$-symmetric space-times is presented in [17], along with a proof that the Gowdy time always exists globally for these space-times. All metric components depend on two coordinates: the Gowdy time t and spatial coordinate $\theta \in S^1$ with $\partial/\partial x$ and $\partial/\partial y$ generating the $\mathbb{T}^2$ isometry. By choosing X and Y to be suitable linear combinations of the generators, we may always assume without loss of generality that $\kappa_X = 0$. We then drop the subscript from κ_Y. We now focus on the subclass of polarized space-times, in which the metric takes the form

$$\begin{aligned} ds^2 = {} & e^{2(\nu-u)}(-\alpha dt^2 + d\theta^2) \\ & + \lambda e^{2u}(dx + G_1 d\theta + M_1 dt)^2 + \lambda e^{-2u} t^2 (dy + G_2 d\theta + M_2 dt)^2, \end{aligned} \tag{8.15}$$

where λ is a positive constant and the functions u, ν, α, G_1, M_1, G_2, M_2 depend on t and θ. The Gowdy case is recovered if $\kappa = 0$, $\alpha = 1$, and $G_1 = G_2 = M_1 = M_2 = 0$. Writing $D = t\partial_t$ and $m = \lambda\kappa^2$, the vacuum field equations take the form

$$D^2u - t^2\alpha u_{\theta\theta} = \frac{1}{2\alpha}D\alpha Du + \frac{t^2}{2}\alpha_\theta u_\theta, \tag{8.16a}$$

$$D\alpha = -\frac{\alpha^2}{t^2}me^{2\nu}, \tag{8.16b}$$

$$D\nu = (Du)^2 + t^2\alpha u_\theta^2 + \frac{\alpha}{4t^2}me^{2\nu}, \tag{8.16c}$$

$$\partial_\theta\nu = 2u_\theta Du - \frac{\alpha_\theta}{2\alpha}, \tag{8.16d}$$

$$G_{1,t} = M_{1,\theta}, \tag{8.16e}$$

$$G_{2,t} = M_{2,\theta} + \frac{\kappa\alpha^{1/2}}{t^3}e^{2\nu}, \tag{8.16f}$$

$$\kappa_t = 0, \tag{8.16g}$$

$$\kappa_\theta = 0. \tag{8.16h}$$

Equations (8.16) constitute an initial-value problem for the polarized space-times, in which the equations (8.16a–d) decouple from the other equations. They form an independent system for $\{u, \alpha, \nu\}$. Once these three functions are known, the other equations can be solved easily. Equations (8.16b–d) in particular—three of the four equations that constitute the heart of the Cauchy problem for these space-times—actually derive from the constraint equations of Einstein's theory. In contrast to the Gowdy case, the wave equation (8.16a) does not decouple from the constraints, since it contains the function α. We therefore take (8.16a–d) as our basic equations, treating (8.16a–c) as evolution equations, and (8.16d) as the only effective constraint. Also, since equation (8.16e) gives $G_{1,t} = M_{1,\theta}$, $G_1 d\theta + M_1 dt$ is locally an exact differential $d\varphi$; replacing x by $x + \varphi$, we may assume locally that $G_1 = M_1 = 0$. Similarly, one can set $M_2 = 0$ by redefining y. Since these reductions are only local and may be incompatible with global requirements, we do not consider them further, even though they do make the geometric "degrees of freedom" more clear.

The local well-posedness of the initial-value problem away from the singularity at $t = 0$ is not quite straightforward, for we must prove that equation (8.16d) propagates. This is not an immediate consequence of standard results because we are not using any of the standard setups for the initial-value problem. It nevertheless does hold; see Problem 8.2 for the analytic case, and Problem 8.3 for the nonanalytic Cauchy problem.

As far as the number of singularity data is concerned, observe that the initial data for (8.16a–c) consist of four functions $\{u, u_t, \alpha, \nu\}$; they are constrained by one relation, (8.16d). Similarly, we will obtain a family of *singular* solutions of (8.16a–c) depending on four arbitrary functions occurring in its singular expansion, and will show that if these singularity data are constrained by one relation, the constraint (8.16d) holds for all time.

8.3.2 Reduced system and its solution

We follow the usual pattern of reduction, except that the first reduction is sufficient to conclude; we therefore go directly from Step D to Step G:

Step A: t is the expansion variable.

Step B: Leading-order asymptotics.

Since we expect Kasner-like behavior at the singularity, and since u and ν appear in the metric exponentially, we choose logarithmic leading terms for u and ν:

$$u \approx k(\theta) \ln t + u_0(\theta) + \ldots, \tag{8.17a}$$

$$\nu \approx (1 + \sigma(\theta)) \ln t + \nu_0(\theta) + \ldots, \tag{8.17b}$$

$$\alpha \approx \alpha_0(\theta) + \ldots. \tag{8.17c}$$

For equation (8.16b) to hold at leading order, it is sufficient that $\sigma > 0$. For (8.16c) to hold at leading order, one needs $D\nu$ and $(Du)^2$ to balance each other, which requires that

$$k^2 = 1 + \sigma, \tag{8.18}$$

which we assume from now on. The function α_0 should be positive, to ensure that the metric has the correct signature. There are four free functions in these leading-term expansions, namely $(k, u_0, \alpha_0, \nu_0)$; they are the singularity data for this system. They are 2π-periodic; furthermore, α_0 and $\sigma = k^2 - 1$ are positive. It is natural that there should be four singularity data, given that equations (8.16a–c) require four Cauchy data. Constraint (8.16d) will be taken into account in Step G below.

These asymptotics may be compared with those of the solutions obtained in the Gowdy case. If k_G denotes the parameter called k in the Gowdy case, the correspondence is $\pm k_G = 2k - 1$. The solutions we obtain here, with $k^2 > 1$, are similar to the "high-velocity" Gowdy solutions, for which $k_G > 1$. The asymptotics (8.17a–c) are not compatible with equations (8.16) if $0 < k < 1$, unless $m = 0$, which is the Gowdy case. Indeed, (8.16b) implies that α is of order $t^{2\sigma}$, which is singular if $\sigma = k^2 - 1$ is negative. This makes the term $D\alpha Du/(2\alpha)$ in (8.16a) more singular than all the other terms in this equation, so that (8.16a) cannot hold. There are two ways to circumvent this: (1) take $k = 0$, so that Du vanishes to leading order, giving a consistent balance, at the expense of losing the freedom to vary k; (2) add terms to the field equations that would compensate the most singular term in (8.16a)—which is possible by going over to the nonpolarized field equations.

Step C. Renormalized unknown and reduction.

We now introduce new unknowns that will provide an exact form for the remainders indicated with "..." in (8.17a–c). Because of the $e^{2\nu}$ term, it is not possible to assume that the remainder terms are of order t. We do expect them to be of order t^ε if ε is small compared to the minimum of σ. We therefore define the renormalized unknowns (v, μ, β) by

$$u(\theta, t) = k(\theta) \ln t + u_0(\theta) + t^\varepsilon v(\theta, t), \tag{8.19a}$$
$$\nu(\theta, t) = k^2(\theta) \ln t + \nu_0(\theta) + t^\varepsilon \mu(\theta, t), \tag{8.19b}$$
$$\alpha(\theta, t) = \alpha_0 + t^\varepsilon \beta(\theta, t). \tag{8.19c}$$

We now show that the renormalized field variables solve a Fuchsian problem. To achieve this, let us first introduce first-order derivatives of v as new unknowns:

$$\mathbf{v} = (v_1, v_2, v_3, v_4, v_5) := (v, Dv, t^\varepsilon v_\theta, \beta, \mu).$$

Let us also introduce the abbreviation $E = m \exp(2\nu_0 + 2t^\varepsilon \mu)$. It is helpful to remove the t-derivatives of α in the right-hand side of (8.16a) by using

$$\frac{D\alpha}{\alpha} = -\alpha t^{2\sigma(\theta)} E, \tag{8.20}$$

which follows from (8.16b) and (8.19c). We then obtain the following evolution equations for $\mathbf{v}$:

$$Dv_1 = v_2, \tag{8.21a}$$
$$\begin{aligned} Dv_2 + 2\varepsilon v_2 + \varepsilon^2 v_1 = {} & t^{2-\varepsilon}(\alpha_0 + t^\varepsilon \beta)(k_{\theta\theta} \ln t + u_{0,\theta\theta} + v_{3,\theta}) \\ & - \frac{1}{2} E \alpha t^{2\sigma - \varepsilon}(k + t^\varepsilon(v_2 + \varepsilon v_1)) \\ & + \frac{1}{2} t^{2-\varepsilon}(\alpha_0 + t^\varepsilon \beta)(k_\theta \ln t + u_{0,\theta} + v_3), \end{aligned} \tag{8.21b}$$
$$Dv_3 = t^\varepsilon \partial_\theta(\varepsilon v_1 + v_2), \tag{8.21c}$$
$$(D + \varepsilon) v_4 = -t^{2\sigma - \varepsilon}(\alpha_0 + t^\varepsilon \beta)^2 E, \tag{8.21d}$$
$$\begin{aligned} (D + \varepsilon) v_5 = {} & 2k(v_2 + \varepsilon v_1) + t^\varepsilon (v_2 + \varepsilon v_1)^2 + \frac{1}{4} E t^{2\sigma - \varepsilon}(\alpha_0 + t^\varepsilon \beta) \\ & + \alpha t^{2-\varepsilon}(k_\theta \ln t + u_{0,\theta} + v_3)^2. \end{aligned} \tag{8.21e}$$

This system has the general form

$$(D + A)\mathbf{v} = t^\varepsilon \mathbf{f}(t, x, \mathbf{v}, \partial_\theta \mathbf{v}),$$

where

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ \varepsilon^2 & 2\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 \\ -2k\varepsilon & -2k & 0 & 0 & \varepsilon \end{pmatrix},$$

and $\mathbf{f}$ is a five-component object containing all the terms in the system that are not already included in the right-hand side.

Step D: Choice of ε.

By taking ε small (less than the smaller of 1 and any possible value of σ), we can ensure that $\mathbf{f}$ is continuous in t and analytic in all the remaining variables. Since the eigenvalues of A are ε and 0, of multiplicities four and one respectively, the boundedness condition of Theorem 4.3 holds. Explicitly, we have $\sigma^A = P\sigma^{A_0}P^{-1}$, where

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & 1 \\ 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 2k \\ 0 & 0 & 0 & 0 & \varepsilon \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

so that

$$\sigma^{A_0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^\varepsilon & 0 & 0 & \sigma^\varepsilon \ln\sigma \\ 0 & 0 & \sigma^\varepsilon & 0 & 0 \\ 0 & 0 & 0 & \sigma^\varepsilon & 2k\sigma^\varepsilon \ln\sigma \\ 0 & 0 & 0 & 0 & \sigma^\varepsilon \end{pmatrix}.$$

Since σ^A is bounded, we may skip Steps E and F.

Step G: Asymptotics determine a unique solution.

The main result is the following Theorem:

Theorem 8.3. *For any choice of the singularity data $k(\theta)$, $u_0(\theta)$, $\nu_0(\theta)$, and $\alpha_0(\theta)$, subject to condition (8.22), the $\mathbb{T}^2$-symmetric vacuum Einstein equations have a solution of the form (8.19), where β, v, and ν are bounded near $t = 0$. It is unique once the twist constant κ has been fixed, except for the freedom in the functions G_1, G_2, M_1, and M_2. Each of these solutions generates space-times with AVD asymptotics.*

Thus, we find a family of singular $\mathbb{T}^2$-symmetric space-times with precise asymptotics at the singularity of AVD type, depending on as many singularity data as there are Cauchy data for solutions away from the singularity. Fuchsian techniques therefore apply even if the constraints do not decouple from the "evolution" equations as in the Gowdy case.

We conclude from Theorem 4.3 that there is a unique solution of the Fuchsian system (8.21) vanishing as t tends to zero, analytic in θ and continuous in time. If we construct u, ν, and α from (8.19a–c) with $v = v_1$, $\mu = v_5$, and $\beta = v_4$, then (u, ν, α) is a solution of equations (8.16a–c). Indeed, (8.21a–c) imply that

$$D(v_3 - t^\varepsilon v_{1,\theta}) = 0,$$

so that any solution that tends to zero with t also satisfies $v_2 = tv_{1,t}$ and $v_3 = t^\varepsilon v_{1,\theta}$.

8.3.3 Propagation of constraints

We now wish to show that by imposing a constraint on the singularity data $(k, u_0, \alpha_0, \nu_0)$, we can guarantee that the solution (u, ν, α) of (8.16a–c) will satisfy the constraint (8.16d) as well, and solve Einstein's vacuum equations. We achieve this using (14.3) (see Problem 8.3) which in turn is derived using only (8.16a–c).

First of all, since $\mathbf{f}$ is bounded, we know that $(D+A)\mathbf{v}$ is actually $\mathcal{O}(t^\varepsilon)$, so that α and $D\alpha$ are of order one and t^ε respectively. In particular, $D\alpha/\alpha = t\alpha_t/\alpha = \mathcal{O}(t^\varepsilon)$. This means, using (14.3), that

$$\frac{\partial_t N}{N} = \frac{\alpha_t}{2\alpha} = \mathcal{O}(t^{\varepsilon-1}),$$

which is integrable up to $t=0$. (One could also have estimated $D\alpha/\alpha$ directly from (8.20).) Letting $z(t,\theta)$ be the integral of this function from 0 to t, we find that

$$N(t,\theta) \propto \exp z(t,\theta).$$

Thus, if we can choose the data so that $N \to 0$ as $t \to 0$ for fixed θ, we will know that N is in fact identically zero, and therefore that the constraint is satisfied. Now

$$N = \nu_\theta - 2u_\theta Du + \frac{\alpha_\theta}{2\alpha} = \nu_{0,\theta} - 2ku_{0,\theta} + \frac{\alpha_{0\theta}}{2\alpha_0} + o(1),$$

where $o(1)$ tends to zero with t. We conclude that the constraint is satisfied if and only if the singularity data satisfy

$$\nu_{0,\theta} - 2ku_{0,\theta} + \frac{\alpha_{0\theta}}{2\alpha_0} = 0. \tag{8.22}$$

We now know that once the functions $(k, u_0, \alpha_0, \nu_0)$ have been specified, and ε has been chosen small enough, the metric is uniquely determined. This completes the proof. All the considerations in this section are local in θ, and therefore allow in principle for other spatial topologies.

Problems

8.1. Solve the Gowdy equations for nonanalytic data using the methods of Chap. 5. Apply the argument of Section 10.2 to prove their stability.

8.2. Solve the initial-value problem for (8.16a–d) away from the singularity.

8.3. Obtain solutions to (8.16a–d) with nonanalytic initial data, by introducing an appropriate Fuchsian hyperbolic system. This problem contains a difficulty that is not found in the previous one, namely that the constraint equations are not decoupled from the "evolution" equations. The equations need to be split in a way that does not correspond to the one suggested by the standard approach to the nonanalytic Cauchy problem in general relativity (for the latter, see [104]).

8.4. Apply the analysis of the Gowdy problem to Einstein's equations with axial symmetry.

9

Applications in Differential Geometry

We solve two problems in conformal geometry: the convergence of the ambient metric construction [112] and the regularity of the hyperbolic radius [113, 114]. We first show that the ambient metric $\mathbf{g}$ generally involves arbitrary functions over and above the coefficients of the initial metric $\mathbf{h}$, and show how nevertheless to associate to $\mathbf{g}$ and any given conformal factor $\exp(2\sigma)$ a definite ambient metric corresponding to the initial metric $\exp(2\sigma)\mathbf{h}$ under conformal changes of the initial metric. We then prove that the hyperbolic radius of a $C^{2+\alpha}$ domain is of class $C^{2+\alpha}$, and outline several applications; see also Problem 9.1 and Chap. 11.

9.1 Fefferman–Graham metrics

As in the previous chapter, we construct Ricci-flat metrics with singular behavior, but with a different structure, and a different interpretation. Most results are taken from [112]; Theorem 9.14 is new.

In 1985, Fefferman and Graham [60] considered the question of locally embedding a real-analytic Riemannian manifold $(M, \mathbf{h})$ of dimension n into a Lorentzian, Ricci-flat, $(n+2)$-dimensional manifold $(\tilde{G}, \tilde{\mathbf{g}})$ admitting a global homothety. This is a generalization of the case of the n-dimensional sphere realized as a section of the light cone in $(n+2)$-dimensional Minkowski space; isometries in Minkowski space then induce conformal self-maps of the sphere. After various choices of local coordinates, they were led to the following PDE problem: Let (x^i) denote local coordinates on M, where indices i, j, etc. run from 1 to n, and use the Einstein summation convention. The problem is to find a tensor-valued solution $\mathbf{g}(\rho) = g_{ij}(x, \rho)dx^i \otimes dx^j$, defined for ρ small and positive, to the system

$$\rho\partial_\rho\left(\rho\partial_\rho - \frac{n}{2}\right)g_{ij} + \frac{1}{2}g^{kl}(\rho\partial_\rho g_{kl})(\rho\partial_\rho - 1)g_{ij} - g^{kl}(\rho\partial_\rho g_{ik})(\rho\partial_\rho g_{jl}) + \rho R_{ij} = 0, \tag{9.1}$$

$$g^{ik}[\nabla_k A_{ij} - \nabla_j A_{ik}] = 0, \tag{9.2}$$

$$g^{ik}[\partial_\rho A_{ik} - g^{jl}A_{jk}A_{il}] = 0, \tag{9.3}$$

$$g_{ij}(x,0) = h_{ij}, \tag{9.4}$$

where R_{ij} is the Ricci curvature of $g_{ij}(x,\rho)$, ∇ denotes the Riemannian connection on M associated to g_{ij} for fixed ρ, $h_{ij}(x)dx^i \otimes dx^j = \mathbf{h}$, and

$$A_{ij} := \frac{1}{2}\partial_\rho g_{ij}(x,\rho). \tag{9.5}$$

The desired Lorentz metric is then

$$\tilde{\mathbf{g}} = t^2 g_{ij}dx^i \otimes dx^j + 2\rho\, dt \otimes dt + 2t\, dt \otimes d\rho. \tag{9.6}$$

One refers to such a metric as an embedding metric associated to $\mathbf{h}$. Indeed, if we restrict $\tilde{\mathbf{g}}$ to the set $\{t = 1, \rho = 0\}$, we recover $\mathbf{h}$.

They found solutions if n is odd. If n is even, they conjectured that there are, in general, formal solutions involving logarithms of ρ, and raised the question of the convergence of such series.

We solve this problem by determining the degree of nonuniqueness of the series, and by proving its convergence in all cases. Theorem 9.14 shows that manifolds $\tilde{G}$ generated by different, conformally related metrics on M are locally diffeomorphic; in this sense, the space $\tilde{G}$ is a conformally invariant object associated with M, thus realizing the initial plan of Fefferman et al. More precisely [112], we have the following result:

Theorem 9.1. *Let $n \geq 1$.*

1. *Equation (9.1) admits formal solutions involving logarithms and square roots of ρ.*
2. *The series converge: solutions are holomorphic functions of r and $r\ln r$, with $r^2 = \rho$, when r is small.*
3. *The series contain $n(n+1)$ arbitrary coefficients, namely*
   (a) the $n(n+1)/2$ components of the metric tensor of M;
   (b) the $(n^2+n-2)/2$ independent components γ_{ij} of the trace-free part of the coefficient of $\rho^{n/2}$; and
   (c) the trace τ of the coefficient of ρ^n,
   where the trace is taken with respect to $\mathbf{h}$.
4. *Equations (9.2–9.3) reduce the arbitrariness in the solution: they determine τ and $\nabla^i_{\mathbf{h}}\gamma_{ij}$, where $\nabla_{\mathbf{h}}$ is the covariant derivative on M determined by $\mathbf{h}$.*

No effort has been made to reduce the number of arbitrary functions by further special choices of coordinates. The statements admit of some simplification in small dimensions; see Sect. 9.5. These results have the following consequences.

1. Given $\mathbf{h}$, there are $n(n+1)/2$ coefficients that cannot be determined by local means alone, subject to $n+1$ local conditions. By contrast, the coefficient of the first logarithmic term can be determined from $\mathbf{h}$ alone by local computations. This echoes classical results on the asymptotics of the Bergman kernel and of the solutions of the complex Monge–Ampère equation [26, 59, 72].
2. Despite the presence of logarithms, the metric admits of local uniformization by the introduction of variables of the form $r(\ln r)^k$.

Remark 9.2. The observation that (9.2–9.3) reduce the arbitrariness in the solution, and rule out logarithms in some cases, can be found already in [60, pp. 112–113].

Remark 9.3. The question solved here goes back further, in particular to the work of Schouten and Haantjes [163], who investigated formal solutions in integer powers when the determinant of $\mathbf{h}$ is normalized to unity. Logarithms, or convergence issues, were not considered by them.

We now describe the strategy carried out in the following sections. We write $\mathbf{g}$ for $g_{ij}dx^i \otimes dx^j$, and more generally use boldface letters for symmetric tensors on M, and italic letters for their local coordinate components. The leading-order behavior is $\mathbf{g} \sim \mathbf{h}$. The expansion variable, which plays the role of T in the general theory, is $r = \sqrt{\rho}$. We define the first renormalized unknown $\mathbf{p}$ by

$$\mathbf{g} = \mathbf{h}(x) + r^2\mathbf{p}(x,r),$$

and derive the first reduced Fuchsian system for $\mathbf{p}$. We then seek an integer ℓ such that there is a formal solution $\mathbf{p} \in A_\ell$; we work with $\ell = 1$. As a result, one obtains *formal solutions* in which $\mathbf{g}$ involves terms $r^p(\ln r)^q$ with $q \leq p-2$. The arbitrary functions are identified at this stage.

To obtain the second reduced equation, we *truncate the expansion* after the exponent where the last arbitrary function appears: we write

$$\mathbf{g} = \mathbf{q}(x,r) + r^2 \sum_{0\leq k\leq 2n-1} \boldsymbol{u}_k(x,r,r\ln r) r^{2n-1}(\ln r)^k,$$

where the $\boldsymbol{u}_k$ form a new set of unknowns. Substituting into the "spatial" components $\tilde{R}_{ij}$ $(1 \leq i,j \leq n)$ of the Ricci tensor of $\tilde{\mathbf{g}}$, we show that the $\boldsymbol{u}_k$ and suitable combinations of their first-order derivatives solve a *generalized Fuchsian system*, with two "time" variables, namely r and $r\ln r$. As a result, there is a *unique local solution* that is holomorphic in r and $r\ln r$.

It remains to show that the other components of $\mathrm{Ric}[\tilde{\mathbf{g}}]$ vanish as well. This is achieved by showing that they satisfy, thanks to the contracted Bianchi identities, a linear Fuchsian system that admits only the trivial solution precisely when the coefficients of $\rho^{n/2}$ and ρ^n in the metric satisfy the constraints in the theorem. The conclusions of the theorem follow.

We conclude with an analysis of the identification of solutions corresponding to conformally equivalent manifolds M. For background information on Riemannian geometry, see [104].

Setup

We collect some information on $\tilde{\mathbf{g}}$. Local coordinates on $\tilde{G}$ are denoted by $(x^0, \dots, x^{n+1})$, where $x^0 = t$ and $x^{n+1} = \rho = r^2$. It is convenient to write $m = n + 1$. The variable r already occurs naturally in [60] in relation to the associated generalized Poincaré metric (see also [137] for generalized Poincaré metrics in a different setup). We use the operator $D = r\partial_r = 2\rho\partial_\rho$. Recall that indices i, j, etc. run from 1 to n. Indices a, b, etc. can also take the values 0 and m. Greek indices take only the values 0 and m. The usual conventions of Riemannian geometry, including the summation convention on repeated indices, are used throughout. Thus, the metric coefficients $\tilde{g}_{ab}$ read

$$\begin{aligned} \tilde{g}_{ij} &= t^2 g_{ij}; & \tilde{g}_{i\alpha} &= 0;\\ \tilde{g}_{mm} &= 0; & \tilde{g}_{00} &= 2\rho;\\ \tilde{g}_{0m} &= t. \end{aligned}$$

Similarly,

$$\begin{aligned} \tilde{g}^{ij} &= t^{-2} g^{ij}; & \tilde{g}^{i\alpha} &= 0;\\ \tilde{g}^{mm} &= -2\rho/t^2; & \tilde{g}^{00} &= 0;\\ \tilde{g}^{0m} &= 1/t. \end{aligned}$$

The special form of the metric ensures that

1. the ρ coordinate lines are geodesics;
2. $\tilde{G}$ is Lorentzian and the hypersurface $\rho = 0$ is null;
3. $\tilde{G}$ admits a global homothety in the t direction.[1]

One can interpret the hypersurface $\rho = 0$ in $\tilde{G}$ as the bundle of metrics proportional to h_{ij}; a section corresponds to a particular relation $t^2 = \varphi(x) > 0$, thus to a conformal factor.

We let

$$\operatorname{tr}(\boldsymbol{u}) := h^{kl} u_{kl}$$

denote the trace with respect to $\mathbf{h}$ of any tensor u.

Next, the Christoffel coefficients $\tilde{\Gamma}^a_{bc}$ of $\tilde{g}_{ab}$ are related to the coefficients Γ^i_{jk} of g_{ij} via

$$\begin{aligned} &\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk};\\ &\tilde{\Gamma}^0_{ij} = -\frac{1}{2} t\partial_\rho g_{ij}(x,\rho); && \tilde{\Gamma}^m_{ij} = (\rho\partial_\rho - 1) g_{ij};\\ &\tilde{\Gamma}^i_{0j} = \frac{1}{t}\delta^i_j; && \tilde{\Gamma}^i_{mj} = \frac{1}{2} g^{ik}\partial_\rho g_{jk};\\ &\tilde{\Gamma}^\alpha_{\beta j} = \tilde{\Gamma}^j_{\alpha\beta} = 0; && \tilde{\Gamma}^0_{\alpha\beta} = \tilde{\Gamma}^m_{\alpha\alpha} = 0; && \tilde{\Gamma}^m_{0m} = 1/t. \end{aligned}$$

[1] A homothety is a vector field X such that $\mathcal{L}_X\tilde{g} = \lambda\tilde{g}$ for some constant λ, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

Furthermore,

$$\tilde{\Gamma}^a_{0a} = (n+1)/t; \qquad \tilde{\Gamma}^a_{ia} = \tilde{\Gamma}^k_{ik} = \Gamma^k_{ik}; \qquad \tilde{\Gamma}^a_{ma} = \frac{1}{2} g^{ik} \partial_\rho g_{ik}.$$

Let us write $\tilde{R}_{ab}$ for the components of the Ricci tensor of $\mathrm{Ric}[\tilde{\mathbf{g}}]$, and $R_{ij}(x,\rho)$ for the Ricci tensor associated with $g_{ij}(x,\rho)$ considered as a metric on M, for fixed ρ. Thus, R_{ij} reduces to the Ricci curvature of M for $\rho = 0$. The expressions (9.1), (9.2), and (9.3) represent $\tilde{R}_{ij}$, $\tilde{R}_{mj}$, and $-\tilde{R}_{mm}$ respectively. They are in particular independent of t. On the other hand, the components $\tilde{R}_{a0}$ vanish identically.

Our task is therefore to solve the system

$$\tilde{R}_{ij} = 0, \tag{9.7}$$

$$\tilde{R}_{mj} = \tilde{R}_{mm} = 0, \tag{9.8}$$

$$g_{ij}(x,0) = h_{ij}(x), \tag{9.9}$$

which is identical to (9.1–9.4). The solutions corresponding to different coordinate patches of M can be patched together after checking, as usual, their compatibility at the intersections; for (9.7), this is an easy consequence of the fact that the solutions will be uniquely determined by certain tensors on M, which play the role of "singularity data." We therefore limit ourselves in the rest of this chapter to a single coordinate patch, covering a small neighborhood of $\{x^i = 0\}$.

9.2 First Fuchsian reduction and construction of formal solutions

We seek an integer l such that the equations $\tilde{R}_{ij} = 0$ have formal solutions in power series in $r(\ln r)^k$, $k = 0, 1, \dots, l$. We find that $l = 1$ suffices. This will be an easy consequence of a reduction to a Fuchsian system, (9.14) below, from which the formal solution can be found inductively.

Remark 9.4. It is not convenient to work with variables of the form $r^p \ln r$, because it makes keeping track of the degree of homogeneity of the terms in the series more cumbersome. If desired, one could, after the solution has been obtained, see whether it can be rearranged as a series in r and $r^p \ln r$ for some p.

Equation (9.7) reads

$$E[\mathbf{g}] := D(D-n)\mathbf{g} + \frac{1}{2}\alpha(D-2)\mathbf{g} - \mathbf{g}^{-1}(D\mathbf{g}, D\mathbf{g}) + 4r^2 \mathrm{Ric}[\mathbf{g}] = 0, \tag{9.10}$$

where

$$\alpha := \mathbf{g}^{-1} D\mathbf{g} = g^{kl} D g_{kl}, \tag{9.11}$$

$$\mathbf{g}^{-1}(D\mathbf{g}, D\mathbf{g}) := g^{kl} D g_{ik} D g_{jl}. \tag{9.12}$$

If $n = 1$, it is shown in Section 9.5 that the solution is entirely free from logarithms. The structure of formal solutions to (9.10) for $n \geq 2$ is given by the following result.

Theorem 9.5. *Assume $n \geq 2$. Equation (9.10) has formal solutions*

$$\mathbf{g} = \mathbf{h} + r^2 \mathbf{p}, \tag{9.13}$$

where $\mathbf{p} \in A_1$. The formal expansion is completely determined as soon as

- *the trace-free part of the coefficient of r^n and*
- *the trace of the coefficient of r^{2n}*

are prescribed.

Proof. Let us write the inverse of $\mathbf{g}$,

$$\mathbf{g}^{-1} = \mathbf{h}^{-1} + r^2 \mathbf{P},$$

where $\mathbf{P}$ can be computed from $\mathbf{p}$. We then obtain

$$\begin{aligned}
\frac{1}{2}\alpha(D-2)\mathbf{g} &= \frac{1}{2}\left[(\mathbf{h}^{-1} + r^2\mathbf{P}) D(r^2\mathbf{p})\right](-2\mathbf{h} + r^2 D\mathbf{p}) \\
&= -\mathbf{h}\,\mathrm{tr}(D(r^2\mathbf{p})) + r^4 \varphi_1[\mathbf{p}, D\mathbf{p}], \\
\mathbf{g}^{-1}(D\mathbf{g}, D\mathbf{g}) &= r^4 \varphi_2[\mathbf{p}, D\mathbf{p}], \\
\mathrm{Ric}(\mathbf{g}) &= \mathrm{Ric}[\mathbf{h}] + r^2 \varphi_3[\mathbf{p}, \partial\mathbf{p}, \partial^2\mathbf{p}],
\end{aligned}$$

where ∂ stands for partial derivatives with respect to the coordinates x^i. It follows that

$$E[\mathbf{g}]/r^2 = L(D+2)\mathbf{p} + 4\mathrm{Ric}[\mathbf{h}] + r^2 \Phi_1[\mathbf{p}, D\mathbf{p}, \partial\mathbf{p}, \partial^2\mathbf{p}], \tag{9.14}$$

where

$$\begin{aligned}
L(D)\boldsymbol{u} &= D(D-n)\boldsymbol{u} - \mathbf{h}\,\mathrm{tr}(D\boldsymbol{u}) \\
&= D(D-n)\left(\boldsymbol{u} - \frac{1}{n}\mathbf{h}\,\mathrm{tr}(\boldsymbol{u})\right) + \frac{1}{n}\mathbf{h} D(D-2n)\,\mathrm{tr}(\boldsymbol{u})
\end{aligned} \tag{9.15}$$

(recall that the trace is being taken with respect to $\mathbf{h}$). Equating the right-hand side of (9.14) to zero gives the *first Fuchsian equation* obtained by reduction of (9.1); equation (9.1) was not Fuchsian because its nonlinear part was not divisible by r. As a consequence of (9.14), we shall derive formal solutions to all orders. Theorem 2.2 yields the following:

Lemma 9.6. *If $L(D)\boldsymbol{u} = r^\mu \mathbf{f}$, where $\mu > 0$, $\mathbf{f}$ is a nonzero polynomial in $\ln r$, and $\boldsymbol{u} = r^\mu \tilde{\mathbf{u}}(\ln r)$, then*

1. *if* $\mu \neq n$, $2n$, *then* $\deg \tilde{\mathbf{u}} = \deg \mathbf{f}$, *where* deg *denotes the degree in* $\ln r$*;*
2. *if* $\mu = n$, $\deg \tilde{\mathbf{u}} \leq \deg \mathbf{f} + 1$, *with equality if the traceless part of* $\mathbf{f}$ *is nonzero; the traceless part of* $\tilde{\mathbf{u}}$ *remains arbitrary;*
3. *if* $\mu = 2n$, $\deg \tilde{\mathbf{u}} \leq \deg \mathbf{f} + 1$, *with equality if the trace of* $\mathbf{f}$ *is nonzero; the trace of* $\tilde{\mathbf{u}}$ *remains arbitrary.*

Proof. Using the decomposition (9.15), it suffices to consider the equations $D(D-k)\tilde{\mathbf{u}} = r^\mu \mathbf{f}$ with $k = n$ or $2n$. One then applies Theorem 2.2. □

Coming back to the proof of Theorem 9.5, let us seek $\mathbf{g}$ in the form

$$\mathbf{g} = \mathbf{h} + \sum_{p \geq 2} \overset{(p)}{\mathbf{g}} r^p.$$

We wish to show that the degree of $\overset{(p)}{\mathbf{g}}$ with respect to $\ln r$ does not exceed $p-2$. We find that $\overset{(2)}{\mathbf{g}}$ is determined by

$$L(D+2)\overset{(2)}{\mathbf{g}} + 4\mathrm{Ric}(\mathbf{h}) = 0.$$

If $n \geq 3$, $L(2)$ is invertible and $\overset{(2)}{\mathbf{g}}$ is uniquely determined, without logarithms. If $n = 2$, the Ricci tensor is diagonal, and $\overset{(2)}{\mathbf{g}}$ still does not involve logarithms. Now, equation (9.14) shows that $\mathbf{p} - \overset{(2)}{\mathbf{g}}$ satisfies an equation to which Theorem 2.14 and Remark 2.15 apply, with $a = 2$ and $b = 0$. Therefore, $\mathbf{p}$ can be found in A_l provided that $2l \geq 1$, which is the maximum order of a positive zero of $x(x-n)$ or $x(x-2n)$. □

Remark 9.7. Logarithms may first occur in $\overset{(p)}{\mathbf{g}}$ for $p = n$ (in fact, only if n is even, since otherwise $\overset{(p)}{\mathbf{g}} = 0$ for p odd and less than n, and therefore $L(D)\overset{(n)}{\mathbf{g}} r^n = 0$). If $\overset{(p)}{\mathbf{g}}$ is free from logarithms for $p \leq 2n$, the series contains no logarithms at all.

Remark 9.8. One obtains, if $n \geq 3$,

$$\overset{(2)}{\mathbf{g}} = \frac{2}{n-2}\left[\mathrm{Ric}(\mathbf{h}) - \frac{R}{2(n-1)}\mathbf{h}\right].$$

9.3 Second Fuchsian reduction and convergence of formal solutions

We now have a formal solution (9.13) in $\mathbf{h} + r^2 A_1$. Let $\mathbf{h} + r^2\mathbf{q}$ be a truncation of this solution such that $E[\mathbf{h} + r^2\mathbf{q}]$ has no term of total degree (in r) less than $2n+1$. Since

$$E[\mathbf{h} + r^2\mathbf{q}] = r^2\Phi_2[\mathbf{p}, D\mathbf{p}, \partial\mathbf{p}, \partial^2\mathbf{p}],$$

this remainder contains a factor r^2. Let us define a new set of unknowns $\mathbf{u} := (\boldsymbol{u}_k(r_0, r_1))$ such that

$$\mathbf{p} = \mathbf{q} + \sum_{k=0}^{2n-1} r^{2n-1}(\ln r)^k \boldsymbol{u}_k(r, r\ln r). \tag{9.16}$$

We expect that there is a unique set of expressions $\boldsymbol{u}_k(r, r\ln r)$ such that $\mathbf{p}$ satisfies (9.14), since all the arbitrary functions in the metric have already been incorporated into $\mathbf{q}$. To achieve this goal, we use the following strategy:

- First view $\boldsymbol{u}_k$ as functions of r_0 and r_1, viewed as independent variables. This will change D into N, and makes all coefficients analytic in (r_0, r_1).
- Next, substitute (9.16) into the equation and use Lemmas 2.11 and 6.13 to handle nonlinear and linear terms respectively. This produces a new generalized Fuchsian system in which the indices all have positive real parts. This achieves the second reduction.
- To conclude, one casts this system into a first-order system in which a Fuchs index zero may appear. This system falls within the scope of Theorem 4.5, and has precisely one holomorphic local solution vanishing at the origin. It yields the desired $\boldsymbol{u}_k$.

We now implement this program. Substituting into (9.14) and applying Lemmas 2.11 and 6.13, we obtain a system of the form

$$L(N + A + 2n + 1)\mathbf{u} = r_0^2\mathbf{f}_0, \tag{9.17}$$

where $\mathbf{f}_0$ depends only on r_0, r_1, $\mathbf{u}$, its first and second x-derivatives, and $N\mathbf{u}$. Writing the $\boldsymbol{u}_k$ as a sum of a trace-free part and a part proportional to $\mathbf{h}$, we obtain a decomposition $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. System (9.17) then decouples into

$$N'(N' - n)\mathbf{u}_1 = r_0^2\mathbf{f}_{01}, \tag{9.18}$$

$$N'(N' - 2n)\mathbf{u}_2 = r_0^2\mathbf{f}_{02}, \tag{9.19}$$

where N' is short for $N + A + 2n + 1$, and N is given by (2.9). We now reduce this system to a first-order Fuchsian system for the unknown

$$\mathbf{v} = (v_K)_{1\le K\le 2n+4} := (\mathbf{u}_1, N\mathbf{u}_1, (r_0\partial_i\mathbf{u}_1)_{1\le i\le n}, \mathbf{u}_2, N\mathbf{u}_2, (r_0\partial_i\mathbf{u}_2)_{1\le i\le n}).$$

Indeed,

$$\begin{aligned}
Nv_1 &= v_2;\\
Nv_2 &= -(3n + 2 + 2A)v_2 - (A + 2n + 1)(A + n + 1)v_1 + r_0^2\mathbf{f}_{01};\\
Nv_{i+2} &= r_0\partial_i(v_1 + v_2);\\
Nv_{n+3} &= v_{n+4};\\
Nv_{n+4} &= -(2n + 2 + 2A)v_{n+4} - (A + 2n + 1)(A + 1)v_{n+3} + r_0^2\mathbf{f}_{01};\\
Nv_{n+i+4} &= r_0\partial_i(v_{n+3} + v_{n+4}).
\end{aligned}$$

Now, since $\mathbf{f}_0$ depends linearly on second-order spatial derivatives of $\mathbf{u}$, the nonlinear terms in the right-hand sides of the above equations are expressible in the form $r_0\Phi_3(\mathbf{u}, D\mathbf{u}, r_0\partial\mathbf{u}, \partial r_0\partial\mathbf{u})$. Therefore the above system for the components of $\mathbf{v}$ has the form

$$(N + B)\mathbf{v} = r_0\Phi_4(x, r_0, r_1, \mathbf{v}, \partial\mathbf{v}). \tag{9.20}$$

By Theorem 4.5 this system has a unique holomorphic solution, defined for x, r_0, r_1 small, and which vanishes for $r_0 = r_1 = 0$ if (i) the eigenvalues of B all have nonnegative real parts, (ii) Φ_4 is holomorphic in all its arguments near 0, and (iii) it depends linearly on $\partial\mathbf{v}$. The truncation of the formal solution has precisely been performed at a sufficiently high level for condition (i) to hold, and the others follow from the construction of the system. Therefore the above system has a unique holomorphic solution $\mathbf{v}$.

It remains to check that this solution does provide a solution to the original problem. We must therefore check that $v_2 = Nv_1$, $v_{2+i} = r_0\partial_i v_1$, $v_{n+4} = Nv_{n+3}$ and $v_{n+i+4} = r_0\partial_i v_{n+3}$, given that these relations hold for $r_0 = r_1 = 0$. The first and the third of these relations are contained in the system for $\mathbf{v}$. One then computes from the system that

$$N(v_{2+i} - r_0\partial_i v_1) = 0.$$

It follows that $v_{2+i} - r_0\partial_i v_1$ vanishes identically since it does at the origin. A similar argument applies to v_{n+i+4}. We have therefore proved that the formal solutions do converge, QED. □

9.4 Propagation of constraint equations

We have now proved that all components of the Ricci tensor of $\tilde{\mathbf{g}}$, *except possibly* $\tilde{R}_{mj}$ and $\tilde{R}_{mm}$, vanish. However, since they can be expressed in terms of g^{ij} and $\partial_\rho g_{ij}$, we do know that they belong to A_1, and that they do not depend on t. Since $\tilde{R}_{a0}$ and $\tilde{R}_{ij}$ are zero, the scalar curvature reduces to $\tilde{R} = \tilde{g}^{mm}\tilde{R}_{mm}$.

9.4.1 Solution of Bianchi identities

The contracted Bianchi identities $\tilde{\nabla}^c\tilde{R}_{ac} - \frac{1}{2}\partial_a\tilde{R} = 0$, for $a = j$ and $a = m$ respectively, yield, taking into account the relations in Sect. 9.1,

$$(D + 2 - n)\tilde{R}_{mj} - 2r^2g^{ik}A_{ik}\tilde{R}_{mj} - r^2\partial_j\tilde{R}_{mm} = 0; \tag{9.21}$$

$$(D + 4 - 2n)\tilde{R}_{mm} + 4r^2g^{ik}A_{ik}\tilde{R}_{mm} - 2g^{ij}[\partial_j\tilde{R}_{mi} - \Gamma^k_{ij}\tilde{R}_{mk}] = 0. \tag{9.22}$$

This is a linear system for the quantities $\tilde{R}_{mj}$ and $\tilde{R}_{mm}$. The contracted Bianchi identity for $a = 0$ sets no further restriction on $\tilde{\mathbf{g}}$.

Remark 9.9. System (9.21–9.22) is not Fuchsian, because the ∂_j term in (9.22) does not have a factor of r. However, $r^{-1}\tilde{R}_{mj}$ and $\tilde{R}_{mm}$ do solve a Fuchsian system with indices $n-3$ and $2n-4$.

It suffices to establish that these quantities vanish to all orders in r to ensure that $\tilde{g}$ is Ricci-flat. Since $\mathbf{p} \in A_1$, the right-hand sides of (9.21) and (9.22) have the form $\sum_{0\leq q\leq p} a_{jpq}(x)r^p(\ln r)^q$ and $\sum_{0\leq q\leq p} b_{pq}(x)r^p(\ln r)^q$. We therefore obtain a sequence of conditions: $a_{jpq} = b_{pq} = 0$.

Let us assume $n \geq 3$. Letting $r = 0$ in the system shows that $\tilde{R}_{mj}$ and $\tilde{R}_{mm}$ have no constant term. If we have established that they have no term containing a power of r less than p, (9.21) implies that $\tilde{R}_{mj}$ contains no power less than $p+2$, *provided that* $p+2 \neq n-2$. If $p+2 = n-2$, this is true only if the coefficient of r^{n-2} in $\tilde{R}_{mj}$ vanishes, for r^{n-2} is annihilated by $D+2-n$. Fortunately, this coefficient involves the traceless part γ_{ij} of $\overset{(n)}{g}$, which is still arbitrary. More precisely, it follows from (9.2) that the coefficient of r^{n-2} in $\tilde{R}_{mj}$ is proportional to

$$\nabla^i_{\mathbf{h}}\gamma_{ij} - \varphi_j, \tag{9.23}$$

where φ_j is determined by the $\overset{(p)}{\mathbf{g}}$ with $p < n$. We show in Section 9.4.2 that this quantity can always be made to vanish by a suitable choice of γ_{ij}.

Similarly, if we know that $\tilde{R}_{jm}$ vanishes to order $p+2$ inclusive, equation (9.22) establishes that $\tilde{R}_{mm}$ vanishes to the same order, *provided that* $p+2 \neq 2n-4$. If $p+2 = 2n-4$, this is true only if the coefficient of r^{2n-4} in $\tilde{R}_{mm}$ vanishes. Now, this coefficient involves the trace τ of $\overset{(2n)}{g}$, which again, is still arbitrary. More precisely, it follows from (9.3) that the coefficient of r^{2n-4} in $\tilde{R}_{mm}$ is proportional to

$$\tau - \psi, \tag{9.24}$$

where ψ is determined by the $\overset{(p)}{\mathbf{g}}$ with $p < 2n$. We therefore first determine γ_{ij} so as to make the expression (9.23) zero. We then choose τ equal to ψ. This ends the proof of Theorem 9.1 if $n \geq 3$.

If $n = 2$, it suffices to show that the Ricci tensor vanishes to least order, provided again that (9.24) holds. One can then conclude, using the above argument, that it is zero to all orders, hence vanishes since it is given by a convergent expansion. Now this vanishing leads to a system for the trace-free part of $\overset{(2)}{\mathbf{g}}$, which is treated in the same way as (9.23). It is in fact somewhat simpler here, see Section 9.5. □

Remark 9.10. Using Remark 9.8, one may check directly that $\tilde{R}_{jm} = 0$ for $\rho = 0$ in case $n \geq 3$: this relation expresses the identity ${C^h}_{hj} = 0$ for the Cotton tensor.

Remark 9.11. As was pointed out in [60], in case n is odd and no fractional powers are allowed, equation (9.3) precludes logarithms in the solution. Indeed, the first logarithmic term in A_{ij} would be proportional to $\rho^{n-1}\ln\rho$, and would contribute to its trace alone. This would generate a logarithmic term in (9.3) that cannot be compensated by any other term in the equation.

9.4.2 Determination of γ_{ij}

That γ_{ij} can be found follows from the following lemma.

Lemma 9.12. *For any given covector φ_j on M, one can find locally γ_{ij} symmetric and trace-free such that $\nabla^i_{\mathbf{h}}\gamma_{ij} = \varphi_j$.*

Proof. In this proof only, we write ∇ for $\nabla_{\mathbf{h}}$. It is natural to seek a solution in the form

$$\gamma_{ij} = \nabla_i X_j + \nabla_j X_i - \frac{2}{n}(\nabla^k X_k)h_{ij}.$$

This ensures that γ_{ij} is symmetric and trace-free. Substituting, one finds that X_j solves an elliptic system, which therefore certainly has a local analytic solution, by applying the Cauchy–Kovalevskaya theorem with respect to the initial surface $\{x^1 = 0\}$. This completes the proof. □

9.5 Special cases

We now deal with the cases $n = 1$ or 2, so that $\tilde{G}$ has dimension 3 or 4.

9.5.1 Case $n = 1$

If $n = 1$, $\tilde{R}_{jm}$ vanishes identically, so that the solution of the Fuchsian system does give a Ricci-flat metric on $\tilde{G}$. In fact, one can solve the problem explicitly, and show in particular that logarithmic terms never appear. Since there are no nonzero traceless tensors in one dimension, we expect no fractional powers of ρ to appear.

Theorem 9.13. *In one space dimension, the solutions of (9.1) read*

$$t^2(h_0(x) + \rho h_1(x))^2 dx \otimes dx + 2\rho dt \otimes dt + 2t dt \otimes d\rho, \tag{9.25}$$

with h_0 and h_1 arbitrary. Furthermore, $\tilde{G}$ is flat if $h_1 = 0$.

Proof. The solution is determined by a single function $u(x, r)$, which represents the only component of g_{ij}. Equation (9.1) reduces to

$$D(D-1)u + \frac{Du}{2u}(D-2)u - \frac{(Du)^2}{u} = 0.$$

Letting $u = v^2$, we obtain

$$2vD(D-2)v = 0.$$

The form of u follows. One then checks directly that (9.2) and (9.3) hold identically if $h_1 = 0$. Now, $\tilde{G}$ is a three-dimensional Ricci-flat manifold. Therefore, its curvature tensor vanishes as well. □

9.5.2 Case $n = 2$

The case $n = 2$ has already been treated as part of the general argument, but some expressions become simpler: Using the relation $R_{ij} = \frac{1}{2} R g_{ij}$, one finds that

$$\overset{(2)}{\mathbf{g}} = \mathrm{Ric}(\mathbf{h}) + \gamma,$$

where γ is trace-free, symmetric, and satisfies

$$\nabla^i_{\mathbf{h}} \gamma_{ij} = \frac{1}{2} \partial_j \mathrm{Scal}(\mathbf{h}),$$

where $\mathrm{Scal}(\mathbf{h})$ denotes the scalar curvature of $\mathbf{h}$. Now we may assume, without loss of generality, that $\mathbf{h}$ is conformally flat: $h_{ij} = \exp 2\sigma(x,y)\delta_{ij}$. One then obtains, since γ is trace-free,

$$\delta^{ik} \partial_k \gamma_{ij} = \frac{1}{2} e^{2\sigma} \partial_j \mathrm{Scal}(\mathbf{h}).$$

Since γ has only two independent components, namely $u = \gamma_{11} = -\gamma_{22}$ and $v = \gamma_{12}$, we end up with

$$u_x + v_y = a(x,y); \qquad v_x - u_y = b(x,y),$$

where a and b are known. We obtain solutions with $b = 0$ by requiring (u,v) to be a gradient, and solutions with $a = 0$ by requiring $(v, -u)$ to be a gradient. Adding the two gives us a particular solution. The general solution is obtained by adding to it a solution of the Cauchy–Riemann equations.

The remaining arbitrary coefficient, namely the trace $\overset{(4)}{\tau}$ of $\overset{(4)}{\mathbf{g}}$, may be computed from (9.3).

9.6 Conformal changes of metric

We prove that given any embedding metric (9.6), $\tilde{\mathbf{g}}$ reducing to $\mathbf{h}$ on M, and given any analytic conformal factor $\exp 2\sigma_0(x)$ on M, then $\tilde{G}$ is locally diffeomorphic to an embedding space associated to $\exp 2\sigma\mathbf{h}$.

Theorem 9.14. *Given any germ of analytic function $\sigma_0(y)$, one can find, near $\rho = 0$, a germ of diffeomorphisms*

$$\Phi : (y^i, s, \tau) \mapsto (x^i, \rho, t)$$

and coefficients $k_{ij}(y,s)$ such that

$$\Phi^*(\tilde{\mathbf{g}}) = \tau^2 k_{ij} dy^i \otimes dy^j + 2\rho\, d\tau \otimes d\tau + 2\tau\, d\tau \otimes ds,$$

where $k_{ij}(y,0) = h_{ij}(y) e^{2\sigma_0(y)}$ and $\alpha^i(y,0) = 0$.

Remark 9.15. The geometric picture is this: Consider the unit 2-sphere viewed as the intersection of the light cone in Minkowski 4-space with a hyperplane of equation $\{x^0 = 1\}$. If one cuts the light cone by the set defined by $\{x^0 = \exp\sigma\}$, one obtains a topological sphere, which is clearly conformally equivalent to the 2-sphere; the theorem explains how to make sure that the form (9.6) of the metric may be preserved by such a process.

Proof. We work in a single coordinate system $y = (y^i)$ and seek the desired transformation in the form

$$t = \tau e^{\sigma(y,s)}; \quad \rho = se^{-2\sigma(y,s)}; \quad x = y - \alpha(y,s),$$

where α stands for (α^i); we require $\alpha^i(y,0) = 0$ and $\sigma(y,0) = \sigma_0(y)$. We let $q_{ij}(y,s) = g_{ij}(y-\alpha(y,s), se^{-2\sigma})$. To emphasize that q_{ij} is a nonlinear function of α and σ, we write $q_{ij} = q_{ij}[\alpha,\sigma]$. We will determine σ and the α^i by solving a PDE of Cauchy–Kovalevskaya type.

First, noting that $2\rho\, d\tau \otimes d\tau + 2\tau\, d\tau \otimes ds = 2d\tau \otimes d(\tau s)$, we compute

$$\begin{aligned} 2dt \otimes d(t\rho) - 2d\tau \otimes d(\tau s) &= 2(d\tau + \tau d\sigma) \otimes (d(\tau s) - \tau s d\sigma) - 2d\tau \otimes d(\tau s) \\ &= 2[\tau d\sigma \otimes \{d(\tau s) - sd\tau\} - \tau^2 s d\sigma \otimes d\sigma] \\ &= 2\tau^2(ds - sd\sigma) \otimes d\sigma. \end{aligned}$$

Next, since $dx^j = dy^j - d\alpha^j$,

$$t^2 g_{ij} dx^i \otimes dx^j = \tau^2 e^{2\sigma} q_{ij} dy^i \otimes dy^j - 2q_{ij} dy^i \otimes d\alpha^j + q_{ij} d\alpha^i \otimes d\alpha^j.$$

Taking into account the relations $d\alpha^j = \alpha^j_k dy^k + \alpha^j_s ds$, $d\sigma = \sigma_k dy^k + \sigma_s ds$, where the subscripts k and s represent $\partial/\partial y^k$ and $\partial/\partial s$ respectively, we find that

$$\Phi^*(\tilde{\mathbf{g}}) - [\tau^2 q_{ij} dy^i \otimes dy^j + 2s\, d\tau \otimes d\tau + 2\tau\, d\tau \otimes ds]$$

is equal to

$$\begin{aligned} &[q_{ij} e^{2\sigma} \partial_s \alpha^i \partial_s \alpha^j + 2\sigma_s(1 - s\sigma_s)]\tau^2 ds \otimes ds \\ &+2\tau^2[(q_{ij}\alpha^j_k - q_{ik})\alpha^i_s e^{2\sigma} + \sigma_k(1 - 2s\sigma_s)]ds \otimes dy^k + \tau^2 \tilde{q}_{ij}[\alpha] dy^i \otimes dy^j, \end{aligned} \tag{9.26}$$

where

$$\begin{aligned} \tilde{q}_{ij}[\alpha,\sigma] dy^i \otimes dy^j = {} & e^{2\sigma}\{q_{ij} d'\alpha^i d'\alpha^j - q_{ij}(dy^i \otimes d'\alpha^j + d'\alpha^i \otimes dy^j)\} \\ & -2sd'\sigma \otimes d'\sigma, \end{aligned}$$

and d' denotes the differential with respect to the y variables alone. Since $\alpha^i = 0$ for $s = 0$, the same is true of $d'\alpha^i$, so that $\tilde{q}_{ij} = 0$ for $s = 0$. We therefore let

$$k_{ij} = q_{ij}[\alpha,\sigma]e^{2\sigma} + \tilde{q}_{ij}[\alpha,\sigma].$$

To complete the proof, it suffices to choose α and σ such that the terms in $ds \otimes ds$ and in $ds \otimes dy^k$ in equation (9.26) vanish. This leads to the system

$$2\sigma_s(1 - s\sigma_s) = -q_{ij}[\alpha, \sigma]e^{2\sigma}\partial_s\alpha^i\partial_s\alpha^j, \tag{9.27}$$

$$(q_{ij}[\alpha, \sigma]\alpha^j_k - q_{ik})\alpha^i_s = -\sigma_k(1 - 2s\sigma_s)e^{-2\sigma}, \tag{9.28}$$

$$\sigma(y, 0) = \sigma_0(y), \tag{9.29}$$

$$\alpha^i(y, 0) = 0, \tag{9.30}$$

which is the desired system of Cauchy–Kovalevskaya type, where s is the evolution variable; it should be reduced to a system in which derivatives enter linearly (Problem 4.1); of course, the nonlinearity is not smooth in s because g_{ij} involves logarithmic terms in its expansion.

9.7 Loewner–Nirenberg metrics

We prove the boundary regularity of the hyperbolic radius in higher dimensions by performing a reduction of the Loewner–Nirenberg equation and by developing a boundary regularity theory for the reduced equation. For the corresponding result in two dimensions, see Problem 9.1.

9.7.1 Main result

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain of class $C^{2+\alpha}$, where $0 < \alpha < 1$. Consider the Loewner–Nirenberg equation

$$-\Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0. \tag{9.31}$$

This equation admits [137] a maximal solution u_Ω, positive and smooth inside Ω; it is the limit of the increasing sequence $(u_m)_{m\geq 1}$ of solutions of (9.31) that are equal to m on the boundary. The *hyperbolic radius* of Ω is the function

$$v_\Omega := u_\Omega^{-2/(n-2)};$$

it vanishes on $\partial\Omega$. Let $d(x)$ denote the distance of x to $\partial\Omega$. It is of class $C^{2+\alpha}$ near $\partial\Omega$. The main result is the following:

Theorem 9.16. *If Ω is of class $C^{2+\alpha}$, then $v_\Omega \in C^{2+\alpha}(\overline{\Omega})$, and*

$$v_\Omega(x) = 2d(x) - d(x)^2[H(x) + o(1)]$$

as $d(x) \to 0$, where $H(x)$ is the mean curvature at the point of $\partial\Omega$ closest to x.

This result is optimal, since H is of class C^α on the boundary. It follows from Theorem 9.16 that v_Ω is a *classical solution* of

$$v_\Omega \Delta v_\Omega = \frac{n}{2}(|\nabla v_\Omega|^2 - 4),$$

even though u_Ω cannot be interpreted as a weak solution of (9.31), insofar as $u^{\frac{n+2}{n-2}} \sim (2d)^{-1-n/2} \notin L^1(\Omega)$.

Remark 9.17. The number n plays a double role: it determines the space dimension as well as the nonlinearity. It is the latter that is essential here. Consider for instance the problem

$$-\Delta u + 24u^2 = 0 \tag{9.32}$$

in three dimensions. It should be handled by the methods of this section with $n = 6$. Since this problem has an interpretation in terms of super-diffusions, it would be interesting to know whether our results, at least in this special case, admit of a probabilistic interpretation.

9.7.2 Motivation

The main reasons for studying u_Ω are as follows:

- u_Ω dominates all classical solutions, and therefore provides a uniform interior bound, independent of boundary data (see [100, 153, 137]).
- The metric
$$v_\Omega^{-2}(dx_1^2 + \cdots + dx_n^2)$$
is complete, and has constant negative scalar curvature; it generalizes the Poincaré metric on the unit disk and provides an intrinsic geometry on Ω. Furthermore, (9.31) admits a partial conformal invariance property. This was the motivation of Loewner and Nirenberg [137].
- The minima of v_Ω, known as hyperbolic centers, are close to the points of concentration arising in several variational problems; see [8].
- The numerical computation of v_Ω proceeds by computing the solution of the Dirichlet problem for (9.31) on a set of the form $\{d(x) > h\}$, where h is small, and the Dirichlet data are given by the boundary asymptotics of u_Ω.

Earlier results on the boundary behavior of u_Ω, summarized below, yield

$$v_\Omega = 2d + \mathcal{O}(d^2) \text{ and } |\nabla v_\Omega| \to 2 \tag{9.33}$$

as $d(x) \to 0$. Motivated by this, Bandle and Flucher conjectured Theorem 9.16 ([8, p. 204]).

9.7.3 Earlier results

Loewner and Nirenberg showed the existence of u_Ω and proved that

- if $\Omega \subset \Omega'$, then any classical solution in Ω' restricts to a classical solution in Ω, so that
$$u_{\Omega'} \leq u_\Omega; \tag{9.34}$$
- $u \sim (2d)^{1-\frac{n}{2}}$ as $d \to 0$.

It follows from [128, 131] that $u = (2d)^{1-n/2}(1 + \mathcal{O}(d))$ as $d \to 0$.

It follows from [7, pp. 95–96] and [9] that

$$|\nabla u_\Omega|(2d)^{n/2} \to (n-2).$$

From this information, equation (9.33) follows.

Thus, earlier results give the leading-order behavior of u. For the purposes of reduction, it is preferable to consider the leading behavior of v_Ω: $v_\Omega \sim 2d$. We introduce the renormalized unknown

$$w := (v_\Omega - 2d)/d^2.$$

It follows from Chap. 3 that the equation for w has Fuchsian structure: the coefficient of the derivatives of order k is divisible by d^k for $k = 0$, 1, and 2, and the nonlinear terms all contain a factor of d. Explicitly,

$$\frac{2v^{n/2}}{n-2}\left\{-\Delta u_\Omega + n(n-2)u_\Omega^{(n+2)/(n-2)}\right\} = Lw + 2\Delta d - M_w(w), \qquad (9.35)$$

where

$$L := d^2\Delta + (4-n)d\nabla d \cdot \nabla + (2-2n),$$

and M_w is a linear operator with w-dependent coefficients, defined by

$$M_w(f) := \frac{nd^2}{2(2+dw)}[2f\nabla d \cdot \nabla w + d\nabla w \cdot \nabla f] - 2df\Delta d.$$

The issue is no longer to find a solution of this equation—the solution is already known to exist—but to analyze its boundary regularity. The proof consists in a careful bootstrap argument in which better and better information on w results in better and better properties of the degenerate linear operator $L - M_w$. A key step is the inversion of the analogue of L in the half-space, which plays the role of the Laplacian in the usual Schauder theory.

Equation (9.35) needs only to be studied in a neighborhood of the boundary. Let us therefore introduce $C^{2+\alpha}$ thin domains $\Omega_\delta = \{0 < d < \delta\}$ such that $d \in C^{2+\alpha}(\overline{\Omega}_\delta)$, and $\partial\Omega_\delta = \partial\Omega \cup \Gamma$ consists of two hypersurfaces of class $C^{2+\alpha}$. Recall that

$$\|u\|_{C^{k+\alpha}_{\#}(\overline{\Omega}_\delta)} := \sum_{j=0}^{k} \|d^j u\|_{C^{j+\alpha}(\overline{\Omega}_\delta)}.$$

The proof rests on three intermediate steps, corresponding to three theorems:

Theorem 9.18. *w and $d^2\nabla w$ are bounded near $\partial\Omega$.*

Theorem 9.18 ensures that $L - M_w$ is of type (I). Theorem 6.7 then implies that $d\nabla w$ is bounded near the boundary; going back to the definition of M_w, we obtain $M_w(w) = \mathcal{O}(d)$; this yields the next theorem:

Theorem 9.19. *$d\nabla w$ and $M_w(w)/d$ are bounded near $\partial\Omega$.*

At this stage, we have $Lw + 2\Delta w = \mathcal{O}(d)$. In order to use Theorem 6.7, we need to subtract from w a function w_0 such that $Lw_0 + 2\Delta = 0$ with controlled boundary behavior, and $w - w_0 = \mathcal{O}(d)$; the construction of the function w_0 has already been given in Theorem 6.10. The third step is to bound $w - w_0$:

Theorem 9.20. *Near the boundary,*

$$\tilde{w} := w - w_0 = \mathcal{O}(d).$$

Postponing the proofs of Theorems 9.18 and 9.20, let us now complete the proof of Theorem 9.16. At this stage, we know that

$$L\tilde{w} = \mathcal{O}(d) \text{ and } \tilde{w} = \mathcal{O}(d)$$

near $\partial\Omega$. Theorem 6.8 yields that $\tilde{w}$ is in $C^{1+\alpha}_{\#}(\overline{\Omega}_\delta)$, for δ small enough. It follows that $M_w(w) \in C^\alpha(\overline{\Omega}_\delta)$. We now appeal to Theorem 6.9 to conclude that $d^2 w$ is of class $C^{2+\alpha}$ near the boundary. Since $\tilde{w} = \mathcal{O}(d)$, w is equal to $-H$ on $\partial\Omega$. This completes the proof of Theorem 9.16.

It remains to prove Theorems 9.18 and 9.20. Theorem 9.18 is proved by a first comparison argument combined with regularity estimates. Theorem 9.20 is proved by a second comparison argument. We write henceforth u and v for u_Ω and v_Ω respectively.

First comparison argument

Since $\partial\Omega$ is $C^{2+\alpha}$, it satisfies a uniform interior and exterior sphere condition, and there is a positive r_0 such that any $P \in \Omega$ such that $d(P) \le r_0$ admits a unique nearest point Q on the boundary, such that there are two points C and C' on the line determined by P and Q with

$$B_{r_0}(C) \subset \Omega \subset \mathbb{R}^n \setminus B_{r_0}(C'),$$

these two balls being tangent to $\partial\Omega$ at Q. We now define two functions u_i and u_e. Let

$$u_i(M) = \left(r_0 - \frac{CM^2}{r_0}\right)^{1-n/2} \text{ and } u_e(M) = \left(\frac{C'M^2}{r_0} - r_0\right)^{1-n/2}.$$

Then u_i and u_e are solutions of equation (9.31) in $B_{r_0}(C)$ and $\mathbb{R} \setminus B_{r_0}(C')$ respectively. If we replace r_0 by $r_0 - \varepsilon$ in the definition of u_e, we obtain a classical solution of (9.31) in Ω, which is therefore dominated by u_Ω. It follows that

$$u_e \le u_\Omega \text{ in } \Omega.$$

The monotonicity property (9.34) yields

$$u_\Omega \le u_i \text{ in } B_{r_0}(C).$$

In particular, the inequality

$$u_e(M) \le u_\Omega(M) \le u_i(M)$$

holds if M lies on the semiopen segment $[P, Q)$. Since Q is then also the point of the boundary closest to M, we have $QM = d(M)$, $CM = r_0 - d$, and $C'M = r_0 + d$, it follows that

$$\left(2d + \frac{d^2}{r_0}\right)^{1-n/2} \le u_\Omega(M) \le \left(2d - \frac{d^2}{r_0}\right)^{1-n/2}.$$

Since $u_\Omega = (2d + d^2 w)^{1-n/2}$, it follows that

$$|w| \le \frac{1}{r_0} \text{ if } d \le r_0.$$

Next, consider $P \in \Omega$ such that $d(P) = 2\sigma$, with $3\sigma < r_0$. For x in the closed unit ball $\overline{B}_1$, let

$$P_\sigma := P + \sigma x; \quad u_\sigma(x) := \sigma^{(n-2)/2} u(P_\sigma).$$

Then u_σ is a classical solution of (9.31) in $\overline{B}_1$. Since $d \mapsto 2d \pm \frac{1}{r_0} d^2$ is increasing for $d < r_0$, and $d(P_\sigma)$ varies between σ and 3σ if x varies in $\overline{B}_1$, we have

$$\left(6 + \frac{9\sigma}{r_0}\right)^{1-n/2} \le u_\sigma(M) \le \left(2 - \frac{\sigma}{r_0}\right)^{1-n/2}.$$

This provides a uniform bound for u_σ on B_1. Applying interior regularity estimates as in [102, 9], we find that ∇u_σ is uniformly bounded for $x = 0$. Recalling that $\sigma = \frac{1}{2} d(P)$, we find that

$$d^{\frac{n}{2}-1} u \text{ and } d^{\frac{n}{2}} \nabla u \text{ are bounded near } \partial\Omega.$$

It follows that $u^{-n/(n-2)} = \mathcal{O}(d^{n/2})$, and since $d^2 w = -2d + u^{-2/(n-2)}$, we have

$$d^2 \nabla w = -2(1 + dw)\nabla d - \frac{2}{n-2} u^{-n/(n-2)} \nabla u;$$

hence $d^2 \nabla w$ is bounded near $\partial\Omega$. This completes the proof of Theorem 9.18.

Second comparison argument

At this stage, we have the following information, where $\Omega_\delta = \{x : 0 < d(x) < \delta\}$, for δ small enough:

1. w and $d\nabla w$ are bounded near $\partial\Omega$;
2. $w = w_0 + \tilde{w}$, where $L\tilde{w} = M_w(w) = \mathcal{O}(d)$, and
3. w_0 is of class $C^{2+\alpha}_{\#}(\overline{\Omega}_\delta)$ for δ small enough.

We wish to estimate $\tilde{w}$. Write $|M_w(w)| \le cd$, where c is constant.

For any constant $A > 0$, define

$$w_A := w_0 + Ad.$$

Since $L(d) = 3(2-n)d + d^2\Delta d$, we have

$$L(w_A - w) = L(Ad - \tilde{w}) \le Ad[3(2-n) + d\Delta d] + cd.$$

Choose δ such that, say, $2(2-n) - d\Delta d \le 0$ for $d \le \delta$. Then, choose A so large that (i) $w_0 + A\delta \ge w$ for $d = \delta$, and (ii) $(2-n)A + c \le 0$. We then have

$$L(w_A - w) \le 0 \text{ in } \Omega_\delta \text{ and } w_A - w \ge 0 \text{ for } d = \delta.$$

Next, choose δ and a constant B such that $nB + (2 + Bd)\Delta d \ge 0$ on Ω_δ. We have, by direct computation,

$$L(d^{-2} + Bd^{-1}) = -(nB + 2\Delta d)d^{-1} - B\Delta d \le 0$$

on Ω_δ. Therefore, for any $\varepsilon > 0$, $z_\varepsilon := \varepsilon[d^{-2} + Bd^{-1}] + w_A - w$ satisfies $Lz_\varepsilon \le 0$, and the maximum principle ensures that z_ε has no negative minimum in Ω_δ. Now, z_ε tends to $+\infty$ as $d \to 0$. Therefore, z_ε is bounded below by the least value of its negative part restricted to $d = \delta$. In other words, for $d \le \delta$, we have, since $w_A - w \ge 0$ for $d = \delta$,

$$w_A - w + \varepsilon[d^{-2} + Bd^{-1}] \ge \varepsilon \min(\delta^{-2} + B\delta^{-1}, 0).$$

Letting $\varepsilon \to 0$, we obtain $w_A - w \ge 0$ in Ω_δ. Similarly, for suitable δ and A, $w - w_{-A} \ge 0$ in Ω_δ. We now know that w lies between $w_0 + Ad$ and $w_0 - Ad$ near $\partial\Omega$; hence $|w - w_0| = \mathcal{O}(d)$. This completes the proof of Theorem 9.20.

Problems

9.1. Let u_Ω be the maximal solution of Liouville's equation (1.15), and $v_\Omega := \exp(-u_\Omega)$, on a domain of class $C^{2+\alpha}$. (a) Show that v_Ω is of class $C^{2+\alpha}$ up to the boundary of Ω [113, 115].

(b) Show that

$$v_\Omega(x) = 2d(x) - d(x)^2[H(x) + o(1)]$$

as $d(x) \to 0$, where $H(x)$ is the mean curvature of $\partial\Omega$ at the point of $\partial\Omega$ closest to x. In addition, v_Ω is a classical solution of

$$v_\Omega \Delta v_\Omega = |\nabla v_\Omega|^2 - 4. \tag{9.36}$$

Remark: In two dimensions, the hyperbolic radius is defined by $v_\Omega = \exp(-u_\Omega)$, where u_Ω solves the Liouville equation (1.15) and $v_\Omega = \exp(-u_\Omega)$. For background information on the two-dimensional case, see [8, 26]; if Ω is simply connected, the hyperbolic radius coincides with the conformal or mapping radius, and with the harmonic radius.

9.2. Show that the system

$$u_T = v, \tag{9.37a}$$

$$Tv_T = \lambda v + cT + F(x, T, u, v), \tag{9.37b}$$

has a unique solution $(u(T, x), v(T, x))$ that vanishes for $T = 0$, and is analytic for small (x, T), provided that (i) c and λ are real; (ii) λ is not an integer; (iii) F is analytic, and contains only monomials of the form $x^a T^b u^c v^d$ with $a + b + c + d \geq 2$ and $b + c + d \geq 1$ in its Taylor expansion at the origin.

The system may be viewed as a PDE in which s plays the role of space variable, but in which no x-derivatives occur. This result is the basis of the construction of infinitely many incongruent minimal embeddings of $\mathbb{S}^3$ into $\mathbb{S}^4$, see [87] and [57, p. 98].

Are the restrictions on λ and F essential?

10

Applications to Nonlinear Waves

This chapter is devoted to applications to hyperbolic equations. Applications to soliton equations, which are often classified under nonlinear wave equations, are also included.

After a general introduction to the issue of blowup, stressing the emergence of the notions of blowup pattern and blowup stability [110], we prove the most detailed result to date on the correspondence between singularity data and Cauchy data [106]. We then present the applications of reduction to laser collapse [34, 35], the weak detonation problem, and the WTC problem in soliton theory [120, 124, 109]. We conclude with two simple examples: the explicit solution of the Liouville equation [136, 109], which, with the WTC problem, gave the initial impetus to the development of reduction techniques, and Nirenberg's example, which is perhaps the simplest case of blowup in higher dimensions for which a complete analysis in elementary terms is possible. The results on the detonation problem are new.

10.1 From blowup time to blowup pattern

Around the 1950s, it was widely held that for wavelike equations, say of second order, and for most practical purposes, (i) wave propagation generalizing the propagation of sound corresponds to a discontinuity of the second derivatives across a moving wave front; (ii) for nonlinear equations, there may also exist shock waves, which correspond to discontinuities of first-order derivatives; (iii) in both cases, the propagation of singularities should be intimately related to the geometry of the solutions of an appropriate eikonal equation determined by the leading part of the equation. The intuitive argument is that if u is singular in some sense, surely its derivatives must be "even more singular."

The mid-fifties saw the emergence of a different class of phenomena: Keller [101] gave the first systematic study of singular solutions for equations $\Box u = f(u)$, and their elliptic counterparts (see also Osserman [153]). These works showed that there are singularities near which the linear part does not dominate. Rather, as we approach the singularity, $\Box u$ and $f(u)$ both become singular, while their difference remains zero: the linear and the nonlinear terms balance each other. This new phenomenon generally came to be called *blowup*; of course, whether the solution or one of its derivatives becomes unbounded depends on the problem.

At the time, it was not possible to go any further, because the basic existence theory for nonlinear wave equations, although initiated by Schauder in the 1930s, had not been sufficiently developed. The period 1960–1980, in rough numbers, saw the emergence of systematic study of the spaces in which nonlinear wave equations could be solved; strong impetus was received, as always, by the desire to understand nonanalytic solutions of Einstein's equations, which, in a very precise sense, contain all of the nonlinear wave equations of classical mathematical physics (for this point, see [104], Chap. 6). Two types of results were developed: (i) local existence–uniqueness results for sufficiently smooth data, (ii) local existence results for rough data, (iii) global existence results for small data or equations of restricted form. The results of type (ii) were strongly stimulated by the discovery of L^p estimates for the wave equation (Strichartz).[1]

The upshot is that in many simple cases, one can find the smallest s such that the Cauchy problem for $\Box u = f(u)$ is well-posed in $H^s \times H^{s-1}$ in the whole space. Keller's solutions do not remain in this space.

Since John's discovery of the anomalously large time of existence in problems motivated by nonlinear elasticity [96], much effort has been developed to understand the behavior of solutions of nonlinear wave equations near blowup [93, 94, 96, 86, 172, 134, 104]. These efforts are based on a small-amplitude limit in which one implicitly assumes that the characteristic cone for the wave equation should play the major role; reduction shows that the blowup set is noncharacteristic for the wave operator, but is characteristic for the reduced Fuchsian PDE. Other authors have tried to construct augmented systems in which an eikonal-type equation is added to the problem; this leads to problems with double characteristics that, unlike Fuchsian PDEs, are ill-posed for nonanalytic data in general.

In a different direction, since the wave equation admits a comparison principle in limited cases, especially in one space dimension, it is tempting to apply free-boundary techniques to the study of the regularity of the blowup surface [37, 38]. One can obtain C^1 regularity for special classes of data in this manner but apparently not the higher regularity results we obtain via the use of the Nash–Moser inverse function theorem. The upshot of these studies is that

[1] It would be interesting to extend these results to Fuchsian equations. We merely note that the EPD equation in two dimensions, which is solved by spherical means, admits a smoothing effect with half a derivative, which is a very special case of theorems on regularity of averages for transport equations.

- the asymptotics of the first time of blowup in the limit when the data are small may be computed in some cases;
- there is a blowup set, but its higher regularity, suggested by free-boundary techniques, remains out of reach;
- it is not possible to continue the solutions as weak solutions in Sobolev spaces.

Reduction addresses these issues as follows: it is possible to continue the regular part of the solution; in addition, two expressions become smoother and smoother as s becomes very large: (i) the equation of the blowup surface $\{t = \psi(x)\}$ and (ii) the renormalized unknown $v = (u - u_0)/(t - \psi)^m$, where u_0 is determined, usually in closed form, by ψ. The so-called blowup time, i.e., the time of appearance of the first singularity, is then given explicitly by

$$t_* = \min_x \psi(x).$$

Since the data are not small, these results cannot be recovered from a perturbative analysis of small-data solutions. All considerations are local, and therefore apply to "finite-energy" data, such as data with compact support, as well as data that may not be small at infinity.

10.1.1 Blowup patterns, blowup mechanism

The determination of the blowup time cannot be made the main goal of the study of blowup for two reasons.

First, the blowup time is not a Lorentz invariant. As a result, if $t = \psi(x)$ is the equation of the singular set for a solution of an equation of the form $\Box u = f(u)$, in one space dimension to fix ideas, an observer at $x = 0$ will believe that the singularity originates at the space-time points (x, t), where x is any minimum of ψ, and propagates from there, while an observer along $x = vt$ will describe the singularity locus in the form

$$\frac{t' + vx'}{\sqrt{1 - v^2}} = \psi\left(\frac{x' + vt'}{\sqrt{1 - v^2}}\right),$$

which, when solved for t', gives a new representation $t' = \tilde{\psi}(x')$ of the same blowup set. It is easy to see that the minima of $\tilde{\psi}$ do not correspond to the points (x, t) in space-time found by an observer at rest; see Problem 10.7. In other words, the event "first singularity" does not have the same meaning for the two observers.

Second, due to finite speed of propagation, singularities usually do not appear simultaneously at all points in space. The locus of singular points at any given time therefore defines a specific evolving pattern that forms spontaneously. This pattern is a collective result of the evolution of the solution as a whole, for one singularity is not necessarily causally related to nearby singularities in space-time. Such patterns are similar to, but distinct from, wave propagation, in which a definite physical quantity is being tracked as it propagates gradually and causally.

10.1.2 Stability of blowup

If a singularity pattern is to be significant, it is necessary that it should be stable under perturbations of the initial data, as well as the equation—for most equations are models in which various effects have been neglected. It should therefore be possible to embed a singular solution in a family of solutions with the maximum number of free functions or parameters. The parameters should be directly related to the asymptotics of the solution at its singularity. If we can establish this stability property, we will have established in particular an explicit description of singularities for all solutions in this class.

In this sense, blowup is a singularity but not an instability: for stable blowup patterns, a small change in the Cauchy data corresponds to a small change in these parameters: the "singularity data."

The definition of the stability of a singular solution is comparable with the case of orbital stability of solitary waves in translation-invariant problems: a solitary wave u is (orbitally) stable if any perturbation of the initial condition for a solitary wave generates a solution that remains close to the orbit of u under translations. Thus, in the case of the Korteweg–de Vries (KdV) equation, an initial condition close to a one-soliton leads to a solution that is not close in the sup-norm to the unperturbed soliton, but does remain close to the set of all translates of this soliton (see Strauss [172, 173], Bona, Souganidis, and Strauss [20] for the KdV case, and their references; further results for KdV-like equations are also found in [156]). Similarly, a singular solution will be stable if a small perturbation leads to a singular solution with the same type of singularity, but possibly with a different, slightly displaced, perturbed singularity locus.

The problem can now be decomposed into two separate issues:

1. Find a reference blowup pattern.
2. Prove its stability under general perturbations.

If we establish stability, we conclude that the *blowup mechanism* is the same as in the reference solution: We will know that the blowup corresponds to a regime in which the equation is close to a linear Fuchsian equation, for which the singular set is characteristic, even though it may not be characteristic for the problem one started from. We will also be able to determine, from the explicit form of the expansion, which combinations of the solution and its derivatives remain finite at blowup, which may be useful in numerical computation, since the explicit expansion takes over precisely in those places where numerical computation becomes inaccurate.[2] If not, we conclude that the reference solution is not representative of the general solution. For a possible scenario leading to lack of stability, see Sect. 3.5.2.

Reduction therefore establishes a very detailed asymptotic representation of the solution at the outset. One first class of examples is given by those cases in which the singularity corresponds to a balance between top-order

[2] There are still few general-purpose numerical schemes for Fuchsian PDEs.

derivatives and nonlinear terms. The most general statement that covers all our examples is the following: if a singularity pattern is stable, it should be possible to characterize solutions by their asymptotics at the singularity, provided that they are pushed to sufficiently high order.

10.2 Semilinear wave equations

We prove, following [106], that solutions of $\Box u = e^u$ with blowup surface close to a hyperplane are stable, via an inverse function theorem establishing a correspondence between the Cauchy data and a pair of "singularity data" that completely describe the blowup. This result provides the prototype for applications of reduction to nonlinear waves, and shows that the blowup mechanism predicted by reduction is generic: perturbation of Cauchy data corresponds to perturbation of the singularity data, in particular, of the blowup surface. It also shows that the more regular the data, the more regular the blowup surface.

The goal of this section is the following result:

Theorem 10.1. *Any solution of* $\Box u = e^u$ *with Cauchy data on* $t = 1$ *close to* $(\ln 2, -2)$ *in* $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$, *where* s *is a large enough integer, must blow up on a spacelike hypersurface defined by an equation* $t = \psi(x)$ *with* $\psi \in H^{s-146-9[n/2]}(\mathbb{R}^n)$. *Furthermore, the solution has an asymptotic expansion* $\ln(2/T^2) + \sum_{j,k} u_{jk}(x) T^{j+k} \ (\ln T)^k$, *where* $T = t - \psi(x)$, *valid up to order* $s - 151 - 10[n/2]$. *Logarithmic terms are absent if and only if the blowup surface has vanishing scalar curvature. The blowup time is the infimum of the function* ψ.

The upshot of the theorem is a complete description of blowup, with an expansion that enables one to compute which functions of the solution and its derivatives blow up or not. Even for infinitely smooth data, there is no method other than the present one to prove that the blowup surface is better than C^1, even in low dimensions. Solutions are positive near blowup. It is easy to localize the result in space, because of finite speed of propagation for hyperbolic problems, which remains true for nonlinear equations [104].

The proof is indirect: one first constructs solutions with prescribed blowup surface, then maps the arbitrary functions in this solution to Cauchy data, and finally shows that this map is invertible. Only at the end of the argument will one know that solutions generated by the Cauchy data do correspond to a regular blowup surface.

10.2.1 What is known?

General references on blowup include [172, 104, 96, 86, 176, 187], which also give information on the complementary issue of global existence.

F. John [93] proved that all solutions of $\Box u = u^2$ with compactly supported data in three space dimensions must become singular in finite time. There are related results for other power nonlinearities and other dimensions that can be found in [172]. Yet, the solution $6/(t+1)^2$ is free of singularities for all positive time. Several other results for power nonlinearities can be found in [172].

Caffarelli and Friedman [37, 38] showed that under appropriate restrictions on the data, in 1, 2, or 3 space dimensions, there exists a C^1 spacelike blowup surface on which the solution becomes infinite. Thus, u is finite if and only if $t < \psi(x)$. For the solutions considered in this section, we prove that the blowup surface is more regular if the data are.

Several authors (see [86, 94, 96, 172] and their references) focused on the estimation of the *blowup time*, which is the time of the first singularity. It is equal to the infimum of the function defining the blowup surface:

$$T_* = \inf_x \psi(x).$$

Two lines of thought led to precise estimates of the asymptotics of T_* in the limit of small data. On the one hand, John [95, 96] (with refinements by Hörmander) proved that for a class of *quasilinear* equations, the blowup time could be computed asymptotically in the limit of small data, in three dimensions; it becomes infinite when the equation satisfies the null condition. On the other hand [133], for the *semilinear* equation, $\Box u = u^2$, if data are proportional to ε, then $T_* \approx \varepsilon^{-2}$; furthermore, one can define a rescaling of the solution that converges as $\varepsilon \to 0$. In such a limit, of course, the singular set is rejected at future infinity. As explained, for instance, in [86], these methods generalize the argument leading to the universality of the equation $u_t + uu_x = 0$ as a model for shock wave formation. Some results on the classification of possible singularities are due to Caflisch et al.; see [39].

Our objective is to understand more precisely the behavior of solutions with a nonempty blowup set. Reduction shows that blowup is regulated by a degenerate hyperbolic model, which is not the leading part of the equation, and for which the blowup surface is characteristic, although it is not characteristic for the wave operator.

10.2.2 Outline of the argument

Let us consider singular solutions of

$$\Box u = e^u \tag{10.1}$$

in n space dimensions that blow up for $t = \psi(x)$. The notation and basic reductions have been written out in Sect. 1.5.6 and are not repeated.

We first label the solution in two ways: first by a pair of Cauchy data on $t = 1$, second by a pair of "singularity data" $(w^{(0)}, \psi)$. To define the function $w^{(0)}$ entering in the singularity data, we introduce coordinates (X,T) by $T = t - \psi(x)$, $X^i = x^i$, and define $w(X,T)$; we then let $w^{(0)} = w(X,0)$.

It is often convenient to use the same letter to denote a function in the (x,t) or the (X,T) coordinates. Whenever this may lead to confusion, we distinguish them by using tildes in the (x,t) coordinates: $u(X,T) = \tilde{u}(x,t)$. The same convention applies to other functions. Now u and w can be thought of as the first components of suitable first-order systems for vector-valued unknowns $\mathbf{u}$ and $\mathbf{w}$ respectively. The system for $\mathbf{u}$ is the usual symmetric-hyperbolic system associated with $\Box u = e^u$ in the coordinates (X,T). The system for $\mathbf{w}$ is a Fuchsian symmetric system; see Sect. 5.5. This means that given the singularity data $(w^{(0)},\psi)$, we construct a singular solution, and then read off its Cauchy data (u_0,u_1). Note that w is, as a function of $(X,T,T\ln T)$, as smooth as the data permit, even on the blowup surface.

We wish now to *invert this process,* constructing singularity data from Cauchy data. It will follow that blowup takes place precisely on $t = \psi(x)$, and, using the Taylor expansion of w, the existence of the first few terms of an asymptotic expansion of the solution near the blowup surface will follow.

We achieve this for data close to the reference solution $u(x,t) = \ln(2/t^2)$, Thus, Cauchy data on $t = 1$ are close to $(\ln 2, -2)$, and the singularity data are close to $(0,0)$. Other nearly constant data can be handled in a similar fashion. This setup suggests the use of an implicit function theorem. We use the Nash–Moser theorem, in a form recalled in Chap. 13. The main point is the proof of the invertibility of the linearization of the map K from singularity data to Cauchy data. The inverse of this linearization is computed by comparing two expansions of a solution to the linearization of (10.1).

10.2.3 Basic definitions

We define here the function w and the map K. We also introduce notation for the linearizations of the various maps used in the proof. We consistently use capital letters for the arguments of differentials: U and W solve the linearized u and w equations, while $(W^{(0)},\Psi)$ represents a generic tangent vector to the space of singularity data, and (U_0,U_1) are linearized Cauchy data. Throughout this section, $\psi \in H^r(\mathbb{R}^n)$, and $\|\psi\|_\infty < \frac{1}{4}$. All Sobolev indices will be rounded off to their integer part, for simplicity, and will be assumed large enough. Recall that $\tilde{u}(x,t) = u(x,t-\psi(x))$. However, the tilde may be omitted whenever the meaning is clear from the context.

If $\tilde{u}(x,t)$ solves

$$\Box\tilde{u} = e^{\tilde{u}}, \tag{10.2}$$

and if $T = t - \psi(x)$, $X^i = x^i$, we have

$$\gamma u_{TT} - \Delta u + 2\psi^i\partial_i u_T + (\Delta\psi)u_T = e^u, \tag{10.3}$$

where $\gamma = 1 - |\nabla\psi|^2$. Note that $\partial_t = \partial_T$ and $\nabla_x = \nabla_X - (\nabla\psi)\partial_T$. We now let R denote the scalar curvature of the hypersurface $t = \psi(x)$ (see the proof of Lemma 1.6):

$$R = [\psi^{il}\psi_{il} - (\Delta\psi)^2]/\gamma + 2\left\{(\psi^\rho\psi_{i\rho})(\psi_\sigma\psi^{i\sigma}) - \psi^\rho\psi^\sigma\psi_{\rho\sigma}\Delta\psi\right\}/\gamma^2. \quad (10.4)$$

The renormalized unknown w is given by (1.13) We also define

$$g = G(\psi, w) := u - \ln(2/t^2). \quad (10.5)$$

Recall from Chap. 1 that w solves the Fuchsian equation (1.14). The singularity data are $\psi(X)$ and $w^{(0)}(X) := w(X, 0)$. Solving (1.14) and substituting into (1.13), we see that they determine u uniquely. The situation is summarized in Fig. 10.1.

Definition of mappings S, Z, E, and K

It will be understood that the operators of this section are defined only in a neighborhood of the origin in their respective spaces; recall that $\|\psi\|_{L^\infty} < \frac{1}{4}$.

The operator S gives the regular part of the solution u on $\frac{1}{4} \le T \le 2$. It is obtained by composition of the operator mapping $(w^{(0)}, \psi)$ to the solution of (1.14) with initial condition $w^{(0)}$ for $T = 0$, with the operator G defined by (10.5), which involves of the reference solution $\ln(2/t^2)$:

$$\begin{aligned} S : H^{r-3} \times H^r &\to H^{r-6-n/2}((1/4, 2) \times \mathbb{R}^n), \\ (w^{(0)}, \psi) &\mapsto G(\psi, w). \end{aligned}$$

Let

$$Z : (\psi, g) \mapsto \tilde{g}.$$

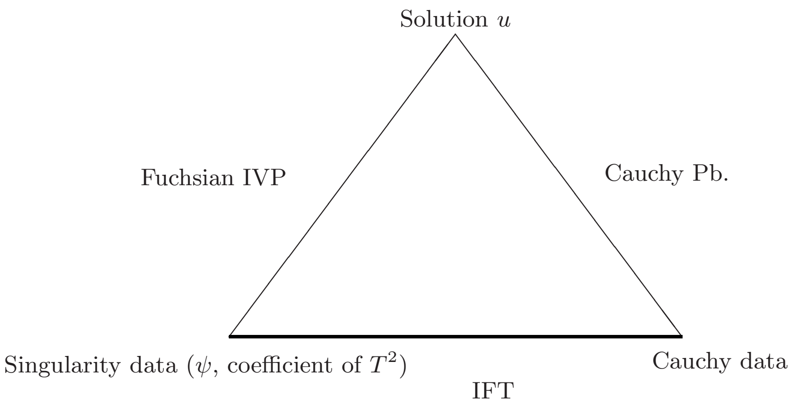

Fig. 10.1. Solution u may be determined by Cauchy data or singularity data, by solving the Cauchy problem, or the Fuchsian initial-value problem (IVP). The two data are related by an inverse function theorem (IFT)

This operator performs the conversion of the function g from the coordinates (X,T) to the coordinates (x,t). To define E, we compose $Z \circ S$ with the evaluation of the Cauchy data of $\tilde{u}$ on $t = 1$:

$$E : H^s((1/2,3/2)\times\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^n)\times H^{s-3/2}(\mathbb{R}^n), \qquad (10.6)$$
$$\tilde{g} \mapsto (\tilde{g}(.,1),\tilde{g}_t(.,1)) = (\tilde{u}(.,1)-\ln 2, \tilde{u}_t(.,1)+2).$$

By construction, $K(w^{(0)},\psi)+(\ln 2,-2)$ represents Cauchy data on $t=1$, for a solution close to the reference solution $\ln(2/t^2)$. Finally, we let

$$K = E\circ Z\circ S : H^{r-3}\times H^r \to H^{r-9-n/2}\times H^{r-10-n/2}. \qquad (10.7)$$

Thus, we define operator K by $K(w^{(0)},\psi)=(u_0,u_1)$. The goal is to invert K. The regularity of these operators is studied next.

Properties of S

To study S, we begin by recasting (10.3) in the form of a Fuchsian system, for which we set up an initial-value problem. We use the reduction to a system developed in Sect. 5.5. The first component of the unknown in this Fuchsian system is the unknown w in (1.14). We find in this way solutions $\tilde{u}$ of $\Box\tilde{u}=e^{\tilde{u}}$ that are continuous in T with values in some Sobolev space. We summarize the results in the following theorem:

Theorem 10.2. *There are symmetric matrices Q and A^j, and a constant matrix A as well as a function f, such that if $\mathbf{t}=(t_0,t_1)$ and $\mathbf{w}$ solves*

$$Q(N+A)\mathbf{w} = t_0A^j\partial_j\mathbf{w} + t\cdot f(t,X,\mathbf{w}), \qquad (10.8)$$

with $w=0$ for $t_0=t_1=0$, then the first component w of $\mathbf{w}$ generates, via (1.13), a singular solution u of $\Box u = e^u$ that blows up for $T=0$. Furthermore, if $\psi\in H^r(\mathbb{R}^n)$, Q and A^j belong to H^{r-1}_{loc}, while f maps H^{r-1} to H^{r-4} smoothly if $r>n/2+4$, and is smooth in t.

We now solve the equation for w. Denote by (u,v) both the Euclidean scalar product on $\mathbb{R}^{n+2}$ and the associated L^2 scalar product. We must check that Q, A, A^j, and f satisfy the assumptions of Theorem 5.12. Here, we may take $V = \operatorname{diag}(2\gamma,1,I_n)$ to satisfy (A3). The other assumptions are readily checked if $s>n/2$. The following existence theorem is a simple consequence of Theorem 5.12. The statement on the domain of existence is proved in the following section.

Theorem 10.3. *Assume that the coefficients and nonlinearity satisfy the assumptions (A1)–(A3). For $s>n/2+1$, (10.8) has exactly one solution $(X,T)\mapsto\mathbf{w}(T,T\ln T,X)$, continuous in T, with values in H^s, which vanishes for $T=0$. The solution is defined for $|T|\le 2$ if ψ and $w^{(0)}$ are small enough.*

In particular, $\mathbf{w}$ is defined for $|t-1| \leq \frac{1}{2}$ at least, since $\|\psi\|_{L^\infty} < \frac{1}{4}$.

Three issues need to be settled about S: Between which spaces does it act in such a way that u is defined up to $t=1$? Does it produce functions of high Sobolev regularity in both space and time variables? Is it a C^2 mapping on these spaces?

For the first question, we note that the nonlinearities g_1 and g_2 satisfy, if $s \leq r-4$,

$$|g_i|_s \leq C|\psi|_r(1+|w|_s+|w_0|_s)+C|t|\,|w|_s.$$

It follows that the time of existence in T can be made arbitrarily large if ψ and $w^{(0)}$ are small enough, by the energy estimates of Chap. 5. Let us therefore assume that the solutions are all defined for $|T|<2$ throughout the rest of the proof.

For the second question, we must consider the smoothness of the time derivatives of the solution. The Fuchsian system for $\mathbf{w}(t_0,t_1)$ can be viewed as a system for $\mathbf{w}(T, T\ln T)$, by replacing N by $T\partial_T$ and t_0 and t_1 in terms of T; it therefore contains information on $(T\partial_T)w$, By applying $T\partial_T$ repeatedly, we see that for any k such that $s-k>n/2$, the derivative $(T\partial_T)^k w$ belongs to H^{s-k}. This ensures that for $\frac{1}{4}<|T|<2$, and therefore, for $\frac{1}{2} \leq t \leq \frac{3}{2}$, w belongs to $H^{s-n/2-1}$ in space and time.

For the third question, we need to study the first two derivatives of w with respect to $w^{(0)}$ and ψ. Since the mapping G defined by (10.5) is manifestly smooth, the real question is about the smoothness of $\mathbf{w}$. We bound its differentials up to third order, thereby ensuring that it is twice continuously differentiable. Two observations are helpful here: First, if an operator P between Banach spaces is such that for any u and h the function $P(u+\varepsilon h)$ is C^1 as a function of ε, and its derivative is uniformly bounded by $C\|h\|$ uniformly in u, it follows that P is continuous and Fréchet differentiable at u. This argument can be transposed immediately to obtain a criterion for P to be C^1, C^2,.... We are therefore led to the consideration of $S(w^{(0)}+\varepsilon W^{(0)}, \psi+\varepsilon\Psi)$. The second observation is that we may take ε as an additional space variable in our Fuchsian equations, and consider instead the function

$$\varphi(\varepsilon)S(w^{(0)}+\varepsilon W^{(0)}, \psi+\varepsilon\Psi),$$

where φ is a smooth cutoff function equal to 1 near $\varepsilon=0$. It is immediate that the corresponding Fuchsian system has a solution $\mathbf{w}$ of class $H^{r-n/2-5}$ in (X,T,ε). In particular, if $r>n+7$, say, it will be a function of class C^1 in ε, and we may differentiate the equation with respect to ε. We have therefore proved that the Gâteaux differential of S may be computed by formal differentiation of the equation. Now, the linearization of the Fuchsian system for w is another Fuchsian system, with coefficients of the same degree of regularity. Since it is a linear system, the solution exists up to $T=2$. Similarly, the second variation is computed by linearizing again. Since $\mathbf{w}(\varepsilon)-\mathbf{w}(\varepsilon=0)$ can be expressed using Taylor's formula, we obtain a bound on the first differential of S. We then repeat the argument for the second and third differentials. The final result is that if the solution is in H^q, the differential of S is defined with values in H^{q-1}, and so on, because each linearization of the equation leads to a Fuchsian equation in which the time derivative of the previously computed differentials occur, leading to a loss of one derivative for each linearization. If we allow for a loss of 3 derivatives, we see that we can achieve S of class C^2. To summarize:

Theorem 10.4. *$w \in H^{s-4}$ for fixed T, but $w \in H^{r-6-n/2}\left(\left(\frac{1}{4}, 2\right) \times \mathbb{R}^n\right)$. It is also of class C^2 with values in $H^{r-9-n/2}\left(\left(\frac{1}{4}, 2\right) \times \mathbb{R}^n\right)$.*

Smoothness of *Z* and definition of *K*

The operator K is obtained from the solution u given by S by (i) changing variables from (X, T) to (x, t) and (ii) restricting $\tilde{u}$ and $\tilde{u}_t$ to $t = 0$. We study the domain and smoothness of the first operation, namely Z. We let $s = r - 9 - n/2$, which we assume to be greater than $n/2$.

Theorem 10.5. *Z maps $H^r(\mathbb{R}^n) \times H^s(\mathbb{R}^n \times (\frac{1}{4}, 2))$ to $H^s(\mathbb{R}^n \times (\frac{-1}{2}, \frac{1}{2}))$. It is of class C^2 with values in $H^{s-2}(\mathbb{R}^n \times (\frac{-1}{2}, \frac{1}{2}))$.*

Proof. We estimate the space-time regularity of $\tilde{u}(x, t) = u(x, t - \psi(x))$. The idea is, as usual, to differentiate and estimate the products of derivatives of ψ using the Gagliardo–Nirenberg inequalities. Since there is an asymmetry between the x and t variables here, we provide the details in a form convenient for the rest of the argument. We first note that for any function f,

$$\iint_{1/2 \le t \le 3/2} |f(x, t - \psi(x))|^2 \, dx \, dt \le \iint_{1/4 \le T \le 2} |f(X, T)|^2 \, dX \, dT,$$

since $\|\psi\|_\infty \le \frac{1}{4}$. We therefore obtain

$$\|\partial_t^k u(x, t - \psi(x))\|_{L^2(1/2<t<3/2)} \le \|\tilde{u}(X, T)\|_{H^s(1/4<T<2)}$$

for $k \le s$. It therefore suffices to estimate pure x derivatives of $\tilde{u}$, and then apply the above argument to estimate mixed space-time derivatives. Now, for any integer s, a generic derivative in x of order s has the form

$$\nabla_x^s[u(x, t - \psi(x))] = \sum c_{a,b,k_1,\ldots,k_j} (\nabla_X^a \nabla_T^b u)(x, t - \psi(x)) \nabla^{k_1}\psi \cdots \nabla^{k_j}\psi,$$

where $k_1 + \cdots + k_j = b = s - a$ (this statement may be checked by induction on s). The Gagliardo–Nirenberg inequality gives quite generally, for functions of $n + 1$ variables,

$$\|\nabla^j \psi\|_{L^{2s/j}} \le C \|\psi\|_{H^s}^{j/s} \|\psi\|_{L^\infty}^{(s-j)/s},$$

provided that $s > (n + 1)/2$. There are now two cases. Either $j \ge 1$ and the Gagliardo–Nirenberg inequality gives that

$$\int |\nabla_x^s [u(x, t - \psi(x))]|^2 \, dx \le C|\psi|_s (1 + \|\psi\|_\infty^{s-1}),$$

uniformly in t, so that the integral of this quantity over time is *a fortiori* bounded, or otherwise $j = 0$, and we are dealing with the pure x derivative $\nabla^s u$, which is estimated in the same way as the time derivatives, since $\nabla_x = \nabla_X$.

To complete the construction of K, it suffices to compose $Z \circ S$ with E, which is linear. Since $u(x, t - \psi(x))$ is of class H^s, we compute $u(x, 1 - \psi(x)) \in H^{s-1/2}$ and, for $t = 1$,

$$\partial_t [u(x, t - \psi(x))] = u_T(x, 1 - \psi(x)) \in H^{s-3/2},$$

by the trace theorem. □

The twice continuous differentiability of K follows again from the consideration of Gâteaux derivatives up to order 3. We find therefore that K is defined near the origin and is C^2 with values in $H^{r-9-n/2-1/2} \times H^{r-10-n/2-1/2}$. This completes the construction of the evaluation at $t = 1$.

10.2.4 Linearization of K and characterization of K'

Since K is the composition of three C^2 maps, it is itself C^2. We must now characterize solutions of the linearization of K in order to be able to identify its inverse in the following section. We first characterize the linearization in two different ways:

Theorem 10.6. *Let* $K'(w^{(0)}, \psi)[W^{(0)}, \Psi] = (U_0, U_1)$ *and consider* $U = (Z \circ S)'(w^{(0)}, \psi)[W^{(0)}, \Psi]$. *Then* U *can be computed in two different ways:*

1. *Compute the solution of the linearization of (1.14), linearized with respect to* ψ *and* w*, and substitute into the linearization of (1.13);*
2. *Let* u *be the solution* $S(w^{(0)}, \psi)$*; then* U *solves*

$$\Box \tilde{U} = e^{\tilde{u}} \tilde{U}$$

with data (U_0, U_1).

As usual, $\tilde{U}(x, t) = U(X, T)$ and $\tilde{u}(x, t) = u(X, T)$.

Proof. Since S is the composition of the solution operator associated with the Fuchsian equation (1.14) with the operator G, its differential is simply the composition of the differentials of these two operators. We have already seen that since all differentials can be computed as Gâteaux derivatives, we are allowed to compute them in the natural way, by linearizing all the equations used to compute u. This proves (1).

For statement (2), we note that the functions $\ln(2/t^2) + (Z \circ S)(w^{(0)}, \psi)$, as $w^{(0)}$ and ψ vary, all solve the same equation, namely $\Box u = e^u$. The differential can again be evaluated as a Gâteaux derivative. Since the reference solution $\ln(2/t^2)$ is independent of $(w^{(0)}, \psi)$, the second statement follows. □

We now compute an expansion of U in powers of T and $T \ln T$ by each of the two methods. By comparing the results, we will be able to define the inverse of K' and to estimate its regularity.

10.2.5 First expansion of U

The linearization S' of S is obtained by linearizing the Fuchsian equation for w. We write this linearization as

$$S'(w^{(0)}, \psi) \quad : \quad (W^{(0)}, \Psi) \mapsto (W, W_T) \mapsto (U, U_T), \tag{10.9}$$

using capitals for solutions of linearized equations. Similarly, it follows from the definition of Z that

$$Z'(\psi, u) \quad : \quad (\Psi, U) \mapsto \tilde{U} - \tilde{u}_t \Psi, \tag{10.10}$$

since $u_T(x, t - \psi(x)) = \tilde{u}_t(x, t)$. Finally, E is linear.

Now, the successive derivatives of w with respect to t_0 and t_1 exist up to order three if we assume, say, $r - 11 - n/2 > 0$. In fact, if we let $\mathbf{w} = t_0\mathbf{w}' + t_1\mathbf{w}''$, one can, as in Sect. 5.5, define a Fuchsian system for $(\mathbf{w}', \mathbf{w}'')$ that implies the original system for $\mathbf{w}$. These systems contain derivatives of w. By iterating the process, we establish the existence of an expansion of the solution in powers of T and $T \ln T$, at least as long as the nonlinearities in these derived Fuchsian systems continue to act on a Sobolev space of order greater than $n/2 + 1$:

$$w = w^{(0)} + w^{(1)}T + w^{(1,1)}T \ln T + \cdots + w^{(j,k)}T^j (\ln T)^k + \cdots ,$$

with $k \leq j$. The same considerations apply to the linearization of (1.14), or rather the associated Fuchsian system, and its solution W, corresponding to the initial value $W^{(0)}$, giving

$$W = W^{(0)} + W^{(1)}T + W^{(1,1)}T \ln T + \cdots + W^{(j,k)}T^j (\ln T)^k + \cdots .$$

Since each term in these series entails a loss of one derivative, these expansions remain valid up to order j as long as $r - 4 - j > n/2$. The coefficients $w^{(j,k)}$ are *known*, since they are the coefficients of the expansion of the reference solution; they are computed by substitution of the expansion into (1.14). We give the result for $w^{(1)}$ and $w^{(1,1)}$, for later use:

$$w^{(1,1)} = -\gamma^{-1} \sum_i \partial_i (R_1 \partial_i \psi);$$

$$4\gamma w^{(1)} = 4 \sum_i (\psi_i \partial_i - \partial_i \psi_i) w^{(0)} + \sum_i (3\psi_i \partial_i + 4\partial_i \psi_i) R_1 + \Delta v^{(1)} + \frac{1}{3}\gamma [v^{(1)}]^3.$$

Computation of U

We now compute U by linearization of (1.13). This is accomplished in two steps: first, we linearize w, which produces

$$U = V^{(0)} + V^{(1)}T + R_1'T^2 \ln T + T^2(W^{(0)} + W^{(1)}T + W^{(1,1)}T \ln T) + \mathcal{O}(T^2(\ln T)^2),$$

where $V^{(0)}$, $V^{(1)}$, and R_1' are the linearizations of $v^{(0)}$, $v^{(1)}$, and R_1 respectively, with respect to ψ. We are interested in

$$\tilde{U} = (Z \circ S)'(w^{(0)}, \psi)[W^{(0)}, \Psi].$$

By (10.10), the linearization of $Z \circ S$ is computed by replacing T by $t - \psi$ in the above expression for U, and by adding

$$-\Psi\left(-\frac{2}{T} + v^{(1)} + R_1T(1 + 2\ln T) + 2Tw + T^2 w_T\right)$$

to the result. Therefore, we obtain an expression for $\tilde{U}$ of the form

$$\tilde{U} = \frac{U^{(-1)}}{T} + U^{(0)} + U^{(1,1)}T \ln T + U^{(1)}T + U^{(2,1)}T^2 \ln T + U^{(2)}T^2 + \cdots, \tag{10.11}$$

where the higher-order terms have at least a factor of T^3 (possibly multiplied by powers of $\ln T$). The first few coefficients are

$$\begin{aligned} U^{(-1)} &= 2\Psi, &(10.12)\\ U^{(0)} &= V^{(0)} - \Psi v^{(1)},\\ U^{(1,1)} &= -2\Psi R_1,\\ U^{(1)} &= V^{(1)} - \Psi(R_1 + 2w^{(0)}),\\ U^{(2,1)} &= R_1' - 3\Psi w^{(1,1)},\\ U^{(2)} &= W^{(0)} - \Psi(3w^{(1)} + w^{(1,1)}). &(10.13)\end{aligned}$$

Observe that Ψ and $W^{(0)}$ can be recovered from $U^{(-1)}$ and $U^{(2)}$ if w and ψ are known. Note also the absence of a pure $\ln T$ term.

Inversion of K'

To compute the inverse of K', we consider a reference solution $u = S(w^{(0)}, \psi)$ such that $(u_0, u_1) = K(w^{(0)}, \psi)$, and a pair (U_0, U_1). We then find a pair $(W^{(0)}, \Psi)$ such that $K'(w^{(0)}, \psi)[W^{(0)}, \Psi] = (U_0, U_1)$. We assume that these data are in $H^\sigma \times H^{\sigma-1}$, where $\sigma = r - 10 - n/2$.

From the characterization of K', we know that we must first define U by solving $\Box\tilde{U} = e^{\tilde{u}}\tilde{U}$ with data (U_0, U_1), and study the behavior of the function

$U = \tilde{U}(X, T + \psi(X))$ as $T \to 0$. We first show that the linearization of (10.2) is itself again a Fuchsian equation. We next show that a solution of a linear Fuchsian equation cannot have a singularity worse than a power of T. We then prove iteratively that the solution of the linearized equation has in fact an expansion in powers of T and $T \ln T$ to all orders. Finally, by comparing this expansion with (10.12) and (10.13), we identify the desired values of $W^{(0)}$ and Ψ.

We provide below the expansion of e^u in terms of T; its existence is a consequence of the representation (1.13), combined with the properties of w:

$$\begin{aligned} e^u &= \frac{2\gamma}{T^2}\{v^{(1)}T + R_1 T^2 \ln T + T^2 w\} \qquad (10.14)\\ &= \frac{2\gamma}{T^2}\left\{1 + v^{(1)}T + R_1 T^2 \ln T + T^2\left(w + \frac{1}{2}[v^{(1)}]^2\right) + \mathcal{O}(T^3 \ln T)\right\}. \end{aligned}$$

Note that there is no $T \ln T$ term in the braces. The equation $\Box U = \exp(u)U$ therefore reads, in the (X, T) variables,

$$T^2(\gamma U_{TT} - \Delta U + 2\psi^i \partial_i u_T + (\Delta\psi)U_T) = \gamma(2 + a_1 T + a_{21} T^2 \ln T + \cdots)U.$$

The equation for U is now converted into Fuchsian form by letting

$$\mathbf{U} = (U, U_0, U_i) := (U, T\partial_T U, T\nabla_x U).$$

We obtain

$$Q(T\partial_T + B)\mathbf{U} = TA^i\partial_i \mathbf{U} + T\begin{pmatrix} 0 \\ [\exp(Tv^{(1)} + T^2 w) - 1]U/T - U_0 \Delta\psi \\ 0 \end{pmatrix},$$

where

$$B = \begin{pmatrix} 0 & -1 & \\ -2 & -1 & \\ & & 0_n \end{pmatrix}.$$

We now show that a solution of a linear Fuchsian equation cannot have a singularity worse than a power of T and that the solution of the linearized equation has an expansion in powers of T and $T \ln T$ to all orders. Finally, we identify from the terms of this expansion the desired values of $W^{(0)}$ and Ψ.

Let us therefore start with a general Fuchsian system, and show that its solutions have only power singularities in T. We then apply the argument to U. This generalizes results of Tahara [174] for the linear C^∞ case. In our case, we need in addition to track the number of derivatives involved carefully. Let $\mathbf{w}$ solve

$$Q(T\partial_T + A)\mathbf{w} = T(\mathcal{B}\mathbf{w} + f(\mathbf{w}))$$

for $T > 0$, where $\mathcal{B} = \sum_j A_j \partial_j$, f is linear (or sublinear), and is only assumed to be continuous in T (it might therefore involve terms in $T \ln T$). The dependence of f on space and time coordinates is suppressed. We find by multiplication that if $e(T) = (\mathbf{w}, VQ\mathbf{w})(T)$, then

$$T\partial_T e + \alpha e \geq -CT(1+e),$$

where α can always be taken to be positive. It follows that

$$(T^\alpha e)_T \geq -CT^\alpha(1+e) \geq -C(1+T^\alpha e),$$

so that we get, by integration, say from T to 1,

$$1 + e(T)T^\alpha \leq \text{const}.$$

Therefore, $\|\mathbf{w}\|_{L^2}$ cannot grow faster than a power of T. In fact, from Chap. 5, $(1-\Delta)^{\sigma/2}$ solves again an equation of the same form as $\mathbf{w}$, and therefore, $T^\alpha\|\mathbf{w}\|_{H^\sigma}$ remains bounded.

Existence of an expansion for U

We now apply these general facts to the Fuchsian equation for U. We find that we may take $\alpha = 1$; indeed, α is determined as the smallest value that makes the inequality $(VQB\mathbf{w}, \mathbf{w}) \leq \alpha(VQ\mathbf{w}, \mathbf{w})$ hold. We now prove the existence of an expansion of U in powers of T and $T\ln T$, to be identified with the expansion (10.11). Since $(T\partial_T + 1)(T\partial_T - 2)U = g(t)$ implies

$$U = \frac{c_{-1}}{T} + c_2T^2 + \int^T \frac{T^3 - s^3}{3s^3T} g(s)\, ds,$$

with c_1 and c_2 independent of T, we see that whenever $g(s) = \mathcal{O}(s^a)$, where a is not an integer, there is a particular solution that is $\mathcal{O}(T^a)$ as well. Indeed, for $a < 0$, we take the lower limit of integration to be 1. For $a > 0$, we split the integrand into two parts, one of order s^{a-3}, and the other of order s^a; we then choose different constants of integration for these two terms. It follows that there is a function U_0 in $H^{\sigma-2}$ such that the linearization U satisfies

$$U = \frac{U^{(-1)}(X)}{T} + \mathcal{O}(1).$$

We may now insert $U = U^{(-1)}T^{-1} + V$ in the left hand side of the equation for U. Integrating again produces the next term, so that

$$U = U^{(-1)}T^{-1} + U^{(0)} + \mathcal{O}(T\ln T).$$

The T^{-1} term obtained at this step must clearly be the same as that obtained before. In addition, we find that $U^{(0)} \in H^{\sigma-1}$. The process can be iterated. Quite generally, since the second derivative terms are multiplied by T^2, we find that terms at level j have two derivatives fewer than those at level $j-2$. The existence of a logarithmic series for U follows; the source of the logarithmic terms is to be found already in the logarithms in the expansion of e^u.

Definition of K'^{-1}

We have found two different ways of computing the expansion of U in powers of T and $T \ln T$. Comparing with (10.12) and (10.13), we obtain

$$\Psi = U^{(-1)}/2$$

and

$$W^{(0)} = U^{(2)} + \Psi(3w^{(1)} + w^{(1,1)}).$$

Note that $\Psi \in H^{\sigma-2}$, and $U^{(2)} \in H^{\sigma-8}$. The above procedure gave us a left inverse of K'. In fact, this operator is also a right inverse, as we proceed to show. Let us apply our inverse to a given pair (U_0, U_1). We obtain a pair $(W^{(0)}, \Psi)$. We want to show that $K'(w^{(0)}, \psi)[W^{(0)}, \Psi]$ coincides with (U_0, U_1). The given pair (U_0, U_1) generates a solution U of the linearized equation, and it would suffice to show that this U coincides with the solution U' generated by $(W^{(0)}, \Psi)$. However, both generate solutions of the same Fuchsian equation and have the same expansion up to order two at least, because these coefficients are completely determined by the value of $(W^{(0)}, \Psi)$. But the coefficients of the expansion of U are determined recursively after order T^2, and therefore U and U' coincide to all orders. We then note that we may, by the process already used to derive the system for $\mathbf{w}$ from that satisfied by $\mathbf{u}$, write $U = U^{-1}/T + \cdots + (U^{(2)} + Y)T^2$, where Y is the first component of a generalized Fuchsian system. Similarly, U' is associated with a function Y' *that solves the same equation.* Since this system has, by Theorem 5.12, only one solution that vanishes for $t_0 = t_1 = 0$, we conclude that $U' = U$, as desired.

Application of the Nash–Moser theorem, end of proof

We wish to use the Nash–Moser theorem with smoothing to invert the mapping K. We use the form given in Sect. 13.1. With the notation of that section, we take $X^r = H^{r-3} \times H^r$ and

$$Y^r = H^{r-10-[n/2]} \times H^{r-11-[n/2]}.$$

For simplicity, all Sobolev spaces will be taken to have integer order; $F(w^{(0)}, \psi) := K(w^{(0)}, \psi) - (u_0 - \ln 2, u_1 + 2)$, and $a = 6 + [n/2]$. We also assume $r > 11 + [n/2]$. We want to apply the Nash–Moser theorem with $s = r$. We have seen that

$$K \in C^2(X^r; Y^r).$$

The solution w generated by a pair $(w^{(0)}, \psi)$ in X^r belongs to H^{r-4} for fixed T, and we have seen that the coefficients $w^{(1)}$ and $w^{(1,1)}$ of the expansion of w are in H^{r-6} at least. On the other hand, the inverse of K' sends $H^\sigma \times H^{\sigma-1}$ to $(W^{(0)}, \Psi)$ as given in equations (10.12–13) it therefore takes values in $H^{r-18-[n/2]} \times H^{r-10-[n/2]}$. We conclude that $L(u)$ exists and is bounded from Y^r to X^{r-a}, where

$$a = 15 + [n/2].$$

Since we want $b > 8a$, we take $b = 8a + 1 = 121 + 8[n/2]$ to fix ideas. To guarantee (F2), we need F to map X^{r+a+b} to Y^{r+a+b}; this imposes a regularity condition on u_0 and u_1, namely

$$(u_0 - \ln 2, u_1 + 2) \in H^{r+146+9[n/2]} \times H^{r+145+9[n/2]}.$$

To ensure that all Sobolev indices appearing in the calculations are greater than $n/2$, we require $r > 11 + [n/2]$. In terms of the Cauchy data, it means that they are taken in $H^s \times H^{s-1}$ with

$$s > 167 + 10[n/2].$$

The Nash–Moser theorem ensures that if Cauchy data have this regularity and are close to $(\ln 2, -2)$, the corresponding solution must blow up on a spacelike hypersurface of class H^r with $r = s - 146 - 9[n/2]$. □

This proves the announced result.

10.2.6 Conclusions

We have therefore proved that any solution with data close to those of $\ln(2/t^2)$ must blow up on a spacelike hypersurface near which it has logarithmic behavior. It is described by the first few terms of the formal expansion, truncated to allow for the limited regularity of the solution. From the knowledge of ψ, one can read off the blowup time, which may not be attained at any finite x, as the case of a bell-shaped ψ shows; see also Problem 10.3. Also, because the Fuchsian equation (1.14), or the associated Fuchsian system, can be solved in a full neighborhood of $T = 0$, the singular solutions are at once defined on both sides of the blowup surface. One therefore reaches the conclusion that singular solutions have a meaningful continuation after blowup. The present approach applies whenever we are given a reference solution (other than $\ln(2/t^2)$), and consider data close to those of this solution. Indeed, our argument for the invertibility of the linearization of K did not use in any essential way the properties of this reference solution. This means that the set of data leading to blowup is *open.* One may perhaps allow for more general blowup surfaces, which do not become flat at infinity, by working in uniformly local Sobolev spaces.

10.2.7 Practical issues: self-similar and renormalized asymptotics

We conclude this set of results on the nonlinear wave equation with exponential nonlinearity with two observations on the practical use of expansions near blowup.

First, they may be used to obtain self-similar asymptotics. Assume, to fix ideas, that $\psi(x) = \alpha|x|^2$ near its minimum. Then

$$u = \ln \frac{2}{(\alpha|x|^2 - t)^2} + h(x,t),$$

where h is bounded, and the blowup time t_* is zero. Let $x = y\sqrt{|t|}$. For $t < 0$, we have

$$u = \ln \frac{2}{t^2} + \ln \frac{1}{1 + \alpha|y|^2} + h(y\sqrt{-t}, t).$$

We therefore see that the introduction of self-similar variables $y = x/\sqrt{|t|}$ suggests that u behaves like $\ln(2/t^2)$ rather than $\ln(2/T^2)$ near the blowup time. This conclusion is correct in a region where both statements are equivalent, namely $|x| \leq M\sqrt{|t|}$ and $t < 0$. Therefore, self-similar asymptotics are recovered, but they are valid on a domain that is strictly smaller than the domain of validity of expansions given by reduction.

Second, we may eliminate singular terms between the expansion of u and that of its derivatives. This produces quantities that remain finite at blowup. Thus, as $T \to 0$,

$$\frac{e^u}{u_{tt}} \to 1 - |\nabla\psi|^2 \quad \text{and} \quad \frac{\nabla u}{\sqrt{2u_{tt}}} \to \nabla\psi,$$

where ∇ denotes spatial gradient. This provides a direct estimation of $\nabla\psi$. If ψ vanishes at infinity, one knows ψ in principle, once its gradient is known. It would be interesting to implement this numerically. Other relations may be obtained by differentiation and elimination, but they rapidly become cumbersome. The above method is not limited in scope to the particular example treated here because the exact form of the reference solution is not used.

10.3 Nonlinear optics and lasers

Nonlinear self-focusing of optical beams may lead to the formation of singularities, resulting in damage in the medium in which the beam propagates. Until the 1980s, this phenomenon was modeled by the nonlinear Schrödinger (NLS) equation

$$iu_z + u_{xx} + u_{yy} + u|u|^2 = 0,$$

where constants have been scaled away for clarity, and u is a complex-valued function of (x, y, z). The equation was viewed as an evolution equation in which z plays the role of evolution variable. The cubic nonlinearity reflects the Kerr effect, according to which the index of refraction of the medium may depend on the field; u represents an envelope of a wave train, in a retarded frame traveling at group velocity. This singular behavior "is not only unphysical but it also prevents examination of the beam's behavior beyond the self-focus" [61]. In addition, once the (rather difficult) question of the blowup rate for NLS had been clarified, it was realized that this NLS regime "describes a very far asymptotic behavior which is not reached even with field amplification of several million." In fact, NLS was derived neglecting time dispersion, and assuming beam paraxiality. Beam paraxiality seems to arrest singularity formation [61], and time dispersion could allow it [162, 63]. Also, NLS is a stationary equation, and therefore does not describe dynamic behavior. A more fundamental difficulty with the NLS model is that it requires that z be taken as an evolution variable. Now, for the wave equation, the initial-value problem with z as evolution variable is linearly ill-posed. Therefore, taking z as evolution variables seems more appropriate for stationary rather than dynamic situations. It was therefore suggested to improve the model and to take into account terms u_{zz} and u_{tt}. This leads to the equation

$$-\epsilon_1 u_{tt} + \epsilon_2 u_{zz} + iu_z + u_{xx} + u_{yy} + u|u|^2 = 0.$$

We consider here the formation of singularities for the equation

$$\Box u + \alpha \frac{\partial u}{\partial z} = 2u|u|^2 \tag{10.15}$$

in three space dimensions, where u is complex-valued. This equation, written here after scaling variables, so that the leading part is the wave operator with speed one, has been proposed as an envelope equation for laser propagation in Kerr media. It differs from the NLS equation in two respects: it contains a term $\partial_{tt}u$, accounting for normal time dispersion, and $\partial_{zz}u$, for deviations from paraxiality. Fuchsian analysis also appears to be able to deal with models that incorporate vectorial effects.

The objective of this section is to suggest a mathematical mechanism to account for some of the qualitative features of blowup which, in this context, corresponds to laser breakdown. One would like to account for the possibility that there may be more that one point (in space-time) at which breakdown might occur. This phenomenon is called "pulse-splitting." The analysis in this section shows how to construct such solutions. It also shows that (i) singular solutions are generic: they may be embedded in a family of solutions parameterized by four real-valued functions of three variables, which is also the number of functions needed to encode the Cauchy data for a complex-valued u; (ii) the expansion of the solution is related to the local geometry of the singular set in space-time. In the process, we justify formal computations by Papanicolaou, Fibich, and Malkin in particular.

As usual with reduction, the results give a complete expansion of the solution near singularities, so that one can read off which quantities become infinite and which do not. Also, self-similar asymptotics, namely estimates when $|x-x_0|/|t-t_0| \leq \text{const}$, where the first singularity occurs at (x_0, t_0), may be recovered, as in Sect. 10.2.7. However, here again, self-similar asymptotics are strictly weaker than estimates in terms of the quantity $t - \psi(x)$, where $t = \psi(x)$ is the equation of the blowup set, see Problem 10.6. Indeed, in the latter case, we obtain information on the limit of u as $t \to t_0$ for fixed $x \neq x_0$, which is not possible with self-similar asymptotics.

We begin this section with a detailed analysis of the real cubic nonlinear wave equation, which contains most of the salient features of the analysis, and then turn to the general case. Results are mostly from [35, 34].

10.3.1 Cubic nonlinearity

Consider the equation

$$\Box u = 2u^3. \tag{10.16}$$

The argument follows the general pattern of the example in Sect. 1.5.6. We seek real solutions of (10.16) that blow up on a prescribed hypersurface $\Sigma = \{t = \psi(x)\}$ of Minkowski space-time, where ψ is given, such that $|\nabla\psi| < 1$. Therefore Σ is spacelike.

Leading-order analysis

Perform the change of variables (1.10) and let

$$D = T\partial_T.$$

The wave operator is given by equation (1.11). We seek the first term of the expansion of u in the form $u_0(X)T^\nu$. The only consistent choice for a singular solution is $\nu = -1$ and

$$u_0^2 = 1 - |\nabla\psi|^2; \tag{10.17}$$

u_0 is uniquely determined up to sign. Since $-u$ is a solution of (10.16) whenever u is, it suffices to consider $u_0 > 0$ if we are interested in solutions that blow up to $+\infty$.

Remark 10.7. Numerical computations strongly suggest that in one space dimension, solutions of the same type in which u_0 changes sign are possible; such solutions tend to $+\infty$ in an interval, and to $-\infty$ in another.

We consider in the sequel the case in which $u_0 > 0$.

First reduction

We take $\varepsilon = 1$, and let

$$u(X,T) = \frac{u_0(X)}{T} + v(X,T).$$

Substituting in (10.16), we obtain

$$\begin{aligned} u_0^2(D+2)(D-3)v = \varDelta\psi u_0 + 2\psi^i u_{0i} \\ + T(6u_0v^2 + \varDelta_X u_0 - \varDelta\psi Dv - 2\psi^i D\partial_{X_i} v) \\ + T^2(2v^3 + \varDelta_X v) \end{aligned} \tag{10.18}$$

The resonance polynomial is given by $P(D+\varepsilon) = (D+2)(D-3)$; since $\varepsilon = 1$, we conclude that $P(D) = (D+1)(D-4)$. The resonances are -1 and 4. The form of the solutions may now be determined.

Theorem 10.8. *Equation (10.16) has infinitely many formal solutions u blowing up exactly on Σ. They are of the form*

$$\pm T^{-1}\left\{ u_0(X) + u_1(X)T + u_2(X)T^2 + u_3(X)T^3 \right. \tag{10.19}$$

$$\left. + \sum_{\substack{j\geq 4\\ 0\leq k\leq j/4}} u_{j,k}(X)T^j(\ln T)^k \right\}.$$

They are entirely determined by the function $u_{4,0}$.

Proof. We are in the framework of Theorem 2.14. A formal solution for v may be found in the form of a series in T and $T\ln T$; since 4 is the only positive resonance, the formal solution is uniquely determined by the coefficient of T^4 in the expansion of Tu. This undetermined function may also be thought of as the coefficient of T^3 in the expansion of v. The more specific result in the theorem may be derived by inspection, or by applying Corrollary 2.22. □

The coefficients of the series admit of geometric interpretation; see Problem 10.7. Also, if $u_{4,1} = 0$, then no logarithmic terms appear in the series (10.19). However, this case is exceptional, since it requires ψ to satisfy a PDE.

Second reduction and existence of solutions

We prove that the formal solutions correspond to actual solutions. We present two results: one for analytic data, the other for data with limited regularity.

Theorem 10.9. *Let $\Sigma = \{t - \psi(x) = 0\}$, with $|\nabla\psi| < 1$, with ψ analytic; fix $(x_0, t_0) \in \Sigma$. Then (10.16) admits infinitely many solutions defined close to (x_0, t_0) that blow up exactly on Σ. They are of the form (10.19). The coefficients u_0, u_1, … are analytic and uniquely determined by the choice of an arbitrary analytic function $u_{4,0}$ on Σ.*

Proof. We have determined the formal solution up to the last positive resonance, namely 4. We perform a second reduction by taking as new renormalized unknown the remainder of the formal solution beyond the last resonance, divided by T^3:

$$u = \frac{u_0}{T} + u_1 + u_2 T + u_3 T^2 + u_{4,1} T^3 \ln T + f(X, T, T \ln T) T^3, \qquad (10.20)$$

where u_0, u_1, u_2, u_3, $u_{4,1}$ are given by (14.10a–d). Since we have only one logarithmic variable to introduce, define $Y = T \ln T$ and $N = T\partial_T + (T + Y)\partial_Y$. This leads to the generalized Fuchsian equation for f:

$$\begin{aligned}
&(1 - |\nabla\psi|^2) N(N+5) f - T^2 \Delta_X f + T \Delta\psi (3 u_{4,1} \ln T + 3f + u_{4,1} + Nf) \\
&+ 2T\psi^i (3 u_{4,1i} \ln T + 3\, \partial_{X_i} f + u_{4,1i} + N \partial_{X_i} f) - T(\Delta u_3 + T \ln T \Delta u_{4,1}) \\
&\quad = 2T[(6 u_0 u_1 u_{4,1} \ln T + 6 u_0 u_1 f + 6 u_0 u_2 u_3 + 3 u_2^2 u_1) + \cdots].
\end{aligned}$$

It has the general form

$$(1 - |\nabla\psi|^2) N(N+5) f = T h_1[X, T, Y, f, Nf] + Y h_2[X, T, Y, f, Nf], \quad (10.21)$$

where h_1 and h_2 are analytic functions of X, T, Y, f, polynomial in Nf and the spatial derivatives of f and Nf. Let z be the column vector with components

$$(z_1, \ldots, z_{n+2}) = (f,\, Nf,\, T\partial_{X_1} f, \ldots,\, T\partial_{X_n} f);$$

we obtain

$$\begin{aligned}
N z_1 &= z_2, \\
(1 - |\nabla\psi|^2)(N+5) z_2 &= T h_1[X, T, Y, z_1, z_2] + Y h_2[X, T, Y, z_1, z_2], \\
N z_{2+i} &= (T\partial_T + (T+Y)\partial_Y)(T \partial_{X_i} f) \\
&= T\partial_{X_i}(Nf) + T\partial_{X_i} f = T\partial_{X_i}(z_1 + z_2).
\end{aligned}$$

In a more compact form,

$$(N + A) z = T \tilde{h}_1[X, T, Y, z] + Y \tilde{h}_2[X, T, Y, z], \qquad (10.22)$$

where

$$A = \left[\begin{array}{cc|c} 0 & -1 & \\ 0 & 5 & \\ \hline & & \mathrm{O}_n \end{array}\right].$$

In view of the definition of z, we are interested in solutions of (10.22) that satisfy

$$\begin{aligned} z_1(X,0,0) &= f(X,0,0) = u_{4,0}, \\ z_2(X,0,0) &= 0, \\ z_{2+i}(X,0,0) &= 0. \end{aligned}$$

This leads to the problem

$$(N+A)z = T\tilde{h}_1[X,T,Y,z] + Y\tilde{h}_2[X,T,Y,z], \tag{10.23}$$
$${}^t z(X,0,0) = [\, u_{4,0}(X,0,0),\, 0,\, 0, \ldots,\, 0\,]. \tag{10.24}$$

Note that $z(X,0,0) \in \mathrm{Ker}(A)$. Theorem 4.5 applied to $z(X,T,Y) - z(X,0,0)$ shows that there is only one solution that reduces to $f(X,0,0)$ for $T = Y = 0$. Conversely, every solution of (10.23)–(10.24) satisfies

$$\begin{aligned} Nz_1 &= z_2, \\ (1-|\nabla\psi|^2)(N+5)z_2 &= (1-|\nabla\psi|^2)N(N+5)z_1 \\ &= Th_1[X,T,Y,z_1,Nz_1] + Yh_2[X,T,Y,z_1,Nz_1], \\ Nz_{2+i} &= T\partial_{X_i}(z_1+z_2). \end{aligned} \tag{10.25}$$

The first and third equations imply

$$N(T\partial_{X_i} z_1) = T\partial_{X_i}(N+1)z_1 = T\partial_{X_i}(z_1+z_2),$$

so that

$$N(z_{2+i} - T\partial_{X_i} z_1) = 0.$$

Taking the initial condition for $T = Y = 0$ into account, we find that $z_{2+i} = T\partial_{X_i} z_1$. Every solution of (10.23)–(10.24) is therefore of the form

$$z_2 = Nz_1, \quad z_{2+i} = T\partial_{X_i} z_1 \text{ for } 1 \le i \le n,$$

where z_1 is solution of (10.21). Letting $f = z_1$ ends the proof. □

We now turn to the nonanalytic case. We wish to apply Theorem 5.9.

Theorem 10.10. *There are symmetric matrices Q and A^j, $1 \le j \le n$, a constant matrix A, and a function f such that if $\boldsymbol{t} = (t_0, t_1)$ and*

$$Q(N+A)\mathbf{w} = t_0 A^j \partial_{X_j}\mathbf{w} + \boldsymbol{t}\cdot f(t,X,\mathbf{w}), \tag{10.26}$$

then the first component w of $\mathbf{w}$ generates a singular solution u of (10.16) that blows up for $T = 0$, provided that

$$\begin{aligned} \mathbf{w} &= (w, w_{(0)}, w_{(i)}), \\ t_0 &= T, \quad t_1 = T\ln T, \\ u &= \frac{u_0}{t_0} + u_1 + u_2 t_0 + u_3 t_0^2 + u_{4,1} t_0^2 t_1 + w(t_0,t_1,X) t_0^3, \end{aligned}$$

and that the $\{w_{(i)}\}$ *are, for* $t_0 = t_1 = 0$*, the components of a gradient. If* $\psi \in H^r(\mathbb{R}^n)$*, then* Q *and* $A^j \in H^{r-1}_{loc}(\mathbb{R}^n)$*, while* f *maps* H^{r-1} *to* H^{r-6} *if* $r > \frac{n}{2} + 6$.

Proof. Let $\gamma = 1 - |\nabla\psi(x)|^2$. We define Q, A^j, A, and f as follows:

$$Q = \begin{bmatrix} 1 & & \\ & \gamma & \\ & & I_n \end{bmatrix} \quad \text{and} \quad A^j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\psi^j & e^j \\ 0 & {}^t e^j & 0 \end{bmatrix},$$

where e^j is the jth vector of the canonical basis,

$$A = \left[\begin{array}{cc|ccc} 3 & -1 & 0 & \cdots & 0 \\ -6 & 2 & 0 & \cdots & 0 \\ \hline 0 & 0 & 2 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 2 \end{array}\right],$$

and

$$f(t_0, t_1, X, \mathbf{w}) = [\, g_0(X, t_0, t_1, \mathbf{w}), g_1(X, t_0, t_1, \mathbf{w}) \,],$$

with

$$\begin{aligned} g_0(X, t_0, t_1, \mathbf{w})^T &= [\, 0,\ b_0(X, t_0, t_1, \mathbf{w}),\ \partial_{X_1} u_{4,1}, \ldots,\ \partial_{X_n} u_{4,1} \,], \\ g_1(X, t_0, t_1, \mathbf{w})^T &= [\, 0,\ b_1(X, t_0, t_1, \mathbf{w}),\ 3\,\partial_{X_1} u_{4,1}, \ldots,\ 3\,\partial_{X_n} u_{4,1} \,] \end{aligned}$$

where ^{T} denotes transposition. Here, b_0 and b_1 are defined by

$$\begin{aligned} b_0(X, t_0, t_1, \mathbf{w}) = {} & 6u_1u_2^2 + 6u_1^2u_3 + 12u_0u_2u_3 + 12u_0u_1w \\ & + (2u_2^3 + 6u_0u_3^2 + 12u_1u_2u_3 + 6u_1^2w + 12u_0u_2w)t_0 \\ & + (6u_2^2u_3 + 6u_1u_3^2 + 12u_1u_2w + 12u_0u_3w)t_0^2 \\ & + (6u_2u_3^2 + 12u_1u_3w + 6u_0w^2 + 6u_2^2w)t_0^3 \\ & + (2u_3^3 + 12u_2u_3w + 6u_1w^2)t_0^4 + (6u_3^2w + 6u_2w^2)t_0^5 \\ & + 6u_3w^2t_0^6 + w^3t_0^7 \end{aligned}$$

and

$$\begin{aligned} b_1(X, t_0, t_1, \mathbf{w}) = {} & -6\psi^i u_{4,1i} - 3\Delta\psi u_{4,1} + 12u_0u_1u_{4,1} + (6u_1^2u_{4,1} \\ & + 12u_0u_2u_{4,1})t_0 + (12u_1u_2u_{4,1} + 12u_0u_3u_{4,1})t_0^2 \\ & + 6u_0u_{4,1}^2t_0^2t_1 + (6u_2^2u_{4,1} + 12u_1u_3u_{4,1} + 12u_0u_{4,1}w)t_0^3 \\ & + 6u_1u_{4,1}^2t_0^3t_1 + (12u_2u_3u_{4,1} + 12u_1u_{4,1}w)t_0^4 \\ & + 6u_2u_{4,1}^2t_0^4t_1 + (6u_3^2u_{4,1} + 12u_2u_{4,1}w)t_0^5 \\ & + 6u_3u_{4,1}^2t_0^5t_1 + 2u_{4,1}^3t_0^5t_1^2 + 12u_3u_{4,1}wt_0^6 \\ & + 12u_3u_{4,1}wt_0^6 + 6u_{4,1}^2wt_0^6t_1 + 6u_{4,1}w^2t_0^7. \end{aligned}$$

Now, let

$$\begin{aligned} u &= \frac{u_0}{t_0} + u_1 + u_2 t_0 + u_3 t_0^2 + u_{4,1} t_0^2 t_1 + w(t_0, t_1, X) t_0^3, \\ u_{(0)} &= -\frac{u_0}{t_0^2} + u_2 + 2u_3 t_0 + (t_0^2 + 3t_0 t_1) u_{4,1} + w_{(0)}(t_0, t_1, X) t_0^2, \\ u_{(i)} &= \frac{u_{0i}}{t_0} + u_{1i} + u_{2i} t_0 + u_{3i} t_0^2 + w_{(i)}(t_0, t_1, X) t_0^2. \end{aligned} \tag{10.27}$$

Substitution shows that if $\mathbf{w}$ solves (10.26), then $\mathbf{u} := (u, u_{(0)}, u_{(i)})$ solves

$$\begin{aligned} \partial_T u &= u_{(0)}, \\ (1 - |\nabla\psi|^2)\partial_T u_{(0)} &= \sum_i (\partial_{X_i} u_{(i)} - 2\psi_i \partial_{X_i} u_{(0)}) - \Delta\psi u_{(0)} + 2u^3, \\ \partial_T u_{(i)} &= \partial_{X_i} u_{(0)}. \end{aligned} \tag{10.28}$$

This system is the usual symmetric-hyperbolic first-order system associated with equation (10.16) in the new variables X and T. In particular, u solves (10.16).

□

Equation (10.16) has been reduced to the generalized Fuchsian problem (10.26). Applying Theorem 5.9, we obtain the following:

Theorem 10.11. *Let $\psi \in H^r$, $r > \frac{n}{2} + 1$, with $|\nabla\psi| < 1$, and let $\Sigma = \{t = \psi(x)\}$. If $r > \frac{n}{2} + 7$, there are infinitely many solutions u of (10.16) blowing up on Σ, they have the form*

$$u = \frac{u_0}{T} + u_1 + u_2 T + u_3 T^2 + u_{4,1} T^3 \ln T + w(X, T, T\ln T) T^3,$$

where $w(X, T, T\ln T)$ is continuous in T, with values in H^{r-6} for T small, and the coefficients u_0, u_1, u_2, u_3, $u_{4,1}$ are as in Sect. 10.3.1.

Proof. Observe that (10.26) reads

$$Q(N + A)\mathbf{w} = \sum_{k=0}^{l} t_k (B_k + f_k(t, \mathbf{w})), \tag{10.29}$$

for $l = 1$,

$$B_k = \sum_{j=1}^{n} A_{jk} \partial_{X_j} = \begin{cases} \sum_{j=1}^{n} A^j \partial_{X_j} & \text{if } k = 0, \\ 0 & \text{if } k = 1, \end{cases}$$

$$N = \sum_{i,j=0}^{l} m_{ij} t_j \frac{\partial}{\partial t_i} = t_0 \frac{\partial}{\partial t_0} + (t_0 + t_1)\frac{\partial}{\partial t_1},$$

and

$$f_k(t, \mathbf{w}) = g_k(X, t_0, t_1, \mathbf{w}), \quad k = 0, 1,$$

using the notation of the proof of Theorem 10.10. Equation (10.29) satisfies hypotheses (A1–A4) in Theorem 5.9. Indeed, (i) all the eigenvalues of A have nonnegative real parts (they are 0, 2, 5); (ii) $f = (f_0, \dots, f_l)$ is a C^∞ function in w and defines a map from $\mathbb{R}^{l+1} \times H^s$ to H^s, provided that $\psi \in H^r$, $r > n/2+6$, and $s = r-6$, furthermore, $f \equiv 0$ if $\|\mathbf{w}\|_{L^\infty}$ or $|t|$ is large enough; (iii) for (A4), $V = \operatorname{diag}(\gamma, \gamma, 1, 1, I_n, I_n)$ is a convenient choice, and multiplication by V is a bounded operator in H^s if $\psi \in H^r$, $r > n/2 + 6$, and $s = r - 6$. Theorem 10.11 follows. □

10.3.2 Kerr nonlinearity

Let us now turn to a slight generalization of equation (10.15):

$$\Box u + \alpha \frac{\partial u}{\partial z} = 2u|u|^2 + \beta u, \tag{10.30}$$

where $\alpha \in i\mathbb{R}$ and $\beta \in \mathbb{R}$; $u = u(x_1, x_2, x_3 = z, t)$ is complex-valued. This problem may be slightly simplified. If v is a solution of

$$\Box v = 2v|v|^2 + (\beta - \alpha^2/4)u$$

and $u = v\, e^{\alpha z/2}$, then u solves (10.15). Therefore, we may assume $\alpha = 0$. Let us seek formal solutions of (10.15) that blow up on a prescribed spacelike hypersurface $\Sigma = \{t = \psi(x)\}$ of Minkowski space-time. We define T and X as before. The main result is the following:

Theorem 10.12. *Equation (10.30) admits infinitely many formal solutions blowing up exactly on Σ. They have the form*

$$\frac{u_0(X)}{T} \left\{ 1 + v_1(X)T + v_2(X)T^2 + \sum_{\substack{j \geq 3 \\ 0 \leq k \leq j/3}} v_{j,k}(X) T^j (\ln T)^k \right\}. \tag{10.31}$$

The coefficients u_0, v_1, $v_2, \dots$ may be complex; they are completely determined by the four singularity data: ψ, the argument of u_0, the imaginary part of $v_{3,0}$, and the real part of $v_{4,0}$.

Proof. Leading-order analysis yields

$$u(X) = u_0(X)\Big(\frac{1}{T} + v(X,T)\Big),$$

with

$$|u_0|^2 = 1 - |\nabla\psi|^2.$$

The phase of u_0 remains arbitrary. Substitution into (10.30) with $\alpha = i$ gives

$$|u_0|^4\Big(D(D-1)v - 4v - 2\overline{v}\Big) = \Delta\psi|u_0|^2 + 2\psi^i u_{0i}\overline{u}_0 + Tg[X,T,v],$$

where

$$\begin{aligned} g[X,T,v] =& 2|u_0|^4(v^2+2|v|^2) + \beta|u_0|^2 - |u_0|^2\Delta\psi Dv \\ & - 2\psi^i u_{0i}\overline{u}_0 Dv - 2|u_0|^2\psi^i D\partial_{X_i} v + \Delta u_0\overline{u}_0 \\ & + T\Big[\Delta u_0\overline{u}_0 v + 2u_0^i\overline{u}_0\partial_{X_i} v + |u_0|^2\Delta_X v \\ & + 2|u_0|^4 v|v|^2 + \beta|u_0|^2 v\Big]. \end{aligned}$$

Write $v = a+b$ with a real and b pure imaginary. Separating real and imaginary parts in the previous equation leads to

$$\begin{aligned} |u_0|^4(D+2)(D-3)a =& \Delta\psi|u_0|^2 + \psi^i u_{0i}\overline{u}_0 + T\Big[2|u_0|^4(3a^2-b^2) \\ & + \beta|u_0|^2 - |u_0|^2\Delta\psi Da - 2|u_0|^2\psi^i D\partial_{X_i} a \\ & + \frac{1}{2}(\Delta u_0\overline{u}_0 + \Delta\overline{u}_0 u_0) - (\psi^i u_{0i}\overline{u}_0 - \psi^i\overline{u}_{0i}u_0)Db \\ & - (\psi^i u_{0i}\overline{u}_0 + \psi^i\overline{u}_{0i}u_0)Da\Big] \\ & + T^2\Big[(u_0^i\overline{u}_0 + \overline{u}_0^i u_0)\partial_{X_i} a + (u_0^i\overline{u}_0 - \overline{u}_0^i u_0)\partial_{X_i} b \\ & + \frac{1}{2}(\Delta u_0\overline{u}_0 + \Delta\overline{u}_0 u_0)a + \frac{1}{2}(\Delta u_0\overline{u}_0 - \Delta\overline{u}_0 u_0)b \\ & + 2|u_0|^4 a(a^2-b^2) + |u_0|^2\Delta_X a + \beta|u_0|^2 a\Big], \end{aligned} \tag{10.32}$$

and

$$\begin{aligned} |u_0|^4(D+1)(D-2)b = & \psi^i u_{0i}\overline{u}_0 + T\Big[(\psi^i u_{0i}\overline{u}_0 - \psi^i\overline{u}_{0i}u_0)Da \\ & + (\psi^i u_{0i}\overline{u}_0 + \psi^i\overline{u}_{0i}u_0)Db - \frac{1}{2}(\Delta u_0\overline{u}_0 - \Delta\overline{u}_0 u_0) \\ & - 4|u_0|^4 ab + 2|u_0|^2\psi^i D\partial_{X_i} b + |u_0|^2\Delta\psi Db\Big] \\ & + T^2\Big[(u_0^i\overline{u}_0 + \overline{u}_0^i u_0)\partial_{X_i} b + (u_0^i\overline{u}_0 - \overline{u}_0^i u_0)\partial_{X_i} a \\ & + \frac{1}{2}(\Delta u_0\overline{u}_0 + \Delta\overline{u}_0 u_0)b + \frac{1}{2}(\Delta u_0\overline{u}_0 - \Delta\overline{u}_0 u_0)a \\ & + \beta|u_0|^2 b + 2|u_0|^4 b(a^2-b^2) + |u_0|^2\Delta_X b\Big]. \end{aligned} \tag{10.33}$$

Thus, if we write

$$\begin{aligned} a(X,T) &= a_1 + a_2T + a_3T^2 + \cdots, \\ b(X,T) &= b_1 + b_2T + b_3T^2 + \cdots, \end{aligned}$$

where the coefficients a_j and b_j are polynomials in $\ln T$, the arguments of Chap. 2 yield the announced results. □

We now turn to the relation of the series to actual solutions. In the analytic case, we have the following theorem:

Theorem 10.13. *Let $\Sigma = \{t = \psi(x)\}$, with $|\nabla\psi(x)| < 1$, and (x_0, t_0) a point on Σ. Then equation (10.30) admits infinitely many solutions defined close to (x_0, t_0) and that blow up exactly on Σ. These solutions are of the form (10.31).*

The coefficients u_0, v_1, $v_2, \ldots$ are analytic and uniquely determined by the choice the argument of u_0, the imaginary part of $v_{3,0}$, and the real part of $v_{4,0}$.

Proof. Let $Y = T \ln T$. If $N = T\partial_T + (T + Y)\partial_Y$ and $w = P + iQ$, then P and Q satisfy:

$$\begin{aligned}(N+1)(N+6)P &= f_1(X) + Tg_1[X,T,Y,P,Q] + Yh_1[X,T,Y,P,Q],\\ (N+2)(N+5)Q &= f_2(X) + Tg_2[X,T,Y,P,Q] + Yh_2[X,T,Y,P,Q],\end{aligned} \tag{10.34}$$

where f_1, f_2, g_1, g_2, h_1, h_2 are analytic in their arguments, and involve also the derivatives of P and Q. Define z by

$$z^T = [\,P,\ NP,\ T\nabla_X P,\ Q,\ NQ,\ T\nabla_X Q\,].$$

We obtain

$$\begin{aligned} Nz_1 &= z_2,\\ (N+7)z_2 + 6\,z_1 &= f_1 + Tg_1[T,Y,z] + Yh_1[T,Y,z],\\ Nz_{2+i} &= T(\partial_{X_i} z_1 + \partial_{X_i} z_2) \quad \forall 1 \le i \le n,\\ Nz_{2+n+1} &= z_{2+n+2},\\ (N+7)z_{2+n+2} + 10\,z_{2+n+1} &= f_2 + Tg_2[T,Y,z] + Yh_2[T,Y,z],\\ Nz_{2+n+2+i} &= T(\partial_{X_i} z_{2+n+1} + \partial_{X_i} z_{2+n+2}) \quad \forall 1 \le i \le n,\end{aligned}$$

where the dependence on the spatial coordinates X has been suppressed. Thus, z solves

$$(N + A)z = f(X) + Tg[X,T,Y,z] + Yh[X,T,Y,z], \tag{10.35}$$

with g and h analytic in their arguments, and

$$A = \left[\begin{array}{cc|c|cc|c} 0 & -1 & & & & \\ 6 & 7 & & & & \\ \hline & & 0_n & & & \\ \hline & & & 0 & -1 & \\ & & & 10 & 7 & \\ \hline & & & & & 0_n \end{array}\right],$$

$$f(X)^T = [\,0, f_1(X), 0, \ldots, 0,\ 0, f_2(X), 0, \ldots, 0\,].$$

We are therefore led to the system

$$(N + A)z = f(X) + Tg[X, T, Y, z] + Yh[X, T, Y, z], \tag{10.36}$$

$$z(X, 0, 0)^T = [\,P(X, 0, 0),\, 0, \ldots,\, 0,\, Q(X, 0, 0),\, 0, \ldots,\, 0\,]. \tag{10.37}$$

The eigenvalues of A are 1, 2, 5, 6 with multiplicity 1, and 0 with multiplicity $2n$. Moreover, $f(X)$ belongs to the range of A. Thus (10.36)–(10.37) admits a unique solution by Theorem 4.5. Conversely, let z be a solution of (10.36)–(10.37). It satisfies

$$z_2 = Nz_1, \tag{10.38}$$

$$z_{2+i} = T\partial_{X_i} z_1, \text{ for } 1 \le i \le n, \tag{10.39}$$

$$z_{2+n+2} = Nz_{2+n+1}, \tag{10.40}$$

$$z_{2+n+2+i} = T\partial_{X_i} z_{2+n+1}, \text{ for } 1 \le i \le n. \tag{10.41}$$

Indeed, since z is solution of (10.36),

$$\begin{aligned} Nz_1 &= z_2, \\ Nz_{2+i} &= T\partial_{X_i}(z_1 + z_2), \end{aligned} \tag{10.42}$$

and the analysis of equation (10.25) applies here, and shows that if z is a solution, then necessarily

$$z^T = [\,z_1,\, Nz_1,\, T\nabla_X z_1,\, z_{n+3},\, Nz_{n+3},\, T\nabla_X z_{n+3}\,],$$

where (z_1, z_{n+3}) is a solution of (10.34). Defining (P, Q) as (z_1, z_2) ends the proof. □

For the nonanalytic case, the main result is our next theorem:

Theorem 10.14. *If $r > \frac{n}{2} + 7$, one can find infinitely many solutions u of (10.15) blowing up exactly on Σ,*

$$\begin{aligned} u(x,t) = u_0(X)\Big[\frac{1}{T} &+ v_1(X) + v_2(X)T + [v_{3,0}(X) + v_{3,1}(X)\ln T]T^2 \\ &+ [v_{4,0}(X) + v_{4,1}(X)\ln T]T^3 + [w(X, T, T\ln T) \\ &+ \lambda(X)\ln T]T^4\Big], \end{aligned}$$

where $w(X, T, T\ln T)$ is continuous in T, with values in H^{r-6} for T small.

To prove this result, we reduce the problem to a generalized Fuchsian system:

Theorem 10.15. *There are symmetric matrices Q and A^j, $1 \le j \le n$, a constant matrix A, and a function f such that if $\boldsymbol{t} = (t_0, t_1)$ and*

$$Q(N + A)\mathbf{w} = t_0 A^j \partial_{X_j}\mathbf{w} + \boldsymbol{t} \cdot f(t, X, \mathbf{w}), \tag{10.43}$$

then the two first components P and Q of $\mathbf{w}$ generate a singular solution u of (10.15) that blows up for $T = 0$, provided that

$$\begin{aligned}
\mathbf{w} =&(P, Q, P_{(0)}, Q_{(0)}, P_{(i)}, Q_{(i)}),\\
w =&P + iQ,\\
w_{(j)} =&P_{(j)} + iQ_{(j)},\\
u =&u_0 \left\{ \frac{1}{t_0} + v_1 + v_2 t_0 + v_{3,0} t_0^2 + v_{3,1} t_0 t_1 + v_{4,0} t_0^3 + v_{4,1} t_0^2 t_1 \right.\\
&\left. + w(t_0, t_1, X) t_0^4 + \lambda t_0^3 t_1 \right\},
\end{aligned}$$

and that the $\{w_{(j)}\}$ *are, for* $t_0 = t_1 = 0$*, the components of a gradient. Moreover, if* $\psi \in H^r(\mathbb{R}^n)$ *with* $r > \frac{n}{2} + 6$*, then* Q *and* $A^j \in H^{r-1}_{loc}(\mathbb{R}^n)$*, while* f *maps* H^{r-1} *to* H^{r-6}*.*

Proof. The argument being similar to the case of the cubic wave equation, we merely give the choices of Q, etc., and check that they have the desired properties. Let $\gamma = 1 - |\nabla\psi(x)|^2$. Define Q and A^j as follows:

$$Q = \begin{bmatrix} I_2 & & \\ & \gamma I_2 & \\ & & I_{2n} \end{bmatrix},$$

$$A^j = \left[\begin{array}{cc|cc|cc} 0 & 0 & & & & \\ 0 & 0 & & & & \\ \hline & & -2\psi^j & 0 & e^j & 0 \\ & & 0 & -2\psi^j & 0 & e^j \\ \hline & & {}^t e^j & 0 & & \\ & & 0 & {}^t e^j & & \end{array}\right],$$

where e^j is the jth vector of the canonical basis.

Let also

$$A = \left[\begin{array}{cccc|cc} 4 & 0 & -1 & 0 & & \\ 0 & 4 & 0 & -1 & & \\ -6 & 0 & 3 & 0 & & \\ 0 & -2 & 0 & 3 & & \\ \hline & & & & 3I_n & 0 \\ & & & & 0 & 3I_n \end{array}\right],$$

and $f(t_0, t_1, X, \mathbf{w}) = (g_0(X, t_0, t_1, \mathbf{w}), g_1(X, t_0, t_1, \mathbf{w}))$, with

$$g_0 = \begin{bmatrix} 0 \\ 0 \\ \mathrm{Re}\{b_0\} \\ \mathrm{Im}\{b_0\} \\ \mathrm{Re}\{\frac{\nabla_X u_0}{u_0}(w_0 + \lambda)\} \\ \mathrm{Im}\{\frac{\nabla_X u_0}{u_0}(w_0 + \lambda)\} \end{bmatrix}$$

and

$$g_1 = \begin{bmatrix} 0 \\ 0 \\ \mathrm{Re}\{b_1\} \\ \mathrm{Im}\{b_1\} \\ 4\ \mathrm{Re}\{\nabla_X \lambda\} \\ 4\ \mathrm{Im}\{\nabla_X \lambda\} \end{bmatrix},$$

where b_0 and b_1 are defined by

$$\begin{aligned} u_0\, b_0(X, t_0, t_1, \mathbf{w}) =& v_{4,0} \Delta u_0 + 2u_{0i} v^i_{4,0} + u_0 \Delta v_{4,0} + u^i_0 w_i - 2\psi^i u_{0i} w_0 \\ & - 2\psi^i u_{0i} \lambda - 2u_0 \psi^i \lambda_i - u_0 w_0 \Delta\psi - u_0 \lambda \Delta\psi \\ & + h_0(t_0,\, t_1,\, w,\, \overline{w},\, v_1,\, v_2, \ldots,\, \overline{v_1},\, \overline{v_2}, \ldots), \\ u_0\, b_1(X, t_0, t_1, \mathbf{w}) =& v_{4,1} \Delta u_0 + 2u_{0i} v^i_{4,1} + u_0 \Delta v_{4,1} - 2\psi^i u_{0i} \lambda - 2u_0 \psi^i \lambda_i \\ & - 4u_0 \lambda \Delta\psi + h_1(t_0,\, t_1,\, w,\, \overline{w},\, v_1,\, v_2, \ldots,\, \overline{v_1},\, \overline{v_2}, \ldots), \end{aligned}$$

where h_0 and h_1 are polynomial functions in their arguments. Let finally

$$\begin{aligned} u =& u_0 \left\{ \frac{1}{t_0} + v_1 + v_2 t_0 + v_{3,0} t_0^2 + v_{3,1} t_0 t_1 + v_{4,0} t_0^3 + v_{4,1} t_0^2 t_1 \right. \\ & \left. + w(t_0, t_1, X) t_0^4 + \lambda t_0^3 t_1 \right\}, \\ u_{(0)} =& u_0 \left\{ -\frac{1}{t_0^2} + v_2 + 2v_{3,0} t_0 + v_{3,1}(t_0 + 2t_1) + 3v_{4,0} t_0^2 + v_{4,1}(t_0^2 + 3t_0 t_1) \right. \\ & \left. + w_{(0)}(t_0, t_1, X) t_0^3 + \lambda(t_0^3 + 4t_0^2 t_1) \right\}, \\ u_{(i)} =& \frac{u_{0i}}{t_0} + (u_{0i} v_1 + u_0 v_{1i}) + (u_{0i} v_2 + u_0 v_{2i}) t_0 + (u_{0i} v_{3,0} + u_0 v_{3,0_i}) t_0^2 \\ & + (u_{0i} v_{3,1} + u_0 v_{3,1_i}) t_0 t_1 + (u_{0i} v_{4,0} + u_0 v_{4,0_i}) t_0^3 \\ & + (u_{0i} v_{4,1} + u_0 v_{4,1_i}) t_0^2 t_1 + u_0 w_{(i)}(t_0, t_1, X) t_0^3, \end{aligned}$$

where $w_{(0)} = P_{(0)} + iQ_{(0)}$ and $w_{(j)} = P_{(j)} + iQ_{(j)}$. If $\mathbf{w}$ is solution of (10.43), then $\mathbf{u} := (u, u_{(0)}, u_{(i)})$ solves

$$\begin{aligned} \partial_T u &= u_{(0)}, \\ (1 - |\nabla\psi|^2) \partial_T u_{(0)} &= \sum_i (\partial_{X_i} u_{(i)} - 2\psi_i \partial_{X_i} u_{(0)}) - u_{(0)} \Delta\psi + 2u|u|^2, \\ \partial_T u_{(i)} &= \partial_{X_i} u_{(0)}. \end{aligned} \tag{10.44}$$

This system is the usual symmetric-hyperbolic first-order system associated with equation (10.15), in the new variables X and T. Thus u is clearly a solution for (10.15), which concludes the proof. □

Since equation (10.43) written as system (10.29) satisfies the hypotheses (A1), (A2), (A3), and (A4) of Theorem 5.12, Theorem 10.14 follows.

10.3.3 Application to concentration of energy

Consider a solution of (10.16) decaying at spatial infinity, blowing up on a C^∞ n-dimensional spacelike manifold $\Sigma = \{t - \psi(x) = 0\} \subset \mathbb{R}^n$, with ψ of class C^∞ and satisfying $|\nabla\psi| < 1$. Suppose also that $\psi(0) = 0$. The so-called energy integral

$$\frac{1}{2}\int [u_t^2 + |\nabla u|^2 - u^4](x,t)\,dx \tag{10.45}$$

is finite and independent of t. We study

$$E(\alpha) = \frac{1}{2}\int_{B_\alpha} [u_t^2 + |\nabla u|^2 - u^4](x,-\alpha)\,dx,$$

the energy integral of u at time $-\alpha$ over B_α, the ball of center 0 and radius α.

Theorem 10.16. *As* $\alpha \to 0$,

(i) $E(\alpha) = \mathcal{O}(\alpha^{n-4})$,
(ii) if $\nabla\psi = 0$, *then* $E(\alpha) \sim -\frac{1}{3}\mathrm{meas}(B_1)\Delta\psi(0)\alpha^{n-3}$,
(iii) if the second-order derivatives of ψ *also vanish, then* $E(\alpha) = \mathcal{O}(\alpha^{n-1})$.

Remark 10.17. Condition (ii) necessarily holds at the first point of blowup.

Proof. Let

$$e(x,t) = u_t^2(x,t) + u_i u^i(x,t) - u^4(x,t);$$

using the expansion of u, we obtain

$$\begin{aligned} e(x,t) = &\frac{2u_0^2\psi_i\psi^i}{T^4} + \frac{2\psi^i u_{0i} - 4u_0^3 u_1}{T^3} \\ &+ \frac{u_{0i}u_0^i + 2\psi^i u_{1i}u_0 - 2\psi^i\psi_i u_0 u_2 - 2u_0u_2 - 6u_0^2u_1^2 - 4u_0^3u_2}{T^2} \\ &+ \mathcal{O}\left(\frac{1}{T}\right). \end{aligned} \tag{10.46}$$

Replacing u_0, u_1, u_2 in (10.46) by their explicit expressions (14.10a-d) leads, for $\mathbf{s} \in B_1$ and $\alpha > 0$, to

$$e(\alpha\,\mathbf{s}, -\alpha) = \frac{2M^2(1-M^2)}{(1+\psi^i(0)s_i)^4\alpha^4} + \mathcal{O}\left(\frac{1}{\alpha^3}\right),$$

where $M = \sqrt{1 - |\nabla\psi(0)|^2}$. Indeed, if $(x,t) = (\alpha\,\mathbf{s}, -\alpha)$, then, since $\psi(0) = 0$,

$$T = -\alpha - \psi(\alpha\,\mathbf{s}), = -\alpha - \alpha\,\mathbf{s}\cdot\nabla\psi(0) + \mathcal{O}(\alpha^2).$$

Furthermore,

$$u_0(\alpha\,\mathbf{s}) = \sqrt{1 - |\nabla\psi(\alpha\mathbf{s})|^2} = \sqrt{1 - |\nabla\psi(0)|^2 + \mathcal{O}(\alpha)} = M + \mathcal{O}(\alpha),$$

while the other coefficients u_{0i}, $u_1, \dots$ are C^∞ and thus $\mathcal{O}(1)$. The most singular term in $e(x,t)$ is $2u_0^2\psi_i\psi^i/T^4$. Therefore,

$$E(\alpha) = \frac{\alpha^n}{2}\int_{B_1} e(\alpha\,\mathbf{s}, -\alpha)\,d\mathbf{s} = \alpha^{n-4}\int_{B_1}\frac{M^2(1-M^2)}{(1+\psi^i(0)s_i)^4}\,d\mathbf{s} + \mathcal{O}(\alpha^{n-3}),$$

which proves (i).

Suppose now that $\nabla\psi(0) = 0$. Then for $(x,t) = (\alpha\,\mathbf{s}, -\alpha)$,

$$T_{|(x,t)=(\alpha\,\mathbf{s},-\alpha)} = -\alpha - \frac{1}{2}\alpha^2\psi^{ij}(0)s_is_j + \mathcal{O}(\alpha^3),$$
$$\psi^i(x) = \alpha\,\psi^{ij}s_j + \mathcal{O}(\alpha^2).$$

The first few terms of the series for u satisfy

$$\begin{aligned} u_0(x) &= 1 + \mathcal{O}(\alpha^2), \\ u_{0i}(x) &= \mathcal{O}(\alpha), \\ u_1(x) &= -\frac{\Delta\psi(0)}{6} + \mathcal{O}(\alpha^2), \\ u_2(x) &= -\frac{\Delta\psi(0)^2}{36} + \frac{\psi^{ij}(0)\psi_{ij}(0)}{6} + \mathcal{O}(\alpha). \end{aligned}$$

Therefore, for α small,

$$\begin{aligned} e(x,t) = &-\frac{2\Delta\psi(0)}{3\alpha^3} \\ &+ \frac{1}{\alpha^2}\Big[2\psi^{ij}(0)\psi_i^k(0)s_js_k + \Delta\psi(0)\psi^{ij}(0)s_is_j - \psi^{ij}(0)\psi_{ij}(0)\Big] \\ &+ \mathcal{O}\left(\frac{1}{\alpha}\right), \end{aligned}$$

and by integration,

$$\begin{aligned} E(\alpha) &= \frac{\alpha^n}{2}\int_{B_1} e(\alpha\,\mathbf{s}, -\alpha)\,d\mathbf{s} \\ &= -\frac{1}{3}\mathrm{meas}(B_1)\Delta\psi(0)\alpha^{n-3} + C\alpha^{n-2} + \mathcal{O}(\alpha^{n-1}), \end{aligned}$$

where $C = 0$ when the second-order derivatives of ψ vanish at the origin. This completes the proof of (ii) and (iii). □

Remark 10.18. If $\nabla\psi(0) = 0$ and $\Delta\psi(0) = 0$, but $|\nabla^2\psi(0)| \neq 0$, then $E(\alpha) = \mathcal{O}(\alpha^{n-2})$ for α close to 0.

Theorem 10.16 shows that if ψ is flat enough at the first blowup point, the energy does not focus at this point; for instance, in case $n = 3$ and $\nabla\psi = 0$, $E(\alpha)$ remains bounded. Moreover, in case ψ is smooth and admits a local extremum at the origin, one can predict the value of $\Delta\psi(0)$ using only integrals on the backward light-cone.

10.3.4 Behavior of L^p norms

In 1984, Brezis [27] raised the question of finding, for a given solution of the cubic nonlinear wave equation

$$u_{tt} - \Delta u = 2u^3 \tag{10.47}$$

with Dirichlet boundary condition on a smooth bounded domain $\Omega \subset \mathbb{R}^n$, with $n \geq 1$, the least upper bound $q_0(u)$ of the set of exponents p such that the L^p norm of u blows up. More precisely, if $n \leq 3$, one has local existence and uniqueness of classical solutions for data in $H^2 \times H_0^1(\Omega)$; they persist as long as they do not blow up in $H_0^1 \times L^2(\Omega)$; the integral

$$\frac{1}{2} \int_\Omega [u_t^2 + |\nabla u|^2 - u^4] dx \tag{10.48}$$

is bounded (in fact, constant); the Gagliardo–Nirenberg inequality shows that blowup in $H_0^1 \times L^2(\Omega)$ implies blowup in L^p if $p > n$. Thus, $q_0(u) \leq n$ if $n \leq 3$. We restrict our attention to the case in which Ω is the unit ball B and the first singularity first appears for $t = 0$, at the center of B.

It is both natural and convenient to consider the forced problem

$$u_{tt} - \Delta u - 2u^3 = F(x,t), \quad x \in B \text{ and } -1 < t < 0, \tag{10.49}$$

$$u = \varphi(x,t), \quad x \in \partial B \text{ and } -1 < t < 0, \tag{10.50}$$

$$(u, u_t) = (f(x), g(x)) \quad \text{for } t = -1. \tag{10.51}$$

Indeed, in applications or numerical work one must often allow for the presence of "noise" of limited regularity on the right hand side F, and, as we shall see, such noise may affect the blowup behavior.

In a nutshell, the consequences of reduction are that $q_0 = n$ for data and F of low regularity, but $q_0 = n/2$ for smooth and "generic" data and F. In all our examples, $\varphi(x,t)$ is at least C^2 and u blows up for $x = 0$, $t = 0$. By defining F through $F = u_{tt} - \Delta u - 2u^3$, one generates explicit examples demonstrating in particular that:

1. If the blowup surface is smooth, $q_0 \leq n/2$.
2. If $n > 12$, one can find $F(x,t)$ continuous with values in L^2, and classical solutions with $q_0 = n/(1+\varepsilon)$ for every $\varepsilon \in (0,1)$.

Both classes of examples are obtained by truncating the series (10.19). with $T = t - \psi(x)$, $|\nabla\psi| < 1$, given by reduction. One can again give a geometric interpretation for the first few coefficients of expansion at the first blowup point.

Remark 10.19. Lindblad and Sogge [134] solve the problem in $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^n)$, with $s = \frac{n}{2} - 1$ if $n \geq 4$, and find uniqueness under an additional condition in spaces of the form $L_t^r(L_x^s)$ (with $r < \infty$); here, $\dot{H}^s$ is the set of u such that $(-\Delta)^{s/2} u$ is square integrable. Reduction yields better results because the regular part of the solution has better smoothness properties, and can be separated from the singular part in a systematic fashion.

The main results are as follows [34,35].

Theorem 10.20. *For every integer $m \geq 1$, there is a problem of the form (10.49–10.49) with f, g, F, φ of class C^∞ for which $q_0 = n/2m$. There are also problems for which u blows up uniformly on a proper subset of B with nonempty interior, and for which $q_0 = 0$.*

Note that $u(x,t) = u_0(x)/T + v(x,t)$, with $T = t - \psi(x)$, where $\psi \geq 0$ and $v \in C^2$. On the other hand, u_0/T is not in L^1 for any positive t, so that u has no continuation after blowup, but v does.

We now turn to solutions of limited regularity. We say that "u is a classical solution" if

1. $u \in \bigcap_{k=0}^{2} C^k([-1,0); H^{2-k}(B))$,
2. u^3 is uniformly bounded in $L^2(B)$ and $F \in C([-1,0]; L^2(B))$,
3. the equation holds in the sense of distributions, and
4. u has finite energy.

Theorem 10.21. *If $\varepsilon \in (0,1)$ and $n > \max(12 - 12\varepsilon, 6 + 6\varepsilon)$, there is a problem of the form (10.49–10.49) with a classical solution such that $q_0 = n/(1+\varepsilon)$.*

Theorem 10.22. *If $\varepsilon \in (0,1)$ and $n > 6 + 6\varepsilon$, there is also a problem of the form (10.49–10.49) with a classical solution that is bounded for every $t < 0$, and such that $q_0 = n/(1+\varepsilon)$.*

Finally, we consider still weaker types of solution Let us say that "u is a solution in H^1" if

1. $u \in \bigcap_{k=0}^{1} C^k([-1,0); H^{1-k}(B))$,
2. u^3, Δu and u_{tt} are uniformly bounded in $L^1(B)$,
3. $F \in C([-1,0]; L^1 \cap H^{-1}(B))$, and
4. u solves equation (10.49).

Theorem 10.23. *If $\varepsilon \in (0,1)$ and $n > 4 + 4\varepsilon$, there is a problem of the form (10.49–10.49) with a solution in H^1 that is bounded in space for every $t < 0$, and for which $q_0 = n/(1+\varepsilon)$; it blows up in H^2 if $n = 5$ and $\varepsilon < \frac{1}{4}$.*

10.4 Weak detonations

We construct explosive solutions to Short's 3D generalization of Clarke's equations, under the sole assumption that the detonation front is supersonic. As usual, we give a complete expansion of solutions and specify the terms in the expansion that determine all the others. This shows how to relate the variation of the data to the variation of the (nonplanar) detonation front.

10.4.1 Background information

The so-called weak detonation, which terminates on the supersonic branch of the Hugoniot curve, has been considered only recently, because no steady traveling weak detonation wave can exist [64]. However, it appears that a "quasisteady" form of weak detonation is relevant in the description of both the shock-induced and initial-disturbance-induced transition to detonation in an explosive material modeled by a one-step Arrhenius reaction with large activation energy [98, 53]. The Arrhenius factor $\exp A/\varphi$, where φ is proportional to temperature, is approximated by an exponential, and the blowup represents a phase of very rapid increase in temperature. Although the temperature does not actually become infinite, it is believed that the blowup set for the resulting model does represent an approximation of the time evolution of the detonation front. There seems to be some agreement about the scenario proposed by Kapila and Dold [98], which is based on a long series of contributions on the possible mechanisms leading to the ignition of a Zel'dovich–von Neumann–Döring detonation, in ignition induced by a piston creating a shock; in particular, one of the phases involves a weak, shockless supersonic detonation wave that moves away from the piston, "as neighboring fluid elements undergo thermal ignition at different times at different locations in space." The reader will recognize the close similarity between this description and the blowup mechanism for nonlinear wave equations.

The problem is therefore to show how the equations for a weak detonation may support an explosive solution in which a supersonic detonation front determines ignition at different points at different times. Since the detonation path is not a straight line, the detonation wave is not a steady wave; in this sense, it is quasisteady, because it is nevertheless expected to admit a tangent plane. We now describe the mathematical problem, and solve it by reduction.

10.4.2 Mathematical problem

The mathematical task is to construct singular solutions to Clarke's reactive-acoustic equations singular on the detonation front; we describe Short's three-dimensional generalization [165], which is not more difficult to handle than the one-dimensional case. The equations are derived from the reactive Euler equations by (i) nondimensionalization; (ii) introduction of the inverse activation energy parameter $\varepsilon \ll 1$; (iii) definition of induction zone unknowns $(\varphi, p, \boldsymbol{u}, v)$ related to temperature, pressure, velocity, and specific volume by $T = 1 + \varepsilon\varphi + \mathcal{O}(\varepsilon^2)$, $P = 1 + \varepsilon p + \mathcal{O}(\varepsilon^2)$, $\boldsymbol{U} = 1 + \varepsilon \boldsymbol{u} + \mathcal{O}(\varepsilon^2)$, and $V = 1 + \varepsilon v + \mathcal{O}(\varepsilon^2)$. The scaled reactant mass fraction $y = 1 - \varepsilon w + \mathcal{O}(\varepsilon^2)$ is determined by $w_t = \exp \varphi$ at first order. Expanding and retaining the leading-order terms in one dimension, corresponding to a mass-weighted Lagrangian coordinate, leads to Clarke's equations. Its three-dimensional generalization reads

$$\begin{aligned} v_t &= \operatorname{div} \boldsymbol{u}, \\ \boldsymbol{u}_t &= -\nabla p, \\ \varphi_t - (\gamma - 1)p_t &= e^{\varphi}, \end{aligned} \tag{10.52}$$

where

$$\gamma p = \varphi - v.$$

The system (10.52) will be referred to as the weak detonation equations. It contains five equations in five unknowns: v, φ, and the three components of $\boldsymbol{u}$. The last equation of the system is equivalent to

$$\varphi_t = (1 - \gamma)v_t + \gamma e^{\varphi}. \tag{10.53}$$

We are interested in solutions that become infinite on some hypersurface $t = \psi(x)$ representing the evolution of the detonation front. More precisely, since the solutions are meaningful only in a domain in which the unknowns remain $\mathcal{O}(1/\varepsilon)$, the detonation path corresponds to the set on which this limitation is barely violated. The detonation path is close to the blowup set, because of the expansion of the solution.

Remark 10.24. We may derive a single third-order equation for φ as follows. Since $v_{tt} = -\Delta p = \gamma^{-1}\Delta(\varphi - v)$, we have

$$\Box_\gamma v := \gamma v_{tt} - \Delta v = -\Delta\varphi. \tag{10.54}$$

Applying $\Box_\gamma$ to equation (10.53) and rearranging terms and using (10.54), we obtain

$$\Box(\varphi_t) = \Box_\gamma(e^{\varphi}) \tag{10.55}$$

with $\Box = \partial_{tt} - \Delta$. This suggests logarithmic leading-order asymptotics for the three unknowns. We now show that this is indeed the case.

Change of independent variables

Let us take as new independent variables x, y, z, and $T = \psi(x) - t$, and view the five unknowns as functions of these variables. Here $T = 0$ represents the detonation front. Replacing ∂_t by $-\partial_T$ and ∇ by $\nabla + (\nabla\psi)\partial_T$, the system takes the form

$$v_T + \operatorname{div} \boldsymbol{u} + \nabla\psi \cdot \boldsymbol{u}_T = 0, \tag{10.56a}$$

$$\boldsymbol{u}_T = \nabla p + p_T \nabla\psi, \tag{10.56b}$$

$$\varphi_T - (\gamma - 1)p_T + e^{\varphi} = 0. \tag{10.56c}$$

Before we proceed to the first reduction, let us eliminate p from this system:

Lemma 10.25. *System (10.56) is equivalent to*

$$(1 - |\nabla\psi|^2)v_T + A - |\nabla\psi|^2 e^{\varphi} = 0, \tag{10.57a}$$

$$\boldsymbol{u}_T = \nabla p + \frac{A - \exp(\varphi)}{1 - |\nabla\psi|^2}\nabla\psi, \tag{10.57b}$$

$$(1 - |\nabla\psi|^2)\varphi_T + (\gamma - |\nabla\psi|^2)e^{\varphi} = +(\gamma - 1)A, \tag{10.57c}$$

where

$$A = A[v, \boldsymbol{u}, \varphi] = \operatorname{div} \boldsymbol{u} + \gamma^{-1}\nabla\psi \cdot \nabla(\varphi - v).$$

Proof. From the third equation, we obtain

$$\gamma(\varphi_T + e^{\varphi}) = (\gamma - 1)(\gamma p)_T = (\gamma - 1)(\varphi_T - v_T),$$

hence, using the expression of v_T,

$$\varphi_T + e^{\varphi} = (1 - \gamma)v_T = (\gamma - 1)[A + p_T|\nabla\psi|^2].$$

Since

$$\begin{aligned}\gamma p_T &= \frac{\gamma}{\gamma - 1}(\varphi_T + e^{\varphi}) = \frac{\gamma}{\gamma - 1}(\varphi_T + \gamma e^{\varphi} + (1 - \gamma)e^{\varphi}) \\ &= \gamma[A + p_T|\nabla\psi|^2] - \gamma e^{\varphi},\end{aligned}$$

we obtain

$$p_T = \frac{A - \exp(\varphi)}{1 - |\nabla\psi|^2},$$

and hence

$$\varphi_T + e^{\varphi} = (\gamma - 1)\frac{A - |\nabla\psi|^2 \exp(\varphi)}{1 - |\nabla\psi|^2}.$$

Equation (10.57c) follows. Equation (10.57b) results from the expression for p_T. Substitution of the value of $\boldsymbol{u}_T$ into the equation for v gives (10.57a). □

Leading order and first reduction

Theorem 10.26. *The weak detonation equations admit of reduction of the form*

$$\varphi = \ln\frac{b}{T} + T\Phi; \quad v = v_0 \ln T + v_1 + TV; \quad \boldsymbol{u} = \boldsymbol{u}_0 \ln T + \boldsymbol{u}_1 + T\mathbf{U}, \tag{10.58}$$

where

$$b = \frac{1 - |\nabla\psi|^2}{\gamma - |\nabla\psi|^2}; \quad v_0 = \frac{|\nabla\psi|^2}{\gamma - |\nabla\psi|^2}; \quad \boldsymbol{u}_0 = \frac{\nabla\psi}{\gamma - |\nabla\psi|^2}. \tag{10.59}$$

If $|\nabla\psi| < 1$, the five unknowns $(\varphi, \boldsymbol{u}, v)$ are uniquely determined by the five singularity data ψ, v_1, and the three components of $\boldsymbol{u}_1$. This provides a supersonic detonation front of arbitrary shape.

Proof. Direct substitution and identification of the leading terms, which are $\mathcal{O}(1/T)$, yields

$$v_0 = \boldsymbol{u}_0 \cdot \nabla\psi; \quad \boldsymbol{u}_0 = (k-1)\frac{\nabla\psi}{\gamma}; \quad (b-1) = \frac{\gamma-1}{\gamma}(k-1).$$

The relations (10.59) follow. □

Second reduction is obtained by the procedure of Sects. 2.2.4 and 3.2. The determination of solutions from the singularity data follows from the existence theorems in Chap. 4 and Chap. 5 in the usual way. The variation of the solution when the singularity data are varied may now be now computed by solving the linearization of the reduced Fuchsian equation. This is possible because this linearization is again Fuchsian; the procedure parallels that of Sect. 10.2.5.

10.5 Soliton theory

Soliton equations are nonlinear wave equations with traveling wave solutions having remarkable superposition properties [104, Chaps. 4 and 5]. Their modern mathematical theory, the method of inverse scattering, may often be understood as a procedure to compute a complete set of first integrals for an infinite-dimensional Hamiltonian system; for this reason, soliton equations are often referred to as integrable systems.[3] Although of very special form, soliton equations may be derived from extremely general nonlinear wave equations by a perturbative modeling method (the “reductive perturbation method”), characterized by a specific choice of scaled independent variables adapted to the dominant wave number in the problem at hand; see [104, pp. 137–145]. The theory of solitons is useful because it replaces the search for approximate solutions of an exact model equation by the search for exact solutions of an approximate model with rich mathematical structure.

We prove that reduction techniques justify the so-called ARS-WTC expansions for solutions of integrable systems, and show how to generalize them to nonintegrable systems [109,120,124]. Let us begin with some background information.

10.5.1 ARS-WTC expansions

Exact solutions given by the method of inverse scattering are, in many cases of interest to applications, rational functions of exponentials. As such, they admit complex singularities that travel at the same speed as the wave. For instance, the Korteweg–de Vries equation (KdV)

$$u_t + 6uu_x + u_{xxx} = 0$$

[3] The rigorous construction of a complete set of integrals may be delicate; see [12].

admits the traveling wave solutions $2k^2 \operatorname{sech}^2(kx - 4k^3t)$, which is singular for $kx - k^3t = i\pi/2 + in\pi$, where n is an integer. Ablowitz, Ramani, and Segur (ARS) [4, 2] observed that ODE reductions of integrable equations are very often ODEs of Painlevé type, that is, ODEs the solutions of which are not ramified around their movable singularities.[4] Weiss et al., in a series of papers, starting from [180], made the basic observation that all the integrable PDES considered by Ablowitz et al. appeared to admit formal solutions to all orders, called WTC expansions, involving only pure powers, of the form

$$u = \phi^{-m} \sum_{j \geq 0} u_j(x,t)\phi^j,$$

where $\phi(x,t)$ is an arbitrary analytic function subject only to the condition that the singularity surface $\phi = 0$ is noncharacteristic (for KdV, this amounts to $\phi_x \neq 0$). Furthermore, the functions u_j may be chosen arbitrarily for certain values of j; for example, for the KdV equation, one finds that $m = 2$ and that u_j is arbitrary for $j = 4$ and 6; these values of j were called the "resonances," and we have kept this terminology. Since both ϕ and the u_j are allowed to depend on x and t, several expansions may correspond to the same function;[5] however, if we take ϕ of the form $x - \psi(t)$ ("reduced ansatz"), and require the u_j to depend only on t, a given function can have at most one expansion if it converges. Unfortunately, should the series diverge, it is conceivable that several solutions have the same expansion: recall that, using the function $\exp(-1/x^2)$, one can construct infinitely many functions asymptotic to a given divergent series. Weiss et al. [180] therefore suggested that the functions $\psi(t)$, $u_4(t)$, and $u_6(t)$ determine a unique solution of the KdV equation. This result will follow from reduction. Since integrable systems generally require only pure powers, it was widely held that expansions with logarithms were not relevant to the theory of solitons. However, as a rule, perturbation of soliton equations does lead to expansions with logarithms.

10.5.2 The impact of reduction

We therefore have two problems before us: (i) Do WTC expansions converge? This would prove that the arbitrary functions in the expansion determine only one solution of the corresponding PDE. (ii) What is the rigorous analogue of WTC expansions for nonintegrable problems? Reduction techniques give answers to both questions. In fact, the examples considered by Weiss et al. fall within the scope of the general methods of Chap. 3. Take for instance the case of KdV.

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[4] A movable singularity is one that depends on the solution: it moves when the Cauchy data are perturbed.

[5] For instance, we may replace u_1 by $u_1 + \phi$ provided we replace u_2 by $u_2 - 1$, leaving all the other coefficients u_j unaltered.

Theorem 10.27. *If $\phi = x - \psi(t)$, $u_4(t)$, and $u_6(t)$ are analytic near the origin, the KdV equation admits a unique local solution given by a convergent expansion $\phi^{-2}\sum_{j=0}^{\infty} u_j(t)\phi^j$. Therefore, the WTC expansion determines a unique solution of KdV for every choice of the arbitrary coefficients.*

Proof. Let us associate a system to the KdV equation. For convenience, we scale variables so that KdV reads $u_t + uu_x - u_{xxx} = 0$. Define new unknowns $u_j = (x\partial_x)^j u$ for $j = 0, 1, 2$. We obtain

$$\begin{aligned} x\partial_x u_0 &= u_1; \\ x\partial_x u_1 &= u_2; \\ x\partial_x u_2 &= 3u_2 - 2u_1 + x^3 u_{0t} + x^2 u_0 u_1. \end{aligned}$$

Assume that the singularity surface is given by $x = \psi(t)$. Take $x - \psi(t)$ as the new space variable, still denoted by x for convenience, so that we should replace u_{0t} by $u_{0t} - \psi'(t)u_{0x} = u_{0t} - \psi' u_1/x$. Now, if the u_j's behave like x^ν, we find that both sides of the equation balance each other at leading order if $\nu = -2$. This suggests the substitution $u_j = v_j x^{-2}$. We find that $v = (v_j)$ solves the system

$$\begin{aligned} (x\partial_x - 2)v_0 &= v_1; \\ (x\partial_x - 2)v_1 &= v_2; \\ (x\partial_x - 2)v_2 &= 3v_2 - 2v_1 + v_0 v_1 - \psi' x^2 v_1 + x^3 v_{0t}. \end{aligned}$$

For $x = 0$, we find that if $v \neq 0$, necessarily $v = (12, -24, 48)$. Setting $v_0 = 12 + xw_0$, $v_1 = -24 + xw_1$, $v_2 = 48 + xw_2$, we obtain the first reduced equation, with $\varepsilon = 1$:

$$\begin{aligned} (x\partial_x - 1)w_0 - w_1 &= 0; \\ (x\partial_x - 1)w_1 - w_2 &= 0; \\ (x\partial_x - 4)w_2 - 10w_1 + 24w_0 &= x(w_0 w_1 - \psi' x w_1 + 24\psi' + x^2 w_{0t}). \end{aligned}$$

This has the general form $(x\partial_x + A)w = \mathcal{O}(x)$. Computing the eigenvalues of A, we find that the positive resonances are 4 and 6, each corresponding to one arbitrary coefficient in the expansion of u. The third resonance, -1, corresponds to the translation-invariance of the problem; see Sect. 3.5.1. We find that one may take $\ell = 0$ (no logarithms). Performing the second reduction and appealing to Theorem 4.5, we obtain the convergence of the expansion, parameterized by ψ and two arbitrary functions. □

Let us now outline the corresponding results for other integrable equations. Results are in all respects similar to those for KdV, except that the exponent m, and the values j for which u_j is arbitrary, vary from one equation to another. The modified KdV equation

$$u_t + 3u^2 u_x - 2u_{xxx} = 0$$

admits convergent series solutions in which $m = 1$, and u_3 and u_4 are arbitrary. The sine–Gordon equation in the form

$$u_{xt} = \sin u$$

is such that $\exp(iu)$ admits a convergent series solution with $m = 2$ and u_2 arbitrary. The Kadomtsev–Petviashvili equation

$$u_{tx} + u_x^2 + uu_{xx} + \delta u_{xxx} + u_{yy} = 0,$$

in which δ is a positive constant, admits series solutions with $m = 2$ and arbitrary functions u_4, u_5, and u_6. The fifth-order KdV equation

$$4u_t = \partial_x[u_{xxxx} + 5u_x^2 + 10uu_{xx} + 10u^3]$$

admits two types of series solutions, both having $m = 2$. The first has $u_0 = -2\phi_x^2$ and arbitrary u_j for $j = 2$, 5, 6, and 8; the second has $u_0 = -2\phi_x^2$ and arbitrary u_j for $j = 6$, 8, and 10. Other examples may be found in the problems. For a class of nonintegrable fifth-order equations, see Problem 10.12.

The general pattern of WTC analysis is that the coefficients u_j of the expansion of the solution should satisfy a recurrence of the form

$$P(j)u_j = f_j(u_0, \dots, u_{j-1}),$$

where the zeros of the polynomial P give the resonances. In integrable cases, it turns out that whenever j is a resonance, the expression f_j also happens to be identically zero. What about nonintegrable problems? In general, f_j has no reason to be zero if $P(j) = 0$. Nevertheless, if we perturb KdV by terms that do not affect the leading balance, the general results of Chap. 3 yield a result similar to Theorem 10.27, except that the expansion may now contain logarithms; see Problem 10.13. For nonintegrable perturbations of integrable systems, even though the solution is generally not analytic, it may be uniformized by the introduction of finitely many new variables of the form ϕ, $\phi \ln \phi$,...,$\phi(\ln \phi)^\ell$. In other words, even the nonintegrable case corresponds to a function of several complex variables that is free from branching.

10.5.3 Resonances and Poincaré–Dulac expansions

We justify the term "resonance" for the indices j such that u_j is arbitrary by showing that resonances in the sense of WTC theory are closely related to the resonance of frequencies in an ODE near a stationary solution, in the sense of Poincaré–Dulac theory. Consider an ODE

$$x_t = F(x),$$

where x has n components, and $F(x) = Ax + Q(x)(x, x)$ near $x = 0$. The eigenvalues λ_j are said to exhibit resonance, in the sense of Poincaré and Dulac, if there exists a nontrivial relation of the form

$$\lambda_{j_0} = m_1\lambda_1 + \cdots + m_n\lambda_n,$$

where the coefficients m_k are nonnegative integers. Let us seek a solution of this equation in the form

$$x(t) = x(z_1, \ldots, z_n),$$

where $z_k = \exp(\lambda_k t)$. We find that x must solve

$$Nx = F(x),$$

where

$$N = \lambda_1 z_1 \frac{\partial}{\partial z_1} + \cdots + \lambda_n z_n \frac{\partial}{\partial z_n}.$$

Consider the ansatz

$$x(t) = \sum_k z_k y_k(t),$$

where, to fix ideas, $y_1 \to 1$ as t tends to $-\infty$, but $y_k \to 0$ for $k > 1$. The equation for x yields

$$\sum_k z_k E_k = 0$$

with

$$E_k = (N + \lambda_k - A)y_k - \sum_l z_l Q(x)(y_k, y_l).$$

Therefore, it *suffices* to solve the generalized Fuchsian system $E_1 = \cdots = E_n = 0$. If we wish to compute a solution of this system as a power series in the z_k's, we need to substitute for y_k a sum of homogeneous monomials $X z_1^{a_1} \cdots z_n^{a_n}$, where X is a constant vector. Now, the operator $(N + \lambda_k - A)$ transforms this monomial into

$$(\lambda_1 a_1 + \cdots + \lambda_n a_n + \lambda_k - A) X z_1^{a_1} \cdots z_n^{a_n}.$$

Therefore, we can recursively compute the coefficients of y unless $\lambda_1 a_1 + \cdots \lambda_n a_n + \lambda_k$ is equal to an eigenvalue λ_{j_0} of A. This provides a resonance whenever the a_k are not all zero. Thus, the resonances in the sense of dynamical systems correspond to the possible arbitrary terms in a series of the solution in powers of exponentials. The occurrence of logarithmic terms in this context corresponds to the familiar terms $t^j \exp(\lambda_k t)$ in the solution of an equation with multiple eigenvalues.

As an application, consider the problem of finding solitary waves to the fifth-order water-wave model:

$$v_t + \alpha v_{xxxxx} + \mu v_{xxx} + \gamma \partial_x [2 v v_{xx} + v_x^2] + 2 q v v_x + 3 r v^2 v_x = 0. \tag{10.60}$$

This equation has exact solitary waves (with tails decaying to zero) and negative speed when $\alpha > 0$, $r \le 0$, $q \ne 0$, and $\mu > 0$ (see [107] for details and other situations in which existence holds). This model has been studied extensively; see [121, 107]. It arises as a model for water waves, and as a general model for the interaction of long and short waves [15]. There are several arguments to the effect that there exists *no* such solution in the KdV limit, which would correspond here to waves with positive speed. Reduction can be used to prove that the solution actually has a convergent expansion in powers of exponentials [107]. This generalizes the result of [121], which was based on a direct study of the series.

10.5.4 Truncated expansions and Lax pairs

By choosing the expansion variable suitably, it is possible to collapse the infinite series for the solution into a *finite sum*; this may be understood as follows: if u is the desired solution, and if u admits a simple pole along some hypersurface, we may simply take $T = 1/u$, so that the solution takes the trivial form

$$u = \frac{1}{T}.$$

A less trivial computation is possible in some cases, and may lead to the construction of a Lax pair (see [104], Chap. 4, for a discussion of Lax pairs and the method of inverse scattering).

The results below on truncated expansions are essentially due to Weiss, although a complete proof that the truncated expansions do solve KdV seems difficult to find. Similar manipulations have been performed by other authors on several examples, but the underlying mechanism is still not well understood. We give in the rest of the section the details of the relation between truncated singular expansions and Lax pairs. The point is that the function ϕ in the WTC expansion should be taken equal to the ratio of two eigenfunctions of the scattering problem associated to another solution of KdV.

Let us define

$$K[u] = u_t + \partial_x \left\{ u_{xx} + \frac{1}{2}u^2 \right\}.$$

Thus, the equation $K[u] = 0$ is equivalent, after scaling u, to the KdV equation.

We solve the following *problem:* Find $u_0(x,t)$, $u_1(x,t)$, $u(x,t)$, and $\phi(x,t)$ such that

$$K[\tilde{u}] = 0,$$

where

$$\tilde{u} = \phi^{-2}(u_0 + u_1\phi) + u. \tag{10.61}$$

The *difficulty* to be overcome is that direct substitution appears to lead to five conditions for the four unknowns ϕ, $u_0(x,t)$, $u_1(x,t)$, $u(x,t)$. More precisely, if we write out $K[\tilde{u}]$ and collect powers of ϕ, we obtain an expression of the form

$$K[\tilde{u}] = \phi^{-5}\left[k_0(x,t) + \phi k_1(x,t) + \cdots + \phi^5 k_5(x,t)\right]; \tag{10.62}$$

if we require $k_0 = \cdots = k_5 = 0$, we obtain five independent equations (not six, because the fifth condition turns out to be the derivative of the sixth). The first and second conditions give $u_0 = -12\phi_x^2$ and $u_1 = 12\phi_{xx}$, and the last gives

$$K[u] = 0,$$

so that both u and $\tilde{u}$ should solve KdV.

There remain two conditions to be satisfied by ϕ (the equations $E_2 = 0$ and $E_3 = 0$ below; see (10.71–10.72). A priori, it is not clear that they are compatible with $K[u] = 0$. Weiss [179] noticed that $E_2 = \partial_x E_3 = 0$, provided that ϕ_x is the square of a solution of the Lax pair equations associated with some solution of KdV. This specifies ϕ up to the addition of a function of t alone. He also proved that if ϕ is the quotient of solutions of *some* Lax pair, then $E_2 = 0$ iff $E_3 = 0$. We identify this latter solution of KdV: u and $\tilde{u}$ are both obtained by a Bäcklund transformation from a *third* solution q of KdV, and a complete, self-contained proof may be obtained by working with the Lax pair equations associated with q:

Theorem 10.28. *Let λ be a constant, and assume (i) $K[q] = 0$ and (ii) $\phi = v_1/v_2$, where v_1 and v_2 are independent solutions of the Lax pair equations associated with v:*

$$v_{xx} + \frac{1}{6}(q+\lambda)v = 0, \tag{10.63}$$

$$v_t = \left(-\frac{1}{3}q + \frac{2}{3}\lambda\right) v_x + \frac{1}{6}q_x v. \tag{10.64}$$

Define $\tilde{u}$ by (10.61), with

$$u_0 = -12\phi_x^2, \tag{10.65}$$

$$u_1 = 12\phi_{xx}, \tag{10.66}$$

$$u = q - 6\partial_{xx} \ln \phi_x. \tag{10.67}$$

We then have

$$K[u] = K[\tilde{u}] = 0.$$

We list without comment some further properties that may be checked directly.

- $u = q + 12\partial_{xx} \ln v_2$ and $\tilde{u} = u + 12\partial_{xx} \ln \phi$.
- If we replace ϕ by $1/\phi$, u and $\tilde{u}$ are exchanged.
- ϕ_x is the square of an eigenfunction of the eigenvalue problem associated with u (it is proportional to $1/v_2^2$).
- ϕ_{xx} solves the linearization of $K[u] = 0$.
- Define

  $$S = \phi_t/\phi_x + \{\phi; x\},$$

  where $\{\phi; x\}$ is the Schwarzian derivative

$$\{\phi; x\} = \partial_x \left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\frac{\phi_{xx}}{\phi_x}.$$

We then have

$$\{\phi; x\} = \frac{1}{3}(q+\lambda), \tag{10.68}$$

$$S = \lambda. \tag{10.69}$$

- A priori, $\tilde{u}$ admits singularities whenever ϕ or $\phi_x = 0$.

Main steps of proof:

The proof is completely elementary, but slightly tricky. One first checks directly the relations (10.68–10.69). Next let $\pi = \phi_{xx}/\phi_x$. One obtains

$$\phi_{xxx}/\phi_x = \pi_x + \pi^2$$

and

$$\pi_x = \{\phi; x\} + \frac{1}{2}\pi^2 = \frac{1}{3}(q+\lambda) + \frac{1}{2}\pi^2.$$

One computes $K[u]$, which simplifies to $-K[q] = 0$. One then computes

$$K[\tilde{u}] - K[u] = 12\partial_x \left[\frac{E_2}{\phi} - \frac{E_3}{\phi^2}\right], \tag{10.70}$$

where

$$E_2 = \phi_{xt} + u\phi_{xx} + \phi_{xxxx}, \tag{10.71}$$

$$E_3 = \phi_x\phi_t + 4\phi_x\phi_{xxx} - 3\phi_{xx}^2 + u\phi_x^2. \tag{10.72}$$

One then checks

$$E_2 - (\phi_x S)_x = \phi_{xx}\left[u - (q+\lambda) + 6\partial_{xx}\ln\phi_x\right], \tag{10.73}$$

$$\phi_x^2 E_2 - \phi_{xx}E_3 = \phi_x^3 S_x. \tag{10.74}$$

Since $S = \lambda$, and u satisfies (10.67), we obtain $E_2 = E_3 = 0$, hence the result.

10.6 The Liouville equation

We consider here the equation

$$u_{tt} - u_{xx} = e^u,$$

for which the general solution, in the analytic case, was obtained by Liouville in 1853 [136]. Generalizing the features of its exact solution gave the first impetus for developing reduction techniques. The exact solution also allows very sharp regularity results. It is equivalent to

$$u_{\xi\eta} = e^u, \tag{10.75}$$

by the change of variables

$$\begin{aligned} \xi &= (t+x)/2, & \eta &= (t-x)/2; \\ \partial_\xi &= \partial_t + \partial_x, & \partial_\eta &= \partial_t - \partial_x. \end{aligned}$$

Here, $t = 0$ corresponds to $\xi + \eta = 0$. We describe in this section the structure of the general solution, and derive its consequences, following [109]. There is a large literature on the applications of the 1+1 Liouville equation both in physics and geometry; see [97, 68, 160] and their references.

Theorem 10.29. *If u is a solution of (10.75) of class C^k near (ξ_0, η_0), with $k \geq 3$, there are two functions f and g of class C^{k+1} such that*

$$u = \ln \frac{2f'(\xi)g'(\eta)}{(f(\xi)+g(\eta))^2} = \ln[-2\partial_{\xi\eta} \ln(f+g)]. \tag{10.76}$$

The functions f and g are determined up to homographies: all other representations of u are given by this formula, with f and g replaced by

$$\frac{af+b}{cf+d} \quad \textit{and} \quad \frac{-ag+b}{cg-d}$$

respectively, where $ad - bc \neq 0$.

Remark 10.30. From the representation (10.76), it follows that there are two types of characteristic singularities: those coming from singularities of f or g, and those coming from the vanishing of f' or g'. We are interested here in the second type, which we relate to the degeneration of noncharacteristic (WTC) singularities.

Proof. The existence of the representation is classical; it is related to the existence of a Bäcklund transformation between the Liouville and wave equations [160]. Of the possible ways to prove it, we choose one that gives some insight into the degree of indeterminacy of the functions f and g: We first need a few formulas involving Schwarzian derivatives. For any function u, let $E = u_{\xi\eta} - e^u$. We have

$$\partial_\xi E - u_\xi E = \partial_\eta \left(u_{\xi\xi} - \frac{1}{2}u_\xi^2 \right).$$

Therefore, if $E = 0$, $u_{\xi\xi} - \frac{1}{2}u_\xi^2 = F(\xi)$, and, similarly, $u_{\eta\eta} - \frac{1}{2}u_\eta^2 = G(\eta)$. If the representation formula holds, F and G are given by Schwarzian derivatives:

$$F(\xi) = \{f;\xi\} \quad \text{and} \quad G(\eta) = \{g;\eta\}, \tag{10.77}$$

where

$$\{f;\xi\} := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

To prove the theorem, consider a given solution u of $E = 0$. We may take $\xi_0 = \eta_0 = 0$ without loss of generality. Let us determine f and g by the third-order equations $\{f;\xi\} = u_{\xi\xi} - \frac{1}{2}u_\xi^2$ and $\{g;\eta\} = u_{\eta\eta} - \frac{1}{2}u_\eta^2$. We need three conditions for f and g, and we require, for definiteness, $f(0) = g(0) = g'(0) = 1$ and

$$\frac{1}{2}f'(0) = e^{u(0,0)}; \quad \frac{f''(0)}{f'(0)} - f'(0) = u_\xi(0,0); \quad g''(0) - 1 = u_\eta(0,0).$$

These conditions express that the representation (10.76) gives the correct values for $u(0,0)$, $u_\xi(0,0)$, and $u_\eta(0,0)$. They determine $f'(0)$, $f''(0)$, and $g''(0)$ respectively.

Let $w = \ln[2f'(\xi)g'(\eta)/(f(\xi) + g(\eta))^2]$. We now prove that $u = w$. We know that $u_{\xi\xi} - \frac{1}{2}u_\xi^2 = w_{\xi\xi} - \frac{1}{2}w_\xi^2$. Since $u_\xi(0,0) = w_\xi(0,0)$, we have $u_\xi(\xi,0) = w_\xi(\xi,0)$. Since $u(0,0) = w(0,0)$, we have $u(\xi,0) = w(\xi,0)$. Since $\partial_\xi(u_\eta)(\xi,0) = \exp u(\xi,0) = \exp w(\xi,0) = \partial_\xi(w_\eta)(\xi,0)$, we also have $u_\eta(\xi,0) = w_\eta(\xi,0)$. Similarly, we also know that $u_{\eta\eta} - \frac{1}{2}u_\eta^2 = w_{\eta\eta} - \frac{1}{2}w_\eta^2$. We conclude that since $u - w$ and $u_\eta - w_\eta$ both vanish for $\eta = 0$, they vanish everywhere. This means that $u = w$, which proves the representation formula.

As for the arbitrariness in f, it follows from the invariance of the Schwarzian derivative under homographies that f may be replaced by $(af + b)/(cf + d)$. Since the functions u, f, and g are assumed to be regular near the origin, it is easily seen that we must have $f(\xi) + g(\eta)$, f', and g' nonzero. Therefore, if f is given, the three values $u_\xi(0,0)$, $u(0,0)$, and $u_\eta(0,0)$ determine respectively $g(0)$, $g'(0)$, and $g''(0)$. Therefore, g is uniquely determined, since its Schwarzian derivative is also known. If we replace f by $(af+b)/(cf+d)$, it is a simple matter to check that g can be replaced by $(-ag+b)/(cg-d)$, and this is the only possible choice for the second function entering in the representation, as we just proved. We have therefore obtained a complete description of the desired solutions. This argument proves that if u is of class C^k, then f and g are of class C^{k+1}. □

The occurrence of homographies suggests that f and g may have isolated poles even when u has no singularity near the origin. This and other special features are confirmed by the following examples.

Examples

1. $f = \xi$, $g = -1/\eta$, $u = \ln[2/(1 - \xi\eta)^2]$. Even though g has a singularity for $\eta = 0$, the solution u has *no such singularity.*
2. $f = \xi$, $g = -\sum_{k=1}^m 1/(\eta - x_k)$. Here, the singular set has $m + 1$ connected components.
3. $f = \xi$, $g = -1/\eta^3$, $u = \ln[6\eta^2/(1 - \xi\eta^3)^2]$ has three singular surfaces, two of them noncharacteristic, and one characteristic ($\eta = 0$).

4. A general recipe for producing solutions with prescribed blowup curve is
$$u = \ln\left[-\frac{2f'(\xi)f'(\psi(\eta))\psi'(\eta)}{[f(\xi) - f(\psi(\eta))]^2}\right],$$
where f is monotone and ψ is nonincreasing, possibly with isolated singularities.
5. The case $f = \xi^2$, $g = \eta^2 - 1$ leads to the solution $\ln[8\xi\eta/(\xi^2 + \eta^2 - 1)^2]$, defined only for $\xi\eta > 0$, and singular inside this domain on two quarter-circles. The singularity surface becomes characteristic precisely when it meets the boundary of the domain of definition, which is itself characteristic.

We now compute the singularity data for noncharacteristic singularities of real-valued solutions.

Theorem 10.31. *Let u be real-valued with the above representation. Any noncharacteristic singularity is given by $\xi = \psi(\eta)$, with ψ decreasing, and $g(\eta) = -f(\psi(\eta))$ locally. Furthermore, if $T = \xi - \psi(\eta)$, the solution has the expansion*
$$u = \ln\left(-\frac{2\psi'}{T^2}\right) + \frac{T^2}{6}S(\psi(x)) + \mathcal{O}(T^3),$$
where $S(x) = \{f; x\}$. The singularity data are ψ and $\frac{1}{6}S(\psi(\xi))$.

Proof. Noncharacteristic singularities occur on surfaces that are not parallel to the axes, which means that $f'g'$ remains positive. The singularity is therefore described by $f(\xi) + g(\eta) = 0$. By the implicit function theorem, we may write locally $\xi = \psi(\eta)$. This gives $g(\eta) = -f(\psi(\eta))$, and $\psi' = -g'(\eta)/f'(\psi(\eta)) < 0$. Substituting $\xi = T + \psi(\eta)$ into the representation formula, we obtain the expansion of u after a short calculation. The proof of the theorem is complete. □

We now show that the requirement that the solution have *no initial singularities* provides information on the number of components of the blowup surface:

Theorem 10.32. *If $u \in C^3$ for $\xi + \eta$ small, then the singular set Σ, if nonempty, is the graph of a decreasing function of ξ, and has at most two components, one on each side of the second diagonal.*

Proof. See Problem 10.10. □

Remark 10.33. The blowup set in this case has 0, 1, or 2 components. Examples are as follows: (a) no singularities: any example where $f + g$ never vanishes, such as $f = \arctan\xi$, $g = \arctan\eta + \pi$; (b) one singularity curve: take f and g linear; (c) two singularity curves: $\ln[2/(1 - \xi\eta)^2]$.

Finally, we show that characteristic singularities may be emitted when the singular set is deformed. The result is the following.

Theorem 10.34. *Generically, the crossing of two characteristic singularities can be perturbed into (a) a solution without singularity or (b) a noncharacteristic singularity.*

Proof. See Problem 10.11. □

10.7 Nirenberg's example

If $\Box u = 0$, then $v = \ln u$ solves

$$\Box v + v_t^2 - |D_x v|^2 = 0. \tag{10.78}$$

This example, pointed out by Nirenberg in connection with the problem of global existence, is the simplest equation that satisfies the "null condition" (see [104]). By taking u close to 1, we see that the solution of (10.78) with small initial data in an appropriate norm will have no singularity at all. On the other hand, there are singular solutions, which correspond precisely to solutions of the wave equation that take the value zero. In this case, the singularity data that determine the WTC series are directly related to suitable Cauchy data. Indeed, consider the solution of the Cauchy problem

$$\Box u = 0; \quad u = 0 \text{ and } u_t = u_1(x) \text{ for } t = \psi(x),$$

where u_1 and ψ are, say, in C^∞ to fix ideas. This solution has the behavior

$$u(x,t) = (t - \psi(x))u_1(x) + (t - \psi(x))^2 u_2(x) + \dots,$$

and is completely determined by the pair $\{\psi, u_1\}$. Let us now relate this solution to the WTC solutions for the equation for v.

Theorem 10.35. *The solution $v = \ln u$ of equation (10.78) has a logarithmic noncharacteristic singularity provided that u_1 does not vanish. Its singularity expansion is entirely determined by the data $\{\psi, u_1\}$.*

Proof. Letting $T = t - \psi(x)$, we obtain

$$v(x,t) = \ln T + \ln u_1(x) + \mathcal{O}(T).$$

The series is entirely determined by its first two terms. The singularity data are therefore $\{\psi, \ln u_1\}$, as announced. □

Remark 10.36. If u is a solution of the wave equation that changes sign, the theorem yields solutions that are real-valued only on part of space-time. Thus, the solution $v = \ln(2x^2 - y^2 + t^2)$ remains real on the outside of a cone, and its singularity surface contains two lines of characteristic points, but is otherwise noncharacteristic.

Remark 10.37. Nonnegative solutions of the wave equation, when they exist, provide simple examples of solutions that are defined and real in a full neighborhood of their singular locus.

Problems

10.1. Show that each of the following problems admits a convergent series solution of the form $\phi^{-m} \sum_{j \geq 0} u_j(x,t)\phi^j$; find in each case the arbitrary coefficients.

1. (Caudrey–Dodd–Gibbon) $u_t + \partial_x[u_{xxxx} + 30uu_{xx} + 60u^3] = 0$.
2. (Kaup–Kuperschmidt) $u_t = \partial_x[u_{xxxx} + \frac{45}{2}u_x^2 + 30uu_{xx} + 60u^3] = 0$.
3. (Hirota–Satsuma) $u_t = \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x$; $v_t = -w_{xxx} - 3uv_x$.
4. (Nonlinear Schrödinger (NLS)) $iu_t + u_{xx} + 2u|u|^2 = 0$.
5. (Boussinesq) $u_{tt} + \partial_x^2[\frac{1}{3} + u^2] = 0$.
6. (modified KdV (mKdV)) $u_t - 3u^2u_x + 2\sigma^2 u_{xxx} = 0$.
7. (Fifth-order KdV (KdV5)) $4u_t = \partial_x(u_{xxxx} + 5u_x^2 + 10uu_{xx} + 10u^3)$.
8. (Kadomtsev–Petviashvili (KP)) $\partial_x[u_t + uu_x + \delta u_{xxx}] + u_{yy} = 0$.
9. (sine–Gordon) $u_{xt} = \sin u$ (in this case, expand $\exp iu$ rather than u).

10.2. Analyze the singular series solutions of the H. Dym equation $u_t = \partial_x^3 u^{-1/2}$. (Let $v = 1/\sqrt{u}$ and seek a solution in powers of $\phi^{1/3}$.)

10.3. Throughout this problem we are interested in solutions u of various nonlinear PDEs in n space dimensions that blow up on a hypersurface of equation $t = \psi(x)$; the blowup time is $t_* = \inf_x \psi(x)$.

(a) Find singular solutions of $\Box u = u^m$ for m integer ≥ 2 by reduction. What happens if m is odd? List all possible ways to continue the solution after blowup, as a real or a complex solution.

(b) Same question if $m = p/q$, and for $m > 1$ irrational.

(c) Show that the condition for the absence of the logarithms in the series is given by the vanishing of the coefficient of the first logarithmic term, and that this condition depends only on the geometry of the blowup surface as a hypersurface of Minkowski space.

(d) Write out this condition explicitly for $m = 2$ and one space dimension.

(e) By choosing ψ suitably, construct examples of solutions with smooth data (hence of locally[6] finite "energy" in any reasonable sense) that (i) blow up at precisely N points for $t = t_*$, for any given N; (ii) blow up on the surface of a sphere for $t = t_*$; (iii) are finite for every x if $t = t_*$.

(f) In one space dimension, find examples such that at any time $t > t_*$, the solution is finite except at two points $x_1(t)$, $x_2(t)$, which become equal for $t = t_*$ and move apart as t increases. If the growth of ψ as $|x| \to \infty$ is sublinear, show that this structure of the blowup points in space as a function of time is not preserved by Lorentz transformations.

10.4. Find singular solutions for $\Box u = u_t^m$, for $m > 1$.

10.5. Find singular solutions for the "ϕ^4 equation" $u_{xt} + u - u^3 = 0$.

[6] Behavior at infinity is irrelevant for the study of local behavior of equations admitting finite speed of propagation.

10.6. On any of the examples of problem 10.3, derive self-similar asymptotics from the expansions given by reduction, in the spirit of Sect. 10.2.7, and show that the domain of validity of self-similar asymptotics is always smaller than the domain of validity of expansions given by reduction.

10.7. (a) Show that for nonlinear wave equations of the form $\Box u = f(u)$, where f is a polynomial, the condition that no logarithms occur in the expansion of singular solutions is invariant under Lorentz transformation.

(b) In the expansion of singular solutions of (10.16), find the expression for the coefficients u_1, u_2, u_3, and $u_{4,0}$. Check that logarithms are absent if and only if $u_{4,0} = 0$.

(c) Consider a particular point P on the blowup surface, corresponding to $X = X_0$. Show that after performing a Lorentz transformation, one may assume that $\nabla\psi(X_0) = 0$. How does this simplify the formulas in (b).

(d) Can one give a geometric interpretation of the coefficients in (b)? In particular, can they be expressed in terms of the first and second fundamental forms of Σ? Work out the case of low dimensions.

(e) Compute the first few terms of series (10.31) and give a geometric interpretation for them along the lines of (d).

(f) Discuss the continuation of solutions of the nonlinear wave equations in this chapter, in the real and the complex domains. One may, for instance, consider functions of the unknown that are smooth up to the blowup set.

10.8. What is the smallest ℓ such that the renormalized unknown v in the solution of (10.15) belongs to A_ℓ? Same question for equation (10.16). *Hint:* Compare with Problem 2.3.

10.9. Perfom reduction directly on equation (10.55).

10.10. Prove Theorem 10.32

10.11. Prove Theorem 10.34

10.12. Perform a reduction analysis of equation

$$u_t + \partial_x\{\alpha u_{xxxx} + \beta u u_{xx} + \gamma u_x^2 + \mu u_{xx} + q u^2 + r u^3\} = 0,$$

where α, β, γ, μ, q, and r are constants. List all cases in which there is one branch of singular solutions with four nonnegative resonances. In each case, estimate the number ℓ such that the reduced equation may be solved in the algebra A_ℓ. (This is solved in [180, Sect. 5]. This equation arises as a higher-order model for water waves, beyond the KdV approximation; see [121], which also mentions other applications.)

10.13. Perform a reduction analysis of the forced KdV equation

$$u_t + u u_x - u_{xxx} = f(x, t),$$

where f is, say, a polynomial, and prove the analogue of Theorem 10.27.

11

Boundary Blowup for Nonlinear Elliptic Equations

Thanks to the detailed asymptotics on solutions of elliptic PDEs with boundary blowup proved in Chap. 9, we give a variational characterization of these singular solutions. It relies on an inequality that contains both Hardy's and Trudinger's inequalities. As a result, a global bound on the maximal solution is obtained. Results are from [118].

Let Ω be an arbitrary domain in $\mathbb{R}^N$, $N \geq 2$, of class $C^{2+\alpha}$; in this chapter, we let $\delta(x)$ denote the distance of x from $\partial\Omega$. The nonlinear PDE

$$-\Delta u + f(u) = 0$$

on Ω, with f monotone, with power or exponential growth to fix ideas, admits a maximal solution [100, 153] that dominates all solutions with bounded boundary data. Since this maximal solution tends uniformly to $+\infty$ as one approaches the boundary, it is said to "blow up at the boundary." This has been proved on typical examples in Sect. 9.7 and Problem 9.1. This remarkable fact provides a uniform interior bound for solutions, which depends on the distance to the boundary and not at all on the boundary data. There is an extensive literature on the issue of boundary blowup; see [7, 8, 9, 10, 100, 128, 131, 137, 140, 153] and their references for details.

Even though the equation is formally the Euler–Lagrange equation of a first-order Lagrangian, the integral of this Lagrangian is divergent for the maximal solution. We show how to circumvent this problem. This leads to a renormalized functional that will be shown to be finite on $H_0^1(\Omega)$ thanks to a synthesis of Hardy's and Trudinger's inequalities.

We begin with background information and the statement of the main results. The rest of the chapter is devoted to the proof of these results. Sect. 11.2 states and proves two auxiliary results (Theorems 11.9 and 11.12), from which generalizations of Hardy's and Trudinger's inequalities (Theorems 11.5 and 11.6) follow as special cases. The main results, Theorem 11.1 and Corollary 11.3, are both proved in Sect. 11.3. The proofs of Sect. 11.2 require the construction of a partition of unity with special properties, carried out in Sect. 11.4.

11.1 A renormalized energy for boundary blowup

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{2+\alpha}$, with $0 < \alpha < 1$. The distance function $\delta(x)$ is of class $C^{2+\alpha}$ near and up to the boundary, but is merely Lipschitz over Ω in general. It is therefore convenient to introduce a function $d(x) \in C^{2+\alpha}(\overline{\Omega})$, which coincides with $\delta(x)$ near $\partial\Omega$, and is positive inside Ω.[1]

Consider the maximal solution u_Ω of the Liouville equation

$$-\Delta u + 4e^{2u} = 0 \tag{11.1}$$

in Ω. It is known that u_Ω is the supremum of all solutions of the Dirichlet problem with smooth boundary data, and that it is equivalent to $-\ln(2d)$ near the boundary. Even though the equation is formally the Euler–Lagrange equation derived from the Lagrangian $L[u] := |\nabla u|^2 + 4e^{2u}$, a direct variational approach is impossible, because $L[u_\Omega] \notin L^1(\Omega)$. Nevertheless, Fuchsian reduction [111, 113, 115] enables one to decompose u_Ω into an explicit singular part v and a more regular function w,

$$u_\Omega = v + w,$$

where the following properties hold:

(P1) $w \in C^{1+\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;
(P2) $w = \mathcal{O}(d)$ as $d \to 0$;
(P3) $e^v = \mathcal{O}(1/d)$ as $d \to 0$;
(P4) $r[v] := -\Delta v + 4e^{2v} = \mathcal{O}(1/d)$ as $d \to 0$.

One may, for instance, take $v = -\ln(2d)$ [113].

For $\phi \in H_0^1$, let us define

$$R[\phi, v] := \int_\Omega |\nabla\phi|^2 + 4e^{2v}[e^{2\phi} - 1 - 2\phi] + 2r[v]\phi, \tag{11.2}$$

which is well defined thanks to (11.4), properties (P3–P4), and Hardy's inequality. The volume element in integrals is omitted for convenience. We then have a variational characterization of u_Ω:

Theorem 11.1. *The infimum*

$$\mathrm{Inf}\{R[\psi - v, v] : \psi \in v + H_0^1(\Omega)\}$$

is attained precisely for $\psi = u_\Omega$.

Since v is given, this provides a characterization of u_Ω.

[1] One may simply take $d = F(\delta)$ for an appropriate F.

Remark 11.2. The result may be stated equivalently as follows: if $\phi \in H_0^1$, then

$$R[\phi, v] \geq R[w, v], \tag{11.3}$$

with equality if and only if $\phi = w$.

As a consequence of the variational characterization, we derive a new global *a priori* bound on u_Ω:

Corollary 11.3. *The maximal solution u_Ω of Liouville's equation satisfies*

$$\|u_\Omega + \ln(2d)\|_{H_0^1} \leq 2H\|\Delta d\|_{L^2(\Omega)}.$$

Remark 11.4. The boundary blowup problem appears at first sight to be beyond the reach of critical-point theory. We have shown that it is in fact equivalent to the minimization of a convex functional in H_0^1. How is this functional related to the usual functional $E[u] := \int_\Omega L[u]$? Take $v = -\ln(2d)$ to fix ideas. Even though $E[u]$ is infinite for $u = u_\Omega$, it is easy to see that if ϕ is smooth and sufficiently flat near the boundary, $\tilde{E}[\phi] := \int_\Omega L[\phi + v] - L[v]$ is well defined. But this does not provide a satisfactory variational principle for two reasons: (i) $u_\Omega - v$ is not very flat at the boundary; it is only $\mathcal{O}(d)$; (ii) $\tilde{E}[\phi]$ is not well defined if ϕ is merely $\mathcal{O}(d)$ (indeed, the term $2\nabla\phi \cdot \nabla v$ is not necessarily integrable). However, subtracting $\int_\Omega 2\,\mathrm{div}(\phi\nabla v)$ from $\tilde{E}[\phi]$, and using the equation satisfied by v, one recovers an expression equivalent to our functional R. Note that the integral of this divergence term is not zero, even if ϕ is smooth, because ∇v blows up at the boundary.

11.2 Hardy–Trudinger inequalities

In this section, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is an arbitrary domain with $\partial\Omega \neq \emptyset$.

11.2.1 Background results and motivation

If Ω is bounded with Lipschitz boundary, and $u \in H_0^1(\Omega)$, the generalized Hardy's inequality states that

$$\left\|\frac{u}{\delta}\right\|_{L^2(\Omega)} \leq H\|\nabla u\|_{L^2(\Omega)}.$$

The optimal value of the "Hardy constant" H, as well as possible generalizations and improvements of this inequality, have been the object of much attention; see [139, 141, 152]. Hardy's inequality arises naturally in several variational problems, as well as in the proof of decay estimates [9, 105].

On the other hand, if $N = 2$, Trudinger's inequality implies that $\exp(u^2) - 1$ is integrable. This suggests our first result:

Theorem 11.5. *Let $\Omega \subset \mathbb{R}^2$ be such that Hardy's inequality holds. Then, for any $u \in H_0^1(\Omega)$,*

$$[\exp(u^2) - 1]/\delta^2 \in L^1(\Omega).$$

This theorem will be derived from a result of independent interest:[2]

Theorem 11.6. *Let $\Omega \subset \mathbb{R}^2$ be an arbitrary domain with $\partial\Omega \neq \emptyset$. Then, if $u \in H_0^1(\Omega)$, and $q > 2$,*

$$\left(\int_\Omega \frac{u^q}{\delta^2} \right)^{1/q} \leq \Sigma_q \left(\int_\Omega |\nabla u|^2 + \frac{u^2}{\delta^2} \right)^{1/2},$$

where $\Sigma_q = \mathcal{O}(q^{1/2+1/q})$ as $q \to \infty$.

Remark 11.7. These results admit natural generalizations to higher dimensions, which are stated and proved in Sect. 11.2. Theorem 11.6 may be considered as trivially true if $\partial\Omega = \emptyset$ with the convention that $\delta \equiv +\infty$ in this case. For background results on Trudinger's inequality, see [177, 145, 6, 29]; our argument is closer to Trudinger's than to Moser's, because the distance function does not transform in a convenient manner under symmetrization.

Remark 11.8. No regularity or boundedness assumptions on Ω are required. This is somewhat surprising in view of the fact [6, p. 120] that elements of $H^1(\Omega)$ are not necessarily L^q for $q > 2$ if Ω is unbounded and with finite volume. Note that for domains with thin "ends" at infinity, δ is very small, and the r.h.s. of our inequality is not equivalent to the H^1 norm.

In the situation of Theorem 11.5, we find in particular that

$$\frac{e^{2u} - 1 - 2u}{\delta^2} \in L^1(\Omega). \tag{11.4}$$

The N-dimensional analogues of Theorems 11.5 and 11.6 are stated in Sect. 11.2.2, and proved in Sects. 11.2.3 and 11.2.4 respectively.

11.2.2 A synthesis of Hardy's and Trudinger's inequalities

Let $N' = N/(N-1)$ and define

$$\Phi_N(u) := \sum_{k \geq N-1} \frac{|u|^{kN'}}{k!},$$

and, for $1 \leq p < \infty$,

$$M_p(u) := \left(\int_\Omega |\nabla u|^p + \frac{|u|^p}{\delta^p} \right)^{1/p}.$$

[2] The Lebesgue measure dx is understood in all integrals in this chapter.

Theorem 11.9. *If $\partial\Omega \neq \emptyset$, there are constants c_1 and c_2, which depend only on the dimension N, such that for any $u \in W_0^{1,N}(\Omega)$ with $M_N(u) = 1$,*

$$\int_\Omega \frac{\Phi_N(u/c_1)}{\delta^N} \leq c_2.$$

Remark 11.10. Note that no smoothness or boundedness assumptions on Ω are required, and that $M_N(u)$ is not necessarily equivalent to the $W_0^{1,N}$ norm. This result implies Theorem 11.6.

Remark 11.11. If Ω is bounded and Lipschitz, Hardy's inequality holds, and we claim that $\Phi_N(u)$ is integrable for any $u \in W_0^{1,N}(\Omega)$: write $u = f+g$, where f is smooth with compact support; since $(|f|+|g|)^{kN'} \leq 2^{kN'}(|f|^{kN'}+|g|^{kN'})$, we have $\Phi_N(f+g) \leq \Phi_N(2f) + \Phi_N(2g)$. The result follows if g is small in $W_0^{1,N}$. For $N = 2$ and $u \in H_0^1(\Omega)$, we recover Theorem 11.5.

11.2.3 An auxiliary result

Let $1 \leq p \leq N$, $p^* = Np/(N-p)$ if $p < N$ (respectively $p^* = +\infty$ if $p = N$).

Theorem 11.12. *If $N \geq 2$, Ω is a domain in $\mathbb{R}^N$, $1 \leq q < \infty$, and $1 \leq p < q \leq p^*$, there is a constant $\Sigma_q(N,p)$ such that*

$$\left(\int_\Omega \frac{|u|^q}{\delta^N}\right)^{1/q} \leq \Sigma_q \left(\int_\Omega \frac{|\nabla u|^p}{\delta^{N-p}} + \frac{|u|^p}{\delta^N}\right)^{1/p}$$

for any $u \in W^{1,p}(\Omega)$.

Remark 11.13. In general, the right-hand side may be infinite. If $p = N$ and Hardy's inequality holds in Ω, the right hand side is finite for $u \in W_0^{1,p}(\Omega)$.

11.2.4 Proof of Theorem 11.12

Step 1: Partition of unity. Denote by $Q(x,s)$ the cube of center x and side s. We prove in Sect. 11.4 that there is a smooth partition of unity $(\phi_k)_{k\geq 0}$ in Ω with the following properties:

(PU1) For every k, ϕ_k is supported in a cube $Q_k = Q(x_k, s_k) \subset \Omega$ and $\sum_k \phi_k = 1$.
(PU2) For every k, $0 \leq \phi_k \leq 1$ and $|\nabla\phi_k| \leq c_3/s_k$, where c_3 depends only on N.
(PU3) There are two positive constants λ and μ such that on $\operatorname{supp}\phi_k$, $\lambda \leq \delta/s_k \leq \mu$.
(PU4) There is an integer P that depends only on the dimension N such that for every $x \in \Omega$, $\phi_k(x)$ is nonzero for at most P values of k.

Simple consequences of these properties are:

1. For every $q \geq 1$, $\sum_k \phi_k^q \leq 1$.
2. $\sum_k \chi_{Q_k} \leq P$, where χ_{Q_k} denotes the characteristic function of Q_k.
3. $(\sum_k \phi_k)^q \leq P^q \sum_k \phi_k^q$. Indeed, for any x, $\phi_k(x) \neq 0$ for at most P values of k, so that $(\sum_k \phi_k(x))^q \leq P^q \max_k \phi_k(x)^q$.

We also need an elementary observation: for any collection of nonnegative numbers b_k, and any $r \geq 1$,

$$\sum_k b_k^r \leq \left(\sum_k b_k\right)^r . \tag{11.5}$$

This may be seen for finite sequences by induction, starting from the inequality $x^r + y^r \leq (x+y)^r$. Recall also that $(x+y)^r \leq 2^r(x^r + y^r)$.

Step 2: Decomposition of u. For any $u \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega \frac{|u|^q}{\delta^N} &= \int_\Omega |\sum_k u\phi_k|^q \delta^{-N} \\
&\leq P^q \sum_k \int_{Q_k} |u\phi_k|^q \delta^{-N} \\
&\leq P^q \sum_k (\lambda s_k)^{-N} \|u\phi_k\|^q_{L^q(Q_k)}.
\end{aligned}$$

Write $Q(s)$ for $Q(0,s)$, and let $S_q = S_q(N,p)$ denote the norm of the embedding of $W_0^{1,p}(Q(1))$ into $L^q(Q(1))$. If $v \in W_0^{1,p}(Q(s))$, the Sobolev inequality applied to $v(sx) \in W_0^{1,p}(Q(1))$ gives

$$\|v\|_{L^q(Q(s))} \leq S_q s^{\frac{N}{q}+1-\frac{N}{p}} \|\nabla v\|_{L^p(Q(s))}. \tag{11.6}$$

It follows that for every k,

$$(\lambda s_k)^{-N} \|u\phi_k\|^q_{L^q(Q_k)} \leq S_q^q \lambda^{-N} s_k^{q(p-N)/p} \|\nabla(u\phi_k)\|^q_{L^p(Q_k)}.$$

It follows that

$$\begin{aligned}
\left(\int_\Omega \frac{|u|^q}{\delta^N}\right)^{p/q} &\leq (PS_q)^p \left(\sum_k \lambda^{-N} s_k^{q(p-N)/p} \|\nabla(u\phi_k)\|^q_{L^p(Q_k)}\right)^{p/q} \\
&\leq (PS_q\lambda^{-N/q})^p \sum_k s_k^{p-N} \|\nabla(u\phi_k)\|^p_{L^p(Q_k)} \\
&\leq (2PS_q\lambda^{-N/q})^p \sum_k s_k^{p-N} \left[\|\phi_k \nabla u\|^p_{L^p(Q_k)} + \|u\nabla\phi_k\|^p_{L^p(Q_k)}\right],
\end{aligned}$$

where we used (11.5) to obtain the second inequality. Now,

$$\int_{\Omega}\sum_k s_k^{p-N}\phi_k^p|\nabla u|^p \le \mu^{N-p}\int_{\Omega}\frac{|\nabla u|^p}{\delta^{N-p}}$$

and

$$\sum_k s_k^{p-N}\|u\nabla\phi_k\|^p_{L^p(Q_k)} \le c_3^p\int_{\Omega}(\sum_k \chi_{Q_k}(x))\frac{|u|^p}{s_k^N} \le Pc_3^p\mu^N\int_{\Omega}\frac{|u|^p}{\delta^N}.$$

We have therefore the desired inequality, with

$$\Sigma_q = 2PS_q(N,p)\lambda^{-N/q}[\mu^{N-p}+Pc_3^p\mu^N]^{1/p}. \tag{11.7}$$

This completes the proof.

11.2.5 Proof of Theorem 11.9

We now consider the case $p = N$, so that $p^* = +\infty$, and q can take arbitrarily large values. From [69, Lemma 7.12 and (7.37)], it follows that

$$S_q(N,N) \le (\omega_N q)^{1-1/N+1/q} \text{ if } q \ge N.$$

If $q \ge N-1$, we have $N'q \ge N$, and therefore,

$$S_{N'q}(N,N)^{N'q} \le (N'q\omega_N)^{q+1} \quad \text{ if } q \ge N-1.$$

For any $c_1 > 0$, we therefore obtain

$$\int_{\Omega}\sum_{q\ge N-1}\frac{|u|^{N'q}}{q!c_1^{N'q}\delta^N} \le c_2 := \sum_{q\ge N-1}\lambda^{-N}N'\omega_N\left(\frac{N'\omega_N A}{c_1^{N'}}\right)^q\frac{q^q}{(q-1)!},$$

where $A = \{2P[1+P(c_3\mu)^N]^{1/N}\}^{N'}$. The series defining c_2 converges if $c_1^{N'} > e\omega_N N'A$. This completes the proof.

11.3 Variational characterization of solutions with boundary blowup

11.3.1 Proof of Theorem 11.1

Let $\phi \in H_0^1(\Omega)$. We wish to prove inequality (11.3). Since $u_\Omega = v + w$ solves Liouville's equation,

$$\Delta w = -\Delta v + 4e^{2v} + 4e^{2v}(e^{2w}-1) = r[v] + 4e^{2v}(e^{2w}-1). \tag{11.8}$$

It follows from (P2–P4) that

$$\Delta w = \mathcal{O}(1/d).$$

Since $w \in C_0^1(\overline{\Omega}) \cap C^2(\Omega)$, we have, for any $\psi \in C_0^1(\overline{\Omega})$ and any $\varepsilon > 0$ small enough,

$$\int_{d>\varepsilon} \nabla\psi \cdot \nabla w = \int_{d=\varepsilon} \psi(\nabla w \cdot \nabla d)\, ds - \int_{d>\varepsilon} \psi \Delta w.$$

Letting first $\varepsilon \to 0$, and then approximating ϕ by ψ in the H_0^1 norm, we obtain

$$\int_\Omega \nabla\phi \cdot \nabla w + \int_\Omega \phi \Delta w = 0. \tag{11.9}$$

Using equation (11.8), we obtain

$$\int_\Omega \nabla\phi \cdot \nabla w + 4e^{2v}(e^{2w} - 1)\phi + r[v]\phi = 0. \tag{11.10}$$

Since

$$R[\phi+w, v] - R[w, v] = \int_\Omega |\nabla\phi|^2 + 2\nabla\phi\cdot\nabla w + 4e^{2v}[e^{2w+2\phi} - e^{2w} - 2\phi] + 2r[v]\phi,$$

we obtain

$$R[\phi + w, v] - R[w, v] = \int_\Omega |\nabla\phi|^2 + 4e^{2(v+w)}(e^{2\phi} - 1 - 2\phi), \tag{11.11}$$

which is manifestly nonnegative, and vanishes precisely if $\phi = 0$. Q.E.D.

Remark 11.14. Since $v+w = u_\Omega$ and $r[u_\Omega] = 0$, the right hand side of equation (11.11) is equal to $R[\phi, u_\Omega]$.

11.3.2 Proof of Corollary 11.3

Property (P4) and Hardy's inequality ensure that there is a constant $K[v]$ such that

$$-\int_\Omega 2r[v]\phi \le K[v]\|\phi\|_{H_0^1}.$$

Expressing that $R[w, v] \le R[0, v] = 0$, we obtain

$$\|w\|_{H_0^1} \le K[v].$$

If $v = -\ln(2d)$, one obtains $r[-\ln(2d)] = (\Delta d)/d$. Hölder's and Hardy's inequalities yield $K[v] \le 2H\|\Delta d\|_{L^2}$. Since $w = u_\Omega - v$, the announced *a priori* H^1 bound on u_Ω follows.

11.4 Construction of the partition of unity

We construct the partition of unity used in the proof of Theorem 11.12; the basic ideas go back to Whitney [185], and many variants may be found in the literature; see, e.g., [171, 141].

First of all, choose two constants η and η' such that

$$\eta/\sqrt{N} > \eta' > 1.$$

Recall that $Q(x,s)$ is the closed cube of center x and side s. For any $\sigma > 0$, we write $Q_\sigma(x,s)$ for $Q(x,\sigma s)$. The following observation will be useful: for any $x \in \Omega$,

$$\delta(x) > \frac{s}{2}\sqrt{N} \Rightarrow Q(x,s) \subset \Omega \Rightarrow \delta(x) > \frac{s}{2}.$$

Conversely,

$$\delta(x) \le \frac{s}{2} \Rightarrow Q(x,s) \not\subset \Omega \Rightarrow \delta(x) \le \frac{s}{2}\sqrt{N}.$$

Covering by dyadic cubes

For $k \in \mathbb{Z}$, let $\mathcal{F}_k$ denote the family of closed cubes of the form

$$[0, 2^{-k}]^N + (m_1, \dots, m_N)2^{-k},$$

where the m_j are signed integers. The cubes in $\mathcal{F}_{k+1}$ are obtained by dyadic subdivision of the cubes in $\mathcal{F}_k$; in particular, for any k, every $Q(x,s) \in \mathcal{F}_{k+1}$ is included in a unique cube $\tilde{Q}(\tilde{x}, \tilde{s}) \in \mathcal{F}_k$, with $\tilde{s} = 2s$. Since Q is obtained by dyadic division of $\tilde{Q}$, $\tilde{x}$ must be a vertex of Q; it follows that $|x - \tilde{x}| = \frac{1}{2}s\sqrt{N}$. Also, if $Q(x,s) \in \mathcal{F}_k$, $s = 2^{-k}$ and x/s has half-integer coordinates.

Let $\mathcal{F} = \bigcup_{k=-\infty}^{+\infty} \mathcal{F}_k$. Define a set $\mathcal{Q} \subset \mathcal{F}$ as follows: $Q \in \mathcal{Q}$ if and only if

$$Q_\eta \subset \Omega \text{ and } \tilde{Q}_\eta \not\subset \Omega. \tag{11.12}$$

The set $\mathcal{F}$ is not empty, since $\partial\Omega \ne \emptyset$ by assumption.

Lemma 11.15. *Ω is the union of the cubes $Q \in \mathcal{Q}$.*

Proof. Let $y \in \Omega$. Consider the set of numbers k for which there is a cube $Q(x, 2^{-k}) \in \mathcal{F}_k$ that contains y and that satisfies $Q_\eta \subset \Omega$. This set is not empty: if k is large enough, we have $\delta(x) \ge \delta(y) - \frac{1}{2}2^{-k}\sqrt{N} > \frac{1}{2}2^{-k}\eta\sqrt{N}$, and $Q_\eta \subset \Omega$. It is bounded below because $\partial\Omega \ne \emptyset$. Let k_0 be the smallest integer in that set, and consider a cube Q with the above property with $k = k_0$. Since k_0 is minimal, $\tilde{Q}_\eta \not\subset \Omega$. Therefore, $y \in Q \in \mathcal{Q}$, as desired. □

Since for each $Q \in \mathcal{Q}$, $Q \subset Q_{\eta'} \subset \Omega$, we have *a fortiori*

$$\Omega = \bigcup_{Q \in \mathcal{Q}} Q_{\eta'}.$$

Properties of the cube decomposition

We now prove that the covering of Ω by the cubes $Q_{\eta'}$ has the additional property that on each of them, the function δ is comparable to the side of Q:

Lemma 11.16. *There are positive constants c_4 and c_5, independent of Ω, such that if $Q(x,s) \in \mathcal{Q}$ and $y \in Q_{\eta'}$, then*

$$c_4 \leq \frac{\delta(y)}{s} \leq c_5.$$

Proof. Since $Q_\eta \subset \Omega$, $\delta(x) > \eta s/2$. If $y \in Q_{\eta'}$, $\delta(y) \geq \delta(x) - \frac{1}{2}\eta' s\sqrt{N} > \frac{1}{2}(\eta - \eta'\sqrt{N})s$. Therefore,

$$\frac{1}{2}(\eta - \sqrt{N}\eta') < \frac{\delta(y)}{s}.$$

To establish an upper bound, we first estimate $\delta(x)/s$. Since $Q(x, \eta s) \subset \Omega$ and $\tilde{Q}(\tilde{x}, 2\eta s) \not\subset \Omega$, $\delta(x) > \frac{1}{2}\eta s$ and $\delta(\tilde{x}) \leq \frac{1}{2}(2\eta s)\sqrt{N}$. Therefore,

$$\delta(x) \leq \delta(\tilde{x}) + |x - \tilde{x}| \leq \left(\eta + \frac{1}{2}\right) s\sqrt{N}.$$

Therefore, if $Q(x,s) \in \mathcal{Q}$,

$$\frac{1}{2}\eta < \frac{\delta(x)}{s} \leq \left(\eta + \frac{1}{2}\right)\sqrt{N}. \tag{11.13}$$

We now estimate $\delta(y)$:

$$\delta(y) \leq \delta(x) + \frac{1}{2}\eta' s\sqrt{N} \leq \left(\eta + \frac{1}{2} + \frac{1}{2}\eta'\right) s\sqrt{N}.$$

It follows that $\delta(y)/s$ lies between positive bounds that do not depend on Ω, as desired. □

Finite intersection property

The cubes $Q_{\eta'}$ are not disjoint; nevertheless, two such cubes may intersect only if the ratio of their sides lies between fixed bounds; this implies that a given cube may intersect only finitely many others.

Lemma 11.17. *There is a constant c_6, independent of Ω, such that if $Q(x,s)$ and $Q'(x',s')$ belong to $\mathcal{Q}$, and if $Q(x,\eta' s) \cap Q'(x', \eta' s') \neq \emptyset$, with $s < s'$, necessarily*

$$s < s' < c_6 s.$$

Furthermore, there is a number P, independent of Ω, such that given $Q \in \mathcal{Q}$, there are at most P cubes $Q' \in \mathcal{Q}$ such that $Q_{\eta'} \cap Q'_{\eta'} \neq \emptyset$.

Proof. Let $y \in Q_{\eta'} \cap Q'_{\eta'}$. We have $|x-x'| \le |x-y|+|y-x'| \le \frac{1}{2}(s+s')\eta'\sqrt{N}$. Since Q and Q' both belong to $\mathcal{Q}$, inequality (11.13) yields $\delta(x') > \frac{1}{2}\eta s'$ and $\delta(x) \le (\frac{1}{2}+\eta)s\sqrt{N}$. Since δ is 1-Lipschitz,

$$\frac{1}{2}\eta s' < \delta(x') \le \delta(x) + |x-x'| \le \left[\left(\frac{1}{2}+\eta+\frac{1}{2}\eta'\right)s + \frac{1}{2}\eta' s'\right]\sqrt{N}.$$

It follows that $s' < c_6 s$ with

$$c_6 = \frac{2\eta+1+\eta'}{\eta-\eta'\sqrt{N}}\sqrt{N}.$$

Therefore, if $s = 2^{-k}$ and $s' = 2^{-k'}$, we have $|k-k'| \le J_1 := \ln c_6/\ln 2$. On the other hand, $|x-x'| \le \frac{1}{2}(s+s')\eta'\sqrt{N} \le J_2 s$, with $J_2 = \frac{1}{2}(1+c_6)\eta'\sqrt{N}$. Scaling the cubes by the factor $1/s$, and taking x as the origin of coordinates, we are led to the question, "how many cubes $Q(x', 2^{-j}) \in \mathcal{F}_j$ can one find subject to the restrictions $|j| \le J_1$ and $|x'| \le J_2$?" The answer is a finite number P, which depends only on N, η, and η'. □

The partition of unity

Take a smooth function $\varphi(x)$ with support in $Q(0,\eta')$, equal to 1 on $Q(0,1)$, and such that $0 \le \varphi(x) \le 1$ for all x. Consider

$$\psi(x) = \sum_{Q=Q(x_Q,s_Q)\in\mathcal{Q}} \varphi\left(\frac{x-x_Q}{s_Q}\right).$$

Since the cubes $Q \in \mathcal{Q}$ cover Ω, $\psi(x) \ge 1$ for all $x \in \Omega$. Since any point belongs to at most P of the cubes $Q_{\eta'}$, we have $\psi(x) \le P$ for all $x \in \Omega$. It follows that the functions $\phi_Q(x) := \varphi((x-x_Q)/s_Q)/\psi(x)$ form a partition of unity, and are supported in the cubes $Q_{\eta'}$.

We now prove that properties (PU1–4) of Sect. 11.2.4 hold. (PU1) and (PU2) are immediate. If $x \in Q_{\eta'}$, Lemma 11.16 ensures that $\delta(x)/s_Q$ is bounded above and below by positive bounds that depend only on N. This proves property (PU3). Finally, since $\varphi((x-x_Q)/s_Q)$ is supported in Q'_η, we find, thanks to Lemma 11.17, that (PU4) holds. This completes the construction of the partition of unity.

Problems

11.1. Let $\alpha > 0$ and $p > \alpha+1$. Fix $R > 0$. Apply reduction to find a solution of

$$-(|u'|^{p-2}u')' + \frac{n-1}{r}|u'|^{p-2}u' + u^\alpha = 0, \tag{11.14}$$

for $R - r$ small and positive such that $u(r) \sim u_0(R-r)^\sigma(1 + \mathcal{O}((R-r)^\varepsilon)$ with $\sigma > 1$ and $\varepsilon > 0$; show that $\sigma = p/(p - \alpha - 1)$. *Remark:* This problem shows how to construct solutions of equations of p-Laplacian type with "dead zones" [110, 123, 52]; u extended by zero for $r > R$ satisfies the equation in the weak sense. These ideas go back further, in particular to suggestions by Brezis on the possible existence of solutions with compact support for nonlinear problems.

Part IV

Background Results

12

Distance Function and Hölder Spaces

We prove the basic properties of the distance to the boundary of a $C^{2+\alpha}$ domain that are required in the study of boundary blowup. We also give several characterizations of Hölder spaces, and the proof of the interior Schauder estimates; further details may be found in [117, 69].

12.1 The distance function

We prove a few properties of the function $d(x) = \mathrm{dist}(x, \partial\Omega)$ when $\Omega \subset \mathbb{R}^n$ is bounded with boundary of class $C^{2+\alpha}$. Without a smoothness assumption on the boundary, all we can say is that d is Lipschitz; indeed, since the boundary is compact, there is, for every x, a $z \in \partial\Omega$ such that $d(x) = |x - z|$. If y is any other point in Ω, we have $d(y) \leq |y - z| \leq |y - x| + |x - z| = |y - x| + d(x)$. It follows that $|d(x) - d(y)| \leq |x - y|$. For more regular $\partial\Omega$, we have the following results:

Theorem 12.1. *If $\partial\Omega$ is bounded of class $C^{2+\alpha}$,*

1. *There is a $\delta > 0$ such that every point such that $d(x) < \delta$ has a unique nearest point on the boundary.*
2. *In this domain, d is of class $C^{2+\alpha}$; furthermore, $|\nabla d| = 1$, and*

$$-\Delta d = \sum_j \frac{\kappa_j}{1 - \kappa_j d},$$

where $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$. In particular, $-\Delta d/$
$(n-1)$ is equal to the mean curvature of the boundary.

Proof. We work near the origin, which we may take on $\partial\Omega$. Our proofs will give local information near the origin, which can be made global by a standard compactness argument.

Choose the coordinate axes so that Ω is locally represented by $\{x_n > h(x')\}$, where $x' = (x_1, \dots, x_{n-1})$ and h is of class $C^{2+\alpha}$ with $h(0) = 0$ and $\nabla h(0) = 0$. We may also assume that the axes have been rotated so that the Hessian $(\partial_{ij}h(0))$ is diagonal. Its eigenvalues are, by definition, the principal curvatures $\kappa_1, \dots, \kappa_{n-1}$ of the boundary. Their average is, again by definition, the mean curvature of the boundary. At any boundary point, the vector with components

$$\nu(x') = (\nu_i) = (-\partial_1 h, \dots, -\partial_{n-1}h, 1)/\sqrt{1 + |\nabla h|^2}$$

is the *inward* normal to $\partial\Omega$ at that point. One checks that $\partial_j \nu_i(0) = -\partial_{ij}h(0) = \kappa_j \delta_{ij}$ for i and j less than n. Thus, ν is of class C^1. For any $T > 0$ and $Y \in \mathbb{R}^{n-1}$, both small, consider the point $x(Y,T) = (Y, h(Y)) + T\nu(Y)$; this represents the point obtained by traveling the distance T into Ω along the normal, starting from the boundary point $(Y, h(Y))$. We write

$$\Phi : (Y,T) \mapsto x(Y,T).$$

We want to prove that all points in a neighborhood of the boundary are obtained by this process, in a unique manner: in other words, $(Y, h(Y))$ is the unique closest point from $x(Y,T)$ on the boundary, provided that T is positive and small. It suffices to argue for $Y = 0$; in that case, since h is C^2, it is bounded below by an expression of the form $a|Y|^2$, which implies that for T sufficiently small, the sphere of radius T about $x(Y,T)$ contains no point of the boundary except the origin.[1] We may now consider the new coordinate system (Y,T) thus defined. We compute, for $Y = 0$, but T not necessarily zero,

$$\frac{\partial x_i}{\partial Y_j} = \delta_{ij}(1 - \kappa_j T)$$

for i and $j < n$, while

$$\frac{\partial x_n}{\partial Y_j} = \frac{\partial x_i}{\partial T} = 0; \qquad \frac{\partial x_n}{\partial T} = 1.$$

The inverse function theorem shows that near the origin, the map Φ and its inverse are of class C^1, and that the Jacobian of Φ^{-1} is, for $Y = 0$,

$$\frac{\partial(Y,T)}{\partial x} = \operatorname{diag}\left(\frac{1}{1-\kappa_1 T}, \dots, \frac{1}{1-\kappa_{n-1}T}, 1\right).$$

In fact, Φ^{-1} is of class $C^{1+\alpha}$. Indeed, Φ has this regularity, the differential of Φ^{-1} is given by $[\Phi' \circ \Phi^{-1}]^{-1}$, and the map $A \mapsto A^{-1}$ on invertible matrices is a smooth map. Since $\nu(Y)$, which is equal to the gradient of d, is a $C^{1+\alpha}$ function of Y, we see that it is also a $C^{1+\alpha}$ function of the x coordinates. It

[1] Indeed, the equation of this sphere is $x_n = T - \sqrt{T^2 - |Y|^2}$, which, by inspection, is bounded below by $a|Y|^2$ for $2aT < 1$.

follows that d is of class $C^{2+\alpha}$. The computation of the second derivatives of d now results from that of the first-order derivatives of ν.

It follows from this discussion that $T = d$ near the boundary, $|\nabla d| = 1$, and $\nabla d = \nu$. □

Since $\partial\Omega$ is compact, there is a positive r_0 such that in any ball of radius r_0 centered at a point of $\partial\Omega$, one may introduce a coordinate system (Y, T) in which $T = d$ is the last coordinate. We may assume that the domain of this coordinate system contains a set of the form

$$0 < T < \theta \text{ and } |Y_j| < \theta \text{ for } j \le n-1.$$

Let $\partial_j = \partial_{x_j}$, and write d_n and d_j for $\partial d/\partial x_n$ and $\partial d/\partial x_j$ respectively. Primes denote derivatives with respect to the Y variables: $\partial'_j = \partial_{Y_j}$, $\nabla' = \nabla_Y$, $\Delta' = \sum_{j<n} \partial'^2_j$, etc. We write $\tilde{\nabla} d = (d_1, \ldots, d_{n-1})$. We let throughout

$$D = T\partial_T.$$

The transformation formulas are

$$T = d(x_1, \ldots, x_n); \qquad Y_j = x_j \text{ for } j < n;$$
$$\partial_n = d_n \partial_T; \qquad \partial_j = d_j \partial_T + \partial'_j.$$

We recall that $\Delta d = (1-n)H$, where H is the mean curvature of $\partial\Omega$. We further have

$$\begin{aligned} d\nabla d \cdot \nabla w &= (D + T\tilde{\nabla} d \cdot \nabla')w, \\ |\nabla w|^2 &= w_T^2 + |\nabla' w|^2 + 2 w_T \tilde{\nabla} d \cdot \nabla' w, \\ \Delta w &= w_{TT} + \Delta' w + 2\tilde{\nabla} d \cdot \nabla' w_T + w_T \Delta d. \end{aligned}$$

12.2 Hölder spaces on $C^{2+\alpha}$ domains

12.2.1 First definitions

Let $\Omega \subset \mathbb{R}^n$ be a domain (*i.e.*, an open and connected set).

Definition 12.2. *A function u is Hölder continuous at the point P of Ω, with exponent $\alpha \in (0, 1)$, if*

$$[u]_{\alpha,\Omega,P} := \sup_{Q\in\Omega, Q\neq P} \frac{|u(P) - u(Q)|}{|P - Q|^\alpha} < \infty.$$

It is Hölder continuous over Ω, or of class $C^\alpha(\Omega)$, *if it satisfies this condition for every $P \in \Omega$. We write $[u]_{\alpha,\Omega} := \sup_P [u]_{\alpha,\Omega}$.*

It is of class $C^\alpha(\Omega)$ if

$$\|u\|_{C^\alpha(\Omega)} := \sup_\Omega |u| + [u]_{\alpha,\Omega}.$$

Functions of class C^α are in particular uniformly continuous. If $\partial\Omega$ is smooth, one can extend u by continuity to a continuous function on $\overline{\Omega}$; for this reason, it is sometimes convenient to write $C^\alpha(\overline{\Omega})$ for $C^\alpha(\Omega)$ in this case, to emphasize that u is continuous up to the boundary. It is easy to check that

$$[uv]_{\alpha,\Omega} \le \|u\|_{C^\alpha(\Omega)}\|v\|_{C^\alpha(\Omega)}.$$

Higher-order Hölder spaces $C^{k+\alpha}(\Omega)$ are defined in the natural way: first, write $|\nabla^k u|$ for the sum of the absolute values of the derivatives of u of order k, and define $[\nabla^k u]_{\alpha,\Omega}$ similarly; then, define

$$\|u\|_{C^k(\Omega)} := \max_{0\le j\le k} \sup_{\Omega} |\nabla^j u|$$

and

$$\|u\|_{C^{k+\alpha}(\Omega)} := \|u\|_{C^k(\Omega)} + [\nabla^k u]_{\alpha,\Omega}.$$

In all these norms, the reference domain Ω will be omitted whenever it is clear from the context.

12.2.2 Dyadic decomposition

The Hölder spaces defined above are all Banach spaces, but smooth functions are not dense in them: even in one dimension, if (f_m) is a sequence of smooth functions and $f \in C^\alpha(\mathbb{R})$ is such that $\|f - f_m\|_{C^\alpha(\mathbb{R})} \to 0$, one proves easily that for any P and any $\varepsilon > 0$, there is a neighborhood of Q on which $|f(P) - f(Q)| \le \varepsilon|P - Q|^\alpha$. In other words, $\lim_{Q\to P} |f(P) - f(Q)||P - Q|^{-\alpha} = 0$. Any function f that does not satisfy this property cannot be approximated by smooth functions in the C^α norm. Nevertheless, there is a systematic way to decompose Hölder-continuous functions on $\mathbb{R}^n$ into a uniformly convergent sum of smooth functions: define the Fourier transform of u by

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx$$

and consider $\varphi \in C_0^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi = 1$ for $|x| \le 1$, $\varphi = 0$ for $|x| \ge 0$. Define

$$\hat{u}_0 = \varphi(|\xi|)\hat{u}(\xi); \quad \hat{u}_j = [\varphi(2^{-j}|\xi|) - \varphi(2^{-(j-1)}|\xi|)]\hat{u}(\xi) \text{ for } j \ge 1.$$

We let $\hat{v}_j = \hat{u}_0 + \cdots + \hat{u}_j$.

Definition 12.3. *The decomposition*

$$u = \sum_{j\ge 0} u_j$$

is the Littlewood–Paley (LP), or dyadic decomposition, of u [171].

By Fourier inversion, we have

$$u_j = \psi_j * u \text{ with } \psi_j(x) = 2^{jn}\psi(2^j x),$$

where $\psi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} [\varphi(|\xi|/2) - \varphi(|\xi|)] \exp(ix\cdot\xi) d\xi$. Note that $\hat{\psi}$ vanishes near the origin; in particular, $\hat{\psi}_j(0) = \int_{\mathbb{R}^n} \psi_j dx = 0$.

Theorem 12.4. *Let* $0 < \alpha < 1$.

1. *(Bernstein's inequality) There is a constant* C *such that for any* k, $\sup_x(|\nabla^k u_j| + |\nabla^k v_j|) \le C 2^{jk} \sup_x |u(x)|$.
2. *If* $u \in C^\alpha(\mathbb{R}^n)$*, there is a constant* C *independent of* j *such that*

$$\sup_x |u_j(x)| \le C 2^{-j\alpha} \|u\|_{C^\alpha}.$$

3. *Conversely, if the above inequality holds for every* $j \ge 1$*, then* $u \in C^\alpha(\mathbb{R}^n)$.

Proof. (1) On the one hand, we have $|u_j(x)| \le \|\psi\|_{L^1} \sup |u|$ and $|v_j(x)| \le \|\phi\|_{L^1} \sup |u|$. On the other hand, if a is a multi-index of length k,

$$|\nabla^a u_j(x)| = \left| \int u(y) 2^{jk} \nabla^a \psi[2^j(x-y)] 2^{jn} dy \right| = C 2^{jk} \sup |u|.$$

The result follows.

(2) Since $\int \psi(y)dy = 0$, u_j may be written, for $j \ge 1$, as

$$u_j(x) = \int [u(x-y) - u(x)] 2^{jn} \psi(2^j y) dy = \int [u(x - z/2^j) - u(x)] \psi(z) dz.$$

If $u \in C^\alpha$, it follows that

$$|u_j(x)| \le 2^{-j\alpha} [u]_\alpha \int |z|^\alpha |\psi(z)| dz,$$

QED.

(3) Conversely, if the u_j are of order $2^{-j\alpha}$, the series $u_0 + u_1 + \cdots$ converges uniformly. Call its sum u; it is readily seen that the u_j do give its LP decomposition. We may apply (1) to $u_{j-1} + u_j + u_{j+1}$, and obtain

$$\sup_x |\nabla u_j(x)| \le C 2^{j(1-\alpha)}.$$

Writing $u = v_{j-1} + w_j$, where $w_j = u_j + u_{j+1} + \cdots$, we find that

$$\begin{aligned} |u(x) - u(y)| &\le \sum_{j>k} |x-y| \sup |\nabla u_j| + 2 \sup |w_j| \\ &\le C|x-y|(1 + \cdots + 2^{(j-1)(1-\alpha)}) + C 2^{-j\alpha} \\ &\le C[2^{-j\alpha} + |x-y| 2^{j(1-\alpha)}]. \end{aligned}$$

Choose j such that $2^{-j} \le |x-y| \le 2^{-(j-1)}$. A bound on $[u]_\alpha$ follows. □

12.2.3 Weighted norms

Several of the results we shall prove estimate the Hölder norm of a function u on a ball of radius R in terms of bounds on the ball of radius $2R$ with the same center. In order to exploit these inequalities in a systematic fashion, it is useful to define Hölder norms weighted by the distance to the boundary. Let $\Omega \neq \mathbb{R}^n$ and let $d(P)$ denote the distance from P to $\partial\Omega$, and

$$d_{P,Q} = \min(d(P), d(Q)).$$

Let also δ be a smooth function in all of Ω that is equivalent to d for d sufficiently small.[2] Define, for $k = 0, 1, \dots,$

$$\|u\|^{\#}_{k,\Omega} = \sum_{j=0}^{k} \sup_{\Omega} d^j |\nabla^j u|$$

and

$$\|u\|^{\#}_{k+\alpha,\Omega} = \sum_{j=0}^{k} \|\delta^j u\|_{C^{j+\alpha}(\Omega)}.$$

The spaces corresponding to these norms are called $C^k_{\#}(\Omega)$, $C^{k+\alpha}_{\#}(\Omega)$. The space $C^{k+\alpha}_{*}(\Omega)$ has the norm

$$\|u\|^{*}_{k+\alpha,\Omega} = \|u\|^{*}_{k,\Omega} + [u]^{*}_{k+\alpha,\Omega},$$

where

$$\|u\|^{*}_{k,\Omega} = \sum_{j=0}^{k} [u]^{*}_{j,\Omega},$$

with $[u]^{*}_{k,\Omega} = \sup_{\Omega} d^k |\nabla^k u|$ and

$$[u]^{*}_{k+\alpha,\Omega} = \sup_{P,Q\in\Omega} d^{k+\alpha}_{P,Q} \frac{|\nabla^k u(P) - \nabla^k u(Q)|}{|P-Q|^{\alpha}}.$$

We also need the further definitions

$$[u]^{(\sigma)}_{\alpha,\Omega} = \sup_{P,Q\in\Omega} d^{\alpha+\sigma}_{P,Q} \frac{|u(P) - u(Q)|}{|P-Q|^{\alpha}}; \qquad \|u\|^{(\sigma)}_{\alpha,\Omega} = \sup_{\Omega} |d^{\sigma} u| + [u]^{(\sigma)}_{\alpha,\Omega}.$$

As before, the mention of Ω will be omitted whenever possible.

[2] Such a function is easy to construct if Ω is bounded and smooth. Note that even in this case, d is smooth only near the boundary; see Sect. 11.1.

12.2.4 Interpolation inequalities

Theorem 12.5. *For any $\varepsilon > 0$, there is a constant C_ε such that*

$$\begin{aligned}
[u]_1^* &\le \varepsilon [u]_2^* + C_\varepsilon \sup |u|,\\
[u]_1^* &\le \varepsilon [u]_{1+\alpha}^* + C_\varepsilon \sup |u|,\\
[u]_2^* &\le \varepsilon [u]_{2+\alpha}^* + C_\varepsilon [u]_1^*,\\
[u]_{1+\alpha}^* &\le \varepsilon [u]_2^* + C_\varepsilon \sup |u|.
\end{aligned}$$

Proof. Recall the elementary inequality, for C^2 functions of one variable $t \in [a,b]$,[3]

$$\sup |f| \le \frac{2}{b-a} \sup |f| + (b-a) \sup |f|.$$

Fix $\theta \in \left(0, \frac{1}{2}\right)$, and $P \in \Omega$. Let $r = \theta d(P)$. If $Q \in B_r(P)$ and $Z \in \partial\Omega$, we have

$$|Z-Q| \ge |Z-P| - |P-Q| \ge d(P)(1-\theta) \ge \frac{1}{2} d(P) \ge r \ge |P-Q|.$$

It follows in particular that $d(Q) \ge d(P)(1-\theta) \ge \frac{1}{2} d(P)$; hence

$$d_{P,Q} \ge \frac{1}{2} d(P).$$

Applying the elementary inequality to u restricted to the segment $[P, P+re_i]$,[4] where e_i is the ith basis vector, we obtain

$$|\partial_i u(P)| \le \frac{2}{r} \sup_{B_r} |u| + r \sup_{B_r} |\partial_{ii} u|.$$

It follows that

$$\sup_{B_r} |\partial_{ii} u| \le \sup d(Q)^{-2} \sup d(Q)^2 |\partial_{ii} u| \le \frac{[u]_2^*}{d(P)^2 (1-\theta)^2}.$$

Therefore,

$$[u]_1^* = \sup_{B_r} |d(Q) \partial_i u(Q)| \le \frac{2}{\theta} \sup |u| + \frac{\theta}{(1-\theta)^2} [u]_2^*.$$

If we choose θ such that $\theta(1-\theta)^{-2} \le \varepsilon$, we arrive at the first of the desired inequalities. For the second, we note that, using again the mean-value theorem, there is on the segment $[P, P+re_i]$ some $\tilde{P}$ such that $|\partial_i u(\tilde{P})| \le (2/r) \sup_{B_r} |u|$. It follows that

[3] For the proof, write $f'(t) = f'(s) + \int_s^t f''(\tau) d\tau$, where s satisfies $f'(s) = (f(b) - f(a))/(b-a)$.

[4] By the choice of r, this segment lies entirely within Ω.

$$\begin{aligned}
|\partial_i u(P)| &\le |\partial_i u(\tilde{P})| + |\partial_i u(P) - \partial_i u(\tilde{P})| \\
&\le \frac{2}{r} \sup_{\Omega} |u| \\
&\times \left(\sup_{Q \in B_r(P)} d_{P,Q}^{-1-\alpha} \right) |P - \tilde{P}|^{\alpha} \sup_{Q \in B_r(P)} d_{P,Q}^{1+\alpha} \frac{|\nabla u(P) - \nabla u(Q)|}{|P-Q|^{\alpha}} \\
&\le \frac{2}{r} \sup_{\Omega} |u| + (2/d(P))^{1+\alpha} (\theta d(P))^{\alpha} [u]^*_{1+\alpha}.
\end{aligned}$$

Multiplying through by $d(P) = r/\theta$, we obtain the second inequality. A similar argument gives the third and fourth inequalities. □

12.2.5 Integral characterization of Hölder continuity

Let Ω be a bounded domain. Write $\Omega(x,r)$ for $\Omega \cap B_r(x)$. We assume that the measure of $\Omega(x,r)$ is at least Ar^n for some positive constant A, if $x \in \Omega$ and $r \le 1$. This condition is easily verified if Ω has a smooth boundary. Define the average of u:

$$u_{x,r} = |\Omega(x,r)|^{-1} \int_{\Omega(x,r)} u \, dx.$$

Theorem 12.6. *The space $C^{\alpha}(\Omega)$ coincides with the space of (classes of) measurable functions that satisfy*

$$\int_{\Omega(x,r)} |u(y) - u_{x,r}|^2 \, dy \le C r^{n+2\alpha}$$

for $0 < r < \operatorname{diam} \Omega$. The smallest constant C, denoted by $\|u\|_{\mathcal{L}^{2,n+2\alpha}}$, is equivalent to the $C^{\alpha}(\Omega)$ norm.

Remark 12.7. If one defines $\mathcal{L}^{p,\lambda}$ by the property $\int_{\Omega(x,r)} |u(y) - u_{x,r}|^p \, dy \le Cr^{\lambda}$, with $n < \lambda < n+p$, one obtains a characterization of the space $C^{(\lambda-n)/p}$.

Proof. The integral estimate is clearly true for Hölder continuous functions. Let us therefore focus on the converse. We first prove that u is uniformly approximated by its averages, and then derive a modulus of continuity for u.

If $x_0 \in \Omega$ and $0 < \rho < r \le 1$, we have

$$\begin{aligned}
A\rho^n |u_{x_0,\rho} - u_{x_0,r}|^2 &\le \int_{\Omega(x_0,\rho)} |u_{x_0,\rho} - u_{x_0,r}|^2 dx \\
&\le 2 \left(\int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^2 dx + \int_{\Omega(x_0,r)} |u - u_{x_0,r}|^2 dx \right) \\
&\le C(r^{\lambda} + \rho^{\lambda}).
\end{aligned}$$

Letting $r_j = r2^{-j}$ and $u_j = u_{x_0,r_j}$ for $j \ge 0$, we obtain

$$|u_{j+1} - u_j| \leq C 2^{j(n-\lambda)/2} r^{(\lambda-n)/2} = C 2^{-j\alpha} r^\alpha.$$

For almost every x_0, the Lebesgue differentiation theorem ensures that $u_j \to u(x_0)$ as $j \to \infty$. It follows that

$$|u(x_0) - u_{x_0,r}| \leq \sum_j |u_{j+1} - u_j| \leq C r^\alpha.$$

Since $u_{x,r}$ is continuous in x and converges uniformly as $r \to 0$, it follows that u may be identified, after modification on a null set, with a continuous function. To conclude the proof of Theorem 12.7, it suffices to estimate $|u(x) - u(y)|$ by $|u(x) - u_{x,r}| + |u_{x,r} - u_{y,r}| + |u_{y,r} - u(y)| \leq 2Cr^\alpha + |u_{x,r} - u_{y,r}|$, which is possible thanks to the following Lemma. □

Lemma 12.8. *Let $u \in \mathcal{L}^{2,n+2\alpha}$, x, y two points in Ω, and $r = |x - y|$; we have*

$$|u_{x,r} - u_{y,r}| \leq C r^\alpha.$$

Proof. We may assume $r = |x-y| \leq 1$. If $z \in B_r(x)$, we have $|z-y| \leq r + |x-y| \leq 2r$. Therefore $\Omega(y, 2r) \supset \Omega(x, r)$. It follows that $\Omega(x, 2r) \cap \Omega(y, 2r) \supset \Omega(x, r)$ has measure Ar^n at least. We therefore have

$$\begin{aligned}
&|\Omega(x, 2r) \cap \Omega(y, 2r)||u_{x,2r} - u_{y,2r}| \\
&\qquad \leq \int_{\Omega(x,2r)} |u(z) - u_{x,2r}| dz + \int_{\Omega(y,2r)} |u(z) - u_{y,2r}| dz \\
&\qquad \leq \left[\int_{\Omega(x,2r)} |u(z) - u_{x,2r}|^2 dz\right]^{1/2} |\Omega(x, 2r)|^{1/2} \\
&\qquad\qquad + \left[\int_{\Omega(y,2r)} |u(z) - u_{y,2r}|^2 dz\right]^{1/2} |\Omega(y, 2r)|^{1/2} \\
&\qquad \leq C r^{\alpha + n/2} r^{n/2}.
\end{aligned}$$

It follows that

$$|u_{x,2r} - u_{y,2r}| \leq C A^{-1} r^\alpha.$$

□

This completes the proof of Lemma 12.8.

12.3 Interior estimates for the Laplacian

12.3.1 Direct arguments from potential theory

Let $n \geq 2$, and let $B_R(P)$ denote the open ball of radius R about P. Mention of the point P is omitted whenever this does not create confusion. The volume of B_R is $\omega_n R^n$, and its surface area, $n\omega_n R^{n-1}$. The Newtonian potential in n dimensions is

$$g(P,Q) = \frac{|P-Q|^{2-n}}{(2-n)n\omega_n} \text{ for } n \geq 3$$

and

$$\frac{1}{2\pi} \ln |P-Q| \text{ for } n = 2.$$

It is helpful to note that:

1. The derivatives of g of order $k \geq 1$ with respect to P are $O(|P-Q|^{2-n-k})$.
2. The average of each of these second derivatives over any sphere of center Q vanishes.[5]

Next consider, for $f \in L^1 \cap L^\infty(\mathbb{R}^n)$, the integral

$$u(P) = \int_{\mathbb{R}^n} g(P,Q) f(Q)\, dQ.$$

We wish to estimate u and its derivatives in terms of bounds on f. Because of the behavior of g as $P \to Q$, g and its first derivatives are locally integrable, but its second derivative is not.

It is easy to see that if the point P lies outside the support of f, u is smooth near P and satisfies $\Delta u = 0$. For this reason, it suffices to study the case in which the density f is supported in a neighborhood of P.

We prove three theorems (i) a pointwise bound on u and its first-order derivatives; (ii) a representation of the second-order derivatives that involves only locally integrable functions; (iii) a direct estimate of $\nabla^2 u(P) - \nabla^2 u(Q)$ using this representation.

Theorem 12.9. *If f vanishes outside $B_R(0)$, we have*

$$\sup_{B_R}(|u| + |\nabla u|) \leq CR^2 \sup |f|,$$

and ∇u is given by formally differentiating the integral defining u.

Proof. Consider a cutoff function $\varphi_\varepsilon(P,Q) := \varphi(|P-Q|/\varepsilon)$, where $\varphi(t)$ is smooth, takes its values between 0 and 1, vanishes for $t \leq 1$, and equals 1 for $t \geq 2$. Considering the functions

$$u_\varepsilon(P) = \int g(P,Q)\varphi_\varepsilon(P,Q) f(Q) dQ,$$

which are smooth, it is easy to see that the $\partial_i u_\varepsilon$ converge uniformly, as $\varepsilon \downarrow 0$, to $\int \partial_i g(P,Q) f(Q) dQ$. Similarly, u_ε converges to u. Therefore, u is continuously

[5] To check this, it is useful to note that the average of x_i^2/r^2 over the unit sphere $\{r = 1\}$ is equal to $\frac{1}{n}$, and similarly, using symmetry, the average of $(x_i - y_i)^2/|x-y|^2$ over the sphere $\{|x-y| = \text{const}\}$ has the same value.

differentiable. Using the growth properties of g and its derivatives, we may estimate $\partial_i u(P)$ by

$$C \int_{B_{2R}(P)} C|P-Q|^{1-n} \sup |f| dQ,$$

because $B_R(0) \subset B_{2R}(P)$. Taking polar coordinates centered at P, the result follows. □

The case of second derivatives is more delicate, since the second derivatives of g are not locally integrable. We know (since Poisson) that the integral defining u is smooth near P if f is constant in a neighborhood of P. This suggests a reduction to the case in which f vanishes at P. We therefore first prove, for such f, a representation of the second-order derivatives in terms of $f - f(P)$.

Theorem 12.10. *If f has support in a bounded neighborhood Ω of the origin, with smooth boundary, and if $f \in C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0,1)$, then all second-order derivatives of u exist, and are equal to*

$$w_{ij} := \int_\Omega \partial_{ij} g(P,Q)[f(Q) - f(P)]dQ - f(P) \int_{\partial\Omega} \partial_i g \, n_j ds(Q),$$

where derivatives of g are taken with respect to its first argument, and n_j are the components of the outward normal to $\partial\Omega$.

Proof. To establish the existence of second derivatives, we consider

$$v_{i\varepsilon}(P) = \int \partial_i g(P,Q)\varphi_\varepsilon(P,Q)f(Q)dQ,$$

which converges pointwise to $\partial_i u(P)$; in fact, since $1 - \varphi_\varepsilon$ is supported by a ball of radius 2ε, a direct computation yields $|u_i - v_{i\varepsilon}|(P) = O(\varepsilon \sup |f|)$. Writing $P = (x_i)$ and $Q = (y_i)$, we have

$$\begin{aligned} \partial_j v_{i\varepsilon}(P) = &\int_\Omega \partial_{x_j}(\varphi_\varepsilon \partial_{x_i} g)(P,Q)[f(Q) - f(P)]dQ \\ &+ f(P) \int_\Omega (\varphi_\varepsilon \partial_{x_i} g)(P,Q)dQ. \end{aligned}$$

Since φ_ε and g depend only on $|P - Q|$, we may replace $\partial/\partial x_j$ by $-\partial/\partial y_j$ and integrate by parts. This yields

$$\begin{aligned} \partial_j v_{i\varepsilon}(P) = &\int_\Omega \partial_{x_j}(\varphi_\varepsilon \partial_{x_i} g)(P,Q)[f(Q) - f(P)]dQ \\ &- f(P) \int_{\partial\Omega} \varphi_\varepsilon \partial_{x_i} g(P,Q) n_j(Q) ds(Q). \end{aligned}$$

We may now estimate the difference $\partial_j v_{i\varepsilon} - w_{ij}$ using the same method as for the first-order derivatives. It follows that $\partial_{ij} u = w_{ij}$. □

We now give the main estimate for second-order derivatives.

Theorem 12.11. *Let*

$$u(P) = \int_{B_{2R}(0)} g(P,Q)f(Q)\,dQ,$$

where $f \in C^\alpha(B_{2R})$, *with* $0 < \alpha < 1$. *Then*

$$\sup_{B_R} |\nabla^2 u| + [\nabla^2 u]_{\alpha,B_R} \le C(\sup_{B_{2R}} |f| + R^\alpha [f]_{\alpha,B_{2R}}). \tag{12.1}$$

Proof. To estimate the regularity of $\partial_{ij}u$, we study $|\partial_{ij}u(P) - \partial_{ij}u(P')|$, for P, P' in $B_R(0)$, where the second derivatives are given by the expressions in the previous theorem. The main step is to decompose the first integrand in the resulting expression for $w_{ij}(P) - w_{ij}(P')$ into

$$[f(Q) - f(P')][\partial_{ij}g(P,Q) - \partial_{ij}g(P',Q)] + [f(P') - f(P)]\partial_{ij}g(P,Q).$$

We therefore need to estimate the following quantities:

(I) $f(P)[\partial_i g(P,Q) - \partial_i g(P',Q)]$ for $Q \in \partial B_{2R}$.
(II) $[f(P) - f(P')]\partial_i g(P',Q)$ for $Q \in \partial B_{2R}$.
(III) $[f(P') - f(P)]\partial_{ij} g(P,Q)$ for $Q \in B_{2R}$.
(IV) $[f(Q) - f(P')][\partial_{ij}g(P,Q) - \partial_{ij}g(P',Q)]$ for $Q \in B_{2R}$.

The first boundary term (I) is easy to estimate using the mean-value theorem:

$$|\partial_i g(P,Q) - \partial_i g(P',Q)| \le |P - P'| \sup_{\xi \in [P,P']} |\nabla \partial_i g(\xi, Q)|.$$

Since $Q \in \partial B_{2R}$ and $\xi \in B_R$, we have $|\xi - Q| \ge 2R - R = R$; therefore, the supremum in the above formula is bounded by a multiple of R^{-n}. Integrating, we get a contribution $O(|P - P'|/R)$, which is *a fortiori* $O(|P - P'|^\alpha / R^\alpha)$.

Expression (II) is $O(|P - P'|^\alpha)$, since f is of class C^α.

To estimate (III) and (IV), let $r_0 = |P - P'|$ and M the midpoint of $[P, P']$. We distinguish two cases: (i) When $|Q - M| > r_0$, the distance from Q to any point on the segment $[P, P']$ is comparable to $|Q - M|$; this will enable a direct estimation of (IV) using the mean-value theorem, and of (III) by integration by parts. (ii) On the set on which $|Q - M| \le r_0$, we may directly estimate the sum of (III) and (IV); the smallness of the region of integration compensates the singularity of the derivatives of g. We begin with the first case: consider first the integral of (III) over the set

$$A := \{Q \in B_{2R} : |Q - M| > r_0\}.$$

Its boundary is included in $\partial B_{2R}(0) \cup \partial B_{r_0}(M)$. Integrating by parts and using the fact that on this set, $|P - Q|$ is bounded below by $\min(R, r_0/2)$, we find that (III) $= O(|P - P'|^\alpha)$. For the term (IV), integrated over the same set, we estimate $\partial_{ij}g(P,Q) - \partial_{ij}g(P',Q)$ by $C|P-P'||\xi-Q|^{-n-1}$, for some $\xi \in [P,P']$. Using the Hölder continuity of f, the integral of (IV) is estimated by

$$Cr_0 \frac{|Q-P'|^\alpha}{|Q-\xi|^{n+1}}.$$

Its integral over A is estimated by its integral over

$$A' := \{Q : |Q-M| > r_0\}.$$

On A',

$$|Q-P'| \le |Q-M| + |M-P'| = |Q-M| + \frac{1}{2}r_0 \le \frac{3}{2}|Q-M|.$$

On the other hand,

$$|Q-\xi| \ge |Q-M| - |M-\xi| \ge |Q-M| - \frac{1}{2}r_0 \ge \frac{1}{2}|Q-M|.$$

Combining the two pieces of information, we obtain

$$\begin{aligned}\int_{A'} Cr_0 \frac{|Q-P'|^\alpha}{|Q-\xi|^{n+1}} dQ &\le Cr_0 \int_{A'} |Q-M|^{\alpha-n-1} dQ \\ &= Cr_0 \int_{r_0}^{\infty} r^{\alpha-2} dr = C|P-P'|^\alpha.\end{aligned}$$

This completes the analysis of the integrals of (III) and (IV) over A.

It remains to consider (III) and (IV) over the part of B_{2R} on which $|Q-M| \le r_0$. In this case, $|P-Q| \le |P-M| + |M-Q| \le \frac{3}{2}r_0$, and similarly for $|P'-Q|$. We therefore estimate directly the sum of (III) and (IV), namely

$$[f(Q)-f(P)]\partial_{ij}g(P,Q) - [f(Q)-f(P')]\partial_{ij}g(P',Q),$$

by

$$\begin{aligned}C[f]_{\alpha,B_{2R}} &\int_{|Q-M|<r_0} (|Q-P|^{\alpha-n} + |Q-P'|^{\alpha-n}) dQ \\ &\le C[f]_{\alpha,B_{2R}} \int_0^{3r_0/2} |Q-P|^{\alpha-1} d|Q-P| \le Cr_0^\alpha.\end{aligned}$$

Since $r_0 = |P-P'|$, this completes the proof. □

12.4 Perturbation of coefficients

12.4.1 Basic a priori estimate

Working on a relatively compact subset Ω' of Ω, we may assume that $[u]^*_{2+\alpha} < \infty$; since the constants in the various inequalities will not depend on the choice of Ω, the full result will follow.

Consider $x_0 \in \Omega$ and let $r = \theta d(x_0)$ with $\theta \leq \frac{1}{2}$. Let $L_0 = \sum_{ij} a^{ij}(x_0)\partial_{ij}$ (the "tangential operator," with coefficients "frozen" at x_0). We define

$$F := L_0 u = \sum_{ij}(a^{ij}(x_0) - a^{ij}(x))\partial_{ij}u - \sum_i b^i \partial_i u - cu + f.$$

We apply the constant-coefficient interior estimates on the ball $B_r(x_0)$. Let $y_0 \neq x_0$ be such that $d(y_0) \geq d(x_0)$.

If $|x_0 - y_0| < r/2$, we have

$$\left(\frac{r}{2}\right)^{2+\alpha}[\nabla^2 u]_{\alpha,x_0,y_0} \leq C\left(\sup |u| + \sup_{B_r}|r^2 F| + \sup_{B_r \times B_r} r^{2+\alpha}\frac{|F(x) - F(y)|}{|x-y|^\alpha}\right).$$

Therefore,

$$d(x_0)^{2+\alpha}[\nabla^2 u]_{\alpha,x_0,y_0} \leq C\theta^{-2-\alpha}\left(\sup |u| + \|F\|^{(2)}_{\alpha,B_r}\right). \tag{12.2}$$

If $|x_0 - y_0| \geq r/2$, we have

$$d(x_0)^{2+\alpha}[\nabla^2 u]_{\alpha,x_0,y_0} \leq 2[u]^*_2 \frac{d(x_0)^\alpha}{|x_0 - y_0|^\alpha} \leq 2[u]^*_2 \left(\frac{2}{\theta}\right)^\alpha. \tag{12.3}$$

The issue is therefore the estimation of $\|F\|^{(2)}_{\alpha,B_r}$ in terms of norms of u and its derivatives over Ω.

For clarity, we begin with three lemmas. The first is proved by direct verification.

Lemma 12.12. $\|uv\|^{(s+t)}_{\alpha,\Omega} \leq \|u\|^{(s)}_{\alpha,\Omega}\|v\|^{(t)}_{\alpha,\Omega}$.

Lemma 12.13. *If* $r = \theta d(x, \partial\Omega)$, *with* $0 < \theta \leq \frac{1}{2}$ *(so that* $B_r(x) \subset \Omega$*), we have*

$$\|\nabla^2 u\|^{(2)}_{\alpha,B_r} \leq 8\left[\theta^2\|\nabla^2 u\|^*_{2,\Omega} + \theta^{2+\alpha}[u]^*_{2+\alpha,\Omega}\right], \tag{12.4}$$

$$\|f\|^{(2)}_{\alpha,B_r} \leq 8\theta^2\|f\|^{(2)}_{\alpha,\Omega}. \tag{12.5}$$

Proof. We need to estimate, for $y \in B_r(x)$, $d(y, \partial B_r(x))$ and d_{x,y,B_r} in terms of the corresponding distances relative to Ω. On the one hand, $d(y, \partial B_r) \leq r - |x-y| \leq r = \theta d(x)$. On the other hand, if $z \in B_r(x)$ and $d(y, \partial B_r(x)) \leq d(z, \partial B_r(x))$, we have $d_{y,z,B_r} \leq \theta d(x)$ and also $d(y) \geq d(y, \partial B_r(x)) \geq (1-\theta)d(x)$; it follows that $d(x) \leq (1-\theta)^{-1}d_{x,y,\Omega}$. Therefore,

$$d(y, \partial B_r) \leq \theta d(x)$$

and

$$d_{y,z,B_r} \leq \frac{\theta}{1-\theta}d_{y,z,\Omega}.$$

The inequalities (12.4–5) follow. □

Lemma 12.14. *If $x \in B_r(x_0)$ with $r = \theta d(x_0)$, with $0 < \theta \le \frac{1}{2}$, we have*

$$\|a(x) - a(x_0)\|^{(0)}_{\alpha,B_r} \le C\theta^\alpha [a]^*_{\alpha,\Omega}.$$

Proof. If $d(x) \le d(y)$ and $|x-y| \le r = \theta d(x_0)$ with $\theta \le 1$, then

$$|a(x) - a(y)| \le d(x)^\alpha \frac{|a(x) - a(y)|}{|x-y|^\alpha} (\frac{|x-y|}{d(x)})^\alpha \le C\theta^\alpha [a]^{(0)}_\alpha,$$

since $(1-\theta)d(x_0) \le d(x) \le (1+\theta)d(x_0)$. Therefore, estimating $|a(x) - a(x_0)|$ by $r^\alpha [a]^*_{\alpha,\Omega}$, we obtain the announced inequality. □

We now resume the proof of the estimate of $[\nabla^2 u]_\alpha$: first,

$$\begin{aligned}\|(a(x) - a(x_0))\nabla^2 u(x)\|^{(2)}_{\alpha,B_r} &\le \|a(x) - a(x_0)\|^{(0)}_{\alpha,B_r} \|\nabla^2 u\|^{(2)}_{\alpha,B_r} \\ &\le C\theta^{2+\alpha} \|a\|^{(0)}_{\alpha,\Omega} (\|\nabla^2 u(x)\|^*_{\alpha,\Omega} + \theta^\alpha [u]^*_{2+\alpha,\Omega}).\end{aligned}$$

Similarly,

$$\begin{aligned}\|b\nabla u(x)\|^{(2)}_{\alpha,B_r} &\le 8\theta^2 \|b\nabla u\|^{(2)}_{\alpha,\Omega} \\ &\le 8\theta^2 \|b\|^{(1)}_{\alpha,\Omega} \|\nabla u\|^{(1)}_{\alpha,\Omega} \\ &\le C\theta^2 \|b\|^{(1)}_{\alpha,\Omega} \{\theta^{2\alpha}[u]^*_{2+\alpha,\Omega} + \sup|u|\}.\end{aligned}$$

Finally,

$$\begin{aligned}\|cu\|^{(2)}_{\alpha,B_r} &\le 8\theta^2 \|cu\|^{(2)}_{\alpha,\Omega} \le 8\theta^2 \|c\|^{(2)}_{\alpha,\Omega} \|u\|^{(0)}_{\alpha,\Omega} \\ &\le 8\theta^2 \{\theta^{2\alpha}[u]^*_{2+\alpha,\Omega} + \sup|u|\}.\end{aligned}$$

It follows that

$$\|F\|^{(2)}_{\alpha,B_r} \le C\theta^{2+2\alpha}[u]^*_{2+\alpha,\Omega} + c(\theta)(\sup|u| + \|f\|^{(2)}_{\alpha,\Omega}).$$

Therefore, using this inequality in (12.2) and (12.3), we obtain

$$d(x_0)^{2+\alpha}[u]^*_{2+\alpha,\Omega} \le C\theta^\alpha [u]^*_{2+\alpha,\Omega} + c'(\theta)(\sup|u| + \|f\|^{(2)}_{\alpha,\Omega}).$$

The desired estimate on $[u]^*_{2+\alpha,\Omega}$ follows.

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Nash–Moser Inverse Function Theorem

We present two forms of the Nash–Moser or "hard" inverse function theorem (IFT) [144]. Both are based on Newton-type iteration in a scale of Banach spaces. We present two varieties: one in spaces of analytic functions [148, 149], the other in Sobolev or C^k spaces[164].

13.1 Nash–Moser theorem without smoothing

Let $\{X_s\}$ and $\{Y_s\}$ be two scales of Banach spaces, with respective norms $\| \quad \|_s$ and $| \quad |_s$, where $s \in [0, 1]$. We assume that $X_{s'} \supset X_s$ and $Y_{s'} \supset Y_s$ if $s' < s$. A typical example is the case in which X_s consists of functions analytic in a strip $|\operatorname{Im} z| \leq s$, the norm being the uniform norm.

We wish to solve the equation

$$F(u) = 0,$$

where F satisfies the following assumptions, in which s, C, and R are given:

(H1) F is defined for $\|u\|_s < R$, and sends X_s to $Y_{s'}$ for every $s' < s$.

(H2) There is a mapping $L(u) : X_\sigma \to Y_{\sigma'}$, for any $\sigma' < \sigma \leq s$, such that

$$|F(v) - F(u) - L(u)(v - u)|_{\sigma'} \leq C(\sigma - \sigma')^{-p} \|u - v\|_\sigma^{1+\delta},$$

for some $\delta \in (0, 1]$.

(H3) For any $f \in Y_{\sigma'}$, one can find $w \in Y_\sigma$ such that $L(u)w = f$, and

$$\|w\|_\sigma \leq c(\sigma - \sigma')^{-q} |f|_{\sigma'}.$$

By abuse of notation, we write $w = L(u)^{-1} f$.

The result can now be stated.

Theorem 13.1. *There is a function $\varepsilon_0(s)$ such that if $|F(0)|_1 < \varepsilon_0(s)$, $F(u) = 0$ has a solution in X_s.*

Proof. We define an iteration by $u_0 = 0$, and

$$u_{k+1} = u_k - L(u_k)^{-1} F(u_k). \tag{13.1}$$

We recognize Newton's method. Since we assume $F(0) \in Y_1$, we can find $u_1 \in X_\sigma$, $\sigma < 1$, *but not in* X_1 in general. Therefore, we cannot iterate in a fixed space. To tackle this problem, we define a sequence $\{s_k\}$ with $s_k \downarrow s$, and estimate $\|u_k\|_{s_k}$. It will be convenient to define another sequence $\{t_k\}$ and to track $|F(u_k)|_{t_k}$.

More precisely, we let

$$s_0 = 1; \quad s_{k-1} - s_k = \rho k^{-2},$$

for $k \geq 2$, and

$$t_k = \frac{1}{2}(s_k - s_{k+1}) = s_k - \frac{1}{2}\rho(k+1)^{-2},$$

for $k \geq 0$. For s_k to tend to s, we must take $(1-s) = \rho \sum_{k=1}^{\infty} k^{-2}$. We also let $\varepsilon = |F(0)|_1$ and $a_k = |F(u_k)|_{t_k}$. Note that $s_k > t_k > s_{k+1}$. In the following, the letter C denotes various positive constants independent of k. We prove by induction on k that $\|u_k\|_{s_k} < R/2$,

$$a_k \leq \varepsilon (k+1)^{-r}, \tag{13.2}$$

with $r = 1 + \frac{2}{\delta}(p + q + (1+\delta))$, and

$$\|u_{k+1} - u_k\|_{s_{k+1}} \leq C\varepsilon (k+1)^{-(1+2(p+q)/\delta)}, \tag{13.3}$$

if ε is small enough. Using (H2), the definition of u_{k+1}, and (H3), we obtain for any $\sigma \in (t_{k+1}, t_k)$,

$$\begin{aligned} a_{k+1} &\leq c\|L(u_k)^{-1}F(u_k)\|_\sigma^{1+\delta}(\sigma - t_{k+1})^{-p} \\ &\leq C[a_k(t_k - \sigma)^{-q}]^{1+\delta}(\sigma - t_{k+1})^{-p} \\ &= C a_k^{1+\delta}(t_k - \sigma)^{-q(1+\delta)}(\sigma - t_{k+1})^{-p}. \end{aligned}$$

We minimize this quantity by taking

$$\sigma = t_{k+1} - \frac{p(t_k - t_{k+1})}{p + q(1+\delta)}.$$

We obtain

$$a_{k+1} \leq C a_k^{1+\delta}(t_k - t_{k+1})^{-p-q(1+\delta)}.$$

It now follows from the induction hypothesis that

$$a_{k+1} \leq \varepsilon (k+2)^{-r}$$

for ε small enough, using the property $r > 2(p + q(1+\delta))/\delta$.

As for the last estimate, we have, using the recurrence relation and (H3),

$$\begin{aligned}\|u_{k+1} - u_k\|_{s_{k+1}} &\le Ca_k(t_k - s_{k+1})^{-q} \\ &= Ca_k[2(k+1)^2/\rho]^q \\ &\le C\varepsilon(k+1)^{-(1+2(p+q)/\delta)}.\end{aligned}$$

Finally,

$$\|u_{k+1}\|_{s_{k+1}} \le C\varepsilon \sum_1^\infty (k+1)^{-(1+2(p+q)/\delta)}$$

is less than $R < 2$ if ε has been chosen small enough. This proves the desired estimates. Thus, the sequence $\{u_k\}$ certainly converges in X_s, and since $a_k \to 0$, its limit is a solution. □

13.2 Nash–Moser theorem with smoothing

The preceding result assumes that the inverse of the linearization of F sends X_σ to $X_{\sigma'}$ for any $\sigma' < \sigma$. This mimics the behavior of the operator d/dz on $X_s = \{u : |u(x+iy)| \text{ is bounded for } |y| < s\}$. It is clearly inadequate in Sobolev-type spaces, where differential operators induce a definite *loss of derivatives.* This loss forces us to modify the iteration (13.1) into

$$u_{k+1} = u_k - S_k L(u_k)^{-1} F(u_k), \tag{13.4}$$

where S_k is a smoothing operator. This new iteration procedure will be shown to converge, under a new set of hypotheses.

It is convenient not to restrict the index s to lie between 0 and 1.

There are two sets of hypotheses: one pertaining to the smoothing, the other to F. In all, $M \ge 1$ is a fixed constant and $\sigma < s$.

One usually assumes that there is a family $S(t)$ of smoothing operators such that

- (S1) $\|S(t)u\|_{s+\sigma} \le Mt^\sigma \|u\|_s$ if $u \in X_s$.
- (S2) $\lim_{t\to\infty} \|(I - S(t))u\|_s = 0$ if $u \in X_s$.
- (S3) $\|(I - S(t))u\|_{s-\sigma} \le Mt^{-\sigma}\|u\|_s$ if $u \in X_\sigma$.
- (S4) $\|(d/dt)S(t)u\|_s \le Mt^{s-\sigma-1}\|u\|_s$ if $u \in X_\sigma$.

We will not make use of (S4), but it usually holds, and does enter in other proofs. We also assume that the map F has the following properties:

- (F1) F is of class C^2 from X_s to Y_s, with first and second derivatives bounded by $M \ge 1$.
- (F2) $F'(u)$ has a right inverse $L(u)$ that sends Y_s to X_{s-a} and that satisfies the estimate
$$\|L(u)F(u)\|_{s+b} \le M(1 + \|u\|_{s+a+b})$$
for some $b > 8a$.

Thus, if s is identified with a degree of differentiability, F' fails to be invertible because $L(u)$ lands in X_{s-a} instead of X_s.

Theorem 13.2. *Under assumptions (S1)–(S4) and (F1)–(F2), the equation $F(u) = 0$ has a solution if $F(0)$ is small enough in X_s.*

Remark 13.3. The assumptions are stronger than requiring that F' have an unbounded inverse. In fact, this latter assumption alone would lead to an incorrect result, as the case of the mapping $F(u) = u - u_0$ from H^s to H^{s-1} shows, if $u_0 \notin H^s$. What happens here is that (F2) requires $L(u)F(u)$ to be smoother and smoother if u is. This excludes the counterexample we just gave. Also, in practice, it is much easier to find $L(0)$ than to construct $L(u)$ for $u \neq 0$. At the formal level, the existence of $L(0)$ suffices to construct a formal series to all orders. This series may, however, be completely meaningless.

Remark 13.4. We can construct a family of smoothing operators by choosing a function $\phi(\xi)$ that equals 1 for $|\xi| < 1$ and 0 for $|\xi| > 2$, with $0 \le \phi \le 1$, and by letting $S(t)u = \mathcal{F}^{-1}\phi(\xi/t)\mathcal{F}u$, where $\mathcal{F}$ denotes the Fourier transform. Indeed, we then have $\phi(\xi/t)(1 + |\xi|^2)^\sigma \le (1 + 4t^2)^\sigma$ and $(1 - \phi(\xi/t))(1 + |\xi|^2)^{-\sigma} \le (1 + t^2)^{-\sigma}$.

Proof. Let $q_k = \exp[\lambda\rho^k]$ and $S_k = S(q_k)$, where $\rho = \frac{3}{2}$. We define $\{u_k\}_{k\ge0}$ by $u_0 = 0$ and

$$u_{k+1} = u_k - S_k L(u_k)F(u_k).$$

We prove by induction that there exist positive constants μ and ν such that

$$\|u_k - u_{k-1}\|_s \le q_k^{-\mu a}; \quad 1 + \|u_{k-1}\|_{s+a+b} \le q_k^{\nu a}. \tag{13.5}$$

These estimates will ensure that the iteration is well defined, and converges in the s-norm.

We first estimate the difference of two consecutive approximations:

$$\begin{aligned}
&\|u_{k+1} - u_k\|_s \\
&\qquad \le M q_k^a \|L(u_k)F(u_k)\|_{s-a} \\
&\qquad \le M^2 q_k^a |F(u_k)|_s \\
&\qquad = M^2 q_k^a \Big[|F(u_{k-1}) - F'(u_{k-1})S_{k-1}L(u_{k-1})F(u_{k-1})|_s \\
&\qquad\quad + \left| \int_0^1 (1-\sigma)F''((1-\sigma)u_{k-1} + \sigma u_k)\cdot(u_k - u_{k-1})^2 \right|_s \Big] \\
&\qquad \le M^2 q_k^a \left\{ |F'(u_{k-1})(I - S_{k-1})L(u_{k-1})F(u_{k-1})|_s + M q_k^{-2a\mu} \right\}.
\end{aligned}$$

Now, using (S3) and (F2), we obtain

$$\|L(u_{k-1})F(u_{k-1})\|_{s+b} \le M(1 + \|u_{k-1}\|_{s+b+a}) \le M q_{k-1}^{a\nu},$$

while (F1) and (S3) imply

$$|F'(u_{k-1})(I - S_{k-1})v|_s \le M^2 q_{k-1}^{-b} \|v\|_{s+b}.$$

Therefore

$$\|u_{k+1} - u_k\|_s \le M^5 q_k^a q_{k-1}^{a\nu - b} + M^3 q_k^{a-2\mu}.$$

Since $M \ge 1$, it suffices to have

$$M^5 (q_k^a q_{k-1}^{a\nu-b} + q_k^{a(1-2\mu)}) \le q_{k+1}^{-a\mu} \tag{13.6}$$

to ensure the first estimate in (13.5).

As for the second estimate, we have

$$\begin{aligned} 1 + \|u_{k+1}\|_{s+b+a} &\le 1 + \sum_{j=0}^{k} \|S_j L(u_j) F(u_j)\|_{s+b+a} \\ &\le 1 + M \sum_j q_j^a \|L(u_j) F(u_j)\|_{s+b} \\ &\le 1 + M^2 \sum_j q_j^a (1 + \|u_j\|_{s+b+a}) \\ &\le 1 + M^2 \sum_j q_j^{a(1+\nu)}. \end{aligned}$$

We therefore need

$$1 + M^2 \sum_{j=1}^{k} q_j^{a(1+\nu)} \le q_{k+1}^{a\nu}. \tag{13.7}$$

This amounts to requiring, since $q_{k+1} \ge q_{j+1}$,

$$1 \ge q_{k+1}^{-a\nu}(1 + M^2 e^{\lambda a(1+\nu)}) + M^2 \sum_{j=2}^{k} e^{\lambda \rho^j a[(1+\nu) - \rho\nu]},$$

which, since $\rho = \frac{3}{2}$, holds if $\nu > 2$ and λ is large enough. As for (13.6), it suffices to require

$$\lambda \rho^{k-1}(a\nu - b + a\rho + a\mu\rho^2) \le -5 \ln M - \ln 2$$

and

$$\lambda \rho^k (a(1 - 2\mu) + a\mu\rho) \le -5 \ln M - \ln 2.$$

These hold for λ large as well, provided that

$$\mu > 2 \quad \text{and} \quad b \ge a\nu + 3a/2 + 9a\mu/4.$$

This estimate also ensures that $\|u_k\|_s$ remains less than $R/2$, so that the iterations are well defined.

To summarize, we need to choose $\mu > 2$, $\nu > 2$, and $b > a\nu + 3a/2 + 9a\mu/4$. This is possible if b is greater than $2a + 3a/2 + 18a/4 = 8a$, which is the case by assumption.

To start the induction, consider $u_1 = -S_0 L(0)F(0)$. Since from (S2),

$$\|S_0 L(0)F(0)\|_{s+a+b} \le M q_0^a \|L(0)F(0)\|_{s+b} \le M^2 q_0^a,$$

we require

$$M^2 q_0^a \le q_1^{\nu a}. \tag{13.8}$$

For the first part of (13.5), we write

$$\|u_1\|_s \le M q_0^a \|L(0)F(0)\|_{s+a} \le M^2 q_0^a |F(0)|_s,$$

which leads to the condition

$$|F(0)|_s \le q_0^{-a} q_1^{-\mu a}/M^2. \tag{13.9}$$

We therefore choose λ large enough to satisfy (13.8), and then check the smallness condition on $F(0)$.

If $F(0)$ is small enough, the iterations remain in a small neighborhood of 0 (and are therefore well defined), and converge in X^s norm. It follows from the continuity of F and the existence of a uniform bound on $L(u)$ that $L(u_k)F(u_k)$ converges in X^{s-a}, and since the smoothing operators approximate the identity, we may write

$$\|(I - S_k)L(u_k)F(u_k)\|_{s-a-1} \le C q_k^{-1} \to 0$$

as $k \to \infty$. We conclude that $u_\infty = \lim_{k\to\infty} u_k$ solves

$$L(u_\infty)F(u_\infty) = 0,$$

in the space X^{s-a-1}. Applying $F'(u_\infty)$, we conclude that

$$F(u_\infty) = 0,$$

QED. □

Solutions

This section gives answers to, or detailed hints for the solution of, selected problems. Most of these are further prototypes of reduction; accordingly, comments on the general rules for reduction suggested by these problems are included. In some cases, further details or references on the context of these problems are also given.

Chapter 2

2.6 Let $u = 1 + tv$. We obtain

$$(t\partial_t - 1)v = t(1 - 3v^2 g(t, tv)),$$

since $u^{1/3} = 1 + tv/3 + t^2v^2 g(t, tv)$. For $t = 0$, $v(0)$ must be zero. However, it is easy to check that there is no solution that is analytic in t. In fact, if we let $v = t\tilde{v}$, the equation takes the form $t\partial_t \tilde{v} = 1 + \mathcal{O}(t)$, which suggests that the leading order for v is $t_1 := t \ln t$. Write $v = t_1 + z$. We obtain

$$(t\partial_t - 1)z = -3tg(t, tt_1 + tz)(z^2 + 2t_1 z + t_1^2).$$

Introduce two new variables and *two* unknowns w_0 and w_1 by the ansatz

$$z = w_0 t_0 + w_1 t_1,$$

where $t_0 = t$ and $t_1 = t \ln t$. We have

$$t\partial_t = N := t_0 \partial/\partial t_0 + (t_0 + t_1)\partial/\partial t_1.$$

The equation for z can therefore be written in the form

$$t_0 E_0 + t_1 E_1 = 0,$$

where

$$E_0 = Nw_0 + w_1 + 3t_0 g[w_0^2 t_0 + 2w_0(w_1+1)t_1]$$

and

$$E_1 = Nw_1 + 3t_0 t_1 g(w_1+1)^2.$$

Therefore, in order to find z, it *suffices* to solve the generalized Fuchsian system

$$E_0 = 0,$$
$$E_1 = 0.$$

This splitting is not unique. For instance, $2t_0^2 t_1$ may be split as $t_0(2t_0t_1)$, or as $t_0(t_0t_1) + t_1(t_0)^2$.

We have therefore achieved the reduction to a Fuchsian system with one logarithmic variable, and the general results of Chap. 4 ensure that there are convergent series solutions of this system in which the value of w_0 for $t_0 = t_1 = 0$ can be prescribed arbitrarily (indeed, the system forces $w_1 = 0$, but leaves w_0 undetermined). In terms of the original equation, this means that there are solutions of the form

$$u = 1 + t\ln t + [a_2t^2 + b_2t^2\ln t + c_2(t\ln t)^2] + [a_3t^3 + \cdots] + \cdots,$$

in which all the coefficients are determined in terms of a_2; the series converges when t and $t\ln t$ are sufficiently small. The arbitrariness of the coefficient of t^2 corresponds to the fact that 2 is a resonance.

2.7 A convenient set of sufficient conditions, which covers most applications, is (i) $A(T, \boldsymbol{u}) = A(0, \boldsymbol{u}_0) + T^\sigma G[T, \boldsymbol{u}] + H[\boldsymbol{u}]$, where $H[\boldsymbol{u}]$ has no constant term for $T = 0$, (ii) $A(0, \boldsymbol{u}_0)\boldsymbol{u}_0 + H[\boldsymbol{u}_0] = 0$, and (iii) identities of the form

$$H[\boldsymbol{u}_0 + T^\varepsilon v] = H_0 + T^{\eta_1} K_1[T, \boldsymbol{v}]$$

and

$$F[T, \boldsymbol{u}_0 + T^\varepsilon v] = T^{\eta_2} K_2[T, \boldsymbol{v}].$$

It may be necessary to enlarge the set of formal series to include T^ε, T^σ, T^{η_1}, and T^{η_2}. Substitution yields

$$(D+A)\boldsymbol{u} = A(0, \boldsymbol{u}_0)\boldsymbol{u}_0 + T^\varepsilon(D + A(0, \boldsymbol{u}_0) + \varepsilon)\boldsymbol{v} + T^\sigma G[T, \boldsymbol{u}] + H_0 + T^{\eta_1} K_1[T, \boldsymbol{v}].$$

Dividing through by T^ε, we obtain a Fuchsian system of the form

$$(D + A(0, \boldsymbol{u}_0))\boldsymbol{v} = \tilde{F}[T, \boldsymbol{v}],$$

provided that $\varepsilon < \min(\sigma, \eta_1, \eta_2)$. Of course, the space of formal series must again be extended to contain powers such as $T^{\eta_1 - \varepsilon}$ if need be.

Chapter 3

3.2 (b) *Hint:* Seeking solutions of the form $u = \sum_{j \geq 0} u_j x^j$, one is led to a recurrence of the form $j(j-2)(j-9)u_j = f_j(u_0, \ldots, u_{j-1})$.

3.4 In all cases, the solution has leading order $u \sim u_0 T^\nu$, where $T = t - a$, with a constant. Resonances are those values of r for which the coefficient of $T^{\nu+r}$ in the expansion of u is arbitrary. In each case, we give ν, u_0, and the resonances. All solutions with the same value of u_0 are said to belong to the same branch. The first reduced equation is the equation for the renormalized unknown w defined by $u = u_0 T^\nu(1 + T^\varepsilon w)$, where ε is positive, but less than the least positive exponent in the expansion of $1 + T^\epsilon w$ (or the minimum of the real parts of the resonances with positive real parts whichever is least). Similarly, the second reduced equation is the equation satisfied by z, defined by $u = u_0 T^\nu(1 + \cdots + T^m w)$, where m is greater than the largest positive resonance (or the maximum of the real parts of all resonances), and the dots stand for the formal solution up to order $m + 1$, in which all the arbitrary constants have been incorporated.

1: $u = -1/T + v$ leads to $(D+2)v = Tv^2$; the only resonance is -1.

2: $\nu = -1$, $u_0 = 1$ or 2. The transformation $u = v_t/v$ leads to $v_{ttt} = 0$. The general solution is therefore $u = (t-a)^{-1} + (t-b)^{-1}$, where a and b are arbitrary; The second branch corresponds to the case $a = b$, and is therefore a limiting case of solutions of the first branch.

3: It is convenient to work with $v = \exp u$. One finds three leading behaviors: $v \sim 2/T^2$, $v \sim v_0$ with $v_0 \neq 0$ or $v \sim \pm T$. In each case, one may compute a reduced equation with $\varepsilon = 1$.

5: $\nu = -2/3$, $u_0 = (10/9)^{1/3}$. This suggests using $s = T^{1/3}$ as independent variable. One obtains $ds/s = \frac{1}{3} dT/T$, hence $T\partial_T = \frac{1}{3} D_s$, where $D_s = s\partial_s$. If $u = u_0 s^{-2}\left(v_f + ws^{11}\right)$, where v_f is a formal solution up to order 11, one obtains a second reduced equation of the form $(D_s+1)(D_s+14)w = sh[s, w]$. The coefficient of s^{10} in the expansion of $s^2 u$ is arbitrary.

6: $\nu = -1$, $r = -1$, $2 \pm i\sqrt{2}$.

7: $\nu = -2/p$, $r = -1$, $2 + 2/p$, $2 + 4/p$.

3.5 Proof of Theorem 3.18: Let y be a solution of (3.27). Let

$$y = \frac{u'}{2u}.$$

We find that u satisfies

$$u^3 u^{(4)} - 5u^2 u' u^{(3)} - \frac{3}{2} u^2 u''^2 + 12uu'^2 u'' - \frac{13}{2} u'^4 = 0. \tag{14.1}$$

If u is a solution, so is

$$(cx+d)^{-12} u\left(\frac{ax+b}{cx+d}\right).$$

Since e^x is an exact solution of this equation, we seek solutions of the form $e^x v(e^x)$. Note that $u = \exp(2bx)$ leads to $y = b$, that is, to constant solutions of (3.26).

We make the change of variables $z = e^x$, and let $D := zd/dz = d/dx$. This turns (14.1) into a Fuchsian equation for $v(z) = u(z)/z$:

$$v^3(D+1)^4 v - 5v^2(D+1)v(D+1)^3 v - \frac{3}{2}v^2(D+1)^2 v$$
$$+ 12v[(D+1)v]^2(D+1)^2 v - \tfrac{13}{2}[(D+1)v]^4 = 0.$$

Letting $v(z) = 1 + zw(z)$, we find that w satisfies an equation of the form

$$(D+1)^3 Dw = zG[z, w, Dw, D^2w, D^3w].$$

It follows that there is exactly one solution with $w(0) = a$, given by a convergent series in z near $z = 0$.

Coming back to x, we have obtained a solution of the desired form, given by a series of exponentials that converges at least for $\operatorname{Re} x < -\rho$ for some finite ρ. This completes the proof of the claims regarding the family of exponential solutions.

Next, consider a solution y with the real axis as natural boundary, to fix ideas. The transformation (3.28) generated by $x \mapsto \varepsilon x/(x - i\varepsilon)$, which maps the real axis to the circle (Γ_ε) of center $\varepsilon/2$ and radius ε, yields the one-parameter family of solutions

$$y(x;\varepsilon) = -\frac{6}{x - i\varepsilon} - \frac{i\varepsilon^2}{(x-i\varepsilon)^2} y\left(\frac{\varepsilon x}{x - i\varepsilon}\right),$$

which are defined *outside* (Γ_ε). As $\varepsilon \to 0$, the natural boundary shrinks to a point, and the solutions $y(x;\varepsilon)$ converge, uniformly on any disk at positive distance from the origin, to the solution $-6/x$. However, the limits $\varepsilon \to 0$ and $x \to 0$ do not commute; in fact, $y(x;\varepsilon)$ is not defined in a full neighborhood of $x = 0$ for all small values of ε. This completes the proof of Theorem 3.18.

Proof of Theorem 3.19: Let y be a solution of (3.27). Let

$$y = \frac{k-6}{2}\frac{u'}{u}.$$

We find that u satisfies

$$uu^{(4)} - (k-2)u'u''' + \frac{3k(k-2)}{2(k+6)}u''^2 = 0. \tag{14.2}$$

If u is a solution, so is

$$(cx+d)^{12/(6-k)} u\left(\frac{ax+b}{cx+d}\right).$$

The first part of the theorem follows from general results on nonlinear Fuchsian equations. Let us seek y in the form

$$y(x) = x^{-1}(a + bz + w(z)z^2),$$

where $z = x^k$, $a = (k-6)/2$, and b is arbitrary. Letting $D = zd/dz$, we find, after substitution into the equation and some algebra, that w satisfies an equation of the form

$$(D+1)(k(D+2)+1)(k(D+2)-1)w = zF[z, w, Dw, D^2w].$$

It follows that there is precisely one solution of the form $y = h(x^k)/x$ if we specify $h(0) = (k-6)/2$ and $h'(0) = b$. This proves the first part of the theorem. If $b = 0$, we obtain $w \equiv 0$.

Let us now focus on $k = 2, 3, 4, 5$. In each case, there is a polynomial solution of (14.2), which generates the desired solutions using the SL(2) action [43]. In fact, we have

$$u = (x - a_1) \cdots (x - a_N)$$

and

$$y(x) = \frac{1}{2}(k-6)\sum_{j=1}^{N} \frac{1}{x - a_j} = \frac{(k-6)}{2x} \sum_{n \geq 0} \frac{\sum_j a_j^n}{x^n},$$

with $N = 1 + (k+6)/(6-k) = 12/(6-k)$. Note that u is analytic near $x = 0$ even when the a_j tend to zero. The relation between linearized solutions and possible confluence patterns is given by the following:

Lemma 14.1. *If we choose the pole locations such that* $a_j = \varepsilon^{1/m} b_j$*, where* $\sum_j b_j^q$ *vanishes for* $q < m$*, but is nonzero if* $q = m$*, then*

$$y(x;\varepsilon) = -\frac{6}{x} + \frac{\text{const}\;\varepsilon}{x^{1+m}}(1 + o(1)).$$

In other words, we have a resonance at $-m$. Since the possible pole locations are obtained by applying homographic transformations to the zeros of a fixed function, not all pole configurations are possible. The lemma follows by direct computation.

If all the poles are equal to zero, we recover $y = -6/x$; if they are all zero except for one that we let tend to infinity, we obtain the solution $-(k+6)/x$. If all poles but one are sent to infinity, we obtain the solution $(k-6)/x$. It is apparent that the first two solutions are unstable.

Let us now show that there cannot be any other type of confluence. Assume that there is a family of homographic transformations, depending on a parameter ε, under which two distinct poles a_1 and a_2 tend to zero while two other poles a_3 and a_4 remain fixed at nonzero (distinct) locations. We do not restrict the location of any additional poles. The anharmonic ratio of (a_1, a_2, a_3, a_4) tends to 1. But it is also independent of ε; it is therefore identically equal to 1. This implies that $a_1 = a_2$ for all ε: a contradiction. Therefore, if there is such a confluence, all poles except one at most must cluster at the same point.

Proof of Theorem 3.20: The stability statements have already been proved in the course of the proof of the previous theorem. Since the solution $(k-6)/2x$ has two positive resonances, namely 1 and k, we expect to be able to conclude using Theorem 3.21. There are in fact no logarithms in the pole expansion, but this does not follow from Theorem 3.18, which only generates a one-parameter solution corresponding to the resonance k: to generate the complete solution, we need to check that the resonance 1 is compatible. It is convenient to do so using the group action. More precisely, the solution $y = x^{-1}h(x^k)$ generates the solutions

$$-\frac{6\varepsilon}{1+\varepsilon x} + \frac{h(x^k/(1+\varepsilon x)^k)}{x(1+\varepsilon x)},$$

which contain the additional parameter ε. Adding the translation parameter, we obtain a three-parameter family to which Theorem 3.21 applies; this completes the proof.

3.6 (a) Let $v = u$, $v_0 = tu_t$, $v_1 = tu_x$ and $v_2 = tu_y$; (v, v_0, v_1, v_2) solves the system

$$\begin{aligned} t\partial_t v &= v_0, \\ t\partial_t v_0 &= \lambda v_0 + t\partial_x v_1 + t\partial_y v_2, \\ t\partial_t v_1 &= t\partial_x v + t\partial_x v_0, \\ t\partial_t v_2 &= t\partial_y v + t\partial_y v_0. \end{aligned}$$

Remark 14.2. It is essential that the space derivatives in the Fuchsian equation contain a factor of t. This concern is the main guide in the choice of variables in more complicated examples. For $\lambda = 0$, the EPD equation is satisfied by the mean of any function of x and y, over the circle of radius t about (x, y). Note that the most general *smooth* solution in this case is entirely determined by only one function, namely the value of u for $t = 0$, even though the equation is of second order. For the n-dimensional case, see [104].

(b) The method used for case (a) is not appropriate, for if we let $(v, v_0, v_1) = (u, tu_t, tu_x)$, we obtain the system

$$\begin{aligned} t\partial_t v &= v_0, \\ t\partial_t v_0 &= \partial_x v_1, \\ t\partial_t v_1 &= v_1 + t\partial_x v_0, \end{aligned}$$

in which the term $\partial_x v_1$ does not have a factor of t. We obviate this problem by letting $t = s^2$. The original equation then becomes

$$(s\partial_s)^2 u - 4s^2 u_{xx} = 0;$$

expanding and dividing through by s^2, we recover the Euler–Poisson–Darboux equation, up to a factor of 4 which may be scaled away.

3.7 By leading-order analysis, we expect the leading terms to be proportional to $1/t$; We therefore let $x = t^{-1}X$ and $y = t^{-1}Y$. This leads to

$$(D-1)X = X(ta + cY),$$
$$(D-1)Y = Y(tb + dX),$$

where $D = td/dt$. For $t = 0$, we find that $(X, Y) = -(1/d, 1/c)$. We therefore try $X = -1/d + tu(t)$, $Y = -1/c + tv(t)$, which leads to the system

$$DU + AU = t\begin{pmatrix}(a+cv)u\\(b+du)v\end{pmatrix} + X_0,$$

where

$$U = \begin{pmatrix}u\\v\end{pmatrix}, \quad X_0 = \begin{pmatrix}-a/d\\-b/c\end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix}1 & c/d\\d/c & 1\end{pmatrix}.$$

Unfortunately, this system has no solution which remains bounded as $t \to 0$ if $X_0 \notin \mathrm{RanA}$. Therefore, let us replace U by $U + U_0 + U_1 \ln t$, where U_0 and U_1 are constant vectors to be determined presently. The equation for U becomes

$$DU + AU + AU_0 + U_1 - X_0 + AU_1 \ln t = F,$$

where every term in the components of F has a factor t, t $\ln t$, or $t(\ln t)^2$. The eigenvalues of A are 0 and 2. We now solve the system

$$AU_0 + U_1 = X_0,$$
$$AU_1 = 0.$$

To see that there always is a solution, it suffices to consider the case that X_0 is an eigenvector of A, since any vector is a linear combination of eigenvectors. If $AX_0 = 2X_0$, we take $U_1 = 0$ and $U_0 = X_0/2$. If $AX_0 = 0$, we take $U_0 = 0$ and $U_1 = X_0$. Note that U_0 is determined up to the addition of an element in the null space of A: this introduces one free parameter into the solution. The second parameter comes from the translations in time.

We have therefore reduced the problem to a generalized Fuchsian system $(D + A)U = F$ involving three time variables. The existence and convergence of expansions with logarithms now follows.

Remark 14.3. The Lotka–Volterra system was given in [129] as an example of a system that one would like to call "integrable," because it has a (transcendental) first integral, but which does not pass the so-called Painlevé test, which requires singular solutions to have expansions free of logarithmic terms; see the discussion in Sect. 10.5. A similar reduction argument would also apply to the Lorenz system, of which the formal theory was considered in detail in [132]. More general quadratic systems have been extensively studied in recent years.

3.8 The proof is by inspection. One first finds, from the second reduction, that $3\gamma R_1 + \alpha = 0$, where

$$\alpha(X) = v^{(1)} \Delta\psi - \Delta v^{(0)} + 2\psi^i \partial_i v^{(1)} - (1 - |\nabla\psi|^2)[v^{(1)}]^2.$$

We wish to prove that

$$\alpha(X) = 2R,$$

where R is the scalar curvature of Σ. Using the expressions for $v^{(0)}$ and $v^{(1)}$, we find, writing $\psi_i\psi^i$ for $|\nabla\psi|^2$, that[1]

$$\begin{aligned}
\alpha &= -2(\Delta\psi)^2/(1-\psi_i\psi^i) + \partial_j \left[\frac{2\psi^{jk}\psi_k}{(1-\psi_i\psi^i)}\right] - 2\psi^j\partial_j\left[\Delta\psi/(1-\psi_i\psi^i)\right] \\
&= (1-\psi_i\psi^i)^{-2}\left\{4\psi^{jk}\psi_k\psi_{ji}\psi^i - 4\psi^j\psi_{ij}\psi^i\Delta\psi\right\} \\
&+ (1-\psi_i\psi^i)^{-1}\left\{-2\psi^j{\psi^k}_{jk} + 2{\psi^{jk}}_j\psi_k + 2\psi^{jk}\psi_{jk} - 2(\Delta\psi)^2\right\} \\
&= 4(1-\psi^i\psi_i)^{-2}\psi^{jk}\psi_k(\psi_{jl}\psi^l - \psi_j\Delta\psi) \\
&\quad - 2(1-\psi^i\psi_i)^{-1}[(\Delta\psi)^2 - \psi^{jk}\psi_{jk}].
\end{aligned}$$

Let us now relate this quantity to the scalar curvature. The induced metric on Σ is

$$g_{ij} = \delta_{ij} - \psi_i\psi_j.$$

The Christoffel symbols are

$$\begin{aligned}
\Gamma^k_{ij} &= (1/2)g^{km}\{\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\} \\
&= (1/2)\left(\delta^{km} + \psi^k\psi^m/(1-|\nabla\psi|^2)\right) \\
&\quad \{-\partial_i(\psi_j\psi_m) - \partial_j(\psi_i\psi_m) + \partial_m(\psi_i\psi_j)\} \\
&= (1/2)\left(\delta^{km} + \psi^k\psi^m/(1-|\nabla\psi|^2)\right)\{-2\psi_m\psi_{ij}\} \\
&= -\left\{\psi^k\psi_{ij} + \psi^k\psi_{ij}\psi^m\psi_m/(1-|\nabla\psi|^2)\right\} \\
&= -\psi^k\psi_{ij}/(1-|\nabla\psi|^2).
\end{aligned}$$

It follows that for any covariant field ω_j,

$$\nabla_i\omega_j = \partial_i\omega_j + \psi^k\omega_k\psi_{ij}/(1-|\nabla\psi|^2).$$

We compute the curvature tensor from the relation

$$(\nabla_i\nabla_j\omega_k - \nabla_j\nabla_i\omega_k) = {R_{ijk}}^l\omega_l.$$

[1] Recall that the summation convention is used throughout, and that indices are raised and lowered using the Kronecker δ.

First,

$$\begin{aligned}\nabla_i\nabla_j\omega_k &= \partial_i(\nabla_j\omega_k) - \Gamma^l_{ij}\nabla_l\omega_k - \Gamma^l_{ik}\nabla_j\omega_l\\ &= \partial_i\left\{\partial_j\omega_k + \psi_{jk}\psi^h\omega_h/(1-\psi^m\psi_m)\right\}\\ &\quad+[\partial_l\omega_k + \psi_{lk}\psi^h\omega_h/(1-\psi^m\psi_m)]\psi^l\psi_{ij}/(1-\psi^m\psi_m)\\ &\quad+[\partial_j\omega_l + \psi_{jl}\psi^h\omega_h/(1-\psi^m\psi_m)]\psi^l\psi_{ik}/(1-\psi^m\psi_m).\end{aligned}$$

Since we know that all the derivatives of ω must cancel in the final result, we have, letting $M = 1-\psi^m\psi_m$,

$$\begin{aligned}R_{ijk}{}^l\omega_l &= \partial_i(\psi_{jk}\psi^l/M)\omega_l - \partial_j(\psi_{ik}\psi^l/M)\omega_l\\ &\quad+\psi^l\psi_{ik}\psi_{jl}\psi^\rho\omega_\rho/M^2 - \psi^l\psi_{jk}\psi_{il}\psi^\rho\omega_\rho/M^2,\end{aligned}$$

and therefore,

$$R_{ijk}{}^l = \partial_i[\psi_{jk}\psi^l/M] - \partial_j[\psi_{ik}\psi^l/M] + (\psi_{ik}\psi_{j\rho}\psi^\rho\psi^l - \psi_{jk}\psi^\rho\psi_{i\rho}\psi^l)/M^2.$$

The derivatives of M give rise to -2 times the last two terms. The net result is

$$R_{ijk}{}^l = [\psi_{jk}\psi_i{}^l - \psi_{ik}\psi_j{}^l]/M + [\psi_{jk}\psi^\rho\psi_{i\rho}\psi^l - \psi_{ik}\psi_{j\rho}\psi^\rho\psi^l]/M^2.$$

Therefore,

$$R_{ik} = [\psi_i{}^l\psi_{lk} - \psi_{ik}\Delta\psi]/M + [(\psi_{lk}\psi^l)(\psi_{\rho i}\psi^\rho) - \psi_{ik}\psi_{l\rho}\psi^\rho\psi^l]/M^2.$$

Finally, the scalar curvature is equal to

$$\begin{aligned}R &= R_{ik}(\delta^{ik} + \psi^i\psi^k/M)\\ &= [\psi^{il}\psi_{il} - (\Delta\psi)^2]/M + [(\psi_{lk}\psi^l)(\psi_{kl}\psi^k) - \psi^i\psi^k\psi_{ik}\Delta\psi]/M^2\\ &\quad+[(\psi_{li}\psi^l)(\psi^{\rho i}\psi_\rho) - (\Delta\psi)\psi_{l\rho}\psi^l\psi^\rho]/M^2\\ &\quad+\left\{(\psi_{lk}\psi^l\psi^k)(\psi_{i\rho}\psi^i\psi^\rho) - (\psi_{ik}\psi^i\psi^k)(\psi_{l\rho}\psi^l\psi^\rho)\right\}/M^3\\ &= [\psi^{il}\psi_{il} - (\Delta\psi)^2]/M + 2\left\{(\psi^\rho\psi_{i\rho})(\psi_\sigma\psi^{i\sigma}) - \psi^\rho\psi^\sigma\psi_{\rho\sigma}\Delta\psi\right\}/M^2.\end{aligned}$$

The relation $\alpha = 2R$ follows.

Chapter 4

4.2 Such problems arise as soon as one mixes different powers in the expansions. To take a simple example, if u is expected to have an expansion in

$t_0 = T$, $t_1 = T\ln T$, and $t_2 = T^{\sqrt{2}}$, one should write $u = u(x, t_0, t_1, t_2)$, and $T\partial_T u = Nu$ with

$$Nu = t_0\partial_{t_0} + (t_0 + t_1)\partial_{t_1} + \sqrt{2}t_2\partial_{t_2}.$$

4.3 The proofs of (a) and (b) will be carried out together, in three steps: (1) multiply v by a suitable function to make the leading part of a vanish; (2) make a change of variables to force $b_{-1} = b_{\varepsilon-1} = 0$; (3) multiply the unknown by a second function to remove b_{-2}.

Step 1: First change of unknown

If $v(y) = e^{\psi(y)}z(y)$, with ψ satisfying

$$2D\psi + \frac{a_{-1}}{y} + a_0 + a_\varepsilon y^\varepsilon = 0,$$

the equation satisfied by z becomes

$$D^2 z + A(y)Dz + B(y)z = 0,$$

with

$$A(y) = y^{\varepsilon+\nu}\alpha(y), \quad B(y) = \gamma_{-2}y^{-2} + \gamma_{-1}y^{-1} + \gamma_{\varepsilon-1}y^{\varepsilon-1} + \gamma_0 + \; y^\delta\gamma(y),$$

$$\gamma_{-2} = b_{-2} - \frac{a_{-1}^2}{4}, \qquad \gamma_{-1} = b_{-1} + \frac{a_{-1}}{2}(1 - a_0),$$
$$\gamma_{\varepsilon-1} = b_{\varepsilon-1} - \frac{a_{-1}a_\varepsilon}{2}, \qquad \gamma_0 = \left(b_0 - \frac{a_0^2}{4}\right),$$

where $\delta = \min(\varepsilon, \nu+\varepsilon-1, \sigma) \in (0, \varepsilon]$, and the function γ is continuous near 0. We take $\psi_1 = \psi$. If $\gamma_{-1} = \gamma_{\varepsilon-1} = 0$, we may go directly to Step 3. Otherwise, we must modify these coefficients by a change of variables.

Step 2: Change of variables

Let $w(s) = z(\theta(s))$, where θ is invertible near $s = 0$. Then w is a solution of

$$D_s^2 w(s) + A_1(s)D_s w(s) + B_1(s)w(s) = 0,$$

where

$$A_1(s) = (1 + A(\theta(s)))\frac{D_s\theta}{\theta} - \frac{D_s^2\theta}{D_s\theta},$$
$$B_1(s) = \left(\frac{D_s\theta}{\theta}\right)^2 B(\theta(s)).$$

Lemma 14.4. *One can choose θ such that*

$$\begin{aligned} A_1(s) &= \mathcal{O}(s^\mu), \\ B_1(s) &= \gamma_{-2}/s^2 + \gamma_0 + \mathcal{O}(s^\mu), \\ \theta(s) &= s + as^2\ln(s) + \mathcal{O}(s^{2+\mu}), \end{aligned}$$

with $\mu > 0$ and $a = -\gamma_{-1}/(2\gamma_{-2})$. The asymptotics obtained by applying D_s and D_s^2 to the equation for θ are valid.

Proof. We will find $\theta \sim s$ such that

$$\left(\frac{D_s\theta}{\theta}\right)^2 (B(\theta(s)) - \gamma_0 - \theta(s)^\delta\gamma(\theta(s))) = \gamma_{-2}s^{-2} + 0(r(s)),$$

where $r(s) = s^\mu \ln s$. Inserting a formal asymptotic expansion, one finds that θ should have the form

$$\theta = s + as^2\ln s + \frac{b}{\varepsilon}s^{2+\varepsilon} + s^2\theta_1(s),$$

where $b = -\gamma_{\varepsilon-1}/(2\gamma_{-2})$, and θ_1 should tend to zero as $s \to 0$. We find that it suffices to have θ_1 solve

$$D_s\theta_1 = s(a\ln(s) + \theta_1(s))^2 + 3as(a\ln(s) + \theta_1(s)) + 3a^2s/2.$$

Now Theorem 5.3 applies to this equation, with $M = 1$ and, for instance, $m = 1$, $\zeta = \psi = s^{1/2}$. We conclude that there is a unique solution that is $\mathcal{O}(s^{1/2})$. The equation shows that we have in fact $\theta_1 = \mathcal{O}(s(\ln s)^2)$. The equation may also be differentiated, showing that $D\theta_1$ and $D^2\theta_1$ have a similar behavior. It follows that

$$D_s\theta(s)/\theta(s) = 1 + as\ln(s) + o(s\ln(s))$$

and

$$D_s^2\theta(s)/D_s\theta(s) = 1 + 2as\ln(s) + o(s\ln(s)),$$

whence $A_1 = 0(s^\mu)$ for μ small. Furthermore, $B_1(s)$ differs from $\gamma_{-2}s^{-2}$ by

$$(s\theta'/\theta(s))^2 (\gamma_0 + \theta(s)^\delta\gamma(\theta(s))) + \mathcal{O}(r(s)).$$

This expression behaves like

$$\begin{aligned} \gamma_0 + \mathcal{O}(s\ln(s)) &\text{ if } \delta = 1, \\ \gamma_0 + \mathcal{O}(s^\delta) &\text{ if } \delta < 1. \end{aligned}$$

The coefficient of $sw'(s)$ is equal to

$$\begin{aligned} &\left[1 - D_s^2\theta/D_s\theta(s) + (1 + A(\theta(s)))s\theta'/\theta(s)\right] \\ &= (\mathcal{O}(s\ln(s)) + (1 + \mathcal{O}(s^{\varepsilon+\nu}))(1 + \mathcal{O}(s\ln(s)) = 1 + \mathcal{O}(s\ln(s)), \end{aligned}$$

for $\varepsilon+\nu > 1$. Let μ further satisfy $\mu < \min(1, \varepsilon)$ and $\mu \leq \delta$. Then all (possible) residual terms

$$s^\varepsilon \ln(s), \quad s^\delta, \quad s\ln(s)$$

are $\mathcal{O}(s^\mu)$. The equation satisfied by w therefore has the form

$$s^2 w''(s) + (1 + \mathcal{O}(s^\mu))sw'(s) + \left(\gamma_{-2}s^{-2} + \gamma_0 + \mathcal{O}(s^\mu)\right) = 0.$$

We may in fact write $a_0(s)s^\mu$, $a_1(s)s^\mu$, with a_0, a_1 continuous instead of $\mathcal{O}(s^\mu)$. □

Remark 14.5. By letting $T = s^{1/3}$ and $Y = s^{1/3}\ln s$, one could reduce the search for θ_1 to a generalized Fuchsian PDE to which Theorem 4.3 applies with $\ell = 1$, yielding θ_1 in the form of a holomorphic function of T and Y near $(0, 0)$. One could also seek an expansion in powers of $(s, s\ln s, s(\ln s)^2)$ and use Theorem 4.3 with $\ell = 2$.

Step 3: Second change of unknown

The function w solves

$$D^2 w(s) + (s^\mu a_1(s))Dw(s) + \left(\gamma_{-2}s^{-2} + \gamma_0 + s^\mu a_0(s)\right) w(s) = 0.$$

Let

$$w(s) = e^{\psi_2(s)} f(s),$$

where $\psi_2 = -\gamma/s + \frac{1}{2}\ln s$. Then the function f satisfies

$$D^2 f + [2\gamma s^{-1} + 1 + s^\mu a_1(s)]Df + [(\gamma^2 + \gamma_{-2})s^{-2} + s^{\mu-1}a_3(s)]f = 0,$$

with

$$a_3(s) = \left(\gamma_0 + \frac{1}{4}\right)s^{1-\mu} + \left(a_0 + \frac{1}{2}a_1\right)s + \gamma a_1.$$

Let us choose γ such that $\gamma^2 + \gamma_{-2} = 0$, $\mathrm{Re}(\gamma) \geq 0$. Letting $\alpha_0 = a_3$, $\alpha_1 = a_1$, we obtain the desired simplified form of the equation. We now turn to questions (c) and (d) of Problem 4.2.

(c) Let $u(x) = v(y)$, $y = 1/\sinh(x)$; v satisfies

$$D^2 v + \frac{y^2}{1+y^2}Dv + \frac{\lambda}{1+y^2}v - \frac{c^2}{y^2(1+y^2)}v = 0.$$

It has the form studied above with

$$a_0 = a_{-1} = a_\varepsilon = 0, \qquad \nu = 1, \quad \sigma = 2,$$
$$b_{-2} = -c^2, \quad b_{-1} = 0, \qquad b_0 = c^2 + \lambda.$$

It follows that

$$\gamma = c \text{ and } a = a_1 = 0.$$

Therefore, there exists only one solution v such that

$$v(y) = y^{1/2} e^{-c/y} [1 + h],$$

with h tending to zero. A similar method yields an expansion to all orders, in terms of $s = 1/y$:

$$v(s) = s^{1/2} e^{-a/s} \left(1 + \sum_1^\infty A_k s^{2k} \right).$$

Remark 14.6. This is the first example used by Jeffreys [92] in one of his papers on what we now call WKBJ expansions [151]. By reducing the irregular singularity at infinity to a regular singularity, we obtain directly the sum associated with the (divergent) WKBJ expansion for the eigenfunction with fastest decay.

(d) It suffices to let $\gamma = -ik$, $a_1 = 0$, $a = \frac{c}{2}\lambda$.

Remark 14.7. This example is motivated in part by the recent interest in scattering theory on curved space-time, where the Regge–Wheeler coordinate on Schwarzschild space plays a role. This coordinate has the form $r + a \ln r$ for large r. If we consider radial solutions of

$$\Delta u + V(r)u + k^2 u = 0,$$

where the spectral parameter k is fixed, it is well known that there are solutions that behave like e^{ikr}/r at infinity if V has sufficient fast decay, but not for $V = c/r$ with c constant (Coulomb potential): the phase must be corrected by a logarithmic term. This is interpreted as meaning that the influence of the potential may be felt "even at infinity," hence "long-range" behavior. This problem shows that equations with long-range potentials may be reduced to short-range problems, in which coefficients have better decay, by introducing a new radial coordinate. But unlike the Regge–Wheeler coordinate, this coordinate cannot be independent of k, or otherwise it would lead to a unitary equivalence of the two types of operators, in contradiction to their known properties. The specific potential of the form $a/r + b/r^{1+\varepsilon}$ has been proposed [130] as effective interaction potential to represent clusters of charged particles.

4.4 Consider $A = z^{-2} A_0 + A_1$, where

$$A_0 = \begin{pmatrix} 0 & 0 \\ -3/16 & 0 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The system $\boldsymbol{u}' = A\boldsymbol{u}$ has the fundamental system of solutions Sz^P, where

$$S = \begin{pmatrix} z & z \\ 1/4 & 3/4 \end{pmatrix}; \quad P = \begin{pmatrix} -3/4 & 0 \\ 0 & -1/4 \end{pmatrix}.$$

Chapter 5

5.2 It is easily verified that there is a constant C_A such that

$$\|\sigma^A\| \leq C_A(1 + |\ln \sigma|^{M-1}) \text{ for } 0 < \sigma < 1.$$

Then (H1) follows if $\alpha > 0$ and $\alpha > M - 1$. Next, let $s = t\sigma$, $0 < \sigma < 1$ and $\tau = |\ln(\sigma)||\ln(t)|^{-1}$. We obtain, if $M > 0$,

$$\begin{aligned} K(t) &\leq \int_0^t [1 + |\ln(s/t)|]^{M-1} \psi(s)\zeta(s)\zeta(t)^{-1}\, s^{-1} ds \\ &= \text{const} \int_0^\infty [1 + \tau|\ln(t)|]^{M-1} |\ln(t)|^{-a} [1 + \tau]^{-a-\alpha}\, |\ln(t)| d\tau. \end{aligned}$$

Estimating $[1 + \tau|\ln(t)|]^{M-1}$ by $[1 + \tau]^{M-1}[1 + |\ln(t)|]^{M-1}$, we obtain

$$K(t) \leq [1 + |\ln(t)|]^{M-1} |\ln(t)|^{1-a} \int_0^\infty [1 + \tau]^{M-1-a-\alpha}\, d\tau.$$

The integral converges if $a > M - \alpha$, and K tends to zero as $t \to 0$ if $a > M$. If $M = 0$, Hypothesis (H4) follows for $a > 0$ and $\alpha > 0$.

For (H2), it suffices to estimate an integral obtained from the above expression for K by replacing a by M. The desired properties follow, since $M > M - \alpha$. The case of power weights is treated similarly.

Chapter 8

8.2 Away from $t = 0$, system (8.16a–c) is of Cauchy–Kovalevskaya type. More precisely, we can reduce it to the following first-order system for $(z_0, z_1, z_2, \alpha, \nu) := (u, u_t, u_\theta, \alpha, \nu)$:

$$\begin{aligned} \partial_t z_0 &= z_1, \\ \partial_t z_1 &= \alpha \partial_\theta z_2 - \frac{z_1}{t} - \frac{m}{2t^3} z_1 e^{2\nu} + \frac{1}{2} z_2 \alpha_\theta, \\ \partial_t z_2 &= \partial_\theta z_1, \\ \partial_t \alpha &= -\frac{\alpha^2}{t^3} m e^{2\nu}, \\ \partial_t \nu &= t z_1^2 + t\alpha z_2^2 + \frac{\alpha}{4t^3} m e^{2\nu}. \end{aligned}$$

In particular, ignoring the constraint (8.16d), we obtain a unique solution of the remaining equations by prescribing the data $\{u, u_t, \alpha, \nu, \dots\}$ for $t = t_0$. Now let

$$N := \nu_\theta - 2u_\theta Du + \frac{\alpha_\theta}{2\alpha}.$$

Since

$$0 = D\nu_\theta - \partial_\theta D\nu = DN + D\left(2u_\theta Du - \frac{\alpha_\theta}{2\alpha}\right) - \partial_\theta D\nu,$$

we obtain, using (8.16a–c),

$$DN - \frac{1}{2\alpha} N D\alpha = 0. \tag{14.3}$$

This is a linear *ordinary* differential equation for N (there are no θ-derivatives). Hence if we choose data $\{u, u_t, \alpha, \nu, \dots\}$ for $t = t_0$ so that $N(t_0) = 0$, the uniqueness theorem for ODEs guarantees that N is identically zero for all time. We therefore have solved the initial-value problem. The results of [17] ensure that the solution remains bounded for $t > \rho$, where $\rho \geq 0$ is independent of θ. It is expected that $\rho > 0$ in special cases only, such as exact Kasner space-times.

8.3 The strategy is as follows: We first treat α_θ as a new field variable $\zeta := \alpha_\theta$, and produce an evolution equation for ζ by differentiating (8.16b) with respect to θ. We then use (8.16b) to eliminate $D\alpha$ from (8.16a), and (8.16d) to express $\partial_\theta \nu$ in terms of the other field variables. This gives us a symmetric-hyperbolic system (14.4) for $(z_0, z_1, z_2, \alpha, \zeta, \nu)$. Standard theorems then ensure that (14.4) admits a unique solution, defined in a small time interval, for nonanalytic, but sufficiently smooth, initial data. We then show that the constraints $\zeta = \alpha_\theta$ and $N = 0$ do propagate, by a variant of the argument used for the propagation of the constraint $N = 0$. This will establish that we do obtain solutions to (8.16a–d) with nonanalytic initial data. We proceed with the details of this argument. The symmetric-hyperbolic system is

$$\partial_t z_0 = z_1, \tag{14.4a}$$

$$\partial_t z_1 = \alpha \partial_\theta z_2 - \frac{z_1}{t} - \frac{m}{2t^3} z_1 e^{2\nu} + \frac{1}{2} z_2 \zeta, \tag{14.4b}$$

$$\alpha \partial_t z_2 = \alpha \partial_\theta z_1, \tag{14.4c}$$

$$\partial_t \alpha = -\frac{\alpha^2}{t^3} m e^{2\nu}, \tag{14.4d}$$

$$\partial_t \nu = t z_1^2 + t\alpha z_2^2 + \frac{\alpha}{4t^3} m e^{2\nu}, \tag{14.4e}$$

$$\partial_t \zeta = -\frac{2m\alpha}{t^3} e^{2\nu} \left[\zeta + \alpha\left(2t z_1 z_2 - \frac{\zeta}{2\alpha}\right)\right]. \tag{14.4f}$$

This system is symmetric-hyperbolic, so that if we prescribe sufficiently smooth initial data $\{u, u_t, \alpha, \zeta, \nu\}$ for $t = t_0$, it has a unique solution. The first and third equations ensure $z_1 = \partial_t z_0$ and $\partial_t(z_2 - \partial_\theta z_0) = 0$ respectively; we may thus set $z_0 = u$, $z_1 = u_t$, and $z_2 = u_\theta$. Equations (8.16a–c) therefore hold, with α_θ replaced by ζ in (8.16a).

Now let

$$R := \zeta - \alpha_\theta \quad \text{and} \quad N' := \nu_\theta - 2u_\theta Du + \frac{\zeta}{2\alpha}. \tag{14.5}$$

We proceed to derive a first-order system of ODEs for R and N'. For the rest of this argument, we write N for N', for convenience. First of all, using equations (14.4d) and (14.4f),

$$\begin{aligned} DR &= D(\zeta - \alpha_\theta) \\ &= -\frac{2m\alpha}{t^2}e^{2\nu}\left[\zeta + \alpha\left(2u_\theta Du - \frac{\zeta}{2\alpha}\right)\right] - \partial_\theta\left(-\frac{\alpha^2}{t^2}me^{2\nu}\right) \\ &= -\frac{2m\alpha}{t^2}e^{2\nu}[R - \alpha N] \\ &= 2\frac{D\alpha}{\alpha}[R - \alpha N]. \end{aligned} \tag{14.6}$$

Using the expression for N from (14.5), taking the relation $\zeta = \alpha_\theta + R$ into account, we have

$$DN = (D\nu)_\theta - 2DuDu_\theta - 2u_\theta D^2u + D\left(\frac{\alpha_\theta + R}{2\alpha}\right),$$

or

$$DN - D\left(\frac{R}{2\alpha}\right) = (D\nu)_\theta - 2DuDu_\theta - 2u_\theta D^2u + D\left(\frac{\alpha_\theta}{2\alpha}\right).$$

From (14.4a,b,d) and the definition of R, we obtain

$$\begin{aligned} DN - D\left(\frac{R}{2\alpha}\right) &= \partial_\theta\left(-\frac{D\alpha}{4\alpha}\right) - \frac{D\alpha}{\alpha}u_\theta Du - t^2u_\theta^2R + D\left(\frac{\alpha_\theta}{2\alpha}\right) \\ &= -t^2u_\theta^2R + \partial_\theta\left(\frac{D\alpha}{4\alpha}\right) - \frac{D\alpha}{2\alpha}(2u_\theta Du). \end{aligned}$$

From (8.16b),

$$\left(\frac{D\alpha}{4\alpha}\right)_\theta = \frac{D\alpha}{2\alpha}\left(\nu_\theta + \frac{\alpha_\theta}{2\alpha}\right).$$

It follows that

$$DN - D\left(\frac{R}{2\alpha}\right) + t^2u_\theta^2R = \frac{D\alpha}{2\alpha}\left(N - \frac{R}{2\alpha}\right). \tag{14.7}$$

Combining (14.6) and (14.7), we have

$$\begin{aligned} DN &= N\frac{D\alpha}{2\alpha} + R\left(D\left(\frac{1}{2\alpha}\right) - t^2u_\theta^2\right) - \frac{RD\alpha}{4\alpha^2} + \frac{D\alpha}{\alpha^2}[R - \alpha N] \\ &= R\left[\frac{D\alpha}{4\alpha^2} - t^2u_\theta^2\right] - N\frac{D\alpha}{2\alpha}. \end{aligned} \tag{14.8}$$

Equations (14.6) and (14.8) constitute a linear homogeneous system of ODEs for R and N. Therefore, if the initial data are such that these quantities are zero for $t = t_0$, they remain so for all time, QED.

Chapter 9

9.1 Leading-order analysis suggests that $u \sim \ln(2d)$. We therefore define $v = \exp(-u)$, which solves

$$v\Delta v = |\nabla v|^2 - 4.$$

We then define the renormalized unknown w by $v = 2d + d^2 w$, where d is the distance to the boundary. One obtains

$$Lw + 2\Delta d = M_w(w),$$

where

$$L := d^2\Delta + 2d\nabla d \cdot \nabla - 2,$$

and M_w is a linear operator with w-dependent coefficients, defined by

$$M_w(f) := \frac{d^2}{2 + dw}[2f\nabla d \cdot \nabla w + d\nabla w \cdot \nabla f] - 2df\Delta d.$$

The assumptions on $\partial\Omega$ ensure that d is of class $C^{2+\alpha}$ near the boundary. We wish to prove that $2d + d^2 w$ is of class $C^{2+\alpha}$ near (and up to) the boundary.

Step I. One first proves, by a comparison argument combined with regularity estimates, that w and $d^2\nabla w$ are bounded near $\partial\Omega$; it follows that the operator $L - M_w$ is of type (I).

Step II. Since w and $(L - M_w)w$ are both bounded near $\partial\Omega$, Theorem 6.7 shows that $d\nabla w$ is bounded near $\partial\Omega$, so that $M_w(w) = \mathcal{O}(d)$ as $d \to 0$.

Step III. One finds w_0, defined near the boundary, such that

$$Lw_0 + 2\Delta d = 0 \tag{14.9}$$

and

$$d^k w_0 \in C^{k+\alpha} \text{ for } k = 0, 1, 2,$$

and proves that one can formally set $d = 0$ in equation (14.9), so that

$$w_0 = \Delta d/(n-1) = -H \,\text{on}\, \partial\Omega,$$

where H is the curvature of the boundary.

Step IV. Let $Z = w - w_0$. One proves, using comparison functions involving d, that $Z = \mathcal{O}(d\ln(1/d))$.

Step V. Since $Z = \mathcal{O}(d^\alpha)$ and $LZ = \mathcal{O}(d)$, one first gets, by the "type (I)" Theorem 6.8, that Z and $d\nabla Z$ are of class C^α. It follows, by inspection of the definition of $M_w(w)$, that LZ is in fact of class C^α near and up to the boundary.

Step VI. Since LZ, Z, and $d\nabla Z$ are C^α near and up to the boundary, Theorem 6.9 gives that $d^2 Z$ is of class $C^{2+\alpha}$. Since $w = w_0 + Z$, we find that $2d + d^2 w$ is of class $C^{2+\alpha}$ near the boundary, QED.

Chapter 10

10.1 We use the notation of the solution of Problem 3.4. We give the positive indices only; -1 is a resonance in all cases.

Caudrey–Dodd–Gibbon: $\nu = -2$. If $u_0 = -\phi_x^{-2}$, the indices are 2, 3, 6, and 10. If $u_0 = -2\phi_x^{-2}$, they are 5, 6, and 12.

Hirota–Satsuma: $u \sim u_0\phi^\mu$, $v \sim v_0\phi^\nu$, where $(\mu,\nu) = (-2,-1)$ or $(-2,-2)$.

Boussinesq: $\nu = -2$, indices 4, 5, and 6.

modified KdV: $\nu = -1$, indices 3 and 4.

KdV5: $\nu = -2$. If $u_0 = -2\phi_x^{-2}$, the indices are 2, 5, 6, and 8. If $u_0 = -6\phi_x^{-2}$, they are 6, 8, and 10; the other resonances are -1 and -3. The presence of the resonance 6 in both branches is natural: the linearization of the equation at u_0x^ν has the form $w_t = \partial_x(x^{-4}P(D)w)$, where $D = x\partial_x$. The resonances are the values of r for which there is a solution w with $w \sim w_0x^{\nu+r}$ with $w_0 \neq 0$. Now, since $\nu = -2$, $\partial_x(x^{-4}P(D)w_0x^{\nu+r}) = w_0\partial_x(x^{r-6}P(\nu+r)) = (r-6)w_0P(\nu+r)x^{r-7}$. The resonance equation is therefore divisible by $(r-6)$.

KP: $\nu = -2$, indices 4, 5, and 6.

10.5 This equation is well known in field theory, and is one of the most widely studied nonlinear perturbations of the wave equation. It derives from a fourth-order variational principle, whence the name. We are writing the equation in characteristic coordinates (x,t): this makes some of the computations slightly easier.

Let us seek solutions that become singular for $t = g(x)$, where $g' < 0$. We let $u = u(X,T)$, where $X = x$ and $T = t - g(x)$. We obtain

$$u_{TT} = a(X)[u_{XT} + u - u^3],$$

where $a = 1/g'$. Multiplying by T^2, this becomes

$$D(D-1)u = a[TDu_X + T^2(u - u^3)],$$

with $D = Td/dT$. Leading-order analysis suggests a first reduction with $\varepsilon = 1$: $u = T^{-1}(u_0 + Tv)$, where $u_0^2 = -2/a$. We obtain

$$(D+2)(D-3)v = -a\partial_X u_0 + aT[Dv_X + u_0(1-3v^2) + Tv(1-v^2)].$$

Replacing v by $v + a\partial_X u_0/6$, we can cancel the first term in the right-hand side. This equation is converted into a Fuchsian system for the unknowns $w_0 = v$, $w_1 = Dv$, and $w_2 = Tv_X$. The matrix A in the Fuchsian system has eigenvalues -2 and 3, corresponding to the resonances -1 and 4 of the equation for u. The general theory enables us to conclude that since the resonance 3 is simple, v is represented by a convergent series in T and $T\ln T$, and is entirely determined by the choice of the coefficient of T^3.

If one wishes to obtain explicitly a generalized Fuchsian system with a matrix A with nonnegative eigenvalues, one could use the following second reduction: Let

$$v = v_0 + v_1 t_0 + v_2 t_0^2 + z_0 t_0^3 + z_1 t_0^2 t_1 + z_2 t_0 t_1^2 + z_3 t_1^3,$$

where $t_0 = T$ and $t_1 = T \ln T$, and $z_0, \ldots, z_3$ are new renormalized unknowns. Substitute this series into the equation, and compute the coefficients v_0, v_1, v_2. The equation takes the form

$$E_0 t_0^3 + E_1 t_0^2 t_1 + E_2 t_0 t_1^2 + E_3 t_1^3 = 0.$$

This decomposition is not unique. However, as in Sect. 2.2.4, it is always possible to choose the decomposition so that the system

$$E_0 = E_1 = E_2 = E_3 = 0$$

is a generalized Fuchsian system for the z_k's. Its solution will generate a solution v of the original problem. The existence of an expansion for z (hence for v and u) to all orders and its convergence for small t follow from the general results.

Thus, it is not necessary to go beyond the computation of the second term in the expansion of u in order to conclude that there exists a convergent logarithmic series for the solution, because we can ascertain already at this level that (1) the equation can be cast in Fuchsian form; (2) there is only one resonance that is simple and greater than 1. In particular, there is no need to compute explicitly the compatibility (or "no-logarithm") condition for the resonance. If more detailed information on the formal properties of the series is available, it is of course possible to use it to simplify the reduction to Fuchsian form, using the general results from Sect. 2.2.4.

10.7 (b) One obtains

$$u_0 = \sqrt{1 - |\nabla\psi|^2}, \tag{14.10a}$$

$$u_1 = \frac{\psi^{ij}\psi_i\psi_j}{3(1-\psi^i\psi_i)^{3/2}} - \frac{\Delta\psi}{6\sqrt{1-\psi^i\psi_i}}, \tag{14.10b}$$

$$u_2 = -\frac{(\psi^{ij}\psi_i\psi_j)^2}{9(1-\psi^i\psi_i)^{7/2}} + \frac{2\,(\psi^{ij}\psi_i\psi_j)\Delta\psi + 3\,\psi^{ij}\psi_j\psi_i^k\psi_k}{18(1-\psi^i\psi_i)^{5/2}} + \frac{6\,\Delta\psi^i\psi_i + 6\,\psi^{ij}\psi_{ij} - (\Delta\psi)^2}{36(1-\psi^i\psi_i)^{3/2}}, \tag{14.10c}$$

$$u_3 = \frac{\lambda_5}{27(1-\psi^i\psi_i)^{11/2}} + \frac{\lambda_4}{9(1-\psi^i\psi_i)^{9/2}} + \frac{\lambda_3}{36(1-\psi^i\psi_i)^{7/2}} + \frac{\lambda_2}{108(1-\psi^i\psi_i)^{5/2}} + \frac{\lambda_1}{24(1-\psi^i\psi_i)^{3/2}}, \tag{14.10d}$$

where the coefficients λ_1, λ_2, λ_3, λ_4, and λ_5 are as follows:

$$\begin{aligned}
\lambda_1 =& \Delta^2\psi,\\
\lambda_2 =& -2\,(\Delta\psi)^3 + 18\,\Delta\psi(\psi^{ij}\psi_{ij}) + 15\,\Delta\psi(\Delta\psi^i\psi_i) - 18\,\psi^{ij}\psi_i^k\psi_{jk}\\
& - 36\,\psi^{ijk}\psi_i\psi_j\psi_k,\\
\lambda_3 =& 2\,(\Delta\psi)^2(\psi^{ij}\psi_i\psi_j) - 4\,(\psi^{ij}\psi_i\psi_j)(\Delta\psi^i\psi_i) + 13\,\Delta\psi(\psi^{ij}\psi_j\psi_i^k\psi_k)\\
& + 2\,(\psi^{ijk}\psi_i\psi_j\psi_k) - 4\,(\psi^{ij}\psi_i\psi_j)(\psi^{ij}\psi_{ij}) - 12\,\psi^{ijk}\psi_j\psi_k\psi_k^l\psi_l\\
& - 30\,\psi^{ij}\psi_i^k\psi_j\psi_k^l\psi_l,\\
\lambda_4 =& \Delta\psi(\psi^{ij}\psi_i\psi_j)^2 - 11\,(\psi^{ij}\psi_i\psi_j)(\psi^{ij}\psi_j\psi_i^k\psi_k)\\
& - (\psi^{ij}\psi_i\psi_j)(\psi^{ijk}\psi_i\psi_j\psi_k),\\
\lambda_5 =& -8\,(\psi^{ij}\psi_i\psi_j)^3.
\end{aligned}$$

(d) Let x be such that $\nabla\psi(x) = 0$. Then (14.10a–d) become

$$\begin{aligned}
u_0(x) =& 1\,,\\
u_1(x) =& -\frac{\Delta\psi}{6}\,,\\
u_2(x) =& -\frac{1}{36}(\Delta\psi)^2 + \frac{1}{6}\psi^{ij}\psi_{ij}\,,\\
u_3(x) =& -\frac{1}{54}(\Delta\psi)^3 + \frac{1}{6}\Delta\psi\psi^{ij}\psi_{ij} + \frac{1}{24}\Delta^2\psi - \frac{1}{6}\psi^{ij}\psi_j^k\psi_{ik},
\end{aligned}$$

while $u_{4,0}$ is arbitrary and

$$\begin{aligned}
u_{4,1}(x) =& -\frac{2}{27}(\Delta\psi)^4 - \frac{2}{3}\Delta\psi^{ij}\psi_{ij} + \frac{7}{9}(\Delta\psi)^2\psi^{ij}\psi_{ij}\\
& - \frac{8}{9}\Delta\psi\psi^{ij}\psi_j^k\psi_{ik} - \frac{1}{9}\Delta\psi^i\Delta\psi_i - \frac{2}{3}(\psi^{ij}\psi_{ij})^2\\
& + \frac{2}{9}\Delta\psi\Delta^2\psi - \frac{1}{3}\psi^{ijk}\psi_{ijk} - \frac{1}{3}\psi^{ij}\psi^{kl}\psi_{ik}\psi_{jl}.
\end{aligned}$$

In one space dimension,

$$\begin{aligned}
u_0(x) &= 1\,,\\
u_1(x) &= -\frac{H(x)}{6}\,,\\
u_2(x) &= \frac{5}{36}H^2(x)\,,\\
u_3(x) &= \frac{1}{24}\frac{\mathrm{d}^2H}{\mathrm{d}s^2}(x) - \frac{31}{216}H^3(x)\,,
\end{aligned}$$

while the condition $u_{4,1} = 0$ reads, whether $\nabla\psi(x)$ vanishes or not,

$$\frac{\mathrm{d}^2(H^2)}{\mathrm{d}s^2}(x) = \frac{2}{3}H^4(x),$$

where $H(x)$ is the curvature of Σ at the point x and s is arc length on Σ. In two space dimensions, these expressions are replaced by

$$u_0(x) = 1\,,$$
$$u_1(x) = -\frac{H(x)}{6}\,,$$
$$u_2(x) = \frac{5}{36}H^2(x) - \frac{1}{3}K(x)\,,$$
$$u_3(x) = \frac{1}{24}\Delta_g H(x) - \frac{31}{216}H^3(x) + \frac{1}{2}H(x)K(x)\,,$$

where x satisfies $\nabla\psi(x) = 0$. The equation $u_{4,1} = 0$ is always equivalent to

$$-\frac{37}{54}H^4 + \frac{23}{9}H^2K - \frac{4}{3}K^2 + \frac{1}{6}H\Delta_g H + \frac{1}{36}\Delta_g(H^2) + \frac{1}{6}\Delta_g^2\gamma = 0,$$

where $H(x)$ represents the mean curvature of Σ at x, $K(x)$ is the total (Gauss) curvature, $\gamma = 1 - |\nabla\psi(x)|^2$, and Δ_g stands for the Laplace–Beltrami operator on Σ, with respect to the metric induced on Σ by the Minkowski metric on $\mathbb{R}^2 \times \mathbb{R}$.

Remark 14.8. In higher dimensions, one finds for the no-logarithm condition

$$-\frac{37}{54}H^4 + \frac{23}{9}H^2K_2 - \frac{4}{3}K_2^2 - \frac{5}{3}HK_3 + \frac{1}{6}H\Delta_g H$$
$$+\frac{1}{36}\Delta_g(H^2) - \frac{4}{3}K_4 + \frac{1}{6}\Delta_g^2\gamma = 0,$$

where H is the mean curvature of Σ, while K_2, K_3, and K_4 are symmetric functions of the principal curvatures of the blowup surface.

10.10 The first task is to prove that such a solution has a global description valid over all of the curve $\xi + \eta = 0$. Indeed, Theorem 10.30 shows that the initial line $\xi = -\eta$ can be covered by countably many open intervals $\{I_k\}_{k=0,\pm1,\dots}$, each of which gives rise to a representation with functions f_k, g_k, which remain finite in the interval I_k. We may always assume that each interval overlaps its immediate neighbors only. Now, we know that there are homographies $r_{k+1,k}$ such that $f_{k+1} = r_{k+1,k}(f_k)$, and associated homographies acting on g_k. We can now define f globally by letting $f = f_0$ in I_0, then $r_{1,0}(f) = f_1$ in I_1, $r_{2,1}(r_{1,0}(f)) = f_2$ in I_2, etc., and similarly for $I_{-1},\dots$. This function f takes its values in $\mathbf{R}\cup\{\infty\}$, but its singularities can be removed locally by a homographic transformation. Once this is done, there will be homographies r_k such that $f_k = r_k(f)$ in I_k. This completes the first part of the proof.

Next, consider a given point where u is not regular. We may always assume that f and g are regular near this point, because any pole singularity can be removed as above. This means that one of the following happens: $f'(\xi) = 0$, $g'(\eta) = 0$, or $f(\xi) + g(\eta) = 0$. But the first two would generate a *line of* *characteristic singularities.* This line would have to cross the line $(\xi + \eta = 0)$, which contradicts the assumption that this line contains no singularity. Since $f'g' \neq 0$, we can solve the equation $f + g = 0$ using the implicit function theorem to obtain a relation $\xi = \psi(\eta)$. We therefore know that the singular set is given by an equation $\xi = \psi(\eta)$, where ψ is decreasing in every component of its domain. On the other hand, ψ cannot admit two vertical asymptotes, since it would have to decrease from $+\infty$ to $-\infty$ between them, and in the process create a forbidden singularity when its graph crosses the line $\xi + \eta = 0$. Therefore ψ is a decreasing function defined at most on two intervals of the form $(-\infty, a)$ and $(b, +\infty)$. This means that the blowup set has at most two components, which completes the proof.

10.11 Recall that characteristic singularities occur on lines parallel to the ξ and η axes, and correspond to places where f' or g' vanishes. We again consider the singularities near $\xi = \eta = 0$. The situation that two characteristic singularities cross, while the solution remains defined on both sides of the singular lines, corresponds generically to the case in which f' and g' vanish up to *second* order, and f' and g' remain nonnegative for ξ and η small. Therefore, we may write

$$f(\xi) = a_0 + a_3\xi^3 + \cdots ; \quad g(\eta) = b_0 + b_3\eta^3 + \cdots ,$$

with a_3 and b_3 positive. We now introduce a deformation parameter ε, and consider the solution u_ε generated by the functions

$$f_\varepsilon(\xi) = a_0 + \varepsilon a_1\xi + a_3\xi^3 + \cdots ; \quad g_\varepsilon(\eta) = b_0 + \varepsilon b_1\eta + b_3\eta^3 + \cdots ,$$

where $a_1 b_1 > 0$. We are ready to describe the singular set of u_ε near the origin. If a_0 or b_0 is nonzero, u_ε does not have any singularity at all near the origin, since $f_\varepsilon + g_\varepsilon \neq 0$. If $a_0 = b_0 = 0$, u_ε has a singularity on a curve that is noncharacteristic near the origin (on it, $\xi \sim -(b_1/a_1)\eta$). The second of the singularity data becomes singular in the limit of small ε, since $\{f_\varepsilon; \xi\} = 6a_3/a_1\varepsilon$ when $\xi = 0$.

Chapter 11

11.1 Let us first identify the leading-order term. Let $T = R - r$ and substitute $u \sim u_0 T^\sigma$. At leading order, the terms of significance in (11.14) are independent of n. Let us substitute into (11.14). We obtain

$$\sigma = \frac{p}{p - \alpha - 1} \tag{14.11}$$

and

$$u_0^{p-1}\sigma^{p-1}(\sigma - 1)(p - 1) = 1.$$

We now introduce a renormalized unknown v by writing

$$u = u_0 T^\sigma (1 + vT^\varepsilon), \tag{14.12}$$

where ε will be chosen small enough. Substitution of (14.12) into (11.14) leads, after a lengthy calculation, to a Fuchsian equation of the form

$$(D + \varepsilon + 1)(D + \varepsilon + \sigma(\alpha + 1))v = T^\varepsilon F(v, Dv, T), \tag{14.13}$$

where $D = T\partial_T$, provided that $\varepsilon < \frac{1}{2}$. For $F \equiv 0$, (14.13) has no solution that remains bounded as $T \to 0$, because (14.11) guarantees that $\sigma > 1$. It follows that (14.13) has precisely one solution that remains bounded as $T \to 0$.

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